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# A variational approach to nonlocal interactions: discrete-to-continuum analysis, ground states and geometric evolutions

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# Contents

# Introduction

This thesis deals with variational models for systems governed by nonlocal interactions. In particular, we analyze systems of hard spheres governed by attractive Riesz potentials, surface energies related to fractional perimeters and gradient flows of such energies leading to local and nonlocal geometric evolutions; eventually, we consider similar problems for densities governed by Gagliardo-type seminorms, focussing on fractional heat flows.

The thesis is constituted by three - almost self-contained - chapters, corresponding to the three articles written during my PhD studies, see [54], [36] and [31]. In the first chapter we introduce a model for hard spheres interacting through attractive Riesz type potentials, and we study its thermodynamic limit. In the second chapter we consider a core-radius approach to nonlocal perimeters governed by kernels having critical and supercritical exponents, extending the notion of $s$-fractional perimeter, defined for $0 < s < 1$, to the case $s \geq 1$. We study the $\Gamma$-convergence and the convergence of the corresponding nonlocal geometric flows, as the core-radius vanishes. In the third chapter we study the limit cases, as $s \to 0^+$ and $s \to 1^-$, for $s$-fractional heat flows with homogeneous Dirichlet boundary conditions.

In the following we describe in more details the content of the three chapters of the thesis.

In the first chapter we introduce and analyze variational models for hard spheres interacting through Riesz type attractive potentials. The model consists in minimizing nonlocal energies of the type

$$\sum_{i\neq j} K^p(|x_i - x_j|), \tag{0.0.1}$$

over all configurations of $N$ points $\{x_1, \ldots, x_N\} \subset \mathbb{R}^d$ satisfying $|x_i - x_j| \geq 2$ for all $i \neq j$; here $K^p : \mathbb{R}^+ \to (-\infty, 0]$ is a power-law attractive potential $K^p(r) \approx -\frac{1}{r^p}$ for large $r$, with $p \in (0, d+1)$. Eventually, we consider the thermodynamic limit $N \to +\infty$.

The thermodynamic limit is described by a nonlocal energy that is a Riesz type continuous counterpart of (0.0.1) for $p \in (0, d)$; in the case $p \in [d, d+1)$ fractional perimeters arise in the limit energy. In both cases $p \in (0, d)$ and $p \in [d, d+1)$, the optimal asymptotic shape is given (after scaling) by the Euclidean ball, and this is a consequence of the Riesz rearrangement inequality and of the fractional isoperimetric inequality, respectively. These results are obtained by providing a $\Gamma$-convergence expansion of the energy.

The combination of the attractive potential together with the hard sphere constraint provides a basic example of long range attractive/short range repulsive interactions. In this respect, the proposed model fits in the class of *aggregation* [42, 20, 25] and *crystallization* [16] problems, but with a substantial change of perspective due to the fundamental role played in our model by the tail of the interaction energy. This is the case for both integrable and non-integrable tails, referred to as *unstable potentials* in the crystallization community [16].

This is why in our model crystallization is replaced by the related but different concept of *optimal packing*, while the microscopic structure does not affect the macroscopic shape, that turns out to be the Euclidean ball.

To explain these new phenomena, we first provide an overview of the classical crystallization problem, focussing on two basic models in two dimensions. They are based on minimization of an interaction energy as in (0.0.1), for some potential $K$ that tends to infinity as $r \to 0$, has a well at a specific length enforcing crystallization and fixing the lattice spacing (and structure), and rapidly decays to 0 as $r \to +\infty$. The basic potential is provided by the Heitmann-Radin model [51] which consists in systems of hard spheres whose pair-interaction energy is $+\infty$ if two balls overlap, it is equal to $-1$ if the balls touch each other, and 0 otherwise. In two dimensions, and for fixed number $N$ of discs, minimizers exhibit crystalline order: the centers of the discs lie on a subset of an equilateral triangular lattice. Moreover, for large $N$ the discs fit a large hexagon. The first phenomenon is referred to as *crystallization*, the second as *macroscopic Wulff-shape*. Crystallization is due to local optimization of the potential around its well: almost each particle tends to maximize the number of nearest neighbor particles. In view of the hard disc constraint, such a number is 6. The macroscopic Wulff shape is the result of the minimization of the number of *boundary* particles that have the wrong number of nearest neighbors. In this respect, the macroscopic shape minimizes an anisotropic perimeter energy; under a volume constraint, this is nothing but the anisotropic isoperimetric problem, whose minimizer is the Wulff shape [44]. Recently, these phenomena have been analyzed in details in the solid formalism of Γ-convergence [11, 37, 45].

A less rigid and most popular model in elasticity is given by the polynomial Lennard-Jones type potential; the hard sphere constraint is replaced by a repulsive term which is infinite only at 0; the only negative value in the Heitmann-Radin potential is replaced by a narrow well, while the zero-long range interaction of the Heitmann-Radin potential is replaced by a rapidly decaying tail energy. In [70] it is proved that, if the well of the potential is very narrow and the tail is a small enough lower order term, then the crystallization property is preserved in average, namely the regular triangular lattice is energetically optimal as the number of particles diverges; furthermore, under Dirichlet or periodic type boundary conditions, the minimality of the regular triangular lattice is proved, while the Wulff shape problem is still open. Recently, it has been proved [15] that a slightly wider well in the potential favours the square lattice rather than the triangular one, while for three body potential also hexagonal lattices may arise as energy minimizers [72, 41]. In higher dimensions the picture is much less clear (see [43] for a relevant contribution in three dimensions).

We pass to describe our model; since the tail energy will be predominant, it is convenient to change length-scale, introducing a parameter $\varepsilon > 0$, whose inverse $\frac{1}{\varepsilon}$ represents the size of the body filled by the hard spheres. Then, in order to deal, in the thermodynamic limit, with a finite macroscopic body, we scale the spheres with $\varepsilon$. After this scaling the potential $K^p$ becomes integrable if and only if $p \in (0, d)$. We discuss first the integrable case: we write $p = d + \sigma$ for some $\sigma \in (-d, 0)$, and we introduce the corresponding potential which, up to a prefactor, becomes the function $f^\sigma_\varepsilon : \mathbb{R}^+ \to \mathbb{R} \cup \{-\infty, +\infty\}$ defined by

$$f^\sigma_\varepsilon(r) := \begin{cases} +\infty & \text{for } r \in [0, 2\varepsilon)\,, \\ -\dfrac{1}{r^{d+\sigma}} & \text{for } r \in [2\varepsilon, \infty)\,. \end{cases} \tag{0.0.2}$$

In this case the Γ-limit as $\varepsilon \to 0$ of the discrete energy (0.0.1), with $K^p = f^\sigma_\varepsilon$, is nothing but its continuous counterpart, defined on absolutely continuous measures, whose density is bounded from above by the density of the optimal packing problem in $\mathbb{R}^d$ (see Theorem 1.2.3). This $\Gamma$-convergence result can be completed with suitable confining volume forcing terms, ensuring compactness properties for minimizers. We prove that minimizers consist, in the limit as $\varepsilon \to 0$, in optimal packed configurations of balls filling a macroscopic set $E$, which is a ball whenever the volume term is radial.

The non-integrable case is much more involved. In this case both the tail and the core of the energy blow up as $\varepsilon \to 0$, the first being the leading term. In order to provide a first order expansion of the energy in terms of $\Gamma$-convergence, we need to regularize the potential, neglecting the core energy. More precisely, we introduce a mesoscopic length-scale $r_\varepsilon \gg \varepsilon$ with $r_\varepsilon \to 0$ as $\varepsilon \to 0$ (see (1.3.1)), and we regularize the potential cutting-off all short range interactions at scales smaller than $r_\varepsilon$. The corresponding regularized Riesz type $p$-power-law potentials, with $p = d + \sigma$ and $\sigma \in [0, 1)$, are defined by

$$f_\varepsilon^\sigma(r) := \begin{cases} +\infty & \text{for } r \in [0, 2\varepsilon)\,, \\ 0 & \text{for } r \in [2\varepsilon, r_\varepsilon)\,, \\ -\frac{1}{r^{d+\sigma}} & \text{for } r \in [r_\varepsilon, +\infty)\,. \end{cases}$$

Then, only the tail of the interaction energy remains, and the microscopic details of the potential are neglected in the limit as $\varepsilon \to 0$. This is consistent with the integrable case (0.0.2), where the core contribution vanishes as a consequence of the only integrability of the potential. Dividing the energy by the diverging tail contribution, we obtain the zero order term in the $\Gamma$-convergence expansion of the energy. This zero order $\Gamma$-limit still enforces optimal packing on minimizing sequences, but does not determine the macroscopic limit shape. Then, we look at the next term in the $\Gamma$-convergence expansion. This consists in removing from the total energy the infinite volume-term energy per particle, so that a finite quantity remains, which turns out to detect the macroscopic shape. In fact, the first order $\Gamma$-limit, provided in Theorem 1.3.3, is nothing but the $\sigma$-fractional perimeter, introduced in [21] for $\sigma \in (0, 1)$, and $\sigma = 0$-perimeter, introduced in [38]. Such an analysis has first been provided in a continuous setting in [38]; our results represent its discrete counterpart. Since fractional perimeters are minimized, under a volume constraint, by Euclidean balls, we deduce that, as $\varepsilon \to 0$, minimizers are given by optimal packed configurations of $\varepsilon$-spheres filling a macroscopic ball. In this respect, our analysis shows how the tail energy plays against the formation of macroscopic faceted crystals.

In the second chapter we have studied strongly attractive nonlocal potentials. Our analysis is geometrical, so that the energy functionals are defined on measurable sets rather than on empirical measures or densities, and can be understood as nonlocal perimeters, whose first variation are nonlocal curvatures, driving the corresponding geometric flows.

We focus on power law pair potentials acting on measurable sets $E \subset \mathbb{R}^d$, whose corresponding nonlocal energy is of the type

$$J^s(E) := \int_E \int_E -\frac{1}{|x-y|^{d+s}} \,\mathrm{d}y\,\mathrm{d}x. \tag{0.0.3}$$

For $-d < s < 0$ the interaction kernel is nothing but Riesz potential; in such a case, the functionals $J^s$ are nonlocal perimeters in the sense of [28]. Such a geometric interpretation is supported by the fact that, as a consequence of Riesz inequality, balls are minimizers of $J^s$ under volume constraints; moreover, the first variation of $J^s$, referred to as nonlocal curvature, is monotone with respect to set inclusion. The latter provides a parabolic maximum principle which yields global existence and uniqueness of level set solutions to the corresponding geometric evolutions [28, 26].

For positive $s$ the kernel in (0.0.3) is not integrable, and the corresponding energy is infinite. Nevertheless, for $0 < s < 1$, changing sign to the interaction and letting $E$ interact with its complementary set instead of itself, gives a finite quantity: the well-known fractional perimeter [21]

$$\tilde{J}^s(E) := \int_E \int_{E^c} \frac{1}{|x-y|^{d+s}} \, \mathrm{d}y \, \mathrm{d}x. \tag{0.0.4}$$

In fact, fractional perimeters can been rigorously obtained as limits of renormalized Riesz energies by removing the infinite core energy and letting the core radius tend to zero. This has been done in [38], showing that the energies in (0.0.3) and (0.0.4) belong to a one parameter family of nonlocal $s$-perimeters, with $-d < s < 1$ (see also [54]); in particular, for $s = 0$ the 0-fractional perimeter.

Remarkably, as $s \to 1^-$, $s$-fractional perimeters, suitably scaled, converge to the standard perimeter [17, 18, 34, 66, 9, 27], and the corresponding (reparametrized in time) geometric flows converge to the standard mean curvature flow [49, 26].

For $s \geq 1$ fractional perimeters are always infinite. Nevertheless, as discussed above, the critical case $s = 1$ corresponds, at least formally, to the Euclidean perimeter. Notice that for $s = 1$ the fractional perimeter can be seen, again formally, as the square of the (infinite) $\dot{H}^{\frac{1}{2}}$ Gagliardo seminorm of the characteristic function of $E$. This fractional energy is particularly relevant in Materials Science, for instance in the theory of dislocations. This is why much effort has been done to derive the Euclidean perimeter directly as the limit of suitable regularizations of the $\dot{H}^{\frac{1}{2}}$ seminorm, mainly through phase field approximations [2].

We have introduced a core-radius approach to renormalize by scaling the generalized $s$-fractional perimeters and curvatures in the critical and supercritical cases $s \geq 1$. We show that, as the core-radius tends to zero, the $\Gamma$-limit of the nonlocal perimeters is the Euclidean perimeter, the nonlocal curvatures converge to the standard mean curvature, and the corresponding geometric flows converge to the mean curvature flow. Moreover, we consider also the anisotropic variants of such perimeters, with applications to dislocation dynamics. Now we discuss our results in more detail.

In Section 2.1 we introduce the core-radius regularized critical and supercritical perimeters (see (2.1.4)). In Theorem 2.1.5 we show that, suitably scaled, they $\Gamma$-converge to the Euclidean perimeter. This analysis is very related with, and in some respects generalizes, many results scattered in the literature, mainly for $s > 1$ [62, 14, 65].

Sections 2.2 and 2.3 are devoted to the proof of Theorem 2.1.5. The proof of the lower bounds providing compactness and $\Gamma$-liminf inequality rely on techniques developed in [1] and for $s = 1$ in [46]. As a byproduct of our $\Gamma$-convergence analysis, we provide a characterization of finite perimeter sets (Theorem 2.3.4) in terms of uniformly bounded renormalized supercritical fractional perimeters. Analogous results for $0 < s < 1$ have been obtained in [17, 34, 66, 55].

In Section 2.4 we compute the first variations of the renormalized critical and supercritical perimeters, and we show that they converge, as the core-radius vanishes, to the standard mean curvature. The estimates are robust enough to apply the theory of stability for geomertic flows developed in [26]. As a consequence, in Theorem 2.4.7 we prove that the level-set solutions of supercritical fractional geometric flows, suitably reparametrized in time, locally uniformly converge to the level set solution of the classical mean curvature flow. This result extends somehow the analysis done in [35] for $s = 1$ and in [23] for $1 \leq s < 2$; in the latter, the authors consider a threshold dynamics based on the $s$-parabolic flow and, in turn, on the notion of $s$-Laplacian, which is well defined only for $0 < s < 2$.

In Section 2.5 we show that our renormalization procedures are robust enough to treat also the double limit as $s \to 1^+$ and the core-radius vanishes simultaneously (see Theorem 2.5.1).

In Section 2.6 we generalize our results to the case of possibly anisotropic kernels (Subsection 2.6.1) and we present a relevant application to dislocation dynamics (Subsection 2.6.2). It is well known that planar dislocation loops formally induce an infinite elastic energy that can be seen as an anisotropic version of the (squared) $\dot{H}^{\frac{1}{2}}$ seminorm of the characteristic function of the slip region enclosed by the dislocation curve. As mentioned above, renormalization procedures are needed to cut off the infinite core energy. In [35, 5, 22], the authors consider the geometric evolution of dislocation loops and face the corresponding renormalization issues: their approach consists in formally computing the first variation of the infinite energy induced by dislocations, deriving a nonlocal infinite curvature. Then, they regularize such a curvature through convolution kernels, obtaining a finite curvature driving the dynamics. As the convolution regularization kernel concentrates to a Dirac mass, they recover in the limit a local anisotropic mean curvature flow. The main issue in their analysis is that the convolution regularization produces a positive part in the nonlocal curvature (corresponding to a negative contribution in the normal velocity), concentrated on the scale of the core of the dislocation, giving back an evolution which does not satisfy the inclusion principle. Therefore, solutions exist only for short time. Moreover, adding strong enough forcing terms, or assuming that the positive part of the curvature is already concentrated on a point (instead of being diffused on the core region), they show that the curvature is in fact monotone with respect to inclusion of sets; as a consequence, they get a globally defined dynamics, converging, as the core-radius vanishes, to the correct anisotropic local mean curvature flow. Here we show that, if one first regularizes the nonlocal perimeters removing the core energy and then computes the corresponding first variation, then the positive part of the curvature is actually concentrated on a point (see Remark 2.6.3), so that the mathematical assumption in [35] is physically correct and fully justified through the solid core-radius formalism.

Finally, in Subsection 2.6.2 we show that the convergence analysis of the geometric flows done in [35] using the approach [69] can be directly deduced from the analysis developed in Section 2.4 and Subsection 2.6.1, providing then a self-contained proof relying on the general theory of nonlocal evolutions and their stability developed in [28, 26].

In the third chapter we have studied the fractional heat equation

$$u_t + C(s)(-\Delta^s)u, \qquad s \in (0,1)$$

posed in a bounded set $\Omega \subset \mathbb{R}^d$ with homogeneous Dirichlet conditions, and its asymptotic analysis as $s \to 0^+$ and $s \to 1^-$.

The fractional heat equation may be in fact seen as the $L^2$-gradient flow of the $s$-Gagliardo seminorm

$$[u]_s := \left[ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^2}{|x - y|^{d+2s}} \, \mathrm{d}y \, \mathrm{d}x \right]^{\frac{1}{2}},$$

with the support of $u$ contained in $\Omega$ when the equation is posed in a bounded domain. The asymptotic behavior of $s$-Gagliardo seminorms has been studied by several authors. The case

$s \to 1^-$ has been first considered in [17], where it is proven that the pointwise limit of the squared $s$-Gagliardo seminorms multiplied by $(1-s)$ is given by (a multiple of) the Dirichlet integral. Such a result is indeed proven for every exponent $1 < p < +\infty$ ($[\cdot]_s$ corresponds to $p = 2$). For $p = 1$ only a control of the limit in terms of the total variation is provided, allowing to characterize BV space; this has been extended in several directions, first by [66, 55] for more general kernels, and then by [34], showing that the pointwise limit is exactly (a multiple of) the total variation.

For what concerns the limit as $s \to 0^+$, in [61] the authors show that, as $s \to 0^+$, the squared $s$-fractional Gagliardo seminorms multiplied by $s$ pointwise converge to (a multiple of) the squared $L^2$-norm (see also [39] for a similar result in the context of $s$-fractional perimeters). The corresponding asymptotic analysis in terms of $\Gamma$-convergence has been developed in [38] in the context of fractional perimeters (that is, restricting to characteristic functions, as recalled above). A functional with more interesting properties is obtained in the limit $s \to 0^+$ by studying the next order term in the asymptotic expansion of the squared $s$-fractional Gagliardo seminorms: in [38] it is shown, still restricting to fractional perimeters, that the corresponding $\Gamma$-limit a nonlocal energy the 0-perimeter.

We have extended the results in [38] to the seminorms. In fact, we remove the constraint on the admissible functions to be characteristic functions; in order to obtain a $\Gamma$-convergence result with respect to the $L^2$ topology we consider functions whose support is in a bounded set $\Omega$. The next order result is Theorem 3.1.4, while the convergence of rescaled seminorms to the squared $L^2$ norm is proven in Theorem 3.1.2.

The analysis of the asymptotics of the $s$-Gagliardo seminorms is completed by Theorem 3.2.1: we study the $\Gamma$-convergence of the $s$-Gagliardo seminorms multiplied by $(1-s)$ to (a multiple of) the Dirichlet integral, thus giving the $\Gamma$-convergence version of the result in[34].

These convergences, which are of independent interest, are employed here to study the stability of the corresponding parabolic flows. Stability of gradient flows with respect to the $\Gamma$-convergence of the corresponding energies is a classical problem, which has been widely investigated in recent years in increasing generality (we refer, for instance, to [68, 67, 10]). In the present framework, we take advantage of the properties of the underlying energies. In fact, we are able to prove that in all the three regimes we consider (zero order for $s \to 0^+$, first order for $s \to 0^+$, zero order for $s \to 1^-$) the functionals are $\lambda$-convex uniformly with respect to $s$.

The gradient flows of $\lambda$-convex energies, namely energies which are convex up to a quadratic perturbation multiplied by $\lambda$, are uniquely determined. Moreover, they are well approximated in terms of discrete time solutions, that is they coincide with the (unique) minimizing movement solution: this is obtained, for every fixed $s \in (0, 1)$, by considering an implicit Euler scheme for the $s$-fractional Gagliardo seminorm and passing to the limit as the time step of the scheme vanishes. Basing on the general theory of minimizing movements (see for instance [7, 33, 64]), we provide, in Theorem 3.3.6, an abstract stability result for gradient flows in Hilbert spaces with respect to sequences of $\Gamma$-converging uniformly $\lambda$-convex functionals, suited for our purposes. In this respect, it is crucial that the quadratic perturbation giving $\lambda$-convexity is of $L^2$ type, since the $L^2$ topology is that for which the gradient flows of the $s$-Gagliardo seminorm is the $s$-fractional heat equation.

Actually, the abstract existence result for $\lambda$-convex functions is in general expressed as a differential inclusion of $u_t$ in the subdifferential of the underlying energy evaluated in $u$, which could be multivalued; in our problems, we get exactly the fractional heat equations since the $s$-Gagliardo seminorms are differentiable in the fractional Sobolev spaces $\mathcal{H}^s_0(\Omega)$, which are dense in $L^2(\Omega)$ (cf. Proposition 3.3.7). This enforces uniqueness for the fractional heat flows, assuming that the initial datum is just in the natural energy space for the problems. In the setting of nonnegative solutions for fractional heat equation in $\mathbb{R}^d$, uniqueness has been shown, in the context of a general Widder theory [73], in e.g. [12, 19, 71], with even not regular (but nonnegative) initial datum.

Besides uniqueness, we also obtain in the abstract theorem an explicit expression of the distance of minimizing movements from discrete time evolutions, in terms only of $\lambda$ and of the time interval. This is a key point to guarantee stability for families of minimizing movements associated to $\lambda$-convex functionals.

For the zero order convergences, the $\lambda$-convexity is direct for every $\lambda > 0$, the Gagliardo seminorms being convex; in the case of the first order convergence as $s \to 0^+$, this follows by differentiating twice the functionals on lines, and it strongly relies on the inclusion of $L^2$ into $L^1$. Moreover, without this inclusion at disposal, we are able to prove the $\Gamma$-convergence result for the first order in terms of $L^1 \cap L^2$-topology. Due to this technical issue, we chose to set our problems in a bounded (Lipschitz) domain $\Omega$.

In this framework, the energies in the three regimes fit in the abstract setting, so that we get stability of their parabolic flows in $s$, in the enhanced formulation where the subdifferential reduces to a singleton. This is contained in our main results, Theorems 3.4.4, 3.4.5, 3.4.6: the limit evolutions are an exponential growth for the 0-th order as $s \to 0^+$, a 0-fractional heat equation for the first order as $s \to 0^+$, and the classical local heat equation as $s \to 1^-$. The stability consists in a weak convergence $H^1$ in time, which is proven to be strong if the limit initial datum is well prepared, namely the approximating initial data are a recovery sequence for the limit datum with respect to the $\Gamma$-converging energies. Furthermore, in this case for every time $t$ the approximating evolutions $u_{s_n}(t)$ are recovery sequences for $u(t)$ with respect to the $\Gamma$-converging energies, namely there is convergence of the energies for every $t$.

# Notation

We work in the space $\mathbb{R}^d$ where $d \geq 2$ and we denote by $\{e_j\}_{j=1,\dots,d}$ the canonical basis of $\mathbb{R}^d$. We denote by $\mathbb{R}^{d\times p}$ the set of the matrices having $d$ rows and $p$ columns. The symbol $|\cdot|$ stands for the Lebesgue measure in $\mathbb{R}^d$, $\mathrm{M}(\mathbb{R}^d)$ is the family of measurable subsets of $\mathbb{R}^d$, whereas $\mathrm{M}_f(\mathbb{R}^d) \subset \mathrm{M}(\mathbb{R}^d)$ is the family of subsets if $\mathbb{R}^d$ having finite measure. We will always assume that every measurable set $E$ coincides with its Lebesgue representative, i.e., with the set of points at which $E$ has density equal to one. Moreover, for every $p > 0$, we denote by $\mathcal{H}^p$ the $p$-Hausdorff measure. $\mathcal{M}_b(\mathbb{R}^d)$ denotes the space of (non negative) finite Radon measures in $\mathbb{R}^d$. The Dirac delta measure centered in $x$ is denoted by $\delta_x$, while the Lebesgue measure by $\mathcal{L}^d$.

For every $x \in \mathbb{R}^d$ and for every $r > 0$, we denote by $B(x,r)$ the open ball of radius $r$ centered at $x$ and by $\overline{B}(x,r)$ its closure. Moreover, we set $\mathbb{S}^{d-1} := \partial B(0,1)$. Following the standard convention, we set $\omega_d := |B(x,1)|$ and we recall that $\mathcal{H}^{d-1}(\partial B(x,r)) = d\omega_d r^{d-1}$. Sometimes, we will consider also subsets of $\mathbb{R}^{d-1}$ . In such a case, we denote by $B'(\xi,\rho)$ the ball centered at $\xi \in \mathbb{R}^{d-1}$ and having radius equal to $\rho > 0$ ; we set $\omega_{d-1} := \mathcal{H}^{d-1}(B'(\xi,1))$ so that $\mathcal{H}^{d-1}(B'(\xi,\rho)) = \omega_{d-1}\rho^{d-1}$ and $\mathcal{H}^{d-2}(\partial B'(\xi,\rho)) = (d-1)\omega_{d-1}\rho^{d-2}$ . Furthermore, we set $Q := [-\frac{1}{2}, \frac{1}{2})^d$ and for every $\nu \in \mathbb{S}^{d-1}$ we set $Q^\nu := R^\nu Q$ , where $R^\nu$ is a (arbitrarily chosen) rotation such that $R^\nu e_d = \nu$ .

For every set $E \in \mathrm{M}(\mathbb{R}^d)$ we denote by $\mathrm{Per}(E)$ the De Giorgi perimeter of $E$ defined by

$$\mathrm{Per}(E) := \sup\left\{ \int_E \mathrm{Div}\Phi(x)\,\mathrm{d}x : \ \Phi \in C_0^1(\mathbb{R}^d;\ \mathbb{R}^d),\ \|\Phi\|_\infty \leq 1 \right\}.$$

For every $E \in \mathrm{M}(\mathbb{R}^d)$ , the set $\partial^* E$ identifies the reduced boundary of $E$ and $\nu_E : \partial^* E \to \mathbb{R}^d$ the outer normal vector field. For all $y \in \mathbb{R}^d$ and for every $\nu \in \mathbb{S}^{d-1}$ we set

$$H_\nu^-(x) := \{y \in \mathbb{R}^d :\ \nu \cdot (y-x) \leq 0\}\,, \tag{0.0.5}$$

$$H_\nu^+(x) := \{y \in \mathbb{R}^d :\ \nu \cdot (y-x) \geq 0\}\,, \tag{0.0.6}$$

$$H_\nu^0(x) := \{y \in \mathbb{R}^d :\ \nu \cdot (y-x) = 0\}\,. \tag{0.0.7}$$

For every subset $E \subset \mathbb{R}^d$ the symbol $E^c$ denotes its complementary set in $\mathbb{R}^d$ , i.e., $E^c := \mathbb{R}^d \setminus E$ .

Finally, we denote by $C(*,\cdots,*)$ a constant that depends on $*,\cdots,*$; such a constant may change from line to line.

# Chapter 1

# Attractive Riesz potentials acting on hard spheres

In this chapter we introduce a model for hard spheres interacting through attractive Riesz type potentials. We see the hard spheres as the finite subset of $\mathbb{R}^d$ such that $|x_i - x_j| \geq 2$ for all $i \neq j$. The potentials is of the type

$$\sum_{i\neq j} K^p(|x_i - x_j|),$$

where the asymptotic behaviour of $K^p(r) \approx -\frac{1}{r^p}$ for large $r$, and we study its thermodynamic limit. We show that the tail energy enforces optimal packing and round macroscopic shapes.

The reference for the following results is [54], joint work with Marcello Ponsiglione.

## 1.1 Hard spheres, optimal packing and empirical measures

Here we introduce the admissible configurations of the variational model proposed in this chapter, and revisit some concepts on optimal packed configurations we will need in our analysis.

### 1.1.1 Density of optimal packing

*Definition* 1.1.1. We denote by $\mathrm{Ad^d}$ be the class of sets $X \subset \mathbb{R}^d$ such that $|x_i - x_j| \geq 2$ for all $x_i$, $x_j \in X$ with $x_i \neq x_j$. The volume density of optimal ball packings in $\mathbb{R}^d$ is the constant $C^d$ defined by

$$C^d := \sup_{X\in \mathrm{Ad^d}} \limsup_{r\to+\infty} \frac{\#(X\cap rQ)\omega_d}{r^d}\,, \tag{1.1.1}$$

where $Q := [0, 1)^d$. Moreover, we say that $\mathrm{T}^d \subset \mathbb{R}^d$ is an optimal configuration for the (centers for the unit ball) optimal packing problem if $\mathrm{T}^d \in \mathrm{Ad^d}$ and

$$\lim_{r\to+\infty} \frac{\#(\mathrm{T}^d\cap rQ)\omega_d}{r^d} = C^d. \tag{1.1.2}$$

In [48] it is proved the existence of an optimal configuration, and that in defining $C^d$ and $\mathrm{T}^d$, $Q$ can be replaced by any open bounded set $A \subset \mathbb{R}^d$ with $A \neq \emptyset$.

Now we want to provide a rate of convergence in (1.1.1). To this purpose, for every $r > 0$ let $\mathrm{Ad}^{\mathrm{d}}(rQ)$ be the class of sets $X \subset rQ$ such that $|x_i - x_j| \geq 2$ for all $x_i,\, x_j \in X$ with $x_i \neq x_j$, and set

$$C^d_r := \sup_{X \in \mathrm{Ad}^{\mathrm{d}}(rQ)} \frac{\#(X)\omega_d}{r^d}\,. \tag{1.1.3}$$

It is easy to see that for all $r > 0$ there exists a maximizer, denoted by $T^d_r$.

**Lemma 1.1.2.** *There exists $C(d) > 0$ such that $C^d \leq C^d_r \leq C^d + \frac{C(d)}{r}$ for all $r > 0$.*

*Proof.* For every $r > 0$ we have $2rQ = \cup_{i=1}^{2^d} rQ_i$ where $Q_i = Q + v_i$, $v_i \in \{0,1\}^d$. Let $T^d_r$ be any maximizer of (1.1.3), and set

$$\hat T^d_r := \{x \in T^d_r :\, \mathrm{dist}\,(x, r\partial Q) \geq 1\}, \tag{1.1.4}$$
$$\tilde T^d_{2r} := \cup_{i=1}^{2^d} \hat T^d_r + v_i,\; v_i \in \{0,1\}^d.$$

It is easy to see that there exists a constant $c(d)$ such that $\#T^d_r - \#\hat T^d_r \leq c(d) r^{d-1}$. Moreover,

$$\max\{\#T^d_{2r} \cap rQ_i,\, i = 1, \cdots, 2^d\} \geq \frac{\#T^d_{2r}}{2^d}\,.$$

Then we have

$$r^d C^d_{2r} = \frac{\#T^d_{2r}}{2^d} \leq r^d C^d_r = \#T^d_r \leq \frac{\#\tilde T^d_{2r}}{2^d} + c(d) r^{d-1} \leq r^d C^d_{2r} + c(d) r^{d-1}.$$

Therefore, for every $r > 0$, $n \in \mathbb{N}$ we have

$$C^d_{2^n r} \leq C^d_{2^{n-1} r} \leq C^d_{2^n r} + \frac{c(d)}{2^{n-1} r},$$

which by iteration over $n$ yields

$$C^d_{2^n r} \leq C^d_r, \qquad C^d_r \leq C^d_{2^n r} + \sum_{k=1}^n \frac{c(d)}{2^{k-1} r}.$$

Sending $n \to +\infty$ we deduce the claim. □

### 1.1.2 The empirical measures

We introduce the family of empirical measures

$$\mathcal{EM} := \left\{ \sum_{i=1}^N \delta_{x_i} : x_i \neq x_j \text{ for } i \neq j, N \in \mathbb{N} \right\} \subset \mathcal{M}_b(\mathbb{R}^d).$$

We consider the space $\mathcal{M}_b(\mathbb{R}^d)$ endowed with the tight topology.

*Definition* 1.1.3 (Tight convergence). We say that a sequence $\{\mu_\varepsilon\}_{\varepsilon \in (0,1)} \subset \mathcal{M}_b(\mathbb{R}^d)$ tightly converges to $\mu \in \mathcal{M}_b(\mathbb{R}^d)$ if $\mu_\varepsilon \overset{*}{\rightharpoonup} \mu$ and $\mu_\varepsilon(\mathbb{R}^d) \to \mu(\mathbb{R}^d)$, as $\varepsilon \to 0^+$.

*Definition* 1.1.4. Let $\varepsilon > 0$, we define the set $\mathcal{EM}_\varepsilon \subset \mathcal{EM}$ as

$$\mathcal{EM}_\varepsilon := \left\{ \mu \in \mathcal{EM} :\, \mu = \sum_{i=1}^N \delta_{x_i} \text{ with } |x_i - x_j| \geq 2\varepsilon \text{ for all } i \neq j \right\}.$$

**Lemma 1.1.5.** *Let $\{\mu_\varepsilon\}_{\varepsilon\in(0,1)}\subset\mathcal{EM}$ with $\mu_\varepsilon\in\mathcal{EM}_\varepsilon$ for all $\varepsilon\in(0,1)$ be such that $\frac{\varepsilon^d\omega_d}{C^d}\mu_\varepsilon\overset{*}{\rightharpoonup}\mu$ for some $\mu\in\mathcal{M}_b(\mathbb{R}^d)$, as $\varepsilon\to0^+$ (where $C^d$ is defined in* (1.1.1)*). Then, there exists $\rho\in\mathrm{L}^1(\mathbb{R}^d,[0,1])$ such that $\mu=\rho\mathcal{L}^d$.*

*Proof.* It is sufficient to prove that $\mu(A)\le|A|$ for all open set $A$. By the lower semi-continuity of the total variation with respect to weak-star convergence, we have

$$\mu(A)\le\liminf_{\varepsilon\to0^+}\frac{\varepsilon^d\omega_d}{C^d}\mu_\varepsilon(A)=\liminf_{\varepsilon\to0^+}\frac{|A|}{C^d}\frac{\omega_d\#\{A\cap\operatorname{supp}(\mu_\varepsilon)\}}{|\frac{A}{\varepsilon}|}$$
$$\le|A|\lim_{\varepsilon\to0^+}\frac{\omega_d\#(\mathrm{T}^d\cap\frac{A}{\varepsilon})}{C_d|\frac{A}{\varepsilon}|}=|A|,$$

where the last inequality follows by (1.1.1) and (1.1.2) with $Q$ replaced by $A$. □

**Lemma 1.1.6.** *For every $\rho\in\mathrm{L}^1(\mathbb{R}^d,[0,1])$ there exists a sequence $\{\mu_\varepsilon\}_{\varepsilon\in(0,1)}\subset\mathcal{EM}$ with $\mu_\varepsilon\in\mathcal{EM}_\varepsilon$ for all $\varepsilon\in(0,1)$ such that $\frac{\varepsilon^d\omega_d}{C^d}\mu_\varepsilon\to\rho\mathcal{L}^d$ tightly in $\mathcal{M}_b(\mathbb{R}^d)$.*

*Proof.* By a standard density argument, it is enough to prove the claim for $\rho=a\chi_A$ for some $a\in(0,1)$ and some open set $A\subset\mathbb{R}^d$. Let $\mu_\varepsilon:=\sum_{i\in I_\varepsilon}\delta_{x_i}$ where $I_\varepsilon:=\varepsilon a^{\frac{-1}{d}}\mathrm{T}^d\cap A$. Then, it is easy to check that $\frac{\varepsilon^d\omega_d}{C^d}\mu_\varepsilon\to a\chi_A\mathcal{L}^d$ tightly in $\mathcal{M}_b(\mathbb{R}^d)$. □

For all $\mu:=\sum_{i=1}^N\delta_{x_i}$ in $\mathcal{EM}_\varepsilon$ we set

$$\hat\mu:=\frac{1}{C^d}\sum_{i=1}^N\chi_{B_\varepsilon(x_i)}. \tag{1.1.5}$$

**Lemma 1.1.7.** *Let $\{\mu_\varepsilon\}_{\varepsilon\in(0,1)}\subset\mathcal{EM}$ with $\mu_\varepsilon\in\mathcal{EM}_\varepsilon$ for all $\varepsilon\in(0,1)$, and let $\rho\in\mathrm{L}^1(\mathbb{R}^d,[0,1])$ be such that $\frac{\varepsilon^d\omega_d}{C^d}\mu_\varepsilon\to\rho\mathcal{L}^d$ tightly in $\mathcal{M}_b(\mathbb{R}^d)$. Then, $\hat\mu_\varepsilon\to\rho\mathcal{L}^d$ tightly.*

*Proof.* We observe that

$$\lim_{\varepsilon\to0^+}\hat\mu_\varepsilon(\mathbb{R}^d)=\lim_{\varepsilon\to0^+}\frac{\varepsilon^d\omega_d}{C^d}\mu_\varepsilon(\mathbb{R}^d)=\int_{\mathbb{R}^d}\rho(x)dx.$$

Therefore, up to a subsequence $\hat\mu_\varepsilon\overset{*}{\rightharpoonup}g$ for some $g\in\mathcal{M}_b(\mathbb{R}^d)$. We have to prove that $g=\rho\mathcal{L}^d$. To this purpose, notice that for all $\varphi\in\mathcal{C}^1_c(\mathbb{R}^d)$ we have

$$\begin{aligned}|\hat\mu_\varepsilon(\varphi)-\rho\mathcal{L}^d(\varphi)|&\le\Big|\hat\mu_\varepsilon(\varphi)-\frac{\varepsilon^d\omega_d}{C^d}\mu_\varepsilon(\varphi)\Big|+\Big|\frac{\varepsilon^d\omega_d}{C^d}\mu_\varepsilon(\varphi)-\rho\mathcal{L}^d(\varphi)\Big|\\&=\Big|\sum_{x\in\operatorname{supp}\mu_\varepsilon}\frac{1}{C^d}\int_{B_\varepsilon(x)}\varphi(y)-\varphi(x)\,dy\Big|+\Big|\frac{\varepsilon^d\omega_d}{C^d}\mu_\varepsilon(\varphi)-\rho\mathcal{L}^d(\varphi)\Big|\\&\le\sum_{x\in\operatorname{supp}\mu_\varepsilon}\frac{1}{C^d}\int_{B_\varepsilon(x)}|\varphi(y)-\varphi(x)|\,dy+\Big|\frac{\varepsilon^d\omega_d}{C^d}\mu_\varepsilon(\varphi)-\rho\mathcal{L}^d(\varphi)\Big|\\&\le2\varepsilon\frac{\varepsilon^d\omega_d}{C^d}\mu_\varepsilon(\mathbb{R}^d)\|\nabla\varphi\|_{L^\infty}+\Big|\frac{\varepsilon^d\omega_d}{C^d}\mu_\varepsilon(\varphi)-\rho\mathcal{L}^d(\varphi)\Big|.\end{aligned}$$

Since $\frac{\varepsilon^d\omega_d}{C^d}\mu_\varepsilon\overset{*}{\rightharpoonup}\rho\mathcal{L}^d$, the claim follows. □

## 1.2 Riesz interactions for $\sigma \in (-d, 0)$

Here we introduce and analyze the Riesz interaction functionals in the integrable case $\sigma \in (-d, 0)$.

### 1.2.1 The energy functionals

For every $\varepsilon > 0$ and $\sigma \in (-d, 0)$, let $f_\varepsilon^\sigma : [0, +\infty) \to \mathbb{R} \cup \{-\infty, +\infty\}$ be defined by

$$f_\varepsilon^\sigma(r) := \begin{cases} +\infty & \text{for } r \in [0, 2\varepsilon)\,, \\ -\dfrac{1}{r^{d+\sigma}} & \text{for } r \in [2\varepsilon, \infty)\,. \end{cases}$$

Let $C^d$ be the volume density of the optimal ball packing in $\mathbb{R}^d$ defined in (1.1.1).

Let $X = \{x_1, \cdots, x_N\}$ be a finite subset of $\mathbb{R}^d$. The corresponding energy $F_\varepsilon^\sigma(X)$ is defined as

$$F_\varepsilon^\sigma(X) := \sum_{i \neq j} f_\varepsilon^\sigma(|x_i - x_j|)\Big(\frac{\omega_d \varepsilon^d}{C_d}\Big)^2.$$

Clearly, there is a one-to-one correspondence, that we denote by $\mathcal{A}$, between the family of empirical measures and the family of finite subsets of $\mathbb{R}^d$ . We introduce the energy $\mathcal{F}_\varepsilon^\sigma : \mathcal{M}_b(\mathbb{R}^d) \to \mathbb{R} \cup \{-\infty, +\infty\}$ as a function of the empirical measure as follows:

$$\mathcal{F}_\varepsilon^\sigma(\mu) := \begin{cases} F_\varepsilon^\sigma(\mathcal{A}(\mu)) & \text{if } \mu \in \mathcal{EM}_\varepsilon, \\ +\infty & \text{elsewhere.} \end{cases} \tag{1.2.1}$$

The functional $\mathcal{F}_\varepsilon^\sigma$ may also be rewritten as

$$\mathcal{F}_\varepsilon^\sigma(\mu) = \begin{cases} \displaystyle\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f_\varepsilon^\sigma(|x-y|)\Big(\frac{\varepsilon^d \omega_d}{C_d}\Big)^2 d\mu \otimes \mu & \text{if } \mu \in \mathcal{EM}_\varepsilon, \\ +\infty & \text{elsewhere.} \end{cases}$$

We observe that the range of the functionals $\mathcal{F}_\varepsilon^\sigma$ is $(-\infty, 0] \cup \{+\infty\}$. Therefore, we do not expect compactness properties for sequences with bounded energy. In fact, it is easy to construct, adding more and more masses, a sequence $\{\mu_\varepsilon\}_{\varepsilon \in (0,1)} \subset \mathcal{EM}_\varepsilon$ with $\varepsilon^d \mu_\varepsilon(\mathbb{R}^d) \to +\infty$ and $\mathcal{F}_\varepsilon^\sigma(\mu_\varepsilon) \to -\infty$ as $\varepsilon \to 0$. Moreover, tight convergence can also fail by loss of mass at infinity, also for sequences with $\varepsilon^d \mu_\varepsilon(\mathbb{R}^d) \le C$. Indeed, let $T^d$ be an optimal configuration for the optimal packing, as in Definition 1.1.1. Let $\{z_\varepsilon\}_{\varepsilon \in (0,1)} \subset \mathbb{R}^d$ with $|z_\varepsilon| \to +\infty$ as $\varepsilon \to 0$. Setting $\mu_\varepsilon = \sum_{x \in \varepsilon \mathrm{T}^d \cap B(z_\varepsilon, 1)} \delta_x$, we have that $\varepsilon^d \mu_\varepsilon(\mathbb{R}^d) \le C$ for some $C$ independent of $\varepsilon$, but in general $\varepsilon^d \mu_\varepsilon$ does not admit converging subsequences in the tight topology.

Now we perturb the energy functionals by adding suitable confining forcing terms that yield the desired compactness properties.

Let $g \in C^0(\mathbb{R}^d)$. Recalling that $C^d$ is the volume density defined in (1.1.1), for all $\varepsilon \in (0, 1)$ we introduce the functionals $\mathcal{T}_\varepsilon^\sigma : \mathcal{M}_b(\mathbb{R}^d) \to \mathbb{R} \cup \{-\infty, +\infty\}$ defined as

$$\mathcal{T}_\varepsilon^\sigma(\mu) := \mathcal{F}_\varepsilon^\sigma(\mu) + \mathcal{G}_\varepsilon^\sigma(\mu), \tag{1.2.2}$$

where

$$\mathcal{G}_\varepsilon^\sigma(\mu) := \int_{\mathbb{R}^d} g(x) \frac{\varepsilon^d \omega_d}{C^d} d\mu\,.$$

### 1.2.2 Compactness

In this section we study compactness properties for the functionals $\mathcal{T}_\varepsilon^\sigma$ introduced in (1.2.2). We assume that

$$g(x) \ge C_1 + C_2|x|^{-\sigma}, \qquad \text{for some } C_1 \in \mathbb{R},\, C_2 > 0. \tag{1.2.3}$$

**Theorem 1.2.1** (Compactness for $\mathcal{T}_\epsilon^\sigma$)**.** *There exists a constant $C^*(\sigma, d) > 0$ such that, if $g$ satisfies* (1.2.3) *with $C_2 > C^*(\sigma, d)$, then the following compactness property hold: let $M > 0$ and let $\{\mu_\varepsilon\}_{\varepsilon\in(0,1)} \subset \mathcal{M}_b(\mathbb{R}^d)$ be such that*

$$\mathcal{T}_\varepsilon^\sigma(\mu_\varepsilon) \le M, \qquad \text{for all } \varepsilon > 0\,.$$

*Then, up to a subsequence, $\frac{\varepsilon^d\omega_d}{C^d}\mu_\varepsilon \to \rho\mathcal{L}^d$ tightly in $\mathcal{M}_b(\mathbb{R}^d)$, for some $\rho \in \mathrm{L}^1(\mathbb{R}^d, [0,1])$.*

*Proof.* In view of (1.2.3), it is enough to prove the theorem for $g(x) = C_1 + C_2|x|^{-\sigma}$ with $C_2 > C^*(\sigma, d)$ for some $C^*(\sigma, d) > 0$. We divide the proof in several steps.

*Step 1.* For all $\mu \in \mathcal{EM}_\varepsilon$ set $K_\varepsilon(\mu) := \varepsilon^d\omega_d\mu(\mathbb{R}^d)$ and let $R_\varepsilon(\mu) > 0$ be such that $R_\varepsilon(\mu)^d\omega_d = K_\varepsilon(\mu)$. In this step we prove that there exists $\tilde{C}(\sigma, d) > 0$ such that for all $\mu := \sum_{i=1}^N \delta_{x_i} \in \mathcal{EM}_\varepsilon(\mathbb{R}^d)$ we have

$$\sum_{i=1}^N \frac{\varepsilon^d\omega_d}{C^d}|x_i|^{-\sigma} \ge \tilde{C}(\sigma, d)\big(K_\varepsilon(\mu)\big)^{\frac{d-\sigma}{d}}\,.$$

Here and later on we will assume without loss of generality (and whenever it will be convenient) that $|x_i| \ge \varepsilon$ for all $x_i \in \mathrm{supp}(\mu)$. Indeed, it is easy to see that $\mathcal{T}_\varepsilon^\sigma(\mu_\varepsilon)$ is uniformly bounded if and only if $\mathcal{T}_\varepsilon^\sigma(\mu_\varepsilon\lfloor_{\mathbb{R}^d\setminus B(0,\varepsilon)})$ is uniformly bounded, and that $\frac{\varepsilon^d\omega_d}{C^d}\mu_\varepsilon \to \rho\mathcal{L}^d$ tightly if and only if $\frac{\varepsilon^d\omega_d}{C^d}\mu_\varepsilon\lfloor_{\mathbb{R}^d\setminus B(0,\varepsilon)} \to \rho\mathcal{L}^d$ tightly. By triangular inequality we have $|y| \le |x_i| + \varepsilon \le 2|x_i|$ for all $y \in B(x_i, \varepsilon)$. Then,

$$\omega_d\varepsilon^d|x_i|^{-\sigma} = \int_{B(x_i,\varepsilon)} |x_i|^{-\sigma}\,dy \ge \frac{1}{2^{-\sigma}}\int_{B(x_i,\varepsilon)} |y|^{-\sigma}\,dy.$$

Let $A_\varepsilon$ be the union of all the balls $B(x_i, \varepsilon)$. We have

$$\begin{aligned}
\sum_{i=1}^N \frac{\varepsilon^d\omega_d}{C^d}|x_i|^{-\sigma} &\ge \sum_{i=1}^N \frac{1}{2^{-\sigma}C^d}\int_{B(x_i,\varepsilon)} |y|^{-\sigma}\,dy \\
&= \frac{1}{2^{-\sigma}C^d}\int_{A_\varepsilon} |y|^{-\sigma}\,dy = \frac{1}{2^{-\sigma}C^d}\int_{A_\varepsilon\cap B(0,R_\varepsilon(\mu))} |y|^{-\sigma}\,dy + \frac{1}{2^{-\sigma}C^d}\int_{A_\varepsilon\setminus B(0,R_\varepsilon(\mu))} |y|^{-\sigma}\,dy \\
&\ge \frac{1}{2^{-\sigma}C^d}\int_{B(0,R_\varepsilon(\mu))} |y|^{-\sigma}\,dy = \tilde{C}(\sigma, d)\big(K_\varepsilon(\mu)\big)^{1-\frac{\sigma}{d}}\,,
\end{aligned}$$

where in the last inequality we have used that $|A_\varepsilon| = K_\varepsilon(\mu) = |B(0, R_\varepsilon(\mu))|$, and that $|y_1|^{-\sigma} \ge |y_2|^{-\sigma}$ for all $y_1 \in A_\varepsilon \setminus B(0, R_\varepsilon(\mu))$, $y_2 \in B(0, R_\varepsilon(\mu))$.

*Step 2.* Here we prove that there exists $\hat{C}(\sigma, d) > 0$ such that, for all $\mu \in \mathcal{EM}_\varepsilon$,

$$\frac{1}{(C^d)^2}\sum_{i\neq j}\frac{(\varepsilon^d\omega_d)^2}{|x_i - x_j|^{d+\sigma}} \le \hat{C}(\sigma, d)\big(K_\varepsilon(\mu)\big)^{1-\frac{\sigma}{d}}.$$

First, we observe that by triangular inequality $|x_i - x_j| \geq \frac{1}{3}|x-y|$ for all $(x,y) \in B(x_i,\varepsilon) \times B(x_j,\varepsilon)$. Then, there exists $\hat{C}(\sigma,d) > 0$ such that

$$\begin{aligned}\frac{1}{(C^d)^2}\sum_{i\neq j}\frac{(\varepsilon^d\omega_d)^2}{|x_i-x_j|^{d+\sigma}} &\leq \hat{C}(\sigma,d)\sum_{i\neq j}\int_{B(x_i,\varepsilon)}\int_{B(x_j,\varepsilon)}\frac{1}{|x-y|^{d+\sigma}}dxdy\\ &\leq \hat{C}(\sigma,d)\int_{B(0,R_\varepsilon(\mu_\varepsilon))}\int_{B(0,R_\varepsilon(\mu_\varepsilon))}\frac{1}{|x-y|^{d+\sigma}}dxdy\\ &\leq \hat{C}(\sigma,d)\int_{B(0,R_\varepsilon(\mu_\varepsilon))}dx\int_{B(0,2R_\varepsilon(\mu_\varepsilon))}\frac{1}{|z|^{d+\sigma}}dz = \hat{C}(\sigma,d)(K_\varepsilon(\mu))^{1-\frac{\sigma}{d}}, \end{aligned} \quad (1.2.4)$$

where the second inequality is nothing but Riesz inequality, see [57].

*Step 3.* Here we prove that there exists $C^*(\sigma,d) > 0$ such that, if $C_2 > C^*(\sigma,d)$, then the following implication holds:

$$\text{if } \limsup_{\varepsilon} \mathcal{T}^\sigma_\varepsilon(\mu_\varepsilon) < +\infty, \text{ then } \limsup_{\varepsilon} \frac{\varepsilon^d\omega_d}{C^d}\mu_\varepsilon(\mathbb{R}^d) < +\infty.$$

By *Step 1* and *Step 2* we have obtained that

$$\mathcal{T}^\sigma_\varepsilon(\mu_\varepsilon) \geq \frac{C_1}{C^d}K_\varepsilon(\mu_\varepsilon) + (-\hat{C}(\sigma,d) + C_2\tilde{C}(\sigma,d))(K_\varepsilon(\mu_\varepsilon))^{1-\frac{\sigma}{d}}.$$

It is then sufficient to choose $C_2$ large enough, so that $(-\hat{C}(\sigma,d) + C_2\tilde{C}(\sigma,d)) > 0$.

*Step 4.* We now prove the tight converge, up to a subsequence, of sequences $\{\mu_\varepsilon\}_{\varepsilon\in(0,1)}$ with bounded energy. In view of Lemma 1.1.5, this step concludes the proof of the theorem. By *Step 3* we have that $\frac{\varepsilon^d\omega_d}{C^d}\mu_\varepsilon(\mathbb{R}^d) \leq \tilde{M}$ for all $\varepsilon \in (0,1)$ and some $\tilde{M} > 0$. Arguing by contradiction, assume that there exists $\delta > 0$, $\varepsilon_n \to 0^+$ and $R_n \to +\infty$ as $n \to +\infty$, such that

$$\frac{\varepsilon_n^d\omega_d}{C^d}\mu_{\varepsilon_n}(\mathbb{R}^d \setminus B(0,R_n)) \geq \delta \quad \forall n. \quad (1.2.5)$$

Now let us split $\mu_{\varepsilon_n}$ into two components: $\mu^1_{\varepsilon_n} := \mu_{\varepsilon_n}\llcorner_{B(0,R_n)}$ and $\mu^2_{\varepsilon_n} := \mu_{\varepsilon_n}\llcorner_{\mathbb{R}^d\setminus B(0,R_n)}$; then

$$\begin{aligned}\mathcal{T}^\sigma_{\varepsilon_n}(\mu_{\varepsilon_n}) =& \mathcal{T}^\sigma_{\varepsilon_n}(\mu^1_{\varepsilon_n}) + \mathcal{T}^\sigma_{\varepsilon_n}(\mu^2_{\varepsilon_n})\\ &- 2\int_{B(0,R_n)}\int_{\mathbb{R}^d\setminus B(0,R_n)}\frac{1}{|x-y|^{\sigma+d}}\left(\frac{\varepsilon_n^d\omega_d}{C^d}\right)^2 d\mu_{\varepsilon_n}\otimes\mu_{\varepsilon_n}.\end{aligned} \quad (1.2.6)$$

From *Step 2* we have that there exists $C > 0$ independent of $n$ such that

$$\mathcal{T}^\sigma_{\varepsilon_n}(\mu^1_{\varepsilon_n}) \geq -\hat{C}(\sigma,d)(K_{\varepsilon_n}(\mu^1_{\varepsilon_n}))^{1-\frac{\sigma}{d}} \geq -C. \quad (1.2.7)$$

Again by *Step 2*, applied now to $\mu^2_{\varepsilon_n}$, we have that there exists $C > 0$ independent of $n$ such that

$$\int_{\mathbb{R}^d\setminus B(0,R_n)}\int_{\mathbb{R}^d\setminus B(0,R_n)}\frac{1}{|x-y|^{d+\sigma}}\left(\frac{\varepsilon_n^d\omega_d}{C^d}\right)^2 d\mu_{\varepsilon_n}\otimes\mu_{\varepsilon_n} \leq C.$$

Therefore, by (1.2.5) we have

$$\mathcal{T}_{\varepsilon_n}(\mu^2_{\varepsilon_n}) \geq -C - |C_1|\tilde{M} + C_2\int_{\mathbb{R}^d\setminus B(0,R_n)} R_n^{-\sigma}\frac{\varepsilon_n^d\omega_d}{C^d}d\mu_{\varepsilon_n} \geq -C + C_2\delta R_n^{-\sigma}. \quad (1.2.8)$$

Finally, by Riesz inequality (or equivalently, arguing as in (1.2.4)) we have that there exists $C > 0$ independent of $n$ such that

$$\int_{B(0,R_n)} \int_{\mathbb{R}^d \setminus B(0,R_n)} \frac{-1}{|x-y|^{d+\sigma}} \left(\frac{\varepsilon_n^d \omega_d}{C^d}\right)^2 d\mu_{\varepsilon_n} \otimes \mu_{\varepsilon_n}(x,y) \geq -C \tag{1.2.9}$$

Now plugging (1.2.7),(1.2.8) and (1.2.9) into (1.2.6), we deduce that

$$M \geq \mathcal{T}^\sigma_{\varepsilon_n}(\mu_{\varepsilon_n}) \geq -C + C_2 \delta R_n^{-\sigma},$$

for some $C$ independent of $n$, which clearly provides a contradiction for $n$ large enough. □

### 1.2.3 Γ-convergence

In this section we study the Γ-convergence of the energy functionals defined in (1.2.1) and (1.2.2).

**Proposition 1.2.2.** *Let $\{\mu_\varepsilon\}_{\varepsilon \in (0,1)} \subset \mathcal{M}_b(\mathbb{R}^d)$ with $\mu_\varepsilon \in \mathcal{EM}_\varepsilon$ for all $\varepsilon \in (0,1)$ and let $\rho \in L^1(\mathbb{R}^d; [0,1])$ be such that $\frac{\varepsilon^d \omega_d}{C^d}\mu_\varepsilon \to \rho\mathcal{L}^d$ tightly. Let moreover $h(x,y) := \frac{1}{|x-y|^{d+\sigma}}$ for all $x$, $y \in \mathbb{R}^d$ with $x \neq y$. Then,*

$$\left(\frac{\varepsilon^d \omega_d}{C^d}\right)^2 \mu_\varepsilon \otimes \mu_\varepsilon(h) \to \rho\mathcal{L}^d \otimes \rho\mathcal{L}^d(h), \qquad \text{as } \varepsilon \to 0^+.$$

*Proof.* The proof is divided in several steps:

*Step 1.* Here we prove that

$$\hat{\mu}_\varepsilon \otimes \hat{\mu}_\varepsilon(h) \to \rho\mathcal{L}^d \otimes \rho\mathcal{L}^d(h), \quad \text{as } \varepsilon \to 0^+,$$

where $\hat{\mu}_\varepsilon$ are defined as in (1.1.5) (with $\mu$ replaced by $\mu_\varepsilon$).

For all $R > 0$ we set

$$D(R) := \bigcup_{x \in \mathbb{R}^d} (\{x\} \times B(x,R)). \tag{1.2.10}$$

We have

$$\left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{|x-y|^{d+\sigma}} d\hat{\mu}_\varepsilon \otimes \hat{\mu}_\varepsilon - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{|x-y|^{d+\sigma}} \rho(x)\rho(y)dxdy \right|$$

$$\leq \int_{D(R)} \frac{1}{|x-y|^{d+\sigma}} d\hat{\mu}_\varepsilon \otimes \hat{\mu}_\varepsilon \tag{1.2.11}$$

$$+ \int_{D(R)} \frac{1}{|x-y|^{d+\sigma}} \rho(x)\rho(y)dxdy \tag{1.2.12}$$

$$+ \left| \int_{\mathbb{R}^{2d} \setminus D(R)} \frac{1}{|x-y|^{d+\sigma}} d\hat{\mu}_\varepsilon \otimes \hat{\mu}_\varepsilon - \int_{\mathbb{R}^{2d} \setminus D(R)} \frac{1}{|x-y|^{d+\sigma}} \rho(x)\rho(y)dxdy \right|. \tag{1.2.13}$$

Moreover, we have

$$\begin{aligned}\int_{D(R)} \frac{1}{|x-y|^{d+\sigma}} d\hat{\mu}_\varepsilon \otimes \hat{\mu}_\varepsilon &= \int_{\mathbb{R}^d} d\hat{\mu}_\varepsilon \int_{B(x,R)} \frac{1}{|x-y|^{d+\sigma}} d\hat{\mu}_\varepsilon \\ &= \int_{\mathbb{R}^d} d\hat{\mu}_\varepsilon(x) \int_{B(x,R)} \sum_{i=1}^{N_\varepsilon} \frac{1}{C^d} \chi_{B(x_i,\varepsilon)}(y) \frac{1}{|x-y|^{d+\sigma}} dy \\ &\leq \int_{\mathbb{R}^d} d\hat{\mu}_\varepsilon \frac{1}{C^d} \int_{B(x,R)} \frac{1}{|x-y|^{d+\sigma}}\, dy = \hat{\mu}_\varepsilon(\mathbb{R}^d)\omega(R),\end{aligned}$$

where $\omega(R) \to 0$ as $R \to 0$. This proves that the quantity in (1.2.11) tends to 0 as $R \to 0$, uniformly in $\varepsilon$; a fully analogous argument shows that the same holds true also for the quantity in (1.2.12). Finally, the quantity in (1.2.13) tends to 0 as $\varepsilon \to 0$ (for fixed $R$) since $\frac{1}{|x-y|^{d+\sigma}}$ is continuous and bounded in $\mathbb{R}^{2d} \setminus D(R)$, and by Lemma 1.1.7 we have that $\hat{\mu}_e \otimes \hat{\mu}_\varepsilon \to \rho\mathcal{L}^d \otimes \rho\mathcal{L}^d$ tightly in $\mathbb{R}^{2d}$, and hence also in $\mathbb{R}^{2d} \setminus D(R)$.

*Step 2.* Here we prove that

$$\left(\frac{\varepsilon^d\omega_d}{C^d}\right)^2 \mu_\varepsilon \otimes \mu_\varepsilon(h) - \hat{\mu}_\varepsilon \otimes \hat{\mu}_\varepsilon(h) \to 0 \quad \varepsilon \to 0^+.$$

Let $x_i, x_j \in \operatorname{supp}(\mu_\varepsilon)$, with $i \neq j$; for all $x \in B(x_i,\varepsilon)$, $y \in B(x_j,\varepsilon)$, by triangular inequality we have $|x-y| \leq 2|x_i - x_j|$, and hence

$$\left(\frac{\varepsilon^d\omega_d}{C^d}\right)^2 \frac{1}{|x_i-x_j|^{d+\sigma}} \leq 2^{d+\sigma} \int_{B(x_i,\varepsilon)} \int_{B(x_j,\varepsilon)} \frac{1}{(C^d)^2} \frac{1}{|x-y|^{d+\sigma}} dxdy. \tag{1.2.14}$$

Let $D(R)$ be the set defined in (1.2.10). We obtain that

$$\begin{aligned}
&\left| \int_{\mathbb{R}^{2d}} \frac{1}{|x-y|^{d+\sigma}} \left(\frac{\varepsilon^d\omega_d}{C^d}\right)^2 d\mu_\varepsilon \otimes \mu_\varepsilon - \int_{\mathbb{R}^{2d}} \frac{1}{|x-y|^{d+\sigma}} d\hat{\mu}_\varepsilon \otimes \hat{\mu}_\varepsilon \right| \\
\leq& \left| \int_{D(R)} \frac{1}{|x-y|^{d+\sigma}} \left(\frac{\varepsilon^d\omega_d}{C^d}\right)^2 d\mu_\varepsilon \otimes \mu_\varepsilon \right| \qquad (1.2.15)\\
&+ \left| \int_{D(R)} \frac{1}{|x-y|^{d+\sigma}} d\hat{\mu}_\varepsilon \otimes \hat{\mu}_\varepsilon \right| \qquad (1.2.16)\\
&+ \left| \int_{\mathbb{R}^{2d}\setminus D(R)} \frac{1}{|x-y|^{d+\sigma}} \left(\frac{\varepsilon^d\omega_d}{C^d}\right)^2 d\mu_\varepsilon \otimes \mu_\varepsilon \right. \qquad (1.2.17)\\
&\qquad \left. - \int_{\mathbb{R}^{2d}\setminus D(R)} \frac{1}{|x-y|^{d+\sigma}} d\hat{\mu}_\varepsilon \otimes \hat{\mu}_\varepsilon \right|.
\end{aligned}$$

By (1.2.14) we deduce that the quantiy in (1.2.15) is, up to a prefactor, less than or equal to the quantity in (1.2.16), which, as proved in *Step 1*, tends to zero as $R \to 0$, uniformly with respect to $\varepsilon$. Finally, since $\frac{1}{|x-y|^{d+\sigma}}$ is continuous and bounded in $\mathbb{R}^{2d} \setminus D(R)$, by Lemma 1.1.7 we easily deduce that, for any fixed $R > 0$, the quantity in (1.2.17) tends to zero as $\varepsilon \to 0$. This concludes the proof of *Step 2.*

The proof of the claim is clearly a consequence of *Step 1* and *Step 2*. □

We now introduce the candidate $\Gamma$-limit $\mathcal{F}^\sigma : \mathcal{M}_b(\mathbb{R}^d) \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\mathcal{F}^\sigma(\mu) := \begin{cases} \displaystyle\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} -\frac{1}{|x-y|^{d+\sigma}} d\mu \otimes \mu & \text{if } \mu \leq \mathcal{L}^d, \\ +\infty & \text{elsewhere.} \end{cases}$$

**Theorem 1.2.3.** *Let $\sigma \in (-d, 0)$. The following $\Gamma$-convergence result holds true.*

1. *($\Gamma$-liminf inequality) For every $\rho \in \mathrm{L}^1(\mathbb{R}^d, [0,1])$ and for every sequence $\{\mu_\varepsilon\}_{\varepsilon\in(0,1)} \subset \mathcal{M}_b(\mathbb{R}^d)$ with $\frac{\varepsilon^d\omega_d}{C^d}\mu_\varepsilon \to \rho\mathcal{L}^d$ tightly in $\mathcal{M}_b(\mathbb{R}^d)$ it holds*

$$\mathcal{F}^\sigma(\rho\mathcal{L}^d) \leq \liminf_{\varepsilon\to 0^+} \mathcal{F}^\sigma_\varepsilon(\mu_\varepsilon).$$

2. *($\Gamma$-limsup inequality) For every $\rho \in \mathrm{L}^1(\mathbb{R}^d,[0,1])$, there exists a sequence $\{\mu_\varepsilon\}_{\varepsilon\in(0,1)} \subset \mathcal{M}_b(\mathbb{R}^d)$ such that $\frac{\varepsilon^d\omega_d}{C^d}\mu_\varepsilon \to \rho\mathcal{L}^d$ tightly in $\mathcal{M}_b(\mathbb{R}^d)$ and*

$$\mathcal{F}^\sigma(\rho\mathcal{L}^d) \geq \limsup_{\varepsilon\to0^+} \mathcal{F}^\sigma_\varepsilon(\mu_\varepsilon).$$

*Proof.* The $\Gamma$-liminf inequality is a direct consequence of Proposition 1.2.2 while the $\Gamma$-limsup inequaility is a direct consequence of Lemma 1.1.6 and again of Proposition 1.2.2. $\square$

Now we introduce the $\Gamma$-limit $\mathcal{T}^\sigma : \mathcal{M}_b(\mathbb{R}^d) \to \mathbb{R}\cup\{-\infty,+\infty\}$ of the functionals $\mathcal{T}^\sigma_\varepsilon$ introduced in (1.2.2), defined by

$$\mathcal{T}^\sigma(\mu) := \begin{cases} \mathcal{F}^\sigma(\mu) + \int_{\mathbb{R}^d} g(x)d\mu(x) & \text{if } \mu \leq \mathcal{L}^d, \\ +\infty & \text{elsewhere.} \end{cases}$$

**Theorem 1.2.4.** *Let $\sigma \in (-d,0)$, let $g \in \mathcal{C}^0(\mathbb{R}^d)$ satisfying $g(x) \geq 0$ for $|x|$ large enough, and let $\mathcal{T}^\sigma_\varepsilon$ be defined in (1.2.2). The following $\Gamma$-convergence result holds true.*

1. *($\Gamma$-liminf inequality) For every $\rho \in \mathrm{L}^1(\mathbb{R}^d,[0,1])$ and for every sequence $\{\mu_\varepsilon\}_{\varepsilon\in(0,1)} \subset \mathcal{M}_b(\mathbb{R}^d)$ with $\frac{\varepsilon^d\omega_d}{C^d}\mu_\varepsilon \to \rho\mathcal{L}^d$ tightly in $\mathcal{M}_b(\mathbb{R}^d)$ it holds*

$$\mathcal{T}^\sigma(\rho\mathcal{L}^d) \leq \liminf_{\varepsilon\to0^+} \mathcal{T}^\sigma_\varepsilon(\mu_\varepsilon).$$

2. *($\Gamma$-limsup inequality) For every $\rho \in \mathrm{L}^1(\mathbb{R}^d,[0,1])$ there exists a sequence $\{\mu_\varepsilon\}_{\varepsilon\in(0,1)}$ such that $\frac{\varepsilon^d\omega_d}{C^d}\mu_\varepsilon \to \rho\mathcal{L}^d$ tightly in $\mathcal{M}_b(\mathbb{R}^d)$ and*

$$\mathcal{T}^\sigma(\rho\mathcal{L}^d) \geq \limsup_{\varepsilon\to0^+} \mathcal{T}^\sigma_\varepsilon(\mu_\varepsilon).$$

*Proof.* We start by proving (1). It is easy to prove (see [8, Proposition 1.62]) that the term $\int_{\mathbb{R}^d} g(x)d\mu$ is lower semicontinuous with respect to tight convergence. Then, by Theorem 1.2.3 we obtain that

$$\begin{aligned}\mathcal{T}^\sigma(\rho\mathcal{L}^d) &= \mathcal{F}^\sigma(\rho\mathcal{L}^d) + \int_{\mathbb{R}^d} g(x)\rho(x)dx \\ &\leq \liminf_{\varepsilon\to0^+} \mathcal{F}^\sigma_\varepsilon(\mu_\varepsilon) + \liminf_{\varepsilon\to0^+} \int_{\mathbb{R}^d} g(x)\frac{\varepsilon^d\omega_d}{C^d}d\mu_\varepsilon(x) \\ &\leq \liminf_{\varepsilon\to0^+} \Big(\mathcal{F}^\sigma_\varepsilon(\mu_\varepsilon) + \int_{\mathbb{R}^d} g(x)\frac{\varepsilon^d\omega_d}{C^d}d\mu_\varepsilon(x)\Big) = \liminf_{\varepsilon\to0^+} \mathcal{T}^\sigma_\varepsilon(\mu_\varepsilon).\end{aligned}$$

We now prove (2). First consider the case $\rho \in C^0_c(\mathbb{R}^d)$. Let $R > 0$ be such that $\mathrm{supp}(\rho) \subset B_R$ and let $\{\mu_\varepsilon\}_{\varepsilon\in(0,1)}$ be the recovery sequence provided by Theorem 1.2.3; then, it is easy to see that $\{\mu_\varepsilon\chi_{B_R}\}_{\varepsilon\in(0,1)}$ provides a recovery sequence also for the functionals $\mathcal{T}^\sigma_\varepsilon$. The general case follows by a standard diagonalization argument. Indeed, for any sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subset C^0(\mathbb{R}^d;[0,1])$ converging to $\varphi$ in $L^1$ we have $\mathcal{F}^\sigma(\varphi_n\mathcal{L}^d) \to \mathcal{F}^\sigma(\varphi\mathcal{L}^d)$ (see for instance the proof of Proposition 1.2.2). Then, for any sequence $\{\rho_n\}_{n\in\mathbb{N}} \subset C^0_c(B(0,R);[0,1])$

converging to $\rho\chi_{B(0,r)}$ in $L^1$ we have $\mathcal{T}^\sigma(\rho_n\mathcal{L}^d) \to \mathcal{T}^\sigma(\rho\chi_{B(0,R)}\mathcal{L}^d)$. Moreover, since $\rho$ is nonnegative and $g(x)$ is positive for $|x|$ large enough, we have that

$$\int_{\mathbb{R}^d} g(x)\rho(x)\chi_{B_R}(x)dx \to \int_{\mathbb{R}^d} g(x)\rho(x)(x)dx$$

as $R \to +\infty$. We deduce that $\mathcal{T}^\sigma(\rho\chi_{B(0,R)}\mathcal{L}^d) \to \mathcal{T}^\sigma(\rho\mathcal{L}^d)$ as $R \to +\infty$. Therefore, there exists a sequence $\{\rho_m\}_{m\in\mathbb{N}} \subset \mathcal{C}^0_c(\mathbb{R}^d)$ such that $\rho_m \to \rho$ in $L^1(\mathbb{R}^d)$ and $\mathcal{T}^\sigma(\rho_m\mathcal{L}^d) \to \mathcal{T}^\sigma(\rho\mathcal{L}^d)$ as $m \to +\infty$. □

### 1.2.4 Asymptotic behaviour of minimizers

Here we analyze the asymptotic behaviour of minimizers of the functionals $\mathcal{T}^\sigma_\varepsilon$ defined in (1.2.2).

**Proposition 1.2.5** (First variation). *Let $\rho\mathcal{L}^d$ be a minimizer of $\mathcal{T}^\sigma$. For almost every $x \in \mathbb{R}^d$ such that $0 < \rho(x) < 1$ we have*

$$g(x) - 2\int_{\mathbb{R}^d} \frac{1}{|x-y|^{d+\sigma}}\rho(y)\,dy = 0\,. \tag{1.2.18}$$

*Proof.* Let $h(x,y) := |x-y|^{-d-\sigma}$. Let $0 < \alpha < \beta < 1$ and set

$$E_{\alpha,\beta} := \{x \in \mathbb{R}^d : \alpha < \rho(x) < \beta\}.$$

Let $E \subseteq E_{\alpha,\beta}$, and set $u := \chi_E$. Then, for $\varepsilon$ small enough the function $\rho + \varepsilon u$ takes values in $(0,1)$. By minimality of $\rho$ we deduce that

$$\begin{aligned}0 &\le \mathcal{T}^\sigma(\rho+\varepsilon u) - \mathcal{T}^\sigma(\rho)\\ &= \varepsilon\int_{\mathbb{R}^d} g(x)u(x)\,dx - 2\varepsilon\int_{\mathbb{R}^{2d}} h(x,y)\rho(y)u(x)\,dy\,dx + o(\varepsilon),\end{aligned}$$

where $o(\varepsilon)/\varepsilon \to 0$ as $\varepsilon \to 0$. We deduce that

$$\int_{\mathbb{R}^d} g(x)u(x)\,dx - 2\int_{\mathbb{R}^{2d}} h(x,y)\rho(y)u(x)\,dy\,dx = 0\,.$$

Since the above inequality holds for $u = \chi_E$ where $E$ is any measurable set contained in $\{x \in \mathbb{R}^d : 0 < \rho(x) < 1\}$, by the fundamental lemma in the calculus of variations and an easy density argument we deduce the claim. □

**Theorem 1.2.6** (Behaviour of minimizers). *Let $\mathcal{T}^\sigma_\varepsilon$ be defined in* (1.2.2) *with $g$ satisfying* (1.2.3) *for some $C_2 > C^*(\sigma,d)$, where $C^*(\sigma,d)$ is the constant provided by Theorem 1.2.1. Let moreover $\mu_\varepsilon$ be minimizers of $\mathcal{T}^\sigma_\varepsilon$ for all $\varepsilon > 0$.*

*Then, up to a subsequence, $\frac{\varepsilon^d\omega_d}{C^d}\mu_\varepsilon \to \chi_E\mathcal{L}^d$ tightly in $\mathcal{M}_b(\mathbb{R}^d)$, for some set $E \in \mathrm{M}_f(\mathbb{R}^d)$. Moreover, $\chi_E\mathcal{L}^d$ is a minimizer of $\mathcal{T}^\sigma$. Finally, if $g(x) := G(|x|)$ for some increasing function $G : \mathbb{R}^+ \to \mathbb{R}$, then $E$ is a ball.*

*Proof.* By Theorem 1.2.1, up to a subsequence, $\frac{\varepsilon^d\omega_d}{C^d}\mu_\varepsilon \to \rho\mathcal{L}^d$ tightly in $\mathcal{M}_b(\mathbb{R}^d)$, for some $\rho \in L^1(\mathbb{R}^d;[0,1])$. Moreover, as a consequence of the $\Gamma$-convergence result established in Theorem 1.2.4, $\rho\mathcal{L}^d$ is a minimizer of $\mathcal{T}^\sigma$; we have to prove that $\rho$ is a characteristic function.

Let now $\tilde{\rho} := \chi_{\mathrm{supp}(\rho)}$ and let $u := \tilde{\rho} - \rho$. By (1.2.18) we have

$$
\begin{aligned}
0 \le \mathcal{T}^\sigma(\rho+u) - \mathcal{T}^\sigma(\rho) \\
&= \int_{\mathbb{R}^d} g(x)u(x)\,dx - 2\int_{\mathbb{R}^{2d}} h(x,y)\rho(y)u(x)\,dy\,dx - \int_{\mathbb{R}^{2d}} h(x,y)u(y)u(x)\,dy\,dx \\
&= \int_{\mathbb{R}^d} u(x)\Big[g(x) - 2\int_{\mathbb{R}^d} \rho(y)\frac{1}{|x-y|^{d+\sigma}}\,dy\Big]dx - \int_{\mathbb{R}^{2d}} u(x)u(y)h(x,y)\,dy\,dx \\
&= -\int_{\mathbb{R}^{2d}} h(x,y)u(y)u(x)\,dy\,dx \le 0.
\end{aligned}
$$

We conlcude that the above inequalities are in fact all equalities, which in turns implies $u = 0$, i.e., $\tilde{\rho} = \rho$ and $\rho$ is a characteristic function.

Finally, if $g$ is radial and increasing with respect to $|x|$, then denoted by $E^*$ the ball centered at 0 with $|E^*| = |E|$, we have

$$
\mathcal{F}^\sigma(E^*\mathcal{L}^d) \le \mathcal{F}^\sigma(E\mathcal{L}^d), \qquad \int_{E^*} g(x)dx \le \int_E g(x)dx, \tag{1.2.19}
$$

where the first inequality is strict for every set $E \in \mathrm{M}_f(\mathbb{R}^d)$ that is not ball; this is a consequence of the uniqueness of the ball in the Riesz inequality for characteristic functions interacting through strict increasing potentials (see for instance [38, Theorem A4]). From (1.2.19) we easily conclude that $E$ must be a ball. □

*Remark* 1.2.7. Theorem 1.2.6 establishes that minimizers of $\mathcal{T}^\sigma_\varepsilon$ tightly converge to a minimizer of $\mathcal{T}^\sigma$, which is a characteristic function of some set $E$, and that such a set $E$ is a ball whenever the volume force term $g$ is radial and increasing with respect to $|x|$. This means that minimizers of $\mathcal{T}^\sigma_\varepsilon$, for $\varepsilon$ small, consist in almost optimally packed configurations filling a macroscopic set E, which is a ball whenever the volume term is radially increasing.

## 1.3 Riesz interactions for $\sigma \in [0,1)$

Here we introduce and analyze regularized Riesz interaction functionals in the non-integrable case $\sigma \in [0,1)$.

### 1.3.1 The energy functionals

Let $\sigma \in [0,1)$. For every $\varepsilon > 0$ let $r_\varepsilon > 0$ be such that $r_\varepsilon \to 0$ as $\varepsilon \to 0$ and

$$
\frac{\varepsilon^{\frac{1}{2\sigma+1}}}{r_\varepsilon} \to 0 \qquad \text{as } \varepsilon \to 0 \qquad \text{for } \sigma \in (0,1); \tag{1.3.1}
$$

$$
\frac{\varepsilon|\log(r_\varepsilon)|^2}{r_\varepsilon} \to 0 \qquad \text{as } \varepsilon \to 0 \qquad \text{for } \sigma = 0. \tag{1.3.2}
$$

The regularized potentials are defined by

$$
f^\sigma_\varepsilon(r) := \begin{cases} +\infty & \text{for } r \in [0, 2\varepsilon)\,, \\ 0 & \text{for } r \in [2\varepsilon, r_\varepsilon)\,, \\ -\frac{1}{r^{d+\sigma}} & \text{for } r \in [r_\varepsilon, +\infty)\,, \end{cases}
$$

As in (1.2.1), we introduce the energy functionals

$$\mathcal{F}^\sigma_\varepsilon(\mu) := \begin{cases} F^\sigma_\varepsilon(\mathcal{A}(\mu)) & \text{if } \mu\in\mathcal{EM}_\varepsilon,\\ +\infty & \text{elsewhere.}\end{cases}$$

We will also introduce suitable renormalized energy functionals. To this purpose, for all $\sigma\in[0,1)$ and $r\in(0,1]$ we set

$$\gamma^\sigma_r := -\int_{B(0,1)\setminus B(0,r)} \frac{1}{|z|^{d+\sigma}}dz. \tag{1.3.3}$$

Notice that

$$\gamma^\sigma_r := \begin{cases} d\omega_d\dfrac{1-r^{-\sigma}}{\sigma} & \text{if } \sigma\neq 0\,,\\ d\omega_d\log r & \text{if } \sigma=0\,.\end{cases} \tag{1.3.4}$$

For $\sigma\in[0,1)$ the renormalized energy functionals $\hat{\mathcal{F}}^\sigma_\varepsilon:\mathcal{M}_b(\mathbb{R}^d)\to\mathbb{R}\cup\{-\infty,+\infty\}$ are defined by

$$\hat{\mathcal{F}}^\sigma_\varepsilon(\mu) := \begin{cases} \mathcal{F}^\sigma_\varepsilon(\mathcal{A}(\mu)) - \gamma^\sigma_{r_\varepsilon}\dfrac{\varepsilon^d\omega_d}{C_d}\mu(\mathbb{R}^d) & \text{if } \mu\in\mathcal{EM}_\varepsilon,\\ +\infty & \text{elsewhere.}\end{cases}$$

The functional $\hat{\mathcal{F}}^\sigma_\varepsilon$ may be also rewritten as

$$\hat{\mathcal{F}}^\sigma_\varepsilon(\mu) = \begin{cases} \displaystyle\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f^\sigma_\varepsilon(|x-y|)\Big(\frac{\varepsilon^d\omega_d}{C_d}\Big)^2 d\mu\otimes\mu - \gamma^\sigma_{r_\varepsilon}\frac{\varepsilon^d\omega_d}{C_d}\mu(\mathbb{R}^d) & \text{if } \mu\in\mathcal{EM}_\varepsilon,\\ +\infty & \text{elsewhere.}\end{cases}$$

### 1.3.2 The continuous model

Here we give a short overview of the Γ-convergence analysis of the continuous model for non-integrable Riesz potentials developed in [38].

First, we introduce the fractional perimeters; for all $\sigma\in(0,1)$, the $\sigma$-fractional perimeter of $E\in\mathrm{M}(\mathbb{R}^d)$ is defined by

$$P^\sigma(E) = \int_E\int_{\mathbb{R}^d\setminus E}\frac{1}{|x-y|^{d+\sigma}}dxdy.$$

For $\sigma=0$, a notion of 0-fractional perimeter has been introduced in [38] as follows.

First, for all $R>1$ we set

$$\gamma^0_R := \int_{B(0,R)\setminus B(0,1)}\frac{1}{|z|^d}dz.$$

Then, the following definition is well posed (namely, the following limit exists, [38])

$$P^0(E) := \lim_{R\to+\infty}\int_E\int_{B(x,R)\setminus E}\frac{1}{|x-y|^d}\,dx\,dy\, - \gamma^0_R|E|.$$

Now, we introduce the continuous Riesz functionals. For all $r \in (0,1)$ let $J_r^\sigma : \mathrm{M}_f(\mathbb{R}^d) \to \mathbb{R} \cup \{-\infty, +\infty\}$ be the functionals defined by

$$J_r^\sigma(E) := \int_E \int_{E \setminus B(x,r)} \frac{-1}{|x-y|^{d+\sigma}} dxdy.$$

The renormalized functionals $\hat{J}_r^\sigma : \mathrm{M}_f(\mathbb{R}^d) \to \mathbb{R} \cup \{-\infty, +\infty\}$ are defined by

$$\hat{J}_r^\sigma(E) := J_r^\sigma(E) - \gamma_r^\sigma |E|,$$

where $\gamma_r^\sigma$ is the constant defined in (1.3.3).

Now we introduce the candidate $\Gamma$-limits. For $\sigma \in (0,1)$ we define the functional $\hat{\mathcal{F}}^\sigma : \mathcal{M}_b(\mathbb{R}^d) \to \mathbb{R} \cup \{-\infty, +\infty\}$ as

$$\hat{\mathcal{F}}^\sigma(\mu) := \begin{cases} P^\sigma(E) - \gamma^\sigma |E| & \text{if } \mu = \chi_E \mathcal{L}^d, \\ +\infty & \text{elsewhere,} \end{cases} \tag{1.3.5}$$

where $\gamma^\sigma = \int_{\mathbb{R}^d \setminus B(0,1)} \frac{1}{|z|^{d+\sigma}} dz$.

Moreover, for $\sigma = 0$ we define $\mathcal{F}^0 : \mathcal{M}_b(\mathbb{R}^d) \to \mathbb{R} \cup \{-\infty, +\infty\}$ as

$$\hat{\mathcal{F}}^0(\mu) := \begin{cases} P^0(E) & \text{if } \mu = \chi_E \mathcal{L}^d, \\ +\infty & \text{elsewhere.} \end{cases} \tag{1.3.6}$$

The following theorem has been proved in [38, Sections 5 & 6].

**Theorem 1.3.1.** *The following compacntess and $\Gamma$-convergence results hold.*

***Compactness:*** *Let $\sigma \in [0,1)$ and let $r_n \to 0^+$. Let $U \subset \mathbb{R}^d$ be an open bounded set and let $\{E_n\}_{n\in\mathbb{N}} \subset \mathrm{M}_f(\mathbb{R}^d)$ be such that $E_n \subset U$ for all $n \in \mathbb{N}$. Finally, let $C > 0$.*

*If $\hat{J}_{r_n}^\sigma(E_n) \le C$ for all $n \in \mathbb{N}$, then, up to a subsequence, $\chi_{E_n} \to \chi_E$ in $L^1(\mathbb{R}^d)$ for some $E \in \mathrm{M}_f(\mathbb{R}^d)$.*

***$\Gamma$-convergence:*** *The following $\Gamma$-convergence result holds true.*

*(i) ($\Gamma$-liminf inequality) For every $E \in \mathrm{M}_f(\mathbb{R}^d)$ and for every sequence $\{E_n\}_{n\in\mathbb{N}}$ with $\chi_{E_n} \to \chi_E$ strongly in $L^1(\mathbb{R}^d)$ it holds*

$$\hat{\mathcal{F}}^\sigma(E) \le \liminf_{n\to+\infty} \hat{J}_{r_n}^\sigma(E_n).$$

*(ii) ($\Gamma$-limsup inequality) For every $E \in \mathrm{M}_f(\mathbb{R}^d)$, there exists a sequence $\{E_n\}_{n\in\mathbb{N}}$ such that $\chi_{E_n} \to \chi_E$ strongly in $L^1(\mathbb{R}^d)$ and*

$$\hat{\mathcal{F}}^\sigma(E) \ge \limsup_{n\to+\infty} \hat{J}_{r_n}^\sigma(E_n).$$

### 1.3.3 Error estimates

Next proposition provides error estimates comparing the discrete functionals $\mathcal{F}^\sigma_\varepsilon$ with its continuous counterpart $J^\sigma_{r_\varepsilon}$.

**Proposition 1.3.2.** *Let $\sigma \in [0,1)$, and let $\{\mu_\varepsilon\}_{\varepsilon\in(0,1)} \subset \mathcal{EM}$ be such that $\mu_\varepsilon \in \mathcal{EM}_\varepsilon$ for all $\varepsilon \in (0,1)$ and $\frac{\varepsilon^d\omega_d}{C^d}\mu_\varepsilon(\mathbb{R}^d) \le M$ for some $M > 0$.*

*Then, there exists $\{E_\varepsilon\}_{\varepsilon\in(0,1)} \subset \mathrm{M}_f(\mathbb{R}^d)$ such that the following properties hold:*

*(i)* $\frac{\varepsilon^d\omega_d}{C^d}\mu_\varepsilon - \chi_{E_\varepsilon} \overset{*}{\rightharpoonup} 0$ *as* $\varepsilon \to 0$*;*

*(ii)* $||E_\varepsilon| - \frac{\varepsilon^d\omega_d}{C^d}\mu_\varepsilon(\mathbb{R}^d)| \le C(M,d)\frac{\sqrt{\varepsilon}}{\sqrt{r_\varepsilon}}$*;*

*(iii)* $|\mathcal{F}^\sigma_\varepsilon(\mu_\varepsilon) - J^\sigma_{r_\varepsilon}(E_\varepsilon)| \le C(\sigma,d,M)|\gamma^\sigma_{r_\varepsilon}|\frac{\sqrt{\varepsilon}}{\sqrt{r_\varepsilon}}$.

*In particular, as a consequence of* (1.3.1)*, we have*

*(iii')* $|\hat{\mathcal{F}}^\sigma_\varepsilon(\mu_\varepsilon) - \hat{J}^\sigma_{r_\varepsilon}(E_\varepsilon)| \to 0 \quad$ *as* $\varepsilon \to 0$.

*Vice-versa, if $\{E_\varepsilon\}_{\varepsilon\in(0,1)} \subset \mathrm{M}_f(\mathbb{R}^d)$ is such that $|E_\varepsilon| \le M$ for some $M > 0$, then there exists $\{\mu_\varepsilon\}_{\varepsilon\in(0,1)} \subset \mathcal{EM}$ with $\mu_\varepsilon \in \mathcal{EM}_\varepsilon$ for all $\varepsilon \in (0,1)$ and such that (i), (ii), (iii) and (iii') hold.*

*Proof.* For every $\varepsilon > 0$, set $\rho_\varepsilon := \sqrt{\varepsilon r_\varepsilon}$. Let $Q := [0,1)^d$ and set

$$\mathfrak{Q}^{\rho_\varepsilon} := \{\rho_\varepsilon(Q+v),\ v \in \mathbb{Z}^d\}.$$

Let moreover

$$\mathfrak{P}^{\rho_\varepsilon} := \{q \in \mathfrak{Q}^{\rho_\varepsilon} : \frac{\varepsilon^d\omega_d}{C^d}\mu_\varepsilon(q) \ge \rho_\varepsilon^d\}.$$

For all $q \in \mathfrak{Q}^{\rho_\varepsilon}$ we denote by $\tilde{q}$ the square concentric to $q$ and such that $\tilde{q} = q$ if $q \in \mathfrak{P}^{\rho_\varepsilon}$, while $|\tilde{q}| = \frac{\varepsilon^d\omega_d}{C^d}\mu_\varepsilon(q)$ if $q \in \mathfrak{Q}^{\rho_\varepsilon} \setminus \mathfrak{P}^{\rho_\varepsilon}$. By Lemma 1.1.2 and by easy scaling arguments we deduce that

$$\#\mathfrak{P}^{\rho_\varepsilon} \le M\rho_\varepsilon^{-d}, \qquad 0 \le \frac{\varepsilon^d\omega_d}{C^d}\mu_\varepsilon(q) - |\tilde{q}| \le C(d)\varepsilon\rho_\varepsilon^{d-1} \quad \text{for all } q \in \mathfrak{Q}^{\rho_\varepsilon}. \tag{1.3.7}$$

Indeed, since $\frac{\varepsilon^d\omega_d}{C^d}\mu_\varepsilon(\mathbb{R}^d) \le M$, we have

$$\rho_\varepsilon^d\#\mathfrak{P}^{\rho_\varepsilon} \le \frac{\varepsilon^d\omega_d}{C^d}\mu_\varepsilon(\bigcup_{q\in\mathfrak{P}^{\rho_\varepsilon}} q) \le M,$$

and the first formula in (1.3.7) follows. The second formula in (1.3.7) is trivial if $q \in \mathfrak{Q}^{\rho_\varepsilon} \setminus \mathfrak{P}^{\rho_\varepsilon}$; for $q \in \mathfrak{P}^{\rho_\varepsilon}$ we define $\tilde{X} := \{\frac{x}{\varepsilon} :\ x \in \mathrm{supp}(\mu_\varepsilon)\} \cap \frac{q}{\varepsilon} \in \mathrm{Ad}^{\mathrm{d}}(\frac{q}{\varepsilon})$, then, by the Lemma 1.1.2, we obtain

$$\begin{aligned} 0 \le \frac{\varepsilon^d\omega_d}{C^d}\mu_\varepsilon(q) - |q| = \rho_\varepsilon^d\Big[\frac{1}{C^d}\frac{\omega_d\#(\tilde{X})}{(\frac{\rho_\varepsilon}{\varepsilon})^d} - 1\Big] \\ \le \rho_\varepsilon^d\Big[\frac{1}{C^d}\Big(C^d + \frac{C(d)}{\frac{\rho_\varepsilon}{\varepsilon}}\Big) - 1\Big] = C(d)\varepsilon\rho_\varepsilon^{d-1}. \end{aligned}$$

We define $E_\varepsilon := \cup_{q\in\mathfrak{Q}^{\rho_\varepsilon}} \tilde{q}$. By (1.3.7) we have that

$$||E_\varepsilon| - \frac{\varepsilon^d \omega_d}{C^d}\mu_\varepsilon(\mathbb{R}^d)| \le MC(d)\frac{\varepsilon}{\rho_\varepsilon} = MC(d)\frac{\sqrt{\varepsilon}}{\sqrt{r}_\varepsilon},$$

which proves property (ii).

Let us pass to the proof of (i). Given $\varphi \in \mathcal{C}^1_c(\mathbb{R}^d)$, by (1.3.7) we have

$$\Big|\langle \frac{\varepsilon^d\omega_d}{C^d}\mu_\varepsilon - \chi_{E_\varepsilon}, \varphi\rangle\Big| \le C(d,M)\|\nabla\varphi\|_{L^\infty}\rho_\varepsilon + \|\varphi\|_{L^\infty} C(d,M)\frac{\varepsilon}{\rho_\varepsilon},$$

which tends to 0 as $\varepsilon \to 0$.

We pass to the proof of (iii). First notice that by construction $|E_\varepsilon| \le M+1$ for $\varepsilon$ small enough. Then, by rearrangement (see for instance Lemma A.6 of [38]) it is easy to see that $-J^\sigma_{r_\varepsilon}(E_\varepsilon) \le C(\sigma,d,M)|\gamma^\sigma_{r_\varepsilon}|$. Therefore, in order to prove (iii) it is enough to show that

$$-J^\sigma_{r_\varepsilon}(E_\varepsilon) \le -\mathcal{F}^\sigma_\varepsilon(\mu_\varepsilon)(1 + C(\sigma,d)\frac{\sqrt{\varepsilon}}{\sqrt{r}_\varepsilon}) + C(\sigma,d)|\gamma^\sigma_{r_\varepsilon}|\frac{\sqrt{\varepsilon}}{\sqrt{r}_\varepsilon}, \tag{1.3.8}$$

$$-\mathcal{F}^\sigma_\varepsilon(\mu_\varepsilon) \le -J^\sigma_{r_\varepsilon}(E_\varepsilon)(1 + C(\sigma,d)\frac{\sqrt{\varepsilon}}{\sqrt{r}_\varepsilon}) + C(\sigma,d)|\gamma^\sigma_{r_\varepsilon}|\frac{\sqrt{\varepsilon}}{\sqrt{r}_\varepsilon}. \tag{1.3.9}$$

We will prove only (1.3.9), the proof of (1.3.8) being fully analogous. For all $p$, $q \in \mathfrak{Q}^{\rho_\varepsilon}$ with $p \neq q$, set

$$I(p,q) := \{(x,y) \in \mathrm{supp}(\mu)\times\mathrm{supp}(\mu) \cap p\times q\},$$

$$R_\varepsilon(p,q) := \mathrm{dist}(p,q), \qquad \tilde{R}_\varepsilon(p,q) := \max_{x\in p, y\in q}\mathrm{dist}(x,y), \qquad m_\varepsilon(q) := \frac{\varepsilon^d\omega_d}{C^d}\mu_\varepsilon(q).$$

By (1.3.7) we have that

$$1 \le \frac{m_\varepsilon(q)}{|\tilde{q}|} \le 1 + C(d)\frac{\varepsilon}{\rho_\varepsilon} \qquad \text{for all } q \in \mathfrak{Q}^{\rho_\varepsilon}. \tag{1.3.10}$$

Moreover, since $\tilde{R}_\varepsilon(p,q) \le R_\varepsilon(p,q) + C(d)\rho_\varepsilon$, it follows that there exists $C(\sigma,d) > 0$ such that, for $\varepsilon$ small enough,

$$\Big(\frac{\tilde{R}_\varepsilon(p,q)}{R_\varepsilon(p,q)}\Big)^{d+\sigma} \le (1 + C(\sigma,d)\frac{\rho_\varepsilon}{R_\varepsilon(p,q)}) \quad \text{for all } q,p \in \mathfrak{Q}^{\rho_\varepsilon} : R_\varepsilon(p,q) \neq 0. \tag{1.3.11}$$

Moreover, let

$$\mathscr{Q}^+ := \{(p,q) \in \mathfrak{Q}^{\rho_\varepsilon}\times\mathfrak{Q}^{\rho_\varepsilon} : R_\varepsilon(p,q) > r_\varepsilon\};$$
$$\mathscr{Q}^- := \{(p,q) \in \mathfrak{Q}^{\rho_\varepsilon}\times\mathfrak{Q}^{\rho_\varepsilon} : \tilde{R}_\varepsilon(p,q) < r_\varepsilon\};$$
$$\mathscr{Q}^= := \mathfrak{Q}^{\rho_\varepsilon}\times\mathfrak{Q}^{\rho_\varepsilon} \setminus (\mathscr{Q}^+ \cup \mathscr{Q}^-).$$

Recalling that $\frac{\varepsilon^d\omega_d}{C^d}\mu_\varepsilon(\mathbb{R}^d) \le M$ and (1.3.7), it easily follows that, for $\varepsilon$ small enough

$$(\frac{\varepsilon^d\omega_d}{C^d})^2 \sum_{(p,q)\in\mathscr{Q}^=}\sum_{(x,y)\in I(p,q)} |x-y|^{-d-\sigma}$$
$$\le C(\sigma,d)r_\varepsilon^{-d-\sigma}r_\varepsilon^{d-1}\rho_\varepsilon = C(\sigma,d)r_\varepsilon^{-\sigma}\frac{\sqrt{\varepsilon}}{\sqrt{r}_\varepsilon} \le C(\sigma,d)|\gamma^\sigma_{r_\varepsilon}|\frac{\sqrt{\varepsilon}}{\sqrt{r}_\varepsilon}. \tag{1.3.12}$$

By (1.3.10), (1.3.11) and (1.3.12) we have that, for $\varepsilon$ small enough,

$$
\begin{aligned}
-\mathcal{F}^\sigma_\varepsilon(\mu_\varepsilon) \le (\frac{\varepsilon^d\omega_d}{C^d})^2 \sum_{(p,q)\in\mathscr{Q}^+}\sum_{(x,y)\in I(p,q)} |x-y|^{-d-\sigma} + C(\sigma,d)|\gamma^\sigma_{r_\varepsilon}|\frac{\sqrt{\varepsilon}}{\sqrt{r_\varepsilon}} \\
\le \sum_{(p,q)\in\mathscr{Q}^+} m_\varepsilon(p)m_\varepsilon(q)R_\varepsilon(p,q)^{-d-\sigma} + C(\sigma,d)|\gamma^\sigma_{r_\varepsilon}|\frac{\sqrt{\varepsilon}}{\sqrt{r_\varepsilon}} \\
\le \sum_{(p,q)\in\mathscr{Q}^+}\Big(1+C(d)\frac{\varepsilon}{\rho_\varepsilon}\Big)^2\Big(1+C(\sigma,d)\frac{\rho_\varepsilon}{R_\varepsilon(p,q)}\Big)|\tilde p||\tilde q|\tilde R_\varepsilon(p,q)^{-d-\sigma} + C(\sigma,d)|\gamma^\sigma_{r_\varepsilon}|\frac{\sqrt{\varepsilon}}{\sqrt{r_\varepsilon}} \\
\le \Big(1+C(\sigma,d)\frac{\sqrt{\varepsilon}}{\sqrt{r_\varepsilon}}\Big)(-J^\sigma_{r_\varepsilon}(E_\varepsilon)) + C(\sigma,d,M)|\gamma^\sigma_{r_\varepsilon}|\frac{\sqrt{\varepsilon}}{\sqrt{r_\varepsilon}}.
\end{aligned}
$$

Finally, property (iii') is an easy consequence of properties (ii), (iii) and of (1.3.1), (1.3.2), (1.3.4).

The proof of the final claim of the proposition is fully analogoug to the proof of the first part of the proposition; we only describe how to define the measure $\mu_\varepsilon$, corresponding to the set $E_\varepsilon$. For all $\varepsilon\in(0,1)$ and for all $q\in\mathfrak{Q}^{\rho_\varepsilon}$ let

$$n^\varepsilon(q) := \min\Big\{\Big\lfloor\frac{C^d|q\cap E_\varepsilon|}{\varepsilon^d\omega_d}\Big\rfloor, \#(\hat T^d_{\frac{\rho_\varepsilon}{\varepsilon}})\Big\}$$

where $\hat T^d_{\frac{\rho_\varepsilon}{\varepsilon}}$ is the set defined in (1.1.4). Notice that the number of $q\in\mathfrak{Q}^{\rho_\varepsilon}$ such that $n^\varepsilon(q)\neq 0$ is finite. By the very definition of $n^\varepsilon(q)$ it is always possible to find a set $X(q)$ of $n^\varepsilon(q)$ points contained in $q$ with the following properties: For all $x_i,x_j\in X(q)$, $i\neq j$, we have $|x_i-x_j|\ge 2\varepsilon$; $\cup_{i=1}^{n^\varepsilon(q)} B(x_i,\varepsilon)\subset q$. Finally we define the measure

$$\mu_\varepsilon := \sum_{q\in\mathfrak{Q}^{\rho_\varepsilon}}\sum_{x\in X(q)}\delta_x.$$

□

### 1.3.4 Compactness and Γ-convergence

Here we prove Γ-convergence and compactness properties for the functionals $\hat{\mathcal{F}}^\sigma_\varepsilon$ defined in (1.3.5) and (1.3.6). Conversely to what done for the integrable case $\sigma\in(-d,0)$, here we will present only the basic case, assuming as in [38] that there are no forcing terms; we enforce compactness assuming that the empirical measures have uniformly bounded support.

**Theorem 1.3.3.** *Let $\sigma\in[0,1)$. The following compactness and Γ-convergence results hold.*

***Compactness:*** *Let $U\subset\mathbb{R}^d$ be an open bounded set and let $M>0$. Let $\{\mu_\varepsilon\}_{\varepsilon\in(0,1)}\subset\mathcal{M}_b(\mathbb{R}^d)$ be such that*

$$\hat{\mathcal{F}}^\sigma_\varepsilon(\mu_\varepsilon)\le M, \qquad \mathrm{supp}(\mu_\varepsilon)\subset U \qquad \forall\varepsilon\in(0,1). \tag{1.3.13}$$

*Then, $\frac{\varepsilon^d\omega_d}{C^d}\mu_\varepsilon\to\chi_E\mathcal{L}^d$ tightly, as $\varepsilon\to 0^+$, for some measurable set $E\subset U$.*

***Γ-convergence:*** *The following Γ-convergence result holds true.*

1. *($\Gamma$-liminf inequality) For every $E \in \mathrm{M}_f(\mathbb{R}^d)$ and for every $\{\mu_\varepsilon\}_{\varepsilon\in(0,1)} \subset \mathcal{M}_b(\mathbb{R}^d)$ with $\frac{\varepsilon^d\omega}{C^d}\mu_\varepsilon \to \chi_E\mathcal{L}^d$ tightly in $\mathcal{M}_b(\mathbb{R}^d)$, we have*
$$\hat{\mathcal{F}}^\sigma(\chi_E\mathcal{L}^d) \leq \liminf_{\varepsilon\to 0^+} \hat{\mathcal{F}}^\sigma_\varepsilon(\mu_\varepsilon).$$
2. *($\Gamma$-limsup inequality) For every $E \in \mathrm{M}_f(\mathbb{R}^d)$, there exists a sequence $\{\mu_\varepsilon\}_{\varepsilon\in(0,1)}$ with $\mu_\varepsilon \in \mathcal{EM}_\varepsilon$ for all $\varepsilon \in (0,1)$ such that $\frac{\varepsilon^d\omega_d}{C^d}\mu_\varepsilon \to \chi_E\mathcal{L}^d$ tightly in $\mathcal{M}_b(\mathbb{R}^d)$ and*
$$\hat{\mathcal{F}}^\sigma(\chi_E\mathcal{L}^d) \geq \limsup_{\varepsilon\to 0^+} \hat{\mathcal{F}}^\sigma_\varepsilon(\mu_\varepsilon).$$

*Proof.* In order to prove the compactness property, first notice that by (1.3.13) we deduce that $\mu_\varepsilon \in \mathcal{EM}_\varepsilon$ for all $\varepsilon \in (0,1)$. From Proposition 1.3.2 we obtain that there exists $\varepsilon_0 \in (0,1)$ such that
$$|\hat{\mathcal{F}}^\sigma_\varepsilon(\mu_\varepsilon) - \hat{J}^\sigma_{r_\varepsilon}(E_\varepsilon)| < 1 \quad \forall \varepsilon < \varepsilon_0,$$
where $\{E_\varepsilon\}_{\varepsilon\in(0,1)}$ is exactly the sequence of sets provided by Proposition 1.3.2. We deduce that $\hat{J}^\sigma_{r_\varepsilon}(E_\varepsilon)$ is bounded; by Theorem 1.3.1 there exists $E \in \mathrm{M}_f(\mathbb{R}^d)$ such that, up to a subsequence, $\chi_{E_\varepsilon} \to \chi_E$ in $\mathrm{L}^1$ for $\varepsilon \to 0^+$. Therefore, again by Proposition 1.3.2 $\frac{\varepsilon^d\omega_d}{C^d}\mu_\varepsilon \to \chi_E$ tightly as $\varepsilon \to 0^+$.

Let us pass to the proof of the $\Gamma$-liminf inequality. By Proposition 1.3.2 and by Theorem 1.3.1 we obtain that
$$\begin{aligned}\hat{\mathcal{F}}^\sigma(\chi_E\mathcal{L}^d) &\leq \liminf_{\varepsilon\to 0^+} \hat{J}^\sigma_{r_\varepsilon}(E_\varepsilon)\\ &\leq \liminf_{\varepsilon\to 0^+}(\hat{J}^\sigma_{r_\varepsilon}(E_\varepsilon) - \hat{\mathcal{F}}^\sigma_\varepsilon(\mu_\varepsilon)) + \liminf_{\varepsilon\to 0^+} \hat{\mathcal{F}}^\sigma_\varepsilon(\mu_\varepsilon)\\ &\leq \liminf_{\varepsilon\to 0^+} \hat{\mathcal{F}}^\sigma_\varepsilon(\mu_\varepsilon).\end{aligned}$$
Hence the $\Gamma$-liminf inequality holds.

We now prove the $\Gamma$-limsup inequality.

Let $\{E_\varepsilon\}_{\varepsilon\in(0,1)}$ be the recovery sequence provided by Theorem 1.3.1; we have
$$\hat{J}^\sigma_{r_\varepsilon}(E_\varepsilon) \to \hat{\mathcal{F}}^\sigma(\chi_E\mathcal{L}^d) \qquad \text{as } \varepsilon \to 0.$$
Let now $\{\mu_\varepsilon\}_{\varepsilon\in(0,1)}$ be the sequence provided by the second part of Proposition 1.3.2. Then, we have
$$|\hat{\mathcal{F}}^\sigma_\varepsilon(\mu_\varepsilon) - \hat{\mathcal{F}}^\sigma(\chi_E\mathcal{L}^d)| \leq |\hat{\mathcal{F}}^\sigma_\varepsilon(\mu_\varepsilon) - \hat{J}^\sigma_{r_\varepsilon}(E_\varepsilon)| + |\hat{J}^\sigma_{r_\varepsilon}(E) - \hat{\mathcal{F}}^\sigma(\chi_E\mathcal{L}^d)|,$$
which, in view of Proposition 1.3.2(iii'), tends to 0 as $\varepsilon \to 0$. □

*Remark* 1.3.4. We have considered in this chapter the first order $\Gamma$-convergence of the functionals $\mathcal{F}^\sigma_\varepsilon$. The zero order analysis, i.e., the $\Gamma$-limit of the functionals $\frac{1}{\gamma_{r^\sigma_\varepsilon}}\mathcal{F}^\sigma_\varepsilon$ would give back less information on the asymptotic behaviour of minimizers; one could show that sequences with bounded energy converge (up to a subsequence) to some characteristic function $\chi_E$, while the $\Gamma$-limit is nothing but the measure of $E$. In this respect, the zero order $\Gamma$-limit still enforces optimal packing on minimizing sequences, but does not determine the macroscopic limit shape.

# Chapter 2

# The core-radius approach to supercritical fractional perimeters, curvatures and geometric flows

In this chapter we consider a core-radius approach to nonlocal perimeters governed by isotropic kernels having critical and supercritical exponents, extending the the notion of $s$-fractional perimeter to the case $s \geq 1$.

We show that, as the core-radius vanishes, such core-radius regularized $s$-fractional perimeters, suitably scaled, $\Gamma$-converge to the standard Euclidean perimeter. Under the same scaling, the first variation of such nonlocal perimeters gives back regularized $s$-fractional curvatures which, as the core radius vanishes, converge to the standard mean curvature; as a consequence, we show that the level set solutions to the corresponding nonlocal geometric flows, suitably reparametrized in time, converge to the standard mean curvature flow.

Finally, we prove analogous results in the case of anisotropic kernels with applications to dislocation dynamics.

The reference for the following results is [36], joint work with Lucia De Luca and Marcello Ponsiglione.

## 2.1 Supercritical perimeters

Let $s \geq 1$. For every $r > 0$, we define the interaction kernel $k_r^s : [0, +\infty) \to [0, +\infty)$ as

$$k_r^s(t) := \begin{cases} \dfrac{1}{r^{d+s}} & \text{for } 0 \leq t \leq r \ , \\ \dfrac{1}{t^{d+s}} & \text{for } t > r \,, \end{cases} \tag{2.1.1}$$

We note that

$$k_r^s(lt) = l^{-d-s} k_{\frac{r}{l}}^s(t) \qquad \text{for every } r, l, t > 0. \tag{2.1.2}$$

For all $r > 0$, we define the functional $J_r^s : \mathrm{M}(\mathbb{R}^d) \to [-\infty, 0]$ as

$$J_r^s(E) := \int_E \int_E -k_r^s(|x-y|) \, \mathrm{d}y \, \mathrm{d}x$$

and for every $E \in \mathrm{M}_f(\mathbb{R}^d)$ we set

$$\tilde{J}_r^s(E) := J_r^s(E) + \lambda_r^s |E| \,, \tag{2.1.3}$$

where

$$\lambda_r^s := \int_{\mathbb{R}^d} k_r^s(|z|)\,\mathrm{d}z = \frac{(d+s)\omega_d}{s r^s} \,.$$

Notice that for every $E \in \mathrm{M}_f(\mathbb{R}^d)$

$$J_r^s(E) \geq -\int_E \int_{\mathbb{R}^d} k_r^s(|x-y|)\,\mathrm{d}y\,\mathrm{d}x = -\lambda_r^s |E| \,,$$

and hence $\tilde{J}_r^s : \mathrm{M}_f(\mathbb{R}^d) \to [0, +\infty)$. Moreover, by the very definition of $\tilde{J}_r^s$ in (2.1.3), for every $E \in \mathrm{M}_f(\mathbb{R}^d)$ we have

$$\tilde{J}_r^s(E) = \int_E \int_{E^c} k_r^s(|x-y|)\,\mathrm{d}y\,\mathrm{d}x \,. \tag{2.1.4}$$

We first state the following result concerning the pointwise limit of the functionals $\tilde{J}_r^s$ as $r \to 0^+$. To this purpose, for every $s \geq 1$ we set

$$\sigma^s(r) := \begin{cases} |\log r| & \text{if } s = 1 \\ \dfrac{d+s}{d+1} \dfrac{r^{1-s}}{s-1} & \text{if } s > 1 \,. \end{cases} \tag{2.1.5}$$

**Proposition 2.1.1.** *Let $s \geq 1$ and let $E \in \mathrm{M}_f(\mathbb{R}^d)$ be a smooth set. Then,*

$$\lim_{r \to 0^+} \frac{\tilde{J}_r^s(E)}{\sigma^s(r)} = \omega_{d-1} \mathrm{Per}(E) \,, \tag{2.1.6}$$

*where $\sigma^s$ is defined in* (2.1.5)*. In fact, for $s > 1$ formula* (2.1.6) *holds for every set $E \in \mathrm{M}_f(\mathbb{R}^d)$ of finite perimeter.*

The proof of Proposition 2.1.1 is postponed and will use, in particular, Proposition 2.1.4 below. For every $E \in \mathrm{M}_f(\mathbb{R}^d)$ we define the functionals

$$F_1^s(E) := \int_E \int_{E^c \cap B^c(x,1)} \frac{1}{|x-y|^{d+s}}\,\mathrm{d}y\,\mathrm{d}x \,, \tag{2.1.7}$$

$$G_r^s(E) := \int_E \int_{E^c \cap B(x,1)} k_r^s(|x-y|)\,\mathrm{d}y\,\mathrm{d}x \,, \tag{2.1.8}$$

and we notice that for every $0 < r < 1$ it holds

$$\tilde{J}_r^s(E) = F_1^s(E) + G_r^s(E) \,. \tag{2.1.9}$$

*Remark* 2.1.2. It is easy to see that, for every $E \in \mathrm{M}_f(\mathbb{R}^d)$, it holds

$$F_1^s(E) \leq \int_E \int_{B^c(0,1)} \frac{1}{|z|^{d+s}}\,\mathrm{d}z = \frac{d\omega_d}{s} |E| \,.$$

Let $s \geq 1$. For all $r > 0$ we define the function $T_r^s : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}^d$ as

$$T_r^s(x) := \begin{cases} -\dfrac{1}{s}\dfrac{x}{|x|^{d+s}} & \text{if } |x| \in [r, +\infty)\,, \\ \dfrac{x}{dr^{d+s}} - \dfrac{d+s}{dsr^s}\dfrac{x}{|x|^d} & \text{if } |x| \in (0, r)\,. \end{cases} \tag{2.1.10}$$

A direct computation shows that

$$\mathrm{Div}(T_r^s(x)) = k_r^s(|x|). \tag{2.1.11}$$

**Lemma 2.1.3.** *Let $E \in \mathrm{M}_f(\mathbb{R}^d)$ be a set of finite perimeter. Then, for every $0 < r < 1$, we have*

$$\begin{aligned} G_r^s(E) =& \frac{d+s}{dsr^s}\int_{\partial^* E} \mathrm{d}\mathcal{H}^{d-1}(y) \int_{E\cap B(y,r)} \frac{y-x}{|x-y|^d}\cdot \nu_E(y)\,\mathrm{d}x \\ &- \frac{1}{dr^{d+s}}\int_{\partial^* E} \mathrm{d}\mathcal{H}^{d-1}(y) \int_{E\cap B(y,r)} (y-x)\cdot \nu_E(y)\,\mathrm{d}x \\ &+ \frac{1}{s}\int_{\partial^* E} \mathrm{d}\mathcal{H}^{d-1}(y) \int_{E\cap(B(y,1)\setminus B(y,r))} \frac{y-x}{|x-y|^{d+s}}\cdot \nu_E(y)\,\mathrm{d}x \\ &- \frac{1}{s}\int_E \mathcal{H}^{d-1}(E^c \cap \partial B(x,1))\,\mathrm{d}x\,, \end{aligned} \tag{2.1.12}$$

*where in the last addendum we recall that $E$ coincides with its Lebesgue representative.*

*Proof.* Let $T_r^s$ be the function defined in (2.1.10); then, by Gauss-Green formula and equation (2.1.11), for every $x \in E$ (and, in fact, for every $x \in \mathbb{R}^d$) we have

$$\begin{aligned} \int_{E^c\cap B(x,1)} k_r^s(|x-y|)\,\mathrm{d}y =& \int_{E^c\cap B(x,1)} \mathrm{Div}(T_r^s(y-x))\,\mathrm{d}y \\ =& \frac{1}{s}\int_{\partial^* E\cap(B(x,1)\setminus B(x,r))} \frac{(y-x)\cdot\nu_E(y)}{|x-y|^{d+s}}\,\mathrm{d}\mathcal{H}^{d-1}(y) \\ &+ \frac{d+s}{dsr^s}\int_{\partial^* E\cap B(x,r)} \frac{(y-x)\cdot\nu_E(y)}{|x-y|^{d}}\,\mathrm{d}\mathcal{H}^{d-1}(y) \\ &- \frac{1}{dr^{d+s}}\int_{\partial^* E\cap B(x,r)} (y-x)\cdot\nu_E(y)\,\mathrm{d}\mathcal{H}^{d-1}(y) \\ &- \frac{1}{s}\mathcal{H}^{d-1}(E^c\cap\partial B(x,1))\,. \end{aligned}$$

The conclusion comes by integrating with respect to $x \in E$, noticing that $\chi_{B(x,R)}(y) = \chi_{B(y,R)}(x)$ for all $x, y \in \mathbb{R}^d$, $R > 0$ and exchanging the order of integration. □

For every $s \geq 1$ we set

$$\boldsymbol{\alpha}^s := \begin{cases} \frac{d+2}{d+1} & \text{if } s = 1 \\ -\frac{1}{s-1} & \text{if } s > 1\,. \end{cases} \tag{2.1.13}$$

**Lemma 2.1.4.** *Let $E \in \mathrm{M}_f(\mathbb{R}^d)$ be a set of finite perimeter. Then, for every $0 < r < 1$, the following formula holds true*

$$\begin{aligned}
\tilde{J}^s_r(E) =&\omega_{d-1}\mathrm{Per}(E)\big(\sigma^s(r) + \boldsymbol{\alpha}^s\big) + F^s_1(E)\\
&- \frac{d+s}{dsr^s}\int_{\partial^* E} \mathrm{d}\mathcal{H}^{d-1}(y) \int_{\left(E\triangle H^-_{\nu_E(y)}(y)\right)\cap B(y,r)} \frac{|(y-x)\cdot\nu_E(y)|}{|x-y|^d}\,\mathrm{d}x\\
&+ \frac{1}{dr^{d+s}}\int_{\partial^* E} \mathrm{d}\mathcal{H}^{d-1}(y) \int_{\left(E\triangle H^-_{\nu_E(y)}(y)\right)\cap B(y,r)} |(y-x)\cdot\nu_E(y)|\,\mathrm{d}x\\
&- \frac{1}{s}\int_{\partial^* E} \mathrm{d}\mathcal{H}^{d-1}(y) \int_{\left(E\triangle H^-_{\nu_E(y)}(y)\right)\cap\left(B(y,1)\setminus B(y,r)\right)} \frac{|(y-x)\cdot\nu_E(y)|}{|x-y|^{d+s}}\,\mathrm{d}x\\
&- \frac{1}{s}\int_E \mathcal{H}^{d-1}(E^c\cap\partial B(x,1))\,\mathrm{d}x\,.
\end{aligned}$$

*Proof.* First, we notice that, using polar coordinates, for every $0 < r < 1$ and for every $\nu \in \mathbb{S}^{d-1}$, it holds

$$\int_{H^-_\nu(0)\cap B(0,r)} \frac{x}{|x|^d}\cdot\nu\,\mathrm{d}x = -\omega_{d-1} r\,, \tag{2.1.14}$$

$$\int_{H^-_\nu(0)\cap B(0,r)} (x)\cdot\nu\,\mathrm{d}x = -\frac{\omega_{d-1}}{d+1} r^{d+1}\,, \tag{2.1.15}$$

$$\int_{H^-_\nu(0)\cap(B(0,1)\setminus B(0,r))} \frac{x}{|x|^{d+s}}\cdot\nu\,\mathrm{d}x = -\omega_{d-1}\gamma^s(r)\,, \tag{2.1.16}$$

where

$$\gamma^s(r) := \begin{cases} |\log r| & \text{if } s = 1\,,\\ \frac{r^{1-s}-1}{s-1} & \text{if } s > 1\,. \end{cases}$$

Now we rewrite in a more convenient way the first three addends in the righthand side of (2.1.12). By (2.1.14), we get

$$\begin{aligned}
&\frac{d+s}{dsr^s}\int_{\partial^* E} \mathrm{d}\mathcal{H}^{d-1}(y) \int_{E\cap B(y,r)} \frac{(y-x)\cdot\nu_E(y)}{|x-y|^d}\,\mathrm{d}x\\
=&\frac{d+s}{dsr^s}\int_{\partial^* E} \mathrm{d}\mathcal{H}^{d-1}(y) \int_{\left(E\setminus H^-_{\nu_E(y)}(y)\right)\cap B(y,r)} \frac{(y-x)\cdot\nu_E(y)}{|x-y|^d}\,\mathrm{d}x\\
&- \frac{d+s}{dsr^s}\int_{\partial^* E} \mathrm{d}\mathcal{H}^{d-1}(y) \int_{\left(H^-_{\nu_E(y)}(y)\setminus E\right)\cap B(y,r)} \frac{(y-x)\cdot\nu_E(y)}{|x-y|^d}\,\mathrm{d}x\\
&+ \frac{d+s}{dsr^s}\int_{\partial^* E} \mathrm{d}\mathcal{H}^{d-1}(y) \int_{H^-_{\nu_E(y)}(y)\cap B(y,r)} \frac{(y-x)\cdot\nu_E(y)}{|x-y|^d}\,\mathrm{d}x\\
=&- \frac{d+s}{dsr^s}\int_{\partial^* E} \mathrm{d}\mathcal{H}^{d-1}(y) \int_{\left(E\triangle H^-_{\nu_E(y)}(y)\right)\cap B(y,r)} \frac{|(y-x)\cdot\nu_E(y)|}{|x-y|^d}\,\mathrm{d}x\\
&+ \omega_{d-1}\mathrm{Per}(E)\frac{d+s}{ds} r^{1-s}\,.
\end{aligned} \tag{2.1.17}$$

Analogously, by (2.1.15), we have

$$
\begin{aligned}
&-\frac{1}{dr^{d+s}}\int_{\partial^* E}\mathrm{d}\mathcal{H}^{d-1}(y)\int_{E\cap B(y,r)}(y-x)\cdot\nu_E(y)\,\mathrm{d}x\\
=&\frac{1}{dr^{d+s}}\int_{\partial^* E}\mathrm{d}\mathcal{H}^{d-1}(y)\int_{\left(E\triangle H^-_{\nu_E(y)}(y)\right)\cap B(y,r)}|(y-x)\cdot\nu_E(y)|\,\mathrm{d}x\\
&-\omega_{d-1}\mathrm{Per}(E)\frac{1}{d(d+1)}r^{1-s}.
\end{aligned}\tag{2.1.18}
$$

Furthermore, by using (2.1.16), we obtain

$$
\begin{aligned}
&\frac{1}{s}\int_{\partial^* E}\mathrm{d}\mathcal{H}^{d-1}(y)\int_{E\cap\left(B(y,1)\setminus B(y,r)\right)}\frac{(y-x)\cdot\nu_E(y)}{|x-y|^{d+s}}\,\mathrm{d}x\\
=&-\frac{1}{s}\int_{\partial^* E}\mathrm{d}\mathcal{H}^{d-1}(y)\int_{\left(E\triangle H^-_{\nu_E(y)}(y)\right)\cap\left(B(y,1)\setminus B(y,r)\right)}\frac{|(y-x)\cdot\nu_E(y)|}{|x-y|^{d+s}}\,\mathrm{d}x\\
&+\omega_{d-1}\mathrm{Per}(E)\frac{1}{s}\gamma^s(r)\,.
\end{aligned}\tag{2.1.19}
$$

We notice that
$$
\frac{1}{s}\gamma^s(r)+\frac{d+s}{ds}r^{1-s}-\frac{1}{d(d+1)}r^{1-s}=\sigma^s(r)+\boldsymbol{\alpha}^s\,;
$$
therefore, plugging (2.1.17), (2.1.18), (2.1.19) into (2.1.12), and using (2.1.9), we obtain the claim. □

We are now in a position to prove Proposition 2.1.1.

*Proof of Proposition 2.1.1.* We prove the claim under the assumption that $E$ is smooth. For $s>1$, the same proof, with $\partial E$ replaced by $\partial^* E$, works also for sets $E\in\mathrm{M}_f(\mathbb{R}^d)$ having finite perimeter. We will use the decomposition of $\tilde{J}^s_r$ in Lemma 2.1.4. Clearly the first contribution $\omega_{d-1}\mathrm{Per}(E)\big(\sigma^s(r)+\boldsymbol{\alpha}^s\big)$, once scaled by $\sigma^s(r)$ converges to $\omega_{d-1}\mathrm{Per}(E)$. Now we will prove that all the other contributions, scaled by $\sigma^s(r)$, vanish as $r\to 0^+$.

$2^{\mathrm{nd}}$ *addend:* By Remark 2.1.2, we have that
$$
\lim_{r\to 0^+}\frac{F^s_1(E)}{\sigma^s(r)}=0\,.
$$

$3^{\mathrm{rd}}$ *addend.* By the very definition of $\sigma^s(r)$ in (2.1.2) we have that $\sigma^s(r)r^{s-1}$ is uniformly bounded from below by a positive constant for every $0<r<\frac{1}{2}$, so that by the change of variable $z=\frac{x-y}{r}$, we have

$$
\begin{aligned}
&\frac{1}{\sigma^s(r)}\frac{d+s}{dsr^s}\int_{\partial E}\mathrm{d}\mathcal{H}^{d-1}(y)\int_{\left(E\triangle H^-_{\nu_E(y)}(y)\right)\cap B(y,r)}\frac{|(y-x)\cdot\nu_E(y)|}{|x-y|^d}\,\mathrm{d}x\\
\le&C(d,s)\int_{\partial E}\mathrm{d}\mathcal{H}^{d-1}(y)\int_{\left(E\triangle H^-_{\nu_E(y)}(y)\right)\cap B(y,r)}\frac{1}{r}\frac{|(y-x)\cdot\nu_E(y)|}{|x-y|^d}\,\mathrm{d}x\\
=&C(d,s)\int_{\partial E}\mathrm{d}\mathcal{H}^{d-1}(y)\int_{\left(\frac{E-y}{r}\triangle(H^-_{\nu_E(y)}(y)-\frac{y}{r})\right)\cap B(0,1)}\frac{|z\cdot\nu_E(y)|}{|z|^d}\,\mathrm{d}z\,,
\end{aligned}
$$

where the last integral vanishes as $r \to 0^+$ in virtue of the Lebesgue's Dominated Convergence Theorem since $\chi_{\frac{E-y}{r}} \to \chi_{H^-_{\nu_E(y)}(y)-\frac{y}{r}}$ in $L^1_{\mathrm{loc}}$.

$4^{\text{th}}$ *addend.* Trivially, we have

$$
\begin{aligned}
&\frac{1}{\sigma^s(r)}\frac{1}{dr^s}\int_{\partial E}\,\mathrm{d}\mathcal{H}^{d-1}(y)\int_{\left(E\triangle H^-_{\nu_E(y)}(y)\right)\cap B(y,r)}\frac{|(y-x)\cdot\nu_E(y)|}{r^d}\,\mathrm{d}x\\
\le&\frac{1}{\sigma^s(r)}\frac{1}{dr^s}\int_{\partial E}\,\mathrm{d}\mathcal{H}^{d-1}(y)\int_{\left(E\triangle H^-_{\nu_E(y)}(y)\right)\cap B(y,r)}\frac{|(y-x)\cdot\nu_E(y)|}{|x-y|^d}\,\mathrm{d}x,
\end{aligned}
$$

where the last integral vanishes as shown above.

$5^{\text{th}}$ *addend.* We first discuss the simpler case $s>1$. In such a case, for every $y\in\partial E$, using again the change of variable $z=\frac{x-y}{r}$, we have

$$
\begin{aligned}
&\frac{1}{\sigma^s(r)}\frac{1}{s}\int_{\partial E}\,\mathrm{d}\mathcal{H}^{d-1}(y)\int_{\left(E\triangle H^-_{\nu_E(y)}(y)\right)\cap\left(B(y,1)\setminus B(y,r)\right)}\frac{|(y-x)\cdot\nu_E(y)|}{|x-y|^{d+s}}\,\mathrm{d}x\\
=&\frac{r^{1-s}}{\sigma^s(r)}\frac{1}{s}\int_{\partial E}\,\mathrm{d}\mathcal{H}^{d-1}(y)\int_{\left(\frac{E-y}{r}\triangle(H^-_{\nu_E(y)}(y)-\frac{y}{r})\right)\cap\left(B(0,\frac{1}{r})\setminus B(0,1)\right)}\frac{|z\cdot\nu_E(y)|}{|z|^{d+s}}\,\mathrm{d}z\\
\le&C(d,s)\int_{\partial E}\,\mathrm{d}\mathcal{H}^{d-1}(y)\int_{\left(\frac{E-y}{r}\triangle(H^-_{\nu_E(y)}(y)-\frac{y}{r})\right)\setminus B(0,1)}\frac{1}{|z|^{d+s-1}}\,\mathrm{d}z\,,
\end{aligned}
$$

where the last double integral vanishes as $r\to 0^+$ in virtue of the Lebesgue's Dominated Convergence Theorem using that $\chi_{\frac{E-y}{r}}\to\chi_{H^-_{\nu_E(y)}(y)-\frac{y}{r}}$ in $\mathrm{L}^1_{\mathrm{loc}}$ as $r\to 0^+$ and the fact that the function $h(z):=\frac{1}{|z|^{d+s-1}}$ is in $\mathrm{L}^1(\mathbb{R}^d\setminus B(0,1))$ for $s>1$.

Notice that the reasoning above does not apply to the case $s=1$ since for $s=1$ the function $h(z)=\frac{1}{|z|^d}$ is not in $\mathrm{L}^1(\mathbb{R}^d\setminus B(0,1))$. Let now $s=1$ and recall that $\sigma^1(r)=|\log r|$. Since $E$ has smooth boundary, there exists $0<\delta<1$ such that for all $y\in\partial E$ the sets $B^-:=B(y-\delta\nu_E(y),\delta)$ and $B^+:=B(y+\delta\nu_E(y),\delta)$ satisfy

$$
B^-\subset E\setminus\partial E\,,\qquad B^+\subset E^c\setminus\partial E\,,\qquad y\in\partial B^-\cap\partial B^+\,.
$$

Therefore, we have that

$$
E\triangle H^-_{\nu_E(y)}(y)\subset(H^-_{\nu_E(y)}(y)\setminus B^-)\cup(H^+_{\nu_E(y)}(y)\setminus B^+)\,,\tag{2.1.20}
$$

where $H^\pm_\nu(y)$ are defined in (0.0.6) and (0.0.5). Fix $y\in\partial E$ and let $R_y$ be a rotation of $\mathbb{R}^d$ such that $R_y\nu_E(y)=e_d$. Moreover, we denote by $z=(z',z_d)$ the points in $\mathbb{R}^d$, so that $z'=(z_1,\dots,z_{d-1})\in\mathbb{R}^{d-1}$. Furthermore, we set $\mathbb{R}^d_+:=\{z\in\mathbb{R}^d\,:\,z_d\ge 0\}$. By (2.1.20) we have

$$
\begin{aligned}
&\frac{1}{|\log r|}\int_{\partial E}\,\mathrm{d}\mathcal{H}^{d-1}(y)\int_{\left(E\triangle H^-_{\nu_E(y)}(y)\right)\cap\left(B(y,1)\setminus B(y,r)\right)}\frac{|(y-x)\cdot\nu_E(y)|}{|x-y|^{d+1}}\,\mathrm{d}x\\
\le&\frac{1}{|\log r|}\int_{\partial E}\,\mathrm{d}\mathcal{H}^{d-1}(y)\int_{\left(H^-_{\nu_E(y)}(y)\setminus B^-\right)\cap B(y,1)}\frac{|(y-x)\cdot\nu_E(y)|}{|x-y|^{d+1}}\,\mathrm{d}x\\
&+\frac{1}{|\log r|}\int_{\partial E}\,\mathrm{d}\mathcal{H}^{d-1}(y)\int_{\left(H^+_{\nu_E(y)}(y)\setminus B^+\right)\cap B(y,1)}\frac{|(y-x)\cdot\nu_E(y)|}{|x-y|^{d+1}}\,\mathrm{d}x\\
=&\frac{2}{|\log r|}\mathrm{Per}(E)\int_{\mathbb{R}^d_+\cap\left(B(0,1)\setminus B(\delta e_d,\delta)\right)}\frac{z_d}{(|z'|^2+z_d^2)^{\frac{d+1}{2}}}\,\mathrm{d}z_d\,.
\end{aligned}\tag{2.1.21}
$$

Therefore, in order to prove that the first double integral in (2.1.21) vanishes as $r \to 0^+$, it is enough to show that

$$\int_{\mathbb{R}^d_+\cap\left(B(0,1)\setminus B(\delta e_d,\delta)\right)} \frac{z_d}{(|z'|^2+z_d^2)^{\frac{d+1}{2}}}\,\mathrm{d}z_d \le C(d,\delta)\,, \tag{2.1.22}$$

for some finite constant $C(d,\delta) > 0$. To this purpose, setting

$$A_\delta := \{z = (z', z_d) \in \mathbb{R}^d_+ \setminus B(\delta e_d, \delta)\,:\, |z'| < \delta\,, z_d < \delta\}\,,$$

we notice that

$$\mathbb{R}^d_+ \cap \left(B(0,1) \setminus B(\delta e_d,\delta)\right) \subset \left(B(0,1)\setminus B(0,\delta)\right) \cup A_\delta\,. \tag{2.1.23}$$

Moreover, there exists a constant $c_\delta$ (take, for instance, $c_\delta = \frac{1}{\delta}$) such that

$$A_\delta \subset \tilde{A}_\delta := \{z = (z', z_d) \in \mathbb{R}^d_+\,:\, |z'| < \delta\,,\ z_d < c_\delta |z'|^2\}\,. \tag{2.1.24}$$

Therefore, by (2.1.23) and (2.1.24), we get

$$\begin{aligned}
&\int_{\mathbb{R}^d_+\cap\left(B(0,1)\setminus B(\delta e_d,\delta)\right)} \frac{z_d}{(|z'|^2+z_d^2)^{\frac{d+1}{2}}}\,\mathrm{d}z_d\\
\le&\int_{\tilde{A}_\delta} \frac{z_d}{(|z'|^2+z_d^2)^{\frac{d+1}{2}}}\,\mathrm{d}z_d + \int_{B(0,1)\setminus B(0,\delta)} \frac{z_d}{(|z'|^2+z_d^2)^{\frac{d+1}{2}}}\,\mathrm{d}z_d\\
\le&\int_{B'(0,\delta)} \mathrm{d}z' \int_0^{\delta-\sqrt{\delta^2-|z'|^2}} \frac{c_\delta|z'|^2}{|z'|^{d+1}}\,\mathrm{d}z_d + \int_{B(0,1)\setminus B(0,\delta)} \frac{1}{|z|^d}\,\mathrm{d}z\\
\le&\frac{c_\delta}{\delta}\int_{B'(0,\delta)} |z'|^{3-d}\,\mathrm{d}z' + |\log\delta| =: C(d,\delta)\,,
\end{aligned}$$

i.e., (2.1.22).

$6^{\text{th}}$ *addend:* We have that

$$\frac{1}{\sigma^s(r)}\int_E \mathcal{H}^{d-1}(E^c\cap\partial B(x,1))\,\mathrm{d}x \le \frac{1}{\sigma^s(r)} d\omega_d|E| \to 0 \qquad \text{as } r\to 0^+\,.$$

Thus, the proof of Lemma 2.1.1 is concluded. □

We will show that the limit (2.1.6) is actually a Γ-limit.

**Theorem 2.1.5.** *Let $s \ge 1$ and let $\{r_n\}_{n\in\mathbb{N}} \subset (0,+\infty)$ be such that $r_n \to 0^+$ as $n \to +\infty$. The following Γ-convergence result holds true.*

*(i) (Compactness) Let $U \subset \mathbb{R}^d$ be an open bounded set and let $\{E_n\}_{n\in\mathbb{N}} \subset \mathrm{M}(\mathbb{R}^d)$ be such that $E_n \subset U$ for every $n \in \mathbb{N}$ and*

$$\tilde{J}^s_{r_n}(E_n) \le M\sigma^s(r_n) \qquad \text{for every } n\in\mathbb{N}, \tag{2.1.25}$$

*for some constant $M$ independent of $n$. Then, up to a subsequence, $\chi_{E_n} \to \chi_E$ strongly in $L^1(\mathbb{R}^d)$ for some set $E \in \mathrm{M}_f(\mathbb{R}^d)$ with $\mathrm{Per}(E) < +\infty$.*

*(ii) (Lower bound) Let $E \in \mathrm{M}_f(\mathbb{R}^d)$. For every $\{E_n\}_{n\in\mathbb{N}} \subset \mathrm{M}_f(\mathbb{R}^d)$ with $\chi_{E_n} \to \chi_E$ strongly in $L^1(\mathbb{R}^d)$ it holds*

$$\omega_{d-1}\mathrm{Per}(E) \le \liminf_{n\to+\infty} \frac{\tilde{J}^s_{r_n}(E_n)}{\sigma^s(r_n)}\,. \tag{2.1.26}$$

*(iii) (Upper bound) For every $E \in \mathrm{M}_f(\mathbb{R}^d)$ there exists $\{E_n\}_{n\in\mathbb{N}} \subset \mathrm{M}_f(\mathbb{R}^d)$ such that $\chi_{E_n} \to \chi_E$ strongly in $L^1(\mathbb{R}^d)$ and*

$$\omega_{d-1}\mathrm{Per}(E) = \lim_{n\to+\infty} \frac{\tilde{J}^s_{r_n}(E_n)}{\sigma^s(r_n)}\,.$$

The proof of Theorem 2.1.5 will be done in Sections 2.2 and 2.3 below.

To ease notation, for every $r > 0$ we set $\bar{J}^s_r(\cdot) := \frac{\tilde{J}^s_r(\cdot)}{\sigma^s(r)}$. In view of (2.1.4), for every $E \in \mathrm{M}_f(\mathbb{R}^d)$ we have

$$\begin{aligned}\bar{J}^s_r(E) =& \frac{1}{\sigma^s(r)} \int_E \int_{E^c} k^s_r(|x-y|)\,\mathrm{d}y\,\mathrm{d}x\\ =& \frac{1}{2\sigma^s(r)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} k^s_r(|x-y|)|\chi_E(x) - \chi_E(y)|\,\mathrm{d}y\,\mathrm{d}x\,.\end{aligned}$$

## 2.2 Proof of Compactness

This section is devoted to the proof of Theorem 2.1.5(i). To accomplish this task we will need some preliminary results that are collected in Subsection 2.2.1 below.

### 2.2.1 Preliminary results

We first recall the following classical result (see also [8, Theorem 3.23]).

**Theorem 2.2.1** (Compactness in BV)**.** *Let $\Omega \subset \mathbb{R}^d$ be an open set and let $\{u_n\}_{n\in\mathbb{N}} \subset \mathrm{BV}_{\mathrm{loc}}(\Omega)$ with*

$$\sup_{n\in\mathbb{N}} \Big\{ \int_A |u_n(x)|\,\mathrm{d}x + |Du_n|(A) \Big\} < +\infty \quad \forall A \subset\subset \Omega \text{ open}.$$

*Then, there exist a subsequence $\{n_k\}_{k\in\mathbb{N}}$ and a function $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$ such that $u_{n_k} \to u$ in $\mathrm{L}^1_{\mathrm{loc}}(\Omega)$ as $k \to +\infty$.*

Now we prove a non-local Poincaré-Wirtinger type inequality.

**Lemma 2.2.2.** *Let $0 < r < l$ be such that $\omega_d r^d \le \frac{l^d}{2}$. Let $\xi \in \mathbb{R}^d$ and let $u \in \mathrm{L}^1(lQ + \xi)$. Then, for every $s \ge 1$ we have*

$$\begin{aligned}&\int_{lQ+\xi} \Big| u(y) - \frac{1}{|(lQ+\xi) \setminus B(y,r)|} \int_{(lQ+\xi)\setminus B(y,r)} u(x)\,\mathrm{d}x \Big|\,\mathrm{d}y\\ &\qquad \le 2d^{\frac{d+s}{2}} l^s \int_{lQ+\xi} \int_{lQ+\xi} |u(y) - u(x)| k^s_r(|x-y|)\,\mathrm{d}y\,\mathrm{d}x\,.\end{aligned} \tag{2.2.1}$$

*Proof.* By translational invariance, it is enough to prove the claim only for $\xi = 0$. By assumption, for every $y \in lQ$ we have

$$|lQ \setminus B(y,r)| \ge l^d - \omega_d r^d \ge \frac{l^d}{2}.$$

As a consequence, we have

$$
\begin{aligned}
&\int_{lQ}\left|u(y)-\frac{1}{|lQ\setminus B(y,r)|}\int_{Q\setminus B(y,r)}u(x)\,\mathrm{d}x\right|\mathrm{d}y\\
\le&\int_{lQ}\frac{1}{|lQ\setminus B(y,r)|}\int_{lQ\setminus B(y,r)}|u(y)-u(x)|\,\mathrm{d}x\,\mathrm{d}y\\
=&\int_{lQ}\frac{1}{|lQ\setminus B(y,r)|}\int_{lQ\setminus B(y,r)}\frac{|u(y)-u(x)|}{|y-x|^{d+s}}|y-x|^{d+s}\,\mathrm{d}x\,\mathrm{d}y\\
\le&\int_{lQ}\frac{2d^{\frac{d+s}{2}}}{l^d}\int_{lQ\setminus B(y,r)}\frac{|u(y)-u(x)|}{|y-x|^{d+s}}l^{d+s}\,\mathrm{d}x\,\mathrm{d}y\\
\le&2d^{\frac{d+s}{2}}l^s\int_{lQ}\int_{lQ}|u(y)-u(x)|k_r^s(|y-x|)\,\mathrm{d}y\,\mathrm{d}x,
\end{aligned}
$$

i.e., (2.2.1). □

**Lemma 2.2.3.** *Let $0<r<l$ be such that $\omega_d r^d<\frac{l^d}{4}$. For every $\xi\in\mathbb{R}^d$ and for every $E\in \mathrm{M}_f(\mathbb{R}^d)$, it holds*

$$
\begin{aligned}
&\frac{1}{l^d}|(lQ+\xi)\setminus E||(lQ+\xi)\cap E|\\
&\qquad\le\int_{lQ+\xi}\left|\chi_E(x)-\frac{1}{|(lQ+\xi)\setminus B(x,r)|}\int_{(lQ+\xi)\setminus B(x,r)}\chi_E(y)\,\mathrm{d}y\right|\mathrm{d}x\,. \quad (2.2.2)
\end{aligned}
$$

*Proof.* We can assume without loss of generality that $\xi=0$. It is enough to prove (2.2.2) only in the case $|lQ\cap E|\ge\frac{l^d}{2}$; indeed, once proven the inequality (2.2.2) in such a case, if $|lQ\setminus E|\ge\frac{l^d}{2}$, then the set $\tilde E=lQ\setminus E$ satisfies $|lQ\cap\tilde E|\ge\frac{l^d}{2}$, and hence $\tilde E$ and, in turn, $E$ satisfy (2.2.2).

Let $|lQ\cap E|\ge\frac{l^d}{2}$; then, for every $x\in\mathbb{R}^d$ we have

$$
|(lQ\cap E)\setminus B(x,r)|\ge|lQ\cap E|-\omega_d r^d\ge|lQ\cap E|-\frac{l^d}{4}\ge\frac{|lQ\cap E|}{2}\,, \quad (2.2.3)
$$

so that

$$
\begin{aligned}
&\int_{lQ}\left|\chi_E(x)-\frac{1}{|lQ\setminus B(x,r)|}\int_{lQ\setminus B(x,r)}\chi_E(y)\,\mathrm{d}y\right|\mathrm{d}x\\
=&\int_{lQ\cap E}\left|1-\frac{|(lQ\setminus B(x,r))\cap E|}{|lQ\setminus B(x,r)|}\right|\mathrm{d}x+\int_{lQ\setminus E}\frac{|(lQ\setminus B(x,r))\cap E|}{|lQ\setminus B(x,r)|}\,\mathrm{d}x\\
\ge&\frac{1}{l^d}\left(\int_{lQ\cap E}|(lQ\setminus B(x,r))\setminus E|\,\mathrm{d}x+\int_{lQ\setminus E}|(lQ\setminus B(x,r))\cap E|\,\mathrm{d}x\right)\\
=&\frac{2}{l^d}\int_{lQ\setminus E}|(lQ\cap E)\setminus B(x,r)|\,\mathrm{d}x\\
\ge&\frac{1}{l^d}|lQ\cap E|\,|lQ\setminus E|\,,
\end{aligned}
$$

where in the last inequality we have used formula (2.2.3). □

The following result is a localized isoperimetric inequality for the non-local perimeters $\tilde J_r^s$.

**Lemma 2.2.4.** *Let $s \geq 1$ and let $\Omega \in \mathrm{M}_f(\mathbb{R}^d)$ be a bounded set with Lipschitz continuous boundary and $|\Omega| = 1$. For every $\eta \in (0,1)$ there exist a constant $C(\eta, d, s) > 0$ and $r_0 > 0$ such that for every measurable set $A \subset \Omega$ with $\eta \leq |A| \leq 1 - \eta$, it holds*

$$\int_A \int_{\Omega \setminus A} k_r^s(|x-y|)\, \mathrm{d}y\, \mathrm{d}x \geq C(\Omega, d, s, \eta)\sigma^s(r) \qquad \text{for every } r \in (0, r_0). \tag{2.2.4}$$

The proof of Lemma 2.2.4 follows along the lines of [46, Lemma 15], with slight differences due to the core radius approach adopted in this chapter. Before proving Lemma 2.2.4, we state the following result which is a consequence of [**?**, Theorem 1.4].

**Lemma 2.2.5** ([46])**.** *Let $\Omega \in \mathrm{M}_f(\mathbb{R}^d)$ be a bounded set with Lipschitz continuous boundary and $|\Omega| = 1$ and let $\phi \in C_c^\infty(B(0,1); [0, +\infty))$ be such that $\int \phi\, \mathrm{d}x = 1$ and $\phi > 0$ in $B(0, \frac{1}{2})$. For every $\delta > 0$ we set $\phi_\delta(\cdot) := \frac{1}{\delta^d}\phi(\frac{\cdot}{\delta})$. For every $\eta \in (0,1)$ there exists a constant $C(\phi, \eta) > 0$ such that for every measurable set $A \subset \Omega$ with $\eta \leq |A| \leq 1 - \eta$ and for every $\delta \in (0,1)$ it holds*

$$\frac{1}{\delta} \int_A \int_{\Omega \setminus A} \phi_\delta(|x-y|)\, \mathrm{d}y\, \mathrm{d}x \geq C(\Omega, \phi, \eta).$$

The above lemma has been stated and proven in [46, Proposition 14] in the case $d = 2$ with $\Omega = (-\frac{1}{2}, \frac{1}{2})^2$ but in fact the same proof is not affected neither by the dimension $d$ nor by the specific shape of $\Omega$. We are now in a position to prove Lemma 2.2.4.

*Proof of Lemma 2.2.4.* Fix $\eta \in (0,1)$, $r \in (0,1)$ and let $I \in \mathbb{N}$ be such that $2^{-I-1} \leq r \leq 2^{-I}$. Notice that

$$k_r^s(|z|) \geq (2^{d+s})^{\min\{i, I\}} \qquad \text{if } 0 \leq |z| \leq 2^{-i}\,, \text{ with } i \in \mathbb{N}\,. \tag{2.2.5}$$

Let $\phi$ and $\phi_\delta$ (for every $\delta > 0$) be as in Lemma 2.2.5. Now we claim that there exists a constant $C(\phi, d, s)$ such that

$$k_r^s(|z|) \geq C(\phi, d, s) \sum_{i=0}^{I} (2^s)^i \phi_{2^{-i}}(z) \quad \text{for every } z \in \mathbb{R}^d\,. \tag{2.2.6}$$

Before proving the claim we show that (2.2.6) implies (2.2.4). Indeed, first notice that

$$\frac{|\log r|}{\log 2} - 1 \leq I \leq \frac{|\log r|}{\log 2}$$

and hence

$$\sum_{i=0}^{I} (2^{s-1})^i = \begin{cases} I + 1 \geq \frac{|\log r|}{\log 2} & \text{if } s = 1\,, \\ \frac{(2^{I+1})^{s-1} - 1}{2^{s-1} - 1} \geq \frac{r^{1-s} - 1}{2^{s-1} - 1} & \text{if } s > 1\,. \end{cases}$$

so that, recalling the very definition of $\sigma^s(r)$ in (2.1.5), for $r$ small enough we have

$$\sum_{i=0}^{I} (2^{s-1})^i \geq C(d, s)\sigma^s(r)\,. \tag{2.2.7}$$

Therefore, by applying (2.2.6) and Lemma 2.2.5 with $\delta$ replaced by $2^{-i}$, we get

$$\begin{aligned} &\int_A \int_{\Omega \setminus A} k_r^s(|x-y|)\, \mathrm{d}y\, \mathrm{d}x \\ \geq &C(\Omega, \phi, d, s) \sum_{i=0}^{I} (2^{s-1})^i\, 2^i \int_A \int_{\Omega \setminus A} \phi_{2^{-i}}(x-y)\, \mathrm{d}y\, \mathrm{d}x \\ \geq &C(\Omega, \phi, d, s, \eta) \sum_{i=0}^{I} (2^{s-1})^i \geq C(\phi, d, s, \eta)\sigma^s(r)\,, \end{aligned} \tag{2.2.8}$$

where the last inequality follows from (2.2.7).

Now we prove the claim (2.2.6). Suppose first that $0 \le |z| \le 2^{-I}$. By applying (2.2.5) with $i = I$, we get

$$\begin{aligned}\sum_{i=0}^{I}(2^s)^i\phi_{2^{-i}}(z) \le& \sup\phi\sum_{i=0}^{I}(2^{d+s})^i = \sup\phi\sum_{i=0}^{I}\frac{1}{(2^{d+s})^{I-i}}(2^{d+s})^I\\ \le& \sup\phi\sum_{j=0}^{+\infty}\frac{1}{(2^{d+s})^j}(2^{d+s})^I = \frac{2^{d+s}}{2^{d+s}-1}\sup\phi\,(2^{d+s})^I\\ \le& C(\phi,d,s)k_r^s(|z|)\,.\end{aligned}\tag{2.2.9}$$

Analogously, if $2^{-\bar{\imath}-1} < |z| \le 2^{-\bar{\imath}}$ for some $\bar{\imath} = 0, 1, \dots, I-1$, using that $\phi_{2^{-i}}(z) = 0$ for every $i = \bar{\imath}+1, \dots, I$, we have

$$\begin{aligned}\sum_{i=0}^{I}(2^s)^i\phi_{2^{-i}}(z) =& \sum_{i=0}^{\bar{\imath}}(2^s)^i\phi_{2^{-i}}(z) \le \sup\phi\sum_{i=0}^{\bar{\imath}}(2^{d+s})^i\\ \le& \sup\phi\sum_{j=0}^{+\infty}\frac{1}{(2^{d+s})^j}(2^{d+s})^{\bar{\imath}} = \frac{2^{d+s}}{2^{d+s}-1}\sup\phi\,(2^{d+s})^{\bar{\imath}}\\ \le& C(\phi,d,s)k_r^s(|z|)\,,\end{aligned}\tag{2.2.10}$$

where the last inequality is a consequence of (2.2.5).

Finally, if $|z| \ge 1$ we have that $\phi_{2^{-i}}(z) = 0$ for every $i$ so that

$$\sum_{i=0}^{I}(2^s)^i\phi_{2^{-i}}(z) = 0 \le k_r^s(|z|)\,.\tag{2.2.11}$$

Therefore, by (2.2.9), (2.2.10) and (2.2.11), we deduce (2.2.6), thus concluding the proof of the lemma. □

### 2.2.2 Proof of Theorem 2.1.5(i)

We are now in a position to prove Theorem 2.1.5(i).

*Proof.* We divide the proof into three steps.

*Step 1.* Let $\alpha \in (0,1)$ and set $l_n := r_n^\alpha$ for every $n \in \mathbb{N}$. Let $\{Q_h^n\}_{h\in\mathbb{N}}$ be a disjoint family of cubes of sidelength $l_n$ such that $\bigcup_{h\in\mathbb{N}} Q_h^n = \mathbb{R}^d$. Since $|E_n| \le |U|$, there exists $H(n) \in \mathbb{N}$, such that, up to permutation of indices,

$$\begin{aligned}|Q_h^n \cap E_n| \ge \frac{l_n^d}{2} \qquad &\text{for every } h = 1, \cdots, H(n),\\ |Q_h^n \setminus E_n| > \frac{l_n^d}{2} \qquad &\text{for every } h \ge H(n)+1\,.\end{aligned}\tag{2.2.12}$$

For every $n \in \mathbb{N}$, we set

$$\tilde{E}_n := \bigcup_{h=1}^{H(n)} Q_h^n\,.$$

Let $\tilde{n} \in \mathbb{N}$ be such that for all $n > \tilde{n}$ the pair $(r_n, l_n)$ satisfies the hypothesis of Lemmas 2.2.2 and 2.2.3. We claim that there exists a constant $C(d,s) > 0$ such that

$$|\tilde{E}_n \triangle E_n| \le C(d,s) l_n^s \sigma^s(r_n) M \qquad \text{for every } n \ge \tilde{n}, \tag{2.2.13}$$

where $M$ is the constant in (2.1.25). Indeed,

$$\begin{aligned}
|E_n \triangle \tilde{E}_n| =& |\tilde{E}_n \setminus E_n| + |E_n \setminus \tilde{E}_n| \\
=& \sum_{h=1}^{H(n)} |Q_h^n \setminus E_n| + \sum_{h=H(n)+1}^{\infty} |E_n \cap Q_h^n| \\
=& 2 \sum_{h=1}^{H(n)} \frac{1}{l_n^d} |Q_h^n \setminus E_n| \frac{l_n^d}{2} + 2 \sum_{h=H(n)+1}^{\infty} \frac{1}{l_n^d} |E_n \cap Q_h^n| \frac{l_n^d}{2} \\
\le& 2 \sum_{h=1}^{+\infty} \frac{1}{l_n^d} |Q_h^n \setminus E_n| |Q_h^n \cap E_n| \\
\le& 2 \sum_{h=1}^{+\infty} \int_{Q_h^n} \left| \chi_{E_n}(x) - \frac{1}{|Q_h^n \setminus B(x, r_n)|} \int_{Q_h^n \setminus B(x,r_n)} \chi_{E_n}(y) \, \mathrm{d}y \right| \mathrm{d}x \\
\le& \sum_{h=1}^{+\infty} 8 d^{\frac{d+s}{2}} l_n^s \int_{Q_h^n \cap E_n} \int_{Q_h^n \setminus E_n} k_{r_n}^s(|x-y|) \, \mathrm{d}y \, \mathrm{d}x \\
\le& C(d,s) l_n^s \tilde{J}_{r_n}^s(E_n) \le C(d,s) l_n^s \sigma^s(r_n) M,
\end{aligned}$$

where the second inequality follows by formula (2.2.2), the third inequality is a consequence of (2.2.1), whereas the last one follows directly by (2.1.25).

*Step 2.* For every $n \in \mathbb{N}$ let $l_n$ and $\tilde{E}_n := \bigcup_{h=1}^{H(n)} Q_h^n$ be as in Step 1. We claim that there exists a constant $C(\alpha, d, s)$ such that for $n$ large enough

$$\mathrm{Per}(\tilde{E}_n) \le C(\alpha, d, s) \bar{J}_{r_n}^s(E_n) \,. \tag{2.2.14}$$

To ease notation, we omit the dependence on $n$ by setting $r := r_n$, $l := l_n$, $E := E_n$, $Q_h := Q_h^n$, $H := H(n)$, and $\tilde{E} := \tilde{E}_n$.

We define the family $\mathcal{R}$ of rectangles $R = \tilde{Q} \cup \hat{Q}$ such that $\tilde{Q}$ and $\hat{Q}$ are adjacent cubes (of the type $Q_h$ introduced above), $\tilde{Q} \subset \tilde{E}$ and $\hat{Q} \subset \tilde{E}^c$.

Notice that

$$\begin{aligned}
\mathrm{Per}(\tilde{E}) \le& 2 d l^{d-1} \sharp \mathcal{R} \,, \\
\bar{J}_r^s(E) \ge& \frac{1}{2d\, \sigma^s(r)} \sum_{R \in \mathcal{R}} \int_{R \cap E} \int_{R \setminus E} k_r^s(|x-y|) \, \mathrm{d}y \, \mathrm{d}x \,.
\end{aligned} \tag{2.2.15}$$

We recall that, by Lemma 2.2.4, for every rectangle $\bar{R}$ given by the union of two adjacent unitary cubes in $\mathbb{R}^d$, there exists $\rho_0 > 0$ such that

$$\begin{aligned}
C(d,s) := \inf \Big\{ & \frac{1}{\sigma^s(\rho)} \int_F \int_{\bar{R} \setminus F} k_\rho^s(|x-y|) \, \mathrm{d}y \, \mathrm{d}x : \\
& 0 < \rho < \rho_0, \ F \in \mathrm{M}_f(\mathbb{R}^d) \,, \ F \subset \bar{R} \,, \ \frac{1}{2} \le |F| \le \frac{3}{2} \Big\} > 0 \,.
\end{aligned} \tag{2.2.16}$$

Furthermore, by the very definition of $\sigma^s(r)$ in (2.1.5), using that $l = r^\alpha$ we have

$$\frac{\sigma^s(r)}{l^{1-s}} = \begin{cases} \dfrac{|\log(r^{1-\alpha})|}{1-\alpha} & \text{if } s = 1 \\ \dfrac{d+s}{d+1}\dfrac{r^{(1-\alpha)(1-s)}}{s-1} & \text{if } s > 1 \end{cases} = \begin{cases} \dfrac{1}{1-\alpha}\sigma^s(r^{1-\alpha}) & \text{if } s = 1 \\ \sigma^s(r^{1-\alpha}) & \text{if } s > 1\,, \end{cases}$$

so that

$$\frac{l^{1-s}}{\sigma^s(r)} \geq C(\alpha)\frac{1}{\sigma^s(r^{1-\alpha})} = C(\alpha)\frac{1}{\sigma^s(\frac{r}{l})}\,. \tag{2.2.17}$$

For every set $O \in \mathrm{M}_f(\mathbb{R}^d)$ we set $O^l := \frac{O}{l}$. By (2.2.15), (2.1.2), (2.2.17) and by applying (2.2.16) with $\bar{R} = R^l$ for every $R \in \mathcal{R}$, for $r$ small enough we obtain

$$\begin{aligned} \bar{J}^s_r(E) \geq & \frac{C(d)}{\sigma^s(r)} l^{2d} \sum_{R\in\mathcal{R}} \int_{R^l\cap E^l}\int_{R^l\setminus E^l} k^s_r(|l(x-y)|)\,\mathrm{d}y\,\mathrm{d}x \\ = & C(d)\frac{l^{1-s}}{\sigma^s(r)} l^{d-1} \sum_{R\in\mathcal{R}} \int_{R^l\cap E^l}\int_{R^l\setminus E^l} k^s_{\frac{r}{l}}(|x-y|)\,\mathrm{d}y\,\mathrm{d}x \\ \geq & C(\alpha,d) l^{d-1} \sum_{R\in\mathcal{R}} \frac{1}{\sigma^s(\frac{r}{l})} \int_{R^l\cap E^l}\int_{R^l\setminus E^l} k^s_{\frac{r}{l}}(|x-y|)\,\mathrm{d}y\,\mathrm{d}x \\ \geq & C(\alpha,d) l^{d-1} \sharp\mathcal{R}\, C(d,s) \geq C(\alpha,d,s)\mathrm{Per}(\tilde{E})\,, \end{aligned}$$

i.e., (2.2.14).

*Step 3.* Here we conclude the proof of the compactness result. We fix $\alpha \in (1 - \frac{1}{s}, 1)$ so that, by (2.2.13), $|E_n \triangle \tilde{E}_n| \to 0$ as $n \to +\infty$.

By assumption and by the very definition of $\tilde{E}_n$ in Step 1, we have that $\tilde{E}_n \subset U$ for all $n \in \mathbb{N}$. Moreover, by formula (2.2.14) and by (2.1.25) for $n$ large enough we have

$$\mathrm{Per}(\tilde{E}_n) \leq C(\alpha,d,s)\bar{J}^s_{r_n}(E_n) \leq C(\alpha,d,s)M\,.$$

It follows that the sequence $\{\chi_{\tilde{E}_n}\}_{n\in\mathbb{N}}$ satisfies the assumption of Theorem 2.2.1, and hence there exists a set $E \subset \mathbb{R}^d$ with $\mathrm{Per}(E) < +\infty$ such that, up to a subsequence, $\chi_{\tilde{E}_n} \to \chi_E$ in $\mathrm{L}^1(\mathbb{R}^d)$ as $n \to +\infty$. Since $|E_n \triangle \tilde{E}_n| \to 0$ as $n \to +\infty$ we obtain that $\chi_{E_n} \to \chi_E$ in $\mathrm{L}^1(U)$, i.e., the claim of Theorem 2.1.5(i). □

The following result follows by the proof of Theorem 2.1.5(i).

**Proposition 2.2.6.** *Let $s \geq 1$. Let $\{r_n\}_{n\in\mathbb{N}} \subset (0,+\infty)$ be such that $r_n \to 0^+$ as $n \to +\infty$. Let $\{E_n\}_{n\in\mathbb{N}} \subset \mathrm{M}_f(\mathbb{R}^d)$ be such that $\chi_{E_n} \to \chi_E$ in $\mathrm{L}^1(\mathbb{R}^d)$ as $n \to +\infty$, for some $E \in \mathrm{M}_f(\mathbb{R}^d)$. If*

$$\limsup_{n\to+\infty} \frac{\tilde{J}^s_{r_n}(E_n)}{\sigma^s(r_n)} \leq M\,,$$

*then $E$ has finite perimeter.*

*Proof.* The proof of this corollary is fully analogous to the proof of Theorem 2.1.5(i), and we adopt the same notation introduced there. Arguing as in the proof of Steps 1 and 2 we have that for $n$ large enough

$$\mathrm{Per}(\tilde{E}_n) \leq C(\alpha, d, s) \limsup_{n\to+\infty} \frac{\tilde{J}^s_{r_n}(E_n)}{\sigma^s(r_n)} \leq C(\alpha, d, s)M\,,$$

and that if $\alpha \in (1-\frac{1}{s}, 1)$ , then $|\tilde{E}_n \triangle E_n| \to 0$ as $n \to +\infty$. By assumption, this implies that

$$\chi_{\tilde{E}_n} \to \chi_E, \quad \text{in } \mathrm{L}^1(\mathbb{R}^d) \quad n \to +\infty\,,$$

and by the lower semicontinuity of the perimeter,

$$\mathrm{Per}(E) \leq \liminf_{n\to+\infty} \mathrm{Per}(\tilde{E}_n) \leq C(\alpha, d, s)M\,.$$

□

## 2.3 Proof of the Γ-limit

This section is devoted to the proofs of Theorem 2.1.5(ii) and (iii), which are the content of Subsections 2.3.1 and 2.3.2 respectively.

### 2.3.1 Proof of the lower bound

The proof of Theorem 2.1.5(ii) closely follows the strategy used in [46]. We recall that for every $\nu \in \mathbb{S}^{d-1}$ , $Q^\nu$ is a unit square centered at the origin with one face orthogonal to $\nu$. Moreover, we recall that $H^+_\nu(0) = \{x \in \mathbb{R}^d \,:\, x\cdot\nu \geq 0\}$.

The following result is the adaptation to our setting of [46, Lemma 18].

**Lemma 2.3.1.** *Let $s \geq 1$ . For every $\varepsilon > 0$, there exist $r_0, \delta_0 > 0$ such that for every $\nu \in \mathbb{S}^1$, for every $E \in \mathrm{M}_f(\mathbb{R}^d)$ with*

$$|(E \triangle H^-_\nu(0)) \cap Q^\nu| \leq \delta_0\,, \tag{2.3.1}$$

*and for every $r < r_0$ it holds*

$$\int_{Q^\nu\cap E}\int_{Q^\nu\cap E^c} k^s_r(|x-y|)\,\mathrm{d}y\,\mathrm{d}x \geq \omega_{d-1}(1-\varepsilon)\sigma^s(r)\,. \tag{2.3.2}$$

*Proof.* Up to a rotation, we can assume that $\nu = -e_d$ so that $Q^\nu \equiv Q = [-\frac{1}{2}, \frac{1}{2})^d$ and $H^-_\nu(0) =: \mathbb{R}^d_+$. Let $0 < r < 1$. We can assume without loss of generality that $E \subset Q$ . Using the change of variable $y = x + z$ we have

$$\begin{aligned}
&\int_{Q\cap E^c} \mathrm{d}x \int_{Q\cap E} k^s_r(|x-y|)\,\mathrm{d}y \\
&= \int_{Q\cap E^c} \mathrm{d}x \int_{\{z\in\mathbb{R}^d\,:\,x+z\in E\}} k^s_r(|-z|)\,\mathrm{d}z \\
&= \int_{Q\cap E^c} \mathrm{d}x \int_{\mathbb{R}^d} k^s_r(|z|)\chi_E(x+z)\,\mathrm{d}z \\
&= \int_{\mathbb{R}^d} k^s_r(|z|) \int_{\mathbb{R}^d} \chi_{E^c\cap Q}(x)\chi_E(x+z)\,\mathrm{d}x\,\mathrm{d}z \\
&= \int_{\mathbb{R}^d} k^s_r(|z|)|E^c\cap(E-z)\cap Q|\,\mathrm{d}z = \int_{\mathbb{R}^d} k^s_r(|z|)m(z)\,\mathrm{d}z\,,
\end{aligned} \tag{2.3.3}$$

where we have set $m(z) := |E^c \cap (E - z) \cap Q|$.

Let $\frac{1}{2} < \lambda < 1$ and let $z \in \mathbb{R}^d$ be such that $|z|_\infty \le \frac{1-\lambda}{2}$ and $z_d > 0$. Since $|(E - z) \cap \lambda Q| = |E \cap (\lambda Q + z)|$, by triangular inequality, we get

$$\begin{aligned}
|(E - z) \cap \lambda Q| - |E \cap \lambda Q| &= \int_{\lambda Q + z} \chi_E \, \mathrm{d}x - \int_{\lambda Q} \chi_E \, \mathrm{d}x \\
\ge& \int_{\lambda Q + z} \chi_{\mathbb{R}^d_+} \, \mathrm{d}x - \int_{\lambda Q} \chi_{\mathbb{R}^d_+} \, \mathrm{d}x - \int_{(\lambda Q + z)\triangle \lambda Q} |\chi_E - \chi_{\mathbb{R}^d_+}| \, \mathrm{d}x \\
\ge& \lambda^{d-1} z_d - \int_{U_{\lambda,z}} |\chi_E - \chi_{\mathbb{R}^d_+}| \, \mathrm{d}x \,,
\end{aligned}$$

where we have set $U_{\lambda,z} := (\lambda + |z|_\infty)Q \setminus (\lambda - |z|_\infty)Q$ and we have used that $(\lambda Q + z)\triangle \lambda Q \subset U_{\lambda,z}$. As a consequence, we deduce that

$$\begin{aligned}
m(z) =& |E^c \cap (E - z) \cap Q| \ge |E^c \cap (E - z) \cap \lambda Q| \\
\ge& |E^c \cap \lambda Q| + |(E - z) \cap \lambda Q| - |\lambda Q| \\
\ge& |E^c \cap \lambda Q| + |E \cap \lambda Q| + \lambda^{d-1} z_d - \int_{U_{\lambda,z}} |\chi_E - \chi_{\mathbb{R}^d_+}| \, \mathrm{d}x - |\lambda Q| \\
=& \lambda^{d-1} z_d - \int_{U_{\lambda,z}} |\chi_E - \chi_{\mathbb{R}^d_+}| \, \mathrm{d}x \,,
\end{aligned} \tag{2.3.4}$$

where the last equality follows by noticing that $|E \cap \lambda Q| + |E^c \cap \lambda Q| = |\lambda Q|$.

Let now $0 < \delta_0 < \frac{1}{64}$ to be chosen later on and set

$$A^+_{\sqrt{\delta_0}} := \Big\{ z \in \mathbb{R}^d \,:\, |z|_\infty \le \frac{\sqrt{\delta_0}}{2},\, z_d > 0 \Big\}.$$

We fix $z \in A^+_{\sqrt{\delta_0}}$ and we set $J := \lfloor \frac{\sqrt{\delta_0}}{|z|_\infty} \rfloor$. We set $\lambda_0 := 1 - 4\sqrt{\delta_0}$ and we cover $(\lambda_0 + 2J|z|_\infty)Q \setminus \lambda_0 Q$ with $J$ squared annuli of thickness $2|z|_\infty$, namely we set $\lambda_j := \lambda_0 + 2j|z|_\infty$ and $U_j := \lambda_j Q \setminus \lambda_{j-1} Q$ for $j = 1, \ldots, J$. Moreover, we set $\tilde\lambda_j := \lambda_0 + (2j - 1)|z|_\infty$ for every $j = 1, \ldots, J$ and we notice that $\frac{1}{2} < \tilde\lambda_j < 1$ for every $j = 1, \ldots, J$. Since $z \in A^+_{\sqrt{\delta_0}}$, we have that $|z|_\infty \le \frac{1-\tilde\lambda_J}{2} \le \frac{1-\tilde\lambda_j}{2}$ for every $j = 1, \ldots, J$. Therefore, for every $j = 1, \ldots, J$ we can apply (2.3.4) with $\lambda = \tilde\lambda_j$ in order to get

$$\begin{aligned}
m(z) \ge& z_d \tilde\lambda_j^{d-1} - \int_{U_{\tilde\lambda_j,z}} |\chi_E - \chi_{\mathbb{R}^d_+}| \, \mathrm{d}x \\
\ge& z_d \lambda_{j-1}^{d-1} - \int_{U_j} |\chi_E - \chi_{\mathbb{R}^d_+}| \, \mathrm{d}x \,,
\end{aligned} \tag{2.3.5}$$

where we have used also that $\tilde\lambda_j - |z|_\infty = \lambda_{j-1}$ and $\tilde\lambda_j + |z|_\infty = \lambda_j$ so that $U_{\tilde\lambda_j,z} = U_j$. Summing (2.3.5) over $j = 1, \ldots, J$ we get

$$Jm(z) \ge z_d \sum_{j=1}^{J} \lambda_{j-1}^{d-1} - \int_Q |\chi_E - \chi_{\mathbb{R}^d_+}| \, \mathrm{d}x \,,$$

which, dividing by $J$ and using discrete Jensen inequality (namely, convextiy), yields

$$m(z) \ge z_d \Big( \frac{1}{J} \sum_{j=1}^{J} \lambda_{j-1} \Big)^{d-1} - \frac{1}{J} \int_Q |\chi_E - \chi_{\mathbb{R}^d_+}| \, \mathrm{d}x \ge z_d \lambda_0^{d-1} - 2|z|_\infty \sqrt{\delta_0} \,, \tag{2.3.6}$$

where in the last inequality we have used (2.3.1) and the fact that $J \geq \frac{\sqrt{\delta_0}}{|z|_\infty} - 1$. Therefore, we have proven that (2.3.6) holds true whenever $z \in A^+_{\sqrt{\delta_0}}$, which combined with (2.3.3), yields

$$\int_{Q\cap E^c} \mathrm{d}x \int_{Q\cap E} k^s_r(|x-y|)\,\mathrm{d}y \geq \lambda_0^{d-1}\int_{A^+_{\sqrt{\delta_0}}} z_d k^s_r(|z|)\,\mathrm{d}z - 2\sqrt{\delta_0}\int_{A^+_{\sqrt{\delta_0}}} |z|_\infty k^s_r(|z|)\,\mathrm{d}z\,. \quad (2.3.7)$$

As for the first integral on the right hand side of (2.3.7), by using polar coordinates $z = \rho\theta$ with $\rho > 0$ and $\theta \in \mathbb{S}^{d-1}$ and using the very definition of $\sigma^s(r)$ in (2.1.5), for $\delta_0$ small enough and for all $r < \delta_0$ we have

$$\begin{aligned}\int_{A^+_{\sqrt{\delta_0}}} z_d k^s_r(|z|)\,\mathrm{d}z \geq& \int_{B(0,r)\cap\mathbb{R}^d_+} \frac{z_d}{r^{d+s}}\,\mathrm{d}z + \int_{(B(0,\delta_0)\setminus B(0,r))\cap\mathbb{R}^d_+} \frac{z_d}{|z|^{d+s}}\,\mathrm{d}z\\ =&\frac{1}{r^{d+s}}\int_0^r \rho^d\,\mathrm{d}\rho\int_{\mathbb{S}^{d-1}\cap\mathbb{R}^d_+}\theta_d\,\mathrm{d}\mathcal{H}^{d-1}(\theta)\\ &+\int_r^{\delta_0}\rho^{-s}\,\mathrm{d}\rho\int_{\mathbb{S}^{d-1}\cap\mathbb{R}^d_+}\theta_d\,\mathrm{d}\mathcal{H}^{d-1}(\theta)\\ =&\omega_{d-1}\frac{r^{1-s}}{d+1} + \omega_{d-1}\int_r^{\delta_0}\rho^{-s}\,\mathrm{d}\rho\\ \geq&\omega_{d-1}\sigma^s(r) - \omega_{d-1}C(\delta_0,s)\,,\end{aligned} \quad (2.3.8)$$

where

$$C(\delta_0,s) := \begin{cases} |\log\delta_0| & \text{if } s = 1\\ \dfrac{\delta_0^{1-s}}{s-1} & \text{if } s > 1\,.\end{cases}$$

Moreover, since $|z|_\infty \leq |z|$, it holds

$$\begin{aligned}\int_{A^+_{\sqrt{\delta_0}}} |z|_\infty k^s_r(|z|)\,\mathrm{d}z \leq& \int_{B(0,1)} |z| k^s_r(|z|)\,\mathrm{d}z\\ =&\frac{1}{r^{d+s}}\int_{B(0,r)}|z|\,\mathrm{d}z + \int_{B(0,1)\setminus B(0,r)}\frac{1}{|z|^{d+s-1}}\,\mathrm{d}z \leq \quad C(d,s)\sigma^s(r)\,,\end{aligned} \quad (2.3.9)$$

for some $C(d,s) > 0$.

Now we define the function $\eta(t) := 1 - (1-4\sqrt{t})^{d-1}$, and we notice that $\eta(t) \to 0$ as $t \to 0^+$. Therefore, by (2.3.7), (2.3.8) and (2.3.9), using that $\lambda_0^{d-1} = 1 - \eta(\delta_0)$, we deduce that

$$\begin{aligned}&\int_{Q\cap E^c}\mathrm{d}x\int_{Q\cap E} k^s_r(|x-y|)\,\mathrm{d}y\\ \geq&\omega_{d-1}\sigma^s(r)\Big(1-\eta(\delta_0) - (1-\eta(\delta_0))\frac{C(\delta_0,s)}{\sigma^s(r)} - 2\sqrt{\delta_0}\frac{C(d,s)}{\omega_{d-1}}\Big)\,,\end{aligned} \quad (2.3.10)$$

so that, choosing $\delta_0 > 0$ such that

$$\eta(\delta_0) + 2\sqrt{\delta_0}\frac{C(d,s)}{\omega_{d-1}} \leq \frac{\varepsilon}{2}$$

and $r_0 > 0$ such that (for every $0 < r < r_0$)

$$(1-\eta(\delta_0))\frac{C(\delta_0,s)}{\sigma^s(r)} \le (1-\eta(\delta_0))\frac{C(\delta_0,s)}{\sigma^s(r_0)} \le \frac{\varepsilon}{2}\,,$$

by (2.3.10) we deduce (2.3.2), thus concluding the proof of the lemma. □

We are now in a position to prove the Γ-liminf inequality in Theorem 2.1.5.

*Proof of Theorem 2.1.5(ii).* We can assume without loss of generality that

$$\bar{J}^s_{r_n}(E_n) = \frac{1}{2\sigma^s(r_n)}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k^s_{r_n}(|x-y|)|\chi_{E_n}(x)-\chi_{E_n}(y)|\,\mathrm{d}y\,\mathrm{d}x \le C\,, \tag{2.3.11}$$

for some constant $C > 0$ independent of $n$. Then, by Corollary 2.2.6 we have that $E$ has finite perimeter. For every $n \in \mathbb{N}$ let $\mu_n$ be the measure on the product space $\mathbb{R}^d \times \mathbb{R}^d$ defined by

$$\mu^s_n(A\times B) := \frac{1}{2\sigma^s(r_n)}\int_A\int_B k^s_{r_n}(|x-y|)|\chi_{E_n}(x)-\chi_{E_n}(y)|\,\mathrm{d}y\,\mathrm{d}x$$

for every $A, B \in \mathrm{M}(\mathbb{R}^d)$. Then by (2.3.11), up to a subsequence, $\mu^s_n \overset{*}{\rightharpoonup} \mu^s$ for some measure $\mu^s$. Now we show that $\mu^s$ is concentrated on the set $D := \{(x,x) : x \in \mathbb{R}^d\}$, i.e., that $\mu^s(\Omega) = 0$ if $\Omega \cap D = \emptyset$. Indeed, let $\varphi \in C_c(\mathbb{R}^d \times \mathbb{R}^d; [0,+\infty))$ be such that $\mathrm{dist}(\mathrm{supp}\,\varphi, D) = \delta$ for some $\delta > 0$; then

$$\begin{aligned}
&\int_{\mathbb{R}^d\times\mathbb{R}^d}\varphi(x,y)\,\mathrm{d}\mu^s(x,y)\\
&= \lim_{n\to+\infty}\frac{1}{2\sigma^s(r_n)}\int_{\mathbb{R}^d\times\mathbb{R}^d}\varphi(x,y)k^s_{r_n}(|x-y|)|\chi_{E_n}(x)-\chi_{E_n}(y)|\,\mathrm{d}y\,\mathrm{d}x\\
&\le \lim_{n\to+\infty}\frac{1}{2\sigma^s(r_n)}\frac{1}{\delta^{d+s}}\int_{\mathbb{R}^d\times\mathbb{R}^d}\varphi(x,y)\,\mathrm{d}y\,\mathrm{d}x = 0\,.
\end{aligned}$$

Now we define the measure $\lambda^s$ on $\mathbb{R}^d$ as $\lambda^s(A) = \mu^s(\{(x,x) : x \in A\})$ and we claim that for $\mathcal{H}^{d-1}$ - a.e. $x_0 \in \partial^* E$ it holds

$$\liminf_{l\to 0^+}\frac{\lambda^s(\overline{Q}^\nu_l(x_0))}{l^{d-1}} \ge \liminf_{l\to 0^+}\liminf_{n\to+\infty}\frac{\mu^s_n(Q^\nu_l(x_0)\times Q^\nu_l(x_0))}{l^{d-1}} \ge \omega_{d-1}\,, \tag{2.3.12}$$

where we have set $\nu = \nu_E(x_0)$ and $Q^\nu_l(x_0) = x_0 + lQ^\nu$. By (2.3.12) and Radon-Nikodym Theorem, using the lower semicontinuity of the total variation of measures with respect to the weak star convergence, we get (2.1.26).

We conclude by proving the claim (2.3.12). We preliminarily notice that the first inequality is a consequence of the upper semicontinuity of the total variation of measures on compact sets with respect to the weak star convergence. We pass to prove the second inequality in (2.3.12). For all $x_0 \in \partial^* E$, we have

$$\lim_{l\to 0^+}\int_{Q^\nu}|\chi_E(x_0+lx) - \chi_{H^-_\nu(0)}(x)|\,\mathrm{d}x = 0\,. \tag{2.3.13}$$

Fix such a $x_0 \in \partial^* E$. We will adopt a blow-up argument. Consider the sequence of sets $\{F_{n,l}\}_{n\in\mathbb{N}}$ defined by $F_{n,l} = x_0 + lE_n$. By the change of variable $x = x_0 + l\xi$ and $y = x_0 + l\eta$ we have

$$\begin{aligned}
&\frac{1}{l^{d-1}}\mu_n^s(Q_l^\nu(x_0)\times Q_l^\nu(x_0))\\
=&\frac{1}{2\sigma^s(r_n)}\int_{Q^\nu}\int_{Q^\nu} l^{d+1}k^s_{r_n}(|l\xi - l\eta|)|\chi_{F_{n,l}}(\xi)-\chi_{F_{n,l}}(\eta)|\,\mathrm{d}\xi\,\mathrm{d}\eta\\
=&\frac{l^{1-s}}{2\sigma^s(r_n)}\int_{Q^\nu}\int_{Q^\nu} k^s_{\frac{r_n}{l}}(|\xi-\eta|)|\chi_{F_{n,l}}(\xi)-\chi_{F_{n,l}}(\eta)|\,\mathrm{d}\xi\,\mathrm{d}\eta\,,
\end{aligned} \tag{2.3.14}$$

where in the last equality we have used (2.1.2). Let $0<\varepsilon<1$ and let $\delta_0, r_0 > 0$ be the constants provided by Lemma 2.3.1. In view of (2.3.13) for $l$ small enough we have

$$\int_{Q^\nu} |\chi_E(x_0+lx) - \chi_{H^-_\nu(0)}(x)|\,\mathrm{d}x \le \frac{\delta_0}{2}\,. \tag{2.3.15}$$

Fix such an $l$; then, there exists $n(l)\in\mathbb{N}$ such that for $n \ge n(l)$, it holds

$$\int_{Q^\nu} |\chi_{F_{n,l}}(x) - \chi_E(x_0+lx)|\,\mathrm{d}x = \frac{1}{l^d}\int_{Q^\nu_l(x_0)} |\chi_{E_n}(x)-\chi_E(x)|\,\mathrm{d}x \le \frac{\delta_0}{2}\,. \tag{2.3.16}$$

By (2.3.15) and (2.3.16), using triangular inequality, we obtain

$$|(F_{n,l}\triangle H^-_\nu(0))\cap Q^\nu| = \int_{Q^\nu}|\chi_{F_{n,l}} - \chi_{H^-_\nu(0)}|\,\mathrm{d}x \le \delta_0\,.$$

Therefore, by applying Lemma 2.3.1 with $k^s_r = k^s_{\frac{r_n}{l}}$ and $E = F_{n,l}$, for $n$ large enough (i.e., in such a way that $n \ge n(l)$ and $r_n < r_0 l$) we have that

$$\frac{1}{2}\int_{Q^\nu}\int_{Q^\nu} k^s_{\frac{r_n}{l}}(|\xi-\eta|)|\chi_{F_{n,l}}(\xi)-\chi_{F_{n,l}}(\eta)|\,\mathrm{d}\xi\,\mathrm{d}\eta \ge \omega_{d-1}(1-\varepsilon)\sigma^s\Big(\frac{r_n}{l}\Big)\,. \tag{2.3.17}$$

Now, by the very definition of $\sigma^s$ in (2.1.5), we have that

$$\frac{l^{1-s}}{\sigma^s(r_n)}\sigma^s\Big(\frac{r_n}{l}\Big) = \begin{cases} \frac{\log l + |\log r_n|}{|\log r_n|} & \text{if } s=1\\ 1 & \text{if } s>1\,,\end{cases}$$

so that, in view of (2.3.14) and (2.3.17), we deduce that for every $0<\varepsilon<1$ and for every $l$ small enough (depending on $\varepsilon$), it holds

$$\liminf_{n\to+\infty}\frac{1}{l^{d-1}}\mu_n^s(Q_l^\nu(x_0)\times Q_l^\nu(x_0)) \ge \omega_{d-1}(1-\varepsilon)\,,$$

whence the second inequality in claim (2.3.12) follows by the arbitrariness of $\varepsilon$. □

### 2.3.2 Proof of the upper bound

The Γ-limsup inequality will be a consequence of Proposition 2.1.1 and of standard density results for sets of finite perimeter.

We first recall the following fundamental approximation theorem (see, for instance, [60, Theorem 13.8]).

**Theorem 2.3.2** (Approximation of set with finite perimeter by smooth sets)**.** *A set* $E \in \mathrm{M}_f(\mathbb{R}^d)$ *has finite perimeter if and only if there exists a sequence* $\{F_k\}_{k\in\mathbb{N}} \subset \mathrm{M}_f(\mathbb{R}^d)$ *of open bounded sets with smooth boundary, such that*

$$\begin{aligned} &\chi_{F_k} \to \chi_E \qquad \textit{(strongly) in } \mathrm{L}^1(\mathbb{R}^d) \textit{ as } k \to +\infty, \\ &\mathrm{Per}(F_k) \to \mathrm{Per}(E) \quad \textit{as } k \to +\infty. \end{aligned} \tag{2.3.18}$$

*Proof of Theorem 2.1.5(iii).* Let $E \in \mathrm{M}_f(\mathbb{R}^d)$ be a set with finite perimeter. By Theorem 2.3.2, there exists a sequence $\{F_k\}_{k\in\mathbb{N}}$ of open bounded sets with smooth boundary satisfying (2.3.18) . In view of Proposition 2.1.1 we have that

$$\lim_{n\to+\infty} \frac{\tilde{J}^s_{r_n}(F_k)}{\sigma^s(r_n)} = \omega_{d-1}\mathrm{Per}(F_k) \qquad \text{for every } k \in \mathbb{N}\,.$$

Therefore, by a standard diagonal argument there exists a sequence $\{E_n\}_{n\in\mathbb{N}}$ with $E_n = F_{k(n)}$ for every $n \in \mathbb{N}$ satisfying the desired properties. □

### 2.3.3 Characterization of sets of finite perimeter

As a byproduct of our Γ-convergence analysis, we prove that a set $E \in \mathrm{M}_f(\mathbb{R}^d)$ has finite perimeter if and only if for all $s \geq 1$

$$\limsup_{r\to0^+} \frac{\tilde{J}^s_r(E)}{\sigma^s(r)} < +\infty.$$

We recall the following classical theorem.

**Theorem 2.3.3** (Characterization via difference quotients)**.** *Let* $E \in \mathrm{M}_f(\mathbb{R}^d)$ *. Then* $E$ *has finite perimeter if and only if there exists* $C > 0$ *such that*

$$\int_{\mathbb{R}^d} |\chi_E(x+z) - \chi_E(x)|\,\mathrm{d}x \leq C|z| \qquad \textit{for every } z \in \mathbb{R}^d.$$

*Specifically, it is possible to choose* $C = \mathrm{Per}(E)$*.*

**Theorem 2.3.4.** *Let* $E \in \mathrm{M}_f(\mathbb{R}^d)$ *. The following statements hold true.*

*(i) If* $\limsup_{r\to0^+} \frac{\tilde{J}^s_r(E)}{\sigma^s(r)} < +\infty$ *for some* $s \geq 1$ *, then* $E$ *is a set of finite perimeter.*

*(ii) If* $E$ *is a set of finite perimeter then* $\limsup_{r\to0^+} \frac{\tilde{J}^s_r(E)}{\sigma^s(r)} < +\infty$ *for every* $s \geq 1$ *. More precisely,*

$$\omega_{d-1}\mathrm{Per}(E) \leq \liminf_{r\to0^+} \frac{\tilde{J}^s_r(E)}{\sigma^s(r)} \leq \limsup_{r\to0^+} \frac{\tilde{J}^s_r(E)}{\sigma^s(r)} \leq M(s,d)\mathrm{Per}(E), \tag{2.3.19}$$

*where*

$$M(s,d) = \begin{cases} \frac{d\omega_d}{2} & \textit{if } s = 1 \\ \omega_{d-1} & \textit{if } s > 1\,. \end{cases}$$

*In particular, for* $s > 1$ *we have that*

$$\lim_{r\to0^+} \frac{\tilde{J}^s_r(E)}{\sigma^s(r)} = \omega_{d-1}\mathrm{Per}(E)\,. \tag{2.3.20}$$

*Remark* 2.3.5. We notice that in the case $s = 1$ the constant $M(1, d) = \frac{d\omega_d}{2} > \omega_{d-1}$ , so that the existence of the limit (2.3.20) is not proven in such a case.

*Proof Theorem 2.3.4:* We notice that (i) is an immediate consequence of Proposition 2.2.6 taking $E_n \equiv E$ for every $n \in \mathbb{N}$. We prove (ii). The Γ-liminf inequality Theorem 2.1.5(ii) implies the first inequality in (2.3.19). Being the second inequality obvious we pass to the proof of the last one. If $s > 1$ then, by Proposition 2.1.1, we have

$$\lim_{r\to 0^+} \frac{\tilde{J}^s_r(E)}{\sigma^s(r)} = \omega_{d-1}\mathrm{Per}(E). \tag{2.3.21}$$

Let now $s = 1$. Let $G^1_r$ be the functional defined in (2.1.8); by Theorem 2.3.3 we obtain

$$\begin{aligned}
\frac{G^1_r(E)}{\sigma^1(r)} =&\frac{1}{|\log r|}\int_E\int_{E^c\cap B(x,1)} k^1_r(|x-y|)\,\mathrm{d}y\,\mathrm{d}x\\
=&\frac{1}{2|\log r|}\int_{\mathbb{R}^d}\int_{B(x,1)} |\chi_E(x)-\chi_E(y)|k^1_r(|x-y|)\,\mathrm{d}y\,\mathrm{d}x\\
=&\frac{1}{2|\log r|}\int_{B(0,1)} k^1_r(|h|)\int_{\mathbb{R}^d}|\chi_E(x+h)-\chi_E(x)|\,\mathrm{d}x\,\mathrm{d}h\\
\le&\frac{1}{2|\log r|}\mathrm{Per}(E)\int_{B(0,1)}|h|k^1_r(|h|)\,\mathrm{d}h\\
=&\frac{d\omega_d}{2}\mathrm{Per}(E)\Big(1+\frac{1}{(d+1)|\log r|}\Big).
\end{aligned} \tag{2.3.22}$$

Moreover, by Remark 2.1.2 we have that

$$\lim_{r\to 0^+}\frac{F^1_1(E)}{\sigma^1(r)} = 0\,, \tag{2.3.23}$$

where $F^1_1$ is the functional defined in formula (2.1.7).

Therefore by formulas (2.1.9), (2.3.22), and (2.3.23) we have

$$\limsup_{r\to 0^+}\frac{\tilde{J}^1_r(E)}{\sigma^1(r)} = \limsup_{r\to 0^+}\frac{G^1_r(E)}{\sigma^1(r)} + \lim_{r\to 0^+}\frac{F^s_r(E)}{\sigma^1(r)} \le \frac{d\omega_d}{2}\mathrm{Per}(E)\,. \tag{2.3.24}$$

thus concluding the proof of (ii). By (2.3.21) and (2.3.24) we conclude the proof of (ii). □

## 2.4 Convergence of curvatures and mean curvature flows

In this section we study the behavior of the non-local curvatures corresponding to the functionals $\tilde{J}^s_r$ and of the corresponding geometric flows. Using the approach in [28, 26], it is enough to focus on smooth enough sets. To this purpose, we introduce the class $\mathfrak{C}$ as the class of the subsets of $\mathbb{R}^d$, which are closures of open sets with compact $C^2$ boundary. Moreover, we define a notion of convergence in $\mathfrak{C}$ as follows. Let $\{E_n\}_{n\in\mathbb{N}} \subset \mathfrak{C}$ we say that $E_n \to E$ in $\mathfrak{C}$ as $n \to +\infty$, for some $E \in \mathfrak{C}$, if there exists a sequence of diffeomorphisms $\{\Phi_n\}_{n\in\mathbb{N}}$ converging to the identity in $C^2$ as $n \to +\infty$, such that $\Phi_n(E) = E_n$ for every $n \in \mathbb{N}$. In the following, we will extend this notion of convergence (in the obvious way) to families of sets $\{E_\rho\}_{\rho\in(0,1)} \subset \mathfrak{C}$ as the parameter $\rho \to 0^+$ .

Notice that if $E \in \mathfrak{C}$, then either $E$ or $E^c$ is compact. Therefore, in order to define the supercritical perimeters and the corresponding curvatures on the whole $\mathfrak{C}$, it is convenient to set $\tilde{J}^s_r(E) := \tilde{J}^s_r(E^c)$ for every set $E \in \mathrm{M}(\mathbb{R}^d)$ with $E^c \in \mathrm{M}_f(\mathbb{R}^d)$.

### 2.4.1 Non-local $k_r^s$-curvatures

Let $s \geq 1$, $r > 0$ and $E \in \mathfrak{C}$. For every $x \in \partial E$ we define the *$k_r^s$-curvature of $E$ at $x$* as

$$\mathcal{K}_r^s(x, E) := \int_{\mathbb{R}^d} (\chi_{E^c}(y) - \chi_E(y)) k_r^s(|x - y|) \, \mathrm{d}y. \tag{2.4.1}$$

Although this fact may be immediate for the experts, we show that $\mathcal{K}_r^s$ is the first variation of the functional $\tilde{J}_r^s$ in the sense specified by the following proposition.

**Proposition 2.4.1** (First variation)**.** *Let $s \geq 1$, $r > 0$, and $E \in \mathfrak{C}$. Let $\Phi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ be a smooth function, and let $\{\Phi_t\}_{t\in\mathbb{R}}$ be defined by $\Phi_t(\cdot) := \Phi(t, \cdot)$ for every $t \in \mathbb{R}$. Assume that $\{\Phi_t\}_{t\in\mathbb{R}}$ is a family of diffeomorphisms with $\Phi_0 = \mathrm{Id}$ and that there exists an open bounded set $A \subset \mathbb{R}^d$ such that*

$$\{x \in \mathbb{R}^d : \; x \neq \Phi_t(x)\} \subset A \quad \text{for all } t \in \mathbb{R}. \tag{2.4.2}$$

*Setting $E_t := \Phi_t(E)$ and $\Psi(\cdot) := \frac{\partial}{\partial t}\Phi_t(\cdot)\big|_{t=0}$, we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{J}_r^s(E_t)\bigg|_{t=0} = \int_{\partial E} \mathcal{K}_r^s(x, E)\Psi(x) \cdot \nu_E(x) \, \mathrm{d}\mathcal{H}^{d-1}(x). \tag{2.4.3}$$

*Proof.* By Taylor expansion for every $x \in \mathbb{R}^d$ we have that $\Phi_t(x) = x + t\Psi(x) + \mathrm{o}(t)$. Therefore the Jacobian $\mathcal{J}\Phi_t$ of $\Phi_t$ is equal to

$$\mathcal{J}\Phi_t(x) := \sqrt{\det(\nabla\Phi_t(x)\, \nabla\Phi_t(x)^*)} = 1 + t\mathrm{Div}(\Psi(x)) + \mathrm{o}(t),$$

where, for every $A \in \mathbb{R}^{m\times k}$ $(m, k \in \mathbb{N})$, the symbol $A^*$ denotes the transpose of the matrix $A$. By change of variable, it follows that

$$\begin{aligned} \tilde{J}_r^s(E_t) &= \int_{\Phi_t(E)} \int_{\Phi_t(E^c)} k_r^s(|x - y|) \, \mathrm{d}y \, \mathrm{d}x \\ &= \int_E \int_{E^c} k_r^s(|\Phi_t(x) - \Phi_t(y)|) \mathcal{J}\Phi_t(x) \mathcal{J}\Phi_t(y) \, \mathrm{d}y \, \mathrm{d}x \\ &= \int_E \int_{E^c} k_r^s(|\Phi_t(x) - \Phi_t(y)|) \, \mathrm{d}y \, \mathrm{d}x \\ &\quad + t \int_E \int_{E^c} k_r^s(|\Phi_t(x) - \Phi_t(y)|) \Big(\mathrm{Div}\Psi(x) + \mathrm{Div}\Psi(y)\Big) \, \mathrm{d}y \, \mathrm{d}x \\ &\quad + \mathrm{o}(t) \int_E \int_{E^c} k_r^s(|\Phi_t(x) - \Phi_t(y)|) \, \mathrm{d}y \, \mathrm{d}x. \end{aligned} \tag{2.4.4}$$

Let $(k_r^s)' : (0, +\infty) \to \mathbb{R}$ be the weak derivative of $k_r^s : (0, +\infty) \to \mathbb{R}$, that is equal a.e. to

$$(k_r^s)'(h) := \begin{cases} 0 & \text{for } 0 < h < r, \\ -(d + s)\dfrac{1}{h^{d+s+1}} & \text{for } h > r. \end{cases}$$

Notice that $k_r^s \in W^{1,1}(\mathbb{R})$. We set

$$\begin{aligned} K(t) := \int_E \int_{E^c} \Big( & k_r^s(|\Phi_t(x) - \Phi_t(y)|) - k_r^s(|x - y|) \\ & - t(k_r^s)'(|x - y|)\frac{x - y}{|x - y|} \cdot (\Psi(x) - \Psi(y))\Big) \, \mathrm{d}y \, \mathrm{d}x \end{aligned}$$

and we claim that

$$\lim_{t\to 0}\frac{K(t)}{t}=0\,. \tag{2.4.5}$$

By the fundamental theorem of calculus, we have

$$\begin{aligned}&\int_E\int_{E^c}\Big(k_r^s(|\Phi_t(x)-\Phi_t(y)|)-k_r^s(|x-y|)\Big)\,\mathrm{d}y\,\mathrm{d}x\\&=\int_0^t\mathrm{d}\tau\bigg[\int_E\int_{E^c}(k_r^s)'(|\Phi_\tau(x)-\Phi_\tau(y)|)\frac{\Phi_\tau(x)-\Phi_\tau(y)}{|\Phi_\tau(x)-\Phi_\tau(y)|}\cdot\Big(\frac{\partial\Phi_\tau}{\partial\tau}(x)-\frac{\partial\Phi_\tau}{\partial\tau}(y)\Big)\,\mathrm{d}y\,\mathrm{d}x\bigg]\,,\end{aligned}$$

so that

$$\begin{aligned}\left|\frac{K(t)}{t}\right|=&\left|\frac{1}{t}\int_0^t\bigg[\int_E\int_{E^c}(k_r^s)'(|\Phi_\tau(x)-\Phi_\tau(y)|)\frac{\Phi_\tau(x)-\Phi_\tau(y)}{|\Phi_\tau(x)-\Phi_\tau(y)|}\cdot\Big(\frac{\partial\Phi_\tau}{\partial\tau}(x)-\frac{\partial\Phi_\tau}{\partial\tau}(y)\Big)\right.\\&\left.-(k_r^s)'(|x-y|)\frac{x-y}{|x-y|}\cdot(\Psi(x)-\Psi(y))\,\mathrm{d}y\,\mathrm{d}x\bigg]\,\mathrm{d}\tau\right|\\\le&\frac{1}{|t|}\int_0^{|t|}\bigg|\int_E\int_{E^c}(k_r^s)'(|\Phi_\tau(x)-\Phi_\tau(y)|)\frac{\Phi_\tau(x)-\Phi_\tau(y)}{|\Phi_\tau(x)-\Phi_\tau(y)|}\cdot\Big(\frac{\partial\Phi_\tau}{\partial\tau}(x)-\frac{\partial\Phi_\tau}{\partial\tau}(y)\Big)\\&-(k_r^s)'(|x-y|)\frac{x-y}{|x-y|}\cdot(\Psi(x)-\Psi(y))\,\mathrm{d}y\,\mathrm{d}x\bigg|\,\mathrm{d}\tau\\=:&\frac{1}{|t|}\int_0^{|t|}\bigg|\int_E\int_{E^c}(f_\tau(x,y)-f_0(x,y))\,\mathrm{d}y\,\mathrm{d}x\bigg|\,\mathrm{d}\tau\,,\end{aligned}$$

where in the last line for every $\tau\in\mathbb{R}$ and for every $(x,y)\in\mathbb{R}^d\times\mathbb{R}^d$ we have set

$$f_\tau(x,y):=(k_r^s)'(|\Phi_\tau(x)-\Phi_\tau(y)|)\frac{\Phi_\tau(x)-\Phi_\tau(y)}{|\Phi_\tau(x)-\Phi_\tau(y)|}\cdot\Big(\frac{\partial\Phi_\tau}{\partial\tau}(x)-\frac{\partial\Phi_\tau}{\partial\tau}(y)\Big)\,.$$

Notice that (2.4.5) follows if we show that

$$\int_E\int_{E^c}f_\tau(x,y)\,\mathrm{d}y\,\mathrm{d}x\to\int_E\int_{E^c}f_0(x,y)\,\mathrm{d}y\,\mathrm{d}x\qquad\text{as }\tau\to0\,. \tag{2.4.6}$$

By change of variable, we get

$$\begin{aligned}&\int_E\int_{E^c}f_\tau(x,y)\,\mathrm{d}y\,\mathrm{d}x\\&\qquad=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}f_0(x,y)\mathcal{J}\Phi_\tau^{-1}(x)\mathcal{J}\Phi_\tau^{-1}(y)\chi_{E\times E^c}(\Phi_\tau^{-1}(x),\Phi_\tau^{-1}(y))\,\mathrm{d}y\,\mathrm{d}x\,.\end{aligned} \tag{2.4.7}$$

Setting $C(\mathcal{J}):=\sup_{\tau\in(0,1)}\ \|\mathcal{J}\Phi_\tau^{-1}\|_{L^\infty}$ , by (2.4.2), we have that

$$\begin{aligned}&|f_0(x,y)|\mathcal{J}\Phi_\tau^{-1}(x)\mathcal{J}\Phi_\tau^{-1}(y)\chi_{E\times E^c}(\Phi_\tau^{-1}(x),\Phi_\tau^{-1}(y))\\&\qquad\le[C(\mathcal{J})]^2|f_0(x,y)|[\chi_{E\cap A}(\Phi_\tau^{-1}(x))+\chi_{E\cap A^c}(x)][\chi_{E^c\cap A}(\Phi_\tau^{-1}(y))+\chi_{E^c\cap A^c}(y)]\\&\qquad\qquad\le[C(\mathcal{J})]^2|f_0(x,y)|[\chi_A(x)+\chi_E(x)][\chi_A(y)+\chi_{E^c}(y)]\,,\end{aligned}$$

where the right hand side term is clearly in $\mathrm{L}^1(\mathbb{R}^{2d})$ . This fact together with (2.4.7) and

$$\mathcal{J}\Phi_\tau^{-1}(x)\mathcal{J}\Phi_\tau^{-1}(y)\chi_{E\times E^c}(\Phi_\tau^{-1}(x),\Phi_\tau^{-1}(y))\to\chi_{E\times E^c}(x,y)\quad\text{a.e. as }\tau\to0\,,$$

yields by Lebesgue Dominated Convergence Theorem, (2.4.6) and, in turn, (2.4.5). By (2.4.4) and (2.4.5), and using the divergence theorem, we obtain that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\tilde{J}^s_r(E_t)\Big|_{t=0} &= \int_E\int_{E^c}(k^s_r)'(|x-y|)\frac{x-y}{|x-y|}\cdot(\Psi(x)-\Psi(y))\,\mathrm{d}y\,\mathrm{d}x \\
&\quad+\int_E\int_{E^c}k^s_r(|x-y|)(\mathrm{Div}\Psi(x)+\mathrm{Div}\Psi(y))\,\mathrm{d}y\,\mathrm{d}x \\
&=\int_E\int_{E^c}(k^s_r)'(|x-y|)\frac{x-y}{|x-y|}\cdot(\Psi(x)-\Psi(y))\,\mathrm{d}y\,\mathrm{d}x \\
&\quad+\int_{E^c}\Big[-\int_E(k^s_r)'(|x-y|)\,\Psi(x)\cdot\frac{x-y}{|x-y|}\,\mathrm{d}x \\
&\quad+\int_{\partial E}k^s_r(|x-y|)\,\Psi(x)\cdot\nu_E(x)\,\mathrm{d}\mathcal{H}^{d-1}(x)\Big]\,\mathrm{d}y \\
&\quad+\int_E\Big[\int_{E^c}(k^s_r)'(|x-y|)\,\Psi(y)\cdot\frac{x-y}{|x-y|}\,\mathrm{d}y \\
&\quad-\int_{\partial E}k^s_r(|x-y|)\,\Psi(y)\cdot\nu_E(y)\,\mathrm{d}\mathcal{H}^{d-1}(y)\Big]\,\mathrm{d}x \\
&=\int_{\partial E}\Psi(x)\cdot\nu_E(x)\int_{\mathbb{R}^d}(\chi_{E^c}(y)-\chi_E(y))k^s_r(|x-y|)\,\mathrm{d}y\,\mathrm{d}\mathcal{H}^{d-1}(x) \\
&=\int_{\partial E}\mathcal{K}^s_r(x,E)\Psi(x)\cdot\nu_E(x)\,\mathrm{d}\mathcal{H}^{d-1}(x),
\end{aligned}$$

whence (2.4.3) follows. □

In Proposition 2.4.2 we prove some qualitative properties of the curvatures $\mathcal{K}^s_r$ defined in (2.4.1), which imply in particular that $\mathcal{K}^s_r$ are non-local curvatures in the sense of [28, 26].

**Proposition 2.4.2.** *For every $s\ge 1$, $0<r<1$ the functionals $\mathcal{K}^s_r$ defined in* (2.4.1) *satisfy the following properties:*

*(M) Monotonicity: If $E,F\in\mathfrak{C}$ with $E\subseteq F$, and if $x\in\partial F\cap\partial E$, then $\mathcal{K}^s_r(x,F)\le\mathcal{K}^s_r(x,E)$;*

*(T) Translational invariance: for any $E\in\mathfrak{C}$, $x\in\partial E$, $y\in\mathbb{R}^d$, $\mathcal{K}^s_r(x,E)=\mathcal{K}^s_r(x+y,E+y)$;*

*(S) Symmetry: For all $E\in\mathfrak{C}$ and for every $x\in\partial E$ it holds*

$$\mathcal{K}^s_r(x,E)=-\mathcal{K}^s_r(x,\mathbb{R}^d\setminus\mathring{E}),$$

*where $\mathring{E}$ denotes the interior of $E$.*

*(B) Lower bound on the curvature of the balls:*

$$\mathcal{K}^s_r(x,\overline{B}(0,\rho))\ge 0\qquad\text{for all } x\in\partial B(0,\rho)\,,\ \rho>0\,; \tag{2.4.8}$$

*(UC) Uniform continuity: There exists a modulus of continuity $\omega_r$ such that the following holds. For every $E\in\mathfrak{C}$, $x\in\partial E$, and for every diffeomorphism $\Phi:\mathbb{R}^d\to\mathbb{R}^d$ of class $C^2$, with $\Phi=\mathrm{Id}$ in $\mathbb{R}^d\setminus B(0,1)$, we have*

$$|\mathcal{K}^s_r(x,E)-\mathcal{K}^s_r(\Phi(x),\Phi(E))|\le\omega_r(\|\Phi-\mathrm{Id}\|_{C^2})\,.$$

*Proof.* We prove separately the properties above.

*Property* (M): Let $E, F \in \mathfrak{C}$ such that $E \subseteq F$, then $-\chi_F \leq -\chi_E$ and $\chi_{F^c} \leq \chi_{E^c}$. Therefore for all $x \in \partial E \cap \partial F$, we have

$$\begin{aligned}\mathcal{K}_r^s(x, F) &= \int_{\mathbb{R}^d} (\chi_{F^c}(y) - \chi_F(y)) k_r^s(|x-y|)\,\mathrm{d}y \\ &\leq \int_{\mathbb{R}^d} (\chi_{E^c}(y) - \chi_E(y)) k_r^s(|x-y|)\,\mathrm{d}y = \mathcal{K}_r^s(x, E)\,.\end{aligned}$$

*Property* (T): Let $E \in \mathfrak{C}$, $x \in \partial E$ and $y \in \mathbb{R}^d$. By the change of variable $\zeta = \eta - y$, we obtain

$$\begin{aligned}\mathcal{K}_r^s(x+y, E+y) &= \int_{\mathbb{R}^d} (\chi_{E^c+y}(\eta) - \chi_{E+y}(\eta)) k_r^s(|x+y-\eta|)\,\mathrm{d}\eta \\ &= \int_{\mathbb{R}^d} (\chi_{E^c}(\zeta) - \chi_E(\zeta)) k_r^s(|x-\zeta|)\,\mathrm{d}\zeta = \mathcal{K}_r^s(x, E)\,.\end{aligned}$$

*Property* (S): Let $E \in \mathfrak{C}$ and $x \in \partial E$, then we have

$$\begin{aligned}\mathcal{K}_r^s(x, E) &= \int_{\mathbb{R}^d} (\chi_{E^c}(y) - \chi_E(y)) k_r^s(|y-x|)\,\mathrm{d}y \\ &= -\int_{\mathbb{R}^d} (\chi_E(y) - \chi_{E^c}(y)) k_r^s(|x-y|)\,\mathrm{d}y = -\mathcal{K}_r^s(x, \mathbb{R}^d \setminus \mathring{E}).\end{aligned}$$

*Property* (B): Let $\rho > 0$ and $\bar{x} \in \partial B(0,\rho)$. Since $B(2\bar{x}, \rho) \subset B^c(0,\rho) = \mathbb{R}^d \setminus \overline{B}(0,\rho)$, we get

$$\begin{aligned}\mathcal{K}_r^s(\bar{x}, \overline{B}(0,\rho)) &= \int_{\mathbb{R}^d} (\chi_{B^c(0,\rho)}(y) - \chi_{B(0,\rho)}(y)) k_r^s(|\bar{x}-y|)\,\mathrm{d}y \\ &\geq \int_{\mathbb{R}^d} (\chi_{B(2\bar{x},\rho)}(y) - \chi_{B(0,\rho)}(y)) k_r^s(|\bar{x}-y|)\,\mathrm{d}y = 0,\end{aligned} \tag{2.4.9}$$

where in the last equality we have used the change of variable $z = 2\bar{x} - y$ and the radial symmetry of $k_r^s$ to deduce that

$$\int_{\mathbb{R}^d} \chi_{B(2\bar{x},\rho)}(y) k_r^s(|\bar{x}-y|)\,\mathrm{d}y = \int_{\mathbb{R}^d} \chi_{B(0,\rho)}(z) k_r^s(|\bar{x}-z|)\,\mathrm{d}z\,.$$

Hence, by formula (2.4.9), (2.4.8) follows.

*Property* (UC): Let $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ be a diffeomorphism of class $C^2$, with $\Phi(y) = y$ for all $|y - x| \geq 1$. We set $\mathcal{E} := \Phi(E)$. Let moreover $\theta_{k_r^s} : [0, +\infty) \to \mathbb{R}$ be the function defined by $\theta_{k_r^s}(\eta) := \int_{B(0,\eta)} k_r^s(|z|)\,\mathrm{d}z$. Fix $\varepsilon > 0$ and let $\eta_\varepsilon > 0$ be small enough such that

$$2\theta_{k_r^s}(\eta_\varepsilon),\ \theta_{k_r^s}(2\eta_\varepsilon) \leq \frac{\varepsilon}{3}. \tag{2.4.10}$$

Notice that

$$\left|\int_{B(x,\eta_\varepsilon)} (\chi_{E^c}(y) - \chi_E(y)) k_r^s(|x-y|)\,\mathrm{d}y\right| \leq \theta_{k_r^s}(\eta_\varepsilon)\,, \tag{2.4.11}$$

$$\left|\int_{B(\Phi(x),\eta_\varepsilon)} (\chi_{\mathcal{E}^c}(y) - \chi_{\mathcal{E}}(y)) k_r^s(|\Phi(x)-y|)\,\mathrm{d}y\right| \leq \theta_{k_r^s}(\eta_\varepsilon)\,, \tag{2.4.12}$$

$$\left|\int_{B(\Phi(x),2\eta_\varepsilon)} (\chi_{\mathcal{E}^c}(y) - \chi_{\mathcal{E}}(y)) k_r^s(|\Phi(x)-y|)\,\mathrm{d}y\right| \leq \theta_{k_r^s}(2\eta_\varepsilon)\,. \tag{2.4.13}$$

By (2.4.11), (2.4.12), and (2.4.10), using triangular inequality, we have

$$\begin{aligned}
&|\mathcal{K}_r^s(x,E)-\mathcal{K}_r^s(\Phi(x),\Phi(E))| \\
\leq&\frac{\varepsilon}{3}+\Big|\int_{B^c(x,\eta_\varepsilon)}(\chi_{E^c}(y)-\chi_E(y))k_r^s(|x-y|)\,\mathrm{d}y \\
&-\int_{B^c(\Phi(x),\eta_\varepsilon)}(\chi_{\mathcal{E}^c}(y)-\chi_{\mathcal{E}}(y))k_r^s(|\Phi(x)-y|)\,\mathrm{d}y\Big| \\
\leq&\frac{\varepsilon}{3}+\Big|\int_{B^c(x,\eta_\varepsilon)}(\chi_{E^c}(y)-\chi_E(y))k_r^s(|x-y|)\,\mathrm{d}y \\
&-\int_{\Phi(B^c(x,\eta_\varepsilon))}(\chi_{\mathcal{E}^c}(y)-\chi_{\mathcal{E}}(y))k_r^s(|\Phi(x)-y|)\,\mathrm{d}y\Big| \\
&+\Big|\int_{\Phi(B^c(x,\eta_\varepsilon))}(\chi_{\mathcal{E}^c}(y)-\chi_{\mathcal{E}}(y))k_r^s(|\Phi(x)-y|)\,\mathrm{d}y \\
&-\int_{B^c(\Phi(x),\eta_\varepsilon)}(\chi_{\mathcal{E}^c}(y)-\chi_{\mathcal{E}}(y))k_r^s(|\Phi(x)-y|)\,\mathrm{d}y\Big|.
\end{aligned} \tag{2.4.14}$$

By the change of variable $z=\Phi(y)$ and using that $\Phi(y)=y$ if $|y-x|\geq 1$, we have

$$\begin{aligned}
&\Big|\int_{B^c(x,\eta_\varepsilon)}(\chi_{E^c}(y)-\chi_E(y))k_r^s(|x-y|)\,\mathrm{d}y \\
&-\int_{\Phi(B^c(x,\eta_\varepsilon))}(\chi_{\mathcal{E}^c}(y)-\chi_{\mathcal{E}}(y))k_r^s(|\Phi(x)-y|)\,\mathrm{d}y\Big| \\
=&\Big|\int_{B^c(x,\eta_\varepsilon)}(\chi_{E^c}(y)-\chi_E(y))k_r^s(|x-y|)\,\mathrm{d}y \\
&-\int_{B^c(x,\eta_\varepsilon)}(\chi_{E^c}(z)-\chi_E(z))k_r^s(|\Phi(x)-\Phi(z)|)\mathcal{J}\Phi(z)\,\mathrm{d}z\Big| \\
\leq&\int_{B^c(x,\eta_\varepsilon)}\Big|k_r^s(|x-y|)-k_r^s(|\Phi(x)-\Phi(y)|)\mathcal{J}\Phi(y)\Big|\,\mathrm{d}y \\
=&\int_{B^c(x,1)}\Big|k_r^s(|x-y|)-k_r^s(|\Phi(x)-y|)\Big|\,\mathrm{d}y && (2.4.15)\\
&+\int_{B(x,1)\setminus B(x,\eta_\varepsilon)}\Big|k_r^s(|x-y|)-k_r^s(|\Phi(x)-\Phi(y)|)\mathcal{J}\Phi(y)\Big|\,\mathrm{d}y\,. && (2.4.16)
\end{aligned}$$

Now, assuming that $\|\Phi-\mathrm{Id}\|_{C^2}$ is small enough, by using Lagrange Theorem one can show that

$$\begin{aligned}
&\int_{B^c(x,1)}\Big|k_r^s(|x-y|)-k_r^s(|\Phi(x)-y|)\Big|\,\mathrm{d}y \\
\leq&\omega(\|\Phi-\mathrm{Id}\|_{C^2})\int_{B^c(x,1)}\frac{1}{|x-y|^{d+s+1}}\,\mathrm{d}y\leq\frac{\varepsilon}{6}\,,
\end{aligned} \tag{2.4.17}$$

for some modulus of continuity $\omega$. Analogously, for $\|\Phi-\mathrm{Id}\|_{C^2}$ small enough, one can easily check that

$$\begin{aligned}
&\int_{B(x,1)\setminus B(x,\eta_\varepsilon)}\Big|k_r^s(|x-y|)-k_r^s(|\Phi(x)-\Phi(y)|)\mathcal{J}\Phi(y)\Big|\,\mathrm{d}y \\
\leq&\int_{B(x,1)\setminus B(x,\eta_\varepsilon)}\Big|k_r^s(|x-y|)-k_r^s(|\Phi(x)-\Phi(y)|)\Big|\,\mathrm{d}y \\
&+\int_{B(x,1)\setminus B(x,\eta_\varepsilon)}k_r^s(|\Phi(x)-\Phi(y)|)|1-\mathcal{J}\Phi(y)|\,\mathrm{d}y\leq\frac{\varepsilon}{6}\,.
\end{aligned} \tag{2.4.18}$$

Therefore, by (2.4.15), (2.4.16), (2.4.17), (2.4.18) we deduce that

$$\begin{aligned}\Big|\int_{B^c(x,\eta_\varepsilon)}&(\chi_{E^c}(y)-\chi_E(y))k_r^s(|x-y|)\,\mathrm{d}y\\ -\int_{\Phi(B^c(x,\eta_\varepsilon))}&(\chi_{\mathcal{E}^c}(y)-\chi_{\mathcal{E}}(y))k_r^s(|\Phi(x)-y|)\,\mathrm{d}y\Big|\le\frac{\varepsilon}{3}\,.\end{aligned}\tag{2.4.19}$$

In the end, we observe that, for $\|\Phi-\mathrm{Id}\|_{C^2}$ small enough, it holds

$$\Phi(B^c(x,\eta_\varepsilon))\triangle B^c(\Phi(x),\eta_\varepsilon)\subset B(\Phi(x),2\eta_\varepsilon)\,,$$

which, in view of (2.4.13), yields

$$\begin{aligned}\Big|\int_{\Phi(B^c(x,\eta_\varepsilon))}&(\chi_{\mathcal{E}^c}(y)-\chi_{\mathcal{E}}(y))k_r^s(|\Phi(x)-y|)\,\mathrm{d}y\\ -\int_{B^c(\Phi(x),\eta_\varepsilon)}&(\chi_{\mathcal{E}^c}(y)-\chi_{\mathcal{E}}(y))k_r^s(|\Phi(x)-y|)\,\mathrm{d}y\Big|\le\theta_{k_r^s}(2\eta_\varepsilon)\le\frac{\varepsilon}{3}.\end{aligned}\tag{2.4.20}$$

Plugging (2.4.19) and (2.4.20) into (2.4.14) , we conclude the proof of property (UC) and of the whole proposition. □

### 2.4.2 The classical mean curvature

For every $E\in\mathfrak{C}$, and for every $x\in\partial E$, we denote by $\mathcal{K}^1(x,E)$ the (scalar) mean curvature of $\partial E$ at $x$, i.e., the sum of the principal curvatures of $\partial E$ at $x$. It is well-known that $\mathcal{K}^1$ is nothing but the first variation of the perimeter. Let $E\in\mathfrak{C}$ , $x\in\partial E$ and assume that $\nu_E(x)=e_d$ ; then in a neighborhood of $x=(x',x_d)\in\partial E$ we have that $\partial E$ is the graph of a $C^2$- function $f:B'(x',r)\to\mathbb{R}$, for some $r>0$ with $\mathrm{D}f(x')=0$ so that $B(x,r)\cap E=\{(x',x_d)\in B(x,r)\,:\,x_d\le f(x')\}$. In this case the mean curvature of $\partial E$ at $x$ is given by

$$\begin{aligned}\mathcal{K}^1(x,E)=&\mathrm{Div}\Big(\frac{-\mathrm{D}f}{\sqrt{1+|\mathrm{D}f|^2}}\Big)(x')=-\sum_{i=1}^{d-1}\frac{\partial^2}{\partial_{x_i^2}}f(x')\\ =&-\frac{1}{\omega_{d-1}}\int_{\mathbb{S}^{d-2}}\theta^*\mathrm{D}^2f(x')\theta\,\mathrm{d}\mathcal{H}^{d-2}(\theta)\,,\end{aligned}\tag{2.4.21}$$

where $\theta^*$ is the row vector obtained by transposing the (column) vector $\theta$ and $\mathrm{D}^2f(x')$ denotes the Hessian matrix of $f$ evaluated at $x'$.

**Proposition 2.4.3.** *The standard mean curvature $\mathcal{K}^1$ satisfies the following properties:*

*(M) Monotonicity: If $E,F\in\mathfrak{C}$ with $E\subseteq F$, and if $x\in\partial F\cap\partial E$, then $\mathcal{K}^1(x,F)\le\mathcal{K}^1(x,E)$;*

*(T) Translational invariance: For every $E\in\mathfrak{C}$, $x\in\partial E$, $y\in\mathbb{R}^d$, it holds: $\mathcal{K}^1(x,E)=\mathcal{K}^1(x+y,E+y)$;*

*(B) Lower bound on the curvature of the balls:*

$$\mathcal{K}^1(x,\overline{B}(0,\rho))\ge0\qquad\text{for all } x\in\partial B(0,\rho)\,,\ \rho>0\,;$$

*(S) Symmetry: For every $E \in \mathfrak{C}$ and for every $x \in \partial E$ it holds*

$$\mathcal{K}^1(x, E) = -\mathcal{K}^1(x, \mathbb{R}^d \setminus \overset{\circ}{E}) .$$

*(UC') Uniform continuity: Given $R > 0$, there exists a modulus of continuity $\omega_R$ such that the following holds. For every $E \in \mathfrak{C}$, $x \in \partial E$, such that $E$ has both an internal and external ball condition of radius $R$ at $x$, and for every diffeomorphism $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ of class $C^2$, with $\Phi = \mathrm{Id}$ in $\mathbb{R}^d \setminus B(0, 1)$, we have*

$$|\mathcal{K}^1(x, E) - \mathcal{K}^1(\Phi(x), \Phi(E))| \le \omega_R(\|\Phi - \mathrm{Id}\|_{C^2}) . \tag{2.4.22}$$

*Proof.* We prove only the property (UC'), since the check of the remaining properties is straightforward. Let $R > 0$ and let $E \in \mathfrak{C}$ be such that $E$ satisfies both an internal and external ball condition of radius $R$ at a point $x \in \partial E$. In order to get the claim, we can always assume without loss of generality that $\|\Phi - \mathrm{Id}\|_{C^2} \le 1$.

By the Implicit Function Theorem we have that $E \cap B(x, r) = \{z \in B(x, r) : \ g(z) < 0\}$, for some $r > 0$ and $g \in C^2(B(x, r))$. Moreover, in suitable coordinates we have that $x = 0$, $\mathrm{D}g(0) = e_d$ and, for all $i \neq j$, with $i$, $j = 1, \cdots, d$, $\frac{\partial^2 g}{\partial z_i \partial z_j}(0) = 0$. Then, it is well known that

$$\mathcal{K}^1(0, E) = \mathrm{Div}_\tau\Big(\frac{\mathrm{D}g}{|\mathrm{D}g|}\Big)(0) = \sum_{i=1}^{d-1} \frac{\partial^2 g}{\partial z_i^2}(0) , \tag{2.4.23}$$

where $\mathrm{Div}_\tau$ denotes the tangential divergence operator. Since the mean curvature is invariant by translations and rotations, up to small perturbations of $\Phi$ in $C^2$ we may assume, without loss of generality, that $\Phi(0) = 0$ and that the normal to $\Phi(E)$ at $\Phi(0) = 0$ is still $e_d$. Since

$$\Phi(E) \cap B(0, \tilde r) = \{y \in B(0, \tilde r) : \ g(\Phi^{-1}(y)) < 0\}$$

for some $\tilde r > 0$, setting $h := g \circ \Phi^{-1}$, we have

$$\mathcal{K}^1(0, \Phi(E)) = \frac{1}{|\mathrm{D}h(0)|} \sum_{j=1}^{d-1} \frac{\partial^2 h}{\partial y_j^2}(0) - \frac{1}{|\mathrm{D}h(0)|^2} \sum_{j=1}^{d-1} \frac{\partial h}{\partial y_j}(0)\, \frac{\partial |\mathrm{D}h|}{\partial y_j}(0) . \tag{2.4.24}$$

Therefore, using that

$$\begin{aligned} \mathrm{D}h(0) =& \mathrm{D}g(0)\, \mathrm{D}\Phi^{-1}(0) = e_d\, \mathrm{D}\Phi^{-1}(0) , \\ \frac{\partial^2 h}{\partial y_j \partial y_k}(0) =& \sum_{i=1}^{d-1} \frac{\partial^2 g}{\partial z_i^2}(0)\, \frac{\partial \Phi_i^{-1}}{\partial y_j}(0)\, \frac{\partial \Phi_i^{-1}}{\partial y_k}(0) + \frac{\partial^2 \Phi_d^{-1}}{\partial y_j \partial y_k}(0) , \end{aligned}$$

we have

$$\begin{aligned} &\frac{1}{|\mathrm{D}h(0)|}\Big| \sum_{j=1}^{d-1} \frac{\partial^2 h}{\partial y_j^2}(0) - \sum_{i=1}^{d-1} \frac{\partial^2 g}{\partial z_i^2}(0)\Big| \\ \le& \frac{1}{|\mathrm{D}h(0)|}\Big| \sum_{j=1}^{d-1} \frac{\partial^2 \Phi_d^{-1}}{\partial y_j^2}(0) + \sum_{i=1}^{d-1} \frac{\partial^2 g}{\partial z_i^2}(0) \Big( \sum_{j=1}^{d-1} \Big(\frac{\partial \Phi_i^{-1}}{\partial y_j}(0)\Big)^2 - 1\Big)\Big| \\ &+ \frac{|\mathrm{D}h(0) - 1|}{|\mathrm{D}h(0)|}\Big| \sum_{i=1}^{d-1} \frac{\partial^2 g}{\partial z_i^2}(0)\Big| \\ \le& C\Big[\|\mathrm{D}^2\Phi^{-1}\|_{C^0} + \|\mathrm{D}^2 g\|_{C^0}\|\mathrm{Id} - (\mathrm{D}\Phi^{-1})^2\|_{C^0} + \|\mathrm{D}^2 g\|_{C^0}\|\mathrm{Id} - \mathrm{D}\Phi^{-1}\|_{C^0}\Big] \\ \le& C\Big(1 + \frac{1}{R}\Big)\|\mathrm{Id} - \Phi\|_{C^2} . \end{aligned} \tag{2.4.25}$$

Similar computations (that are left to the reader) show that

$$\frac{1}{|\mathrm{D}h(0)|^2}\Big|\sum_{j=1}^{d-1}\frac{\partial h}{\partial y_j}(0)\,\frac{\partial|\mathrm{D}h|}{\partial y_j}(0)\Big| \le C\Big(1+\frac{1}{R}\Big)\|\mathrm{Id}-\Phi\|_{C^2}\,. \tag{2.4.26}$$

Therefore, (2.4.22) follows from (2.4.23), (2.4.24), (2.4.25) and (2.4.26). □

### 2.4.3 Convergence of $k_r^s$-curvature flow to mean curvature flow

We now prove that the viscosity solutions to the $k_r^s$-curvature flow converge to the classical mean curvature flow as $r\to 0^+$. To this end, we will adopt notation and we will use results in [26].

We preliminarily notice that since the curvatures $\mathcal{K}_r^s$ defined in (2.4.1) satisfy property (UC) in Proposition 2.4.2, they also satisfy property (UC') in Proposition 2.4.3 (with $\mathcal{K}^1$ replaced by $\mathcal{K}_r^s$). As a consequence $\mathcal{K}^1$ and $\mathcal{K}_r^s$ (for every $0<r<1$ and $s\ge 1$) satisfy the following continuity property:

(C) Continuity: If $\{E_n\}_{n\in\mathbb{N}}\subset\mathfrak{C}$, $E\in\mathfrak{C}$ and $E_n\to E$ in $\mathfrak{C}$, then the corresponding curvatures of $E_n$ at $x$ converge to the curvature of $E$ at $x$ for every $x\in\partial E_n\cap\partial E$.

Such a property, together with properties (M) and (T) (see Propositions 2.4.3 and 2.4.2), implies that the functionals $\mathcal{K}^1$ and $\mathcal{K}_r^s$ (for every $s\ge 1$ and for every $r\in(0,1)$) are non-local curvatures in the sense of [26, Definition 2.1] (see also [28]).

Moreover, since by Propositions 2.4.3 and 2.4.2, $\mathcal{K}^1$ and $\mathcal{K}_r^s$ satisfy also properties (B) and (UC') (referred to as (C') in [26]), they both satisfy the assumptions of [26, Theorem 2.9] that guarantee existence and uniqueness of suitably defined viscosity solutions of the corresponding geometric flows. We refer to [26, Definition 2.3] for the precise definition of viscosity solution in this setting.

**Proposition 2.4.4.** *Let $s\ge 1$ and $r>0$. Let $u_0\in C(\mathbb{R}^d)$ be a uniformly continuous function with $u_0=C_0$ in $\mathbb{R}^d\setminus B(0,R_0)$ for some $C_0,R_0\in\mathbb{R}$ with $R_0>0$. Then, there exists a unique viscosity solution - in the sense of [26, Definition 2.3] - $u_r^s:\mathbb{R}^d\times[0,+\infty)\to\mathbb{R}$ to the Cauchy problem*

$$\begin{cases}\partial_t u(x,t)+|\mathrm{D}u(x,t)|\mathcal{K}_r^s(x,\{y\,:\,u(y,t)\ge u(x,t)\})=0\\ u(x,0)=u_0(x)\,.\end{cases} \tag{2.4.27}$$

*Moreover, the same result holds true if $\mathcal{K}_r^s$ is replaced by (any multiple of) $\mathcal{K}^1$.*

We will show that, as $r\to 0^+$, the scaled $k_r^s$-curvatures $\frac{1}{\sigma^s(r)}\mathcal{K}_r^s$ converge to $\omega_{d-1}\mathcal{K}^1$ on regular sets. In view of [26, Theorem 3.2], such a result will be crucial in order to prove the convergence of the corresponding geometric flows. We first prove the following result by adopting the same strategy used in [49, Proposition 2].

**Lemma 2.4.5.** *Let $M,N\in\mathbb{R}^{(d-1)\times(d-1)}$ and let $\{M_r\}_{r>0},\{N_r\}_{r>0}\subset\mathbb{R}^{(d-1)\times(d-1)}$ be such that $M_r\to M$, $N_r\to N$ as $r\to 0^+$. Then, for every $\delta>0$ it holds*

$$\begin{aligned}&\lim_{r\to 0^+}\Big(\frac{1}{\sigma^s(r)}\Big(\int_{\mathcal{F}^1_{r,\delta}}k_r^s(|y|)\,\mathrm{d}y-\int_{\mathcal{F}^2_{r,\delta}}k_r^s(|y|)\,\mathrm{d}y\Big)\Big)\\ &=\int_{\mathbb{S}^{d-2}}\theta^*(N-M)\theta\,\mathrm{d}\mathcal{H}^{d-2}(\theta),\end{aligned} \tag{2.4.28}$$

*where*

$$\mathcal{F}^1_{r,\delta} := \{y = (y', y_d) \in B(0,\delta) \,:\, (y')^* M_r y' \le y_d \le (y')^* N_r y'\}$$
$$\mathcal{F}^2_{r,\delta} := \{y = (y', y_d) \in B(0,\delta) \,:\, (y')^* N_r y' \le y_d \le (y')^* M_r y'\}\,.$$

*Proof.* For every $\alpha > 0$ we set

$$\mathcal{G}^1_\alpha := \{y = (y', y_d) \in \mathbb{R}^{d-1} \times \mathbb{R} \,:\, y' = \rho\theta,\ 0 \le \rho \le \alpha,\ \theta \in \mathbb{S}^{d-2}\,,\ \rho^2\theta^* M_r\theta \le y_d \le \rho^2\theta^* N_r\theta\}$$

Therefore, for $r$ small enough,

$$\mathcal{F}^1_{r,\delta} = \mathcal{G}^1_\delta \cap B(0,\delta)\,, \qquad \mathcal{F}^1_{r,\delta} \cap B(0,r) = \mathcal{G}^1_r \cap B(0,r)\,,$$
$$\mathcal{F}^1_{r,\delta} \setminus B(0,r) = ((\mathcal{G}^1_\delta \setminus \mathcal{G}^1_r) \cap (B(0,\delta)) \cup (\mathcal{G}^1_r \setminus B(0,r))\,.$$

It follows that

$$\begin{aligned}
&\int_{\mathcal{F}^1_{r,\delta}} k^s_r(|y|)\,\mathrm{d}y \\
&= \int_{\mathcal{G}^1_\delta} k^s_r(|y|)\,\mathrm{d}y - \int_{\mathcal{G}^1_\delta \setminus B(0,\delta)} k^s_r(|y|)\,\mathrm{d}y \\
&= \int_{\mathcal{G}^1_r} k^s_r(|y|)\,\mathrm{d}y + \int_{\mathcal{G}^1_\delta \setminus \mathcal{G}^1_r} k^s_r(|y|)\,\mathrm{d}y - \int_{\mathcal{G}^1_\delta \setminus B(0,\delta)} \frac{1}{|y|^{d+s}}\,\mathrm{d}y \\
&= \int_{\mathcal{G}^1_r \cap B(0,r)} k^s_r(|y|)\,\mathrm{d}y + \int_{\mathcal{G}^1_r \setminus B(0,r)} k^s_r(|y|)\,\mathrm{d}y + \int_{\mathcal{G}^1_\delta \setminus \mathcal{G}_r} \frac{1}{|y|^{d+s}}\,\mathrm{d}y \\
&\quad - \int_{\mathcal{G}^1_\delta \setminus B(0,\delta)} \frac{1}{|y|^{d+s}}\,\mathrm{d}y \\
&= \int_{\mathcal{G}^1_r \cap B(0,r)} \frac{1}{r^{d+s}}\,\mathrm{d}y + \int_{\mathcal{G}^1_r \setminus B(0,r)} \frac{1}{|y|^{d+s}}\,\mathrm{d}y + \int_{\mathcal{G}^1_\delta \setminus \mathcal{G}^1_r} \frac{1}{|y|^{d+s}}\,\mathrm{d}y \\
&\quad - \int_{\mathcal{G}^1_\delta \setminus B(0,\delta)} \frac{1}{|y|^{d+s}}\,\mathrm{d}y \\
&= \int_{\mathcal{G}^1_r} \frac{1}{r^{d+s}}\,\mathrm{d}y + \int_{\mathcal{G}^1_\delta \setminus \mathcal{G}^1_r} \frac{1}{|y|^{d+s}}\,\mathrm{d}y \\
&\quad - \int_{\mathcal{G}^1_\delta \setminus B(0,\delta)} \frac{1}{|y|^{d+s}}\,\mathrm{d}y - \int_{\mathcal{G}^1_r \setminus B(0,r)} \frac{1}{r^{d+s}}\,\mathrm{d}y + \int_{\mathcal{G}^1_r \setminus B(0,r)} \frac{1}{|y|^{d+s}}\,\mathrm{d}y\,.
\end{aligned} \tag{2.4.29}$$

We set

$$A^1_r := \{\theta \in \mathbb{S}^{d-2} \,:\, \theta^*(M_r - N_r)\theta \le 0\}$$

and we notice that

$$
\begin{aligned}
&\int_{\mathcal{G}^1_r}\frac{1}{r^{d+s}}\,\mathrm{d}y+\int_{\mathcal{G}^1_\delta\setminus\mathcal{G}^1_r}\frac{1}{|y|^{d+s}}\,\mathrm{d}y\\
=&\frac{1}{r^{d+s}}\int_{A^1_r}\mathrm{d}\mathcal{H}^{d-2}(\theta)\int_0^r\mathrm{d}\rho\,\rho^{d-2}\int_{\rho^2\theta^*M_r\theta}^{\rho^2\theta^*N_r\theta}\mathrm{d}y_d\\
&+\int_{A^1_r}\mathrm{d}\mathcal{H}^{d-2}(\theta)\int_r^\delta\mathrm{d}\rho\,\rho^{d-2}\int_{\rho^2\theta^*M_r\theta}^{\rho^2\theta^*N_r\theta}\frac{1}{(\rho^2+y_d^2)^{\frac{d+s}{2}}}\,\mathrm{d}y_d\\
=&\frac{1}{d+1}\frac{r^{d+1}}{r^{d+s}}\int_{A^1_r}\theta^*(N_r-M_r)\theta\,\mathrm{d}\mathcal{H}^{d-2}(\theta)\\
&+\int_{A^1_r}\mathrm{d}\mathcal{H}^{d-2}(\theta)\int_r^\delta\mathrm{d}\rho\,\rho^{d-2}\int_{\theta^*M_r\theta}^{\theta^*N_r\theta}\frac{\rho^2}{(\rho^2+\rho^4t^2)^{\frac{d+s}{2}}}\,\mathrm{d}t\\
=&\frac{r^{1-s}}{d+1}\int_{A^1_r}\theta^*(N_r-M_r)\theta\,\mathrm{d}\mathcal{H}^{d-2}(\theta)\\
&+\int_{A^1_r}\mathrm{d}\mathcal{H}^{d-2}(\theta)\int_r^\delta\mathrm{d}\rho\frac{1}{\rho^s}\int_{\theta^*M_r\theta}^{\theta^*N_r\theta}\frac{1}{(1+\rho^2t^2)^{\frac{d+s}{2}}}\,\mathrm{d}t\,,
\end{aligned}
\tag{2.4.30}
$$

where in the last but one equality we have used the change of variable $y_d=\rho^2t$.

Moreover, trivially we have

$$
\int_{\mathcal{G}^1_\delta\setminus B(0,\delta)}\frac{1}{|y|^{d+s}}\,\mathrm{d}y\le C(\delta)\,,
\tag{2.4.31}
$$

for some $C(\delta)>0$. Furthermore, it is easy to see that, for $r$ small enough,

$$
\mathcal{G}^1_r\setminus B(0,r)\subset(B'(0,r)\setminus B'(0,r-cr^2))\times[-cr^2,cr^2]
$$

for some constant $c>0$ independent of $r$; as a consequence,

$$
|\mathcal{G}^1_r\setminus B(0,r)|\le Cr^{d+2}\,,
$$

whence we deduce that

$$
\begin{aligned}
&\int_{\mathcal{G}^1_r\setminus B(0,r)}\frac{1}{r^{d+s}}\,\mathrm{d}y\le Cr^{2-s}\,,\\
&\int_{\mathcal{G}^1_r\setminus B(0,r)}\frac{1}{|y|^{d+s}}\,\mathrm{d}y\le\int_{\mathcal{G}^1_r\setminus B(0,r)}\frac{1}{r^{d+s}}\,\mathrm{d}y\le Cr^{2-s}\,.
\end{aligned}
\tag{2.4.32}
$$

Therefore, by (2.4.29), (2.4.30), (2.4.31) and (2.4.32), we obtain

$$
\begin{aligned}
&\frac{1}{\sigma^s(r)}\int_{\mathcal{F}^1_{r,\delta}}k^s_r(|y|)\,\mathrm{d}y\\
=&\frac{r^{1-s}}{(d+1)\sigma^s(r)}\int_{A^1_r}\theta^*(N_r-M_r)\theta\,\mathrm{d}\mathcal{H}^{d-2}(\theta)\\
&+\frac{1}{\sigma^s(r)}\int_{A^1_r}\mathrm{d}\mathcal{H}^{d-2}(\theta)\int_r^\delta\mathrm{d}\rho\frac{1}{\rho^s}\int_{\theta^*M_r\theta}^{\theta^*N_r\theta}\frac{1}{(1+\rho^2t^2)^{\frac{d+s}{2}}}\,\mathrm{d}t\\
&+f^1(r)\,,
\end{aligned}
\tag{2.4.33}
$$

where $f^1(r) \to 0$ as $r \to 0^+$.

Now we set

$$A_r^2 := \{\theta \in \mathbb{S}^{d-2} \, : \, \theta^*(N_r - M_r)\theta \leq 0\} \, ;$$

by arguing as in the proof of (2.4.33) we obtain

$$\begin{aligned} &\frac{1}{\sigma^s(r)} \int_{\mathcal{F}^2_{r,\delta}} k_r^s(|y|) \, \mathrm{d}y \\ =&\frac{r^{1-s}}{(d+1)\sigma^s(r)} \int_{A_r^2} \theta^*(M_r - N_r)\theta \, \mathrm{d}\mathcal{H}^{d-2}(\theta) \\ &+ \frac{1}{\sigma^s(r)} \int_{A_r^2} \mathrm{d}\mathcal{H}^{d-2}(\theta) \int_r^\delta \mathrm{d}\rho \frac{1}{\rho^s} \int_{\theta^* N_r \theta}^{\theta^* M_r \theta} \frac{1}{(1+\rho^2 t^2)^{\frac{d+s}{2}}} \, \mathrm{d}t \\ &+ f^2(r) \, , \end{aligned} \tag{2.4.34}$$

where $f^2(r) \to 0$ as $r \to 0^+$. Therefore by formulas (2.4.33) and (2.4.34), using that $A_r^1 \cup A_r^2 = \mathbb{S}^{d-2}$, we get

$$\begin{aligned} &\left(\frac{1}{\sigma^s(r)}\Big(\int_{\mathcal{F}^1_{r,\delta}} k_r^s(|y|) \, \mathrm{d}y - \int_{\mathcal{F}^2_{r,\delta}} k_r^s(|y|) \, \mathrm{d}y\Big)\right) \\ =&\left[\frac{r^{1-s}}{(d+1)\sigma^s(r)}\right] \int_{\mathbb{S}^{d-2}} \theta^*(N_r - M_r)\theta \, \mathrm{d}\mathcal{H}^{d-2}(\theta) \\ &+ \frac{1}{\sigma^s(r)} \int_{\mathbb{S}^{d-2}} \mathrm{d}\mathcal{H}^{d-2}(\theta) \int_r^\delta \mathrm{d}\rho \frac{1}{\rho^s} \int_{\theta^* M_r \theta}^{\theta^* N_r \theta} \frac{1}{(1+\rho^2 t^2)^{\frac{d+s}{2}}} \, \mathrm{d}t \\ &+ f^1(r) - f^2(r) \, . \end{aligned} \tag{2.4.35}$$

Since

$$\frac{r^{1-s}}{(d+1)\sigma^s(r)} = \begin{cases} \frac{1}{(d+1)|\log r|} & \text{if } s = 1 \\ \frac{s-1}{d+s} & \text{if } s > 1 \, , \end{cases}$$

and recalling that $M_r$ and $N_r$ converge to $M$ and $N$, respectively, we get

$$\begin{aligned} &\lim_{r\to 0^+} \frac{r^{1-s}}{(d+1)\sigma^s(r)} \int_{\mathbb{S}^{d-2}} \theta^*(N_r - M_r)\theta \, \mathrm{d}\mathcal{H}^{d-2}(\theta) \\ &= \begin{cases} 0 & \text{if } s = 1, \\ \dfrac{s-1}{d+s} \displaystyle\int_{\mathbb{S}^{d-2}} \theta^*(N - M)\theta \, \mathrm{d}\mathcal{H}^{d-1}(\theta) & \text{if } s > 1 \, . \end{cases} \end{aligned} \tag{2.4.36}$$

Moreover, for every $s \geq 1$, using de l'Hôpital rule (i.e., differentiating both terms in the product below with respect to $r$) and the very definition of $\sigma^s(r)$ in (2.1.5), we have

$$\lim_{r\to 0^+} \frac{1}{\sigma^s(r)} \int_r^\delta \frac{1}{\rho^s} \frac{1}{(1+\rho^2 t^2)^{\frac{d+s}{2}}} \, \mathrm{d}\rho = \frac{d+1}{d+s} \, ,$$

which, by the Dominate Convergence Theorem, yields

$$\begin{aligned} &\lim_{r\to 0^+} \frac{1}{\sigma^s(r)} \int_{\mathbb{S}^{d-2}} \mathrm{d}\mathcal{H}^{d-2}(\theta) \int_r^\delta \mathrm{d}\rho \frac{1}{\rho^s} \int_{\theta^* M_r \theta}^{\theta^* N_r \theta} \frac{1}{(1+\rho^2 t^2)^{\frac{d+s}{2}}} \, \mathrm{d}t \\ =& \lim_{r\to 0^+} \int_{\mathbb{S}^{d-2}} \mathrm{d}\mathcal{H}^{d-2}(\theta) \int_{\theta^* M_r \theta}^{\theta^* N_r \theta} \mathrm{d}t \frac{1}{\sigma^s(r)} \int_r^\delta \frac{1}{\rho^s} \frac{1}{(1+\rho^2 t^2)^{\frac{d+s}{2}}} \, \mathrm{d}\rho \\ =&\frac{d+1}{d+s} \int_{\mathbb{S}^{d-2}} \theta^*(N - M)\theta \, \mathrm{d}\mathcal{H}^{d-2}(\theta) \, . \end{aligned} \tag{2.4.37}$$

By formulas (2.4.35), (2.4.36) and (2.4.37) we obtain (2.4.28). □

**Theorem 2.4.6.** *Let $s \geq 1$. Let $\{E_r\}_{r>0} \subset \mathfrak{C}$ be such that $E_r \to E$ in $\mathfrak{C}$ as $r \to 0^+$, for some $E \in \mathfrak{C}$. Then, for every $x \in \partial E \cap \partial E_r$ for every $r > 0$, it holds*

$$\lim_{r\to 0^+} \frac{\mathcal{K}_r^s(x, E_r)}{\sigma^s(r)} = \omega_{d-1}\mathcal{K}^1(x, E).$$

*Proof.* Let $x \in \partial E \cap \partial E_r$ for all $r > 0$. By Proposition 2.4.2 we have that the curvatures $\mathcal{K}_r^s$ satisfy properties (S) and (T); moreover, it is easy to check that $\mathcal{K}_r^s$ are invariant by rotations. Therefore, we can assume without loss of generality that $E$ and $\{E_r\}_{r>0}$ are compact, and that $x = 0$, $\nu_E(0) = \nu_{E_r}(0) = e_d$ for all $r > 0$. Since $E_r \to E$ in $\mathfrak{C}$ as $r \to 0^+$ we have that there exist $\delta > 0$ and functions $\phi, \phi_r : B'(0, \delta) \to \mathbb{R}$ such that $\phi_r \to \phi$ in $C^2$ as $r \to 0^+$, $\phi(0) = \phi_r(0) = 0$, $\mathrm{D}\phi(0) = \mathrm{D}\phi_r(0) = 0$ and

$$\begin{aligned}
\partial E \cap B(0,\delta) &= \{(y', \phi(y')) :\ y' \in B'(0,\delta)\} \cap B(0,\delta)\,,\\
\partial E_r \cap B(0,\delta) &= \{(y', \phi_r(y')) :\ y' \in B'(0,\delta)\} \cap B(0,\delta)\,,\\
E \cap B(0,\delta) &= \{(y', y_d) :\ y' \in B'(0,\delta)\,,\ y_d \leq \phi(y')\} \cap B(0,\delta)\,,\\
E_r \cap B(0,\delta) &= \{(y', y_d) :\ y' \in B'(0,\delta)\,,\ y_d \leq \phi_r(y')\} \cap B(0,\delta)\,.
\end{aligned}$$

Let $\eta > 0$ be fixed; for $\delta$ small enough we have

$$\left|\phi_r(y') - \frac{1}{2}(y')^*\mathrm{D}^2\phi_r(0)y'\right| \leq \eta|y'|^2 \quad \text{for every } 0 < r < 1\,,\ y' \in B'(0,\delta). \tag{2.4.38}$$

We define the following sets

$$\begin{aligned}
A(r) :=& \{y = (y', y_d) \in B(0,\delta) \,:\, -\phi_r(y') \leq y_d \leq \phi_r(y')\}\,,\\
B(r) :=& \{y = (y', y_d) \in B(0,\delta) \,:\, \phi_r(y') \leq y_d \leq -\phi_r(y')\}\,,\\
C(r) :=& (E_r^c \setminus B(r)) \cap B(0,\delta)\\
=& \{y = (y', y_d) \in B(0,\delta) \,:\, y_d \geq \max\{\phi_r(y'), -\phi_r(y')\}\}\,,\\
D(r) :=& (E_r \setminus A(r)) \cap B(0,\delta)\\
=& \{y = (y', y_d) \in B(0,\delta) \,:\, y_d \leq \min\{\phi_r(y'), -\phi_r(y')\}\}\,,
\end{aligned}$$

where the equalities above are understood in the sense of measurable sets, i.e., up to negligible sets. We notice that

$$E_r \cap B(0,\delta) = A(r) \cup D(r)\,, \qquad E_r^c \cap B(0,\delta) = B(r) \cup C(r)\,,$$
$$\int_{C(r)} k_r^s(|y|)\,\mathrm{d}y = \int_{D(r)} k_r^s(|y|)\,\mathrm{d}y\,,$$

whence we deduce that

$$
\begin{aligned}
\mathcal{K}^s_r(0,E_r) &= \int_{B(0,\delta)} (\chi_{E^c_r}(y) - \chi_{E_r}(y)) k^s_r(|y|)\, \mathrm{d}y \\
&\quad + \int_{B^c(0,\delta)} (\chi_{E^c_r}(y) - \chi_{E_r}(y)) k^s_r(|y|)\, \mathrm{d}y \\
&= \int_{\mathbb{R}^d} (\chi_{B(r)}(y) - \chi_{A(r)}(y)) k^s_r(|y|)\, \mathrm{d}y \\
&\quad + \int_{\mathbb{R}^d} (\chi_{C(r)}(y) - \chi_{D(r)}(y)) k^s_r(|y|)\, \mathrm{d}y \\
&\quad + \int_{B^c(0,\delta)} (\chi_{E^c_r}(y) - \chi_{E_r}(y)) k^s_r(|y|)\, \mathrm{d}y \\
&= \int_{\mathbb{R}^d} (\chi_{B(r)}(y) - \chi_{A(r)}(y)) k^s_r(|y|)\, \mathrm{d}y \\
&\quad + \int_{B^c(0,\delta)} (\chi_{E^c_r}(y) - \chi_{E_r}(y)) k^s_r(|y|)\, \mathrm{d}y\,.
\end{aligned}
\tag{2.4.39}
$$

Trivially,

$$
\Big| \int_{B^c(0,\delta)} (\chi_{E^c_r}(y) - \chi_{E_r}(y)) k^s_r(|y|)\, \mathrm{d}y \Big| \le d\omega_d \frac{\delta^{-s}}{s}\,.
$$

In order to study the limit

$$
\lim_{r\to 0^+} \frac{1}{\sigma^s(r)} \int_{\mathbb{R}^d} (\chi_{B(r)}(y) - \chi_{A(r)}(y)) k^s_r(|y|)\, \mathrm{d}y\,,
$$

we define the following sets

$$
\begin{aligned}
A^-(r) := \Big\{ & y = (y', y_d) \in B(0,\delta)\ : \\
& -\frac{1}{2}(y')^* \mathrm{D}^2\phi_r(0) y' + \eta |y'|^2 \le y_d \le \frac{1}{2}(y')^* \mathrm{D}^2\phi_r(0) y' - \eta |y'|^2 \Big\}, \\
A^+(r) := \Big\{ & y = (y', y_d) \in B(0,\delta)\ : \\
& -\frac{1}{2}(y')^* \mathrm{D}^2\phi_r(0) y' - \eta |y'|^2 \le y_d \le \frac{1}{2}(y')^* \mathrm{D}^2\phi_r(0) y' + \eta |y'|^2 \Big\}, \\
B^-(r) := \Big\{ & y = (y', y_d) \in B(0,\delta)\ : \\
& \frac{1}{2}(y')^* \mathrm{D}^2\phi_r(0) y' + \eta |y'|^2 \le y_d \le -\frac{1}{2}(y')^* \mathrm{D}^2\phi_r(0) y' - \eta |y'|^2 \Big\}, \\
B^+(r) := \Big\{ & y = (y', y_d) \in B(0,\delta)\ : \\
& \frac{1}{2}(y')^* \mathrm{D}^2\phi_r(0) y' - \eta |y'|^2 \le y_d \le -\frac{1}{2}(y')^* \mathrm{D}^2\phi_r(0) y' + \eta |y'|^2 \Big\}.
\end{aligned}
$$

By (2.4.38) we have that

$$
A^-(r) \subset A(r) \subset A^+(r), \quad B^-(r) \subset B(r) \subset B^+(r)\,,
$$

and hence

$$\begin{aligned}&\int_{\mathbb{R}^d}(\chi_{B^-(r)}(y)-\chi_{A^+(r)}(y))k_r^s(|y|)\,\mathrm{d}y\\ \le&\int_{\mathbb{R}^d}(\chi_{B(r)}(y)-\chi_{A(r)}(y))k_r^s(|y|)\,\mathrm{d}y\\ \le&\int_{\mathbb{R}^d}(\chi_{B^+(r)}(y)-\chi_{A^-(r)}(y))k_r^s(|y|)\,\mathrm{d}y.\end{aligned}\tag{2.4.40}$$

Then, by applying Lemma 2.4.5 and using (2.4.40), we obtain

$$\begin{aligned}&-\int_{\mathbb{S}^{d-2}}\theta^*(\mathrm{D}^2\phi(0)+2\eta\mathrm{Id})\theta\,\mathrm{d}\mathcal{H}^{d-2}(\theta)\\ \le&\liminf_{r\to0^+}\frac{1}{\sigma^s(r)}\int_{\mathbb{R}^d}(\chi_{B(r)}(y)-\chi_{A(r)}(y))k_r^s(|y|)\,\mathrm{d}y\\ \le&\limsup_{r\to0^+}\frac{1}{\sigma^s(r)}\int_{\mathbb{R}^d}(\chi_{B(r)}(y)-\chi_{A(r)}(y))k_r^s(|y|)\,\mathrm{d}y\\ \le&-\int_{\mathbb{S}^{d-2}}\theta^*(\mathrm{D}^2\phi(0)-2\eta\mathrm{Id})\theta\,\mathrm{d}\mathcal{H}^{d-2}(\theta)\,.\end{aligned}\tag{2.4.41}$$

Therefore, by (2.4.39) and (2.4.41), we get

$$\begin{aligned}&-\int_{\mathbb{S}^{d-2}}\theta^*(\mathrm{D}^2\phi(0)+2\eta\mathrm{Id})\theta\,\mathrm{d}\mathcal{H}^{d-2}(\theta)\\ \le&\liminf_{r\to0^+}\frac{1}{\sigma^s(r)}\mathcal{K}_r^s(0,E_r)\le\limsup_{r\to0^+}\frac{1}{\sigma^s(r)}\mathcal{K}_r^s(0,E_r)\\ \le&-\int_{\mathbb{S}^{d-2}}\theta^*(\mathrm{D}^2\phi(0)-2\eta\mathrm{Id})\theta\,\mathrm{d}\mathcal{H}^{d-2}(\theta)\,,\end{aligned}$$

which, sending $\eta$ to 0 and using (2.4.21) implies the claim. □

We are now in a position to state the main result of this section.

**Theorem 2.4.7.** *Let $s\ge1$ be fixed. Let $u_0\in C(\mathbb{R}^d)$ be a uniformly continuous function with $u_0=C_0$ in $\mathbb{R}^d\setminus B(0,R_0)$ for some $C_0,R_0\in\mathbb{R}$ with $R_0>0$. For every $r>0$, let $u_r^s:\mathbb{R}^d\times[0,+\infty)\to\mathbb{R}$ be the viscosity solution to the Cauchy problem* (2.4.27)*. Then, setting $v_r^s(x,t):=u_r^s(x,\frac{t}{\sigma^s(r)})$ for all $x\in\mathbb{R}^d$, $t\ge0$, we have that, for every $T>0$, $v_r^s$ uniformly converge to $u$ as $r\to0^+$ in $\mathbb{R}^d\times[0,T]$, where $u:\mathbb{R}^d\times[0,+\infty)\to\mathbb{R}$ is the viscosity solution to the classical mean curvature flow*

$$\begin{cases}\partial_t u(x,t)+|\mathrm{D}u(x,t)|\omega_{d-1}\mathcal{K}^1(x,\{y\,:\,u(y,t)\ge u(x,t)\})=0\\ u(x,0)=u_0(x)\,.\end{cases}\tag{2.4.42}$$

*Proof.* We preliminarily notice that, by an easy scaling argument, the functions $v_r^s$ are viscosity solution to

$$\begin{cases}\partial_t v(x,t)+|\mathrm{D}v(x,t)|\frac{1}{\sigma^s(r)}\mathcal{K}_r^s(x,\{y\,:\,v(y,t)\ge v(x,t)\})=0\\ v(x,0)=u_0(x)\,.\end{cases}$$

By Theorem 2.4.6 we have that , as $r\to0^+$ the scaled $k_r^s$-curvatures $\frac{1}{\sigma^s(r)}\mathcal{K}_r^s$ converge to $\omega_{d-1}\mathcal{K}^1$ on regular sets. Moreover, by Propositions 2.4.2 and 2.4.3, $\mathcal{K}_r^s$ (for every $r\in(0,1)$) and $\mathcal{K}^1$ satisfy properties (M), (T), (S), (UC'). Furthermore, for every $\rho>0$ and for every $x\in\partial B(0,\rho)$, by Proposition 2.4.2, we have that $\mathcal{K}_r^s(x,\overline{B}(0,\rho)\ge0$ whereas, by Theorem 2.4.6 we get that $\sup_{r\in(0,1)}\mathcal{K}_r^s(x,\overline{B}(0,\rho))<+\infty$. This trivially implies the following property:

(UB) There exists $K > 0$ such that $\inf_{r\in(0,1)} \mathcal{K}_r^s(x, \overline{B}(0,\rho)) \geq -K\rho$ for all $\rho > 1$, $x \in \partial B(0,\rho)$ and $\sup_{r\in(0,1)} \mathcal{K}_r^s(x, \overline{B}(0,\rho)) < +\infty$ for all $\rho > 0$, $x \in \partial B(0,\rho)$.

Properties (M), (T), (S), (UC') (referred to as (C') in [26]) and (UB), together with the convergence of the curvatures on regular sets, are exactly the assumptions of [26, Theorem 3.2], which, in our case, establishes the convergence of $v_r^s$ to $u$ locally uniformly in $\mathbb{R}^d \times [0,T]$ for every $T > 0$. □

## 2.5 Stability as $r \to 0^+$ and $s \to 1^+$ simultaneously

In this section we study Γ-convergence and compactness properties for the $s$-fractional perimeters $\tilde{J}_r^s$ defined in (2.1.4) when $r \to 0^+$ and $s \to \bar{s}$ (with $\bar{s} \geq 1$) simultaneously. Moreover, we study the convergence of the corresponding geometric flows in such a case. In fact, we will consider only the critical case $\bar{s} = 1$, the case $\bar{s} > 1$ being easier.

Let $\{r_n\}_{n\in\mathbb{N}} \subset (0,1)$ and $\{s_n\}_{n\in\mathbb{N}} \subset (1,+\infty)$ be such that $r_n \to 0^+$ and $s_n \to 1^+$ as $n \to +\infty$. Recalling the definitions of $\sigma^s(r)$ in (2.1.5) and $\boldsymbol{\alpha}^s$ in (2.1.13), we set

$$\boldsymbol{\beta}(r_n, s_n) := \sigma^{s_n}(r_n) + \boldsymbol{\alpha}^{s_n} = \frac{d+s_n}{d+1}\frac{r_n^{1-s_n}-1}{s_n-1} + \frac{1}{d+1} \tag{2.5.1}$$

and we notice that

$$\begin{aligned}\lim_{n\to+\infty} \boldsymbol{\beta}(r_n,s_n) &\geq \lim_{n\to+\infty} \frac{r_n^{1-s_n}-1}{s_n-1} = \lim_{n\to+\infty} \int_{r_n}^1 \rho^{-s_n}\,\mathrm{d}\rho \\ &\geq \lim_{n\to+\infty} \int_{r_n}^1 \rho^{-1}\,\mathrm{d}\rho = \lim_{n\to+\infty} |\log r_n| = +\infty\,.\end{aligned} \tag{2.5.2}$$

### 2.5.1 Γ-convergence and compactness

In Theorem 2.5.1 below we study the Γ-convergence of the functionals $\frac{1}{\boldsymbol{\beta}(r_n,s_n)}\tilde{J}_{r_n}^{s_n}$ as $n \to +\infty$.

**Theorem 2.5.1.** *Let $\{r_n\}_{n\in\mathbb{N}} \subset (0,1)$ and $\{s_n\}_{n\in\mathbb{N}} \subset (1,+\infty)$ be such that $r_n \to 0^+$ and $s_n \to 1^+$ as $n \to +\infty$. The following Γ-convergence result holds true.*

*(i) (Compactness) Let $U \subset \mathbb{R}^d$ be an open bounded set and let $\{E_n\}_{n\in\mathbb{N}} \subset \mathrm{M}(\mathbb{R}^d)$ be such that $E_n \subset U$ for every $n \in \mathbb{N}$ and*

$$\tilde{J}_{r_n}^{s_n}(E_n) \leq M\boldsymbol{\beta}(r_n,s_n) \quad \text{for every } n \in \mathbb{N},$$

*for some constant $M$ independent of $n$. Then up to a subsequence, $\chi_{E_n} \to \chi_E$ strongly in $\mathrm{L}^1(\mathbb{R}^d)$ for some set $E \in \mathrm{M}_f(\mathbb{R}^d)$ with $\mathrm{Per}(E) < +\infty$.*

*(ii) (Lower bound) Let $E \in \mathrm{M}_f(\mathbb{R}^d)$. For every $\{E_n\}_{n\in\mathbb{N}} \subset \mathrm{M}_f(\mathbb{R}^d)$ with $\chi_{E_n} \to \chi_E$ strongly in $L^1(\mathbb{R}^d)$ it holds*

$$\omega_{d-1}\mathrm{Per}(E) \leq \liminf_{n\to+\infty} \frac{\tilde{J}_{r_n}^{s_n}(E_n)}{\boldsymbol{\beta}(r_n,s_n)}\,.$$

*(iii) (Upper bound) For every $E \in \mathrm{M}_f(\mathbb{R}^d)$ there exists $\{E_n\}_{n\in\mathbb{N}} \subset \mathrm{M}_f(\mathbb{R}^d)$ such that $\chi_{E_n} \to \chi_E$ strongly in $L^1(\mathbb{R}^d)$ and*

$$\omega_{d-1}\mathrm{Per}(E) = \limsup_{n\to+\infty} \frac{\tilde{J}_{r_n}^{s_n}(E_n)}{\boldsymbol{\beta}(r_n,s_n)}\,.$$

### Proof of compactness

We start by proving the compactness property Theorem 2.5.1(i). To this purpose, we first prove the following lemma which corresponds to Lemma 2.2.4 when both $r$ and $s$ vary.

**Lemma 2.5.2.** *Let $\{r_n\}_{n\in\mathbb{N}} \subset (0,1)$ and $\{s_n\}_{n\in\mathbb{N}} \subset (1,+\infty)$ be such that $r_n \to 0^+$ and $s_n \to 1^+$ as $n \to +\infty$. Let $\Omega \in \mathrm{M}_f(\mathbb{R}^d)$ be a bounded set with Lipschitz continuous boundary and $|\Omega| = 1$. For every $\eta \in (0,1)$ there exist a constant $C(\Omega, d, S, \eta) > 0$ and $\bar{n} \in \mathbb{N}$ such that for every measurable set $A \subset \Omega$ with $\eta \le |A| \le 1-\eta$ it holds*

$$\int_A \int_{\Omega\setminus A} k_{r_n}^{s_n}(|x-y|)\,\mathrm{d}y\,\mathrm{d}x \ge C(\Omega, d, S, \eta)\boldsymbol{\beta}(r_n, s_n) \qquad \text{for every } n \ge \bar{n}\,,$$

*where $S := \sup_{n\in\mathbb{N}} s_n$.*

*Proof.* The proof is fully analogous to the one of Lemma 2.2.4; we sketch only the main differences. For every $n \in \mathbb{N}$, let $I_n \in \mathbb{N}$ be such that $2^{-I_n-1} \le r_n \le 2^{-I_n}$. Let $\phi$ and $\phi_\delta$ (for every $\delta > 0$) be as in Lemma 2.2.5. By arguing verbatim as in the proof of (2.2.6) (see (2.2.9), (2.2.10), and (2.2.11)), for every $n \in \mathbb{N}$ and for every $z \in \mathbb{R}^d$ we have

$$\begin{aligned} k_{r_n}^{s_n}(|z|) \ge& \frac{2^{d+s_n}-1}{2^{d+s_n}}\frac{1}{\sup\phi}\sum_{i=0}^{I_n}(2^{s_n})^i\phi_{2^{-i}}(z)\\ \ge&\frac{2^{d+1}}{2^{d+1}-1}\frac{1}{\sup\phi}\sum_{i=0}^{I_n}(2^{s_n})^i\phi_{2^{-i}}(z)\,. \end{aligned} \tag{2.5.3}$$

Moreover, since

$$\frac{|\log r_n|}{\log 2} - 1 \le I_n \le \frac{|\log r_n|}{\log 2}\,,$$

setting $m(S) := \inf_{s\in(1,S]} \frac{s-1}{2^{s-1}-1}$,we get

$$\begin{aligned} \sum_{i=0}^{I_n}(2^{s_n-1})^i =& \frac{\left(2^{s_n-1}\right)^{I_n+1}-1}{2^{s_n-1}-1} \ge \frac{r_n^{1-s_n}-1}{2^{s_n-1}-1}\\ =&\frac{r_n^{1-s_n}-1}{s_n-1}\frac{s_n-1}{2^{s_n-1}-1} \ge \frac{m(S)}{2}\frac{d+1}{d+s_n}\Big(2\frac{d+s_n}{d+1}\frac{r_n^{1-s_n}-1}{s_n-1}\Big)\\ \ge&\frac{m(S)}{2}\frac{d+1}{d+S}\Big(\frac{d+s_n}{d+1}\frac{r_n^{1-s_n}-1}{s_n-1} + \frac{1}{d+1}\frac{r_n^{1-s_n}-1}{s_n-1}\Big)\\ \ge&\frac{m(S)}{2}\frac{d+1}{d+S}\boldsymbol{\beta}(r_n,s_n)\,, \end{aligned} \tag{2.5.4}$$

where in the last inequality we have used that, in view of (2.5.1), $\frac{r_n^{1-s_n}-1}{s_n-1} \ge 1$. Therefore, by arguing as in (2.2.8), using (2.5.3) and (2.5.4), we get the claim. □

With Lemma 2.5.2 in hand, we can prove Theorem 2.5.1(i), whose proof closely follows the one of Theorem 2.1.5(i). We sketch only the main differences.

*Proof of Theorem 2.5.1(i).* We preliminarily notice that, up to a subsequence, the following limit exists

$$\lim_{n\to+\infty}(s_n-1)|\log r_n| =: \lambda\,; \tag{2.5.5}$$

clearly, $\lambda \in [0, +\infty]$. We first prove the claim under the assumption $\lambda \neq 0$. Let $\alpha \in (0,1)$ and for every $n \in \mathbb{N}$ we set $l_n := r_n^\alpha (s_n - 1)$; therefore, since $\lambda \in (0, +\infty]$,

$$\lim_{n\to+\infty} \frac{r_n}{l_n} = \lim_{n\to+\infty} \frac{r_n^{1-\alpha}}{s_n - 1} = 0\,.$$

By adopting the same notation as in Subsection 2.2.2 we set

$$\tilde{E}_n := \bigcup_{h=1}^{H(n)} Q_h^n\,,$$

where $\{Q_h^n\}_{h\in\mathbb{N}}$ is a family of pairwise disjoint cubes of sidelength $l_n$ which covers the whole $\mathbb{R}^d$ and satisfies (2.2.12).

By arguing verbatim as in the proof of (2.2.13) one can show that there exists $n' \in \mathbb{N}$ such that

$$|\tilde{E}_n \triangle E_n| \le 4 l_n^{s_n} \boldsymbol{\beta}(r_n, s_n) M \quad \text{for every } n \ge n'. \tag{2.5.6}$$

We observe that

$$\begin{aligned} \lim_{n\to+\infty} l_n^{s_n} \boldsymbol{\beta}(r_n, s_n) &= \lim_{n\to+\infty} r_n^{\alpha s_n} (s_n - 1)^{s_n} \Big( \frac{d + s_n}{d+1} \frac{r_n^{1-s_n} - 1}{s_n - 1} + \frac{1}{d+1} \Big) \\ &= \lim_{n\to+\infty} r_n^{1 - s_n + \alpha s_n} \frac{d + s_n}{d+1} (s_n - 1)^{s_n - 1} = 0\,. \end{aligned} \tag{2.5.7}$$

Now, setting $S := \sup_{n\in\mathbb{N}} s_n$, we claim that there exists a constant $C(\alpha, d, S)$ such that for $n$ large enough

$$\mathrm{Per}(\tilde{E}_n) \le C(\alpha, d, S) \frac{\tilde{J}_{r_n}^{s_n}(E_n)}{\boldsymbol{\beta}(r_n, s_n)}\,. \tag{2.5.8}$$

In order to prove (2.5.8), we argue as in Step 2 in Subsection 2.2.2. We define the family $\mathcal{R}$ of rectangles $R = \tilde{Q}_h^n \cup \hat{Q}_h^n$ such that $\tilde{Q}_h^n$ and $\hat{Q}_h^n$ are adjacent, $\tilde{Q}_h^n \subset \tilde{E}_n$ and $\hat{Q}_h^n \subset \tilde{E}_n^c$.

Notice that

$$\begin{aligned} \mathrm{Per}(\tilde{E}_n) \le& 2 d l_n^{d-1} \sharp \mathcal{R}\,, \\ \frac{\tilde{J}_{r_n}^{s_n}(E_n)}{\boldsymbol{\beta}(r_n, s_n)} \ge& \frac{1}{2d\, \boldsymbol{\beta}(r_n, s_n)} \sum_{R\in\mathcal{R}} \int_{R\cap E_n} \int_{R\setminus E_n} k_{r_n}^{s_n}(|x-y|) \,\mathrm{d}y \,\mathrm{d}x\,. \end{aligned} \tag{2.5.9}$$

Moreover, by Lemma 2.5.2, for every rectangle $\bar{R}$ given by the union of two adjacent unitary cubes in $\mathbb{R}^d$, there exists $\bar{n} \in \mathbb{N}$ such that

$$\begin{aligned} C(d, \lambda) := \inf \Big\{ & \frac{1}{\boldsymbol{\beta}(\rho_n, s_n)} \int_F \int_{\bar{R}\setminus F} k_{\rho_n}^{s_n}(|x-y|) \,\mathrm{d}y \,\mathrm{d}x : \\ & n \ge \bar{n},\ F \in \mathrm{M}_f(\mathbb{R}^d)\,,\ F \subset \bar{R}\,,\ \frac{1}{2} \le |F| \le \frac{3}{2} \Big\} > 0\,. \end{aligned} \tag{2.5.10}$$

Furthermore, by the very definition of $\boldsymbol{\beta}(r_n, s_n)$ in (2.5.1), we have

$$\frac{\boldsymbol{\beta}(r_n, s_n)}{l_n^{1-s_n} \boldsymbol{\beta}(\frac{r_n}{l_n}, s_n)} = 1 + \frac{(d+1)(l_n^{1-s_n} - 1)}{(d + s_n)(r_n^{1-s_n} - l_n^{1-s_n}) + (s_n - 1) l_n^{1-s_n}}\,,$$

whence, using that $l_n = r_n^\alpha(s_n-1)$ and (2.5.5), we deduce

$$\lim_{n\to+\infty} \frac{\boldsymbol{\beta}(\frac{r_n}{l_n}, s_n)}{\frac{\boldsymbol{\beta}(r_n,s_n)}{l_n^{1-s_n}}} = \begin{cases} 1+\frac{e^{\lambda\alpha}-1}{e^{\lambda}-e^{\lambda\alpha}} & \text{if } \lambda\neq+\infty \\ 1 & \text{if } \lambda=+\infty\,. \end{cases} \tag{2.5.11}$$

For every set $O \in \mathrm{M}_f(\mathbb{R}^d)$ we set $O^{l_n} := \frac{O}{l_n}$. By (2.5.9), (2.1.2), (2.5.11) and by applying (2.5.10) with $\bar{R} = R^{l_n}$ for every $R \in \mathcal{R}$, for $n$ large enough we obtain

$$\begin{aligned}
&\frac{\tilde{J}^{s_n}_{r_n}(E_n)}{\boldsymbol{\beta}(r_n,s_n)}\\
\geq&\frac{C(d)}{\boldsymbol{\beta}(r_n,s_n)} l_n^{2d}\sum_{R\in\mathcal{R}}\int_{R^{l_n}\cap E^{l_n}}\int_{R^{l_n}\setminus E^{l_n}} k^{s_n}_{r_n}(|l_n(x-y)|)\,\mathrm{d}y\,\mathrm{d}x\\
=&C(d)\frac{l_n^{1-s_n}}{\boldsymbol{\beta}(r_n,s_n)} l_n^{d-1}\sum_{R\in\mathcal{R}}\int_{R^{l_n}\cap E^{l_n}}\int_{R^{l_n}\setminus E^{l_n}} k^{s_n}_{\frac{r_n}{l_n}}(|x-y|)\,\mathrm{d}y\,\mathrm{d}x\\
\geq&C(\alpha,d,\lambda) l_n^{d-1}\sum_{R\in\mathcal{R}}\frac{1}{\boldsymbol{\beta}(\frac{r_n}{l_n},s_n)}\int_{R^{l_n}\cap E^{l_n}}\int_{R^{l_n}\setminus E^{l_n}} k^{s_n}_{\frac{r_n}{l_n}}(|x-y|)\,\mathrm{d}y\,\mathrm{d}x\\
\geq&C(\alpha,d,\lambda) l_n^{d-1}\sharp\mathcal{R}\,C(d,\lambda)\geq C(\alpha,d,\lambda)\mathrm{Per}(\tilde{E}_n)\,,
\end{aligned}$$

i.e., (2.5.8). Therefore, using (2.5.6), (2.5.7) and (2.5.8), by arguing as in Step 3 of the proof of Theorem 2.1.5(i), we get the claim whenever (2.5.5) is satisfied.

Finally, if

$$\lim_{n\to+\infty}(s_n-1)|\log r_n| = 0\,,$$

taking $l_n = r_n^\alpha$ (with $\alpha\in(0,1)$), one can show that

$$\begin{aligned}
\lim_{n\to+\infty} l_n^{s_n}\boldsymbol{\beta}(r_n,s_n) &= 0\,,\\
\lim_{n\to+\infty}\frac{\boldsymbol{\beta}(r_n,s_n)}{l_n^{1-s_n}\boldsymbol{\beta}(\frac{r_n}{l_n},s_n)} &= \frac{1}{1-\alpha}\,,
\end{aligned}$$

which used in the above proof, in place of (2.5.7) and (2.5.11), respectively, imply the claim also in this case. □

The following result follows by arguing as in the proof of Proposition 2.2.6, using now the estimates in the proof of Theorem 2.5.1(i) instead of the ones in the proof of Theorem 2.1.5 (i).

**Proposition 2.5.3.** *Let $\{E_n\}_{n\in\mathbb{N}}\subset\mathrm{M}_f(\mathbb{R}^d)$ be such that $\chi_{E_n}\to\chi_E$ in $\mathrm{L}^1(\mathbb{R}^d)$ as $n\to+\infty$, for some $E\in\mathrm{M}_f(\mathbb{R}^d)$. If*

$$\limsup_{n\to+\infty}\frac{\tilde{J}^{s_n}_{r_n}(E_n)}{\boldsymbol{\beta}(r_n,s_n)}<+\infty,$$

*then $E$ is a set of finite perimeter.*

### Proof of the lower bound

In order to prove the Γ-liminf inequality Theorem 2.5.1(ii), we first need the following result, which is the analogous to Lemma 2.3.1 under our assumptions on $\{s_n\}_{n\in\mathbb{N}}$ and $\{r_n\}_{n\in\mathbb{N}}$.

**Lemma 2.5.4.** *Let $\{r_n\}_{n\in\mathbb{N}} \subset (0,1)$ and $\{s_n\}_{n\in\mathbb{N}} \subset (1,+\infty)$ be such that $r_n \to 0^+$ and $s_n \to 1^+$ as $n \to +\infty$. For every $\varepsilon > 0$ there exist $\delta_0 > 0$ and $\bar n \in \mathbb{N}$ such that for every $\nu \in \mathbb{S}^{d-1}$, for every $E \in \mathrm{M}_f(\mathbb{R}^d)$ with*

$$|(E \triangle H^-_\nu(0)) \cap Q^\nu| < \delta_0$$

*and for every $n \geq \bar n$ it holds*

$$\int_{Q^\nu\cap E}\int_{Q^\nu\cap E^c} k^{s_n}_{r_n}(|x-y|)\,\mathrm{d}y\,\mathrm{d}x \geq \omega_{d-1}(1-\varepsilon)\boldsymbol{\beta}(r_n,s_n)\,.$$

*Proof.* By arguing as in the proof of Lemma 2.3.1 (see (2.3.10)) one can prove that

$$\begin{aligned}&\int_{Q^\nu\cap E}\int_{Q^\nu\cap E^c} k^{s_n}_{r_n}(|x-y|)\,\mathrm{d}y\,\mathrm{d}x\\ \geq&\,\omega_{d-1}\boldsymbol{\beta}(r_n,s_n)\Big(1-\eta(\delta_0)-\frac{1-\eta(\delta_0)}{\boldsymbol{\beta}(r_n,s_n)}\frac{\delta_0^{1-s_n}-1}{s_n-1}-2C(d)\sqrt{\delta_0}\Big)\,,\end{aligned} \tag{2.5.12}$$

where $\eta(t) \to 0$ as $t \to 0$. Notice that we can choose $0 < \delta_0 < 1$ such that

$$\eta(\delta_0) + 2C(d)\sqrt{\delta_0} \leq \frac{\varepsilon}{2}\,. \tag{2.5.13}$$

Furthermore, since

$$\lim_{n\to+\infty}\frac{\delta_0^{1-s_n}-1}{s_n-1} = |\log\delta_0| \quad \text{and} \quad \lim_{n\to+\infty}\boldsymbol{\beta}(r_n,s_n) = +\infty\,,$$

we have that there exists $\bar n \in \mathbb{N}$ such that

$$\frac{1-\eta(\delta_0)}{\boldsymbol{\beta}(r_n,s_n)}\frac{\delta_0^{1-s_n}-1}{s_n-1} \leq \frac{\varepsilon}{2} \quad \text{for } n \geq \bar n\,. \tag{2.5.14}$$

Therefore, by (2.5.12), (2.5.13) and (2.5.14), we get the claim. □

*Proof of Theorem 2.5.1(ii).* The proof closely follows the one of Theorem 2.1.5(ii). We can assume without loss of generality that

$$\frac{1}{2\boldsymbol{\beta}(r_n,s_n)}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k^{s_n}_{r_n}(|x-y|)|\chi_{E_n}(x)-\chi_{E_n}(y)|\,\mathrm{d}y\,\mathrm{d}x \leq C\,, \tag{2.5.15}$$

for some constant $C > 0$ independent of $n$. Then, by Corollary 2.5.3 we have that $E$ has finite perimeter. For every $n \in \mathbb{N}$ let $\mu_n$ be the measure on the product space $\mathbb{R}^d \times \mathbb{R}^d$ defined by

$$\mu_n(A\times B) := \frac{1}{2\boldsymbol{\beta}(r_n,s_n)}\int_A\int_B k^{s_n}_{r_n}(|x-y|)|\chi_{E_n}(x)-\chi_{E_n}(y)|\,\mathrm{d}y\,\mathrm{d}x$$

for every $A, B \in \mathrm{M}(\mathbb{R}^d)$. By arguing as in the proof of Theorem 2.1.5(ii) we have that, up to a subsequence, $\mu_n \to \mu$ as $n \to +\infty$ for some measure $\mu$ concentrated on the set $D := \{(x,x) : x \in \mathbb{R}^d\}$. Therefore, by using the same Radon-Nykodym argument exploited in the proof of Theorem 2.1.5(ii), it is enough to show that for $\mathcal{H}^{d-1}$ - a.e. $x_0 \in \partial^* E$

$$\liminf_{l\to0^+}\frac{\mu(Q^\nu_l(x_0)\times Q^\nu_l(x_0)}{l^{d-1}} \geq \liminf_{l\to0^+}\liminf_{n\to+\infty}\frac{\mu_n(Q^\nu_l(x_0)\times Q^\nu_l(x_0))}{l^{d-1}} \geq \omega_{d-1}\,, \tag{2.5.16}$$

where we have set $\nu := \nu_E(x_0)$ and $Q^l_\nu(x_0) := x_0 + lQ^\nu$ . In order to prove (2.5.16) we adopt the same strategy used to prove (2.3.12). More precisely, setting $F_{n,l} = x_0 + lE_n$ , in place of (2.3.14) we have

$$\begin{aligned}&\frac{1}{l^{d-1}}\mu_n(Q^\nu_l(x_0)\times Q^\nu_l(x_0))\\&=\frac{l^{1-s_n}}{2\boldsymbol{\beta}(r_n,s_n)}\int_{Q^\nu}\int_{Q^\nu}k^{s_n}_{\frac{r_n}{l}}(|\xi-\eta|)|\chi_{F_{n,l}}(\xi)-\chi_{F_{n,l}}(\eta)|\,\mathrm{d}\xi\,\mathrm{d}\eta\,,\end{aligned}\tag{2.5.17}$$

and, in place of (2.3.17),

$$\begin{aligned}\frac{1}{2}\int_{Q^\nu}\int_{Q^\nu}k^{s_n}_{\frac{r_n}{l}}(|\xi-\eta|)|\chi_{F_{n,l}}(\xi)-\chi_{F_{n,l}}(\eta)|\,\mathrm{d}\xi\,\mathrm{d}\eta&\\\geq\omega_{d-1}(1-\varepsilon)\boldsymbol{\beta}\Big(\frac{r_n}{l},s_n\Big)\,,&\end{aligned}\tag{2.5.18}$$

which is a consequence of Lemma 2.5.4. Therefore, since

$$\lim_{n\to+\infty}\frac{l^{1-s_n}}{\boldsymbol{\beta}(r_n,s_n)}\boldsymbol{\beta}\Big(\frac{r_n}{l},s_n\Big)=1\,,$$

by (2.5.17) and (2.5.18), we get

$$\liminf_{l\to0^+}\frac{\mu(Q^\nu_l(x_0)\times Q^\nu_l(x_0))}{l^{d-1}}\geq(1-\varepsilon)\omega_{d-1}\,,$$

whence (2.5.16) follows by the arbitrariness of $\varepsilon$ . □

### Proof of the upper bound

In order to prove the Γ-limsup inequality, we need the following result which is the analogous of Proposition 2.1.1 when both $r$ and $s$ vary.

**Proposition 2.5.5.** *Let $E\in\mathrm{M}_f(\mathbb{R}^d)$ be a smooth set. Then*

$$\lim_{n\to+\infty}\frac{\tilde J^{s_n}_{r_n}(E)}{\boldsymbol{\beta}(r_n,s_n)}=\omega_{d-1}\mathrm{Per}(E).$$

*Proof.* By Lemma 2.1.4 and by formula (2.1.9) we have

$$\begin{aligned}\frac{\tilde J^{s_n}_{r_n}(E)}{\boldsymbol{\beta}(r_n,s_n)}&=\omega_{d-1}\mathrm{Per}(E)+\frac{1}{\boldsymbol{\beta}(r_n,s_n)}F^{s_n}_1(E)\\&-\frac{1}{\boldsymbol{\beta}(r_n,s_n)}\int_{\partial E}\mathrm{d}\mathcal{H}^{d-1}(y)\int_{(E\triangle H^-_{\nu_E(y)}(y))\cap B(y,r_n)}\frac{1}{{r_n}^{s_n}}\frac{|(y-x)\cdot\nu_E(y)|}{|x-y|^d}\Big(\frac{d+s_n}{ds_n}\Big)\,\mathrm{d}x\\&+\frac{1}{\boldsymbol{\beta}(r_n,s_n)}\int_{\partial E}\mathrm{d}\mathcal{H}^{d-1}(y)\int_{(E\triangle H^-_{\nu_E(y)}(y))\cap B(y,r_n)}\frac{1}{{r_n}^{s_n}}\frac{|(y-x)\cdot\nu_E(y)|}{{r_n}^d}\frac{1}{d}\,\mathrm{d}x\\&-\frac{1}{\boldsymbol{\beta}(r_n,s_n)}\frac{1}{s_n}\int_{\partial E}\mathrm{d}\mathcal{H}^{d-1}(y)\int_{(E\triangle H^-_{\nu_E(y)}(y))\cap(B(y,1)\setminus B(y,r_n))}\frac{|(y-x)\cdot\nu_E(y)|}{|x-y|^{d+s_n}}\,\mathrm{d}x\\&-\frac{1}{\boldsymbol{\beta}(r_n,s_n)}\frac{1}{s_n}\int_E\mathcal{H}^{d-1}(E^c\cap\partial B(x,1))\,\mathrm{d}x\,,\end{aligned}$$

where $H^-_\nu(y)$ is the set defined in (0.0.5). Therefore, by arguing verbatim as in the proof of Proposition 2.1.1 and using (2.5.2), we deduce the claim. □

With Proposition 2.5.5 in hand, the proof of Theorem 2.5.1(iii) is fully analogous to the one of Theorem 2.1.5(iii) and is omitted.

### 2.5.2 Convergence of the $k_{r_n}^{s_n}$-curvature flows to the mean curvature flow

Here we study the convergence of the curvatures $\mathcal{K}_{r_n}^{s_n}$ defined in (2.4.1) to the classical mean curvature $\mathcal{K}^1$ in (2.4.21) when $r_n \to 0^+$ and $s_n \to 1^+$ simultaneously. As in Subsection 2.4.3 we use such a result to deduce the convergence of the corresponding geometric flows.

First we prove the following lemma which is the analogous of Lemma 2.4.5 in the case treated in this section.

**Lemma 2.5.6.** *Let $\{s_n\}_{n\in\mathbb{N}} \subset (1,+\infty)$ and $\{r_n\}_{n\in\mathbb{N}} \subset (0,1)$ be such that $r_n \to 0^+$ and $s_n \to 1^+$ as $n \to +\infty$. Let $M, N \in \mathbb{R}^{(d-1)\times(d-1)}$ and let $\{M_n\}_{n\in\mathbb{N}}, \{N_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^{(d-1)\times(d-1)}$ be such that $M_n \to M$, $N_n \to N$ as $n \to +\infty$. Then, for every $\delta > 0$, it holds*

$$\begin{aligned} \lim_{n\to+\infty}\Big(\frac{1}{\beta(r_n,s_n)}\Big(\int_{\mathcal{F}^1_{n,\delta}} k_{r_n}^{s_n}(|y|)\,\mathrm{d}y - \int_{\mathcal{F}^2_{n,\delta}} k_{r_n}^{s_n}(|y|)\,\mathrm{d}y\Big)\Big) \\ = \int_{\mathbb{S}^{d-2}} \theta^*(N-M)\theta\,\mathrm{d}\mathcal{H}^{d-2}(\theta)\,, \end{aligned} \tag{2.5.19}$$

*where*

$$\begin{aligned} \mathcal{F}^1_{n,\delta} :=& \{y = (y', y_d) \in B(0,\delta)\,:\,(y')^* M_n y' \le y_d \le (y')^* N_n y'\} \\ \mathcal{F}^2_{n,\delta} :=& \{y = (y', y_d) \in B(0,\delta)\,:\,(y')^* N_n y' \le y_d \le (y')^* M_n y'\}\,. \end{aligned}$$

*Proof.* By arguing verbatim as in the proof of (2.4.35) one can show that

$$\begin{aligned} &\frac{1}{\beta(r_n,s_n)}\int_{\mathcal{F}^1_{n,\delta}} k_{r_n}^{s_n}(|y|)\,\mathrm{d}y - \frac{1}{\beta(r_n,s_n)}\int_{\mathcal{F}^2_{n,\delta}} k_{r_n}^{s_n}(|y|)\,\mathrm{d}y \\ =&\frac{{r_n}^{1-s_n}}{(d+1)\beta(r_n,s_n)}\int_{\mathbb{S}^{d-2}}\theta^*(N_n-M_n)\theta\,\mathrm{d}\mathcal{H}^{d-2}(\theta) \\ &+\frac{1}{\beta(r_n,s_n)}\int_{\mathbb{S}^{d-2}}\mathrm{d}\mathcal{H}^{d-2}(\theta)\int_{r_n}^{\delta}\mathrm{d}\rho\frac{1}{\rho^{s_n}}\int_{\theta^*M_n\theta}^{\theta^*N_n\theta}\frac{1}{(1+\rho^2t^2)^{\frac{d+s_n}{2}}}\,\mathrm{d}t\,, \\ &+\eta_n\,, \end{aligned} \tag{2.5.20}$$

where $\eta_n \to 0$ as $n \to +\infty$. It is easy to see that

$$\lim_{n\to+\infty}\frac{{r_n}^{1-s_n}}{\beta(r_n,s_n)} = 0\,,$$

whence we get

$$\lim_{n\to+\infty}\frac{r_n^{1-s_n}}{(d+1)\beta(r_n,s_n)}\int_{\mathbb{S}^{d-2}}\theta^*(N_n-M_n)\theta\,\mathrm{d}\mathcal{H}^{d-2}(\theta) = 0\,. \tag{2.5.21}$$

Now we claim that for every $t \in \mathbb{R}$

$$\lim_{n\to+\infty}\frac{1}{\beta(r_n,s_n)}\int_{r_n}^{\delta}\frac{1}{\rho^{s_n}}\frac{1}{(1+\rho^2t^2)^{\frac{d+s_n}{2}}}\,\mathrm{d}\rho = 1\,, \tag{2.5.22}$$

which in view of (2.5.20) and (2.5.21) and of the Dominate Convergence Theorem, implies (2.5.19). In order to prove (2.5.22), we first notice that

$$\begin{aligned}
&\frac{1}{\boldsymbol{\beta}(r_n,s_n)}\int_{r_n}^{\delta}\frac{1}{\rho^{s_n}}\frac{1}{(1+\rho^2t^2)^{\frac{d+s_n}{2}}}\\
=&\frac{1}{\boldsymbol{\beta}(r_n,s_n)}\int_{r_n}^{\delta}\frac{1}{\rho^{s_n}}\,\mathrm{d}\rho-\frac{1}{\boldsymbol{\beta}(r_n,s_n)}\int_{r_n}^{\delta}\frac{1}{\rho^{s_n}}\Big(1-\frac{1}{(1+\rho^2t^2)^{\frac{d+s_n}{2}}}\Big)\,\mathrm{d}\rho\\
=&\frac{1}{\boldsymbol{\beta}(r_n,s_n)}\frac{r_n^{1-s_n}-\delta^{1-s_n}}{s_n-1}-\frac{1}{\boldsymbol{\beta}(r_n,s_n)}\int_{r_n}^{\delta}\frac{1}{\rho^{s_n}}\Big(1-\frac{1}{(1+\rho^2t^2)^{\frac{d+s_n}{2}}}\Big)\,\mathrm{d}\rho\,,
\end{aligned}$$

so that, by the very definition of $\boldsymbol{\beta}$ in (2.5.1), it is enough to show that

$$\limsup_{n\to+\infty}\frac{1}{\boldsymbol{\beta}(r_n,s_n)}\int_{r_n}^{\delta}\frac{1}{\rho^{s_n}}\Big(1-\frac{1}{(1+\rho^2t^2)^{\frac{d+s_n}{2}}}\Big)\,\mathrm{d}\rho=0\,. \tag{2.5.23}$$

As for the proof of (2.5.23) we argue as follows. First we notice that, setting $S:=\sup_{n\in\mathbb{N}}s_n$,

$$(1+\rho^2t^2)^{\frac{d+s_n}{2}}\le 1+C(d,S,t)\rho^2\,,$$

so that, for $n$ large enough,

$$\begin{aligned}
&\frac{1}{\boldsymbol{\beta}(r_n,s_n)}\int_{r_n}^{\delta}\frac{1}{\rho^{s_n}}\Big(1-\frac{1}{(1+\rho^2t^2)^{\frac{d+s_n}{2}}}\Big)\,\mathrm{d}\rho\\
=&\frac{1}{\boldsymbol{\beta}(r_n,s_n)}\int_{r_n}^{\delta}\frac{1}{\rho^{s_n}}\frac{(1+\rho^2t^2)^{\frac{d+s_n}{2}}-1}{(1+\rho^2t^2)^{\frac{d+s_n}{2}}}\,\mathrm{d}\rho\\
\le&\frac{1}{\boldsymbol{\beta}(r_n,s_n)}\int_{r_n}^{\delta}C(d,S,t)\rho^{2-s_n}\,\mathrm{d}\rho\to 0\qquad\text{as } n\to+\infty\,,
\end{aligned}$$

thus concluding the proof of (2.5.23) and of the whole lemma. □

By using Lemma 2.5.6 in place of Lemma 2.4.5 in the proof of Theorem 2.4.6, one can prove the following result.

**Theorem 2.5.7.** *Let $\{r_n\}_{n\in\mathbb{N}}\subset(0,1)$ and $s_n\subset(1,+\infty)$ be such that $r_n\to0^+$ and $s_n\to1^+$ as $n\to+\infty$. Let $\{E_n\}_{n\in\mathbb{N}}\subset\mathfrak{C}$ such that $E_n\to E$ in $\mathfrak{C}$ as $n\to+\infty$, for some $E\in\mathfrak{C}$. Then for every $x\in\partial E\cap\partial E_n$ for every $n\in\mathbb{N}$,*

$$\lim_{n\to+\infty}\frac{\mathcal{K}^{s_n}_{r_n}(x,E_n)}{\boldsymbol{\beta}(r_n,s_n)}=\omega_{d-1}\mathcal{K}^1(x,E).$$

Finally, by using Theorem 2.5.7 in place of Theorem 2.4.6 in the proof of Theorem 2.4.7, one can prove the following result which provides the convergence of the $k^{s_n}_{r_n}$-nonlocal curvature flows when $r_n\to0^+$ and $s_n\to1^+$.

**Theorem 2.5.8.** *Let $\{r_n\}_{n\in\mathbb{N}}\subset(0,1)$ and $s_n\subset(1,+\infty)$ be such that $r_n\to0^+$ and $s_n\to1^+$ as $n\to+\infty$. Let $u_0\in C(\mathbb{R}^d)$ be a uniformly continuous function with $u_0=C_0$ in $\mathbb{R}^d\setminus B(0,R_0)$ for some $C_0,R_0\in\mathbb{R}$ with $R_0>0$. For every $n\in\mathbb{N}$, let $u^{s_n}_{r_n}:\mathbb{R}^d\times[0,+\infty)\to\mathbb{R}$ be the viscosity solution to the Cauchy problem (2.4.27) (with $r$ and $s$ replaced by $r_n$ and $s_n$, respectively). Then, setting $v^{s_n}_{r_n}(x,t):=u^{s_n}_{r_n}(x,\frac{t}{\boldsymbol{\beta}(r_n,s_n)})$ for all $x\in\mathbb{R}^d$, $t\ge0$, we have that, for every $T>0$, $v^{s_n}_{r_n}$ uniformly converge to $u$ as $n\to+\infty$ in $\mathbb{R}^d\times[0,T]$, where $u:\mathbb{R}^d\times[0,+\infty)\to\mathbb{R}$ is the viscosity solution to the classical mean curvature flow (2.4.42).*

## 2.6 Anisotropic kernels and applications to dislocation dynamics

In this section we study the asymptotic behavior of supercritical nonlocal perimeters and the corresponding geometric flows in the case of anisotropic kernels. Moreover, we present an application to the dynamics of dislocation curves in two dimensions.

### 2.6.1 Anisotropic kernels

Let $g \in C(\mathbb{S}^{d-1};(0,+\infty))$ be such that $g(\xi)=g(-\xi)$ for every $\xi \in \mathbb{S}^{d-1}$. For every $s \geq 1$ and for every $r>0$ we define the function $k_r^{g,s}:\mathbb{R}^d\setminus\{0\}\to(0,+\infty)$ as $k_r^{g,s}(x):=g(\frac{x}{|x|})k_r^s(|x|)$, where $k_r^s$ is defined in (2.1.1). Here we study the asymptotic behavior, as $r\to 0^+$ of the functionals $\tilde{J}_r^{g,s}:\mathrm{M}_f(\mathbb{R}^d)\to[0,+\infty)$ defined by

$$\tilde{J}_r^{g,s}(E):=\int_E\int_{E^c}k_r^{g,s}(y-x)\,\mathrm{d}y\,\mathrm{d}x\,. \tag{2.6.1}$$

In Proposition 2.6.1 below we will show that the functionals $\tilde{J}_r^{g,s}$ scaled by $\sigma^s(r)$ converge as $r\to 0^+$ to the anisotropic perimeter $\mathrm{Per}^g$ defined on finite perimeter sets as

$$\mathrm{Per}^g(E):=\int_{\partial^* E}\varphi^g(\nu_E(x))\,\mathrm{d}\mathcal{H}^{d-1}(x)\,, \tag{2.6.2}$$

where the density $\varphi^g$ is given by

$$\varphi^g(\nu):=\int_{\{\xi\in\mathbb{S}^{d-1}\,:\,\xi\cdot\nu\geq 0\}}g(\xi)\,\xi\cdot\nu\,\mathrm{d}\mathcal{H}^{d-1}(\xi)\,,\qquad\text{for every }\nu\in\mathbb{S}^{d-1}\,. \tag{2.6.3}$$

**Proposition 2.6.1.** *For every $s\geq 1$ and for every set $E\in\mathrm{M}_f(\mathbb{R}^d)$ of finite perimeter it holds*

$$\lim_{r\to 0^+}\frac{\tilde{J}_r^{g,s}(E)}{\sigma^s(r)}=\mathrm{Per}^g(E)\,.$$

*Proof.* First we claim the following anisotropic version of formula (2.1.12):

$$\begin{aligned}
&\int_E\int_{E^c\cap B(x,1)}k_r^{g,s}(y-x)\,\mathrm{d}y\,\mathrm{d}x\\
=&\frac{d+s}{dsr^s}\int_{\partial^* E}\mathrm{d}\mathcal{H}^{d-1}(y)\int_{E\cap B(y,r)}g\Big(\frac{y-x}{|x-y|}\Big)\frac{(y-x)}{|x-y|^d}\cdot\nu_E(y)\,\mathrm{d}x\\
&-\frac{1}{dr^{d+s}}\int_{\partial^* E}\mathrm{d}\mathcal{H}^{d-1}(y)\int_{E\cap B(y,r)}g\Big(\frac{y-x}{|x-y|}\Big)(y-x)\cdot\nu_E(y)\,\mathrm{d}x\\
&+\frac{1}{s}\int_{\partial^* E}\mathrm{d}\mathcal{H}^{d-1}(y)\int_{E\cap(B(y,1)\setminus B(y,r))}g\Big(\frac{y-x}{|x-y|}\Big)\frac{(y-x)\cdot\nu_E(y)}{|x-y|^{d+s}}\,\mathrm{d}x\\
&-\frac{1}{s}\int_E\mathrm{d}x\int_{E^c\cap\partial B(x,1)}g\Big(\frac{y-x}{|x-y|}\Big)\,\mathrm{d}\mathcal{H}^{d-1}(y)\,.
\end{aligned} \tag{2.6.4}$$

If $g\in C^1(\mathbb{S}^{d-1})$, the proof of (2.6.4) is identical to the proof of (2.1.12), noticing that $\nabla g(\frac{x}{|x|})\cdot T_r^s(x)=0$ for every $x\in\mathbb{R}^d\setminus\{0\}$, with $T_r^s$ defined in (2.1.10). The case $g\in C(\mathbb{S}^{d-1})$ follows by standard density arguments. □

In Theorem 2.6.2 below we will see that the functionals $\tilde{J}_r^{g,s}$ actually $\Gamma$-converge, as $r \to 0^+$, to $\mathrm{Per}^g$ .

**Theorem 2.6.2.** *Let $s \geq 1$ and let $\{r_n\}_{n\in\mathbb{N}} \subset (0, +\infty)$ be such that $r_n \to 0$ as $n \to +\infty$. The following $\Gamma$-convergence result holds true.*

*(i) (Compactness) Let $U \subset \mathbb{R}^d$ be an open bounded set and let $\{E_n\}_{n\in\mathbb{N}} \subset \mathrm{M}(\mathbb{R}^d)$ be such that $E_n \subset U$ for every $n \in \mathbb{N}$ and*

$$\tilde{J}_{r_n}^{g,s}(E_n) \leq M\sigma^s(r_n) \qquad \text{for every } n \in \mathbb{N},$$

*for some constant $M$ independent of $n$. Then, up to a subsequence, $\chi_{E_n} \to \chi_E$ strongly in $L^1(\mathbb{R}^d)$ for some set $E \in \mathrm{M}_f(\mathbb{R}^d)$ with $\mathrm{Per}(E) < +\infty$.*

*(ii) (Lower bound) Let $E \in \mathrm{M}_f(\mathbb{R}^d)$. For every $\{E_n\}_{n\in\mathbb{N}} \subset \mathrm{M}_f(\mathbb{R}^d)$ with $\chi_{E_n} \to \chi_E$ strongly in $L^1(\mathbb{R}^d)$ it holds*

$$\mathrm{Per}^g(E) \leq \liminf_{n\to+\infty} \frac{\tilde{J}_{r_n}^{g,s}(E_n)}{\sigma^s(r_n)} .$$

*(iii) (Upper bound) For every $E \in \mathrm{M}_f(\mathbb{R}^d)$ there exists $\{E_n\}_{n\in\mathbb{N}} \subset \mathrm{M}_f(\mathbb{R}^d)$ such that $\chi_{E_n} \to \chi_E$ strongly in $L^1(\mathbb{R}^d)$ and*

$$\mathrm{Per}^g(E) = \lim_{n\to+\infty} \frac{\tilde{J}_{r_n}^{g,s}(E_n)}{\sigma^s(r_n)} .$$

*Proof.* The proof of the compactness property (i) follows by Theorem 2.1.5(i), once noticed that there exist two positive constants $c_1 < c_2$ such that $c_1 \leq g(\theta) \leq c_2$ for every $\theta \in \mathbb{S}^{d-1}$. As for the proof of the $\Gamma$-liminf inequality in (ii) one can argue verbatim as in the proof of Theorem 2.1.5(ii), using the following inequality

$$\int_{Q^\nu}\int_{Q^\nu} g\Big(\frac{x-y}{|x-y|}\Big) k_r^s(|x-y|)|\chi_E(x) - \chi_E(y)| \,\mathrm{d}y \,\mathrm{d}x \geq (1-\varepsilon)\sigma^s(r)\varphi^g(\nu) , \tag{2.6.5}$$

in place of (2.3.2). The proof of (2.6.5) under the assumptions of Lemma 2.3.1 is identical to the proof of Lemma 2.3.1 (see (2.3.8)).

Finally, the $\Gamma$-limsup inequality (iii) follows as in the isotropic case Theorem 2.1.5(iii) using Proposition 2.6.1 in place of Proposition 2.1.1. □

We introduce the notion of $k_r^{g,s}$ curvature and we study the convergence as $r \to 0^+$ of the corresponding geometric flows.

Let $s \geq 1$, $r > 0$ and $E \in \mathfrak{C}$. For every $x \in \partial E$ we define the $k_r^{g,s}$-*curvature of $E$ at $x$* as

$$\mathcal{K}_r^{g,s}(x, E) := \int_{\mathbb{R}^d} (\chi_{E^c}(y) - \chi_E(y)) k_r^{g,s}(x-y) \,\mathrm{d}y. \tag{2.6.6}$$

*Remark* 2.6.3. We notice that for every $E \in \mathfrak{C}$ and for every $x \in \partial E$ it holds

$$\begin{aligned}
\mathcal{K}_r^{g,s}(x, E) &= \int_{\mathbb{R}^d} k_r^{g,s}(x-y) \,\mathrm{d}y - 2\int_E k_r^{g,s}(x-y) \,\mathrm{d}y \\
&= \int_{\mathbb{R}^d} k_r^{g,s}(z) \,\mathrm{d}z - 2k_r^{g,s} * \chi_E(x) \\
&= \Big(-2k_r^{g,s} + \int_{\mathbb{R}^d} k_r^{g,s}(z) \,\mathrm{d}z \,\boldsymbol{\delta}_0\Big) * \chi_E(x) ,
\end{aligned} \tag{2.6.7}$$

where $*$ denotes the convolution operator and $\boldsymbol{\delta}_0$ is the Dirac delta centered at $0$. By (2.6.7) we have that $\mathcal{K}_r^{g,s}$ is exactly the type of curvatures considered in [35, formula (1.4)]. We remark that the positive part of the curvature $\mathcal{K}_r^{g,s}$ is concentrated on a point. This is why, as already observed in [35], the curvature $\mathcal{K}_r^{g,s}$, although having a positive contribution, still satisfies the desired monotonicity property with respect to set inclusion (see the proof of (M) in Proposition 2.4.2).

We first show that $\mathcal{K}_r^{g,s}$ is the first variation of $\tilde{J}_r^{g,s}$ in the sense specified by the following proposition, which is the anisotropic analogous of Proposition 2.4.1.

**Proposition 2.6.4.** *Let $s \geq 1$, $r > 0$, and $E \in \mathfrak{C}$. Let $\Phi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ be as in Proposition 2.4.1. Setting $E_t := \Phi_t(E)$ and $\Psi(\cdot) := \frac{\partial}{\partial t}\Phi_t(\cdot)\big|_{t=0}$, we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{J}_r^{g,s}(E_t)\bigg|_{t=0} = \int_{\partial E} \mathcal{K}_r^{g,s}(x,E)\Psi(x)\cdot \nu_E(x)\,\mathrm{d}\mathcal{H}^{d-1}(x)\,.$$

*Proof.* If $g \in C^1$, then the proof is fully analogous to the proof of Proposition 2.4.1. The case when $g \in C^0$ follows by a density argument, using that if $\{g_n\}_{n\in\mathbb{N}} \subset C^1(\mathbb{S}^{d-1};(0,+\infty))$ uniformly converges to $g$, $E_n \to E$ in $\mathfrak{C}$ and $x_n \to x$, then $\mathcal{K}_r^{g_n,s}(x_n,E_n)$ converge to $\mathcal{K}_r^{g,s}(x,E)$. Such a continuity property can be proved as in Proposition 2.4.2 (UC). □

By arguing as in the proof of Proposition 2.4.2 one can show that the curvatures $\mathcal{K}_r^{g,s}$ satisfy properties (M), (T), (S), (B), (UC). Now we introduce the (local) anisotropic curvatures $\mathcal{K}^{g,1}$ defined as follows. Let $E \in \mathfrak{C}$ and $x \in \partial E$; in a neighborhood of $x$, $\partial E$ is the graph of function $f \in C^2(H^0_{\nu_E(x)}(x))$ (see (0.0.7) for the definition of $H^0_\nu(x)$) with $\mathrm{D}f(x) = 0$ (here and below $\mathrm{D}f$ and $\mathrm{D}^2 f$ are computed with respect to a given system of orthogonal coordinates on $H^0_{\nu_E(x)}(x)$). The anisotropic mean curvature of $\partial E$ at $x$ is given by

$$\mathcal{K}^{g,1}(x,E) = -\int_{H^0_{\nu_E(x)}(x)\cap\mathbb{S}^{d-1}} g(\xi)\xi^*\mathrm{D}^2 f(x)\xi\,\mathrm{d}\mathcal{H}^{d-2}(\xi)\,. \tag{2.6.8}$$

One can check that $\mathcal{K}^{g,1}$ is the first variation of $\mathrm{Per}^g$ in the sense specified by Proposition 2.4.1 (we refer to [13] for the first variation formula of generic anisotropic perimeters, while we leave to the reader the computations for the specific anisotropic density $\varphi^g$ considered here, defined in (2.6.3)). Notice that if $g \equiv 1$, then $\mathcal{K}^{g,1} = \omega_{d-1}\mathcal{K}^1$ where $\mathcal{K}^1$ is the classical mean curvature defined in (2.4.21). Moreover, one can check that $\mathcal{K}^{g,1}$ satisfies properties (M), (T), (S), (B), (UC') in Proposition 2.4.3.

In Proposition 2.6.5 below we show that the curvatures $\mathcal{K}_r^{g,s}$ converge, as $r \to 0^+$, to the anisotropic curvature $\mathcal{K}^{g,1}$.

**Theorem 2.6.5.** *Let $s \geq 1$. Let $\{E_r\}_{r>0} \subset \mathfrak{C}$ be such that $E_r \to E$ in $\mathfrak{C}$ as $r \to 0^+$, for some $E \in \mathfrak{C}$. Then, for every $x \in \partial E \cap \partial E_r$ for all $r > 0$, it holds*

$$\lim_{r\to 0^+} \frac{\mathcal{K}_r^{g,s}(x,E_r)}{\sigma^s(r)} = \mathcal{K}^{g,1}(x,E). \tag{2.6.9}$$

*Proof.* The proof of (2.6.9) is fully analogous to the one of Theorem 2.4.6 and in particular it is based on a suitable anisotropic variant of Lemma 2.4.5. In fact, Lemma 2.4.5 can be extended also to the anisotropic case with (2.4.28) replaced by

$$\begin{aligned}&\lim_{r\to 0^+}\Big(\frac{1}{\sigma^s(r)}\Big(\int_{\mathcal{F}^1_{r,\delta}} g\Big(\frac{y}{|y|}\Big)k^s_r(|y|)\,\mathrm{d}y-\int_{\mathcal{F}^2_{r,\delta}} g\Big(\frac{y}{|y|}\Big)k^s_r(|y|)\,\mathrm{d}y\Big)\Big)\\&\quad=\int_{\mathbb{S}^{d-2}} g(\theta,0)\theta^*(N-M)\theta\,\mathrm{d}\mathcal{H}^{d-2}(\theta)\,.\end{aligned} \tag{2.6.10}$$

If $\nu_E(x)=e_d$, one can argue verbatim as in the proof of Theorem 2.4.6, clearly using (2.6.10) in place of (2.4.28). The same proof with only minor notational changes can be adapted also to the case $\nu_E(x)\neq e_d$. □

We are now in a position to state our result on the convergence of the geometric flows of $\mathcal{K}^{g,s}_r$ as $r\to 0^+$, whose proof is omitted, being fully analogous to the one of Theorem 2.4.7.

**Theorem 2.6.6.** *Let $s\geq 1$ be fixed. Let $u_0\in C(\mathbb{R}^d)$ be a uniformly continuous function with $u_0=C_0$ in $\mathbb{R}^d\setminus B(0,R_0)$ for some $C_0,R_0\in\mathbb{R}$ with $R_0>0$. For every $r>0$, let $u^s_r:\mathbb{R}^d\times[0,+\infty)\to\mathbb{R}$ be the viscosity solution to the Cauchy problem*

$$\begin{cases}\partial_t u(x,t)+|\mathrm{D}u(x,t)|\mathcal{K}^{g,s}_r(x,\{y\,:\,u(y,t)\geq u(x,t)\})=0\\ u(x,0)=u_0(x)\,.\end{cases}$$

*Then, setting $v^s_r(x,t):=u^s_r(x,\frac{t}{\sigma^s(r)})$ for all $x\in\mathbb{R}^d$, $t\geq 0$, we have that, for every $T>0$, $v^s_r$ uniformly converge to $u$ as $r\to 0^+$ in $\mathbb{R}^d\times[0,T]$, where $u:\mathbb{R}^d\times[0,+\infty)\to\mathbb{R}$ is the viscosity solution to the anisotropic mean curvature flow*

$$\begin{cases}\partial_t u(x,t)+|\mathrm{D}u(x,t)|\mathcal{K}^{g,1}(x,\{y\,:\,u(y,t)\geq u(x,t)\})=0\\ u(x,0)=u_0(x)\,.\end{cases}$$

### 2.6.2 Applications to dislocation dynamics

Here we apply the results in Subsection 2.6.1 to the motion of curved dislocations in the plane. To this purpose, we briefly recall and describe, in an informal way, some notions about the isotropic linearized elastic energy induced by planar dislocations; such notions are well known to experts and we refer to classic books such as [50] for an exhaustive monography on this subject.

Let $E$ be a bounded region of the plane $\mathbb{R}^2=\mathbb{R}^3\cap\{z\in\mathbb{R}^3\,:\,z_3=0\}$, representing a plastic slip region with Burgers vector $b=e_1=(1,0,0)$. Formally, the elastic energy induced by such a dislocation is given by

$$J(E):=\frac{\boldsymbol{\mu}}{8\pi}\int_E\int_{E^c}\frac{1}{|x-y|^5}\Big(\frac{1+\boldsymbol{\nu}}{1-\boldsymbol{\nu}}x_1^2+\frac{1-2\boldsymbol{\nu}}{1-\boldsymbol{\nu}}x_2^2\Big)\,\mathrm{d}y\,\mathrm{d}x\,, \tag{2.6.11}$$

where $\boldsymbol{\mu}>0$ and $\boldsymbol{\nu}\in(-1,\frac12)$ are the shear modulus and the Poisson's ratio, respectively. Formula (2.6.11) can be deduced by [50, formula (4-44)], by integrating by parts. Clearly, the energy $J$ in (2.6.11) is always infinite whenever $E$ is non-empty. It is well understood that such an infinite energy should be suitably truncated through ad hoc core regularizations, specific of the microscopic details of the underlying crystal. The specific choice of the core regularization, giving back the physically relevant (finite) elastic energy induced by the dislocation is, for our purposes, irrelevant. Here we adopt the energy-renormalization procedure introduced in (2.6.1). First we set

$$g(\xi) := \frac{\boldsymbol{\mu}}{8\pi}\Big(\frac{1+\boldsymbol{\nu}}{1-\boldsymbol{\nu}}\xi_1^2 + \frac{1-2\boldsymbol{\nu}}{1-\boldsymbol{\nu}}\xi_2^2\Big), \quad \text{for every } \xi \in \mathbb{S}^1, \tag{2.6.12}$$

and we notice that the energy in (2.6.11) can be (formally) rewritten as

$$J(E) = \int_E \int_{E^c} g\Big(\frac{x-y}{|x-y|}\Big)\frac{1}{|x-y|^3}\,\mathrm{d}y\,\mathrm{d}x = \int_E \int_{E^c} k^g(x-y)\,\mathrm{d}y\,\mathrm{d}x\,,$$

where $k^g$ is defined by $k^g(z) := g(\frac{z}{|z|})\frac{1}{|z|^3}$. The core-regularization of $J$ is given by the functional $\tilde{J}^{g,1}_r$ defined by (2.6.1), where the parameter $r > 0$ plays the role of the core-size. Now, consider the dynamics of a dislocation curve, enclosing a (moving) bounded set $E$, with Burgers vector equal to $e_1$, governed by a self-energy release mechanism. We consider a geometric evolution, that can be formally understood as the gradient flow of the self-energy $\tilde{J}^{g,1}_r$ with respect to an $\mathrm{L}^2$ structure on the (graphs locally describing the) evolving dislocation curve. If the energy were the standard perimeter, this evolution would be nothing but the standard mean curvature flow. Notice that the energy considered here is nonlocal; moreover, although it is derived from isotropic linearized elasticity, it has in fact an anisotropic dependence (induced by the direction of the given Burgers vector) on the normal to the curve. Another possible source of anisotropy is the so-called mobility, depending on the microscopic details of the underlying crystalline lattice; here, for simplicity, we assume that such a mobility is in fact isotropic, equal to one. The dynamics discussed above corresponds to the geometric evolution where the normal velocity of the evolving dislocation curve at any point $x$ is given by $-\mathcal{K}^{g,1}_r$, defined in (2.6.6).

In order to study the asymptotic behavior, as $r \to 0^+$, of the dynamics described above we use the results developed in Subsection 2.6.1. First, we notice that that the function $g$ defined in (2.6.12) is continuous (actually, it is smooth) and even, so that it satisfies the assumptions required in Subsection 2.6.1. Moreover, recalling (2.6.3) and (2.6.8), for the choice of $g$ in (2.6.12), an easy computation shows that

$$\begin{aligned}
\varphi^g(\nu) &= \frac{\boldsymbol{\mu}}{12\pi}\Big(\frac{1+\boldsymbol{\nu}}{1-\boldsymbol{\nu}}(1+\nu_1^2) + \frac{1-2\boldsymbol{\nu}}{1-\boldsymbol{\nu}}(1+\nu_2^2)\Big), \text{ for every } \nu \in \mathbb{S}^1, \\
\mathcal{K}^{g,1}(x,E) &= \frac{\boldsymbol{\mu}}{8\pi}\Big(\frac{1+\boldsymbol{\nu}}{1-\boldsymbol{\nu}}(\nu_E(x))_2^2 + \frac{1-2\boldsymbol{\nu}}{1-\boldsymbol{\nu}}(\nu_E(x))_1^2\Big), \text{ for every } E \in \mathfrak{C}, x \in \partial E.
\end{aligned}$$

Therefore, by Theorem 2.6.6, the unique (in the level set sense) dislocation dynamics described above, converges, as $r \to 0^+$, to a degenerate evolution where the dislocation disappear instantaneously. After a logarithmic in time reparametrization, the evolution converges to the anisotropic mean curvature flow governed by the release of the line tension energy $\mathrm{Per}^g$ (2.6.2), corresponding to the anisotropic energy density $\varphi^g$ defined above. Such a dynamics is nothing but the evolution $t \mapsto \partial E_t$, where the normal velocity of the evolving dislocation curve $\partial E_t$ at any point $x \in \partial E_t$ is given by the (opposite of the) anisotropic curvature $\mathcal{K}^{g,1}(x, E_t)$ defined above.

# Chapter 3

# The variational approach to $s$-fractional heat flows and the limit cases $s \to 0^+$ and $s \to 1^-$

In this chapter we study the limit cases for $s$-fractional heat flows in a cylindrical domain, with homogeneous Dirichlet boundary conditions, as $s \to 0^+$ and $s \to 1^-$.
We describe the fractional heat flows as minimizing movements of the corresponding Gagliardo seminorms, with respect to the $L^2$ metric. We provide an abstract stability result for minimizing movements in Hilbert spaces, with respect to a sequence of $\Gamma$-converging uniformly $\lambda$-convex energy functionals. Then, we provide the $\Gamma$-convergence analysis of the $s$-Gagliardo seminorms as $s \to 0^+$ and $s \to 1^-$, and apply the general stability result to such specific cases. As a consequence, we prove that $s$-fractional heat flows (suitably scaled in time) converge to the standard heat flow as $s \to 1^-$, and to a degenerate ODE type flow as $s \to 0^+$. Moreover, looking at the next order term in the asymptotic expansion of the $s$-fractional Gagliardo seminorm, we show that suitably forced $s$-fractional heat flows converge, as $s \to 0^+$, to the parabolic flow of an energy functional that can be seen as a sort of renormalized 0-Gagliardo seminorm: the resulting parabolic equation involves the first variation of such an energy, that can be understood as a zero (or logarithmic) Laplacian.

The reference for the following results is [31], joint work with Vito Crismale, Lucia De Luca, Angelo Ninno and Marcello Ponsiglione.

## 3.1 $\Gamma$-convergence of $F^s$ as $s \to 0^+$

In this section we study the convergence of $s$-Gagliardo seminorms as $s \to 0^+$, both in the 0-th and in the first order.

### 3.1.1 0-th order $\Gamma$-convergence for the functionals $F^s$ as $s \to 0^+$

Let $d \in \mathbb{N}$, $d \geq 1$, and let $s \in (0,1)$. The Gagliardo $s$-seminorm of a measurable function $u : \mathbb{R}^d \to \mathbb{R}$ is defined by

$$[u]_s := \left[ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^2}{|x-y|^{d+2s}} \,\mathrm{d}y \,\mathrm{d}x \right]^{\frac{1}{2}},$$

whenever the double integral above is finite. Let $\Omega$ be an open bounded subset of $\mathbb{R}^d$ with Lipschitz continuous boundary. We denote by $\mathcal{H}^s_0(\Omega)$ the completion of $C^\infty_c(\Omega)$ with respect to the Gagliardo $s$-seminorm defined above. For every measurable function $u : \Omega \to \mathbb{R}$ we denote by $\tilde{u}$ its extension to 0 on the whole $\mathbb{R}^d$, i.e., defined by $\tilde{u} = u$ in $\Omega$ and $\tilde{u} = 0$ in $\mathbb{R}^d \setminus \Omega$.

In [61, Theorem 2] it has been proven that there exists a constant $C(d)$ depending only on the dimension $d$, such that for $d > 2s$

$$\int_\Omega \frac{|u(x)|^2}{|x|^{2s}}\,\mathrm{d}x \le C(d)\frac{s(1-s)}{(d-2s)^2}[\tilde{u}]^2_s \quad \text{for every } u \in \mathcal{H}^s_0(\Omega)\,.$$

It follows that $\mathcal{H}^s_0(\Omega) \subset \mathrm{L}^2(\Omega)$ for every $d \ge 1$ and every $s \in (0,1)$: for $2s < d$ this comes from the above estimate, being $\Omega$ bounded; for $d \le 2s$ it is enough to pass to suitable $s' < s$ with $2s' < d$, recalling that $[\tilde{u}]_{s_1} \le C(d,s)[\tilde{u}]_{s_2}$ for $0 < s_1 \le s_2 < 1$ (see e.g. [40, Proposition 2.1]).

Along with [47, Theorem 1.4.2.2] (see also [63, Theorem 3.29]), the inclusion $\mathcal{H}^s_0(\Omega) \subset \mathrm{L}^2(\Omega)$ gives that

$$\mathcal{H}^s_0(\Omega) = \left\{ u \in \mathrm{L}^2(\Omega) \ : \ [\tilde{u}]_s < +\infty \right\}.$$

For every $s \in (0,1)$, we define the functional $F^s : \mathrm{L}^2(\Omega) \to [0,+\infty]$ as

$$F^s(u) := [\tilde{u}]^2_s \tag{3.1.1}$$

and the functional $F^0 : \mathrm{L}^2(\Omega) \to [0,+\infty)$ as

$$F^0(u) := \frac{d\omega_d}{2}\|u\|^2_{\mathrm{L}^2}, \tag{3.1.2}$$

where $\omega_d$ is the measure of the unit ball of $\mathbb{R}^d$. The following result is a trivial consequence of [61, formula (9)].

**Theorem 3.1.1.** *Let $\delta \in (0,1)$. For every $s \in (0, \frac{\delta^2}{8})$ and for every $u \in \mathcal{H}^s_0(\Omega)$ we have*

$$\frac{d\omega_d}{2}\int_\Omega \frac{|u(x)|^2}{|x|^{2s}}\,\mathrm{d}x \le s\frac{2^{2s}}{(1-\delta)^2}F^s(u)\,. \tag{3.1.3}$$

The following theorem follows easily from the above estimate.

**Theorem 3.1.2.** *Let $\{s_n\}_{n\in\mathbb{N}} \subset (0,1)$ be such that $s_n \to 0^+$ as $n \to +\infty$.*

*(i) (Compactness) Let $\{u^n\}_{n\in\mathbb{N}} \subset \mathrm{L}^2(\Omega)$ be such that*

$$\sup_{n\in\mathbb{N}} s_n F^{s_n}(u^n) \le C,$$

*for some constant $C \in \mathbb{R}$. Then, up to a subsequence, $u^n \rightharpoonup u$ in $\mathrm{L}^2(\Omega)$ for some $u \in \mathrm{L}^2(\Omega)$.*

*(ii) ($\Gamma$-liminf inequality) For every $u \in \mathrm{L}^2(\Omega)$ and for every $\{u^n\}_{n\in\mathbb{N}} \subset \mathrm{L}^2(\Omega)$ with $u^n \rightharpoonup u$ in $\mathrm{L}^2(\Omega)$, it holds*

$$F^0(u) \le \liminf_{n\to+\infty} s_n F^{s_n}(u^n). \tag{3.1.4}$$

*(iii) ($\Gamma$-limsup inequality) For every $u \in \mathrm{L}^2(\Omega)$ there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathrm{L}^2(\Omega)$ with $u_n \to u$ in $\mathrm{L}^2(\Omega)$ such that*

$$F^0(u) = \lim_{n\to+\infty} s_n F^{s_n}(u^n). \tag{3.1.5}$$

*Proof.* Since $\Omega$ is bounded, there exists $0 < R < +\infty$ such that $\Omega \subset B_R$. Therefore, in view of (3.1.3) and of the energy bound, for $n$ large enough, we have that

$$\frac{1}{R^{2s_n}} \int_\Omega |u^n(x)|^2 \, \mathrm{d}x \le \int_\Omega \frac{|u^n(x)|^2}{|x|^{2s_n}} \, \mathrm{d}x \le C(d) . \tag{3.1.6}$$

It follows that $\|u^n\|_{\mathrm{L}^2(\Omega)}$ is uniformly bounded and hence, up to a subsequence, $u_n \rightharpoonup u$ in $\mathrm{L}^2(\Omega)$ for some $u \in \mathrm{L}^2(\Omega)$, proving (i).

Let us pass to the proof of (ii).
Let $\delta \in (0,1)$ be fixed. Using again
(3.1.3), for $n$ large enough, we have

$$s_n F^{s_n}(u^n) \ge \frac{(1-\delta)^2}{2^{2s_n}} \frac{d\omega_d}{2} \int_\Omega \frac{|u^n(x)|^2}{|x|^{2s_n}} \, \mathrm{d}x \ge \frac{(1-\delta)^2}{2^{2s_n} R^{2s_n}} \frac{d\omega_d}{2} \int_\Omega |u^n(x)|^2 \, \mathrm{d}x,$$

which, passing to the limit as $n \to +\infty$ and using the weak lower semicontinuity of the $\mathrm{L}^2$ norm, yields

$$\liminf_{n \to +\infty} s_n F^{s_n}(u^n) \ge (1-\delta)^2 \frac{d\omega_d}{2} \int_\Omega |u(x)|^2 \, \mathrm{d}x;$$

by the arbitrariness of $\delta$, the claim (ii) follows.

Now we show that also (iii) holds true. If $u \in \mathrm{C}^\infty_\mathrm{c}(\Omega)$, the claim is proven in [61, Theorem 3], with $u_n \equiv u$. Since $\mathrm{C}^\infty_\mathrm{c}(\Omega)$ is dense in $\mathrm{L}^2(\Omega)$, the general case follows by a standard diagonal argument. □

### 3.1.2 The first order Γ-limit of the functionals $F^s$ as $s \to 0^+$

In order to compute the Γ-limit of the renormalized functionals $F^s - \frac{1}{s}F^0$ as $s \to 0^+$ we need to rewrite the functional $F^s$ in a different manner.

Let $s \in [0,1)$. We define the functional $G^s_1 : \mathrm{L}^2(\Omega) \to [0, +\infty]$ as

$$G^s_1(u) := \frac{1}{2} \iint_{\mathcal{B}_1} \frac{|\tilde{u}(x) - \tilde{u}(y)|^2}{|x-y|^{d+2s}} \, \mathrm{d}y \, \mathrm{d}x , \tag{3.1.7}$$

where $\mathcal{B}_1 := \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d \, : \, |x-y| < 1\}$, and the functional $J^s_1 : \mathrm{L}^2(\Omega) \to (-\infty, +\infty)$ as

$$J^s_1(u) := -\iint_{\mathbb{R}^{2d} \setminus \overline{\mathcal{B}}_1} \frac{\tilde{u}(x)\tilde{u}(y)}{|x-y|^{d+2s}} \, \mathrm{d}y \, \mathrm{d}x . \tag{3.1.8}$$

We notice that the functionals $J^s_1$ are well-defined in $\mathrm{L}^2(\Omega)$ since, by Hölder inequality,

$$|J^s_1(u)| \le \|u\|^2_{\mathrm{L}^1(\Omega)} \le |\Omega| \|u\|^2_{\mathrm{L}^2(\Omega)} . \tag{3.1.9}$$

It is easy to check that for every $s \in (0,1)$

$$\hat{F}^s(u) := F^s(u) - \frac{1}{s} F^0(u) = G^s_1(u) + J^s_1(u) \qquad \text{for every } u \in \mathrm{L}^2(\Omega). \tag{3.1.10}$$

In analogy with (3.1.10), we define the functionals $\hat{F}^0 : \mathrm{L}^2(\Omega) \to (-\infty, +\infty]$ as

$$\hat{F}^0(u) := G^0_1(u) + J^0_1(u) , \tag{3.1.11}$$

and we introduce the space

$$\mathcal{H}^0_0(\Omega) := \{u \in \mathrm{L}^2(\Omega) \, : \, G^0_1(u) < +\infty\} .$$

*Remark* 3.1.3. It is natural to endow the space $\mathcal{H}_0^0(\Omega)$ with a *0-Gagliardo* type norm

$$[u]_0 := (2G_1^0(u))^{\frac{1}{2}} .$$

We are now in a position to state our Γ-convergence result for the functionals $\hat{F}^s$ defined in (3.1.10) .

**Theorem 3.1.4.** *Let $\{s_n\}_{n\in\mathbb{N}} \subset (0,1)$ be such that $s_n \to 0^+$ as $n \to +\infty$ . The following Γ-convergence result holds true.*

*(i) (Compactness) Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathrm{L}^2(\Omega)$ be such that*

$$\hat{F}^{s_n}(u^n) + 2|\Omega| \|u^n\|^2_{\mathrm{L}^2(\Omega)} \le M, \tag{3.1.12}$$

*for some constant $M$ independent of $n$ . Then, up to a subsequence, $u^n \to u$ strongly in $\mathrm{L}^2(\Omega)$ for some $u \in \mathcal{H}_0^0(\Omega)$ .*

*(ii) (Γ-liminf inequality) For every $u \in \mathrm{L}^2(\Omega)$ and for every $\{u^n\}_{n\in\mathbb{N}} \subset \mathrm{L}^2(\Omega)$ with $u_n \to u$ in $\mathrm{L}^2(\Omega)$ , it holds*

$$\hat{F}^0(u) \le \liminf_{n\to+\infty} \hat{F}^{s_n}(u^n) .$$

*(iii) (Γ-limsup inequality) For every $u \in \mathcal{H}_0^0(\Omega)$ there exists $\{u^n\}_{n\in\mathbb{N}} \subset \mathrm{L}^2(\Omega)$ with $u_n \to u$ in $\mathrm{L}^2(\Omega)$ such that*

$$\hat{F}^0(u) = \lim_{n\to+\infty} \hat{F}^{s_n}(u^n) .$$

*Remark* 3.1.5. By the Dominated Convergence Theorem, for every $s \in [0,1)$ the functionals $J_1^s$ are continuous with respect to the strong $\mathrm{L}^1$ convergence, and hence also with respect to the strong $\mathrm{L}^2$ convergence.

### 3.1.3 Compactness and Γ-liminf inequality

In order to prove (i) of Theorem 3.1.4, we recall the following result proven in [52] .

**Theorem 3.1.6** (Local compactness [52])**.** *Let $k : \mathbb{R}^d \to [0,+\infty]$ be a radially symmetric kernel such that*

$$\int_{\mathbb{R}^d} k(z)\,\mathrm{d}z = +\infty \qquad \text{and} \qquad \int_{\mathbb{R}^d} \min\{1,|z|^2\} k(z)\,\mathrm{d}z < +\infty$$

*and let*

$$\mathcal{W}^k(\Omega) := \left\{ u \in \mathrm{L}^2(\Omega) \,:\, \iint_{\mathbb{R}^{2d}} |\tilde{u}(x) - \tilde{u}(y)|^2 k(x-y)\,\mathrm{d}y\,\mathrm{d}x < +\infty \right\}$$

*be the Banach space endowed with the norm*

$$\|u\|_{\mathcal{W}^k(\Omega)} := \|u\|_{\mathrm{L}^2(\Omega)} + \Big( \iint_{\mathbb{R}^{2d}} |\tilde{u}(x) - \tilde{u}(y)|^2 k(x-y)\,\mathrm{d}y\,\mathrm{d}x \Big)^{\frac{1}{2}} .$$

*Then, the embedding $\mathcal{W}^k(\Omega) \hookrightarrow \mathrm{L}^2(\Omega)$ is compact.*

With Theorem 3.1.6 in hand, we are in a position to prove compactness.

*Proof of Theorem 3.1.4(i).* By (3.1.12), (3.1.10) and (3.1.9), we have that

$$M \geq \hat{F}^{s_n}(u^n) + 2|\Omega| \|u^n\|^2_{\mathrm{L}^2(\Omega)} \geq -|\Omega| \|u^n\|^2_{\mathrm{L}^2(\Omega)} + 2|\Omega| \|u^n\|^2_{\mathrm{L}^2(\Omega)} = |\Omega| \|u^n\|^2_{\mathrm{L}^2(\Omega)} \,,$$

i.e., that $\|u_n\|_{\mathrm{L}^2(\Omega)}$ is uniformly bounded. Therefore, by (3.1.10) we deduce

$$\begin{aligned} G^0_1(u^n) \leq & G^{s_n}_1(u^n) \leq M + \iint_{\mathbb{R}^{2d} \setminus \overline{\mathcal{B}}_1} \frac{|\tilde{u}^n(x)||\tilde{u}^n(y)|}{|x-y|^{d+2s_n}} \, \mathrm{d}y \, \mathrm{d}x \leq M + |\Omega| \|u^n\|^2_{\mathrm{L}^2(\Omega)} \\ \leq & 2M \,, \end{aligned} \tag{3.1.13}$$

whence, by applying Theorem 3.1.6 with $k(z) := \frac{\chi_{B_1}(z)}{|z|^d}$, we deduce that, up to a subsequence, $u^n \to u$ in $\mathrm{L}^2(\Omega)$ for some $u \in \mathrm{L}^2(\Omega)$. Finally, by (3.1.13) and by the lower semicontinuity of the functional $G^0_1$ with respect to the strong $\mathrm{L}^2$ convergence, we get that $u \in \mathcal{H}^0_0(\Omega)$. □

Now we prove the Γ-liminf inequality.

*Proof of Theorem 3.1.4(ii).* By Fatou lemma we have

$$G^0_1(u) \leq \liminf_{n \to +\infty} G^{s_n}_1(u^n) \,; \tag{3.1.14}$$

moreover, by the Dominated Convergence Theorem we get

$$J^0_1(u) = \lim_{n \to +\infty} J^{s_n}_1(u^n) \,. \tag{3.1.15}$$

In view of (3.1.10) and (3.1.11), we get the claim. □

### 3.1.4 Γ-limsup inequality

Here we construct the recovery sequence for the functionals $\hat{F}^s$. We start by showing that, for smooth functions, the pointwise limit of the functional $\hat{F}^s$ as $s \to 0^+$ coincides with the functionals $\hat{F}^0$.

**Lemma 3.1.7.** *For every $u \in \mathrm{C}^\infty_\mathrm{c}(\Omega)$ we have that*

$$\lim_{s \to 0^+} \hat{F}^s(u) = \hat{F}^0(u) \,.$$

*Proof.* In view of the definition of $\hat{F}^s$ in (3.1.10) it is enough to show

$$\lim_{s \to 0^+} G^s_1(u) = G^0_1(u) \,, \tag{3.1.16}$$

$$\lim_{s \to 0^+} J^s_1(u) = J^0_1(u) \,. \tag{3.1.17}$$

We start by proving (3.1.16). To this end, we note that, since $\tilde{u} \in \mathrm{C}^\infty(\mathbb{R}^d)$, for every $x$, $y \in \mathbb{R}^d$ we have

$$|\tilde{u}(y) - \tilde{u}(x)|^2 \leq \|\nabla \tilde{u}\|^2_{\mathrm{L}^\infty} |x-y|^2 \,.$$

Let $U \subset \mathbb{R}^d$ be an open set such that $\mathrm{dist}(\Omega, \mathbb{R}^d \setminus U) > 1$ and let $\varepsilon \in (0,1)$; we have

$$
\begin{aligned}
|G_1^s(u) - G_1^0(u)| \leq & \frac{1}{2}\int_U \mathrm{d}x \int_{B_\varepsilon(x)} |\tilde{u}(x)-\tilde{u}(y)|^2 \left| \frac{1}{|x-y|^{d+2s}} - \frac{1}{|x-y|^d}\right| \mathrm{d}y \\
& + \frac{1}{2}\iint_{\mathcal{B}_1\setminus\overline{\mathcal{B}}_\varepsilon} |\tilde{u}(x)-\tilde{u}(y)|^2 \left| \frac{1}{|x-y|^{d+2s}} - \frac{1}{|x-y|^d}\right| \mathrm{d}y\,\mathrm{d}x.
\end{aligned}
$$

By Dominated Convergence Theorem the second addend in the righthand side tends to zero (for fixed $\varepsilon$) as $s \to 0^+$, while the first addend is bounded from above by $|U|\|\nabla\tilde{u}\|^2_{\mathrm{L}^\infty}\int_{B_\varepsilon}\frac{1}{|z|^{d+2s-2}}\,\mathrm{d}z$, which tends to zero as $\varepsilon \to 0^+$. This clearly yields (3.1.16). Finally, (3.1.17) is a trivial consequence of the Dominated Convergence Theorem, once noticed that

$$
J_1^s(u) = -\int_\Omega u(x)\int_{\Omega\setminus\overline{B}_1(x)} \frac{u(y)}{|x-y|^{d+2s}}\,\mathrm{d}y\,\mathrm{d}x\,.
$$

□

**Lemma 3.1.8** (Density of smooth functions)**.** *For every $u \in \mathcal{H}_0^0(\Omega)$ there exists $\{u_k\}_{k\in\mathbb{N}} \subset \mathrm{C}_\mathrm{c}^\infty(\Omega)$ such that $u_k \to u$ (strongly) in* $\mathrm{L}^2$
*and*

$$
\lim_{k\to+\infty} J_1^0(u_k) = J_1^0(u) \qquad \textit{and} \qquad \lim_{k\to+\infty} G_1^0(u_k) = G_1^0(u). \tag{3.1.18}
$$

*Proof.* This result is proven in [63, Theorem 3.29], for domains with a continuous boundary. Up to a partition of the unity argument, one may assume $\Omega$ to be the subgraph of a continuous function: thus it is enough to approximate first with $u_\delta(x) := u(x', x_n+\delta)$, for small $\delta$, whose support is well contained in $\Omega$, and then to take $u_\delta * \phi_\varepsilon$, for a family of mollifiers $\{\phi_\varepsilon\}_\varepsilon$ and small $\varepsilon$. □

The limsup inequality in Theorem 3.1.4 follows directly from the density proved above.

*Proof of Theorem 3.1.4(iii).* Let $u \in \mathcal{H}^0(\Omega)$. By Lemma 3.1.8 there exists a sequence of functions $\{u^k\}_{k\in\mathbb{N}} \subset \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^d)$ such that $u^k \to u$ in $\mathrm{L}^2$ and

$$
\limsup_{k\to+\infty} \hat{F}^0(u^k) = \hat{F}^0(u)\,.
$$

In view of Lemma 3.1.7 we have

$$
\lim_{n\to+\infty} \hat{F}^{s_n}(u^k) = \hat{F}^0(u^k) \qquad \text{for every } k \in \mathbb{N}\,.
$$

Therefore, by a standard diagonal argument, there exists a sequence $\{u^n\}_{n\in\mathbb{N}} \subset \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^d)$ with $u^n = u^{k(n)}$ for every $n \in \mathbb{N}$ satisfying the desired properties. □

## 3.2 Γ-convergence of $F^s$ as $s \to 1^-$

Here we study the Γ-convergence of the functionals $(1-s)F^s$ as $s \to 1^-$, where $F^s$ is defined in (3.1.1). The candidate Γ-limit is the functional $F^1 : \mathrm{L}^2(\Omega) \to \mathbb{R}\cup\{+\infty\}$ defined by

$$
F^1(u) := \begin{cases} \dfrac{\omega_d}{2}\displaystyle\int_\Omega |\nabla u(x)|^2\,\mathrm{d}x & \text{if } u \in H_0^1(\Omega)\,, \\ +\infty & \text{elsewhere in } \mathrm{L}^2(\Omega)\,. \end{cases} \tag{3.2.1}
$$

**Theorem 3.2.1.** *Let $\{s_n\}_{n\in\mathbb{N}} \subset (0,1)$ be such that $s_n \to 1^-$ as $n \to +\infty$. The following Γ-convergence result holds true.*

*(i) (Compactness) Let $\{u^n\}_{n\in\mathbb{N}} \subset \mathrm{L}^2(\Omega)$ be such that*

$$\sup_{n\in\mathbb{N}}(1-s_n)F^{s_n}(u^n) + \|u^n\|^2_{\mathrm{L}^2(\Omega)} \leq M, \tag{3.2.2}$$

*for some constant $M$ independent of $n$. Then, up to a subsequence, $u^n \to u$ strongly in $\mathrm{L}^2(\Omega)$ for some $u \in H^1_0(\Omega)$.*

*(ii) (Γ-liminf inequality) For every $u \in \mathrm{L}^2(\Omega)$ and for every $\{u^n\}_{n\in\mathbb{N}} \subset \mathrm{L}^2(\Omega)$ with $u^n \to u$ in $\mathrm{L}^2(\Omega)$, it holds*

$$F^1(u) \leq \liminf_{n\to+\infty}(1-s_n)F^{s_n}(u^n)\,. \tag{3.2.3}$$

*(iii) (Γ-limsup inequality) For every $u \in \mathrm{L}^2(\Omega)$ there exists $\{u^n\}_{n\in\mathbb{N}} \subset \mathrm{L}^2(\Omega)$ with $u^n \to u$ in $\mathrm{L}^2(\Omega)$ such that*

$$F^1(u) = \lim_{n\to+\infty}(1-s_n)F^{s_n}(u^n)\,. \tag{3.2.4}$$

### 3.2.1 Proof of Compactness

This subsection is devoted to the proof of Theorem 3.2.1(i). To accomplish this task, we adopt the strategy in [9] adapting it to our case. We first recall that for every function $v \in \mathrm{L}^2(\Omega)$ and for every $h \in \mathbb{R}^d$ the shift $\tau_h v$ of $v$ by $h$ is defined by $\tau_h v(\cdot) := v(\cdot + h)$ .We recall the following two classical results.

**Theorem 3.2.2** (Fréchet-Kolmogorov)**.** *Let $\{v^n\}_{n\in\mathbb{N}} \subset \mathrm{L}^2(\Omega)$ be such tha $\sup_{n\in\mathbb{N}}\|v^n\|_{\mathrm{L}^2(\mathbb{R}^d)} \leq M$, for some constant $M$ independent of $n$. If*

$$\lim_{|h|\to 0^+}\sup_{n\in\mathbb{N}}\|\tau_h v^n - v^n\|_{\mathrm{L}^2(\mathbb{R}^d)} = 0\,,$$

*then $\{v^n\}_{n\in\mathbb{N}}$ is pre-compact in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d)$.*

**Theorem 3.2.3.** *Let $v \in \mathrm{L}^2(\mathbb{R}^d)$. Then $v \in H^1(\mathbb{R}^d)$ if and only if there exists $C > 0$ such that*

$$\|\tau_h v - v\|_{\mathrm{L}^2(\Omega')} \leq C|h| \quad \textit{for every open bounded set } \Omega' \subset \mathbb{R}^d \textit{ and for every } h \in \mathbb{R}^d\,.$$

For every $A \subset \mathbb{R}^d$ and for every $t > 0$ we define the set

$$A_t := \{x \in \mathbb{R}^d : \ \mathrm{dist}(x, A) < t\}. \tag{3.2.5}$$

The following result which allows to estimate the $\mathrm{L}^2$ distance of a function from its shift has been proven in [9, Proposition 5] in $\mathrm{L}^1$; for the sake of completeness, we state and prove it also in our case.

**Proposition 3.2.4.** *There exists a constant $C(d) > 0$ such that the following holds true: for every $v \in \mathrm{L}^2(\mathbb{R}^d)$, for every $h \in \mathbb{R}^d$ and for every open bounded set $\Omega' \subset \mathbb{R}^d$ we have*

$$\|\tau_h v - v\|^2_{\mathrm{L}^2(\Omega')} \leq C(d)\frac{|h|^2}{\rho^{d+2}}\int_{B_\rho}\|\tau_y v - v\|^2_{\mathrm{L}^2(\Omega'_{|h|})}\,\mathrm{d}y \quad \textit{for every } \rho \in (0, |h|], \tag{3.2.6}$$

*with $\Omega'_{|h|}$ defined in (3.2.5).*

*Proof.* The proof closely resembles the one of [9, Proposition 5]. Let $\varphi \in \mathrm{C}^1_{\mathrm{c}}(B_1)$ be a fixed function with $\varphi \geq 0$ and $\int_{B_1} \varphi(x)\,\mathrm{d}x = 1$. For every $\rho > 0$ we define the functions $U_\rho, V_\rho : \mathbb{R}^d \to \mathbb{R}$ as

$$U_\rho(x) := \frac{1}{\rho^d}\int_{B_\rho} v(x+y)\varphi\Big(\frac{y}{\rho}\Big)\,\mathrm{d}y\,, \qquad V_\rho(x) := \frac{1}{\rho^d}\int_{B_\rho} (v(x) - v(x+y))\varphi\Big(\frac{y}{\rho}\Big)\,\mathrm{d}y\,;$$

clearly, for every $\rho > 0$ and for every $x \in \mathbb{R}^d$

$$v(x) = U_\rho(x) + V_\rho(x)\,, \tag{3.2.7}$$

and hence

$$|\tau_h v(x) - v(x)|^2 \leq 3|U_\rho(x+h) - U_\rho(x)|^2 + 3|V_\rho(x)|^2 + 3|V_\rho(x+h)|^2\,. \tag{3.2.8}$$

By Jensen inequality, for every $\xi \in \mathbb{R}^d$ we have

$$|V_\rho(\xi)|^2 \leq \frac{\omega_d}{\rho^d}\|\varphi\|_{\mathrm{L}^\infty(B_1)}\int_{B_\rho} |v(\xi) - \tau_y(v)(\xi)|^2\,\mathrm{d}y. \tag{3.2.9}$$

Moreover, by the change of variable $z = x + y$, we have that

$$U_\rho(x) = \frac{1}{\rho^d}\int_{B_\rho(x)} v(z)\varphi\Big(\frac{z-x}{\rho}\Big)\,\mathrm{d}z$$

whence we deduce that

$$\begin{aligned}
\mathrm{D}\,U_\rho(x) &= -\frac{1}{\rho^{d+1}}\int_{B_\rho(x)} v(z)\mathrm{D}\varphi\Big(\frac{z-x}{\rho}\Big)\,\mathrm{d}z \\
&= -\frac{1}{\rho^{d+1}}\int_{B_\rho(x)} (v(z) - v(x))\mathrm{D}\varphi\Big(\frac{z-x}{\rho}\Big)\,\mathrm{d}z \\
&= -\frac{1}{\rho^{d+1}}\int_{B_\rho} (v(x+y) - v(x))\mathrm{D}\varphi\Big(\frac{y}{\rho}\Big)\,\mathrm{d}y\,;
\end{aligned}$$

therefore, by the fundamental Theorem of Calculus and by Jensen inequality, we obtain

$$\begin{aligned}
|U_\rho(x+h) - U_\rho(x)|^2 &\leq |h|^2\int_0^1 |\mathrm{D}\,U_\rho(x+th)|^2\,\mathrm{d}t \\
&\leq \omega_d\frac{|h|^2}{\rho^{d+2}}\|\mathrm{D}\varphi\|^2_{\mathrm{L}^\infty(B_1)}\int_0^1\int_{B_\rho} |\tau_y v(x+th) - v(x+th)|^2\,\mathrm{d}y\,\mathrm{d}t.
\end{aligned} \tag{3.2.10}$$

Now, by (3.2.8), (3.2.9), and (3.2.10), taking $\rho < |h|$, we have

$$\begin{aligned}
|\tau_h v(x) - v(x)|^2 \leq & 3\omega_d\frac{|h|^2}{\rho^{d+2}}\|\mathrm{D}\varphi\|^2_\infty\int_0^1\int_{B_\rho} |\tau_y v(x+th) - v(x+th)|^2\,\mathrm{d}y\,\mathrm{d}t \\
&+ 3\omega_d\frac{|h|^2}{\rho^{d+2}}\|\varphi\|^2_\infty\int_{B_\rho} |\tau_y v(x) - v(x)|^2 dy \\
&+ 3\omega_d\frac{|h|^2}{\rho^{d+2}}\|\varphi\|^2_\infty\int_{B_\rho} |\tau_y v(x+h) - v(x+h)|^2\,\mathrm{d}y\,.
\end{aligned} \tag{3.2.11}$$

Finally, by integrating (3.2.11) on $\Omega'$, by Fubini theorem, we obtain (3.2.6) with $C(d) := 3\omega_d(2\|\varphi\|^2_{\mathrm{L}^\infty(B_1)} + \|\mathrm{D}\varphi\|^2_{\mathrm{L}^\infty(B_1)})$. □

We recall the following version of Hardy's inequality, that is proven in [9, Proposition 6].

**Lemma 3.2.5.** *Let $g : \mathbb{R} \to [0, +\infty)$ be a Borel measurable function. Then for all $l \geq 0$ we have*

$$\int_0^r \frac{1}{\rho^{d+l+1}} \int_0^\rho g(t)\,\mathrm{d}t\,\mathrm{d}\rho \leq \frac{1}{d+l} \int_0^r \frac{g(t)}{t^{d+l}}\,\mathrm{d}t \quad \textit{for every } r \geq 0.$$

The following result will be used in the proof of Theorem 3.2.1(i). It is the $\mathrm{L}^2$ analog of [9, Proposition 4].

**Proposition 3.2.6.** *There exists a constant $\bar{C}(d) > 0$ such that for every $v \in \mathrm{L}^2(\mathbb{R}^d)$, for every open bounded set $\Omega' \subset \mathbb{R}^d$, for every $s \in (0,1)$, and for every $h \in \mathbb{R}^d$, we have*

$$\|\tau_h v - v\|^2_{\mathrm{L}^2(\Omega')} \leq |h|^{2s} \bar{C}(d)(1-s) \int_{B_{|h|}} \frac{\|\tau_y v - v\|^2_{\mathrm{L}^2(\Omega'_{|h|})}}{|y|^{d+2s}}\,\mathrm{d}y\,. \tag{3.2.12}$$

*Proof.* For a fixed $v \in \mathrm{L}^2(\mathbb{R}^d)$, we define the function $g : [0, |h|] \to \mathbb{R}$ as

$$g(t) := \int_{\partial B_t} \|\tau_y(v) - v\|^2_{\mathrm{L}^2(\Omega'_{|h|})}\,\mathrm{d}\mathcal{H}^{d-1}(y)\,.$$

By integrating in polar coordinates formula (3.2.6) we thus have

$$\|\tau_h v - v\|^2_{\mathrm{L}^2(\Omega')} \leq C(d) \frac{|h|^2}{\rho^{d+2}} \int_0^\rho g(t)\,\mathrm{d}t. \tag{3.2.13}$$

By multiplying both sides of (3.2.13) by $\rho^{1-2s}$ and integrating in the interval $[0, |h|]$, using Lemma 3.2.5 and the very definition of $g$, we obtain

$$\begin{aligned}
\|\tau_h v - v\|^2_{\mathrm{L}^2(\Omega')} \leq & 2C(d)(1-s)|h|^{2s} \int_0^{|h|} \frac{1}{\rho^{d+2s+1}} \int_0^\rho g(t)\,\mathrm{d}t\,\mathrm{d}z \\
\leq & 2C(d)(1-s)|h|^{2s} \int_0^{|h|} \frac{g(t)}{t^{d+2s}}\,\mathrm{d}t \\
= & 2C(d)(1-s)|h|^{2s} \int_{B_{|h|}} \frac{\|\tau_y v - v\|^2_{\mathrm{L}^2(\Omega'_{|h|})}}{|y|^{d+2s}}\,\mathrm{d}y\,,
\end{aligned} \tag{3.2.14}$$

which concludes the proof. □

We are now in position to prove Theorem 3.2.1(i).

*Proof of Theorem 3.2.1(i).* By Proposition 3.2.6 and by the upper bound (3.2.2) we obtain that for every open bounded set $\Omega' \subset \mathbb{R}^d$ and for every $h \in \mathbb{R}^d$

$$\|\tau_h \tilde{u}^n - \tilde{u}^n\|_{\mathrm{L}^2(\Omega')} \leq C(d, M)|h|^{s_n}\,, \tag{3.2.15}$$

where we recall $\tilde{u}^n$ is the extension of $u^n$ to 0 in $\mathbb{R}^d \setminus \Omega$. Therefore, the sequence $\{\tilde{u}^n\}_{n\in\mathbb{N}}$ satisfies the assumption of Theorem 3.2.2, and hence there exists a function $v \in \mathrm{L}^2(\mathbb{R}^d)$ with $v = 0$ in $\mathbb{R}^d \setminus \Omega$, such that, up to a subsequence, $\tilde{u}^n \to v$ in $\mathrm{L}^2(\mathbb{R}^d)$. Now, sending $n \to +\infty$ in (3.2.15), we obtain that for every open bounded set $\Omega' \subset \mathbb{R}^d$

$$\|\tau_h v - v\|_{\mathrm{L}^2(\Omega')} \leq C(d, M)|h| \quad \text{for every } h \in \mathbb{R}^d\,,$$

and hence by Theorem 3.2.3 we obtain that $\mathrm{D}v \in \mathrm{L}^2(\mathbb{R}^d)$. Since $v = 0$ in $\mathbb{R}^d \setminus \Omega$, by the regularity of $\partial\Omega$, we have that $v$ is the extension to 0 in $\mathbb{R}^d \setminus \Omega$ of a function $u \in H^1_0(\Omega)$, thus concluding the proof. □

### 3.2.2 Proof of the Γ-liminf inequality

Here we prove the Γ-liminf inequality in Theorem 3.2.1.

*Proof of Theorem 3.2.1(ii).* We can assume without loss of generality that (3.2.2) holds true so that the function $u$ is actually in $H^1_0(\Omega)$. *Claim 1. Let* $\eta \in \mathrm{C}^\infty_\mathrm{c}(B_1)$ *be a standard mollifier, i.e.,* $\eta \geq 0$ *and* $\int_{B_1} \eta(x)\,\mathrm{d}x = 1$ . *For every* $\varepsilon > 0$ , *we set* $\eta_\varepsilon(\cdot) := \eta(\frac{\cdot}{\varepsilon})$ . *For every* $s \in (0,1)$

$$F^s(v_\varepsilon) \leq F^s(v) \qquad \text{for every } v \in \mathrm{L}^2(\Omega) \text{ and for every } \varepsilon > 0\,, \tag{3.2.16}$$

*where* $v_\varepsilon := v * \eta_\varepsilon$ .

Indeed, setting $\tilde{v}_\varepsilon := \tilde{v} * \eta_\varepsilon$ and $\Omega_\varepsilon := \{x \in \mathbb{R}^d \ : \ \mathrm{dist}(x,\Omega) \leq \varepsilon\}$ , we have that $\tilde{v}_\varepsilon = 0$ in $\mathbb{R}^d \setminus \Omega_\varepsilon$ ; therefore, by applying Jensen inequality to the probability measure $\frac{1}{\varepsilon^d}\eta_\varepsilon\,\mathrm{d}z$ , we get

$$\begin{aligned}
F^s(v_\varepsilon) \leq& \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|\tilde{v}(x-z)-\tilde{v}(y-z)|^2}{|x-y|^{d+2s}}\eta_\varepsilon(z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x\\
=& \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{1}{\varepsilon^d}\int_{\mathbb{R}^d} \frac{|\tilde{v}(x-z)-\tilde{v}(y-z)|^2}{|x-z-(y-z)|^{d+2s}}\eta_\varepsilon(z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x\\
=&F^s(v)\,.
\end{aligned}$$

*Claim 2. For every* $\varepsilon > 0$ *and for every* $R > 0$, *it holds*

$$\frac{\omega_d}{2}\liminf_{n\to+\infty}\int_{B_R}|\nabla\tilde{u}^n_\varepsilon|^2(\mathrm{dist}(x,\partial B_R))^{2(1-s_n)}\,\mathrm{d}x \leq \liminf_{n\to+\infty}(1-s_n)F^{s_n}(u^n_\varepsilon)\,, \tag{3.2.17}$$

*with* $\eta_\varepsilon$ *as in Claim 1.*

Indeed, by Taylor expansion, using that $\sup_{n\in\mathbb{N}}\|u^n\|^2_{\mathrm{L}^2} \leq M$ we have that

$$|\tilde{u}^n_\varepsilon(x)-\tilde{u}^n_\varepsilon(y)|^2 \geq \Big|\nabla\tilde{u}^n_\varepsilon(x)\cdot\frac{x-y}{|x-y|}\Big|^2|x-y|^2 - C(\varepsilon,M)\mathrm{o}(|x-y|^2)\,.$$

Therefore, setting $\delta := \mathrm{dist}(x,\partial B_R)$ , we get

$$\begin{aligned}
&(1-s_n)\int_{B_R}\frac{|\tilde{u}^n_\varepsilon(x)-\tilde{u}^n_\varepsilon(y)|^2}{|x-y|^{d+2s_n}}\,\mathrm{d}y \geq (1-s_n)\int_{B_{\delta}(x)}\frac{|\tilde{u}^n_\varepsilon(x)-\tilde{u}^n_\varepsilon(y)|^2}{|x-y|^{d+2s_n}}\,\mathrm{d}y\\
\geq&(1-s_n)\int_{B_\delta(x)}\Big|\nabla\tilde{u}^n_\varepsilon(x)\cdot\frac{x-y}{|x-y|}\Big|^2|x-y|^{2(1-s_n)-d}\,\mathrm{d}y\\
&-C(\varepsilon,M)(1-s_n)\,\mathrm{o}\Big(\int_{B_\delta(x)}|x-y|^{2(1-s_n)-d}\,\mathrm{d}y\Big)\\
=&\frac{\omega_d}{2}\delta^{2(1-s_n)}|\nabla\tilde{u}^n_\varepsilon(x)|^2 - C(\varepsilon,M,d)\mathrm{o}(1)\,,
\end{aligned}\tag{3.2.18}$$

where in the last equality we integrated over spherical boundaries from 0 to $\delta$, using that $\int_{\mathbb{S}^{d-1}}|\nabla\tilde{u}^n_\varepsilon(x)\cdot\theta|^2\,\mathrm{d}\theta = \omega_d$ . By integrating (3.2.18) over $B_R$, we get (3.2.17).

By Claim 1 and Claim 2, for every $\varepsilon > 0$ and for every $R > 0$ we have that

$$\liminf_{n\to+\infty}(1-s_n)F^{s_n}(u^n) \geq \frac{\omega_d}{2}\liminf_{n\to+\infty}\int_{B_R}|\nabla\tilde{u}^n_\varepsilon(x)|^2(\mathrm{dist}(x,\partial B_R))^{2(1-s_n)}\,\mathrm{d}x\,, \tag{3.2.19}$$

whence, using that for every $\varepsilon > 0$ the sequence $\{\tilde{u}^n_\varepsilon\}_{n\in\mathbb{N}}$ is equi-Lipschitz, and applying the Dominated Convergence Theorem and Fatou lemma, we get that, up to a (not relabeled) subsequence,

$$\liminf_{n\to+\infty}(1-s_n)F^{s_n}(u^n) \geq \frac{\omega_d}{2}\liminf_{n\to+\infty}\int_{B_R}|\nabla\tilde{u}^n_\varepsilon(x)|^2\,\mathrm{d}x\,.$$

Therefore, since we have assumed that (3.2.2) holds true, we have that, up to a further subsequence, $\tilde{u}_\varepsilon^n \rightharpoonup v_\varepsilon$ in $H^1(B_R)$ for some $v_\varepsilon \in H^1(B_R)$. In particular, $\tilde{u}_\varepsilon^n \to v_\varepsilon$ in $\mathrm{L}^2(B_R)$ and hence $v_\varepsilon = \tilde{u}_\varepsilon$ a.e. in $B_R$. In conclusion, by (3.2.19), using that $\tilde{u}_\varepsilon \to \tilde{u}$ in $H^1(\mathbb{R}^d)$ as $\varepsilon \to 0$, we deduce that

$$\liminf_{n\to+\infty}(1-s_n)F^{s_n}(u^n) \geq \liminf_{\varepsilon\to 0}\frac{\omega_d}{2}\int_{B_R}|\nabla\tilde{u}_\varepsilon(x)|^2\,\mathrm{d}x \geq \frac{\omega_d}{2}\int_{B_R}|\nabla\tilde{u}(x)|^2\,\mathrm{d}x\,,$$

whence (3.2.3) follows sending $R \to +\infty$. □

### 3.2.3 Proof of the Γ-limsup inequality

The proof of the Γ-limsup inequality relies on the pointwise convergence of $(1-s)F^s$ to $F^1$ (as $s \to 1$) for smooth functions with compact support and on the density of smooth functions in $H_0^1(\Omega)$. As for the pointwise convergence we recall the following result, proved in [59] in a more general setting.

**Theorem 3.2.7.** *For every $v \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^d)$ it holds*

$$\lim_{s\to 1^-}(1-s)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|v(x)-v(y)|^2}{|x-y|^{d+2s}}\,\mathrm{d}x\,\mathrm{d}y = \frac{\omega_d}{2}\int_{\mathbb{R}^d}|\nabla v(x)|^2\,\mathrm{d}x\,.$$

With Theorem 3.2.7 in hand we can prove Theorem 3.2.1(iii) using standard density arguments in Γ-convergence.

*Proof of Theorem 3.2.1(iii).* It is enough to prove the claim only for $u \in H_0^1(\Omega)$. For every $u \in H_0^1(\Omega)$ there exists $\{u^k\}_{k\in\mathbb{N}} \subset \mathrm{C}_\mathrm{c}^\infty(\Omega)$ such that $u^k \to u$ (as $k \to +\infty$) in $H^1(\Omega)$. In view of Theorem 3.2.7 we have that for every $k \in \mathbb{N}$

$$\begin{aligned}\lim_{n\to+\infty}(1-s_n)F^{s_n}(u^k) &= \lim_{n\to+\infty}(1-s_n)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|\tilde{u}^k(x)-\tilde{u}^k(y)|^2}{|x-y|^{d+2s_n}}\,\mathrm{d}x\,\mathrm{d}y\\ &=\frac{\omega_d}{2}\int_{\mathbb{R}^d}|\nabla\tilde{u}^k(x)|^2\,\mathrm{d}x = \frac{\omega_d}{2}\int_\Omega|\nabla u^k(x)|^2\,\mathrm{d}x\,.\end{aligned}$$

Therefore by a standard diagonal argument there exists $\{k_n\}_{n\in\mathbb{N}}$ such that

$$\lim_{n\to+\infty}u^{k_n} = u,\quad \limsup_{n\to+\infty}(1-s_n)F^{s_n}(u^{k_n}) \leq \frac{\omega_d}{2}\int_\Omega|\nabla u(x)|^2\,\mathrm{d}x = F^1(u)\,,$$

i.e., (3.2.4). □

## 3.3 Minimizing movements for λ-convex functionals defined on a Hilbert space

In this section we develop the general theory that will allow us to study the stability of the $s$-fractional heat flow as $s \to 0^+$ and $s \to 1^-$. Throughout this section $\mathscr{H}$ is a generic Hilbert space, $\langle\cdot,\cdot\rangle_{\mathscr{H}}$ is the inner product of $\mathscr{H}$ and $|\cdot|_{\mathscr{H}}$ is the norm induced by such a scalar product. In the abstract setting of this section, we denote by $\dot{v}$ the time derivative of any function $v$ from a time interval with values in $\mathscr{H}$.

*Definition* 3.3.1 ($\lambda$-convexity, $\lambda$-positivity, $\lambda$-coercivity). Let $\lambda > 0$. We say that a function $\mathcal{F} : \mathscr{H} \to (-\infty, +\infty]$ is $\lambda$-convex if the function $f(\cdot) + \frac{\lambda}{2}|\cdot|^2_{\mathscr{H}}$ is convex. Moreover, we say that $\mathcal{F}$ is $\lambda$-positive if $\mathcal{F}(x) + \frac{\lambda}{2}|x|^2_{\mathscr{H}} \geq 0$ for every $x \in \mathscr{H}$, and we say that $\mathcal{F}$ is $\lambda$-coercive if the sublevels of the function $\mathcal{F}(\cdot) + \frac{\lambda}{2}|\cdot|^2_{\mathscr{H}}$ are bounded.

*Remark* 3.3.2. We notice that if $\mathcal{F}$ is $\lambda$-positive, then $\mathcal{F}$ is $\tilde{\lambda}$-coercive for every $\tilde{\lambda} > \lambda$.

**Proposition 3.3.3.** *Let $\mathcal{F} : \mathscr{H} \to (-\infty, +\infty]$ be a proper, strongly lower semicontinuous function which is $\lambda$-convex and $\lambda$-positive for some $\lambda > 0$. Then for every $0 < \tau < \frac{1}{2\lambda}$ and for every $y \in \mathscr{H}$ the problem*

$$\min\Big\{\mathcal{F}(x) + \frac{1}{2\tau}|x - y|^2_{\mathscr{H}} : \, x \in \mathscr{H}\Big\} \tag{3.3.1}$$

*admits a unique solution.*

*Proof.* We preliminarily notice that, since $\mathcal{F}$ is $\lambda$-convex and strongly lower semicontinuous, then the function $\mathcal{F}(\cdot) + \frac{1}{2\tau}|\cdot|^2_{\mathscr{H}}$ is strictly convex and strongly lower semicontinuous and, in turn, weakly lower semicontinuous. Clearly, this implies that also $\mathcal{F}(\cdot) + \frac{1}{2\tau}|\cdot - y|^2_{\mathscr{H}}$ is weakly lower semicontinuous. Moreover, by Remark 3.3.2, we have that $\mathcal{F}$ is $\frac{1}{2\tau}$-coercive.

Since $\mathcal{F}$ is proper,

$$0 \leq \inf\Big\{\mathcal{F}(x) + \frac{1}{2\tau}|x - y|^2_{\mathscr{H}} : \, x \in \mathscr{H}\Big\} \leq M \,,$$

for some $M > 0$. Let $\{x_k\}_{k\in\mathbb{N}} \subset \mathscr{H}$ be a sequence such that

$$\lim_{k\to+\infty} \mathcal{F}(x_k) + \frac{1}{2\tau}|x_k - y|^2_{\mathscr{H}} = \inf\Big\{\mathcal{F}(x) + \frac{1}{2\tau}|x - y|^2_{\mathscr{H}} : \, x \in \mathscr{H}\Big\} . \tag{3.3.2}$$

By triangular inequality, for $k$ sufficiently large, we have

$$2M \geq \mathcal{F}(x_k) + \frac{1}{2\tau}|x_k - y|^2_{\mathscr{H}} \geq \mathcal{F}(x_k) + \frac{1}{4\tau}|x_k|^2_{\mathscr{H}} - \frac{1}{2\tau}|y|^2_{\mathscr{H}} \tag{3.3.3}$$

whence, in view of the $\frac{1}{2\tau}$-coercivity of the function $\mathcal{F}$, we deduce that, up to a subsequence, $x_k \overset{\mathscr{H}}{\rightharpoonup} x_\infty$ for some $x_\infty \in \mathscr{H}$. Therefore, by (3.3.2) and by the weak lower semicontinuity of the function $\mathcal{F}(\cdot) + \frac{1}{2\tau}|\cdot - y|^2_{\mathscr{H}}$, we obtain

$$\begin{aligned}\inf\Big\{\mathcal{F}(x) + \frac{1}{2\tau}|x - y|^2_{\mathscr{H}} : \, x \in \mathscr{H}\Big\} &= \lim_{k\to+\infty} \mathcal{F}(x_k) + \frac{1}{2\tau}|x_k - y|^2_{\mathscr{H}} \\ &\geq \mathcal{F}(x_\infty) + \frac{1}{2\tau}|x_\infty - y|^2_{\mathscr{H}} ,\end{aligned} \tag{3.3.4}$$

i.e., that $x_\infty$ is a minimizer of the problem in (3.3.1).

Finally, the uniqueness of the solution is a consequence of the strict convexity of the functional $\mathcal{F}(\cdot) + \frac{1}{2\tau}|\cdot - y|^2$. □

For every function $\mathcal{F} : \mathscr{H} \to (-\infty, +\infty]$ we denote by $D(\mathcal{F})$ the set of all $x \in \mathscr{H}$ such that $\mathcal{F}(x) \in \mathbb{R}$.

*Definition* 3.3.4 (Fréchet subdifferential). For $\mathcal{F} : \mathscr{H} \to (-\infty, +\infty]$ and $x \in D(\mathcal{F})$, the Fréchet subdifferential of $\mathcal{F}$ at $x$ is defined as

$$\partial\mathcal{F}(x) := \left\{ v \in \mathscr{H} : \liminf_{y \to x} \frac{\mathcal{F}(y) - \mathcal{F}(x) - \langle v, y - x \rangle_{\mathscr{H}}}{|y - x|_{\mathscr{H}}} \geq 0 \right\}.$$

*Remark* 3.3.5. Whenever $\mathcal{F}$ is a λ-convex function it holds that

$$\partial\mathcal{F}(x) = \left\{ v \in \mathscr{H} : \mathcal{F}(y) - \mathcal{F}(x) - \langle v, y - x \rangle_{\mathscr{H}} \geq -\lambda \, |y - x|^2_{\mathscr{H}} \quad \text{for every } y \in \mathscr{H} \right\}. \quad (3.3.5)$$

Indeed, for a convex function $\phi$, $v \in \partial\phi(x)$ if and only if $\phi(y) - \phi(x) - \langle v, y - x \rangle_{\mathscr{H}} \geq 0$ for every $y \in \mathscr{H}$, namely the FrÃĺchet subdifferential coincides with the usual subdifferential of convex analysis. Then, being $\mathcal{F}$ λ-convex and since $\partial\Big(\phi + \lambda |\cdot|^2_{\mathscr{H}}\Big) = \partial\phi + 2\lambda\,\cdot$, it holds that $v \in \partial\mathcal{F}(x)$ if and only if

$$\mathcal{F}(y) + \lambda|y|^2_{\mathscr{H}} - \mathcal{F}(x) - \lambda|x|^2_{\mathscr{H}} - \langle v + 2\lambda x, y - x \rangle_{\mathscr{H}} \geq 0 \quad \text{for every } y \in \mathscr{H},$$

which coincides with the condition in (3.3.5) since $|y - x|^2_{\mathscr{H}} = |y|^2_{\mathscr{H}} - |x|^2_{\mathscr{H}} - 2\langle x, y - x \rangle_{\mathscr{H}}$.

Let $\mathcal{F} : \mathscr{H} \to (-\infty, +\infty]$ be a proper, strongly lower semicontinuous function which is λ-positive and λ-convex, for some $\lambda > 0$, and let $x_0 \in D(\mathcal{F})$. For every $0 < \tau < \frac{1}{2\lambda}$, we denote by $\{x_k^\tau\}_{k \in \mathbb{N}}$ the discrete-in-time evolution for $\mathcal{F}$ with initial datum $x_0$, defined by

$$x_0^\tau := x_0, \qquad x_{k+1}^\tau \in \operatorname{argmin}\left\{ \mathcal{F}(x) + \frac{1}{2\tau}|x - x_k^\tau|^2_{\mathscr{H}} \right\} \quad \text{for every } k \in \mathbb{N} \cup \{0\}. \qquad (3.3.6)$$

Since $x_0 \in D(\mathcal{F})$, then $x_k^\tau \in D(\mathcal{F})$ for every $k \in \mathbb{N}$. Furthermore, we define the piecewise-affine interpolation $x^\tau : [0, +\infty) \to \mathscr{H}$ of $\{x_k^\tau\}_{k \in \mathbb{N}}$ as

$$x^\tau(t) := x_k^\tau + \frac{x_{k+1}^\tau - x_k^\tau}{\tau}(t - k\tau), \quad t \in [k\tau, (k+1)\tau). \qquad (3.3.7)$$

**Theorem 3.3.6.** *Let $\mathcal{F} : \mathscr{H} \to (-\infty, +\infty]$ be a proper, strongly lower semicontinuous function which is λ-convex and λ-positive, for some $\lambda > 0$. Let moreover $x_0 \in D(\mathcal{F})$. Then, there exists a unique solution $x \in H^1([0, +\infty); \mathscr{H})$ to the following Cauchy problem*

$$\begin{cases} \dot{x}(t) \in -\partial\mathcal{F}(x(t)) & \text{for a.e. } t \in [0, +\infty), \\ x(0) = x_0. \end{cases} \qquad (3.3.8)$$

*Moreover, for every $T > 0$, $x^\tau \rightharpoonup x$ in $H^1([0, T]; \mathscr{H})$, where $x^\tau$ is defined in* (3.3.7) *for $0 < \tau < \frac{1}{2\lambda}$. Furthermore,*

$$\|\dot{x}\|^2_{L^2((0,T);\mathscr{H})} \leq 4^{8\lambda T + 4}(\mathcal{F}(x_0) + \lambda|x_0|^2_{\mathscr{H}}) \text{ for every } T > 0, \qquad (3.3.9)$$

$$|x(t) - x^\tau(t)|^2_{\mathscr{H}} \leq C\tau\, 4^{8\lambda t}(\mathcal{F}(x_0) + \lambda|x_0|^2_{\mathscr{H}})(1 + e^{8\lambda t}) \text{ for every } t \geq 0,\ \tau < \frac{1}{16\lambda} \qquad (3.3.10)$$

*for a universal constant $C > 0$.*

*Proof.* *Uniqueness.* Let $T > 0$ and let $x_1, x_2 \in H^1([0,T]; \mathscr{H})$ satisfy the Cauchy problem (3.3.8) up to time $T$.

$$\langle y_1 - y_2, v_1 - v_2\rangle_{\mathscr{H}} \le 2\lambda\, |y_1 - y_2|^2_{\mathscr{H}} \quad \text{for every } y_1, y_2 \in \mathscr{H}\,,\ -v_i \in \partial\mathcal{F}(y_i) \text{ for } i = 1, 2\,. \tag{3.3.11}$$

Indeed, by (3.3.5), we have

$$\mathcal{F}(y) - \mathcal{F}(y_1) + \langle v_1, y - y_1\rangle_{\mathscr{H}} \ge -\lambda\, |y - y_1|^2_{\mathscr{H}}\,, \quad y \in \mathscr{H}\,,$$

which, for $y = y_2$ implies

$$\mathcal{F}(y_2) - \mathcal{F}(y_1) + \langle v_1, y_2 - y_1\rangle_{\mathscr{H}} \ge -\lambda\, |y_2 - y_1|^2_{\mathscr{H}}; \tag{3.3.12}$$

analogously

$$\mathcal{F}(y_1) - \mathcal{F}(y_2) + \langle v_2, y_1 - y_2\rangle_{\mathscr{H}} \ge -\lambda\, |y_2 - y_1|^2_{\mathscr{H}}\,. \tag{3.3.13}$$

Therefore, (3.3.11) follows by summing (3.3.12) and (3.3.13).

Finally, by formula (3.3.11) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}|x_1(t) - x_2(t)|^2_{\mathscr{H}} = 2\langle \dot{x}_1(t) - \dot{x}_2(t), x_1(t) - x_2(t)\rangle_{\mathscr{H}} \le 4\lambda\, |x_1(t) - x_2(t)|^2_{\mathscr{H}} \quad \text{for a.e. } t \in [0,T],$$

which, by Gronwall's Lemma, implies

$$|x_1(t) - x_2(t)|^2_{\mathscr{H}} \le |x_0 - x_0|_{\mathscr{H}} \exp(4\lambda t) = 0 \quad \text{for a.e. } t \in [0,T]\,,$$

i.e., $x_1(t) = x_2(t)$ a.e. $t \in [0,T]$. We notice that the solution is in $C^{0,\frac{1}{2}}([0,T]; \mathscr{H})$ by the Sobolev embedding of $H^1([0,T]; \mathscr{H})$ into $C^{0,\frac{1}{2}}([0,T]; \mathscr{H})$, so that $x_1(t) = x_2(t)$ when passing to the continuous representatives.

*Existence.* We first prove that for every $T > 0$ the functions $x^\tau$ defined in (3.3.7) converge (as $\tau \to 0$) weakly in $H^1([0,T]; \mathscr{H})$ to some function $x \in H^1([0,T]; \mathscr{H})$ and then we show that the limit $x$ satisfies (3.3.8) up to time $T$.

By (3.3.6) we have that

$$\mathcal{F}(x^\tau_{k+1}) + \frac{1}{2\tau}|x^\tau_{k+1} - x^\tau_k|^2_{\mathscr{H}} \le \mathcal{F}(x^\tau_k), \quad \text{for every } k \in \mathbb{N}\,, \tag{3.3.14}$$

which together with the $\lambda$-positivity of $\mathcal{F}$ implies that

$$\begin{aligned}
\sum_{k=0}^{K} \frac{1}{\tau}\,|x^\tau_{k+1} - x^\tau_k|^2_{\mathscr{H}} &\le 2\sum_{k=0}^{K}\left(\mathcal{F}(x^\tau_k) - \mathcal{F}(x^\tau_{k+1})\right)\\
&= 2(\mathcal{F}(x^\tau_0) - \mathcal{F}(x^\tau_{K+1}))\\
&= 2\Big(\mathcal{F}(x^\tau_0) + \frac{\lambda}{2}|x^\tau_{K+1}|^2_{\mathscr{H}} - \frac{\lambda}{2}|x^\tau_{K+1}|^2_{\mathscr{H}} - \mathcal{F}(x^\tau_{K+1})\Big)\\
&\le 2\left(\mathcal{F}(x_0) + \frac{\lambda}{2}|x^\tau_{K+1}|^2_{\mathscr{H}}\right) \qquad \text{for every } K \in \mathbb{N}\,.
\end{aligned} \tag{3.3.15}$$

Set $\hat{T} = \frac{1}{8\lambda}$ and let $0 < \tau \le \frac{1}{16\lambda}$. We set $\hat{K} := \left\lceil \frac{\hat{T}}{\tau} \right\rceil$; by (3.3.15), we have

$$\int_0^{\hat{T}} |\dot{x}^\tau(t)|^2_{\mathscr{H}}\,\mathrm{d}t \le \sum_{k=0}^{\hat{K}} \frac{1}{\tau}\,|x^\tau_{k+1} - x^\tau_k|^2_{\mathscr{H}} \le 2\left(\mathcal{F}(x_0) + \frac{\lambda}{2}|x^\tau_{\hat{K}+1}|^2_{\mathscr{H}}\right)\,. \tag{3.3.16}$$

Moreover, by triangular and Jensen inequalities and using again (3.3.15), we get

$$
\begin{aligned}
&\frac{1}{2}|x^\tau_{\hat K+1}|^2_{\mathscr H} - |x_0|^2_{\mathscr H} \le |x^\tau_{\hat K+1} - x_0|^2_{\mathscr H} \\
\le& \tau(\hat K+1)\sum_{k=1}^{\hat K+1}\frac{1}{\tau}\left|x^\tau_k - x^\tau_{k-1}\right|^2_{\mathscr H} = \tau(\hat K+1)\sum_{k=0}^{\hat K}\frac{1}{\tau}\left|x^\tau_{k+1} - x^\tau_k\right|^2_{\mathscr H} \\
\le& 2(\hat T + 2\tau)\Big(\mathcal F(x_0) + \frac{\lambda}{2}|x^\tau_{\hat K+1}|^2_{\mathscr H}\Big)\,,
\end{aligned}
\tag{3.3.17}
$$

which, recalling that $0 < 2\tau \le \frac{1}{8\lambda} = \hat T$, implies that

$$
|x^\tau_{\hat K+1}|^2_{\mathscr H} \le \frac{2}{\lambda}\mathcal F(x_0) + 4|x_0|^2_{\mathscr H}\,. \tag{3.3.18}
$$

By (3.3.16) and (3.3.18), we have that, for every $\tau$ small enough,

$$
\|\dot x^\tau\|^2_{\mathrm L^2((0,\hat T);\mathscr H)} \le 4(\mathcal F(x_0) + \lambda|x_0|^2_{\mathscr H})\,. \tag{3.3.19}
$$

Iterating the estimates in (3.3.18) and (3.3.19), we deduce that for every $j \in \mathbb N$

$$
\begin{aligned}
|x^\tau(j\hat T)|^2_{\mathscr H} &\le 4^j\Big(\frac{1}{\lambda}\mathcal F(x_0) + |x_0|^2_{\mathscr H}\Big), \\
\|\dot x^\tau\|^2_{\mathrm L^2((0,J\hat T);\mathscr H)} &\le 4^{J+3}(\mathcal F(x_0) + \lambda|x_0|^2_{\mathscr H})\,.
\end{aligned}
$$

In particular, for every $T > 0$, we have that

$$
\|\dot x^\tau\|^2_{\mathrm L^2((0,T);\mathscr H)} \le 4^{8\lambda T+4}(\mathcal F(x_0) + \lambda|x_0|^2_{\mathscr H})\,. \tag{3.3.20}
$$

Therefore, for every $T > 0$, $\|x^\tau\|_{H^1([0,T];\mathscr H)}$ is uniformly bounded and hence, up to a subsequence, $x^\tau \rightharpoonup x$ in $H^1([0,T];\mathscr H)$ for some $x \in H^1([0,T];\mathscr H)$; this, in particular, implies the convergence in $C^{0,\frac{1}{2}}([0,T];\mathscr H)$ and hence that $x(0) = x_0$. Passing to the limit in (3.3.20) we readily get (3.3.9).

Now we aim at proving that $x$ solves (3.3.8) up to time $T$,
for every $T > 0$, that is

$$
\dot x(t) \in -\partial\mathcal F(x(t)) \qquad \text{for almost every } t \in (0,T)\,. \tag{3.3.21}
$$

To this end, we define the piecewise-constant interpolation $\tilde x^\tau : [0,+\infty) \to \mathscr H$ of $\{x^\tau_k\}_{k\in\mathbb N}$ as

$$
\tilde x^\tau(t) := x^\tau_{k+1}, \quad t \in [k\tau,(k+1)\tau)\,, \tag{3.3.22}
$$

and we notice that, by minimality, for $\tau$ small enough,

$$
\dot x^\tau(t) \in -\partial\mathcal F(\tilde x^\tau(t)) \qquad \text{for almost every } t \in [0,+\infty)\,. \tag{3.3.23}
$$

We claim that

$$
\tilde x^\tau \overset{\mathscr H}{\to} x \quad \text{in } \mathrm L^2((0,T);\mathscr H)\,, \quad \text{for every } T > 0\,. \tag{3.3.24}
$$

Indeed, by triangular inequality, we have that

$$\begin{aligned}\|\tilde{x}^\tau - x\|^2_{\mathrm{L}^2((0,T);\mathscr{H})} &\le 2\|x^\tau - x\|^2_{\mathrm{L}^2((0,T);\mathscr{H})} + 2\|\tilde{x}^\tau - x^\tau\|^2_{\mathrm{L}^2((0,T);\mathscr{H})} \\ &\le 2\|x^\tau - x\|^2_{\mathrm{L}^2((0,T);\mathscr{H})} + 2\tau^2\|\dot{x}^\tau\|^2_{\mathrm{L}^2((0,T);\mathscr{H})}\,,\end{aligned} \tag{3.3.25}$$

where in the last inequality we have used that

$$x^\tau(t) - \tilde{x}^\tau(t) = \frac{x^\tau_{k+1} - x^\tau_k}{\tau}(t-(k+1)\tau) = \dot{x}^\tau(t)(t-(k+1)\tau), \quad \text{for every } t \in (k\tau,(k+1)\tau)\,.$$

Therefore, by (3.3.25), (3.3.10) and (3.3.9), we get

$$\|\tilde{x}^\tau - x\|^2_{\mathrm{L}^2((0,T);\mathscr{H})} \le 16\tau T 4^{8\lambda T+4}(\mathcal{F}(x_0)+\lambda|x_0|^2_{\mathscr{H}})(1+e^{4\lambda T}) + 2\tau^2 4^{8\lambda T+4}(\mathcal{F}(x_0)+\lambda|x_0|^2_{\mathscr{H}})\,,$$

which, sending $\tau \to 0$, implies (3.3.24).

With (3.3.24) in hand, we are in a position to prove (3.3.21). Let $t_0 \in (0,T)$ be a Lebesgue point of the function $\dot{x} : [0,T) \to \mathscr{H}$. By (3.3.23), we have that

$$\mathcal{F}(y) \ge \mathcal{F}(\tilde{x}^\tau(t)) - \langle \dot{x}^\tau(t), y - \tilde{x}^\tau(t)\rangle_{\mathscr{H}} - \frac{\lambda}{2}|y - \tilde{x}^\tau(t)|^2_{\mathscr{H}} \qquad \text{for every } y \in \mathscr{H}\,. \tag{3.3.26}$$

Let $y \in \mathscr{H}$ and $h > 0$; by integrating (3.3.26) in the interval $(t_0, t_0+h)$ and dividing by $h$, we obtain

$$\mathcal{F}(y) \ge \frac{1}{h}\int_{t_0}^{t_0+h} \mathcal{F}(\tilde{x}^\tau(t))\,\mathrm{d}t - \frac{1}{h}\int_{t_0}^{t_0+h}\langle \dot{x}^\tau(t), y - \tilde{x}^\tau(t)\rangle_{\mathscr{H}}\,\mathrm{d}t - \frac{1}{h}\int_{t_0}^{t_0+h}\frac{\lambda}{2}|y - \tilde{x}^\tau(t)|^2_{\mathscr{H}}\,\mathrm{d}t\,,$$

which, sending $\tau \to 0$, and using the strong lower semicontinuity of $\mathcal{F}$, the weak $\mathrm{L}^2$-convergence of $\dot{x}^\tau$ to $\dot{x}$, and (3.3.24), yields

$$\mathcal{F}(y) \ge \frac{1}{h}\int_{t_0}^{t_0+h} \mathcal{F}(x(t))\,\mathrm{d}t - \frac{1}{h}\int_{t_0}^{t_0+h}\langle \dot{x}(t), y - x(t)\rangle_{\mathscr{H}}\,\mathrm{d}t - \frac{1}{h}\int_{t_0}^{t_0+h}\frac{\lambda}{2}|y - x(t)|^2_{\mathscr{H}}\,\mathrm{d}t\,.$$

Now, since $x \in C^{0,\frac{1}{2}}([0,T];\mathscr{H})$ and since $t_0$ is a Lebesgue point for $\dot{x}$, sending $h \to 0$ in the formula above, and using again that $\mathcal{F}$ is strongly lower semicontinuous, by the arbitrariness of $y$, we get (3.3.21).

Finally, we prove that (3.3.10) holds true. Let $\eta^\tau : [0,+\infty) \to (0,\tau]$ be the function defined by $\eta^\tau(t) = (k+1)\tau - t$ for every $t \in [k\tau,(k+1)\tau)$. By (3.3.23),

$$\dot{x}^\tau(t) \in -\partial\mathcal{F}(x^\tau(t+\eta^\tau(t))) \qquad \text{for every } t > 0\,, \tag{3.3.27}$$

which, using (3.3.8) and (3.3.11), yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|x(t) - x^\tau(t)|^2_{\mathscr{H}} &= 2\langle x(t) - x^\tau(t), \dot{x}(t) - \dot{x}^\tau(t)\rangle_{\mathscr{H}} \\ &= 2\langle x(t) - x^\tau(t+\eta^\tau(t)), \dot{x}(t) - \dot{x}^\tau(t)\rangle_{\mathscr{H}} \\ &\quad +2\langle x^\tau(t+\eta^\tau(t)) - x^\tau(t), \dot{x}(t) - \dot{x}^\tau(t)\rangle_{\mathscr{H}} \\ &\le 4\lambda|x(t) - x^\tau(t+\eta^\tau(t))|^2_{\mathscr{H}} + 2|x^\tau(t+\eta^\tau(t)) - x^\tau(t)|_{\mathscr{H}}|\dot{x}(t) - \dot{x}^\tau(t)|_{\mathscr{H}} \\ &\le 8\lambda|x(t) - x^\tau(t)|^2_{\mathscr{H}} + 8\lambda|x^\tau(t+\eta^\tau(t)) - x^\tau(t)|^2_{\mathscr{H}} \\ &\quad +2|x^\tau(t+\eta^\tau(t)) - x^\tau(t)|_{\mathscr{H}}(|\dot{x}(t)|_{\mathscr{H}} + |\dot{x}^\tau(t)|_{\mathscr{H}}) \\ &\le 8\lambda|x(t) - x^\tau(t)|^2_{\mathscr{H}} + \tau(8\lambda\tau+3)(|\dot{x}^\tau(t)|^2_{\mathscr{H}} + |\dot{x}(t)|^2_{\mathscr{H}})\,, \\ &\le 8\lambda|x(t) - x^\tau(t)|^2_{\mathscr{H}} + 5\tau(|\dot{x}^\tau(t)|^2_{\mathscr{H}} + |\dot{x}(t)|^2_{\mathscr{H}})\,,\end{aligned}$$

where in the last inequality we have used that $\tau \leq \frac{1}{16\lambda}$ (recall also $|x^\tau(t+\eta^\tau(t)) - x^\tau(t)|_{\mathscr{H}} = \eta^\tau(t)|\dot{x}^\tau(t)|_{\mathscr{H}} \leq \tau|\dot{x}^\tau(t)|_{\mathscr{H}}$).

We now apply Gronwall Lemma with (3.3.9) and (3.3.20). For

$$\alpha(t) := 10\tau \cdot 4^{8\lambda t+4}(\mathcal{F}(x_0) + \lambda|x_0|^2_{\mathscr{H}}),$$

we get

$$|x(t) - x^\tau(t)|^2_{\mathscr{H}} \leq \alpha(t) + 8\lambda e^{8\lambda} \int_0^t \alpha(s)\,\mathrm{d}s \leq C\,\alpha(t)(1 + e^{8\lambda t})$$

for a universal constant $C > 0$, which gives (3.3.10). □

Let $\mathscr{V}$ be a vector space, with a standard notation we denote with $\mathscr{V}^*$ the algebraic dual space of $\mathscr{V}$.

**Proposition 3.3.7.** *Let $\mathcal{F} : \mathscr{H} \to (-\infty, +\infty]$ be a proper lower semicontinuous function which is $\lambda$-convex, for some $\lambda > 0$ and let $x \in D(\mathcal{F})$. Let $\hat{\mathscr{H}}$ be a dense subspace of $\mathscr{H}$. If there exists $T \in (\hat{\mathscr{H}})^*$ such that*

$$\lim_{t\to 0} \frac{\mathcal{F}(x + t\varphi) - \mathcal{F}(x)}{t} = T(\varphi) \qquad \text{for every } \varphi \in \hat{\mathscr{H}}, \tag{3.3.28}$$

*then, either $\partial\mathcal{F}(x) = \emptyset$ or $\partial\mathcal{F}(x) = \{v\}$, where $v$ is the (unique) element in $\mathscr{H}$ satisfying $T(\varphi) = \langle v, \varphi\rangle_{\mathscr{H}}$ for every $\varphi \in \hat{\mathscr{H}}$. In particular, $T \in (\hat{\mathscr{H}})'$ and $v$ is its unique continuous extension to $\mathscr{H}'$.*

*Proof.* Since $\hat{\mathscr{H}}$ is dense in $\mathscr{H}$, in order to get the claim it is enough to prove that for every $v \in \partial\mathcal{F}(x)$

$$\langle v, \varphi\rangle_{\mathscr{H}} = T(\varphi) \qquad \text{for every } \varphi \in \hat{\mathscr{H}}. \tag{3.3.29}$$

To this purpose, we notice that every $v \in \partial\mathcal{F}(x)$ satisfies

$$\mathcal{F}(x + t\varphi) - \mathcal{F}(x) - t\langle v, \varphi\rangle_{\mathscr{H}} \geq -t^2\frac{\lambda}{2}|\varphi|^2_{\mathscr{H}} \qquad \text{for every } \varphi \in \hat{\mathscr{H}},\ t \in \mathbb{R},$$

which, dividing by $t$, yields

$$\lim_{t\to 0} \frac{\mathcal{F}(x + t\varphi) - \mathcal{F}(x)}{t} = \langle v, \varphi\rangle_{\mathscr{H}} \qquad \text{for every } \varphi \in \hat{\mathscr{H}};$$

therefore, in view of (3.3.28), we get (3.3.29). □

We conclude this section by showing how we can use the results here collected in order to prove a convergence result for the *Minimizing Movement* type solutions to gradient-flow equations associated to a $\Gamma$-converging sequence of functions satisfying the assumptions of Theorem 3.3.6.

**Theorem 3.3.8.** *Let $\{\mathcal{F}^n\}_{n\in\mathbb{N}}$ with $\mathcal{F}^n : \mathscr{H} \to (-\infty, +\infty]$ for every $n \in \mathbb{N}$ be a sequence of proper, strongly lower semicontinuous functions which are $\lambda$-convex and $\lambda$-positive, for some $\lambda > 0$ independent of $n$. Let $\{x_0^n\}_{n\in\mathbb{N}} \subset \mathscr{H}$ be such that $x_0^n \in D(\mathcal{F}^n)$ for every $n \in \mathbb{N}$, $S := \sup_{n\in\mathbb{N}} \mathcal{F}^n(x_0^n) < +\infty$ and $x_0^n \to x_0^\infty$ for some $x_0^\infty \in \mathscr{H}$. Assume that one of the following statements is satisfied:*

*(a) The functions $\mathcal{F}^n$ $\Gamma$-converge to some proper function $\mathcal{F}^\infty$ with respect to the weak $\mathscr{H}$-convergence (as $n \to +\infty$). Moreover, the $\Gamma$-limsup inequality is satisfied with respect to the strong $\mathscr{H}$-convergence, i.e., for every $y \in \mathscr{H}$ there exists a sequence $\{y^n\}_{n\in\mathbb{N}}$ with $y^n \stackrel{\mathscr{H}}{\to} y$ such that $\mathcal{F}^n(y^n) \to \mathcal{F}^\infty(y)$ as $n \to +\infty$.*

*(b) The functions $\mathcal{F}^n$ $\Gamma$-converge to some proper function $\mathcal{F}^\infty$ with respect to the strong $\mathscr{H}$-convergence (as $n \to +\infty$) and every sequence $\{y^n\}_{n\in\mathbb{N}} \subset \mathscr{H}$ with $\sup_{n\in\mathbb{N}} \mathcal{F}^n(y^n) + \frac{\lambda}{2}|y^n|^2_{\mathscr{H}} < +\infty$, admits a strongly convergent subsequence.*

*Then, $x_0^\infty \in D(\mathcal{F}^\infty)$ and, for every $T > 0$, the solutions $x^n$ to the Cauchy problem*

$$\begin{cases} \dot{x}(t) \in -\partial\mathcal{F}^n(x(t)) & \text{for a.e. } t \in (0,T)\,, \\ x(0) = x_0^n \end{cases} \tag{3.3.30}$$

*weakly converge, as $n \to +\infty$, in $H^1([0,T];\mathscr{H})$ to the unique solution $x^\infty$ to the problem*

$$\begin{cases} \dot{x}(t) \in -\partial\mathcal{F}^\infty(x(t)) & \text{for a.e. } t \in (0,T)\,, \\ x(0) = x_0^\infty\,. \end{cases} \tag{3.3.31}$$

*Furthermore, if*

$$\lim_{n\to+\infty} \mathcal{F}^n(x_0^n) = \mathcal{F}^\infty(x_0^\infty)\,, \tag{3.3.32}$$

*then, we have that*

$$x^n \to x^\infty \qquad \text{(strongly) in } H^1([0,T];\mathscr{H}) \qquad \text{for every } T > 0\,, \tag{3.3.33}$$

$$x^n(t) \stackrel{\mathscr{H}}{\to} x^\infty(t) \quad \text{and} \quad \mathcal{F}^n(x^n(t)) \to \mathcal{F}^\infty(x^\infty(t)) \qquad \text{for every } t \geq 0\,. \tag{3.3.34}$$

*Proof.* We preliminarily notice that, if either (a) or (b) is satisfied, then the function $\mathcal{F}^\infty$ is strongly lower semicontinuous, $\lambda$-convex and $\lambda$-positive and $x_0^\infty \in D(\mathcal{F}^\infty)$. Moreover, by Theorem 3.3.6, for every $n \in \mathbb{N}$ there exists a unique solution $x^n$ to (3.3.30).

Let $0 < \tau < \frac{1}{2\lambda}$ and let $\{x_k^{\infty,\tau}\}_{k\in\mathbb{N}}$ denote the discrete-in-time evolution in (3.3.6) for $x_0 := x_0^\infty$ and $\mathcal{F} := \mathcal{F}^\infty$. Analogously, for every $n \in \mathbb{N}$, let $\{x_k^{n,\tau}\}_{k\in\mathbb{N}}$ denote the discrete-in-time evolution in (3.3.6) for $x_0 := x_0^n$ and $\mathcal{F} := \mathcal{F}^n$. By Proposition 3.3.3, $\{x_k^{\infty,\tau}\}_{k\in\mathbb{N}}$ and $\{x_k^{n,\tau}\}_{k\in\mathbb{N}}$ are uniquely determined. Furthermore, for every $k \in \mathbb{N}$ we set

$$\begin{aligned} \mathcal{I}_k^{n,\tau}(\cdot) &:= \mathcal{F}^n(\cdot) + \frac{1}{2\tau}|\cdot - x_{k-1}^{n,\tau}|^2_{\mathscr{H}} \qquad \text{for every } n \in \mathbb{N}\,, \\ \mathcal{I}_k^{\infty,\tau}(\cdot) &:= \mathcal{F}^\infty(\cdot) + \frac{1}{2\tau}|\cdot - x_{k-1}^{\infty,\tau}|^2_{\mathscr{H}}\,. \end{aligned}$$

We first show that, if either (a) or (b) is satisfied, then for every $k \in \mathbb{N}$

$$\mathcal{F}^n(x_k^{n,\tau}) \to \mathcal{F}^\infty(x_k^{\infty,\tau}) \quad \text{and} \quad |x_k^{n,\tau} - x_k^{\infty,\tau}|_{\mathscr{H}} \to 0 \qquad \text{as } n \to +\infty\,. \tag{3.3.35}$$

By finite induction, it is enough to show (3.3.35) for $k = 1$. We distinguish the two cases in which either (a) or (b) holds true.

Assume first that (a) holds true. By the assumptions on $x_0^n$, we have that

$$\mathcal{I}_1^{n,\tau}(x_1^{n,\tau}) \leq \mathcal{F}^n(x_0^n) \leq S\,,$$

whence, using that for $\frac{1}{2\tau} > \lambda$ the functions $\mathcal{I}_1^{n,\tau}(\cdot)$ are weakly equi-coercive, we deduce that, up to a subsequence, $x_1^{n,\tau} \overset{\mathscr{H}}{\rightharpoonup} y_1$ for some $y_1 \in \mathscr{H}$. Moreover, since $|x_0^n - x_0^\infty|_{\mathscr{H}} \to 0$ as $n \to +\infty$ and since the functions $\mathcal{F}^n$ $\Gamma$-converge to the function $\mathcal{F}^\infty$ with respect to the weak $\mathscr{H}$-convergence, we have that

$$\mathcal{I}_1^{\infty,\tau}(y_1) \le \liminf_{n\to+\infty} \mathcal{F}^n(x_1^{n,\tau}) + \frac{1}{2\tau} \liminf_{n\to+\infty} |x_1^{n,\tau} - x_0^n|^2_{\mathscr{H}} \le \liminf_{n\to+\infty} \mathcal{I}_1^{n,\tau}(x_1^{n,\tau}) . \tag{3.3.36}$$

Furthermore, since the $\Gamma$-limsup inequality is satisfied with respect to the strong $\mathscr{H}$-convergence, there exists $\{\bar{x}_1^{n,\tau}\}_{n\in\mathbb{N}} \subset \mathscr{H}$ such that

$$\bar{x}_1^{n,\tau} \overset{\mathscr{H}}{\to} x_1^{\infty,\tau} \quad \text{and} \quad \mathcal{F}^n(\bar{x}_1^{n,\tau}) \to \mathcal{F}(x_1^{\infty,\tau}) , \tag{3.3.37}$$

where $x_1^{\infty,\tau}$ is the unique solution to the problem (3.3.6) with $\mathcal{F} = \mathcal{F}^\infty$ and $k = 1$. Therefore, by (3.3.36) and (3.3.37), we get

$$\begin{aligned} \mathcal{I}_1^{\infty,\tau}(y_1) &\le \liminf_{n\to+\infty} \mathcal{F}^n(x_1^{n,\tau}) + \frac{1}{2\tau} \liminf_{n\to+\infty} |x_1^{n,\tau} - x_0^n|^2_{\mathscr{H}} \\ &\le \liminf_{n\to+\infty} \mathcal{I}_1^{n,\tau}(x_1^{n,\tau}) \le \limsup_{n\to+\infty} \mathcal{I}_1^{n,\tau}(x_1^{n,\tau}) \\ &\le \lim_{n\to+\infty} \mathcal{I}_1^{n,\tau}(\bar{x}_1^{n,\tau}) = \mathcal{I}_1^{\infty,\tau}(x_1^{\infty,\tau}) , \end{aligned} \tag{3.3.38}$$

whence, by the minimality of $x_1^{\infty,\tau}$, we deduce that all the inequalities above are in fact equalities and, in particular, that $y_1$ is a minimizer of $\mathcal{I}_1^{\infty,\tau}$; in view of the uniqueness of the minimizer of $\mathcal{I}_1^{\infty,\tau}$, we deduce that $y_1 = x_1^{\infty,\tau}$. By Urysohn Lemma, this implies that the whole sequence $\{x_1^{n,\tau}\}_{n\in\mathbb{N}}$ weakly converges to $x_1^{\infty,\tau}$. Moreover, using that

$$\mathcal{F}^\infty(x_1^{\infty,\tau}) + \frac{1}{2\tau}|x_1^{\infty,\tau} - x_0^\infty|^2_{\mathscr{H}} = \liminf_{n\to+\infty} \mathcal{F}^n(x_1^{n,\tau}) + \frac{1}{2\tau} \liminf_{n\to+\infty} |x_1^{n,\tau} - x_0^n|^2_{\mathscr{H}} ,$$

since

$$\mathcal{F}^\infty(x_1^{\infty,\tau}) \le \liminf_{n\to+\infty} \mathcal{F}^n(x_1^{n,\tau}) \quad \text{and} \quad \frac{1}{2\tau}|x_1^{\infty,\tau} - x_0^\infty|^2_{\mathscr{H}} \le \frac{1}{2\tau} \liminf_{n\to+\infty} |x_1^{n,\tau} - x_0^n|^2_{\mathscr{H}} ,$$

we deduce that

$$\mathcal{F}^\infty(x_1^{\infty,\tau}) = \lim_{n\to+\infty} \mathcal{F}^n(x_1^{n,\tau}) \quad \text{and} \quad |x_1^{\infty,\tau} - x_0^\infty|_{\mathscr{H}} = \lim_{n\to+\infty} |x_1^{n,\tau} - x_0^n|_{\mathscr{H}} ,$$

which implies (3.3.35) (for $k = 1$ and then for all $k \in \mathbb{N}$).

Assume now that (b) holds true.

As above (recall $\lambda < \frac{1}{2\tau}$) we have that

$$\mathcal{F}^n(x_1^{n,\tau}) + \lambda|x_1^{n,\tau} - x_0^n|^2_{\mathscr{H}} \le \mathcal{I}_1^{n,\tau}(x_1^{n,\tau}) \le S ,$$

whence, by the strong compactness property of the functions $\mathcal{F}^n(\cdot) + \frac{\lambda}{2}|\cdot|^2_{\mathscr{H}}$ we deduce that, up to a subsequence, $x_1^{n,\tau} \overset{\mathscr{H}}{\to} y_1$ for some $y_1 \in \mathscr{H}$. Moreover, since $|x_0^n - x_0^\infty|_{\mathscr{H}} \to 0$ as $n \to +\infty$, we have that the functionals $\mathcal{I}_1^{n,\tau}$ $\Gamma$-converge with respect to the strong-$\mathscr{H}$ convergence to the functional $\mathcal{I}_1^{\infty,\tau}$. By the fundamental theorem of $\Gamma$-convergence and by the uniqueness of the minimizer of the problem (3.3.6) with $\mathcal{F} = \mathcal{F}^\infty$ and $k = 1$, we get that $y_1 = x_1^{\infty,\tau}$, that the whole sequence $\{x_1^{n,\tau}\}_{n\in\mathbb{N}}$ strongly converges to $x_1^{\infty,\tau}$ , and that (3.3.35) is satisfied for $k=1$ . This concludes the proof of (3.3.35) for both the cases (a) and (b).

Now we show that for every $T>0$ , $x^n \rightharpoonup x^\infty$ in $H^1([0,T];\mathscr{H})$ , where $x^\infty$ is the unique solution to (3.3.31) . To this end, we first notice that by (3.3.9), for $n$ large enough,

$$\|\dot{x}^n\|^2_{L^2((0,T);\mathscr{H})} \le 4^{8\lambda T+4}(S+2\lambda|x_0|^2_{\mathscr{H}}),$$

so that, up to a subsequence, $x^n \rightharpoonup \bar{x}$ in $H^1([0,T];\mathscr{H})$ , for some $\bar{x}\in H^1([0,T];\mathscr{H})$ . Now we show that $\bar{x}=x^\infty$ .

For every $0<\tau<\frac{1}{2\lambda}$ , let $x^{\infty,\tau}$ and $x^{n,\tau}$ $(n\in\mathbb{N})$ denote the piecewise affine interpolations defined in (3.3.7), of $\{x_k^{\infty,\tau}\}_{k\in\mathbb{N}}$ and $\{x_k^{n,\tau}\}_{k\in\mathbb{N}}$, respectively. By (3.3.35), we have that

$$\lim_{n\to+\infty}|x^{n,\tau}(t)-x^{\infty,\tau}(t)|_{\mathscr{H}}=0 \qquad \text{for every } t>0\,, 0<\tau<\frac{1}{2\lambda}\,. \tag{3.3.39}$$

Let $t>0$ . For every $0<\tau<\frac{1}{2\lambda}$ , by triangular inequality and by (3.3.10), for a universal constant $C>0$ we have that

$$\begin{aligned}|x^n(t)-x^\infty(t)|_{\mathscr{H}} \le&|x^n(t)-x^{n,\tau}(t)|_{\mathscr{H}}+|x^{n,\tau}(t)-x^{\infty,\tau}(t)|_{\mathscr{H}}+|x^{\infty,\tau}(t)-x^\infty(t)|_{\mathscr{H}}\\ \le&C\tau 4^{8\lambda t}(S+2\lambda|x_0|^2_{\mathscr{H}})(1+e^{8\lambda t})+|x^{n,\tau}(t)-x^{\infty,\tau}(t)|_{\mathscr{H}}\,;\end{aligned} \tag{3.3.40}$$

therefore, sending first $n\to+\infty$ and then $\tau\to 0$ in (3.3.40) and using (3.3.39), we get that $x^n(t)\overset{\mathscr{H}}{\to}x^\infty(t)$ as $n\to+\infty$ . By the uniqueness of the limit we deduce that $\bar{x}=x^\infty$ and that the whole sequence $\{x^n\}_{n\in\mathbb{N}}$ weakly converges in $H^1([0,T];\mathscr{H})$ to $x^\infty$ .

Finally, we prove that (3.3.33) and (3.3.34) hold true. By (3.3.40), the first part of (3.3.34) is satisfied. Moreover, by [67, formula (1.10)] (notice that, as observed in [67], the formula applies also for $\lambda$-convex energies), we have that, for every $t>0$ ,

$$\begin{aligned}&\mathcal{F}^n(x_0^n(t))-\mathcal{F}^n(x^n(t))=\frac{1}{2}\int_0^t|\dot{x}^n(s)|^2_{\mathscr{H}}\,\mathrm{d}s \qquad \text{for every } n\in\mathbb{N}\,,\\ &\mathcal{F}^\infty(x_0^\infty(t))-\mathcal{F}^\infty(x^\infty(t))=\frac{1}{2}\int_0^t|\dot{x}^\infty(s)|^2_{\mathscr{H}}\,\mathrm{d}s\,,\end{aligned} \tag{3.3.41}$$

which, using (3.3.32), the $\Gamma$-liminf inequality (that holds true in both the cases (a) and (b)) and the weak $H^1$-convergence of $x^n$ to $x^\infty$, implies

$$\begin{aligned}&\mathcal{F}^\infty(x_0^\infty(t))-\liminf_{n\to+\infty}\mathcal{F}^n(x^n(t))\le\mathcal{F}^\infty(x_0^\infty(t))-\mathcal{F}^\infty(x^\infty(t))=\frac{1}{2}\int_0^t|\dot{x}^\infty(s)|^2_{\mathscr{H}}\,\mathrm{d}s\\ \le&\liminf_{n\to+\infty}\frac{1}{2}\int_0^t|\dot{x}^n(s)|^2_{\mathscr{H}}\,\mathrm{d}s\le\limsup_{n\to+\infty}\frac{1}{2}\int_0^t|\dot{x}^n(s)|^2_{\mathscr{H}}\,\mathrm{d}s\\ \le&\limsup_{n\to+\infty}\mathcal{F}^n(x_0^n(t))-\liminf_{n\to+\infty}\mathcal{F}^n(x^n(t))=\mathcal{F}^\infty(x_0^\infty(t))-\liminf_{n\to+\infty}\mathcal{F}^n(x^n(t))\,.\end{aligned} \tag{3.3.42}$$

Therefore, all the inequalities above are actually equalities; in particular,

$$\frac{1}{2}\int_0^t|\dot{x}^\infty(s)|^2_{\mathscr{H}}\,\mathrm{d}s=\lim_{n\to+\infty}\frac{1}{2}\int_0^t|\dot{x}^n(s)|^2_{\mathscr{H}}\,\mathrm{d}s\,, \tag{3.3.43}$$

which, together (3.3.32) and (3.3.41), yields

$$\mathcal{F}^\infty(x^\infty(t))=\lim_{n\to+\infty}\mathcal{F}^n(x^n(t))\,,$$

thus obtaining also the second part of (3.3.34). Finally, by (3.3.43), we obtain also (3.3.33), thus concluding the proof of the theorem. □

## 3.4 Convergence of the $s$-fractional heat flows

This section is devoted to the proof of the stability of the $s$-fractional heat flows as $s \to 0^+$ and $s \to 1^-$. In the first part, we define the $s$-fractional laplacian for $s \in (0,1)$ and for $s = 0$. The second part contains the convergence theorems, which are the main results of the chapter.

### 3.4.1 The $s$-fractional laplacian for $s \in (0,1)$ and for $s = 0$

For every $s \in (0,1)$ and for every $\psi \in \mathrm{C}^\infty_{\mathrm{c}}(\mathbb{R}^d)$ the $s$-fractional laplacian of $\psi$ is defined by

$$(-\Delta)^s\psi(x) := \int_{\mathbb{R}^d} \frac{2\psi(x) - \psi(x+z) - \psi(x-z)}{|z|^{d+2s}}\,\mathrm{d}z, \qquad x \in \mathbb{R}^d. \tag{3.4.1}$$

In [40, Lemma 3.2] it is proven that the above integral is finite, that $(-\Delta)^s\psi \in \mathrm{L}^\infty(\mathbb{R}^d)$, and that

$$(-\Delta)^s\psi(x) = 2\lim_{r\to 0^+} \int_{\mathbb{R}^d\setminus B_r(0)} \frac{\psi(x) - \psi(x+z)}{|z|^{d+2s}}\,\mathrm{d}z\,. \tag{3.4.2}$$

For every $u \in \mathcal{H}^s_0(\Omega)$ we define the $s$-fractional laplacian of $u$ by duality as

$$\langle(-\Delta)^s u, \varphi\rangle := \langle u, (-\Delta)^s\tilde{\varphi}\rangle, \qquad \text{for all } \varphi \in \mathrm{C}^\infty_{\mathrm{c}}(\Omega). \tag{3.4.3}$$

Here and below $\langle\cdot,\cdot\rangle$ denotes the standard scalar product in $\mathrm{L}^2$.

Clearly, the $s$-fractional laplacian is nothing but the first variation of the squared Gagliardo $s$-norm, as shown below.

**Proposition 3.4.1.** *Let $s \in (0,1)$. For every $u \in \mathcal{H}^s_0(\Omega)$ and for every $\varphi \in \mathrm{C}^\infty_{\mathrm{c}}(\Omega)$ we have*

$$\lim_{t\to 0} \frac{F^s(u+t\varphi) - F^s(u)}{t} = \langle(-\Delta)^s u, \varphi\rangle. \tag{3.4.4}$$

*Proof.* We have

$$\begin{aligned}
\lim_{t\to 0} \frac{F^s(u+t\varphi) - F^s(u)}{t} &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{(\tilde{u}(x) - \tilde{u}(y))(\tilde{\varphi}(x) - \tilde{\varphi}(y))}{|x-y|^{d+2s}}\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_\Omega u(x) \lim_{r\to 0^+} \int_{\mathbb{R}^d\setminus B_r(0)} \frac{\tilde{\varphi}(x) - \tilde{\varphi}(x+z)}{|z|^{d+2s}}\,\mathrm{d}z\,\mathrm{d}x \\
&\quad + \int_\Omega u(y) \lim_{r\to 0^+} \int_{\mathbb{R}^d\setminus B_r(0)} \frac{\tilde{\varphi}(y) - \tilde{\varphi}(y-z)}{|z|^{d+2s}}\,\mathrm{d}z\,\mathrm{d}y \\
&= \langle u, (-\Delta)^s\tilde{\varphi}\rangle = \langle(-\Delta)^s u, \varphi\rangle,
\end{aligned}$$

where we have used the change of variable $z = y - x$, (3.4.2) and (3.4.3). □

For every $\psi \in \mathrm{C}^\infty_{\mathrm{c}}(\mathbb{R}^d)$ we define the 0-fractional laplacian of $\psi$ as

$$(-\Delta)^0\psi(x) := \int_{B_1} \frac{2\psi(x) - \psi(x+z) - \psi(x-z)}{|z|^d}\,\mathrm{d}z - 2\int_{\mathbb{R}^d\setminus\overline{B}_1} \frac{\psi(x+z)}{|z|^d}\,\mathrm{d}z\,, \quad x \in \mathbb{R}^d. \tag{3.4.5}$$

We notice that $(-\Delta)^0\psi$ is well-defined for every $\psi \in \mathrm{C}^\infty_{\mathrm{c}}(\mathbb{R}^d)$ since

$$\int_{B_1} \frac{|2\psi(x) - \psi(x+z) - \psi(x-z)|}{|z|^d}\,\mathrm{d}z \le 2\int_{B_1} \frac{|\psi(x+z) - \psi(x)|}{|z|^d}\,\mathrm{d}z \le C[\psi]_{0,1}\,.$$

and

$$\int_{\mathbb{R}^d \setminus \overline{B}_1} \frac{|\psi(x+z)|}{|z|^d}\,\mathrm{d}z \le \|\psi\|_{\mathrm{L}^1}\,.$$

*Remark* 3.4.2. In [29] the notion of *logarithmic laplacian* $\mathrm{L}_\Delta$ has been introduced as follows

$$\mathrm{L}_\Delta \psi(x) := c_{d,1}(-\Delta)^0\psi(x) + c_{d,2}\psi(x)\,,$$

where $c_{d,1}$ and $c_{d,2}$ are specific constant depending only on the dimension $d$. Such a logarithmic laplacian would correspond to renormalizing the Gagliardo $s$-seminorm of $\psi$ by removing all but a finite amount of the blowing up quantity $\frac{\|\psi\|^2_{\mathrm{L}^2}}{s}$.

For every $u \in \mathcal{H}^0_0(\Omega)$ we define 0-fractional laplacian of $u$ by duality as

$$\langle(-\Delta)^0 u, \varphi\rangle := \langle u, (-\Delta)^0\tilde{\varphi}\rangle\,, \qquad \text{for all } \varphi \in \mathrm{C}^\infty_{\mathrm{c}}(\Omega)\,. \tag{3.4.6}$$

Clearly, the 0-fractional laplacian is the first variation of the functional $\hat{F}^0$, as shown in the following result.

**Proposition 3.4.3.** *For every $u \in \mathcal{H}^0_0(\Omega)$ and for every $\varphi \in \mathrm{C}^\infty_{\mathrm{c}}(\Omega)$ we have*

$$\lim_{t\to 0}\frac{G^0_1(u+t\varphi) - G^0_1(u)}{t} = \Big\langle u, \int_{B_1} \frac{2\tilde{\varphi}(x) - \tilde{\varphi}(x+z) - \tilde{\varphi}(x-z)}{|z|^d}\,\mathrm{d}z\Big\rangle \tag{3.4.7}$$

$$\lim_{t\to 0}\frac{J^0_1(u+t\varphi) - J^0_1(u)}{t} = \Big\langle u, -2\int_{\mathbb{R}^d\setminus\overline{B}_1} \frac{\tilde{\varphi}(x+z)}{|z|^d}\,\mathrm{d}z\Big\rangle\,, \tag{3.4.8}$$

*so that*

$$\lim_{t\to 0}\frac{\hat{F}^0(u+t\varphi) - \hat{F}^0(u)}{t} = \langle(-\Delta)^0 u, \varphi\rangle\,. \tag{3.4.9}$$

*Proof.* Fix $u \in \mathcal{H}^0_0(\Omega)$ and $\varphi \in \mathrm{C}^\infty_{\mathrm{c}}(\Omega)$. Then, using the change of variable $z = y - x$, we have

$$\begin{aligned}
&\lim_{t\to 0}\frac{G^0_1(u+t\varphi) - G^0_1(u)}{t} = \iint_{\mathcal{B}_1} \frac{(\tilde{u}(x)-\tilde{u}(y))(\tilde{\varphi}(x)-\tilde{\varphi}(y))}{|x-y|^d}\,\mathrm{d}y\,\mathrm{d}x\\
&= \int_\Omega u(x)\int_{B_1}\frac{\tilde{\varphi}(x)-\tilde{\varphi}(x+z)}{|z|^d}\,\mathrm{d}z\,\mathrm{d}x + \int_\Omega u(y)\int_{B_1}\frac{\tilde{\varphi}(y)-\tilde{\varphi}(y-z)}{|z|^d}\,\mathrm{d}z\,\mathrm{d}y\\
&= \Big\langle u, \int_{B_1}\frac{2\tilde{\varphi}(x)-\tilde{\varphi}(x+z)-\tilde{\varphi}(x-z)}{|z|^d}\,\mathrm{d}z\Big\rangle\,,
\end{aligned}$$

i.e., (3.4.7). Moreover, using again the change of variable $z = y - x$, we obtain

$$\begin{aligned}
&\lim_{t\to 0}\frac{J^0_1(u+t\varphi) - J^0_1(u)}{t} = -2\iint_{\mathbb{R}^{2d}\setminus\overline{\mathcal{B}}_1}\frac{\tilde{u}(x)\tilde{\varphi}(y)}{|x-y|^d}\,\mathrm{d}y\,\mathrm{d}x\\
&= \int_\Omega u(x)\Big(-2\int_{\mathbb{R}^d\setminus\overline{B}_1(x)}\frac{\tilde{\varphi}(y)}{|x-y|^d}\,\mathrm{d}y\Big)\,\mathrm{d}x\,,\\
&= \Big\langle u, -2\int_{\mathbb{R}^d\setminus\overline{B}_1}\frac{\tilde{\varphi}(x+z)}{|z|^d}\,\mathrm{d}z\Big\rangle\,,
\end{aligned}$$

namely, (3.4.8). Finally, (3.4.9) follows from (3.4.7) and (3.4.8), using (3.4.6). □

### 3.4.2 The main results

Here we state and prove the convergence of the parabolic flows corresponding to the rescaling of the $s$-Gagliardo seminorms. These follow by collecting the preparatory results of the previous sections, and Lemma 3.4.7 for the first order convergence as $s \to 0^+$.

We start with the convergences as $s \to 0^+$.

**Theorem 3.4.4.** *Let $\{s_n\}_{n\in\mathbb{N}} \subset (0,1)$ be such that $s_n \to 0^+$ as $n \to +\infty$. Let $u_0^0 \in \mathrm{L}^2(\Omega)$ and let $\{u_0^n\}_{n\in\mathbb{N}} \subset \mathrm{L}^2(\Omega)$ be such that $u_0^n \in \mathcal{H}_0^{s_n}(\Omega)$, $S := \sup_{n\in\mathbb{N}} s_n F^{s_n}(u_0^n) < +\infty$ and $u_0^n \to u_0^0$ in $\mathrm{L}^2(\Omega)$. Then, for every $n \in \mathbb{N}$ there exists a unique solution $u^n \in H^1([0,+\infty);\mathrm{L}^2(\Omega))$ to*

$$\begin{cases} u_t(t) = -s_n(-\Delta)^{s_n} u(t) & \text{for a.e. } t \in [0,+\infty) \\ u(0) = u_0^n\,, \end{cases} \tag{3.4.10}$$

*satisfying $(-\Delta)^{s_n} u^n(t) \in \mathrm{L}^2(\Omega)$ for every $t \geq 0$. Moreover, for every $T > 0$, $u^n \rightharpoonup u^0$ in $H^1([0,T];\mathrm{L}^2(\Omega))$ as $n \to +\infty$, where $u^0 \in H^1([0,T];\mathrm{L}^2(\Omega))$ is the unique solution to*

$$\begin{cases} u_t(t) = -d\omega_d u(t) & \text{for a.e. } t \in (0,T)\,, \\ u(0) = u_0^0\,. \end{cases} \tag{3.4.11}$$

*Furthermore, if*

$$\lim_{n\to+\infty} s_n F^{s_n}(u_0^n) = F^0(u_0^0)\,,$$

*then, $u^n \to u^0$ (strongly) in $H^1([0,T];\mathrm{L}^2(\Omega))$ for every $T > 0$, and*

$$\|u^n(t) - u^0(t)\|_{\mathrm{L}^2(\Omega)} \to 0 \quad \text{and} \quad s_n F^{s_n}(u^n(t)) \to F^0(u^0(t)) \qquad \text{for every } t \geq 0\,.$$

**Theorem 3.4.5.** *Let $\{s_n\}_{n\in\mathbb{N}} \subset (0,1)$ be such that $s_n \to 0^+$ as $n \to +\infty$. Let $u_0^0 \in \mathrm{L}^2(\Omega)$ and let $\{u_0^n\}_{n\in\mathbb{N}} \subset \mathrm{L}^2(\Omega)$ be such that $u_0^n \in \mathcal{H}_0^{s_n}(\Omega)$, $S := \sup_{n\in\mathbb{N}} \hat{F}^{s_n}(u_0^n) < +\infty$ and $u_0^n \to u_0^0$ in $\mathrm{L}^2(\Omega)$. Then, for every $n \in \mathbb{N}$ there exists a unique solution $u^n \in H^1([0,+\infty);\mathcal{H}_0^{s_n}(\Omega))$ to*

$$\begin{cases} u_t(t) = -\Big[(-\Delta)^{s_n} u(t) - \dfrac{d\omega_d}{s_n} u(t)\Big] & \text{for a.e. } t \in (0,T)\,, \\ u(0) = u_0^n\,, \end{cases} \tag{3.4.12}$$

*satisfying $(-\Delta)^{s_n} u(t) \in \mathrm{L}^2(\Omega)$ for every $t \geq 0$. Moreover, $u_0^0 \in \mathcal{H}_0^0(\Omega)$ and, for every $T > 0$, $u^n \to u^0$ in $H^1([0,T];\mathrm{L}^2(\Omega))$ as $n \to +\infty$, where $u^0 \in H^1([0,T];\mathcal{H}_0^0(\Omega))$ is the unique (distributional) solution to*

$$\begin{cases} u_t(t) = -(-\Delta)^0 u(t) & \text{for a.e. } t \in (0,T) \\ u(0) = u_0^0\,, \end{cases} \tag{3.4.13}$$

*satisfying $(-\Delta)^0 u^0(t) \in L^2(\Omega)$ for every $t \geq 0$. Furthermore, if*

$$\lim_{n\to+\infty} \hat{F}^{s_n}(u_0^n) = \hat{F}^0(u_0^0)\,,$$

*then, $u^n \to u^0$ (strongly) in $H^1([0,T];\mathrm{L}^2(\Omega))$ for every $T > 0$, and*

$$\|u^n(t) - u^0(t)\|_{\mathrm{L}^2(\Omega)} \to 0 \quad \text{and} \quad \hat{F}^{s_n}(u^n(t)) \to \hat{F}^0(u^0(t)) \qquad \text{for every } t \geq 0\,.$$

The result below shows the convergence toward the classical heat equation as $s \to 1^-$ of the rescaled in time $s$-fractional heat equations.

**Theorem 3.4.6.** *Let $\{s_n\}_{n\in\mathbb{N}} \subset (0,1)$ be such that $s_n \to 1^-$ as $n \to +\infty$. Let $u_0^\infty \in \mathrm{L}^2(\Omega)$ and let $\{u_0^n\}_{n\in\mathbb{N}} \subset \mathrm{L}^2(\Omega)$ be such that $u_0^n \in \mathcal{H}_0^{s_n}(\Omega)$, $S := \sup_{n\in\mathbb{N}}(1-s_n)F^{s_n}(u_0^n) < +\infty$ and $u_0^n \to u_0^\infty$ in $\mathrm{L}^2(\Omega)$. Then, for every $n \in \mathbb{N}$ there exists a unique solution $u^n \in H^1([0,+\infty);\mathrm{L}^2(\Omega))$ to*

$$\begin{cases} u_t(t) = -(1-s_n)(-\Delta)^{s_n}u(t) & \text{for a.e. } t \in [0,+\infty) \\ u(0) = u_0^n\,, \end{cases} \tag{3.4.14}$$

*satisfying $(-\Delta)^{s_n}u^n(t) \in \mathrm{L}^2(\Omega)$ for every $t \geq 0$. Moreover, $u_0^\infty \in H_0^1(\Omega)$, and, for every $T > 0$, $u^n \rightharpoonup u^\infty$ in $H^1([0,T];\mathrm{L}^2(\Omega))$ as $n \to +\infty$, where $u^\infty \in H^1([0,T];H_0^1(\Omega))$ is the unique (distributional) solution to*

$$\begin{cases} u_t(t) = \omega_d \Delta u(t) & \text{for a.e. } t \in [0,+\infty) \\ u(0) = u_0^\infty\,. \end{cases} \tag{3.4.15}$$

*Furthermore, if*

$$\lim_{n\to+\infty}(1-s_n)F^{s_n}(u_0^n) = F^1(u_0^\infty)\,, \tag{3.4.16}$$

*then, $u^n \to u^\infty$ (strongly) in $H^1([0,T];\mathrm{L}^2(\Omega))$ for every $T > 0$, and*

$$\|u^n(t) - u^\infty(t)\|_{\mathrm{L}^2(\Omega)} \to 0 \quad \text{and} \quad (1-s_n)F^{s_n}(u^n(t)) \to F^1(u^\infty(t)) \qquad \text{for every } t \geq 0\,.$$

We first prove Theorem 3.4.4.

*Proof of Theorem 3.4.4.* By the very definition of $F^s$ in (3.1.1), we have that for every $n \in \mathbb{N}$ $D(s_nF^{s_n}) = \mathcal{H}_0^{s_n}(\Omega) \neq \emptyset$ and that the functionals $s_nF^{s_n}$ are strongly lower semicontinuous, $\lambda$-positive and $\lambda$-convex for every $\lambda > 0$. Moreover, by combining Proposition 3.4.1 with Proposition 3.3.7 for $\mathcal{F} = s_nF^{s_n}$, $\mathscr{H} = \mathrm{L}^2(\Omega)$, and $\hat{\mathscr{H}} = \mathrm{C}_\mathrm{c}^\infty(\Omega)$, we have that for every $u \in \mathcal{H}_0^{s_n}(\Omega)$, either $\partial(s_nF^{s_n})(u) = \emptyset$ of $\partial(s_nF^{s_n})(u) = \{(-\Delta)^{s_n}u\}$ with $s_n(-\Delta)^{s_n}u \in \mathrm{L}^2(\Omega)$. Therefore, by Theorem 3.3.6, there exists a unique solution to the Cauchy problem (3.4.10), for every $n \in \mathbb{N}$. Furthermore, for every $u \in \mathrm{L}^2(\Omega)$ we have that

$$\lim_{t\to 0}\frac{F^0(u+t\varphi) - F^0(u)}{t} = d\omega_d\langle u,\varphi\rangle_{\mathrm{L}^2(\Omega)} \qquad \text{for every } \varphi \in \mathrm{L}^2(\Omega)\,, \tag{3.4.17}$$

whence we deduce that $\partial F^0(u) = \{d\omega_d\, u\}$. As a consequence, there exists a unique solution to the problem (3.4.11). Finally, the stability claims follow by applying Theorem 3.3.8 with $\mathcal{F}^n = s_nF^{s_n}$ and $\mathcal{F}^\infty = F^0$, once noticed that, in view of Theorem 3.1.2, assumption (a) is satisfied. □

In order to prove Theorem 3.4.5, we provide below a lemma showing uniform $\lambda$-convexity of the underlying functionals.

**Lemma 3.4.7.** *For every $\lambda > 2|\Omega|$, the functionals $\hat{F}^s$ are $\lambda$-positive and $\lambda$-convex for every $s \in [0,1)$.*

*Proof.* As for the $\lambda$-positivity it is enough to notice that, by the very definition of $\hat{F}^s$ in (3.1.10) and (3.1.11) and by (3.1.9), recalling that $G_1^s \geq 0$ for every $s \in [0,1)$, we have that

$$\hat{F}^s(u) + \frac{\lambda}{2}\|u\|^2_{\mathrm{L}^2(\Omega)} \geq G_1^s(u) + \left(\frac{\lambda}{2} - |\Omega|\right)\|u\|^2_{\mathrm{L}^2(\Omega)} \geq 0 .$$

Now we show that the functionals $\hat{F}^s$ are $\lambda$-convex for every $s \in [0,1)$. We preliminarily notice that the functionals $G_1^s$ are convex for every $s \in [0,1)$. Therefore, it is enough to show that the functionals $J_1^s$ are $\lambda$-convex. To this end, for every $u, v \in \mathcal{H}_0^0(\Omega)$ we define the function

$$f : \mathbb{R} \to \mathbb{R}, \qquad f(t) := J_1^s(u+tv) + \frac{\lambda}{2}\|u+tv\|^2_{\mathrm{L}^2(\Omega)}$$

and we claim that $\frac{\mathrm{d}^2}{\mathrm{d}t^2} f(t) \geq 0$ for every $t \in \mathbb{R}$. Indeed, since

$$J_1^s(u+tv) = J_1^s(u) - 2t \iint_{\mathbb{R}^{2d}\setminus\mathcal{B}_1} \frac{\tilde{u}(x)\tilde{v}(y)}{|x-y|^{d+2s}}\,\mathrm{d}x\,\mathrm{d}y + t^2 J_1^s(v) \tag{3.4.18}$$

and

$$\|u+tv\|^2_{\mathrm{L}^2(\Omega)} = \|u\|^2_{\mathrm{L}^2(\Omega)} + 2t\int_\Omega u(x)v(x)\,\mathrm{d}x + t^2\|v\|^2_{\mathrm{L}^2(\Omega)} , \tag{3.4.19}$$

by (3.1.9) we have

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} f(t) = 2J_1^s(v) + \lambda\|v\|^2_{\mathrm{L}^2(\Omega)} \geq (-2|\Omega| + \lambda)\|v\|^2_{\mathrm{L}^2(\Omega)} \geq 0 ,$$

which implies the $\lambda$-convexity of the functional $J_1^s$ and then the $\lambda$-convexity of $\hat{F}^s$. □

*Proof of Theorem 3.4.5.* Let $\lambda > 2|\Omega|$ be fixed. Then, by the very definition of $\hat{F}^s$ in (3.1.10) for every $n \in \mathbb{N}$ we have that $D(\hat{F}^{s_n}) = \mathcal{H}_0^{s_n}(\Omega) \neq \emptyset$ and, by Remark 3.1.5 and Lemma 3.4.7, that the functionals $\hat{F}^{s_n}$ are strongly lower semicontinuous, $\lambda$-positive and $\lambda$-convex. Moreover, by (3.4.4) and by (3.4.17), for every $u \in \mathcal{H}_0^{s_n}(\Omega)$ and for every $\varphi \in \mathrm{C}_\mathrm{c}^\infty(\Omega)$ we have

$$\lim_{t\to 0} \frac{\hat{F}^{s_n}(u+t\varphi) - \hat{F}^{s_n}(u)}{t} = \langle(-\Delta)^{s_n}u - \frac{d\omega_d}{s_n}u, \varphi\rangle_{\mathrm{L}^2(\Omega)} ,$$

which, by applying Proposition 3.3.7 with $\mathcal{F} = \hat{F}^{s_n}$, $\mathscr{H} = \mathrm{L}^2(\Omega)$ and $\hat{\mathscr{H}} = \mathrm{C}_\mathrm{c}^\infty(\Omega)$, implies that for every $u \in \mathcal{H}^{s_n}(\Omega)$ either $\partial\hat{F}^{s_n}(u) = \emptyset$ or $\partial\hat{F}^{s_n}(u) = \{(-\Delta)^{s_n}u - \frac{d\omega_d}{s_n}u\}$ with $(-\Delta)^{s_n}u - \frac{d\omega_d}{s_n}u \in \mathrm{L}^2(\Omega)$. Analogously, by Lemma 3.4.3 and by Proposition 3.3.7, we have that for every $u \in \mathcal{H}_0^0(\Omega)$ either $\partial\hat{F}^0(u) = \emptyset$ or $\partial\hat{F}^0(u) = \{(-\Delta)^0 u\}$ with $(-\Delta)^0 u \in \mathrm{L}^2(\Omega)$.

Therefore, by Theorem 3.3.6, the solutions to the problems (3.4.12) ($n \in \mathbb{N}$) and (3.4.13) are uniquely determined. Finally, the stability claim follows by applying Theorem 3.3.8 with $\mathcal{F}^n = \hat{F}^{s_n}$ and $\mathcal{F}^\infty = \hat{F}^0$, once noticed that, in view of Theorem 3.1.4, assumption (b) is satisfied. □

It lasts to prove Theorem 3.4.6. Also in this case, this follows from the general results already discussed.

*Proof of Theorem 3.4.6.* By the very definition of $F^s$ in (3.1.1), we have that for all $n \in \mathbb{N}$ the set $D((1-s_n)F^{s_n}) = \mathcal{H}_0^{s_n}(\Omega) \neq \emptyset$ and that the functional $(1-s_n)F^{s_n}$ is strongly lower semicontinuous, $\lambda$-positive and $\lambda$-convex for every $\lambda > 0$. Now, by combining Proposition 3.4.1 with Proposition 3.3.7 for $\mathcal{F} = (1-s_n)F^{s_n}$, $\mathscr{H} = \mathrm{L}^2(\Omega)$ and $\hat{\mathscr{H}} = \mathrm{C}_\mathrm{c}^\infty(\Omega)$, we have that for every $u \in \mathcal{H}_0^{s_n}(\Omega)$, either $(1-s_n)\partial F^{s_n}(u) = \emptyset$ or $(1-s_n)\partial F^{s_n}(u) = \{(1-s_n)(-\Delta)^{s_n}u\}$ with $(1-s_n)(-\Delta)^{s_n}u \in \mathrm{L}^2(\Omega)$. Therefore, by Theorem 3.3.6, for every $n \in \mathbb{N}$, there exists a unique solution to the Cauchy problem (3.4.14). Furthermore, for every $u \in H_0^1(\Omega)$ and for all $\varphi \in \mathrm{C}_\mathrm{c}^\infty(\Omega)$ we have that

$$\lim_{h \to 0} \frac{F^1(u+h\varphi) - F^1(u)}{h} = \omega_d \langle \nabla u, \nabla \varphi \rangle_{\mathrm{L}^2(\Omega)} =: \omega_d \langle (-\Delta)u, \varphi \rangle_{\mathrm{L}^2(\Omega)} \,;$$

therefore, by applying Proposition 3.3.7 with $\mathcal{F} = F^1$, $\mathscr{H} = \mathrm{L}^2(\Omega)$ and $\hat{\mathscr{H}} = \mathrm{C}_\mathrm{c}^\infty(\Omega)$ we have that either $\partial F^1(u) = \emptyset$ or $\partial F^1(u) = \{(-\Delta u)\}$ with $(-\Delta u) \in \mathrm{L}^2(\Omega)$. Finally, the stability claim follows by applying Theorem 3.3.8 with $\mathcal{F}^n = (1-s_n)F^{s_n}$ and $\mathcal{F}^\infty = F^1$, once noticed that, in view of Theorem 3.2.1, assumption (b) is satisfied. □

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