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Generalized Hardy operators

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Abstract

Consider the operator on $L^2(\mathbb{R}^d)$ $\mathcal{L}_a = (-\Delta)^{\alpha/2} + a|x|^{-\alpha}$ with $0 < \alpha < \min\{2, d\}$. Under the condition $a \geqslant -\frac{2^\alpha \Gamma((d+\alpha)/4)^2}{\Gamma((d-\alpha)/4)^2}$ the operator is non negative and selfadjoint. We prove that fractional powers $\mathcal{L}_a^{s/2}$ for $s \in (0, 2]$ satisfy the estimates $\left\|\mathcal{L}_a^{s/2} f\right\|_{L^p} \lesssim \left\|(-\Delta)^{\alpha s/4} f\right\|_{L^p}$, $\left\|(-\Delta)^{s/2} f\right\|_{L^p} \lesssim \left\|\mathcal{L}_a^{\alpha s/4} f\right\|_{L^p}$ for suitable ranges of p. Our result fills the remaining gap in earlier results from Killip *et al* (2018 *Math. Z.* **288** 1273–98); Merz (2021 *Math. Z.* **299** 101–21); Frank *et al* (*Int. Math. Res. Not.* **2021** 2284–303). The method of proof is based on square function estimates for operators whose heat kernel has a weak decay.

Keywords: Fractional Laplacian, Hardy inequality, Hardy operator, heat kernel, square function

Mathematics Subject Classification numbers: 35A23, 46E3, 35K08, 42B20.

1. Introduction

A non negative selfadjoint differential operator $\mathcal{L}$ on $L^2(\mathbb{R}^n)$ generates in a natural way a scale of Sobolev type norms $\left\|\mathcal{L}^{s/2}u\right\|_{L^p}$. Establishing the equivalence of such norms with standard Bessel potential norms $\|u\|_{\dot{H}^r_p} = \left\|(-\Delta)^{r/2}u\right\|_{L^p}$ is an important problem, with many applications in spectral theory, linear and nonlinear dispersive equations, and probability. We mention for instance the equivalence $\left\|\mathcal{L}^{1/2}u\right\|_{L^2} \simeq \|u\|_{\dot{H}^1}$ for second order elliptic operators in divergence form, known for a long time as the Kato square root problem and now a theorem [2]; and the calculus of Schrödinger operators with electromagnetic potentials, which is an essential tool to investigate scattering for the corresponding time dependent problem [9, 10, 14, 15, 20, 25].

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In this paper, we consider the following generalized Schrödinger operators on $L^2(\mathbb{R}^d)$

$$\mathcal{L}_a = (-\Delta)^{\alpha/2} + a|x|^{-\alpha} \quad \text{with } \alpha \in (0, 2 \wedge d). \tag{1}$$

Throughout the paper, we use the notations $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$.

The two terms in this operator have the same scaling, thus powers $\mathcal{L}_a^{s/2}$ are a natural candidate to generate homogeneous Sobolev type norms. The case $\alpha = 2$ is well studied, and in recent years the fractional case $\alpha \in (0, 2 \wedge d)$ has attracted extensive attention.

The operator $\mathcal{L}_a$ admits a non negative Friedrichs selfadjoint extension provided

$$a \geqslant a^* = -\frac{2^\alpha \Gamma((d+\alpha)/4)^2}{\Gamma((d-\alpha)/4)^2}.$$

This follows from Herbst' estimate [16]: for $0 < \alpha < 1$ and $p \in (1, n/\alpha)$, we have

$$\left\| (-\Delta)^{\alpha/2} u \right\|_{L^p} + \Psi_{\alpha,d}(n/p') \left\| |x|^{-\alpha} u \right\|_{L^p} \geqslant 0.$$

Here the constant $\Psi_{\alpha,d}(n/p')$ is sharp, and is expressed in terms of the function $\Psi_{\alpha,d}(\delta)$, defined as follows: $\Psi_{\alpha,d}(0) = 0$, $\Psi_{\alpha,d}(d-\alpha) = 0$, and for other values of $\delta \in (-\alpha, d-\alpha)$

$$\Psi_{\alpha,d}(\delta) = -2^\alpha \frac{\Gamma\left(\frac{\delta+\alpha}{2}\right)\Gamma\left(\frac{d-\delta}{2}\right)}{\Gamma\left(\frac{d-\delta-\alpha}{2}\right)\Gamma\left(\frac{\delta}{2}\right)}.$$

We have $\Psi_{\alpha,d}(\delta) = \Psi_{\alpha,d}(d-\delta)$, $\Psi_{\alpha,d}(\delta) \to +\infty$ as $\delta \to -\alpha$ or $\delta \to d-\alpha$, and $\Psi_{\alpha,d}(\delta)$ is strictly decreasing for $\delta < \frac{d-\alpha}{2}$ and strictly increasing for $\delta > \frac{d-\alpha}{2}$ with a minimum at $\delta = \frac{d-\alpha}{2}$ where it takes the value a^* (see [5, 23] with a slightly different notation; see also [4, 6, 7] for additional information).

Since $\Psi_{\alpha,d}(\delta) : (-\alpha, \frac{d-\alpha}{2}] \to (a^*, +\infty]$ is strictly decreasing it is invertible. In the following theorem, which is our main result, we shall denote its inverse by

$$\sigma := \Psi_{\alpha,d}^{-1}(a), \qquad \sigma : (a^*, +\infty] \to (-\alpha, \frac{d-\alpha}{2}]. \tag{2}$$

Theorem 1.1. *Let $d \in \mathbb{N}$, $\alpha \in (0, 2 \wedge d)$ and $s \in (0, 2]$. Let $a \geqslant a^*$ and $\sigma = \sigma(a)$ be defined by (2).*

(a) *If $\frac{d}{d-\sigma\vee 0} < p < \frac{d}{(\sigma+\alpha s/2)\vee 0}$ with convention $\frac{d}{0} = \infty$, then we have*

$$\left\| (-\Delta)^{\alpha s/2} f \right\|_p \lesssim \left\| \mathcal{L}_a^{\alpha s/4} f \right\|_p. \tag{3}$$

(b) *If $1 < p < \infty$ with $\frac{d}{d-\sigma\vee 0} < p < \frac{d}{(\alpha s/2)\vee\sigma}$, then we have*

$$\left\| \mathcal{L}_a^{s/2} f \right\|_p \lesssim \left\| (-\Delta)^{\alpha s/4} f \right\|_p. \tag{4}$$

We recall that estimates (3) and (4) for the case $\alpha = 2$ were proved in [18]. The range $\alpha \in (0, d \wedge 2)$ has been investigated recently in [22], limited to the case $p = 2$, and in [21] for general p but with $a \geqslant 0$. Therefore, theorem 1.1 fills the remaining gap $a \in [a^*, 0)$ and general p. The main reason for the restriction $a \geqslant 0$ in [21] is the essential use of the spectral multiplier theorem from [15], which requires the Poisson upper bound on the heat kernel. We note indeed that when $a < 0$, the heat kernel fails to enjoy the Poisson upper bound.

In order to overcome the weak decay of the kernel, we employ a new approach, based on square function estimates (see section 3). Although our approach is quite similar to that in

[8], here we encounter several additional obstructions. The method in [8] was built upon the following heat kernel estimate, valid for $\mathcal{L}_a$ in the Hardy case $\alpha = 2$:

$$\left\|e^{-t\mathcal{L}_a} f\right\|_{L^q(F)} \leqslant Ct^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} e^{-\frac{d(E,F)^2}{ct}} \|f\|_{L^p(E)}$$

for all measurable subsets $E, F \subset \mathbb{R}^d$, all $f \in L^p(E)$, and suitable $1 \leqslant p \leqslant q \leqslant \infty$. For the approach in [8] (see also [18]), the exponential term in the above estimate plays an essential role. Unfortunately, this type of estimate fails to be true in our setting with $\alpha < 2$ (see section 3). To deal with this difficulty, we consider the $L^p - L^q$ off-diagonal estimates on balls and their corresponding annuli. This difference requires dedicated heat kernel and square functions estimates, which form the main bulk of the paper.

The paper is organized as follows. Section 2 is devoted to auxiliary lemmas and estimates. The heat kernel is analysed in detail in section 3, where sharp estimates of the kernel and its derivatives are proved. Finally, section 4 is dedicated to square function estimates and to the proof of the main result.

2. Preliminaries

We start with some notations which will be used frequently. We always write C and c to denote positive constants that are independent of the main parameters involved but whose values may differ from line to line. We write $A \lesssim B$ if there is a universal constant C so that $A \leqslant CB$ and $A \simeq B$ if $A \lesssim B$ and $B \lesssim A$. For $a, b \in \mathbb{R}$, we use the notations $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$. For $p \in [1, \infty]$, we denote by $p' = \frac{p}{p-1}$ the conjugate exponent of p. The average of a measurable function f over a measurable set E with $0 < |E| < \infty$ will be denoted by

$$\fint_E f(x)\mathrm{d}x = \frac{1}{|E|}\int_E f(x)\mathrm{d}x.$$

Given a ball B, the associated annuli are the sets $S_j(B) = 2^jB \backslash 2^{j-1}B$ for $j = 1, 2, 3, \ldots$, while we write $S_0(B) = B$.

For $r > 0$, the Hardy–Littlewood maximal function $\mathcal{M}_r$ is defined as

$$\mathcal{M}_r f(x) = \sup_{B \ni x}\left(\frac{1}{|B|}\int_B |f(y)|^r \,\mathrm{d}y\right)^{1/r}, \qquad x \in \mathbb{R}^d,$$

where the supremum is taken over all balls B containing x. When $r = 1$, we write simply $\mathcal{M}$ instead of $\mathcal{M}_1$. The following estimate is well known:

Lemma 2.1 ([25]). *Let $0 < r < \infty$. Then for $p > r$, we have*

$$\|\mathcal{M}_r f\|_p \lesssim \|f\|_p.$$

The following estimates are elementary and we omit their proof.

Lemma 2.2.

(*a*) *Let $\kappa \in (-\infty, d)$. Then there exists $C > 0$ so that for all $r > 0$*

$$\int_{B(0,t)} \left(\frac{t}{|x|}\right)^{\kappa} \mathrm{d}x \leqslant Ct^d.$$

(*b*) *For* $\epsilon > 0$, *there exists* $C > 0$ *such that*

$$\int_{\mathbb{R}^d} \frac{1}{t^{d/\alpha}} \left(\frac{t^{1/\alpha} + |x-y|}{t^{1/\alpha}} \right)^{-d-\epsilon} \mathrm{d}y \leqslant C$$

uniformly in $x \in \mathbb{R}^d$.

(*c*) *For* $\epsilon > 0$, *there exists* $C > 0$ *such that*

$$\int_{\mathbb{R}^d} \frac{1}{t^{d/\alpha}} \left(\frac{t^{1/\alpha} + |x-y|}{t^{1/\alpha}} \right)^{-d-\epsilon} |f(y)| \mathrm{d}y \leqslant C\mathcal{M}f(x)$$

uniformly in $x \in \mathbb{R}^d$.

We next recall two criteria for singular integrals to be bounded on Lebesgue spaces, which will play an important role in the proof of the boundedness of the square functions. The first theorem gives a criterion on the boundedness on $L^p(\mathbb{R}^d)$ spaces with $p \in (1,2)$, while the second one covers the range $p > 2$.

Theorem 2.3. *Let* $1 \leqslant p_0 < 2$ *and let* T *be a sublinear operator which is bounded on* $L^2(\mathbb{R}^d)$. *Assume that there exists a family of operators* $\{\mathcal{A}_t\}_{t>0}$ *satisfying that for* $j \geqslant 2$ *and every ball* B

$$\left(\fint_{S_j(B)} |T(I - \mathcal{A}_{r_B})f|^2 \right)^{1/2} \leqslant \alpha(j) \left(\fint_B |f|^{p_0} \right)^{1/p_0}, \tag{5}$$

and

$$\left(\fint_{S_j(B)} |\mathcal{A}_{r_B} f|^2 \right)^{1/2} \leqslant \alpha(j) \left(\fint_B |f|^{p_0} \right)^{1/p_0}, \tag{6}$$

for all f *supported in* B. *If* $\sum_j \alpha(j) 2^{jd} < \infty$, *then* T *is bounded on* $L^p(\mathbb{R}^d)$ *for all* $p \in (p_0, 2)$.

Theorem 2.4. *Let* $2 < q_0 \leqslant \infty$. *Let* T *be a bounded sublinear operator on* $L^2(\mathbb{R}^d)$. *Assume that there exists a family of operators* $\{\mathcal{A}_t\}_{t>0}$ *satisfying that*

$$\left(\fint_B |T(I - \mathcal{A}_{r_B})f|^2 \mathrm{d}x \right)^{1/2} \leqslant C\mathcal{M}_2(f)(x), \tag{7}$$

and

$$\left(\fint_B |T\mathcal{A}_{r_B} f|^{q_0} \mathrm{d}x \right)^{1/q_0} \leqslant C\mathcal{M}_2(|Tf|)(x), \tag{8}$$

for all balls B *with radius* r_B, *all* $f \in C_c^\infty(\mathbb{R}^d)$ *and all* $x \in B$. *Then* T *is bounded on* $L^p(\mathbb{R}^d)$ *for all* $2 < p < q_0$.

For the proof of theorems 2.3 and 2.4, see [1].

3. Some kernel estimates

Given $\theta \in (-\infty, d]$, we write

$$d_\theta = \begin{cases} \dfrac{d}{\theta}, & \theta > 0 \\ \infty, & \theta \leqslant 0, \end{cases}$$

and d'_θ is the conjugate exponent of d_θ. It is easy to see that $d'_\theta = d_{d-\theta\vee 0}$ for $\theta \in (-\infty, d]$.

For $\theta \in \mathbb{R}$, we also denote

$$D_\theta(x,t) = \left(1 + \frac{t^{1/\alpha}}{|x|}\right)^\theta$$

for $t > 0$ and $x \in \mathbb{R}^d$.

Theorem 3.1. *Let $\{T_t\}_{t>0}$ be a family of linear operators on $L^2(\mathbb{R}^d)$ with their associated kernels $T_t(x,y)$. Assume that there exist $C, c > 0$ and $\theta \in (-\infty, d]$ such that for all $t > 0$ and $x, y \in \mathbb{R}^d \backslash \{0\}$,*

$$|T_t(x,y)| \leqslant Ct^{-d/\alpha}\left(\frac{t^{1/\alpha} + |x-y|}{t^{1/\alpha}}\right)^{-d-\alpha} D_\theta(x,t)D_\theta(y,t). \tag{9}$$

Assume that $d'_\theta < p \leqslant q < d_\theta$. Then for any ball B, for every $t > 0$ and $j \in \mathbb{N}$, we have:

$$\left(\fint_{S_j(B)} |T_t f|^q\right)^{1/q} \leqslant C \max\left\{\left(\frac{r_B}{t^{1/\alpha}}\right)^{d/p}, \left(\frac{r_B}{t^{1/\alpha}}\right)^d\right\}\left(1 + \frac{t^{1/\alpha}}{2^j r_B}\right)^{d/q}\left(1 + \frac{2^j r_B}{t^{1/\alpha}}\right)^{-d-\alpha}\left(\fint_B |f|^p\right)^{1/p} \tag{10}$$

for all $f \in L^p(\mathbb{R}^d)$ supported in B, and

$$\left(\fint_B |T_t f|^q\right)^{1/q} \leqslant C \max\left\{\left(\frac{2^j r_B}{t^{1/\alpha}}\right)^d, \left(\frac{2^j r_B}{t^{1/\alpha}}\right)^{d/p}\right\}\left(1 + \frac{t^{1/\alpha}}{r_B}\right)^{d/q}\left(1 + \frac{2^j r_B}{t^{1/\alpha}}\right)^{-d-\alpha}\left(\fint_{S_j(B)} |f|^p\right)^{1/p} \tag{11}$$

for all $f \in L^p(S_j(B))$.

Proof. We will prove only (10) since the proof of (11) is completely analogous. For convenience, we set $F = S_j(B)$ and $E = B$. We have obviously

$$\begin{aligned} \|T_t f\|_{L^q(F)} &\leqslant \left\{\int_F \left[\int_E t^{-d/\alpha}\left(\frac{t^{1/\alpha} + |x-y|}{t^{1/\alpha}}\right)^{-d-\alpha} D_\theta(x,t)D_\theta(y,t)|f(y)|\mathrm{d}y\right]^q \mathrm{d}x\right\}^{1/q} \\ &\leqslant I_1 + I_2 + I_3 + I_4, \end{aligned}$$

where

$$I_1 = \left\{ \int_{F\cap B(0,t^{1/\alpha})} \left[\int_{E\cap B(0,t^{1/\alpha})} t^{-d/\alpha} \left(\frac{t^{1/\alpha} + |x-y|}{t^{1/\alpha}} \right)^{-d-\alpha} D_\theta(x,t) D_\theta(y,t) |f(y)| \mathrm{d}y \right]^q \mathrm{d}x \right\}^{1/q},$$

$$I_2 = \left\{ \int_{F\cap B(0,t^{1/\alpha})} \left[\int_{E\backslash B(0,t^{1/\alpha})} t^{-d/\alpha} \left(\frac{t^{1/\alpha} + |x-y|}{t^{1/\alpha}} \right)^{-d-\alpha} D_\theta(x,t) D_\theta(y,t) |f(y)| \mathrm{d}y \right]^q \mathrm{d}x \right\}^{1/q},$$

$$I_3 = \left\{ \int_{F\backslash B(0,t^{1/\alpha})} \left[\int_{E\cap B(0,t^{1/\alpha})} t^{-d/\alpha} \left(\frac{t^{1/\alpha} + |x-y|}{t^{1/\alpha}} \right)^{-d-\alpha} D_\theta(x,t) D_\theta(y,t) |f(y)| \mathrm{d}y \right]^q \mathrm{d}x \right\}^{1/q},$$

and

$$I_4 = \left\{ \int_{F\backslash B(0,t^{1/\alpha})} \left[\int_{E\backslash B(0,t^{1/\alpha})} t^{-d/\alpha} \left(\frac{t^{1/\alpha} + |x-y|}{t^{1/\alpha}} \right)^{-d-\alpha} D_\theta(x,t) D_\theta(y,t) |f(y)| \mathrm{d}y \right]^q \mathrm{d}x \right\}^{1/q}.$$

We now consider the first term I_1. By Hölder's inequality,

$$I_1 \leqslant Ct^{-d/\alpha} \left(1 + \frac{d(E,F)}{t^{1/\alpha}}\right)^{-d-\alpha} \left[\int_{F\cap B(0,t^{1/\alpha})} D_{q\theta}(x,t)\mathrm{d}x \right]^{1/q} \left(\int_E |f|^p \right)^{1/p} \left[\int_{B(0,t^{1/\alpha})} D_{p'\theta}(y,t)\mathrm{d}y \right]^{1/p'}.$$

Note that

$$\begin{aligned} \int_{F\cap B(0,t^{1/\alpha})} D_{q\theta}(x,t)\mathrm{d}x &\simeq \int_{F\cap B(0,t^{1/\alpha})} \frac{t^{q\theta/\alpha}}{|x|^{q\theta}} \mathrm{d}x \\ &\leqslant \int_{B(0,t^{1/\alpha})} \frac{t^{q\theta/\alpha}}{|x|^{q\theta}} \mathrm{d}x \lesssim t^{q\theta/\alpha} t^{(d-q\theta)/\alpha} = t^{d/\alpha}, \end{aligned}$$

where in the last inequality we used lemma 2.2 since $q\theta < d$. For the same reason,

$$\int_{B(0,t^{1/\alpha})} D_{p'\theta}(y,t)\mathrm{d}y \lesssim t^{d/\alpha}.$$

Substituting the two estimates into the bound for I_1 we get

$$\begin{aligned} I_1 &\leqslant Ct^{-\frac{d}{\alpha}(1-\frac{1}{q}-\frac{1}{p'})} \left(1 + \frac{d(E,F)}{t^{1/\alpha}}\right)^{-d-\alpha} \|f\|_{L^p(E)} \\ &\leqslant Ct^{-\frac{d}{\alpha}(\frac{1}{p}-\frac{1}{q})} \left(1 + \frac{d(E,F)}{t^{1/\alpha}}\right)^{-d-\alpha} \|f\|_{L^p(E)}, \end{aligned}$$

which implies

$$\begin{aligned} |S_j(B)|^{-1/q} \times I_1 &\lesssim 2^{-jd/q} \left(\frac{r_B}{t^{1/\alpha}}\right)^{d(\frac{1}{p}-\frac{1}{q})} \left(1 + \frac{2^j r_B}{t^{1/\alpha}}\right)^{-d-\alpha} \left(\fint_B |f|^p \right)^{1/p} \\ &\lesssim \left(\frac{r_B}{t^{1/\alpha}}\right)^{\frac{d}{p}} \left(1 + \frac{t^{1/\alpha}}{2^j r_B}\right)^{d/q} \left(1 + \frac{2^j r_B}{t^{1/\alpha}}\right)^{-d-\alpha} \left(\fint_B |f|^p \right)^{1/p}. \end{aligned} \tag{12}$$

For the second term I_2, in this situation we have $D_\theta(y,t) \simeq 1$. Hence,

$$
\begin{aligned}
I_2 &\lesssim \left\{ \int_{F\cap B(0,t^{1/\alpha})} \left[\int_{E\setminus B(0,t^{1/\alpha})} D_\theta(x,t) t^{-d/\alpha} \right.\right.\\
&\qquad \left.\left. \times \left(\frac{t^{1/\alpha} + |x-y|}{t^{1/\alpha}} \right)^{-d-\alpha} |f(y)| \mathrm{d}y \right]^q \mathrm{d}x \right\}^{1/q}\\
&\lesssim t^{-\frac{d}{\alpha}} \left(1 + \frac{d(E,F)}{t^{1/\alpha}} \right)^{-d-\alpha} \|f\|_{L^1(E)} \left(\int_{B(0,t^{1/\alpha})} D_{\theta q}(x,t) \mathrm{d}x \right)^{1/q}\\
&\lesssim t^{-\frac{d}{\alpha}(1-\frac{1}{q})} \left(1 + \frac{d(E,F)}{t^{1/\alpha}} \right)^{-d-\alpha} |E|^{1/p'} \|f\|_{L^p(E)},
\end{aligned}
$$

where in the last inequality we used lemma 2.2 and Hölder's inequality. It follows that

$$
\begin{aligned}
|S_j(B)|^{-1/q} \times I_2 &\lesssim t^{-\frac{d}{\alpha}(1-\frac{1}{q})} |B| |2^j B|^{-1/q} \left(1 + \frac{2^j r_B}{t^{1/\alpha}} \right)^{-d-\alpha} \left(\fint_B |f|^p \right)^{1/p}\\
&\lesssim 2^{-jd/q} \left(\frac{r_B}{t^{1/\alpha}} \right)^{d(1-\frac{1}{q})} \left(1 + \frac{2^j r_B}{t^{1/\alpha}} \right)^{-d-\alpha} \left(\fint_B |f|^p \right)^{1/p}\\
&\lesssim \left(\frac{r_B}{t^{1/\alpha}} \right)^{d} \left(1 + \frac{t^{1/\alpha}}{2^j r_B} \right)^{d/q} \left(1 + \frac{2^j r_B}{t^{1/\alpha}} \right)^{-d-\alpha} \left(\fint_B |f|^p \right)^{1/p}. \qquad (13)
\end{aligned}
$$

For the third term I_3, using the fact that $D_\theta(x,t) \simeq 1$ we have

$$
\begin{aligned}
I_3 &\lesssim \left\{ \int_{F\setminus B(0,t^{1/\alpha})} \left[\int_{E\cap B(0,t^{1/\alpha})} t^{-d/\alpha} \left(\frac{t^{1/\alpha} + |x-y|}{t^{1/\alpha}} \right)^{-d-\alpha} D_\theta(y,t) |f(y)| \mathrm{d}y \right]^q \mathrm{d}x \right\}^{1/q}\\
&\lesssim t^{-\frac{d}{\alpha}} \left(1 + \frac{d(E,F)}{t^{1/\alpha}} \right)^{-d-\alpha} |F|^{1/q} \int_{E\cap B(0,t^{1/\alpha})} D_\theta(y,t) |f(y)| \mathrm{d}y.
\end{aligned}
$$

By lemma 2.2 and Hölder's inequality,

$$
\begin{aligned}
\int_{E\cap B(0,t^{1/\alpha})} D_\theta(y,t) |f(y)| \mathrm{d}y &\lesssim \left(\int_{B(0,t^{1/\alpha})} D_{p'\theta}(y,t) \mathrm{d}y \right)^{1/p'} \|f\|_{L^p(E)}\\
&\lesssim t^{\frac{d}{p'\alpha}} \|f\|_{L^p(E)}.
\end{aligned}
$$

As a consequence,

$$
I_3 \lesssim t^{-\frac{d}{\alpha p}} \left(1 + \frac{d(E,F)}{t^{1/\alpha}} \right)^{-d-\alpha} |F|^{1/q} \|f\|_{L^p(E)},
$$

which implies

$$
\begin{aligned}
|S_j(B)|^{-1/q} \times I_3 &\lesssim t^{-\frac{d}{\alpha p}} |B|^{1/p} \left(1 + \frac{2^j r_B}{t^{1/\alpha}}\right)^{-d-\alpha} \left(\fint_B |f|^p\right)^{1/p} \\
&\simeq \left(\frac{r_B}{t^{1/\alpha}}\right)^{\frac{d}{p}} \left(1 + \frac{2^j r_B}{t^{1/\alpha}}\right)^{-d-\alpha} \left(\fint_B |f|^p\right)^{1/p}. \qquad (14)
\end{aligned}
$$

It remains to estimate the term I_4. Note that in this case $D_\theta(x,t) \simeq D_\theta(y,t) \simeq 1$. Hence,

$$
\begin{aligned}
I_4 &\lesssim \left\{ \int_F \left[\int_E t^{-d/\alpha} \left(\frac{t^{1/\alpha} + |x-y|}{t^{1/\alpha}} \right)^{-d-\alpha} |f(y)| \mathrm{d}y \right]^q \mathrm{d}x \right\}^{1/q} \\
&\lesssim t^{-d/\alpha} \left(1 + \frac{d(E,F)}{t^{1/\alpha}}\right)^{-d-\alpha} |F|^{1/q} \|f\|_{L^1(E)} \\
&\lesssim t^{-d/\alpha} \left(1 + \frac{d(E,F)}{t^{1/\alpha}}\right)^{-d-\alpha} |F|^{1/q} |E|^{1/p'} \|f\|_{L^p(E)},
\end{aligned}
$$

where in the last inequality we used Hölder's inequality. Thus we obtain

$$
\begin{aligned}
|S_j(B)|^{-1/q} \times I_4 &\lesssim t^{-\frac{d}{\alpha}} |B| \left(1 + \frac{2^j r_B}{t^{1/\alpha}}\right)^{-d-\alpha} \left(\fint_B |f|^p\right)^{1/p} \\
&\lesssim \left(\frac{r_B}{t^{1/\alpha}}\right)^{d} \left(1 + \frac{2^j r_B}{t^{1/\alpha}}\right)^{-d-\alpha} \left(\fint_B |f|^p\right)^{1/p}, \qquad (15)
\end{aligned}
$$

and this completes the proof of (10). □

Theorem 3.2. *Let $\{T_t\}_{t>0}$ be a family of linear operators on $L^2(\mathbb{R}^d)$ with their associated kernels $T_t(x,y)$ satisfying the estimate (9). Then for every $d'_\theta < q < d_\theta$, there exists C such that*

$$\|T_t\|_{q\to q} \leqslant C$$

uniformly in $t > 0$.

Proof. Using the same notations as in the proof of theorem 3.1, we have

$$\|T_t f\|_q \lesssim I_1 + I_2 + I_3 + I_4,$$

where I_1, I_2, I_3 and I_4 are terms defined similarly to those in the proof of theorem 3.1 with respect to $E = F = \mathbb{R}^d$.

Similarly to the proof of theorem 3.1, we have

$$I_1 \leqslant C\|f\|_q.$$

Similarly to the estimate of the term I_2 in the proof of theorem 3.1, we also have

$$I_2 \lesssim \left\{ \int_{B(0,t^{1/\alpha})} D_{q\theta}(x,t) \left[\int_{\mathbb{R}^d \setminus B(0,t^{1/\alpha})} t^{-d/\alpha} \left(\frac{t^{1/\alpha} + |x-y|}{t^{1/\alpha}} \right)^{-d-\alpha} |f(y)| \mathrm{d}y \right]^q \mathrm{d}x \right\}^{1/q}.$$

Using Hölder's inequality and lemma 2.2 (b),

$$\begin{aligned}\int_{\mathbb{R}^d\backslash B(0,t^{1/\alpha})} & t^{-d/\alpha}\left(\frac{t^{1/\alpha}+|x-y|}{t^{1/\alpha}}\right)^{-d-\alpha}|f(y)|\mathrm{d}y \\ &\lesssim t^{-\frac{d}{\alpha q}}\|f\|_q\left[\int_{\mathbb{R}^d}t^{-d/\alpha}\left(\frac{t^{1/\alpha}+|x-y|}{t^{1/\alpha}}\right)^{-(d+\alpha)q'}\mathrm{d}y\right]^{1/q'} \\ &\lesssim t^{-\frac{d}{\alpha q}}\|f\|_q.\end{aligned}$$

This gives

$$\begin{aligned}I_2 &\lesssim t^{-\frac{d}{\alpha q}}\|f\|_q\left[\int_{B(0,t^{1/\alpha})}D_{q\theta}(x,t)\mathrm{d}x\right]^{1/q} \\ &\lesssim \|f\|_q,\end{aligned}$$

where in the last inequality we used lemma 2.2 (a).

In a similar way, we have

$$I_3 \lesssim \|f\|_q,$$

and using the same argument as in the proof of theorem 3.1,

$$\begin{aligned}I_4 &\lesssim \left\{\int_{\mathbb{R}^d}\left[\int_{\mathbb{R}^d}t^{-d/\alpha}\left(\frac{t^{1/\alpha}+|x-y|}{t^{1/\alpha}}\right)^{-d-\alpha}|f(y)|\mathrm{d}y\right]^q\mathrm{d}x\right\}^{1/q} \\ &\lesssim \left\{\int_{\mathbb{R}^d}\mathcal{M}f(x)^q\mathrm{d}x\right\}^{1/q} \\ &\lesssim \|f\|_q,\end{aligned}$$

where we used lemma 2.2 (c) in the second inequality and the L^q-boundedness of the maximal operator $\mathcal{M}$ in the last inequality. □

The following heat kernel estimates are taken from [3, 5, 11, 17, 26].

Theorem 3.3. *Let $d\in\mathbb{N}$, $0<\alpha<2\wedge d$, $a\geqslant a^*$, and let σ be defined by (2). Let $p_t(x,y)$ be the kernel associated to the semigroups $e^{-t\mathcal{L}_a}$. Then for all $t>0$ and $x,y\in\mathbb{R}^d\backslash\{0\}$,*

$$p_t(x,y) \simeq D_\sigma(x,t)D_\sigma(y,t)t^{-d/\alpha}\left(\frac{t^{1/\alpha}+|x-y|}{t^{1/\alpha}}\right)^{-d-\alpha}.$$

The following result show that the upper bound for the kernel $p_t(x,y)$ can be extended to the complex heat kernel $p_z(x,y)$ for $z\in\mathbb{C}_{\pi/4}:=\{z\in\mathbb{C}:|\arg z|<\pi/4\}$.

Proposition 3.4. *Let $d\in\mathbb{N}$, $0<\alpha<2\wedge d$, $a\geqslant a^*$, and let σ be defined by (2). Let $p_z(x,y)$ be the kernels associated to the semigroups $e^{-z\mathcal{L}_a}$ with $z\in\mathbb{C}_{\pi/4}:=\{z\in\mathbb{C}:|\arg z|<\pi/4\}$. Then there exists a constant C such that*

$$|p_z(x,y)| \leqslant CD_\sigma(x,|z|)D_\sigma(y,|z|)|z|^{-d/\alpha}\left(\frac{|z|^{1/\alpha}+|x-y|}{|z|^{1/\alpha}}\right)^{-d-\alpha} \tag{16}$$

Proof. We adapt the standard argument in [12, lemma 3.4.6] to our present situation. Thus it is sufficient to prove that

$$|w(x)p_z(x,y)w(y)| \leqslant \frac{C}{|z|^{d/\alpha}}, \tag{17}$$

where $w(x) = D_{-\sigma}(x,|z|)$.

Now for $f : \mathbb{R}^d \to \mathbb{R}$, we define the norm

$$|f|_{wL^\infty} = \sup_x |f(x)w(x)|$$

so that (17) is equivalent to

$$\left\|e^{-z\mathcal{L}_a}\right\|_{L^1_{w^{-1}}\to wL^\infty} \leqslant \frac{C}{|z|^{d/\alpha}}.$$

Write $z = 2t + is$ for some $t \geqslant 0$ and $s \in \mathbb{R}$. Since $|\arg z| < \pi/4$, we have $t \simeq |z|$ and hence

$$\left\|e^{-z\mathcal{L}_a}\right\|_{L^1_{w^{-1}}\to wL^\infty} \leqslant \left\|e^{-t\mathcal{L}_a}\right\|_{L^2\to wL^\infty}\left\|\mathrm{e}^{-\mathrm{i}s\mathcal{L}_a}\right\|_{L^2\to L^2}\left\|e^{-t\mathcal{L}_a}\right\|_{L^1_{w^{-1}}\to L^2}.$$

Since $\mathcal{L}_a$ is nonnegative and self-adjoint, we have $\left\|\mathrm{e}^{-\mathrm{i}s\mathcal{L}_a}\right\|_{L^2\to L^2} \leqslant 1$, and the claim will follow as soon as we prove that

$$\left\|e^{-t\mathcal{L}_a}\right\|_{L^1_{w^{-1}}\to L^2} \lesssim t^{-\frac{d}{2\alpha}} \quad \text{and} \quad \left\|e^{-t\mathcal{L}_a}\right\|_{L^2\to wL^\infty} \lesssim t^{-\frac{d}{2\alpha}}.$$

We shall prove that $\left\|e^{-t\mathcal{L}_a}\right\|_{L^1_{w^{-1}}\to L^2} \lesssim t^{-\frac{d}{2\alpha}}$; the second inequality $\left\|e^{-t\mathcal{L}_a}\right\|_{L^2\to wL^\infty} \lesssim t^{-\frac{d}{2\alpha}}$ can be proved in the same manner. Indeed, for $f \in L^1_{w^{-1}}$ we have, by theorem 3.3,

$$\begin{aligned}
\left\|e^{-t\mathcal{L}_a}f\right\|_{L^2} &\lesssim \left[\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d} t^{-d/\alpha}D_\sigma(x,t)D_\sigma(y,t)\left(\frac{t^{1/\alpha}+|x-y|}{t^{1/\alpha}}\right)^{-d-\alpha}|f(y)|\mathrm{d}y\right|^2\mathrm{d}x\right]^{1/2}\\
&\lesssim \int_{\mathbb{R}^d}\left[\int_{\mathbb{R}}\left|t^{-d/\alpha}D_\sigma(x,t)\left(\frac{t^{1/\alpha}+|x-y|}{t^{1/\alpha}}\right)^{-d-\alpha}\right|^2\mathrm{d}x\right]^{1/2} D_\sigma(y,t)|f(y)|\mathrm{d}y\\
&\lesssim \sup_{y\in\mathbb{R}^d}\left[\int_{\mathbb{R}^d}\left|t^{-d/\alpha}D_\sigma(x,t)\left(\frac{t^{1/\alpha}+|x-y|}{t^{1/\alpha}}\right)^{-d-\alpha}\right|^2\mathrm{d}x\right]^{1/2}\|f\|_{L^1_{w^{-1}}}.
\end{aligned}$$

Hence, it suffices to prove that

$$\sup_{y\in\mathbb{R}^d}\left[\int_{\mathbb{R}^d}\left|t^{-d/\alpha}D_\sigma(x,t)\left(\frac{t^{1/\alpha}+|x-y|}{t^{1/\alpha}}\right)^{-d-\alpha}\right|^2\mathrm{d}x\right]^{1/2} \lesssim t^{-\frac{d}{2\alpha}}. \tag{18}$$

Indeed, we break the integral into two parts: the first part corresponding to the integral over $B(0,t^{1/\alpha})$ and the second corresponding to the integral over $B(0,t^{1/\alpha})^c$, and denote them by I_1

and I_2, respectively. For the first integral, we use the fact that $2\sigma \leqslant d-\alpha < d$ and lemma 2.2 (a) to bound it by

$$\left[\int_{B(0,t^{1/\alpha})} \left|t^{-d/\alpha}D_\sigma(x,t)\right|^2 \mathrm{d}x\right]^{1/2} = t^{-d/\alpha}\left[\int_{B(0,t^{1/\alpha})} D_{2\sigma}(x,t)\mathrm{d}x\right]^{1/2} \lesssim t^{-\frac{d}{2\alpha}}.$$

Using the fact that $D_\sigma(x,t) \simeq 1$ as $x \in B(0,t^{1/\alpha})^c$ and lemma 2.2 we can dominate the second integral by

$$\left[\int_{B(0,t^{1/\alpha})^c} \left|t^{-2d/\alpha}\left(\frac{t^{1/\alpha}+|x-y|}{t^{1/\alpha}}\right)^{-2(d+\alpha)}\mathrm{d}x\right|\right]^{1/2} \lesssim t^{-\frac{d}{2\alpha}}.$$

These two estimates ensure (18). Hence,

$$\left\|e^{-t\mathcal{L}_a}f\right\|_{L^2} \lesssim t^{-\frac{d}{2\alpha}}\|f\|_{L^1_{w^{-1}}}$$

which implies

$$\left\|e^{-t\mathcal{L}_a}\right\|_{L^1_{w^{-1}}\to L^2} \lesssim t^{-\frac{d}{2\alpha}}.$$

This completes our proof. □

As a direct consequence of proposition 3.4 and Cauchy formula, we obtain the following result.

Proposition 3.5. *Let $d \in \mathbb{N}$, $0<\alpha<2\wedge d$, $a \geqslant a^*$ and let σ be defined by (2). For any $k \in \mathbb{N}$, there exists $C_k > 0$ such that for all $t>0$ and $x, y \in \mathbb{R}^d\backslash\{0\}$,*

$$|p_{t,k}(x,y)| \leqslant C_k t^{-(k+d/\alpha)}D_\sigma(x,t)D_\sigma(y,t)\left(\frac{t^{1/\alpha}+|x-y|}{t^{1/\alpha}}\right)^{-d-\alpha},$$

where $p_{t,k}(x,y)$ is an associated kernel to $\mathcal{L}_a^k e^{-t\mathcal{L}_a}$.

Proof. Applying Cauchy formula, for every $t>0$ and $k \in \mathbb{N}$,

$$\mathcal{L}_a^k e^{-t\mathcal{L}_a} = \frac{(-1)^k k!}{2\pi i}\int_{|\xi-t|=\eta t} e^{-\xi\mathcal{L}_a}\frac{\mathrm{d}\xi}{(\xi-t)^k},$$

where $\eta > 0$ is small enough so that $\{\xi : |\xi - t| = \eta t\} \subset \mathbb{C}_{\pi/2}$, and the integral does not depend on the choice of η.

We now apply proposition 3.4 and the fact that $|\xi| \simeq |\xi-t| \simeq t$ to deduce that

$$|p_{t,k}(x,y)| \leqslant C_k t^{-(k+d/\alpha)}D_\sigma(x,t)D_\sigma(y,t)\left(\frac{t^{1/\alpha}+|x-y|}{t^{1/\alpha}}\right)^{-d-\alpha}$$

for all $x, y \in \mathbb{R}^d$ and $t>0$.

This completes our proof. □

In the previous estimates k is an integer. By a suitable interpolation argument, we now proceed to extend the estimates to the fractional case:

Proposition 3.6. *Let $d \in \mathbb{N}$, $0 < \alpha < 2 \wedge d$, $a \geqslant a^*$ and let σ be defined by* (2).

(a) *For any $s \in \mathbb{R}, s > 0$ and $d'_\sigma < p < d_\sigma$, there exists C such that*

$$\left\|(t\mathcal{L}_a)^s e^{-t\mathcal{L}_a}\right\|_{p\to p} \leqslant C$$

for all $t > 0$.

(b) *For any $s \in \mathbb{R}, s > 1 + \max\{0, \sigma/\alpha\}$, there exists $C_s > 0$ such that for all $t > 0$ and $x, y \in \mathbb{R}^d\backslash\{0\}$,*

$$|p_{t,s}(x,y)| \leqslant C_k t^{-(s+d/\alpha)} D_\sigma(x,t) D_\sigma(y,t) \left(\frac{t^{1/\alpha} + |x-y|}{t^{1/\alpha}}\right)^{-d-\alpha},$$

where $p_{t,s}(x,y)$ is an associated kernel to $\mathcal{L}_a^s e^{-t\mathcal{L}_a}$.

Proof.

(a) Due to proposition 3.5 and theorem 3.2, we need only to verify (a) for $s \notin \mathbb{N}$. To do this, we write $s = k - \gamma$, where $k \in \mathbb{N}$ and $\gamma \in (0, 1)$. Using the following subordination formula

$$\mathcal{L}_a^{-\gamma} = \frac{1}{\Gamma(\gamma)} \int_0^\infty u^\gamma e^{-u\mathcal{L}_a} \frac{\mathrm{d}u}{u},$$

we have

$$(t\mathcal{L}_a)^s e^{-t\mathcal{L}_a} = \frac{1}{\Gamma(\gamma)} \int_0^\infty t^s u^\gamma \mathcal{L}_a^k e^{-(u+t)\mathcal{L}_a} \frac{\mathrm{d}u}{u},$$

which implies that

$$(t\mathcal{L}_a)^s e^{-t\mathcal{L}_a} = \frac{1}{\Gamma(\gamma)} \int_0^\infty \frac{t^s u^\gamma}{(u+t)^k} (u+t)^k \mathcal{L}_a^k e^{-(u+t)\mathcal{L}_a} \frac{\mathrm{d}u}{u}.$$

This, along with proposition 3.5 and theorem 3.2, yields

$$\begin{aligned}\left\|(t\mathcal{L}_a)^s e^{-t\mathcal{L}_a}\right\|_p &= \frac{1}{\Gamma(\gamma)} \int_0^\infty \frac{t^s u^\gamma}{(u+t)^k} \left\|(u+t)^k \mathcal{L}_a^k e^{-(u+t)\mathcal{L}_a}\right\|_p \frac{\mathrm{d}u}{u} \\ &\lesssim \int_0^\infty \frac{t^s u^\gamma}{(u+t)^k} \frac{\mathrm{d}u}{u} = \int_0^\infty \frac{t^s u^\gamma}{(u+t)^{s+\gamma}} \frac{\mathrm{d}u}{u} \\ &\lesssim 1.\end{aligned}$$

This completes our proof.

(b) Due to proposition 3.5, it suffices to prove proposition 3.6 for $s \notin \mathbb{N}$. In this situation, we can find $k \in \mathbb{N}$ and $\gamma \in (0, 1)$ such that $s = k - \gamma$. Using the above subordination formula, we can write

$$\mathcal{L}_a^s e^{-t\mathcal{L}_a} = \frac{1}{\Gamma(\gamma)} \int_0^\infty u^\gamma \mathcal{L}_a^k e^{-(u+t)\mathcal{L}_a} \frac{\mathrm{d}u}{u}.$$

It follows, by proposition 3.5, that

$$
\begin{aligned}
|p_{t,s}(x,y)| &\lesssim \int_0^\infty \frac{u^\gamma}{(t+u)^{k+d/\alpha}} D_\sigma(x,t+u) D_\sigma(y,t+u) \quad \left(\frac{(t+u)^{1/\alpha}+|x-y|}{(t+u)^{1/\alpha}}\right)^{-d-\alpha} \frac{\mathrm{d}u}{u} \\
&= \int_0^t \ldots + \int_t^\infty \ldots \\
&=: I_1 + I_2.
\end{aligned}
$$

For the term I_1, using the fact that $t+u \simeq t$ as $u \leqslant t$,

$$
\begin{aligned}
I_1 &\lesssim \int_0^t \frac{u^\gamma}{t^{k+d/\alpha}} D_\sigma(x,t) D_\sigma(y,t) \left(\frac{t^{1/\alpha}+|x-y|}{t^{1/\alpha}}\right)^{-d-\alpha} \frac{\mathrm{d}u}{u} \\
&\lesssim \frac{t^\gamma}{t^{k+d/\alpha}} D_\sigma(x,t) D_\sigma(y,t) \left(\frac{t^{1/\alpha}+|x-y|}{t^{1/\alpha}}\right)^{-d-\alpha} \\
&\lesssim \frac{1}{t^{s+d/\alpha}} D_\sigma(x,t) D_\sigma(y,t) \left(\frac{t^{1/\alpha}+|x-y|}{t^{1/\alpha}}\right)^{-d-\alpha}.
\end{aligned}
$$

Likewise, for the term I_2, since $t+u \simeq u$ as $u \geqslant t$, we have

$$
\begin{aligned}
I_2 &\lesssim \int_t^\infty \frac{u^\gamma}{u^{k+d/\alpha}} D_\sigma(x,u) D_\sigma(y,u) \left(\frac{u^{1/\alpha}+|x-y|}{u^{1/\alpha}}\right)^{-d-\alpha} \frac{\mathrm{d}u}{u} \\
&\lesssim \int_t^\infty \frac{1}{u^{s+d/\alpha}} D_\sigma(x,u) D_\sigma(y,u) \left(\frac{u^{1/\alpha}+|x-y|}{u^{1/\alpha}}\right)^{-d-\alpha} \frac{\mathrm{d}u}{u}.
\end{aligned}
$$

This, in combination with the following inequalities

$$
D_\sigma(x,u) D_\sigma(y,u) \lesssim D_\sigma(x,t) D_\sigma(y,t) \left(\frac{u}{t}\right)^{\max\{0,\sigma/\alpha\}}
$$

and

$$
\left(\frac{u^{1/\alpha}+|x-y|}{u^{1/\alpha}}\right)^{-d-\alpha} \lesssim \left(\frac{t^{1/\alpha}+|x-y|}{t^{1/\alpha}}\right)^{-d-\alpha} \left(\frac{u}{t}\right)^{1+d/\alpha},
$$

implies that

$$
\begin{aligned}
I_2 &\lesssim D_\sigma(x,t) D_\sigma(y,t) \left(\frac{t^{1/\alpha}+|x-y|}{t^{1/\alpha}}\right)^{-d-\alpha} \int_t^\infty \frac{1}{u^{s+d/\alpha}} \left(\frac{u}{t}\right)^{1+d/\alpha+\max\{0,\sigma/\alpha\}} \frac{\mathrm{d}u}{u} \\
&\lesssim \frac{1}{t^{s+d/\alpha}} D_\sigma(x,t) D_\sigma(y,t) \left(\frac{t^{1/\alpha}+|x-y|}{t^{1/\alpha}}\right)^{-d-\alpha},
\end{aligned}
$$

as long as $s > 1 + \max\{0, \sigma/\alpha\}$.

This completes our proof. □

4. Equivalence of Sobolev norms involving $\mathcal{L}_a$

This section is devoted to prove theorem 1.1. Before coming to the proof of the main result, we need some estimates for square functions which play a key role in the argument.

Given $\gamma > 0$, we consider the following square function

$$S_{\mathcal{L}_a,\gamma} f(x) = \left(\int_0^\infty |(t\mathcal{L}_a)^\gamma e^{-t\mathcal{L}_a} f|^2 \frac{\mathrm{d}t}{t} \right)^{1/2}.$$

Note that by the functional calculus in [20], the square function $S_{\mathcal{L}_a,\gamma}$ is bounded on L^2. In the following theorem, we prove the L^p estimates for $S_{\mathcal{L}_a,\gamma}$.

Theorem 4.1. *Let $d \in \mathbb{N}$, $0 < \alpha < 2 \wedge d$, $a \geqslant a^*$ and let σ be defined by (2). Let $\gamma \in (0, 1]$. Then for all $d'_\sigma < p < d_\sigma$,*

$$\|S_{\mathcal{L}_a,\gamma} f\|_p \sim \|f\|_p.$$

As a consequence, for $s \in (0, 2]$ and $d'_\sigma < p < d_\sigma$,

$$\left\| \left(\int_0^\infty t^{-s} |t\mathcal{L}_a e^{-t\mathcal{L}_a} f|^2 \frac{\mathrm{d}t}{t} \right)^{1/2} \right\|_p \simeq \left\| \mathcal{L}_a^{s/2} f \right\|_p.$$

Proof. We divide the proof into two steps.

Step 1 Proof of the boundedness of $S_{\mathcal{L}_a,\gamma}$ on L^p for $d'_\sigma < p \leqslant 2$.

Fix $r \in (d'_\sigma, 2]$ and $m > 1 + \frac{d}{\alpha r'}$. For every ball B we define

$$\mathcal{A}_{r_B} = I - (I - e^{-r_B^\alpha \mathcal{L}_a})^m.$$

We will claim that

$$\left(\fint_{S_j(B)} |S_{\mathcal{L}_a,\gamma}(I - \mathcal{A}_{r_B}) f|^2 \mathrm{d}x \right)^{1/2} \lesssim 2^{-j(d+\alpha)} \left(\fint_B |f|^r \mathrm{d}x \right)^{1/r}, \quad j \geqslant 2, \tag{19}$$

and

$$\left(\fint_{S_j(B)} |\mathcal{A}_{r_B} f|^2 \right)^{1/2} \lesssim 2^{-j(d+\alpha)} \left(\fint_B |f|^r d \right)^{1/r} \tag{20}$$

for all balls B and all $f \in C^\infty$ supported in B. Once these two estimates have been proved, the boundedness of $S_{\mathcal{L}_a,\gamma}$ on L^p for $d'_\sigma < p \leqslant 2$ will follow directly from theorem 2.3.

We rewrite

$$\left(\int_{S_j(B)} |S_{\mathcal{L}_a,\gamma}(I - \mathcal{A}_{r_B}) f|^2 \mathrm{d}x \right)^{1/2} = \left(\int_0^\infty \left\| (t\mathcal{L}_a)^\gamma e^{-t\mathcal{L}_a} (I - \mathcal{A}_{r_B}) f \right\|^2_{L^2(S_j(B))} \frac{\mathrm{d}t}{t} \right)^{1/2}.$$

Hence, we are reduced to prove the following inequality:

$$\begin{aligned} |S_j(B)|^{-1/2} &\left(\int_0^\infty \left\| (t\mathcal{L}_a)^\gamma e^{-t\mathcal{L}_a} (I - \mathcal{A}_{r_B}) f \right\|^2_{L^2(S_j(B))} \frac{\mathrm{d}t}{t} \right)^{1/2} \\ &\lesssim 2^{-j(d+\alpha)} \left(\fint_B |f|^r \mathrm{d}x \right)^{1/r} \end{aligned} \tag{21}$$

for $j \geqslant 2$, all balls B, and all $f \in C^\infty$ supported in B.

We note that for every $g \in L^2$ and $s > 0$,

$$\int_t^u \mathcal{L}_a^{s+1} e^{-\tau\mathcal{L}_a} g \, \mathrm{d}\tau = \mathcal{L}_a^s e^{-t\mathcal{L}_a} g - \mathcal{L}_a^s e^{-u\mathcal{L}_a} g.$$

This, along with Minkowski's inequality, implies

$$\big\| \mathcal{L}_a^s e^{-t\mathcal{L}_a} g \big\|_{L^2(S_j(B))} \leqslant \int_t^\infty \big\| \mathcal{L}_a^{s+1} e^{-\tau\mathcal{L}_a} g \big\|_{L^2(S_j(B))} \mathrm{d}\tau.$$

Thus we have

$$\begin{aligned}\left(\int_0^\infty \big\| (t\mathcal{L}_a)^\gamma e^{-t\mathcal{L}_a} g \big\|^2_{L^2(S_j(B))} \frac{\mathrm{d}t}{t} \right)^{1/2} &\leqslant \left[\int_0^\infty t^{2\gamma} \left(\int_t^\infty \big\| \mathcal{L}_a^{\gamma+1} e^{-\tau\mathcal{L}_a} g \big\|_{L^2(S_j(B))} \mathrm{d}\tau \right)^2 \frac{\mathrm{d}t}{t} \right]^{1/2} \\ &\lesssim \left[\int_0^\infty \big\| (t\mathcal{L}_a)^{\gamma+1} e^{-t\mathcal{L}_a} g \big\|^2_{L^2(S_j(B))} \frac{\mathrm{d}t}{t} \right]^{1/2},\end{aligned}$$

where in the last inequality we used Hardy's inequality.

By iteration we have, for every $N \in \mathbb{N}$,

$$\left(\int_0^\infty \big\| (t\mathcal{L}_a)^\gamma e^{-t\mathcal{L}_a} g \big\|^2_{L^2(S_j(B))} \frac{\mathrm{d}t}{t} \right)^{1/2} \leqslant C_N \left[\int_0^\infty \big\| (t\mathcal{L}_a)^{\gamma+N} e^{-t\mathcal{L}_a} g \big\|^2_{L^2(S_j(B))} \frac{\mathrm{d}t}{t} \right]^{1/2}.$$

Taking $g = (I - \mathcal{A}_{r_B}) f$, we have, for every $N \in \mathbb{N}$,

$$\begin{aligned}&\left(\int_0^\infty \big\| (t\mathcal{L}_a)^\gamma e^{-t\mathcal{L}_a} (I - \mathcal{A}_{r_B}) f \big\|^2_{L^2(S_j(B))} \frac{\mathrm{d}t}{t} \right)^{1/2} \\ &\quad \leqslant C_N \left[\int_0^\infty \big\| (t\mathcal{L}_a)^{\gamma+N} e^{-t\mathcal{L}_a} (I - \mathcal{A}_{r_B}) f \big\|^2_{L^2(S_j(B))} \frac{\mathrm{d}t}{t} \right]^{1/2}.\end{aligned}$$

Due to this inequality it suffices to prove (21) assuming $\gamma > 1 + \max\{0, \sigma/\alpha\}$. To do this, we write

$$\begin{aligned}\left(\int_0^\infty \big\| (t\mathcal{L}_a)^\gamma e^{-t\mathcal{L}_a} (I - \mathcal{A}_{r_B}) f \big\|^2_{L^2(S_j(B))} \frac{\mathrm{d}t}{t} \right)^{1/2} &\leqslant \left(\int_0^{r_B^\alpha} \dots \right)^{1/2} + \left(\int_{r_B^\alpha}^\infty \dots \right)^{1/2} \\ &=: E_1 + E_2.\end{aligned}$$

We now estimate E_1. Since

$$\begin{aligned}(t\mathcal{L}_a)^\gamma e^{-t\mathcal{L}_a} (I - \mathcal{A}_{r_B}) = (t\mathcal{L}_a)^\gamma e^{-t\mathcal{L}_a} (I - e^{-r_B^\alpha \mathcal{L}_a})^m &= \sum_{k=0}^m C_k^m (t\mathcal{L}_a)^\gamma e^{-(t+kr_B^\alpha)\mathcal{L}_a} \\ &= (t\mathcal{L}_a)^\gamma e^{-t\mathcal{L}_a} + \sum_{k=1}^m C_k^m (t\mathcal{L}_a)^\gamma e^{-(t+kr_B^\alpha)\mathcal{L}_a},\end{aligned}$$

applying proposition 3.6 with $\gamma > 1 + \max\{0, \sigma/\alpha\}$ and the fact that $t + kr_B^\alpha \simeq r_B^\alpha$ for $t \in (0, r_B^\alpha)$ and $k \geqslant 1$, we get the following upper bound for the kernel of $(t\mathcal{L}_a)^\gamma e^{-t\mathcal{L}_a}(I - e^{-r_B^\alpha \mathcal{L}_a})^m$:

$$t^{-d/\alpha} D_\sigma(x,t) D_\sigma(y,t) \left(\frac{t^{1/\alpha} + |x-y|}{t^{1/\alpha}} \right)^{-d-\alpha}$$
$$+ \left(\frac{t}{r_B^\alpha} \right)^\gamma r_B^{-d} D_\sigma(x, r_B^\alpha) D_\sigma(y, r_B^\alpha) \left(\frac{r_B + |x-y|}{r_B} \right)^{-d-\alpha}.$$

At this stage, using (10) from theorem 3.1 and the fact that $t \leqslant r_B^\alpha$, we can write, for $d'_\sigma < r \leqslant 2$,

$$|S_j(B)|^{-1/2} \left\| (t\mathcal{L}_a)^\gamma e^{-t\mathcal{L}_a}(I - \mathcal{A}_{r_B}) f \right\|_{L^2(S_j(B))}$$
$$\lesssim \left(\frac{r_B}{t^{1/\alpha}} \right)^d \left(\frac{2^j r_B}{t^{1/\alpha}} \right)^{-d-\alpha} \left(\fint_B |f|^r \right)^{1/r} + \left(\frac{t}{r_B^\alpha} \right)^\gamma 2^{-j(d+\alpha)} \left(\fint_B |f|^r \right)^{1/r}.$$

Plugging this into the expression of E_1 we obtain

$$|S_j(B)|^{-1/2} \times E_1 \lesssim 2^{-j(d+\alpha)} \left(\fint_B |f|^r \right)^{1/r}.$$

For the second term E_2, we note that

$$I - \mathcal{A}_{r_B} = (I - e^{-r_B^\alpha \mathcal{L}_a})^m = \int_0^{r_B^\alpha} \dots \int_0^{r_B^\alpha} \mathcal{L}_a^m e^{-(s_1 + \dots + s_m)\mathcal{L}_a} \, \mathrm{d}\vec{s}, \tag{22}$$

where $\mathrm{d}\vec{s} = \mathrm{d}s_1 \dots \mathrm{d}s_m$. Therefore,

$$\left\| (t\mathcal{L}_a)^\gamma e^{-t\mathcal{L}_a}(I - \mathcal{A}_{r_B}) f \right\|_{L^2(S_j(B))} \leqslant \int_{[0, r_B^\alpha]^m} \left\| t^\gamma \mathcal{L}_a^{\gamma+m} e^{-(t+s_1+\dots+s_m)\mathcal{L}_a} f \right\|_{L^2(S_j(B))} \mathrm{d}\vec{s}$$

Hence, using proposition 3.5 and (10) in theorem 3.1 with the fact that $t + s_1 + \dots + s_m \simeq t$ as $\vec{s} \in [0, r_B^\alpha]^m$ and $t \geqslant r_B^\alpha$, we have

$$|S_j(B)|^{-1/2} \left\| (t\mathcal{L}_a)^\gamma e^{-t\mathcal{L}_a}(I - \mathcal{A}_{r_B}) f \right\|_{L^2(S_j(B))}$$
$$\lesssim \int_{[0, r_B^\alpha]^m} t^{-m} \left(\frac{r_B}{t^{1/\alpha}} \right)^{d/r} \left(1 + \frac{t^{1/\alpha}}{2^j r_B} \right)^{d/2} \left(1 + \frac{2^j r_B}{t^{1/\alpha}} \right)^{-d-\alpha} \left(\fint_B |f|^r \right)^{1/r} \mathrm{d}\vec{s}$$
$$\lesssim \left(\frac{r_B^\alpha}{t} \right)^m \left(\frac{r_B}{t^{1/\alpha}} \right)^{d/r} \left(1 + \frac{t^{1/\alpha}}{2^j r_B} \right)^{d/2} \left(1 + \frac{2^j r_B}{t^{1/\alpha}} \right)^{-d-\alpha} \left(\fint_B |f|^r \right)^{1/r}.$$

Plugging this into the expression of E_2,

$$|S_j(B)|^{-1/2} \times E_2 \lesssim \int_{r_B^\alpha}^\infty \left(\frac{r_B^\alpha}{t} \right)^m \left(\frac{r_B}{t^{1/\alpha}} \right)^{d/r} \left(1 + \frac{t^{1/\alpha}}{2^j r_B} \right)^{d/2}$$
$$\times \left(1 + \frac{2^j r_B}{t^{1/\alpha}} \right)^{-d-\alpha} \frac{\mathrm{d}t}{t} \left(\fint_B |f|^r \right)^{1/r}.$$

We break the integral in dt into two pieces, i.e. the integral over the interval $(r_B^\alpha, 2^{j\alpha}r_B^\alpha)$ and the integral over $(2^{j\alpha}r_B^\alpha, \infty)$:

$$\begin{aligned}|S_j(B)|^{-1/2}\times E_2 \lesssim &\int_{r_B^\alpha}^{2^{j\alpha}r_B^\alpha}\left(\frac{r_B^\alpha}{t}\right)^m\left(\frac{r_B}{t^{1/\alpha}}\right)^{d/r}\left(\frac{2^jr_B}{t^{1/\alpha}}\right)^{-d-\alpha}\frac{\mathrm{d}t}{t}\times\left(\fint_B|f|^r\right)^{1/r}\\ &+\int_{2^{j\alpha}r_B^\alpha}^{\infty}\left(\frac{r_B^\alpha}{t}\right)^m\left(\frac{r_B}{t^{1/\alpha}}\right)^{d/r}\left(\frac{t^{1/\alpha}}{2^jr_B}\right)^{d/2}\frac{\mathrm{d}t}{t}\times\left(\fint_B|f|^r\right)^{1/r}.\end{aligned}$$

Keeping in mind that $m > 1 + \frac{d}{\alpha r'}$ and using a straightforward calculation, we obtain

$$\begin{aligned}|S_j(B)|^{-1/2}\times E_2 &\lesssim \left[2^{-j(d+\alpha)}+2^{-j(m+d/2)}\right]\left(\fint_B|f|^r\right)^{1/r}\\ &\lesssim 2^{-j(d+\alpha)}\left(\fint_B|f|^r\right)^{1/r}.\end{aligned}$$

From the estimates of E_1 and E_2 we obtain (19).

It remains to prove (20). Since

$$\mathcal{A}_{r_B} = \sum_{k=1}^{m} C_k^m e^{-kr_B^\alpha\mathcal{L}_a},$$

by using theorem 3.3, we can dominate the kernel $\mathcal{A}_{r_B}(x,y)$ of $\mathcal{A}_{r_B}$ by

$$\text{constant}\times\left(1+\frac{r_B}{|x|}\right)^\sigma\left(1+\frac{r_B}{|y|}\right)^\sigma r_B^{-d}\left(\frac{r_B+|x-y|}{r_B}\right)^{-d-\alpha}.$$

Therefore, applying (10) in theorem 3.1,

$$\left(\fint_{S_j(B)}|\mathcal{A}_{r_B}f(x)|^2\mathrm{d}x\right)^{1/2}\lesssim 2^{-j(d+\alpha)}\left(\fint_B|f(x)|^r\mathrm{d}x\right)^{1/r},$$

which proves (20). This completes the proof of step 1.

Step 2 Proof of the boundedness of $S_{\mathcal{L}_a,\gamma}$ on L^p for $2 < p < d_\sigma$.

By theorem 2.4, for any $q \in (2, d_\sigma)$ it suffices to prove that

$$\left(\fint_B|S_{\mathcal{L}_a,\gamma}(I-\mathcal{A}_{r_B})f|^2\mathrm{d}x\right)^{1/2}\leqslant C\mathcal{M}_2(f)(x), \tag{23}$$

and

$$\left(\fint_B|S_{\mathcal{L}_a,\gamma}\mathcal{A}_{r_B}f|^q\mathrm{d}x\right)^{1/q}\leqslant C\mathcal{M}_2(|S_{\mathcal{L}_a,\gamma}f|)(x) \tag{24}$$

for all balls B with radius r_B, all $f \in C_c^\infty(\mathbb{R}^d)$ and all $x \in B$ with $\mathcal{A}_{r_B} = I - (I - e^{-r_B^\alpha\mathcal{L}_a})^m$, $m > 1 + d/2$. The proof of these two inequalities is quite similar to that of step 1; however, since the decay is different, we give full details of the argument.

To prove (23), we write

$$\left(\fint_B |S_{\mathcal{L}_a,\gamma}(I-\mathcal{A}_{r_B})f|^2 \mathrm{d}x\right)^{1/2} \leqslant \sum_{j=0}^{\infty}\left(\fint_B |S_{\mathcal{L}_a,\gamma}(I-\mathcal{A}_{r_B})f_j|^2 \mathrm{d}x\right)^{1/2}$$
$$:= \sum_{j=0}^{\infty} I_j,$$

where $f_j = f\chi_{S_j(B)}$.

For $j = 0, 1$, using the L^2-boundedness of $S_{\mathcal{L}_a,\gamma}$ and $\mathcal{A}_{r_B}$ we have

$$I_j \lesssim \mathcal{M}_2(f)(x).$$

Hence, it suffices to prove that

$$I_j \lesssim 2^{-j\alpha}\left(\fint_{S_j(B)} |f|^2 \mathrm{d}x\right)^{1/2}$$

for $j \geqslant 2$.

Arguing similarly to the proof of step 1, we are reduced to prove the following inequality

$$|B|^{-1/2}\left(\int_0^\infty \left\|(t\mathcal{L}_a)^\gamma e^{-t\mathcal{L}_a}(I-\mathcal{A}_{r_B})f_j\right\|^2_{L^2(B)}\frac{\mathrm{d}t}{t}\right)^{1/2} \lesssim 2^{-j\alpha}\left(\fint_{S_j(B)} |f|^2 \mathrm{d}x\right)^{1/2} \tag{25}$$

for $\gamma > 1 + \max\{0, \sigma/\alpha\}$ and $j \geqslant 2$.

To do this, we write

$$\left(\int_0^\infty \left\|(t\mathcal{L}_a)^\gamma e^{-t\mathcal{L}_a}(I-\mathcal{A}_{r_B})f_j\right\|^2_{L^2(B)}\frac{\mathrm{d}t}{t}\right)^{1/2} \leqslant \left(\int_0^{r_B^\alpha}\dots\right)^{1/2} + \left(\int_{r_B^\alpha}^\infty\dots\right)^{1/2}$$
$$=: F_1 + F_2.$$

Arguing similarly to the estimate of E_1 in step 1, but using (11) instead of (10) for $t \leqslant r_B^\alpha$,

$$|B|^{-1/2}\left\|(t\mathcal{L}_a)^\gamma e^{-t\mathcal{L}_a}(I-\mathcal{A}_{r_B})f_j\right\|_{L^2(B)} \lesssim \left(\frac{t}{r_B^\alpha}\right)^\gamma 2^{-j\alpha}\left(\fint_{S_j(B)} |f|^2\right)^{1/2}$$
$$+ \left(\frac{2^j r_B}{t^{1/\alpha}}\right)^d\left(1+\frac{2^j r_B}{t^{1/\alpha}}\right)^{-d-\alpha}\left(\fint_{S_j(B)} |f|^2\right)^{1/2}.$$

Inserting this into the expression of F_1, we obtain

$$|B|^{-1/2} \times F_1 \lesssim 2^{-j\alpha}\left(\fint_{S_j(B)} |f|^2\right)^{1/2}.$$

For the second term F_2, using an argument similar to that of the estimate for E_2 in step 1, but using (11) instead of (10) from theorem 3.1, we have

$$
\begin{aligned}
&|B|^{-1/2}\big\|(t\mathcal{L}_a)^{\gamma}e^{-t\mathcal{L}_a}(I-\mathcal{A}_{r_B})f_j\big\|_{L^2(B)}\\
&\quad\lesssim \int_{[0,r_B^{\alpha}]^m} t^{-m}\,\max\left\{\left(\frac{2^j r_B}{t^{1/\alpha}}\right)^{d},\left(\frac{2^j r_B}{t^{1/\alpha}}\right)^{d/2}\right\}\\
&\qquad\times\left(1+\frac{t^{1/\alpha}}{r_B}\right)^{d/2}\left(1+\frac{2^j r_B}{t^{1/\alpha}}\right)^{-d-\alpha}\mathrm{d}\vec{s}\left(\fint_{S_j(B)}|f|^2\right)^{1/2}\\
&\quad\lesssim \left(\frac{r_B^{\alpha}}{t}\right)^{m}\max\left\{\left(\frac{2^j r_B}{t^{1/\alpha}}\right)^{d},\left(\frac{2^j r_B}{t^{1/\alpha}}\right)^{d/2}\right\}\left(\frac{t^{1/\alpha}}{r_B}\right)^{d/2}\\
&\qquad\times\left(1+\frac{2^j r_B}{t^{1/\alpha}}\right)^{-d-\alpha}\left(\fint_{S_j(B)}|f|^2\right)^{1/2}
\end{aligned}
$$

as long as $t \geqslant r_B^{\alpha}$. Inserting this into the expression of F_2 we obtain

$$
\begin{aligned}
|B|^{-1/2}\times F_2 \lesssim{}& \int_{r_B^{\alpha}}^{\infty}\left(\frac{r_B^{\alpha}}{t}\right)^{m}\max\left\{\left(\frac{2^j r_B}{t^{1/\alpha}}\right)^{d},\left(\frac{2^j r_B}{t^{1/\alpha}}\right)^{d/2}\right\}\left(\frac{t^{1/\alpha}}{r_B}\right)^{d/2}\\
&\times\left(1+\frac{2^j r_B}{t^{1/\alpha}}\right)^{-d-\alpha}\frac{\mathrm{d}t}{t}\left(\fint_{S_j(B)}|f|^2\right)^{1/2}.
\end{aligned}
$$

Splitting the integral in $\mathrm{d}t$ into the integral over the interval $(r_B^{\alpha}, 2^{j\alpha}r_B^{\alpha})$ and the integral over $(2^{j\alpha}r_B^{\alpha}, \infty)$, we find that

$$
\begin{aligned}
|S_j(B)|^{-1/2}\times F_2 \lesssim{}& \int_{r_B^{\alpha}}^{2^{j\alpha}r_B^{\alpha}}\left(\frac{r_B^{\alpha}}{t}\right)^{m}\left(\frac{2^j r_B}{t^{1/\alpha}}\right)^{d}\left(\frac{r_B}{t^{1/\alpha}}\right)^{d/2}\left(\frac{2^j r_B}{t^{1/\alpha}}\right)^{-d-\alpha}\frac{\mathrm{d}t}{t}\\
&\times\left(\fint_{S_j(B)}|f|^2\right)^{1/2}+\int_{2^{j\alpha}r_B^{\alpha}}^{\infty}\left(\frac{r_B^{\alpha}}{t}\right)^{m}\left(\frac{2^j r_B}{t^{1/\alpha}}\right)^{d/2}\\
&\times\left(\frac{t^{1/\alpha}}{r_B}\right)^{d/2}\frac{\mathrm{d}t}{t}\times\left(\fint_{S_j(B)}|f|^2\right)^{1/2}\\
\lesssim{}& 2^{-j\alpha}\left(\fint_{S_j(B)}|f|^2\right)^{1/2},
\end{aligned}
$$

as long as $m > 1+\frac{d}{\alpha r'}$. Collecting the estimates of F_1 and F_2 we obtain (25).

It remains to prove (24). We first write

$$
\begin{aligned}
&\left(\int_B |S_{\mathcal{L}_a,\gamma}\mathcal{A}_{r_B}f(x)|^q \mathrm{d}x\right)^{1/q}\\
&\quad = \left[\int_B\left(\int_0^\infty \left|\sum_{k=1}^m C_k^m e^{-kr_B^\alpha \mathcal{L}_a}(t\mathcal{L}_a)^\gamma e^{-t\mathcal{L}_a}f(x)\right|^2\frac{\mathrm{d}t}{t}\right)^{q/2}\mathrm{d}x\right]^{1/q}\\
&\quad \lesssim \sum_{j\geqslant 0}\left[\int_B\left(\int_0^\infty \left|\sum_{k=1}^m C_k^m e^{-kr_B^\alpha \mathcal{L}_a}[(t\mathcal{L}_a)^\gamma e^{-t\mathcal{L}_a}f\chi_{S_j(B)}](x)\right|^2\frac{\mathrm{d}t}{t}\right)^{q/2}\mathrm{d}x\right]^{1/q}
\end{aligned}
$$

which, along with Minkowski's inequality, theorem 3.3, and (11) in theorem 3.1, gives

$$
\begin{aligned}
&\left(\fint_B |S_{\mathcal{L}_a,\gamma}\mathcal{A}_{r_B}f(x)|^q \mathrm{d}x\right)^{1/q}\\
&\quad \lesssim \sum_{j\geqslant 0}|B|^{-1/q}\left(\int_0^\infty \left\|\sum_{k=1}^m e^{-kr_B^\alpha \mathcal{L}_a}[(t\mathcal{L}_a)^\gamma e^{-t\mathcal{L}_a}f\chi_{S_j(B)}]\right\|^2_{L^q(B)}\frac{\mathrm{d}t}{t}\right)^{1/2}\\
&\quad \lesssim \sum_{j\geqslant 0}2^{-j\alpha}|S_j(B)|^{-1/2}\left(\int_0^\infty \|(t\mathcal{L}_a)^\gamma e^{-t\mathcal{L}_a}f\|^2_{L^2(S_j(B))}\frac{\mathrm{d}t}{t}\right)^{1/2}\\
&\quad \lesssim \sum_{j\geqslant 0}2^{-j\alpha}\left(\fint_{2^jB}|S_{\mathcal{L}_a,\gamma}f(x)|^2\mathrm{d}x\right)^{1/2}.
\end{aligned}
$$

This implies (24). Hence the proof of step 2 is completed.

Thus we have proved that the square function $S_{\mathcal{L}_a,\gamma}$ is bounded on $L^p(\mathbb{R}^d)$ for all $d'_\sigma < p < d_\sigma$, that is to say

$$\|S_{\mathcal{L}_a,\gamma}f\|_p \lesssim \|f\|_p.$$

To prove the converse inequality we use duality. By functional calculus, for any $g \in L^{p'}(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} f(x)g(x)\mathrm{d}x = c(\alpha)\int_{\mathbb{R}^d}\int_0^\infty (t\mathcal{L}_a)^{2(\gamma)}e^{-2t\mathcal{L}_a}f(x)g(x)\frac{\mathrm{d}t}{t}\,\mathrm{d}x,$$

where $c(\alpha) = \int_0^\infty t^{2(\gamma)}e^{-2t}\frac{\mathrm{d}t}{t}$. Using Hölder's inequality we obtain

$$
\begin{aligned}
\int_{\mathbb{R}^d} f(x)g(x)\mathrm{d}x &= c(\alpha)\int_{\mathbb{R}^d}\int_0^\infty (t\mathcal{L}_a)^\gamma e^{-t\mathcal{L}_a}f(x)(t\mathcal{L}_a)^\gamma e^{-t\mathcal{L}_a}g(x)\frac{\mathrm{d}t}{t}\,\mathrm{d}x\\
&\lesssim \int_{\mathbb{R}^d} S_{\mathcal{L}_a,\gamma}f(x)S_{\mathcal{L}_a,\gamma}g(x)\mathrm{d}x\\
&\lesssim \|S_{\mathcal{L}_a,\gamma}f\|_p\|S_{\mathcal{L}_a,\gamma}g\|_{p'}.
\end{aligned}
$$

By using the direct inequality $\|S_{\mathcal{L}_a,\gamma}g\|_{p'} \lesssim \|g\|_{p'}$, we get

$$\int_{\mathbb{R}^d} f(x)g(x)\mathrm{d}x \lesssim \|S_{\mathcal{L}_a,\gamma}f\|_p\|g\|_{p'}.$$

As a consequence,

$$\|f\|_p \lesssim \|S_{\mathcal{L}_a,\gamma}f\|_p$$

which completes the proof. □

The following result on the boundedness of square functions involving the difference $t\mathcal{L}_a e^{-t\mathcal{L}_a} - t(-\Delta)^{\alpha/2}e^{-t(-\Delta)^{\alpha/2}}$ will play an essential role in the proofs of the main results.

Theorem 4.2. *Let $d \in \mathbb{N}$, $0 < \alpha < 2 \wedge d$, and $s \in (0, 2]$. Let $a \geqslant a^*$ and σ be defined by (2). We have the following estimate*

$$\left\| \left(\int_0^\infty t^{-s} \left| \left(t\mathcal{L}_a e^{-t\mathcal{L}_a} - t(-\Delta)^{\alpha/2}e^{-t(-\Delta)^{\alpha/2}} \right) f \right|^2 \frac{\mathrm{d}t}{t} \right)^{1/2} \right\|_{L^p} \lesssim \left\| \frac{f}{|x|^{s\alpha/2}} \right\|_{L^p}$$

provided that $d'_\sigma < p < d_\sigma$.

We will provide the proof of theorem 4.2 later on. We now introduce the following two functions for the bound of the kernel of $t(-\Delta)^{\alpha/2}e^{-t(-\Delta)^{\alpha/2}} - t\mathcal{L}_a e^{-t\mathcal{L}_a}$:

$$L_t^\alpha(x, y) := \mathbf{1}_{\{(|x|\vee|y|)^\alpha \leqslant t\}} t^{-d/\alpha} \left(\frac{t^{2/\alpha}}{|x||y|} \right)^{\sigma+} + \mathbf{1}_{\{(|x|\vee|y|)^\alpha \geqslant t\}} \frac{t}{(|x| \vee |y|)^{d+\alpha}} \left(1 \vee \frac{t^{1/\alpha}}{|x| \wedge |y|} \right)^{\sigma+},$$

and

$$M_t^\alpha(x, y) := \mathbf{1}_{\{(|x|\vee|y|)^\alpha \geqslant t\}} \mathbf{1}_{\{\frac{1}{2}|x| \leqslant |y| \leqslant 2|x|\}} \frac{t^{1-d/\alpha}}{(|x| \wedge |y|)^\alpha} \left(1 \wedge \frac{t^{1+d/\alpha}}{|x - y|^{d+\alpha}} \right),$$

where $\sigma_+ = \max\{0, \delta\}$, that is, $\sigma_+ = 0$ if $\sigma \geqslant 0$ and $\sigma_+ = \sigma$ if $\sigma < 0$.

These two functions were used in [22] in the proof of the upper bound for the kernel of the difference $e^{-t\mathcal{L}_a} - e^{-t(-\Delta)^{\alpha/2}}$.

We have the following lemma.

Lemma 4.3. *Let $\sigma \in (-\alpha, (d - \alpha)/2]$. Suppose that $T_t(x, y)$ and $H_t(x, y)$ are two measurable functions defined by*

$$T_t(x, y) = t^{-d/\alpha} \left(\frac{t^{1/\alpha} + |x - y|}{t^{1/\alpha}} \right)^{-d-\alpha} D_\sigma(x, t)D_\sigma(y, t),$$

and

$$H_t(x, y) = t^{-d/\alpha} \left(\frac{t^{1/\alpha} + |x - y|}{t^{1/\alpha}} \right)^{-d-\alpha}.$$

for all $t > 0$ and $x, y \in \mathbb{R}^d \backslash \{0\}$.

Then there exists $C > 0$ such that

$$U_t(x,y) := \int_{\mathbb{R}^d}\int_0^t H_{t-s}(x,z)|z|^{-\alpha}T_s(z,y)\mathrm{d}s\,\mathrm{d}z \leqslant C[L_t^\alpha(x,y) + M_t^\alpha(x,y)],$$

whenever $|y| \geqslant t^{1/\alpha}$ *and* $\frac{1}{2}|y| \leqslant |x| \leqslant 2|y|$.

Proof. The proof of this lemma has the same spirit as that of lemma 3.1 [22]. For the sake of completeness, we sketch it here.

Since $D_\sigma(y,s) \simeq 1$ as $|y| \geqslant t^{1/\alpha} \geqslant s^{1/\alpha}$, we have

$$\begin{aligned} U_t(x,y) \lesssim & \int_{\mathbb{R}^d}\int_0^t s^{-d/\alpha}(t-s)^{-d/\alpha}|z|^{-\alpha}\left(\frac{(t-s)^{1/\alpha}+|x-z|}{(t-s)^{1/\alpha}}\right)^{-d-\alpha} \\ & \left(\frac{s^{1/\alpha}+|z-y|}{s^{1/\alpha}}\right)^{-d-\alpha} D_\sigma(z,s)\mathrm{d}s\,\mathrm{d}z \\ & = \int_{B(0,|x|/8)}\int_0^t \ldots + \int_{B(0,|x|/8)^c}\int_0^t \ldots \\ & = E_1 + E_2. \end{aligned}$$

For the term E_2, in this case $s^{1/\alpha} \leqslant t^{1/\alpha} \leqslant |y| \lesssim |x| \lesssim |z|$, and hence $D_\sigma(z,s) \simeq 1$. This, along with the fact that $|z|^{-\alpha} \lesssim |x|^{-\alpha}$, implies

$$\begin{aligned} E_2 \lesssim & |x|^{-\alpha}\int_{\mathbb{R}^d}\int_0^t s^{-d/\alpha}(t-s)^{-d/\alpha}\left(\frac{(t-s)^{1/\alpha}+|x-z|}{(t-s)^{1/\alpha}}\right)^{-d-\alpha} \\ & \left(\frac{s^{1/\alpha}+|z-y|}{s^{1/\alpha}}\right)^{-d-\alpha}\mathrm{d}s\,\mathrm{d}z. \end{aligned}$$

Denote by $\tilde{p}_t(x,y)$ the kernel of $e^{-t(-\Delta)^{\alpha/2}}$. Then we have

$$\begin{aligned} & \int_{\mathbb{R}^d} s^{-d/\alpha}(t-s)^{-d/\alpha}\left(\frac{(t-s)^{1/\alpha}+|x-z|}{(t-s)^{1/\alpha}}\right)^{-d-\alpha}\left(\frac{s^{1/\alpha}+|z-y|}{s^{1/\alpha}}\right)^{-d-\alpha}\mathrm{d}z \\ & \simeq \int_{\mathbb{R}^d}\tilde{p}_{t-s}(x,z)\tilde{p}_s(z,y)\mathrm{d}z \\ & = \tilde{p}_t(x,y) \simeq t^{-d/\alpha}\left(\frac{t^{1/\alpha}+|x-y|}{t^{1/\alpha}}\right)^{-d-\alpha}, \end{aligned}$$

where in the second line we use theorem 3.3 for $a = 0$ and in the last line we used the semigroup property. Substituting this into the bound of E_2 we get

$$\begin{aligned} E_2 & \lesssim |x|^{-\alpha}t^{-d/\alpha}\left(\frac{t^{1/\alpha}+|x-y|}{t^{1/\alpha}}\right)^{-d-\alpha}\int_0^t \mathrm{d}s \\ & \lesssim \frac{t}{|x|^\alpha}t^{-d/\alpha}\left(\frac{t^{1/\alpha}+|x-y|}{t^{1/\alpha}}\right)^{-d-\alpha} \\ & \lesssim M_t^\alpha(x,y). \end{aligned}$$

Let us move on the term E_1. In this situation, we have $|x-z| \simeq |x|$ and $|y-z| \simeq |y|$ as $|z| \leqslant |x|/8$ and $\frac{1}{2}|y| \leqslant |x| \leqslant |y|$. Hence,

$$
\begin{aligned}
E_1 \simeq & \int_{B(0,|x/8|)} \int_0^t s^{-d/\alpha}(t-s)^{-d/\alpha}|z|^{-\alpha}\left(\frac{|x|}{(t-s)^{1/\alpha}}\right)^{-d-\alpha} \\
& \left(\frac{|y|}{s^{1/\alpha}}\right)^{-d-\alpha} D_\sigma(z,s)\mathrm{d}s\,\mathrm{d}z \\
\simeq & \frac{1}{(|x||y|)^{d+\alpha}} \int_0^t s(t-s) \int_{B(0,|x/8|)} |z|^{-\alpha} D_\sigma(z,s)\mathrm{d}z\,\mathrm{d}s.
\end{aligned}
$$

By lemma 2.2,

$$
\begin{aligned}
\int_{B(0,|x/8|)} |z|^{-\alpha} D_\sigma(z,s)\mathrm{d}z & \lesssim \int_{B(0,|x/8|)} \left(|z|^{-\alpha} + |s|^{\frac{\sigma^+}{\alpha}}|z|^{-\alpha-\sigma^+}\right)\mathrm{d}z \\
& \lesssim |x|^{d-\alpha} + s^{\frac{\sigma^+}{\alpha}}|x|^{d-\alpha-\sigma^+} \simeq |x|^{d-\alpha}.
\end{aligned}
$$

Plugging this into the bound of E_1,

$$
\begin{aligned}
E_1 & \lesssim \frac{|x|^{d-\alpha}}{(|x||y|)^{d+\alpha}} \int_0^t s(t-s)\mathrm{d}s \\
& \lesssim \frac{t^3|x|^{d-\alpha}}{(|x||y|)^{d+\alpha}}.
\end{aligned}
$$

Since in this situation $|x| \simeq |y|$ and $|x|, |y| \gtrsim t^{1/\alpha}$, we have

$$
\begin{aligned}
E_1 & \lesssim \frac{t^3}{|x|^{2\alpha}|y|^{d+\alpha}} \\
& \lesssim \frac{t}{|y|^{d+\alpha}} \\
& \lesssim L_t^\alpha(x,y).
\end{aligned}
$$

This completes our proof. □

Let $Q_t(x,y)$ be the kernel of $t(-\Delta)^{\alpha/2}e^{t(-\Delta)^{\alpha/2}} - t\mathcal{L}_a e^{-t\mathcal{L}_a}$. We have the following estimates:

Proposition 4.4. *Let $d \in \mathbb{N}$, $0 < \alpha < 2 \wedge d$, and $s \in (0,2]$. Let $a \geqslant a^*$ and σ be defined by (2). Then for all $x, y \in \mathbb{R}^d$ and $t > 0$,*

$$
|Q_t(x,y)| \lesssim L_t^\alpha(x,y) + M_t^\alpha(x,y). \tag{26}
$$

Proof. Let $T_t(x,y)$ and $H_t(x,y)$ be two functions as in lemma 4.3. Denote $\tilde{p}_t(x,y)$ and $\tilde{p}_{t,1}(x,y)$ by the kernels of $e^{-t(-\Delta)^{\alpha/2}}$ and $(-\Delta)^{\alpha/2}e^{-t(-\Delta)^{\alpha/2}}$, respectively. Then from theorem 3.3 and proposition 3.5 for $a = 0$, we have

$$
|\tilde{p}_t(x,y)| + t|\tilde{p}_{t,1}(x,y)| \lesssim H_t(x,y)
$$

for all $t > 0$ and $x, y \in \mathbb{R}^d$.

Moreover, from proposition 3.5,

$$|p_t(x,y)| + t|p_{t,1}(x,y)| \lesssim T_t(x,y)$$

for all $t > 0$ and $x, y \in \mathbb{R}^d$.

By symmetry we may assume that $|x| \leqslant |y|$. We now consider two cases.

Case 1: $|y| \leqslant t^{1/\alpha}$, or $|y| \geqslant t^{1/\alpha}$ and $|x| \leqslant |y|/2$. Then, if $\sigma \leqslant 0$ we have

$$\begin{aligned} |Q_t(x,y)| &\lesssim H_t(x,y) = t^{-d/\alpha}\left(\frac{t^{1/\alpha} + |x-y|}{t^{1/\alpha}}\right)^{-d-\alpha} \simeq t^{-d/\alpha} \\ &\lesssim L_t^\alpha(x,y). \end{aligned}$$

On the other hand, if $\sigma > 0$ we have

$$\begin{aligned} |Q_t(x,y)| &\lesssim T_t(x,y) = t^{-d/\alpha}\left(\frac{t^{1/\alpha} + |x-y|}{t^{1/\alpha}}\right)^{-d-\alpha} D_\sigma(x,t) D_\sigma(y,t) \\ &\simeq t^{-d/\alpha}\left(\frac{t^{\frac{2}{\alpha}}}{|x||y|}\right)^{\sigma} \\ &\lesssim L_t^\alpha(x,y). \end{aligned}$$

Hence, in this case

$$|Q_t(x,y)| \lesssim L_t^\alpha(x,y).$$

Case 2: $|y| \geqslant t^{1/\alpha}$ and $|y|/2 \leqslant |x| \leqslant |y|$. In this case by Duhamel's formula,

$$\begin{aligned} &\tilde{p}_t(x,y) - p_t(x,y) \\ &\quad = a\int_0^t \int_{\mathbb{R}^d} \tilde{p}_{t-s}(x,z)|z|^{-\alpha} p_s(z,y)\mathrm{d}z\,\mathrm{d}s \\ &\quad = a\int_0^{t/2} \int_{\mathbb{R}^d} \tilde{p}_{t-s}(x,z)|z|^{-\alpha} p_s(z,y)\mathrm{d}z\,\mathrm{d}s \\ &\qquad + a\int_0^{t/2} \int_{\mathbb{R}^d} \tilde{p}_{s}(x,z)|z|^{-\alpha} p_{t-s}(z,y)\mathrm{d}z\,\mathrm{d}s. \end{aligned}$$

Differentiating both sides with respect to t and then multiplying by t, we come up with, by simple manipulations,

$$\begin{aligned} Q_t(x,y) &= at\int_{\mathbb{R}^d} \tilde{p}_{t/2}(x,z)|z|^{-\alpha} p_{t/2}(z,y)\mathrm{d}z \\ &\quad + at\int_0^{t/2} \int_{\mathbb{R}^d} \tilde{p}_{t-s,1}(x,z)|z|^{-\alpha} p_s(z,y)\mathrm{d}z\,\mathrm{d}s \\ &\quad + at\int_{t/2}^{t} \int_{\mathbb{R}^d} \tilde{p}_{t-s}(x,z)|z|^{-\alpha} p_{s,1}(z,y)\mathrm{d}z\,\mathrm{d}s \\ &= I_1 + I_2 + I_3. \end{aligned} \tag{27}$$

We can estimate the term I_1 as follows:

$$t\int_{\mathbb{R}^d}\tilde{p}_{t/2}(x,z)|z|^{-\alpha}p_{t/2}(z,y)\mathrm{d}z = 6\int_{t/3}^{t/2}\int_{\mathbb{R}^d}\tilde{p}_{t/2}(x,z)|z|^{-\alpha}p_{t/2}(z,y)\mathrm{d}z\,\mathrm{d}s.$$

Note that for $s\in[t/3,t/2]$, $H_{t-s}(\cdot,\cdot)\simeq H_{t/2}(\cdot,\cdot)\gtrsim\tilde{p}_{t/2}(\cdot,\cdot)$ and $T_s(\cdot,\cdot)\simeq T_{t/2}(\cdot,\cdot)\gtrsim p_{t/2}(\cdot,\cdot)$. Consequently,

$$\begin{aligned} t\int_{\mathbb{R}^d}\tilde{p}_{t/2}(x,z)|z|^{-\alpha}p_{t/2}(z,y)\mathrm{d}z &\lesssim \int_{t/3}^{t/2}\int_{\mathbb{R}^d}H_{t-s}(x,z)|z|^{-\alpha}T_s(z,y)\mathrm{d}z\,\mathrm{d}s \\ &\lesssim \int_0^t\int_{\mathbb{R}^d}H_{t-s}(x,z)|z|^{-\alpha}T_s(z,y)\mathrm{d}z\,\mathrm{d}s. \end{aligned}$$

Then applying lemma 4.3,

$$I_1\leqslant C[L_t^\alpha(x,y)+M_t^\alpha(x,y)].$$

For the second term, note that for $s\in(0,t/2)$,

$$t|\tilde{p}_{t-s,1}(x,z)|\simeq(t-s)|\tilde{p}_{t-s,1}(x,z)|\lesssim H_{t-s}(x,z),$$

which implies that

$$\begin{aligned} I_2 &\lesssim \int_0^{t/2}\int_{\mathbb{R}^d}H_{t-s}(x,z)|z|^{-\alpha}T_s(z,y)\mathrm{d}z\,\mathrm{d}s \\ &\leqslant C[L_t^\alpha(x,y)+M_t^\alpha(x,y)], \end{aligned}$$

where in the second inequality we used lemma 4.3.

Likewise,

$$\begin{aligned} I_3 &\lesssim \int_{t/2}^{t}\int_{\mathbb{R}^d}H_{t-s}(x,z)|z|^{-2}T_s(z,y)\mathrm{d}z\,\mathrm{d}s \\ &\leqslant C[L_t^\alpha(x,y)+M_t^\alpha(x,y)]. \end{aligned}$$

This completes our proof. □

Remark 4.5. In [22], a similar upper bound was obtained for the kernel of the difference $e^{t(-\Delta)^{\alpha/2}}-e^{-t\mathcal{L}_a}$. However, this is not suitable for our purpose since the square functions used here are different.

We now turn to the proof theorem 4.2.

Proof of theorem 4.2. Using proposition 4.4

$$\begin{aligned}
&\left(\int_0^\infty t^{-s}\left|\left(t\mathcal{L}_a e^{-t\mathcal{L}_a} - t(-\Delta)^{\alpha/2}e^{-t(-\Delta)^{\alpha/2}}\right)f(x)\right|^2\frac{\mathrm{d}t}{t}\right)^{1/2}\\
&\leqslant \left[\sum_{j\in\mathbb{Z}}\int_{2^{2j}}^{2^{2(j+1)}} t^{-s}\left|\left(t\mathcal{L}_a e^{-t\mathcal{L}_a} - t(-\Delta)^{\alpha/2}e^{-t(-\Delta)^{\alpha/2}}\right)f(x)\right|^2\frac{\mathrm{d}t}{t}\right]^{1/2}\\
&\leqslant \left[\sum_{j\in\mathbb{Z}}\int_{2^{\alpha j}}^{2^{\alpha(j+1)}} t^{-s}\left(\int_{\mathbb{R}^d}[L_t^\alpha(x,y)+M_t^\alpha(x,y)]|f(y)|\mathrm{d}y\right)^2\frac{\mathrm{d}t}{t}\right]^{1/2}\\
&\leqslant \sum_{j\in\mathbb{Z}} 2^{-js\alpha/2}\int_{\mathbb{R}^d}\left[L^\alpha_{2^{\alpha(j+1)}}(x,y)+M^\alpha_{2^{j\alpha}}(x,y)\right]|f(y)|\mathrm{d}y
\end{aligned}$$

where in the last inequality we used the fact that $\ell_1 \hookrightarrow \ell_2$. At this stage, arguing similarly to the proof of proposition 3 in [21] we deduce the desired result.

This completes our proof. □

We now recall the Hardy's inequality for the operator $\mathcal{L}_a$ in [21].

Theorem 4.6. *Let $d\in\mathbb{N}$, $0<\alpha<2\wedge d$, $a\geqslant a^*$ and let σ be defined by (2). Suppose $0<s\alpha/2<d$. Then for $d'_\sigma<p<d_{s\alpha/2+\sigma}$ we have*

$$\left\||x|^{-\alpha s/2}f\right\|_p \lesssim \left\|\mathcal{L}_a^{s/2}f\right\|_p.$$

Finally, we are ready to give the proof of theorem 1.1.

Proof of theorem 1.1. Fix $0<s\leqslant 2$, $d'_\sigma<p<d_{s\alpha/2+\sigma\vee 0}$. Then by theorems 4.1, 4.2 and 4.6 we have

$$\begin{aligned}
\left\|(-\Delta)^{s\alpha/2}f\right\|_p &\lesssim \left\|\left(\int_0^\infty t^{-s}|t(-\Delta)^{\alpha/2}e^{-t(-\Delta)^{\alpha/2}}f|^2\frac{\mathrm{d}t}{t}\right)^{1/2}\right\|_p\\
&\lesssim \left\|\left(\int_0^\infty t^{-s}\left|(t\mathcal{L}_a e^{-t\mathcal{L}_a}-t(-\Delta)^{\alpha/2}e^{-t(-\Delta)^{\alpha/2}})f\right|^2\frac{\mathrm{d}t}{t}\right)^{1/2}\right\|_p\\
&\quad+\left\|\left(\int_0^\infty t^{-s}|t\mathcal{L}_a e^{-t\mathcal{L}_a}f|^2\frac{\mathrm{d}t}{t}\right)^{1/2}\right\|_p\\
&\lesssim \left\|\frac{f}{|x|^{s\alpha/2}}\right\|_p+\left\|\mathcal{L}_a^{s/2}f\right\|_p\\
&\lesssim \left\|\mathcal{L}_a^{s/2}f\right\|_p,
\end{aligned}$$

where in the first inequality we using theorem 4.1 for $a=0$.

Conversely, for $d'_\sigma < p < \frac{d}{(s\alpha/2)\vee\sigma}$, we have

$$\begin{aligned}
\left\|\mathcal{L}_a^{s/2} f\right\|_p &\lesssim \left\|\left(\int_0^\infty t^{-s}|t\mathcal{L}_a e^{-t\mathcal{L}_a} f|^2 \frac{\mathrm{d}t}{t}\right)^{1/2}\right\|_p \\
&\lesssim \left\|\left(\int_0^\infty t^{-s}\left|(t\mathcal{L}_a e^{-t\mathcal{L}_a} - t(-\Delta)^{\alpha/2} e^{-t(-\Delta)^{\alpha/2}}) f\right|^2 \frac{\mathrm{d}t}{t}\right)^{1/2}\right\|_p \\
&\quad + \left\|\left(\int_0^\infty t^{-s}|t(-\Delta)^{\alpha/2} e^{-t(-\Delta)^{\alpha/2}} f|^2 \frac{\mathrm{d}t}{t}\right)^{1/2}\right\|_p \\
&\lesssim \left\|\frac{f}{|x|^{s\alpha/2}}\right\|_p + \left\|(-\Delta)^{s\alpha/4} f\right\|_p \\
&\lesssim \left\|(-\Delta)^{s\alpha/4} f\right\|_p,
\end{aligned}$$

where in the last inequality we used theorem 4.6 for $a = 0$.

This completes our proof. □

Remark 4.7. It is possible to extend the results in theorem 1.1 to the operator $(-\Delta)^{\alpha/2} + V$, where V is a function satisfying

$$\frac{a}{|x|^\alpha} \leqslant V(x) \leqslant \frac{\tilde{a}}{|x|^\alpha}$$

with $a^* \leqslant a \leqslant \tilde{a} < \infty$.

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