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Twisting quantum groups at the roots of unity.

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A mia nonna, Francesca Romana... ci vediamo poi!

Abstract

Let G be a simply connected Lie group and $\mathfrak{g}$ be its complexified Lie algebra. Building on the work of Wenzl in [Wen98], we present a weak tensor structure on the unitary modular categories arising from representation categories of quantum groups $\mathcal{U}_q(\mathfrak{g})$ when q is specialised at roots of unity, following [CCP21]. The theory therein developed allows one to reconstruct these categories as representation categories of a discrete unitary coboundary weak Hopf algebra.

Then, we consider the twisted categories obtained by modifying the associator by means of 3-cocycles on the dual of the centre of G and reconstruct them as representation categories of suitable discrete unitary weak Hopf algebras; this is done by adaptation of the work in [NY15] in the analogous scenario of the compact quantum group corresponding to $\mathcal{U}_q(\mathfrak{g})$ specialised at $q > 1$.

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I am especially glad to thank S. Neshveyev and M. Yamashita for their beautiful article [NY15], whence the present thesis draws its origin. Finally, I wish to credit P. Aschieri and F. D'Andrea for their work in refereeing the thesis, affording several useful remarks and appropriate corrections.

Contents

Introduction

Motivation

The main focus of the present thesis are representation categories arising from quantum groups at roots of 1, and Tannakian reconstruction of a certain twisted version of them, as we outline deeper in the introduction. Beforehand however, we would like to spend some words about WZW models and their remarkable relation with our topic, in order to insert our work in such an interesting context.

Modular tensor categories arise naturally in the context of conformal field theories (CFT), more specifically the Wess-Zumino-Witten (WZW) models. More precisely, as first shown in [MSe88] and [MSe89], chiral rational CFT satisfy certain polynomial equations that lead to the Verlinde conjecture; in the case of WZW models this in turn produces the surprising Verlinde formula for the dimensions of the spaces of sections of the "generalized theta divisors".

Moreover, Moore and Seiberg noticed in [MSe89] that their polynomial equations seem to reflect the properties of a special sort of tensor categories. Modular tensor categories were actually formalised in [Tur92] by axiomatisation of some of these properties, and an abstract modular category may be obtained from the polynomial equations.

The later work of Huang about rational CFT in the context of vertex operator algebras (VOA) granted the proper rigour to the discourse of Moore and Seiberg, while also clarifying some notable examples. Less vaguely, WZW models (see [DMS12] as a comprehensive treatise) are associated to affine Lie algebras and to the relative universal affine VOA (the ones defined e.g. in [Kac98]); Huang reformulates the Verlinde conjecture in this context, and proves it in [Hua05], under suitable requirements on the VOA. Therefore the Verlinde formula on the associate CFT yields very strong information about a certain category of modules of the affine Lie algebras, including in particular modularity.

Let us elaborate on such categories down to some detail, following [Hua18]. Given a simple complex Lie algebra $\mathfrak{g}$ with dual Coxeter number $\check{h}$ and invariant symmetric bilinear form $(\cdot|\cdot)$, the affine Lie algebra $\hat{\mathfrak{g}}$ is $\big(\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]\big) \oplus \mathbb{C}K$ as a vector space; we write

$$\hat{\mathfrak{g}}_\pm := \mathfrak{g} \otimes t^{\pm 1}\mathbb{C}[t^{\pm 1}] \ , \quad \text{so that} \quad \hat{\mathfrak{g}} = \hat{\mathfrak{g}}_- \oplus \mathfrak{g} \oplus \mathbb{C}K \oplus \hat{\mathfrak{g}}_+ \ .$$

The bracket is determined by

$$[a \otimes t^m, b \otimes t^n] = [a, b] \otimes t^{m+n} + m\delta_{m+n,0}(a|b)K \qquad [K, a \otimes t^n] = 0$$

for a, b in $\mathfrak{g}$ and m, n in $\mathbb{Z}$. It corresponds to the operator product expansion (see [Kac98])

$$a(z)b(w) \sim \frac{[a,b](w)}{z-w} + \frac{(a|b)K}{(z-w)^2} ,$$

where $a(z) = \sum_{n\in\mathbb{Z}} at^n z^{-1-n}$ for a in $\mathfrak{g}$; thus, in the language of [Kac98], $\hat{\mathfrak{g}}$ is a formal distribution Lie algebra.

Let M be a $\mathfrak{g}$-module, decomposable into generalised eigenspaces of the Casimir operator. The action on M is extended to $\mathfrak{g} \oplus \mathbb{C}K \oplus \hat{\mathfrak{g}}_+$ by letting K act as a scalar k, called level, and $\hat{\mathfrak{g}}_+$ as 0, and we have the induced $\hat{\mathfrak{g}}$-module $\widehat{M}_k := U(\mathfrak{g}) \otimes_{U(\mathfrak{g}\oplus\mathbb{C}K\oplus\mathfrak{g}_+)} M$. If $\check{h} + k \neq 0$, M is endowed with a $\mathbb{C}$-grading that extends to one on the whole $\widehat{M}_k$ using the $\mathbb{C}$-grading of $\hat{\mathfrak{g}}$ itself.

In the case when M is the irreducible highest weight module $L(\lambda)$ of highest weight λ, one finds by usual cyclicity arguments that $\widehat{M}_k$ admits a unique irreducible quotient, denoted by $L(k, \lambda)$, which is the unique irreducible $\hat{\mathfrak{g}}$-module such that K acts as k and the space of all elements annihilated by $\hat{\mathfrak{g}}$ is isomorphic to $L(\lambda)$ as a $\mathfrak{g}$-module. Now $L(k, 0)$ has a natural structure of VOA, the universal affine VOA $V^k(\mathfrak{g})$ of [Kac98], and $L(k, \lambda)$ becomes an $L(k, 0)$-module for dominant integral λ. For k non-negative integer, consider the category $\tilde{\mathcal{O}}_k$ of $\hat{\mathfrak{g}}$-modules of level k that are isomorphic to direct sums of irreducible $\hat{\mathfrak{g}}$-modules of the form $L(k, \lambda)$ for λ a dominant integral weight such that $(\lambda, \theta) \leq k$, where θ is the highest root of $\mathfrak{g}$.

The categories $\tilde{\mathcal{O}}_k$ are the ones proved to possess a structure of modular tensor category in [Hua05]. Even more interestingly, and of special relevance to our main focus, they are equivalent as ribbon tensor categories to certain subquotients $\overline{\mathcal{T}_q(\mathfrak{g})}$ of the representation categories of quantum enveloping algebras specialised at roots of unity q. More precisely, as proved in [Fin96] and its correction [Fin13], $\tilde{\mathcal{O}}_k$ is equivalent to $\overline{\mathcal{T}_q(\mathfrak{g})}$ for q of the following form:

$$e^{\frac{\imath\pi}{\ell}} \qquad \text{with} \quad D \mid \ell \, , \tag{1}$$

where D is the ratio between the squares of a long root and a short root, if $\ell/D - \check{h} = k$. This theorem was proved by Finkelberg, using a previous equivalence due to Kazhdan and Lusztig between $\overline{\mathcal{T}_q(\mathfrak{g})}$ and a different category of $\hat{\mathfrak{g}}$-modules; we report however, as pointed out in [Hua18], that there are actually a few cases not covered, e.g. when $\mathfrak{g}$ is of type E_8 and $k = 2$.

It is appropriate to note that the categories $\overline{\mathcal{T}_q(\mathfrak{g})}$ for q of the form (1) can be proved to be modular in a more direct fashion, using the results of [Bru00]. In this case $\overline{\mathcal{T}_q(\mathfrak{g})}$ is also unitary (so it is a C^* ribbon category in the sense of the forthcoming Definition 1.5C); this result becomes apparent through the treatment of [Wen98] reviewed in subsection 4.3 and is usually referred to as Wenzl-Xu theorem, e.g. in [Row06]. We also refer to the latter article for an account about modularity of $\overline{\mathcal{T}_q(\mathfrak{g})}$ considering general roots of 1.

Quantum groups at roots of 1 and weak Hopf algebras

We now elaborate briefly about the categories $\overline{\mathcal{T}_q(\mathfrak{g})}$ for q as in (1), which are the main concrete mathematical object the present thesis deals with. Let $\mathfrak{g}$ be a simple complex Lie algebra; the quantized universal enveloping algebra $\mathcal{U}_q(\mathfrak{g})$ is obtained from a formal deformation of the classical universal enveloping algebra of $\mathfrak{g}$ by specialisation to q using the so called "restricted integral form" (see [CP95]).

Referring to section 3 for details, here we content ourselves with saying that $\mathcal{U}_q(\mathfrak{g})$ is a Hopf algebra endowed with an involution $\cdot^*$, compatible with its product just as in usual $*$-algebras. On the other hand the compatibility with the coproduct Δ is less of a standard one. In fact, contrary to what happens for Hopf $*$-algebras, Δ does not commute with $\cdot^*$, but we rather have $\Delta \circ {}^* = ({}^* \otimes {}^*) \circ \Delta^{\mathrm{op}}$, where Δ^{op} is the switched coproduct, i.e. $\cdot^*$ is anticomultiplicative. Moreover, the R-matrix, defining a braiding on $\mathrm{Rep}\,(\mathcal{U}_q(\mathfrak{g}))$, satisfies $R^* = R_{21}^{-1}$. Returning to $\overline{\mathcal{T}_q(\mathfrak{g})}$, $\mathcal{T}_q(\mathfrak{g})$ stands for the subcategory of tilting modules (see e.g. [Lus93] for a comprehensive treatment and references to the original work of H. H. Andersen and others) in $\mathrm{Rep}\,(\mathcal{U}_q(\mathfrak{g}))$; on the other hand the overline refers to a categorical quotient by the tensor ideal of "negligible modules".

It is implicitly expressed in [Wen98] that $\overline{\mathcal{T}_q(\mathfrak{g})}$ can be presented as a linear subcategory $\mathcal{G}_q(\mathfrak{g}) \subset \mathcal{T}_q(\mathfrak{g})$, using a new truncated tensor product that eliminates the negligible summands that may appear; hence the forgetful functor $\mathcal{W} : \mathcal{G}_q(\mathfrak{g}) \to \mathrm{Hilb}$ is well defined. Besides, $\mathcal{G}_q(\mathfrak{g})$ is actually strictly monoidal, i.e. its tensor product is associative and the associator is trivial. Even though $\mathcal{W}$, due to the truncation procedure, cannot be endowed with a tensor structure as it trivially happens for the forgetful functor on $\mathrm{Rep}\,(\mathcal{U}_q(g))$, it admits a particularly interesting weakened structure, called "weak tensor"; the notion is introduced in [CCP21], providing the theoretical basis for the thesis. This means that we have natural transformations

$$F^2_{\rho,\sigma} : \mathcal{W}(\rho) \otimes \mathcal{W}(\sigma) \to \mathcal{W}(\rho \otimes \sigma)\ , \qquad G^2_{\rho,\sigma} : \mathcal{W}(\rho \otimes \sigma) \to \mathcal{W}(\rho) \otimes \mathcal{W}(\sigma)$$

such that $F^2_{\rho,\sigma} \circ G^2_{\rho,\sigma} = \mathcal{W}(\rho \otimes \sigma)$, the identical transformation, and F^2, G^2 have a certain weak compatibility with associators, in our case trivial, of $\mathcal{G}_q(\mathfrak{g})$ and Hilb:

$$\begin{aligned}
&F^2_{\rho,\sigma\otimes\tau} \circ (\mathcal{W}(\rho) \otimes F^2_{\sigma,\tau}) \circ (G^2_{\rho,\sigma} \otimes \mathcal{W}(\tau)) \circ G^2_{\rho\otimes\sigma,\tau} = \mathcal{W}(\rho \otimes \sigma \otimes \tau)\ ,\\
&F^2_{\rho\otimes\sigma,\tau} \circ (F^2_{\rho,\sigma} \otimes \mathcal{W}(\tau)) \circ (\mathcal{W}(\rho) \otimes G^2_{\sigma,\tau}) \circ G^2_{\rho,\sigma\otimes\tau} = \mathcal{W}(\rho \otimes \sigma \otimes \tau)\ .
\end{aligned}$$

Such structure is especially relevant in view of Tannakian theory, which produces a complex algebra equipped with a non-unital coproduct (A, Δ) and a tensor equivalence $\mathcal{E} : \mathcal{C} \to \mathrm{Rep}(A)$, such that $\mathcal{E}$ sends each ρ in a representation on $\mathcal{W}(\rho)$. The properties of Δ mirror the weak tensor structure of $\mathcal{W}$; in particular Δ is not associative, but we still have

$$(\mathrm{id} \otimes \Delta)(\Delta(1))(\Delta \otimes \mathrm{id})(\Delta(a)) = (\mathrm{id} \otimes \Delta)(\Delta(a))(\Delta \otimes \mathrm{id})(\Delta(1)) \qquad \forall\, a \in A\ , \quad (2)$$

i.e, according to the nomenclature of [CCP21], A is a weak bialgebra; moreover the rigidity of $\mathcal{G}_q(\mathfrak{g})$ results in the definition of a (unique) antipode for A, turning it into a weak Hopf algebra. Besides, $\mathcal{W}$ being a $*$-functor, it induces on A a $*$-algebra structure, and the properties of the involution of $\mathcal{U}_x(\mathfrak{g})$ sprout in a remarkable compatibility of the involution of A with its coproduct, given by the quasi-triangular structure induced on (A, Δ) by the R-matrix of $U_x(\mathfrak{g})$. The situation is worded in [CCP21] saying that A is unitary coboundary, where an abstract theory is modelled on the very case of the functor $\mathcal{W}$. More generally, [CCP21] introduces the wider notion of unitary weak Hopf algebras.

Among the notable properties of weak Hopf algebras, they admit a simple notion of 2-cocycles, by which the coproduct may be modified to obtain new weak Hopf algebras. To begin with, a 2-cocycle on A is a partial isomorphism in $A \otimes A$ form $\Delta(1)$ to some other idempotent of $A \otimes A$, and one defines the new coproduct $\Delta_F(\cdot) = F\Delta(\cdot)F^{-1}$; so $\Delta_F(1)$ is the final domain of F. Then (A, Δ_F) is still a weak Hopf algebra if and only if F fulfils the following 2-cocycle identity:

$$\begin{aligned} (\mathrm{id} \otimes \Delta)(F^{-1})(1 \otimes F^{-1})(F \otimes 1)(\Delta \otimes \mathrm{id})(F) &= (\mathrm{id} \otimes \Delta)\big(\Delta(1)\big)(\Delta \otimes \mathrm{id})\big(\Delta(1)\big) \\ (\Delta \otimes \mathrm{id})(F^{-1})(F^{-1} \otimes 1)(1 \otimes F)(\mathrm{id} \otimes \Delta)(F) &= (\Delta \otimes \mathrm{id})\big(\Delta(1)\big)(\mathrm{id} \otimes \Delta)\big(\Delta(1)\big) \end{aligned} \tag{3}$$

2-cocycles are also employed in order to express the compatibility of involution and coproduct in the case of a unitary weak Hopf algebra: by definition, $(A, \cdot^*, \Delta, \Omega)$ is a unitary weak Hopf algebra if $(A, \cdot^*)$ is a $*$-algebra, (A, Δ) is a weak Hopf algebra and Ω is a positive 2-cocycle such that $\Delta_\Omega = \Delta(\cdot^*)^*$. This way $(A, \cdot, \Delta_F, \Omega_F)$, having put $\Omega_F := (F^*)^{-1}\Omega F^{-1}$, is still a unitary weak Hopf algebra.

In absence of condition (3), F is simply called a "twist", and Δ_F gets to induce the further generalised "unitary weak quasi-Hopf algebra" structure on A; the main difference lies in the fact that property (2) is replaced by

$$\Phi(\Delta \otimes \mathrm{id})\big(\Delta(a)\big) = (\mathrm{id} \otimes \Delta)\big(\Delta(a)\big)\Phi \ ,$$

where Φ is a partial isomorphism between the idempotents $(\Delta \otimes \mathrm{id})\big(\Delta(1)\big)$ and $(\mathrm{id} \otimes \Delta)\big(\Delta(1)\big)$; Φ encodes the associator of $\mathrm{Rep}(A)$. In the case of a weak Hopf algebra, $\Phi = (\mathrm{id}\otimes\Delta)\big(\Delta(1)\big)(\Delta\otimes\mathrm{id})\big(\Delta(1)\big)$, with inverse $\Phi^{-1} = (\Delta\otimes\mathrm{id})\big(\Delta(1)\big)(\mathrm{id}\otimes \Delta)\big(\Delta(1)\big)$. As an upside of weak quasi-Hopf algebras, a given (Δ, Φ) can be modified by arbitrary twists to new weak quasi-coalgebra structures (Δ_F, Φ_F) on A, where Φ_F depends on Φ and F.

Twisted associators and QUE algebras

The main result of the thesis arises from the work by S. Neshveyev and M. Yamashita in [NY15]. The authors consider the specialised QUE algebra $\mathcal{U}_q(\mathfrak{g})$ for $q > 1$; in this case $\mathrm{Rep}\,\big(\mathcal{U}_q(\mathfrak{g})\big)$ is the representation category of a compact quantum group G_q and the situation is much more similar to the classical one. Now, one may utilise usual 3-cocycles on the integral weight lattice for $\mathfrak{g}$, or equivalently on the dual of the centre of the corresponding compact simply connected Lie group G, with values in the circle, to modify the associator of $\mathrm{Rep}(G_q)$; we thus have a new category $\mathrm{Rep}(G_q)^\Phi$. The authors prove it to be unitarily equivalent to the category of representations of a particular compact quantum group G_q^τ, obtained modifying the coproduct of $\mathcal{U}_q(\mathfrak{g})$ by an l-tuple τ of elements of the centre of G, l being the rank of $\mathfrak{g}$

The thesis deals with the same problem in the scenario where G_q is replaced by the unitary coboundary weak Hopf algebra A constructed from the weak tensor $*$-functor $\mathcal{W}$. The theory of [CCP21] naturally applies to the problem, once the weak tensor structure of $\mathcal{W}$ is explicitly written down in a suitable form. In particular, the analogue of $\mathrm{Rep}(G_q)^\Phi$ turns out to be the representation category of a unitary weak quasi-Hopf algebra. We prove that such algebra can be twisted to a unitary weak Hopf algebra, though not a coboundary one as A is; the employed twist is defined from an n-tuple τ as in [NY15], but inserting idempotents suited to the non-unitality of the coproduct.

Subsequently, the twisted algebras are tracked back to appropriate twists of $\mathcal{U}_q(\mathfrak{g})$, coinciding with the ones introduced in [NY15] except for the difference in the involution, due to the different values of q. Writing $\mathcal{U}_q(\mathfrak{g})'$ for the new QUE algebra, we may again consider the tilting modules in Rep $(\mathcal{U}_q(\mathfrak{g})')$ and the quotient by the tensor ideal of negligible morphisms, obtaining a C^* tensor category analogue to $\mathcal{G}_q(\mathfrak{g})$ along the same lines followed to define the latter. We prove that the two C^* tensor categories are isomorphic.

The stated results might assume greater interest in view of the equivalence found in [Fin96] between $\overline{\mathcal{T}_q(\mathfrak{g})}$ and the category $\tilde{\mathcal{O}}_k$ mentioned above. A conceivable development of the work of the thesis is the research of vertex algebras yielding module categories equivalent to the twisted versions of $\overline{\mathcal{T}_q(\mathfrak{g})}$.

Summary structure of the thesis

The first chapter develops the Tannakian reconstruction theory for a weak quasi-tensor $*$-functor on a C^* tensor category treated in [CCP21].

The second introduces quantum groups at roots of unity and the fusion categories arising from them; the theory of the first chapter is subsequently applied to a weak tensor $*$-functor constructed building on the work of [Wen98].

The third chapter adapts the reconstruction result of [NY15] to the scenario presented in the second chapter. We refer to the introductions at the beginning of chapters and sections for more detailed summaries.

Note on cross-references We adopt the convention of [Hum12]. So, Proposition 1.3 is the (unique) Proposition in subsection 1.3, and is referred to as "the Proposition" within the subsection. Similarly, formula 2.2(1) is the expression labelled by (1) in 2.2, just called (1) within the subsection.

General theory

The present chapter is devoted to the development of the abstract algebraic structure providing our general framework, following [CCP21]. Section 1 mainly recalls some basic categorical notions centred around tensor categories and functors between them. Section 2 focuses on the situation when a tensor category is endowed with a faithful functor into the category of vector spaces. Tannaka-Krein duality then associates to the functor a suitable algebra, whose properties shall be discussed there.

1 Tensor categories

We start by fixing the terminology and notation from general category theory. Subsections 1.1 and 1.2 are designed to introduce tensor categories, providing the basic categorical framework of the thesis; in this regard we basically follow the approach of the first four chapters of [EGNO15]. However, we forewarn that our Definition 1.2E of tensor categories is given in the more restrictive context of semi-simple abelian categories (see the comments below it for a more detailed comparison with the definition at the beginning of the fourth chapter of [EGNO15]).

Subsections 1.3, 1.4 and 1.5 deal with possible richer structure, i.e. categorical duals, generalised coboundaries (a generalisation of braidings we are going to need later on), and involutions respectively.

Throughout the whole thesis every category, unless otherwise stated, will be meant to be $\mathbb{C}$-linear, i.e. for each pair of objects a, b the corresponding set of morphisms (a, b) will actually be a vector space, and the composition maps will be bilinear. The identity morphism in (a, a) will be denoted by a. Every functor between $\mathbb{C}$-linear categories will be taken to be linear on each morphism space.

1.1 Additive structure

Since we are mainly focused on representation categories of quantum groups, we are particularly interested in semi-simple abelian categories.

Definition A. A category $\mathcal{A}$ is said to be *abelian* if

i) $\mathcal{C}$ contains a null object;

ii) $\mathcal{C}$ has finite direct sums;

iii) every morphism of $\mathcal{C}$ admits a kernel and a cokernel;

iv) every monomorphism of $\mathcal{C}$ is a kernel and every epimorpism is a cokernel.

Remark A. We recall that, given objects $\rho_1, \dots, \rho_n$, a direct sum of them is an object $\rho_1 \oplus \dots \oplus \rho_n$ with morphism i_k in $(\rho_k, \rho_1 \oplus \dots \oplus \rho_n)$ and p_k in $(\rho_1 \oplus \dots \oplus \rho_n, \rho_k)$ such that $p_k i_k = \rho_k$, for $k = 1, \dots, n$ and $\sum_{k=1}^{n} i_k p_k = \rho_1 \oplus \dots \oplus \rho_n$.

It is also useful to note that the existence of kernels and cokernels imply that $\mathcal{C}$ has addends. Namely if e is an idempotent element in some (ρ, ρ) then there is an object π and morphism i in (π, ρ), p in (ρ, π) such that $pi = \pi$ and $ip = e$.

We refrain from going into greater detail about abelianness, referring to [Mac71] for a proper treatment, and contenting ourselves with a couple of simple observations appropriate to polish our terminology.

Proposition-Definition. Let ρ be a non-null object in an abelian category. Then, up to isomorphisms, ρ admits exactly two subobjects if and only if it admits exactly two quotients. If this is the case, we say that ρ is *simple*.

Proof. Suppose the only quotients of ρ are 0, the null object, and ρ itself; consider a monomorphism f in (π, ρ). Then $f = \ker(p)$, where p is an epimorphism in (ρ, σ) for some object σ. So either $p = 0$ or $p = \rho$, and accordingly $f = 0$ or $f = \rho$. The converse is dual. □

Proposition. *Let ρ be a simple object in an abelian category. Then, for any morphism f in (π, ρ), f is an epimorphism or $f = 0$; dually, for any g in (ρ, σ), g is a monomorphism or $g = 0$.*

Proof. We just prove the first assertion. Since our category is abelian, we may write $f = me$, with m a monomorphism and e an epimorphism. Then, by the Proposition-Definition, either $m = \rho$ up to isomorphism or $m = 0$, and $f = e$ or $f = 0$ accordingly. □

Remark B. In an abelian category a morphism that is both mono and epi is actually an isomorphism; so the Proposition implies Shur's lemma, which is of fundamental importance in the forthcoming development.

Let us turn to semi-simplicity.

Definition B. An abelian category is called *semi-simple* if each of its objects can be written as a direct sum of a finite number of simple objects.

We observe that, thanks to Shur's lemma, the morphism spaces of a semi-simple abelian category are finite-dimensional. We also point out that any exact sequence splits. To sum up, these categories are very well behaved under the "additive" aspect, which indeed we are not going to further investigate in itself.

1.2 Monoidal structure

Definition A. A *monoidal* category is the datum of a triple $(\mathcal{C}, \otimes, a)$, where

- $\mathcal{C}$ is a category;
- $\otimes$ is a functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$, called *tensor product*;

- $a_{\rho,\sigma,\tau} : (\rho \otimes \sigma) \otimes \tau \to \rho \otimes (\sigma \otimes \tau)$, with ρ, σ, τ running through $\mathcal{C}$, are natural isomorphisms satisfying the pentagon axiom

$$(\rho \otimes a_{\sigma,\tau,\upsilon}) \circ a_{\rho,\sigma\otimes\tau,\upsilon} \circ (a_{\rho,\sigma,\tau} \otimes \upsilon) = a_{\rho,\sigma,\tau\otimes\upsilon} \circ a_{\rho\otimes\sigma,\tau} \qquad \forall\ \rho, \sigma, \tau, \upsilon \in \mathcal{C}\ .$$

The natural isomorphism a is called *associator*.

Furthermore $(\mathcal{C}, \otimes, a)$ is assumed to possess units, which we introduce in next definition. We will often trim $\otimes$ and/or a out of our notation, writing e.g. $(\mathcal{C}, \otimes)$ or $\mathcal{C}$, when they will not be specifically relevant or will result clearly from the context.

We adopt the definition of monoidal units originally given by Saavedra, following the treatment in [Koc08].

Definition B. A *unit* for a monoidal category is a pair $(\mathbb{1}, i)$, where

- the object $\mathbb{1}$ is *cancellable*, i.e. the functors given, for each morphism f, by

$$f \mapsto \mathbb{1} \otimes f\ , \qquad f \mapsto f \otimes \mathbb{1}$$

are equivalences;

- i is an isomorphism in $(\mathbb{1} \otimes \mathbb{1}, \mathbb{1})$.

The units of a given monoidal category $\mathcal{C}$ are the objects of a new category $\mathcal{U}$. A morphism ϕ from $(\mathbb{1}, i)$ to $(\mathbb{1}', i')$, is a morphism of $\mathcal{C}$ in $(\mathbb{1}, \mathbb{1}')$ such that

- for all morphism f, $\phi \otimes f = 0$ and $f \otimes \phi = 0$ both imply $f = 0$; this is usually worded by saying that the morphism ϕ is *cancellable*;
- $i' \circ (\phi \otimes \phi) = \phi \otimes i$.

We have to point out that the category $\mathcal{U}$ in previous definition is actually not $\mathbb{C}$-linear; indeed for any pair of objects in $\mathcal{U}$ the corresponding morphism space contains exactly one isomorphism, i.e. $\mathcal{U}$ is contractible. It is also in order to remark that Definition B does not rely on the associator, contrary to the more traditional definition in terms of natural isomorphisms $l : \mathbb{1} \otimes \rho \to \rho$ and $r : \rho \otimes \mathbb{1} \to \rho$, called *unitors*. However, for any given associator, the assignment

$$(\mathbb{1}, l, r) \mapsto (\mathbb{1}, i)\ , \qquad i := l_{\mathbb{1}} = r_{\mathbb{1}}$$

actually gives a bijection between “traditional” and “Saavedra” units. This becomes particularly useful when it comes to monoidal functors.

Definition C. Let $(\mathcal{C}, \otimes, a)$ and $(\mathcal{C}', \otimes', a')$ be monoidal categories. A monoidal functor from the former to the latter is the datum of a functor $\mathcal{F} : \mathcal{C} \to \mathcal{C}'$ plus natural isomorphisms $F_{\rho,\sigma} : \mathcal{F}(\rho) \otimes \mathcal{F}(\sigma) \to \mathcal{F}(\rho \otimes \sigma)$ such that

$$\begin{array}{ccccc}
(\mathcal{F}(\rho) \otimes \mathcal{F}(\sigma)) \otimes \mathcal{F}(\tau) & \xrightarrow{F_{\rho,\sigma}\otimes\mathcal{F}(\tau)} & \mathcal{F}(\rho\otimes\sigma) \otimes \mathcal{F}(\tau) & \xrightarrow{F_{\rho\otimes\sigma,\tau}} & \mathcal{F}((\rho\otimes\sigma)\otimes\tau) \\
\downarrow{\scriptstyle a'_{\mathcal{F}(\rho),\mathcal{F}(\sigma),\mathcal{F}(\tau)}} & & & & \downarrow{\scriptstyle \mathcal{F}(a_{\rho,\sigma,\tau})} \\
\mathcal{F}(\rho) \otimes (\mathcal{F}(\sigma) \otimes \mathcal{F}(\tau)) & \xrightarrow{\mathcal{F}(\rho)\otimes F_{\sigma,\tau}} & \mathcal{F}(\rho) \otimes \mathcal{F}(\sigma\otimes\tau) & \xrightarrow{F_{\rho,\sigma\otimes\tau}} & \mathcal{F}(\rho\otimes(\sigma\otimes\tau))
\end{array}$$

commutes for all ρ, σ, τ objects in $\mathcal{C}$. The natural isomorphism F is called the *monoidal structure* of $\mathcal{F}$. Of course, the identity functor $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is a monoidal functor with the trivial monoidal structure.

In the above situation, consider a unit for $\mathcal{C}$, say $(\mathbb{1}, i)$; then we have the isomorphism $i' := \mathcal{F}(i) \circ F_{\mathbb{1},\mathbb{1}}$. So as soon as $F(\mathbb{1})$ is cancellable, e.g. it is isomorphic to some unit as an object of $\mathcal{C}'$, $\big(\mathcal{F}(\mathbb{1}), i'\big)$ is a unit for $\mathcal{C}'$; moreover, by contractibility, $F(\mathbb{1})$ is cancellable if and only if $F(u)$ is, for any unit (u, j) of $\mathcal{C}$. To sum up, we have

Proposition-Definition. Let $\mathcal{F}$ be as in Definition C; for each unit (u, j) for $\mathcal{C}$, consider the pair $\big(\mathcal{F}(u), j'\big)$. Then all of the latter pairs are units for $\mathcal{C}'$ if and only if $\mathcal{F}(\mathbb{1})$ is cancellable for some unit $(\mathbb{1}, i)$ of $\mathcal{C}$. If this is the case, $\mathcal{F}$ is said to be *compatible with units.*

Now, as it is shown in [Koc08], it turns out that this notion of compatibility with units is equivalent to the traditional one. More precisely, we have the following

Lemma. *Let $\mathcal{F} : (\mathcal{C}, \otimes) \to (\mathcal{C}', \otimes')$ be a monoidal functor compatible with units. Consider a unit $(\mathbb{1}, i)$ of $\mathcal{C}$, a unit $(\mathbb{1}', i')$ of $\mathcal{C}'$, and the corresponding traditional units $(\mathbb{1}, l, r)$, $(\mathbb{1}', l', r')$; we denote by ψ the unique isomorphism from $(\mathbb{1}', i')$ to $(F(\mathbb{1}), \mathcal{F}(i) \circ F_{\mathbb{1},\mathbb{1}})$. Then ψ is also the unique isomorphism from $\mathbb{1}'$ to $\mathcal{F}(\mathbb{1})$ such that the following commute for each ρ in $\mathcal{C}$:*

$$
\begin{array}{ccc}
\mathbb{1}' \otimes' \mathcal{F}(\rho) & \xrightarrow{l'_{\mathcal{F}(\rho)}} & \mathcal{F}(\rho) \\
\downarrow {\scriptstyle \psi \otimes \mathcal{F}(\rho)} & & \uparrow {\scriptstyle \mathcal{F}(l_\rho)} \\
\mathcal{F}(\mathbb{1}) \otimes' \mathcal{F}(\rho) & \xrightarrow{F_{\mathbb{1},\rho}} & \mathcal{F}(\mathbb{1} \otimes \rho)
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{F}(\rho) \otimes' \mathbb{1}' & \xrightarrow{r'_{\mathcal{F}(\rho)}} & \mathcal{F}(\rho) \\
\downarrow {\scriptstyle \mathcal{F}(\rho) \otimes \psi} & & \uparrow {\scriptstyle \mathcal{F}(r_\rho)} \\
\mathcal{F}(\rho) \otimes' \mathcal{F}(\mathbb{1}) & \xrightarrow{F_{\rho,\mathbb{1}}} & \mathcal{F}(\rho \otimes \mathbb{1})
\end{array}
$$

To summarize, compatibility with units is a property rather than a further piece of information and we will always assume it for every monoidal functor.

We conclude our review of general monoidal categories by the following

Definition D. Let $(\mathcal{F}, F), (\mathcal{G}, G)$ be monoidal functors between $(\mathcal{C}, \otimes)$ and $(\mathcal{C}', \otimes')$. A natural transformation given by morphisms η_ρ in $\big(\mathcal{F}(\rho), \mathcal{G}(\rho)\big)$ is said to be *monoidal* if

$$G_{\rho,\sigma} \circ (\eta_\rho \otimes' \eta_\sigma) = \eta_{\rho\otimes\sigma} \circ F_{\rho,\sigma} \qquad \forall\ \rho, \sigma \in \mathcal{C}$$

and $\eta_{\mathbb{1}}$ is an isomorphism for any $\mathbb{1}$ unit of $\mathcal{C}$.

Clearly the last condition actually holds for all units by naturality; it is also easy to see that $\eta_{\mathbb{1}} \circ \phi = \psi$, where ϕ and ψ express the compatibilities with units of $\mathcal{F}$ and $\mathcal{G}$.

Quotient categories For future use (in the forthcoming subsection 4.2), we record a simple categorical construction (the standard more general approach may be found in II.8 of [Mac71]).

Proposition. *Let $(\mathcal{C}, \otimes, a)$ be a monoidal category and $\mathcal{I}$ a collection of vector subspaces $\mathcal{I}_{\rho,\sigma} \subset (\rho, \sigma)$ for ρ, σ objects of $\mathcal{C}$; the quotient maps will be marked by overlines.*

We assume that $\mathcal{I}$ is an ideal of $\mathcal{C}$, which means that $\overline{g} = 0$ implies $\overline{fgh} = 0$ for all morphisms f, g, h such that the composition is defined. Then $\overline{f}\overline{g} := \overline{fg}$ defines a *composition for a new category $\overline{C}$ whose objects are the objects of $\mathcal{C}$, with morphism space $\overline{(\rho,\sigma)}$ for each ρ,σ.*

Furthermore, if $\overline{f}=0$ implies $\overline{f\otimes g}=0=\overline{g\otimes f}$ for all morphisms f,g then $\overline{f}\overline{\otimes}\overline{g}:=\overline{f\otimes g}$ defines a tensor product $\overline{\otimes}$ coinciding with $\otimes$ on objects, and $(\overline{\mathcal{C}},\overline{\otimes},\overline{a})$ is a monoidal category.

We finally settle down a basic terminology issue about the main notion of current section.

Definition E. By a *tensor category* we will mean a monoidal category $(\mathcal{C},\otimes,a)$ where $\mathcal{C}$ is a semisimple abelian category. We also require the units to be simple objects.

A quick word of warning: the above definition is tailored on the theoretical development of section 2, but it is not really standard and actually quite a few variations are spread through the literature, see e.g. [EGNO15] and [CP95]. In the notable case of [EGNO15], the authors contemplate locally finite categories rather than semi-simple ones, and assume them to be rigid (see Definition 1.3A).

In the present thesis we will use “tensor” instead of “monoidal” for the relative notions as well; e.g. we will speak of tensor functors rather than of monoidal ones. This is also consistent with our forthcoming nomenclature for functors from a tensor category to Vec or Hilb, which follows [CCP21].

Finally, we will generally use $\mathbb{1}$ to denote a generic unit, refraining from making a specific choice; however, we will often make use of this freedom for normalisation purposes (e.g., in Definition 2.1A).

1.3 Rigidity

We now briefly recall the notion of dual objects in a tensor category. We limit ourselves to the case where it is strict as a monoidal category, i.e. the tensor product is associative, whence the associator is taken to be trivial, and we have a unit $\mathbb{1}$ such that $\mathbb{1}\otimes\rho=\rho=\rho\otimes\mathbb{1}$ for every object ρ. The general definitions are recovered by just inserting associators and unitors where needed; moreover, thanks to the strictness theorem (see [Mac71]) any property valid for a strict category extends to the general case just in the same way.

Definition A. Let $\mathcal{C}$ be a strict tensor category. Given objects ρ,σ, we say that σ is a *right dual* of ρ, or that ρ is a *left dual* of σ, if there is a pair (b,d) with b in $(\mathbb{1},\rho\otimes\sigma)$ and d in $(\sigma\otimes\rho,\mathbb{1})$ such that the compositions

$$\rho\xrightarrow{b\otimes\rho}\rho\otimes\sigma\otimes\rho\xrightarrow{\rho\otimes d}\rho\qquad\text{and}\qquad\sigma\xrightarrow{\sigma\otimes b}\sigma\otimes\rho\otimes\sigma\xrightarrow{d\otimes\sigma}\sigma$$

equal respectively ρ and σ. In this is the case, we say that (b,d) is a *duality pair* with left object ρ and right object σ; $\mathcal{C}$ is said to be *rigid* if every object possesses both left and right duals.

Assuming that $\mathcal{C}$ is rigid, suppose we have an assignment $\rho\mapsto(b_\rho,d_\rho)$ where the latter is a duality pair with left object ρ and right object $\rho^\vee$ for all ρ in $\mathcal{C}$. Then this extends to a contravariant functor $\mathcal{D}$ in a unique way. Namely, given f in (ρ,σ),

$$(f\otimes\rho^\vee)\circ b_\rho=(\sigma\otimes f^\vee)\circ b_\sigma,\ \text{or equivalently,}\ d_\rho\circ(\rho\otimes f^\vee)=d_\sigma\circ(f\otimes\sigma^\vee).$$

Moreover the choice of the duality pairs is in fact not essential, thanks to the following basic observation.

Proposition. *Consider a duality pair* (b,d) *with left object* ρ *and right object* σ*. Then*

- *if* $u:\sigma\to\tilde\sigma$ *is an isomorphism, then* $\big((\rho\otimes u)\circ b, d\circ(u^{-1}\otimes\rho)\big)$ *is still a duality pair with left object* ρ*;*
- *conversely, if* $(\tilde b,\tilde d)$ *is a duality pair with left object* ρ *and right object* $\tilde\sigma$*, then* $(\tilde b,\tilde d)=\big((\rho\otimes u)\circ b, d\circ(u^{-1}\otimes\rho)\big)$ *with* $u=(d\otimes\tilde\sigma)\circ(\sigma\otimes\tilde b)$*.*

Indeed, supposing we have choices (b_ρ,d_ρ) and $(b_{\tilde\rho},d_{\tilde\rho})$, the above second point provides us with isomorphisms u_ρ and one can easily verify that this defines a natural isomorphism between the two right duality functors.

Finally, the $\cdot^\vee$ functors are actually tensor between $\mathcal{C}$ and $\mathcal{C}^{\mathrm{op}}$ (see the beginning of next subsection for $\mathcal{C}^{\mathrm{op}}$), in such a way that the natural isomorphisms u become tensor as well. More precisely, given objects ρ and σ, $\sigma^\vee\otimes\rho^\vee$ is a right dual of $\rho\otimes\sigma$ by

$$\mathbb{1}\xrightarrow{b_\rho}\rho\otimes\rho^\vee\xrightarrow{\rho\otimes b_\sigma\otimes\rho^\vee}\rho\otimes\sigma\otimes\sigma^\vee\otimes\rho^\vee\ ,$$
$$\sigma^\vee\otimes\rho^\vee\otimes\rho\otimes\sigma\xrightarrow{\sigma^\vee\otimes d_\rho\otimes\sigma}\sigma^\vee\otimes\sigma\xrightarrow{d_\sigma}\mathbb{1}\ ,$$

and again the second point of the Proposition yields isomorphisms $D_{\rho,\sigma}$ in $\big((\rho\otimes\sigma)^\vee,\sigma^\vee\otimes\rho^\vee\big)$ (keep in mind that $\mathcal{D}$ is contravariant). The triviality of the associator makes it easy to check tensoriality for both the $\mathcal{D}$ functors and the u isomorphisms. We proceed to recall the notion of compatibility with duality of an isomorphism of the identity functor for later use.

Definition B. Let $\mathcal{C}$ be a rigid category and suppose we have right and left duality functors both having the value $\rho^\vee$ on each object ρ. Given natural morphisms η_ρ in (ρ,ρ), we say that the natural transformation η is *compatible with duality* if $\eta_{\rho^\vee}=\eta_\rho^\vee$ for all ρ.

In fact, compatibility with duality does not depend on the choice of the right duals; indeed, in the situation of Definition B, if $\cdot^{\vee'}$ is another right duality functor consider natural isomorphisms u_ρ in $(\rho^\vee,\rho^{\vee'})$. We have

$$\eta_{\rho^{\vee'}}=u\eta_{\rho^\vee}u^{-1}=u\eta_\rho^\vee u^{-1}=\eta_\rho^{\vee'}\ .$$

Remark A. What we have said since the definition of $\mathcal{D}$ obviously goes through for left duals as well. Let us now further suppose that the right and left duality functors we chose for our rigid category $\mathcal{C}$ coincide on objects. Then $\rho^{\vee\vee}$ and ρ are both right duals for $\rho^\vee$ and again we obtain a natural isomorphism ω from the identity functor $\mathrm{id}_\mathcal{C}$ to $\mathcal{D}^2$. Conversely, given right duals $\rho^\vee$ and a natural isomorphisms ω_ρ in $(\rho,\rho^{\vee\vee})$, we apply the first point of the Proposition to pass from dualities with left object $\rho^\vee$ and right object $\rho^{\vee\vee}$ to dualities with the same left object and right object ρ, so $\rho^\vee$ becomes also a left dual of ρ. The two passages are clearly inverse to each other.

To sum up, given a right duality functor $\mathcal{D}$, we have a bijective correspondence between left duality functors coinciding on objects with $\mathcal{D}$ and natural isomorphisms from $\mathrm{id}_\mathcal{C}$ to $\mathcal{D}^2$.

We have to stress that the natural isomorphism ω_ρ of Remark A will generally fail to be tensor. It therefore makes sense to introduce the following more specific notion.

Definition C. Given a right duality functor $\mathcal{D}$ on $\mathcal{C}$, a *pivotal structure* for $\mathcal{D}$ is a monoidal isomorphism ω from $\mathrm{id}_\mathcal{C}$ to $\mathcal{D}^2$.

The scenario of Definition C further specifies if one considers the categorical traces. We recall that for every object ρ of $\mathcal{C}$ one defines $\mathrm{Tr}^L_\rho, \mathrm{Tr}^R_\rho : (\rho,\rho) \to \mathbb{C}$ by

$$\mathrm{Tr}^L_\rho(f) = d_\rho \circ (\rho^\vee \otimes f\omega^{-1}) \circ b_{\rho^\vee} \ , \qquad \mathrm{Tr}^R_\rho(f) = d_{\rho^\vee} \circ (\omega f \otimes \rho^\vee) \circ b_\rho \ .$$

Definition D. A pivotal structure on $\mathcal{C}$ is said to be a *spherical structure* if $\mathrm{Tr}^L_\rho = \mathrm{Tr}^R_\rho$ for all objects ρ. Denoting the common value by Tr_ρ, the *categorical dimension* is defined for each ρ by $\mathrm{Tr}_\rho(\rho)$.

The categorical traces are clearly linear; in the pivotal case they are also multiplicative, namely $\mathrm{Tr}^L_{\rho\otimes\sigma}(f \otimes g) = \mathrm{Tr}^L_\rho(f)\,\mathrm{Tr}^L_\sigma(g)$ for all f in (ρ,ρ), g in (σ,σ), and similarly for Tr^R_ρ. Finally, in the spherical case one also has $\mathrm{Tr}_\rho(gf) = \mathrm{Tr}_\sigma(fg)$ for all f in (ρ,σ), g in (σ,ρ).

For later use, we now consider yet another kind of a special case in the choice of left and right duals.

Remark B. Let $(\mathcal{C},\otimes,a)$ be a rigid tensor category, and suppose we have ${}^\vee\cdot$ and $\cdot^\vee$ left and right duality functors inverse to each other. Then we may define a new monoidal structure on $\mathcal{C}$:

$$\rho\check{\otimes}\sigma := {}^\vee(\sigma^\vee \otimes \rho^\vee) \ , \qquad \check{a}_{\rho,\sigma,\tau} = {}^\vee a_{\tau^\vee,\sigma^\vee,\rho^\vee} \ .$$

The verifications are straightforward; for instance, $a_{\tau,\sigma,\rho}$ is in $\big((\tau\otimes\sigma)\otimes\rho, \tau\otimes(\sigma\otimes\rho)\big)$, so

$$\begin{aligned} {}^\vee a_{\tau^\vee,\sigma^\vee,\rho^\vee} \quad \text{is in} \quad & \Big({}^\vee(\tau^\vee \otimes (\sigma^\vee \otimes \rho^\vee)), {}^\vee((\tau^\vee \otimes \sigma^\vee) \otimes \rho^\vee)\Big) = \\ & \Big({}^\vee(\tau^\vee \otimes (\rho\check{\otimes}\sigma)^\vee), {}^\vee((\sigma\check{\otimes}\tau)^\vee \otimes \rho^\vee)\Big) = ((\rho\check{\otimes}\sigma)\check{\otimes}\tau, \rho\check{\otimes}(\sigma\check{\otimes}\tau)) \ . \end{aligned}$$

We also remark that $D_{\rho,\sigma} : (\rho\otimes\sigma)^\vee \to \sigma^\vee \otimes \rho^\vee$ is a tensor structure on the right duality functor $\mathcal{D}$ exactly if ${}^\vee D_{\rho,\sigma} : \rho\check{\otimes}\sigma \to \rho\otimes\sigma$ is one on the identity functor considered from $(\mathcal{C},\otimes,a)$ to $(\mathcal{C},\check{\otimes},\check{a})$; this can be seen by just applying ${}^\vee\cdot$ or $\cdot^\vee$ to the diagram for each of the tensor structures.

1.4 Braidings and generalised coboundaries

General coboundaries were introduced in [CCP21] and provide a simple generalisation of braidings in terms of tensor functors. Since they offer a convenient tool for our forthcoming developments (see 2.7), we go ahead and treat them here, while also fixing our notation and definitions for a few related more usual categorical notions.

Definition A. Let $\mathcal{C}$ be a tensor category. A *generalised coboundary* on it is a tensor structure on the identity functor considered from $\mathcal{C}^{\mathrm{swap}}$ to $\mathcal{C}$.

Here $\mathcal{C}^{\mathrm{swap}}$ denotes the tensor category obtained by composing the tensor product with the swap functor on $\mathcal{C}\times\mathcal{C}$ and modifying the associator accordingly; namely, if we start with $(\mathcal{C}, \otimes, a)$ then $\mathcal{C}^{\mathrm{swap}} = \mathcal{C}$, $f \otimes^{\mathrm{op}} g = g \otimes f$ for all f, g morphisms of $\mathcal{C}$ and $a^{\mathrm{op}}_{\rho,\sigma,\tau} = a^{-1}_{\tau,\sigma,\rho}$ for all ρ, σ, τ objects of $\mathcal{C}$. So c is a generalised coboundary for $\mathcal{C}$ if

$$\begin{array}{ccccc}
(\rho\otimes\sigma)\otimes\tau & \xrightarrow{c_{\rho,\sigma}\otimes\tau} & (\sigma\otimes\rho)\otimes\tau & \xrightarrow{c_{\sigma\otimes\rho,\tau}} & \tau\otimes(\sigma\otimes\rho) \\
\downarrow{\scriptstyle a_{\rho,\sigma,\tau}} & & & & \uparrow{\scriptstyle a_{\tau,\sigma,\rho}} \\
\rho\otimes(\sigma\otimes\tau) & \xrightarrow{\rho\otimes c_{\sigma,\tau}} & \rho\otimes(\tau\otimes\sigma) & \xrightarrow{c_{\rho,\tau\otimes\sigma}} & (\tau\otimes\sigma)\otimes\rho
\end{array} \tag{1}$$

commutes for all ρ, σ, τ in $\mathcal{C}$. We observe that since the units are simple $c_{\mathbb{1},\mathbb{1}}$ is a non-zero scalar, so up to dividing c by it we may assume $c_{\mathbb{1},\mathbb{1}} = 1$. Hence the compatibility of c with units implies

$$l_\rho = r_\rho \circ c_{\mathbb{1},\rho}\ , \quad r_\rho = l_\rho \circ c_{\rho,\mathbb{1}} \qquad \forall\ \rho \in \mathcal{C}\ .$$

Remark A. By naturality of c, the diagrams

$$\begin{array}{ccc}
(\rho\otimes\sigma)\otimes\tau & \xrightarrow{c_{\rho,\sigma}\otimes\tau} & (\sigma\otimes\rho)\otimes\tau \\
\downarrow{\scriptstyle c_{\rho\otimes\sigma,\tau}} & & \downarrow{\scriptstyle c_{\sigma\otimes\rho,\tau}} \\
\tau\otimes(\rho\otimes\sigma) & \xrightarrow{\tau\otimes c_{\rho,\sigma}} & \tau\otimes(\sigma\otimes\rho)
\end{array}
\qquad
\begin{array}{ccc}
\rho\otimes(\sigma\otimes\tau) & \xrightarrow{\rho\otimes c_{\sigma,\tau}} & \rho\otimes(\tau\otimes\sigma) \\
\downarrow{\scriptstyle c_{\rho,\sigma\otimes\tau}} & & \downarrow{\scriptstyle c_{\rho,\tau\otimes\sigma}} \\
(\sigma\otimes\tau)\otimes\rho & \xrightarrow{c_{\sigma,\tau}\otimes\rho} & (\tau\otimes\sigma)\otimes\rho
\end{array}$$

commute for all ρ, σ, τ in $\mathcal{C}$. If we apply them respectively to the first and the second row of (1) and rotate the resulting diagram by π, we obtain that $c'_{\rho,\sigma} := c^{-1}_{\sigma,\rho}$ is a general coboundary as well; in keeping with the nomenclature for braidings, it is called the *reversed* generalised coboundary.

Example. Diagram (1) commutes if c is a braiding. This follows from the braided version of the coherence theorem (see [Mac71]), since the two rows of (1) give the same element of B_2, the braid group with two generators. Moreover, if c is a braiding $c_{\mathbb{1},\mathbb{1}} = 1$ is automatic.

Now, let us suppose that $\mathcal{C}$ is rigid, and consider right duals $\rho^\vee$ for each object ρ with duality pairs (b_ρ, d_ρ); we may see them as left duals for $\mathcal{C}^{\mathrm{swap}}$. Then, viewing $\mathrm{id}_\mathcal{C}$ as a tensor isomorphism from $\mathcal{C}^{\mathrm{swap}}$ to $\mathcal{C}$ with tensor structure c, we get duality pairs (b'_ρ, d'_ρ) with right object ρ and left object $\rho^\vee$. Explicitly,

$$b'_\rho = c(\rho^\vee, \rho)^{-1} \circ b_\rho \qquad \text{and} \qquad d'_\rho = d_\rho \circ c(\rho, \rho^\vee)\ .$$

The corresponding isomorphism (see Remark 1.3A) from $\mathrm{id}_\mathcal{C}$ to $\mathcal{D}^2$ is called the *Drinfel'd isomorphism* of c; we denote it by u. Now from c one can define two interesting tensor structures on $\mathrm{id}_\mathcal{C}$:

c^2) this is given by $c_{\sigma,\rho} \circ c_{\rho,\sigma}$ for each ρ, σ; in other words we go from $\mathcal{C}$ to $\mathcal{C}^{\mathrm{swap}}$ and then back to $\mathcal{C}$ with the identity functors equipped respectively with $c_{\sigma,\rho}$ and $c_{\rho,\sigma}$ and c^2 is the tensor structure induced on the composition.

c_2) Since u is a natural isomorphism from $\mathrm{id}_{\mathcal{C}}$ to $\mathcal{D}^2$, we may pull back the tensor structure of $\mathcal{D}^2$ (see 1.3) to $\mathrm{id}_{\mathcal{C}}$; this is c_2.

Remark B. We observe that c_2 does not depend on the particular realization of $\mathcal{D}$. To see this, let us consider a second right duality functor $\mathcal{D}'$ and denote the relative quantities adding a $'$ to the notation for the matches relative to $\mathcal{D}$. Since $\mathcal{D}$ and $\mathcal{D}'$ are tensor isomorphic, so are $\mathcal{D}^2$ and $\mathcal{D}'^2$, say by ϑ; therefore the right cell of

$$\begin{array}{ccccc} \rho\otimes\sigma & \xrightarrow{u_\rho\otimes u_\sigma} & \rho^{\vee\vee}\otimes\sigma^{\vee\vee} & \xrightarrow{\vartheta_\rho\otimes\vartheta_\sigma} & \rho^{\vee'\vee'}\otimes\sigma^{\vee'\vee'} \\ \downarrow{\scriptstyle (c_2)_{\rho,\sigma}} & & \downarrow{\scriptstyle D^2_{\rho,\sigma}} & & \downarrow{\scriptstyle D'_{\rho,\sigma}} \\ \rho\otimes\sigma & \xrightarrow{u_{\rho\otimes\sigma}} & (\rho\otimes\sigma)^{\vee\vee} & \xrightarrow{\vartheta_{\rho\otimes\sigma}} & (\rho\otimes\sigma)^{\vee'\vee'} \end{array}$$

commutes and so does the left one by definition of c_2, hence the outer cell commutes as well. Moreover the rows of the diagram actually compose to $u'_\rho\otimes u'_\sigma$ and $u'_{\rho\otimes\sigma}$. So $(c_2)_{\rho,\sigma}$ must equal $(c'_2)_{\rho,\sigma}$ by definition of the latter.

The following two theorems provide a useful characterization of the possible pivotal structures on $\mathcal{C}$ in terms of automorphisms of $\mathrm{id}_{\mathcal{C}}$ and a sufficient condition for them to be actually spherical.

Definition B. Consider a tensor category $\mathcal{C}$ and a tensor structure b for $\mathrm{id}_{\mathcal{C}}$. A *balancing structure* for b is a tensor isomorphism from $\mathrm{id}_{\mathcal{C}}$ with b to $\mathrm{id}_{\mathcal{C}}$ with the trivial tensor structure. Explicitly,

$$v_\rho\otimes v_\sigma = v_{\rho\otimes\sigma}\circ b_{\rho,\sigma}\ .$$

Since $\mathbb{1}$ is simple we may, and will, assume $v_{\mathbb{1}}=1$. Furthermore, in the case where $\mathcal{C}$ is rigid, a balancing structure for b compatible with duality will be called a *ribbon structure.*

Theorem. *Let $\mathcal{C}$ be a rigid tensor category with a generalised coboundary c. We consider a right duality functor $\mathcal{D}$, and the corresponding Drinfel'd isomorphism u. Then*

- *if v is a ribbon structure for c^2, then the left and right categorical traces for uv^{-1} coincide;*
- *there is a bijective correspondence between pivotal structures for $\mathcal{D}$ and balancing structures for c_2, given by*

$$\omega\leftrightarrow w \qquad \omega = uw^{-1}\ .$$

Proof. The second point is obvious by definition of c_2 and of balancing structure. The first point on the other hand requires quite a bit of calculation, for which we refer to [CCP21] (Theorem 21.13). ☐

Remark C. We also recall (see Proposition 8.9.3 of [EGNO15] for a proof) that if c is a braiding we have $c^2=c_2$. So in this case the Theorem asserts that if v is a ribbon structure for c^2 then uv^{-1} is a spherical structure.

Before concluding the subsection, we return to the general scenario of a tensor category $\mathcal{C}$ with a generalised coboundary c on it, and introduce a simple suitable notion of deformation. Given a natural isomorphism η from the identity functor $\mathrm{id}_{\mathcal{C}}$ to itself with $\eta_{\mathbb{1}} = 1$, we put

$$c^{\eta}_{\rho,\sigma} := c_{\rho,\sigma} \circ (\eta_{\rho}^{-1} \otimes \eta_{\sigma}^{-1}) \circ \eta_{\rho\otimes\sigma} \ , \tag{2}$$

namely we compose c, a tensor structure on $\mathrm{id}_{\mathcal{C}} : \mathcal{C}^{\mathrm{swap}} \to \mathcal{C}$, with the unique tensor structure T_{η} on $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ such that η becomes tensor from the identical tensor structure to T_{η}. Equivalently, by naturality of c and η, we may rewrite (2) as

$$c^{\eta}_{\rho,\sigma} = \eta_{\sigma\otimes\rho} \circ c_{\rho\sigma} \circ (\eta_{\rho}^{-1} \otimes \eta_{\sigma}^{-1}) \ ,$$

i.e. c^{η} is the unique generalised coboundary on $\mathcal{C}$ such that η becomes tensor from $\mathrm{id}_{\mathcal{C}} : \mathcal{C}^{\mathrm{swap}} \to \mathcal{C}$ with c to $\mathrm{id}_{\mathcal{C}} : \mathcal{C}^{\mathrm{swap}} \to \mathcal{C}$ with c^{η}; in particular c^{η} is a braiding if c is. By similar computations one sees that $(c^{\eta})' = (c')^{\eta^{-1}}$.

Finally, we have the following nice result "intertwining" the bijection of the Theorem with deformation.

Lemma. *We consider a rigid tensor category $\mathcal{C}$ and fix a right duality functor $\mathcal{D}$. Let c be a generalised coboundary and u its Drinfel'd isomorphism relative to $\mathcal{D}$. Then, given a natural isomorphism η from $\mathrm{id}_{\mathcal{C}}$ to itself with $\eta_{\mathbb{1}} = 1$ and compatible with duality, the Drinfel'd isomorphism for c^{η} is*

$$u \circ \eta^{-2} =: u^{\eta} \ . \tag{3}$$

Moreover, the assignment $v \mapsto v \circ \eta^{-2}$ defines a bijection from balancing structures for c^2 to balancing structures for $(c^{\eta})^2$. The same is true for c_2 and $(c^{\eta})_2$.

Proof. Formula (3) is easily computed applying the definition of the Drinfel'd element (see also Remark 1.3A) using compatibility of η with duality. Therefore

$$u \circ v^{-1} = u^{\eta} \circ (v \circ \eta^{-2})^{-1}$$

for all v isomorphisms of $\mathrm{id}_{\mathcal{C}}$; so, by the second point of the Theorem, v is balancing for c_2 if and only if $v \circ \eta^{-2}$ is for $(c^{\eta})_2$. The same fact for c^2 and $(c^{\eta})^2$ follows from the readily checked formula

$$(c^{\eta})^2_{\rho,\sigma} = \eta^2_{\rho\otimes\sigma} \circ (c^2)_{\rho,\sigma} \circ (\eta_{\rho}^{-2} \otimes \eta_{\sigma}^{-2}) \ . \qquad \square$$

1.5 C^* categories and ribbon tensor categories

Definition A. A C^* *category* is a pair $(\mathcal{C}, \cdot^*)$ where $\mathcal{C}$ is a $\mathbb{C}$-linear category and $\cdot^*$ is a *conjugation* on $\mathcal{C}$. This in turn means that we have antilinear maps

$$\cdot^* : (\rho, \sigma) \to (\sigma, \rho) \qquad f \mapsto f^*$$

such that $f^{**} = f$ and $(g \circ f)^* = f^* \circ g^*$ whenever the composition is defined. Moreover we require the morphism spaces to be Banach spaces, whose norms, denoted by $\| \cdot \|$, satisfy the following properties:

- $\|g \circ f\| \leq \|g\|\|f\|$; (sub-multiplicativity)
- $\|f^*f\| = \|f\|^2$. (C^* identity)

Finally, f^*f is required to be positive for every morphism f (note that (a, a) is a C^*-algebra for every object a).

If $\mathcal{C}$ is a semi-simple abelian category, which is the situation of our interest, the portion of Definition A concerning norms is actually redundant.

Proposition. *Let $\mathcal{C}$ be a semi-simple abelian category, with an involution $\cdot^*$ such that f^*f is positive for every f morphism of $\mathcal{C}$. Then $\mathcal{C}$ admits a unique structure of a C^* category.*

Proof. To begin with, the morphism spaces are finite-dimensional, hence complete with respect to every norm. Furthermore, any (ρ, ρ) admits a unique C^* norm. More explicitly, $\|f\| = \sqrt{r(f^*f)}$ for all f in (ρ, ρ), where r stands for "spectral radius of"; so by the C^* identity there is at most one choice of norms satisfying Definition A. On the other hand it is easy to see that $\|f\| := \sqrt{r(f^*f)}$ defines norms as required, provided f^*f to be positive for all morphism f. □

Remark. If, again, $\mathcal{C}$ is semi-simple abelian, the positivity requirement allows us to upgrade general direct sums to orthogonal ones. Explicitly, let us consider i_k in $(\rho_k, \rho_1 \oplus \cdots \oplus \rho_n)$ and p_k in $(\rho_1 \oplus \cdots \oplus \rho_n, \rho_k)$ as in Remark 1.1A; up to further decomposing each ρ_k, we may assume them to be simple. Then $i_k^* i_k$ is id_{ρ_k} times a positive scalar, so up to dividing the square root, we may assume $i_k^* i_k = \mathrm{id}_{\rho_k}$, for all k. To sum up, we may assume the i_k to be isometries, and put $p_k = i_k^*$.

Conversely the positivity requirement is automatically met if orthogonal sums are available. To see this, let us consider f in (ρ, σ). By functional calculus we find self-adjoint idempotents p_+, p_- commuting with f^*f such that $f^*fp_+ \geq 0$ and $f^*fp_- \leq 0$; we need to prove $f^*fp_- = 0$. Now $f^*fp_- = p_-f^*fp_- = (fp_-)^*(fp_-)$, so we are reduced to prove that $f^*f \leq 0$ implies $f^*f = 0$. To this aim we consider an orthogonal direct sum τ with isometries i in (ρ, τ), j in (σ, τ) such that $ii^* + jj^* = \tau$. We put $g := jfi^*$ so that $g^*g = if^*fi^*$; now $f^*f = -h^*h$ for some h in (ρ, ρ), hence

$$g^*g = if^*fi^* = -ih^*hi^* = -(ihi^*)^*(ihi^*) \leq 0 \ ,$$

since ihi^* is in (τ, τ). On the other hand $g^*g \geq 0$, so $g^*g = 0$ and $f^*f = i^*g^*gi = 0$.

Definition B. Given C^* categories $(\mathcal{C}, \cdot^*)$, $(\mathcal{C}', \cdot^{*'})$, a functor $\mathcal{F} : \mathcal{C} \to \mathcal{C}'$ is said to be a $*$-functor if $\mathcal{F}(f^*) = \mathcal{F}(f)^{*'}$ for all f morphism of $\mathcal{C}$. Given $*$-functors $\mathcal{F}, \mathcal{G} : \mathcal{C} \to \mathcal{C}'$, a natural isomorphism η from $\mathcal{F}$ to $\mathcal{G}$ is said to be *unitary* if so is η_ρ for all ρ object of $\mathcal{C}$.

All kinds of categories that can be defined combining structures from previous subsections admit a C^* version, which is generally obtained by requiring suitable compatibility conditions with the conjugation. The most elaborate instance of such an arrangement occurs for ribbon tensor categories, which we proceed to define in both the "algebraic" and C^* versions.

Definition C. A *ribbon category* is a rigid tensor category $(\mathcal{C}, \otimes, a)$, say with duality pairs (b_ρ, d_ρ) with left object ρ and right object $\rho^\vee$ for each ρ, which is further equipped with a braiding c and a ribbon element v.

Then $\mathcal{C}$ is a C^* *ribbon category* if it is also a C^* category, say with conjugation $\cdot^*$, and the following are verified:

- $(f \otimes g)^* = f^* \otimes g^*$;
- $a^*_{\rho,\sigma,\tau} = a^{-1}_{\rho,\sigma,\tau}$, $c^*_{\rho,\sigma} = c^{-1}_{\rho,\sigma}$, $v^*_\rho = v^{-1}_\rho$;
- $b^*_\rho = d_\rho \circ c_{\rho,\rho^\vee} \circ (v^{-1}_\rho \otimes \rho^\vee)$;
- $d^*_\rho = (\rho^\vee \otimes v_\rho) \circ c^{-1}_{\rho^\vee,\rho} \circ b_\rho$.

The compatibility conditions of the first and the second points in Definition 1.5C just require $\otimes$ to be a $*$-functor and the natural isomorphisms involved to be unitary. The third and the fourth point deal with the new duality pairs obtained by conjugating (b_ρ, d_ρ); they yield a left duality functor coinciding on objects with the given right duality functor $\mathcal{D}$. The condition is that the corresponding isomorphism ω from $\mathrm{id}_\mathcal{C}$ to $\mathcal{D}^2$ actually equal uv^{-1}, where u is the Drinfel'd isomorphism for c.

So ω is a spherical structure, which coincides with the one coming from c and v. Hence the categorical trace may be computed, for each object ρ, as follows:

$$\mathrm{Tr}_\rho(f) = d_\rho \circ (\rho^\vee \otimes f) \circ d^*_\rho = b^*_\rho \circ (f \otimes \rho^\vee) \circ b_\rho \qquad \forall\ f \in (\rho, \rho)\ .$$

In particular, we see that categorical dimensions are positive. Moreover $\mathrm{Tr}(fg^*)$ defines a scalar product on each morphism space (ρ, σ), turning it into a Hilbert space.

2 Weak quasi-tensor functors and Tannaka-Krein duality

Most notable examples of a tensor category are offered by representation categories of an algebra endowed with some form of coproduct; one then has the forgetful functor, with values in Vec. Throughout the present section, we will mainly adopt the reverse point of view. More precisely our basic starting point will be a weak quasi-tensor functor, a remarkably flexible generalised version of a tensor functor, from some tensor category into Vec, say $\mathcal{F} : \mathcal{C} \to \mathrm{Vec}$.

As it is well known in the case of tensor functors, Tannaka-Krein duality allows one to use $\mathcal{F}$ to reconstruct $\mathcal{C}$ as a representation category of a suitable algebraic object, and the latter is determined by $\mathcal{F}$ up to isomorphism. So we may take $\mathcal{F}$ as a convenient form to present our algebraic object, in that it is evidently an enrichment of the categorical datum. This way our category is clearly displayed and any further structure attached comes effectively encoded by the properties of the particular functor we are looking at.

Subsection 2.1 is devoted to the introduction of discrete weak quasi-bialgebras, the algebraic object needed for the reconstruction results we mentioned, and specifically to their construction from weak quasi-tensor functors. Indeed, weak quasi-Hopf algebras were introduced in [MS92] and [Sch95] with this exact purpose in the context of fusion categories arising from conformal field theory; their methods were subsequently developed and conveniently formalised in [Här97], and extended beyond fusion categories in [CCP21]. The actual Tannaka-Krein duality result we present, in Theorem 2.2, follows Theorem 5.6 of [CCP21].

Subsection 2.3 refines the reconstruction theorem by reexpressing antipodes and/or braided symmetries our category may be endowed with in terms of additional structure of the relative discrete weak quasi-bialgebra.

Moving on, subsection 2.4 refines the treatment of 2.1 and 2.2 in order to deal with the case of a $*$-functor on a C^* tensor category $\mathcal{F} : \mathcal{C} \to \mathrm{Hilb}$; accordingly, the corresponding weak quasi-bialgebra A will be endowed with an involution. Likewise, subsection 2.6 completes 2.3 considering some further arising compatibility issue.

Generally speaking such adaptations and refinements are often most conveniently treated under the algebraic aspect in terms of certain elements of $A \otimes A$ called "twists". Roughly speaking, they can be thought of as morphisms between the possible weak quasi-coalgebra structures on A, or, in categorical terms, between the possible weak quasi-tensor structures on $\mathcal{F}$. Twists will be treated in subsection 2.5 and will play a fundamental role in the last chapter.

Finally, subsections 2.7 and 2.8 focus on the properties of two special types of weak quasi-Hopf algebras. In the case of the unitary coboundary type, such properties arise from the C^*-ribbon structure of $\mathcal{C}$ (see Definition 1.5C); on the other hand, weak Hopf algebras correspond to the case when the weak quasi-tensor functor $\mathcal{F}$ belong to a class more alike the one of usual tensor functors, though still more general. Both such special situations are met in the examples arising from quantum groups at roots of 1, which motivate their formalisation, as discussed in next chapter.

2.1 Weak quasi-bialgebras

Following [CCP21], we introduce the main notion of the present section. In order to keep our notation agile, we will treat Vec, the category of complex vector spaces of finite dimension, as if it were strict with unit $\mathbb{C}$, by just implying its associator and unitors.

Definition A. Let $(\mathcal{C}, \otimes, a)$ be a tensor category (see the paragraph at the end of 1.2) and $\mathcal{F} : \mathcal{C} \to \mathrm{Vec}$ a faithful functor. A *weak quasi-tensor structure* on $\mathcal{F}$ is a pair (F, G) of natural transformations given by

$$F_{\rho,\sigma} : \mathcal{F}(\rho) \otimes \mathcal{F}(\sigma) \to \mathcal{F}(\rho \otimes \sigma) \ , \qquad G_{\rho,\sigma} : \mathcal{F}(\rho \otimes \sigma) \to \mathcal{F}(\rho) \otimes \mathcal{F}(\sigma)$$

such that $F \circ G$ is the identity. We further require $\mathcal{F}$ to be compatible with units (see the Remark below), and assume the normalisation condition $\mathcal{F}(\mathbb{1}) = \mathbb{C}$.

Finally, in keeping with Definition 1.2D, given faithful functors $\mathcal{F}_i : \mathcal{C} \to \mathrm{Vec}$ with weak quasi-tensor structures (F_i, G_i) where $i = 1, 2$, an isomorphism u from $\mathcal{F}_1$ to $\mathcal{F}_2$ will be called a *tensor isomorphism* if

$$(F_2)_{\rho,\sigma} \circ (u_\rho \otimes u_\sigma) = u_{\rho\otimes\sigma} \circ (F_1)_{\rho,\sigma} \quad \text{and} \quad (u_\rho \otimes u_\sigma) \circ (G_1)_{\rho,\sigma} = (G_2)_{\rho,\sigma} \circ u_{\rho\otimes\sigma} \ ;$$

we also require the condition $u_{\mathbb{1}} = 1$.

Remark. A weak quasi-tensor structure is obviously far weaker than a tensor one. On the other hand we just gave the notion of compatibility with units for tensor functors so to this regard Definition A needs to be made clear.

To this aim, we take the agreement that, in the first place, for $\mathcal{F}$ to be "compatible with units" means that $G_{\rho,\sigma}F_{\rho,\sigma}$ is also identical whenever any of ρ,σ is a unit, so that $G_{\rho,\sigma}$ and $F_{\rho,\sigma}$ are inverse to each other. Moreover $(\mathcal{F},F)$ is assumed to make the diagram in Definition 1.2C commutative whenever any two of ρ,σ,τ are units.

With this understanding settled, a weak quasi-tensor functor is as good as a usual tensor one for what pertains to units, and the discussion and results of 1.2 apply as well.

We shall now proceed to construct out of $\mathcal{F}$ a few pieces of algebraic structure that will add up to a certain discrete weak quasi-bialgebra, whose general notion we introduce later on (Definition C).

The algebra We let $A := \mathrm{End}(\mathcal{F})$, the unital associative $\mathbb{C}$-algebra of natural endomorphisms of $\mathcal{F}$, and also define the generalised tensor powers $A^{\otimes n} := \mathrm{End}(\mathcal{F}^{\otimes n})$ (see the forthcoming Definition B for a formalisation); namely, a generic element of $A^{\otimes n}$ is given by natural maps

$$\eta_{\rho_1,\dots,\rho_n} : \quad \mathcal{F}(\rho_1)\otimes\cdots\otimes\mathcal{F}(\rho_n)\to\mathcal{F}(\rho_1)\otimes\cdots\otimes\mathcal{F}(\rho_n) \ .$$

The identities are just the identical natural endomorphism, which we will usually just denote by 1 for every $A^{\otimes n}$, unless this causes ambiguity.

It is important to note that the $A^{\otimes n}$ algebras are isomorphic to a direct product of matrix algebras. More explicitly, any choice of a complete collection of mutually non-equivalent simple objects $I=\{\iota\}$ yields an isomorphism

$$A^{\otimes n}\to\prod_{\iota_1,\dots,\iota_n\in I}\bigotimes_{k=1}^{n}\mathrm{End}\left(\mathcal{F}(\iota_k)\right) \ ,$$

obtained by considering the values $\eta_{\iota_1,\dots,\iota_n}$ of η in $A^{\otimes n}$ on n-tuples of objects in I.

The coproduct Given η in A, $\Delta(\eta)$ is defined by $\Delta(\eta)_{\rho,\sigma} := G_{\rho,\sigma}\circ\eta_{\rho\otimes\sigma}\circ F_{\rho,\sigma}$ for each ρ,σ objects of $\mathcal{C}$, namely the diagram

$$\begin{array}{ccc}
\mathcal{F}(\rho)\otimes\mathcal{F}(\sigma) & \xrightarrow{\Delta(\eta)_{\rho,\sigma}} & \mathcal{F}(\rho)\otimes\mathcal{F}(\sigma) \\
\downarrow{\scriptstyle F_{\rho,\sigma}} & & \uparrow{\scriptstyle G_{\rho,\sigma}} \\
\mathcal{F}(\rho\otimes\sigma) & \xrightarrow{\eta_{\rho\otimes\sigma}} & \mathcal{F}(\rho\otimes\sigma)
\end{array} \tag{1}$$

is commutative; Δ is multiplicative, since $\Delta(\eta\theta)_{\rho,\sigma} = G_{\rho,\sigma}\circ\eta_{\rho\otimes\sigma}\circ\theta_{\rho\otimes\sigma}\circ F_{\rho,\sigma} = G_{\rho,\sigma}\circ\eta_{\rho\otimes\sigma}\circ F_{\rho,\sigma}\circ G_{\rho,\sigma}\circ\theta_{\rho\otimes\sigma}\circ F_{\rho,\sigma} = (\Delta\eta)_{\rho,\sigma}(\Delta\theta)_{\rho,\sigma}$. In a similar fashion, we may take the coproduct on some factor of a higher tensor power, defining e.g. the maps $(\mathrm{id}\otimes\Delta)$ and $(\Delta\otimes\mathrm{id})$ from $A\otimes A$ to $A\otimes A\otimes A$.

We note that generally Δ will be neither coassociative nor unital, whereas it would be both had we considered a tensor functor. Instead, we have $\Delta(1)_{\rho,\sigma} =$

$G_{\rho,\sigma}F_{\rho,\sigma}$, which is indeed idempotent since $FG = 1$. However, we still have a (unique) counit ϵ, given by $\epsilon(\eta) = \eta_{\mathbb{1}}$; like in the case of Δ, we may also take ϵ on some factor of a tensor power, e.g. if θ is in $A^{\otimes 2}$ then $\big((\mathrm{id} \otimes \epsilon)\theta\big)_\rho = \theta_{\rho,\mathbb{1}}$. This said, the counit properties hold just as usual:

$$(\epsilon \otimes \mathrm{id})\big(\Delta(\eta)\big) = \eta = (\mathrm{id} \otimes \epsilon)\big(\Delta(\eta)\big) \ ;$$

indeed, by compatibility of $\mathcal{F}$ with units we have

$$\mathcal{F}(l_\rho) = G_{\mathbb{1},\rho} \ , \quad \mathcal{F}(r_\rho) = G_{\rho,\mathbb{1}} \qquad \forall \ \rho \ , \tag{2}$$

so for example $\Delta(\eta)_{\mathbb{1},\rho} = G_{\mathbb{1},\rho}\eta_{\mathbb{1},\rho}F_{\mathbb{1},\rho} = \mathcal{F}(l_\rho)\eta_{\mathbb{1},\rho}\mathcal{F}(l_\rho^{-1}) = \eta_\rho$, having used the naturality of η for the last equality.

The associator Even if Δ is not expected to be coassociative, $(\Delta \otimes \mathrm{id}) \circ \Delta$ and $(\mathrm{id} \otimes \Delta) \circ \Delta$ are in fact "equivalent". We define Φ in $A \otimes A \otimes A$ by commutativity of

$$\begin{array}{ccc}
\mathcal{F}(\rho) \otimes \mathcal{F}(\sigma) \otimes \mathcal{F}(\tau) & \xrightarrow{\ \Phi_{\rho,\sigma,\tau}\ } & \mathcal{F}(\rho) \otimes \mathcal{F}(\sigma) \otimes \mathcal{F}(\tau) \\
\Big\downarrow {\scriptstyle F_{\rho,\sigma} \otimes \mathcal{F}(\tau)} & & \Big\uparrow {\scriptstyle \mathcal{F}(\rho) \otimes G_{\sigma,\tau}} \\
\mathcal{F}(\rho \otimes \sigma) \otimes \mathcal{F}(\tau) & & \mathcal{F}(\rho) \otimes \mathcal{F}(\sigma \otimes \tau) \\
\Big\downarrow {\scriptstyle F_{\rho \otimes \sigma,\tau}} & & \Big\uparrow {\scriptstyle G_{\rho,\sigma \otimes \tau}} \\
\mathcal{F}\big((\rho \otimes \sigma) \otimes \tau\big) & \xrightarrow{\ \mathcal{F}(a_{\rho,\sigma,\tau})\ } & \mathcal{F}\big(\rho \otimes (\sigma \otimes \tau)\big)
\end{array} \tag{3}$$

for each ρ, σ, τ objects in $\mathcal{C}$. It is easy to see that Φ is a partial isomorphism (see see Appendix I) in $\big((\Delta \otimes \mathrm{id})(\Delta 1), (\mathrm{id} \otimes \Delta)(\Delta 1)\big)$, and that

$$\Phi(\Delta \otimes \mathrm{id})(\Delta a) = (\mathrm{id} \otimes \Delta)(\Delta a)\Phi \qquad \forall \ a \in A \ .$$

Furthermore, since $(\rho \otimes l_\tau) \circ a_{\rho,\mathbb{1},\tau} \circ (r_\rho^{-1} \otimes \tau) = \rho \otimes \tau$ by coherence,

$$\begin{aligned}
\Phi_{\rho,\mathbb{1},\tau} &= \big(\mathcal{F}(\rho) \otimes G_{\mathbb{1},\tau}\big) \circ G_{\rho,\mathbb{1}\otimes\tau} \circ \mathcal{F}(a_{\rho,\sigma,\tau}) \circ F_{\rho\otimes\mathbb{1},\tau} \circ \big(F_{\rho,\mathbb{1}} \otimes \mathcal{F}(\tau)\big) \\
&= \big(\mathcal{F}(\rho) \otimes \mathcal{F}(l_\tau)\big) \circ G_{\rho,\mathbb{1}\otimes\tau} \circ \mathcal{F}(a_{\rho,\sigma,\tau}) \circ F_{\rho\otimes\mathbb{1},\tau} \circ \big(\mathcal{F}(r_\rho^{-1}) \otimes \mathcal{F}(\tau)\big) \\
&= G_{\rho,\tau} \circ \mathcal{F}(\rho \otimes l_\tau) \circ \mathcal{F}(a_{\rho,\sigma,\tau}) \circ \mathcal{F}(r_\rho^{-1} \otimes \tau) \circ F_{\rho,\tau} = G_{\rho,\tau}F_{\rho,\tau} \ ,
\end{aligned}$$

having used (2) for the first equality and naturality of F and G for the second; so $(\mathrm{id} \otimes \epsilon \otimes \mathrm{id})\Phi = \Delta(1)$. Finally, the pentagon axiom for a results in the following cocycle condition for Φ:

$$(\mathrm{id} \otimes \Phi)(\mathrm{id} \otimes \Delta \otimes \mathrm{id})\Phi(\Phi \otimes \mathrm{id}) = (\mathrm{id} \otimes \mathrm{id} \otimes \Delta)\Phi(\Phi \otimes \mathrm{id} \otimes \mathrm{id})\Phi \ .$$

We may now axiomatise the triple (A, Δ, Φ); prior to doing that, we clarify what we mean by a *discrete algebra*.

Definition B. We say that a unital associative $\mathbb{C}$-algebra A is *discrete* if it is isomorphic to a direct product of matrix algebras, i.e. an algebra of the form

$$M = \prod_{\iota \in I} \mathrm{End}(V_\iota) \ ,$$

where each V_ι is a complex vector space of finite dimension, with the indexing set I possibly infinite. We also consider the algebras

$$M^{\underline{\otimes} n} := \prod_{\iota_1,\ldots,\iota_n} \bigotimes_{k=1}^{n} \mathrm{End}(V_{i_k}) \ .$$

For each n, $M^{\underline{\otimes} n}$ is endowed with the topology of "pointwise convergence", which is the weakest topology such that the projection map onto each $\bigotimes_{k=1}^{n} \mathrm{End}(V_{i_k})$ is continuous for all choice of $i_1, \ldots, i_n$; we observe that $M^{\underline{\otimes} n}$ is complete and the usual tensor product $M^{\otimes n}$ is dense in $M^{\underline{\otimes} n}$, for all n. We transfer the induced topology of $M^{\otimes n}$ on $A^{\otimes n}$ by isomorphism; it is easy to see that the topology on A does not depend on which isomorphism we started with. For each n, we define the *generalised tensor power* $A^{\underline{\otimes} n}$ as the completion of $A^{\otimes n}$, which is isomorphic to $M^{\underline{\otimes} n}$.

A coproduct on the discrete algebra A is allowed to take values in $A^{\underline{\otimes} 2}$; however, any time we define a linear map having some $A^{\otimes n}$ as its domain we may consider its unique extension to $A^{\underline{\otimes} n}$ (the original map is automatically uniformly continuous). For this reason we drop the underlined symbol $\underline{\otimes}$, and just use $\otimes$.

Example. Let G be any discrete group. Then $M := \mathrm{Map}(G, \mathbb{C})$, the set of complex valued functions on G, may be viewed as a discrete algebra. Indeed $M = \prod_{g \in G} \mathbb{C}$, so this is the special case when all matrix are 1×1.

Moreover the n-th generalised tensor power is just $\mathrm{Map}(I^n, \mathbb{C})$, and a coproduct is obtained using the composition of G, e.g.:

$$(\mathrm{id} \otimes \Delta) : M \otimes M \to M \otimes M \otimes M \qquad ((\mathrm{id} \otimes \Delta) f)(g, h, k) = f(g, hk) \ .$$

We note that in the case of $\mathrm{End}(\mathcal{F})$ introduced above the generalised tensor powers were concretely realized as $\mathrm{End}(\mathcal{F}^{\otimes n})$, so we preferred to define Δ, ϵ and Φ directly in these terms. We also point out that Definition B is just aimed to make more rigorous the treatment of the concrete example presented in this subsection. For a proper abstract treatment in the due generality, we refer to [VDa96].

Definition C. A *discrete weak quasi-bialgebra* is a triple (A, Δ, Φ) where:

i) A is a discrete unital associative $\mathbb{C}$-algebra;

ii) $\Delta : A \to A \otimes A$ is linear and multiplicative, not necessarily unital; moreover Δ admits a counit, i.e. a homomorphism of unital algebras $\epsilon : A \to \mathbb{C}$ such that $(\epsilon \otimes \mathrm{id})(\Delta a) = a = (\mathrm{id} \otimes \epsilon)(\Delta a)$ for all a in A.

iii) Φ is a partial isomorphism in $A \otimes A \otimes A$ (see Appendix I), with domain $(\Delta \otimes \mathrm{id})(\Delta 1)$ and codomain $(\mathrm{id} \otimes \Delta)(\Delta 1)$, such that

$$\Phi(\Delta \otimes \mathrm{id})(\Delta a) = (\mathrm{id} \otimes \Delta)(\Delta a)\Phi \qquad \forall\ a \in A \ . \tag{4}$$

Moreover Φ satisfies the normalisation condition $(\mathrm{id} \otimes \epsilon \otimes \mathrm{id})\Phi = \Delta(1)$ and the cocycle identity

$$(1 \otimes \Phi)(\mathrm{id} \otimes \Delta \otimes \mathrm{id})\Phi(\Phi \otimes 1) = (\mathrm{id} \otimes \mathrm{id} \otimes \Delta)\Phi(\Delta \otimes \mathrm{id} \otimes \mathrm{id})\Phi \ . \tag{5}$$

We need to take care of a couple of bits of notation. Firstly, when we write some element η of the generalised tensor power $A^{\otimes n}$ as $\eta_1 \otimes \cdots \otimes \eta_n$, we actually refer to the value $\eta_{\rho_1,\ldots,\rho_n}$ for generic $\rho_1, \ldots, \rho_n$ objects of $\mathcal{C}$, while also implying a finite summation symbol; one may think we are approximating η by regular tensor powers, so that, given a finite-dimensional representation of $A^{\otimes n}$, we find an expression $\sum \eta_1 \otimes \cdots \otimes \eta_n$ equalling η when both are evaluated on the representation we are considering.

This notation is also convenient in order to express multiplication of some components of an element η in $A^{\otimes n}$. For instance, let us consider $\eta = \eta_1 \otimes \eta_2 \otimes \eta_3$ in $A^{\otimes 3}$; then, for each ρ, σ, $(\eta_3\eta_1 \otimes \eta_2)_{\rho,\sigma}$ is obtained by considering the value $\eta_{\rho,\sigma,\rho}$ in $\mathrm{End}\,(\mathcal{F}(\rho)) \otimes \mathrm{End}\,(\mathcal{F}(\sigma)) \otimes \mathrm{End}\,(\mathcal{F}(\rho))$ and multiplying the first and the third factor, putting the latter on the left.

Furthermore, we will sometimes make use of Sweedler's notation for coproducts, even though generally, due to non-coassociativity,

$$a_{(1)(1)} \otimes a_{(1)(2)} \otimes a_{(2)} \neq a_{(1)} \otimes a_{(2)(1)} \otimes a_{(2)(2)} \ ,$$

so expressions like $a_{(1)} \otimes a_{(2)} \otimes a_{(3)}$ make no sense.

2.2 Tannakian reconstruction

Given the discrete weak quasi bialgebra (A, Δ, Φ), we may consider the category $\mathrm{Rep}(A)$ of finite-dimensional A-modules; as usual, the morphisms are the linear maps intertwining the action of A, which we also refer to as "A-linear": given A-modules V and W a $\mathbb{C}$-linear map is in (V, W) if

$$f(av) = af(v) \qquad \forall\ v \in V,\ a \in A.$$

We define the tensor product $V \otimes^A W$ by the usual formula

$$a(v \otimes w) := a_{(1)}v \otimes a_{(2)}w \qquad \forall\ a \in A\ ,\ v \in V\ ,\ w \in W\ ;$$

however, as a vector space $V \otimes^A W$ is not the whole tensor product $V \otimes W$ but it is rather the subspace $\Delta(1)(V \otimes W)$. Furthermore, given morphisms f in (V_1, V_2) and g in (W_1, W_2), by A-linearity we have

$$f \otimes g\Delta(a)_{V_1,W_1} = \Delta(a)_{V_2,W_2} f \otimes g\ ,$$

where $\Delta(a)_{V_i,W_i}$ denotes the action of $\Delta(a)$ as an element of $A \otimes A$ on $V_i \otimes W_i$ $(i = 1, 2)$; in particular $f \otimes g$ maps $V_1 \otimes^A W_1$ into $V_2 \otimes^A W_2$, so we can put

$$f \otimes^A g := f \otimes g\ , \tag{1}$$

understanding the right-hand side as a map from $V_1 \otimes^A W_1$ to $V_2 \otimes^A W_2$. Turning to the associator, we put

$$(a^A)_{U,V,W}(u \otimes v \otimes w) := \phi u \otimes \varphi v \otimes \psi w\ ,$$

where we wrote $\Phi = \phi \otimes \varphi \otimes \psi$ (see the end of 2.1); the properties of Φ, listed in point iii) of Definition 2.1C, ensure that a^A is a legitimate associator for $\mathrm{Rep}(A)$. Finally, the counit ϵ defines a unit of $\mathrm{Rep}(A)$, and points ii) and iii) in Definition 2.1C grant that ϵ is a (traditional) unit, with the usual unitors of Vec. Summarizing,

Lemma. *The semi-simple abelian category* $\mathrm{Rep}(A)$*, endowed with the tensor product* $\otimes^A$ *and the associator* a^A*, is a tensor category.*

Remark. To see that a^A is an associator, it is very appropriate to keep in mind the categories $\mathcal{P}(A\otimes A\otimes A)$ and $\mathcal{P}\big(\operatorname{End}(U\otimes V\otimes W)\big)$ of appendix I, and the functor $\mathcal{S}$ of formula I(3) relative to $U\otimes V\otimes W$.

Indeed Φ is a partial isomorphism from $(\mathrm{id}\otimes\Delta)(\Delta 1)$ to $(\Delta\otimes\mathrm{id})(\Delta 1)$, so if we apply $\mathcal{S}$ to its action on $U\otimes V\otimes W$ as an element of $A\otimes A\otimes A$ we obtain an isomorphism between the subspaces $(\mathrm{id}\otimes\Delta)(\Delta 1)(U\otimes V\otimes W)$ and $(\Delta\otimes\mathrm{id})(\Delta 1)((U\otimes V\otimes W))$; moreover $(a^A)_{U,V,W}$ is A-linear by 2.1(4), and a^A is natural in U,V,W by definition of intertwiners. Let us look more closely at 2.1(5). This is to be interpreted as an identity between partial isomorphisms from $(\Delta\otimes\mathrm{id}\otimes\mathrm{id})(\Delta\otimes\mathrm{id})(\Delta 1)=:p$ to $(\mathrm{id}\otimes\mathrm{id}\otimes\Delta)(\mathrm{id}\otimes\Delta)(\Delta 1)=:q$. More in detail, $\Phi\otimes 1$ is a partial isomorphism from $\big((\Delta\otimes\mathrm{id})(\Delta 1)\big)\otimes 1$ to $\big((\mathrm{id}\otimes\Delta)(\Delta 1)\big)\otimes 1$; now

$$\big((\Delta\otimes\mathrm{id})(\Delta 1)\big)\otimes 1=\big((\Delta\otimes\mathrm{id}\otimes\mathrm{id})(\Delta\otimes\mathrm{id})\big)(1\otimes 1)\ ,$$

so we may consider the restriction of $\Phi\otimes 1$ from p onto $(\Phi\otimes 1)p(\Phi^{-1}\otimes 1)=(\mathrm{id}\otimes\Delta\otimes\mathrm{id})(\Delta\otimes\mathrm{id})(\Delta 1)$, by 2.1(4). Then, $(\mathrm{id}\otimes\Delta\otimes\mathrm{id})\Phi$ goes from $(\mathrm{id}\otimes\Delta\otimes\mathrm{id})(\Delta\otimes\mathrm{id})(\Delta 1)$ to $(\mathrm{id}\otimes\Delta\otimes\mathrm{id})(\mathrm{id}\otimes\Delta)(\Delta 1)$, and we see like for $\Phi\otimes 1$ that $1\otimes\Phi$ in turn goes from $(\mathrm{id}\otimes\Delta\otimes\mathrm{id})(\mathrm{id}\otimes\Delta)(\Delta 1)$ to q. By similar considerations, we see that the right-hand side of 2.1(5) is a composition of partial isomorphisms too, from p to q on the whole.

Now, if we consider the action of both sides on a generic $U\otimes V\otimes W\otimes X$, and apply the functor $\mathcal{S}$ relative to $\mathcal{P}\big(\operatorname{End}(U\otimes V\otimes W\otimes X)\big)$, it is clear that 2.1(5) becomes exactly the pentagon axiom for a^A.

We also remark that the defining formula (1) is nicely interpreted in terms of $\mathcal{P}\Big(\operatorname{End}\big((V_1\otimes W_1)\oplus(V_2\otimes W_2)\big)\Big)$: $f\otimes g$ is a morphism from $\Delta(a)_{V_1,W_1}$ to $\Delta(a)_{V_2,W_2}$, so the understanding we established after the formula is nothing but the implication at the right-hand side of the functor $\mathcal{S}$ relative to $(V_1\otimes W_1)\oplus(V_2\otimes W_2)$.

We are finally ready to state the following basic Tannakian reconstruction result:

Theorem. *Let* $(\mathcal{C},\otimes,a)$ *be a tensor category and* $\mathcal{F}:\mathcal{C}\to\mathrm{Vec}$ *a faithful functor with weak quasi-tensor structure* (F,G)*; we also consider the discrete weak quasi-bialgebra* (A,Δ,Φ) *constructed as in 2.1.*

Then there exists a tensor equivalence $\mathcal{E}:\mathcal{C}\to\mathrm{Rep}(A)$ *such that* $\mathcal{F}=\mathcal{F}_A\circ\mathcal{E}$*, where* $\mathcal{F}_A:\mathrm{Rep}(A)\to\mathrm{Vec}$ *is the forgetful functor.*

Conversely, let A' *be a discrete quasi bialgebra, and* $\mathcal{E}':\mathcal{C}\to\mathrm{Rep}(A')$ *a tensor equivalence such that* $\mathcal{F}$ *and* $\mathcal{F}_{A'}\circ\mathcal{E}'$ *are tensor isomorphic (see the end of Definition 2.1A). Then* A *and* A' *are isomorphic.*

Proof. Consider the vector space $\mathcal{F}(\rho)$, with ρ an object of $\mathcal{C}$; we upgrade it to an A-module, which we call $\mathcal{E}(\rho)$, by introducing the action

$$\eta v:=\eta_\rho(v)\qquad\forall\,\eta\in A\ .$$

Then, given objects ρ,σ and a morphism f in (ρ,σ), the map $\mathcal{F}(f)$ is also A-linear, by naturality. The assignment

$$f\in(\rho,\sigma)\ \mapsto\ \mathcal{E}(f)\in\big(\mathcal{E}(\rho),\mathcal{E}(\sigma)\big)\qquad\mathcal{E}(f)=\mathcal{F}(f)\ ,$$

together with the previous introduction of $\mathcal{E}(\rho)$, clearly defines a functor $\mathcal{E} : \mathcal{C} \to \mathrm{Rep}(A)$; it is also clear that $\mathcal{F} = \mathcal{F}_A \circ \mathcal{E}$.

Then, since $F_{\rho,\sigma} G_{\rho,\sigma} = \mathcal{F}(\rho \otimes \sigma)$ and $G_{\rho,\sigma} F_{\rho,\sigma} = \Delta(1)\mathcal{F}(\rho) \otimes \mathcal{F}(\sigma)$, $F_{\rho,\sigma}$ and $G_{\rho,\sigma}$ are actually partial isomorphisms inverse to each other between $\mathcal{E}(\rho) \otimes \mathcal{E}(\sigma)$ and $\mathcal{E}(\rho \otimes \sigma)$. They are also A-linear by 2.1(1), and 2.1(3) is exactly the compatibility with the associators (see the diagram in Definition 1.2C).

We are left to verify that $\mathcal{E}$ is an equivalence and prove the second assertion of the Theorem. In order to proceed, we make a simple observation about representations of discrete algebras.

Given vector spaces V_ι, with ι in I possibly infinite, let us consider the direct product and $M = \prod_{\iota \in I} \mathrm{End}(V_\iota)$; we denote the projection map onto the ι-th component by $p_\iota : M \to \mathrm{End}(V_\iota)$. Then it is easy to see that $\{p_\iota\}_{\iota \in I}$ is a complete collection of mutually non-equivalent simple objects (we abbreviate to c.n.e.s. from now to the end of the proof) for $\mathrm{Rep}(M)$.

The fact that $\mathcal{E}$ is an equivalence follows at once, because given a c.n.e.s. $\{\iota\}$ for $\mathcal{C}$, $\mathcal{E}$ sends each ι to the corresponding projection for A.

Now, consider A' and $\mathcal{E}'$ as in the last assertion, but with $\mathcal{F} = \mathcal{F}_A \circ \mathcal{E}'$; then $\{\mathcal{E}(\iota)\}$ is a c.n.e.s. for $\mathrm{Rep}(A')$, and $\mathcal{E}(\iota)$ is $\mathcal{F}(\iota)$ as a vector space, so, again by the observation, A' is isomorphic to A. Finally, suppose we have $\mathcal{F}' : \mathcal{C} \to \mathrm{Vec}$ with a weak quasi tensor structure (F', G') and natural isomorphisms $u_\rho : \mathcal{F}(\rho) \to \mathcal{F}'(\rho)$ as in the end of Definition 2.1A; we apply the constructions of 2.1 for $\mathcal{F}'$ to get (A', Δ', Φ'). Then it is readily verified that

$$U : A \to A' \qquad U(\eta)_\rho = u_\rho \eta_\rho u_\rho^{-1}$$

is an isomorphisms from (A, Δ, Φ) to (A', Δ', Φ'). □

2.3 More structure on (A, Δ, Φ)

We remain in the scenario of Theorem 2.2. If $\mathcal{C}$ carries any structure on top of its monoidal datum $(\otimes, a)$, the additional data will transfer to $\mathrm{Rep}(A)$ thanks to the tensor equivalence $\mathcal{E}$, hence it will result in more structure on top of (A, Δ, Φ) itself.

Antipodes Suppose that $\mathcal{C}$ is rigid, and consider right duals $\rho^\vee$ for each object ρ, with duality pairs (b_ρ, d_ρ); we further assume $\dim \mathcal{F}(\rho^\vee) = \dim \mathcal{F}(\rho)$ for all ρ object of $\mathcal{C}$.

Given a complete collection of mutually non-equivalent simple objects $I = \{\iota\}$, we choose isomorphisms $U_\iota : \mathcal{F}(\iota)' \to \mathcal{F}(\iota^\vee)$, denoting by $\cdot'$ the usual dual on Vec, and turn $\mathcal{F}(\iota)'$ into an A-module by upgrading U_ι to an intertwiner, for each ι.

This way $\mathcal{F}(\iota)$ and $\mathcal{F}(\iota)'$ become left and right duals as A-modules; indeed so are $\mathcal{F}(\iota)$ and $\mathcal{F}(\iota^\vee)$, since $\mathcal{E} : \mathcal{C} \to \mathrm{Rep}(A)$ is a tensor equivalence, so we shall apply Proposition 1.3. Therefore we have A-linear maps

$$b_\iota : \mathbb{C} \to \mathcal{F}(\iota) \otimes^A \mathcal{F}(\iota)' \quad \text{and} \quad d_\iota : \mathcal{F}(\iota)' \otimes^A \mathcal{F}(\iota) \to \mathbb{C} \ , \tag{1}$$

where $\otimes^A$ is the tensor product of $\mathrm{Rep}(A)$, forming a duality pair. We define an antiautomorphism $S : A \to A$ by

$$\langle f, S(\eta)_\iota v \rangle := \langle \eta_\iota f, v \rangle \qquad \forall \iota \in I \ , \quad v \in \mathcal{F}(\iota) \ , \quad f \in \mathcal{F}(\iota)' \ , \tag{2}$$

where $\langle\cdot,\cdot\rangle$ denotes the usual duality pairing in Vec. Now, we translate the properties of the pairs (1) into properties of S.

To begin with, there exist unique α, β in A such that b_ι sends 1 into β_ι and $d_\iota = \mathrm{Tr}(\alpha_\iota \cdot)$, where Tr is the usual trace on Vec, for all ι in I. Then, A-linearity of d_ι means that, for all ι,

$$\langle f, S(\eta_{(1)})\alpha\eta_{(2)}v\rangle = \epsilon(\eta)\langle f, \alpha v\rangle \qquad \forall\ v \in \mathcal{F}(\iota)\ ,\quad f \in \mathcal{F}(\iota)'\ ,$$

so $S(\eta_{(1)})\alpha\eta_{(2)} = \epsilon(\eta)\alpha$, for all η in A. Similarly, each η in A acts on β just by multiplication by $\epsilon(\eta)$, hence $\eta_{(1)}\beta S(\eta_{(2)}) = \epsilon(\eta)\beta$ for all η in A.

Finally, let us unravel the equations for the duality pairs (b_ι, d_ι). The first duality identity states that the composition

$$\begin{aligned}\mathcal{F}(\iota) \to \mathbb{C}\otimes\mathcal{F}(\iota) \xrightarrow{b_\iota\otimes\mathrm{id}} (\mathcal{F}(\iota)\otimes\mathcal{F}(\iota)')\otimes\mathcal{F}(\iota)\cdots\cdots \\ \xrightarrow{\Phi}\mathcal{F}(\iota)\otimes(\mathcal{F}(\iota)'\otimes\mathcal{F}(\iota)) \xrightarrow{\mathrm{id}\otimes d_\iota} \mathcal{F}(\iota)\otimes\mathbb{C}\to\mathcal{F}(\iota)\end{aligned}$$

equals $\mathcal{F}(\iota)$; we put $\phi\otimes\varphi\otimes\psi := \Phi$ and $\phi'\otimes\varphi'\otimes\psi' := \Phi^{-1}$ (see the end of 2.1). By tracking a generic v in $\mathcal{F}(\iota)$ through the composition and applying f in $\mathcal{F}(\iota)'$, we get $\langle f, \phi\beta S(\varphi)\alpha\psi v\rangle = \langle f, v\rangle$. So we have $\phi\beta S(\varphi)\alpha\psi = 1$; treating the second duality identity analogously, we get $S(\phi')\alpha\varphi'\beta S(\psi') = 1$ as well.

Summarizing, we have defined a weak antipode on A.

Definition A. A *weak antipode* on a discrete weak quasi-bialgebra (A, Δ, Φ) is a triple (S, α, β) where $S : A \to A$ is an antiautomorphism and α, β are invertible elements in A such that:

- $S(\eta_{(1)})\alpha\eta_{(2)} = \epsilon(\eta)\alpha$, for all η in A;
- $\eta_{(1)}\beta S(\eta_{(2)}) = \epsilon(\eta)\beta$ for all η in A;
- $\phi\beta S(\varphi)\alpha\psi = 1$ and $S(\phi')\alpha\varphi'\beta S(\psi') = 1$, having put $\phi\otimes\varphi\otimes\psi := \Phi$ and $\phi'\otimes\varphi'\otimes\psi' := \Phi^{-1}$ as above.

By uniqueness of the counit, we also have $\epsilon\circ S = S$. A discrete weak quasi-bialgebra admitting a weak antipode is called a *discrete weak quasi-Hopf algebra.*

Remark A. A weak antipode (S, α, β) defines right duals on Rep(A) in the usual way. Namely, given an A-module V, the dual vector space V' becomes an A-module by

$$\langle \eta f, v\rangle := \langle f, S(\eta)v\rangle \qquad \forall\ \eta \in A\ ;$$

moreover if we put $b_V(1) = \beta.$ and $d_V = \mathrm{Tr}(\alpha.\cdot)$, where $\alpha.$ and $\beta.$ denote the actions of α and β on V, we obtain a duality pair. The corresponding right duality functor $\cdot^\vee$ is just the usual transpose in Vec on morphisms.

The verifications retrace the passages we went through to prove the properties of (S, α, β); indeed a weak antipode for (A, Δ, Φ) is exactly equivalent to A-actions on all V' and duality pairs (b_V, d_V) with left object V and right object V' for each A-module V. From this fact and Proposition 1.3 the following simple result readily follows.

Proposition-Definition. Let (S, α, β) be a weak antipode for the weak quasi-Hopf algebra (A, Δ, Φ). For u in A invertible, we put $S_u := uS(\cdot)u^{-1}$, $\alpha_u := u\alpha$ and $\beta_u := \beta u^{-1}$; then (S_u, α_u, β_u) is another weak antipode, and all weak antipodes for (A, Δ, Φ) are of this form.

If $(S, 1, 1)$ is a weak antipode, S is called a *strong antipode*. So (A, Δ, Φ) admits a strong antipode if and only if all weak antipodes are of the form (S_u, u, u^{-1}); we also see that in this case there is just one strong antipode.

Remark B. By (S, α, β) we may define left duals too. This time, given an A-module V, its dual V' becomes an A-module by

$$\langle \eta f, v\rangle := \langle f, S^{-1}(\eta)v\rangle \qquad \forall\, \eta \in A\ ,$$

while $b'_V(1) = S^{-1}(\beta)$ and $d'_V = \mathrm{Tr}\,(S^{-1}(\alpha)\,\cdot\,)$ form the relative duality pair; again, the corresponding left duality functor ${}^\vee\cdot$ is the Vec transpose on morphisms. Moreover up to identifying the vector spaces V and V'', ${}^\vee\cdot$ and $\cdot^\vee$ are inverse to each other, so Remark 1.3B applies.

It is easily checked that the new monoidal structure on $\mathrm{Rep}(A)$ is given by coproduct and associator

$$\check{\Delta} = (S \otimes S) \circ \Delta^{\mathrm{op}} \circ S^{-1}\ , \qquad \check{\Phi} = (S \otimes S \otimes S)\Phi_{321}\ ,$$

where $\Delta^{\mathrm{op}}(\eta) = \Delta(\eta)_{21}$ (see appendix II). Moreover, we have a tensor structure on the identity functor considered from $\big(\mathrm{Rep}(A), \otimes^A, a^A\big)$ to $\big(\mathrm{Rep}(A), \check{\otimes}^A, \check{a}^A\big)$, which is equivalent to a twist f from (Δ, Φ) to $(\check{\Delta}, \check{\Phi})$ (see Lemma 2.5B). We may look at the situation as a weak form of anticomultiplicativity of S, as

$$f\big(S(\eta)_{(1)} \otimes S(\eta)_{(2)}\big)f^{-1} = S(\eta_{(2)}) \otimes S(\eta_{(1)}) \qquad \forall\, \eta \in A\ .$$

Finally the tensor structure on $\mathcal{D}^2$ may be expressed in terms of f:

$$D^2 = f^{-1}(S \otimes S)(f_{21})\ , \tag{3}$$

of course implying the due double duals of Vec, so that $D^2_{V,W} : V^{\vee\vee} \otimes W^{\vee\vee} \to (V \otimes W)^{\vee\vee}$.

***R*-matrix** Suppose that $\mathcal{C}$ has a generalised coboundary c (Definition 1.4A); we define R in $A \otimes A$ by

$$G_{\sigma,\rho}\mathcal{F}(c_{\rho,\sigma})F_{\rho,\sigma} =: \Sigma\big(\mathcal{F}(\rho), \mathcal{F}(\sigma)\big)R_{\rho,\sigma} \qquad \forall\, \rho, \sigma \text{ objects of } \mathcal{C}\ , \tag{4}$$

where $\Sigma(V, W) : V \otimes W \to W \otimes V$ is the flip map. Now c is a generalised coboundary on $\mathcal{C}$ exactly if $G_{\sigma,\rho}\mathcal{F}(c_{\rho,\sigma})F_{\rho,\sigma}$ is one on $\mathrm{Rep}(A)$, since $\mathcal{E}$ is a tensor equivalence. This is in turn equivalent to the fact that R is a twist from (Δ, Φ) to $(\Delta^{\mathrm{op}}, \Phi^{\mathrm{op}})$, where $\Delta^{\mathrm{op}}(\eta) = \Delta(\eta)_{21}$, and $\Phi^{\mathrm{op}} = \Phi^{-1}_{321}$; the compatibility of c with units plus the further condition $c_{\mathbb{1},\mathbb{1}} = 1$ (see 1.4) result in the normalisation conditions $(\epsilon \otimes \mathrm{id})R = 1 = (\mathrm{id} \otimes \epsilon)R$.

As anticipated at the beginning of present section 2, twists are to be treated in 2.5, and introduced in Definition 2.5A. For the time being however, we provide a working definition right away, also considering that another twist occurs, just as naturally as here, in 2.4.

Definition B. Consider (Δ_1, Φ_1) and (Δ_2, Φ_2) both making the discrete algebra A into a weak quasi-bialgebra. Then a *twist* from (Δ_1, Φ_1) to (Δ_2, Φ_2) is a partially invertible (see Appendix I) element U of $A \otimes A$ from $\Delta_1(1)$ to $\Delta_2(1)$ such that

$$U\Delta_1(\cdot) = \Delta_2(\cdot)U \qquad (1 \otimes U)(\mathrm{id} \otimes \Delta_1)(U)\Phi_1 = \Phi_2(\Delta_2 \otimes \mathrm{id})(U)(U \otimes 1) \ .$$

Coming back to the generalised coboundary c, we also note that is a braiding exactly if R further satisfies

$$(\mathrm{id} \otimes \Delta)R = \Phi_{231}^{-1} R_{13} \Phi_{213} R_{12} \Phi_{123}^{-1} \ , \ (\Delta \otimes \mathrm{id})R = \Phi_{312} R_{13} \Phi_{132}^{-1} R_{23} \Phi_{123} \ . \qquad (5)$$

Summarizing the remarks about c, we have the following

Proposition. *The assignment* (4) *defines a bijection between generalised coboundaries on* $\mathcal{C}$ *and twists from* (Δ, Φ) *to* $(\Delta^{\mathrm{op}}, \Phi^{\mathrm{op}})$*; moreover,* c *is a braiding if and only if* R *further satisfies* (5)*.*

We therefore establish the following terminology:

Definition C. An *almost cocommutative structure* on a discrete weak quasi-bialgebra (A, Δ, Φ) is a twist R from (Δ, Φ) to $(\Delta^{\mathrm{op}}, \Phi^{\mathrm{op}})$; if R further satisfies (5), it is called a *quasi-triangular structure*. Accordingly, the quadruple (A, Δ, Φ, R) is referred to as a discrete almost cocommutative, or quasi-triangular, weak quasi-bialgebra.

One could also say, as it is more usual, that given R in $A \otimes A$, the natural isomorphisms $\Sigma(\mathcal{F}(\rho), \mathcal{F}(\sigma))R_{\rho,\sigma}$ amount to a braiding if and only if

$$R\Delta(a)R^{-1} = \Delta^{\mathrm{op}}(a) \qquad \forall\, a \in A \ . \qquad (6)$$

and (5) holds. From this properties one deduces the compatibility of R with both associator and counit (see Definition 2.5A), by Example 1.4. We also observe that if the twist R corresponds to the generalised coboundary c then R_{21}^{-1} corresponds to its reverse.

We now assume to have a fixed braiding c, and the corresponding quasi-triangular structure R.

Lemma. *If* A *admits a (unique by the Proposition-Definition) strong antipode* S *then the Drinfel'd isomorphism* u *and its inverse are related to* $R = r \otimes t$ *and* $R^{-1} = \overline{r} \otimes \overline{t}$ *by*

$$u = S(t)r \ , \qquad u^{-1} = S^{-1}(\overline{t})\overline{r} \ .$$

Proof. This is a straightforward generalisation of the corresponding proof for quasi-triangular Hopf algebras (see for example Proposition 4.2.3 in [CP95]); indeed, even though the defining property of a usual antipode, which is satisfied by our strong antipodes, is applied, coassociativity of the coproduct is not. □

Here of course we implied the double dual in Vec as in (3), so that u in an invertible element of A with

$$S^2(a) = uau^{-1} \qquad \forall\, a \in A \ .$$

With the expression of D^2 in hand, we observe that the identity $c_2 = c^2$ (see Remark 1.4C) may be expressed as it follows

$$R_{21}R\Delta(u) = \Delta(u)R_{21}R = f^{-1}(S \otimes S)(f_{21})f(u \otimes u) \ ,$$

where we added the first equality by just using (6). Let us now turn to the ribbon structures of c.

Definition D. Let (A, Δ, Φ) be a discrete quasi-Hopf algebra, with a weak antipode (S, α, β). Given a quasi-triangular structure R, a *ribbon element* for R is a central invertible element v in A such that

$$R_{21}R = \Delta(v^{-1})v \otimes v \ , \tag{7}$$

$S(v) = v$ and $\epsilon(v) = 1$.

Ribbon elements of A are nothing but ribbon structures of $\mathcal{C}$ seen as elements of A. Indeed, by definition of A, the algebra of natural isomorphisms from $\mathrm{id}_{\mathcal{C}}$ to itself embeds into A as its centre; besides, (7) is the fact that v is balancing (see Definition 1.4B) for $c^2 = c_2$; the other two conditions are respectively compatibility with duality and the normalisation condition $v_{\mathbb{1}} = 1$.

We point out that $\epsilon(v) = 1$ can actually be obtained applying $\epsilon \otimes \mathrm{id}$ to both sides of (7), since $(\epsilon \otimes \mathrm{id})R = 1 = (\mathrm{id} \otimes \epsilon)R$. Furthermore, in the situation of the Lemma, we have $v^2 = uS(u)$; this is achieved by applying $(S \otimes \mathrm{id})$ to both sides of (7) and then multiplying, making use of the strong antipode property of S.

Remark C. Let us look back to the bijection of the second point of Theorem 1.4: balancing structures w correspond to pivotal ones ω, which we can now see as invertible elements of A such that

$$\omega a \omega^{-1} = S^2(a) \ , \qquad \Delta(\omega) = f^{-1}(S \otimes S)(f_{21})\omega \otimes \omega \ .$$

Now, if we take w to be a ribbon element v then ω also verifies

$$S(\omega) = \omega^{-1} \ , \qquad r\omega^{-1}s = s\omega r \ , \tag{8}$$

where $R =: r \otimes s$; conversely, if a pivotal element ω satisfies the identities (8) then $v = u\omega^{-1}$ is ribbon.

This remark is an easy adaptation of [Pan97], where the results are proved in the case of a finite dimensional quasi-triangular Hopf algebra. The author defines a *charmed element* to be an invertible ω such that $\omega a \omega^{-1} = S^2(a)$ holds for all a in the algebra and the further identities (8) are verified; he also reexpresses the second of those as $S(u) = \omega^{-1}u\omega^{-1}$, provided that the first holds.

He then proves that v is ribbon if and only if $\omega = uv^{-1}$ is charmed and group-like. Indeed the pivotal condition $\Delta(\omega) = f^{-1}(S \otimes S)(f_{21})\omega \otimes \omega$ reduces to just "ω group-like" when $f = 1$, as it is for a Hopf algebra.

To sum up, we have the following specialisation of the second point of Theorem 1.4: v is ribbon if and only if $\omega = uv^{-1}$ is charmed and pivotal.

2.4 Unitary weak quasi-bialgebras

We now enrich the constructions of 2.1 to treat the case of a weak quasi-tensor functor $\mathcal{F} : \mathcal{C} \to \text{Hilb}$. Definition 2.1A still applies by just replacing "Vec" with "Hilb" and "tensor category" with "C^* tensor category"; we also expand the normalisation condition $\mathcal{F}(\mathbb{1}) = \mathbb{C}$ implying the trivial scalar product on $\mathbb{C}$.

The algebra We equip the discrete algebra $A = \text{End}(\mathcal{F})$ and its generalised tensor powers with the following involutions:

$$(\eta^\dagger)_{\rho_1,\dots,\rho_n} := \eta^*_{\rho_1,\dots,\rho_n} \qquad \forall\ \rho_1,\dots,\rho_n \text{ objects of } \mathcal{C}\ , \tag{1}$$

where the $\cdot^*$ in the right-hand side is the Hilbert adjoint relative to the $\otimes$ scalar product on $\mathcal{F}(\rho_1) \otimes \cdots \otimes \mathcal{F}(\rho_n)$; antilinearity, antimultiplicativity and $\cdot^{\dagger\dagger} = \text{id}$ are evident. Any choice of a complete collection of mutually non-equivalent simple objects $I = \{\iota\}$ still yields an isomorphism

$$A^{\otimes n} \to \prod_{\iota_1,\dots,\iota_n \in I} \bigotimes_{k=1}^{n} \text{End}\left(\mathcal{F}(\iota_k)\right)$$

as in 2.1. However now the $\mathcal{F}(\iota)$ are Hilbert spaces, so Definition 2.1B needs to be adapted.

Definition A. We say that a unital associative $\mathbb{C}$-algebra A with an involution $\cdot^\dagger$ is a *discrete unitary algebra* if it is $*$-isomorphic to an algebra of the form $M = \prod_{\iota \in I} \text{End}(V_\iota)$, where each V_ι is a Hilbert space of finite dimension. The algebras

$$M^{\underline{\otimes} n} := \prod_{\iota_1,\dots,\iota_n} \bigotimes_{k=1}^{n} \text{End}(V_{i_k})$$

are equipped with the direct product of the tensor products of the involutions relative to the scalar products.

The generalised tensor powers are then introduced just as in Definition 2.1B; we will also drop the underlined symbol $\underline{\otimes}$ for the same reasons.

The coproduct and the associator Given the original weak quasi-tensor structure (F, G), we readily see that (G^*, F^*) is a weak quasi-tensor structure as well, to which we may apply the constructions of 2.1. This way we obtain a new pair $(\tilde{\Delta}, \tilde{\Phi})$, and by the definitions we have

$$\tilde{\Delta} = \Delta(\cdot^\dagger)^\dagger\ , \qquad \tilde{\Phi} = (\Phi^\dagger)^{-1}\ .$$

Moreover $\Omega := F^*F$ is a twist from (Δ, Φ) to $(\tilde{\Delta}, \tilde{\Phi})$, with partial inverse $\Omega^{-1} = GG^*$, so Ω and Ω^{-1} are both positive. We recall that *twists* will be introduced in next subsection, more precisely in Definition 2.5A; the fact that Ω twists (Δ, Φ) to $(\tilde{\Delta}, \tilde{\Phi})$ is a particular case of Remark 2.5.

To sum up, we have defined a discrete unitary weak quasi-bialgebra.

Definition B. A *discrete unitary weak quasi-bialgebra* is a quintuple $(A, \cdot^\dagger, \Delta, \Phi, \Omega)$ such that:

- $(A, \cdot^\dagger)$ is a discrete unitary algebra;
- (A, Δ, Φ) is a discrete weak quasi-bialgebra;
- Ω is a twist from (Δ, Φ) to $(\tilde{\Delta}, \tilde{\Phi})$, where $\tilde{\Delta} = \Delta(\cdot^\dagger)^\dagger$ and $\tilde{\Phi} = (\Phi^\dagger)^{-1}$, positive as an element of $A \otimes A$.

By uniqueness of the counit, we also have $\epsilon(a^\dagger) = \overline{\epsilon(a)}$ for all a in A; indeed, putting $\tilde{\epsilon}(a) := \overline{\epsilon(a^\dagger)}$, we have

$$(\mathrm{id} \otimes \tilde{\epsilon})\Delta(a) = a_{(1)}\overline{\epsilon(a^\dagger_{(2)})} = (\mathrm{id} \otimes \epsilon)\Delta(a^\dagger)^\dagger = (\mathrm{id} \otimes \epsilon)\big(\Omega\Delta(a)\Omega^{-1}\big) = a \ ,$$

since $(\mathrm{id} \otimes \epsilon)\Omega = 1$; $(\tilde{\epsilon} \otimes \mathrm{id})\Delta(a) = a$ is verified analogously, so $\tilde{\epsilon}$ is a counit.

Remark. The condition $\tilde{\Delta} = \Omega\Delta(\cdot)\Omega^{-1}$, may be expanded as

$$(a^\dagger)_{(1)} \otimes (a^\dagger)_{(2)} = \Omega^{-1}(a^\dagger_{(1)} \otimes a^\dagger_{(2)})\Omega \qquad \forall\, a \in A \ , \tag{2}$$

so Δ is a $*$-homomorphism if $\Omega = \Delta(1)$; then we would also have $\Phi^\dagger = \Phi^{-1}$. An apparently weaker situation occurs when Ω is a trivial twist (see Definition 2.5A), namely $\Omega = \Delta(1)^\dagger\Delta(1)$ and $\Omega^{-1} = \Delta(1)\Delta(1)^\dagger$; this is the case exactly if G and F^* are isometries, i.e. G^*G and FF^* are both identical.

However, it is shown in section 2 of [CCP21] that this is in turn equivalent to the apparently stronger condition $F^* = G$, i.e. $\Omega = F^*F = GF = \Delta(1)$, so nothing changed.

Tannakian reconstruction

We proceed to adapt Lemma 2.2 and Theorem 2.2 to the present scenario, and consider the category $\mathrm{Rep}^+(A)$ of C^* A-modules; the generic object is a finite-dimensional A-module V which is also equipped with a complex valued A-invariant scalar product $(\cdot,\cdot)_V$, namely

$$(v, aw)_V = (a^\dagger v, w)_V \qquad \forall\, v, w \in V \ .$$

In other words the objects of $\mathrm{Rep}^+(A)$ are the $*$-representations of A on finite-dimensional Hilbert spaces. Given C^* A-modules V and W, if a map $f : V \to W$ is A-linear so is the Hilbert adjoint $f^* : W \to V$; with this conjugation, $\mathrm{Rep}^+(A)$ is clearly a C^* tensor category.

It is also clear that if we forget the scalar products on the C^* A-modules, we obtain a forgetful functor $\mathcal{O}$ from $\mathrm{Rep}^+(A)$ to $\mathrm{Rep}(A)$, identical on morphisms; we require the tensor product and associator on $\mathrm{Rep}^+(A)$ to make the identity maps of each $\mathcal{O}(V) \otimes^A \mathcal{O}(W)$ into a tensor structure. This means that the tensor products are defined just like in 2.2 as A-modules, and the associator is given by the same maps; hence we still denote them by $\otimes^A$ and a^A respectively.

We are left to define the scalar products on the tensor products. To this aim, given C^* A-modules V and W, we consider the categories $\mathcal{P}(A \otimes A, \cdot^\dagger)$ and $\mathcal{P}\big(\mathrm{End}(V \otimes W), \cdot^\dagger\big)$ (see appendix I), where the latter $\cdot^\dagger$ comes from the usual $\otimes$ scalar product, which we denote by $(\cdot,\cdot)_p$. By considering the action of $A \otimes A$ on

$V \otimes W$ we obtain a $*$-functor from $\mathcal{P}(A \otimes A, \cdot^\dagger)$ to $\mathcal{P}\big(\operatorname{End}(V \otimes W), \cdot^\dagger\big)$, since $(\cdot,\cdot)_V$ and $(\cdot,\cdot)_W$ are A-invariant.

We endow $V \otimes^A W$ with a scalar product by enriching the object $\Delta(1)$ of $\mathcal{P}(A)$ to the object $\big(\Delta(1), \Omega\big)$ of $\mathcal{P}(A, \cdot^\dagger)$; we then apply the $\mathcal{S}$ functor (defined in formula I(3)) to their actions on the full tensor product $V \otimes W$. Indeed, $(\cdot,\cdot)_{V \otimes^A W}$ is A-invariant: for all x, y in $V \otimes^A W$,

$$\begin{gathered}(x, \Delta(a)y)_{V\otimes^A W} = \big(x, \Omega\Delta(a)y\big)_p = \big(\Delta(a)^\dagger \Omega x, y\big)_p = \\ \big(\Omega\Delta(a^\dagger)x, y\big)_p = \big(\Delta(a^\dagger)x, \Omega y\big)_p = \big(\Delta(a^\dagger)x, y\big)_{V\otimes^A W} ,\end{gathered} \tag{3}$$

where we used (2) and the A-invariance of $(\cdot,\cdot)_V$ and $(\cdot,\cdot)_W$; so $V \otimes^A W$ is a C^* A-module as well.

Lemma. *The C^* semi-simple abelian category $\operatorname{Rep}^+(A)$, endowed with the tensor product $\otimes^A$ and the associator a^A, is a C^* tensor category.*

Proof. We need to verify that $\otimes^A$ is a $*$-functor and that the associator maps are unitary. With regard to the fist issue, let us consider morphisms f in (V_1, V_2), g in (W_1, W_2) and denote by Ω_{V_i,W_i} the action of Ω as an element of $A \otimes A$ on $V_i \otimes W_i$ $(i = 1, 2)$; then

$$(f \otimes g)^* = \Omega^{-1}_{V_1,W_1} f^* \otimes g^* \Omega_{V_2,W_2} = f^* \otimes g^* ,$$

by A-linearity of f^* and g^* (see formula 2.2(1) and the end of Remark 2.2).

Turning to the associator, we consider the category $\mathcal{P}(A \otimes A \otimes A, \cdot^\dagger)$; Φ is a partial isomorphism from $\big((\Delta \otimes \mathrm{id})(\Delta 1), (\Omega \otimes 1)(\Delta \otimes \mathrm{id})\Omega\big)$ to $\big((\mathrm{id} \otimes \Delta)(\Delta 1), (1 \otimes \Omega)(\mathrm{id} \otimes \Delta)\Omega\big)$, so

$$\begin{gathered}\Phi^* = (\Delta \otimes \mathrm{id})\Omega^{-1}(\Omega^{-1} \otimes 1)\Phi^\dagger(1 \otimes \Omega)(\mathrm{id} \otimes \Delta)\Omega = \\ (\Omega^{-1} \otimes 1)(\tilde{\Delta} \otimes \mathrm{id})\Omega^{-1}\Phi^\dagger(\mathrm{id} \otimes \tilde{\Delta})\Omega(1 \otimes \Omega) = \\ \big((1 \otimes \Omega)(\mathrm{id} \otimes \Delta)\Omega\Phi(\Delta \otimes \mathrm{id})\Omega^{-1}(\Omega^{-1} \otimes 1)\big)^\dagger = \big((\Phi^\dagger)^{-1}\big)^\dagger = \Phi^{-1} ,\end{gathered}$$

where we used (2) in the first passage, self-adjointness of Ω in the second and compatibility of the twist Ω with associators in the third. The desired unitarity is obtained by letting both sides act on a generic full tensor product $U \otimes V \otimes W$ and then applying the $*$-functor $\mathcal{S}$ relative to $U \otimes V \otimes W$ with the $\otimes$ scalar product. □

Theorem. *Let $(\mathcal{C}, \cdot^*, \otimes, a)$ be a C^* tensor category and $\mathcal{F} : \mathcal{C} \to \mathrm{Hilb}$ a faithful $*$-functor with weak quasi-tensor structure (F, G); we also consider the discrete unitary weak quasi-bialgebra $(A, \cdot^\dagger, \Delta, \Phi, \Omega)$ constructed above.*

Then there exists an equivalence of C^* tensor categories $\mathcal{E} : \mathcal{C} \to \operatorname{Rep}^+(A)$ such that $\mathcal{F} = \mathcal{F}_A \circ \mathcal{E}$, where $\mathcal{F}_A : \operatorname{Rep}^+(A) \to \mathrm{Hilb}$ is the forgetful functor.

Conversely, let A' be a discrete unitary weak quasi-bialgebra and $\mathcal{E}' : \mathcal{C} \to \operatorname{Rep}^+(A')$ an equivalence of C^* tensor categories such that $\mathcal{F}$ and $\mathcal{F}_{A'} \circ \mathcal{E}'$ are unitarily tensor isomorphic (see Definitions 1.5B and 2.1A). Then A and A' are isomorphic.

Proof. We upgrade each Hilbert space $\mathcal{F}(\rho)$ to an A-module as in the proof of Theorem 2.2; the scalar product on $\mathcal{F}(\rho)$ is A-invariant by the defining formula (1), hence it makes $\mathcal{F}(\rho)$ into a C^* A-module $\mathcal{E}(\rho)$.

Then, again, by defining $\mathcal{E}$ to coincide with $\mathcal{F}$ on morphisms we obtain a functor with $\mathcal{F} = \mathcal{F}_A \circ \mathcal{E}$; and $\mathcal{E}$ is actually a *-functor because $\mathcal{F}$ is.

The tensor structure E of $\mathcal{E}$ is defined as in the proof of Theorem 2.2; but now each map $E_{\rho,\sigma}$ is unitary, since $\Omega^{-1}F^* = G$ by definition of Ω. However the verification that $\mathcal{E}$ is an equivalence of C^* categories proceeds otherwise analogously, replacing the theory of tensor categories with that of C^* tensor categories.

We content ourselves with making explicit the properties of $\mathcal{E}$ for the sake of clarity: the tensor structure of $\mathcal{E}$ is unitary; furthermore there exists a "quasi-inverse" $\tilde{\mathcal{E}} : \mathrm{Rep}^+(A) \to \mathcal{C}$ which is also a $*$-functor with unitary tensor structure, such that $\mathrm{id}_\mathcal{C}$ and $\tilde{\mathcal{E}}\mathcal{E}$ are unitarily monoidally isomorphic (see Definitions 1.2D and 1.5B) and so are $\mathrm{id}_{\mathrm{Rep}^+(A)}$ and $\mathcal{E}\tilde{\mathcal{E}}$.

The proof of the final assertion also proceeds analogously to the case of Theorem 2.2; the only new feature is that since every isomorphism u_ρ is unitary, U is a $*$-isomorphism too. □

2.5 Twists

Throughout present subsection $\mathcal{C}$ will be a C^* tensor category. Rather than considering weak quasi-tensor $*$-functors from $\mathcal{C}$ to Hilb one at a time, we will be mainly concerned with the relation between their isomorphism classes. We recall that the Grothendieck ring of $\mathcal{C}$ is the associative ring $\mathrm{Gr}(\mathcal{C})$ of isomorphism classes of objects of $\mathcal{C}$;

$$[\rho] + [\sigma] = [\rho + \sigma]\ , \qquad [\rho] \cdot [\sigma] = [\rho \otimes \sigma]\ ,$$

where $[\rho]$ denotes the isomorphism class of the object ρ.

The following simple proposition provides a complete linear invariant; here and in the sequel "linear" is just a shortcut for "not necessarily unitary" (remember that we generally assume all functors to be $\mathbb{C}$-linear).

Proposition A. *Given a functor $\mathcal{F} : \mathcal{C} \to \mathrm{Vec}$, we introduce its dimension function*

$$d : \mathrm{Gr}(\mathcal{C}) \to \mathbb{N} \qquad d([\rho]) = \dim\left(\mathcal{F}(\rho)\right)\ .$$

Then $\mathcal{F}$ is faithful if and only if d vanishes nowhere.

Furthermore, consider weak quasi-tensor $*$-functors $\mathcal{F}_1, \mathcal{F}_2 : \mathcal{C} \to \mathrm{Hilb}$ and their dimension functions d_1, d_2; then $\mathcal{F}_1$ and $\mathcal{F}_2$ are linearly isomorphic if and only if $d_1 = d_2$.

Proof. By semi-simplicity, f is faithful exactly if $\mathcal{F}(\rho)$ is not the null vector space for all ρ simple object of $\mathcal{C}$, which is in turn equivalent to d vanishing nowhere.

Moving on, suppose $d_1 = d_2$, and consider a complete collection of mutually non-equivalent simple objects $I = \{\iota\}$. Since $\dim\left(\mathcal{F}_1(\iota)\right) = \dim\left(\mathcal{F}_2(\iota)\right)$, we may choose a linear isomorphism $\eta_\iota : \mathcal{F}_1(\iota) \to \mathcal{F}_2(\iota)$, for each ι; the unique natural transformation η taking the values η_ι on I is then a linear isomorphism η from $\mathcal{F}_1$ to $\mathcal{F}_2$. The other direction is trivial. □

Now, let us denote by $\mathcal{V}$ the forgetful functor from Hilb to Vec. Two weak quasi-tensor $*$-functors $\mathcal{F}_1, \mathcal{F}_2$ may well have the same dimension function, or even satisfy $\mathcal{V} \circ \mathcal{F}_1 = \mathcal{V} \circ \mathcal{F}_2$, without being unitarily isomorphic. More explicitly, a given a

$*$-functor $\mathcal{F}$, may be modified by perturbing the scalar product $(\cdot,\cdot)$ on the generic $\mathcal{F}(\rho)$ by means of any natural transformation t, positive and invertible: we define new scalar products on each $\mathcal{F}(\rho)$ by

$$(v,w)_t := (v, t_\rho w) \qquad \forall\ v,w \in \mathcal{F}(\rho)\ ;$$

we also need $t_{\mathbb{1}} = 1$ to grant the normalisation condition (see the beginning of 2.4). If this is the case, the above modification defines a new weak quasi-tensor $*$-functor $\mathcal{F}_t$, coinciding with $\mathcal{F}$ on morphisms; $\mathcal{F}_t$ is still a $*$-functor by naturality of t. This is actually all that can happen up to unitary isomorphism.

Proposition B. *Let $\mathcal{F} : \mathcal{C} \to \mathrm{Hilb}$ be a weak quasi-tensor $*$-functor, with dimension function d and consider the discrete $*$-algebra $(A, \cdot^\dagger)$ corresponding to $\mathcal{F}$ (see the beginning of 2.4). We introduce the following equivalence relation on the set $A_1^+ := \{t \in A$ positive definite with $\epsilon(t) = 1\}$:*

$$t_1 \sim t_2 \qquad \textit{if} \qquad t_1 = u^\dagger t_2 u$$

for some invertible u in A with $\epsilon(u) = 1$. Then the assignment $t \mapsto \mathcal{F}_t$ defines a bijection from the quotient $A_1^+/\sim$ to the set of unitary isomorphism classes of weak quasi-tensor $*$-functors from $\mathcal{C}$ to Hilb *with dimension function d.*

Proof. To begin with, we note that $t_1 = u^\dagger t_2 u$ amounts exactly to "u is a unitary isomorphism from $\mathcal{F}_{t_1}$ to $\mathcal{F}_{t_2}$"; and any such isomorphism may be assumed to satisfy $\epsilon(u) = 1$ by rescaling if needed. So all we have to prove is that any weak quasi-tensor $*$-functor $\mathcal{G} : \mathcal{C} \to \mathrm{Hilb}$ with dimension function d is unitarily isomorphic to some $\mathcal{F}_t$.

By Proposition A, we have a linear isomorphism v from $\mathcal{F}$ to $\mathcal{G}$ (we may take $\epsilon(v) = 1$), and we may consider the unique $*$-functor $\tilde{\mathcal{F}}$ such that $\mathcal{V}\circ\tilde{\mathcal{F}} = \mathcal{V}\circ\mathcal{F}$ which also upgrades v to a unitary isomorphism: we just pull back the scalar products on the generic object $\mathcal{G}(\rho)$ by v.

Now, let us say that $\mathcal{F}(\rho)$ is the vector space $\mathcal{V}(\mathcal{F}(\rho))$ with scalar product $(\cdot,\cdot)$; the scalar product relative to $\tilde{\mathcal{F}}(\rho)$ is of the form $(\cdot, t_\rho \cdot)$ for a unique positive definite t_ρ in $\mathrm{End}\big(\mathcal{F}(\rho)\big)$, and $t_{\mathbb{1}} = 1$ by normalisation of the functors involved. Moreover the t_ρs are natural in ρ by the fact that $\tilde{\mathcal{F}}$ is a $*$-functor. So they define t in A_1^+ and $\mathcal{F}_t = \tilde{\mathcal{F}}$ which is unitarily isomorphic to $\mathcal{G}$. □

Lemma A. *Let $\mathcal{F} : \mathcal{C} \to \mathrm{Hilb}$ be a weak quasi-tensor $*$-functor, and consider the discrete unitary weak quasi-bialgebra $(A, \cdot^\dagger, \Delta, \Phi, \Omega)$ constructed as in 2.4. Then the discrete unitary weak quasi-bialgebra corresponding to $\mathcal{F}_t$ is $(A, \cdot^{\dagger_t}, \Delta, \Phi, \Omega_t)$, with $\cdot^{\dagger_t} = t^{-1} \cdot^\dagger t$ and $\Omega_t = (t^{-1} \otimes t^{-1})\Omega\Delta(t)$.*

Furthermore, if we send the generic C^* A-module into the vector space with the same A-action, but with scalar product perturbed by the action of t, we obtain a tensor $*$-isomorphism

$$\mathcal{E}_t : \mathrm{Rep}^+(A, \cdot^\dagger, \Delta, \Phi, \Omega) \to \mathrm{Rep}^+(A, \cdot^{\dagger_t}, \Delta, \Phi, \Omega_t)$$

identical on morphisms, with tensor structure given by the identical maps.

Finally, we have the commutative diagram

$$\begin{array}{ccc} \mathrm{Rep}^+(A, \cdot^\dagger, \Delta, \Phi, \Omega) & \xrightarrow{\mathcal{E}_t} & \mathrm{Rep}^+(A, \cdot^{\dagger_t}, \Delta, \Phi, \Omega_t) \\ & {\scriptstyle (\mathcal{F}_{A,\cdot^\dagger,\Delta,\Phi,\Omega})_t} \searrow \quad \swarrow {\scriptstyle \mathcal{F}_{A,\cdot^{\dagger_t},\Delta,\Phi,\Omega_t}} & \\ & \mathrm{Hilb} & \end{array} \ ,$$

where $\mathcal{F}_{A,\cdot^\dagger,\Delta,\Phi,\Omega}$ *and* $\mathcal{F}_{A,\cdot^{\dagger_t},\Delta,\Phi,\Omega_t}$ *denote the suitable forgetful functors. The equality holds also with respect to the weak quasi-tensor structures involved.*

Proof. The first assertion is easily verified: the coproduct and the associator depend only on (F, G), whereas $\cdot^{\dagger_t}$ and Ω_t have to be computed using the Hilbert adjoints relative to the perturbed scalar products, yielding the stated formulas.

Moving on, by naturality of t,

$$\mathcal{E}_t(f)^* = t_V^{-1} f^* t_W = f^* = \mathcal{E}_t(f^*) \qquad \forall\ f \in (V, W)\ ,$$

where t_V and t_W are the actions of t on V and W, so $\mathcal{E}_t$ is a $*$-functor. Furthermore, the scalar products on $\mathcal{E}_t(V) \otimes \mathcal{E}_t(W)$ and $\mathcal{E}_t(V \otimes W)$ coincide:

$$(t \otimes t)\Omega_t = (t^{-1} \otimes t^{-1})(t \otimes t)\Omega\Delta(t) = \Omega\Delta(t)\ ,$$

hence the identical maps are a legitimate tensor structure on $\mathcal{E}_t$.

The final assertion is now self-evident; it may be taken as an example of the situation of Theorem 2.4. □

Proposition B refines Proposition A by considering unitary isomorphisms rather than just linear ones, but we are still overlooking the weak quasi-tensor structures. The following refining step is achieved by a modification analogous to "$\mathcal{F} \to \mathcal{F}_t$", where the role of t will be played by certain elements in $A \otimes A$ called "twists", the most relevant algebraic objects in the thesis. In order to introduce them, we briefly divert from the refining progression just outlined.

The following definition relies on the basic notion of partial isomorphisms in an associative algebra, already encountered in quite a few cases. They are separately treated in Appendix I, where their immediate definition is followed by a simple discussion about categories of idempotents of associative algebras.

Definition A. Let (A, Δ_i, Φ_i) be a discrete weak quasi-bialgebra, for $i = 1, 2$. A *twist* of A from (Δ_1, Φ_1) to (Δ_2, Φ_2) is a partially invertible element U of $A \otimes A$ with $(\epsilon \otimes \mathrm{id})U = 1 = (\mathrm{id} \otimes \epsilon)U$ and the following further properties:

- $U^{-1}U = \Delta_1(1)$, $UU^{-1} = \Delta_2(1)$;
- $U\Delta_1(a) = \Delta_2(a)U$ for all a in A;
- $(1 \otimes U)(\mathrm{id} \otimes \Delta_1)(U)\Phi_1 = \Phi_2(\Delta_2 \otimes \mathrm{id})(U)(U \otimes 1)$.

If $U = \Delta_2(1)\Delta_1(1)$ and $U^{-1} = \Delta_1(1)\Delta_2(1)$ then U is called a *trivial twist.* In this case the second and the third point simplify to:

- $\Delta_2(1)\Delta_1(a) = \Delta_2(a)\Delta_1(1)$ for all a in A;

- $(\mathrm{id} \otimes \Delta_2)(\Delta_2 1)\Phi_1 = \Phi_2(\Delta_1 \otimes \mathrm{id})(\Delta_1 1)$.

It is clear that the twists of a discrete algebra form a groupoid; however, if we are given a certain discrete weak quasi-bialgebra (A, Δ, Φ) it is most immediate to focus on twists from the known pair (Δ, Φ). To this regard we observe that if U is a partially invertible element of $A \otimes A$ with domain $\Delta(1)$ and $(\epsilon \otimes \mathrm{id})U = 1 = (\mathrm{id} \otimes \epsilon)U$, then if we put

$$\Delta_U = U\Delta(\cdot)U^{-1} \ , \quad \Phi_U = (1 \otimes U)(\mathrm{id} \otimes \Delta)(U)\Phi(\Delta \otimes \mathrm{id})(U^{-1})(U^{-1} \otimes 1)$$

the new triple (A, Δ_U, Φ_U) is still a discrete weak quasi-bialgebra and U is a twist from (Δ, Φ) to (Δ_U, Φ_U).

As an example, if u is an invertible in A with $\epsilon(u) = 1$ then $\delta(u) := (u \otimes u)\Delta(u^{-1})$ is a twist from (Δ, Φ): its codomain is $(u \otimes u)\Delta(1)(u^{-1} \otimes u^{-1})$, the partial inverse is $\Delta(u)(u^{-1} \otimes u^{-1})$ and

$$(\epsilon \otimes \mathrm{id})\delta(u) = (\mathrm{id} \otimes \epsilon)\delta(u) = \epsilon(u)uu^{-1} = 1 \ .$$

Definition B. Let (A, Δ, Φ) be a discrete weak quasi-bialgebra. A twist of A from (Δ, Φ) of the form $\delta(u)$ is called a *coboundary*.

Two twists U_1, U_2 of A from (Δ, Φ) are said to be *cohomologous* if

$$U_2 = (u \otimes u)U_1\Delta(u^{-1})$$

for some invertible u in A with $\epsilon(u) = 1$.

Equivalently, $U_2 = (u \otimes u)\Delta_{U_1}(u^{-1})U_1$, so U_2 is cohomologous to U_1 exactly if it can be obtained by multiplying U_1 by a coboundary on the left side in the groupoid of twists. We may also write $U_2 = (u \otimes u)U_1(u^{-1} \otimes u^{-1})(u \otimes u)\Delta(u^{-1})$, and $(u \otimes u)U_1(u^{-1} \otimes u^{-1})$ is checked to be a twist from $(\Delta_{\delta(u)}, \Phi_{\delta(u)})$.

Remark. Let $\mathcal{F} : \mathcal{C} \to \mathrm{Vec}$ be a faithful functor, and (F, G), (F', G') two weak quasi-tensor structures on it; we write (Δ, Φ) and (Δ', Φ') for the coproduct and the associator that they define on $\mathrm{End}(\mathcal{F})$. We note that $U := G'F$ is a twist from (Δ, Φ) to (Δ', Φ'), with inverse GF'; so we also have $(F', G') = (FU^{-1}, UG)$.

We are ready to set about refining Proposition B, as anticipated. For the rest of the subsection $\mathcal{F} : \mathcal{C} \to \mathrm{Hilb}$ will be a faithful $*$-functor with weak quasi-tensor structure (F, G), and d its dimension function; the relative discrete unitary weak quasi-bialgebra will be denoted by $(A, \cdot^\dagger, \Delta, \Phi, \Omega)$.

$T(A, \Delta, \Phi)$ will be the set of twists of A from (Δ, Φ) and A_1^+ will be as in Proposition B. For all twist U from (Δ, Φ), we write $(F_U, G_U) := (FU^{-1}, UG)$, which clearly is still a weak quasi-tensor structure on $\mathcal{F}$.

Proposition C. *Consider the set* $A_1^+ \times T(A, \Delta, \Phi)$ *and the following equivalence relation on it:* $(t_1, U_1) \sim (t_2, U_2)$ *if there is an invertible* u *in* A *with* $\epsilon(u) = 1$ *such that*

$$t_1 = u^\dagger t_2 u \quad \textit{and} \quad U_2 = (u \otimes u)U_1\Delta(u^{-1}) \ .$$

Then the assignment $(t, U) \mapsto (\mathcal{F}_t, (F_U, G_U))$ *defines a bijection from the quotient* $(A_1^+ \times T(A, \Delta, \Phi))/\sim$ *to the set of unitary tensor isomorphism classes of weak quasi-tensor* $*$*-functors from* $\mathcal{C}$ *to* Hilb *with dimension function* d*.*

Proof. It retraces for the most part the proof of Proposition B. The only new fact to be shown is that, for all t in A_1^+, any weak quasi-tensor structure (H, I) on $\mathcal{F}_t$ is of the form (F_U, G_U) for some twist U from (Δ, Φ). This follows form the Remark. $\square$

Lemma B. *For every twist U of A from (Δ, Φ), the discrete unitary weak quasi-bialgebra corresponding to $\mathcal{F}$ with tensor structure (F_U, G_U) is $(A, \cdot^\dagger, \Delta_U, \Phi_U, \Omega_U)$, with $\Omega_U = (U^\dagger)^{-1}\Omega U^{-1}$.*

Furthermore the action of U^{-1} on the generic $\mathcal{F}(\rho) \otimes \mathcal{F}(\sigma)$ is a tensor structure on the identical functor from $\mathrm{Rep}^+(A, \cdot^\dagger, \Delta, \Phi, \Omega)$ to $\mathrm{Rep}^+(A, \cdot^\dagger, \Delta_U, \Phi_U, \Omega_U)$, which thus becomes a tensor $*$-isomorphism

$$\mathcal{E}_U : \mathrm{Rep}^+(A, \cdot^\dagger, \Delta, \Phi, \Omega) \to \mathrm{Rep}^+(A, \cdot^\dagger, \Delta_U, \Phi_U, \Omega_U) \ .$$

Finally, with respect to the weak quasi-tensor structures involved, we have the commutative diagram

$$\begin{array}{ccc} \mathrm{Rep}^+(A, \cdot^\dagger, \Delta, \Phi, \Omega) & \xrightarrow{\ \mathcal{E}_U\ } & \mathrm{Rep}^+(A, \cdot^\dagger, \Delta_U, \Phi_U, \Omega_U) \\ & {\scriptstyle (\mathcal{F}_{A,\cdot^\dagger,\Delta,\Phi,\Omega})_U} \searrow \quad \swarrow {\scriptstyle \mathcal{F}_{A,\cdot^\dagger,\Delta_U,\Phi_U,\Omega_U}} & \\ & \mathrm{Hilb} & \end{array} \ ,$$

where $\mathcal{F}_{A,\cdot^\dagger,\Delta,\Phi,\Omega}$ and $\mathcal{F}_{A,\cdot^\dagger,\Delta_U,\Phi_U,\Omega_U}$ denote the suitable forgetful functors; the subscript U on the former specifies that its weak quasi-tensor structure is twisted by U, whereas the latter is taken with its standard weak quasi-tensor structure.

Proof. As in the case of Lemma A, the first assertion in verified going through the construction of subsections 2.1 and 2.4; twisting the weak quasi-tensor structure does not affect the involution $\cdot^\dagger$.

Moving on, U^{-1} is a partial isomorphism from $\Delta_U(1)$ to $\Delta(1)$, hence, implying the functor $\mathcal{S}$ of formula I(3) relative to $V \otimes W$ on the action of U^{-1} on $V \otimes W$, it defines an isomorphism $(E_U)_{V,W} : V \otimes_{\Delta_U} W \to V \otimes_\Delta W$, for all A-modules V, W. Now, the second point of Definition A for U is exactly A-linearity of all $(E_U)_{V,W}$; the third one is compatibility of E_U with the associators (see the diagram in Definition 1.2C); finally the normalisation condition $(\epsilon \otimes \mathrm{id})U = 1 = (\mathrm{id} \otimes \epsilon)U$ is compatibility of E_U with units.

As in the case of Lemma A, the last assertion is self-evident, and it provides an example of the situation of Theorem 2.4. $\square$

The modifications $\mathcal{F} \to \mathcal{F}_t$ and $(F, G) \to (F_U, G_U)$, to which we will refer simply as "twistings" by t and U respectively, are obviously independent. Hence the corresponding twistings of the relative discrete unitary weak quasi-tensor bialgebras commute with each other; in particular $(\Omega_t)_U = (\Omega_U)_t$ (the analogous identities for $\cdot^\dagger$ and (Δ, Φ) are obvious since both are affected by just one of the twistings).

This could also be easily checked just in terms of unitary discrete weak quasi-bialgebras. In fact, even though we preferred to focus on discrete algebras in view of their relevance in Tannakian reconstruction, all the structure introduced makes just as much sense in terms of usual algebras and may be treated without reference to category theory; for instance, a unitary weak quasi-bialgebra $(A, \cdot^\dagger, \Delta, \Phi, \Omega)$ may be twisted to new ones using the quintuples in Lemmas A and B as definitions.

Coming back to the commutativity of the two twistings, Definitions A and B could have been applied to the simpler case of a weak quasi-tensor functor from a tensor category, just like we made no reference to weak quasi-tensor structures in Proposition B; in its place we would have proved

Proposition D. *Let $\mathcal{F} : \mathcal{C} \to \mathrm{Vec}$ a faithful functor with weak quasi-tensor structure (F, G), say with dimension function d, and consider the relative discrete weak quasi-bialgebra (A, Δ, Φ). We denote by $T(A, \Delta, \Phi)$ the set of twists of A from (Δ, Φ) and the cohomologousness relation by $\sim$.*

Then the assignment $U \mapsto (\mathcal{F}, (F_U, G_U))$ defines a bijection from the quotient $T(A, \Delta, \Phi)/\!\sim$ to the set of tensor isomorphism classes of weak quasi-tensor functors from $\mathcal{C}$ to Vec with dimension function d.

However, we would have then still come to Proposition C passing from linear to $*$ weak quasi-tensor functors.

2.6 More structure on $(A, \dagger, \Delta, \Phi, \Omega)$

Let us return to the scenario of Theorem 2.4. Even though we may just apply the arguments of 2.3 to obtain weak antipodes and/or quasi-triangular structures on (A, Δ, Φ), some adaptations are in order in view of the further structure given by $(\cdot^\dagger, \Omega)$. We will come to compatibility conditions suitable for the whole quintuple $(A, \cdot^\dagger, \Delta, \Phi, \Omega)$.

Antipodes We refer to the homonymous paragraph in 2.3. The condition on dimensions $\dim \mathcal{F}(\rho^\vee) = \dim \mathcal{F}(\rho)$ is still needed, but here we replace $\mathcal{F}(\iota)'$ with the conjugate vector space $\overline{\mathcal{F}(\iota)}$, inheriting the scalar product of $\mathcal{F}(\iota)$, and we take the isomorphisms U_ι to be unitary, so that the scalar product of $\overline{\mathcal{F}(\iota)}$ is A-linear.

Then we translate the defining formula 2.3(2) to conjugate spaces using the usual isomorphisms between dual and conjugate Hilbert spaces. So, denoting by $\phi : A \to \mathrm{End}(V)$ the homomorphism for a generic A-module V and by ϕ^c the one for the right dual $\overline{V}$, we have

$$\phi^c(\eta) = \overline{\phi(S\eta)^*} \; ; \tag{1}$$

given a $\mathbb{C}$-linear map T, $\overline{T}$ is the same map between the conjugate vector spaces. Now, since ϕ and ϕ^c are both $*$-homomorphisms,

$$\phi(S(\eta^\dagger)) = \overline{\phi^c(\eta^\dagger)^*} = \overline{\phi^c(\eta)} = \phi(S\eta)^* = \phi((S\eta)^\dagger) \; ,$$

so the weak antipode (S, α, β) has the additional property $S(\cdot^\dagger) = (S\cdot)^\dagger$.

Before dealing with compatibility with Ω, we need the following more general fact about twists and weak antipodes.

Proposition A. *Let (A, Δ, Φ) a discrete weak quasi-Hopf algebra and (S, α, β) a weak antipode. Given a twist U from (Δ, Φ), the twisted algebra (A, Δ_U, Φ_U) admits the weak antipode (S, α_U, β_U), where*

$$\alpha_U = S(\overline{u})\alpha\overline{v} \quad \text{and} \quad \beta_U = u\beta S(v) \; ,$$

having put $U = u \otimes v$ and $U^{-1} = \overline{u} \otimes \overline{v}$ (see the end of 2.1).

Proof. This can be checked directly. Alternatively one can take into account the first assertion of Lemma 2.5B (dropping the unitary structure) and apply the construction of 2.3 to compute an antipode for the twisted algebra. □

Coming back to our quintuple $(A, \cdot^\dagger, \Delta, \Phi, \Omega)$ we note that the the weak quasi-bialgebra $(A, \tilde{\Delta}, \tilde{\Phi})$ (introduced just before Definition 2.4B) admits the weak antipode $(\tilde{S}, \tilde{\alpha}, \tilde{\beta})$ with

$$\tilde{S} = S^{-1}(\cdot^\dagger)^\dagger \ , \quad \tilde{\alpha} = S^{-1}(\beta)^\dagger \ , \quad \tilde{\beta} = S^{-1}(\alpha)^\dagger \ ;$$

this is obtained applying † and S^{-1} to the three points of Definition 2.3A. On the other hand we have the weak antipode $(S, \alpha_\Omega, \beta_\Omega)$ yielded by Proposition A; so Proposition-Definition 2.3 provides us with a unique invertible ω in A such that

$$S = \omega S^{-1}(\cdot^\dagger)^\dagger \omega^{-1} \ , \quad \alpha_\Omega = \omega S^{-1}(\beta)^\dagger \ , \quad \beta_\Omega = S^{-1}(\alpha)^\dagger \omega^{-1} \ .$$

R-matrix Like in the case of weak antipodes, we preliminarily state a basic fact about twists in presence of almost cocommutative structures (see Definition 2.3C).

Proposition-Definition. Let $(A, \Delta_i, \Phi_i, R_i)$ be a discrete almost cocommutative weak quasi-bialgebra, for $i = 1, 2$.

Then, for all twist U from (Δ_1, Φ_1) to (Δ_2, Φ_2), $(R_1)_U := U_{21} R_1 U^{-1} = R_2$ is an almost cocommutative structure on (A_2, Δ_2, Φ_2), quasi-triangular exactly if so is R_1 for (A_1, Δ_1, Φ_1). If $(R_1)_U = R_2$ holds, U is said to be a *cocommutative twist* from (Δ_1, Φ_1, R_1) to (Δ_2, Φ_2, R_2).

If U is a cocommutative twist from (Δ_1, Φ_1, R_1) to (Δ_2, Φ_2, R_2) and there is another cocommutative twist V from (Δ_2, Φ_2, R_2) to (Δ_3, Φ_3, R_3) then VU is a cocommutative twist from (Δ_1, Φ_1, R_1) to (Δ_3, Φ_3, R_3).

Returning to our discrete unitary quasi-bialgebra $(A, \cdot^\dagger, \Delta, \Phi, \Omega)$, if the triple (A, Δ, Φ) possesses an almost cocommutative structure, we readily obtain one for $(A, \tilde{\Delta}, \tilde{\Phi})$, namely $\tilde{R} = (R^\dagger)^{-1}$, and if R is quasi-triangular, so is $\tilde{R}$. The comparison between $\tilde{R}$ and R_Ω, yields the following useful result.

Lemma. *In, the scenario of Theorem 2.4, suppose $\mathcal{C}$ possesses a generalised coboundary c, with corresponding almost cocommutative structure R. Then c is unitary if and only if Ω is a cocommutative twist from (Δ, Φ, R) to $(\tilde{\Delta}, \tilde{\Phi}, \tilde{R})$, i.e. $R_\Omega = \tilde{R}$.*

Proof. By definition of R (see formula 2.3(4)) and since $\mathcal{E}$ is an equivalence of C^* tensor categories, c is unitary exactly if

$$\Sigma(\mathcal{F}(\rho), \mathcal{F}(\sigma)) R_{\rho,\sigma} : \mathcal{E}(\rho) \otimes^A \mathcal{E}(\sigma) \to \mathcal{E}(\sigma) \otimes^A \mathcal{E}(\rho)$$

is for all ρ, σ objects of $\mathcal{C}$, where $\otimes^A$ denotes the tensor product of $\mathrm{Rep}^+(A)$. Now, keeping in mind the category $\mathcal{P}\Big(\mathrm{End}\left(\mathcal{F}(\rho) \otimes \mathcal{F}(\sigma)\right)\Big)$, with the tensor product of the involutions coming from the scalar products of $\mathcal{F}(\rho)$ and $\mathcal{F}(\sigma)$, we compute

$$\Big(\Sigma(\mathcal{F}(\rho), \mathcal{F}(\sigma)) R_{\rho,\sigma}\Big)^* = \Omega^{-1}_{\rho,\sigma} R^\dagger_{\rho,\sigma} \Sigma(\mathcal{F}(\sigma), \mathcal{F}(\rho)) \Omega_{\sigma,\rho} =$$
$$(\Omega^{-1} R^\dagger \Omega_{21})_{\rho,\sigma} \Sigma(\mathcal{F}(\sigma), \mathcal{F}(\rho)) = (R^\dagger_\Omega)_{\rho,\sigma} \Sigma(\mathcal{F}(\sigma), \mathcal{F}(\rho)) \ ,$$

having used that the flip map Σ is of course unitary with respect to the mentioned tensor product involution. On the other hand

$$\Big(\Sigma(\mathcal{F}(\rho),\mathcal{F}(\sigma))R_{\rho,\sigma}\Big)^{-1} = R^{-1}_{\rho,\sigma}\Sigma(\mathcal{F}(\sigma),\mathcal{F}(\rho)) \ .$$

Therefore c is unitary exactly if $R^\dagger_\Omega = R^{-1}$, i.e. $R_\Omega = \tilde{R}$. □

2.7 Unitary coboundary weak quasi-Hopf algebras

Some discrete quasi-triangular quasi-Hopf algebras, notably the ones arising from quantum groups at roots of 1 to be treated in next chapter, possess an involution featuring a special kind of compatibility with the quasi-triangular structure. As we shall see, this circumstance, together with the existence of a certain root of a ribbon element, provides such algebras with an intrinsic structure of unitary weak quasi-Hopf algebras and their representation category is C^* ribbon (see Definition 1.5C).

Definition. A *discrete unitary coboundary weak quasi-Hopf algebra* is a sextuple $(A, \cdot^\dagger, \Delta, \Phi, R, w)$ where:

- $(A, \cdot^\dagger)$ is a discrete unitary algebra;
- (A, Δ, Φ) is a discrete weak quasi-Hopf algebra, with a weak antipode (S, α, β) such that $S(\cdot^\dagger) = S(\cdot)^\dagger$;
- R is a quasi-triangular structure for (A, Δ, Φ) and there is a trivial twist E from $(\Delta^{\mathrm{op}}, \Phi^{\mathrm{op}})$ to $(\tilde{\Delta}, \tilde{\Phi})$ such that $(R^\dagger)^{-1} = E_{21}R_{21}E^{-1}$;
- w is a central unitary element of A, with $\epsilon(w) = 1$ and $S(w) = w$; moreover $v := w^2$ is a ribbon element for R. Finally, writing $T_w := \Delta(w)(w^{-1} \otimes w^{-1})$, $\Omega_w := ERT_w$ is positive.

We point out that the condition on S in the second point is actually free, since an antipode commuting with $\cdot^\dagger$ may be obtained by the argument presented in 2.6, which does not rely on Ω. Given (S, α, β) with $S(\cdot^\dagger) = S(\cdot)^\dagger$, all other antipodes satisfying the same condition are of the form

$$(xS(\cdot)x^\dagger, x\alpha, \beta x^\dagger) \ , \qquad \text{for } x \in A \text{ unitary.}$$

Proposition. *The quintuple $(A, \cdot^\dagger, \Delta, \Phi, \Omega_w)$ is a discrete unitary weak quasi-Hopf algebra, such that Ω_w is a cocommutative twist from (Δ, Φ, R) to $(\tilde{\Delta}, \tilde{\Phi}, \tilde{R})$, where $\tilde{R} := (R^\dagger)^{-1}$.*

Proof. We observe that $R_{21} =: R^{\mathrm{op}}$ is a quasi-triangular structure on the triple $(A, \Delta^{\mathrm{op}}, \Phi^{\mathrm{op}})$ and $(R^\dagger)^{-1} = \tilde{R}$ is one on $(A, \tilde{\Delta}, \tilde{\Phi})$ (see 2.6). Moreover, the identity $E_{21}R_{21}E^{-1} = (R^\dagger)^{-1}$ states that E is a cocommutative twist (see Proposition-Definition 2.6) from $(\Delta^{\mathrm{op}}, \Phi^{\mathrm{op}}, R^{\mathrm{op}})$ to $(\tilde{\Delta}, \tilde{\Phi}, \tilde{R})$.

Let us now look more closely at the central unitary element w. Given its properties, we may use it to perform a deformation (see the end of 1.4) on the braiding on

$\mathrm{Rep}^+(A)$ corresponding to R, thus obtaining the new almost cocommutative structure $\overline{R} := R\Delta(w)(w^{-1} \otimes w^{-1}) = RT_w$. We also note that T_w is a cocommutative twist from (Δ, Φ, R) to itself; since R is clearly one from (Δ, Φ, R) to $(\Delta^{\mathrm{op}}, \Phi^{\mathrm{op}}, R^{\mathrm{op}})$, we conclude that $\Omega_w = E\overline{R} = ERT_w$ is a cocommutative twist from (Δ, Φ, R) to $(\tilde{\Delta}, \tilde{\Phi}, \tilde{R})$.

Finally, since Ω_w is also assumed to be positive, $(A, \cdot^\dagger, \Delta, \Phi, \Omega)$ is a discrete unitary weak quasi-bialgebra. □

Remark. In the Proposition, it is actually possible to drop the assumption that Ω is positive and still deduce $\Omega_w^\dagger = \Omega_w$ from the other axioms, as it is found in [CCP21]. However, this is only useful when the first point in the Definition is relaxed to "$(A, \cdot^\dagger)$ is a discrete $*$-algebra", which means that the V_ι of Definition 2.4A are allowed to be equipped with non-degenerate Hermitian forms rather than necessarily scalar products. Then one considers the category $\mathrm{Rep}_h(A)$, where the A-moduli are accordingly equipped with non-degenerate Hermitian forms, so that $\Omega^\dagger = \Omega$ is good enough to define the forms on tensor products, and the treatment of 2.4 still goes through.

This more general approach is carried over in [CCP21], where the more general notion of "discrete Hermitian coboundary weak quasi-Hopf algebra" replaces the unitary case of the Definition.

It is also interesting to note that $\overline{R}_{21}\overline{R} = \Delta(1)$:

$$\begin{aligned}\overline{R}_{21}\overline{R} &= R_{21}\Delta^{\mathrm{op}}(w)(w^{-1} \otimes w^{-1})R\Delta(w)(w^{-1} \otimes w^{-1}) \\ &= R_{21}R\Delta(w^2)(w^{-2} \otimes w^{-2}) = R_{21}R\Delta(v)(v^{-1} \otimes v^{-1}) = \Delta(1) \ .\end{aligned}$$

This is equivalent to $\overline{c}^2 = 1$, where $\overline{c}$ is the generalised coboundary corresponding to $\overline{R}$; so $\overline{c}$ is what is usually called a coboundary, which explains the nomenclature of the Definition. Let us now turn to $\mathrm{Rep}^+(A)$.

Lemma. *Consider the C^* tensor category $\mathrm{Rep}^+(A)$ endowed with the right duals defined by (S, α, β) and the ribbon structure defined by (R, v) (see Remark 2.3A, formula 2.6(1) and Proposition 2.3).*

Then $\mathrm{Rep}^+(A)$ is a C^* ribbon category (see Definition 1.5C) if and only if $\beta = \alpha^\dagger$.

Proof. Since Ω_w is a cocommutative twist from (Δ, Φ, R) to $(\tilde{\Delta}, \tilde{\Phi}, \tilde{R})$, the corresponding braiding c is unitary by Lemma 2.6; moreover the ribbon element $v = w^2$ is unitary because w is. Thus, in order to prove $\mathrm{Rep}^+(A)$ to be C^* ribbon, one just has to check the identities

$$b_V^* = d_V \circ c_{V,\overline{V}} \circ (v_V^{-1} \otimes \mathrm{id}_{\overline{V}}) \ , \qquad d_V^* = (\mathrm{id}_{\overline{V}} \otimes v_V) \circ c^{-1}_{\overline{V},V} \circ b_V \tag{1}$$

for the generic C^* A-module V, which we proceed to do assuming $\beta = \alpha^\dagger$. To this aim, we choose an orthonormal basis $\{e_k\}$ for V and write

$$b_V(1) = \sum_k \beta e_k \overline{e_k} \ , \quad d_V(\overline{v} \otimes w) = (\alpha w, v)_V \ ,$$

where $\overline{u}$ is just the vector u in V considered as an element of $\overline{V}$ and $(\cdot,\cdot)_V$ is the scalar product on V; it induces on $\overline{V}$ the scalar product

$$(\overline{v},\overline{w})_{\overline{V}} := (w,v)_V \ , \qquad \forall\ v,w \in V \ .$$

We recall that the A-action on $\overline{V}$ is given by $a\overline{v} = \overline{S(a)^\dagger v}$. In order to compute $d_V^*(1)$, we consider $\overline{V}\otimes V$ with the scalar product $(\cdot,\cdot)_{\overline{V}}\otimes(\cdot,\cdot)_V$, and denote the corresponding adjoint by $\cdot^{*_p}$. E.g.

$$d_V^{*_p} = \sum_k \overline{e_k}\otimes\alpha^\dagger e_k \ , \qquad b_V^{*_p}(v\otimes\overline{w}) = (\beta^\dagger v,\overline{w}) \ .$$

Finally, we write $\Omega_w =: m\otimes n$ and $\Omega_w^{-1} =: p\otimes q$. Since the action of Ω_w defines the scalar product on $\overline{V}\otimes^A V$, we have

$$\begin{aligned} d_V^*(1) = \Omega_w^{-1} d_V^{*_p}(1) = \Omega_w^{-1}\sum_k \overline{e_k}\otimes\alpha^\dagger e_k = \sum_k p\overline{e_k}\otimes q\alpha^\dagger e_k \\ = \sum_k \overline{S(p)^\dagger e_k}\otimes q\alpha^\dagger e_k = \sum_k \overline{e_k}\otimes q\alpha^\dagger S(p) e_k = \sum_k \overline{e_k}\otimes \alpha_{\Omega_w}^\dagger e_k \ . \end{aligned}$$

In the last passage we used $S(\cdot^\dagger) = S(\cdot)^\dagger$ and applied Proposition 2.6A for the identity $q\alpha^\dagger S(p) = \big(S(p)\alpha q\big)^\dagger = \alpha_{\Omega_w}$. Along the same lines, we compute

$$\begin{aligned} b_V^*(v\otimes\overline{w}) = b_V^{*_p}\big(\Omega_w(v\otimes\overline{w})\big) = b_V^{*_p}(mv\otimes n\overline{w}) = b_V^{*_p}(mv\otimes\overline{S(n)^\dagger w}) \\ = (\beta^\dagger mv, S(n)^\dagger w)_V = (S(n)\beta^\dagger mv, w)_V = (\beta_{\Omega_w}^\dagger v, w)_V \ . \end{aligned}$$

We are now ready to check the first of (1). Writing $R =: r\otimes s$, we evaluate the right-hand side on $v_1\otimes\overline{v_2}$:

$$\begin{aligned} \big(d_V\circ c_{V,\overline{V}}\circ(v_V^{-1}\otimes \mathrm{id}_{\overline{V}})\big)(v_1\otimes\overline{v_2}) = \big(d_V\circ\Sigma_{V,\overline{V}}\big)(rv^{-1}v_1\otimes s\overline{v_2}) = \\ d_V\big(\overline{S(s)^\dagger v_2}\otimes rv^{-1}v_1\big) = \big(\alpha rv^{-1}v_1, S(s)^\dagger v_2\big)_V = \big(S(s)\alpha rv^{-1}v_1, v_2\big)_V \ . \end{aligned} \tag{2}$$

On the other hand, since $\Omega_w = \Omega_w^\dagger = (w\otimes w)\Delta(w)^\dagger R^\dagger\Delta(1)^\dagger$, Proposition 2.6A allows us to compute

$$\begin{aligned} \beta_{\Omega_w}^\dagger &= \big(w(w_{(1)})^\dagger r^\dagger 1_{(1)}\beta S(1_{(2)})S(s^\dagger)S(w_{(2)})^\dagger w\big)^\dagger \\ &= \big(w(w_{(1)})^\dagger r^\dagger\beta S(s^\dagger)S(w_{(2)})^\dagger w\big)^\dagger = S(w_{(2)})S(s)\beta r w_{(1)} v^{-1} \ , \end{aligned}$$

where we used the second weak antipode identity (the second point in Definition 2.3A) for the second equality and the properties of w for the third one. Moreover, denoting by $\mu : A\otimes A\to A$ the multiplication, we have

$$\begin{aligned} S(w_{(2)})S(s)\beta^\dagger r w_{(1)} = \big(\mu\circ(S\otimes\mathrm{id})\big)\big(S(\beta^\dagger)sw_{(2)}\otimes rw_{(1)}\big) = \\ \big(\mu\circ(S\otimes\mathrm{id})\big)\big(S(\beta^\dagger)w_{(1)}s\otimes w_{(2)}r\big) = S(s)S(w_{(1)})\beta^\dagger w_{(2)} r \ , \end{aligned}$$

having used $R_{21}\Delta^{\mathrm{op}}(\cdot) = \Delta(\cdot)R_{21}$. Finally, we can apply $\beta = \alpha^\dagger$, the first weak antipode identity and $\epsilon(w) = 1$ to conclude

$$\beta_{\Omega_w}^\dagger = S(s)S(w_{(1)})\beta^\dagger w_{(2)} rv^{-1} = S(s)S(w_{(1)})\alpha w_{(2)} rv^{-1} = S(s)\alpha rv^{-1} \ .$$

Together with (2), this proves the first of (1); a proof of the second is achieved through similar technicalities, and may be found in [CCP21]. Besides, the cited proof actually shows the identity $\beta = \alpha^\dagger$ to be equivalent to the second of (1). So $\beta = \alpha^\dagger$ is also a necessary condition for $\mathrm{Rep}^+(A)$ to be a C^* ribbon category. □

We conclude the subsection with a useful result that will enable us to construct a discrete unitary coboundary weak quasi-Hopf algebras from a purely categorical datum, by a reconstruction procedure. From a more systematic point of view, the notion of a discrete unitary coboundary weak quasi-Hopf algebra will be shown to arise from a Tannakian result, a situation analogous to the ones of 2.1 and 2.4.

Theorem. *Let $\mathcal{C}^+$ be a C^* semi-simple abelian category, $(\mathcal{C}, \otimes, a)$ a tensor category and $\mathcal{F} : \mathcal{C}^+ \to \mathcal{C}$ a linear equivalence. Furthermore, let $\mathcal{G}^+ : \mathcal{C}^+ \to \mathrm{Hilb}$ a faithful $*$-functor and $\mathcal{G} : \mathcal{C} \to \mathrm{Hilb}$ a faithful functor with weak quasi-tensor structure (G, H) such that the following diagram commutes.*

$$\begin{array}{ccc} \mathcal{C}^+ & \xrightarrow{\mathcal{G}^+} & \mathrm{Hilb} \\ \big\downarrow \mathcal{F} & \nearrow_{\mathcal{G}} & \\ \mathcal{C} & & \end{array} \tag{3}$$

Then $\mathcal{G}(a)$ is unitary (see Definition 1.5B) if and only if $\mathcal{C}^+$ can be upgraded to a C^* tensor category and $\mathcal{F}$ to a tensor equivalence with tensor structure F such that $\mathcal{G}(F)$ is unitary.

If this is the case, let us further suppose $\mathcal{C}$ to be rigid and endowed with a braiding c; we also assume that there is a natural isomorphism w from the identity functor $\mathrm{id}_\mathcal{C}$ to itself, with $w_\mathbb{1} = 1$ and compatible with duality, such that $v := w^2$ is a ribbon structure for c. We pull back all this additional structure to $\mathcal{C}^+$ by means of the tensor equivalence $\mathcal{F}$ and likewise consider the unique weak quasi-tensor structure on $\mathcal{G}^+$ such that (3) also commutes with respect to the weak quasi-tensor structures involved. We denote by $(A^+, \cdot^\dagger, \Delta, \Phi, \Omega)$ the discrete unitary weak quasi-Hopf algebra constructed as in 2.4, with quasi-triangular structure and ribbon element (R, v) constructed as in 2.6.

Then $(A^+, \cdot^\dagger, \Delta, \Phi, R, w)$ is a discrete unitary coboundary weak quasi-Hopf algebra such that $\Omega_w = \Omega$ if and only if $\mathcal{G}(c_{\rho,\sigma})$ and $\mathcal{G}(w_\rho)$ are unitary for all ρ, σ in $\mathcal{C}$ and the following identities hold:

$$G_{\sigma,\rho}\Sigma(\mathcal{G}(\rho), \mathcal{G}(\sigma))G^*_{\rho,\sigma} = \mathcal{G}(c^w_{\rho,\sigma}) \ , \quad H^*_{\rho,\sigma}\Sigma(\mathcal{G}(\sigma), \mathcal{G}(\rho))H_{\sigma,\rho} = \mathcal{G}(c^w_{\rho,\sigma})^{-1} \ ,$$

where c^w is the coboundary obtained from c by deformation by w.

Proof. We consider the discrete unitary algebra $(A^+, \cdot^\dagger)$ and the equivalence of C^* categories $\mathcal{E}^+ : \mathcal{C}^+ \to \mathrm{Rep}^+(A^+)$ constructed from $\mathcal{G}^+$ as in Theorem 2.4, or actually a simpler version where all monoidal data is suppressed.

Similarly, we consider the discrete weak quasi-bialgebra (A, Δ, Φ) and the tensor equivalence $\mathcal{E} : \mathcal{C} \to \mathrm{Rep}(A)$ constructed as in Theorem 2.1 from $(\mathcal{G}, (G, H))$. Since $\mathcal{G}$ takes values in Hilb, we also have the weak quasi tensor structure (H^*, G^*). We denote the corrisponding coproduct and associator by $(\hat{\Delta}, \hat{\Phi})$, and note that

$\Omega := G^*G$ is a twist from (Δ, Φ) to $(\hat{\Delta}, \hat{\Phi})$, with partial inverse $\Omega^{-1} = HH^*$ (see Definition 2.5A).

Since $\mathcal{F}$ is a linear equivalence, we have the linear isomorphism

$$\gamma : A \to A^+ \qquad \gamma(\eta) = \eta \circ \mathcal{E} \ ,$$

where the natural transformation $\eta \circ \mathcal{E}$ takes the values $(\eta \circ \mathcal{E})_\rho = \eta_{\mathcal{E}(\rho)}$; $\gamma^{\otimes k}$ is similarly defined between the k-th generalised tensor powers, for all k. We may use γ to transport (Δ, Φ) and $(\hat{\Delta}, \hat{\Phi})$ to A^+; we use the same notation for the transported pairs and proceed to clarify the relation between $(\hat{\Delta}, \hat{\Phi})$ and the pair $(\tilde{\Delta}, \tilde{\Phi})$ of Definition B.

While $\hat{\Delta} = \Delta(\cdot^\dagger)^\dagger \tilde{\Phi}$ results clearly from the definition, it is not the same for the associators. Indeed

$$\begin{aligned}\hat{\Phi}_{\rho,\sigma,\tau} = \Phi_{\mathcal{F}(\rho),\mathcal{F}(\sigma),\mathcal{F}(\tau)} = \big(\mathcal{G}^+(\rho) \otimes G^*_{\mathcal{F}(\rho),\mathcal{F}(\sigma)}\big) \circ G^*_{\mathcal{F}(\rho),\mathcal{F}(\sigma)\otimes\mathcal{F}(\tau)} \circ \\ \mathcal{G}(a_{\mathcal{F}(\rho),\mathcal{F}(\sigma),\mathcal{F}(\tau)}) \circ H^*_{\mathcal{F}(\rho)\otimes\mathcal{F}(\sigma),\mathcal{F}(\tau)} \circ \big(H^*_{\mathcal{F}(\rho),\mathcal{F}(\sigma)} \otimes \mathcal{G}^+(\tau)\big) \ ,\end{aligned}$$

so $\hat{\Phi} = (\Phi^*)^{-1}$ exactly if $\mathcal{G}(a)$ is unitary. We assume that this is the case and prove the "only if" implication in the first assertion of the statement.

By the above discussion $(A^+, \cdot^\dagger, \Delta, \Phi, \Omega)$ is a discrete unitary weak quasi-bialgebra, hence $\mathrm{Rep}^+(A^+)$ is a C^* tensor category by Lemma 2.4. Then, by general categorical arguments, we may pull back the monoidal data of $\mathrm{Rep}^+(A^+)$ to $\mathcal{C}^+$ using the equivalence of C^* tensor categories $\mathcal{E}^+$, in a way that $\mathcal{C}^+$ becomes a C^* tensor category and $\mathcal{E}^+$ an equivalence of C^* tensor categories. Finally, $\mathcal{F}$ shall be endowed with the unique (keeping in mind that $\mathcal{G}$ and $\mathcal{G}^+$ are faithful) tensor structure F such that (3) also holds with respect to the weak quasi-tensor structures; moreover we may achieve unitarity of $\mathcal{G}(F)$ by polar decomposition, accordingly modifying $\mathcal{C}^+$ and the tensor structure of $\mathcal{E}^+$ if needed.

To see the converse, let us assume $\mathcal{C}^+$ to be a C^* tensor category and $\mathcal{F}$ a tensor equivalence with tensor structure F; we consider the unique weak quasi-tensor structure (G^+, H^+) on $\mathcal{G}^+$ such that (3) also commutes with respect to the weak quasi-tensor structures and apply the full version of Theorem 2.4, upgrading $(A^+, \cdot^\dagger)$ to a discrete unitary weak quasi-bialgebra $(A^+, \cdot^\dagger, \Delta^+, \Phi^+, \Omega)$.

In this situation the above γ becomes an isomorphism of weak quasi-bialgebras; in particular, for all ρ, σ, τ objects of $\mathcal{C}^+$,

$$\Phi_{\mathcal{F}(\rho),\mathcal{F}(\sigma),\mathcal{F}(\tau)} = \Phi^+_{\rho,\sigma,\tau} \ , \tag{4}$$

which is unitary from $(\mathcal{E}^+(\rho) \otimes^{A^+} \mathcal{E}^+(\sigma)) \otimes^{A^+} \mathcal{E}^+(\tau)$ to $\mathcal{E}^+(\rho) \otimes^{A^+} (\mathcal{E}^+(\sigma) \otimes^{A^+} \mathcal{E}^+(\tau))$. Moreover, the unitary tensor structure of $\mathcal{E}^+$ is given by

$$G^+_{\rho,\sigma} = \mathcal{G}(F_{\rho,\sigma}) \circ G_{\mathcal{F}(\rho),\mathcal{F}(\sigma)}$$

restricted to $\mathcal{E}^+(\rho) \otimes^{A^+} \mathcal{E}^+(\sigma)$; therefore unitarity of $\mathcal{G}(F_{\rho,\sigma})$ implies that the restriction of $G_{\mathcal{F}(\rho),\mathcal{F}(\sigma)}$ to $\mathcal{E}^+(\rho) \otimes^{A^+} \mathcal{E}^+(\sigma)$ is unitary as well. Finally, $\mathcal{E}(\rho) \otimes^A \mathcal{E}(\sigma) = \mathcal{E}^+(\rho) \otimes^{A^+} \mathcal{E}^+(\sigma)$ as vector spaces, thus by (4) and the defining diagram (3), $\mathcal{G}(a_{\mathcal{F}(\rho),\mathcal{F}(\sigma),\mathcal{F}(\tau)})$ is unitary for all ρ, σ, τ objects of $\mathcal{C}^+$; hence $\mathcal{G}(a)$ is unitary since $\mathcal{F}$ is an equivalence.

Let us now pass to prove the "only if" implication of the second assertion; as in the statement, we drop the + superscripts from (Δ^+, Φ^+) for simplicity. By unitarity of the braiding on $\mathrm{Rep}^+(A^+)$ (see the proof of the Lemma) and of $\mathcal{G}(F)$, we deduce that $\mathcal{G}(c)$ is unitary too, just as in the case of $\mathcal{G}(a)$; the unitarity of $\mathcal{G}(w)$ is evident.

Furthermore, we have $G^*_{\rho,\sigma} = \Omega_{\rho,\sigma} H_{\rho,\sigma}$ for all ρ, σ in $\mathcal{C}$, and $\Omega = ER_w$ by assumption, where E is the trivial twist $\Delta(1)^\dagger \Delta^{\mathrm{op}}(1)$ and R_w corresponds to the coboundary c^w. Thus $(R_w)_{\rho,\sigma} = \Sigma(\mathcal{G}(\sigma), \mathcal{G}(\rho)) H_{\sigma,\rho} \mathcal{G}(c^w_{\rho,\sigma}) G_{\rho,\sigma}$ and we compute

$$\mathcal{G}(c^w_{\rho,\sigma}) = G_{\sigma,\rho} \Sigma(\mathcal{G}(\rho), \mathcal{G}(\sigma)) E^{-1} G^*_{\rho,\sigma} = G_{\sigma,\rho} \Sigma(\mathcal{G}(\rho), \mathcal{G}(\sigma)) G^*_{\rho,\sigma} \ ,$$

where we cancelled $E^{-1} = \Delta^{\mathrm{op}}(1)\Delta(1)^\dagger$ because

$$\Delta^{\mathrm{op}}(1) = \Sigma(\mathcal{G}(\sigma), \mathcal{G}(\rho)) H_{\sigma,\rho} G_{\sigma,\rho} \Sigma(\mathcal{G}(\rho), \mathcal{G}(\sigma)), \quad \Delta(1)^\dagger = G^*_{\rho,\sigma} H^*_{\rho,\sigma} \tag{5}$$

by the definitions. The second identity at the end of the statement can be derived analogously.

We now turn to the other implication, and write $E := \Delta(1)^\dagger \Delta^{\mathrm{op}}(1)$ and $\underline{E} := \Delta^{\mathrm{op}}(1)\Delta(1)^\dagger$. Still keeping in mind (5), the first of the given identities allows us to compute

$$\begin{aligned}(R_w)_{\rho,\sigma} &= \Sigma(\mathcal{G}(\sigma), \mathcal{G}(\rho)) H_{\sigma,\rho} \mathcal{G}(c^w_{\rho,\sigma}) G_{\rho,\sigma} \\ &= \Sigma(\mathcal{G}(\sigma), \mathcal{G}(\rho)) H_{\sigma,\rho} G_{\sigma,\rho} \Sigma(\mathcal{G}(\rho), \mathcal{G}(\sigma)) G^*_{\rho,\sigma} G_{\rho,\sigma} = \underline{E}_{\rho,\sigma} \Omega_{\rho,\sigma} \ ,\end{aligned}$$

i.e. $R_w = \underline{E}\Omega$; likewise $R_w^{-1} = \Omega^{-1} E$ by the second identity at the end of the statement. So E is actually partially invertible, with inverse $\underline{E}$, and $\Omega_w = ER_w = \Omega$, which incidentally is positive. Furthermore, since $\mathcal{G}(c)$ and $\mathcal{G}(F)$ are unitary, so is the the braiding induced by c on A^+, whence Ω is a cocommutative twist from (Δ, Φ, R) to $(\tilde{\Delta}, \tilde{\Phi}, \tilde{R})$ by Lemma 2.6; therefore E is a cocommutative twist from $(\Delta^{\mathrm{op}}, \Phi^{\mathrm{op}}, R^{\mathrm{op}})$ to $(\tilde{\Delta}, \tilde{\Phi}, \tilde{R})$, as required by the Definition. Finally $\epsilon(w) = 1$, $S(w) = w$ and unitarity of w are equivalent to $w_{\mathbb{1}} = 1$, compatibility of w with duality and unitarity of $\mathcal{G}(w)$ respectively. □

2.8 Weak Hopf algebras

The notion of a weak quasi-tensor functor (Definition 2.1A) features a remarkable special case, closer to an actual tensor functor but still general enough to be useful in the reconstruction of categories arising from quantum groups at roots of 1, which we will treat in next chapter.

Definition A. Let$(\mathcal{C}, \otimes, a)$ be a tensor category and $\mathcal{F} : \mathcal{C} \to \mathrm{Vec}$ a faithful functor with weak quasi-tensor structure (F, G). We say that (F, G) is a *weak tensor structure* if it satisfies:

$$\mathcal{F}(a_{\rho,\sigma,\tau}) = F_{\rho,\sigma\otimes\tau} \circ (\mathcal{F}(\rho) \otimes F_{\sigma,\tau}) \circ (G_{\rho,\sigma} \otimes \mathcal{F}(\tau)) \circ G_{\rho\otimes\sigma,\tau} \ , \tag{1}$$

$$\mathcal{F}(a^{-1}_{\rho,\sigma,\tau}) = F_{\rho\otimes\sigma,\tau} \circ (F_{\rho,\sigma} \otimes \mathcal{F}(\tau)) \circ (\mathcal{F}(\rho) \otimes G_{\sigma,\tau}) \circ G_{\rho,\sigma\otimes\tau} \ . \tag{2}$$

Now, we fix a rigid tensor category $\mathcal{C}$, say with right duals $\rho^\vee$ and duality pairs (b_ρ, d_ρ) for each object ρ, and a weak tensor functor $\mathcal{F}$ as in Definition A. We will exploit the identities (1) and (2) to derive special properties for the weak quasi-Hopf algebra (A, Δ, Φ) constructed from $\mathcal{F}$ (see 2.1).

The most relevant new property is about the form of the associator. Replacing $\mathcal{F}(a_{\rho,\sigma,\tau})$ and $\mathcal{F}(a^{-1}_{\rho,\sigma,\tau})$ in the defining diagram 2.1(3) by means of (1) and (2), we obtain respectively

$$\Phi = (\mathrm{id} \otimes \Delta)(\Delta 1)(\Delta \otimes \mathrm{id})(\Delta 1) \;, \quad \text{and} \quad \Phi^{-1} = (\Delta \otimes \mathrm{id})(\Delta 1)(\mathrm{id} \otimes \Delta)(\Delta 1) \;.$$

Turning to the antipode, thanks to (1), (2) and the good behaviour with respect to units discussed in Remark 2.1, weak tensor functors are as good as tensor ones when it comes to compatibility with duals.

Proposition A. *For each ρ object of $\mathcal{C}$, the pair $(G_{\rho,\rho^\vee} \circ \mathcal{F}(b_\rho), \mathcal{F}(d_\rho) \circ F_{\rho^\vee,\rho})$ is a duality pair in* Vec *with left object $\mathcal{F}(\rho)$ and right object $\mathcal{F}(\rho^\vee)$.*

Proof. We limit ourselves to prove the first duality identity, which in our case reads

$$\big(\mathcal{F}(\rho) \otimes \mathcal{F}(d_\rho)\big) \circ \big(\mathcal{F}(\rho) \otimes F_{\rho^\vee,\rho}\big) \circ \big(G_{\rho,\rho^\vee} \otimes \mathcal{F}(\rho)\big) \circ \big(\mathcal{F}(b_\rho) \otimes \mathcal{F}(\rho)\big) = \mathcal{F}(\rho) \;; \quad (3)$$

the second one is treated in complete analogy. Now, by identities 2.1(2) and naturality of F, $\mathcal{F}(\rho) \otimes \mathcal{F}(d_\rho)$ equals

$$\mathcal{F}(r_\rho) \circ F_{\rho,\mathbb{1}} \circ \big(\mathcal{F}(\rho) \otimes \mathcal{F}(d_\rho)\big) = \mathcal{F}(r_\rho) \circ \mathcal{F}(\rho \otimes d_\rho) \circ F_{\rho,\rho^\vee \otimes \rho} \;;$$

analogously, by 2.1(2) and naturality of G, $\mathcal{F}(b_\rho) \otimes \mathcal{F}(\rho)$ equals

$$\big(\mathcal{F}(b_\rho) \otimes \mathcal{F}(\rho)\big) \circ G_{\mathbb{1},\rho} \circ \mathcal{F}(l_\rho^{-1}) = G_{\rho \otimes \rho^\vee,\rho} \circ \mathcal{F}(b_\rho \otimes \rho) \circ \mathcal{F}(l_\rho^{-1}) \;.$$

Furthermore, identity (1) produces

$$F_{\rho,\rho^\vee \otimes \rho} \circ \big(\mathcal{F}(\rho) \otimes F_{\rho^\vee,\rho}\big) \circ \big(G_{\rho,\rho^\vee} \otimes \mathcal{F}(\rho)\big) \circ G_{\rho \otimes \rho^\vee,\rho} = \mathcal{F}(a_{\rho,\rho^\vee,\rho}) \;.$$

Therefore, recollecting the pieces we see that the left-hand side of (3) equals

$$\begin{gathered}\mathcal{F}(r_\rho) \circ \mathcal{F}(\rho \otimes d_\rho) \circ \mathcal{F}(a_{\rho,\rho^\vee,\rho}) \circ \mathcal{F}(b_\rho \otimes \rho) \circ \mathcal{F}(l_\rho^{-1}) = \\ \mathcal{F}(r_\rho \circ (\rho \otimes d_\rho) \circ a_{\rho,\rho^\vee,\rho} \circ (b_\rho \otimes \rho) \circ l_\rho^{-1}) = \mathcal{F}(\rho) \;,\end{gathered}$$

by the first duality identity for the pair (b_ρ, d_ρ). □

Proposition A allows us to improve the construction of the weak antipode presented in 2.3. Since $\mathcal{F}(\rho)$ and $\mathcal{F}(\rho^\vee)$ are left and right duals in Vec, we may apply Proposition 1.3 to take $U_\iota : \mathcal{F}(\iota)' \to \mathcal{F}(\iota^\vee)$ such that

$$b_\rho^{\mathcal{E}} = (\mathcal{F}(\rho) \otimes U_\rho) \circ b^{\mathrm{Vec}}_{\mathcal{F}(\rho)} \;, \quad \text{and} \quad d_\rho^{\mathcal{E}} = d^{\mathrm{Vec}}_{\mathcal{F}(\rho)} \circ (U_\rho^{-1} \otimes \mathcal{F}(\rho)) \;,$$

where $(b^{\mathcal{E}}, d^{\mathcal{E}}) := (G_{\rho,\rho^\vee} \circ \mathcal{F}(b_\rho), \mathcal{F}(d_\rho) \circ F_{\rho^\vee,\rho})$ and $(b^{\mathrm{Vec}}_{\mathcal{F}(\rho)}, d^{\mathrm{Vec}}_{\mathcal{F}(\rho)})$ is the usual duality pair in Vec for $\mathcal{F}(\rho)$, namely

$$b^{\mathrm{Vec}}_{\mathcal{F}(\rho)}(1) = \mathrm{id}_{\mathcal{F}(\rho)} \quad \text{and} \quad d^{\mathrm{Vec}}_{\mathcal{F}(\rho)} = \mathrm{Tr}_{\mathrm{End}\,(\mathcal{F}(\rho))} \;. \quad (4)$$

Now $\mathcal{E} : \mathcal{C} \to \mathrm{Rep}(A)$ is a tensor equivalence, $b_\rho^{\mathcal{E}}$ and $d_\rho^{\mathcal{E}}$ are morphisms in $\mathrm{Rep}(A)$, whence so are b_ρ^{Vec} and d_ρ^{Vec}. Therefore $(b_{\mathcal{F}(\rho)}^{\mathrm{Vec}}, d_{\mathcal{F}(\rho)}^{\mathrm{Vec}})$ is actually a duality pair in $\mathrm{Rep}(A)$; from these duality pairs we define (S, α, β) as in 2.3, but now $\alpha = 1 = \beta$ by (4), so S is a strong antipode.

Prior to axiomatising the special structure obtained and discussing its properties, we need to introduce some notational shortcuts. Given a coproduct Δ, we put

$$
\begin{aligned}
{}_3\Delta(1) &:= (\Delta \otimes \mathrm{id})(\Delta 1) \ , & \Delta_3(1) &:= (\mathrm{id} \otimes \Delta)(\Delta 1) \ , \\
{}_4\Delta(1) &:= (\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\Delta \otimes \mathrm{id})(\Delta 1) \ , & \Delta_4(1) &:= (\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\mathrm{id} \otimes \Delta)(\Delta 1) \ ,
\end{aligned}
$$

so for instance an associator for Δ is a partially invertible from ${}_3\Delta(1)$ to $\Delta_3(1)$.

Definition B. A pair (A, Δ) is called a *discrete weak Hopf algebra* if $(A, \Delta, \Delta_3(1) {}_3\Delta(1))$ is a weak quasi-Hopf algebra and

$$
(\Delta_3(1) {}_3\Delta(1))^{-1} = {}_3\Delta(1) \Delta_3(1) \ .
$$

Definition B is remarkably simple, in that not only is the associator determined by the pair (A, Δ), but we also have a canonical choice of a weak antipode. Indeed the existence of a, unique by Proposition-Definition 2.3, strong antipode for the above example does not depend on the concrete construction we presented. It rather just follows from Definition B, so we may just take the strong antipode, usually denoted by S.

Proposition B. *Let (A, Δ) be a discrete weak Hopf algebra. Then it admits a strong antipode.*

Proof. Let (S, α, β) be a weak antipode. We write

$$
\begin{aligned}
\Phi &= \big(1_{(1)} \otimes 1_{(2)(1)} \otimes 1_{(2)(2)}\big)\big(1_{(1')(1') \otimes} 1_{(1')(2')} \otimes 1_{(2')}\big) \\
&= 1_{(1)} 1_{(1')(1')} \otimes 1_{(2)(1)} 1_{(1')(2')} \otimes 1_{(2)(2)} 1_{(2')}
\end{aligned}
$$

Therefore the first identity in the third point of Definition 2.3A reads

$$
\begin{aligned}
1 &= 1_{(1)} 1_{(1')(1')} \beta S\big(1_{(2)(1)} 1_{(1')(2')}\big) \alpha 1_{(2)(2)} 1_{(2')} \\
&= 1_{(1)} 1_{(1')(1')} \beta S(1_{(1')(2')}) S(1_{(2)(1)}) \alpha 1_{(2)(2)} 1_{(2')} \\
&= \epsilon(1_{(1')}) \epsilon(1_{(2)}) 1_{(1)} \beta \alpha 1_{(2')} = \beta \alpha \ ,
\end{aligned}
$$

i.e. $\beta = \alpha^{-1}$. The statement follows from Proposition-Definition 2.3. □

We now focus on the special case of weak tensor functors in Theorem 2.1. In view of the arguments developed between Definition A and Proposition A, we just need to point out the following trivial fact:

given a discrete weak Hopf algebra (A, Δ), the forgetful functor $\mathcal{F}_A : \mathrm{Rep}(A) \to \mathrm{Vec}$ admits a natural weak tensor structure. Namely, for each choice of A-modules V, W we set $(\Delta 1)_{V,W} =: G_{V,W} F_{V,W}$, where $(\Delta 1)_{V,W}$ denotes the action of $\Delta(1)$ on $V \otimes W$ and $G_{V,W} : (\Delta 1)_{V,W} \to V \otimes W$, $F_{V,W} : V \otimes W \to (\Delta 1)_{V,W}$ are determined by $F_{V,W} G_{V,W} = \mathrm{id}_{V \otimes W}$.

Corollary A. *We consider the situation of Theorem 2.1, with $\mathcal{C}$ rigid. Then (F, G) is a weak tensor structure if and only if (A, Δ) is a weak Hopf algebra.*

Let us now come back to the scenario we fixed after Definition A, namely we have a rigid tensor category $\mathcal{C}$ and a weak tensor functor $\mathcal{F} : \mathcal{C} \to \mathrm{Vec}$; we also consider the associated discrete weak Hopf algebra (A, Δ).

Now, in view of the classification achieved by Proposition 2.5D, we apply Definition A to express whether the twisted structure (F_U, G_U) is still weak tensor in terms of the twist U. Identities (1) and (2) for (F_U, G_U) translate into

$$(1 \otimes U)(\mathrm{id} \otimes \Delta)(U)(\Delta \otimes \mathrm{id})(U^{-1})(U \otimes 1) = (\Delta_U)_3(1)_3(\Delta_U)(1) \ , \tag{5}$$

$$(U \otimes 1)(\Delta \otimes \mathrm{id})(U)(\mathrm{id} \otimes \Delta)(U^{-1})(1 \otimes U) = {}_3(\Delta_U)(1)(\Delta_U)_3(1) \ , \tag{6}$$

which correspond to the more usual cocycle condition presented, for instance, in [NT14] in the analogous situation for tensor functors. We therefore extend the same terminology to our case.

Definition C. A *cocycle* of a discrete weak Hopf algebra (A, Δ), is a twist U of A from $(\Delta, \Delta_3(1)_3\Delta(1))$ such that (5) and (6) are verified. We denote the set of such cocycles by $Z(A, \Delta)$.

Remark. Given twists U_1, U_2 of A from $(\Delta, \Delta_3(1)_3\Delta(1))$, u is a tensor isomorphism from $(\mathcal{F}, (F_{U_1}, G_{U_1}))$ to $(\mathcal{F}, (F_{U_1}, G_{U_1}))$ exactly if

$$U_2 = u \otimes u U_1 \Delta(u^{-1}) \ .$$

So, since tensor isomorphisms send weak tensor structures into weak tensor structures, we see that U_1 is a cocycle exactly if U_2 is. Thus, as it can also be checked directly, cohomologousness preserves cocycles and any coboundary is a cocycle.

It is now clear that cocycles are exactly what we need in order to formulate a refinement of Proposition 2.5D appropriate to take care of weak tensor structures. Indeed, in view of Corollary A and the arguments since developed, we have the following

Corollary B. *We consider the situation of Proposition 2.5D, with $\mathcal{C}$ rigid and (F, G) weak tensor.*

Then (F_U, G_U) is a weak tensor structure exactly if U is a cocycle of (A, Δ). Moreover, if we restrict the assignment $U \mapsto (\mathcal{F}, (F_U, G_U))$ to cocycles of (A, Δ) we obtain a bijection from $Z(A, \Delta)/\sim$ to the set of tensor isomorphism classes of weak tensor functors from $\mathcal{C}$ to Vec *with dimension function d.*

We do not mind to mention that the results of this subsection may be straightforwardly adapted to deal with the unitary case, e.g. to obtain suitable refinements of Theorem 2.4 and Proposition 2.5C analogous to Corollaries A and B.

Quantum groups

Quantum groups at roots of unity are the actual mathematical object we deal with in the thesis, by means of the general algebraic tools presented in previous chapter. More in detail, we are specially interested in the fusion categories of [Wen98], which the present chapter is therefore designed to introduce. To this aim quantum universal enveloping algebras are introduced in section 3, together with their specialisations and the basic structure results for the relative representation categories. Section 4 deals with the category of tilting modules and their quotient by the tensor ideal of negligible modules (see [CP95] or [Lus93]). The work of [Wen98] endows these fusion categories with a C^* ribbon structure (see Definition 1.5C) and allows us to present a weak tensor functor on them, following [CCP21]. This provides a remarkable application for the theory of previous chapter.

3 Quantum universal enveloping algebras

Since they provide one of the most relevant incarnations of quantum groups, QUE (we will henceforth stick to the acronym) algebras are very well known. However, we will rapidly introduce them in order to fix notation; subsequently, we will adopt the less widely established version presented in [Saw06], which is especially appropriate to the development of next chapter.

3.1 The Hopf algebra $\mathcal{U}_x(\mathfrak{g})$ and its involution

The Lie algebra datum We follow [Hum12] for notation and nomenclature about Lie algebras. Let $\mathfrak{g}$ be a simple complex Lie algebra, $\mathfrak{h}$ a maximal toral subalgebra, $\Phi \subset \mathfrak{h}'$ the relative root system, $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ a base for it. Let $(\cdot, \cdot)$ the unique associative non-degenerate symmetric bilinear form on $\mathfrak{h}'$ such that $(\alpha, \alpha) = 2$ for every α in Φ short and D the ratio between the square lengths of a long root and a short root; for types A, D, E we take $(\alpha, \alpha) = 2$ for all α in Φ and $D = 1$.

For each α in Φ, we write $\check{\alpha} := \frac{2}{(\alpha,\alpha)}\alpha$, so that $\check{\Phi}$ is the root system dual to Φ. We recall that $\check{\Delta}$ is a base for $\check{\Phi}$ and that the partial order $\prec$ on Φ restricted to long roots or to short ones, is preserved by $\check{\cdot}$. Therefore if θ is the highest root in Φ and $\check{\varphi}$ is the highest root in $\check{\Phi}$, then φ is the highest short root in Φ and $\check{\theta}$ is the highest short root in $\check{\Phi}$.

Let E be the euclidean space generated by Φ endowed with $(\cdot, \cdot)$; let $\Lambda := \{\lambda \in E \mid (\lambda, \check{\alpha}) \in \mathbb{Z} \ \forall \ \alpha \in \Phi\}$ be the weight lattice, Λ_r the sublattice generated by Φ

and $\Lambda^+ := \{\lambda \in \Lambda \mid (\lambda, \check{\alpha}_i) \geq 0 \ \forall\ 1 \leq i \leq l\}$ the set of dominant integral weights. Furthermore, we denote by L the least positive integer such that $L(\lambda, \mu)$ is integer for all λ, μ in Λ; e.g. $L = l+1$ for type A_l, and the other values are listed in [Saw06].

The fundamental dominant weights and the Cartan integers are

$$\langle \lambda_i, \check{\alpha}_j \rangle := \delta_{ij} \ , \quad a_{ij} := (\check{\alpha}_i, \alpha_j) \ ;$$

as common in the context of QUE algebras, the latter differ from those of [Hum12] by a switch of i and j. We denote the Weyl group by $\mathcal{W}$ and put $\rho := \frac{1}{2} \sum_{\alpha \succ 0} \alpha$; the translated action of $\mathcal{W}$ on E is defined by

$$\sigma \cdot v + \rho := \sigma(v + \rho) \ , \qquad \forall\ v \in E \ , \sigma \in \mathcal{W} \ . \tag{1}$$

Finally we introduce the dual Coxeter number and the Coxeter number:

$$\check{h} := (\rho, \check{\theta}) + 1 \ , \qquad h := (\rho, \check{\varphi}) + 1 \ .$$

The QUE algebra Let $\mathcal{A} = \mathbb{Q}[x, x^{-1}]$, the ring of Laurent polynomials in x with rational coefficients. The x-integers and x-binomials are defined, for all n integer and k positive integer, by

$$[n]_x := \frac{x^n - x^{-n}}{x - x^{-1}} \ , \quad \begin{bmatrix} n \\ k \end{bmatrix}_x := \prod_{j=1}^{k} \frac{[n+1-j]_x}{[j]_x} \ .$$

Note that $[0]_x = 0$, and $[-n]_x = -[n]_x$, whence $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_x = 0$ for $n \geq 0$ and $k > n$; we also put $\left[\begin{smallmatrix} n \\ 0 \end{smallmatrix}\right]_x = 1$ for all integer n. With this understanding we have, for all $n \geq k \geq 1$, the binomial identity

$$\begin{bmatrix} n \\ k \end{bmatrix}_x = x^{-k} \begin{bmatrix} n-1 \\ k \end{bmatrix}_x + x^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_x \ ;$$

in particular $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_x$ in in $\mathcal{A}$ by induction. Finally, we will write $[n]_x! := \prod_{i=1}^{n} [i]_x$ for all positive integer n and $[0]_x! := 1$.

We proceed to introduce $\mathcal{U}_x(\mathfrak{g})$; as an associative algebra over $\mathbb{Q}(x)$, it has generators E_k, F_k, K_k, K_k^{-1}, for $1 \leq k \leq l$, and relations

$$K_i K_j = K_j K_i \ , \quad K_i K_i^{-1} = 1 = K_i^{-1} K_i \ ;$$

$$K_i E_j K_i^{-1} = x^{(\alpha_i, \alpha_j)} E_j \ , \qquad K_i F_j K_i^{-1} = x^{-(\alpha_i, \alpha_j)} F_j \ ;$$

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{x_i - x_i^{-1}} \ ;$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_x E_i^{1-a_{ij}-k} E_j E_i^k = 0 \qquad \text{for } i \neq j \ ;$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_x F_i^{1-a_{ij}-k} F_j F_i^k = 0 \qquad \text{for } i \neq j \ .$$

Here x_i stands for x if α_i is short and for x^D if α_i is long. The following coproduct turns $\mathcal{U}_x(\mathfrak{g})$ into a Hopf algebra:

$$\Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i \ , \quad \Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i \ , \quad \Delta(K_i) = K_i \otimes K_i \ .$$

We agreed with [Saw06], whereas some authors (e.g. [Wen98]) use $\Delta^{\rm op}$ in place of the above Δ; accordingly, the antipode is given by

$$S(K_i) = K_i^{-1} \ , \qquad S(E_i) = -E_i K_i^{-1} \ , \qquad S(F_i) = -K_i F_i \ ,$$

and the counit ϵ is given by $\epsilon(E_i) = 0$, $\epsilon(F_i) = 0$, $\epsilon(K_i) = 1$.

The involution As in [Wen98], we define an involution $\bar{\cdot}$ on $\mathbb{Q}(x)$ by $\overline{q} = q$ for all q in $\mathbb{Q}$ and $\overline{x} = x^{-1}$. Conjugate vector spaces over $\mathbb{Q}(x)$, or conjugate $\mathcal{A}$-moduli, and antilinear maps are accordingly defined. Now, we define an involution $\cdot^*$ on $\mathcal{U}_x(\mathfrak{g})$, prescribing the following:

$$E_i^* = F_i \ , \quad F_i^* = E_i \ , \quad K_i^* = K_i^{-1} \ ;$$

In fact $\cdot^*$ extends to an antilinear antiautomorphism of $\mathcal{U}_x(\mathfrak{g})$ squaring to the identity. Besides, for all a in $\mathcal{U}_x(\mathfrak{g})$,

$$(a^*)_{(1)} \otimes (a^*)_{(2)} = (a_{(2)})^* \otimes (a_{(1)})^* \ , \quad \epsilon(a^*) = \overline{\epsilon(a)} \ , \quad S(a^*) = S(a)^* \ . \tag{2}$$

Note that this involution differs from the one usually given when x is to be specialised to a real number. Most notably we have $K_i^* = K_i^{-1}$ rather than $K_i^* = K_i$. The reason is that x is to be specialised (see Definition 3.2) to a root of unity; so, since K_i acts on a vector of weight λ as the scalar $x^{(\lambda,\alpha_i)}$, the K_i must be unitary.

3.2 The restricted integral form

Definition. Let $\mathcal{U}_x$ a Hopf algebra over $\mathbb{Q}(x)$, and write $\mathcal{A} = \mathbb{Q}[x, x^{-1}]$. An *integral form* of $\mathcal{U}_x$ is a Hopf $\mathcal{A}$-subalgebra $\mathcal{U}_\mathcal{A} \subset \mathcal{U}_x$ such that $\mathcal{U}_x = \mathcal{U}_\mathcal{A} \otimes_\mathcal{A} \mathbb{Q}(x)$.

The *specialisation* of $\mathcal{U}_x$ to a non-zero complex number q is the tensor product $\mathcal{U}_q := \mathcal{U}_\mathcal{A} \otimes_\phi \mathbb{C}$ defined by the homomorphism

$$\phi : \mathcal{A} \to \mathbb{C} \qquad x \mapsto q \ .$$

In our particular case, we consider the so called “restricted” integral form $\mathcal{U}_\mathcal{A}^{\rm res}(\mathfrak{g})$. This nomenclature stresses the contrast with the equally interesting “non-restricted” integral form. Indeed, the representation theories of the respective specialisation differ significantly both from each other and from the classical one. We refer to sections 11.1 of [CP95] for the non-restricted form, or to their original sources [DK90] and [DKP92], where the irreducible modules are shown to be intimately related to the algebraic variety of characters on the centre.

Coming back to $\mathcal{U}_x^{\rm res}(\mathfrak{g})$, as an $\mathcal{A}$-subalgebra of $\mathcal{U}_x(\mathfrak{g})$ it is generated by the elements

$$E_i^{(r)} := \frac{E_i}{[r]_x!} \ , \quad F_i^{(r)} := \frac{F_i}{[r]_x!} \ , \quad K_i \ , \ K_i^{-1} \qquad 1 \le i \le l \ , \ r \in \mathbb{N} \ .$$

We refer to 11.2 of [CP95] for an account of the main properties of $\mathcal{U}_\mathcal{A}^{\rm res}(\mathfrak{g})$; however, they mostly depend on the following fundamental PBW result, which we report without proof.

Theorem. *Let* $\mathcal{U}_{\mathcal{A}}^{\mathrm{res}\,+}(\mathfrak{g})$, $\mathcal{U}_{\mathcal{A}}^{\mathrm{res}\,-}(\mathfrak{g})$ *and* $\mathcal{U}_{\mathcal{A}}^{\mathrm{res}\,0}(\mathfrak{g})$ *be the subalgebras generated by* $\{E_1,\dots,E_l\}$, $\{F_1,\dots,F_l\}$ *and* $\{K_1,\dots,K_l\}$. *Then, given an enumeration of the positive roots* $\{\beta_1,\dots,\beta_N\}$, *the sets* $\{E_1,\dots,E_l\}$ *and* $\{F_1,\dots,F_l\}$ *may be completed to sets* $\{E_{\beta_1},\dots,E_{\beta_N}\}$ *and* $\{F_{\beta_1},\dots,F_{\beta_N}\}$ *such that*

$$K_i E_{\beta_j} K_i^{-1} = x^{(\alpha_i,\beta_j)} E_{\beta_j} \ , \quad K_i F_{\beta_j} K_i^{-1} = x^{-(\alpha_i,\beta_j)} F_{\beta_j} \ . \tag{1}$$

Moreover, $\mathcal{U}_{\mathcal{A}}^{\mathrm{res}\,+}(\mathfrak{g})$ *and* $\mathcal{U}_{\mathcal{A}}^{\mathrm{res}\,-}(\mathfrak{g})$ *are free* $\mathcal{A}$*-modules with bases formed by the products*

$$E_{\beta_N}^{(t_N)} \cdots E_{\beta_1}^{(t_1)} \ , \quad F_{\beta_N}^{(t_N)} \cdots F_{\beta_1}^{(t_1)} \ ,$$

for $t_1,\dots,t_N$ *non-negative integers such that* $t_j > 0$ *for at least one* j.

The subalgebra $\mathcal{U}_{\mathcal{A}}^{\mathrm{res}\,0}(\mathfrak{g})$ *is a free* $\mathcal{A}$*-module as well, with basis formed by the products*

$$\prod_{i=1}^{l} K_i^{\sigma_i} \begin{bmatrix} K_i;0 \\ s_i \end{bmatrix}_{x^i} \ , \qquad \text{where} \quad \begin{bmatrix} K_i;0 \\ s_i \end{bmatrix}_{x^i} := \prod_{k=1}^{s_i} \frac{K_i x_i^{1-k} - K_i^{-1} x_i^{k-1}}{x_i^k - x_i^{-k}} \ ,$$

for $s_1,\dots,s_l$ *non-negative integers and* σ_i *in* $\{0,1\}$. *Finally, multiplication defines an isomorphism of* $\mathcal{A}$*-modules*

$$\mathcal{U}_{\mathcal{A}}^{\mathrm{res}\,-}(\mathfrak{g}) \otimes \mathcal{U}_{\mathcal{A}}^{\mathrm{res}\,0}(\mathfrak{g}) \otimes \mathcal{U}_{\mathcal{A}}^{\mathrm{res}\,+}(\mathfrak{g}) \quad \longrightarrow \quad \mathcal{U}_{\mathcal{A}}^{\mathrm{res}}(\mathfrak{g}) \ .$$

The fact that $\mathcal{U}_{\mathcal{A}}^{\mathrm{res}}(\mathfrak{g})$ is an integral form of $\mathcal{U}_x(\mathfrak{g})$, for which we have the same basis, follows at once. Furthermore, the representation theory of $\mathcal{U}_x(\mathfrak{g})$ is treated mostly as for the classical enveloping algebra $\mathcal{U}(\mathfrak{g})$, or equivalently, the Lie algebra $\mathfrak{g}$ itself (see § 20 and 21 of [Hum12]). In our case though, the role of $\mathfrak{h}$ is played by $\mathcal{U}_x^0(\mathfrak{g})$, the subalgebra generated by $\{K_1,\dots,K_l\}$.

Proposition-Definition. We consider the inclusion

$$\iota : \Lambda_r \to \mathcal{U}_x^0(\mathfrak{g}) \qquad \alpha_i \mapsto K_i \quad \text{for } 1 \le i \le l \ ,$$

and write $\iota(\beta) =: K_\beta$. Let V be a $\mathcal{U}_x^0(\mathfrak{g})$-module and ω a homomorphism from $\mathcal{U}_x^0(\mathfrak{g})$ to the multiplicative group of $\mathbb{Q}(x)$. A non-zero vector v in V is said to be an *ω-vector* if $K_\beta v = \omega(\beta) v$ for all β in Λ_r. For $1 \le i \le l$,

$$K_\beta E_i v = q^{(\alpha_i,\beta)} \omega(\beta) E_i v \quad \text{and} \quad K_\beta F_i v = q^{-(\alpha_i,\beta)} \omega(\beta) F_i v \ . \tag{2}$$

The representation category Rep $(\mathcal{U}_x(\mathfrak{g}))$ is semi-simple as in the classical case, but now the simple objects are parametrised by homomorphism of the form

$$\omega_{\sigma,\lambda}(K_\beta) = -1^{\sigma(\beta)} x^{(\lambda,\beta)} \ ,$$

where λ is a dominant integral weight and σ is a homomorphism from Λ_r to $\mathbb{Z}/(2\mathbb{Z})$. In detail, the module $V_{\sigma,\lambda}$ contains an $\omega_{\sigma,\lambda}$-vector $v_{\sigma,\lambda}$ and $V_{\sigma,\lambda} = \mathcal{U}_x^-(\mathfrak{g}) v_{\sigma,\lambda}$, where $\mathcal{U}_x^-(\mathfrak{g})$ is the subalgebra generated by $\{F_1,\dots,F_l\}$. Moreover, $V_{\sigma,0}$ is one-dimensional for all σ, with

$$K_i v_{\sigma,0} = -1^{\sigma(\alpha_i)} \ , \quad E_i v_{\sigma,0} = 0 \ , \quad F_i v_{\sigma,0} = 0 \ , \qquad \forall\, 1 \le i \le l \ ,$$

and $V_{\sigma,\lambda} = V_{\sigma,0} \otimes V_{0,\lambda}$ for all σ, λ. For this reason, we are just going to consider the cases $\sigma = 0$ and speak of weight vectors as usual, rather than $\omega_{0,\lambda}$-vectors. We also write $V_{0,\lambda} =: V_\lambda$ and call it the Weyl module of highest weight λ. In fact all its weights are of the form $\lambda - \sum_{i=1}^{l} c_i\alpha_i$, where c_i are non-negative integers, by identities (2); the weight spaces have the same multiplicities as in the classical case, and the tensor products $V_\lambda \otimes V_\mu$ obey the same fusion rules.

Sawin's construction and R-matrix

For all dominant integral weight λ, we consider the $\mathcal{U}_\mathcal{A}^{\mathrm{res}}$-module $V_{\lambda,\mathcal{A}} := \mathcal{U}_\mathcal{A}^{\mathrm{res}\,-} v_\lambda \subset V_\lambda$. Thanks to the Theorem, this is a free $\mathcal{A}$-module and $V_\lambda = V_{\lambda,\mathcal{A}} \otimes_\mathcal{A} \mathbb{Q}(x)$.

Furthermore, we consider the discrete Hopf algebra (see Definition 2.1B) $\mathcal{U}_\mathcal{A}^\dagger(\mathfrak{h}) := \mathcal{A}^\Lambda$, the set of maps from Λ to $\mathcal{A}$. It may be thought as a direct product of one-dimensional matrix algebras indexed by Λ; accordingly, the k-th generalised tensor power is $\mathcal{A}^{(\Lambda^k)}$. The coproduct, counit and antipode are as follows:

$$(\Delta f)(\lambda,\mu) = f(\lambda+\mu)\ , \quad \epsilon(f) = f(0)\ , \quad (Sf)(\lambda) = f(-\lambda) \qquad \forall\ \lambda,\mu \in \Lambda\ .$$

Next, we consider the root space decomposition of $\mathcal{U}_\mathcal{A}^{\mathrm{res}}(\mathfrak{g})$, i.e. the decomposition into weight spaces with respect to the adjoint action: $\mathcal{U}_\mathcal{A}^{\mathrm{res}\,0}(\mathfrak{g})$ is the 0-space, and identities (1) produce the decomposition for $\mathcal{U}_\mathcal{A}^{\mathrm{res}\,+}(\mathfrak{g})$ and $\mathcal{U}_\mathcal{A}^{\mathrm{res}\,-}(\mathfrak{g})$. Multiplication of each λ-space by $f(\lambda)$ defines an action of $\mathcal{U}_\mathcal{A}^\dagger(\mathfrak{h})$ on $\mathcal{U}_\mathcal{A}^{\mathrm{res}}(\mathfrak{g})$, and it is readily checked that $\mathcal{U}_\mathcal{A}^{\mathrm{res}}(\mathfrak{g})$ is a Hopf algebra in the category of $\mathcal{U}_\mathcal{A}^\dagger(\mathfrak{h})$-modules, namely all the operations are $\mathcal{U}_\mathcal{A}^\dagger(\mathfrak{h})$-linear.

Therefore, we may consider the smash product $\mathcal{U}_\mathcal{A}^{\mathrm{res}}(\mathfrak{g}) \rtimes \mathcal{U}_\mathcal{A}^\dagger(\mathfrak{h})$ (see [Mon93]). It is $\mathcal{U}_\mathcal{A}^{\mathrm{res}}(\mathfrak{g}) \otimes \mathcal{U}_\mathcal{A}^\dagger(\mathfrak{h})$ as an $\mathcal{A}$-module and, writing $a \otimes h =: ah$, a product is defined by

$$(ah)(bk) = (ah_{(1)}b)(h_{(2)}k) \qquad \forall\ a,b \in \mathcal{U}_\mathcal{A}^{\mathrm{res}}(\mathfrak{g})\ ,\ h,k \in \mathcal{U}_\mathcal{A}^\dagger(\mathfrak{h})\ .$$

We endow $\mathcal{U}_\mathcal{A}^{\mathrm{res}}(\mathfrak{g}) \rtimes \mathcal{U}_\mathcal{A}^\dagger(\mathfrak{h})$ with the unique coproduct restricting to the coproducts already given on the subalgebras $\mathcal{U}_\mathcal{A}^{\mathrm{res}}(\mathfrak{g})$ and $\mathcal{U}_\mathcal{A}^\dagger(\mathfrak{h})$.

Even though we have to consider the tensor products $\mathcal{U}_\mathcal{A}^{\mathrm{res}}(\mathfrak{g})^{\otimes k} \otimes \mathcal{A}^{(\Lambda^k)}$ rather than the usual tensor powers, $\mathcal{U}_\mathcal{A}^{\mathrm{res}}(\mathfrak{g}) \rtimes \mathcal{U}_\mathcal{A}^\dagger(\mathfrak{h})$ satisfies all the axioms of a Hopf algebra in the category of $\mathcal{A}$-modules. For all f in $\mathcal{U}_\mathcal{A}^{\mathrm{res}}(\mathfrak{g})$, we have the relations

$$fK_i = K_i f\ , \quad fE_i = E_i f(\cdot + \alpha_i)\ , \quad fF_i = F_i f(\cdot - \alpha_i)\ . \tag{3}$$

The involution extends to $\mathcal{U}_\mathcal{A}^\dagger(\mathfrak{h})$ by $f^* = \overline{f}$, preserving properties 3.1(2).

As our next step, we define $\mathcal{U}_\mathcal{A}^\dagger(\mathfrak{g})^{\otimes k}$ as the quotient of each $\mathcal{U}_\mathcal{A}^{\mathrm{res}}(\mathfrak{g})^{\otimes k} \otimes \mathcal{A}^{(\Lambda^k)}$ by the kernel I_k of its action on

$$\bigoplus_{\lambda_1,\dots,\lambda_k \in \Lambda^+} V_{\lambda_1,\mathcal{A}} \otimes \cdots \otimes V_{\lambda_k,\mathcal{A}}\ , \qquad \forall\ k \in \mathbb{N}\ ;$$

in other words we identify the elements of $\mathcal{U}_\mathcal{A}^{\mathrm{res}}(\mathfrak{g})^{\otimes k} \otimes \mathcal{A}^{(\Lambda^k)}$ with their action on all the possible tensor products $V_{\lambda_1,\mathcal{A}} \otimes \cdots \otimes V_{\lambda_k,\mathcal{A}}$, so that $\mathcal{U}_\mathcal{A}^\dagger(\mathfrak{g})$ includes into a discrete algebra:

$$\mathcal{U}_\mathcal{A}^\dagger(\mathfrak{g})^{\otimes k} \quad \hookrightarrow \quad \bigoplus_{\lambda_1,\dots,\lambda_k \in \Lambda^+} \mathrm{End}\left(V_{\lambda_1,\mathcal{A}} \otimes \cdots \otimes V_{\lambda_k,\mathcal{A}}\right) \qquad \forall\ k \in \mathbb{N}\ . \tag{4}$$

Finally, we replace each $\mathcal{U}_{\mathcal{A}}^{\dagger}(\mathfrak{g})^{\otimes k}$ with its closure with respect to the topology induced by the above inclusions.

Since the various $\overbrace{\mathrm{id}\otimes\cdots\otimes\mathrm{id}\otimes\Delta\otimes\mathrm{id}\otimes\cdots\otimes\mathrm{id}}^{k\text{ factors}}$ map I_k into I_{k+1} and are obviously continuous with respect to the given topologies, $\mathcal{U}_{\mathcal{A}}^{\dagger}(\mathfrak{g})$ still satisfies the axioms of a Hopf algebra in the category of $\mathcal{A}$-modules, and we take it as our integral form; likewise, $\cdot^*$ lowers to the quotient and then extends to the closure retaining all its properties.

Remark. By definition of weight vectors,

$$K_i = x^{(\alpha_i,\cdot)} \ , \quad \begin{bmatrix} K_i;0 \\ s_i \end{bmatrix}_{x^i} = \begin{bmatrix} (\check{\alpha}_i,\cdot) \\ s_i \end{bmatrix}_{x^i} \qquad \forall\, 1 \le i \le l \ . \tag{5}$$

So we may think of $\mathcal{U}_{\mathcal{A}}^{\dagger}(\mathfrak{h})$ and its generalised tensor powers $\mathcal{A}^{(\Lambda^k)}$ as extensions of $\mathcal{U}_{\mathcal{A}}^{\mathrm{res}\,0}(\mathfrak{g})^{\otimes k}$: more concretely, instead of being limited to the $\mathcal{A}$-linear span of products of elements of the form (5), all maps $\Lambda^k \to \mathcal{A}$ are available. We also remark that the elements of $\mathcal{U}_{\mathcal{A}}^{\mathrm{res}}(\mathfrak{g})^{\otimes k} \otimes \mathcal{A}^{(\Lambda^k)}$ may be viewed as maps $\Lambda^k \to \mathcal{U}_{\mathcal{A}}^{\mathrm{res}}(\mathfrak{g})^{\otimes k}$.

The upside of $\mathcal{U}_{\mathcal{A}}^{\dagger}(\mathfrak{g})$ over $\mathcal{U}_{\mathcal{A}}^{\mathrm{res}}(\mathfrak{g})$ is that it provides a suitable environment where to define the R-matrix.

To this aim we just still need a slight modification. Let $\mathcal{A}' = \mathbb{Z}[y, y^{-1}]$, define an inclusion $\mathcal{A} \hookrightarrow \mathcal{A}'$ by $x \mapsto y^L$ and put $\mathcal{U}_{\mathcal{A}'}^{\dagger}(\mathfrak{g}) := \mathcal{U}_{\mathcal{A}}^{\dagger}(\mathfrak{g}) \otimes_{\mathcal{A}} \mathcal{A}'$. This amounts to adjoining to $\mathcal{A}$ an L-th root of 1, and we write $x^{\frac{n}{L}} =: y^n$ for all integer n. We are now ready to write the R-matrix, lying in $\mathcal{U}_{\mathcal{A}'}^{\dagger}(\mathfrak{g})^{\otimes 2}$:

$$R(\lambda,\mu) := x^{(\lambda,\mu)} \sum_{t_1,\dots,t_N=0}^{+\infty} \prod_{j=1}^{N} x_j^{\frac{t_j(t_j+1)}{2}} (1-x_j^2)^{t_j} ([t_j]_{x_j}!) \left(E_{\beta_j}^{(t_j)} \otimes F_{\beta_j}^{(t_j)} \right) ,$$

where x_r stands for x if β_r is short and for x^D if it is long. We note that $x^{(\lambda,\mu)}$ is the only possibly fractioned power in the expression, with (λ,μ) lying in $\mathbb{Z}/L$ by definition of L. More importantly, the series is actually Cauchy with respect to the topology induced by (4), because each of the summands vanishes on $V_{\lambda,\mathcal{A}} \otimes V_{\mu,\mathcal{A}}$ for all but finitely many choices of λ, μ; therefore it defines a legitimate element of $\mathcal{U}_{\mathcal{A}'}^{\dagger}(\mathfrak{g})^{\otimes 2}$.

Lemma. *R is invertible in $\mathcal{U}_{\mathcal{A}'}^{\dagger}(\mathfrak{g})^{\otimes 2}$ and it has the following properties:*

$$R\Delta(\cdot)R^{-1} = \Delta^{\mathrm{op}} \ , \quad (\Delta\otimes\mathrm{id})R = R_{13}R_{23} \ , \quad (\mathrm{id}\otimes\Delta)R = R_{13}R_{12} \ ,$$
$$R^* = R_{21}^{-1} \ .$$

Proof. For the first three identities we refer to chapter 32 of [Lus93]. The last one is proved in [Wen98] (Lemma 1.4.1), using $\Delta(\cdot)^* = \Delta^{\mathrm{op}}(\cdot^*)$ (the first of identities (2)) together with the peculiar form of R. □

The Lemma may be rephrased by saying that R is a quasi-triangular structure for $\mathcal{U}_{\mathcal{A}'}^{\dagger}(\mathfrak{g})$, and that R^* is the opposite structure (Definition 2.3C and the subsequent observations straightforwardly adapt to the present scenario).

Finally, $\omega := x^{(2\rho,\cdot)}$ is a group-like charmed element for R (see the end of 2.3). Indeed $\omega a \omega^{-1} = K_{2\rho} a K_{2\rho}^{-1} = S^2(a)$ is easily checked for the generators of $\mathcal{U}_x(\mathfrak{g})$. However, even though we know R explicitly, a verification of the second of identities 2.3(8) would still be quite laborious so we refrain from undertaking it here.

3.3 Roots of 1

We now fix a complex root of 1 q of order ℓ', and specialise (see Definition 3.2) $\mathcal{U}^\dagger_{\mathcal{A}'}(\mathfrak{g})$ to it. Let ℓ be the order of q^2, i.e. $\ell = \ell'/2$ if ℓ' is even and $\ell = \ell'$ if ℓ' is odd, and choose r such that $r^L = q$. We consider the specialisation $\mathcal{U}_q(\mathfrak{g}) := \mathcal{U}^\dagger_{\mathcal{A}'}(\mathfrak{g}) \otimes_{\mathcal{A}'} \mathbb{C}$ obtained by identifying $x^{1/L}$ with r.

Furthermore, rather than considering the category of all $\mathcal{U}_q(\mathfrak{g})$-modules, we limit ourselves to the "admissible" ones; this piece of nomenclature is borrowed from [NY15], and it corresponds to the "type 1" of [CP95].

Definition. A finite dimensional $\mathcal{U}_q(\mathfrak{g})$-module V is said to be *admissible* if, as a vector space, has decomposition $V = \bigoplus_\lambda V^\lambda$ such that for all λ

$$fv = f(\lambda)v \qquad \forall\ f \in \mathcal{U}^\dagger_q(\mathfrak{h})\ ,\ v \in V^\lambda\ ,$$

where $\mathcal{U}^\dagger_q(\mathfrak{h}) = \mathcal{U}^\dagger_{\mathcal{A}'}(\mathfrak{h}) \otimes_{\mathcal{A}'} \mathbb{C}$. From now on, $\operatorname{Rep}\big(\mathcal{U}_q(\mathfrak{g})\big)$ will stand for the category of admissible $\mathcal{U}_q(\mathfrak{g})$-modules.

In other words we are excluding the analogue of the case $\sigma \neq 0$ described before Sawin's construction. Note that, by 3.2(5), each V^λ is a weight space according to Definition 11.2.2 in [CP95]. Of course, the specialised modules $V_\lambda(q) := V_{\lambda,\mathcal{A}'} \otimes_{\mathcal{A}'} \mathbb{C}$ are admissible but, as it is well known, not all of them are simple; more precisely, we rather have the forthcoming Lemma A, known as linkage principle.

Prior to reporting it, we need to introduce the affine Weyl group $\mathcal{W}_\ell$. It is generated by the reflections across the hyperplanes

$$H_{\alpha,k} = \begin{cases} \{(x,\alpha) = k\ell\} & \text{if } D \mid \ell \\ \{(x,\check{\alpha}) = k\ell\} & \text{if } D \nmid \ell \end{cases} \qquad \alpha \in \Phi\ ,\ k \in \mathbb{Z}\ .$$

We list below some basic properties, and establish some nomenclature, for $\mathcal{W}_\ell$ and its translated action (defined as in 3.1(1)). If $D \mid \ell$ ($D \nmid \ell$),

- $\mathcal{W}_\ell = \mathcal{W} \rtimes \ell\check{\Lambda}_r$ ($\mathcal{W} \rtimes \ell\Lambda_r$);
- $\mathcal{W}_\ell$ is generated by the reflections across the hyperplanes orthogonal to the simple roots together with the translation by $\ell\theta/D$ ($\ell\varphi$);
- $\mathcal{W}_\ell$ acts freely and transitively on the set of the connected components of the complement of $\bigcup_{\alpha,k}(H_{\alpha,k} - \rho)$, called *Weyl alcoves*;
- the *principal Weyl alcove* C_ℓ, defined as

  $$\big\{x \in E \mid (x+\rho,\alpha_i) > 0 \text{ for } i = 1,\dots,l\ ,\ (x+\rho,\theta) < \ell\ \big((x+\rho,\varphi) < \ell\big)\big\}\ ,$$

  is a Weyl alcove, and its closure $\overline{C_\ell}$ is a fundamental domain for the translated action of $\mathcal{W}_\ell$.

Lemma A (Linkage principle)**.** *Let λ, μ be dominant integral weights. Then there exists a non-zero homomorphism from a subobject of $V_\lambda(q)$ to a quotient of $V_\mu(q)$ if and only if λ and μ belong to the same $\mathcal{W}_\ell$-orbit.*

Remark. The PBW basis for $\mathcal{U}_{\mathcal{A}}^{\mathrm{res}\,-}(\mathfrak{g})$ given by Theorem 3.2 specialises to a basis for the subalgebra $\mathcal{U}_q^-(\mathfrak{g})$ generated by $\{F_1, \dots, F_l\}$. Therefore, by the arguments of § 20 of [Hum12], any $\mathcal{U}_q(\mathfrak{g})$-module generated by a vector of highest weight λ is indecomposable and admits a unique irreducible quotient, which is the unique irreducible module of maximal weight λ.

So, as a consequence of the linkage principle, if λ lies in $\overline{C_\ell}$ then $V_\lambda(q)$ is irreducible. Indeed for any submodule we can consider the submodule generated by a vector of highest weight $\mu \prec \lambda$ in it; since μ is in the principal Weyl alcove as well, $\mu = \lambda$ because $\overline{C_\ell}$ is a fundamental domain.

We next recall the quantum version of Weyl's formula, which expresses the quantum dimension of the Weyl modules in terms of q and the root system. In the formula $\mathrm{qdim}(V) = \mathrm{qtr}_V(\mathrm{id}_V)$, the categorical trace corresponding to the spherical structure given by $q^{2(\rho,\cdot)}$, i.e. $\mathrm{qtr}_V(f) = \mathrm{Tr}(q^{2(\rho,\cdot)} f)$ for all f in (V, V).

$$\mathrm{qdim}\,(V_\lambda(q)) = \prod_{\Phi \ni \beta \succ 0} \frac{q^{(\lambda+\rho,\beta)} - q^{-(\lambda+\rho,\beta)}}{q^{(\rho,\beta)} - q^{-(\rho,\beta)}}\ , \tag{1}$$

provided that $\ell/D \geq \check{h}$ ($\ell \geq h$) if $D \mid \ell$ ($D \nmid \ell$); this condition is exactly the requirement that each of the above denominators does not vanish.

4 Fusion categories

As it is well-known, $\mathrm{Rep}\,(\mathcal{U}_q(\mathfrak{g}))$ is not a semi-simple category, and a standard way to circumvent the problem is introducing "tilting modules". The representation theory of quantum groups at roots of unity was studied in [APK90], and tilting modules are treated in detail in [And92a].

We report the definition and basic properties of tilting modules in 4.1. Notably, they form a monoidal subcategory $\mathcal{T}_q(\mathfrak{g})$ of $\mathrm{Rep}\,(\mathcal{U}_q(\mathfrak{g}))$, which behaves in many respects like the semi-simple tensor category of classical $\mathfrak{g}$-modules. Indeed one recovers a C^* ribbon category (see Definition 1.5C) with a finite number of objects $\overline{\mathcal{T}_q(\mathfrak{g})}$ after quotienting by a certain "negligible" ideal, through the basic categorical procedure we recalled in Proposition 1.2; this will be done in 4.2.

Finally, 4.4 contains an explicit realisation of the mentioned quotient as a linear subcategory of $\mathrm{Rep}\,(\mathcal{U}_q(\mathfrak{g}))$, by the introduction of a suitable "truncated" tensor product. In particular, we will have the forgetful functor $\mathcal{W} : \overline{\mathcal{T}_q(\mathfrak{g})} \to \mathrm{Hilb}$, which will be shown to possess a natural weak tensor structure in Lemma 4.4A. Therefore, combining the theory of 2.7 and 2.8, we will obtain a unitary coboundary weak Hopf algebra, as shown in Lemma 4.4B.

The two mentioned Lemmas reformulate part of Theorem 26.1 of [CCP21]; on the whole, subsection 4.4 is made possible by the analysis in [Wen98] about invariant forms on tensor products of Weyl modules, which we will outline in 4.3.

4.1 Tilting modules

Definition A. An admissible $\mathcal{U}_q(\mathfrak{g})$-module V has a *Weyl filtration* if there exists a sequence of submodules

$$0 = V_0 \subset \cdots \subset V_p = V$$

with $V_r/V_{r-1} \simeq V_{\lambda_r}(q)$, where λ_r is a dominant integral weight, for $r = 1\ldots, p$.

An admissible $\mathcal{U}_q(\mathfrak{g})$-module V is said to be *tilting* if both V and its dual V' have Weyl filtrations.

We proceed to report some basic properties of tilting modules. The Proposition and Lemma A are Propositions 11.3.3 and 11.3.4 of [CP95], while Lemma B is Theorem 2 of [Saw06].

Proposition.
- *The dual of a tilting module is tilting;*
- *the direct sum of two tilting modules is tilting;*
- *any direct summand of a tilting module is tilting;*
- *the tensor product of two tilting modules is tilting.*

Lemma A. *For all dominant integral weight λ, there exists a unique up to isomorphism indecomposable tilting module $T_q(\lambda)$ such that:*

- *the set of weights of $T_q(\lambda)$ is contained in the convex hull of the $\mathcal{W}$-orbit of λ;*
- *λ is the unique highest weight of $T_q(\lambda)$, and* $\dim\big(T_q(\lambda)^\lambda\big) = 1$*;*
- *$T_q(\lambda)' \simeq T_q\big(-w_0(\lambda)\big)$, where w_0 is the unique element of $\mathcal{W}$ mapping all positive roots to negative roots.*

Lemma B. *For every dominant integral weight λ,* $\operatorname{qdim}\big(T_q(\lambda)\big) \neq 0$ *exactly if λ lies in C_ℓ.*

As reported in [Wen98], tilting modules may be also characterized in terms of fundamental modules, which are the specialisation to q of the classical fundamental modules (see chapter 13 of [Cart05]). Following [Wen98], for each Lie type we choose a certain fundamental module V and let κ be its highest weight. For the classical types, V is chosen as it follows (the vertices in the Dynkin diagrams are numbered as in [Hum12]):

A_l) the natural vector module of $\mathfrak{sl}_{l+1}$, κ is the fundamental dominant weight λ_1;

B_l) the spin module of $\mathfrak{o}_{2l+1}$, $\kappa = \lambda_l$;

C_l) the natural vector module of $\mathfrak{sp}_{2l}$, $\kappa = \lambda_1$;

D_l) any of the two spin modules of $\mathfrak{o}_{2l}$, κ equal to λ_{l-1} and λ_l respectively;

E_6) κ is any of λ_1 or λ_6, i.e. the corresponding vertex in the Dynkin diagram is chosen between the two farthest from the branching point;

E_7, E_8) κ equals respectively, λ_7 and λ_8, i.e. its vertex is the farthest from the branching point;

F_4) $\kappa = \lambda_4$, i.e. its vertex is connected to only one edge and α_4 is short;

G_2) $\kappa = \lambda_1$, i.e. α_1 is short.

Theorem. *Suppose that κ lies in the principal Weyl alcove C_ℓ. Then an admissible $\mathcal{U}_q(\mathfrak{g})$-module is tilting if and only if it is a direct sum of direct summands of tensor powers of $V(q)$.*

In view of the above results, we fix some bits of notation: if $D \mid \ell$ ($D \nmid \ell$) we write $k := \ell/D - \check{h}$ ($l - h$), and $h(\lambda) := (\lambda, \check{\theta})$ $((\lambda, \phi))$ for every dominant integral weight λ. So λ lies in C_l exactly if $h(\lambda) \leq k$. We also write $\Lambda_\ell := \Lambda^+ \cap C_\ell$. The integer k is usually called level (see the introduction for the relation with the level for vertex operator algebra modules).

Example. $h(\kappa)$ is 1 for the classical types, E_6 and E_7; it is 2 for E_8. If $D \mid \ell$ ($D \nmid \ell$), it is 1 (2) for F_4 and G_2.

4.2 Negligible modules and the quotient category

Let $\mathcal{T}_q(\mathfrak{g})$ be the full subcategory of Rep $(\mathcal{U}_q(\mathfrak{g}))$ whose objects are the tilting modules. Thanks to Proposition 4.1, it inherits the monoidal structure of $\mathcal{U}_q(\mathfrak{g})$, and direct sums are well defined. However, we note that $\mathcal{T}_q(\mathfrak{g})$ is not abelian, because some morphisms fail to have a kernel; this may be seen by a basic analysis of the indecomposable tilting modules with highest weight outside C_ℓ for $\mathfrak{g} = \mathfrak{sl}_2$ (see Example 11.3.9 in [CP95]).

In order to recover semi-simplicity, we need to quotient out a certain "negligible" ideal of $\mathcal{T}_q(\mathfrak{g})$ by means of the construction described in Proposition 1.2. As we shall see, this actually yields a fusion category, i.e. we will have just a finite number of non-equivalent simple objects, indexed by Λ_ℓ.

The following definition of negligible morphisms is the one given in 3.3 of 4.3.

Definition. A morphism $f : S \to T$ between tilting modules is said to be *negligible* if $\operatorname{qtr}_S(gf) = 0$ for all morphism $g : T \to S$. A tilting module is said be negligible if its categorical trace is identically zero.

Example. For all dominant integral weight λ, the highest weight indecomposable tilting module $T_q(\lambda)$ (see Lemma 4.1A) is negligible if and only if $\operatorname{qdim}(T_q(\lambda)) = 0$.

To see this, let $\operatorname{qtr}_\lambda$ be the categorical trace on $T_q(\lambda)$ and consider an endomorphism f of $T_q(\lambda)$; writing s for the semi-simple part of f, $\operatorname{qtr}_\lambda(f) = \operatorname{qtr}_\lambda(s)$. Besides, $T_q(\lambda)$ is indecomposable and each of the eigenspaces of s is a submodule, there is just one of them, i.e. s is a multiple of the identity, so the claim about negligibility of $T_q(\lambda)$ follows from Lemma 4.1B.

Remark. As a consequence, given a tilting module T we can apply Proposition 4.1 and Lemma 4.1A to obtain a decomposition $T \simeq \left(\bigoplus_{\lambda \in \Lambda_\ell} n_\lambda T_q(\lambda) \right) \oplus N$, where the n_λ are non-negative integers and N is negligible.

Moreover, by Lemma 3.3 $V_\lambda(q)$ is irreducible for all λ in Λ_ℓ; hence its dual is $V_{-w_0(\lambda)(q)}$ as in the classical case. Since $-w_0(\rho) = \rho$, $V_{-w_0(\lambda)(q)}$ is in turn irreducible, and we conclude that $T_q(\lambda) = V_\lambda(q)$ for all λ in Λ_ℓ. Therefore the above decomposition may be rewritten as

$$T \simeq \Big(\bigoplus_{\lambda\in\Lambda_\ell} n_\lambda V_\lambda(q)\Big) \oplus N \ . \tag{1}$$

Proposition. *Negligible morphisms form an ideal $\mathcal{N}$ of $\mathcal{T}_q(\mathfrak{g})$ (see Proposition 1.2), coinciding with the ideal generated by the negligible tilting modules, or more properly by the identities* id_T *for T negligible. Moreover, if f is a negligible morphism so are $f\otimes g$ and $g\otimes f$ for all morphism g.*

The quotient monoidal category $\overline{\mathcal{T}_q(\mathfrak{g})}$ constructed as in Proposition 1.2 is a semi-simple abelian category and the restriction of the quotient map to the linear subcategory of tilting modules whose negligible part in the decomposition (1) *vanishes is a linear equivalence. Therefore the Weyl modules $V_\lambda(q)$ with λ ranging in Λ_ℓ form a complete collection of non-equivalent simple objects for $\overline{\mathcal{T}_q(\mathfrak{g})}$.*

Finally, the duality pairs, the braiding and the ribbon obtained by taking the suitable quotients turn $\overline{\mathcal{T}_q(\mathfrak{g})}$ into a ribbon category (see Definition 1.5C).

Proof. If $g : T \to U$ is negligible so is any composition fgh, since $\mathrm{qtr}(efgh) = \mathrm{qtr}(hefg)$; it is also clear that if a morphism f factors through a negligible module f itself is negligible, so the ideal composed by the linear spans of such morphism is contained in $\mathcal{N}$.

On the other hand, let $f : T \to U$ be negligible, and consider idempotents $\mathrm{id}_T = p_s + p_n$, $\mathrm{id}_U = q_s + q_n$ providing the decomposition (1) for T and U. Then

$$0 = \mathrm{qtr}_T(gf) = \mathrm{qtr}_T(gq_sfp_s) \qquad \forall\ g : U \to T \ .$$

But the Weyl modules $V_\lambda(q)$ with λ in Λ_ℓ are irreducible and they have non-zero quantum dimension by Lemma 4.1B, so we must have $q_sfp_s = 0$, i.e. $f = q_nf + q_sfp_n$; the first sentence in the statement is proved. Moving on, let us also consider a tensor product $f\otimes g$ for an arbitrary morphism g between tilting modules V and W; as just shown we may write $f = f_1f_2$ with f_1 in (N,U), f_2 in (T,N) and N negligible. So we have

$$\begin{aligned}\mathrm{qtr}_{T\otimes V}\big(h(f\otimes g)\big) &= \mathrm{qtr}_{T\otimes V}\big((f_1\otimes g)(f_2\otimes \mathrm{id}_W)h\big)\\ = \mathrm{qtr}_{N\otimes V}\big((f_2\otimes \mathrm{id}_W)h(f_1\otimes g)\big) &\qquad \forall\ h\in (U\otimes W, T\otimes V)\ .\end{aligned}$$

Now $\mathrm{End}(N\otimes V) \simeq N\otimes (V\otimes V')\otimes N' \simeq N\otimes \mathrm{End}(V)\otimes N'$, and by taking the categorical trace of the middle factor we obtain a homomorphism $\gamma : \mathrm{End}(N\otimes V) \to \mathrm{End}(N)$ (usually called a “contraction”, see 3.2 in [Wen98]) such that $\mathrm{qtr}_{N\otimes V} = \mathrm{qtr}_N\circ\gamma$; in particular $\mathrm{qtr}_{N\otimes V} = 0$ since N is negligible, so $\mathrm{qtr}_{T\otimes V}\big(h(f\otimes g)\big)$ vanishes for all h, i.e. $f\otimes g$ is negligible.

Given that all the assumptions of Proposition 1.2 are met, we construct the quotient monoidal category $\overline{\mathcal{T}_q(\mathfrak{g})}$. By the above discussion, the Weyl modules with highest weights in Λ_ℓ form a complete collection of mutually non-equivalent simple objects for $\overline{\mathcal{T}_q(\mathfrak{g})}$, so the restriction of the statement is indeed a linear equivalence. The final sentence is self-evident. □

4.3 Invariant forms on tensor products

For the rest of current section, we fix a root of 1 of the form $q = e^{\frac{\imath\pi}{\ell}}$ with $D \mid \ell$, and also assume ℓ to be so large that the highest weight of the fundamental module (see 4.1) is contained in Λ_ℓ. By Example 4.1 this amounts to assume the level $k = \ell/D - \check{h}$ to be at least 1 except for type E_8, for which the level is needed to be at least 2.

Lemma. *Let λ be a dominant integral weight and η_λ a highest weight vector in the Weyl module V_λ. Then there is a unique invariant sesquilinear form $(\cdot,\cdot)$ on V_λ with $(v_\lambda, v_\lambda) = 1$, and such form is Hermitian; the adjectives "Hermitian" and "invariant" mean respectively*

$$(v, w) = \overline{(w, v)} \quad \text{and} \quad (av, w) = (v, a^* w) \qquad \forall\ v, w \in V_\lambda\ , a \in \mathcal{U}_x(\mathfrak{g})\ . \tag{1}$$

Furthermore, consider the arc $I_\lambda = \{q_t := e^{\imath\pi t} \mid |t| < \frac{1}{m-D}\}$, where $m = (\lambda+\rho, \theta)$. Then the specialised module $V_\lambda(q_t)$ is irreducible for all q_t in I_λ, and $(\cdot,\cdot)$ specialises to an invariant positive definite Hermitian form.

We also note that λ lies in $\overline{C_\ell}$ exactly if $q = e^{\frac{\imath\pi}{\ell}}$ is in I_λ, and then for all $|t| < \frac{1}{\ell}$ such that q_t is a root of 1 λ also lies in $\overline{C_{\ell_t}}$, where ℓ_t is the order of q_t^2.

Proof. The first part of the statement merges Lemma 2.2 and Proposition 2.3 in [Wen98]. One considers the $\mathcal{U}_x(\mathfrak{g})$-action

$$a\overline{v} := \overline{S^{-1}(a^*)} \qquad a \in \mathcal{U}_x(\mathfrak{g})\ ,\ v \in V_\lambda$$

on the conjugate $\overline{V_\lambda}$ (see the end of 3.1) and provides the dual V'_λ with the action

$$\langle v, af\rangle := \langle S^{-1}(a)v, f\rangle \qquad a \in \mathcal{U}_x(\mathfrak{g})\ ,\ v \in V_\lambda\ ,\ f \in V'_\lambda\ .$$

Then $\overline{V_\lambda}$ and V'_λ are isomorphic because they are both irreducible with highest weight $-w_0(\lambda)$, and the isomorphism is used to define the desired form. We refer to Proposition 2.3 in [Wen98] for the Hermitianity of $(\cdot,\cdot)$; its uniqueness follows by cyclicity of v_λ.

The second part of the statement is Proposition 2.4 in [Wen98]. We claim that if $t \neq 0$ and $|t| < \frac{1}{m-D}$ then

$$(\lambda+\rho, \theta_t) \leq \ell_t\ , \tag{2}$$

where ℓ_t is the order of q_t^2, which we understand to be $+\infty$ if q_t is not a root of 1; $\theta_t = \theta$ if $D \mid \ell$ (we take that to be true if $\ell_t = +\infty$), whereas $\theta_t = \varphi$ if $D \nmid \ell$.

Proof of inequality (2). The claim is empty if q is not a root of 1; otherwise, we note that $|t| < \frac{1}{m-D}$ implies $\ell_t > m - D$, since $2\pi r|t| < 2\pi$ for all $r \leq m - D$. We exhaust the dichotomy $D \mid \ell_t$ or $D \nmid \ell_t$.

($D \mid \ell_t$) We have $\theta_t = \theta$ and we need to check $m \leq \ell_t$. Now $D \mid (\mu, \theta)$ for all integral weight μ; thus both m and ℓ_t are multiples of D and then $m - D < \ell_t$ is equivalent to $m \leq \ell_t$.

($2 = D \nmid \ell_t$) Now $\theta_t = \varphi$ and we need $(\lambda+\rho, \varphi) \leq \ell_t$. We note that

$$(\lambda+\rho, \varphi) = m - (\lambda+\rho, \theta-\varphi) < \ell_t + D - (\lambda+\rho, \theta-\varphi)\ ;$$

but $(\lambda+\rho, \varphi) < (\lambda+\rho, \theta)$ are both integers and $D = 2$, so $(\lambda+\rho, \varphi) < \ell_t + 1$.

($3 = D \nmid \ell_t$) Again $\theta = \varphi$, and we check $(\lambda+\rho, \varphi) \leq \ell_t$. The type of $\mathfrak{g}$ is G_2; $(\lambda+\rho, \theta-\varphi) = 4$ and we conclude as in previous case. □

Therefore λ lies in C_{ℓ_t} if q_t is a root of 1, whence $V_\lambda(q_t)$ is irreducible for all t in I_λ by Remark 3.3 (recall that if q_t is not a root of 1 $V_\lambda(q_t)$ is irreducible).

By $(v_\lambda, v_\lambda) = 1$ and the second of (1), $(\cdot,\cdot)$ restricts to a sesquilinear form on $V_{\lambda,\mathcal{A}}$ with values in $\mathcal{A}$, so it specialises to a non-zero complex valued invariant Hermitian form. The radical $V(q)^\perp$ is a submodule by invariance, so it is the null space for all t in I_λ by irreducibility, namely $(\cdot,\cdot)$ is non-degenerate. Moreover, since $(\cdot,\cdot)$ is positive definite at $q_t = 1$ and its signature is continuous in t, it is also positive definite for all q_t in I_λ.

Finally, using $D \mid \ell$ as shown above, $(\lambda+\rho, \theta) \leq \ell$ is equivalent to $\frac{1}{\ell} < \frac{1}{m-D}$; by the claim, this in turn implies $(\lambda+\rho, \theta_t) \leq \ell_t$ for all $|t| < \frac{1}{\ell}$. □

We now extend the Lemma to tensor products, first in the general case and subsequently for the specialisation to q, which once again requires special care.

The general case

Given Weyl modules with highest weights $\lambda_1, \ldots, \lambda_n$, we consider the tensor product $W := V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$ and endow it with the product form

$$(\xi_1 \otimes \cdots \otimes \xi_n, \eta_1 \otimes \cdots \otimes \eta_n)^p_n := \prod_{k=1}^{n} (\xi_k, \eta_k) \qquad \xi_k, \eta_k \in V_{\lambda_k} \ , \ k = 1, \ldots n \ .$$

The form $(\cdot,\cdot)^p_n$ fails to be invariant, because the coproduct of $\mathcal{U}_x(\mathfrak{g})$ does not commute with its involution. Rather, the first of identities 3.1(2) may be rewritten as $\tilde{\Delta} = \Delta^{\mathrm{op}}$ (see Definition 2.4B for this notation). Therefore, since R twists Δ to Δ^{op}, it may be used inductively on n to perturb $(\cdot,\cdot)^p_n$ to an invariant form $(\cdot,\cdot)'_n$. More precisely, realising W as the tensor product $W_1 \otimes W_2$, where W_1 and W_2 have n_1 and n_2 factors, we put

$$(\xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2)'_n := (\xi_1, r\eta_1)'_{n_1} (\xi_2, s\eta_2)'_{n_2} \qquad \xi_1, \eta_1 \in W_1 \ , \ \xi_2, \eta_2 \in W_2 \ , \tag{3}$$

where we put $R := r \otimes s$; of course we define $(\cdot,\cdot)' = (\cdot,\cdot)^p$ for $n = 1$. The invariance of $(\cdot,\cdot)'_n$ is checked exactly as in 2.4(3). The fact that $(\cdot,\cdot)'_n$ does not depend on the realisation of the tensor product is again checked by induction, using the second and third identities in Lemma 3.2 (namely the fact that R is a quasi-triangular structure) and we also find expressions for the overall element R_n in $\mathcal{U}_x(\mathfrak{g})^{\otimes n}$ such that $(\cdot,\cdot)'_n = (\cdot, R_n \cdot)^p_n$. E.g.

$$(\mathrm{id}_n \otimes \cdots \otimes \mathrm{id}_3 \otimes c_{1,2})(\mathrm{id}_n \otimes \cdots \otimes \mathrm{id}_4 \otimes c_{(1\otimes 2),3}) \cdots c_{(1\otimes\cdots\otimes n-1),n} = \Sigma R_n \ ,$$

where c is the braiding corresponding to R and we wrote k in place of V_{λ_k} for $k = 1, \ldots, n$ for better readability. Σ is the permutation associated to the braid of the composition at the left-hand side. Alternatively, we could have used any other composition with the same braid, such as

$$(c_{n-1,n} \otimes \mathrm{id}_{n-2} \otimes \cdots \otimes \mathrm{id}_1)(c_{n-2,(n-1\otimes n)} \otimes \mathrm{id}_{n-2} \otimes \cdots \otimes \mathrm{id}_1) \cdots c_{1,2\otimes\cdots\otimes n} \ .$$

We recall (see [Dri89] for a proof) that the ribbon element v corresponding to the charmed element ω (see the end of 3.2) acts on each Weyl module V_ν as the scalar $x^{-G(\nu)}$, where $G(\nu) = (\nu, \nu + 2\rho)$.

In order to treat v as an element of $\mathcal{U}_x(\mathfrak{g})$, we map the usual tensor products $\mathcal{U}_x(\mathfrak{g})^{\otimes M}$ to their action on $\prod_{\lambda_1,\ldots,\lambda_M\in\Lambda^+} \mathrm{End}(V_{\lambda_1}) \otimes \cdots \otimes \mathrm{End}(V_{\lambda_M})$, as for Sawin's construction; this defines a homomorphism α from $\mathcal{U}_x(\mathfrak{g})$ to the discrete Hopf $\mathbb{Q}(x)$-algebra of natural transformations of the forgetful functor $\mathcal{F} : \mathrm{Rep}\,(\mathcal{U}_x(\mathfrak{g})) \to \mathrm{Vec}_{\mathbb{Q}(x)}$.

Since the action of R obviously factors through α, the application of $\alpha \otimes \alpha$ to R may be implied in the defining formula (3). Furthermore, by semi-simplicity of $\mathrm{Rep}\,(\mathcal{U}_x(\mathfrak{g}))$, we can define a unitary square root w of v in the mentioned discrete Hopf algebra by requiring w to act as the scalar $x^{-\frac{G(\nu)}{2}}$ on each Weyl module V_ν.

Now, in keeping with 2.7, we define a new form $(\cdot,\cdot)_n$ by replacing R with its deformation $R\Delta(w)w^{-1} \otimes w^{-1}$ in (3), so

$$(\cdot,\cdot)_n := (\cdot, \Xi_n \cdot)'_n \qquad \Xi_n \xi := x^{\frac{G(\nu) - \sum_{k=1}^n G(\lambda_k)}{2}} \xi \tag{4}$$

whenever ξ is in a submodule of $V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$ with highest weight ν. We henceforth drop the subscript n from $(\cdot,\cdot)^p_n$, $(\cdot,\cdot)'_n$ and $(\cdot,\cdot)_n$.

Proposition A. *The form $(\cdot,\cdot)$ is invariant and Hermitian.*

Proof. Invariance follows from invariance of $(\cdot,\cdot)'$ together with the fact that $\Delta(w)w^{-1} \otimes w^{-1}$ commutes with $\Delta(a)$ for all a in $\mathcal{U}_x(\mathfrak{g})$.

Since $(\cdot,\cdot)^p$ is Hermitian, we are left to show that $\Omega_w := R\Delta(w)w^{-1} \otimes w^{-1}$ is self-adjoint. To this aim, we compute

$$\Omega_w = R\Delta(w)w^{-1} \otimes w^{-1} = w^{-1} \otimes w^{-1}\Delta^{\mathrm{op}}(w)R = w^* \otimes w^*\Delta(w^*)^*R\ ,$$

where we used $R\Delta(\cdot) = \Delta^{\mathrm{op}}(\cdot)R$, $\Delta^{\mathrm{op}} = \tilde{\Delta}$ and $w^* = w^{-1}$. Therefore

$$\begin{aligned}\Omega_w^* &= R^*\Delta(w^{-1})w \otimes w = RR^{-1}R^*\Delta(w^{-1})w \otimes w \\ &= R(R_{21}R)^{-1}\Delta(w^{-1})w \otimes w = R\Delta(v)v^{-1} \otimes v^{-1}\Delta(w^{-1})w \otimes w = \Omega_w\ ,\end{aligned}$$

having used $R^* = R_{21}^{-1}$ (Lemma 3.2). □

Remark. Consider the two adjoints $\cdot^*$ and ${}^*\cdot$ defined by the non-Hermitian form $x^{\frac{\sum_{i=1}^r G(\lambda_i)}{2}}(\cdot,\cdot)'$ on the tensor product $V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_r}$: given $V = V_{\mu_1} \otimes \cdots \otimes V_{\mu_s}$, $W = V_{\nu_1} \otimes \cdots \otimes V_{\nu_t}$ and a morphism f in (V, W),

$$\begin{aligned} x^{\frac{\sum_{k=1}^t G(\nu_k)}{2}}(fv, w)' &=: x^{\frac{\sum_{j=1}^s G(\mu_j)}{2}}(v, f^*w)' \\ x^{\frac{\sum_{k=1}^t G(\nu_k)}{2}}(w, fv)' &=: x^{\frac{\sum_{j=1}^s G(\mu_j)}{2}}({}^*fw, v)' \end{aligned}$$

for all v in V and w in W. We observe that $\cdot^* = {}^*\cdot$. Indeed, denoting the adjoint relative to $(\cdot,\cdot)^p$ by $\cdot^\dagger$, we have

$$\begin{aligned} f^* &= x^{\frac{-\sum_{j=1}^s G(\mu_j)}{2}} R_s^{-1} f^\dagger R_t x^{\frac{\sum_{k=1}^t G(\nu_k)}{2}} \quad \text{and} \\ {}^*f &= x^{\frac{\sum_{j=1}^s G(\mu_j)}{2}} (R_s^*)^{-1} f^\dagger R_t^* x^{\frac{-\sum_{k=1}^t G(\nu_k)}{2}} \quad , \text{so} \\ f^* &= x^{-\sum_{j=1}^s G(\mu_j)} (R_s^{-1} R_s^*)({}^*f)(R_t^{-1} R_t^*)^{-1} x^{\sum_{k=1}^t G(\nu_k)} \ . \end{aligned}$$

Now, using $R^* = R_{21}^{-1}$, one computes that $R_r^{-1} R_r^*$ acts on each submodule with highest weight λ of the generic $V_\lambda \otimes \cdots \otimes V_{\lambda_r}$ as the scalar $x^{-G(\lambda)+\sum_{i=1}^r G(\lambda_i)}$, whence our claim follows, also keeping in mind that f^* and ${}^*f$ are intertwiners, because $(\cdot,\cdot)'$ is invariant. It is checked analogously that $\cdot^*$ and ${}^*\cdot$ coincide with the usual adjoint defined by the Hermitian form $(\cdot,\cdot)$.

The specialisation to q

We now deal with the problem of specialising the forms $(\cdot,\cdot)'$ and $(\cdot,\cdot)$ to the root q. As in 3.3, we consider the specialisation $\mathcal{U}_q(\mathfrak{g}) = \mathcal{U}^\dagger_{\mathcal{A}'}(\mathfrak{g}) \otimes_{\mathcal{A}'} \mathbb{C}$ obtained by identifying $x^{1/L}$ with the L-th root of $q\ e^{\frac{\imath\pi}{L\ell}}$; similarly the specialisation to q of a $\mathcal{U}^\dagger_{\mathcal{A}}(\mathfrak{g})$ module V will be denoted $V(q) := V \otimes_{\mathcal{A}'} \mathbb{C}$. Given $\mathcal{U}^\dagger_{\mathcal{A}}(\mathfrak{g})$ modules V_1, V_2 and a morphism f in (V_1, V_2), we write

$$f(q) : V_1(q) \to V_2(q) \quad f(q)(v \otimes p) = f(v) \otimes p(r) \qquad \forall\ v \in V_1\ ,\ p \in \mathcal{A}'\ .$$

As already observed in the proof of the above Lemma, the form $(\cdot,\cdot)$ on a single Weyl module V_λ restricts to a Hermitian form on $V_{\lambda,\mathcal{A}}$ with values in $\mathcal{A}$, for all λ in Λ^+; moreover if λ lies in $\overline{C_\ell}$ the specialisation is a non-degenerate complex form on the irreducible module $V_\lambda(q)$. We therefore have the following close consequence.

Proposition B. *Let $\lambda_1, \ldots, \lambda_n$ be dominant integral weights lying in $\overline{C_\ell}$; consider the tensor product $W := V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$ and the forms $(\cdot,\cdot)^p$ and $(\cdot,\cdot)'$ defined on it. Then*

- $(\cdot,\cdot)^p$ *specialises to a non-degenerate Hermitian form $(\cdot,\cdot)^p_q$ on $W(q)$;*
- $(\cdot,\cdot)'$ *specialises to an invariant non-degenerate form $(\cdot,\cdot)'_q$ on $W(q)$.*

Proof. The first point simply follows from the Lemma. Furthermore R is an invertible element of $\mathcal{U}^\dagger_{\mathcal{A}'}(\mathfrak{g})^{\otimes 2}$ by Lemma 3.2, so it specialises to an invertible element of $\mathcal{U}_q(\mathfrak{g})^{\otimes 2}$; the same is true for each R_n (except that it is in $\mathcal{U}_q(\mathfrak{g})^{\otimes n}$). The second point follows from the first one together with the definition of $(\cdot,\cdot)'$. □

Contrary to the situation of $(\cdot,\cdot)^p$ and $(\cdot,\cdot)'$, the form $(\cdot,\cdot)$ cannot be specialised to q. The reason is that the square root w introduced before the defining formula (4) by semi-simplicity of Rep$(\mathcal{U}_x(\mathfrak{g}))$ is not guaranteed to belong in the integral form $\mathcal{U}^\dagger_{\mathcal{A}'}(\mathfrak{g})$, which would be needed to just proceed as plainly as in Proposition B.

In fact, we see that the integral form cannot contain a square root of the ribbon element: if it did, we could specialise the $(\cdot,\cdot)$ form on any tensor product $V_\lambda \otimes V_\mu$, e.g. for λ, μ in Λ_ℓ; then the specialised form would have to be positive by the continuity argument of the Lemma, whence $V_\lambda(q) \otimes V_\mu(q)$ would be completely reducible. Of course, this is actually false for some λ, μ in Λ_ℓ.

Nevertheless, in the scenario of Proposition B, the square root $w(q)$ may still be defined on any submodule of W (which is a tilting module, as well as all its submodules, by Proposition 4.1) with null negligible part, because such submodules are direct sums of Weyl modules with highest weight in Λ_ℓ. Moreover if p is a self-adjoint idempotent element of $\mathrm{End}(W)$ then ${}^*p(q) = p(q) = p(q)^*$ with respect to $(\cdot,\cdot)'_q$ (see the Remark) so the restriction of the form to $p(q)W(q)$ is still non-degenerate.

Summarizing, if a self-adjoint idempotent element p of $\mathrm{End}(W)$ is such that $p(q)W(q)$ has null negligible part then we can perturb the restriction of $(\cdot,\cdot)'_q$ to $p(q)W(q)$ as in formula (4), and we thus obtain a positive definite invariant form on $p(q)W(q)$. The following Theorem provides idempotents as desired in the case of tensor products of the form $V_\lambda \otimes V$ (V is the fundamental module) for λ in Λ_ℓ; this will be enough for our purposes by Theorem 4.1.

Theorem. *Let λ be a dominant integral weight in Λ_ℓ, and consider the tensor product $W := V_\lambda \otimes V$. Then there exists an idempotent p in $\mathrm{End}(W)$, self-adjoint with respect to the form $(\cdot,\cdot)$ on W, such that:*

- *pW is isomorphic to a direct sum $\bigoplus_{\lambda\in\Lambda_\ell} n_\lambda V_\lambda$, where the n_λ are non-negative integers;*
- *the restriction of $(\cdot,\cdot)'_q$ to $p(q)$ is non-degenerate and its following perturbation is positive definite:*

$$(\cdot,\Xi\cdot)'_q \qquad \Xi\xi := q^{\frac{G(\nu)-G(\lambda)-G(\kappa)}{2}}\xi \tag{5}$$

  whenever ξ is in a submodule of $p(q)W(q)$ with highest weight ν;
- *$(1-p(q))W(q)$ is negligible.*

Moreover, the second point still holds upon replacing q with $q_t := e^{\imath\pi t}$, through the whole arc given by $|t| \le 1/\ell$.

The Theorem corresponds to Lemma 3.6.1, Lemma 3.6.2 and Proposition 3.6 in [Wen98], to which we refer for a proof. We just point out that the final assertion simply follows from the fact that if q_t is a root of 1 then the relative principal Weyl alcove contains the one for q. On the other hand, as the order of q_t increases the negligible part of $W(q_t)$ eventually reduces to zero, as it is of course the case for $q_t = 1$.

4.4 The forgetful functor on the quotient category

Applying iteratively Theorem 4.3, we find for each positive integer n an idempotent p_n in $\mathrm{End}(V^{\otimes n})$, self adjoint with respect to $(\cdot,\cdot)$ such that $p_n(q)V^{\otimes n}(q)$ is isomorphic to a direct sum of specialised Weyl modules with highest weight in the principal alcove and $(1-p_n(q))V^{\otimes n}(q)$ is negligible. Each $p_n(q)V^{\otimes n}(q)$ is endowed with the form $(\cdot,\cdot)_q$ defined as in (5).

The category $\mathcal{G}_q(\mathfrak{g})$ We are now set up to use Theorem 4.1 to realise (see the forthcoming Proposition B) $\overline{\mathcal{T}_q(\mathfrak{g})}$ as a linear subcategory $\mathcal{G}_q(\mathfrak{g})$ of Rep $(\mathcal{U}_q(\mathfrak{g}))$, also endowed with the structure of a C^* tensor category:

- the generic object is an enumeration of a finite set of pairs (n, p) where n is a positive integer and p is an idempotent of End $(V^{\otimes n}(q))$ with $p < p_n(q)$, self-adjoint with respect to $(\cdot,\cdot)_q$;
- the generic morphism $\{(n_j, q_j)\}_{j=1}^N \to \{(m_i, p_i)\}_{i=1}^M$ is a matrix $M \times N$ where the (i,j) entry is an element of $p_i(V^{\otimes j}, V^{\otimes i})q_j$.

The composition for $\mathcal{G}_q(\mathfrak{g})$ is the matrix product. The involution is obtained by taking the Hermitian conjugate matrix, where we mean the $\cdot^*$ operation for the conjugate of each entry.

Turning to the tensor product, given $p < p_m$, $p' < p_{m'}$, $q < p_n$, $q' < p_{n'}$, f in (p, p') and g in (q, q') we put

$$f \underline{\otimes} g := p_{m'+n'} f \otimes g p_{m+n} \ , \tag{1}$$

and extend by bilinearity. The following simple result plays an essential role in the subsequent calculations.

Proposition A. *Let $f : T_1 \to T_2$ be a morphism between tilting modules, both with null negligible part. If f factors through a negligible module then $f = 0$.*

Proof. T_1 and T_2 are each isomorphic to a direct sum of specialised Weyl modules with highest weights in the principal Weyl alcove. Moreover it follows from Weyl's formula 3.3(1) that qdim $(V_\lambda(q))$ is positive for all λ in Λ_ℓ. From this fact it follows that $\mathrm{qtr}(f^* f) = 0$ implies $f = 0$.

By semi-simplicity, it suffices to verify the implication assuming that T_1 and T_2 are both a multiples of a certain Weyl module; then if $f \neq 0$ $\mathrm{qtr}(f^* f)$ is qdim $(V_\lambda(q))$ times the usual trace of a positive definite matrix, so $\mathrm{qtr}(f^* f) > 0$.

Coming back to our statement, let N be a negligible module, g in (T_1, N) and h in (N, T_2) such that $f = hg$; then

$$\mathrm{qtr}_{T_1}(f^* f) = \mathrm{qtr}_{T_1}(g^* h^* h g) = \mathrm{qtr}_N(g g^* h^* h) = 0 \ . \qquad \square$$

With the Proposition at our disposal, $\underline{\otimes}$ is easily checked to be a bifunctor; besides, one also sees that $\mathcal{G}_q$ is a strict monoidal category. More precisely, given self-adjoint idempotents $P^{(n_k)} < p_{n_k}$ for $k = 1, \dots N$,

$$P^{(k_1)} \underline{\otimes} \cdots \underline{\otimes} P^{(k_N)} = p_{n_{k_1} + \cdots + n_{k_N}} P^{(k_1)} \otimes \cdots \otimes P^{(k_N)} p_{n_{k_1} + \cdots + n_{k_N}} \ .$$

Proposition B. *The category $\mathcal{G}_q$ is tensor equivalent to the quotient $\overline{\mathcal{T}_q(\mathfrak{g})}$ by the inclusion of $\mathcal{G}_q$ in $\mathcal{T}_q(\mathfrak{g})$ followed by the quotient functor.*

Proof. As a morphism of $\overline{\mathcal{T}_q(\mathfrak{g})}$, each p_n is the identity of $\mathrm{End}(V^{\otimes n})$; the identical maps are immediately seen to be a tensor structure. In view of Proposition 4.2, the stated functor sends a complete collection of mutually non-equivalent simple objects for $\mathcal{G}_q(\mathfrak{g})$ into one for $\overline{\mathcal{T}_q(\mathfrak{g})}$ and is therefore an equivalence. $\square$

We also mention that, by the very Definition 4.2, the quantum traces lower to the quotients, and we check as in Proposition A that the Hermitian form

$$f, g \quad \mapsto \quad \mathrm{qtr}_U(fg^*) \qquad f, g \in (T, U)$$

is actually a scalar product on the morphism space (T, U), for all T, V tilting modules. Therefore, according to the definition in 2.3 of [Row06], $\overline{\mathcal{T}_q(\mathfrak{g})}$ is a unitary ribbon category for q fixed as in the beginning of 4.3; this is Wenzl-Xu theorem.

It is also worthwhile to mention that as soon as D divides the order of q^2 (since $q^2 = e^{\imath\frac{2\pi}{\ell}}$ and $D \mid \ell$, this is our case) $\overline{\mathcal{T}_q(\mathfrak{g})}$ is modular. We refer to [Row06] for a detailed account on the problem of modularity of $\overline{\mathcal{T}_q(\mathfrak{g})}$. The very interesting relation between modular tensor categories and topological conformal field theories is treated in [Tur92] and [Tur94].

The weak tensor structure We denote by $\mathcal{W} : \mathcal{G}_q(\mathfrak{g}) \to \mathrm{Hilb}$ the forgetful functor and we choose, for each weight λ in Λ_ℓ, a self adjoint idempotent $p_\lambda < p_{n_\lambda}$ such that $p_\lambda(q)V^{\otimes n_\lambda}(q) \simeq V_\lambda(q)$. By Proposition 4.2, they form a complete collection of mutually non-equivalent simple objects for $\mathcal{G}_q(\mathfrak{g})$; therefore we may define structure maps F, G for $\mathcal{W}$ by prescribing

$$F_{\lambda,\mu} := (p_\lambda \underline{\otimes} p_\mu)(p_\lambda \otimes p_\mu) \qquad G_{\lambda,\mu} := (p_\lambda \otimes p_\mu)(p_\lambda \underline{\otimes} p_\mu) \ . \tag{2}$$

We remark that, for each λ, μ in Λ_ℓ,

$$\begin{aligned} F_{\lambda,\mu}G_{\lambda,\mu} &= p_{n_\lambda+n_\mu}(p_\lambda \otimes p_\mu)p_{n_\lambda+n_\mu}(p_\lambda \otimes p_\mu)p_{n_\lambda+n_\mu}(p_\lambda \otimes p_\mu)p_{n_\lambda+n_\mu} \\ &= p_{n_\lambda+n_\mu}(p_\lambda \otimes p_\mu)p_{n_\lambda+n_\mu} = p_\lambda \underline{\otimes} p_\mu \ , \end{aligned}$$

where we used Proposition A to cancel the factors $p_{n_\lambda+n_\mu}$ in the middle of the expression; so FG is indeed identical. Moreover, since F e G are compatible with orthogonal direct sums by naturality, we obtain expressions for them at generic objects $P^{(n)} < p_n$ and $P^{(m)} < p_m$.

To this aim, consider $S_j : p_{\lambda_j} \to P^{(n)}$ and $T_k : p_{\mu_k} \to P^{(m)}$ such that

$$S_j^* S_{j'} = \delta_{j,j'} p_{\lambda_j} \ , \quad \sum_j S_j S_j^* = P^{(n)} \ , \quad T_k^* T_{k'} = \delta_{k,k'} p_{\lambda_k} \ , \quad \sum_k T_k T_k^* = P^{(m)} \ .$$

We note that, being $\otimes$ and $\underline{\otimes}$ both bilinear and commuting with *, the products $S_j \otimes T_k$ and $S_j \underline{\otimes} T_k$ enjoy the same properties, i.e. they provide orthogonal decomposition for $P^{(n)} \otimes P^{(m)}$ and $P^{(n)} \underline{\otimes} P^{(m)}$ respectively (the first with respect to the products form). We therefore have the expressions

$$F_{P^{(n)},P^{(m)}} = \sum_{j,k} S_j \underline{\otimes} T_k F_{\lambda_j,\mu_k} S_j^* \otimes T_k^* \ , \quad G_{P^{(n)},P^{(m)}} = \sum_{j,k} S_j \otimes T_k G_{\lambda_j,\mu_k} S_j^* \underline{\otimes} T_k^* \ .$$

Lemma A. *The maps F and G form a weak tensor structure on W.*

Proof. Being $\mathcal{G}_q$ strict, we have to prove that

$$F_{P,Q\underline{\otimes}R} \circ (P \otimes F_{Q,R}) \circ (G_{P,Q} \otimes R) \circ G_{P\underline{\otimes}Q,R} = P\underline{\otimes}Q\underline{\otimes}R \ , \tag{3}$$

$$F_{P\underline{\otimes}Q,R} \circ (F_{P,Q} \otimes R) \circ (P \otimes G_{Q,R}) \circ G_{P,Q\underline{\otimes}R} = P\underline{\otimes}Q\underline{\otimes}R \ ; \tag{4}$$

by naturality, we may limit ourselves to the case $P = p_\lambda$, $Q = p_\mu$, $R = p_\nu$ con λ, μ, ν in Λ_ℓ.

In order to evaluate, for instance, (3) we have to consider orthogonal decompositions of $p_\lambda \underline{\otimes} p_\mu$ and $p_\mu \underline{\otimes} p_\nu$, say by S_j and T_k; however the left-hand side is an endomorphism of $p_\lambda \underline{\otimes} p_\mu \underline{\otimes} p_\nu = p_{n_\lambda + n_\mu + n_\nu} p_\lambda \otimes p_\mu \otimes p_\nu p_{n_\lambda + n_\mu + n_\nu}$. So, applying Proposition A together with the fact that $f \otimes g$ is negligible as soon as either of f, g is (see Proposition 4.2), we may cancel all the p_k appearing in the middle of the expression. E.g.

$$F_{p_\lambda, p_\mu \underline{\otimes} p_\nu} = \sum_k (p_\lambda \underline{\otimes} T_k) F_{\lambda, \mu_k} (p_\lambda \otimes T_k^*) =$$

$$\sum_k p_{n_\lambda + n_\mu + n_\nu} (p_\lambda \otimes T_k) p_{n_\lambda + n_{\lambda_k}} (p_{n_\lambda} \otimes p_{n_{\lambda_k}}) p_{n_\lambda + n_{\lambda_k}} (p_{n_\lambda} \otimes p_{n_{\lambda_k}}) (p_\lambda \otimes T_k^*)$$

reduces to

$$p_{n_\lambda + n_\mu + n_\nu} \sum_k (p_\lambda \otimes T_k)(p_\lambda \otimes T_k^*) = p_{n_\lambda + n_\mu + n_\nu} p_\lambda \otimes p_\mu \otimes p_\nu \ .$$

Treating the other factors at the left-hand side of (3) likewise we see that it indeed equals $p_\lambda \underline{\otimes} p_\mu \underline{\otimes} p_\nu$. Identity (4) is verified just analogously. □

We can now apply Theorem 2.4 and Corollary 2.8A to obtain a unitary weak Hopf algebra $A(\mathfrak{g}, q, \mathcal{W})$, with a unitary tensor equivalence $\mathcal{G}_q(\mathfrak{g}) \to \mathrm{Rep}\,\big(A(\mathfrak{g}, q, \mathcal{W})\big)$.

Remark. A different choice of the p_λ affects $A(\mathfrak{g}, q, \mathcal{W})$ just by a trivial twist (see Definition 2.5A); in order to outline the easy verification, we call F' and G' the structure maps for the new choice. We have to check

$$G'F = G'F'GF \ , \qquad GF' = GFG'F' \ ,$$

and we can take subscripts among the p_λ. So we would need to use the general (summed) expression for G' ed F'; again Proposition A yields the needed cancellations.

Finally, a different choice of the p_n, say p'_n, produces a category $\mathcal{G}'_q$ unitarily tensorially isomorphic to $\mathcal{G}_q$. The isomorphism, is defined as follows:

$$\mathcal{E} : \mathcal{G}_q \to \mathcal{G}'_q \quad \mathcal{E}(f) = p'_n f p'_m \qquad \text{for } f \in (p, q) \ , \quad p < p_m \ , q < p_n \ . \tag{5}$$

It is clearly a $*$-functor, and, marking the tensor product of $\mathcal{G}'_q$ by $\underline{\otimes}'$,

$$\mathcal{E}(p) \underline{\otimes}' \mathcal{E}(q) = p'_{m+n} (p'_m p p'_m) \otimes (p'_n q p'_n) p'_{m+n} = p'_{m+n} (p \otimes q) p'_{m+n} = \mathcal{E}(p \underline{\otimes} q) \ ,$$

having used Proposition A once again. Therefore $\mathcal{E}$ is a tensor $*$-isomorphism with identical tensor structure, whose inverse is obtained by swapping p_k and p'_k in (5) for $k = m, n$.

We now take into account the quasi-triangular structure of $\mathcal{U}_q(\mathfrak{g})$, and see how it reflects on $A(\mathfrak{g}, q, \mathcal{W})$. As we already observed (in the proof of Proposition 4.3A) the coproduct of $\mathcal{U}_q(\mathfrak{g})$ satisfies $\tilde{\Delta} = \Delta^{\mathrm{op}}$, and $R^* = R_{21}^{-1}$ (Lemma 3.2); these properties actually carry over to $A(\mathfrak{g}, q, \mathcal{W})$ through the truncation procedure we discussed, making it into a unitary coboundary weak Hopf algebra.

Lemma B. *The unitary weak Hopf algebra $A(\mathfrak{g}, q, \mathcal{W})$, endowed with the quasi-triangular structure and the root of the ribbon element arising from $\mathcal{G}_q(\mathfrak{g})$ (considered with the braiding and the root of the ribbon element induced by the equivalence of Proposition B) as in 2.3, is a unitary coboundary weak Hopf algebra.*

Proof. In view of Theorem 2.7, applied to the particular case where $\mathcal{F}$ is the identity functor, we are left to check the identities

$$F_{p,q}\Sigma(p,q)F^*_{p,q} = \mathcal{W}(c^w_{p,q}) \ , \qquad G^*_{p,q}\Sigma(q,p)G_{q,p} = \mathcal{W}(c^w_{p,q})^{-1} \ . \tag{6}$$

We just prove the first identity, forgoing the analogous treatment of the other one. The idempotents $p < p_m$, $q < p_n$ can be taken among the chosen p_λ, so

$$F_{p,q} = (p\underline{\otimes}q)(p \otimes q) = p_{m+n}p \otimes qp_{m+n}p \otimes q \ ;$$

all of the Hilbert spaces we are considering are subspaces of $V^{\otimes m+n}$, and we shall treat them in terms of idempotents and partial adjoints implying the functor $\mathcal{S}$ of I(3) (this is also the reason why we wrote $\Sigma(p,q)$ rather than $\Sigma(\mathcal{W}(p), \mathcal{W}(q))$ in (6)).

In order to compute $F^*_{p,q}$, we note that the domain is taken with the product of the $(\cdot,\cdot)$ forms on $p_m V^{\otimes m}$ and $p_n V^{\otimes n}$, whereas the the codomain is taken with $(\cdot,\cdot)$ on $v^{\otimes m+n}$. We define

$$\Omega_k := \underline{p_k}\Xi_k\overline{p_k}\rho^{\otimes k}(R_k) \ .$$

Here Ξ_k is defined as in Theorem 4.3, while $\underline{p_k}, \overline{p_k}$ are such that $\overline{p_k}\underline{p_k} = \mathrm{id}$, $\underline{p_k}\overline{p_k} = p_k$; R_k is the one for the $(\cdot,\cdot)'$ form (see 4.3) and ρ is the homomorphism for the $\mathcal{U}_q(\mathfrak{g})$-action on V. Therefore, by definition of $(\cdot,\cdot)$,

$$F^*_{p,q} = {\Omega_m}^{-1} \otimes {\Omega_n}^{-1}F^\dagger_{p,q}\Omega_{m+n} \ ,$$

where $\dagger$ denotes the adjoint with respect to $(\cdot,\cdot)^p$ on $V^{\otimes m+n}$; besides p is orthogonal with respect to Ω_m, q is to Ω_n and $p\underline{\otimes}q$ is to Ω_{m+n}, thus

$$p^\dagger = \Omega_m p {\Omega_m}^{-1} \ , \quad q^\dagger = \Omega_n q {\Omega_n}^{-1} \ , \quad (p\underline{\otimes}q)^\dagger = \Omega_{m+n}(p\underline{\otimes}q){\Omega_{m+n}}^{-1} \ .$$

Hence we have

$$\begin{aligned} F^*_{p,q} &= {\Omega_m}^{-1} \otimes {\Omega_n}^{-1}(\Omega_m \otimes \Omega_n)(p \otimes q)({\Omega_m}^{-1} \otimes {\Omega_n}^{-1}) \\ &\quad \Omega_{m+n}(p\underline{\otimes}q){\Omega_{m+n}}^{-1}\Omega_{m+n} \\ &= (p \otimes q)({\Omega_m}^{-1} \otimes {\Omega_n}^{-1})\Omega_{m+n}(p\underline{\otimes}q) \ . \end{aligned}$$

Therefore on the whole

$$\begin{aligned} F_{q,p}\Sigma(p,q)F^*_{p,q} =& (q\underline{\otimes}p)(q \otimes p)\Sigma(p,q)(p \otimes q)({\Omega_m}^{-1} \otimes {\Omega_n}^{-1})\Omega_{m+n}(p\underline{\otimes}q) \\ =& p_{m+n}\Sigma(p,q)(p \otimes q)({\Omega_m}^{-1} \otimes {\Omega_n}^{-1})\Omega_{m+n}(p \otimes q)p_{m+n} \ . \end{aligned}$$

In the final passage we used $\Sigma(p,q)(p \otimes q) = (q \otimes p)\Sigma(p,q)$ and applied Lemma A to eliminate the p_{m+n} in the middle of the expression.

Let us now turn to evaluate the right-hand side of (6). As stated in Proposition B the needed c^w for $\mathcal{G}_q$ is pulled back from the one for $\overline{\mathcal{T}_q(\mathfrak{g})}$. The braiding for the latter is just the one for $\overline{\mathcal{T}_q(\mathfrak{g})}$ seen in the quotient; on the other hand w is determined by its values $q^{-\frac{G(\nu)}{2}}$ on each Weyl module with highest weight ν in Λ_ℓ. The corresponding morphisms in $\mathcal{G}_q$ are just given by the representatives between the appropriate p_k. In conclusion

$$\begin{aligned}W(c^w_{p,q}) =& p_{m+n}\Sigma(p,q)(p\otimes q)\rho^{\otimes m+n}((\Delta_m\otimes\Delta_n)R)\\ &(i_m{\Xi_m}^{-1}j_m)\otimes(i_n{\Xi_n}^{-1}j_n)(p\otimes q)i_{m+n}\Xi_{m+n}j_{m+n}p_{m+n}\ .\end{aligned}$$

Now, we note that

$$\begin{gathered}({\Omega_m}^{-1}\otimes{\Omega_n}^{-1})\Omega_{m+n} =\\ \left(i_m{\Xi_m}^{-1}j_m\rho^{\otimes m}({R_m}^{-1})\right)\otimes\left(i_n{\Xi_n}^{-1}j_n\rho^{\otimes n}({R_n}^{-1}\right)\\ \rho^{\otimes m+n}(R_{m+n})i_{m+n}\Xi_{m+n}j_{m+n} =\\ \left((i_m{\Xi_m}^{-1}j_m)\otimes(i_n{\Xi_n}^{-1}j_n)\right)\rho^{\otimes m+n}\left((\Delta_m\otimes\Delta_n)R\right)\left(i_{m+n}\Xi_{m+n}j_{m+n}\right)\ ,\end{gathered}$$

taking into account that $({R_m}^{-1}\otimes{R_n}^{-1})R_{m+n}=(\Delta_m\otimes\Delta_n)R$ for last passage. Finally the first factor in last line commutes with the second ($i_m{\Xi_m}^{-1}j_m$ and $i_n{\Xi_n}^{-1}j_n$ are intertwiners). Therefore, substituting the new expression of $({\Omega_m}^{-1}\otimes{\Omega_n}^{-1})\Omega_{m+n}$ in the one of $F_{q,p}\Sigma(p,q)F^*_{p,q}$, we see that the latter coincides with the expression of $\mathcal{W}(c^w_{p,q})$. □

Twisted fusion categories

By the quantum Racah formula (see [Saw06]), the fusion rules of the quotients $\overline{\mathcal{T}_q(\mathfrak{g})}$ introduced in Proposition 4.2 depend on q only through the order ℓ of q^2. Very remarkably, as shown in [KW93], one can associate to an arbitrary tensor category $\mathcal{C}$ with the same fusion rules (the product of the Grothendieck ring of $\mathcal{C}$, recalled at the beginning of 2.5) as the mentioned quotients, for $\mathfrak{g} = \mathfrak{sl}_N$ and ℓ fixed, a pair of non-zero complex numbers $(q_\mathcal{C}, \tau_\mathcal{C})$, defined up to replacing $(q_\mathcal{C}, \tau_\mathcal{C})$ with $(q_\mathcal{C}^{-1}, \tau_\mathcal{C}^{-1})$, in such a way that $\mathcal{C}$ is determined by the pair up to tensor equivalence.

Moreover, all possible pairs are actually reached by some twisted versions of $\overline{\mathcal{T}_q(\mathfrak{g})}$ itself. Less vaguely, such twists are obtained perturbing the associativity morphism by a certain "invariant 3-cocycle" defined in terms of an N-th root of 1 w (more details are reported in Example 5.2). Then $q_\mathcal{C} = q^2$ and $\tau_\mathcal{C} = (-1)^N q^{N-1} w$.

The present chapter arises form [NY15], where the authors settle down the problem of reconstructing the mentioned twisted categories as representation categories of quantum groups of their own, for q a positive real number (so $\ell = +\infty$); this is the case when the predual of the specialised form $\mathcal{U}_q\mathfrak{g}$ introduced in 3.3 is actually a compact quantum group in the sense of Woronowicz.

On the other hand, we shall deal with the same problem for q a non-trivial root of 1. The treatment of [NY15] carries over without particular difficulty. However, it is interesting to note that Sawin's presentation of the QUE algebras (see 3.2) turns out to be nicely suited for the adaptation needed, and we will get to take advantage of the explicit knowledge of the weak tensor structure of the forgetful functor $\mathcal{W}$ on the quotient category $\mathcal{G}_q(\mathfrak{g})$ presented in 4.4. As in the computations there discussed, Proposition 4.4A will play a crucial role.

More interestingly, the twisted versions of $\mathcal{G}_q(\mathfrak{g})$ fit immediately in the general framework of our first chapter as representation categories of twists of the discrete weak Hopf algebra $A(\mathfrak{g}, q, \mathcal{W})$ introduced in 4.4, in the sense of 2.5. Furthermore, the twisted algebras are actually still unitary weak Hopf algebras, providing us with more examples of this remarkable algebraic structure. Their representation categories are also granted not to be equivalent to $\mathcal{G}_q(\mathfrak{g})$ since as in the compact case (see Remark 4.4 in [NY15]), as observed in [CCP21], they are not braided.

Finally, the twists of $A(\mathfrak{g}, q, \mathcal{W})$ can be tracked back to a suitable central extension of the QUE algebra we started with, as in 2.1 of [NY15]. Even though in our case one has to go through the categorical quotient and the construction discussed in 4.4, this adaptation does not specially require further work; incidentally, Sawin's presentation of the QUE algebra will still come in very handy.

5 Reconstructing the twisted categories

Throughout this section, unless otherwise stated, $\mathfrak{g}$ will be a simple complex Lie algebra as in 3.1, and we also adopt all further structure and notation therein introduced. Furthermore, q will be a complex root of 1 as at the beginning of 4.3, so that all the constructions leading to the unitary coboundary weak Hopf algebra $A(\mathfrak{g}, q, \mathcal{W})$ go through (see Lemma 4.4B). More generally, we will freely refer to the various objects defined in this scenario through previous chapter.

5.1 Abelian quantum subgroups and cohomology

Let us consider the specialised QUE algebra $\mathcal{U}_q(\mathfrak{g})$ introduced in 3.3, and the category of tilting modules $\mathcal{T}_q(\mathfrak{g})$ (from the beginning of 4.2). Even though the latter is not a semi-simple abelian category, the natural endomorphisms of the forgetful functor $\mathcal{W}_0 : \mathcal{T}_q(\mathfrak{g}) \to \mathrm{Hilb}$ form a complex unital associative algebra $A(\mathfrak{g}, q)$ just as in 2.1; a coassociative coproduct with its counit is defined as well, since $\mathcal{W}_0$ comes with its obvious tensor structure. Of course non-semi-simplicity of $\mathcal{T}_q(\mathfrak{g})$ prevents $A(\mathfrak{g}, q)$ from being a discrete algebra (Definition 2.1B).

The reason why we introduced $A(\mathfrak{g}, q)$ is its immediate relation with the discrete unitary coboundary weak Hopf algebra $A(\mathfrak{g}, q, \mathcal{W})$ constructed from the forgetful functor $\mathcal{W} : \mathcal{G}_q(\mathfrak{g}) \to \mathrm{Hilb}$ in previous section. If η is a natural endomorphism of $\mathcal{W}_0$, we may limit ourselves to consider the values η takes on the linear subcategory $\mathcal{G}_q(\mathfrak{g}) \subset \mathcal{T}_q(\mathfrak{g})$, thus obtaining an algebra morphism

$$A(\mathfrak{g}, q) \to A(\mathfrak{g}, q, \mathcal{W}) \qquad \eta \mapsto [\eta] \ . \tag{1}$$

For greater generalised tensor powers one likewise restricts all the arguments of a given $H_{T_1,\dots,T_n}$ to belong to $\mathcal{G}_q(\mathfrak{g})$. All these morphisms are actually surjective because of the "semisimple+negligible" decomposition discussed in Remark 4.2.

Proposition. *Let Δ and $\underline{\Delta}$ be the coproducts of $A(\mathfrak{g}, q)$ and $A(\mathfrak{g}, q, \mathcal{W})$ respectively. We have*

$$\underline{\Delta}[\eta] = [\Delta\eta]\underline{\Delta}1 = \underline{\Delta}1[\Delta\eta] \qquad \forall\ \eta \in A(\mathfrak{g}, q) \ . \tag{2}$$

Proof. Let $p < p_m$, $q < p_n$ be among the simple objects $p_\lambda < p_{n_\lambda}$ chosen for the definition 4.4(2) of the weak tensor structure of $\mathcal{W}$. Then we apply 2.1(1) to compute $\underline{\Delta}[\eta]_{p,q} =$

$$\begin{aligned} G_{p,q}[\eta]_{p\underline{\otimes}q}F_{p,q} &= (p\otimes q)p_{m+n}(p\otimes q)p_{m+n}[\eta]_{p\underline{\otimes}q}p_{m+n}(p\otimes q)p_{m+n}(p\otimes q) \\ &= (p\otimes q)p_{m+n}(p\otimes q)p_{m+n}(\Delta\eta)_{p,q}p_{m+n}(p\otimes q)p_{m+n}(p\otimes q) \\ &= (p\otimes q)p_{m+n}(p\otimes q)p_{m+n}(p\otimes q)(\Delta\eta)_{p,q}(p\otimes q)p_{m+n}(p\otimes q)p_{m+n}(p\otimes q) \ , \end{aligned}$$

keeping in mind the definition 4.4(1) of the $\underline{\otimes}$ coproduct for the second equality and Proposition 4.4A to move the two inner p_{m+n} in the last passage. Moreover $(\underline{\Delta}1)_{p,q} = (p\otimes q)p_{m+n}(p\otimes q)p_{m+n}(p\otimes q)$; finally $(\Delta\eta)_{p,q}$ commutes with both $p\otimes q$ and p_{m+n} by naturality. □

The Proposition generalises formula (10.9) of [MS92], covering the case $\mathfrak{g} = \mathfrak{sl}_2$. Of course formula (2) still holds for arbitrary compositions of evaluations of $\underline{\Delta}$ on some tensor component, e.g.

$$(\mathrm{id} \otimes \underline{\Delta})(\underline{\Delta}\eta) = (\mathrm{id} \otimes \underline{\Delta})(\underline{\Delta}1)[(\mathrm{id} \otimes \Delta)(\Delta\eta)] = [(\mathrm{id} \otimes \Delta)(\Delta\eta)](\mathrm{id} \otimes \underline{\Delta})(\underline{\Delta}1) \ .$$

By definition of intertwiners, the tensor powers of $\mathcal{U}_q(\mathfrak{g})$ map into the corresponding ones of $A(\mathfrak{g}, q)$. We are particularly interested in the natural isomorphisms coming from the specialisation $\mathcal{U}_q^\dagger(\mathfrak{h})$ of the Hopf subalgebra $\mathcal{U}_\mathcal{A}^\dagger(\mathfrak{h}) \subset \mathcal{U}_\mathcal{A}^\dagger(\mathfrak{g})$ defined in 3.2. We recall that $\mathcal{U}_q^\dagger(\mathfrak{h})$ is isomorphic, as a Hopf algebra, to $\mathrm{Map}(\Lambda, \mathbb{C})$, the abelian algebra of functions on the weight lattice Λ.

Remark. An element f of $\mathcal{U}_q^\dagger(\mathfrak{h})$ acts on a vector of weight λ as the scalar $f(\lambda)$. This is consistent with the notion of weight vector given in [CP95] in the case when the order ℓ' of q is odd, so $\ell = \ell'$, namely

$$K_i v = q_i^{(\lambda, \check{\alpha}_i)} \ , \qquad \begin{bmatrix} K_i; 0 \\ \ell \end{bmatrix}_{q_i} v = \begin{bmatrix} (\lambda, \check{\alpha}_i) \\ \ell \end{bmatrix}_{q_i} v \qquad \text{for } i = 1, \dots, l \ ,$$

where $q_i =$ stands for q if α_i is short and for q^D if α_i is long. We also note that, according with this notion, a tensor product of a λ-vector and a μ-vector is a $(\lambda+\mu)$-vector, which is again consistent with our notion, since $(\Delta f)(\lambda, \mu) = f(\lambda + \mu)$.

The coproduct law $(\Delta f)(\lambda, \mu) = f(\lambda + \mu)$ actually establishes that the natural isomorphisms of the tensor powers of $\mathcal{W}_0$ coming from $\mathcal{U}_q^\dagger(\mathfrak{h})$ form a cochain complex isomorphic to $C^\bullet(\Lambda, \mathbb{C}) = \mathrm{Map}(\Lambda^\bullet, \mathbb{C})$, the trivial cochain complex for the weight lattice Λ.

Example. Consider a simply connected compact Lie group G such that $\mathfrak{g}$ is its complexified Lie algebra, and let T be a maximal torus corresponding to the maximal toral subalgebra $\mathfrak{h}$. Then the Pontryagin dual $\hat{T}$ may be identified with the weight lattice Λ; moreover the characters that are trivial on the centre of G correspond to the root lattice $\Lambda_r \subset \Lambda$, whence $\widehat{Z(G)} \simeq \Lambda/\Lambda_r$. Therefore we have the inclusion

$$T \hookrightarrow \mathcal{U}_q^\dagger(\mathfrak{h}) \qquad \tau \mapsto \langle \tau, \cdot \rangle \ ;$$

the image of T in $\mathcal{U}_q^\dagger(\mathfrak{h})$ is exactly given by the homomorphisms from Λ to the circle $\mathbb{T}$ by Pontryagin duality. Furthermore, by the relations 3.2(3)

$$\mathcal{U}_q^\dagger(\mathfrak{h}) \cap Z(\mathcal{U}_q(\mathfrak{g})) = \{ f : \Lambda \to \mathbb{C} \mid f(\cdot + \alpha) = f \ \forall \ \alpha \in \Lambda_r \} \ .$$

We conclude that the above intersection identifies with $\mathrm{Map}(\Lambda/\Lambda_r, \mathbb{C})$, and $Z(G) = \hat{\Lambda} \cap Z(\mathcal{U}_q(\mathfrak{g}))$, implying the above identifications.

Of course what just said still holds after replacing T with T^n and $\mathcal{U}_q(\mathfrak{g})$ with its n-th generalised tensor power (see 3.2). Namely

- T^n embeds into $\mathcal{U}_q^\dagger(\mathfrak{h})^{\otimes n}$ as $\widehat{\Lambda^n}$;
- $\mathcal{U}_q^\dagger(\mathfrak{h})^{\otimes n} \cap Z(\mathcal{U}_q(\mathfrak{g})^{\otimes n}) = \mathrm{Map}\left((\Lambda/\Lambda_r)^n, \mathbb{C}\right)$;
- $Z(G)^n = \widehat{\Lambda^n} \cap Z(\mathcal{U}_q(\mathfrak{g})^{\otimes n})$.

5.2 Cocycles on the dual of $Z(G)$

By Example 5.1 usual 3-cocycles on Λ/Λ_r correspond to central elements of $\mathcal{U}_q(\mathfrak{g})^{\otimes 3}$. We now use the epimorphism 5.1(1) to construct new associators for the discrete algebra $A(\mathfrak{g}, q, \mathcal{W})$. Indeed, it will be enough to take care of the domain and codomain of the new associator.

Proposition. *Let f in $Z^3(\Lambda/\Lambda_r, \mathbb{C})$ be a normalised (i.e. $f(\cdot, 0, \cdot) = 1$) 3-cocycle, and put*

$$\Phi := (\underline{\Delta}_3)1[f]({}_3\underline{\Delta})1 \ ;$$

here and below we use the notation introduced before Definition 2.8B for compositions of coproducts. Then, spelling out the unitary weak Hopf algebra $A(\mathfrak{g}, q, \mathcal{W})$ as $(A, \cdot^\dagger, \Delta, \Omega)$, the quintuple $(A, \cdot^\dagger, \underline{\Delta}, \Phi, \Omega)$ is still a unitary weak quasi Hopf algebra.

Proof. We need to show that Φ meets the conditions in point iii) of Definition 2.1C applied to $(A, \underline{\Delta}, \Phi)$. The element Φ of $A^{\otimes 3}$ is partially invertible, with inverse $\Phi^{-1} = ({}_3\underline{\Delta})1[f^{-1}](\underline{\Delta}_3)1$. This can be verified using Proposition 4.4A as in the proof of Lemma 4.4A.

We now turn to verify that $\Phi({}_3\underline{\Delta})a = (\underline{\Delta}_3)a\Phi$ for all a in A; we may take $a = [\eta]$ with η in $A(\mathfrak{g}, q)$ thanks to the epimorphism 5.1(1). By Proposition 5.1, we have

$$\begin{aligned}(\underline{\Delta}_3 1)[f]({}_3\underline{\Delta}1)({}_3\underline{\Delta}[\eta]) &= (\underline{\Delta}_3 1)[f]({}_3\underline{\Delta}1)[{}_3\Delta\eta] \\ &= (\underline{\Delta}_3 1)[\Delta_3\eta][f]({}_3\underline{\Delta}1) = (\underline{\Delta}_3[\eta])(\underline{\Delta}_3 1)[f]({}_3\underline{\Delta}1) \ ;\end{aligned}$$

for the second equality, we used coassociativity of Δ. Furthermore, the action of ${}_3\Delta\eta = \Delta_3\eta$ commutes with all endomorphisms of any tensor product of three objects of $\mathcal{T}_q(\mathfrak{g})$.

Now, once evaluated on a generic full tensor product of simple objects p, q, r among the ones chosen for definitions 4.4(2), such are $[f]$ and the idempotents ${}_3\underline{\Delta}1$, $\underline{\Delta}_3 1$. As for the idempotents, this follows from the definition of the coproduct 2.1(1) applied to 4.4(2); on the other hand f lies in $Z(\mathcal{U}_q(\mathfrak{g})^{\otimes 3})$. Therefore the above equalities are actually justified, and we rewrite the proved identity as

$$(\underline{\Delta}_3 1)[f]({}_3\underline{\Delta}a) = (\underline{\Delta}_3 a)[f]({}_3\underline{\Delta}1) \qquad \forall\ a \in A \ . \tag{1}$$

The normalisation condition for Φ follows from those for f and for the weak Hopf algebra associator $(\underline{\Delta}_3 1)({}_3\underline{\Delta}1)$; so, in order to prove that $(A, \underline{\Delta}, \Phi)$ is still a weak quasi-bialgebra, we are left to check the cocycle identity

$$(1 \otimes \Phi)(\mathrm{id} \otimes \underline{\Delta} \otimes \mathrm{id})\Phi(\Phi \otimes 1) = (\mathrm{id} \otimes \mathrm{id} \otimes \underline{\Delta})\Phi(\underline{\Delta} \otimes \mathrm{id} \otimes \mathrm{id})\Phi \ . \tag{2}$$

By (1) the left-hand side amounts to

$$\begin{aligned}&\Big(1 \otimes ((\underline{\Delta}_3 1)[f])\Big)(\mathrm{id} \otimes \underline{\Delta} \otimes \mathrm{id})((\underline{\Delta}_3 1)[f]({}_3\underline{\Delta}1))\Big(([f]({}_3\underline{\Delta}1)) \otimes 1\Big) \\ &= (\underline{\Delta}_4 1)\Big(1 \otimes ([f]({}_3\underline{\Delta}1))\Big)(\mathrm{id} \otimes \underline{\Delta} \otimes \mathrm{id})[f]\Big(((\underline{\Delta}_3 1)[f]) \otimes 1\Big)({}_4\underline{\Delta}1) \\ &= (\underline{\Delta}_4 1)\Big(1 \otimes ([f]({}_3\underline{\Delta}1))\Big)\big[(\mathrm{id} \otimes \Delta \otimes \mathrm{id})f\big]\Big(((\underline{\Delta}_3 1)[f]) \otimes 1\Big)({}_4\underline{\Delta}1) \ ,\end{aligned}$$

where we used Proposition 5.1 for the last passage. Moreover, since ${}_3\Delta 1$ and $\Delta_3 1$ appear between ${}_4\Delta 1$ and $\Delta_4 1$, they may be cancelled by Proposition 4.4A. To sum up the left-hand side of (2) reduces to

$$(\underline{\Delta}_4 1)(1 \otimes [f])\,[(\mathrm{id} \otimes \Delta \otimes \mathrm{id})f]\,([f] \otimes 1)({}_4\underline{\Delta} 1)\ .$$

Along the same lines, we compute $(\mathrm{id} \otimes \mathrm{id} \otimes \underline{\Delta})\Phi(\underline{\Delta} \otimes \mathrm{id} \otimes \mathrm{id})\Phi$ to equal

$$\begin{aligned}&(\underline{\Delta}_4 1)(\mathrm{id} \otimes \mathrm{id} \otimes \underline{\Delta})[f](\underline{\Delta} \otimes \underline{\Delta})(\underline{\Delta} 1)(\underline{\Delta} \otimes \mathrm{id} \otimes \mathrm{id})[f]({}_4\underline{\Delta} 1)\\ &\quad = (\underline{\Delta}_4 1)\,[(\mathrm{id} \otimes \mathrm{id} \otimes \Delta)f]\,[(\Delta \otimes \mathrm{id} \otimes \mathrm{id})f]\,({}_4\underline{\Delta} 1)\ .\end{aligned}$$

Therefore (2) holds by the usual cocycle identity for f. Finally, we have to verify that Ω satisfies the identity

$$(1 \otimes \Omega)(\mathrm{id} \otimes \underline{\Delta})(\Omega)\Phi(\underline{\Delta} \otimes \mathrm{id})(\Omega^{-1})(\Omega^{-1} \otimes 1) = (\Phi^\dagger)^{-1}\ ,$$

which is true when Φ is replaced by the weak bialgebra associator $(\underline{\Delta}_3 1)({}_3\Delta 1)$; but $\Phi = (\underline{\Delta}_3 1)[f]({}_3\Delta 1)$ and $[f]$ commutes with all other factors on both sides, so the above identity holds for Φ as well. □

For the rest of current subsection we set $\mathfrak{g} = \mathfrak{sl}_N$, and we take $\mathfrak{h}$ to be the Lie subalgebra of diagonal matrix of null trace. We also consider the whole vector space $\tilde{\mathfrak{h}}$ of $N \times N$ diagonal matrices, and the basis $\{L_i \mid i = 1, \dots, N\}$ of its dual $\tilde{\mathfrak{h}}'$ dual to the basis of elementary diagonal matrices $\{e_{ii} \mid i = 1, \dots, N\}$.

Therefore $\mathfrak{h}'$ may be viewed as the quotient of $\tilde{\mathfrak{h}}'$ given by the relation $L_1 + \dots + L_N = 0$; implying this quotient, the fundamental dominant weights are given by $\lambda_i = L_1 + \cdots L_i$. Finally, we introduce the homomorphism

$$|\cdot| : \Lambda \to \mathbb{Z} \qquad |L_i| = 1 \text{ for } i = 1, \dots, N-1\ ,\ |L_N| = 1 - N\ . \tag{3}$$

$L_1 + \dots + L_N$ maps to zero, so $|\cdot|$ is well defined. Moreover it is easy to check that $|\lambda| = 0$ exactly if λ is in the root sublattice, so $|\cdot|$ provides an explicit isomorphism $\Lambda/\Lambda_r \simeq \mathbb{Z}/N\mathbb{Z}$.

Example (Kazhdan-Wenzl cocycles)**.** The homomorphism (3) allows us to introduce a very relevant instance of the associators in the Proposition. Namely, given an N-th root of 1 w, we take

$$f^w(\lambda, \mu, \nu) = w^{\gamma(|\lambda|,|\mu|)|\nu|} \qquad \text{where} \quad \gamma(m, n) := \left\lfloor \frac{m+n}{N} \right\rfloor - \left\lfloor \frac{m}{N} \right\rfloor - \left\lfloor \frac{n}{N} \right\rfloor \tag{4}$$

for integer m, n. In fact $H^3(\Lambda/\Lambda_r, \mathbb{T})$ is isomorphic to $\mathbb{Z}/N\mathbb{Z}$ and it is generated by any f_w with w of order N (see the appendix of [NY15]). Moreover the associator of the category $\mathrm{Rep}(A, \underline{\Delta}, \Phi_{f^w})$ differs from the one of the original representation category $\mathcal{C}$ of the weak Hopf algebra $(A, \underline{\Delta})$ by the scalar $f^w(\lambda, \mu, \nu)$ for each triple of simple objects with highest weights λ, μ, ν. Therefore, by Kazhdan-Wenzl theory, the categories $\mathcal{C}^w := \mathrm{Rep}(A, \underline{\Delta}, \Phi_{f^w})$ with w ranging through the N-th roots of (primitive or not) are pairwise non-tensor equivalent; moreover they cover all tensor equivalence classes of any tensor category with the same fusion rules as $\mathcal{C}$.

5.3 Neshveyev-Yamashita twists

Let us return to the more general picture where $\mathfrak{g}$ is any simple Lie algebra. Following 2.2 of [NY15], we introduce a particular type of elements of $C^2(\Lambda, \mathbb{T}) = \mathrm{Map}(\Lambda^2, \mathbb{T})$ whose coboundary are actually 3-cocycles on the dual of $Z(G)$; their definition and properties are reported here for better convenience.

Lemma. *Given $\tau = (\tau_1, \ldots, \tau_l)$ in $Z(G)^l$, let us consider a function $g : \Lambda^2 \to \mathbb{T}$ satisfying*

$$g(\lambda + \Lambda_r, \mu) = g(\lambda, \mu) \ , \quad g(\lambda, \mu + \alpha_i) = \langle \tau_i, \lambda \rangle g(\lambda, \mu) \qquad \forall\ \lambda, \mu \in \Lambda \tag{1}$$

for $i = 1, \ldots, l$. Then the coboundary ∂g is invariant under translation by Λ_r in each of its arguments, and it is a 3-cocycle by construction, so indeed ∂g lies in $Z^3(\Lambda/\Lambda_r, \mathbb{T})$. Moreover:

- *if g' also satisfies conditions* (1) *then the cohomology classes of ∂g and $\partial g'$ coincide, so they represent the same element of the third cohomology group $H^3(\Lambda/\Lambda_r, \mathbb{T})$;*
- *the twisted coproduct $\Delta_g = g\Delta(\cdot)g^{-1}$ for $A(\mathfrak{g}, q)$ does not depend on the choice of g either, so it only depends on τ as the cohomology class of ∂g;*
- *the 3-cocycles of the form ∂g with g satisfying* (1) *exhaust $H^3(\Lambda/\Lambda_r, \mathbb{T})$.*

The last point is proved in Proposition 2.6 of [NY15], and it relies significantly on the assumption that $\mathfrak{g}$ is simple, rather than just semi-simple. For better convenience, we list here the cocycles, twists and associators appearing in the rest of the section:

- $g : \Lambda^2 \to \mathbb{T}$ satisfies conditions (1) for a fixed $\tau = (\tau_1, \ldots, \tau_l)$ in $Z(G)^l$;
- $f = \partial g^{-1}$ is a unitary element of $Z^3(\Lambda/\Lambda_r, \mathbb{T})$ by the Lemma;
- in the case $\mathfrak{g} = \mathfrak{sl}_N$, f^w is the unitary element of $Z^3(\Lambda/\Lambda_r, \mathbb{T})$ defined in 5.2(4); by the Lemma we may choose g such that $f = \partial g^{-1} = f^w$.
- Let $(A, \cdot^\dagger, \underline{\Delta}, \Phi, \Omega)$ be the discrete unitary weak quasi-bialgebra of Proposition 5.2 with f as in the second point, so $\Phi = (\underline{\Delta}_3)1[f]({}_3\underline{\Delta})1$.
- We finally put $F := [g]\underline{\Delta}1$, which is clearly a twist from $(\underline{\Delta}, \Phi)$, and we denote its codomonain by $(\underline{\Delta}_F, \Phi_F)$ (see the remarks following Definition 2.5A).

Proposition A. *We have $\Phi_F = \big((\underline{\Delta}_F)_3 1\big)\big({}_3(\underline{\Delta}_F)1\big)$, i.e. $(A, \underline{\Delta}_F)$ is a weak Hopf algebra. Therefore F twists $(A, \cdot^\dagger, \underline{\Delta}, \Phi, \Omega)$ to a discrete unitary weak Hopf algebra.*

Proof. By definition we have $F^{-1} = \underline{\Delta}1[g^{-1}]$, so $\underline{\Delta}_F = [g]\underline{\Delta}(\cdot)[g^{-1}]$. Thus

$$\begin{aligned}\Phi_F &= (1 \otimes F)(\mathrm{id} \otimes \underline{\Delta}) F \Phi (\underline{\Delta} \otimes \mathrm{id}) F^{-1} (F^{-1} \otimes 1) \\ &= \big(1 \otimes [g]\big)(\mathrm{id} \otimes \underline{\Delta})[g](\underline{\Delta}_3 1)[\partial g^{-1}]({}_3\underline{\Delta}1)(\underline{\Delta} \otimes \mathrm{id})[g^{-1}]\big([g^{-1}] \otimes 1\big) \ .\end{aligned}$$

On the other hand

$$\begin{aligned}(\underline{\Delta}_F)_3 1 &= (1\otimes[g])(\mathrm{id}\otimes\underline{\Delta})([g](\underline{\Delta}1)[g^{-1}])(1\otimes[g^{-1}])\\ &= (1\otimes[g])(\mathrm{id}\otimes\underline{\Delta})[g](\underline{\Delta}_3 1)(\mathrm{id}\otimes\underline{\Delta})[g^{-1}](1\otimes[g^{-1}])\ ,\\ {}_3(\underline{\Delta}_F)1 &= ([g]\otimes 1)(\underline{\Delta}\otimes\mathrm{id})[g]({}_3\underline{\Delta}1)(\underline{\Delta}\otimes\mathrm{id})[g^{-1}]([g^{-1}]\otimes 1)\ ;\end{aligned}$$

moreover, since $\underline{\Delta}_3 1 > (\mathrm{id}\otimes\underline{\Delta})(1\otimes 1)$ and ${}_3\underline{\Delta}1 > (\underline{\Delta}\otimes\mathrm{id})(1\otimes 1)$

$$\begin{aligned}&(\underline{\Delta}_3 1)(\mathrm{id}\otimes\underline{\Delta})[g^{-1}](1\otimes[g^{-1}])([g]\otimes 1)(\underline{\Delta}\otimes\mathrm{id})[g]({}_3\underline{\Delta}1)\\ &= (\underline{\Delta}_3 1)(\mathrm{id}\otimes\Delta)[g^{-1}](1\otimes[g^{-1}])([g]\otimes 1)(\Delta\otimes\mathrm{id})[g]({}_3\underline{\Delta}1)\end{aligned}$$

by Proposition 4.4A. But $(\mathrm{id}\otimes\Delta)[g^{-1}](1\otimes[g^{-1}])([g]\otimes 1)(\Delta\otimes\mathrm{id})[g]$ equals $[\partial g^{-1}]$, therefore we conclude $((\underline{\Delta}_F)_3 1)({}_3(\underline{\Delta}_F)1) = \Phi_F$ as desired. The final assertion just follows from Lemma 2.5B. □

Corollary. *Let $\mathfrak{g} = \mathfrak{sl}_N$. The representation categories of the discrete unitary weak Hopf algebras $(A, \cdot^\dagger, \underline{\Delta}_F, \Omega_F)$ exhaust all tensor equivalence classes of any tensor category with the same fusion rules as the quotient $\mathcal{C} = \overline{\mathcal{T}_q(\mathfrak{g})}$.*

Proof. Since F twists $(A, \cdot^\dagger, \underline{\Delta}, \Phi, \Omega)$ to $(A, \cdot^\dagger, \underline{\Delta}_F, \Omega_F)$ their representation categories are tensor $*$-isomorphic, by Lemma 2.5B. Moreover, for $f = f^w$ (see the third point in the list before Proposition A) the category $\mathrm{Rep}(A, \cdot^\dagger, \underline{\Delta}, \Phi, \Omega)$ is the $\mathcal{C}^w$ of Example 5.2; the statement follows from Kazhdan-Wenzl theory, as recalled at the end of said example. □

We observe that unitarity of every $\mathrm{Rep}(A, \cdot^\dagger, \underline{\Delta}_F, \Omega_F)$ provides a concrete way to verify a fact implied by Proposition 19.12 of [CCP21], namely that pseudounitarity of $\mathcal{C}$ carries over to all its twists $\mathcal{C}^w$.

It is also possible to apply the classification to show that $\mathcal{C}^w$ does not admit braidings unless $w^2 = 1$, as proved in Proposition 19.9 of wqh following the analogous Remark 4.4 of [NY15] for the compact case.

Proposition B. *The twisted category $\mathcal{C}^w$ admits generalised coboundaries only if $w = 1$ for N odd, and only if $w = \pm 1$ for N even.*

Proof. Let w be an N-th root of 1; as observed in Example 5.2, $\mathcal{C}^w$ is tensor equivalent to $\mathrm{Rep}(A, \underline{\Delta}, \Phi)$, where $\Phi = (\underline{\Delta}_3 1)[f^w]({}_3\underline{\Delta}1)$. Writing $\Phi_0 := (\underline{\Delta}_3 1)({}_3\underline{\Delta}1)$ and $[f^w] =: \Upsilon$, we have $\Phi = \Phi_0\Upsilon$.

By Proposition 2.3, generalised coboundaries on $\mathrm{Rep}(A, \underline{\Delta}, \Phi)$ correspond exactly to almost cocommutative structures for $(A, \underline{\Delta}, \Phi)$ (Definition 2.3C); so, given a twist R from $(\underline{\Delta}, \Phi_0)$ to $(\underline{\Delta}^{\mathrm{op}}, (\Phi_0)^{-1}_{321})$, it will be enough to prove that $w^2 = 1$.

We denote by R_0 the quasi-triangular structure for the weak Hopf algebra $(A, \underline{\Delta})$, so R_0 twists $(\underline{\Delta}, \Phi_0)$ to $(\underline{\Delta}^{\mathrm{op}}, (\Phi_0)^{-1}_{321})$ too; moreover, since $\Phi_0 := (\underline{\Delta}_3 1)({}_3\underline{\Delta}1)$ compatibility with the associators (the third point in Definition 2.5A) reduces to

$$(1\otimes R_0)(\mathrm{id}\otimes\underline{\Delta})(R_0)(\underline{\Delta}\otimes\mathrm{id})(R_0^{-1})(R_0^{-1}\otimes 1) = (\Phi_0)^{-1}_{321}\ .$$

By centrality of Υ, the same condition for R is

$$(1\otimes R)(\mathrm{id}\otimes\underline{\Delta})(R)(\underline{\Delta}\otimes\mathrm{id})(R^{-1})(R^{-1}\otimes 1)\Upsilon = (\Phi_0)^{-1}_{321}\Upsilon^{-1}_{321}\ . \tag{2}$$

Now, we define $F := R_0^{-1}R$ and observe that

$$(1\otimes F)(\mathrm{id}\otimes\underline{\Delta})F = \big((1\otimes R_0)(\mathrm{id}\otimes\underline{\Delta})R_0\big)^{-1}(1\otimes R)(\mathrm{id}\otimes\underline{\Delta})R$$
$$(F\otimes 1)(\underline{\Delta}\otimes\mathrm{id})F = \big((R_0\otimes 1)(\underline{\Delta}\otimes\mathrm{id})R_0\big)^{-1}(R\otimes 1)(\underline{\Delta}\otimes\mathrm{id})R\ ,$$

because both R^0 and R twist $\underline{\Delta}$ to $\underline{\Delta}^{\mathrm{op}}$ (the second point in Definition 2.5A), which also implies that F twists $\underline{\Delta}$ to itself. Furthermore, we compute

$$(1\otimes F^{-1})(\mathrm{id}\otimes\underline{\Delta})F^{-1}(\underline{\Delta}\otimes\mathrm{id})F(F\otimes 1) =$$
$$\big((1\otimes F)(\mathrm{id}\otimes\underline{\Delta})F\big)^{-1}(F\otimes 1)(\underline{\Delta}\otimes\mathrm{id})F = (\mathrm{id}\otimes\underline{\Delta})R^{-1}(1\otimes R^{-1})\cdot$$
$$\cdot(1\otimes R_0)(\mathrm{id}\otimes\underline{\Delta})R_0(\underline{\Delta}\otimes\mathrm{id})R_0^{-1}(R_0^{-1}\otimes 1)(R\otimes 1)(\underline{\Delta}\otimes\mathrm{id})R =$$
$$(\mathrm{id}\otimes\underline{\Delta})R^{-1}(1\otimes R^{-1})\Phi^0_{321}(R\otimes 1)(\underline{\Delta}\otimes\mathrm{id})R\ .$$

Finally, applying the inverse of (2) (still keeping in mind that R twists $\underline{\Delta}$ to $\underline{\Delta}^{\mathrm{op}}$), we have

$$(1\otimes F^{-1})(\mathrm{id}\otimes\underline{\Delta})F^{-1}(\underline{\Delta}\otimes\mathrm{id})F(F\otimes 1) = \Upsilon_{321}\Upsilon\Phi_0\ .$$

Summarizing, F twists $(\underline{\Delta},\Upsilon_{321}\Upsilon\Phi_0)$ to $(\underline{\Delta},\Phi_0)$, therefore the representation categories $\mathrm{Rep}(A,\underline{\Delta},\Upsilon_{321}\Upsilon\Phi_0)$ and $\mathrm{Rep}(A,\underline{\Delta})$ are isomorphic by Lemma 2.5B. On the other hand, as observed in Remark 4.4 of [NY15], $f^w_{321}f^w$ and f^{w^2} represent the same element of $H^3(\Lambda/\Lambda_r,\mathbb{T})$, so we conclude that $w^2 = 1$ by Kazhdan-Wenzl classification. □

The twisted QUE algebra Since the 3-cocycle f lies in $Z^3(\Lambda/\Lambda_r,\mathbb{T})$, we may view it as a new associator for the Hopf algebra $\mathcal{U}_q(\mathfrak{g})$, in the sense of Definition 2.1C (which of course makes sense outside the context of discrete algebras as well), thus turning the original QUE into a new quasi-Hopf algebra $\mathcal{U}(\Delta,f)$.

Furthermore, the treatment of 4.3 applies without modifications. One may also realise the quotient $\overline{\mathcal{T}_q(\mathfrak{g})}$ of Proposition 4.2 as a linear subcategory $\mathcal{C}(\Delta,f)\subset\mathcal{T}_q(\mathfrak{g})$; therefore $\mathcal{C}(\Delta,f)$ coincides with the unitary strict tensor category $\mathcal{G}_q(\mathfrak{g})$ of 4.4, except that the trivial associator is multiplied by f.

Finally the forgetful functor $\mathcal{C}(\Delta,f)\to\mathrm{Hilb}$ admits the same weak quasi-tensor structure, but such structure is not actually weak tensor as in the case of $\mathcal{W}$ due to the modification of the associator; more precisely the discrete unitary weak quasi-Hopf algebra provided by the Tannakian theorem 2.4 applied to $\mathcal{C}(\Delta,f)$ is exactly the quintuple $(A,\cdot^\dagger,\underline{\Delta},\Phi,\Omega)$ of Proposition 5.2. Now, following [NY15], we modify the coproduct for $\mathcal{U}^\dagger_{\mathcal{A}}(\mathfrak{g})$ by setting

$$\Delta'(E_i) = E_i\otimes K_i + \tau_i\otimes E_i\ ,\quad \Delta'(F_i) = F_i\otimes 1 + \tau_i^{-1}K_i^{-1}\otimes F_i$$
$$\Delta'(f) = \Delta(f)\qquad \forall\ f\in\mathcal{U}^\dagger_{\mathcal{A}}(\mathfrak{h})\ ,$$

obtaining a new Hopf algebra $U(\Delta_g,\mathrm{id})$.

Proposition C. *The quasi-triangular structure R twists (Δ,f) to $(\tilde{\Delta},f)$ (see Definition 2.5A) and g twists (Δ,f) to (Δ_g,id). Therefore the element $R_g := gRg^{-1}$ twists (Δ_g,id) to $(\tilde{\Delta}_g,\mathrm{id})$.*

Proof. Since R is a quasi-triangular structure for $U_q(\mathfrak{g})$ and $\Delta^{\mathrm{op}} = \Delta(\cdot^*)^*$,

$$R\Delta(\cdot^*) = \Delta(\cdot)^* R \quad , \quad (1 \otimes R)(\mathrm{id} \otimes \Delta)R = (\Delta \otimes \mathrm{id})R(R \otimes 1) \ ,$$

i.e. R twists (Δ, id) to $(\tilde{\Delta}, \tilde{\mathrm{id}} = \mathrm{id})$. This is still true after replacing id with f, because f is central and unitary, so $\tilde{f} = (\overline{f})^{-1} = f$.

Turning to the second assertion, the identity $f\Delta(\cdot)f^{-1} = \Delta_g$ is readily verified using the relations 3.2(3) together with properties (1) of g; compatibility with associators (the third point in Definition 2.5A) reduces to $f = \partial g^{-1}$.

Finally, $g = (\overline{g})^{-1}$ twists $(\tilde{\Delta}, \tilde{f})$ to $(\widetilde{\Delta_g}, \tilde{\mathrm{id}})$, so the last assertion follows from the others by groupoid composition of twists. □

The Proposition may be summarized by saying that g twists the triple $(\cdot^*, \Delta, f, R)$ to $(\cdot^*, \Delta_g, \mathrm{id}, R_g)$, keeping in mind Lemma 2.5B. Even though in our case R is not self adjoint, it still plays the same role of the twist Ω in Definition 2.4B, especially through the introduction of the $(\cdot,\cdot)'$ form of 4.3.

Moreover the constructions for $\mathcal{U}_q(\mathfrak{g})$ of 4.2, 4.3 and 4.4, adapted to $\mathcal{U}(\Delta, f)$ as already discussed, mirror to $\mathcal{U}(\Delta_g, \mathrm{id})$ through g. More precisely:

- $\mathcal{U}(\Delta, f)$ and $\mathcal{U}(\Delta_g, \mathrm{id})$ share the antipode, $S^2(x) = q^{2\rho} x q^{-2\rho}$ and in both cases $q^{2\rho}$ is group-like; so it defines a spherical structure for both representation categories. Hence the category $\mathcal{T}_q(\mathfrak{g})$ and its quotient $\overline{\mathcal{T}_q(\mathfrak{g})}$ may be equally well defined from $\mathcal{U}(\Delta_g, \mathrm{id})$, just as in 4.2.

- The identity functor, endowed with the tensor structure defined by the action of g^{-1}, becomes an isomorphism of monoidal categories from Rep $(\mathcal{U}(\Delta, f))$ to Rep $(\mathcal{U}(\Delta_g, \mathrm{id}))$. Moreover, given dominant integral weights λ, μ, the structure map $g : V_\lambda(q) \otimes_\Delta V_\mu(q) \to V_\lambda(q) \otimes_{\Delta_g} V_\mu(q)$ is unitary with respect to the $(\cdot,\cdot)'$ forms defined by R and R_g respectively (see the situation of Lemma 2.5B).

- By previous point, the $(\cdot,\cdot)'$ form on a tensor product of specialised Weyl modules, e.g. $W := V_\lambda(q) \otimes_{\Delta_g} V_\mu(q)$, may be perturbed to a positive definite form on the semi-simple part W_s just as in 4.3, using the automorphism w of W_s given by the action of $q^{-\frac{G(\nu)}{2}}$ on each simple component of highest weight ν. Indeed, in our basic case of two Weyl modules, $gR\Delta(w)w^{-1} \otimes w^{-1}g^{-1} = R_g \Delta_g(w) w^{-1} \otimes w^{-1}$.

- For each of the idempotents p_n in $\mathrm{End}(V^{\otimes_\Delta n})$, we put

$$p'_n = g_n p_n g_n^{-1} \qquad g_2 = 2 \ , \ g_n = (g_{n-1} \otimes 1)(\Delta_{n-1} \otimes \mathrm{id})g \quad \forall\ n \geq 2 \ .$$

  Here of course $\Delta_2 := \Delta$ and $\Delta_n = (\Delta \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}) \circ \Delta_{n-1}$ (or any other composition of $n-1$ coproducts, since Δ is coassociative); clearly g_n is an isomorphism from $V^{\otimes_\Delta n}$ to $V^{\otimes_{\Delta_g} n}$.

We may use the idempotents p'_n in the last point to realise the quotient $\overline{\mathcal{T}_q(\mathfrak{g})}$ as a linear subcategory $\mathcal{C}(\Delta_g, \mathrm{id}) \subset \mathcal{T}_q(\mathfrak{g})$ as already done for $\mathcal{U}(\Delta, f)$ and $\mathcal{U}_q(\mathfrak{g})$ using the p_n; the scalar product will be marked by $\underline{\otimes}'$. Moreover, given the simple objects $p_\lambda < p_{n_\lambda}$ chosen for definitions 4.4(2), we put $p'_\lambda := g_{n_\lambda} p_\lambda g_{n_\lambda}^{-1}$ accordingly, so that $\{p'_\lambda\}_{\lambda \in \Lambda_\ell}$ is a complete collection of pairwise non-equivalent simple objects for $\mathcal{C}(\Delta_g, \mathrm{id})$. Finally, a weak quasi-tensor structure on the forgetful functor from $\mathcal{C}(\Delta_g, \mathrm{id})$ to Hilb is defined just as in 4.4, of course with the idempotents p'_λ in place of the p_λ.

We are now set to exploit the properties of g to define a unitary tensor isomorphism of C^* tensor categories $\mathcal{C}(\Delta, f)$ and $\mathcal{C}(\Delta_g, \mathrm{id})$.

Proposition D. *Consider the C^* tensor categories $\mathcal{C}(\Delta, f)$, $\mathcal{C}(\Delta_g, \mathrm{id})$. There exists an isomorphism of C^* tensor categories $\mathcal{E} : \mathcal{C}(\Delta, f) \to \mathcal{C}(\Delta_g, \mathrm{id})$.*

Proof. We define $\mathcal{E}$ by setting

$$\mathcal{E}(f) = g_n f g_m^{-1} \qquad \text{for } f \in (p, q) \ , \quad p < p_m \ , q < p_n \tag{3}$$

and extending by linearity. In order to define a tensor structure, we consider the unitary natural isomorphisms

$$p'_{m+n}(g_m \otimes g_n)(\Delta_m \otimes \Delta_n)(g)(p \otimes q)p_{m+n} \ , \qquad p < p_m, \ q < p_n \tag{4}$$

from $p\underline{\otimes}q$ to $\mathcal{E}(p)\underline{\otimes}'\mathcal{E}(q)$. In order to check unitarity, we note that all the Hilbert spaces involved are vector subspaces of $V^{\otimes m+n}$, so they may be treated in terms of idempotents and partial adjoints as in the proof of Lemma 4.4B. The isomorphisms in (4) are the restrictions to $p\underline{\otimes}q = p_{m+n}(p \otimes q)p_{m+n}$ of the ones given by the same expression without the factor $(p \otimes q)$. The adjoints are given by

$$\begin{gathered} p_{m+n}R^{-1}_{m+n}(\Delta_m \otimes \Delta_n)(g^{-1})(g_m^{-1} \otimes g_n^{-1})(R_g)_{m+n}p'_{m+n} \ , \text{ and} \\ p'_{m+n}(g_m \otimes g_n)(\Delta_m \otimes \Delta_n)(g)p_{m+n} \ \cdot \\ \cdot \ p_{m+n}R^{-1}_{m+n}(\Delta_m \otimes \Delta_n)(g^{-1})(g_m^{-1} \otimes g_n^{-1})(R_g)_{m+n}p'_{m+n} \\ = p'_{m+n}(R_g)^{-1}_{m+n}(R_g)_{m+n}p'_{m+n} = p'_{m+n} \ ; \end{gathered}$$

we applied Proposition 4.4A to cancel off p_{m+n} and kept in mind that the restriction of Δ to $\mathcal{U}_q(\mathfrak{h})$ is cocommutative and commutes with $\cdot^*$. Moreover, by Proposition C the trivial associator on $\mathrm{Rep}\left(\mathcal{U}(\Delta_g, \mathrm{id})\right)$ is unitary with respect to the forms defined by R_g on the tensor products; this implies that the perturbing matrix does not depend on how the parentheses are arranged, as it is also true for R itself (this can also be derived by quasi-triangularity of R, see 4.3 below formula (3)). In particular

$$(g_m \otimes g_n)(\Delta_m \otimes \Delta_n)(g)R^{-1}_{m+n}(\Delta_m \otimes \Delta_n)(g^{-1})(g_m^{-1} \otimes g_n^{-1}) = (R_g)^{-1}_{m+n} \ ,$$

and the isomorphisms in (4) are actually unitary. Furthermore, so are

$$g_n p \in (p, \mathcal{E}(p)) \qquad \text{for } p < p_n \ .$$

Therefore, we have the natural unitary isomorphisms

$$p'_{m+n}g_{m+n}(p \otimes q)(\Delta_m \otimes \Delta_n)(g^{-1})(g_m^{-1} \otimes g_n^{-1})p'_{m+n} \ , \qquad p < p_m, \ q < p_n$$

from $\mathcal{E}(p)\underline{\otimes}'\mathcal{E}(q)$ to $\mathcal{E}(p\underline{\otimes}q)$. The verification that they form a tensor structure for $\mathcal{E}$ boils down to the identity $f = \partial g^{-1}$, using once again Proposition 4.4A through the calculation. The inverse of $\mathcal{E}$ is obtained replacing g with g^{-1} and p_k with p'_k in (3). □

Note that, denoting by $\mathcal{W}_{\Delta,f}$ and $\mathcal{W}_{\Delta_g,\mathrm{id}}$ the forgetful functors to Hilb on $\mathcal{C}(\Delta, f)$ and $\mathcal{C}(\Delta_g, \mathrm{id})$ endowed with the respective weak quasi-tensor structures, $\mathcal{W}_{\Delta,f}$ and $\mathcal{W}_{\Delta_g,\mathrm{id}} \circ \mathcal{E}$ are isomorphic only up to a twist (see Proposition 2.5C). Indeed the latter is weak tensor, and produces the discrete unitary weak Hopf algebra $(A, \underline{\Delta}_F)$ of Proposition A.

Appendices

I Idempotent elements of an associative algebra

Let us consider an associative $\mathbb{C}$-algebra A, not necessarily unital. Its idempotents can be seen as the objects of a $\mathbb{C}$-linear category $\mathcal{P}(A)$, whose morphism spaces are given by

$$(p,q) := qAp = \{f \in A \mid fp = f = qf\}$$

for each pair of idempotents p,q in A. From now on we will write $q < p$ if $qp = q = pq$, in which case $p - q$ is idempotent too; we will say that an element of A is *partially invertible*, or that is a *partial isomorphism*, if it is an isomorphism of $\mathcal{P}(A)$. We further note that if $f : p \to q$ is partially invertible then we have the bijection

$$p' < p \iff q' < q \qquad q' = fp'f^{-1} \ , \tag{1}$$

and we may consider the restrictions $fp' = q'f$, that are partial isomorphisms as well, from p' to q'.

Now, let us suppose we have antilinear antiautomorphisms ${}^{\dagger}\cdot$ and $\cdot^{\dagger}$, inverse to each other. We use this additional datum to define an enrichment $\mathcal{P}(A, {}^{\dagger}\cdot, \cdot^{\dagger})$ of $\mathcal{P}(A)$. An object of the new category is a triple (p,ϕ,ψ), where p is an idempotent of A, while ϕ and ψ are partially invertible respectively in $(p, {}^{\dagger}p)$ and $(p, p^{\dagger})$, with $\phi^{\dagger} = \psi$.

Example. An important special case occurs when $\phi = p = \psi$. This means that ${}^{\dagger}p = p$, or equivalently $p = p^{\dagger}$; we call such idempotents *orthogonal projections.*

Morphisms are defined as before, but now ${}^{\dagger}\cdot$ e $\cdot^{\dagger}$ define respective operations ${}^{*}\cdot$ e $\cdot^{*}$. Given objects (p_1,ϕ_1,ψ_1), (p_2,ϕ_2,ψ_2) and f in (p_1,p_2), we put

$${}^{*}f := \phi_1^{-1}({}^{\dagger}f)\phi_2 \ , \qquad f^{*} := \psi_1^{-1}(f^{\dagger})\psi_2 \ . \tag{2}$$

Clearly ${}^{*}\cdot$ and $\cdot^{*}$ are contravariant antilinear automorphisms of $\mathcal{P}(A, {}^{\dagger}\cdot, \cdot^{\dagger})$, inverse to each other. We also remark that in the full subcategory of orthogonal projections, which we denote by $\mathcal{P}(A, {}^{\dagger}\cdot, \cdot^{\dagger})^{\perp}$, we simply have ${}^{*}\cdot = {}^{\dagger}\cdot$ and $\cdot^{*} = \cdot^{\dagger}$.

Remark. Let (p,ϕ,ψ) be an object of $\mathcal{P}(A, {}^{\dagger}\cdot, \cdot^{\dagger})$; the $\mathbb{C}$-algebra $pAp = (p,p)$ may be equipped with the antilinear antiautomorphisms ${}^{*}\cdot$ e $\cdot^{*}$, so we can consider the category $(pAp, {}^{*}\cdot, \cdot^{*})$, and its relative operations ${}^{\star}\cdot$ and $\cdot^{\star}$.

In detail, consider objects (q_i,π_i,ϖ_i) of $(pAp, {}^{*}\cdot, \cdot^{*})$, where $i = 1,2$. This means that π_i is in $\big(q_i, \phi^{-1}({}^{\dagger}q_i)\phi\big)$, ϖ_i is in $\big(q_i, \psi^{-1}(q_i^{\dagger})\psi\big)$ and $\varpi_i = \pi_i^{*} = \psi^{-1}\pi_i^{\dagger}\psi$; furthermore, given g in (q_1,q_2),

$${}^{\star}g = \pi_1^{-1}({}^{*}g)\pi_2 = \pi_1^{-1}\phi^{-1}({}^{\dagger}g)\phi\pi_2 = (\phi\pi_1)^{-1}({}^{\dagger}g)(\phi\pi_2) \ ,$$

and analogously $g^\star = (\psi\varpi_1)^{-1}(g^\dagger)(\psi\varpi_2)$. Finally

$$\psi\varpi_i = \psi\pi_i^* = \psi\psi^{-1}\pi_i^\dagger\psi = \pi_i^\dagger\phi^\dagger = (\phi\pi_i)^\dagger \ .$$

The Remark could be synthesised by saying that if we send $(q,\pi,\varpi) \mapsto (q,\phi\pi,\psi\varpi)$ we obtain an inclusion of $(pAp, {}^*\cdot, \cdot^*)$, which we shall call the category of sub-projections of (p,ϕ,ψ), in $\mathcal{P}(A, {}^\dagger\cdot, \cdot^\dagger)$, identical on morphisms.

A remarkable case occurs when we restrict ourselves to $(pAp, {}^*\cdot, \cdot^*)^\perp$; an idempotent $q < p$ is an orthogonal sub-projection exactly if $\phi q \phi^{-1} = {}^\dagger q$, or equivalently $\psi q \psi^{-1} = q^\dagger$ (see the bijections of the kind of (1)). To compute the operations ${}^*\cdot$ e $\cdot^*$ for morphisms between the corresponding objects of $\mathcal{P}(A, {}^\dagger\cdot, \cdot^\dagger)$ is particularly convenient; for instance, consider objects (p_i, ϕ_i, ψ_i) and orthogonal sub-projections $q_i < p_i$, $i = 1, 2$. Then, given g in (q_1, q_2),

$${}^*g = q_1\phi_1^{-1}({}^\dagger g)\phi_2 q_2 = \phi_1^{-1}({}^\dagger q_1)({}^\dagger g)({}^\dagger q_2)\phi_2 = \phi_1^{-1}({}^\dagger g)\phi_2 \ ,$$

and similarly $g^\dagger = \psi^{-1} g^\dagger \psi_2$; so, though still keeping in mind that we see g as a morphism in (q_1, q_2), we may compute the respective operations as if it were in (p_1, p_2).

Idempotents and subspaces

The category $\mathcal{P}(A)$ is modelled on the case $A = \mathrm{End}(V)$, where V is a $\mathbb{C}$-vector space of finite dimension. Moreover it is natural to consider the functor $\mathcal{S}$ from $\mathcal{P}(A)$ to the category of the subspaces of V: we send the generic idempotent p into pV and, given f in (p,q), we put

$$\mathcal{S}(f) = \underline{q} f \overline{p} \ , \tag{3}$$

where we wrote $p = \overline{p}\underline{p}$ with $\overline{p} : pV \to V$ and $\underline{p} : V \to pV$ such that $\underline{p}\overline{p} = \mathrm{id}_{pV}$, and likewise $q = \overline{q}\underline{q}$. In short we see f as a map from pV to qV.

If V comes with a non-degenerate sesquilinear form $(\cdot,\cdot)$, there are the two adjoints ${}^\dagger\cdot$ e $\cdot^\dagger$: for all f in A

$$(fv, w) =: (v, f^\dagger w) \ , \quad (w, fv) =: ({}^\dagger f w, v) \qquad \forall \ v, w \in V \ .$$

So we can consider $\mathcal{P}(A, {}^\dagger\cdot, \cdot^\dagger)$. Then, given ψ in $(p, p^\dagger)$ partially invertible, it defines the new non-degenerate form on pV

$$(v, w)_\psi := (v, \mathcal{S}(\psi)w) = (\mathcal{S}(\phi)v, w) \qquad \forall \ v, w \in pV \ ;$$

in the case of orthogonal projections one just gets the restriction of $(\cdot,\cdot)$ to pV. If we take f as in (2), the adjoints ${}^*\mathcal{S}(f)$ and $\mathcal{S}(f)^*$ relative to $(\cdot,\cdot)_{\psi_1}$ and $(\cdot,\cdot)_{\psi_2}$ are exactly given by the right-hand sides in (2), with $\mathcal{S}$ applied to each factor.

To sum up, $\mathcal{S}$ may be refined to a functor, compatible with the ${}^*\cdot$ and $\cdot^*$ operations, from $\mathcal{P}(A, {}^\dagger\cdot, \cdot^\dagger)$ to the category of subspaces of V endowed with a non-degenerate sesquilinear form, by sending (p,ϕ,ψ) to pV with $(\cdot,\cdot)_\psi$.

The Hermitian case If $(\cdot,\cdot)$ is Hermitian, ${}^\dagger\cdot = \cdot^\dagger$, in which case we replace the notation $\mathcal{P}(A, {}^\dagger\cdot, \cdot^\dagger)$ with the simpler $\mathcal{P}(A, \cdot^\dagger)$. Then $(\cdot,\cdot)_\psi$ is in turn Hermitian if and only if $\phi = \psi$, i.e. ψ is self-adjoint, in which case we just write (p, ψ) rather than (p, ϕ, ψ).

So, naming $\mathcal{P}(A, \cdot^\dagger)_{\text{herm}}$ the full subcategory of the (p, ψ) objects, $\mathcal{S}$ becomes a $*$-functor from $\mathcal{P}(A, \cdot^\dagger)_{\text{herm}}$ to the category of subspaces of V endowed with a non-degenerate Hermitian form.

Finally, if $(\cdot,\cdot)$ is a scalar product then $(\cdot,\cdot)_\psi$ is also positive definite if and only if ψ is positive in A.

To conclude the appendix, we do not mind to point out that the case of idempotents relative to several vector spaces is covered as well by considering suitable direct sums; see the end of Remark 2.2 for an example.

II Leg notation

Let V be a $\mathbb{C}$-vector space, and consider the n-th tensor power $V^{\otimes n}$. Then

$$\sigma \in S_n \mapsto \sigma. \in \mathrm{GL}(V^{\otimes n}) \qquad \sigma.(v_1 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$$

defines a (left) action of the n-th symmetric group on $V^{\otimes n}$. Now, we denote $\sigma.v$ by $v_{\sigma(1),\dots,\sigma(n)}$; as an example, if σ in S_3 is given by $\sigma(1) = 2$, $\sigma(2) = 3$, $\sigma(3) = 1$, then

$$(u \otimes v \otimes w)_{231} = \sigma.(u \otimes v \otimes w) = w \otimes u \otimes v \ .$$

Namely, the k-th factor of a tensor product moves to the position given by the k-th number in the subscript.

If V is a unital algebra A, then for $m < n$ $A^{\otimes m} \hookrightarrow A^{\otimes n}$, by just inserting 1 as the last $n - m$ factors; when we apply the above notation after such an inclusion, we only specify the final position of the original components of the given element $a_1 \otimes \cdots \otimes a_m$. As an example, with $m = 2$ and $n = 4$,

$$(a \otimes b)_{42} = 1 \otimes b \otimes 1 \otimes a \ .$$

Even though n is not expressed, it is generally clear from the context (e.g. it is 3 in identities 2.3(5)).

The leg notation obviously extends by continuity to the discrete algebras of Definition 2.1B. However, if the algebra is explicitly presented as a direct sum of matrix algebras, as in the case of 2.1, it also applies directly.

In detail, consider η in $A^{\otimes n}$, where A is the discrete algebra of 2.1, and a permutation σ in S_n. Then for each $\rho_1, \dots, \rho_n$, denoting by

$$\Sigma : \mathrm{End}\left(\mathcal{F}(\rho_{\sigma(1)})\right) \otimes \cdots \otimes \mathrm{End}\left(\mathcal{F}(\rho_{\sigma(n)})\right) \to \mathrm{End}\left(\mathcal{F}(\rho_1)\right) \otimes \cdots \otimes \mathrm{End}\left(\mathcal{F}(\rho_n)\right)$$

the appropriate flip map, $(\sigma.\eta)_{\rho_1,\dots,\rho_n} = \Sigma(\eta_{\rho_{\sigma(1)},\dots,\rho_{\sigma(n)}})$. We still have the inclusions $A^{\otimes m} \hookrightarrow A^{\otimes n}$ for $m < n$, and we establish the same rule as above for permutations following these inclusions.

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