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A semilinear elliptic equation with a mild singularity at $u = 0$: Existence and homogenization

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A B S T R A C T

In this paper we consider singular semilinear elliptic equations whose prototype is the following

$$\begin{cases} -div\, A(x)Du = f(x)g(u) + l(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is an open bounded set of $\mathbb{R}^N$, $N \geq 1$, $A \in L^\infty(\Omega)^{N\times N}$ is a coercive matrix, $g : [0, +\infty[\to [0, +\infty]$ is continuous, and $0 \leq g(s) \leq \frac{1}{s^\gamma} + 1$ for every $s > 0$, with $0 < \gamma \leq 1$ and $f, l \in L^r(\Omega)$, $r = \frac{2N}{N+2}$ if $N \geq 3$, $r > 1$ if $N = 2$, $r = 1$ if $N = 1$, $f(x), l(x) \geq 0$ a.e. $x \in \Omega$.
We prove the existence of at least one nonnegative solution as well as a stability result; we also prove uniqueness if $g(s)$ is nonincreasing or "almost nonincreasing". Finally, we study the homogenization of these equations posed in a sequence of domains Ω^ε obtained by removing many small holes from a fixed domain Ω.

R É S U M É

Dans cet article nous étudions des équations elliptiques semi-linéaires singulières dont le prototype est le suivant

$$\begin{cases} -div\, A(x)Du = f(x)g(u) + l(x) & \text{dans } \Omega, \\ u = 0 & \text{sur } \partial\Omega, \end{cases}$$

où Ω est un ouvert borné de $\mathbb{R}^N$, $N \geq 1$, $A \in L^\infty(\Omega)^{N\times N}$ est une matrice coercive, $g : [0, +\infty[\to [0, +\infty]$ est une fonction continue qui vérifie $0 \leq g(s) \leq \frac{1}{s^\gamma} + 1$ pour tout $s > 0$, avec $0 < \gamma \leq 1$ et $f, l \in L^r(\Omega)$ avec $r = \frac{2N}{N+2}$ si $N \geq 3$, $r > 1$ si $N = 2$ et $r = 1$ si $N = 1$, $f(x), l(x) \geq 0$ p.p. $x \in \Omega$.

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Nous démontrons l'existence d'au moins une solution positive de cette équation et un résultat de stabilité ; de plus nous démontrons l'unicité de la solution si $g(s)$ est décroissante ou "presque décroissante".
Nous étudions enfin l'homogénéisation d'une suite de ces équations posées dans des domaines Ω^ε obtenus en perforant un domaine fixe Ω par des trous de plus en plus petits et de plus en plus nombreux.

1. Introduction

We deal in this paper with nonnegative solutions to the following singular semilinear problem

$$\begin{cases} -div\, A(x)Du = F(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where the model for the function $F(x,u)$ is

$$F(x,u) = f(x)g(u) + l(x),$$

for some continuous function $g(s)$ with $0 \le g(s) \le \frac{1}{s^\gamma} + 1$ for every $s > 0$, with $0 < \gamma \le 1$, and some nonnegative functions $f(x)$ and $l(x)$ which belong to suitable Lebesgue spaces.

Note that (except as far as uniqueness is concerned) we do not require g to be nonincreasing, so that functions g like

$$g(s) = \frac{1}{s^\gamma}\big(2 + \sin\frac{1}{s}\big)$$

can be considered.

In the present paper we are first interested in existence, uniqueness and stability results for this kind of problems. After this, we will study the asymptotic behaviour, as ε goes to zero, of a sequence of problems posed in domains Ω^ε obtained by removing many small holes from a fixed domain Ω, in the framework of [4].

As far as existence and regularity results for this kind of problems are concerned, we refer to the classical paper [5] by M.G. Crandall, P.H. Rabinowitz and L. Tartar, and to the paper [2] by L. Boccardo and L. Orsina which inspired our work. We also refer to the references quoted in these papers as well as those quoted in the paper [1] by L. Boccardo and J. Casado-Díaz which deals with the homogenization of this problem for a sequence of matrices $A^\varepsilon(x)$.

In [5] the authors show the existence of a classical positive solution if the matrix $A(x)$, the boundary $\partial\Omega$ and the function $F(x,s)$ are smooth enough; the function $F(x,s)$, which is not supposed to be nonincreasing in s, is bounded from above uniformly for $x \in \overline{\Omega}$ and $s \ge 1$. Boundary behaviour of $u(x)$ and $|Du(x)|$ when x tends to $\partial\Omega$ is also studied.

In [2] the authors study the problem (1.1) with $F(x,u) = \frac{f(x)}{u^\gamma}$, $\gamma > 0$ and f in Lebesgue spaces. They prove existence, uniqueness and regularity results depending on the values of γ and on the summability of f. Specifically, they prove the existence of strictly positive distributional solutions. In order to prove their results, they work by approximation and construct an increasing sequence $(u_n)_{n\in\mathbb{N}}$ of solutions to the

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(nonsingular) problems

$$\begin{cases} -div\, A(x)Du_n = \dfrac{f_n(x)}{(u_n+\frac{1}{n})^\gamma} & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f_n(x) = \min\{f(x), n\}$. This sequence satisfies, for every $\omega \subset\subset \Omega$,

$$u_n(x) \geq u_{n-1}(x) \geq \ldots \geq u_1(x) \geq c_\omega > 0, \ \forall x \in \omega. \tag{1.2}$$

In order to prove this property, it is essential to assume that the nonlinearity $F(x,s)$ is nonincreasing in the s variable and to use, as a main tool, the strong maximum principle. Note that (1.2) provides the existence of a limit function $u = sup_n\, u_n$ which is strictly positive on every compact set ω of Ω; in addition, (1.2) implies that, on every such set ω, the functions $\frac{f_n(x)}{(u_n+\frac{1}{n})^\gamma}$ are uniformly dominated by a function $h_\omega \in L^1(\omega)$. This allow the authors to prove that the function u is a solution in the sense of distributions.

In the present paper, we are interested in giving existence and stability results without assuming that $F(x,s)$ is nonincreasing in the s variable and without using the strong maximum principle in the proofs of these results. The main interest of this lies in the fact that this kind of proofs provides the tools to deal with the homogenization of the problem

$$\begin{cases} -div\, A(x)Du^\varepsilon = F(x,u^\varepsilon) & \text{in } \Omega^\varepsilon, \\ u^\varepsilon = 0 & \text{on } \partial\Omega^\varepsilon, \end{cases}$$

when Ω^ε is obtained by removing many small holes from Ω (see Theorem 5.2). Of course, the existence and stability results (see Theorem 4.1 and Theorem 4.2) have also an autonomous interest, due to the more general assumptions and to a different method of proof.

Moreover, we point out that this method, which avoids using the strong maximum principle, also has a strong interest in other problems where one cannot expect the strict positivity of the solution on every compact set of Ω. Let us briefly describe some of these situations.

A first situation is the case of singular parabolic problems with p-laplacian type principal part, $p > 1$, and nonnegative data u_0 and f, whose model is the following:

$$\begin{cases} u_t - div(|Du|^{p-2}Du) = f(x,t)\big(\dfrac{1}{u^\gamma}+1\big) & \text{in } \Omega\times(0,T), \\ u(x,t) = 0 & \text{on } \partial\Omega\times(0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where $\gamma > 0$. In this case, due to the assumption $p > 1$ and the fact that the initial datum u_0 is not assumed to be strictly positive, the method of expansion of positivity cannot be applied and one cannot guarantee that the solution is strictly positive inside $\Omega \times (0,T)$ (see [3]).

A second situation deals with existence and homogenization for elliptic singular problems in an open domain Q of $\mathbb{R}^N$ which is made of an upper part Q_1^ε and a lower part Q_2^ε separated by an oscillating interface Γ^ε, when the boundary conditions at the interface Γ^ε are the continuity of the flux and the fact that this flux is proportional to the jump of the solution through the interface. Our method also applies in this case (see [9]).

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A third situation where our method applies is the case of a singular semilinear problem which involves a zeroth-order term whose coefficient is a nonnegative bounded measure μ which also belongs to $H^{-1}(\Omega)$, namely

$$\begin{cases} u \geq 0 & \text{in } \Omega, \\ -div\, A(x)Du + \mu u = F(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Problem (1.3) naturally arises when performing the homogenization of (1.1) (where there is no zeroth-order term) posed on a domain Ω^ε obtained from Ω by perforating Ω by many small holes (see Section 5 below). Our method allows us to obtain results of existence, stability, uniqueness and homogenization, even if the strong maximum principle does not hold true in general in such a context (see [10] and [11]) (see also [13]).

In the present paper we consider the case $0 < \gamma \leq 1$. We consider the case $\gamma > 1$ (and more generally the case of a general singularity) in the papers [10] and [11] (see also [12]). Let us point out that in the latest case, no global energy estimate is available for the solutions when the singularity has a strong behaviour. This makes the problem more difficult, in particular from the point of view of homogenization. For this reason, we have to introduce a convenient (even if rather complicated) framework, in which we prove existence, stability, uniqueness and homogenization results. Let us emphasize that despite the changes which are made necessary by this framework, the method of the present paper provides the guide to follow also in the case of a general singularity.

The precise definition of the solution that we use in the present paper is given in Definition 3.1. Note that the solutions are nonnegative.

The keystone in our proofs is the analysis of the behaviour of the singular term near the singularity, which is done in Proposition 6.2 of Section 6.

On the other hand, if we suppose that $F(x,s)$ is "almost nonincreasing" in s (see (2.4)), we prove the uniqueness of the solution (see Theorem 4.4).

Let us now come to the homogenization problem

$$\begin{cases} -div\, A(x)Du^\varepsilon = F(x,u^\varepsilon) & \text{in } \Omega^\varepsilon, \\ u^\varepsilon = 0 & \text{on } \partial\Omega^\varepsilon, \end{cases}$$

when Ω^ε is obtained by removing many small holes from a given domain Ω according to the framework of [4] (for the study of this problem we have to assume that $N \geq 2$, see Remark 5.1 below). The general questions we are concerned with are the following: do the solutions u^ε converge to a limit u when the parameter ε tends to zero? If this limit exists, can it be characterized? Will the result be the same as in the nonsingular case? In principle the answer is not obvious at all since, as ε tends to zero, the number of holes becomes greater and greater and the singular set for the right-hand side (which includes at least the holes' boundary) tends to "invade" the entire Ω.

Actually we will prove that a "strange term" appears in the limit of the singular problem in the same way as in the nonsingular case studied in [4]. This result is a priori not obvious at all, and a very different behaviour could have been expected.

We now describe the plan of the paper. Section 2 deals with the precise assumptions on problem (1.1). In Section 3 we give the precise definition of a solution to problem (1.1) which we will use in the whole of this paper. Section 4 is devoted to the statements of the existence, stability and uniqueness results; in addition a regularity result dealing with the boundedness of solutions is stated in this Section. In Section 5 we give the statement of the homogenization result in a domain with many small holes and Dirichlet boundary condition, as well as a corrector result. In Section 6 we prove a priori estimates. Section 7 is devoted to the proofs of the stability, existence and regularity results stated in Theorems 4.2 and 4.1 and in Proposition 4.3.

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In Section 8 we state and prove a comparison principle and we prove the uniqueness Theorem 4.4. Finally we prove in Section 9 the homogenization Theorem 5.2 and the corrector Theorem 5.5.

2. Assumptions

In this Section, we give the assumptions on problem (1.1).

We assume that Ω is an open bounded set of $\mathbb{R}^N$, $N \geq 1$ (no regularity is assumed on the boundary $\partial\Omega$ of Ω), that the matrix A satisfies

$$\begin{cases} A \in L^\infty(\Omega)^{N\times N}, \\ \exists \alpha > 0,\ A(x) \geq \alpha I \quad \text{a.e. } x \in \Omega, \end{cases} \tag{2.1}$$

and that the function F satisfies

$$\begin{cases} F : \Omega \times [0, +\infty[\to [0, +\infty] \text{ is a Carathéodory function,} \\ \text{i.e. } F \text{ satisfies} \\ i) \text{ for a.e. } x \in \Omega,\ s \in [0, +\infty[\to F(x, s) \in [0, +\infty] \text{ is continuous,} \\ ii) \ \forall s \in [0, +\infty[,\ x \in \Omega \to F(x, s) \in [0, +\infty] \text{ is measurable,} \end{cases} \tag{2.2}$$

$$\begin{cases} \exists \gamma, \exists h \text{ with} \\ i)\ 0 < \gamma \leq 1, \\ ii)\ h \in L^r(\Omega),\ r = \frac{2N}{N+2} \text{ if } N \geq 3,\ r > 1 \text{ if } N = 2,\ r = 1 \text{ if } N = 1, \\ iii)\ h(x) \geq 0 \text{ a.e. } x \in \Omega, \\ \text{such that} \\ iv)\ 0 \leq F(x, s) \leq h(x)\left(\dfrac{1}{s^\gamma} + 1\right) \text{ a.e. } x \in \Omega, \forall s > 0. \end{cases} \tag{2.3}$$

Remark 2.1. The function $F = F(x, s)$ is a nonnegative Carathéodory function with values in $[0, +\infty]$. But, in view of (2.3 *iv*), the function $F(x, s)$ can take the value $+\infty$ only when $s = 0$ (or, in other terms, $F(x, s)$ is always finite when $s > 0$). □

On the other hand, for proving comparison and uniqueness results, we will assume that $F(x, s)$ is "*almost nonincreasing*" in s: denoting by λ_1 the first eigenvalue of the operator $-div\, {}^sA(x)D$ in $H_0^1(\Omega)$, where ${}^sA(x) = (A(x) + {}^tA(x))/2$ is the symmetrized part of the matrix $A(x)$, we will assume that

$$\begin{cases} \text{there exists } \lambda \text{ with } 0 \leq \lambda < \lambda_1 \text{ such that} \\ F(x, s) - \lambda s \leq F(x, t) - \lambda t \text{ a.e. } x \in \Omega,\ \forall s, \forall t,\ 0 \leq t \leq s, \end{cases} \tag{2.4}$$

or in other terms that $F(x, s) - \lambda s$ is nonincreasing in s for some λ such that $0 \leq \lambda < \lambda_1$.

Remark 2.2. Note that (2.4) holds with $\lambda = 0$ when F is assumed to be nonincreasing. But if in place of (2.4) one only assumes that the function

$$s \in [0, +\infty] \to F(x, s) - \lambda_1 s \ \text{ is nonincreasing,} \tag{2.5}$$

uniqueness of the solution to problem (1.1) in general does not hold true, see Remark 8.2 below. □

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Notation. We denote by $\mathcal{D}(\Omega)$ the space of the $C^\infty(\Omega)$ functions whose support is a compact set included in Ω, and by $\mathcal{D}'(\Omega)$ the space of distributions on Ω.

Since Ω is bounded, $\|Dw\|_{L^2(\Omega)^N}$ is a norm equivalent to $\|w\|_{H^1(\Omega)}$ on $H_0^1(\Omega)$. We set

$$\|w\|_{H_0^1(\Omega)} = \|Dw\|_{L^2(\Omega)^N}, \quad \forall w \in H_0^1(\Omega).$$

For every $s \in \mathbb{R}$ and every $k > 0$ we define

$$s^+ = \max\{s, 0\},\ s^- = \max\{0, -s\},$$
$$T_k(s) = \max\{-k, \min\{s, k\}\}, \quad G_k(s) = s - T_k(s).$$

For $l : \Omega \longrightarrow [0, +\infty]$ a measurable function we denote

$$\{l = 0\} = \{x \in \Omega : l(x) = 0\},\ \{l > 0\} = \{x \in \Omega : l(x) > 0\}.$$

3. Definition of a solution to problem (1.1)

We now give a precise definition of a solution to problem (1.1).

Definition 3.1. Assume that the matrix A and the function F satisfy (2.1), (2.2) and (2.3). We will say that u is a solution to problem (1.1) if u satisfies

$$u \in H_0^1(\Omega), \tag{3.1}$$

$$u \geq 0 \quad \text{a.e. in } \Omega, \tag{3.2}$$

$$\begin{cases} \forall \varphi \in H_0^1(\Omega) \text{ with } \varphi \geq 0, \text{ one has} \\ \displaystyle\int_\Omega F(x,u)\varphi < +\infty, \\ \displaystyle\int_\Omega A(x)DuD\varphi = \int_\Omega F(x,u)\varphi. \end{cases} \tag{3.3}$$

Remark 3.2. Given $\varphi \in H_0^1(\Omega)$, one can take φ^+ and φ^- as test functions in (3.3). This implies that (3.3) is actually equivalent to

$$\begin{cases} \forall \varphi \in H_0^1(\Omega), \text{ one has} \\ \displaystyle\int_\Omega F(x,u)|\varphi| < +\infty, \\ \displaystyle\int_\Omega A(x)DuD\varphi = \int_\Omega F(x,u)\varphi. \end{cases} \tag{3.4}$$

This also proves that for every solution u to problem (1.1) in the sense of Definition 3.1 one has $F(x,u)\varphi \in L^1(\Omega)$ for every $\varphi \in H_0^1(\Omega)$ and that

$$F(x,u) \in L^1_{\text{loc}}(\Omega), \quad -div\, A(x)Du = F(x,u) \text{ in } \mathcal{D}'(\Omega). \qquad \square$$

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Remark 3.3. The nonnegative measurable function $F(x, u(x))$ can take infinite values when $u(x) = 0$. The integral $\int_\Omega F(x,u)\varphi$ is therefore correctly defined as a number in $[0, +\infty]$ for every measurable function $\varphi \geq 0$.

In (3.3) we require that this number is finite for every $\varphi \in H_0^1(\Omega)$, $\varphi \geq 0$, when u is a solution to problem (1.1) in the sense of Definition 3.1. This in particular implies that

$$F(x, u(x)) \text{ is finite almost everywhere on } \Omega, \tag{3.5}$$

or in other terms that

$$\text{meas}\{x \in \Omega : u(x) = 0 \text{ and } F(x, 0) = +\infty\} = 0. \tag{3.6}$$

A result which is stronger than (3.6) will be given in Proposition 3.4, and an even stronger result will be given in Proposition 3.5 and Remark 3.7; note however that the strong maximum principle is used to obtain the results of Proposition 3.5 and Remark 3.7. □

Proposition 3.4. *Assume that the matrix A and the function F satisfy (2.1), (2.2) and (2.3). Then every solution u to problem (1.1) in the sense of Definition 3.1 satisfies*

$$F(x, 0) = 0 \quad \textit{for a.e. } x \in \{u = 0\} \tag{3.7}$$

and

$$\int_\Omega F(x,u)\varphi = \int_{\{u>0\}} F(x,u)\varphi \quad \forall \varphi \in H_0^1(\Omega). \tag{3.8}$$

Proof. In Proposition 6.3 below we prove that for every u solution to problem (1.1) in the sense of Definition 3.1 one has

$$\int_{\{u=0\}} F(x,u)\varphi = 0 \quad \forall \varphi \in H_0^1(\Omega), \varphi \geq 0, \tag{3.9}$$

which of course implies (3.8).

Writing

$$\{u = 0\} = \Big(\{u = 0\} \cap \{F(x, 0) = 0\}\Big) \cup \Big(\{u = 0\} \cap \{0 < F(x, 0) \leq +\infty\}\Big)$$

implies that (3.9) is equivalent to

$$\int_{\{u=0\}\cap\{0<F(x,0)\leq+\infty\}} F(x,u)\varphi = 0 \quad \forall \varphi \in H_0^1(\Omega),\ \varphi \geq 0.$$

The latest assertion is equivalent to

$$\text{meas}\{x \in \Omega : u(x) = 0 \text{ and } 0 < F(x, 0) \leq +\infty\} = 0,$$

which is equivalent to (3.7). Proposition 3.4 is therefore proved.

Note that (3.7) is also equivalent to

$$\{x \in \Omega : u(x) = 0\} \subset \{x \in \Omega : F(x, 0) = 0\} \text{ except for a set of zero measure,} \tag{3.10}$$

and also equivalent to

$$\{x \in \Omega : 0 < F(x, 0) \leq +\infty\} \subset \{x \in \Omega : u(x) > 0\} \text{ except for a set of zero measure.} \qquad \square$$

The following Proposition 3.5 and Remark 3.7 assert that for every solution u to problem (1.1) in the sense of Definition 3.1, we can have two possibilities: either $u(x) > 0$ a.e. in Ω or $u \equiv 0$ in Ω.

This assertion is stronger than (3.10), but its proof uses the strong maximum principle.

As pointed out in the Introduction, the strong maximum principle is one of the key tools used in the proofs of the results obtained in [2] by L. Boccardo and L. Orsina, results which inspired the present paper.

Note that, in contrast with the proofs of the results in [2], the proofs of all the results in the present paper do not make use neither of the strong maximum principle nor of the results of Proposition 3.5 and Remark 3.7 below.

Proposition 3.5. *Assume that the matrix A and the function F satisfy (2.1), (2.2) and (2.3). Then every solution u to problem (1.1) in the sense of Definition 3.1 satisfies*

$$\textit{either } u \equiv 0 \textit{ or } meas\{x \in \Omega : u(x) = 0\} = 0. \tag{3.11}$$

Proof. We first recall the statement of the strong maximum principle (Theorem 8.19 of [14]), or more exactly of its variant, where u is replaced by $-u$. In this variant, Theorem 8.19 of [14] becomes

$$\begin{cases} \text{Let } u \in H^1(\Omega) \text{ which satisfies } Lu \leq 0. \\ \text{If for some ball } B \subset\subset \Omega \text{ we have } \inf_B u = \inf_\Omega u \leq 0, \\ \text{then } u \text{ is constant in } \Omega. \end{cases} \tag{3.12}$$

In our situation one has $Lu = div\, A(x)Du$, and $Lu \leq 0$ is nothing but $-div\, A(x)Du \geq 0$. Therefore (3.12) implies the following result

$$\begin{cases} \text{Let } u \in H^1(\Omega) \text{ which satisfies } -div\, A(x)Du \geq 0 \text{ in } \mathcal{D}'(\Omega). \\ \text{If } u \geq 0 \text{ a.e. in } \Omega \text{ and if } \inf_B u = 0 \text{ for some ball } B \subset\subset \Omega, \\ \text{then } u = 0 \text{ in } \Omega, \end{cases} \tag{3.13}$$

since when u is a constant in Ω with $\inf_B u = 0$, then $u = 0$ in Ω.

But one has the alternative:

$$\begin{cases} \text{either } \inf_B u > 0 \text{ for every ball } B \subset\subset \Omega, \\ \text{or there exists a ball } B \subset\subset \Omega \text{ such that } \inf_B u = 0. \end{cases}$$

In the first case, one has $\text{meas}\{x \in \Omega : u(x) = 0\} = 0$; in the second case, (3.13) implies that $u \equiv 0$.

This proves (3.11). $\square$

Remark 3.6. Actually the proof of Proposition 3.5 (which uses the strong maximum principle) provides a result which is much stronger than (3.11), namely

$$\begin{cases} \text{either } u \equiv 0, \\ \text{or for every ball } B \subset\subset \Omega \text{ one has} \\ \inf_B u \geq c(u,B) \text{ for some } c(u,B) \in \mathbb{R},\ c(u,B) > 0. \end{cases} \tag{3.14}$$

Since the strong maximum principle continues to hold if the operator $-div\, A(x)Du$ is replaced by $-div\, A(x)Du + a_0 u$, with $a_0 \in L^\infty(\Omega)$, $a_0 \geq 0$, both (3.11) and (3.14) continue to hold for such an operator.

But when $a_0 \geq 0$ does not belong to $L^\infty(\Omega)$ and is only a nonnegative element of $H^{-1}(\Omega)$ (this can be the case in the result of the homogenization process with many small holes that we will perform in Section 5), the strong maximum principle does not hold anymore for the operator $-div\, A(x)Du + a_0 u$ (see [10] for a counter-example due to G. Dal Maso), and therefore (3.14) does not hold anymore for such an operator. □

Remark 3.7. If $u \equiv 0$ is a solution to problem (1.1) in the sense of Definition 3.1, then Proposition 3.4 implies that $F(x,0) = 0$ for almost every $x \in \Omega$.

Conversely, if $F(x,0) \not\equiv 0$, $u \equiv 0$ is not a solution to problem (1.1) in the sense of Definition 3.1, and Proposition 3.5 (or more exactly (3.14)) then implies that

$$u(x) > 0 \text{ a.e. } x \in \Omega. \qquad \square$$

4. Statements of the existence, stability, uniqueness and regularity results

In this Section we state results of existence, stability and uniqueness of the solution to problem (1.1) in the sense of Definition 3.1. We also state a result (Proposition 4.3) which provides the boundedness of the solutions under a regularity assumption on the function F.

Theorem 4.1 *(Existence). Assume that the matrix A and the function F satisfy (2.1), (2.2) and (2.3). Then there exists at least one solution u to problem (1.1) in the sense of Definition 3.1.*

The proof of Theorem 4.1 will be done in Subsection 7.2 below. It is based on a stability result (see Theorem 4.2 below), and on a priori estimates of $\|u\|_{H^1_0(\Omega)}$ and of $\displaystyle\int_{\{u\leq\delta\}} F(x,u)\varphi$ for every $\varphi \in H^1_0(\Omega)$, $\varphi \geq 0$ and for every $\delta > 0$ (see Propositions 6.1 and 6.2 below); these a priori estimates are satisfied by every solution u to problem (1.1) in the sense of Definition 3.1.

Theorem 4.2 *(Stability). Assume that the matrix A satisfies assumption (2.1). Let F_n be a sequence of functions and F_∞ be a function which both satisfy assumptions (2.2) and (2.3) for the same γ and h. Assume moreover that*

$$a.e.\ x \in \Omega,\ F_n(x,s_n) \to F_\infty(x,s_\infty) \text{ if } s_n \to s_\infty, s_n \geq 0, s_\infty \geq 0. \tag{4.1}$$

Let u_n be any solution to problem $(1.1)_n$ in the sense of Definition 3.1, where $(1.1)_n$ is the problem (1.1) with $F(x,u)$ replaced by $F_n(x,u_n)$.

Then there exists a subsequence, still labeled by n, and a function u_∞, which is a solution to problem $(1.1)_\infty$ in the sense of Definition 3.1, such that

$$u_n \to u_\infty \text{ in } H^1_0(\Omega) \text{ strongly.} \tag{4.2}$$

In the following Proposition we state a regularity result.

Proposition 4.3 *(Boundedness). Assume that the matrix A and the function F satisfy (2.1), (2.2) and (2.3). Assume moreover that the function h which appears in (2.3) satisfies*

$$h \in L^t(\Omega),\ t > \frac{N}{2} \ \text{ if } N \geq 2,\ \ t = 1 \text{ if } N = 1. \tag{4.3}$$

Then every u solution to problem (1.1) in the sense of Definition 3.1 belongs to $L^\infty(\Omega)$ and satisfies the estimate

$$\|u\|_{L^\infty(\Omega)} \leq 1 + \frac{2}{\alpha}\, C(|\Omega|, N, t)\, \|h\|_{L^t(\Omega)}, \tag{4.4}$$

where the constant $C(|\Omega|, N, t)$ depends only on $|\Omega|$, N and t and is a nondecreasing function of $|\Omega|$.

Finally, our uniqueness result is a consequence of the comparison principle stated in Theorem 8.1 below. Note that these two results are the only results where the "almost nonincreasing" character in s of the function $F(x, s)$ is used in the present paper.

Theorem 4.4 *(Uniqueness). Assume that the matrix A and the function F satisfy (2.1), (2.2) and (2.3). Assume moreover that the function F also satisfies assumption (2.4). Then the solution to problem (1.1) in the sense of Definition 3.1 is unique.*

Remark 4.5. When assumptions (2.1), (2.2), (2.3) as well as (2.4) hold true, Theorems 4.1, 4.2 and 4.4 together assert that problem (1.1) is well posed in the sense of Hadamard in the framework of Definition 3.1. □

5. Statement of the homogenization result in a domain with many small holes and Dirichlet boundary condition

In this Section we deal with the asymptotic behaviour, as ε tends to zero, of solutions in the sense of Definition 3.1 to the singular semilinear problem

$$\begin{cases} -div\, A(x) Du^\varepsilon = F(x, u^\varepsilon) & \text{in } \Omega^\varepsilon, \\ u^\varepsilon = 0 & \text{on } \partial\Omega^\varepsilon, \end{cases} \tag{5.0$^\varepsilon$}$$

where u^ε satisfies the homogeneous Dirichlet boundary condition on the whole of the boundary of Ω^ε, when Ω^ε is a perforated domain obtained by removing many small holes from a given open bounded set Ω in $\mathbb{R}^N$, $N \geq 2$, with a repartition of those many small holes producing a "strange term" when ε tends to 0.

We begin by describing in Subsection 5.1 the geometry of the perforated domains and the framework introduced in [4] (see also [6] and [15]) for this problem when the right-hand side is given in $L^2(\Omega)$. We then state in Subsection 5.2 the homogenization and corrector results for the singular semilinear problem (5.0$^\varepsilon$); the proofs of these results will be given in Section 9.

As above we consider in this Section a given matrix A which satisfies (2.1) and a given function F which satisfies (2.2) and (2.3). But in this Section, as well as in Section 9, we assume that

$$N \geq 2, \tag{5.1}$$

(see Remark 5.1 below).

5.1. The perforated domains

Let Ω be a bounded open set of $\mathbb{R}^N$ $(N \geq 2)$ and let us perforate it by closed holes: we obtain an open set Ω^ε. More precisely, consider for every ε, where ε takes its values in a sequence of positive numbers which tends to zero, some closed sets T_i^ε of $\mathbb{R}^N$, $1 \leq i \leq n(\varepsilon)$, which are the holes. The domain Ω^ε is defined by removing the holes T_i^ε from Ω, that is

$$\Omega^\varepsilon = \Omega - \bigcup_{i=1}^{n(\varepsilon)} T_i^\varepsilon.$$

We suppose that the sequence of domains Ω^ε is such that there exist a sequence of functions w^ε, a distribution $\mu \in \mathcal{D}'(\Omega)$ and two sequences of distributions $\mu^\varepsilon \in \mathcal{D}'(\Omega)$ and $\lambda^\varepsilon \in \mathcal{D}'(\Omega)$ such that

$$w^\varepsilon \in H^1(\Omega) \cap L^\infty(\Omega), \tag{5.2}$$

$$0 \leq w^\varepsilon \leq 1 \text{ a.e. } x \in \Omega, \tag{5.3}$$

$$\forall \psi \in H_0^1(\Omega) \cap L^\infty(\Omega),\ w^\varepsilon \psi \in H_0^1(\Omega^\varepsilon), \tag{5.4}$$

$$w^\varepsilon \rightharpoonup 1 \text{ in } H^1(\Omega) \text{ weakly, in } L^\infty(\Omega) \text{ weakly-star and a.e. in } \Omega, \tag{5.5}$$

$$\mu \in H^{-1}(\Omega), \tag{5.6}$$

$$\begin{cases} -div\, {}^tA(x) tDw^\varepsilon = \mu^\varepsilon - \lambda^\varepsilon \text{ in } \mathcal{D}'(\Omega), \\ \text{with } \mu^\varepsilon \in H^{-1}(\Omega),\ \lambda^\varepsilon \in H^{-1}(\Omega), \\ \mu^\varepsilon \geq 0 \text{ in } \mathcal{D}'(\Omega), \\ \mu^\varepsilon \to \mu \text{ in } H^{-1}(\Omega) \text{ strongly}, \\ \langle \lambda^\varepsilon, \tilde{z^\varepsilon} \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = 0 \ \ \forall z^\varepsilon \in H_0^1(\Omega^\varepsilon), \end{cases} \tag{5.7}$$

where, as well as everywhere in the present paper, for every function z^ε in $L^2(\Omega)$, we define $\tilde{z^\varepsilon}$ as the extension by 0 of z^ε to Ω, namely by

$$\tilde{z^\varepsilon}(x) = \begin{cases} z^\varepsilon(x) & \text{in } \Omega^\varepsilon, \\ 0 & \text{in } \displaystyle\bigcup_{i=1}^{n(\varepsilon)} T_i^\varepsilon\,; \end{cases} \tag{5.8}$$

then $\tilde{z^\varepsilon} \in L^2(\Omega)$ and $\|\tilde{z^\varepsilon}\|_{L^2(\Omega)} = \|z^\varepsilon\|_{L^2(\Omega^\varepsilon)}$. Moreover

$$\begin{cases} \text{if } z^\varepsilon \in H_0^1(\Omega^\varepsilon), \text{ then } \tilde{z^\varepsilon} \in H_0^1(\Omega) \\ \text{with } \widetilde{Dz^\varepsilon} = D\tilde{z^\varepsilon} \text{ and } \|\tilde{z^\varepsilon}\|_{H_0^1(\Omega)} = \|z^\varepsilon\|_{H_0^1(\Omega^\varepsilon)}. \end{cases} \tag{5.9}$$

The meaning of assumption (5.4) is that

$$w^\varepsilon = 0 \text{ on } \bigcup_{i=1}^{n(\varepsilon)} T_i^\varepsilon, \tag{5.10}$$

while the meaning of the last statement of (5.7) is that the distribution λ^ε only acts on the holes T_i^ε, $i = 1, \cdots, n(\varepsilon)$, since taking test functions in $\mathcal{D}(\Omega^\varepsilon)$ in the first statement of (5.7) implies that

$$-div\, {}^tA(x)Dw^\varepsilon = \mu^\varepsilon \text{ in } \mathcal{D}'(\Omega^\varepsilon).$$

Taking $z^\varepsilon = w^\varepsilon \phi$, with $\phi \in \mathcal{D}(\Omega)$, $\phi \geq 0$, as test function in (5.7), we have

$$\int_\Omega \phi\, {}^tA(x)Dw^\varepsilon Dw^\varepsilon + \int_\Omega w^\varepsilon\, {}^tA(x)Dw^\varepsilon D\phi = \langle \mu^\varepsilon, w^\varepsilon \phi\rangle_{H^{-1}(\Omega), H_0^1(\Omega)},$$

from which using (5.5) and the fourth statement of (5.7) we easily deduce that

$$\int_\Omega \phi A(x)Dw^\varepsilon Dw^\varepsilon \to \langle \mu, \phi\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \ \forall \phi \in \mathcal{D}(\Omega),\ \phi \geq 0,$$

and therefore that

$$\mu \geq 0.$$

The distribution $\mu \in H^{-1}(\Omega)$ is therefore also a nonnegative Radon measure. Moreover, since

$$\begin{cases} \forall \phi \in \mathcal{D}(\Omega), \phi \geq 0, \\ \displaystyle\int_\Omega \phi d\mu = \langle \mu, \phi\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \lim_\varepsilon \int_\Omega \phi A(x)Dw^\varepsilon Dw^\varepsilon \leq C\|\phi\|_{L^\infty(\Omega)}, \end{cases}$$

the measure μ is a finite Radon measure which satisfies $\displaystyle\int_\Omega d\mu \leq C < +\infty$.

It is then (well) known[1] (see e.g. [7] Section 1 and [8] Subsection 2.2 for more details) that if $z \in H_0^1(\Omega)$, then z (or more exactly its quasi-continuous representative for the $H_0^1(\Omega)$ capacity) satisfies

$$z \in L^1(\Omega; d\mu) \ \text{ with } \ \langle \mu, z\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \int_\Omega z\, d\mu\,; \tag{5.11}$$

moreover if $z \in H_0^1(\Omega) \cap L^\infty(\Omega)$, then z satisfies

$$z \in L^\infty(\Omega; d\mu) \ \text{ with } \ \|z\|_{L^\infty(\Omega; d\mu)} = \|z\|_{L^\infty(\Omega)}; \tag{5.12}$$

therefore when $z \in H_0^1(\Omega) \cap L^\infty(\Omega)$, then z belongs to $L^1(\Omega; d\mu) \cap L^\infty(\Omega; d\mu)$ and therefore to $L^p(\Omega; d\mu)$ for every p, $1 \leq p \leq +\infty$.

When one assumes that the holes T_i^ε, $i = 1, \cdots, n(\varepsilon)$, are such that the assumptions (5.2), (5.3), (5.4), (5.5), (5.6) and (5.7) hold true, then (see [4] or [15], or [6] for a more general framework) for every $f \in L^2(\Omega)$, the (unique) solution y^ε to the linear problem

$$\begin{cases} y^\varepsilon \in H_0^1(\Omega^\varepsilon), \\ -div\, A(x)Dy^\varepsilon = f \text{ in } \mathcal{D}'(\Omega^\varepsilon), \end{cases} \tag{5.13}$$

[1] The reader who would not enter in this theory could continue reading the present paper assuming in (5.6) that μ is a function of $L^r(\Omega)$ (with $r = (2^*)'$ if $N \geq 3$, $r > 1$ if $N = 2$, and $r = 1$ if $N = 1$) and not only an element of $H^{-1}(\Omega)$.

satisfies

$$\tilde{y^\varepsilon} \rightharpoonup y^0 \text{ in } H_0^1(\Omega),$$

where y^0 is the (unique) solution to

$$\begin{cases} y^0 \in H_0^1(\Omega) \cap L^2(\Omega; d\mu), \\ -div\, A(x)Dy^0 + \mu y^0 = f \text{ in } \mathcal{D}'(\Omega), \end{cases}$$

or equivalently to

$$\begin{cases} y^0 \in H_0^1(\Omega) \cap L^2(\Omega; d\mu), \\ \displaystyle\int_\Omega A(x)Dy^0 Dz + \int_\Omega y^0 z\, d\mu = \int_\Omega fz \;\; \forall z \in H_0^1(\Omega) \cap L^2(\Omega; d\mu); \end{cases} \tag{5.14}$$

in (5.14) the "strange term" μy^0 appears; this term is in some sense the asymptotic memory of the fact that $\tilde{y^\varepsilon}$ was zero on the holes.

The prototype of the examples where assumptions (5.2), (5.3), (5.4), (5.5), (5.6) and (5.7) are satisfied is the case where the matrix $A(x)$ is the identity (and therefore where the operator is $-div\, A(x)D = -\Delta$), where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, where the holes T_i^ε are balls of radius r^ε (or more generally sets obtained by a homothety of ratio r^ε from a given bounded closed set $T \subset \mathbb{R}^N$) with r^ε given by

$$\begin{cases} r^\varepsilon = C_0\, \varepsilon^{N/(N-2)} \text{ if } N \geq 3, \\ r^\varepsilon = \exp(-\dfrac{C_0}{\varepsilon^2}) \text{ if } N = 2, \end{cases}$$

which are periodically distributed at the vertices of an N-dimensional lattice of cubes of size 2ε, and where the measure μ is given by

$$\begin{cases} \mu = \dfrac{S_{N-1}(N-2)}{2^N} C_0^{N-2} \text{ if } N \geq 3, \\ \mu = \dfrac{\pi}{2}\dfrac{1}{C_0} \text{ if } N = 2, \end{cases}$$

(see e.g. [4] and [15] for more details, and for other examples, in particular for the case where the holes are distributed on a manifold).

Remark 5.1. In this Remark we prove that in dimension $N = 1$, there is no sequence w^ε which satisfies (5.2), (5.4) and (5.5) whenever for every ε there exists at least one hole $T_{i_\varepsilon}^\varepsilon$ with $T_{i_\varepsilon}^\varepsilon \cap \overline{\Omega} \neq \emptyset$. This is the reason why we assume in this Section, as well as in Section 9, that $N \geq 2$ (see (5.1)).

Indeed let x be any point in $\overline{\Omega}$, and let z^ε be any point in $T_{i_\varepsilon}^\varepsilon \cap \overline{\Omega}$. Assume that there exists a sequence w^ε which satisfies (5.2), (5.4) and (5.5), and let $M > 0$ be such that $\|w^\varepsilon\|_{H^1(\Omega)} \leq M$ for every ε. Since $N = 1$, one has $H^1(\Omega) \subset C^{0,1/2}(\overline{\Omega})$, and since $w^\varepsilon(z^\varepsilon) = 0$, one has

$$\begin{cases} |w^\varepsilon(x)| = |w^\varepsilon(x) - w^\varepsilon(z^\varepsilon)| \leq \|w^\varepsilon\|_{C^{0,1/2}(\overline{\Omega})} \; |x - z^\varepsilon|^{1/2} \leq \\ \leq C\, \|w^\varepsilon\|_{H^1(\Omega)} \; |x - z^\varepsilon|^{1/2} \leq CM \; |x - z^\varepsilon|^{1/2}. \end{cases} \tag{5.15}$$

Since there exists a subsequence, still denoted by ε, such that the point z^ε converges to some point $z \in \overline{\Omega}$, passing to the limit in (5.15) gives in view of (5.5)

$$1 \leq CM \; |x - z|^{1/2},$$

which is a contradiction when $x = z$. $\square$

5.2. The homogenization and corrector results for the singular semilinear problem (5.0$^\varepsilon$)

The existence Theorem 4.1 asserts that when the matrix A and the function F satisfy assumptions (2.1), (2.2) and (2.3), then for given $\varepsilon > 0$, the singular semilinear problem (5.0$^\varepsilon$) posed on Ω^ε has at least one solution u^ε in the sense of Definition 3.1 (this solution is moreover unique if the function $F(x, s)$ also satisfies assumption (2.4)).

The following Theorem asserts that the result of the homogenization process for the singular problem (5.0$^\varepsilon$) is very similar to the homogenization process for the linear problem (5.13). This Theorem will be proved in Subsection 9.1.

Theorem 5.2. *Assume that $N \geq 2$ and that the matrix A and the function F satisfy (2.1), (2.2) and (2.3). Assume also that the sequence of perforated domains Ω^ε is such that (5.2), (5.3), (5.4), (5.5), (5.6) and (5.7) hold true. Finally let u^ε be any solution to problem (5.0$^\varepsilon$) in the sense of Definition 3.1, namely*

$$\begin{cases} i)\, u^\varepsilon \in H_0^1(\Omega^\varepsilon), \\ ii)\, u^\varepsilon(x) \geq 0 \ \ a.e.\ x \in \Omega^\varepsilon, \end{cases} \tag{5.16}$$

$$\begin{cases} \forall \varphi^\varepsilon \in H_0^1(\Omega^\varepsilon) \ \ with\ \varphi^\varepsilon \geq 0, \ \ one\ has \\ \displaystyle\int_{\Omega^\varepsilon} F(x, u^\varepsilon)\varphi^\varepsilon < +\infty, \\ \displaystyle\int_{\Omega^\varepsilon} A(x) Du^\varepsilon D\varphi^\varepsilon = \int_{\Omega^\varepsilon} F(x, u^\varepsilon)\varphi^\varepsilon. \end{cases} \tag{5.17}$$

Then there exists a subsequence, still labeled by ε, such that for this subsequence one has, for $\tilde{u}^\varepsilon$ defined by (5.8),

$$\tilde{u}^\varepsilon \rightharpoonup u^0 \ \ in\ H_0^1(\Omega)\ weakly, \tag{5.18}$$

where u^0 is a solution to

$$\begin{cases} i)\, u^0 \in H_0^1(\Omega) \cap L^2(\Omega; d\mu), \\ ii)\, u^0(x) \geq 0 \ \ a.e.\ x \in \Omega, \end{cases} \tag{5.19}$$

$$\begin{cases} \forall z \in H_0^1(\Omega) \cap L^2(\Omega; d\mu) \ \ with\ z \geq 0, \ \ one\ has \\ \displaystyle\int_{\Omega} F(x, u^0) z < +\infty, \\ \displaystyle\int_{\Omega} A(x) Du^0 Dz + \int_{\Omega} u^0 z d\mu = \int_{\Omega} F(x, u^0) z. \end{cases} \tag{5.20}$$

Remark 5.3. Requirements (5.19) and (5.20) are the adaptation of Definition 3.1 of a solution to problem (1.1) to the case of problem

$$\begin{cases} -div\, A(x) Du^0 + \mu u^0 = F(x, u^0) & \text{in } \Omega, \\ u^0 = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.21}$$

in which there is now a zeroth order term μu^0, where μ is a nonnegative finite Radon measure which belongs to $H^{-1}(\Omega)$. Theorem 5.2 therefore expresses the fact that, when assumptions (5.2), (5.3), (5.4), (5.5), (5.6)

and (5.7) hold true, the result of the homogenization process of the singular semilinear problem (5.0$^\varepsilon$) in Ω^ε with Dirichlet boundary condition on the whole of the boundary $\partial\Omega^\varepsilon$ is the singular semilinear problem (5.21), where the "strange term" μu^0 appears exactly as in the case of the linear problem (5.13) where the right-hand side belongs to $L^2(\Omega)$.

Note nevertheless that the result was not a priori obvious due to the presence of the term $F(x, u^\varepsilon)$, which is singular (at least) on the boundary $\partial\Omega^\varepsilon$ and, in particular, on the boundary of the holes, whose number increases more and more when ε goes to zero, "invading" the entire open set Ω. □

Remark 5.4. If $F(x, s)$ satisfies, in addition to (2.2) and (2.3), the further assumption (2.4), the solution u^ε to (5.16) and (5.17) is unique (see Theorem 4.4 above), and the solution u^0 to (5.19) and (5.20) is also unique, as it is easily seen from a proof very similar to the one made in Section 8 below.

Under this further assumption there is therefore no need to extract a subsequence in Theorem 5.2, and the convergence takes place for the whole sequence ε. □

Further to the homogenization result of Theorem 5.2, we will also prove in Subsection 9.2 the following corrector result, which, under the assumptions that $u^0 \in L^\infty(\Omega)$ and that the matrix A is symmetric, states that $w^\varepsilon u^0$ is a strong approximation in $H^1_0(\Omega)$ of $\tilde{u^\varepsilon}$.

Theorem 5.5. *Assume that $N \geq 2$ and that the matrix A and the function F satisfy (2.1), (2.2) and (2.3). Assume also that the sequence of perforated domains Ω^ε is such that (5.2), (5.3), (5.4), (5.5), (5.6) and (5.7) hold true. Finally let u^ε be the subsequence of solutions to problem (5.0$^\varepsilon$) in the sense of Definition 3.1 (see (5.16) and (5.17)) defined in Theorem 5.2, and let u^0 be its limit defined by (5.18), (5.19) and (5.20). Assume moreover that*

$$A(x) = {}^tA(x), \tag{5.22}$$

$$u^0 \in L^\infty(\Omega). \tag{5.23}$$

Then further to (5.18) one has

$$\tilde{u^\varepsilon} = w^\varepsilon u^0 + r^\varepsilon, \text{ where } r^\varepsilon \to 0 \text{ in } H^1_0(\Omega) \text{ strongly.} \tag{5.24}$$

Remark 5.6. If further to assumptions (2.2) and (2.3), the function F is assumed to satisfy the regularity assumption (4.3), then in view of Proposition 4.3 every solution u^ε to (5.0$^\varepsilon$) in the sense of Definition 3.1 satisfies

$$\|\tilde{u^\varepsilon}\|_{L^\infty(\Omega)} = \|u^\varepsilon\|_{L^\infty(\Omega^\varepsilon)} \leq 1 + \frac{2}{\alpha}\, C(|\Omega^\varepsilon|, N, t)\, \|h\|_{L^t(\Omega^\varepsilon)} \leq 1 + \frac{2}{\alpha}\, C(|\Omega|, N, t)\, \|h\|_{L^t(\Omega)}.$$

In such a case, the limit u^0 satisfies assumption (5.23) with

$$\|u^0\|_{L^\infty(\Omega)} \leq 1 + \frac{2}{\alpha}\, C(|\Omega|, N, t)\, \|h\|_{L^t(\Omega)}. \qquad \square$$

6. A priori estimates

Proposition 6.1 *($H^1_0(\Omega)$ a priori estimate). Assume that the matrix A and the function F satisfy (2.1), (2.2) and (2.3). Then every u solution to problem (1.1) in the sense of Definition 3.1 satisfies*

$$\|u\|_{H^1_0(\Omega)} \leq C(|\Omega|, N, \alpha, \gamma, r)\, \big(\|h\|_{L^r(\Omega)} + \|h\|^{1/2}_{L^1(\Omega)}\big), \tag{6.1}$$

where the constant $C(|\Omega|, N, \alpha, \gamma, r)$ depends only on $|\Omega|$, N, α, γ and r and is a nondecreasing function of $|\Omega|$.

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Proof. We take $\varphi = u$ as test function in (3.3). Using (2.3 *iv*) and Young's inequality with $1/p = 1-\gamma$ and $1/p' = \gamma$ when $0 < \gamma < 1$, which implies that

$$u^{1-\gamma} \le \frac{1}{p} u^{(1-\gamma)p} + \frac{1}{p'} = (1-\gamma)u + \gamma, \tag{6.2}$$

we obtain

$$\begin{cases} \displaystyle\int_\Omega A(x)DuDu = \int_\Omega F(x,u)u \le \int_\Omega h(x)\big(\frac{1}{u^\gamma}+1\big)u \le \\ \displaystyle\le \int_\Omega h(x)\left((1-\gamma)u+\gamma+u\right) = \int_\Omega h(x)\left((2-\gamma)u+\gamma\right). \end{cases} \tag{6.3}$$

When $N \ge 3$, we use Sobolev's embedding Theorem $H_0^1(\Omega) \subset L^{2^*}(\Omega)$, with 2^* defined by $\dfrac{1}{2^*} = \dfrac{1}{2} - \dfrac{1}{N}$, and the Sobolev's inequality

$$\|v\|_{L^{2^*}(\Omega)} \le C_N \|Dv\|_{L^2(\Omega)} \quad \forall v \in H_0^1(\Omega) \text{ when } N \ge 3; \tag{6.4}$$

note that $(2^*)' = 2N/(N+2) = r$ since $N \ge 3$. Using in (6.3) the coercivity (2.1), Hölder's inequality, Sobolev's inequality (6.4) and finally Young's inequality, we get

$$\begin{cases} \displaystyle\alpha \int_\Omega |Du|^2 \le (2-\gamma)\|h\|_{L^r(\Omega)}\|u\|_{L^{2^*}(\Omega)} + \gamma\|h\|_{L^1(\Omega)} \le \\ \displaystyle\le (2-\gamma)C_N\|h\|_{L^r(\Omega)}\|Du\|_{L^2(\Omega)^N} + \gamma\|h\|_{L^1(\Omega)} \le \\ \displaystyle\le \frac{\alpha}{2}\|Du\|^2_{L^2(\Omega)^N} + \frac{1}{2\alpha}(2-\gamma)^2 C_N^2 \|h\|^2_{L^r(\Omega)} + \gamma\|h\|_{L^1(\Omega)}, \end{cases} \tag{6.5}$$

which yields

$$\|Du\|^2_{L^2(\Omega)^N} \le \left(\frac{(2-\gamma)\,C_N}{\alpha}\right)^2 \|h\|^2_{L^r(\Omega)} + \frac{2\gamma}{\alpha}\,\|h\|_{L^1(\Omega)},$$

which finally implies

$$\|Du\|_{L^2(\Omega)^N} \le \frac{(2-\gamma)\,C_N}{\alpha}\|h\|_{L^r(\Omega)} + \left(\frac{2\gamma}{\alpha}\right)^{1/2} \|h\|^{1/2}_{L^1(\Omega)},$$

namely estimate (6.1) with a constant which depends only on N, α and γ.

The proof is similar when $N = 1$ and $N = 2$, but Sobolev's inequality (6.4) has now to be replaced by

$$\|v\|_{L^\infty(\Omega)} \le |\Omega|^{1/2}\,\|Dv\|_{L^2(\Omega)} \quad \forall v \in H_0^1(\Omega) \quad \text{when } N = 1,$$

and by

$$\forall r' > 1, \quad \|v\|_{L^{r'}(\Omega)} \le C(|\Omega|, r)\,\|Dv\|_{L^2(\Omega)} \quad \forall v \in H_0^1(\Omega) \quad \text{when } N = 2,$$

where the constant $C(|\Omega|, r)$ is a nondecreasing function of $|\Omega|$.

This completes the proof of estimate (6.1). □

In the following Proposition we give an estimate of the integral of $F(x,u)\varphi$ near the singular set $\{u=0\}$. To this aim we introduce for $\delta>0$ the function $Z_\delta : [0,+\infty[\to [0,+\infty[$ defined by

$$Z_\delta(s) = \begin{cases} 1, & \text{if } 0 \le s \le \delta, \\ -\frac{s}{\delta}+2, & \text{if } \delta \le s \le 2\delta, \\ 0, & \text{if } 2\delta \le s. \end{cases} \tag{6.6}$$

Proposition 6.2 *(Control of $\int\limits_{\{u\le\delta\}} F(x,u)v$ when δ is small). Assume that the matrix A and the function F satisfy (2.1), (2.2) and (2.3). Then every u solution to problem (1.1) in the sense of Definition 3.1 satisfies*

$$\begin{cases} \forall \varphi \in H_0^1(\Omega),\ \varphi \ge 0,\ \forall \delta > 0, \\ 0 \le \int\limits_{\{u\le\delta\}} F(x,u)\varphi \le \int\limits_\Omega A(x)DuD\varphi Z_\delta(u). \end{cases} \tag{6.7}$$

Proof. The proof consists in taking $T_k(\varphi)Z_\delta(u)$, $\varphi \in H_0^1(\Omega), \varphi \ge 0$ as test function in (3.3). This function belongs to $H_0^1(\Omega)$ and we get

$$\int\limits_\Omega A(x)DuDT_k(\varphi)Z_\delta(u) = \frac{1}{\delta}\int\limits_{\{\delta<u<2\delta\}} A(x)DuDuT_k(\varphi) + \int\limits_\Omega F(x,u)T_k(\varphi)Z_\delta(u).$$

Since $Z_\delta(u)=1$ on $\{u\le\delta\}$, this implies that

$$\begin{cases} \forall \varphi \in H_0^1(\Omega),\ \varphi \ge 0,\ \forall k > 0,\ \forall \delta > 0, \\ 0 \le \int\limits_{\{u\le\delta\}} F(x,u)T_k(\varphi) \le \int\limits_\Omega A(x)DuDT_k(\varphi)Z_\delta(u). \end{cases} \tag{6.8}$$

We now pass to the limit in (6.8) as k tends to infinity, using the strong convergence of $DT_k(u)$ to Du in $L^2(\Omega)^N$ in the right-hand side and Fatou's Lemma for on the left-hand side. This gives (6.7). □

As a consequence of Proposition 6.2 we have:

Proposition 6.3. *Assume that the matrix A and the function F satisfy (2.1), (2.2) and (2.3). Then every u which is solution to problem (1.1) in the sense of Definition 3.1 satisfies*

$$\int\limits_{\{u=0\}} F(x,u)\varphi = 0 \quad \forall \varphi \in H_0^1(\Omega),\ \varphi \ge 0. \tag{6.9}$$

Proof. Since $\{u=0\} \subset \{u\le\delta\}$ for every $\delta>0$, inequality (6.7) implies that

$$\begin{cases} \forall \varphi \in H_0^1(\Omega),\ \varphi \ge 0,\ \forall \delta > 0, \\ 0 \le \int\limits_{\{u=0\}} F(x,u)\varphi \le \int\limits_\Omega A(x)DuD\varphi Z_\delta(u). \end{cases}$$

When δ tends to zero, one has

$$Z_\delta(u) \to \chi_{\{u=0\}} \text{ a.e. in } \Omega;$$

on the other hand, since $u \in H_0^1(\Omega)$, one has

$$Du = 0 \text{ a.e. on } \{u = 0\};$$

therefore, since $A(x)DuD\varphi \in L^1(\Omega)$, one has, by Lebesgue's dominated convergence theorem,

$$\int_\Omega A(x)DuD\varphi Z_\delta(u) \to 0 \text{ as } \delta \to 0,$$

which proves (6.9). □

7. Proofs of the stability, existence and regularity results (Theorems 4.2 and 4.1 and Proposition 4.3)

7.1. Proof of the stability Theorem 4.2

First step

Since all the functions $F_n(x, s)$ satisfy assumptions (2.2) and (2.3) for the same γ and h, every solution u_n to problem $(1.1)_n$ in the sense of Definition 3.1 satisfies the a priori estimates (6.1) and (6.7) of Propositions 6.1 and 6.2.

Therefore there exist a subsequence, still labeled by n, and a function u_∞ such that

$$u_n \rightharpoonup u_\infty \text{ in } H_0^1(\Omega) \text{ weakly and a.e. in } \Omega. \tag{7.1}$$

Since $u_n \geq 0$, we have also $u_\infty \geq 0$.

Since u_n satisfies $(3.3)_n$, we have

$$\int_\Omega A(x)Du_nD\varphi = \int_\Omega F_n(x, u_n)\varphi \ \ \forall \varphi \in H_0^1(\Omega),\ \varphi \geq 0. \tag{7.2}$$

Using in the left-hand side the weak convergence (7.1), and in the right-hand side the almost everywhere convergence (7.1) of u_n to u_∞, assumption (4.1) on the functions F_n and Fatou's Lemma, one obtains

$$\int_\Omega F_\infty(x, u_\infty)\varphi \leq \int_\Omega A(x)Du_\infty D\varphi < +\infty \ \ \forall \varphi \in H_0^1(\Omega),\ \varphi \geq 0, \tag{7.3}$$

which implies the first assertion of $(3.3)_\infty$.

It remains to prove the second assertion of $(3.3)_\infty$ and the strong convergence (4.2).

Second step

We fix a function $\varphi \in H_0^1(\Omega),\ \varphi \geq 0$, and, for every $\delta > 0$, we write (7.2) as

$$\int_\Omega A(x)Du_nD\varphi = \int_{\{u_n \leq \delta\}} F_n(x, u_n)\varphi + \int_{\{u_n > \delta\}} F_n(x, u_n)\varphi. \tag{7.4}$$

We pass now to the limit as n tends to infinity for $\delta > 0$ fixed in (7.4). In the left-hand side we get (as before)

$$\int_\Omega A(x)Du_nD\varphi \to \int_\Omega A(x)Du_\infty D\varphi. \tag{7.5}$$

For what concerns the first term of the right-hand side of (7.4) we use the a priori estimate (6.7). Since $D\varphi Z_\delta(u_n)$ tends to $D\varphi Z_\delta(u_\infty)$ strongly in $L^2(\Omega)^N$ while $A(x)Du_n$ tends to $A(x)Du_\infty$ weakly in $L^2(\Omega)^N$, we obtain

$$\forall \delta > 0,\ \limsup_n \int_{\{u_n \le \delta\}} F_n(x, u_n)\varphi \le \int_\Omega A(x)Du_\infty D\varphi Z_\delta(u_\infty). \tag{7.6}$$

Since

$$Z_\delta(u_\infty) \to \chi_{\{u_\infty = 0\}} \text{ a.e. in } \Omega, \text{ as } \delta \to 0,$$

and since $u_\infty \in H_0^1(\Omega)$ implies that $Du_\infty = 0$ almost everywhere on the set $\{x \in \Omega : u_\infty(x) = 0\}$, the right-hand side of (7.6) tends to 0 when δ tends to 0.

We have proved that

$$\limsup_n \int_{\{u_n \le \delta\}} F_n(x, u_n)\varphi \to 0 \text{ as } \delta \to 0. \tag{7.7}$$

Third step

Let us now observe that for every $\delta > 0$

$$\int_{\{u_\infty = 0\}} F_n(x, u_n)\chi_{\{u_n \le \delta\}}\varphi \le \int_{\{u_n \le \delta\}} F_n(x, u_n)\varphi. \tag{7.8}$$

Since u_n converges almost everywhere to u_∞, one has, for every $\delta > 0$,

$$\chi_{\{u_n \le \delta\}} \to \chi_{\{u_\infty \le \delta\}} \text{ a.e. on } \{x \in \Omega : u_\infty(x) \ne \delta\},$$

and therefore

$$\chi_{\{u_n \le \delta\}} \to 1 \text{ a.e. on } \{x \in \Omega : u_\infty(x) = 0\},$$

while in view of assumption (4.1), one has

$$F_n(x, u_n(x)) \to F_\infty(x, u_\infty(x)) \text{ a.e. } x \in \Omega.$$

Applying Fatou's Lemma to the left-hand side of (7.8), we obtain

$$\forall \delta > 0, \int_{\{u_\infty = 0\}} F_\infty(x, u_\infty)\varphi \le \limsup_n \int_{\{u_n \le \delta\}} F_n(x, u_n)\varphi,$$

which in view of (7.7) implies that

$$\int_{\{u_\infty = 0\}} F_\infty(x, u_\infty)\varphi = 0. \tag{7.9}$$

Fourth step

Let us finally pass to the limit in n for $\delta > 0$ fixed in the second term of the right-hand side of (7.4), namely in

$$\int_{\{u_n>\delta\}} F_n(x,u_n)\varphi = \int_\Omega F_n(x,u_n)\chi_{\{u_n>\delta\}}\varphi.$$

Since in view of (2.3 *iv*)

$$0 \le F_n(x,u_n)\chi_{\{u_n>\delta\}}\varphi \le \ h(x)\left(\frac{1}{\delta^\gamma}+1\right)\varphi \ \text{ a.e. } x\in\Omega,$$

since $h\varphi \in L^1(\Omega)$, since in view of assumption (4.1) and of the almost everywhere convergence (7.1) of u_n to u_∞ one has

$$F_n(x,u_n)\varphi \to F_\infty(x,u_\infty)\varphi \text{ a.e. on } \Omega,$$

and finally since

$$\chi_{\{u_n>\delta\}} \to \chi_{\{u_\infty>\delta\}} \text{ a.e. on } \{x\in\Omega : u_\infty(x)\neq\delta\},$$

defining the set $\mathcal{C} \subset [0,+\infty[$ by

$$\mathcal{C} = \{\delta>0, \ \text{meas}\{x\in\Omega : u_\infty(x)=\delta\}>0\}$$

(note that this set is at most countable), and choosing $\delta \notin \mathcal{C}$, Lebesgue's dominated convergence Theorem implies that

$$\int_{\{u_n>\delta\}} F_n(x,u_n)\varphi \to \int_{\{u_\infty>\delta\}} F_\infty(x,u_\infty)\varphi \ \text{ as } n\to+\infty, \ \forall\delta\notin\mathcal{C}. \tag{7.10}$$

Since the set $\mathcal{C}$ is at most a countable, choosing δ outside of the set $\mathcal{C}$ and using the fact that the set $\{x\in\Omega : u_\infty(x)>\delta\}$ monotonically shrinks to the set $\{x\in\Omega : u_\infty(x)>0\}$ as δ tends to 0, the fact that $F_\infty(x,u_\infty)\varphi$ belongs to $L^1(\Omega)$ (see (7.3)), and finally (7.9), we have proved that

$$\int_{\{u_\infty>\delta\}} F_\infty(x,u_\infty)\varphi \to \int_{\{u_\infty>0\}} F_\infty(x,u_\infty)\varphi = \int_\Omega F_\infty(x,u_\infty)\varphi \text{ as } \delta\to0, \delta\notin\mathcal{C}. \tag{7.11}$$

Fifth step

Passing to the limit in each term of (7.4), first in n for $\delta>0$ fixed with $\delta\notin\mathcal{C}$, and then for $\delta\notin\mathcal{C}$ which tends to 0, and collecting the results obtained in (7.5), (7.7), (7.10) and (7.11), we have proved that

$$\int_\Omega A(x)Du_\infty D\varphi = \int_\Omega F_\infty(x,u_\infty)\varphi \ \ \forall\varphi\in H_0^1(\Omega), \ \varphi\ge0,$$

which is nothing but the second assertion in $(3.3)_\infty$.

We have proved a weaker version of Theorem 4.2, where the strong $H_0^1(\Omega)$ convergence (4.2) is replaced by the weak $H_0^1(\Omega)$ convergence (7.1).

Sixth step

Let us now prove that (4.2) (namely the strong $H_0^1(\Omega)$ convergence) holds true. Indeed, taking u_n as test function in $(3.3)_n$, we have

$$\int_\Omega A(x)Du_nDu_n = \int_\Omega F_n(x,u_n)u_n.$$

Observe that in view of hypothesis (4.1) and of convergence (7.1) we have

$$F_n(x,u_n)u_n \to F_\infty(x,u_\infty)u_\infty \text{ a.e. } x \in \Omega.$$

Observe also that the functions $F_n(x,u_n)u_n$ are equi-integrable: indeed for every measurable set $E \subset \Omega$, we have, using (2.3 *iv*), (6.2), Hölder's and Sobolev's inequalities (see the proof of Proposition 6.1 above) and finally (6.1),

$$\begin{cases} 0 \le \int_E F_n(x,u_n)u_n \le \int_E h(x)\left(\frac{1}{u_n^\gamma}+1\right)u_n \le \\ \le \int_E h(x)\left((1-\gamma)u_n+\gamma+u_n\right) = \int_E h(x)\left((2-\gamma)u_n+\gamma\right) \le \\ \le (2-\gamma)\|h\|_{L^r(E)}\, c(|\Omega|,N,r)\,\|Du_n\|_{L^2(\Omega)^N} + \gamma\|h\|_{L^1(E)} \le \\ \le c\|h\|_{L^r(E)} + \gamma\|h\|_{L^1(E)}, \end{cases} \tag{7.12}$$

where c is a constant which does not depend neither on E nor on n. Therefore by Vitali's Theorem we have

$$F_n(x,u_n)u_n \to F_\infty(x,u_\infty)u_\infty \text{ in } L^1(\Omega) \text{ strongly}.$$

This implies that

$$\int_\Omega A(x)Du_nDu_n = \int_\Omega F_n(x,u_n)u_n \to \int_\Omega F_\infty(x,u_\infty)u_\infty.$$

On the other hand, taking u_∞ as test function in $(3.3)_\infty$, we have

$$\int_\Omega A(x)Du_\infty Du_\infty = \int_\Omega F_\infty(x,u_\infty)u_\infty dx.$$

Therefore

$$\int_\Omega A(x)Du_nDu_n \to \int_\Omega A(x)Du_\infty Du_\infty.$$

Together with (7.1), this implies the strong convergence (4.2).

This completes the proof of the stability Theorem 4.2. □

7.2. Proof of the existence Theorem 4.1

Let u_n be a solution to

$$\begin{cases} u_n \in H_0^1(\Omega), \\ -div\, A(x)Du_n = T_n(F(x,u_n^+)) \quad \text{in}\, \mathcal{D}'(\Omega), \end{cases} \tag{7.13}$$

where T_n is the truncation at height n.

Since $T_n(F(x,s^+))$ is a bounded Carathéodory function defined on $\Omega\times\mathbb{R}$, Schauder's fixed point theorem implies that problem (7.13) has at least one solution. Since $F(x,s^+)\geq 0$, this solution is nonnegative by the weak maximum principle, and therefore $u_n^+ = u_n$.

It is then clear that u_n is a solution to problem $(1.1)_n$ in the sense of Definition 3.1, where $(1.1)_n$ is the problem (1.1) with $F(x,u)$ replaced by $F_n(x,u_n) = T_n(F(x,u_n))$.

Moreover it is easy to see, considering the cases where $s_\infty > 0$ and where $s_\infty = 0$, that the functions $F_n(x,s)$ satisfy assumption (4.1) with

$$F_\infty(x,s) = F(x,s).$$

The stability Theorem 4.2 then implies that there exists a subsequence of u_n whose limit u_∞ is a solution to problem (1.1) in the sense of Definition 3.1.

This proves the existence Theorem 4.1. □

7.3. Proof of the regularity Proposition 4.3

Using $G_k(u)$, $k>0$ as test function in (3.3), we get

$$\int_\Omega A(x)DG_k(u)DG_k(u) = \int_\Omega F(x,u)G_k(u) \quad \forall k>0.$$

Setting $k=j+1$ with $j\geq 0$, this implies, using the coercivity (2.1) and condition (2.3 *iv*), that

$$\begin{cases} \alpha\displaystyle\int_\Omega |DG_{j+1}(u)|^2 \leq \int_\Omega h(x)\left(\frac{1}{u^\gamma}+1\right)G_{j+1}(u) \leq \\ \qquad\qquad \leq \displaystyle\int_{\{u>1\}} h(x)\left(\frac{1}{u^\gamma}+1\right)G_{j+1}(u) \leq 2\int_\Omega h(x)\,G_{j+1}(u),\ \forall j\geq 0. \end{cases} \tag{7.14}$$

Since

$$G_{j+1}(s) = G_j(G_1(s)) \quad \forall s\in\mathbb{R}, \forall j\geq 0,$$

and since $G_1(u)\in H_0^1(\Omega)$, setting

$$\overline{u} = G_1(u),$$

we deduce from (7.14) that $\overline{u}$ satisfies

$$\begin{cases} \overline{u}\in H_0^1(\Omega), \\ \displaystyle\int_\Omega |DG_j(\overline{u})|^2 \leq \frac{2}{\alpha}\int_\Omega h(x)G_j(\overline{u}) \quad \forall j\geq 0. \end{cases}$$

A result of G. Stampacchia (see the proof of Lemma 5.1 and Lemma 4.1 in [16]) (see also Section 5 in [12]) then implies that when $h \in L^t(\Omega)$ (hypothesis (4.3)), the function $\overline{u}$ belongs to $L^\infty(\Omega)$, and that there exists a constant $C(|\Omega|, N, t)$ which is a nondecreasing function of $|\Omega|$ such that

$$\|\overline{u}\|_{L^\infty(\Omega)} \le \frac{2}{\alpha}\, C(|\Omega|, N, t)\, \|h\|_{L^t(\Omega)}.$$

Combined with

$$u = T_1(u) + G_1(u) = T_1(u) + \overline{u},$$

this result implies that

$$\|u\|_{L^\infty(\Omega)} \le 1 + \frac{2}{\alpha}\, C(|\Omega|, N, t)\, \|h\|_{L^t(\Omega)},$$

which proves Proposition 4.3. □

8. Comparison principle and proof of the uniqueness Theorem 4.4

In this Section we prove the following comparison result:

Theorem 8.1 *(Comparison principle). Assume that the matrix A satisfies (2.1). Let $F_1(x,s)$ and $F_2(x,s)$ be two functions satisfying (2.2) and (2.3) (for the same or for different γ and h). Assume moreover that*

$$\text{either } F_1(x,s) \text{ or } F_2(x,s) \text{ satisfies (2.4)}, \tag{8.1}$$

and that

$$F_1(x,s) \le F_2(x,s) \ \text{ a.e. } x \in \Omega, \quad \forall s \ge 0. \tag{8.2}$$

Let u_1 and u_2 be solutions in the sense of Definition 3.1 to problem $(1.1)_1$ and $(1.1)_2$, where $(1.1)_1$ and $(1.1)_2$ stand for (1.1) with $F(x,u)$ replaced by $F_1(x,u_1)$ and $F_2(x,u_2)$. Then

$$u_1(x) \le u_2(x) \ \text{ a.e. } x \in \Omega. \tag{8.3}$$

8.1. Proof of the uniqueness Theorem 4.4

Applying this comparison principle to the case where $F_1(x,s) = F_2(x,s) = F(x,s)$, with $F(x,s)$ satisfying (2.4), immediately proves the uniqueness Theorem 4.4. □

8.2. Proof of Theorem 8.1

Since $(u_1 - u_2)^+ \in H_0^1(\Omega)$, we can take it as test function in $(3.3)_1$ and add to both sides of $(3.3)_1$ the finite term $-\lambda \int_\Omega u_1(u_1-u_2)^+$. The same holds for $(3.3)_2$. This gives

$$\int_\Omega A(x) Du_i D(u_1-u_2)^+ - \lambda \int_\Omega u_i(u_1-u_2)^+ = \int_\Omega (F_i(x,u_i) - \lambda u_i)(u_1-u_2)^+, \ \ i = 1,2.$$

Taking the difference between these two equations it follows that

$$\begin{cases} \displaystyle\int_\Omega A(x)D(u_1-u_2)^+D(u_1-u_2)^+ - \lambda \int_\Omega |(u_1-u_2)^+|^2 = \\ \displaystyle= \int_\Omega \big((F_1(x,u_1)-\lambda u_1)-(F_2(x,u_2)-\lambda u_2)\big)(u_1-u_2)^+. \end{cases}$$

Using the coercivity (2.1) and the characterization of the first eigenvalue λ_1 of the operator $-div\,{}^sA(x)D$ in $H_0^1(\Omega)$, we get

$$(\lambda_1-\lambda)\int_\Omega |(u_1-u_2)^+|^2 \le \int_\Omega \big((F_1(x,u_1)-\lambda u_1)-(F_2(x,u_2)-\lambda u_2)\big)(u_1-u_2)^+. \tag{8.4}$$

Let us now prove that

$$\big((F_1(x,u_1)-\lambda u_1)-(F_2(x,u_2)-\lambda u_2)\big)(u_1-u_2)^+ \le 0 \text{ a.e. } x\in\Omega, \tag{8.5}$$

or equivalently that

$$(F_1(x,u_1)-\lambda u_1)-(F_2(x,u_2)-\lambda u_2) \le 0 \text{ a.e. } x\in\{u_1>u_2\}. \tag{8.6}$$

We first observe that since u_1 and u_2 are solutions to $(1.1)_1$ and $(1.1)_2$ in the sense of Definition 3.1, one has (see (3.5))

$$F_1(x,u_1) \text{ and } F_2(x,u_2) \text{ are nonnegative and finite a.e. } x\in\Omega. \tag{8.7}$$

In order to prove (8.6), let us first consider the case where F_1 satisfies (2.4). In this case we have

$$F_1(x,u_1)-\lambda u_1 \le F_1(x,u_2)-\lambda u_2 \text{ a.e. } x\in\{u_1>u_2\}. \tag{8.8}$$

We observe that hypothesis (8.2) implies that

$$F_1(x,u_2) \le F_2(x,u_2) \text{ a.e. } x\in\Omega,$$

and therefore, using (8.7), that

$$F_1(x,u_2) \text{ is nonnegative and finite a.e. } x\in\Omega.$$

Since there is no indeterminacy of the type $(\infty-\infty)$, it is licit to write that

$$\begin{cases} (F_1(x,u_1)-\lambda u_1)-(F_2(x,u_2)-\lambda u_2) = \\ = (F_1(x,u_1)-\lambda u_1)-(F_1(x,u_2)-\lambda u_2)+ \\ +(F_1(x,u_2)-\lambda u_2)-(F_2(x,u_2)-\lambda u_2) \text{ a.e. } x\in\Omega. \end{cases} \tag{8.9}$$

Since the first line of the right-hand side of (8.9) is nonpositive on $\{u_1>u_2\}$ by (8.8), and since the second line of this right-hand side, namely $F_1(x,u_2)-F_2(x,u_2)$, is nonpositive by (8.2), we have proved (8.6).

Let us now consider the case where F_2 satisfies (2.4). In this case we have

$$F_2(x,u_1)-\lambda u_1 \le F_2(x,u_2)-\lambda u_2 \text{ a.e. } x\in\{u_1>u_2\}. \tag{8.10}$$

We observe that, together with the fact that $F_2(x, u_2)$ is finite almost everywhere on Ω (see (8.7)), this result implies that

$$F_2(x, u_1) \text{ is nonnegative and finite a.e. } x \in \{u_1 > u_2\}.$$

Since there is no indeterminacy of the type $(\infty - \infty)$, it is licit to write that

$$\begin{cases} (F_1(x, u_1) - \lambda u_1) - (F_2(x, u_2) - \lambda u_2) = \\ = (F_1(x, u_1) - \lambda u_1) - (F_2(x, u_1) - \lambda u_1) + \\ + (F_2(x, u_1) - \lambda u_1) - (F_2(x, u_2) - \lambda u_2) \text{ a.e. } x \in \{u_1 > u_2\}. \end{cases} \tag{8.11}$$

Since the second line of the right-hand side of (8.11) is nonpositive on $\{u_1 > u_2\}$ by (8.10), and since the first line of this right-hand side, namely $F_1(x, u_1) - F_2(x, u_1)$, is nonpositive by (8.2), we have again proved (8.6).

In both cases we have proved that the right-hand side of (8.4) is nonpositive when assumptions (8.1) and (8.2) are assumed to hold true. Since $\lambda_1 - \lambda > 0$ by hypothesis (2.4), this implies that $(u_1 - u_2)^+ = 0$.

This proves (8.3). □

Remark 8.2. Consider the case where the matrix A satisfies (2.1) and is symmetric and where the function F is defined by

$$F(x, s) = \lambda_1 T_k(s) \quad \forall s \geq 0, \tag{8.12}$$

where T_k is the truncation at height $k > 0$, for some k fixed, and where λ_1 and ϕ_1 are the first eigenvalue and eigenvector of the operator $-div\, A(x)D$ in $H_0^1(\Omega)$, namely

$$\begin{cases} \phi_1 \in H_0^1(\Omega),\ \phi_1 \geq 0,\ \displaystyle\int_\Omega |\phi_1|^2 = 1, \\ -div\, A(x)D\phi_1 = \lambda_1 \phi_1 \quad \text{in } \mathcal{D}'(\Omega). \end{cases} \tag{8.13}$$

The function F defined by (8.12) satisfies assumptions (2.2), (2.3) and (2.5), but does not satisfy (2.4).

Recall that ϕ_1, the solution to (8.13), belongs to $L^\infty(\Omega)$. Then for every t with $0 \leq t \leq k/\|\phi_1\|_{L^\infty(\Omega)}$, the function

$$u = t\phi_1$$

is a solution to (1.1) with $F(x, s)$ given by (8.12) in the classical weak sense, and therefore in the sense of Definition 3.1.

This proves that uniqueness does not hold if assumption (2.4) is replaced by the weaker assumption (2.5). □

9. Proofs of the homogenization Theorem 5.2 and of the corrector Theorem 5.5

9.1. Proof of the homogenization Theorem 5.2

First step

Theorem 4.1 asserts that for every $\varepsilon > 0$ there exists at least one solution to problem (5.0^ε) in the sense of Definition 3.1, namely a least one u^ε which satisfies (5.16) and (5.17).

Proposition 6.1 and (5.9) imply that every such u^ε satisfies

$$\begin{cases} \|\tilde{u}^\varepsilon\|_{H_0^1(\Omega)} = \|u^\varepsilon\|_{H_0^1(\Omega^\varepsilon)} \leq \\ \leq C(|\Omega^\varepsilon|, N, \alpha, \gamma, r)\left(\|h\|_{L^r(\Omega^\varepsilon)} + \|h\|^{1/2}_{L^1(\Omega^\varepsilon)}\right) \leq C(|\Omega|, N, \alpha, \gamma, r)\left(\|h\|_{L^r(\Omega)} + \|h\|^{1/2}_{L^1(\Omega)}\right). \end{cases} \tag{9.1}$$

Estimate (9.1) implies that there exists a function u^0, and a subsequence $\tilde{u^\varepsilon}$, still labeled by ε, which satisfies

$$\tilde{u^\varepsilon} \rightharpoonup u^0 \text{ in } H_0^1(\Omega) \text{ weakly and a.e. in } \Omega. \tag{9.2}$$

Observe that $u^0(x) \geq 0$ a.e. $x \in \Omega$.

Second step

In view of assumptions (5.2), (5.3) and (5.4), one has

$$w^\varepsilon \psi \in H_0^1(\Omega^\varepsilon) \cap L^\infty(\Omega^\varepsilon) \quad \forall \psi \in H_0^1(\Omega) \cap L^\infty(\Omega),$$

and

$$\begin{cases} \|w^\varepsilon \psi\|_{H_0^1(\Omega^\varepsilon)} = \|w^\varepsilon \psi\|_{H_0^1(\Omega)} \leq \\ \leq \|w^\varepsilon\|_{L^\infty(\Omega)} \|D\psi\|_{L^2(\Omega)^N} + \|\psi\|_{L^\infty(\Omega)} \|Dw^\varepsilon\|_{L^2(\Omega)^N} \leq C^*(\|D\psi\|_{L^2(\Omega)^N} + \|\psi\|_{L^\infty(\Omega)}), \end{cases}$$

where

$$C^* = \max_\varepsilon \{1, \|Dw^\varepsilon\|_{L^2(\Omega)^N}\}.$$

We now fix $\psi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, $\psi \geq 0$, and we use $\varphi^\varepsilon = w^\varepsilon \psi \in H_0^1(\Omega^\varepsilon)$, $\varphi^\varepsilon \geq 0$, as test function in (5.17). We obtain

$$\int_{\Omega^\varepsilon} A(x) Du^\varepsilon D\psi\, w^\varepsilon + \int_{\Omega^\varepsilon} A(x) Du^\varepsilon Dw^\varepsilon\, \psi = \int_{\Omega^\varepsilon} F(x, u^\varepsilon) w^\varepsilon \psi,$$

which using (5.9) implies that

$$\int_{\Omega} A(x) D\tilde{u^\varepsilon} D\psi\, w^\varepsilon + \int_{\Omega} A(x) D\tilde{u^\varepsilon} Dw^\varepsilon\, \psi = \int_{\Omega} \widetilde{F(x, u^\varepsilon)} w^\varepsilon \psi. \tag{9.3}$$

Equation (9.3) in particular implies by (9.1) and (5.5) that

$$\int_{\Omega} \widetilde{F(x, u^\varepsilon)} w^\varepsilon \psi \leq C, \tag{9.4}$$

where C is independent of ε.

We now claim that for a subsequence, still labeled by ε,

$$\chi_{\Omega^\varepsilon} \to 1 \text{ a.e. in } \Omega; \tag{9.5}$$

indeed, from $w^\varepsilon \chi_{\Omega^\varepsilon} = w^\varepsilon$ a.e. in Ω, which results from (5.4) (see also (5.10)), and from (5.5) we get

$$\chi_{\Omega^\varepsilon} = \chi_{\Omega^\varepsilon} w^\varepsilon + \chi_{\Omega^\varepsilon}(1 - w^\varepsilon) = w^\varepsilon + \chi_{\Omega^\varepsilon}(1 - w^\varepsilon) \rightharpoonup 1 \text{ in } L^\infty(\Omega) \text{ weakly-star},$$

which implies that

$$\int_\Omega |\chi_{\Omega^\varepsilon} - 1| = \int_\Omega (1 - \chi_{\Omega^\varepsilon}) \to 0$$

which implies (9.5) (for a subsequence).

From now on, ε will always belong to this subsequence.

We deduce from (9.5) that for almost every x_0 fixed in Ω there exists $\varepsilon_0(x_0)$ such that $\chi_{\Omega^\varepsilon}(x_0) = 1$ for every $\varepsilon \le \varepsilon_0(x_0)$, which means that $x_0 \in \Omega^\varepsilon$ for every $\varepsilon \le \varepsilon_0(x_0)$. This implies that

$$\widetilde{F(x, u^\varepsilon)}(x_0) = F(x, u^\varepsilon)(x_0) = F(x, \tilde{u^\varepsilon})(x_0) \quad \forall \varepsilon \le \varepsilon_0(x_0).$$

Therefore, using (9.2), we get

$$\widetilde{F(x, u^\varepsilon)}(x_0) = F(x, \tilde{u^\varepsilon}(x_0)) \to F(x, u^0(x_0)) \text{ as } \varepsilon \to 0,$$

or in other terms

$$\widetilde{F(x, u^\varepsilon)} \to F(x, u^0) \text{ a.e. } x \in \Omega. \tag{9.6}$$

Using (9.4), (5.5) and (9.6) and applying Fatou's Lemma implies that

$$\int_\Omega F(x, u^0)\psi < +\infty \ \ \forall \psi \in H_0^1(\Omega) \cap L^\infty(\Omega), \ \ \psi \ge 0. \tag{9.7}$$

Third step

Let us now fix $\phi \in \mathcal{D}(\Omega), \phi \ge 0$, and take $\psi = \phi$ in (9.3). Since in view of (5.7) one has

$$\begin{cases} \displaystyle\int_\Omega A(x) D\tilde{u^\varepsilon} Dw^\varepsilon \, \phi = \int_\Omega {}^tA(x) Dw^\varepsilon D(\phi \tilde{u^\varepsilon}) - \int_\Omega {}^tA(x) Dw^\varepsilon D\phi \, \tilde{u^\varepsilon} = \\ \displaystyle = \langle \mu^\varepsilon, \phi \tilde{u^\varepsilon} \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} - \int_\Omega {}^tA(x) Dw^\varepsilon D\phi \, \tilde{u^\varepsilon}, \end{cases}$$

equation (9.3) implies that

$$\begin{cases} \displaystyle\int_\Omega A(x) D\tilde{u^\varepsilon} D\phi \, w^\varepsilon + \langle \mu^\varepsilon, \phi \tilde{u^\varepsilon} \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} - \int_\Omega {}^tA(x) Dw^\varepsilon D\phi \, \tilde{u^\varepsilon} = \\ \displaystyle = \int_\Omega \widetilde{F(x, u^\varepsilon)} w^\varepsilon \phi \quad \forall \phi \in \mathcal{D}(\Omega), \phi \ge 0. \end{cases} \tag{9.8}$$

Using (9.2), (5.4), (5.5) and (5.7), we easily pass to the limit in the left-hand side of (9.8), and we obtain

$$\begin{cases} \displaystyle\int_\Omega A(x) D\tilde{u^\varepsilon} D\phi \, w^\varepsilon + \langle \mu^\varepsilon, \phi \tilde{u^\varepsilon} \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} - \int_\Omega {}^tA(x) Dw^\varepsilon D\phi \, \tilde{u^\varepsilon} \to \\ \displaystyle \to \int_\Omega A(x) Du^0 D\phi + \langle \mu, \phi u^0 \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}. \end{cases} \tag{9.9}$$

As far as the right-hand side of (9.8) is concerned we write it for every $\delta > 0$ as

$$\int_\Omega \widetilde{F(x,u^\varepsilon)} w^\varepsilon \phi = \int_\Omega \widetilde{F(x,u^\varepsilon)} w^\varepsilon \phi \chi_{\{0 \le \tilde{u^\varepsilon} \le \delta\}} + \int_\Omega \widetilde{F(x,u^\varepsilon)} w^\varepsilon \phi \chi_{\{\tilde{u^\varepsilon} > \delta\}}. \tag{9.10}$$

Fourth step

We now use $\varphi^\varepsilon = w^\varepsilon \phi Z_\delta(u^\varepsilon)$ as test function in (5.17), where the function Z_δ is defined by (6.6) and where $\phi \in \mathcal{D}(\Omega)$, $\phi \ge 0$. Note that $\varphi^\varepsilon \in H_0^1(\Omega^\varepsilon) \cap L^\infty(\Omega^\varepsilon)$, $\varphi^\varepsilon \ge 0$ in view of (5.4). We get

$$\begin{cases} \displaystyle\int_{\Omega^\varepsilon} F(x,u^\varepsilon) w^\varepsilon \phi Z_\delta(u^\varepsilon) = \\ \displaystyle= \int_{\Omega^\varepsilon} A(x) Du^\varepsilon Dw^\varepsilon \, \phi Z_\delta(u^\varepsilon) + \int_{\Omega^\varepsilon} A(x) Du^\varepsilon D\phi \, w^\varepsilon Z_\delta(u^\varepsilon) + \int_{\Omega^\varepsilon} A(x) Du^\varepsilon Du^\varepsilon \, Z'_\delta(u^\varepsilon) w^\varepsilon \phi, \end{cases}$$

which implies, since $Z_\delta(s) = 1$ for $0 \le s \le \delta$ and since Z_δ is nonincreasing, that

$$\int_{\Omega^\varepsilon} F(x,u^\varepsilon) w^\varepsilon \phi \chi_{\{0 \le u^\varepsilon \le \delta\}} \le \int_{\Omega^\varepsilon} A(x) Du^\varepsilon Dw^\varepsilon \, \phi Z_\delta(u^\varepsilon) + \int_{\Omega^\varepsilon} A(x) Du^\varepsilon D\phi \, w^\varepsilon Z_\delta(u^\varepsilon).$$

In view of the definition (5.8) of the extension by zero and of (5.9), we get

$$\int_\Omega \widetilde{F(x,u^\varepsilon)} w^\varepsilon \phi \chi_{\{0 \le \tilde{u^\varepsilon} \le \delta\}} \le \int_\Omega A(x) D\tilde{u^\varepsilon} Dw^\varepsilon \, \phi Z_\delta(\tilde{u^\varepsilon}) + \int_\Omega A(x) D\tilde{u^\varepsilon} D\phi \, w^\varepsilon Z_\delta(\tilde{u^\varepsilon}). \tag{9.11}$$

Let us define the function $Y_\delta : [0, +\infty[\to [0, +\infty[$ by

$$Y_\delta(s) = \int_0^s Z_\delta(\sigma) d\sigma, \quad \forall s \ge 0,$$

and observe that $Y_\delta(u^\varepsilon) \in H_0^1(\Omega^\varepsilon)$ and $\widetilde{Y_\delta(u^\varepsilon)} = Y_\delta(\tilde{u^\varepsilon})$. Using (5.7), we have

$$\begin{cases} \displaystyle\int_\Omega A(x) D\tilde{u^\varepsilon} Dw^\varepsilon \phi Z_\delta(\tilde{u^\varepsilon}) = \int_\Omega {}^tA(x) Dw^\varepsilon DY_\delta(\tilde{u^\varepsilon}) \phi = \\ \displaystyle= \int_\Omega {}^tA(x) Dw^\varepsilon D(\phi Y_\delta(\tilde{u^\varepsilon})) - \int_\Omega {}^tA(x) Dw^\varepsilon D\phi \, Y_\delta(\tilde{u^\varepsilon}) = \\ \displaystyle= \langle \mu^\varepsilon, \phi Y_\delta(\tilde{u^\varepsilon}) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} - \int_\Omega {}^tA(x) Dw^\varepsilon D\phi \, Y_\delta(\tilde{u^\varepsilon}). \end{cases} \tag{9.12}$$

Using now (5.7), (9.2), the fact that

$$Y_\delta(\tilde{u^\varepsilon}) \rightharpoonup Y_\delta(u^0) \text{ in } H_0^1(\Omega) \text{ weakly and } L^2(\Omega) \text{ strongly},$$

and (5.5) proves that the right-hand side of (9.12) tends to

$$\langle \mu, \phi Y_\delta(u^0) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}$$

as ε tends to zero for $\delta > 0$ fixed.

Turning back to (9.11), using (9.12) and the latest result, and passing to the limit as ε tends to zero, we have proved that for every fixed $\delta > 0$

$$\limsup_{\varepsilon} \int_{\Omega} \widetilde{F(x,u^\varepsilon)} w^\varepsilon \phi \chi_{\{0 \le \tilde{u^\varepsilon} \le \delta\}} \le \langle \mu, \phi Y_\delta(u^0) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \int_{\Omega} A(x) Du^0 D\phi \, Z_\delta(u^0). \tag{9.13}$$

We now pass to the limit in the right-hand side of (9.13) as δ tends to zero.

For the first term of the right-hand side of (9.13), we use the fact that

$$0 \le Z_\delta(u^0) \le 1, \quad Z_\delta(u^0) \to \chi_{\{u^0=0\}} \text{ a.e. in } \Omega \quad \text{as } \delta \to 0, \tag{9.14}$$

and

$$Du^0 = 0 \quad \text{a.e. } x \in \{u^0 = 0\} \quad \text{since } u^0 \in H_0^1(\Omega), \tag{9.15}$$

imply that

$$DY_\delta(u^0) = Z_\delta(u^0) Du^0 \to \chi_{\{u^0=0\}} Du^0 = 0 \quad \text{strongly in } L^2(\Omega)^N;$$

this implies the strong $H_0^1(\Omega)$ convergence of $Y_\delta(u^0)$ to 0, and therefore that

$$\langle \mu, \phi Y_\delta(u^0) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \to 0 \quad \text{as } \delta \to 0.$$

For the second term of the right-hand side of (9.13) we have, using again (9.14) and (9.15),

$$\int_{\Omega} A(x) Du^0 D\phi \, Z_\delta(u^0) \to \int_{\Omega} A(x) Du^0 D\phi \, \chi_{\{u^0=0\}} = 0 \text{ as } \delta \to 0.$$

As far as the first term of the right-hand side of (9.10) is concerned, we have proved that

$$\limsup_{\varepsilon} \int_{\Omega} \widetilde{F(x,u^\varepsilon)} w^\varepsilon \phi \chi_{\{0 \le \tilde{u^\varepsilon} \le \delta\}} \to 0 \quad \text{as } \delta \to 0. \tag{9.16}$$

Fifth step

Let us now pass to the limit in the second term of the right-hand side of (9.10).

Observe that there is at most a countable set $\mathcal{C}^0$ of values of $\delta > 0$ such that

$$\text{meas}\{x \in \Omega : u^0(x) = \delta\} > 0 \text{ if } \delta \in \mathcal{C}^0.$$

From now on we will often choose $\delta > 0$ outside of this set $\mathcal{C}^0$.

Using (9.6), (5.5), (9.2), the fact that

$$\forall \delta > 0, \ \chi_{\{\tilde{u^\varepsilon} > \delta\}} \to \chi_{\{u^0 > \delta\}} \text{ a.e. } x \in \{u^0 \ne \delta\},$$

and therefore that

$$\forall \delta \notin \mathcal{C}^0, \ \chi_{\{\tilde{u^\varepsilon} > \delta\}} \to \chi_{\{u^0 > \delta\}} \text{ a.e. } x \in \Omega,$$

and the estimate (2.3 *iv*), which yields

$$0 \le \widetilde{F(x,u^\varepsilon)} w^\varepsilon \phi(x) \chi_{\{\tilde{u^\varepsilon}>\delta\}} \le h(x)\left(\frac{1}{(\tilde{u^\varepsilon})^\gamma}+1\right)\phi(x)\chi_{\{\tilde{u^\varepsilon}>\delta\}} \le h(x)\left(\frac{1}{\delta^\gamma}+1\right)\phi(x) \text{ a.e. } x\in\Omega,$$

Lebesgue's dominated convergence Theorem implies that

$$\int_\Omega \widetilde{F(x,u^\varepsilon)} w^\varepsilon \phi \chi_{\{\tilde{u^\varepsilon}>\delta\}} \to \int_\Omega F(x,u^0)\phi\chi_{\{u^0>\delta\}} \text{ as } \varepsilon\to 0, \forall \delta\notin \mathcal{C}^0.$$

Using (9.7) and Lebesgue's dominated convergence Theorem, we pass to the limit in this equality when $\delta \notin \mathcal{C}^0$ tends to zero. We have proved that

$$\lim_\varepsilon \int_\Omega \widetilde{F(x,u^\varepsilon)} w^\varepsilon \phi \chi_{\{\tilde{u^\varepsilon}>\delta\}} \to \int_\Omega F(x,u^0)\phi\chi_{\{u^0>0\}} \text{ as } \delta\to 0, \delta\notin \mathcal{C}^0. \tag{9.17}$$

Sixth step

We now want to prove that

$$\int_{\{u^0=0\}} F(x,u^0)\phi = 0. \tag{9.18}$$

Since $\tilde{u^\varepsilon}$ converges almost everywhere to u^0, one has

$$\tilde{u^\varepsilon}(x_0)\to 0 \text{ as } \varepsilon\to 0, \text{ a.e. } x_0\in\{u^0=0\},$$

and therefore $\tilde{u^\varepsilon}(x_0)<\delta$ for every $\varepsilon<\varepsilon_0(x_0)$. This implies that for every $\delta>0$

$$\chi_{\{0\le \tilde{u^\varepsilon}\le\delta\}} \to 1 \text{ a.e. } x\in\{u^0=0\}.$$

Using this fact, (9.6), (5.5) and Fatou's Lemma for $\delta>0$ fixed we get

$$\int_{\{u^0=0\}} F(x,u^0)\phi \le \liminf_\varepsilon \int_{\{u^0=0\}} \widetilde{F(x,u^\varepsilon)} w^\varepsilon \phi \chi_{\{0\le\tilde{u^\varepsilon}\le\delta\}} \forall\delta>0,$$

which, passing to the limit with δ which tends to zero and using (9.16) gives (9.18). This implies that

$$\int_\Omega F(x,u^0)\phi\chi_{\{u^0>0\}} = \int_\Omega F(x,u^0)\phi. \tag{9.19}$$

Let us come back to (9.8). Collecting together (9.9), (9.10), (9.16), (9.17) and (9.19) we have proved that

$$\begin{cases} \forall\phi\in\mathcal{D}(\Omega),\ \phi\ge 0, \\ \displaystyle\int_\Omega A(x)Du^0D\phi + \langle\mu, u^0\phi\rangle_{H^{-1}(\Omega),H^1_0(\Omega)} = \int_\Omega F(x,u^0)\phi. \end{cases} \tag{9.20}$$

Using (5.11) (see footnote 1), this is equivalent to

$$\begin{cases} \forall\phi\in\mathcal{D}(\Omega),\ \phi\ge 0, \\ \displaystyle\int_\Omega A(x)Du^0D\phi + \int_\Omega u^0\phi d\mu = \int_\Omega F(x,u^0)\phi. \end{cases} \tag{9.21}$$

Seventh step

Let us now fix $\psi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, $\psi \geq 0$.

Consider a sequence ψ_n such that

$$\begin{cases} \psi_n \in \mathcal{D}(\Omega),\ \psi_n \geq 0,\ \|\psi_n\|_{L^\infty(\Omega)} \leq C, \\ \psi_n \to \psi \text{ in } H_0^1(\Omega) \text{ strongly and a.e. } x \in \Omega, \end{cases}$$

and define

$$\hat{\psi}_n = \inf\{\psi_n, \psi\};$$

then

$$\begin{cases} \hat{\psi}_n \in H_0^1(\Omega) \cap L^\infty(\Omega),\ \hat{\psi}_n \geq 0,\ \|\hat{\psi}_n\|_{L^\infty(\Omega)} \leq C, \\ \text{supp } \hat{\psi}_n \subset \text{supp } \psi_n \subset\subset \Omega, \\ \hat{\psi}_n \to \psi \text{ in } H_0^1(\Omega) \text{ strongly and a.e. } x \in \Omega. \end{cases}$$

For the moment let n be fixed and let ρ_η be a sequence of mollifiers. For η sufficiently small the support of $\hat{\psi}_n \star \rho_\eta$ is included in a fixed compact K_n of Ω, and $\hat{\psi}_n \star \rho_\eta \in \mathcal{D}(\Omega)$, $\hat{\psi}_n \star \rho_\eta \geq 0$. We can therefore use $\phi = \hat{\psi}_n \star \rho_\eta$ as test function in (9.21). We get

$$\int_\Omega A(x) Du^0 D(\hat{\psi}_n \star \rho_\eta) + \int_\Omega u^0 (\hat{\psi}_n \star \rho_\eta) d\mu = \int_\Omega F(x, u^0)(\hat{\psi}_n \star \rho_\eta).$$

Let us pass to the limit in each term of this equation for n fixed as η tends to zero. In the right-hand side we use the facts that $F(x, u^0) \in L^1_{\text{loc}}(\Omega)$ (see (9.7)), that supp $(\hat{\psi}_n \star \rho_\eta) \subset K_n$, that $\|\hat{\psi}_n \star \rho_\eta\|_{L^\infty(\Omega)} \leq \|\hat{\psi}_n\|_{L^\infty(\Omega)}$ and the almost convergence of $\hat{\psi}_n \star \rho_\eta$ to $\hat{\psi}_n$, and we apply Lebesgue's dominated convergence Theorem. In the first term of the left-hand side we use the strong convergence of $\hat{\psi}_n \star \rho_\eta$ to $\hat{\psi}_n$ in $H_0^1(\Omega)$. This strong convergence also implies (for a subsequence) the quasi-everywhere convergence for the $H_0^1(\Omega)$ capacity and therefore the μ-almost everywhere convergence of $\hat{\psi}_n \star \rho_\eta$ to $\hat{\psi}_n$; we use again Lebesgue's dominated convergence Theorem, this time in $L^1(\Omega; d\mu)$, together with the facts that (see (5.12))

$$0 \leq u^0(\hat{\psi}_n \star \rho_\eta) \leq u^0 \|\hat{\psi}_n\|_{L^\infty(\Omega; d\mu)} = u^0 \|\hat{\psi}_n\|_{L^\infty(\Omega)} \quad \mu\text{-a.e. } x \in \Omega,$$

and that (see (5.11)) $u^0 \in L^1(\Omega; d\mu)$ in order to pass to the limit in the second term of the left-hand side. We have proved that

$$\int_\Omega A(x) Du^0 D\hat{\psi}_n + \int_\Omega u^0 \hat{\psi}_n d\mu = \int_\Omega F(x, u^0)\hat{\psi}_n. \tag{9.22}$$

We now pass to the limit in each term of (9.22) as n tends to infinity. This is easy in the right-hand side by Lebesgue's dominated convergence Theorem since $\hat{\psi}_n$ tends almost everywhere to ψ, since by the definition of $\hat{\psi}_n$

$$0 \leq F(x, u^0)\hat{\psi}_n \leq F(x, u^0)\psi \quad \text{a.e. } x \in \Omega,$$

and since the latest function belongs to $L^1(\Omega)$ (see (9.7)). This is also easy in the first term of the left-hand side of (9.22) since $\hat{\psi}_n$ tends to ψ strongly in $H_0^1(\Omega)$. This strong convergence also implies (for a subsequence) the quasi-everywhere convergence for the $H_0^1(\Omega)$ capacity and therefore the μ-almost everywhere convergence

of $\hat{\psi}_n$ to ψ; we use again Lebesgue's dominated convergence Theorem in $L^1(\Omega; d\mu)$, together with the facts that (see (5.12))

$$0 \le u^0 \hat{\psi}_n \le u^0 \psi \le u^0 \|\psi\|_{L^\infty(\Omega; d\mu)} = u^0 \|\psi\|_{L^\infty(\Omega)} \ \ \mu\text{-a.e. } x \in \Omega,$$

and that (see (5.11)) $u^0 \in L^1(\Omega; d\mu)$ in order to pass to the limit in the second term of the left-hand side. We have proved that

$$\begin{cases} \forall \psi \in H_0^1(\Omega) \cap L^\infty(\Omega),\ \psi \ge 0, \\ \displaystyle\int_\Omega A(x) Du^0 D\psi + \int_\Omega u^0 \psi d\mu = \int_\Omega F(x, u^0)\psi. \end{cases} \tag{9.23}$$

Eighth step

Let us finally prove that $u^0 \in L^2(\Omega; d\mu)$ and that (5.20) holds true.

Taking $\psi = T_n(u^0) \in H_0^1(\Omega) \cap L^\infty(\Omega)$ in (9.23) we obtain

$$\int_\Omega A(x) Du^0 DT_n(u^0) + \int_\Omega u^0 T_n(u^0) d\mu = \int_\Omega F(x, u^0) T_n(u^0),$$

in which using the coercivity (2.1) of A and condition (2.3 *iv*) on the function F, we obtain

$$\int_\Omega |T_n(u^0)|^2 d\mu \le \int_\Omega F(x, u^0) T_n(u^0) \le \int_\Omega h(x)\Big(\frac{1}{(u^0)^\gamma} + 1\Big) u^0 = \int_\Omega h(x)((u^0)^{(1-\gamma)} + u^0) < +\infty, \tag{9.24}$$

which using Fatou's Lemma implies that

$$u^0 \in L^2(\Omega; d\mu).$$

Fix now $z \in H_0^1(\Omega) \cap L^2(\Omega; d\mu)$, $z \ge 0$. Taking $\psi = T_n(z) \in H_0^1(\Omega) \cap L^\infty(\Omega)$ as test function in (9.23) we have

$$\int_\Omega A(x) Du^0 DT_n(z) + \int_\Omega u^0 T_n(z) d\mu = \int_\Omega F(x, u^0) T_n(z). \tag{9.25}$$

It is easy to pass to the limit in each term of the left-hand side of (9.25), since $T_n(z)$ tends to z in $H_0^1(\Omega) \cap L^2(\Omega; d\mu)$ and since $u^0 \in L^2(\Omega; d\mu)$.

Applying Fatou's Lemma to the right-hand side of (9.25), we obtain

$$\int_\Omega F(x, u^0) z \le \int_\Omega A(x) Du^0 Dz + \int_\Omega u^0 z d\mu < +\infty,$$

which is the first statement of (5.20).

But since

$$0 \le F(x, u^0) T_n(z) \le F(x, u^0) z,$$

and since the latest function belongs to $L^1(\Omega)$, Lebesgue's dominated convergence Theorem implies that

$$\int_\Omega F(x,u^0)T_n(z) \to \int_\Omega F(x,u^0)z,$$

which allows us to pass to the limit in (9.25) and completes the proof of the second statement of (5.20).

The proof of Theorem 5.2 is now complete. □

9.2. Proof of the corrector Theorem 5.5

First step

In view of hypothesis (5.23) and of (5.2), the function $w^\varepsilon u^0$ belongs to $H_0^1(\Omega)\cap L^\infty(\Omega)$, and therefore the function r^ε defined by (5.24) belongs to $H_0^1(\Omega)$. By the coercivity assumption (2.1) and by the symmetry assumption (5.22) on the matrix A, we have

$$\begin{cases} \alpha \displaystyle\int_\Omega |Dr^\varepsilon|^2 \le \int_\Omega A(x)Dr^\varepsilon Dr^\varepsilon = \int_\Omega A(x)(D\tilde{u^\varepsilon} - D(w^\varepsilon u^0))(D\tilde{u^\varepsilon} - D(w^\varepsilon u^0)) = \\ = \displaystyle\int_\Omega A(x)D\tilde{u^\varepsilon}D\tilde{u^\varepsilon} - 2\int_\Omega A(x)D\tilde{u^\varepsilon}D(w^\varepsilon u^0) + \int_\Omega A(x)D(w^\varepsilon u^0)D(w^\varepsilon u^0). \end{cases} \tag{9.26}$$

We will pass to the limit in each term of the right-hand side of (9.26).

Second step

As far as the first term of the right-hand side of (9.26) is concerned, taking $u^\varepsilon \in H_0^1(\Omega^\varepsilon)$ as test function in (5.17) and extending u^ε and $F(x,u^\varepsilon)$ by zero into $\tilde{u^\varepsilon}$ and $\widetilde{F(x,u^\varepsilon)}$ (see (5.9) and (5.8)), we get

$$\int_\Omega A(x)D\tilde{u^\varepsilon}D\tilde{u^\varepsilon} = \int_\Omega \widetilde{F(x,u^\varepsilon)}\tilde{u^\varepsilon}. \tag{9.27}$$

Let us pass to the limit in the right-hand side of (9.27). By (2.3 *iv*) we have

$$0 \le F(x,u^\varepsilon)u^\varepsilon \le h(x)\left(\frac{1}{(u^\varepsilon)^\gamma}+1\right)u^\varepsilon,$$

which by (9.1) and by a computation similar to the one made in (7.12) implies that for every measurable set $E\subset\Omega$ one has

$$0 \le \int_E \widetilde{F(x,u^\varepsilon)}\tilde{u^\varepsilon} \le \int_E h(x)\left(\frac{1}{(\tilde{u^\varepsilon})^\gamma}+1\right)\tilde{u^\varepsilon} \le c\,\|h\|_{L^r(E)} + \gamma\|h\|_{L^1(E)},$$

where c is a constant which does not depend neither on E nor on ε, which implies the equi-integrability of $\widetilde{F(x,u^\varepsilon)}\tilde{u^\varepsilon}$. Then convergences (9.2) and (9.6) and Vitali's Theorem imply that

$$\int_\Omega \widetilde{F(x,u^\varepsilon)}\tilde{u^\varepsilon} \to \int_\Omega F(x,u^0)u^0. \tag{9.28}$$

On the other hand, taking $z=u^0$ as test function in (5.20) implies that

$$\int_\Omega A(x)Du^0Du^0 + \int_\Omega (u^0)^2 d\mu = \int_\Omega F(x,u^0)u^0.$$

By (9.27), (9.28) and the previous equality we have, using (5.11) which holds true since $(u^0)^2 \in H_0^1(\Omega)$ when $u^0 \in L^\infty(\Omega)$,

$$\int_{\Omega^\varepsilon} A(x)D\tilde{u^\varepsilon}D\tilde{u^\varepsilon} \to \int_\Omega A(x)Du^0Du^0 + \int_\Omega (u^0)^2 d\mu = \int_\Omega A(x)Du^0Du^0 + \langle \mu, (u^0)^2\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}. \tag{9.29}$$

Third step

Let us now pass to the limit in the third term of the right-hand side of (9.26). Using (5.7) we obtain

$$\begin{cases} \displaystyle\int_\Omega A(x)D(w^\varepsilon u^0)D(w^\varepsilon u^0) = \\ \displaystyle= \int_\Omega A(x)D(w^\varepsilon u^0)Dw^\varepsilon u^0 + \int_\Omega A(x)D(w^\varepsilon u^0)Du^0 w^\varepsilon = \\ \displaystyle= \int_\Omega {}^tA(x)Dw^\varepsilon D(w^\varepsilon (u^0)^2) - \int_\Omega {}^tA(x)Dw^\varepsilon Du^0 w^\varepsilon u^0 + \int_\Omega A(x)D(w^\varepsilon u^0)Du^0 w^\varepsilon = \\ \displaystyle= \langle \mu^\varepsilon, w^\varepsilon (u^0)^2\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} - \int_\Omega {}^tA(x)Dw^\varepsilon Du^0 w^\varepsilon u^0 + \int_\Omega A(x)D(w^\varepsilon u^0)Du^0 w^\varepsilon, \end{cases}$$

in which it is easy to pass to the limit in each term, obtaining

$$\int_\Omega A(x)D(w^\varepsilon u^0)D(w^\varepsilon u^0) \to \langle \mu, (u^0)^2\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + \int_\Omega A(x)Du^0Du^0. \tag{9.30}$$

Fourth step

Passing to the limit in the second term of the right-hand side of (9.26) is a little bit more delicate (except in the case where the regularity hypothesis (4.3) is made on the function F, see Remark 9.1 below).

Fix $\phi \in \mathcal{D}(\Omega)$ and write

$$\int_\Omega A(x)D\tilde{u^\varepsilon}D(w^\varepsilon u^0) = \int_\Omega A(x)D\tilde{u^\varepsilon}Du^0 w^\varepsilon + \int_\Omega A(x)D\tilde{u^\varepsilon}Dw^\varepsilon \phi + \int_\Omega A(x)D\tilde{u^\varepsilon}Dw^\varepsilon (u^0 - \phi). \tag{9.31}$$

It is easy to pass to the limit in the first term of the right-hand side of (9.31), obtaining

$$\int_\Omega A(x)D\tilde{u^\varepsilon}Du^0 w^\varepsilon \to \int_\Omega A(x)Du^0Du^0. \tag{9.32}$$

For what concerned the second term of the right-hand side of (9.31), we have in view of (5.7)

$$\begin{cases} \displaystyle\int_\Omega A(x)D\tilde{u^\varepsilon}Dw^\varepsilon \phi = \int_\Omega {}^tA(x)Dw^\varepsilon D\tilde{u^\varepsilon}\phi = \\ \displaystyle= \int_\Omega {}^tA(x)Dw^\varepsilon D(\tilde{u^\varepsilon}\phi) - \int_\Omega {}^tA(x)Dw^\varepsilon D\phi\, \tilde{u^\varepsilon} = \langle \mu^\varepsilon, \tilde{u^\varepsilon}\phi\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} - \int_\Omega {}^tA(x)Dw^\varepsilon D\phi\, \tilde{u^\varepsilon}, \end{cases} \tag{9.33}$$

and therefore

$$\int_\Omega A(x)D\tilde{u^\varepsilon}Dw^\varepsilon \phi \to \langle \mu, u^0\phi\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = \langle \mu, (u^0)^2\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + \langle \mu, u^0(\phi - u^0)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}. \tag{9.34}$$

Fifth step

We now use $w^\varepsilon(u^0-\phi)^2$ as test function in (5.7). This gives

$$\int_\Omega {}^tA(x)Dw^\varepsilon Dw^\varepsilon(u^0-\phi)^2 + 2\int_\Omega {}^tA(x)Dw^\varepsilon D(u^0-\phi)(u^0-\phi)w^\varepsilon = \langle \mu^\varepsilon, w^\varepsilon(u^0-\phi)^2\rangle_{H^{-1}(\Omega),H_0^1(\Omega)},$$

which implies that

$$\int_\Omega {}^tA(x)Dw^\varepsilon Dw^\varepsilon(u^0-\phi)^2 \to \langle \mu, (u^0-\phi)^2\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}.$$

By the coercivity (2.1), this implies that, for every $\phi\in\mathcal{D}(\Omega)$,

$$\limsup_\varepsilon \alpha\int_\Omega |Dw^\varepsilon|^2|u^0-\phi|^2 \le \langle \mu, (u^0-\phi)^2\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}.$$

This result together with Hölder's inequality and the bound (9.1) on $\|\tilde{u^\varepsilon}\|_{H_0^1(\Omega)}$ implies that for every $\phi\in\mathcal{D}(\Omega)$

$$\begin{cases} \limsup_\varepsilon \left|\int_\Omega A(x)D\tilde{u^\varepsilon}Dw^\varepsilon(u^0-\phi)\right| \le \\ \le \|A\|_{L^\infty(\Omega)^{N\times N}} \limsup_\varepsilon \|\tilde{u^\varepsilon}\|_{H_0^1(\Omega)} \left(\limsup_\varepsilon \int_\Omega |Dw^\varepsilon|^2|u^0-\phi|^2\right)^{1/2} \le \\ \le \|A\|_{L^\infty(\Omega)^{N\times N}} C(|\Omega|,N,\alpha,\gamma,r)\left(\|h\|_{L^r(\Omega)}+\|h\|_{L^1(\Omega)}^{1/2}\right)\left(\frac{1}{\alpha}\langle \mu,(u^0-\phi)^2\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\right)^{1/2} \le \\ \le c\left(\langle \mu,(u^0-\phi)^2\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\right)^{1/2}, \end{cases} \tag{9.35}$$

where $C(|\Omega|,N,\alpha,\gamma,r)$ is the constant which appears in (6.1), and where the constant c depends only on $|\Omega|$, N, α, $\|A\|_{L^\infty(\Omega)^{N\times N}}$, γ, r, $\|h\|_{L^r(\Omega)}$ and $\|h\|_{L^1(\Omega)}$.

Sixth step

Using in (9.26) the results obtained in (9.29), (9.30), (9.31), (9.32), (9.34) and (9.35), we have proved that for every $\phi\in\mathcal{D}(\Omega)$ one has

$$\begin{cases} \limsup_\varepsilon \alpha\|r^\varepsilon\|^2_{H_0^1(\Omega)} \le \\ \le -2\langle \mu, u^0(\phi-u^0)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + \limsup_\varepsilon\left(-2\int_\Omega A(x)D\tilde{u^\varepsilon}Dw^\varepsilon(u^0-\phi)\right) \le \\ \le -2\langle \mu, u^0(\phi-u^0)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + 2c\left(\langle \mu,(u^0-\phi)^2\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\right)^{1/2}. \end{cases} \tag{9.36}$$

Since the sequence y_n^2 converges to 0 strongly in $H_0^1(\Omega)$ when y_n converges to 0 strongly in $H_0^1(\Omega)$ and weakly-star in $L^\infty(\Omega)$, approximating $u^0\in H_0^1(\Omega)\cap L^\infty(\Omega)$ by a sequence of functions $\phi\in\mathcal{D}(\Omega)$ which converges to u^0 strongly in $H_0^1(\Omega)$ and weakly-star in $L^\infty(\Omega)$ proves that

$$r^\varepsilon \to 0 \text{ in } H_0^1(\Omega) \text{ strongly},$$

i.e. (5.24).

Theorem 5.5 is proved. □

Remark 9.1. The above proof of Theorem 5.5 has been made assuming that (5.23) holds true, namely that $u^0 \in L^\infty(\Omega)$. If we assume that the function F, in addition to assumption (2.2) and (2.3), verifies the regularity condition (4.3), the proof of the corrector Theorem 5.5 becomes simpler.

Indeed under this hypothesis, the solutions $\tilde{u}^\varepsilon$ are bounded in $L^\infty(\Omega)$ (see Remark 5.6) (a result which, by the way, implies (5.23)). We claim that this $L^\infty(\Omega)$ bound on $\tilde{u}^\varepsilon$ allows us to perform the computation in the fourth step above when we replace ϕ by u^0: indeed in this case the third term of the right-hand side of (9.31) vanishes (and therefore the fifth step above becomes useless), and (9.33) becomes

$$\begin{cases} \displaystyle\int_\Omega A(x) D\tilde{u^\varepsilon} Dw^\varepsilon\, u^0 = \int_\Omega {}^tA(x) Dw^\varepsilon D\tilde{u^\varepsilon}\, u^0 = \\ \displaystyle= \int_\Omega {}^tA(x) Dw^\varepsilon D(\tilde{u^\varepsilon} u^0) - \int_\Omega {}^tA(x) Dw^\varepsilon Du^0\, \tilde{u^\varepsilon} = \langle \mu^\varepsilon, \tilde{u^\varepsilon} u^0 \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} - \int_\Omega {}^tA(x) Dw^\varepsilon Du^0\, \tilde{u^\varepsilon}, \end{cases} \tag{9.37}$$

where each term has a meaning since now $\tilde{u}^\varepsilon \in L^\infty(\Omega)$. Using the fact that $\tilde{u}^\varepsilon$ is bounded in $L^\infty(\Omega)$ allows us to pass to the limit in (9.37), obtaining

$$\int_\Omega A(x) D\tilde{u^\varepsilon} Dw^\varepsilon\, u^0 \to \langle \mu, (u^0)^2 \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}.$$

Then (9.36) reads as

$$\limsup_\varepsilon\, \alpha \, \|r^\varepsilon\|^2_{H_0^1(\Omega)} \le 0,$$

which is nothing but the desired result (5.24). □

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