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A Random Number Generator for the Kolmogorov Distribution

Abstract. We discuss an acceptance-rejection algorithm for the random number generation from the Kolmogorov distribution. Since the cumulative distribution function (CDF) is a functions series and we need the density distribution function in our algorithm, we prove that the series of the derivatives converges uniformly in order to can derive term by term the functions series; also we provide a similar proof for showing that the ratio between the target Kolmogorov density and the auxiliary density implemented is bounded. Finally, for the application in the algorithm we propose to approximate the density of Kolmogorov distribution by truncation series where the truncation is posed as far away as possible according to the precision of the calculator, we asses the accuracy of this method by a simulation study.

Keywords: Acceptance-Rejection algorithm, Uniform Convergence, Monte Carlo methods, Logistic distribution

1. Introduction

The Kolmogorov distribution naturally arise in the so-called Kolmogorov-Smirnov test, Kolmogorov (1933), Smirnov (1939), here we briefly describe the case of the so-called one sample test. Let

$$X_1, X_2, \dots, X_n \overset{i.i.d.}{\sim} F(\cdot)\,,$$
$$\hat{F}_n(x) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{I}_{\{X_i \le x\}}(x)\,,$$

where $\mathbb{I}_{\{A\}}(x)$ is the indicator function of the set A, so $\hat{F}_n(\cdot)$ is the empirical CDF associated to the observed sample with size n. Let $F_0(\cdot)$ be an absolutely continuous probability distribution and

$$D_n = \sqrt{n}\,\sup|\hat{F}_n(x) - F_0(x)|\,.$$

Kolmogorov (1933) prove that under the null hypothesis $F(\cdot) = F_0(\cdot)$ the following result holds,

$$\Lambda(x) = \lim_{n\to+\infty} \mathrm{P}\left(D_n \le x\right) = \sum_{k=-\infty}^{+\infty} (-1)^k \exp(-2k^2x^2)\,, \quad x > 0.$$

Therefore the asymptotic distribution of D_n is called Kolmogorov distribution and if the null hypothesis is true, it does not depend from $F_0(\cdot)$ as long as $F_0(\cdot)$ is absolutely continuous. Smirnov (1939) provide 2 alternative and equivalent representation for the CDF of Kolmogorov distribution i.e.

$$\Lambda_1(x) = 1 - 2\sum_{k=1}^{+\infty}(-1)^{k-1}\exp(-2k^2x^2)\,, \quad x > 0,$$
$$\Lambda_2(x) = \frac{\sqrt{2\pi}}{x}\sum_{k=1}^{+\infty}\exp\left(-\frac{(2k-1)^2\pi^2}{8x^2}\right)\,, \quad x > 0,$$

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where $\Lambda(x) = \Lambda_1(x)$ is easy to prove by simply algebraic manipulation and $\Lambda_1(x) = \Lambda_2(x)$ is based on the so-called transformation formula for theta functions, Feller (1948), Smirnov (1939).

The Kolmogorov distribution also arise in an other completely different framework i.e. in a representation of logistic distribution as a scale mixtures of Gaussian random variables. Indeed Andrews and Mallows (1974) and Stefanski (1991) prove the following result, if

$$\begin{aligned} Y|W \sim & N(0, 4W^2)\,, \\ W \sim & \Lambda(\cdot)\,, \end{aligned}$$

where $N(\mu, \sigma^2)$ denote the normal distribution with mean μ and variance σ^2, then

$$Y \sim Logis(0,1)\,,$$

where $Logis(a,b)$ denote the logistic distribution with density

$$\frac{\exp(-\frac{x-a}{b})}{b(1+\exp(-\frac{x-a}{b}))^2}\,.$$

1.1 Simulation from Kolmogorov distribution

There is a stochastic representation for the Kolmogorov distribution, indeed it is know if $B(t)$ is a Brownian bridge and $X = \sup|B(t)|$ then $X \sim \Lambda(\cdot)$, Perman and Wellner (2014). This representation can be useful for the random number generation but to generate a Brownian bridge and to get the supremum absolute value is computationally expensive, especially when one need to generate several values from the Kolmogorov distribution.

Therefore in this paper we propose to use an acceptance-rejection algorithm, let $f(x)$ be a target density with space set S and $g(x)$ an auxiliary density such that there exists a constant M which verifies

$$\frac{f(x)}{Mg(x)} \leq 1\,, \quad x \in S\,, \quad 0 < M < +\infty\,.$$

Hence one can use the following procedure

1. sample x from $g(x)$,
2. compute
$$p = \frac{f(x)}{Mg(x)}\,,$$
3. sample u from an uniform distribution in $(0,1)$,
4. if $u \leq p$ get x as a sample from $f(x)$.

Obviously in our case the target density is the Kolmogorov distribution, we use as auxiliaries densities the Gamma and inverse Gamma distributions. As we will explain, we obtain very fine theoretical acceptance rates equals to 89.04% and 95.23% respectively.

2. The density distribution function

In the acceptance-rejection method we need to compute the density of Kolmogorov distribution. Since in both representations of the CDF there is a functions series we need to prove that the series of derivatives converges uniformly in order to can differentiate the series term by term. Notice that we say $\sum_{k=1}^{+\infty} h_k(x)$ converges uniformly in $\mathrm{A} \subseteq \mathbb{R}$ if

$$\lim_{n\to+\infty} \sup_{x\in \mathrm{A}} \left| \sum_{k=1}^{+\infty} h_k(x) - \sum_{k=1}^{n} h_k(x) \right| = 0\,,$$

furthermore a sufficient condition for the uniform convergence is provided by Weierstrass criterion i.e. if

$$
\begin{aligned}
&|h_k(x)| \leq M_k\,, \quad x \in \mathrm{A}\,,\ k = 1,2,\dots\ ,\\
&\sum_{k=1}^{+\infty} M_k < +\infty\,,
\end{aligned}
$$

then $\sum_{k=1}^{+\infty} h_k(x)$ converges uniformly for in A.

First of all we note that

$$
\frac{d}{dx}\left((-1)^{k-1}\exp(-2k^2x^2)\right) = (-1)^k 4k^2 x \exp(-2k^2x^2)\,, \tag{1}
$$

$$
\frac{d}{dx}\exp\left(-\frac{(2k-1)^2\pi^2}{8x^2}\right) = \frac{(2k-1)^2\pi^2}{4x^3}\exp\left(-\frac{(2k-1)^2\pi^2}{8x^2}\right)\,, \tag{2}
$$

so from (1) we require to prove the following proposition.

Proposition 2.0.1. *Let* $\mathrm{A} = \{x \in \mathbb{R} : x \geq x_0\}$ *for some* $x_0 > 0$ *and*

$$
h_k(x) = (-1)^k 4k^2 x \exp(-2k^2x^2)\,,
$$

then $\sum_{k=1}^{+\infty} h_k(x)$ *converges uniformly in* A.

Proof: Let

$$
a_k(x) = 4k^2 x \exp(-2k^2x^2)\,,
$$

hence $h_k(x) = (-1)^k a_k(x)$. We fix x, with $x \geq x_0 > 0$, so $\{h_k(x)\}_{k=1}^{+\infty}$ is an alternating sequence with $a_k(x) > 0$, it is easy to show that

$$
a_k(x) < a_{k+1}(x) \quad \text{if} \quad k > 1/(x\sqrt{2})
$$

and

$$
\lim_{k\to+\infty} a_k(x) = 0\,.
$$

Therefore the series $\sum_{k=1}^{+\infty} h_k(x)$ converges point-wise by Leibniz criterion and we know

$$
\left|\sum_{k=1}^{+\infty} h_k(x) - \sum_{k=1}^{n} h_k(x)\right| \leq a_{n+1}(x)\,,
$$

which implies

$$
\sup_{x\geq x_0}\left|\sum_{k=1}^{+\infty} h_k(x) - \sum_{k=1}^{n} h_k(x)\right| \leq \sup_{x\geq x_0} a_{n+1}(x)\,. \tag{3}
$$

Hence it is easy to compute

$$
a'_{n+1}(x) = 4(n+1)^2 \exp\left(-2(n+1)^2x^2\right)\left(1 - 4x^2(n+1)^2\right)\,,
$$

so a global maximum exists for $x = \left(2(n+1)\right)^{-1}$, but we have restricted the space to $x \geq x_0$ thus

$$
\arg\max_{x\geq x_0} a_{n+1}(x) = \begin{cases} \left(2(n+1)\right)^{-1} & \text{if} \quad \left(2(n+1)\right)^{-1} > x_0 \\ x_0 & \text{if} \quad \left(2(n+1)\right)^{-1} \leq x_0 \end{cases}\,,
$$

so we have

$$
\max_{x\geq x_0} a_{n+1}(x) = \begin{cases} 2(n+1)\exp\left(-\dfrac{1}{2}\right) & \text{if} \quad \left(2(n+1)\right)^{-1} > x_0 \\ 4(n+1)^2 x_0 \exp\left(-2(n+1)^2 x_0^2\right) & \text{if} \quad \left(2(n+1)\right)^{-1} \leq x_0 \end{cases}\,.
$$

It is straightforward

$$\lim_{n\to+\infty} \arg\max_{x\geq x_0} a_{n+1}(x) = x_0\,,$$

so by taking the limit of (3) we obtain

$$\lim_{n\to+\infty} \sup_{x\geq x_0} \left| \sum_{k=1}^{+\infty} h_k(x) - \sum_{k=1}^{n} h_k(x) \right| \leq \lim_{n\to+\infty} 4(n+1)^2 x_0 \exp\left(-2(n+1)^2 x_0^2\right) = 0\,.$$

Notice that if we extend the domain set A of **proposition 2.0.1** from $x \geq x_0$ to $x \geq 0$ then we have

$$\arg\max_{x\geq 0} a_{n+1}(x) = \frac{1}{2(n+1)}\,,$$
$$\max_{x\geq 0} a_{n+1}(x) = 2(n+1)\exp\left(-\frac{1}{2}\right)\,,$$

but in this case the limit for $n \to +\infty$ is not 0. Therefore if we set the lower bound of A equal to 0, the sufficient condition provided by majorant function of Leibniz criterion fails.

Now we need to prove the following proposition from (2).

Proposition 2.0.2. *Let* $\mathrm{A} = \{x \in \mathbb{R} : 0 < x \leq x_0\}$ *for some* $x_0 > 0$ *and*

$$h_k(x) = \frac{(2k-1)^2\pi^2}{4x^3} \exp\left(-\frac{(2k-1)^2\pi^2}{8x^2}\right)\,,$$

then $\sum_{k=1}^{+\infty} h_k(x)$ *converges uniformly in* A.

Proof: We use the sufficient condition provided by Weierstrass criterion, so we set

$$M_k = \max_{0<x\leq x_0} |h_k(x)| = \max_{0<x\leq x_0} h_k(x)\,,$$

since $h_k(x)$ is always positive for $x > 0$. It is easy to show that

$$h_k'(x) = \frac{(2k-1)^2\pi^2}{4x^4}\left(\frac{(2k-1)^2\pi^2}{4x^2} - 3\right)\exp\left(-\frac{(2k-1)^2\pi^2}{8x^2}\right)\,,$$

hence a global maximum exists for $x = (2k-1)\pi/\sqrt{12}$, but we have restricted the space to $0 < x \leq x_0$ thus

$$\arg\max_{0<x\leq x_0} h_k(x) = \begin{cases} (2k-1)\pi/\sqrt{12} & \text{if} \quad (2k-1)\pi/\sqrt{12} < x_0 \\ x_0 & \text{if} \quad (2k-1)\pi/\sqrt{12} \geq x_0 \end{cases}\,,$$

so we have

$$\max_{0<x\leq x_0} h_k(x) = M_k = \begin{cases} \dfrac{2\sqrt{27}}{(2k-1)\pi}\exp\left(-\dfrac{3}{2}\right) & \text{if} \quad (2k-1)\pi/\sqrt{12} < x_0 \\ \dfrac{(2k-1)^2\pi^2}{4x^3}\exp\left(-\dfrac{(2k-1)^2\pi^2}{8x^2}\right) & \text{if} \quad (2k-1)\pi/\sqrt{12} \geq x_0 \end{cases}\,,$$

and it is easy to prove $\sum_{k=1}^{+\infty} M_k$ is finite.

Notice that if we extend the domain set A of **proposition 2.0.2** from $0 < x \leq x_0$ to $0 < x < +\infty$ then we have

$$\arg\max_{0<x<+\infty} h_k(x) = \frac{(2k-1)\pi}{\sqrt{12}}\,,$$
$$\max_{0<x<+\infty} h_k(x) = M_k = \frac{2\sqrt{27}}{(2k-1)\pi}\exp\left(-\frac{3}{2}\right)\,,$$

but in this case the series $\sum_{k=1}^{+\infty} M_k$ is divergent. Therefore if the definition set is unbounded above, the sufficient condition provided by Weierstrass criterion fails.

Finally, we can obtain the density by deriving the series term by term. Let $\Lambda_1'(x) = \lambda_1(x)$ and $\Lambda_2'(x) = \lambda_2(x)$, we get for some $x_0 > 0$,

$$\lambda_1(x) = 8x \sum_{k=1}^{+\infty} (-1)^{k-1} k^2 \exp\left(-2k^2 x^2\right) , \qquad x \geq x_0 ,$$

$$\lambda_2(x) = \frac{\sqrt{2\pi}}{x^2} \sum_{k=1}^{+\infty} \left(\frac{(2k-1)^2 \pi^2}{4x^2} - 1 \right) \exp\left(-\frac{(2k-1)^2 \pi^2}{8x^2} \right) , \qquad 0 < x \leq x_0 .$$

3. The proposal density

As proposal density we try to use the Gamma and inverse Gamma distribution, so we need to verify if the ratios between Kolmogorov density and these are bounded. Notice that the target density is always finite for $0 < x < +\infty$; hence we only need to verify the limits ratios are bounded for $x \to 0^+$ and $x \to +\infty$ that is

$$\lim_{x \to +\infty} \frac{\lambda_1(x)}{g(x)} < +\infty , \lim_{x \to 0^+} \frac{\lambda_2(x)}{g(x)} < +\infty , \tag{4}$$

where $g(x)$ is the proposal density. Notice that we cannot use

$$\lim_{x \to +\infty} \frac{\lambda_2(x)}{g(x)} < +\infty \text{ or } \lim_{x \to 0^+} \frac{\lambda_1(x)}{g(x)} < +\infty$$

because in section 2 we fail to prove that $\lambda_1(x)$ and $\lambda_2(x)$ are representation of Kolmogorov density when $x \to 0^+$ and $x \to +\infty$ respectively.

3.1 Inverse Gamma proposal

Let

$$g(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{-\alpha-1} \exp\left(-\frac{\beta}{x} \right) , \quad x > 0 ,$$

hence by (4) we must prove

$$\lim_{x \to +\infty} \frac{8\Gamma(\alpha)}{\beta^\alpha} \sum_{k=1}^{+\infty} (-1)^{k-1} k^2 x^{\alpha+2} \exp\left(-2k^2 x^2 + \frac{\beta}{x} \right) < +\infty , \tag{5}$$

$$\lim_{x \to 0^+} \frac{\sqrt{2\pi}\Gamma(\alpha)}{\beta^\alpha} \sum_{k=1}^{+\infty} \left(\frac{(2k-1)^2 \pi^2}{4x^2} - 1 \right) x^{\alpha-1} \exp\left(-\frac{(2k-1)^2 \pi^2}{8x^2} + \frac{\beta}{x} \right) < +\infty . \tag{6}$$

In order to change the order between the limits and summations we need to prove the series uniformly converge.

Proposition 3.1.1. *Let* $\mathrm{A} = \{x \in \mathbb{R} : x \geq x_0\}$ *for some* $x_0 > 0$ *and*

$$h_k(x) = (-1)^{k-1} k^2 x^{\alpha+2} \exp\left(-2k^2 x^2 + \frac{\beta}{x} \right) ,$$

then $\sum_{k=1}^{+\infty} h_k(x)$ *converges uniformly in* A.

Proof: Let

$$a_k(x) = k^2 x^{\alpha+2} \exp\left(-2k^2x^2 + \frac{\beta}{x}\right) ,$$

hence $h_k(x) = (-1)^k a_k(x)$. We fix x, with $x \geq x_0 > 0$, so $\{h_k(x)\}_{k=1}^{+\infty}$ is an alternating sequence with $a_k(x) > 0$, it is easy to show that

$$a_k(x) < a_{k+1}(x) \quad \text{if} \quad k > 1/(x\sqrt{2})$$

and

$$\lim_{k\to+\infty} a_k(x) = 0 \, .$$

Therefore the series $\sum_{k=1}^{+\infty} h_k(x)$ converges point-wise by Leibniz criterion and we know

$$\left| \sum_{k=1}^{+\infty} h_k(x) - \sum_{k=1}^{n} h_k(x) \right| \leq a_{n+1}(x) \, ,$$

which implies

$$\sup_{x\geq x_0} \left| \sum_{k=1}^{+\infty} h_k(x) - \sum_{k=1}^{n} h_k(x) \right| \leq \sup_{x\geq x_0} a_{n+1}(x) \, . \tag{7}$$

We consider

$$a_{n+1}(x) = (n+1)^2 x^{\alpha+2} \exp\left(-2(n+1)^2x^2 + \frac{\beta}{x}\right) ,$$
$$b_{n+1}(x) = (n+1)^2 x^{\alpha+2} \exp\left(-2(n+1)^2x^2 + \frac{\beta}{x_0}\right) ,$$

so $a_{n+1}(x) \leq b_{n+1}(x)$ since $x \geq x_0$. Furthermore

$$\frac{\mathrm{d} \log b_{n+1}(x)}{\mathrm{d}x} = \frac{\alpha+2}{x} - 4(n+1)^2 x$$

and

$$\frac{\mathrm{d} \log b_{n+1}(x)}{\mathrm{d}x} \geq 0 \iff x^2 \leq \frac{\alpha+2}{4(n+1)^2} \, ,$$

hence we obtain

$$\arg\max_{x\geq x_0} b_{n+1}(x) = \begin{cases} \dfrac{\sqrt{\alpha+2}}{2(n+1)} & \text{if} \quad x_0 < \dfrac{\sqrt{\alpha+2}}{2(n+1)} \\ x_0 & \text{if} \quad x_0 \geq \dfrac{\sqrt{\alpha+2}}{2(n+1)} \end{cases} ,$$

It is straightforward

$$\lim_{n\to+\infty} \arg\max_{x\geq x_0} b_{n+1}(x) = x_0 \, ,$$
$$\lim_{n\to+\infty} \max_{x\geq x_0} b_{n+1}(x) = \lim_{n\to+\infty} (n+1)^2 x_0^{\alpha+2} \exp\left(-2(n+1)^2 x_0^2 + \frac{\beta}{x_0}\right) = 0 \, ,$$

so by taking the limit of (7) we obtain

$$\lim_{n\to+\infty} \sup_{x\geq x_0} \left| \sum_{k=1}^{+\infty} h_k(x) - \sum_{k=1}^{n} h_k(x) \right| \leq \lim_{n\to+\infty} \sup_{x\geq x_0} a_{n+1}(x) \leq \lim_{n\to+\infty} \sup_{x\geq x_0} b_{n+1}(x) = 0 \, .$$

Notice that by **proposition 3.1.1** we can compute easily the (5) indeed we change the order between limit and summation so we obtain

$$\frac{8\Gamma(\alpha)}{\beta^\alpha} \sum_{k=1}^{+\infty} \lim_{x\to+\infty} (-1)^{k-1} k^2 x^{\alpha+2} \exp\left(-2k^2x^2 + \frac{\beta}{x}\right) = 0 \, .$$

Proposition 3.1.2. *Let* $\mathrm{A} = \{x \in \mathbb{R} : 0 < x \leq x_0\}$ *for some* $x_0 > 0$ *and*

$$h_k(x) = \sum_{k=1}^{+\infty} \left(\frac{(2k-1)^2\pi^2}{4x^2} - 1 \right) x^{\alpha-1} \exp\left(-\frac{(2k-1)^2\pi^2}{8x^2} + \frac{\beta}{x} \right) ,$$

then $\sum_{k=1}^{+\infty} h_k(x)$ *converges uniformly in* A.

Proof: Let

$$a_k(x) = \frac{(2k-1)^2\pi^2}{4} x^{\alpha-3} \exp\left(-\frac{(2k-1)^2\pi^2}{8x^2} + \frac{\beta}{x} \right) ,$$

thus $h_k(x) \leq a_k(x)$ since $0 < x \leq x_0$, we consider the exponential term of $a_k(x)$, i.e.

$$\exp\left(-\frac{(2k-1)^2\pi^2}{8x^2} + \frac{\beta}{x} \right) = \exp\left(\frac{1}{x} \left(\beta - \frac{(2k-1)^2\pi^2}{8x} \right) \right) ,$$

then we have

$$\beta - \frac{(2k-1)^2\pi^2}{8x} > 0 \iff x > \frac{(2k-1)^2\pi^2}{8\beta} .$$

We have 2 different cases; if $x_0 > (2k-1)^2\pi^2/(8\beta)$ then

$$\exp\left(\frac{1}{x} \left(\beta - \frac{(2k-1)^2\pi^2}{8x} \right) \right) \leq \exp\left(\frac{8\beta}{(2k-1)^2\pi^2} \left(\beta - \frac{(2k-1)^2\pi^2}{8x} \right) \right) ,$$

if $x_0 \leq (2k-1)^2\pi^2/(8\beta)$ then

$$\exp\left(\frac{1}{x} \left(\beta - \frac{(2k-1)^2\pi^2}{8x} \right) \right) \leq \exp\left(\frac{1}{x_0} \left(\beta - \frac{(2k-1)^2\pi^2}{8x} \right) \right) ,$$

therefore, let

$$b_k(x) = \begin{cases} \dfrac{(2k-1)^2\pi^2}{4} x^{\alpha-3} \exp\left(\dfrac{\beta}{x_0} - \dfrac{(2k-1)^2\pi^2}{8x_0 x} \right) & \text{if} \quad x_0 \leq \dfrac{(2k-1)^2\pi^2}{8\beta} \\ \dfrac{(2k-1)^2\pi^2}{4} x^{\alpha-3} \exp\left(\dfrac{8\beta^2}{(2k-1)^2\pi^2} - \dfrac{\beta}{x} \right) & \text{if} \quad x_0 > \dfrac{(2k-1)^2\pi^2}{8\beta} \end{cases} ,$$

thus $h_k(x) \leq a_k(x) \leq b_k(x)$. We use the sufficient condition provided by Weierstrass criterion, so we set

$$M_k = \max_{0<x\leq x_0} b_k(x) \leq \max_{0<x\leq x_0} h_k(x) ,$$

it is easy to obtain

$$\frac{\mathrm{d}\log b_k(x)}{\mathrm{d}x} = \begin{cases} \dfrac{\alpha-3}{x} + \dfrac{(2k-1)^2\pi^2}{8x_0x^2} & \text{if} \quad x_0 \leq \dfrac{(2k-1)^2\pi^2}{8\beta} \\ \dfrac{\alpha-3}{x} + \dfrac{\beta}{x^2} & \text{if} \quad x_0 > \dfrac{(2k-1)^2\pi^2}{8\beta} \end{cases} ,$$

we have 3 different cases; if $\alpha \geq 3$ then

$$\arg \max_{0<x\leq x_0} b_k(x) = x_0 , \; k = 1, 2, \dots ,$$

$$\max_{0<x\leq x_0} b_k(x) = \begin{cases} \dfrac{(2k-1)^2\pi^2}{4} x_0^{\alpha-3} \exp\left(\dfrac{\beta}{x_0} - \dfrac{(2k-1)^2\pi^2}{8x_0^2} \right) & \text{if} \quad x_0 \leq \dfrac{(2k-1)^2\pi^2}{8\beta} \\ \dfrac{(2k-1)^2\pi^2}{4} x_0^{\alpha-3} \exp\left(\dfrac{8\beta^2}{(2k-1)^2\pi^2} - \dfrac{\beta}{x_0} \right) & \text{if} \quad x_0 > \dfrac{(2k-1)^2\pi^2}{8\beta} \end{cases} ,$$

if $\alpha < 3$ and $\beta \le (3-\alpha)x_0$ then

$$\arg\max_{0<x\le x_0} b_k(x) = \begin{cases} x_0 & \text{if} \quad x_0 \le \dfrac{(2k-1)\pi}{2\sqrt{2(3-\alpha)}} \\ \dfrac{(2k-1)^2\pi^2}{(3-\alpha)8x_0} & \text{if} \quad \dfrac{(2k-1)\pi}{2\sqrt{2(3-\alpha)}} < x_0 \le \dfrac{(2k-1)^2\pi^2}{8\beta} \\ \dfrac{\beta}{3-\alpha} & \text{if} \quad x_0 > \dfrac{(2k-1)^2\pi^2}{8\beta} \end{cases},$$

$$\max_{0<x\le x_0} b_k(x) = \begin{cases} \dfrac{(2k-1)^2\pi^2}{4} x_0^{\alpha-3} \exp\left(\dfrac{\beta}{x_0} - \dfrac{(2k-1)^2\pi^2}{8x_0^2}\right) & \text{if} \quad x_0 \le \dfrac{(2k-1)\pi}{2\sqrt{2(3-\alpha)}} \\ \dfrac{(2k-1)^{2\alpha-4}\,\pi^{2\alpha-4}}{(3-\alpha)^{\alpha-3}\,2^{3\alpha-7}\,x_0^{\alpha-3}} \exp\left(\dfrac{\beta}{x_0} + \alpha - 3\right) & \text{if} \quad \dfrac{(2k-1)\pi}{2\sqrt{2(3-\alpha)}} < x_0 \le \dfrac{(2k-1)^2\pi^2}{8\beta} \\ \dfrac{(2k-1)^2\pi^2}{4}\left(\dfrac{\beta}{3-\alpha}\right)^{\alpha-3} \exp\left(\dfrac{8\beta^2}{(2k-1)^2\pi^2} + \alpha - 3\right) & \text{if} \quad x_0 > \dfrac{(2k-1)^2\pi^2}{8\beta} \end{cases},$$

and if $\alpha < 3$ and $\beta > (3-\alpha)x_0$ then

$$\arg\max_{0<x\le x_0} b_k(x) = \begin{cases} x_0 & \text{if} \quad x_0 \le \dfrac{(2k-1)^2\pi^2}{8\beta} \\ \dfrac{\beta}{3-\alpha} & \text{if} \quad x_0 > \dfrac{(2k-1)^2\pi^2}{8\beta} \end{cases},$$

$$\max_{0<x\le x_0} b_k(x) = \begin{cases} \dfrac{(2k-1)^2\pi^2}{4} x_0^{\alpha-3} \exp\left(\dfrac{\beta}{x_0} - \dfrac{(2k-1)^2\pi^2}{8x_0^2}\right) & \text{if} \quad x_0 \le \dfrac{(2k-1)^2\pi^2}{8\beta} \\ \dfrac{(2k-1)^2\pi^2}{4}\left(\dfrac{\beta}{3-\alpha}\right)^{\alpha-3} \exp\left(\dfrac{8\beta^2}{(2k-1)^2\pi^2} + \alpha - 3\right) & \text{if} \quad x_0 > \dfrac{(2k-1)^2\pi^2}{8\beta} \end{cases}.$$

It is straightforward for all 3 cases there exists $\tilde{k}$ such that for $k \ge \tilde{k}$ we have

$$\arg\max_{0<x\le x_0} b_k(x) = x_0\,,$$
$$\max_{0<x\le x_0} b_k(x) = \frac{(2k-1)^2\pi^2}{4} x_0^{\alpha-3} \exp\left(\frac{\beta}{x_0} - \frac{(2k-1)^2\pi^2}{8x_0^2}\right),$$

so it is easy to show

$$\sum_{k=1}^{+\infty} M_k = \sum_{k=1}^{\tilde{k}-1} M_k + \sum_{k=\tilde{k}}^{+\infty} M_k < +\infty\,.$$

Notice that by **proposition 3.1.2** we can compute easily the (6) indeed we change the order between limit and summation so we obtain

$$\frac{\sqrt{2\pi}\Gamma(\alpha)}{\beta^\alpha} \sum_{k=1}^{+\infty} \lim_{x\to 0^+} \left(\frac{(2k-1)^2\pi^2}{4x^2} - 1\right) x^{\alpha-1} \exp\left(-\frac{(2k-1)^2\pi^2}{8x^2} + \frac{\beta}{x}\right) = 0\,.$$

Therefore the inverse Gamma distribution is an admissible proposal density for all values of α, β and x_0.

3.2 Gamma proposal

Let

$$g(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\beta x)\,, \quad x > 0\,,$$

hence by (4) we must prove

$$\lim_{x\to+\infty}\frac{8\Gamma(\alpha)}{\beta^\alpha}\sum_{k=1}^{+\infty}(-1)^{k-1}k^2x^{2-\alpha}\exp\left(-2k^2x^2+\beta x\right)<+\infty\,, \tag{8}$$

$$\lim_{x\to0^+}\frac{\sqrt{2\pi}\Gamma(\alpha)}{\beta^\alpha}\sum_{k=1}^{+\infty}\left(\frac{(2k-1)^2\pi^2}{4x^2}-1\right)x^{-\alpha-1}\exp\left(-\frac{(2k-1)^2\pi^2}{8x^2}+\beta x\right)<+\infty\,. \tag{9}$$

As for the inverse Gamma, in order to change the order between the limits and summations we need to prove the series uniformly converge.

Proposition 3.2.1. *Let* $\mathrm{A}=\{x\in\mathbb{R}:x\geq x_0\}$ *for some* $x_0>0$ *and*

$$h_k(x)=(-1)^{k-1}k^2x^{2-\alpha}\exp\left(-2k^2x^2+\beta x\right)\,,$$

then $\sum_{k=1}^{+\infty}h_k(x)$ *converges uniformly in* A.

Proof: See Appendix.

Hence by **proposition 3.2.1** we can compute easily the (8), indeed we change the order between limit and summation so we obtain

$$\frac{8\Gamma(\alpha)}{\beta^\alpha}\sum_{k=1}^{+\infty}\lim_{x\to+\infty}(-1)^{k-1}k^2x^{2-\alpha}\exp\left(-2k^2x^2+\beta x\right)=0\,.$$

Proposition 3.2.2. *Let* $\mathrm{A}=\{x\in\mathbb{R}:0<x\leq x_0\}$ *for some* $x_0>0$ *and*

$$h_k(x)=\left(\frac{(2k-1)^2\pi^2}{4x^2}-1\right)x^{-\alpha-1}\exp\left(-\frac{(2k-1)^2\pi^2}{8x^2}+\beta x\right)\,,$$

then $\sum_{k=1}^{+\infty}h_k(x)$ *converges uniformly in* A.

Proof: See Appendix.

Hence by **proposition 3.2.2** we can compute easily the (9), indeed we change the order between limit and summation so we obtain

$$\frac{\sqrt{2\pi}\Gamma(\alpha)}{\beta^\alpha}\sum_{k=1}^{+\infty}\lim_{x\to0^+}\left(\frac{(2k-1)^2\pi^2}{4x^2}-1\right)x^{-\alpha-1}\exp\left(-\frac{(2k-1)^2\pi^2}{8x^2}+\beta x\right)=0\,.$$

Therefore the Gamma distribution is an admissible proposal density for all values of α,β and x_0.

4. Numerical Optimization

Once obtained a representation for the Kolmogorov distribution density and admissible proposals for acceptance-rejection algorithm we need to choose some parameters for applications. In particular we need a value k^* in order to truncate the series of $\lambda_1(\cdot)$ and $\lambda_2(\cdot)$, a value x_0 and to tune the parameters α and β of the proposals.

According to precision of calculation there exists a value $\bar{x}$ such that $\exp(-x)$ is approximated with 0 for all values of x greater than $\bar{x}$. Therefore we define

$$\bar{k}_1(x)=\sup\{k\in\mathbb{N}_0:2k^2x^2\leq\bar{x}\}=\left\lfloor\frac{\sqrt{2\bar{x}}}{2x}\right\rfloor\,,$$

$$\bar{k}_2(x)=\sup\{k\in\mathbb{N}_0:(2k-1)^2\pi^2/(8x^2)\leq\bar{x}\}=\left\lfloor\frac{x\sqrt{2\bar{x}}}{\pi}+\frac{1}{2}\right\rfloor\,,$$

where $\lfloor \cdot \rfloor$ is the floor function. The values $\bar{k}_1(x)$ and $\bar{k}_2(x)$ depend from x, for computational reasons we avoid to calculate them for all values of x; since the first is decreasing in x with $x \geq x_0$ and the second is increasing in x with $0 < x \leq x_0$, we use their maximum value by assuming $x = x_0$ i.e. we set

$$k^*(x_0) = \max\left(\bar{k}_1(x_0)\ \bar{k}_2(x_0)\right) . \tag{10}$$

Hence the (10) provide the value of k^* as a function of x_0, so we choice

$$x_0^* = \inf\left\{\arg\min_{x_0>0} k^*(x_0)\right\} ,$$
$$k^* = k^*(x_0^*) .$$

Finally we obtain a numerical approximation $f^*(\cdot)$ of the Kolmogorov density by

$$f^*(x) = \begin{cases} \dfrac{\sqrt{2\pi}}{x^2}\displaystyle\sum_{k=1}^{k^*}\left(\frac{(2k-1)^2\pi^2}{4x^2}-1\right)\exp\left(-\frac{(2k-1)^2\pi^2}{8x^2}\right) & \text{if} \quad 0 < x < x_0^* \\ 8x\displaystyle\sum_{k=1}^{k^*}(-1)^{k-1}k^2\exp\left(-2k^2x^2\right) & \text{if} \quad x \geq x_0^* \end{cases} .$$

In our machine the value of $\bar{x}$ is 745.13321910194116 with an approximation of 14 digits, so we obtain

$$x_0^* = 1.207\,,\ k^* = 15\,.$$

For the parameters of proposal of the acceptance-rejection algorithm we proceed in the following manner: let $g(\cdot;\alpha,\beta)$ be the proposal density depending from parameters α and β, we set

$$M = h(\alpha,\beta) = \sup_{x>0}\frac{f^*(x)}{g(x;\alpha,\beta)} ,$$

so we obtain

$$(\alpha^*,\beta^*) = \arg\min_{\alpha>0,\beta>0} h(\alpha,\beta) ,$$

where the minimization is computed via numerical approximation. For the inverse Gamma proposal we have $\alpha^* = 10.29$, $\beta^* = 8.33$ and $M = 1.05$; for the Gamma proposal we obtain $\alpha^* = 9.21$, $\beta^* = 10.96$ and $M = 1.123$. The values of M lead to a theoretical acceptance rate $1/M$ equals to 95.23% and 89.04% for inverse Gamma and Gamma respectively.

Finally we perform a simulation study in order to evaluate the accuracy and efficiency of our procedure with the inverse Gamma proposal. We generate 100 times 10^6 draws, for each iteration the mean elapsed time is $1.3623\,\text{sec}$; furthermore we set the seed to 1 and with a single iteration of 10^6 draws we obtain an empirical mean and empirical variance equal to

$$\hat{\mu} = 8.687866 \times 10^{-1}\,,\ \hat{\sigma}^2 = 6.76608 \times 10^{-2}\,,$$

while, as reported in Marsaglia, Tsang, and Wang (2003), the true values are:

$$\mu = \sqrt{\pi/2}\log 2 \approx 8.687311 \times 10^{-1}\,,\ \sigma^2 = \pi^2/12 - \mu^2 \approx 6.777320 \times 10^{-2}\,.$$

5. Conclusion

In this paper we have derived rigorously the density function of Kolmogorov distribution by deriving the series term by term; surprisingly the standard sufficient condition for the uniformly convergence fails to hold for both bounds of the domain set, so we have used 2 different representation which converge uniformly in 1 bound a time, furthermore they are useful also for speed up the convergence of the series. We have provided an approximation of the obtained density function via truncation series based on the precision of calculator.

In the same way we have proved rigorously that Gamma and inverse Gamma distribution are admissible proposal for Kolmogorov distribution in acceptance-rejection algorithm and both of them provide very fine theoretical acceptance rate after numerical optimization of the parameters. A simulation study have showed the accuracy and efficiency of our method.

Appendix

Here we provide the proof of **proposition 3.2.1** and **proposition 3.2.2** .

Proposition 3.2.1. *Let* $\mathrm{A} = \{x \in \mathbb{R} : x \geq x_0\}$ *for some* $x_0 > 0$ *and*

$$h_k(x) = (-1)^{k-1} k^2 x^{2-\alpha} \exp\left(-2k^2x^2 + \beta x\right) \ ,$$

then $\sum_{k=1}^{+\infty} h_k(x)$ *converges uniformly in* A.

Proof: Let

$$a_k(x) = k^2 x^{2-\alpha} \exp(-2k^2x^2) \,,$$

hence $h_k(x) = (-1)^k a_k(x)$. We fix x, with $x \geq x_0 > 0$, so $\{h_k(x)\}_{k=1}^{+\infty}$ is an alternating sequence with $a_k(x) > 0$, it is easy to show that

$$a_k(x) < a_{k+1}(x) \quad \text{if} \quad k > 1/(x\sqrt{2})$$

and

$$\lim_{k \to +\infty} a_k(x) = 0 \,.$$

Therefore the series $\sum_{k=1}^{+\infty} h_k(x)$ converges point-wise by Leibniz criterion and we know

$$\left| \sum_{k=1}^{+\infty} h_k(x) - \sum_{k=1}^{n} h_k(x) \right| \leq a_{n+1}(x) \,,$$

which implies

$$\sup_{x \geq x_0} \left| \sum_{k=1}^{+\infty} h_k(x) - \sum_{k=1}^{n} h_k(x) \right| \leq \sup_{x \geq x_0} a_{n+1}(x) \,. \tag{11}$$

We define

$$b_{n+1}(x) = \begin{cases} x_0^{2-\alpha} (n+1)^2 \exp\left(-2(n+1)^2x^2 + \beta x\right) & \text{if} \quad \alpha \geq 2 \\ a_{n+1}(x) & \text{if} \quad \alpha < 2 \end{cases} ,$$

so $a_{n+1}(x) \leq b_{n+1}(x)$ since $x \geq x_0$, furthermore

$$\frac{\mathrm{d} \log b_{n+1}(x)}{\mathrm{d}x} = \begin{cases} -\,4(n+1)^2 x + \beta & \text{if} \quad \alpha \geq 2 \\ \dfrac{2-\alpha}{x} - 4(n+1)^2 x + \beta & \text{if} \quad \alpha < 2 \end{cases} .$$

Therefore, we have 2 different cases, if $\alpha \geq 2$ then

$$\arg\max_{x \geq x_0} b_{n+1} = \begin{cases} \dfrac{\beta}{4(n+1)^2} & \text{if} \quad \dfrac{\beta}{4(n+1)^2} \geq x_0 \\ x_0 & \text{if} \quad \dfrac{\beta}{4(n+1)^2} < x_0 \end{cases} ,$$

if $\alpha < 2$ then

$$\arg\max_{x \geq x_0} b_{n+1} = \begin{cases} \dfrac{\beta + \sqrt{\beta^2 + 16(n+1)^2(2-\alpha)}}{8(n+1)^2} & \text{if} \quad \dfrac{\beta + \sqrt{\beta^2 + 16(n+1)^2(2-\alpha)}}{8(n+1)^2} \geq x_0 \\ x_0 & \text{if} \quad \dfrac{\beta + \sqrt{\beta^2 + 16(n+1)^2(2-\alpha)}}{8(n+1)^2} < x_0 \end{cases} ,$$

In both cases it is straightforward

$$\lim_{n\to+\infty} \arg\max_{x\geq x_0} b_{n+1}(x) = x_0\,,$$
$$\lim_{n\to+\infty} \max_{x\geq x_0} b_{n+1}(x) = \lim_{n\to+\infty} (n+1)^2 x_0^{2-\alpha} \exp\left(-2(n+1)^2 x_0^2 + \beta x_0\right) = 0\,,$$

so by taking the limit of (11) we obtain

$$\lim_{n\to+\infty} \sup_{x\geq x_0} \left| \sum_{k=1}^{+\infty} h_k(x) - \sum_{k=1}^{n} h_k(x) \right| \leq \lim_{n\to+\infty} \sup_{x\geq x_0} a_{n+1}(x) \leq \lim_{n\to+\infty} \sup_{x\geq x_0} b_{n+1}(x) = 0\,.$$

Proposition 3.2.2. *Let* $\mathrm{A} = \{x \in \mathbb{R} : 0 < x \leq x_0\}$ *for some* $x_0 > 0$ *and*

$$h_k(x) = \left(\frac{(2k-1)^2\pi^2}{4x^2} - 1\right) x^{-\alpha-1} \exp\left(-\frac{(2k-1)^2\pi^2}{8x^2} + \beta x\right)\,,$$

then $\sum_{k=1}^{+\infty} h_k(x)$ *converges uniformly in* A.

Proof: Let

$$a_k(x) = \frac{(2k-1)^2\pi^2}{4} x^{-\alpha-1} \exp\left(-\frac{(2k-1)^2\pi^2}{8x^2} + \beta x_0\right)\,,$$

thus

$$h_k(x) \leq a_k(x)\,.$$

Hence

$$\frac{\mathrm{d}\log a_k(x)}{\mathrm{d}x} = -\frac{\alpha+1}{x} + \frac{(2k-1)^2\pi^2}{4x^3}\,,$$

therefore

$$\arg\max_{0<x\leq x_0} a_k(x) = \begin{cases} \dfrac{(2k-1)\pi}{2\sqrt{\alpha+1}} & \text{if} \quad \dfrac{(2k-1)\pi}{2\sqrt{\alpha+1}} < x_0 \\ x_0 & \text{if} \quad \dfrac{(2k-1)\pi}{2\sqrt{\alpha+1}} \geq x_0 \end{cases}\,,$$

$$\max_{0<x\leq x_0} a_k(x) = \begin{cases} \dfrac{(2k-1)^{1-\alpha}\pi^{1-\alpha}}{2^{1-\alpha}(\alpha+1)^{-\frac{\alpha+1}{2}}} \exp\left(-\dfrac{\alpha+1}{2} + \beta x_0\right) & \text{if} \quad \dfrac{(2k-1)\pi}{2\sqrt{\alpha+1}} < x_0 \\ \dfrac{(2k-1)^2\pi^2}{4} x_0^{-\alpha-1} \exp\left(-\dfrac{(2k-1)^2\pi^2}{8x_0^2} + \beta x_0\right) & \text{if} \quad \dfrac{(2k-1)\pi}{2\sqrt{\alpha+1}} \geq x_0 \end{cases}\,.$$

So, let

$$\tilde{k} = \inf\left\{k \in \mathbb{N}_0 : \frac{(2k-1)\pi}{2\sqrt{\alpha+1}} \geq x_0\right\}\,,$$
$$M_k = \max_{0<x\leq x_0} a_k(x) \leq a_k(x) \leq h_k(x)\,,$$

we use the sufficient condition provided by Weierstrass criterion and it is easy to show

$$\sum_{k=1}^{+\infty} M_k = \sum_{k=1}^{\tilde{k}-1} M_k + \sum_{k=\tilde{k}}^{+\infty} M_k < +\infty\,.$$

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