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Spectral Theory of Non-self-adjoint Dirac Operators and Other Dispersive Models

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Contents

Introduction

In the present thesis, we are going to collect results belonging to two lines of research: the first part of the work is devoted to the spectral theory for non-self-adjoint operators, whereas in the second part we consider nonlinear hyperbolic equations with time depending coefficients, and in particular their blow-up phenomena. The first argument is in some sense the lion's share of the thesis, being the main interest of research during my doctoral studies. Nevertheless, both of them have been deeply explored for decades and are still highly topical nowadays, being fascinating both for the mathematical and physical community.

The bulk of the thesis is constituted by five chapters, all almost completely self-contained, mirroring the five independent papers listed at the end of this Introduction. In the following two sections, we are going to present our problems and aims, outlining the results we proved.

Cages for eigenvalues

Since around the dawn of the millennium, there has been a flood of interest in the study of non-self-adjoint operators in Quantum Mechanics. This is due in part for their physical relevance, which relies, inter alia, on the new concept of representing quantomechanic observables by operators which are merely similar to self-adjoint ones. On the other side, the mathematical community is thrilled by the absence of tools such as the spectral theorem and the variational methods, which makes this topic challenging. The difficulty of the non-self-adjoint theory is nicely caught in the following quotation from [Dav07] by E. B. Davies:

> Studying non-self-adjoint operators is like being a vet rather than a doctor: one has to acquire a much wider range of knowledge, and to accept that one cannot expect to have as high a rate of success when confronted with particular cases.

As good sources for the non-self-adjoint operators theory and its developments, we may cite the monographs [GK69,Kat95,Tre08] or the more recent books [Dav02,BGSZ15], where physical applications may also be found.

In particular, a huge attention is paid to the spectral properties of non-self-adjoint operators and to the so-called Keller-type inequalities, id est bounds on the eigenvalues in terms of norms of the potential. Especially in the case of the Schrödinger operator, they can be referred to as well as Lieb-Thirring-type inequalities. Indeed, they constitute somewhat the counterpart of the celebrated inequalities for the self-adjoint Schrödinger operator $-\Delta + V$, exploited by E. H. Lieb and W. E. Thirring in the '70 of the last century to prove the stability of matter (an exciting argument, but here we just cite the monograph [LS10] for an academic treatment of the subject).

The first appearance of a Keller-type inequality for the *non-self-adjoint* Schrödinger operator $-\Delta + V$, where the potential V is a complex-valued function, is due to A. A. Abramov, A. Aslanyan and E. B. Davies in [AAD01], where they observed that the bound

$$|z|^{1/2} \leq \frac{1}{2} \|V\|_{L^1}$$

holds, in dimension $n = 1$, for any eigenvalue $z \in \sigma_p(-\Delta + V)$, and the constant is sharp.

In view of this result, A. Laptev and O. Safronov in [LS09] conjectured that the eigenvalues localization bound

$$|z|^{\gamma} \leq D_{\gamma,n} \|V\|_{L^{\gamma+n/2}}^{\gamma+n/2}$$

should be true for any $0 < \gamma \leq n/2$ and a positive constant $D_{\gamma,n}$. In the seminal work [Fra11], R. Frank proved the conjecture to be true for $0 < \gamma \leq 1/2$, and later in [FS17b], together with B. Simon, extended the range up to the one suggested by Laptev and Safronov under radial symmetry assumptions. The above relation holds also in the case $\gamma = 0$, in the sense that if $D_{0,n} \|V\|_{L^{n/2}}^{n/2} < 1$ for some positive constant $D_{0,n}$, then the point spectrum of $-\Delta + V$ is empty.

The Laptev-Safronov conjecture certainly can not be true for $\gamma > n/2$, as observed originally by Laptev and Safronov themselves (see also S. Bögli [Bög17] for the construction of bounded potentials in $L^{\gamma+n/2}$, $\gamma > n/2$, with infinitely many eigenvalues accumulating to the real non-negative semi-axis). The situation in the range $1/2 < \gamma \leq n/2$ remained unclear for more than a decade. An argument in [FS17b] suggested that, for these values of γ, the Laptev-Safronov conjecture should fail in general, but it was not until very recently that S. Bögli and J.-C. Cuenin completely disproved the conjecture for this range of γ in their new preprint [BC21].

The Lieb-Thirring-type bound in [Fra11] are obtained by Frank exploiting two main tools: the Birman-Schwinger principle and the Kenig-Ruiz-Sogge estimates in [KRS87] on the conjugate line, viz.

$$\left\|(-\Delta - z)^{-1}\right\|_{L^p \to L^{p'}} \leq C|z|^{-n/2+n/p-1}, \qquad \frac{2}{n+1} \leq \frac{1}{p} - \frac{1}{p'} \leq \frac{2}{n},$$

where $1/p + 1/p' = 1$ and C is some positive constant. In fact, the combination of the Birman-Schwinger principle with resolvent estimates for free operators is one of the way to approach the localization problem for eigenvalues: it has been widely employed in the later times (see e.g. [Fra11, CLT14, Enb16, FS17b, Cue17, FKV18b, FK19, CIKŠ20, CPV20] to cite just few recent papers) and it will be the approach we are going to follow in this work too, as we will see. Despite the robustness of the Birman-Schwinger principle, it is not the only tool one could use to obtain spectral enclosures for non-self-adjoint operators: another powerful technique is the method of multipliers, see e.g. [FKV18a, FKV18b, Cos17, CFK20, CK20].

Roughly speaking, the principle states that $z \in \mathbb{C}$ is an eigenvalue of an operator $H := H_0 + B^*A$ if and only if -1 is an eigenvalue of the Birman-Schwinger operator $K_z := A(H_0 - z)^{-1}B^*$. In typical quantum mechanical examples, H_0 is a differential operator representing the kinetic energy of the system, while B^*A is a factorization of a multiplication operator representing an electromagnetic interaction. In this way, the spectral problem for an unbounded differential operator is reduced to a bounded integral operator. In particular, the eigenvalues of the perturbed operator H are confined in the complex region defined by $1 \leq \|K_z\|$ and the point spectrum is empty if $\|K_z\| < 1$ uniformly with respect to z.

It is clear from the definition of the Birman-Schwinger operator that this approach reduces to establishing suitable resolvent estimates for the unperturbed operator H_0. Indeed, once we know how to bound $(H_0 - z)^{-1}$, it is usually an easy matter setting A and B in a suitable normed space, and then obtain an estimate for K_z. Of course, this naïf reasoning is well-known, and can be synthesized claiming that each resolvent estimate corresponds, via the Birman-Schwinger principle, to a localization estimate for the eigenvalues of the perturbed operator.

The aim of the first part of the thesis is to apply this strategy to the non-self-adjoint Dirac operator formally defined by

$$\mathcal{D}_{m,V} := \mathcal{D}_m + V = -ic\hbar \sum_{k=1}^{n} \alpha_k \partial_k + mc^2 \alpha_{n+1} + V$$

where $n \geq 1$ is the dimension, $m \geq 0$ is the mass, c is the speed of light, $\hbar$ is the reduced Planck constant and $\alpha_k \in \mathbb{C}^{N \times N}$, for $k \in \{1, \dots, n+1\}$ and $N := 2^{\lceil n/2 \rceil}$, are the Dirac matrices. The potential $V \colon \mathbb{R}^n \to \mathbb{C}^{N \times N}$ is a possibly non-Hermitian matrix-valued function. The Dirac operator plays a huge role in Quantum Physics, with widespread applications: just to cite the classic ones, it describes the relativistic quantum mechanics of spin-$1/2$ particles both compatibly with the theory of relativity and naturally taking in account the spin of the particle and its magnetic moment. Moreover, it successfully describes the hydrogen atom. An essential reference for the theory of the Dirac operator (in the self-adjoint setting) is the B. Thaller's monography [Tha92].

The spectral studies for $\mathcal{D}_{m,V}$ were started by J.-C. Cuenin, A. Laptev and C. Tretter in their celebrated work [CLT14], for the 1-dimensional case. There they proved that if $V \in \mathbb{C}^{2\times 2}$ is a potential such that

$$\|V\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} |V(x)| dx < 1,$$

where $|V(\cdot)|$ is the operator norm of $V(\cdot)$ in $\mathbb{C}^2$ with the Euclidean norm, then every non-embedded eigenvalue $z \in \mathbb{C} \setminus \{(-\infty, -m] \cup [m, +\infty)\}$ of $\mathcal{D}_{m,V}$ lies in the union

$$z \in \overline{B}_{R_0}(x_0^-) \cup \overline{B}_{R_0}(x_0^+)$$

of two disjoint closed disks in the complex plane, with centers and radius respectively

$$x_0^\pm = \pm m \sqrt{\frac{\|V\|_1^4 - 2\|V\|_1^2 + 2}{4(1 - \|V\|_1^2)} + \frac{1}{2}}, \quad R_0 = m \sqrt{\frac{\|V\|_1^4 - 2\|V\|_1^2 + 2}{4(1 - \|V\|_1^2)} - \frac{1}{2}}.$$

In particular, in the massless case the spectrum is $\sigma(\mathcal{D}_{0,V}) = \mathbb{R}$. Moreover, this inclusion is shown to be optimal. Again, the proof relies on the combination of the Birman-Schwinger principle with a resolvent estimate for the free Dirac operator, namely

$$\left\|(\mathcal{D}_m - z)^{-1}\right\|_{L^1(\mathbb{R})\to L^\infty(\mathbb{R})} \leq \sqrt{\frac{1}{2} + \frac{1}{4}\left|\frac{z+m}{z-m}\right| + \frac{1}{4}\left|\frac{z-m}{z+m}\right|}.$$

In some sense, this is the counterpart for the Dirac operator of the above-cited Abramov-Aslanyan-Davies inequality for the Schrödinger operator in 1-dimension.

One could ask if, in the same fashion of the Frank's argument in [Fra11], one can combine the Birman-Schwinger principle with $L^p - L^{p'}$ resolvent estimates for the free Dirac operator, to derive Keller-type inequalities for the perturbed Dirac operator. Unfortunately, these reasoning can not be straightforwardly applied, since such Kenig-Ruiz-Sogge-type estimates does not exists in the case of Dirac for dimension $n \geq 2$, as observed by Cuenin in [Cue14]. Indeed, due to the Stein-Thomas restriction theorem and standard estimates for Bessel potentials, the resolvent $(\mathcal{D}_m - z)^{-1} \colon L^p(\mathbb{R}^n) \to L^{p'}(\mathbb{R}^n)$ is bounded uniformly for $|z| > 1$ if and only if

$$\frac{2}{n+1} \leq \frac{1}{p} + \frac{1}{p'} \leq \frac{1}{n},$$

hence the only possible choice is $(n, p, p') = (1, 1, \infty)$. For the Schrödinger operator the situation is much better since the right-hand side of the above range is replaced by $2/n$, as per the Kenig-Ruiz-Sogge estimates.

For the high dimensional case $n \geq 2$, we may refer among others to the works [Dub14, CT16, Cue17, FK19] where the eigenvalues are localized in terms of L^p-norm of the potential, but the confinement region is unbounded around $\sigma(\mathcal{D}_m) = (-\infty, -m] \cup [m, +\infty)$, i.e. the spectrum of the free Dirac operator $\mathcal{D}_m$. Instead, we are mainly devoted to the research of a *compact* region in which to localize the point spectrum.

In Chapter 1, corresponding to the paper [S1], we achieve this objective, generalizing in higher dimensions the above result by Cuenin, Laptev and Tretter [CLT14]. Indeed, assuming V small enough respect to a suitable mixed Lebesgue norm, namely

$$\|V\|_Y := \max_{j\in\{1,\dots,n\}} \|V\|_{L^1_{x_j} L^\infty_{\widehat{x}_j}} = \max_{j\in\{1,\dots,n\}} \int_{\mathbb{R}} \|V(x_j, \cdot)\|_{L^\infty(\mathbb{R}^{n-1})}\, dx_j \leq C_0$$

for a positive constant C_0 independent of V, we prove in the massive case $m > 0$ that the eigenvalues of $\mathcal{D}_{m,V}$ are contained in the union of two closed disks in the complex plane with centers and radius depending on $\|V\|_Y$. Instead, in the massless case $m = 0$, the spectrum is the same of the one for the unperturbed operator, viz. $\sigma(\mathcal{D}_{0,\mathcal{V}}) = \mathbb{R}$, and there are no eigenvalues, under the same smallness assumption for the potential. This results are proved combining the Birman-Schwinger principle together with new Agmond-Hörmander-type estimates for the resolvent of the Schrödinger operator and its first derivatives.

In Chapter 2, whose results are proved in [S2], again we take advantage of the main engine of the Birman-Schwinger operator fueled this time with resolvent estimates already published in the literature, but which imply spectral results for the Dirac operator (and for the Klein-Gordon one) worthy of consideration. In particular, in dimension $n \geq 3$ we show again results similar to the previous ones, hence confinement of the eigenvalues in two disks in the massive case and their absence in the massless case, assuming now for the potential the smallness assumption

$$\||x|V\|_{\ell^1 L^\infty} := \sum_{j\in\mathbb{Z}} \||x|V\|_{L^\infty(2^{j-1}\leq|x|<2^j)} < C_1.$$

The constant C_1 can be explicitly showed as a number depending only on the dimension n and, even if far to be optimal, is still valuable in the applications. Moreover, in this chapter the results for the spectrum stability are proved not only in the massless case, but also in the massive one, assuming smallness pointwise assumptions on the weighted potential, namely $\left\||x|\rho^{-2}V\right\|_{L^\infty} < C_2$. The constant C_2 is made explicit in terms of the dimension n and the mass m, and ρ is a positive weight satisfying $\sum_{j\in\mathbb{Z}} \|\rho\|^2_{L^\infty(2^{j-1}\leq|x|<2^j)} < \infty$ and additionally, in the massive case, such that $|x|^{1/2}\rho \in L^\infty(\mathbb{R}^n)$ (prototypes of such kind of weights already appeared e.g. in [BRV97]).

Finally, in Chapter 3, which corresponds to the work [S3], we consider some families of potentials with a peculiar matricial structure satisfying some rigidity assumptions. Employing resolvent estimates for the Schrödinger operator well-established in the literature, we can obtain, among others, the counterpart of the above-mentioned results by Abramov, Aslanyan and Davies [AAD01] and by Frank [Fra11] for the Dirac operator, viz. we prove that, for some positive constant $D_{\gamma,n,m}$,

$$|z^2 - m^2|^\gamma \leq D_{\gamma,n,m} \|V\|^{\gamma+n/2}_{L^{\gamma+n/2}}$$

holds, where $\gamma = 1/2$ if $n = 1$, and $0 < \gamma \leq 1/2$ if $n \in \mathbb{N} \setminus \{2,4\}$ (the exclusion of dimensions $n = 2$ and $n = 4$ are due to the conditions required on the potential). The case $\gamma = 0$ is again included, in the sense that if $D_{0,n,m} \|V\|^{n/2}_{L^{n/2}} < 1$, then there is no eigenvalue. In the massless case, we obtain the spectrum stability of the perturbed Dirac operator for any of our special potentials. What is remarkable in these results (for $\gamma \neq 0$) is the absence of any restriction on the norm size of the potential, contrary to the known results regarding the Dirac operator; however, we dearly pay on the rigidity structure of the potential. Here we underline these results in order to appreciate the parallelism with the Schrödinger case, but many others are presented in this chapter, concerning both the eigenvalues enclosure in (un)bounded regions and the spectrum stability, depending on the rigidity assumptions for the potential and involving different kinds of norms. In one case, no rigidity assumptions at all are required, but a eigenvalues confinement in two complex closed disks is obtained supposing the $L^{n,1}_\rho L^\infty_\theta$-norm of V small enough, in the same fashion of the result in Chapter 1. As already said, the Birman-Schwinger machine is here powered by many well-known Schrödinger resolvent estimates, of which we will depict a complete picture.

A trigger to blow-up

In order to start presenting the topic of Part II of this thesis, we will borrow the words from the Introduction of the monograph [Str89] by W. Strauss:

> Any hyperbolic equation is a wave equation, but there are other wave equation as well, such as the Schrödinger and Korteweg-de Vries equations. The

> solutions of such equations tend to be oscillations which spread out spatially. A nonlinear term such as u^p will tend to magnify the size of u where u is large, and to be negligible when u is small. It can make a solution blow-up in a finite time, it can produce a solitary wave, or (if it involves derivatives of u) it can produce a shock wave.

The Cauchy problem associated to a general nonlinear wave equation with time-depending speed of propagation, damping and mass terms, viz.

$$u_{tt} - a(t)\Delta u + d(t)u_t + m(t)u = F(x, t, u, u_t, \nabla u)$$

with initial data $u(0, x) = u_0(x)$, $u_t(0, x) = u_1(x)$ in suitable initial spaces, have been widely studied during the last half a century, collecting a great interest and a enormous numbers of results. Despite this, a complete theory classifying the results of the above equation according to the properties of its coefficients is still not developed. However, for suitable choices of the coefficients and of the nonlinearity, many progresses have been achieved.

Generally speaking, when addressing the Cauchy problem above, the research focuses on the understanding of the structural properties of the solution (after all, the properties are what define what is a solution, see the nice Section 3.2 in [Tao06]). One is interested in exhibiting a representation formula, deriving $L^p - L^q$ decay estimates, getting an asymptotic descriptions of the solutions, and classifying their behavior according to the behavior of the coefficients. Some of the first questions one can ask are about the wellposedness or illposedness of the problem: there exist (in some sense) solutions of the equation? Are they global with respect to time? Or something dramatic occurs, and we face blow-up, with norms exploding in finite time?

Our investigation will be indeed focused on the blow-up phenomena. When considering a nonlinearity of the type e.g. $|u|^p$ or $|u_t|^p$, typically there exists a *critical exponent* p_{crit} such that, if $p > p_{\text{crit}}$, there exists a unique global-in-time solution to the problem, whereas if $1 < p \leq p_{\text{crit}}$, the solutions blow-up in finite time, that is there exists a time $T \geq 0$ such that beyond it no reasonable kind of solution exists anymore. In this case, one is interested in estimating this *lifespan* T.

In Chapter 4, we consider the above problem with constant speed of propagation $a(t) \equiv 1$, scale-invariant damping and mass terms, nonlinearity of the type $|u|^p$ and small initial data. We proceed recollecting, at the best of our knowledge, the many results achieved during the decades on the widely studied damped wave equation, with and without mass, reorganizing and unifying them, other than proving new results in the massive case (for the purely damped case, we find an improvement in the lifespan estimates in 1-dimension). The main tool we will use is a Kato-type lemma, whose mechanism is essentially based on an inductive argument. Our analysis wants to stress in particular the competition between the "wave-like" and "heat-like" behaviors of the solutions, not only respect to the critical power, but also respect to the lifespan estimates. The precise meaning of what we intend with this terms will be explained later in Subsection 4.1.1. Anyway, making a small digression and trying to leave a cliffhanger, we recall that some wave-like equations behaves indeed more like the heat equation. A classical example is the telegraph equation $u_{tt} - \Delta u + u_t = 0$, whose solution experiences the diffusion phenomenon like the corresponding heat equation $-\Delta + u_t = 0$, as $t \to +\infty$. The fact that this two equations are connected can be seen by a scaling argument: setting $u(t,x) = v(\lambda t, \sqrt{\lambda}x)$, $\lambda t = s$ and $\sqrt{\lambda}x = y$, with a positive parameter λ, we have that $\lambda v_{ss} - \Delta_y v + v_s = 0$. Hence letting $\lambda \to 0^+$, which corresponds to $t \to +\infty$, we get the wave equation $-\Delta_y v + v_s = 0$. In Chapter 4, whose results are collected in [S4], one of the main goals is to explore this "heat versus wave" antagonism in the blow-up context.

Least but not least, in Chapter 5, corresponding to the paper [S5], we consider the generalized Tricomi equation, or Gellerstedt equation (namely the speed of propagation is equal to $a(t) = t^{2m}$ for some positive constant m), with derivative nonlinearity $|u_t|^p$ and small initial data. We do not consider any damping or mass term this time. Very recently this equation catalyzed a lot of attention and many papers appeared about it in a short time, see Section 5.1 for the background. We will study the blow-up of this equation furnishing the *papabili* critical exponent and lifespan estimates. Of course, to confirm that they are indeed the right ones, further consideration should be done demonstrating existence results. An attempt in this direction is done here proving a local existence result by using Fourier estimates for the Taniguchi-Tozaki multipliers. As a consequence, we show the optimality of the lifespan estimates at least in 1-dimension. This time, the main strategy relies on the construction of a suitable test function and hence applying the test function method in order to reach our claimed results.

Articles of the thesis

[S1] P. D'Ancona, L. Fanelli, and N. M. Schiavone. Eigenvalue bounds for non-selfadjoint Dirac operators. *Math. Ann.*, 2021. DOI: 10.1007/s00208-021-02158-x.

[S2] P. D'Ancona, L. Fanelli, D. Krejčiřík, and N. M. Schiavone. Localization of eigenvalues for non-self-adjoint Dirac and Klein-Gordon operators. *Nonlinear Analysis*, 214:112565, 2021. DOI: 10.1016/j.na.2021.112565.

[S3] H. Mizutani, and N. M. Schiavone. Keller-type bounds for Dirac operators perturbed by rigid potentials. *arXiv preprint arXiv:2108.12854*, 2021.

[S4] N.-A. Lai, N. M. Schiavone, and H. Takamura. Heat-like and wave-like lifespan estimates for solutions of semilinear damped wave equations via a Kato's type lemma. *J. Differential Equations*, 269(12):11575–11620, 2020. DOI: 10.1016/j.jde.2020.08.020.

[S5] N.-A. Lai, and N. M. Schiavone. Blow-up and lifespan estimate for generalized Tricomi equations related to Glassey conjecture. *arXiv preprint arXiv:2007.16003v2*, 2020.

Part I

Spectral theory of non-self-adjoint Dirac operators

Science is spectral analysis.
Art is light synthesis.

Karl Kraus, *Pro domo et mundo*, 1912

Chapter 1

Eigenvalue bounds for non-self-adjoint Dirac operators

In this chapter we are goint to prove that the eigenvalues of the massive Dirac operator, perturbed by a possibly non-Hermitian potential V, are enclosed in the union of two disjoint disks of the complex plane, provided V is sufficiently small with respect to the mixed norms $L^1_{x_j} L^\infty_{\widehat{x}_j}$, for any $j \in \{1, \dots, n\}$. In the massless case instead, under the same smallness assumption on V, the spectrum is shown to be the same of that for the unperturbed operator, and the point spectrum is empty. At this aim we establish new Agmon-Hörmander-type resolvent estimates, which will be combined with the Birman-Schwinger principle.

The reference for the following results is [S1], joint work with Piero D'Ancona and Luca Fanelli.

1.1 The Dirac operator

Let us start turning the spotlights on the star of the show: the perturbation of the free Dirac operator $\mathcal{D}_m$ by an eventually non-Hermitian potential, namely

$$\mathcal{D}_{m,V} := \mathcal{D}_m + V.$$

We consider the operator $\mathcal{D}_{m,V}$ acting on the Hilbert space of spinors $\mathfrak{H} = L^2(\mathbb{R}^n; \mathbb{C}^N)$, where n is the dimension, $N := 2^{\lceil n/2 \rceil}$ and $\lceil \cdot \rceil$ is the ceiling function. The perturbed operator $\mathcal{D}_{m,V}$ is only formally defined as a sum of operators; we will be able to properly define it later, thanks to Lemma 1.4.

The free Dirac operator $\mathcal{D}_m$, with non-negative mass m, is defined as

$$\mathcal{D}_m := -ic\hbar\, \boldsymbol{\alpha} \cdot \nabla + mc^2 \alpha_{n+1} = -ic\hbar \sum_{k=1}^{n} \alpha_k \partial_k + mc^2 \alpha_{n+1}, \tag{1.1.1}$$

being c the speed of light, $\hbar$ the reduced Planck constant and $\alpha_k \in \mathbb{C}^{N \times N}$, for $k \in \{1, \dots, n+1\}$, the Dirac matrices. These are Hermitian matrices elements of the Clifford algebra (see e.g. [Obo98]), satisfying the anti-commutation relations

$$\alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_j^k I_N, \quad \text{for } j, k \in \{1, \dots, n+1\}, \tag{1.1.2}$$

where δ_j^k is the Kronecker symbol and I_N the $N \times N$ unit matrix. We will handle in greater details the Dirac matrices later in Section 3.5. For now, it is enough to know that, without loss of generality, we can take

$$\alpha_{n+1} = \begin{pmatrix} I_{N/2} & 0 \\ 0 & -I_{N/2} \end{pmatrix}.$$

Additionally, we can change the unit of measure in such a way that $c = \hbar = 1$. Finally, we recall also that free Dirac operator is self-adjoint with domain

$$\operatorname{dom}(\mathcal{D}_m) = \{\psi \in \mathfrak{H} \colon \nabla\psi \in \mathfrak{H}^n\}$$

and core $C_0^\infty(\mathbb{R}^n; \mathbb{C}^N)$.

The potential $V \colon \mathbb{R}^n \to \mathbb{C}^{N\times N}$ may be any complex matrix-valued function such that $V \in L^2_{\text{loc}}(\mathbb{R}^n; \mathbb{R})$. We will say that $V \in \mathcal{X}$ for a generic space $\mathcal{X}$ if $|V| \in \mathcal{X}$, where $|\cdot| \colon \mathbb{C}^{N\times N} \to \mathbb{R}$ is the operator norm. To make things concrete, here and in the rest of the thesis we will consider $|\cdot|$ as the norm induced by the Euclidean one, viz. $|A| = \sqrt{\rho(A^*A)}$, where $\rho(M)$ is the spectral radius of a matrix M. With the usual slight abuse of notation, the same symbol V denotes both the matrix and the corresponding multiplication operator on $\mathfrak{H}$, with initial domain $\operatorname{dom}(V) = C_0^\infty(\mathbb{R}^n; \mathbb{C}^N)$.

Before to move on presenting our results, let us collect a selection of the known ones. In the Introduction, we already cited the point spectrum enclosure in dimension $n = 1$ proved by Cuenin, Laptev and Tretter [CLT14]. As we said, in that work they show the non-embedded eigenvalues to be confined in two disjoint disks of the complex plane, assuming $\|V\|_{L^1(\mathbb{R})}$ smaller than 1. The study on the spectrum of $\mathcal{D}_{m,V}$ they initiated in the 1-dimensional case was followed by [Cue14, CS18, Enb18]. In the higher dimensional case instead, we may refer to the works [Dub14, CT16, Cue17, FK19, Sam16].

In [Cue17], Cuenin localized the eigenvalues of the perturbed Dirac operator in terms of the L^p-norm of the potential V, but in an unbounded region of the complex plane. Indeed, Theorem 6.1.b of [Cue17] states that, if $n \geq 2$ and $V \in L^p$, with $p \geq n$, then any non-embedded eigenvalue of $\mathcal{D}_{m,V}$ satisfies

$$\left|\frac{\Im z}{\Re z}\right|^{\frac{n-1}{p}} |\Im z|^{1-\frac{n}{p}} \leq C \|V\|_{L^p(\mathbb{R}^n)},$$

for some positive constant C independent of z and V. Similar unbounded enclosing regions were obtained in [CT16], where Cuenin and Tretter study arbitrary non-symmetric perturbations of self-adjoint operators. In particular, for the massless Dirac operator in $\mathbb{R}^2$, if $V \in L^p$ with $p > 2$, they obtain that

$$\sigma(\mathcal{D}_{m,V}) \subset \bigcap_{0<b<1} \left\{ z \in \mathbb{C} \colon |\Im z|^2 \leq \frac{(2\pi(p-2))^{-\frac{2}{p-2}} \|V\|_{L^p(\mathbb{R}^2)}^{\frac{2p}{p-2}} b^{-\frac{4}{p-2}} + b^2|\Re z|^2}{1-b^2} \right\}.$$

Considering instead the massive Dirac operator with Coulomb-like potential in $\mathbb{R}^3$, the authors in [CT16] obtain that, if $|V(x)|^2 \leq C_1^2 + C_2^2|x|^{-2}$ for almost all $x \in \mathbb{R}^3$, where $C_1, C_2 \geq 0$ are constants such that $C_1^2 + 4C_2^2 m^2 < m^2$, then

$$\sigma(\mathcal{D}_{m,V}) \subset \left\{ z \in \mathbb{C} \colon |\Re z| \geq m - \sqrt{C_1^2 + 4C_2^2 m^2},\ |\Im z|^2 \leq \frac{C_1^2 + 4C_2|\Re z|^2}{1-4C_2^2} \right\}.$$

A different result on the localization of eigenvalues in an unbounded region was proved by Fanelli and Krejčiřík in [FK19]: in 3D, if $V \in L^3(\mathbb{R}^3)$ and $z \in \sigma_p(\mathcal{D}_{m,V})$, then

$$\left(1+\frac{(\Re z)^2}{(\Re\sqrt{m^2-z^2})^2}\right)^{-1/2} < \left(\frac{\pi}{2}\right)^{1/3}\sqrt{1+e^{-1}+2e^{-2}}\,\|V\|_{L^3(\mathbb{R}^3)}\,. \tag{1.1.3}$$

The advantage of the last result lies in the explicit condition which is easy to check in the applications. However, also in this result the eigenvalues are localized in an unbounded region around $\sigma(\mathcal{D}_m) = (-\infty, -m] \cup [m, +\infty)$.

In the works [EGG19] by Erdoğan, Goldberg and Green, and [EG21] by Erdoğan and Green, the authors, studying the limiting absorption principle and dispersive bounds, prove that for a bounded, continuous potential V satisfying a mild decaying condition, there are no eigenvalues of the perturbed Dirac operator in a sector of the complex plane containing a portion of the real line sufficiently far from zero energy. However these results are qualitative, in the sense that their bounds does not explicitly depend on some norm of the potential, as in the inequalities object of our study.

Lastly, we mention the recent paper [CFK20] by Cossetti, Fanelli and Krejčiřík, where the authors obtain results on the absence of eigenvalues for the Schrödinger and Pauli operators with a constant magnetic field and non-Hermitian potentials, and for the purely magnetic Dirac operators. However, Dirac operators with electric perturbations can not be treated by the multiplicative techniques of [CFK20]. In fact, the square of a purely magnetic Dirac operator is a diagonal magnetic Laplacian, which allows one to use the multiplier method.

What moved our analysis is the desire of finding some sort of generalization of the result by Cuenin, Laptev and Tretter [CLT14] in higher dimensions. As we saw, in the literature similar results already raised, but often involving eigenvalues confinement in *unbounded* regions wrapping around the real continuous spectrum of the free Dirac operator. We are interested instead in finding compact regions where to cage our eigenvalues. However, one of the major difficulties, as explicit in the Introduction, is the absence of $L^p(\mathbb{R}^n) \to L^{p'}(\mathbb{R}^n)$ resolvent estimates for the free Dirac operator. In their place, we discover and use the Agmon-Hörmander-type estimates in Lemma 1.1, finding in this way our coveted bounds.

1.2 Main results

Before formalizing our results in Theorems 1.1 and 1.2 below, we introduce a few notations used throughout the chapter.

We use the symbols $\sigma(H)$, $\sigma_p(H)$, $\sigma_e(H)$ and $\rho(H)$ respectively for the spectrum, the point spectrum, the essential spectrum and the resolvent of an operator H. More explicitly, we define

$$\sigma_e(H) = \{z \in \mathbb{C} \colon\ H - z \text{ is not a Fredholm operator}\},$$

whereas the discrete spectrum is defined as

$$\sigma_d(H) = \{z \in \mathbb{C} \colon z \text{ is an isolated eigenvalue of } H \text{ of finite multiplicity}\}.$$

Recall that, for non-self-adjoint operators, the essential spectrum defined above is not the complement of the discrete spectrum, see e.g. [EE18]. For $z \in \rho(H)$, we denote with

$R_H(z) := (H - z)^{-1}$ the resolvent operator of H. We recall also that

$$\sigma(-\Delta) = \sigma_e(-\Delta) = [0, +\infty),$$
$$\sigma(\mathcal{D}_m) = \sigma_e(\mathcal{D}_m) = (-\infty, -m] \cup [m, +\infty).$$

For $j \in \{1, \dots, n\}$ and $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we write

$$\widehat{x}_j := (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n) \in \mathbb{R}^{n-1},$$
$$(\overline{x}, \widehat{x}_j) := (x_1, \dots, x_{j-1}, \overline{x}, x_{j+1}, \dots, x_n) \in \mathbb{R}^n.$$

The mixed Lebesgue spaces $L^p_{x_j} L^q_{\widehat{x}_j}(\mathbb{R}^n)$ are the spaces of measurable functions on $\mathbb{R}^n$ such that

$$\|f\|_{L^p_{x_j} L^q_{\widehat{x}_j}} := \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^{n-1}} |f(x_j, \widehat{x}_j)|^q d\widehat{x}_j \right)^{p/q} dx_j \right)^{1/p} < \infty.$$

Obvious modifications occur for $p = \infty$ or $q = \infty$ (see e.g. [BP61] for general properties of such spaces).

For any matrix-valued function $M \colon \mathbb{R}^n \to \mathbb{C}^{N \times N}$, we set

$$\|M\|_{L^p_{x_j} L^q_{\widehat{x}_j}} := \| |M| \|_{L^p_{x_j} L^q_{\widehat{x}_j}}$$

where $|\cdot| \colon \mathbb{C}^{N\times N} \to \mathbb{R}$ denotes the operator norm induced by the Euclidean one. Furthermore, we write

$$[f *_{x_j} g](x) := \int_{\mathbb{R}} f(y_j, \widehat{x}_j) g(x_j - y_j, \widehat{x}_j) dy_j,$$
$$[\mathscr{F}_{x_j} f](\xi_j, \widehat{x}_j) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix_j \xi_j} f(x_j, \widehat{x}_j) dx_j,$$
$$[\mathscr{F}^{-1}_{\xi_j} f](x_j, \widehat{x}_j) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix_j \xi_j} f(\xi_j, \widehat{x}_j) d\xi_j,$$

to denote the partial convolution respect to x_j, the partial Fourier transform with respect to x_j, and its inverse, respectively. The partial (inverse) Fourier transform with respect to $\widehat{x}_j$ and the complete (inverse) Fourier transform with respect to x are defined in a similar way. Finally, we shall need the function spaces

$$X \equiv X(\mathbb{R}^n) := \bigcap_{j=1}^n L^1_{x_j} L^2_{\widehat{x}_j}(\mathbb{R}^n), \qquad Y \equiv Y(\mathbb{R}^n) := \bigcap_{j=1}^n L^1_{x_j} L^\infty_{\widehat{x}_j}(\mathbb{R}^n),$$

with norms defined as follows

$$\|f\|_X := \max_{j \in \{1,\dots,n\}} \|f\|_{L^1_{x_j} L^2_{\widehat{x}_j}}, \qquad \|f\|_Y := \max_{j \in \{1,\dots,n\}} \|f\|_{L^1_{x_j} L^\infty_{\widehat{x}_j}}.$$

The dual space of X and the norm with which is endowed are given by

$$X^* \equiv X^*(\mathbb{R}^n) := \sum_{j=1}^n L^\infty_{x_j} L^2_{\widehat{x}_j}(\mathbb{R}^n), \quad \|f\|_{X^*} := \inf \left\{ \sum_{j=1}^n \|f_j\|_{L^\infty_{x_j} L^2_{\widehat{x}_j}} : f = \sum_{j=1}^n f_j \right\},$$

see e.g. [BL76].

We can finally state our results.

Theorem 1.1. *Let $m > 0$. There exists a constant $C_0 > 0$ such that if*

$$\|V\|_Y < C_0,$$

then all eigenvalues $z \in \sigma_p(\mathcal{D}_{m,V})$ of $\mathcal{D}_{m,V}$ are contained in the union

$$z \in \overline{B}_{R_0}(x_0^-) \cup \overline{B}_{R_0}(x_0^+)$$

of the two closed disks in $\mathbb{C}$ with centers x_0^-, x_0^+ and radius R_0 given by

$$x_0^\pm := \pm m \frac{\nu^2+1}{\nu^2-1}, \quad R_0 := m \frac{2\nu}{\nu^2-1}, \quad \nu \equiv \nu(V) := \left[\frac{(n+1)C_0}{\|V\|_Y} - n\right]^2 > 1.$$

Theorem 1.2. *Let $m = 0$. There exists a constant $C_0 > 0$ such that if*

$$\|V\|_Y < C_0,$$

then $\mathcal{D}_{0,V}$ has no eigenvalues. In this case, we have $\sigma(\mathcal{D}_{0,V}) = \mathbb{R}$.

Remark 1.1. As anticipated, the crucial tool in our proof is a sharp uniform resolvent estimate for the free Dirac operator. This approach is inspired by [Fra11], where the result by Kenig, Ruiz and Sogge [KRS87] was used for the same purpose. In our case, we prove in Section 1.3 the following estimates, of independent interest:

$$\begin{aligned} \|R_{-\Delta}(z)\|_{X\to X^*} &\le C|z|^{-1/2}, \\ \|\partial_k R_{-\Delta}(z)\|_{X\to X^*} &\le C, \end{aligned}$$

and

$$\|R_{\mathcal{D}_m}(z)\|_{X\to X^*} \le C\left[n + \left|\frac{z+m}{z-m}\right|^{\operatorname{sgn}(\Re z)/2}\right].$$

These can be regarded as precised resolvent estimates of Agmon-Hörmander-type. Note also that similar uniform estimates, but in non sharp norms, were proved earlier by D'Ancona and Fanelli in [DF07, DF08, EGG19]. In Section 1.4, we combine our uniform estimates with the Birman-Schwinger principle, enabling us in Section 1.5 to complete the proof of Theorems 1.1 and 1.2.

Remark 1.2. The space Y satisfies the embedding

$$Y \hookrightarrow L^{n,1}(\mathbb{R}^n) \hookrightarrow L^n(\mathbb{R}^n), \tag{1.2.1}$$

where $L^{p,q}(\mathbb{R}^n)$ denotes the Lorentz spaces. Moreover, we have

$$W^{1,1}(\mathbb{R}^n) \hookrightarrow \bigcap_{j=1}^n L^1_{\widehat{x}_j} L^\infty_{x_j}(\mathbb{R}^n) \hookrightarrow L^{n/(n-1),1}(\mathbb{R}^n),$$

where $W^{m,p}(\mathbb{R}^n)$ is the Sobolev space. In particular, in dimension $n = 2$ we obtain

$$W^{1,1}(\mathbb{R}^2) \hookrightarrow Y = L^1_{x_1} L^\infty_{x_2}(\mathbb{R}^2) \cap L^1_{x_2} L^\infty_{x_1}(\mathbb{R}^2) \hookrightarrow L^{2,1}(\mathbb{R}^2) \hookrightarrow L^2(\mathbb{R}^2).$$

We refer to Fournier [Fou87], Blei and Fournier [BF89] and Milman [Mil] for these inclusions.

Remark 1.3. According to the previous remark we have $Y(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3)$. Thus in the massive 3-dimensional case the assumption $\|V\|_Y < C_0$ implies both our result, Theorem 1.1, and the one by Fanelli and Krejčiřík [FK19], i.e. the eigenvalue bound (1.1.3). Although our result improves the latter one for large eigenvalues, bounding them in two compact regions, it may happen that, in a neighbourhood of $z = -m$ and $z = m$, the bound in (1.1.3) improves the one stated in Theorem 1.1. It is not hard to check that, supposing $\|V\|_Y$ sufficiently small, our disks are enclosed in the region found by Fanelli and Krejčiřík if

$$m\frac{\nu^2+1}{\nu^2-1} - \sqrt{\left(m\frac{2\nu}{\nu^2-1}\right)^2 - (\Im z)^2} \geq \sqrt{(1-c^2\,\|V\|_{L^3}^2)m^2 - \left(1-\frac{1}{c^2\,\|V\|_{L^3}^2}\right)(\Im z)^2}, \quad (1.2.2)$$

where

$$c = (\pi/2)^{1/3}\sqrt{1+e^{-1}+2e^{-2}}, \qquad \nu = \left[\frac{4C_0}{\|V\|_Y} - 3\right]^2.$$

This condition may not always be satisfied and depends on the norms of the potential V in the spaces $L^3(\mathbb{R}^3)$ and $Y(\mathbb{R}^3)$. If this happens, the result in Theorem 1.1 and the one in [FK19] should be jointly taken in consideration for the eigenvalues bound. This situation is illustrated in Figure 1.1.

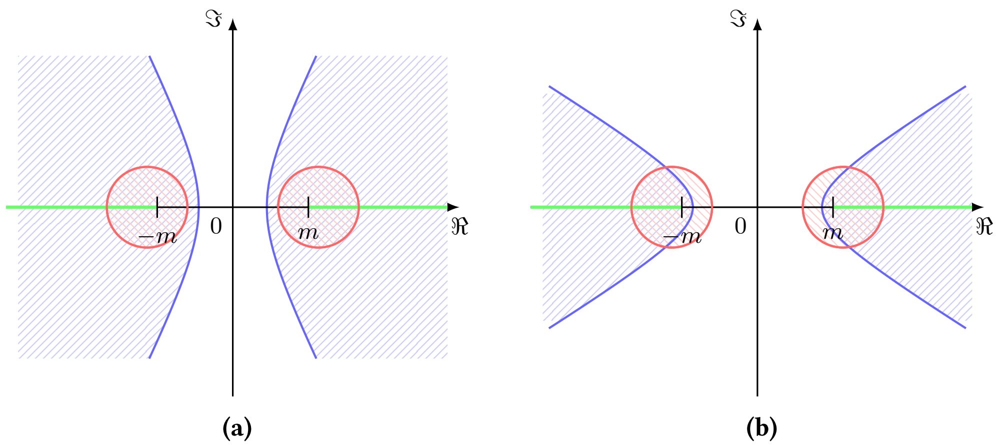


Figure 1.1: The disks in our Theorem 1.1, for $n = 3$, are represented in red; the Fanelli-Krejčiřík region from [FK19] defined by (1.1.3) is in blue; the spectrum of $\mathcal{D}_m$ is in green. When (1.2.2) holds we are in situation (a) and our result implies the result in [FK19]; if (1.2.2) does not hold the two results are not entirely comparable, as shown in picture (b).

1.3 The Agmon-Hörmander-type estimates

Let us fix the constants $r, R, \delta > 0$ such that

$$1 < r < R, \quad \sqrt{R^2-1} < \delta < 1,$$

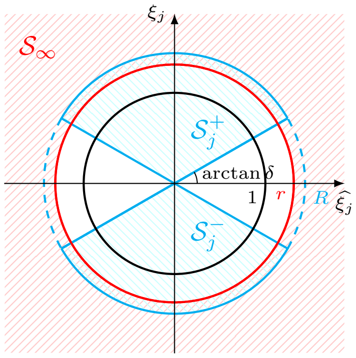


Figure 1.2: In the picture, the set $\mathcal{S}_\infty$ from the cover $\mathcal{S} = \{\mathcal{S}_j^+, \mathcal{S}_j^-, \mathcal{S}_\infty\}_{j\in\{1,\dots,n\}}$ is enlighten in red, while the sets $\mathcal{S}_j^+$ and $\mathcal{S}_j^-$ for fixed $j \in \{1, \dots, n\}$ are colored in blue.

and consider the open cover $\mathcal{S} = \{\mathcal{S}_j^+, \mathcal{S}_j^-, \mathcal{S}_\infty\}_{j\in\{1,\dots,n\}}$ of the space $\mathbb{R}^n$ defined by

$$\mathcal{S}_j^\pm = \{\xi \in \mathbb{R}^n \colon \pm \xi_j > \delta|\widehat{\xi}_j|, \ |\xi| < R\}, \quad \mathcal{S}_\infty = \{\xi \in \mathbb{R}^n \colon |\xi| > r\}.$$

See Figure 1.2 for a graphical representation. Let $\{\chi_j^+, \chi_j^-, \chi_\infty\}_{j\in\{1,\dots,n\}}$ be a smooth partition of unity subordinate to $\mathcal{S}$, that is to say a family of smooth positive functions such that

$$\operatorname{supp} \chi_j^\pm \subset \mathcal{S}_j^\pm, \quad \operatorname{supp} \chi_\infty \subset \mathcal{S}_\infty, \quad \chi_\infty + \sum_{j=1}^n [\chi_j^+ + \chi_j^-] \equiv 1.$$

From these, define the smooth partition of unity $\chi := \{\chi_j\}_{j\in\{1,\dots,n\}}$, with

$$\chi_j := \chi_j^+ + \chi_j^- + \frac{1}{n}\chi_\infty, \tag{1.3.1}$$

and correspondingly, for $j \in \{1, \dots, n\}$, the Fourier multipliers

$$\chi_j(|z|^{-1/2}D)f = \mathscr{F}_\xi^{-1}[\chi_j(|z|^{-1/2}\xi)\, \mathscr{F}_x f].$$

Note in particular that

$$\sum_{j=1}^n \chi_j(|z|^{-1/2}D)f = f. \tag{1.3.2}$$

Therefore, the following estimates hold true.

Lemma 1.1. *For every $z \in \rho(-\Delta) = \mathbb{C} \setminus [0, +\infty)$, $f \in L^1_{x_j} L^2_{\widehat{x}_j}$ and $j, k \in \{1, \dots, n\}$, we have that*

$$\left\| \chi_j \left(|z|^{-1/2} D \right) R_{-\Delta}(z) f \right\|_{L^\infty_{x_j} L^2_{\widehat{x}_j}} \leq C\, |z|^{-1/2}\, \|f\|_{L^1_{x_j} L^2_{\widehat{x}_j}},$$

$$\left\| \chi_j \left(|z|^{-1/2} D \right) \partial_k R_{-\Delta}(z) f \right\|_{L^\infty_{x_j} L^2_{\widehat{x}_j}} \leq C\, \|f\|_{L^1_{x_j} L^2_{\widehat{x}_j}},$$

where $\{\chi_j\}_{j\in\{1,\dots,n\}}$ are defined in (1.3.1) and $C > 0$ does not depend on z. In particular, it follows that

$$\|R_{-\Delta}(z)\|_{X \to X^*} \leq C|z|^{-1/2}, \qquad \|\partial_k R_{-\Delta}(z)\|_{X \to X^*} \leq C.$$

Lemma 1.2. *For every $z \in \rho(\mathcal{D}_m) = \{(-\infty, -m] \cup [m, +\infty)\}$, $f \in L^1_{x_j} L^2_{\widehat{x}_j}$ and $j \in \{1, \ldots, n\}$ we have that*

$$\left\| \chi_j \left(|z^2 - m^2|^{-1/2} D \right) R_{\mathcal{D}_m}(z) f \right\|_{L^\infty_{x_j} L^2_{\widehat{x}_j}} \leq C \left[n + \left| \frac{z+m}{z-m} \right|^{\operatorname{sgn}(\Re z)/2} \right] \|f\|_{L^1_{x_j} L^2_{\widehat{x}_j}},$$

where $\{\chi_j\}_{j \in \{1,\ldots,n\}}$ are defined in (1.3.1) and $C > 0$ is the same as in Lemma 1.1. In particular, it follows that

$$\|R_{\mathcal{D}_m}(z)\|_{X \to X^*} \leq C \left[n + \left| \frac{z+m}{z-m} \right|^{\operatorname{sgn}(\Re z)/2} \right].$$

Remark 1.4. Before we proceed further, we give a heuristic explanation for the choice of the localization in the frequency domain via the Fourier multipliers $\chi_j(|z|^{-1/2} D)$ for $j \in \{1, \ldots, n\}$. Since the symbol $(|\xi|^2 - z)^{-1}$ of the resolvent $R_{-\Delta}(z)$ blows-up as $z \to \zeta$, for every fixed $\zeta \geq 0$, our trick is to use the norms $L^\infty_{x_j} L^2_{\widehat{x}_j}$ for $j \in \{1, \ldots, n\}$, which allows us to restrict the problem from the spherical surface $\{\xi \in \mathbb{R}^n \colon |\xi| = |z|^{-1/2}\}$ to the "equators" given by $\{\xi \in \mathbb{R}^n \colon \xi_j = 0, |\widehat{\xi}_j| = |z|^{-1/2}\}$. We then avoid these regions thanks to the smooth functions χ_j.

Proof of Lemma 1.1. The last two estimates follow trivially from the first two estimates, (1.3.2) and the definitions of the norms on X and X^*.

For simplicity, from now on $C > 0$ will stand for a generic positive constant independent of z and which may change from line to line. Clearly, by scaling, it is sufficient to consider $z \in \mathbb{C}$ such that $|z| = 1$, $z \neq 1$. Thus we boil down to show that

$$\|\chi_j(D) \partial_k^s R_{-\Delta}(z) f\|_{L^\infty_{x_j} L^2_{\widehat{x}_j}} \leq C \|f\|_{L^1_{x_j} L^2_{\widehat{x}_j}},$$

where $|z| = 1$, $s \in \{0, 1\}$, $\partial_k^0 = 1$, $\partial_k^1 = \partial_k$ and $j, k \in \{1, \ldots, n\}$. This is equivalent to

$$\left\| \mathscr{F}_\xi^{-1} \left(\frac{\xi_k^s \, \chi_j(\xi)}{|\xi|^2 - \lambda - i\varepsilon} \mathscr{F}_x f \right) \right\|_{L^\infty_{x_j} L^2_{\widehat{x}_j}} \leq C \|f\|_{L^1_{x_j} L^2_{\widehat{x}_j}}, \tag{1.3.3}$$

where we have written $z = \lambda + i\varepsilon$, with $\lambda^2 + \varepsilon^2 = 1$ and $z \neq 1$. We proceed by splitting χ_j in the functions which appear in its definition (1.3.1), localizing ourselves in the regions of the frequency domain near the unit sphere, i.e. $\mathcal{S}_j^\pm$, and far from it, i.e. $\mathcal{S}_\infty$.

Estimate on $\mathcal{S}_j^\pm$. We want to prove

$$\left\| \mathscr{F}_\xi^{-1} \left(\frac{\xi_k^s \chi_j^\pm(\xi)}{|\xi|^2 - \lambda - i\varepsilon} \mathscr{F}_x f \right) \right\|_{L^\infty_{x_j} L^2_{\widehat{x}_j}} \leq C \|f\|_{L^2_{\widehat{x}_j} L^1_{x_j}}. \tag{1.3.4}$$

Let us define the family of operators

$$T_j^\pm \colon L^p_{x_j} L^2_{\widehat{x}_j} \to L^p_{x_j} L^2_{\widehat{x}_j}, \quad f \mapsto T_j^\pm f := \mathscr{F}_\xi^{-1} \left(\hat{f} \circ \Phi \right),$$

where

$$\Phi(\xi) := (\xi_j + \varphi(\widehat{\xi}_j), \widehat{\xi}_j), \quad \varphi(\widehat{\xi}_j) := \pm\sqrt{|1 - |\widehat{\xi}_j|^2|}.$$

Roughly speaking, the operator $T_j^{\pm}$ flattens the upper half unit sphere in the frequency domain $\{\xi \in \mathbb{R}^n : |\xi| = 1, \pm\xi_j > 0\}$. Writing more explicitly these operators, we have

$$\begin{aligned}
T_j^{\pm} f(x) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{f}(\xi_j + \varphi(\widehat{\xi}_j), \widehat{\xi}_j) d\xi \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \int_{\mathbb{R}^n} f(y) e^{-iy\cdot(\xi_j+\varphi(\widehat{\xi}_j),\widehat{\xi}_j)} dy d\xi \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{n-1}} e^{i\widehat{x}_j\cdot\widehat{\xi}_j} \int_{\mathbb{R}^{n-1}} e^{-i\widehat{y}_j\cdot\widehat{\xi}_j} \int_{\mathbb{R}} \int_{\mathbb{R}} f(y) e^{i(x_j-y_j)\xi_j - iy_j\varphi(\widehat{\xi}_j)} dy_j d\xi_j d\widehat{y}_j d\widehat{\xi}_j \\
&= \frac{1}{2\pi} \mathscr{F}_{\widehat{\xi}_j}^{-1} \mathscr{F}_{\widehat{y}_j} \left(e^{-ix_j\varphi(\widehat{\xi}_j)} \int_{\mathbb{R}} \int_{\mathbb{R}} f(y) e^{i(x_j-y_j)\xi_j} dy_j d\xi_j \right) \\
&= \mathscr{F}_{\widehat{\xi}_j}^{-1} \mathscr{F}_{\widehat{y}_j} \left(e^{-ix_j\varphi(\widehat{\xi}_j)} f(x_j, \widehat{y}_j) \right)
\end{aligned}$$

where we used the substitution $\xi_j \mapsto \xi_j - \varphi(\widehat{\xi}_j)$ in the fourth step. Applying the Plancherel Theorem twice, we obtain that $T_j^{\pm}$ are isometries on $L^p_{x_j} L^2_{\widehat{x}_j}$, viz. for $p \in [1, +\infty]$ we have

$$\left\| T_j^{\pm} f \right\|_{L^p_{x_j} L^2_{\widehat{x}_j}} = \|f\|_{L^p_{x_j} L^2_{\widehat{x}_j}} . \tag{1.3.5}$$

Then we can write

$$\begin{aligned}
\left\| \mathscr{F}_\xi^{-1} \left(\frac{\xi_k^s \chi_j^{\pm}(\xi)}{|\xi|^2 - \lambda - i\varepsilon} \mathscr{F}_x f \right) \right\|_{L^\infty_{x_j} L^2_{\widehat{x}_j}} &= \left\| T_j^{\pm} \mathscr{F}_\xi^{-1} \left(\frac{\xi_k^s \chi_j^{\pm}(\xi)}{|\xi|^2 - \lambda - i\varepsilon} \mathscr{F}_x f \right) \right\|_{L^\infty_{x_j} L^2_{\widehat{x}_j}} \\
&= \left\| \mathscr{F}_\xi^{-1} \left(\frac{(\xi_k^s \chi_j^{\pm}) \circ \Phi}{|\Phi|^2 - \lambda - i\varepsilon} \widehat{T_j^{\pm} f} \right) \right\|_{L^\infty_{x_j} L^2_{\widehat{x}_j}} \\
&= \frac{1}{\sqrt{2\pi}} \left\| a_{\lambda,\varepsilon}(D)\psi *_{x_j} \mathscr{F}_{\xi_j}^{-1} \left(\frac{\widehat{T_j^{\pm} f}}{\xi_j - i|\varepsilon|} \right) \right\|_{L^\infty_{x_j} L^2_{\widehat{\xi}_j}} \\
&\leq \frac{1}{\sqrt{2\pi}} \|a_{\lambda,\varepsilon}(D)\psi\|_{L^1_{x_j} L^\infty_{\widehat{\xi}_j}} \left\| \mathscr{F}_\xi^{-1} \left(\frac{\widehat{T_j^{\pm} f}}{\xi_j - i|\varepsilon|} \right) \right\|_{L^\infty_{x_j} L^2_{\widehat{\xi}_j}}
\end{aligned}$$

where the last inequality follows from Young's inequality and

$$\begin{gathered}
a_{\lambda,\varepsilon}(D)\psi = \mathscr{F}_{\xi_j}^{-1} \left(a_{\lambda,\varepsilon} \mathscr{F}_{x_j}(\psi) \right), \\
a_{\lambda,\varepsilon}(\xi) := \frac{(\xi_j - i|\varepsilon|) \left(\xi_k \pm \delta_{k,j} \sqrt{1 - |\widehat{\xi}_j|^2} \right)^s}{\xi_j \left(\xi_j \pm 2\sqrt{1 - |\widehat{\xi}_j|^2} \right) + 1 - \lambda - i\varepsilon} \sqrt{(\chi_j^{\pm} \circ \Phi)(\xi)}, \\
\psi(x_j, \widehat{\xi}_j) = \mathscr{F}_{\xi_j}^{-1} \left(\sqrt{(\chi_j^{\pm} \circ \Phi)(\xi)} \right).
\end{gathered}$$

Note that we dropped the absolute value appearing in φ, i.e. $\sqrt{|1 - |\widehat{\xi}_j|^2|} = \sqrt{1 - |\widehat{\xi}_j|^2}$, because $\operatorname{supp}\{\chi_j^{\pm} \circ \Phi\} \subset \{\xi \in \mathbb{R}^n : |\widehat{\xi}_j| \leq 1\}$, thanks to the definition of $\mathcal{S}_j^{\pm}$ and the assumption $\delta \geq \sqrt{R^2-1}$. Now, despite the truly cumbersome definition of $a_{\lambda,\varepsilon}$, it is simple to see that $a_{\lambda,\varepsilon}(D)\psi \in \mathscr{S}$, where $\mathscr{S}$ is the space of the Schwartz functions, since $a_{\lambda,\varepsilon}(D)\psi$ is the inverse Fourier transform of a smooth compactly supported function. Moreover, we can consider $a_{\lambda,\varepsilon}(D)\psi$ as a pseudodifferential operator with symbol $a_{\lambda,\varepsilon}$ applied to the Schwartz function ψ; letting $\lambda + i\varepsilon \to 1$ we have the pointwise convergence

$$\lim_{\lambda+i\varepsilon\to 1} a_{\lambda,\varepsilon}(\xi) = \frac{\left(\xi_k \pm \delta_{k,j}\sqrt{1-|\widehat{\xi}_j|^2}\right)^s}{\xi_j \pm 2\sqrt{1-|\widehat{\xi}_j|^2}} \sqrt{\chi_j^{\pm}\left(\xi_j \pm \sqrt{1-|\widehat{\xi}_j|^2}, \widehat{\xi}_j\right)} =: a(\xi) \in \mathscr{S}$$

and hence $a_{\lambda,\varepsilon}(D)\psi \to a(D)\psi$ in $\mathscr{S}$, which implies

$$\lim_{\lambda+i\varepsilon\to 1} \|a_{\lambda,\varepsilon}(D)\psi\|_{L^1_{x_j}L^\infty_{\widehat{\xi}_j}} = \|a(D)\psi\|_{L^1_{x_j}L^\infty_{\widehat{\xi}_j}} < +\infty.$$

Thus, $\|a_{\lambda,\varepsilon}(D)\psi\|_{L^1_{x_j}L^\infty_{\widehat{\xi}_j}}$ is uniformly bounded respect to $z \in \mathbb{C}$ with $|z| = 1$, and we proved

$$\left\| \mathscr{F}_\xi^{-1}\left(\frac{\xi_k^s \chi_j^{\pm}(\xi)}{|\xi|^2-\lambda-i\varepsilon}\mathscr{F}_x f\right)\right\|_{L^\infty_{x_j}L^2_{\widehat{x}_j}} \leq C \left\| \mathscr{F}_\xi^{-1}\left(\frac{\widehat{T_j^{\pm} f}}{\xi_j - i|\varepsilon|}\right)\right\|_{L^\infty_{x_j}L^2_{\widehat{\xi}_j}}. \tag{1.3.6}$$

By Plancherel's Theorem, Young's inequality, and the equality (1.3.5), we get

$$\begin{aligned}
\sqrt{2\pi}\left\| \mathscr{F}_{\xi_j}^{-1}\left(\frac{\widehat{T_j^{\pm} f}}{\xi_j - i|\varepsilon|}\right)\right\|_{L^\infty_{x_j}L^2_{\widehat{\xi}_j}} &= \left\| \mathscr{F}_{\xi_j}^{-1}\left(\frac{1}{\xi_j - i|\varepsilon|}\right) *_{x_j} \mathscr{F}_{\widehat{x}_j}(T_j^{\pm} f)\right\|_{L^\infty_{x_j}L^2_{\widehat{\xi}_j}} \\
&= \left\| ie^{-|\varepsilon|x_j}\Theta *_{x_j} \mathscr{F}_{\widehat{x}_j}(T_j^{\pm} f)\right\|_{L^\infty_{x_j}L^2_{\widehat{\xi}_j}} \\
&\leq \left\| e^{-|\varepsilon|x_j}\Theta *_{x_j} \left\|T_j^{\pm} f\right\|_{L^2_{\widehat{x}_j}}\right\|_{L^\infty_{x_j}} \\
&\leq \left\| e^{-|\varepsilon|x_j}\Theta\right\|_{L^\infty_{x_j}} \|f\|_{L^1_{x_j}L^2_{\widehat{x}_j}} \\
&= \|f\|_{L^1_{x_j}L^2_{\widehat{x}_j}},
\end{aligned}$$

where $\Theta \equiv \Theta(x_j)$ is the Heaviside function. Inserting this inequality in (1.3.6), we finally reach (1.3.4).

Estimate on $\mathcal{S}_\infty$. We shall now prove that

$$\left\| \mathscr{F}_\xi^{-1}\left(\frac{\xi_k^s \chi_\infty(\xi)}{|\xi|^2-\lambda-i\varepsilon}\mathscr{F}_x f\right)\right\|_{L^\infty_{x_j}L^2_{\widehat{x}_j}} \leq C\|f\|_{L^2_{\widehat{x}_j}L^1_{x_j}}. \tag{1.3.7}$$

We consider three cases, depending on whether we are localized in the regions defined by

$$\begin{aligned}
\mathcal{C}^1_{R,j} &:= \{\xi \in \mathbb{R}^n \colon |\widehat{\xi}_j| > R\}, \\
\mathcal{C}^2_{R,j} &:= \{\xi \in \mathbb{R}^n \colon |\widehat{\xi}_j| \leq R, |\xi_j| \leq 2R\}, \\
\mathcal{C}^3_{R,j} &:= \{\xi \in \mathbb{R}^n \colon |\widehat{\xi}_j| \leq R, |\xi_j| > 2R\}.
\end{aligned}$$

We set

$$\chi^1_\infty(\xi) := \begin{cases} 1 & \text{if } |\widehat{\xi}_j| > R, \\ 0 & \text{otherwise,} \end{cases}$$
$$\chi^2_\infty(\xi) := \begin{cases} \chi_\infty(\xi) & \text{if } |\widehat{\xi}_j| \le R \text{ and } |\xi_j| \le 2R, \\ 0 & \text{otherwise,} \end{cases}$$
$$\chi^3_\infty(\xi) := \begin{cases} 1 & \text{if } |\widehat{\xi}_j| \le R \text{ and } |\xi_j| > 2R, \\ 0 & \text{otherwise,} \end{cases}$$

and observe that $\chi_\infty = \chi^1_\infty + \chi^2_\infty + \chi^3_\infty$, since $\chi_\infty \equiv 1$ for $|\xi| > R$, from the assumptions on the cover $\mathcal{S}$ and the partition χ.

By Plancherel's Theorem and Hölder's, Young's and Minkowski's integral inequalities, for $h \in \{1, 2, 3\}$ we infer

$$\left\| \mathscr{F}^{-1}_\xi \left(\frac{\xi^s_k \chi^h_\infty(\xi)}{|\xi|^2 - \lambda - i\varepsilon} \mathscr{F}_x f \right) \right\|_{L^\infty_{x_j} L^2_{\widehat{x}_j}} \le C_h \, \|f\|_{L^1_{x_j} L^2_{\widehat{x}_j}}$$

with

$$C_h := \frac{1}{\sqrt{2\pi}} \left\| \mathscr{F}^{-1}_{\xi_j} \left(\frac{\xi^s_k \, \chi^h_\infty(\xi)}{\xi^2_j + \sigma^2} \right) \right\|_{L^\infty_{x_j} L^\infty_{\widehat{\xi}_j}}, \qquad \sigma := \sqrt{|\widehat{\xi}_j|^2 - \lambda - i\varepsilon}. \tag{1.3.8}$$

Here and below, we always consider the principal branch of the complex square root function. Clearly, if we prove that C_h, for $h \in \{1, 2, 3\}$, are bounded uniformly with respect to λ and ε, we recover (1.3.7).

Estimate on $\mathcal{C}^1_{R,j}$. Observing that $\chi^1_\infty(\xi) \equiv \chi^1_\infty(\widehat{\xi}_j)$ and noting that

$$\Re\sigma = \sqrt{\frac{|\sigma|^2 + |\widehat{\xi}_j|^2 - \lambda}{2}} > 0,$$

we can explicitly compute the Fourier transforms:

- if $k \ne j$, then
$$\begin{aligned} C_1 = \left\| \chi^1_\infty(\widehat{\xi}_j) \xi^s_k \frac{e^{-\sigma|x_j|}}{2\sigma} \right\|_{L^\infty_{x_j} L^\infty_{\widehat{\xi}_j}} &\le \left\| \chi^1_\infty(\widehat{\xi}_j) |\widehat{\xi}_j|^s \frac{e^{-\Re\sigma|x_j|}}{2|\sigma|} \right\|_{L^\infty_{x_j} L^\infty_{\widehat{\xi}_j}} \\ &\le \sup_{|\widehat{\xi}_j| > R} \frac{|\widehat{\xi}_j|^s}{2(|\widehat{\xi}_j|^4 - 2\lambda|\widehat{\xi}_j|^2 + 1)^{1/4}} \\ &\le \begin{cases} \frac{R^s}{2\sqrt{R^2 - 1}} & \text{if } \lambda > 0, \\ 1/2 & \text{if } \lambda \le 0; \end{cases} \end{aligned}$$

- if $s = 1$, $k = j$, then
$$C_1 = \left\| \chi^1_\infty(\widehat{\xi}_j) \frac{i}{2} \operatorname{sgn}(x_j) e^{-\sigma|x_j|} \right\|_{L^\infty_{x_j} L^\infty_{\widehat{\xi}_j}} \le \frac{1}{2}.$$

Estimate on $\mathcal{C}^2_{R,j}$. By the definition of the inverse Fourier transform in (1.3.8) and from the fact that $\chi^2_\infty(\xi) = 0$ when $|\xi| < r$, we see that

$$C_2 \leq \frac{1}{2\pi} \left\| \int_{-\infty}^{+\infty} \frac{|e^{ix_j\xi_j}|\,|\xi_k^s|\,\chi^2_\infty(\xi)}{||\xi|^2 - \lambda|} d\xi_j \right\|_{L^\infty_{x_j} L^\infty_{\widehat{\xi}_j}} \leq \frac{(2R)^s}{2\pi} \left\| \frac{\chi^2_\infty(\xi)}{|\xi|^2 - 1} \right\|_{L^\infty_{\widehat{\xi}_j} L^1_{\xi_j}}$$

which is finite since χ^2_∞ is compactly supported due to its definition.

Estimate on $\mathcal{C}^3_{R,j}$. By the inverse Fourier transform in (1.3.8), recalling the definition of χ^3_∞ and exploiting the substitution $\xi_j \mapsto \operatorname{sgn}(x_j)\xi_j$, we have

$$\begin{aligned} C_3 &= \frac{1}{2\pi} \left\| (1 - \chi^1_\infty)(\widehat{\xi}_j) \int_{|\xi_j| > 2R} e^{i|x_j|\xi_j} \frac{\xi_k^s}{\xi_j^2 + \sigma^2} d\xi_j \right\|_{L^\infty_{x_j} L^\infty_{\widehat{\xi}_j}} \\ &= \frac{1}{2\pi} \left\| \int_{|\xi_j| > 2R} \psi(x_j, \widehat{\xi}_j; \xi_j) d\xi_j \right\|_{L^\infty_{x_j} L^\infty_{\widehat{\xi}_j}} \end{aligned}$$

where, for fixed $\widehat{\xi}_j, x_j$, the complex function $\psi(x_j, \widehat{\xi}_j; \cdot) \colon \mathbb{C} \to \mathbb{C}$ is defined by

$$\psi(x_j, \widehat{\xi}_j; w) := \begin{cases} (1 - \chi^1_\infty)(\widehat{\xi}_j) \dfrac{\xi_k^s}{w^2 + \sigma^2} e^{i|x_j|w} & \text{if } k \neq j, \\ (1 - \chi^1_\infty)(\widehat{\xi}_j) \dfrac{w}{w^2 + \sigma^2} e^{i|x_j|w} & \text{if } s = 1,\, k = j, \end{cases}$$

which is holomorphic in $\mathbb{C} \setminus \{w_-, w_+\}$, where $w_\pm = \pm i\sigma$. Observe that $\psi \equiv 0$ for $|\widehat{\xi}_j| > R$, and if $|\widehat{\xi}_j| \leq R$ we have

$$|w_\pm| = |\sigma| = \sqrt[4]{(|\widehat{\xi}_j|^2 - \lambda)^2 + \varepsilon^2} < \sqrt{2}R. \tag{1.3.9}$$

Define, for a radius $A > 0$, the semicircle $\gamma_A := \{Ae^{i\theta} \colon \theta \in [0, \pi]\}$ in the upper half-complex plane. Fixing $\rho > 2R$, by the Residue Theorem, we get

$$\left(\int_{[-\rho, -2R]} - \int_{\gamma_{2R}} + \int_{[2R, \rho]} + \int_{\gamma_\rho} \right) \psi(x_j, \widehat{\xi}_j; w) dw = 0.$$

Observing that we can consider $x_j \neq 0$, letting $\rho \to +\infty$ we can apply Jordan's lemma to the integral on the curve γ_ρ, finally obtaining

$$C_3 = \frac{1}{2\pi} \left\| \int_{\gamma_{2R}} \psi(x_j, \widehat{\xi}_j; w) dw \right\|_{L^\infty_{x_j} L^\infty_{\widehat{\xi}_j}} \leq \frac{(2R)^s}{2\pi} \left\| \int_0^\pi \frac{(1 - \chi^1_\infty)(\widehat{\xi}_j)}{|4R^2 e^{2i\theta} + \sigma^2|} d\theta \right\|_{L^\infty_{\widehat{\xi}_j}} \leq (2R)^{s-2}$$

where we used the relation (1.3.9).

Summing all up, we can finally recover the desired estimate (1.3.3), where the positive constant C does not depend on λ and ε, but only on R and the partition χ. □

Let us prove now Lemma 1.2, which is a straightforward corollary of Lemma 1.1.

Proof of Lemma 1.2. Again, the last estimate in the statement follows from the first one, (1.3.2) and the definition of the X and X^* norms.

From the anticommutation relations (1.1.2) we can infer, for every $z \in \mathbb{C}$, the identity

$$(\mathcal{D}_m - zI_N)(\mathcal{D}_m + zI_N) = (-\Delta + m^2 - z^2)I_N.$$

Thus, for $z \in \rho(\mathcal{D}_m)$ we can write

$$R_{\mathcal{D}_m}(z) = (\mathcal{D}_m + zI_N)R_{-\Delta}(z^2 - m^2)I_N.$$

Let us set $f_j = \chi_j(|z^2 - m^2|^{-1/2}D)f$ for simplicity. By Lemma 1.1, it is easy to recover

$$\begin{aligned}
\|R_{\mathcal{D}_m}(z)f_j\|_{L^\infty_{x_j}L^2_{\widehat{x}_j}} \leq& \left\| \sum_{k=1}^{n} \alpha_k \partial_k R_{-\Delta}(z^2 - m^2)f_j \right\|_{L^\infty_{x_j}L^2_{\widehat{x}_j}} \\
&+ \left\|(m\alpha_{n+1} + zI_N)R_{-\Delta}(z^2 - m^2)f_j\right\|_{L^\infty_{x_j}L^2_{\widehat{x}_j}} \\
\leq& \sum_{k=1}^{n} \left\|\partial_k R_{-\Delta}(z^2 - m^2)f_j\right\|_{L^\infty_{x_j}L^2_{\widehat{x}_j}} \\
&+ \max\{|z+m|, |z-m|\} \left\|R_{-\Delta}(z^2 - m^2)f_j\right\|_{L^\infty_{x_j}L^2_{\widehat{x}_j}} \\
\leq& C \left[n + \left| \frac{z+m}{z-m} \right|^{\operatorname{sgn}(\Re z)/2} \right] \|f\|_{L^1_{x_j}L^2_{\widehat{x}_j}}
\end{aligned}$$

as claimed. □

1.4 The Birman-Schwinger principle

In this section, following the method of [Kat66] by Kato and [KK66] by Konno and Kuroda, we define in a rigorous way the closed extension of a perturbed operator with a factorizable potential, formally defined as $H_0 + B^*A$, and we will provide an abstract version of the Birman-Schwinger principle. In the recent work [HK20], Hansmann and Krejčiřík use a different approach to establish the Birman-Schwinger principle, establishing it for different kind of spectra, and not only for the point one. In particular, they develop a nice and innovative argument to deal with the embedded eigenvalues, which will be borrowed also in this section. Since both the road are worth of interest, in Section 2.3 of the next chapter we will revive the Birman-Schwinger principle, following there [HK20].

Let $\mathfrak{H}, \mathfrak{H}'$ be Hilbert spaces and consider the densely defined, closed linear operators

$$H_0\colon \operatorname{dom}(H_0) \subseteq \mathfrak{H} \to \mathfrak{H}, \quad A\colon \operatorname{dom}(A) \subseteq \mathfrak{H} \to \mathfrak{H}', \quad B\colon \operatorname{dom}(B) \subseteq \mathfrak{H} \to \mathfrak{H}',$$

such that $\rho(H_0) \neq \varnothing$ and

$$\operatorname{dom}(H_0) \subseteq \operatorname{dom}(A), \quad \operatorname{dom}(H_0^*) \subseteq \operatorname{dom}(B).$$

For simplicity, we assume also that $\sigma(H_0) \subset \mathbb{R}$. By $R_{H_0}(z) = (H_0 - z)^{-1}$, we denote the resolvent operator of H_0 for any $z \in \rho(H_0)$.

The idea of the principle is easy to explain in the case of bounded operators A and B. In this case $H = H_0 + B^*A$ is well defined as a sum of operators, and if $z \in \rho(H_0)$, the *Birman-Schwinger* operator

$$K_z = A(H_0 - z)^{-1}B^*$$

is also a bounded operator. One checks immediately that $z \in \sigma_p(H) \cap \rho(H_0)$ implies $-1 \in \sigma_p(K_z)$, and so $\|K_z\|_{\mathfrak{H}' \to \mathfrak{H}'} \geq 1$. Hence, a bound for the norm of K_z gives information on the localization of the non-embedded eigenvalues of H.

We now return to the general case of an unbounded perturbation B^*A. As in [KK66], we assume the following set of assumptions.

Assumption A. For some, and hence for all, $z \in \rho(H_0)$, the operator $AR_{H_0}(z)B^*$, densely defined on $\mathrm{dom}(B^*)$, has a closed extension K_z in $\mathfrak{H}'$,

$$K_z = \overline{AR_{H_0}(z)B^*},$$

which we call the *Birman-Schwinger operator*, with norm bounded by

$$\|K_z\|_{\mathfrak{H}' \to \mathfrak{H}'} \leq \Lambda(z) \tag{1.4.1}$$

for some function $\Lambda\colon \rho(H_0) \to \mathbb{R}_+$.

Assumption B. There exists $z_0 \in \rho(H_0)$ such that $-1 \in \rho(K_{z_0})$.

Observe that the last assumption is implied by the following one:

Assumption B$'$. There exists $z_0 \in \rho(H_0)$ such that $\Lambda(z_0) < 1$.

Indeed, assuming Assumptions A and B', we get that $\|K_{z_0}\|_{\mathfrak{H}' \to \mathfrak{H}'} < 1$. Thus, expanding in a Neumann series, we see that $(1 + K_{z_0})^{-1}$ exists and hence $-1 \in \rho(K_{z_0})$.

Let us collect some useful facts in the next lemma.

Lemma 1.3. *Suppose Assumptions A and B and let $z, z_1, z_2 \in \rho(H_0)$. Then the following holds true:*

(i) $AR_{H_0}(z) \in \mathcal{B}(\mathfrak{H}, \mathfrak{H}')$, $\quad \overline{R_{H_0}(z)B^*} = [B(H_0^* - \overline{z})^{-1}]^* \in \mathcal{B}(\mathfrak{H}', \mathfrak{H})$;

(ii) $\overline{R_{H_0}(z_1)B^*} - \overline{R_{H_0}(z_2)B^*} = (z_1 - z_2)R_{H_0}(z_i)\overline{R_{H_0}(z_j)B^*}$, *for* $i, j \in \{1, 2\}, i \neq j$;

(iii) $K_z = A\overline{R_{H_0}(z)B^*}$, $\quad K_{\overline{z}}^* = B\overline{R_{H_0}(\overline{z})^*A^*}$;

(iv) $\mathrm{ran}(\overline{R_{H_0}(z)B^*}) \subseteq \mathrm{dom}(A)$, $\quad \mathrm{ran}(\overline{R_{H_0}(\overline{z})^*A^*}) \subseteq \mathrm{dom}(B)$;

(v) $K_{z_1} - K_{z_2} = (z_1 - z_2)AR_{H_0}(z_i)\overline{R_{H_0}(z_j)B^*}$, *for* $i, j \in \{1, 2\}, i \neq j$.

Proof. See Lemma 2.2 in [GLMZ05]. □

We can construct now the extension of the perturbed operator $H_0 + B^*A$.

Lemma 1.4 (Extension of operators with factorizable potential)**.** *Suppose Assumptions A and B. Let $z_0 \in \rho(H_0)$ such that $-1 \in \rho(K_{z_0})$. Then the operator*

$$R_H(z_0) = R_{H_0}(z_0) - \overline{R_{H_0}(z_0)B^*}(1 + K_{z_0})^{-1} A R_{H_0}(z_0) \tag{1.4.2}$$

defines a densely defined, closed, linear operator H in $\mathfrak{H}$ which has $R_H(z_0)$ as resolvent and which extends $H_0 + B^*A$.

Proof. We refer to Theorem 2.3 in [GLMZ05]. See also Kato [Kat66]. □

We can finally formulate the abstract Birman-Schwinger principle.

Lemma 1.5 (Birman-Schwinger principle)**.** *Suppose Assumptions A and B. Let $z_0 \in \rho(H_0)$ such that $-1 \in \rho(K_{z_0})$ and H be the extension of $H_0 + B^*A$ given by Lemma 1.4. Fix $z \in \sigma_p(H)$ with eigenfunction $0 \neq \psi \in \operatorname{dom}(H)$, i.e. $H\psi = z\psi$, and set $\phi := A\psi$.*

Then $\phi \neq 0$, and in addition

(i) if $z \in \rho(H_0)$ then

$$K_z \phi = -\phi$$

and in particular

$$1 \leq \|K_z\|_{\mathfrak{H}' \to \mathfrak{H}'} \leq \Lambda(z);$$

(ii) if $z \in \sigma(H_0) \setminus \sigma_p(H_0)$ and if H_0 is self-adjoint, then

$$\lim_{\varepsilon \to 0^\pm} K_{z+i\varepsilon}\phi = -\phi \quad \textit{weakly,}$$

id est

$$\lim_{\varepsilon \to 0^\pm} (\varphi, K_{z+i\varepsilon}\phi)_{\mathfrak{H}'} = -(\varphi, \phi)_{\mathfrak{H}'} \tag{1.4.3}$$

for every $\varphi \in \mathfrak{H}'$, where $(\cdot,\cdot)_{\mathfrak{H}'}$ is the scalar product on $\mathfrak{H}'$. In particular

$$1 \leq \liminf_{\varepsilon \to 0^\pm} \|K_{z+i\varepsilon}\|_{\mathfrak{H}' \to \mathfrak{H}'} \leq \liminf_{\varepsilon \to 0^\pm} \Lambda(z + i\varepsilon). \tag{1.4.4}$$

Proof. Let $\varepsilon = 0$ if $z \in \rho(H_0)$ and $\varepsilon \neq 0$ if $z \in \sigma(H_0) \setminus \sigma_p(H_0)$. In order to treat the embedded eigenvalues, we will adapt the argument of Lemma 1 in [KK66] together with the limiting argument from Theorem 8 in [HK20].

Note that $H\psi = z\psi$ is equivalent to

$$\psi = (z - z_0) R_H(z_0)\psi, \tag{1.4.5}$$

and hence we obtain from (1.4.2) that

$$\begin{aligned}(H_0 - z - i\varepsilon)&R_{H_0}(z_0)\psi \\ &= -(z - z_0)\overline{R_{H_0}(z_0)B^*}(1 + K_{z_0})^{-1} A R_{H_0}(z_0)\psi - i\varepsilon R_{H_0}(z_0)\psi. \end{aligned}\tag{1.4.6}$$

Define

$$\widetilde{\psi} := (1 + K_{z_0})^{-1} A R_{H_0}(z_0)\psi.$$

If $\widetilde{\psi} = 0$, by (1.4.6) follows $(H_0 - z)R_{H_0}(z_0)\psi = 0$. Since $0 \neq R_{H_0}(z_0)\psi \in \operatorname{dom}(H_0)$, we get $z \in \sigma_p(H_0)$, which contradicts the assumption on z. Thus, we proved $\widetilde{\psi} \neq 0$. Moreover, we can show the identity

$$\phi = A\psi = (z - z_0)(1 + K_{z_0})^{-1} A R_{H_0}(z_0)\psi = (z - z_0)\widetilde{\psi}, \tag{1.4.7}$$

from which in particular $\phi \neq 0$. Indeed, by (1.4.2) and (iii) of Lemma 1.3, it follows that

$$A R_H(z_0) = (1 + K_{z_0})^{-1} A R_{H_0}(z_0),$$

which combined with (1.4.5) gives us (1.4.7).

Multiplying by $(1 + K_{z_0})^{-1} A R_{H_0}(z + i\varepsilon)$ both sides of (1.4.6), we obtain

$$\begin{aligned}\widetilde{\psi} = &-(z - z_0)(1 + K_{z_0})^{-1} A R_{H_0}(z + i\varepsilon)\overline{R_{H_0}(z_0)B^*}\widetilde{\psi} \\ &- i\varepsilon(1 + K_{z_0})^{-1} A R_{H_0}(z + i\varepsilon) R_{H_0}(z_0)\psi\end{aligned}$$

and so, by (v) of Lemma 1.3 and by the resolvent identity, we have

$$\begin{aligned}\widetilde{\psi} = &-\frac{z - z_0}{z - z_0 + i\varepsilon}(1 + K_{z_0})^{-1}[K_{z+i\varepsilon} - K_{z_0}]\widetilde{\psi} \\ &- \frac{i\varepsilon}{z - z_0 + i\varepsilon}(1 + K_{z_0})^{-1} A[R_{H_0}(z + i\varepsilon) - R_{H_0}(z_0)]\psi \\ = &\,\widetilde{\psi} - \frac{z - z_0}{z - z_0 + i\varepsilon}(1 + K_{z_0})^{-1}(1 + K_{z+i\varepsilon})\widetilde{\psi} \\ &- \frac{i\varepsilon}{z - z_0 + i\varepsilon}(1 + K_{z_0})^{-1} A R_{H_0}(z + i\varepsilon)\psi,\end{aligned}$$

from which, using identity (1.4.7), we finally arrive at

$$K_{z+i\varepsilon}\phi = -\phi - i\varepsilon A R_{H_0}(z + i\varepsilon)\psi. \tag{1.4.8}$$

If $z \in \rho(H_0)$, then $\varepsilon = 0$ and we completely proved case (i), the “in particular” part being straightforward.

In the following, we suppose $z \in \sigma(H_0) \setminus \sigma_p(H_0)$ and H_0 self-adjoint. Fixed $\varphi \in \mathfrak{H}'$, we get from (1.4.8) that

$$\begin{aligned}(\varphi, K_{z+i\varepsilon}\phi)_{\mathfrak{H}'} &= -(\varphi, \phi)_{\mathfrak{H}'} - i\varepsilon(\varphi, A R_{H_0}(z + i\varepsilon)\psi)_{\mathfrak{H}'} \\ &=: -(\varphi, \phi)_{\mathfrak{H}'} + I_\varepsilon.\end{aligned}$$

Exploiting the Spectral Theorem and denoting the spectral measure of H_0 as E_0, we have

$$I_\varepsilon = \int_{\sigma(H_0)} f_\varepsilon(\lambda)\, d(\varphi, A E_0(\lambda)\psi)_{\mathfrak{H}'}, \quad \text{where} \quad f_\varepsilon(\lambda) := \frac{-i\varepsilon}{\lambda - z - i\varepsilon}.$$

From the fact that

$$\lim_{\varepsilon \to 0^\pm} f_\varepsilon(\lambda) = \begin{cases} 0 & \text{if } \lambda \neq z, \\ 1 & \text{if } \lambda = z, \end{cases}$$

and $E_0(\{z\}) = 0$ since $z \notin \sigma_p(H_0)$, we infer that $f_\varepsilon \to 0$ as $\varepsilon \to 0^\pm$ almost everywhere with respect to the spectral measure. Moreover

$$|f_\varepsilon(\lambda)| = \frac{|\varepsilon|}{\sqrt{(\lambda - z)^2 + \varepsilon^2}} \leq 1 \quad \text{and} \quad \int_{\sigma(H_0)} d(\varphi, A E_0(\lambda)\psi)_{\mathfrak{H}'} = (\varphi, A\psi)_{\mathfrak{H}'},$$

hence by Dominated Converge Theorem we conclude that $I_\varepsilon \to 0$ as $\varepsilon \to 0^\pm$, proving (1.4.3).

Finally, since by (1.4.3) we have

$$\|\phi\|^2_{\mathfrak{H}'} = |(\phi,\phi)_{\mathfrak{H}'}| = \lim_{\varepsilon\to 0^\pm} |(\phi, K_{z+i\varepsilon}\phi)_{\mathfrak{H}'}| \leq \|\phi\|^2_{\mathfrak{H}'} \liminf_{\varepsilon\to 0^\pm} \|K_{z+i\varepsilon}\|_{\mathfrak{H}'\to\mathfrak{H}'}$$

we get the first inequality in (1.4.4), while the second one is obvious by Assumption A. □

1.5 Proof of the theorems

We can now specialize to our problem the abstract theory developed in the last section. We choose $\mathfrak{H} = \mathfrak{H}' = L^2(\mathbb{R}^n;\mathbb{C}^N)$ and H_0 the free Dirac operator $\mathcal{D}_m$. The factorization of V is given using the polar decomposition $V = UW$ where $W = (V^*V)^{1/2}$ and the unitary matrix U is a partial isometry: then we may set $A = W^{1/2}$ and $B = W^{1/2}U^*$. It is easy to see that Assumption A holds thanks to Lemma 1.2 with

$$\Lambda(z) := nC\,\|V\|_Y \left[n + \left| \frac{z+m}{z-m} \right|^{\operatorname{sgn}(\Re z)/2} \right].$$

Indeed, for $\varphi \in C_0^\infty(\mathbb{R}^n;\mathbb{C}^N)$,

$$\begin{aligned}
\|AR_{\mathcal{D}_m}(z)B^*\varphi\|_{\mathfrak{H}} &\leq \sum_{j=1}^n \left\| A\,\chi_j(|z^2-m^2|^{-1/2}D)R_{\mathcal{D}_m}(z)B^*\varphi \right\|_{\mathfrak{H}} \\
&\leq C\left[n + \left| \frac{z+m}{z-m} \right|^{\operatorname{sgn}(\Re z)/2} \right] \sum_{j=1}^n \|A\|_{L^2_{x_j}L^\infty_{\widehat{x}_j}} \|B^*\|_{L^2_{x_j}L^\infty_{\widehat{x}_j}} \|\varphi\|_{\mathfrak{H}} \\
&\leq \Lambda(z)\,\|\varphi\|_{\mathfrak{H}}\,,
\end{aligned}$$

and hence by density (1.4.1). We used above the s

$$\|A\|_{L^2_{x_j}L^\infty_{\widehat{x}_j}} = \|B^*\|_{L^2_{x_j}L^\infty_{\widehat{x}_j}} = \left\|W^{1/2}\right\|_{L^2_{x_j}L^\infty_{\widehat{x}_j}} = \|V\|^{1/2}_{L^1_{x_j}L^\infty_{\widehat{x}_j}}.$$

We show now that also Assumption B' holds. To find $z_0 \in \rho(\mathcal{D}_m)$ such that $\Lambda(z_0) < 1$, let us define

$$C_0 = [n(n+1)C]^{-1}, \qquad \nu = [(n+1)C_0/\,\|V\|_Y - n]^2.$$

Since from the hypothesis of Theorems 1.1 and 1.2 we have $\|V\|_Y < C_0$ and so $\nu > 1$, the condition $1 \leq \Lambda(z)$ is equivalent to $\nu \leq |z/z|$ if $m = 0$, and to

$$\left(\Re z - \operatorname{sgn}(\Re z) m\,\frac{\nu^2+1}{\nu^2-1} \right)^2 + \Im z^2 \leq \left(m\,\frac{2\nu}{\nu^2-1} \right)^2 \tag{1.5.1}$$

if $m > 0$. Then, if $m = 0$ it is sufficient to choose $z_0 \in \mathbb{C}\setminus\mathbb{R}$, while if $m > 0$ we take $z_0 \in \rho(\mathcal{D}_m)$ outside the disks in the statement of Theorem 1.1.

Thus, we can apply Lemma 1.4 to properly define $\mathcal{D}_{m,V}$, and Lemma 1.5 in combination with the relations (1.5.1) and $\nu \leq |z/z|$ to prove Theorem 1.1 and the absence of eigenvalues in the massless case, respectively. For the final claim in Theorem 1.2, we will follow the argument in [CLT14] to prove that the potential $V \in Y = \bigcap_{j=1}^n L^1_{x_j} L^\infty_{\widehat{x}_j}(\mathbb{R}^n)$ leaves the essential spectrum invariant and that $\sigma(\mathcal{D}_{m,V}) \setminus \sigma_e(\mathcal{D}_{m,V}) = \sigma_d(\mathcal{D}_{m,V})$. The argument hold for any $m \geq 0$, and in particular, in the massless case, we get $\sigma(\mathcal{D}_{0,V}) \setminus \mathbb{R} = \varnothing$.

To get the invariance of the essential spectrum, it is sufficient to prove that, fixed $z \in \rho(\mathcal{D}_m)$ such that $-1 \in \rho(K_z)$, the operator $AR_{\mathcal{D}_m}(z)$ is a Hilbert-Schmidt operator, hence compact. Thus identity (1.4.2) gives

$$R_{\mathcal{D}_{m,V}}(z) - R_{\mathcal{D}_m}(z) = -\overline{R_{\mathcal{D}_m}(z)B^*}(1+K_z)^{-1}AR_{\mathcal{D}_m}(z)$$

from which it follows that $R_{\mathcal{D}_{m,V}}(z) - R_{\mathcal{D}_m}(z)$ is compact and so, by Theorem 9.2.4 in [EE18],

$$\sigma_e(\mathcal{D}_{m,V}) = \sigma_e(\mathcal{D}_m) = (-\infty, -m] \cup [m, +\infty).$$

To see that $AR_{\mathcal{D}_m}(z)$ is a Hilbert-Schmidt operator, we need to prove that its kernel $A(x)\mathscr{K}(z, x-y)$ is in $L^2(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{C}^N)$, where $\mathscr{K}(z, x-y)$ is the kernel of the resolvent $(\mathcal{D}_m - z)^{-1}$. By the Young inequality

$$\left\|A(\mathcal{D}-z)^{-1}\right\|^2_{HS} = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |A(x)|^2 |\mathscr{K}(z,x-y)|^2 dx dy \leq \|V\|_{L^p} \|\mathscr{K}\|^2_{L^{2q}} \tag{1.5.2}$$

where $1/p + 1/q = 2$. Hence we need to find in which Lebesgue space $L^{2q}(\mathbb{R}^n; \mathbb{C}^N)$ the kernel $\mathscr{K}(z, x)$ lies. For $z \in \rho(-\Delta) = \mathbb{C} \setminus [0, \infty)$, it is well-known (see e.g. [GS16]) that the kernel $\mathscr{K}_0(z, x-y)$ of the resolvent operator $(-\Delta - z)^{-1}$ is given by

$$\mathscr{K}_0(z, x-y) = \frac{1}{(2\pi)^{n/2}} \left(\frac{\sqrt{-z}}{|x-y|}\right)^{\frac{n}{2}-1} K_{\frac{n}{2}-1}(\sqrt{-z}|x-y|)$$

where $K_\nu(w)$ is the modified Bessel function of second kind and we consider the principal branch of the complex square root. Fixed now $z \in \rho(\mathcal{D}_m) = \mathbb{C} \setminus \{(-m, -\infty] \cup [m, +\infty)\}$, from the identity

$$(\mathcal{D}_m - zI_N)^{-1} = (\mathcal{D}_m + zI_N)(-\Delta + m^2 - z^2)^{-1} I_N$$

and relations (A.4) and (A.5) for the derivative of the modified Bessel functions, we get

$$\begin{aligned}\mathscr{K}(z, x-y) = &\frac{1}{(2\pi)^{n/2}}\left(\frac{k(z)}{|x-y|}\right)^{\frac{n}{2}} \boldsymbol{\alpha} \cdot (x-y) K_{\frac{n}{2}}(k(z)|x-y|) \\ &+ \frac{1}{(2\pi)^{n/2}}\left(\frac{k(z)}{|x-y|}\right)^{\frac{n}{2}-1} (m\alpha_{n+1} + z) K_{\frac{n}{2}-1}(k(z)|x-y|)\end{aligned}$$

where for simplicity $k(z) = \sqrt{m^2 - z^2}$. From the limiting form for the modified Bessel functions (A.7), (A.8) and (A.10), we obtain that

$$\|\mathscr{K}(z,x)\| \leq C(n,m,z) \begin{cases} |x|^{-(n-1)} & \text{if } |x| \leq x_0(n,m,z) \\ |x|^{-(n-1)/2} e^{-\Re k(z)|x|} & \text{if } |x| \geq x_0(n,m,z) \end{cases}$$

for some positive constants $C(n,m,z)$, $x_0(n,m,z)$ depending on z. Hence it is clear that $\mathscr{K}(z,x) \in L^{2q}(\mathbb{R}^n; \mathbb{C}^N)$ for $2q < n/(n-1)$ and, consequently, from equation (1.5.2) we have that $A(\mathcal{D}_m - z)^{-1}$ is a Hilbert-Schmidt operator if $V \in L^p(\mathbb{R}^n; \mathbb{C}^N)$ for $p > n/2$. Since by (1.2.1) we have $V \in L^n(\mathbb{R}^n; \mathbb{C}^N)$, the proof of the identity $\sigma_e(\mathcal{D}_{m,V}) = \sigma_e(\mathcal{D}_m)$ is complete.

Finally, since $\rho(\mathcal{D}_m) = \mathbb{C} \setminus \sigma_e(\mathcal{D}_m)$ is composed by one, or two in the massless case, connected components which intersect $\rho(\mathcal{D}_{m,V})$ in a non-empty set, by Theorem XVII.2.1 in [GGK90] we have $\sigma(\mathcal{D}_{m,V}) \setminus \sigma_e(\mathcal{D}_{m,V}) = \sigma_d(\mathcal{D}_{m,V})$.

Chapter 2

Localization of eigenvalues for non-self-adjoint Dirac and Klein-Gordon operators

In the Introduction, we already explained in a nutshell which are the gears grinding in the Birman-Schwinger principle, stressing that to each resolvent estimate of a free operator we can correspond, via the principle, a localization estimate for the eigenvalues of the perturbed operator. Since resolvent estimates have been an object of study for a considerably longer time with respect to the eigenvalues confinement for non-selfadjoint operators, it is natural that some results for the latter problem, even if interesting per se, go unnoticed.

The goal of the current chapter is indeed bringing to light some new spectral results for the Dirac and Klein-Gordon operators, by inserting already established resolvent estimates in the main engine of the Birman-Schwinger principle. The assumptions we will impose on the potential are essentially of pointwise smallness and decay near the origin and infinity.

The results in this chapter are contained in [S2], joint work with Piero D'Ancona, Luca Fanelli and David Krejčiřík.

2.1 Main results

In this chapter, together our main protagonist, the spinorial Dirac operator, there will be the scalar Klein-Gordon operator. They are formally defined respectively as

$$\mathcal{D}_{m,V} = \mathcal{D}_m + V \quad \text{and} \quad \mathcal{G}_{m,V} = \mathcal{G}_m + V$$

where, for fixed mass $m \geq 0$, the free Klein-Gordon operator is

$$\mathcal{G}_m = \sqrt{m^2 - \Delta},$$

while the Dirac operator is defined in (1.1.1), where the Dirac matrices $\alpha_k \in \mathbb{C}^{N\times N}$, with $N := 2^{\lceil n/2 \rceil}$, satisfy the anti-commutation relations (1.1.2). If we set for simplicity $N := 1$ when we are dealing with the Klein-Gordon operator, we can say that both the operators

$\mathcal{G}_m$ and $\mathcal{D}_m$ act on $\mathfrak{H} = L^2(\mathbb{R}^n; \mathbb{C}^N)$, have domain $H^1(\mathbb{R}^n; \mathbb{C}^N)$ and are self-adjoint with core $C_0^\infty(\mathbb{R}^n; \mathbb{C}^N)$.

Concerning both perturbed operators, the potential $V : \mathbb{R}^n \to \mathbb{C}^{N\times N}$ is a generic, possibly non-Hermitian, matrix-valued function (respectively scalar valued in the case of Klein-Gordon). Invoking the usual abuse of notation, we denote with the same symbol V the multiplication operator by the matrix V in $\mathfrak{H}$ with initial domain $\operatorname{dom}(V) = C_0^\infty(\mathbb{R}^n; \mathbb{C}^N)$.

Again, for any matrix-valued function $M : \mathbb{R}^n \to \mathbb{C}^{N\times N}$ and norm $\|\cdot\| : \mathbb{C} \to \mathbb{R}_+$, we write $\|M\| := \||M|\|$, where $|M(x)|$ denotes the operator norm of the matrix $M(x)$ induced by the Euclidean norm.

For simplicity, we will say that the spectrum of $\mathcal{G}_{m,V}$ or $\mathcal{D}_{m,V}$ is *stable* (with respect to the corresponding free operator spectrum) if

$$\sigma(\mathcal{G}_{m,V}) = \sigma_c(\mathcal{G}_{m,V}) = \sigma(\mathcal{G}_m) = [m, +\infty) \tag{2.1.1}$$

in the case of the Klein-Gordon operator, whereas

$$\sigma(\mathcal{D}_{0,V}) = \sigma_c(\mathcal{D}_{0,V}) = \sigma(\mathcal{D}_0) = \mathbb{R}\,, \tag{2.1.2}$$

$$\sigma(\mathcal{D}_{m,V}) = \sigma_c(\mathcal{D}_{m,V}) = \sigma(\mathcal{D}_m) = (-\infty, -m] \cup [m, +\infty)\,, \tag{2.1.3}$$

in the case of the massless and massive Dirac operators respectively. In any case, note that this means in particular that the point and residual spectra of the perturbed operator are empty.

Finally, let us introduce the weights defined as

$$\tau_\varepsilon(x) := |x|^{\frac{1}{2}-\varepsilon} + |x| \tag{2.1.4}$$

$$w_\sigma(x) := |x|(1 + |\log|x||)^\sigma, \quad \text{for } \sigma > 1. \tag{2.1.5}$$

We are ready to enunciate our results.

Theorem 2.1. *Let $n \geq 3$. There exist positive constants α and ε, which are independent of V, such that if*

$$\left\|\tau_\varepsilon^2 V\right\|_{L^\infty} < \alpha$$

then the spectrum of $\mathcal{G}_{m,V}$ is stable, viz. (2.1.1) *holds true.*

Theorem 2.2. *Let $n \geq 3$. For $m = 0$, there exists a positive constant α, independent of V, such that if*

$$\|w_\sigma V\|_{L^\infty} < \alpha$$

then the spectrum of $\mathcal{D}_{0,V}$ is stable, viz. (2.1.2) *holds true.*

For $m > 0$, there exist positive constants α and ε, independent of V, such that if

$$\left\|\tau_\varepsilon^2 V\right\|_{L^\infty} < \alpha$$

then the spectrum of $\mathcal{D}_{m,V}$ is stable, viz. (2.1.3) *holds true.*

For the Dirac operator we can improve the above theorem in two ways. Firstly, slightly generalizing the choice of the weights (see also Remark 2.3 below). Secondly, and above all, we can give a quantitative form for the smallness condition of the potential (even if our expression for the constant is probably far from being optimal). With this aim we bring into play the dyadic norms defined as

$$\|u\|^p_{\ell^p L^q} := \sum_{j\in\mathbb{Z}} \|u\|^p_{L^q(2^{j-1}\le|x|<2^j)}, \qquad \|u\|_{\ell^\infty L^q} := \sup_{j\in\mathbb{Z}} \|u\|_{L^q(2^{j-1}\le|x|<2^j)}, \tag{2.1.6}$$

for $1 \le p < \infty$ and $1 \le q \le \infty$.

Theorem 2.3. *Let $n \ge 3$, $m \ge 0$ and $\rho \in \ell^2 L^\infty(\mathbb{R}^n)$ be a positive weight. If $m > 0$, assume in addition that $|x|^{1/2}\rho \in L^\infty(\mathbb{R}^n)$. For $m > 0$, define*

$$C_1 \equiv C_1(n,m,\rho) := 576n\left[\sqrt{n} + (2m+1)\sqrt[4]{64n+324}\right]\|\rho\|^2_{\ell^2 L^\infty} + (2m+1)\sqrt{\frac{\pi}{2(n-2)}}\left\||x|^{1/2}\rho\right\|^2_{L^\infty}$$

whereas if $m = 0$,

$$C_1 \equiv C_1(n,0,\rho) := 2C_2\,\|\rho\|^2_{\ell^2 L^\infty}, \tag{2.1.7}$$

$$C_2 \equiv C_2(n) := 576n\max\{\sqrt{n}, \sqrt[4]{64n+324}\}. \tag{2.1.8}$$

Supposing

$$C_1\left\||x|\rho^{-2}V\right\|_{L^\infty} < 1$$

then the spectrum of $\mathcal{D}_{m,V}$ is stable, viz. (2.1.3) *holds true.*

In the massless case, we can ask for less stringent conditions on the potential in order to still get the spectrum stable.

Theorem 2.4. *Let $n \ge 3$, $m = 0$ and*

$$2C_2\,\||x|V\|_{\ell^1 L^\infty} < 1,$$

where C_2 is defined in (2.1.7)*. Then the spectrum of $\mathcal{D}_{0,V}$ is stable, viz.* (2.1.2) *holds true.*

Last but not least, we prove some results on the eigenvalues confinement in two complex disks for the massive Dirac operator. To this end one can use either the weighted dyadic norm (this gives the counterpart for $m > 0$ of Theorem 2.4), or again the weighted-L^2 norm with weaker conditions on the weight ρ (namely, removing in Theorem 2.3 the assumption $|x|^{1/2}\rho \in L^\infty(\mathbb{R}^n)$ when $m > 0$).

Theorem 2.5. *Let $n \ge 3$, $m > 0$ and*

$$N_1(V) := \||x|V\|_{\ell^1 L^\infty}, \qquad N_2(V) := \|\rho\|^2_{\ell^2 L^\infty}\left\||x|\rho^{-2}V\right\|_{L^\infty}$$

for some positive weight $\rho \in \ell^2 L^\infty(\mathbb{R}^n)$. For fixed $j \in \{1,2\}$, if we assume

$$2C_2 N_j(V) < 1,$$

with C_2 defined in (2.1.7)*, then*

$$\sigma_p(\mathcal{D}_{m,V}) \subset \overline{B}_{r_0}(x_0^-) \cup \overline{B}_{r_0}(x_0^+)$$

where the two closed complex disks have centres x_0^-, x_0^+ and radius r_0 defined by

$$x_0^\pm := \pm m\frac{\nu_j^2+1}{\nu_j^2-1}, \quad r_0 := m\frac{2\nu_j}{\nu_j^2-1}, \quad \textit{with} \quad \nu_j := \left[\frac{1}{C_2N_j(V)} - 1\right]^2 > 1.$$

Remark 2.1. In the above Theorem 2.5, the case $j = 2$ is actually redundant. Indeed, one can easily observe that $N_1(V) \leq N_2(V)$ simply by Hölder's inequality. Thus, if $2C_2N_2(V) < 1$, it follows that $\nu_2 \leq \nu_1$ and the disks obtained for $j = 1$ are enclosed in those obtained for $j = 2$. However, we explicit both the case since, as observed above, Theorem 2.5 is in some sense the counterpart of Theorem 2.3 and Theorem 2.4.

Remark 2.2. In our results, the low dimensional cases $n = 1, 2$ are excluded. This restriction comes from the key resolvent estimates we are going to employ, collected in Lemma 2.1, Lemma 2.2 and Lemma 2.3 and proved in [DF08] and [CDL16] (see Section 2.2 below). Indeed, regarding the last lemma, it holds for $n \geq 3$ since to prove it the multiplier method is exploited, which fails in low dimensions. In the case of the first two lemmata instead, the low dimensions are excluded essentially due to the use of Kato-Yajima's estimates; but there is a deeper reason behind instead of a mere technical one.

In fact, tracing back the computations in [DF08], a key step in the proof of Lemma 2.1 and Lemma 2.2 is equation (2.19) of [DF08] concerning the Schrödinger resolvent, namely

$$\left\|\tau_\varepsilon^{-1}(-\Delta - z)^{-1} f\right\|_{L^2} \leq C(1 + |z|^2)^{-1/2} \left\|\tau_\varepsilon f\right\|_{L^2} \leq C \left\|\tau_\varepsilon f\right\|_{L^2},$$

with some positive constant C and $n \geq 3$. The above inequality is obtained by fusing together results by Barcelo, Ruiz and Vega [BRV97] and by Kato and Yajima [KY89], and it is without any doubt false for $n = 1, 2$. In fact, by contradiction, exploiting computations similar to the ones we will carry on in Section 2.4, one should be able to prove the counterpart of Theorems 2.1 and 2.2 for the Schrödinger operator, in other words the spectrum of $-\Delta + V$ would be stable if $\left\|\tau_\varepsilon^2 V\right\|_{L^\infty} < \alpha$ for some positive constants α and ε. This assertion is true for $n \geq 3$, but certainly impossible for $n = 1, 2$, due to the well-know fact that the Schrödinger operator is critical if, and only if, $n = 1, 2$.

The *criticality* of an operator H_0 means that it is not stable against small perturbations: there exists a compactly supported potential V such that $H_0 + \epsilon V$ possesses a discrete eigenvalue for all small $\epsilon > 0$. For the Schrödinger operator this is equivalent to the lack of Hardy's inequality. On the contrary, the existence of Hardy's inequality in dimension $n \geq 3$ is sometimes referred to as the *subcriticality* of $-\Delta$.

In the light of this argument for the Schrödinger operator, a very interesting question, deserving to be pursued, naturally arises: one can conjecture that also the Klein-Gordon and Dirac operators are critical if and only if $n = 1, 2$, that is Theorems 2.1 and 2.2 are false in low dimensions and their spectra are not stable if perturbed by small compactly supported potentials.

Remark 2.3. In Theorem 2.1 and 2.2 we used the explicit weights τ_ε and w_σ, while in the subsequent statements exploiting the weighted-L^2 norm they are replaced by the weight $|x|\rho^{-2}$ with $\rho \in \ell^2 L^\infty(\mathbb{R}^n)$. We compare these assumptions.

It easy to check that $\rho_1 := (1 + |\log|x||)^{-\sigma/2}$ and $\rho_2 := (|x|^{-\varepsilon} + |x|^\delta)^{-1}$ are weights in $\ell^2 L^\infty(\mathbb{R}^n)$ for any $\sigma > 1$ and $\varepsilon, \delta > 0$. Consequently we can set $|x|\rho^{-2} = w_\sigma(x)$ or $|x|\rho^{-2} = (|x|^{1/2-\varepsilon} + |x|^{1/2+\delta})^2$. The additional condition $|x|^{1/2}\rho \in L^\infty$ can be obtained for ρ_2 if we set $\delta = 1/2$, and hence $\tau_\varepsilon^2 = |x|\rho_2^{-2}$. In other words, w_σ and τ_ε are the prototypes of the class of weights we used, since $|x|^{1/2} w_\sigma^{-1/2}, |x|^{1/2}\tau_\varepsilon^{-1} \in \ell^2 L^\infty(\mathbb{R}^n)$ and $|x|\tau_\varepsilon^{-1} \in L^\infty(\mathbb{R}^n)$.

This generalization gives only a minor improvement in the type of admissible weights, however we think it is useful since it stresses the properties and limiting behaviors required on them.

Finally, we note that the extra condition $|x|^{1/2}\rho \in L^\infty(\mathbb{R}^n)$ affects the behavior of $\rho \in \ell^2 L^\infty(\mathbb{R}^n)$ only at infinity. Indeed, near the origin, say when $|x| \leq 1$,

$$|x|^{1/2}\rho \leq \|\rho\|_{L^\infty} \leq \|\rho\|_{\ell^2 L^\infty} ,$$

so no further requirement is added on the behavior of ρ near $x = 0$; on the contrary

$$\|\rho\|_{\ell^2 L^\infty(|x|\geq 1)} \leq \left\| |x|^{-1/2} \right\|_{\ell^2 L^\infty(|x|\geq 1)} \left\| |x|^{1/2}\rho \right\|_{L^\infty} = \sqrt{2} \left\| |x|^{1/2}\rho \right\|_{L^\infty}$$

when $|x| \geq 1$, so $L^\infty(|x| \geq 1) \subset \ell^2 L^\infty(|x| \geq 1)$.

Remark 2.4. For a concrete example, let us make the constants C_1 and C_2 explicit in a special case. We set $n = 3$, $m \in [0, 1]$ and choose $\rho = |x|^{1/2}\tau_{1/2}^{-1} = (|x|^{-1/2} + |x|^{1/2})^{-1}$, which implies easily $\|\rho\|_{\ell^2 L^\infty} \leq 2$ and $\left\| |x|^{1/2}\rho \right\|_{L^\infty} \leq 1$.

Therefore, it follows that $C_2 \leq 8.24 \cdot 10^3$, $C_1 \leq 1.11 \cdot 10^5$ if $m > 0$ and $C_1 \leq 6.59 \cdot 10^4$ if $m = 0$. Hence the smallness condition on the potential in Theorem 2.3 is implied by

$$\left\|(1 + |x|)^2 V\right\|_{L^\infty} < \begin{cases} 9.00 \cdot 10^{-6} & \text{if } m > 0, \\ 1.51 \cdot 10^{-5} & \text{if } m = 0, \end{cases}$$

and the one in Theorem 2.4 by $\| |x| V \|_{\ell^1 L^\infty} < 6.06 \cdot 10^{-5}$.

Our conditions on the potential V are certainly not sharp. We conjecture that the pointwise smallness conditions of Theorem 2.2 can be replaced by suitable integral hypotheses.

Conjecture. Let $n = 3$. There exists a positive constant α independent of V such that if $\|V\|_{L^3} < \alpha$, then the spectrum of $\mathcal{D}_{0,V}$ is stable, viz. (2.1.2) holds true, whereas if $\|V\|_{L^3} + \|V\|_{L^{3/2}} < \alpha$, then the spectrum of $\mathcal{D}_{m,V}$ is stable, viz. (2.1.3) holds true.

2.2 A bundle of resolvent estimates

As anticipated above, the main ingredients in our proofs are a collection of inequalities already published in the literature. The first two, recalled in the next two lemmata, come from [DF08].

Lemma 2.1. *Let $n \geq 3$ and $z \in \mathbb{C}$. There exist $\varepsilon > 0$ sufficiently small and a constant $C > 0$ such that*

$$\left\| \tau_\varepsilon^{-1} (\sqrt{m^2 - \Delta} - z)^{-1} f \right\|_{L^2} \leq C \|\tau_\varepsilon f\|_{L^2}$$

where the weight τ_ε is defined in (2.1.4).

The massless case for this Klein-Gordon resolvent estimate is obtained by equation (2.39) in [DF08] letting $W = 0$. Instead, equation (2.43) from the same paper gives us the massive case for unitary mass $m = 1$, and for all positive m by a change of variables.

Let us face now the Dirac operator.

Lemma 2.2. *Let $n \geq 3$ and $z \in \mathbb{C}$. There exist $\varepsilon > 0$ sufficiently small and a constant $C > 0$ such that*

$$\left\| w_\sigma^{-1/2} (\mathcal{D}_0 - z I_N)^{-1} f \right\|_{L^2} \leq C \left\| w_\sigma^{1/2} f \right\|_{L^2} , \tag{2.2.1}$$

$$\left\| \tau_\varepsilon^{-1} (\mathcal{D}_m - z I_N)^{-1} f \right\|_{L^2} \leq C \left\| \tau_\varepsilon f \right\|_{L^2} , \tag{2.2.2}$$

in the massless and massive case respectively, where the weights τ_ε and w_σ are defined in (2.1.4) *and* (2.1.5).

These estimates correspond to equation (2.49) and (2.52) from [DF08] respectively, even if the estimate for the massless case was previously proved in [DF07] by the same authors. It should be noted that, in the cited paper, estimates (2.49) and (2.52) are explicated only in the 3-dimensional case, but it can be easily seen that they hold in any dimension $n \geq 3$, since their proofs mostly rely on the well-known identity $\mathcal{D}_m^2 = (-\Delta + m^2) I_N$.

The resolvent estimates just stated are uniform, in the sense that the constant C in the estimates is independent of z. This will imply, as we will see, the total absence of eigenvalues under suitable smallness assumptions on the potential.

For the Dirac operator the above result can be improved. First of all, we can give a non-sharp but explicit estimate for the constant C. Moreover, paying with a constant dependent on z (obtaining then a localization for the eigenvalues instead of their absence in the massless case) we can substitute the weighted-L^2 norms with dyadic ones, or relax the hypothesis on the weights in the massive case.

This step-up will be gained making use of the sharp resolvent estimate for the Schrödinger operator in dimension $n \geq 3$ contained in Theorem 1.1 of [CDL16] (the same estimate can be obtained also e.g. from Theorem 1.2 in [D'A20], but the latter does not provide explicit constants). Setting $a = I_n$, $b = c = 0$, $N = \nu = 1$ and $C_a = C_b = C_c = C_- = C_+ = 0$ in the referred theorem, one immediately obtain the trio of estimates stated below.

Lemma 2.3. *Let $n \geq 3$, $z \in \mathbb{C} \setminus [0, +\infty)$ and $R_0(z) := (-\Delta - z)^{-1}$. Then*

$$\begin{aligned} \|R_0(z) f\|_{\dot X}^2 + \|\nabla R_0(z) f\|_{\dot Y}^2 &\leq (288 n)^2 \|f\|_{\dot Y^*}^2 , \\ |\Re z| \, \|R_0(z) f\|_{\dot Y}^2 &\leq (576 \sqrt{2}\, n^2)^2 \|f\|_{\dot Y^*}^2 , \\ |\Im z| \, \|R_0(z) f\|_{\dot Y}^2 &\leq (864 \sqrt{2}\, n)^2 \|f\|_{\dot Y^*}^2 , \end{aligned}$$

where the $\dot X$ and $\dot Y$ norms are the Morrey-Campanato-type norms defined by

$$\|u\|_{\dot X}^2 := \sup_{R>0} \frac{1}{R^2} \int_{|x|=R} |u|^2 dS, \qquad \|u\|_{\dot Y}^2 := \sup_{R>0} \frac{1}{R} \int_{|x|\leq R} |u|^2 dx,$$

and the $\dot Y^*$ norm is predual to the $\dot Y$ norm.

Since the Morrey-Campanato-type norms above introduced are not so handy, observe that the $\dot X$ norm can be written as a radial-angular norm

$$\|u\|_{\dot X} = \left\| |x|^{-1} u \right\|_{\ell^\infty L^\infty_{|x|} L^2_\theta} := \sup_{j \in \mathbb{Z}} \sup_{R \in [2^{j-1}, 2^j)} \left\| |x|^{-1} u \right\|_{L^2(|x|=R)}$$

whereas the $\dot{Y}$ norm is equivalent to the weighted dyadic norm $\left\| |x|^{-1/2} \cdot \right\|_{\ell^\infty L^2}$, and hence by duality the $\dot{Y}^*$ norm is equivalent to $\left\| |x|^{1/2} \cdot \right\|_{\ell^1 L^2}$ (being $\|\cdot\|_{\ell^p L^q}$ defined in (2.1.6)). More precisely, since we want to show explicit constants, we have that

$$\left\| |x|^{-1/2} u \right\|_{\ell^\infty L^2}^2 = \sup_{j \in \mathbb{Z}} \int_{2^{j-1}}^{2^j} |x|^{-1} |u|^2 dx \le 2 \sup_{j \in \mathbb{Z}} \frac{1}{2^j} \int_{|x| \le 2^j} |u|^2 dx \le 2 \|u\|_{\dot{Y}}^2 ,$$

while from the other side, fixed $R \in [2^{j-1}, 2^j)$ for some $j \in \mathbb{Z}$, we get

$$\frac{1}{R} \int_{|x| \le R} |u|^2 dx \le 2^{1-j} \sum_{n=-\infty}^{j} 2^n \int_{2^{n-1}}^{2^n} |x|^{-1} |u|^2 dx \le 4 \left\| |x|^{-1/2} u \right\|_{\ell^\infty L^2}^2 .$$

Summarizing

$$\begin{aligned} 2^{-1/2} \left\| |x|^{-1/2} u \right\|_{\ell^\infty L^2} \le \ \|u\|_{\dot{Y}} \ \le \ 2 \left\| |x|^{-1/2} u \right\|_{\ell^\infty L^2} \\ 2^{-1} \left\| |x|^{1/2} u \right\|_{\ell^1 L^2} \le \|u\|_{\dot{Y}^*} \le 2^{1/2} \left\| |x|^{1/2} u \right\|_{\ell^1 L^2} . \end{aligned}$$

Inserting the above norm equivalence relations in Lemma 2.3 one can straightforwardly infer the following.

Corollary 2.1. *Under the same assumptions of Lemma 2.3, the estimates*

$$\begin{aligned} \left\| |x|^{-1} R_0(z) f \right\|_{\ell^\infty L^\infty_{|x|} L^2_\theta} &\le 576 n \left\| |x|^{1/2} f \right\|_{\ell^1 L^2} , \\ |z|^{1/2} \left\| |x|^{-1/2} R_0(z) f \right\|_{\ell^\infty L^2} &\le 576 n \sqrt[4]{64 n + 324} \left\| |x|^{1/2} f \right\|_{\ell^1 L^2} , \\ \left\| |x|^{-1/2} \nabla R_0(z) f \right\|_{\ell^\infty L^2} &\le 576 n \left\| |x|^{1/2} f \right\|_{\ell^1 L^2} , \end{aligned}$$

hold true.

Simply applying Hölder's inequality, one can deduce also the weighted-L^2 version of Lemma 2.1. Moreover, this allows us to employ the $-\Delta$-supersmoothness of $|x|^{-1}$ to obtain a homogeneous (in effect even stronger) weighted-L^2 estimate for the Schrödinger resolvent. Namely, we have the following.

Corollary 2.2. *Under the same assumptions of Lemma 2.3, the following estimates hold*

$$\begin{aligned} \left\| |x|^{-3/2} \rho R_0(z) f \right\|_{L^2} &\le 576 n \|\rho\|_{\ell^2 L^\infty}^2 \left\| |x|^{1/2} \rho^{-1} f \right\|_{L^2} , \\ |z|^{1/2} \left\| |x|^{-1/2} \rho R_0(z) f \right\|_{L^2} &\le 576 n \sqrt[4]{64 n + 324} \|\rho\|_{\ell^2 L^\infty}^2 \left\| |x|^{1/2} \rho^{-1} f \right\|_{L^2} , \\ \left\| |x|^{-1/2} \rho \nabla R_0(z) f \right\|_{L^2} &\le 576 n \|\rho\|_{\ell^2 L^\infty}^2 \left\| |x|^{1/2} \rho^{-1} f \right\|_{L^2} , \end{aligned} \tag{2.2.3}$$

for any arbitrary positive weight $\rho \in \ell^2 L^\infty(\mathbb{R}^n)$.

If in addition $|x|^{1/2} \rho \in L^\infty(\mathbb{R}^n)$, then

$$\langle z \rangle^{1/2} \left\| |x|^{-1/2} \rho R_0(z) f \right\|_{L^2} \le C_3 \left\| |x|^{1/2} \rho^{-1} f \right\|_{L^2}$$

where

$$C_3 \equiv C_3(n,\rho) := 576n\sqrt[4]{64n+324}\,\|\rho\|^2_{\ell^2 L^\infty} + \sqrt{\frac{\pi}{2(n-2)}}\left\||x|^{1/2}\rho\right\|^2_{L^\infty}$$

and $\langle x\rangle := \sqrt{1+x^2}$ *are the Japanese brackets.*

Proof. By Hölder's inequality we easily obtain the set of inequalities

$$\begin{aligned}
\left\||x|^{1/2}u\right\|_{\ell^1 L^2} &\leq \|\rho\|_{\ell^2 L^\infty}\left\|\rho^{-1}|x|^{1/2}u\right\|_{L^2}\,,\\
\left\||x|^{-1/2}\rho u\right\|_{L^2} &\leq \|\rho\|_{\ell^2 L^\infty}\left\||x|^{-1/2}u\right\|_{\ell^\infty L^2}\,,\\
\left\||x|^{-3/2}\rho u\right\|_{L^2} &\leq \left\||x|^{-1/2}\rho\right\|_{\ell^2 L^2_{|x|}L^\infty_\theta}\left\||x|^{-1}u\right\|_{\ell^\infty L^\infty_{|x|}L^2_\theta}\\
&\leq \|\rho\|_{\ell^2 L^\infty}\left\||x|^{-1}u\right\|_{\ell^\infty L^\infty_{|x|}L^2_\theta}\,,
\end{aligned}$$

which inserted in Corollary 2.1 give us the first three weighted-L^2 estimates.

The last one is instead obtained making use of the celebrated Kato-Yajima result in [KY89], that is

$$\left\||x|^{-1}R_0(z)f\right\|_{L^2} \leq \sqrt{\frac{\pi}{2(n-2)}}\,\||x|f\|_{L^2}\,,$$

with the best constant furnished by Simon [Sim92], combined with the trivial bounds

$$\begin{aligned}
\||x|u\|_{L^2} &\leq \left\||x|^{1/2}\rho\right\|_{L^\infty}\left\||x|^{1/2}\rho^{-1}u\right\|_{L^2}\,,\\
\left\||x|^{-1/2}\rho u\right\|_{L^2} &\leq \left\||x|^{1/2}\rho\right\|_{L^\infty}\left\||x|^{-1}u\right\|_{L^2}\,,
\end{aligned}$$

given again by Hölder's inequality. □

We can return now to the Dirac operator. As a consequences of Corollaries 2.1 and 2.2 we obtain the following lemma.

Lemma 2.4. *Let* $n \geq 3$ *and* $z \in \mathbb{C}\setminus\{(-\infty,-m]\cup[m,+\infty)\}$*. Then*

$$\left\||x|^{-1/2}(\mathcal{D}_m - z)^{-1}f\right\|_{\ell^\infty L^2} \leq C_2\left[1+\left|\frac{z+m}{z-m}\right|^{\frac{\operatorname{sgn}\Re z}{2}}\right]\left\||x|^{1/2}f\right\|_{\ell^1 L^2}$$

where C_2 *is defined in* (2.1.7)*, and in particular*

$$\left\||x|^{-1/2}\rho(\mathcal{D}_m - z)^{-1}f\right\|_{L^2} \leq C_2\,\|\rho\|^2_{\ell^2 L^\infty}\left[1+\left|\frac{z+m}{z-m}\right|^{\frac{\operatorname{sgn}\Re z}{2}}\right]\left\||x|^{1/2}\rho^{-1}f\right\|_{L^2} \quad (2.2.4)$$

for any positive weight $\rho \in \ell^2 L^\infty(\mathbb{R}^n)$.

If in addition $|x|^{1/2}\rho \in L^\infty(\mathbb{R}^n)$*, then*

$$\left\||x|^{-1/2}\rho(\mathcal{D}_m - z)^{-1}f\right\|_{L^2} \leq C_1\left\||x|^{1/2}\rho^{-1}f\right\|_{L^2} \quad (2.2.5)$$

where C_1 *is defined in the statement of Theorem 2.3.*

Proof. By Corollary 2.1 and the identity

$$(\mathcal{D}_m - z)^{-1} = (\mathcal{D}_m + z)(-\Delta + m^2 - z^2)^{-1} I_N$$

we obtain

$$\begin{aligned}
\left\| |x|^{-1/2}\rho(\mathcal{D}_m - z)^{-1} f \right\|_{\ell^\infty L^2} &\leq \left\| |x|^{-1/2}\rho \sum_{k=1}^{n} \alpha_k \partial_k R_0(z^2 - m^2) f \right\|_{\ell^\infty L^2} \\
&\quad + \left\| |x|^{-1/2}\rho(m\alpha_{n+1} + zI_N) R_0(z^2 - m^2) f \right\|_{\ell^\infty L^2} \\
&\leq \sqrt{n} \left\| |x|^{-1/2}\rho \nabla R_0(z^2 - m^2) f \right\|_{\ell^\infty L^2} \\
&\quad + \max\{|z+m|, |z-m|\} \left\| |x|^{-1/2}\rho R_0(z^2 - m^2) f \right\|_{\ell^\infty L^2} \\
&\leq C_2 \left[1 + \left| \frac{z+m}{z-m} \right|^{\operatorname{sgn}(\Re z)/2} \right] \left\| |x|^{1/2}\rho^{-1} f \right\|_{\ell^1 L^2}.
\end{aligned}$$

Similarly we have the other two inequalities, using Corollary 2.2 and the fact that

$$\max\{|z+m|, |z-m|\}\langle z^2 - m^2\rangle^{-1/2} \leq 2m + 1$$

for the homogenous estimate (2.2.5). Note also that, in the massless case, (2.2.5) is already contained in (2.2.4). □

2.3 The Birman-Schwinger principle, da capo version

In this section, we recall again the technicalities for the Birman-Schwinger principle and for properly define an operator perturbed by a factorizable potential. This time, in contrast with the approach of Section 1.4, we completely rely on the abstract analysis carried out by Hansmann and Krejčiřík in [HK20], to which we refer for more results and background. There, in addition to the point spectrum, appropriate versions of the principle are stated even for the residual, essential and continuous spectra.

Let us start recalling some spectral definitions. The *spectrum* $\sigma(H)$ of a closed operator H in a Hilbert space $\mathfrak{H}$ is the set of the complex numbers z for which $H - z\colon \operatorname{dom}(H) \to \mathfrak{H}$ is not bijective. The *resolvent set* is the complement of the spectrum, $\rho(H) := \mathbb{C} \setminus \sigma(H)$. The *point spectrum* $\sigma_p(H)$ is the set of eigenvalues of H, namely the set of complex number such that $H - z$ is not injective. The *continuous spectrum* $\sigma_c(H)$ is the set of elements of $\sigma(H) \setminus \sigma_p(H)$ such that the closure of the range of $H - z$ equals $\mathfrak{H}$; if instead such closure is a proper subset of H, we speak of the *residual spectrum* $\sigma_r(H)$.

Here we collect the set of hypotheses we need.

Assumption I. Let $\mathfrak{H}$ and $\mathfrak{H}'$ be complex separable Hilbert spaces, H_0 be a self-adjoint operator in $\mathfrak{H}$ and $|H_0| := (H_0^2)^{1/2}$ its absolute value. Also, let $A\colon \operatorname{dom}(A) \subseteq \mathfrak{H} \to \mathfrak{H}'$ and $B\colon \operatorname{dom}(B) \subseteq \mathfrak{H} \to \mathfrak{H}'$ be linear operators such that $\operatorname{dom}(|H_0|^{1/2}) \subseteq \operatorname{dom}(A) \cap \operatorname{dom}(B)$. We assume that for some (and hence for all) $b > 0$ the operators $A(|H_0| + b)^{-1/2}$ and $B(|H_0| + b)^{-1/2}$ are bounded and linear from $\mathfrak{H}$ to $\mathfrak{H}$.

At this point, defining $G_0 := |H_0| + 1$, we can consider, for any $z \in \rho(H_0)$, the *Birman-Schwinger operator*

$$K_z := [AG_0^{-1/2}][G_0(H_0 - z)^{-1}][BG_0^{-1/2}]^*, \tag{2.3.1}$$

which is linear and bounded from $\mathfrak{H}'$ to $\mathfrak{H}'$.

The second assumption we need is stated below.

Assumption II. There exists $z_0 \in \rho(H_0)$ such that $-1 \notin \sigma(K_{z_0})$.

While in general Assumption I is easy to check in the applications, Assumption II is more tricky. Thus, we can replace it with the following one, stronger but more manageable.

Assumption II′. There exists $z_0 \in \rho(H_0)$ such that $\|K_{z_0}\|_{\mathfrak{H}' \to \mathfrak{H}'} < 1$.

That the latter implies Assumption II can be easily proved by observing that the spectral radius is dominated by the operator norm, or recurring to Neumann series. Alternative conditions implying Assumption II are collected in Lemma 1 of [HK20], but for our purposes Assumption II′ will be enough.

Before recalling the Birman-Schwinger principle, we properly define the formal perturbed operator $H_0 + V$ with $V = B^*A$.

Theorem 2.6. *Under Assumptions I and II, there exists a unique closed extension H_V of $H_0 + V$ such that $\mathrm{dom}(H_V) \subseteq \mathrm{dom}(|H_0|^{1/2})$ and the following representation formula holds true:*

$$(\phi, H_V \psi)_{\mathfrak{H} \to \mathfrak{H}} = (G_0^{1/2}\phi, (H_0 G_0^{-1} + [B G_0^{-1/2}]^* A G_0^{-1/2}) G_0^{1/2} \psi)_{\mathfrak{H} \to \mathfrak{H}}$$

for $\phi \in \mathrm{dom}(|H_0|^{1/2})$, $\psi \in \mathrm{dom}(H_V)$.

This result correspond to Theorem 5 in [HK20], where the operator H_V is obtained via the pseudo-Friedrichs extension. Note that following the alternative approach by Kato [Kat66], the extension of $H_0 + B^*A$ is not only closed, but also quasi-selfadjoint. We refer to the paper of Hansmann and Krejčiřík [HK20] for a cost-benefit comparison of the two methods, and for a list of cases when the two extensions coincide.

Finally, we can exhibit the abstract Birman-Schwinger principle, for the proof of which see Theorem 6, 7, 8 and Corollary 4 of [HK20].

Theorem 2.7. *Under Assumption I and II, we have:*

(i) if $z \in \rho(H_0)$, then $z \in \sigma_p(H_V)$ if and only if $-1 \in \sigma_p(K_z)$;

(ii) if $z \in \sigma_c(H_0) \cap \sigma_p(H_V)$ and $H_V \psi = z\psi$ for $0 \neq \psi \in \mathrm{dom}(H_V)$, then $A\psi \neq 0$ and

$$\lim_{\varepsilon \to 0^\pm} (K_{z+i\varepsilon} A\psi, \phi)_{\mathfrak{H}' \to \mathfrak{H}'} = -(A\psi, \phi)_{\mathfrak{H}' \to \mathfrak{H}'}$$

for all $\phi \in \mathfrak{H}'$.

In particular

(i) if $z \in \sigma_p(H_V) \cap \rho(H_0)$, then $\|K_z\|_{\mathfrak{H}' \to \mathfrak{H}'} \geq 1$;

(ii) if $z \in \sigma_p(H_V) \cap \sigma_c(H_0)$, then $\liminf_{\varepsilon \to 0^\pm} \|K_{z+i\varepsilon}\|_{\mathfrak{H} \to \mathfrak{H}'} \geq 1$.

While from the "in particular" part of the previous theorem one could infer a localization for the eigenvalues of H_V, the principle can be employed in a "negative" way to prove their absence when the norm of the Birman-Schwinger operator is strictly less than 1 uniformly respect to $z \in \rho(H_0)$. This is precisely stated in the next concluding result, corresponding to Theorem 3 in [HK20], which is even richer: not only gives information on the absence of the eigenvalues, but also on the invariance of the spectrum of the perturbed operator.

Theorem 2.8. *Suppose Assumption I and that* $\sup_{z\in\rho(H_0)} \|K_z\|_{\mathfrak{H}'\to\mathfrak{H}'} < 1$. *Then we have:*

(i) $\sigma(H_0) = \sigma(H_V)$;

(ii) $\sigma_p(H_V) \cup \sigma_r(H_V) \subseteq \sigma_p(H_0)$ *and* $\sigma_c(H_0) \subseteq \sigma_c(H_V)$.

In particular, if $\sigma(H_0) = \sigma_c(H_0)$, *then* $\sigma(H_V) = \sigma_c(H_V) = \sigma_c(H_0)$.

2.3.1 A concrete case

We now specialize the situation from the abstract to a concrete setting, typical in many common applications and relevant for our analysis.

Suppose that $\mathfrak{H} = \mathfrak{H}' = L^2(\mathbb{R}^n; \mathbb{C}^{N\times N})$, $N \in \mathbb{N}$, and V is the multiplication operator generated in $\mathfrak{H}$ by a matrix-valued (scalar-valued if $N = 1$) function $V\colon \mathbb{R}^n \to \mathbb{C}^{N\times N}$, with initial domain $\operatorname{dom}(V) = C_0^\infty(\mathbb{R}^n; \mathbb{C}^N)$. As customary, we consider the factorization of V given by the polar decomposition $V = UW$, where $W = \sqrt{V^*V}$ and the unitary matrix U is a partial isometry. Therefore we may set $A = \sqrt{W}$, $B = \sqrt{W}U^*$ and consider the corresponding multiplication operators generated by A and B^* in $\mathfrak{H}$ with initial domain $C_0^\infty(\mathbb{R}^n; \mathbb{C}^N)$, denoted by the same symbols. In the end, we can factorize the potential V in two closed operators A and B^*. Via the Closed Graph Theorem, Assumption I is verified.

Furthermore, in general the operator K_z defined in (2.3.1) is a bounded extension of the classical Birman-Schwinger operator $A(H_0 - z)^{-1}B^*$ defined on $\operatorname{dom}(B^*)$. Since in our case the initial domain of B^* is $C_0^\infty(\mathbb{R}^n; \mathbb{C}^N)$, hence dense in $\mathfrak{H}$, we get that K_z is exactly the closure of $A(H_0 - z)^{-1}B^*$.

In conclusion, everything reduces to the study of $\left\|A(H_0 - z)^{-1}B^*\right\|_{\mathfrak{H}\to\mathfrak{H}}$: if there exists $z_0 \in \rho(H_0)$ such that this norm is strictly less than 1, then Theorem 2.7 holds; if this is true uniformly respect to $z \in \rho(H_0)$, then also Theorem 2.8 holds true.

2.4 Proof of the theorems

Taking into account the last subsection and recalling the uniform resolvent estimates from Section 2.2, proving our claimed results on the Klein-Gordon and Dirac operators is now a simple matter.

For $z \in \rho(H_0)$ and $\phi \in C_0^\infty(\mathbb{R}^n)$, from the resolvent estimate in Lemma 2.1, we immediately get

$$\begin{aligned}
\left\|A(\mathcal{G}_m - z)^{-1}B^*\phi\right\|_{L^2} &\le \|A\tau_\varepsilon\|_{L^\infty} \left\|\tau_\varepsilon^{-1}(\mathcal{G}_m - z)^{-1}B^*\phi\right\|_{L^2} \\
&\le C\,\|A\tau_\varepsilon\|_{L^\infty} \|\tau_\varepsilon B^*\phi\|_{L^2} \\
&\le C\left\|\tau_\varepsilon^2 V\right\|_{L^\infty} \|\phi\|_{L^2} \\
&< \alpha C\,\|\phi\|_{L^2}\,.
\end{aligned}$$

If $\alpha = 1/C$, then Theorem 2.1 follows from Theorem 2.8. By analogous computations one obtains Theorem 2.2 making use of the resolvent estimates in Lemma 2.2, and the other theorems concerning the Dirac operator exploiting Lemma 2.4.

Let us just make explicit the computations for Theorem 2.5 with $N_1(V) = \||x|V\|_{\ell^1 L^\infty}$. By Lemma 2.4 we have that

$$\begin{aligned}\left\|A(\mathcal{D}_m - z)^{-1}B^*\phi\right\|_{L^2} &\leq \left\|A|x|^{1/2}\right\|_{\ell^2 L^\infty} \left\||x|^{-1/2}(\mathcal{D}_m - z)^{-1}B^*\phi\right\|_{\ell^\infty L^2} \\ &\leq C_2 \left[1 + \left|\frac{z+m}{z-m}\right|^{\operatorname{sgn}\Re z/2}\right] \left\|A|x|^{1/2}\right\|_{\ell^2 L^\infty} \left\||x|^{1/2}B^*\phi\right\|_{\ell^1 L^2} \\ &\leq C_2 \left[1 + \left|\frac{z+m}{z-m}\right|^{\operatorname{sgn}\Re z/2}\right] \||x|V\|_{\ell^1 L^\infty} \|\phi\|_{L^2}\,.\end{aligned}$$

Setting $\nu_1 := [1/[C_2 N_1(V)] - 1]^2 > 1$, the condition $\left\|A(\mathcal{D}_m - z)^{-1}B^*\phi\right\|_{\mathfrak{H}\to\mathfrak{H}} \geq 1$ turns out to be equivalent to the expression

$$\left(\Re z - \operatorname{sgn}(\Re z) m \frac{\nu_1^2+1}{\nu_1^2-1}\right)^2 + (\Im z)^2 \leq \left(m\frac{2\nu_1}{\nu_1^2-1}\right)^2$$

which define exactly the disks in the statement of the theorem. Just take any $z_0 \in \rho(\mathcal{D}_m)$ outside these two disks to verify Assumption II$'$, and finally we can prove the statement applying the “in particular” part of Theorem 2.7.

Chapter 3

Keller-type bounds for Dirac operators perturbed by rigid potentials

In this chapter we are interested in generalizing Keller-type eigenvalue estimates for the non-self-adjoint Schrödinger operator to the Dirac operator, imposing some suitable rigidity conditions on the matricial structure of the potential. What is relevant is that we obtain results for the Dirac operator without necessarily requiring the smallness of its norm.

The reference for the results in this chapter is [S3], joint work with Haruya Mizutani.

3.1 Keller-type bound for Schrödinger

Let us start recapping in greater details the Keller-type bound for the Schrödinger operator, partly anticipated in the Introduction.

As we know, the first Keller-type inequality for the non-self-adjoint Schrödinger operator $-\Delta + V$ is due to Abramov, Aslanyan and Davies [AAD01] in 1-dimension, viz.

$$|z|^{1/2} \leq \frac{1}{2} \|V\|_{L^1}, \tag{3.1.1}$$

where $z \in \sigma_p(-\Delta + V)$ and the constant is sharp.

Subsequently, Laptev and Safronov [LS09] conjectured that the eigenvalues localization bound $|z|^\gamma \leq D_{\gamma,n} \|V\|^{\gamma+n/2}_{L^{\gamma+n/2}}$ should hold for any $0 < \gamma \leq n/2$ and some constant $D_{\gamma,n} > 0$. Thanks to Frank [Fra11], the conjecture turned out to be true for $0 < \gamma \leq 1/2$, and later Frank and Simon [FS17b] proved it completely under radial symmetry assumptions. Explicitly, in dimension $n \geq 2$ the eigenvalues of $-\Delta + V$ satisfy the estimates

$$|z|^\gamma \leq D_{\gamma,n} \begin{cases} \|V\|^{\gamma+n/2}_{L^{\gamma+n/2}} & \text{for } 0 < \gamma \leq \frac{1}{2}, \\ \|V\|^{\gamma+n/2}_{L^{\gamma+n/2}_\rho L^\infty_\theta} & \text{for } \frac{1}{2} < \gamma < \frac{n}{2}, \\ \|V\|^{n}_{L^{n,1}_\rho L^\infty_\theta} & \text{for } \gamma = \frac{n}{2}, \end{cases} \tag{3.1.2}$$

where the positive constant $D_{\gamma,n}$ is independent of z and V and where the radial-angular spaces $L^p_\rho L^s_\theta$ and $L^{p,q}_\rho L^s_\theta$ are defined as

$$\begin{aligned} L^p_\rho L^s_\theta &:= L^p(\mathbb{R}_+, r^{n-1}dr; L^s(\mathbb{S}^{n-1})) \\ L^{p,q}_\rho L^s_\theta &:= L^{p,q}(\mathbb{R}_+, r^{n-1}dr; L^s(\mathbb{S}^{n-1})) \end{aligned} \tag{3.1.3}$$

being $L^{p,q}$ the Lorentz spaces and $\mathbb{S}^{n-1}$ the n-dimensional unit spherical surface. In the case $1 \leq p, q < \infty$, the respective norms are explicitly given by

$$\begin{aligned} \|f\|_{L^p_\rho L^s_\theta} &:= \left(\int_0^\infty \|f(r\,\cdot)\|^p_{L^s(\mathbb{S}^{n-1})} \, r^{n-1} dr \right)^{1/p} \\ \|f\|_{L^{p,q}_\rho L^s_\theta} &:= \left(p \int_0^\infty t^{q-1} \mu \left\{ r > 0\colon \; \|f(r\,\cdot)\|_{L^s(\mathbb{S}^{n-1})} \geq t \right\}^{q/p} dt \right)^{1/q} \end{aligned} \tag{3.1.4}$$

where μ is the measure $r^{n-1}dr$ on $\mathbb{R}_+ = (0, +\infty)$. The above relations (3.1.2) hold also in the case $\gamma = 0$, in the sense that if $D_{0,n} \|\mathcal{V}\|^{n/2}_{L^{n/2}} < 1$ for some $D_{0,n} > 0$, then the point spectrum of $-\Delta + \mathcal{V}$ is empty (the optimal constant is given by $D_{0,3} = 4/(3^{3/2}\pi^2)$ in 3-dimensions).

The Laptev-Safronov conjecture certainly does not hold for $\gamma > n/2$, as already noted by Laptev and Safronov themselves. For the range $1/2 < \gamma \leq n/2$, an argument in [FS17b] suggested that the conjecture should fail in general, and this was recently confirmed in [BC21] with the construction of a suitable counterexample.

Nevertheless, for $n \geq 1$ and $\gamma > 1/2$, Frank in [Fra18] proved a localization result still involving the $L^{\gamma+n/2}$ norm of the potential, but in an unbounded region of the complex plane around the semi-line $\sigma(-\Delta) = [0, +\infty)$, viz.

$$|z|^{1/2} \operatorname{dist}(z, [0, +\infty))^{\gamma-1/2} \leq D_{\gamma,n} \|V\|^{\gamma+n/2}_{L^{\gamma+n/2}}. \tag{3.1.5}$$

In the limiting case $\gamma = \infty$ one has the trivial bound

$$\operatorname{dist}(z, [0, +\infty)) \leq D_{\infty,n} \|V\|_{L^\infty}. \tag{3.1.6}$$

Thus, it seems that to go beyond the threshold $\gamma = 1/2$, one should ask radial symmetry on the potential, or abandon the idea of localizing the eigenvalues in compact regions (cf. Section 3.3 below).

To conclude the recap on the spectral results for the Schrödinger operator, besides the ones related to the above conjecture, one should refer also to [FLLS06], where bounds on sums of eigenvalues outside a cone around the positive axis were proved, and to the works [DN02, Saf10, Enb16, FS17a, FKV18b, Fra18, LS19, Cue20], where one can find Keller-type inequalities involving not only the L^p norms.

We turn out our attention to the Dirac operator (1.1.1). If we look at the results we proved in the first two chapter of this thesis and at the literature therein mentioned, two situations seems to arise: or the confinement regions are unbounded, containing the continuous spectrum of the free Dirac operator $\mathcal{D}_m$, or the regions are bounded, but the potential is required to be small respect to some "cumbersome" norm.

In the present chapter we recover Keller-type bounds which we believe to be a worthy analogous of the Schrödinger enclosures in (3.1.2), hence exploiting L^p norms at least for

$n/2 \leq p \leq (n+1)/2$; also, we can remove the smallness assumption on the potential (when $p \neq n/2$). Of course, to reach such a nice result, the price to pay is high: we will require to our potentials to be of the form $\mathcal{V} = vV$, where $v : \mathbb{R}^n \to \mathbb{C}$ is a scalar function in the desired space of integrability, whereas V is a constant matrix satisfying some suitable rigidity conditions. Hence, in a way to be clarified later, we will fully take advantage of the matricial structure of the Dirac operator in order to reduce ourselves basically to the Schrödinger case.

3.2 Idea and main results

As anticipated above, the trick of our argument relies completely on the matricial structure of the potential, which in the rest of the chapter will be denoted with the calligraphic letter $\mathcal{V}$. Before to rattle off the hypothesis we are going to impose on it, in order to understand our idea we need to apply the Birman-Schwinger principle in its simplest form. In order to make things work and being formal, just for the moment assume that $\mathcal{V}$ is bounded, such that $\mathcal{D}_{m,\mathcal{V}} = \mathcal{D}_m + \mathcal{V}$ is well defined as sum of operators.

We know that the principle assure us that z is an eigenvalue of $\mathcal{D}_{m,\mathcal{V}}$, where $\mathcal{V} = \mathcal{B}^*\mathcal{A}$ is a factorizable potential, if and only if -1 is an eigenvalue of the Birman-Schwinger operator $K_z := \mathcal{A}(\mathcal{D}_m - z)^{-1}\mathcal{B}^*$. If $-1 \in \sigma_p(K_z)$ then $\|K_z\| \geq 1$, which turns out to be the desired localization bound, if one is able to estimate the Birman-Schwinger operator.

From the well-known identity

$$(\mathcal{D}_m - z)^{-1} = (\mathcal{D}_m + z)R_0(z^2 - m^2)I_N \tag{3.2.1}$$

which links the resolvent for the Dirac operator $(\mathcal{D}_m - z)^{-1}$ with the resolvent for the Schrödinger operator $R_0(z) := (-\Delta - z)^{-1}$, we have that

$$\mathcal{A}(\mathcal{D}_m - z)^{-1}\mathcal{B}^* = -i\sum_{k=1}^{n} \mathcal{A}\alpha_k\partial_k R_0(z^2 - m^2)\mathcal{B}^* + \mathcal{A}(m\alpha_{n+1} + z)R_0(z^2 - m^2)\mathcal{B}^*. \tag{3.2.2}$$

At this point, the receipt one usually cooks (as in the previous two chapters) is the following. First of all, the polar decomposition $\mathcal{V} = \mathcal{U}\mathcal{W}$ of the potential is exhibited, where $\mathcal{W} = \sqrt{\mathcal{V}^*\mathcal{V}}$ and the unitary matrix $\mathcal{U}$ is a partial isometry. Then one takes $\mathcal{A} = \sqrt{\mathcal{W}}$ and $\mathcal{B} = \sqrt{\mathcal{W}}\mathcal{U}^*$; this choice assures a certain symmetry in splitting the potential, since $\mathcal{A}$ and $\mathcal{B}$ are in the same space of integrability. Therefore, making use of resolvent estimates and of the Hölder's inequality, one reaches an estimate of the form $1 \leq \|K_z\| \leq \kappa(z)\,\|\mathcal{V}\|_X$ for some suitable function $\kappa : \mathbb{C} \to \mathbb{R}$ and space X.

Clearly, the main problem is reduced to the research of nice resolvent estimates. For the Schrödinger operator, these have been extensively studied, so if we look at (3.2.2) the main concern comes from the estimates for the derivatives of $R_0(z)$. Our idea here is to choose $\mathcal{A}$ and $\mathcal{B}$ in such a way that the terms $\mathcal{A}\alpha_k\partial_k R_0(z^2 - m^2)\mathcal{B}^*$, for any $k \in \{1, \ldots, n\}$, simply disappear (we will make an exception to this for Theorem 3.9). If additionally we impose also $\mathcal{A}R_0(z^2 - m^2)\mathcal{B}^*$ to be zero, we are also able to remove the smallness assumption on the potential, because it turns out that they originates from this term. Therefore, let us state the following hypothesis.

Rigidity Assumptions. Let us consider a potential of the type $\mathcal{V} = vV = \mathcal{B}^*\mathcal{A}$, with $\mathcal{A} = aA$ and $\mathcal{B} = bB$, in such a way that $v = \bar{b}a$ and $V = B^*A$, where $a, b, v \colon \mathbb{R}^n \to \mathbb{C}$ are complex-valued functions and $A, B, V \in \mathbb{C}^{N\times N}$ are constant matrices.

On the scalar part v, we impose the usual polar decomposition, viz. $a = |v|^{1/2}$ and $b = \overline{\mathrm{sgn}(v)}|v|^{1/2}$, where the sign function is defined as $\mathrm{sgn}(w) = w/|w|$ for $0 \neq w \in \mathbb{C}$ and $\mathrm{sgn}(0) = 0$.

On the matricial part V, we ask the following set of conditions:

$$A\alpha_k B^* = 0 \quad \text{for } k \in \{1, \dots, n\},$$
$$V = B^* A \neq 0.$$

It is not restrictive to assume also that

$$|A| = |B| = 1$$

where $|\cdot|\colon \mathbb{C}^{N\times N} \to \mathbb{R}$ denotes the operator norm induced by the Euclidean norm, viz. $|A| = \sqrt{\rho(A^*A)}$, where $\rho(M)$ is the spectral radius of a matrix M.

In addition to the above stated hypothesis, suppose also one between the next conditions:

(i) $A\alpha_{n+1}B^* \neq 0$ and $AB^* \neq 0$;

(ii) $A\alpha_{n+1}B^* \neq 0$ and $AB^* = 0$;

(iii) $A\alpha_{n+1}B^* = 0$ and $AB^* \neq 0$;

(iv) $A\alpha_{n+1}B^* = 0$ and $AB^* = 0$.

In the following, we will refer to our set of rigidity assumptions as RA(ι), where $\iota \in \{i, ii, iii, iv\}$ depends on which of the four conditions above is considered.

Remark 3.1. Note that we will *not* assume any Rigidity Assumptions in Theorem 3.9, but only in Theorems 3.1–3.8 below.

Remark 3.2. At this point the reader may argue that the assumptions above are not rigorous, since we have not explicitly defined the Dirac matrices α_k, $k \in \{1, \dots, n+1\}$. Moreover, there is not a unique representation for these matrices! The concern is legit, and we will furnish later the exact definitions of our Dirac matrices, in Section 3.5, which will be all devoted to computations with matrices. The choice of a particular representation of the Dirac matrices is not restrictive, see Remark 3.6.

Remark 3.3. As will be proved in Section 3.5, we can find matrices A and B satisfying RA(i) in any dimension $n \geq 1$, whereas there are no matrices satisfying RA(ii) and RA(iii) in dimensions $n = 2, 4$ and no matrices satisfying RA(iv) in dimensions $n = 1, 2$. This explains the dimensions restriction in the statements of the theorems below.

We can state now our main results. Recall, other than the Lebesgue norm, the Lorentz norm and the radial-angular norm introduced in (3.1.4). We refer to Figures 3.1, 3.2 and 3.3 to visualize the boundary curves of the confinement regions described in the various theorems.

Let us start considering the case of RA(ii).

Theorem 3.1. *Let $m > 0$, $n = 1$ and $\mathcal{V} = vB^*A$ satisfying RA(ii). Then*

$$|z^2 - m^2|^{1/2} \leq \frac{1}{2} \|v\|_{L^1}$$

for any $z \in \sigma_p(\mathcal{D}_{m,\mathcal{V}})$.

Theorem 3.2. *Let $m > 0$, $n \in \mathbb{N} \setminus \{1, 2, 4\}$ and $\mathcal{V} = vB^*A$ satisfying RA(ii). There exists $D_{\gamma,n,m} > 0$ such that*

$$|z^2 - m^2|^\gamma \leq D_{\gamma,n,m} \begin{cases} \|v\|_{L^{\gamma+n/2}}^{\gamma+n/2} & \text{for } 0 < \gamma \leq \frac{1}{2}, \\ \|v\|_{L_\rho^{\gamma+n/2} L_\theta^\infty}^{\gamma+n/2} & \text{for } \frac{1}{2} < \gamma < \frac{n}{2}, \\ \|v\|_{L_\rho^{n,1} L_\theta^\infty}^{n} & \text{for } \gamma = \frac{n}{2}, \end{cases}$$

for any $z \in \sigma_p(\mathcal{D}_{m,\mathcal{V}})$.

In the case $\gamma = 0$, there exists $D_{0,n} > 0$ such that, if

$$\|v\|_{L^{n/2}} < D_{0,n,m}$$

then

$$\sigma(\mathcal{D}_{m,\mathcal{V}}) = \sigma_c(\mathcal{D}_{m,\mathcal{V}}) = \sigma(\mathcal{D}_m) = (-\infty, -m] \cup [m, +\infty)$$

and in particular $\sigma_p(\mathcal{D}_{m,\mathcal{V}}) = \varnothing$.

Theorem 3.3. *Let $m > 0$, $n \in \mathbb{N} \setminus \{1, 2, 4\}$, $\gamma > 1/2$ and $\mathcal{V} = vB^*A$ satisfying RA(ii). There exists $D_{\gamma,n,m} > 0$ such that*

$$|z^2 - m^2|^{1/2} \operatorname{dist}(z^2 - m^2, [0, +\infty))^{\gamma - 1/2} \leq D_{\gamma,n,m} \|v\|_{L^{\gamma+n/2}}^{\gamma+n/2}$$

for any $z \in \sigma_p(\mathcal{D}_{m,\mathcal{V}})$. In the case $\gamma = \infty$, the above relation is replaced by

$$\operatorname{dist}(z^2 - m^2, [0, +\infty)) \leq D_{\infty,n,m} \|v\|_{L^\infty}.$$

Remark 3.4. Note that, since

$$\operatorname{dist}(z, [0, +\infty)) = \begin{cases} |\Im z| & \text{if } \Re z \geq 0, \\ |z| & \text{if } \Re z \leq 0, \end{cases}$$

then

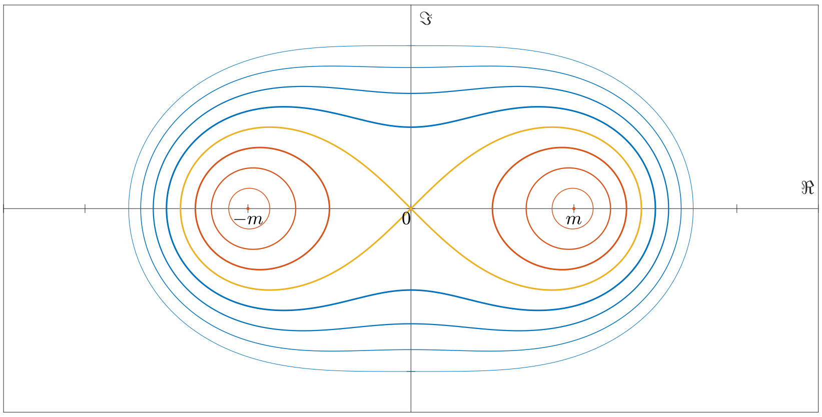
$$\operatorname{dist}(z^2 - m^2, [0, +\infty)) = \begin{cases} 2|\Re z||\Im z| & \text{if } (\Re z)^2 - (\Im z)^2 \geq m^2, \\ |z^2 - m^2| & \text{if } (\Re z)^2 - (\Im z)^2 \leq m^2. \end{cases}$$

The results collected in the three theorems above should be compared with the corresponding ones for the Schrödinger operator, respectively (3.1.1), (3.1.2), (3.1.5) and (3.1.6). We supposed RA(ii) with positive mass $m > 0$, which means, looking (3.2.2), that

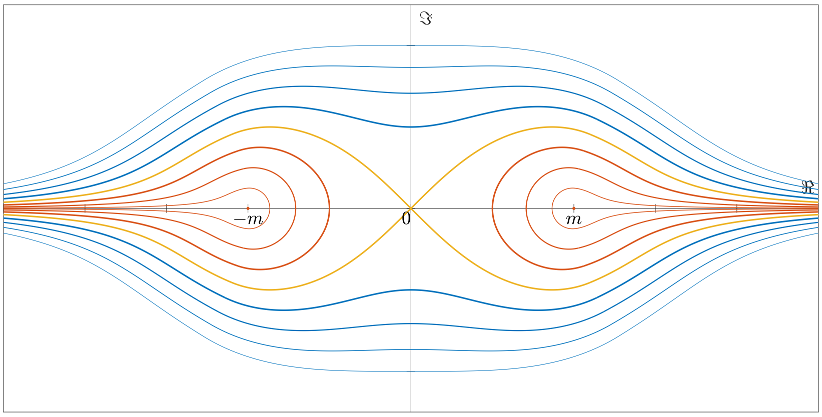
$$\mathcal{A}(\mathcal{D}_m - z)^{-1}\mathcal{B}^* = m[A\alpha_{n+1}B^*][aR_0(z^2 - m^2)\bar{b}].$$

Roughly speaking, the Birman-Schwinger operator for $\mathcal{D}_m + \mathcal{V}$ behaves (more or less) as the Birman-Schwinger operator for $-\Delta + v$. This explains the strict connection between the Dirac and Schrödinger results.

If we consider RA(ii) with $m = 0$, or instead RA(iv), then the Birman-Schwinger operator for Dirac vanish identically, implying the following result of spectral stability.



(a) Case of Theorems 3.1 and 3.2.



(b) Case of Theorem 3.3.

Figure 3.1: The plots of the boundary curves corresponding to the spectral enclosures described in Theorems 3.1, 3.2 and 3.3, for various values of the norm of the potential.
When $\beta := D_{\gamma,n,m} \|v\|^{\gamma+n/2} = 1$, where $D_{1/2,1,m} = 1/2$ and $\|v\|$ is one of the norms appearing in the theorems, we have two regions joined only in the origin (in yellow). If $\beta < 1$ there are two disconnected regions (in red), while if $\beta > 1$ there is one connected region (in blue).
The curves in picture (a) are known as Cassini ovals with foci in m and $-m$.

Theorem 3.4. *Let $n \in \mathbb{N} \setminus \{2,4\}$, $m = 0$ and $\mathcal{V} = vB^*A$ satisfying RA(ii), or alternatively $n \in \mathbb{N} \setminus \{1,2\}$, $m \geq 0$ and $\mathcal{V} = vB^*A$ satisfying RA(iv). Then*

$$\sigma(\mathcal{D}_{m,\mathcal{V}}) = \sigma_c(\mathcal{D}_{m,\mathcal{V}}) = \sigma(\mathcal{D}_m) = (-\infty, -m] \cup [m, +\infty)$$

and in particular $\sigma_p(\mathcal{D}_{m,\mathcal{V}}) = \varnothing$.

We stress out again that the above results does not require any smallness assumption on the potential, even if, of course, the regions of confinement described in Theorems 3.1, 3.2 and 3.3 become larger and larger when the norm of v increases.

Let us wonder now what happens removing the condition $AB^* = 0$. As we see from the following theorems, the requirement that the potential should be small pops up again. Moreover, we find a compact localization for the eigenvalues (or their absence) only respect to the L^1-norm when $n = 1$, and to the $L^{n,1}_\rho L^\infty_\theta$-norm when $n \geq 2$.

About the localization around the continuous spectrum of the free operator, it is not so nice as that in Theorem 3.3, where the region of confinement, even if unbounded, "narrows" around $\sigma(\mathcal{D}_m)$. Denoting for simplicity with $\mathcal{N}$ one of the region described in Theorems 3.6 and 3.8, we have that it "become wider" around $\sigma(\mathcal{D}_m)$, even if the sections $\mathcal{N} \cap \{z \in \mathbb{C} \colon \Re z = x_0\}$ are compact for any fixed $x_0 \in \mathbb{R}$. Also, we need to require $\gamma \geq n/2$, otherwise the region $\mathcal{N}$ would be the complement of a bounded set, and hence not so interesting (see Section 3.4).

Hence, let us state now the results assuming RA(iii) and RA(i) respectively.

Theorem 3.5. *Let $n \in \mathbb{N} \setminus \{2,4\}$, $m \geq 0$ and $\mathcal{V} = vB^*A$ satisfying RA(iii). Moreover, let us set for simplicity*

$$\|\cdot\| := \begin{cases} \|\cdot\|_{L^1} & \text{if } n = 1, \\ \|\cdot\|_{L^{n,1}_\rho L^\infty_\theta} & \text{if } n \geq 2. \end{cases}$$

There exists $C_0 > 0$ such that, if $\|v\| < C_0$ and $m > 0$, then

$$|z^2 - m^2|^{1/2} |z|^{-1} \leq C_0^{-1} \|v\|$$

for any $z \in \sigma_p(\mathcal{D}_{m,\mathcal{V}})$, whereas, if $\|v\| < C_0$ and $m = 0$, then

$$\sigma(\mathcal{D}_{0,\mathcal{V}}) = \sigma_c(\mathcal{D}_{0,\mathcal{V}}) = \sigma(\mathcal{D}_0) = \mathbb{R}$$

and in particular $\sigma_p(\mathcal{D}_{0,\mathcal{V}}) = \varnothing$.

If $n = 1$, we can take $C_0 = 2$.

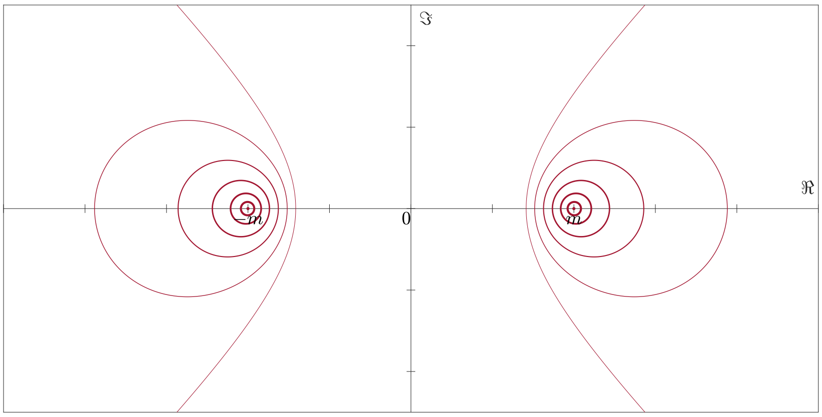
Theorem 3.6. *Let $n \in \mathbb{N} \setminus \{1,2,4\}$, $m \geq 0$, $\mathcal{V} = vB^*A$ satisfying RA(iii) and $\gamma \geq n/2$. Then there exists $C_0 > 0$ such that*

$$|z^2 - m^2|^{1/2} |z|^{-\gamma - n/2} \operatorname{dist}(z^2 - m^2, [0, +\infty))^{\gamma - \frac{1}{2}} \leq C_0^{-1} \|v\|^{\gamma + n/2}_{L^{\gamma + n/2}}$$

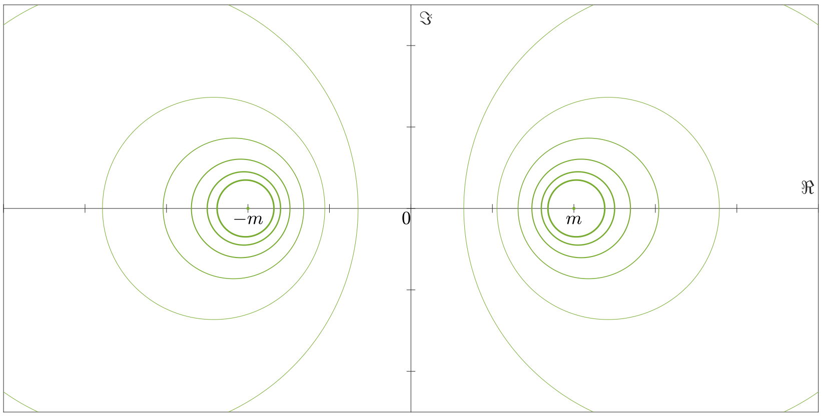
for any $z \in \sigma_p(\mathcal{D}_{m,\mathcal{V}})$. If $\gamma = \infty$, the above relation is substituted by

$$|z|^{-1} \operatorname{dist}(z^2 - m^2, [0, +\infty)) \leq C_0^{-1} \|v\|_{L^\infty}.$$

If $\gamma = n/2$, we should ask also that $\|v\|^{\gamma + n/2}_{L^{\gamma + n/2}} < C_0$.

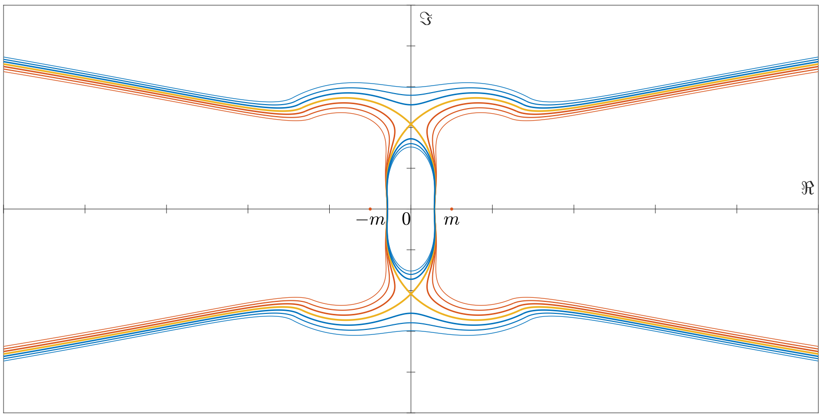


(a) Case of Theorem 3.5.

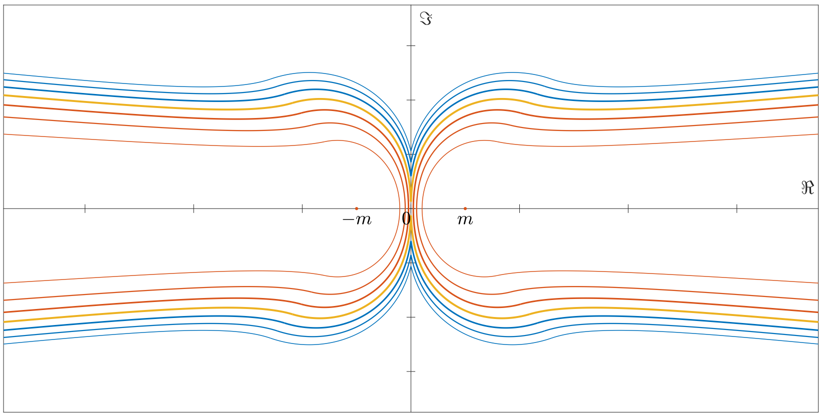


(b) Case of Theorems 3.7 and 3.9.

Figure 3.2: The plots of the boundary curves corresponding to the spectral enclosures described in Theorem 3.5 and in Theorems 3.7 and 3.9, for various values of the norm of the potential. The region is always the union of two disconnected components.



(a) Case of Theorem 3.6.



(b) Case of Theorem 3.8.

Figure 3.3: The plots of the boundary curves corresponding to the spectral enclosures described in Theorems 3.6 and 3.8, for various values of the norm of the potential and for $n/2 < \gamma < \infty$. According to the value of the norm of v, the enclosure region can be composed: by two disconnected components (in red); by two components joining in two points in the case of Theorem 3.6, and in the origin in the case of Theorem 3.8 (in yellow); by one connected region (in blue), which presents a "hole" around the origin in the case of Theorem 3.6.

Theorem 3.7. *Let $n \geq 1$, $m \geq 0$ and $\mathcal{V} = vB^*A$ satisfying RA(i). Moreover, let us set for simplicity*

$$\|\cdot\| := \begin{cases} \|\cdot\|_{L^1} & \text{if } n = 1, \\ \|\cdot\|_{L^{n,1}_\rho L^\infty_\theta} & \text{if } n \geq 2. \end{cases}$$

There exists a constant $C_0 > 0$ such that, if $m > 0$ and $\|v\| < C_0$, then the point spectrum of $\mathcal{D}_{m,\mathcal{V}}$ is confined in the union of the two closed disks

$$\sigma_p(\mathcal{D}_\mathcal{V}) \subseteq \overline{B}_R(c_+) \cup \overline{B}_R(c_-)$$

with centers and radius given by

$$c_\pm = \pm m \frac{C_0^4 + \|v\|^4}{C_0^4 - \|v\|^4}, \qquad R = m \frac{2C_0^2 \|v\|^2}{C_0^4 - \|v\|^4}.$$

Instead, if $m = 0$ and $\|v\| < C_0$, then

$$\sigma(\mathcal{D}_{0,\mathcal{V}}) = \sigma_c(\mathcal{D}_{0,\mathcal{V}}) = \sigma(\mathcal{D}_0) = \mathbb{R}$$

and in particular $\sigma_p(\mathcal{D}_{0,\mathcal{V}}) = \varnothing$.

If $n = 1$, we can take $C_0 = 2$.

Theorem 3.8. *Let $n \geq 2$, $m \geq 0$, $\mathcal{V} = vB^*A$ satisfying RA(i) and $\gamma \geq n/2$. Then there exist $C_0 > 0$ such that*

$$|z^2 - m^2|^{\frac{1}{2}(1-\gamma-\frac{n}{2})} \left| \frac{z+m}{z-m} \right|^{-(\gamma+\frac{n}{2})\frac{\operatorname{sgn} \Re z}{2}} \operatorname{dist}(z^2 - m^2, [0,+\infty))^{\gamma-\frac{1}{2}} \leq C_0^{-1} \|v\|^{\gamma+n/2}_{L^{\gamma+n/2}}$$

for any $z \in \sigma_p(\mathcal{D}_{m,\mathcal{V}})$. If $\gamma = \infty$, the above relation is substituted by

$$|z - m|^{\frac{\operatorname{sgn} \Re z - 1}{2}} |z + m|^{-\frac{\operatorname{sgn} \Re z + 1}{2}} \operatorname{dist}(z^2 - m^2, [0,+\infty)) \leq C_0^{-1} \|v\|_{L^\infty}.$$

If $\gamma = n/2$, we should ask also that $\|v\|^{\gamma+n/2}_{L^{\gamma+n/2}} < C_0$.

As we already explained, the main trick to get the theorems above basically consists of imposing all the term of the type $\mathcal{A}\alpha_k \partial_k R_0(z^2 - m^2)\mathcal{B}^*$ in (3.2.2) to vanish, leaving only the last term:

$$\mathcal{A}(\mathcal{D}_m - z)^{-1}\mathcal{B}^* = A(m\alpha_{n+1} + z)B^* \left[aR_0(z^2 - m^2)\overline{b} \right].$$

This because we want to employ estimates for the resolvent of the Schrödinger operator but not for its derivatives. However, the work [BRV97] furnish us some kind of such estimates for the derivatives of the Schrödinger resolvent (see Lemma 3.3 below). Consequently, we can easily obtain the following confinement result without requiring any special structure on the potential $\mathcal{V}$, but only assuming its smallness respect to the $L^{n,1}_\rho L^\infty_\theta$-norm.

Theorem 3.9. *Let $n \geq 2$, $m \geq 0$ and $\mathcal{V} \colon \mathbb{R}^n \to \mathbb{C}^{N \times N}$ a generic potential. There exists a constant $C_0 > 0$ such that, if $m > 0$ and $\|\mathcal{V}\|_{L^{n,1}_\rho L^\infty_\theta} < C_0$, then the point spectrum of $\mathcal{D}_{m,\mathcal{V}}$ is confined in the union of the two closed disks*

$$\sigma_p(\mathcal{D}_\mathcal{V}) \subseteq \overline{B}_R(c_+) \cup \overline{B}_R(c_-)$$

with centers and radius given by

$$c_\pm = \pm m \frac{\nu^2+1}{\nu^2-1}, \qquad R = m\frac{2\nu}{\nu^2-1}, \qquad \nu := \left[\frac{2C_0}{\|\mathcal{V}\|_{L^{n,1}_\rho L^\infty_\theta}} - 1\right]^2.$$

Instead, if $m=0$ *and* $\|\mathcal{V}\|_{L^{n,1}_\rho L^\infty_\theta} < C_0$*, then*

$$\sigma(\mathcal{D}_{0,\mathcal{V}}) = \sigma_c(\mathcal{D}_{0,\mathcal{V}}) = \sigma(\mathcal{D}_0) = \mathbb{R}$$

and in particular $\sigma_p(\mathcal{D}_{0,\mathcal{V}}) = \varnothing$.

The above theorem is a generalization of Theorem 3.7, dropping the many restrictions on $\mathcal{V}$ and with slightly modified definitions for the centers and the radius of the disks. In some sense, it can be seen as the radial version of the result in Theorem 1.1 from Chapter 1.

3.3 Resolvent estimates for Schrödinger

In this section we collect some well-known resolvent estimates for the Schrödinger operator. For our purposes the estimates on the conjugate line are sufficient, but we think it is nice to look at the complete picture.

For dimension $n \geq 3$, let us define the following endpoints

$$A := \left(\frac{n+1}{2n}, \frac{n-3}{2n}\right), \qquad A' := \left(\frac{n+3}{2n}, \frac{n-1}{2n}\right),$$
$$B := \left(\frac{n+1}{2n}, \frac{(n-1)^2}{2n(n+1)}\right), \qquad B' := \left(\frac{n^2+4n-1}{2n(n+1)}, \frac{n-1}{2n}\right),$$
$$A_0 := \frac{A+A'}{2} = \left(\frac{n+2}{2n}, \frac{n-2}{2n}\right), \qquad B_0 := \frac{B+B'}{2} = \left(\frac{n+3}{2(n+1)}, \frac{n-1}{2(n+1)}\right),$$
$$C := \left(\frac{n+1}{2n}, \frac{n-1}{2n}\right),$$

and the trapezoidal region

$$\begin{aligned}\mathcal{T}_n &:= \left\{\left(\frac{1}{p}, \frac{1}{q}\right) \in \mathcal{Q} \colon \frac{2}{n+1} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{2}{n},\ \frac{1}{p} > \frac{n+1}{2n},\ \frac{1}{q} < \frac{n-1}{2n}\right\}\\ &= [A, B, B', A'] \setminus \{[A,B] \cup [A', B']\}\end{aligned}$$

where $\mathcal{Q}$ is the square $[0,1]\times[0,1]$ and, for any finite set of points $\{p_1, \ldots, p_k\} \subseteq \mathcal{Q}$, we denote with $[p_1, \ldots, p_k]$ its convex hull.

In the 2-dimensional case we define

$$B := \left(\frac{3}{4}, \frac{1}{12}\right), \quad B' := \left(\frac{11}{12}, \frac{1}{4}\right), \quad B_0 := \frac{B+B'}{2} = \left(\frac{5}{6}, \frac{1}{6}\right),$$
$$A_0 := (1,0), \quad C := \left(\frac{3}{4}, \frac{1}{4}\right), \quad D := \left(\frac{3}{4}, 0\right), \quad D' := \left(1, \frac{1}{4}\right)$$

and the diamond region

$$\begin{aligned}\mathcal{T}_2 &:= \left\{\left(\frac{1}{p}, \frac{1}{q}\right) \in \mathcal{Q} \colon \frac{2}{3} \leq \frac{1}{p} - \frac{1}{q} < 1,\ \frac{3}{4} < \frac{1}{p} \leq 1,\ 0 \leq \frac{1}{q} < \frac{1}{4}\right\}\\ &= [B, D, A_0, D', B'] \setminus \{[B,D] \cup \{A_0\} \cup [B', D']\}.\end{aligned}$$

Lemma 3.1. *Let $z \in \mathbb{C} \setminus [0, +\infty)$. If $n = 1$, then*

$$\left\|(-\Delta - z)^{-1}\right\|_{L^1 \to L^\infty} \leq \frac{1}{2}|z|^{-1/2}.$$

If $n \geq 2$, there exists a constant $C > 0$ independent on z such that:

(i) if $(1/p, 1/q) \in \mathcal{T}_n$, then

$$\left\|(-\Delta - z)^{-1}\right\|_{L^p \to L^q} \leq C|z|^{-1+\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}; \tag{3.3.1}$$

(ii) if $(1/p, 1/q) \in \{B, B'\}$ or if, when $n \geq 3$, $(1/p, 1/q) \in \{A, A'\}$, then the restricted weak-type estimate

$$\left\|(-\Delta - z)^{-1}\right\|_{L^{p,1} \to L^{q,\infty}} \leq C|z|^{-1+\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}$$

holds true.

The 1-dimensional estimate immediately follows from the explicit representation for the kernel of the Laplacian resolvent, i.e.

$$(-\Delta - z)^{-1}u(x) = \int_{-\infty}^{+\infty} \frac{i}{2\sqrt{z}} e^{i\sqrt{z}|x-y|}u(y)dy,$$

and from the Young's inequality. This estimate was firstly applied to obtain an eigenvalues localization for the Schrödinger operator by Abramov, Aslanyan and Davies [AAD01].

The estimate in Lemma 3.1.(i) has been proved true on the open segment (A, A') and on the conjugate segment $[A_0, B_0]$ in Lemma 2.2.(b) and Theorem 2.3 of the celebrated paper [KRS87] by Kenig, Ruiz and Sogge. From here comes out the adjective "uniform" with which these kind of estimates are known (even if the multiplicative factor in general shows a dependence on z): the main result in [KRS87] concerns the exponents on the segment (A, A'), on which the exponent in the factor $|z|^{-1+(1/p-1/q)n/2}$ is indeed equal to zero. Nowadays, the term "uniform" is generally used when the multiplicative factor is bounded for large value of $|z|$, which is relevant if we want to localize the eigenvalues in compact sets.

The estimate (3.3.1) was then proved true on the optimal range $(1/p, 1/q) \in \mathcal{T}_n$ by Gutiérrez in Theorem 6 of [Gut04]. In this work the author proved also the inequality at Lemma 3.1.(ii) on the endpoints B and B', whereas the proof for the endpoints A and A' was recently given by Ren, Xi and Zhang in [RXZ18].

It should be noted that both the works [KRS87] and [Gut04] assume $n \geq 3$. The 2-dimensional case seems to have been gone quietly in the literature, nevertheless the arguments in the aforementioned papers can be quite smoothly extended in dimension $n = 2$. This has been observed firstly in Frank [Fra11] concerning the Kenig, Ruiz and Sogge's result, and by Kwon and Lee [KL20] about the work by Gutiérrez.

Now, one question arises naturally: does estimates similar to (3.3.1) hold outside the region $\mathcal{T}_n$? Well yes, but actually no. The range of exponents stated in the above theorem is optimal: estimates (3.3.1) does not hold true if $(1/p, 1/q)$ lies outside $\mathcal{T}_n$. For $n \geq 3$, the constrains $\frac{1}{p} > \frac{n+1}{2n}$ and $\frac{1}{q} < \frac{n-1}{2n}$ are due to considerations from the theory of the

Bochner-Riesz operators of negative orders, the condition $\frac{1}{p}-\frac{1}{q}\geq\frac{2}{n+1}$ comes from the Knapp counterexample (see e.g. [Str77]) and finally $\frac{1}{p}-\frac{1}{q}\leq\frac{2}{n}$ follows by an argument involving the Littlewood-Paley projection. For details on this discussion we refer to [KL20] (and to [KRS87]).

Nonetheless, we can still extend the region of the estimates if we sacrifice something. This is the main theme of the paper [KL20] by Kwon and Lee, where they conjecture that, for $n\geq 2$ and $z\in\mathbb{C}\setminus[0,+\infty)$, the relation

$$\left\|(-\Delta-z)^{-1}\right\|_{L^p\to L^q}\approx|z|^{-1+\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\left(\frac{|z|}{\operatorname{dist}(z,[0,+\infty))}\right)^{\gamma(n,p,q)} \tag{3.3.2}$$

with

$$\gamma(n,p,q):=\max\left\{0,\,1-\frac{n+1}{2}\left(\frac{1}{p}-\frac{1}{q}\right),\,\frac{n+1}{2}-\frac{n}{p},\,\frac{n}{q}-\frac{n-1}{2}\right\} \tag{3.3.3}$$

should hold on the “stripe”

$$\mathcal{S}:=\left\{\left(\frac{1}{p},\frac{1}{q}\right)\in\mathcal{Q}\colon 0\leq\frac{1}{p}-\frac{1}{q}\leq\frac{2}{n}\right\}\setminus\mathcal{S}_0 \tag{3.3.4}$$

where

$$\mathcal{S}_0:=\begin{cases}[A,B]\cup[A',B']\cup[E,E_0)\cup\left(E_0,E'\right]\cup\{F\}\cup\{F'\} & \text{if } n\geq 3,\\ [B,D]\cup[B',D']\cup[E,E_0)\cup\left(E_0,E'\right]\cup\{A_0\} & \text{if } n=2,\end{cases}$$

$$E:=\left(\frac{n-1}{2n},\frac{n-1}{2n}\right),\quad E':=\left(\frac{n+1}{2n},\frac{n+1}{2n}\right),\quad E_0:=\left(\frac{1}{2},\frac{1}{2}\right),$$

$$F:=\left(\frac{2}{n},0\right),\quad F':=\left(1,\frac{n-2}{n}\right).$$

The symbol $A\approx B$ in (3.3.2) means that there exists an absolute constant, independent on z, such that $C^{-1}B\leq A\leq CB$.

Observe that the region $\mathcal{S}$ contains in particular $\mathcal{T}_n$, on which $\gamma(n,p,q)=0$ as one can naturally expect in light of the Kenig-Ruiz-Sogge-Gutiérrez inequalities. In their work, Kwon and Lee prove their conjecture to be indeed true, making exception of the upper bound implicitly contained in (3.3.2) on the region

$$\widetilde{\mathcal{R}}:=\begin{cases}\varnothing & \text{if } n=2,\\ \mathcal{R}\cup\mathcal{R}' & \text{if } n\geq 3,\end{cases} \tag{3.3.5}$$

where

$$\mathcal{R}:=[P_*,P_\circ,E_0]\setminus\{E_0\},\quad \mathcal{R}':=[P'_*,P'_\circ,E_0]\setminus\{E_0\},$$

and the endpoints are defined by

$$P_\circ:=\left(\frac{1}{p_\circ},\frac{1}{q_\circ}\right),\qquad P'_\circ:=\left(1-\frac{1}{q_\circ},1-\frac{1}{p_\circ}\right),$$

$$P_*:=\left(\frac{1}{p_*},\frac{1}{p_*}\right),\quad P'_*:=\left(1-\frac{1}{p_*},1-\frac{1}{p_*}\right),$$

with

$$\frac{1}{p_\circ} := \begin{cases} \dfrac{(n+5)(n-1)}{2(n^2+4n-1)} & \text{if } n \text{ is odd,} \\ \dfrac{n^2+3n-6}{2(n^2+3n-2)} & \text{if } n \text{ is even,} \end{cases} \qquad \frac{1}{q_\circ} := \begin{cases} \dfrac{(n+3)(n-1)}{2(n^2+4n-1)} & \text{if } n \text{ is odd,} \\ \dfrac{(n-1)(n+2)}{2(n^2+3n-2)} & \text{if } n \text{ is even,} \end{cases}$$

$$\frac{1}{p_*} := \begin{cases} \dfrac{3(n-1)}{2(3n+1)} & \text{if } n \text{ is odd,} \\ \dfrac{3n-2}{2(3n+2)} & \text{if } n \text{ is even.} \end{cases}$$

Let us gather the above results by Kwon and Lee [KL20] in the following lemma.

Lemma 3.2. *Let $n \geq 2$ and $z \in \mathbb{C} \setminus [0, +\infty)$. There exists a constant $K > 0$ independent on z such that:*

(i) if $(1/p, 1/q) \in \mathcal{S}$, then

$$\left\|(-\Delta - z)^{-1}\right\|_{L^p \to L^q} \geq K^{-1} |z|^{-1+\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\frac{|z|}{\operatorname{dist}(z, [0, +\infty))}\right)^{\gamma(n,p,q)};$$

(ii) if $(1/p, 1/q) \in \mathcal{S} \setminus \widetilde{\mathcal{R}}$, then

$$\left\|(-\Delta - z)^{-1}\right\|_{L^p \to L^q} \leq K |z|^{-1+\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\frac{|z|}{\operatorname{dist}(z, [0, +\infty))}\right)^{\gamma(n,p,q)}.$$

The regions $\mathcal{S}$ and $\widetilde{\mathcal{R}}$ are described in (3.3.4) and (3.3.5) respectively, while $\gamma(n, p, q)$ is defined in (3.3.3).

The analysis of Kwon and Lee pictures quite clearly the situation outside the so-called "uniform boundedness range" $\mathcal{T}_n$: we can still have $L^p - L^q$ inequalities so long as the factor depending on z explodes when $\Im z \to 0^\pm$, and this can not be improved. If we want to apply these estimates in the eigenvalues localization problem, this means that we can not obtain the eigenvalues confined in a compactly supported region of the complex plane, but in a set containing the continuous spectrum of the unperturbed operator.

In this optic, one can instead try to save the uniformity of the estimates, in the sense that the factor depending on z should be uniformly bounded for $|z|$ sufficiently large. In this way, we can again hope to get the eigenvalues confined inside compact regions. This can be indeed obtained on a smaller region respect to $\mathcal{S}$ if we restrict ourself on considering radial functions.

Define, for $n \geq 2$, the open triangle

$$\begin{aligned} \mathcal{P} &:= \left\{ \left(\frac{1}{p}, \frac{1}{q}\right) \in \mathcal{Q} \colon \frac{1}{n} < \frac{1}{p} - \frac{1}{q} < \frac{2}{n+1}, \ \frac{1}{p} > \frac{n+1}{2n}, \ \frac{1}{q} < \frac{n-1}{2n} \right\} \\ &= [B, C, B'] \setminus \{[B, B'] \cup [B, C] \cup [C, B']\}. \end{aligned}$$

Recall the radial-angular spaces defined in (3.1.3) and their norms (3.1.4). Adopting the terminology and notations of [BRV97] and [FS17b], we introduce also the radial Mizohata-Takeuchi norm

$$\|w\|_{\mathcal{MT}} := \sup_{R>0} \int_R^\infty \frac{r}{\sqrt{r^2 - R^2}} \|w(r\,\cdot)\|_{L^\infty(\mathcal{S}^{n-1})}\, dr$$

and we say that $w \in \mathcal{MT}$ if $\|w\|_{\mathcal{MT}} < \infty$.

Lemma 3.3. *Let $n \geq 2$ and $z \in \mathbb{C} \setminus [0, +\infty)$. There exists a constant $K > 0$ independent on z such that:*

(i) if $(1/p, 1/q) \in (C, B_0)$, then

$$\left\|(-\Delta - z)^{-1}\right\|_{L^p_\rho L^2_\theta \to L^q_\rho L^2_\theta} \leq K|z|^{-1-\frac{n}{2}+\frac{n}{p}};$$

(ii) if $(1/p, 1/q) = C$, then

$$\left\|(-\Delta - z)^{-1}\right\|_{L^{2n/(n+1),1}_\rho L^2_\theta \to L^{2n/(n-1),\infty}_\rho L^2_\theta} \leq K|z|^{-1/2}, \tag{3.3.6}$$

$$\left\|\nabla(-\Delta - z)^{-1}\right\|_{L^{2n/(n+1),1}_\rho L^2_\theta \to L^{2n/(n-1),\infty}_\rho L^2_\theta} \leq K. \tag{3.3.7}$$

If in particular $u \in L^p(\mathbb{R}^n)$ is a radial function, then

$$\left\|(-\Delta - z)^{-1}u\right\|_{L^q} \leq K|z|^{-1+\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|u\|_{L^p}$$

for any $(1/p, 1/q) \in \mathcal{P}$, and

$$\left\|(-\Delta - z)^{-1}u\right\|_{L^{2n/(n-1),\infty}} \leq K|z|^{-1/2} \|u\|_{L^{2n/(n+1),1}}$$

$$\left\|\nabla(-\Delta - z)^{-1}u\right\|_{L^{2n/(n-1),\infty}} \leq K \|u\|_{L^{2n/(n+1),1}}$$

in the case $(1/p, 1/q) = C$.

Proof. The result in Lemma 3.3.(i) is stated in Theorem 4.3 by Frank and Simon [FS17b]. Instead, the case of the endpoint C is essentially due to Theorem 1.(b) and Theorem 2 by Barcelo, Ruiz and Vega [BRV97]. Indeed, let us consider firstly the estimate for $(-\Delta - z)^{-1}$. Observe that, by Hölder's inequality and by duality, the estimate (3.3.6) is equivalent to

$$\left\|w_1^{1/2}(-\Delta - z)^{-1}w_2^{1/2}u\right\|_{L^2} \leq K|z|^{-1/2} \|w_1\|^{1/2}_{L^{n,1}_\rho L^\infty_\theta} \|w_2\|^{1/2}_{L^{n,1}_\rho L^\infty_\theta} \|u\|_{L^2} \tag{3.3.8}$$

for any $w_1, w_2 \in L^{n,1}_\rho L^\infty_\theta$. In fact, that (3.3.6) implies (3.3.8) is obvious by Hölder's inequality for Lorentz spaces. Conversely, we have that

$$\begin{aligned}\left\|(-\Delta - z)^{-1}w_2^{1/2}u\right\|_{L^{2n/(n-1),\infty}_\rho L^2_\theta} &= \sup_{0\neq w_1 \in L^{n,1}_\rho L^\infty_\theta} \frac{\left\|w_1^{1/2}(-\Delta - z)^{-1}w_2^{1/2}u\right\|_{L^2}}{\left\|w_1^{1/2}\right\|_{L^{2n,2}_\rho L^\infty_\theta}} \\ &\leq K|z|^{-1/2} \|w_2\|_{L^{n,1}_\rho L^\infty_\theta} \|u\|_{L^2},\end{aligned}$$

that is to say that, for any fixed $w \in L^{n,1}_\rho L^\infty_\theta$, the operator $(-\Delta - z)^{-1} w^{1/2}$ is bounded from L^2 to $L^{2n/(n-1),\infty}_\rho L^2_\theta$ with norm

$$\left\| (-\Delta - z)^{-1} w^{1/2} \right\|_{L^2 \to L^{2n/(n-1),\infty}_\rho L^2_\theta} \leq K|z|^{-1/2} \|w\|_{L^{n,1}_\rho L^\infty_\theta} .$$

By duality this implies that the operator $w^{1/2}(-\Delta - z)^{-1}$ is bounded from $L^{2n/(n+1),1}_\rho L^2_\theta$ to L^2 with norm

$$\left\| w^{1/2} (-\Delta - z)^{-1} \right\|_{L^{2n/(n+1),1}_\rho L^2_\theta \to L^2} \leq K|z|^{-1/2} \|w\|_{L^{n,1}_\rho L^\infty_\theta} ,$$

from which we finally get

$$\begin{aligned} \|(-\Delta - z)^{-1} u\|_{L^{2n/(n-1),\infty}_\rho L^2_\theta} &= \sup_{0 \neq w_1 \in L^{n,1}_\rho L^\infty_\theta} \frac{\left\| w_1^{1/2} (-\Delta - z)^{-1} u \right\|_{L^2}}{\left\| w_1^{1/2} \right\|_{L^{2n,2}_\rho L^\infty_\theta}} \\ &\leq K|z|^{-1/2} \|u\|_{L^{2n/(n+1),1}_\rho L^2_\theta} . \end{aligned}$$

From Barcelo, Ruiz and Vega [BRV97] we have that

$$\left\| w_1^{1/2} (-\Delta - z)^{-1} w_2^{1/2} u \right\|_{L^2} \leq K|z|^{-1/2} \|w_1\|^{1/2}_{\mathcal{MT}} \|w_2\|^{1/2}_{\mathcal{MT}} \|u\|_{L^2} \tag{3.3.9}$$

which implies (3.3.8). Indeed, we can replace the $\mathcal{MT}$ norm with the $L^{n,1}_\rho L^\infty_\theta$ norm since, as proved in equation (4.2) of [FS17b], the embedding

$$L^{n,1}_\rho L^\infty_\theta \hookrightarrow \mathcal{MT}$$

holds true (cf. Theorem 4.4 in [FS17b]). To be precise, equation (3.3.9) is proved in [BRV97] for $w_1 = w_2 \in \mathcal{MT}$, but the possibility of choosing two different weights follows easily from their proof (see Proposition 2 of the same paper).

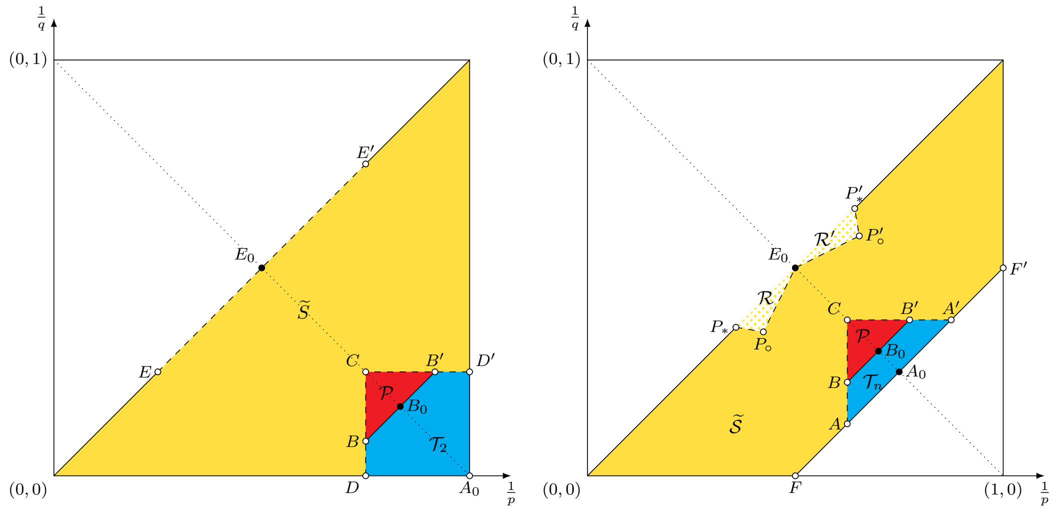
Consider now the estimate for $\nabla(-\Delta - z)^{-1}$ on the endpoint C. From Theorem 2 in [BRV97] we have that

$$\|v\|_{L^2} \leq K \|w\|_{\mathcal{MT}} \left\| w^{1/2} \nabla (-\Delta - z) w^{-1/2} v \right\|_{L^2} \tag{3.3.10}$$

for $z \geq 0$. Supposing this inequality true for any complex number z, we can then obtain estimate (3.3.7) following the same argument as above. The fact that (3.3.10) is true everywhere on the complex plane is implicit in the proof given by Barcelo, Ruiz and Vega. Indeed, the proof of Theorem 2 at pages 373–374 of [BRV97] is still valid for any real z. Then, the argument based on the Phragmén-Lindelöf principle exploited at page 373 to prove Theorem 1.(b) can be adapted also to this situation, proving (3.3.10) for any $z \in \mathbb{C}$.

Finally, for radial functions the radial-angular norms (3.1.4) from [FS17b] reduce simply to the Lebesgue and Lorentz norms. Real interpolation between the estimates on the open segment (C, B_0) and the ones on the open segment (B, B') coming from Lemma 3.1 prove the assertion on $\mathcal{P}$ for radial functions. □

Thus ends our recap on the Schrödinger resolvent estimates. The results in Lemmata 3.1, 3.2 and 3.3 are visually summarized in Figure 3.4. We conclude this section with a direct corollary of Lemma 3.3 concerning the free Dirac resolvent.



(a) Case $n = 2$. **(b)** Case $n \geq 3$.

Figure 3.4: In this picture we visualize the many regions and endpoints appearing in Section 3.3. The Kenig-Ruiz-Sogge-Gutiérrez region $\mathcal{T}_n$ from Lemma 3.1 is highlighted in blue, while in red we show the triangle $\mathcal{P}$ from Lemma 3.3 about the estimates for radial functions. Finally, the yellow region $\widetilde{\mathcal{S}}$ is such that $\mathcal{S} \setminus \widetilde{\mathcal{R}} = \widetilde{\mathcal{S}} \cup \mathcal{P} \cup \mathcal{T}_n$, where $\mathcal{S}$ is the Kwon-Lee region interested by Lemma 3.2 and $\widetilde{\mathcal{R}} = \mathcal{R} \cup \mathcal{R}'$ is pictured dotted.

Corollary 3.1. *Let $n \geq 2$, $m \geq 0$ and $z \in \mathbb{C} \setminus \{(-\infty, -m] \cup [m, +\infty)\}$. There exists a constant $K > 0$ independent on z such that*

$$\left\|(\mathcal{D}_m - z)^{-1}\right\|_{L^{2n/(n+1),1}_\rho L^2_\theta \to L^{2n/(n-1),\infty}_\rho L^2_\theta} \leq K \left[1 + \left|\frac{z+m}{z-m}\right|^{\frac{\operatorname{sgn} \Re z}{2}}\right]$$

and in particular, if $u \in L^{\frac{2n}{n+1},1}(\mathbb{R}^n)$ is a radial function, then

$$\left\|(\mathcal{D}_m - z)^{-1} u\right\|_{L^{2n/(n-1),\infty}} \leq K \left[1 + \left|\frac{z+m}{z-m}\right|^{\frac{\operatorname{sgn} \Re z}{2}}\right] \|u\|_{L^{2n/(n+1),1}} .$$

Proof. By the identity (3.2.1) and the estimates (3.3.6)–(3.3.7), it is immediate to get

$$\begin{aligned}
\left\|(\mathcal{D}_m - z)^{-1} u\right\|_{L^{\frac{2n}{n-1},\infty}} &\leq \left\|\sum_{k=1}^{n} \alpha_k \partial_k (-\Delta + m^2 - z^2)^{-1} u\right\|_{L^{\frac{2n}{n-1},\infty}} \\
&\quad + \left\|(m\alpha_{n+1} + z I_N)(-\Delta + m^2 - z^2)^{-1} u\right\|_{L^{\frac{2n}{n-1},\infty}} \\
&\leq \sqrt{n} \left\|\nabla(-\Delta + m^2 - z^2)^{-1} u\right\|_{L^{\frac{2n}{n-1},\infty}} \\
&\quad + \max\{|z+m|, |z-m|\} \left\|(-\Delta + m^2 - z^2)^{-1} u\right\|_{L^{\frac{2n}{n-1},\infty}} \\
&\leq K \left[1 + \left|\frac{z+m}{z-m}\right|^{\frac{\operatorname{sgn} \Re z}{2}}\right] \|u\|_{L^{2n/(n+1),1}}
\end{aligned}$$

and hence the claimed inequalities. □

Let us combine now the estimates above with the Birman-Schwinger principle to get our claimed results.

3.4 Proof for the theorems

Before to put our hands on the computations for the theorems, we need to bring to mind the abstract technicalities of the Birman-Schwinger principle exploited in Section 2.3, where we also properly defined an operator perturbed by a factorizable potential. In our case, $\mathfrak{H} = \mathfrak{H}' = L^2(\mathbb{R}^n; \mathbb{C}^{N\times N})$ and $\mathcal{V} = \mathcal{B}^*\mathcal{A}$ is a multiplication operator in $\mathfrak{H}$, with initial domain $\operatorname{dom}(\mathcal{V}) = C_0^\infty(\mathbb{R}^n; \mathbb{C}^N)$, generated by a matrix-valued function $\mathcal{V}\colon \mathbb{R}^n \to \mathbb{C}^{N\times N}$ (again, with the customary abuse of notation, we use the same symbol to denote the matrix and the operator). Same thing holds for the operators $\mathcal{A}$ and $\mathcal{B}^*$, which is not restrictive to consider closed. In this way, Assumption I is verified by the Closed Graph Theorem. By the argument exploited in Subsection 2.3.1, since $\operatorname{dom}(\mathcal{B}^*) = C_0^\infty(\mathbb{R}^n; \mathbb{C}^N)$, then $K_z = \overline{\mathcal{A}(H_0 - z)^{-1}\mathcal{B}^*}$. As usual, we reduced to study just $\left\|\mathcal{A}(H_0 - z)^{-1}\mathcal{B}^*\right\|_{\mathfrak{H}\to\mathfrak{H}}$.

Recall the identity (3.2.2). Exploiting the Rigidity Assumptions and setting for simplicity $k^2 \equiv k^2(z) := z^2 - m^2$, (3.2.2) becomes

$$\mathcal{A}(\mathcal{D}_m - z)^{-1}\mathcal{B}^* = (mA\alpha_{n+1}B^* + zAB^*)aR_0(k^2)\bar{b}.$$

In particular, assume that RA(ι) hold, for fixed $\iota \in \{i, ii, iii, iv\}$. Then, since $|A| = |B| = 1$, we get

$$\|A(m\alpha_{n+1} + zI_N)B^*\|_{L^\infty} \le \varkappa$$

where

$$\varkappa \equiv \varkappa(z) := \begin{cases} |k(z)| \left|\dfrac{z+m}{z-m}\right|^{\frac{\operatorname{sgn}\Re z}{2}} & \text{if } \iota = i \text{ and } m > 0, \\ m & \text{if } \iota = ii \text{ and } m > 0, \\ |z| & \text{if } \iota = iii, \text{ or } \iota = i \text{ and } m = 0, \\ 0 & \text{if } \iota = iv, \text{ or } \iota = ii \text{ and } m = 0. \end{cases}$$

By Hölder's inequality,

$$\left\|\mathcal{A}(\mathcal{D}_m - z)^{-1}\mathcal{B}^*\phi\right\|_{L^2} \le \varkappa \|a\|_{L^{\frac{2q}{q-2}}} \|b\|_{L^{\frac{2p}{2-p}}} \left\|R_0(k^2)\right\|_{L^p\to L^q} \|\phi\|_{L^2}$$

and so, recalling that $|a| = |b| = |v|^{1/2}$, setting $q = p'$ and $1/r = 1/p - 1/q$, we get

$$\left\|\mathcal{A}(\mathcal{D}_m - z)^{-1}\mathcal{B}^*\phi\right\|_{L^2} \le \varkappa \|v\|_{L^r} \left\|R_0(k^2)\right\|_{L^p\to L^q} \|\phi\|_{L^2}.$$

Similarly one infers also

$$\left\|\mathcal{A}(\mathcal{D}_m - z)^{-1}\mathcal{B}^*\right\|_{L^2\to L^2} \le \varkappa \|v\|_{L^r_\rho L^\infty_\theta} \left\|R_0(k^2)\right\|_{L^p_\rho L^2_\theta \to L^q_\rho L^2_\theta}$$
$$\left\|\mathcal{A}(\mathcal{D}_m - z)^{-1}\mathcal{B}^*\right\|_{L^2\to L^2} \le \varkappa \|v\|_{L^{r,1}_\rho L^\infty_\theta} \left\|R_0(k^2)\right\|_{L^{p,1}_\rho L^2_\theta \to L^{q,\infty}_\rho L^2_\theta}.$$

From Lemmata 3.1, 3.2 and 3.3 on the conjugate line (hence on the segments $[A_0, B_0]$, $(B_0, C]$ and $(C, E_0]$ respectively), if $n = 1$ we get

$$\left\|\mathcal{A}(\mathcal{D}_m - z)^{-1}\mathcal{B}^*\right\|_{L^2\to L^2} \le \frac{\varkappa}{2}|k|^{-1} \|v\|_{L^1} \tag{3.4.1}$$

whereas, if $n \geq 2$ and $r > 1$, we have

$$\left\|\mathcal{A}(\mathcal{D}_m - z)^{-1}\mathcal{B}^*\right\|_{L^2\to L^2} \lesssim \varkappa|k|^{-2+\frac{n}{r}} \begin{cases} \|v\|_{L^r} & \text{if } r \in \left[\frac{n}{2}, \frac{n+1}{2}\right], \\ \|v\|_{L^r_\rho L^\infty_\theta} & \text{if } r \in \left(\frac{n+1}{2}, n\right), \\ \|v\|_{L^{n,1}_\rho L^\infty_\theta} & \text{if } r = n, \end{cases} \tag{3.4.2}$$

$$\left\|\mathcal{A}(\mathcal{D}_m - z)^{-1}\mathcal{B}^*\right\|_{L^2\to L^2} \lesssim \frac{\varkappa|k|^{-\frac{1}{r}}}{\operatorname{dist}(k^2, [0,\infty))^{1-\frac{n+1}{2r}}} \|v\|_{L^r} \quad \text{if } r \in \left(\frac{n+1}{2}, \infty\right]. \tag{3.4.3}$$

In short, we have found inequalities of the type

$$\left\|\mathcal{A}(\mathcal{D}_m - z)^{-1}\mathcal{B}^*\right\|_{L^2\to L^2} \leq C\kappa(z)\,\|v\|$$

for a suitable norm $\|\cdot\|$ of v, a positive constant C independent on z and where the function κ is either

$$\kappa(z) = \varkappa(z)|k(z)|^{-2+\frac{n}{r}} \quad \text{or} \quad \kappa(z) = \varkappa(z)\frac{|k(z)|^{-1/r}}{\operatorname{dist}(z^2 - m^2, [0,+\infty))^{1-\frac{n+1}{2r}}}.$$

Applying the Birman-Schwinger principle and proving our results is now straightforward and easy, maybe just a bit dazzling due to the fauna of cases. According to the hypothesis assumed in the statements of each of our theorems, observe that the region $\mathcal{S}$ described by

$$\mathcal{S} = \{z \in \mathbb{C} \colon 1 \leq C\kappa(z)\,\|v\|\}$$

in any case covers all the region $\rho(\mathcal{D}_m) = \mathbb{C} \setminus \{(-\infty,-m] \cup [m,+\infty)\}$. Ergo we can always fix a complex number $z_0 \in \rho(\mathcal{D}_m)$ outside $\mathcal{S}$ satisfying $CK(z_0)\,\|v\| < 1$, namely Assumption II$'$ is verified (e.g. one can take $z_0 = iy_0$, for $y_0 \in \mathbb{R}$ sufficiently large). By Theorem 2.7 we can deduce that the point spectrum of the perturbed operator $\mathcal{D}_{m,\mathcal{V}}$ is confined in $\mathcal{S}$. If in particular $\kappa(z)$ is a nonnegative constant smaller than 1 (even 0, in which case the Birman-Schwinger operator is identically zero), we can exploit Theorem 2.8 obtaining that $\sigma(\mathcal{D}_{m,\mathcal{V}}) = \sigma_c(\mathcal{D}_{m,\mathcal{V}}) = \sigma_c(\mathcal{D}_m) = (-\infty,-m] \cup [m,+\infty)$ and in particular $\sigma_p(\mathcal{D}_{m,\mathcal{V}}) = \varnothing$.

In the case of RA(ii) with $m > 0$, we have $\varkappa \equiv 1$ and it is immediate, from the Birman-Schwinger principle and all the above estimates for $\mathcal{A}(\mathcal{D}_m - z)^{-1}\mathcal{B}^*$, to conclude the proofs for Theorems 3.1, 3.2 and 3.3. When we consider RA(ii) with $m = 0$ or instead RA(iv), then $\varkappa \equiv 0$ and hence the Birman-Schwinger operator is identically zero, implying the stability of the spectrum stated in Theorem 3.4.

Now consider the case of RA(i) and $m > 0$. Therefore $\varkappa(z) = |k(z)| \left|\frac{z+m}{z-m}\right|^{\operatorname{sgn}\Re z/2}$ and hence $\kappa(z)$ is either of the form

$$\kappa(z) = |k(z)|^{-1+\frac{n}{r}} \left|\frac{z+m}{z-m}\right|^{\operatorname{sgn}\Re z/2} \tag{3.4.4}$$

or of the form

$$\kappa(z) = \frac{|k(z)|^{1-1/r}}{\operatorname{dist}(z^2 - m^2, [0,+\infty))^{1-\frac{n+1}{2r}}} \left|\frac{z+m}{z-m}\right|^{\operatorname{sgn}\Re z/2} \tag{3.4.5}$$

We are interested in localizing the eigenvalues in compact regions, or at least in neighborhood $\mathcal{N}$ of the continuous spectrum of $\mathcal{D}_m$ such that $\mathcal{N}\cap\{z\in\mathbb{C}\colon \Re z = x_0\}$ is compact for any fixed $x_0\in\mathbb{R}$. At this aim one should ask that $\kappa(z)$ is uniformly bounded as $|z|\to\infty$ in the first case, and that $K(x_0+\Im z)$ is uniformly bounded as $|\Im z|\to\infty$ in the second case. It is easy to check that if $\kappa(z)$ is like in (3.4.4), then

$$\kappa(z)\sim |z|^{-1+n/r} \quad \text{as } |z|\to\infty$$

whereas if $\kappa(z)$ is like in (3.4.5), then

$$\kappa(\Re z + i\Im z)\sim |\Im z|^{-1+\frac{n}{r}} \quad \text{as } |\Im z|\to\infty$$

for fixed $\Re z\in\mathbb{R}$. In both cases, we should ask $r\geq n$ to get an interesting (in the sense specified above) localization for the eigenvalues. The same argument holds in the case of RA(i) and $m=0$, or in the case of RA(iii), namely when $\varkappa(z)=|z|$. For this reason, to get Theorems 3.5–3.8 we only employ the estimates (3.4.1), (3.4.2) for $r=n$ and (3.4.3) for $\gamma := r-n/2\geq n/2$.

In particular, Theorem 3.7 when $\varkappa(z)=|k(z)|\left|\frac{z+m}{z-m}\right|^{\operatorname{sgn}\Re z/2}$ is implied by (3.4.1) and (3.4.2) for $r=n$, taking in account that $\frac{C_0}{\|v\|}\leq\left|\frac{z+m}{z-m}\right|^{\operatorname{sgn}\Re z/2}$ is equivalent to

$$\left(|\Re z| - m\frac{C_0^4+\|v\|^4}{C_0^4-\|v\|^4}\right)^2 + (\Im z)^2 \leq \left(m\frac{2C_0^2\|v\|^2}{C_0^4-\|v\|^4}\right)^2$$

if $\|v\|<C_0$. In the same case, (3.4.3) implies Theorem 3.8. When instead $\varkappa(z)=|z|$ and $m=0$, noting that $\varkappa|k|^{-1}\equiv 1$, thanks to (3.4.1) and (3.4.2) for $r=n$ we can prove the massless cases in Theorems 3.5 and 3.7. The last inequalities are used to prove also Theorem 3.5, in the case of RA(iii) and $m>0$. Finally, Theorem 3.6 is proved exploiting (3.4.3) in the case $\varkappa=|z|$. We conclude noting that in Theorems 3.6 and 3.8, when $\gamma=n/2$, the additional hypothesis $\|v\|^{\gamma+n/2}_{L^{\gamma+n/2}}<C_0$ is necessary, since in this case $K(x_0+i\Im z)\sim 1$ as $|\Im z|\to\infty$ for fixed $x_0\in\mathbb{R}$. Hence, if the norm of the potential is not small enough, the condition $\mathcal{N}\cap\{z\in\mathbb{C}\colon \Re z=x_0\}$ compact would not be satisfied.

Last but not least, we sketch the proof of Theorem 3.9, which is not so different from that of Theorem 3.7. Here we need to use the usual polar decomposition $\mathcal{V}=\mathcal{U}\mathcal{W}=\mathcal{B}^*\mathcal{A}$ with $\mathcal{A}=\sqrt{\mathcal{W}}$ and $\mathcal{B}=\sqrt{\mathcal{W}}\mathcal{U}^*$. Employing Corollary 3.1, by Hölder's inequality we immediately obtain

$$\left\|\mathcal{A}(\mathcal{D}_m - z)^{-1}\mathcal{B}^*\phi\right\|_{L^2}\leq K\left\|\mathcal{V}\right\|_{L^{n,1}_\rho L^\infty_\theta}\left[1+\left|\frac{z+m}{z-m}\right|^{\frac{\operatorname{sgn}\Re z}{2}}\right]\|\phi\|_{L^2}.$$

Assumptions I and II are verified as above, and note that in the massive case the inequality

$$1\leq K\left\|\mathcal{V}\right\|_{L^{n,1}_\rho L^\infty_\theta}\left[1+\left|\frac{z+m}{z-m}\right|^{\frac{\operatorname{sgn}\Re z}{2}}\right]$$

describes the two disks in the statement of Theorem 3.9, letting $C_0=\frac{1}{2K}$. Another application of the Birman-Schwinger principle concludes the proof.

3.5 Game of matrices

The present section is fully dedicated to computations with matrices in order to exhibit some explicit examples of potentials $\mathcal{V} = vV$ such that the matricial part V can be factorized in the product of two matrices B^* and A satisfying the various assumptions stated in Section 3.2.

We will prove that in some low dimensions it is not possible to find the required potential, more precisely they do not exists in dimension $n = 2, 4$ in the case of RA(ii) and RA(iii), and in dimension $n = 1, 2$ in the case of RA(iv). In all the other case, we will exhibit at least a couple of examples. There is no intent here to be exhaustive in finding the suitable matrices, but rather we want to suggest an idea to build them. At this aim we firstly need to show an explicit representation for the Dirac matrices, and then we need to introduce some special "brick" matrices.

3.5.1 The Dirac matrices

First of all, as anticipated in Remark 3.2, let us explicitly define the Dirac matrices we are going to employ in our calculations, or better, one of their possible representations. At this aim we rely on the recursive construction performed by Kalf and Yamada in the Appendix of [KY01].

Let us introduce the Pauli matrices

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Moreover, let us define for two matrices $A = (a_{ij}) \in \mathbb{C}^{r_1 \times c_1}$ and $B = (b_{ij}) \in \mathbb{C}^{r_2 \times c_2}$, with $r_1, c_1, r_2, c_2 \in \mathbb{N}$, the Kronecker product

$$A \otimes B := \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{pmatrix} \in \mathbb{C}^{r_1 r_2 \times c_1 c_2}.$$

Recall that the Kronecker product satisfies, among others, the associative property and the mixed-product property, viz.

$$A_1 \otimes (A_2 \otimes A_3) = (A_1 \otimes A_2) \otimes A_3 = A_1 \otimes A_2 \otimes A_3$$
$$(A_1 \otimes B_1)(A_2 \otimes B_2) = (A_1 A_2) \otimes (B_1 B_2).$$

The Dirac matrices in low dimensions can be chosen to be the Pauli matrices, namely for $n = 1$ we set

$$\alpha_1^{(1)} := \sigma_1, \quad \alpha_2^{(1)} := \sigma_3,$$

and for $n = 2$

$$\alpha_1^{(2)} := \sigma_1, \quad \alpha_2^{(2)} := \sigma_2, \quad \alpha_3^{(2)} := \sigma_3.$$

The apex (n) stands for the dimension; we will omit it when there is no possibility of confusion. Let us start the recursion, after recalling that we defined $N := 2^{\lceil n/2 \rceil}$:

(i) if $n \geq 3$ is odd, we use the matrices $\alpha_1^{(n-1)}, \dots, \alpha_{n+1}^{(n-1)}$ known from the dimension $n-1$ to construct

$$\alpha_k^{(n)} := \sigma_1 \otimes \alpha_k^{(n-1)}, \qquad \alpha_{n+1}^{(n)} := \sigma_3 \otimes I_{N/2}$$

for $k \in \{1, \dots, n\}$;

(ii) if the dimension $n \geq 4$ is even, we define

$$\alpha_1^{(n)} := \sigma_1 \otimes I_{N/2}, \qquad \alpha_{k+1}^{(n)} := \sigma_2 \otimes \alpha_k^{(n-2)}, \qquad \alpha_{n+1}^{(n)} := \sigma_3 \otimes I_{N/2}$$

for $k \in \{1, \dots, n-1\}$.

In any dimension $n \geq 1$, the Dirac matrices $\alpha_1, \dots, \alpha_{n+1}$ just defined are Hermitian, satisfy (1.1.2) and have the structure

$$\alpha_k = \begin{pmatrix} 0 & \beta_k \\ \beta_k^* & 0 \end{pmatrix}, \quad \alpha_{n+1} = \begin{pmatrix} I_{N/2} & 0 \\ 0 & -I_{N/2} \end{pmatrix}$$

for $k \in \{1, \dots, n\}$, where the matrices $\beta_k \in \mathbb{C}^{N/2 \times N/2}$ satisfy

$$\beta_k \beta_j^* + \beta_j \beta_k^* = 2\delta_k^j I_{N/2}$$

and are Hermitian if n is odd.

Remark 3.5. Not only in dimension $n = 1, 2$, but also in dimension $n = 3$, the above representation for the Dirac matrices coincides with the classical one:

$$\alpha_1^{(3)} = \sigma_1 \otimes \sigma_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad \alpha_2^{(3)} = \sigma_1 \otimes \sigma_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix},$$
$$\alpha_3^{(3)} = \sigma_1 \otimes \sigma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \alpha_4^{(3)} = \sigma_3 \otimes I_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Remark 3.6. If $\{\alpha_1, \dots, \alpha_{n+1}\}$ and $\{\widetilde{\alpha}_1, \dots, \widetilde{\alpha}_{n+1}\}$ are a pair of sets of Dirac matrices, then there exists a unitary matrix $U \in \mathbb{C}^{N \times N}$ such that $\widetilde{\alpha}_k = U\alpha_k U^{-1}$ or $\widetilde{\alpha}_k = -U\alpha_k U^{-1}$, for $k \in \{1, \dots, n+1\}$. If n is odd we always fall in the first case; if n is even and we are in the second case, set $\widetilde{U} = U \prod_{k=1}^n \alpha_k$, then

$$\widetilde{\mathcal{D}}_m = -i \sum_{j=1}^n \widetilde{\alpha}_k \partial_k + m\widetilde{\alpha}_{n+1} = \widetilde{U} \left[-i \sum_{j=1}^n \alpha_k \partial_k - m\alpha_{n+1} \right] \widetilde{U}^{-1} = \widetilde{U} \mathcal{D}_{-m} \widetilde{U}^{-1}.$$

Therefore, considering the perturbed operator $\widetilde{\mathcal{D}}_{m,\widetilde{\mathcal{V}}}$, in odd dimension it is unitarily equivalent to $\mathcal{D}_{m,\mathcal{V}}$ with $\widetilde{\mathcal{V}} = \widetilde{U}\mathcal{V}\widetilde{U}^{-1}$, whereas in even dimension it is unitarily equivalent to either $\mathcal{D}_{m,\mathcal{V}}$ or $\mathcal{D}_{-m,\mathcal{V}}$.

In our case, noting that all the results in Section 3.2 are symmetric respect to the imaginary axis (namely they are not effected replacing m with $-m$ in the definition of the Dirac operator), it becomes evident that the choice of a particular representation for the Dirac matrices is not restrictive at all.

The above recursive definition for the matrices may appear too much implicit, but we can go further exploding the representation. Let us define the "Kronecker exponentiation"

$$\begin{aligned} M^{\otimes 0} &= 1 \\ M^{\otimes k} &= \underbrace{M \otimes \cdots \otimes M}_{k \text{ times}} \end{aligned}$$

for any complex matrix M and for any $k \in \mathbb{N}$, imposing the natural identification between $1 \in \mathbb{C}$ and the matrix $(1) \in \mathbb{C}^{1\times 1}$. Therefore, one can explicitly write the Dirac matrices in even dimension $n \geq 2$ as

$$\alpha_k = \begin{cases} \sigma_2^{\otimes k-1} \otimes \sigma_1 \otimes I_2^{\otimes n/2-k} & \text{for } k \in \left\{1, \dots, \frac{n}{2}\right\} \\ \sigma_2^{\otimes n/2} & \text{for } k = \frac{n}{2} + 1 \\ \sigma_2^{\otimes n+1-k} \otimes \sigma_3 \otimes I_2^{\otimes k-n/2-2} & \text{for } k \in \left\{\frac{n}{2} + 2, \dots, n+1\right\} \end{cases} \tag{3.5.1}$$

and in odd dimension $n \geq 3$ as

$$\alpha_k = \begin{cases} \sigma_1 \otimes \sigma_2^{\otimes k-1} \otimes \sigma_1 \otimes I_2^{\otimes (n-1)/2-k} & \text{for } k \in \left\{1, \dots, \frac{n-1}{2}\right\} \\ \sigma_1 \otimes \sigma_2^{\otimes (n-1)/2} & \text{for } k = \frac{n-1}{2} + 1 \\ \sigma_1 \otimes \sigma_2^{\otimes n-k} \otimes \sigma_3 \otimes I_2^{\otimes k-(n-1)/2-2} & \text{for } k \in \left\{\frac{n-1}{2} + 2, \dots, n\right\} \\ \sigma_3 \otimes I_2^{\otimes (n-1)/2} & \text{for } k = n+1. \end{cases}$$

The odd dimensional case follow easily from the recursive definition and from the explicit definition (3.5.1) of the Dirac matrices in the even dimensional case; the latter can be easily verified by induction, and we omit the proof.

For later use, we collect in the following lemma a recursive formula which connects the Dirac matrices associated to two different dimensions.

Lemma 3.4. *Let $n, m \in \mathbb{N}$ such that $2 \leq m \leq n$ and $n - m$ is even. Thus the following identity hold:*

$$\alpha_k^{(n)} = \begin{cases} \alpha_k^{(m)} \otimes I_2^{\otimes (n-m)/2} & \text{for } k \in \left\{1, \dots, \left\lfloor \frac{m}{2} \right\rfloor\right\} \\ \alpha_{\lfloor m/2 \rfloor + 1}^{(m)} \otimes \alpha_{k-\lfloor m/2 \rfloor}^{(n-m)} & \text{for } k \in \left\{\left\lfloor \frac{m}{2} \right\rfloor + 1, \dots, n-m+\left\lfloor \frac{m}{2} \right\rfloor + 1\right\} \\ \alpha_{k-(n-m)}^{(m)} \otimes I_2^{\otimes (n-m)/2} & \text{for } k \in \left\{n-m+\left\lfloor \frac{m}{2} \right\rfloor + 2, \dots, n+1\right\} \end{cases}$$

where $\lfloor \cdot \rfloor$ is the floor function.

Proof. If n, m are both even, we want to prove

$$\alpha_k^{(n)} = \begin{cases} \alpha_k^{(m)} \otimes I_2^{\otimes (n-m)/2} & \text{for } k \in \left\{1, \dots, \frac{m}{2}\right\} \\ \alpha_{m/2+1}^{(m)} \otimes \alpha_{k-m/2}^{(n-m)} & \text{for } k \in \left\{\frac{m}{2} + 1, \dots, n - \frac{m}{2} + 1\right\} \\ \alpha_{k-(n-m)}^{(m)} \otimes I_2^{\otimes (n-m)/2} & \text{for } k \in \left\{n - \frac{m}{2} + 2, \dots, n+1\right\}. \end{cases} \tag{3.5.2}$$

But, from (3.5.1) and setting for simplicity $j := k - m/2$, $h := k - (n-m)$ for any $k \in \mathbb{N}$, we immediately have

$$\alpha_k^{(n)} = \begin{cases} \sigma_2^{\otimes k-1} \otimes \sigma_1 \otimes I_2^{\otimes \frac{m}{2}-k} \otimes I_2^{\otimes \frac{n-m}{2}} & \text{for } k \in \left\{1, \dots, \frac{m}{2}\right\} \\ \sigma_2^{\otimes \frac{m}{2}} \otimes \sigma_2^{j-1} \otimes \sigma_1 \otimes I_2^{\otimes \frac{n-m}{2}-j} & \text{for } k \in \left\{\frac{m}{2}+1, \dots, \frac{n}{2}\right\} \\ \sigma_2^{\otimes \frac{m}{2}} \otimes \sigma_2^{\otimes \frac{n-m}{2}} & \text{for } k = \frac{n}{2}+1 \\ \sigma_2^{\otimes \frac{m}{2}} \otimes \sigma_2^{n-m+1-j} \otimes \sigma_3 \otimes I_2^{\otimes j - \frac{n-m}{2} - 2} & \text{for } k \in \left\{\frac{n}{2}+2, \dots, n - \frac{m}{2}+1\right\} \\ \sigma_2^{\otimes m+1-h} \otimes \sigma_3 \otimes I_2^{\otimes h - \frac{m}{2} - 2} \otimes I_2^{\otimes \frac{n-m}{2}} & \text{for } k \in \left\{n - \frac{m}{2}+2, \dots, n+1\right\} \end{cases}$$

from which our assertion is evident.

If n, m are both odd, exploiting (3.5.2) it follows that

$$\begin{aligned} \alpha_k^{(n)} &= \begin{cases} \sigma_1 \otimes \alpha_k^{(n-1)} & \text{for } k \in \{1, \dots, n\} \\ \sigma_3 \otimes I_2^{\otimes n/2-1} & \text{for } k = n+1 \end{cases} \\ &= \begin{cases} \sigma_1 \otimes \alpha_k^{(m-1)} \otimes I_2^{\otimes (n-m)/2} & \text{for } k \in \left\{1, \dots, \frac{m-1}{2}\right\} \\ \sigma_1 \otimes \alpha_{(m-1)/2+1}^{(m-1)} \otimes \alpha_{k-(m-1)/2}^{(n-m)} & \text{for } k \in \left\{\frac{m-1}{2}+1, \dots, n - \frac{m-1}{2}\right\} \\ \sigma_1 \otimes \alpha_{k-(n-m)}^{(m-1)} \otimes I_2^{\otimes (n-m)/2} & \text{for } k \in \left\{n - \frac{m-1}{2}+1, \dots, n\right\} \\ \sigma_3 \otimes I_2^{\otimes n/2-1} & \text{for } k = n+1 \end{cases} \\ &= \begin{cases} \alpha_k^{(m)} \otimes I_2^{\otimes (n-m)/2} & \text{for } k \in \left\{1, \dots, \frac{m-1}{2}\right\} \\ \alpha_{(m-1)/2+1}^{(m)} \otimes \alpha_{k-(m-1)/2}^{(n-m)} & \text{for } k \in \left\{\frac{m-1}{2}+1, \dots, n - \frac{m-1}{2}\right\} \\ \alpha_{k-(n-m)}^{(m)} \otimes I_2^{\otimes (n-m)/2} & \text{for } k \in \left\{n - \frac{m-1}{2}+1, \dots, n+1\right\} \end{cases} \end{aligned}$$

which concludes the proof. □

To conclude this subsection on the Dirac matrices, it seems interesting to us noting the following relation about their product, even if we are not going to exploit it.

Lemma 3.5. *We have that*

$$\widetilde{\alpha} := (-i)^{\lfloor \frac{n}{2} \rfloor} \prod_{k=1}^{n+1} \alpha_k = \begin{cases} -i\sigma_2 \otimes I_{N/2} & \text{if } n \text{ is odd,} \\ I_N & \text{if } n \text{ is even,} \end{cases} \tag{3.5.3}$$

and in particular

$$\widetilde{\alpha}^2 = (-1)^n I_N, \qquad \widetilde{\alpha}^* = (-1)^n \widetilde{\alpha}, \qquad \alpha_k \widetilde{\alpha} = (-1)^n \widetilde{\alpha} \alpha_k$$

for $k \in \{1, \dots, n+1\}$.

Proof. The three properties follows obviously from the anticommutation relations (1.1.2), so we need just to prove the second equality in (3.5.3). Suppose firstly that n is even. Then the identity follows by inductive argument. If $n = 2$, it is directly verified that

$$-i\alpha_1^{(2)}\alpha_2^{(2)}\alpha_3^{(2)} = -i\sigma_1\sigma_2\sigma_3 = I_2.$$

Fix now $n \geq 4$ even and suppose that

$$(-i)^{n/2-1}\prod_{k=1}^{n-1}\alpha_k^{(n-2)} = I_{N/2}.$$

Exploiting the definitions of the Dirac matrices and the mixed-product property of the Kronecker product, we get

$$\begin{aligned}
(-i)^{n/2}\prod_{k=1}^{n+1}\alpha_k^{(n)} &= (-i)^{n/2}\left[\sigma_1 \otimes I_{N/2}\right]\left[\prod_{k=1}^{n-1}\sigma_2 \otimes \alpha_k^{(n-2)}\right]\left[\sigma_3 \otimes I_{N/2}\right] \\
&= (-i)^{n/2}\left[\sigma_1 \otimes I_{N/2}\right]\left[\sigma_2^{n-1} \otimes \prod_{k=1}^{n-1}\alpha_k^{(n-2)}\right]\left[\sigma_3 \otimes I_{N/2}\right] \\
&= -i\left[\sigma_1 \otimes I_{N/2}\right]\left[\sigma_2 \otimes I_{N/2}\right]\left[\sigma_3 \otimes I_{N/2}\right] \\
&= -i\sigma_1\sigma_2\sigma_3 \otimes I_{N/2} \\
&= I_N
\end{aligned}$$

Finally, let $n \geq 1$ be odd. If $n = 1$, then it is trivially checked that $\alpha_1^{(1)}\alpha_2^{(1)} = \sigma_1\sigma_3 = -i\sigma_2$. If $n \geq 3$, then

$$(-i)^{\frac{n-1}{2}}\prod_{k=1}^{n}\alpha_k^{(n)} = (-i)^{\frac{n-1}{2}}\prod_{k=1}^{n}\sigma_1 \otimes \alpha_k^{(n-1)} = (-i)^{\frac{n-1}{2}}\sigma_1^n \otimes \prod_{k=1}^{n}\alpha_k^{(n-1)} = \sigma_1 \otimes I_{N/2}$$

and hence

$$(-i)^{\frac{n-1}{2}}\prod_{k=1}^{n+1}\alpha_k^{(n)} = [\sigma_1 \otimes I_{N/2}][\sigma_3 \otimes I_{N/2}] = -i\sigma_2 \otimes I_{N/2}$$

concluding the proof of the identity. □

3.5.2 The brick matrices

Before to proceed with the construction of the examples for the potentials, we need to point our attention on some peculiar 2×2 matrices. We want to find $\rho^k, \tau^k \in \mathbb{C}^{2\times 2}$ satisfying the conditions

$$\begin{gathered}
\rho^k\sigma_1(\tau^k)^* = 0 = \rho^k\sigma_k(\tau^k)^* \\
\rho^k\sigma_h(\tau^k)^* \neq 0 \neq (\tau^k)^*\rho^k
\end{gathered}$$

for fixed $k \in \{0, 2, 3\}$ and any $h \in \{0, 2, 3\} \setminus \{k\}$, where we define for simplicity $\sigma_0 := I_2$. Moreover, let us ask also $|\rho^k| = |\tau^k| = 1$, where $|\cdot|$ is the matricial 2-norm, a.k.a. the spectral norm. It is quite simple to find a couple of such matrices for any $k \in \{0, 2, 3\}$, properly combining the Pauli matrices.

In the case $k=0$, we can consider

$$\rho^0_+ = \frac{\sigma_2 + i\sigma_3}{2} = \tau^0_-, \quad \tau^0_+ = \frac{I_2 + \sigma_1}{2} = \rho^0_-,$$

from which, using the anticommutation relations (1.1.2), it easy to check

$$\begin{gathered} \rho^0_\pm \sigma_0 (\tau^0_\pm)^* = 0 = \rho^0_\pm \sigma_1 (\tau^0_\pm)^* \\ \rho^0_\pm \sigma_2 (\tau^0_\pm)^* = \frac{I_2 + \sigma_1}{2} = \mp i \rho^0_\pm \sigma_3 (\tau^0_\pm)^* \\ (\tau^0_\pm)^* \rho^0_\pm = \frac{\sigma_2 \pm i\sigma_3}{2}. \end{gathered}$$

In the case $k=2$, we can set

$$\rho^2_\pm = \frac{\sigma_1 \mp i\sigma_2}{2} = \tau^2_\pm,$$

and thus

$$\begin{gathered} \rho^2_\pm \sigma_1 (\tau^2_\pm)^* = 0 = \rho^2_\pm \sigma_2 (\tau^2_\pm)^* \\ \rho^2_\pm \sigma_0 (\tau^2_\pm)^* = \frac{I_2 \mp \sigma_3}{2} = \pm \rho^2_\pm \sigma_3 (\tau^2_\pm)^* \\ (\tau^2_\pm)^* \rho^2_\pm = \frac{I_2 \pm \sigma_3}{2}. \end{gathered}$$

Finally, in the case $k=3$, we can consider

$$\rho^3_\pm = \frac{I_2 \pm \sigma_2}{2} = \tau^3_\pm$$

and hence

$$\begin{gathered} \rho^3_\pm \sigma_1 (\tau^3_\pm)^* = 0 = \rho^3_\pm \sigma_3 (\tau^3_\pm)^* \\ \rho^3_\pm \sigma_0 (\tau^3_\pm)^* = \frac{I_2 \pm \sigma_2}{2} = \pm \rho^3_\pm \sigma_2 (\tau^3_\pm)^* \\ (\tau^3_\pm)^* \rho^3_\pm = \frac{I_2 \pm \sigma_2}{2}. \end{gathered}$$

The couple of matrices we found for each of the three cases are not the only solutions satisfying the required set of conditions, but for our purposes are enough.

Now, we want to find matrices $A, B \in \mathbb{C}^{\mathbb{N}\times N}$ such that

$$\begin{gathered} A\alpha_k B^* = 0 \\ V = B^* A \neq 0 \end{gathered}$$

for $k \in \{1, \dots, n\}$. In addition, we will also impose, or not, that AB^* and $A\alpha_{n+1}B^*$ are null matrices.

3.5.3 The odd-dimensional case

Let us start with the 1-dimensional case, for which we have basically already found the admissible matricial part for the potentials, thanks to the brick matrices found in the previous subsection.

In fact, to satisfy RA(i) we need to find $V = B^*A \neq 0$ such that $A\sigma_1 B^* = 0$, $AB^* \neq 0$, $A\sigma_3 B^* \neq 0$, thus we can choose $A_\pm = \rho^2_\pm$, $B_\pm = \tau^2_\pm$, obtaining the couple of examples

$$V_\pm = \frac{1}{2}\begin{pmatrix} 1\pm 1 & 0 \\ 0 & 1\mp 1 \end{pmatrix} = \left[\frac{1}{2}\begin{pmatrix} 0 & 1\mp 1 \\ 1\pm 1 & 0 \end{pmatrix}\right]^* \left[\frac{1}{2}\begin{pmatrix} 0 & 1\mp 1 \\ 1\pm 1 & 0 \end{pmatrix}\right] = B^*_\pm A_\pm.$$

Similarly we can proceed for the case of RA(ii) and RA(iii), in which case we use $\rho^0_\pm, \tau^0_\pm$ and $\rho^3_\pm, \tau^3_\pm$ respectively, viz. for RA(ii) we have the couple of examples

$$V_\pm = \frac{i}{2}\begin{pmatrix} \pm 1 & -1 \\ 1 & \mp 1 \end{pmatrix} = \left[\frac{1}{2}\begin{pmatrix} 1 & \pm 1 \\ 1 & \pm 1 \end{pmatrix}\right]^* \left[\frac{i}{2}\begin{pmatrix} 1 & \mp 1 \\ 1 & \mp 1 \end{pmatrix}\right] = B^*_\pm A_\pm,$$

while for RA(iii) we have the couple of examples

$$V_\pm = \frac{1}{2}\begin{pmatrix} 1 & \mp i \\ \pm i & 1 \end{pmatrix} = \left[\frac{1}{2}\begin{pmatrix} 1 & \mp i \\ \pm i & 1 \end{pmatrix}\right]^* \left[\frac{i}{2}\begin{pmatrix} 1 & \mp i \\ \pm i & 1 \end{pmatrix}\right] = B^*_\pm A_\pm.$$

This examples can be easily generalized in any odd dimension, taking in account the following lemma.

Lemma 3.6. *If $V^{(n-2)} = [B^{(n-2)}]^* A^{(n-2)}$ is an admissible matrix in dimension $n-2$, then an admissible matrix in dimension n is given by*

$$V^{(n)} := V^{(n-2)} \otimes I_2 = \left[B^{(n-2)} \otimes M^{-1}\right]^* \left[A^{(n-2)} \otimes M\right] =: [B^{(n)}]^* A^{(n)}$$

for any invertible matrix $M \in \mathbb{C}^{2\times 2}$.

This assertion is a trivial consequence of Lemma 3.4. Thus, in any odd dimension $n \geq 1$, couples of examples satisfying RA(i), RA(ii) and RA(iii) are given respectively by

$$V_\pm = \frac{1}{2}\begin{pmatrix} I_{\frac{N}{2}} \pm I_{\frac{N}{2}} & 0 \\ 0 & I_{\frac{N}{2}} \mp I_{\frac{N}{2}} \end{pmatrix}, \quad V_\pm = \frac{i}{2}\begin{pmatrix} \pm I_{\frac{N}{2}} & -I_{\frac{N}{2}} \\ I_{\frac{N}{2}} & \mp I_{\frac{N}{2}} \end{pmatrix}, \quad V_\pm = \frac{1}{2}\begin{pmatrix} I_{\frac{N}{2}} & \mp i I_{\frac{N}{2}} \\ \pm i I_{\frac{N}{2}} & I_{\frac{N}{2}} \end{pmatrix}.$$

Let us turn now our attention to the case of RA(iv), for which there are no examples of potentials in dimension $n = 1$. Indeed, let us fix $A, B \in \mathbb{C}^{2\times 2}$ and let us denote with $a = (a_1, a_2)$ and $b = (b_1, b_2)$ their respective first rows. Since we are imposing

$$A\sigma_1 B^* = A\sigma_3 B^* = AB^* = 0,$$

in particular we obtain that

$$a_2\overline{b_1} + a_1\overline{b_2} = a_1\overline{b_1} - a_2\overline{b_2} = a_1\overline{b_1} + a_2\overline{b_2} = 0,$$

from which we deduce that if $a \neq 0$, then $b = 0$, and vice versa if $b \neq 0$, then $a = 0$. Therefore, one can easily be convinced that there are no solutions such that both A and B are non-trivial.

Let us consider then the 3-dimensional case. By the definition of the Dirac matrices, we would like to find matrices A, B such that $B^*A \neq 0$ and

$$A(\sigma_1 \otimes \sigma_1)B^* = A(\sigma_1 \otimes \sigma_2)B^* = A(\sigma_1 \otimes \sigma_3)B^* = A(\sigma_3 \otimes I_2)B^* = A(I_2 \otimes I_2)B^* = 0.$$

Anyway, from the properties of our brick matrices and by the mixed-product property of the Kronecker product, it is readily seen that we can choose $A = \rho^k_\pm \otimes \rho^0_\pm$ and $B = \tau^k_\pm \otimes \tau^0_\pm$ for any $k \in \{0, 2, 3\}$. In fact

$$(\rho^k_\pm \otimes \rho^0_\pm)(\sigma_1 \otimes \sigma_j)(\tau^k_\pm \otimes \tau^0_\pm)^* = \rho^k_\pm \sigma_1 (\tau^k_\pm)^* \otimes \rho^0_\pm \sigma_j (\tau^0_\pm)^* = 0 \otimes \rho^0_\pm \sigma_j (\tau^0_\pm)^* = 0$$

for $j \in \{1, 2, 3\}$, and

$$(\rho^k_\pm \otimes \rho^0_\pm)(\sigma_h \otimes I_2)(\tau^k_\pm \otimes \tau^0_\pm)^* = \rho^k_\pm \sigma_h (\tau^k_\pm)^* \otimes \rho^0_\pm I_2 (\tau^0_\pm)^* = \rho^k_\pm \sigma_h (\tau^k_\pm)^* \otimes 0 = 0$$

for $h \in \{0, 3\}$. Essentially, we use the fact that the first tensorial factors appearing in the definitions of A and B kill σ_1, while the second tensorial factors kill I_2. At this point, as above we can extend the 3-dimensional case to any odd dimension $n \geq 5$.

Exempli gratia, letting $k = 0$, we have that a couple of examples of matricial part of potentials satisfying RA(iv) for odd dimension $n \geq 3$ are given by

$$\begin{aligned}
V_\pm &= \frac{1}{4}\begin{pmatrix} -1 & \pm1 & \pm1 & -1 \\ \mp1 & 1 & 1 & \mp1 \\ \mp1 & 1 & 1 & \mp1 \\ -1 & \pm1 & \pm1 & -1 \end{pmatrix} \otimes I_{N/4} \\
&= -\frac{1}{4}\begin{pmatrix} \pm1 & -1 \\ 1 & \mp1 \end{pmatrix}^{\otimes 2} \otimes I_{N/4} \\
&= \left[\frac{1}{4}\begin{pmatrix} 1 & \pm1 \\ 1 & \pm1 \end{pmatrix}^{\otimes 2} \otimes I_{N/4}\right]^* \left[-\frac{1}{4}\begin{pmatrix} 1 & \mp1 \\ 1 & \mp1 \end{pmatrix}^{\otimes 2} \otimes I_{N/4}\right] \\
&= \left[\frac{1}{4}\begin{pmatrix} 1 & \pm1 & \pm1 & 1 \\ 1 & \pm1 & \pm1 & 1 \\ 1 & \pm1 & \pm1 & 1 \\ 1 & \pm1 & \pm1 & 1 \end{pmatrix} \otimes I_{N/4}\right]^* \left[\frac{1}{4}\begin{pmatrix} -1 & \pm1 & \pm1 & -1 \\ -1 & \pm1 & \pm1 & -1 \\ -1 & \pm1 & \pm1 & -1 \\ -1 & \pm1 & \pm1 & -1 \end{pmatrix} \otimes I_{N/4}\right] \\
&= B^*_\pm A_\pm.
\end{aligned}$$

3.5.4 The even-dimensional case

We will consider the situation case by case for RA(i)–RA(iv).

3.5.4.1 Case of RA(i)

Between the four cases, this is the only one for which we can find examples of our desired potentials in any dimension. Indeed, let us start from $n = 2$, for which a couple of examples can be found immediately exploiting our brick matrices, setting $A = \rho^2_\pm$ and $B = \tau^2_\pm$. Therefore, making use of Lemma 3.6, a couple of examples for the matricial part of the potentials satisfying RA(i) for any even dimension $n \geq 2$ is given by

$$\begin{aligned} V_\pm &= \frac{1}{2}\begin{pmatrix} I_{N/2} \pm I_{N/2} & 0 \\ 0 & I_{N/2} \mp I_{N/2} \end{pmatrix} \\ &= \left[\frac{1}{2}\begin{pmatrix} 0 & I_{N/2} \mp I_{N/2} \\ I_{N/2} \pm I_{N/2} & 0 \end{pmatrix}\right]^* \left[\frac{1}{2}\begin{pmatrix} 0 & I_{N/2} \mp I_{N/2} \\ I_{N/2} \pm I_{N/2} & 0 \end{pmatrix}\right] \\ &= B_\pm^* A_\pm. \end{aligned}$$

3.5.4.2 Case of RA(ii)

We can find potentials only for $n \geq 6$. Indeed, in dimension $n = 2$, the situation is similar to the case of RA(iv) for $n = 1$. We are searching matrices $A, B \in \mathbb{C}^{2\times 2}$ such that $V = B^*A \neq 0$, $A\sigma_3 B^* \neq 0$ and

$$A\sigma_1 B^* = A\sigma_2 B^* = AB^* = 0.$$

Denoting with $a = (a_1, a_2)$ and $b = (b_1, b_2)$ the first rows of A and B respectively, from the previous condition we infer

$$a_2\overline{b_1} + a_1\overline{b_2} = a_2\overline{b_1} - a_1\overline{b_2} = a_1\overline{b_1} + a_2\overline{b_2} = 0$$

and therefore $a = 0$ if $b \neq 0$ and on the contrary $b = 0$ if $a \neq 0$. Thus, there are no solutions such that $A \neq 0$ and $B \neq 0$.

Analogously, we can repeat the argument for $n = 4$. In this case the Dirac matrices are

$$\begin{gathered} \alpha_1^{(4)} = \sigma_1 \otimes I_2, \quad \alpha_2^{(4)} = \sigma_2 \otimes \sigma_1, \quad \alpha_3^{(4)} = \sigma_2 \otimes \sigma_2, \\ \alpha_4^{(4)} = \sigma_2 \otimes \sigma_3, \quad \alpha_5^{(4)} = \sigma_3 \otimes I_2. \end{gathered} \tag{3.5.4}$$

We impose

$$\begin{gathered} A\alpha_j^{(4)} B^* = AB^* = 0 \\ A\alpha_5^{(4)} B^* \neq 0 \end{gathered} \tag{3.5.5}$$

for $j \in \{1, 2, 3, 4\}$. Let us denote with $a = (a_1, \dots, a_4)$ and $b = (b_1, \dots, b_4)$ the first rows of A and B respectively. Hence from the conditions (3.5.5) we infer

$$\begin{aligned} a_3\overline{b_1} + a_4\overline{b_2} + a_1\overline{b_3} + a_2\overline{b_4} &= 0 \\ -a_4\overline{b_1} - a_3\overline{b_2} + a_2\overline{b_3} + a_1\overline{b_4} &= 0 \\ a_4\overline{b_1} - a_3\overline{b_2} - a_2\overline{b_3} + a_1\overline{b_4} &= 0 \\ -a_3\overline{b_1} + a_4\overline{b_2} + a_1\overline{b_3} - a_2\overline{b_4} &= 0 \\ a_1\overline{b_1} + a_2\overline{b_2} + a_3\overline{b_3} + a_4\overline{b_4} &= 0 \\ a_1\overline{b_1} + a_2\overline{b_2} - a_3\overline{b_3} - a_4\overline{b_4} &\neq 0 \end{aligned}$$

and equivalently

$$\begin{gathered} a_3\overline{b_1} + a_2\overline{b_4} = a_3\overline{b_2} - a_1\overline{b_4} = a_4\overline{b_1} - a_2\overline{b_3} = a_4\overline{b_2} + a_1\overline{b_3} = 0 \\ a_1\overline{b_1} + a_2\overline{b_2} = -a_3\overline{b_3} - a_4\overline{b_4} \neq 0. \end{gathered}$$

However, this system is impossible. Suppose indeed that $a_1 \neq 0$. Then

$$\overline{b_3} = -\frac{a_4}{a_1}\overline{b_2}, \qquad \overline{b_4} = \frac{a_3}{a_1}\overline{b_2}$$
$$a_4\left(\overline{b_1} + \frac{a_2}{a_1}\overline{b_2}\right) = a_3\left(\overline{b_1} + \frac{a_2}{a_1}\overline{b_2}\right) = 0$$
$$a_1\overline{b_1} + a_2\overline{b_2} = -a_3\overline{b_3} - a_4\overline{b_4} \neq 0.$$

From the first two lines one infers that or $a_1\overline{b_1} + a_2\overline{b_2} = 0$, or $a_3 = a_4 = b_3 = b_4 = 0$. Both the possibilities are incompatible with the last condition. Similarly one can prove that the system is impossible also when $a_1 = 0$.

Now, let us look at the dimension $n = 6$. Here we can build examples by the aid of our brick matrices, but it is not so straightforward as in the odd-dimensional case, and we need to be sneaky. Firstly, recall that

$$\alpha_1^{(6)} = \sigma_1 \otimes I_4, \quad \alpha_{k+1}^{(6)} = \sigma_2 \otimes \alpha_k^{(4)}, \quad \alpha_7^{(6)} = \sigma_3 \otimes I_4$$

for $k \in \{1, \dots, 5\}$. We search matrices $A, B \in \mathbb{C}^{8\times 8}$ such that

$$AB^* = A\alpha_k^{(6)}B^* = 0$$
$$B^*A \neq 0 \neq A\alpha_7^{(6)}B^*$$

for $k \in \{1, \dots, 6\}$. Let us start with the ansatz that A and B have the following structure:

$$A = \frac{1}{2}\begin{pmatrix} \widetilde{A} & -\widetilde{A}(\sigma_1 \otimes \sigma_1) \\ \widetilde{A} & -\widetilde{A}(\sigma_1 \otimes \sigma_1) \end{pmatrix}, \qquad B = \frac{1}{2}\begin{pmatrix} \widetilde{B} & \widetilde{B}(\sigma_1 \otimes \sigma_1) \\ \widetilde{B} & \widetilde{B}(\sigma_1 \otimes \sigma_1) \end{pmatrix}$$

with $\widetilde{A}, \widetilde{B} \in \mathbb{C}^{4\times 4}$. In this way, recalling the definition of the Dirac matrices and observing that $(\sigma_1 \otimes \sigma_1)^2 = I_4$, the conditions $AB^* = 0$ and $A\alpha_1^{(6)}B^* = A(\sigma_1 \otimes I_4)B^* = 0$ are immediately verified and the other ones become

$$A\alpha_{k+1}^{(6)}B^* = -\frac{i}{4}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \otimes \widetilde{A}[(\sigma_1 \otimes \sigma_1)\alpha_k^{(4)} + \alpha_k^{(4)}(\sigma_1 \otimes \sigma_1)]\widetilde{B}^* = 0 \tag{3.5.6}$$

for $k \in \{1, \dots, 5\}$, and

$$A\alpha_7 B^* = A(\sigma_3 \otimes I_4)B^* = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \otimes \widetilde{A}\widetilde{B}^* \neq 0$$
$$B^*A = \frac{1}{2}\begin{pmatrix} \widetilde{B}^*\widetilde{A} & -\widetilde{B}^*\widetilde{A}(\sigma_1 \otimes \sigma_1) \\ (\sigma_1 \otimes \sigma_1)\widetilde{B}^*\widetilde{A} & -(\sigma_1 \otimes \sigma_1)\widetilde{B}^*\widetilde{A}(\sigma_1 \otimes \sigma_1) \end{pmatrix} \neq 0.$$

In (3.5.6), exploiting the definition of the Dirac matrices in dimension $n = 4$, the anticommutation relations (1.1.2) and the identities $\sigma_1\sigma_2 = i\sigma_3$ and $\sigma_1\sigma_3 = -i\sigma_2$, we get that also the identities

$$A\alpha_3^{(6)}B^* = A\alpha_6^{(6)}B^* = 0$$

are immediately satisfied, and the remaining ones reduce to

$$A\alpha_2^{(6)}B^* = -\frac{i}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\otimes \widetilde{A}(I_2\otimes\sigma_1)\widetilde{B}^* = 0$$
$$A\alpha_4^{(6)}B^* = \frac{i}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\otimes \widetilde{A}(\sigma_3\otimes\sigma_3)\widetilde{B}^* = 0$$
$$A\alpha_5^{(6)}B^* = -\frac{i}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\otimes \widetilde{A}(\sigma_3\otimes\sigma_2)\widetilde{B}^* = 0.$$

Thus, it would be enough to find $\widetilde{A},\widetilde{B}\in\mathbb{C}^{4\times 4}$ such that

$$\widetilde{A}(I_2\otimes\sigma_1)\widetilde{B}^* = \widetilde{A}(\sigma_3\otimes\sigma_3)\widetilde{B}^* = \widetilde{A}(\sigma_3\otimes\sigma_2)\widetilde{B}^* = 0$$
$$\widetilde{A}\widetilde{B}^*\neq 0\neq \widetilde{B}^*\widetilde{A}.$$

This step is easily achieved exploiting our brick matrices, indeed we can choose

$$\widetilde{A}_\pm = \rho_\pm^3\otimes\rho_\pm^k,\qquad \widetilde{B}_\pm = \tau_\pm^3\otimes\tau_\pm^k$$

for any fixed $k\in\{0,2,3\}$. In this way we can construct many examples for the 6-dimensional case. If we choose e.g. $k=3$ in the above definition of $\widetilde{A}$ and $\widetilde{B}$, and taking again in account Lemma 3.6, we can exhibit the following couple of examples for matrices satisfying RA(ii) for even dimension $n\geq 6$:

$$\begin{aligned}
V_\pm &= \frac{1}{8}\begin{pmatrix}(I_2\pm\sigma_2)^{\otimes 2} & -(\sigma_1\mp i\sigma_3)^{\otimes 2}\\ (\sigma_1\pm i\sigma_3)^{\otimes 2} & -(I_2\mp\sigma_2)^{\otimes 2}\end{pmatrix}\otimes I_{N/8}\\
&= \frac{1}{8}\begin{pmatrix}
1 & \mp i & \mp i & -1 & 1 & \pm i & \pm i & -1\\
\pm i & 1 & 1 & \mp i & \pm i & -1 & -1 & \mp i\\
\pm i & 1 & 1 & \mp i & \pm i & -1 & -1 & \mp i\\
-1 & \pm i & \pm i & 1 & -1 & \mp i & \mp i & 1\\
-1 & \pm i & \pm i & 1 & -1 & \mp i & \mp i & 1\\
\pm i & 1 & 1 & \mp i & \pm i & -1 & -1 & \mp i\\
\pm i & 1 & 1 & \mp i & \pm i & -1 & -1 & \mp i\\
1 & \mp i & \mp i & -1 & 1 & \pm i & \pm i & -1
\end{pmatrix}\otimes I_{N/8}\\
&= B_\pm^* A_\pm
\end{aligned}$$

where

$$A_\pm = \frac{1}{8}\begin{pmatrix}(I_2\pm\sigma_2)^{\otimes 2} & -(\sigma_1\mp i\sigma_3)^{\otimes 2}\\ (I_2\pm\sigma_2)^{\otimes 2} & -(\sigma_1\mp i\sigma_3)^{\otimes 2}\end{pmatrix}\otimes I_{N/8},$$
$$B_\pm = \frac{1}{8}\begin{pmatrix}(I_2\pm\sigma_2)^{\otimes 2} & (\sigma_1\mp i\sigma_3)^{\otimes 2}\\ (I_2\pm\sigma_2)^{\otimes 2} & (\sigma_1\mp i\sigma_3)^{\otimes 2}\end{pmatrix}\otimes I_{N/8}.$$

3.5.4.3 Case of RA(iii).

Mutatis mutandis, the situation is similar to the the case of RA(ii), hence we skip the computations. As above, one can prove the absence of our desired potentials in dimension $n=2$ and $n=4$. In even dimension $n\geq 6$ instead, we impose to A and B to have the structure

$$A = \frac{1}{2}\begin{pmatrix}\widetilde{A} & \widetilde{A}(\sigma_1\otimes\sigma_1)\\ \widetilde{A} & \widetilde{A}(\sigma_1\otimes\sigma_1)\end{pmatrix}\otimes I_{N/8},\qquad B = \frac{1}{2}\begin{pmatrix}\widetilde{B} & \widetilde{B}(\sigma_1\otimes\sigma_1)\\ \widetilde{B} & \widetilde{B}(\sigma_1\otimes\sigma_1)\end{pmatrix}\otimes I_{N/8},$$

where $\widetilde{A}, \widetilde{B} \in \mathbb{C}^{4\times 4}$ have to satisfy the relations

$$\widetilde{A}(\sigma_1 \otimes \sigma_1)\widetilde{B}^* = \widetilde{A}(\sigma_3 \otimes I_2)\widetilde{B}^* = \widetilde{A}(\sigma_2 \otimes \sigma_1)\widetilde{B}^* = 0$$
$$\widetilde{A}\widetilde{B}^* \neq 0 \neq \widetilde{B}^*\widetilde{A}.$$

For example we can choose again

$$\widetilde{A}_\pm = \rho^3_\pm \otimes \rho^3_\pm, \quad \widetilde{B}_\pm = \tau^3_\pm \otimes \tau^3_\pm,$$

and hence we obtain the following couple of examples of matrices satisfying RA(iii) in even dimension $n \geq 6$:

$$\begin{aligned}
V_\pm &= \frac{1}{8}\begin{pmatrix} (I_2 \pm \sigma_2)^{\otimes 2} & (\sigma_1 \mp i\sigma_3)^{\otimes 2} \\ (\sigma_1 \pm i\sigma_3)^{\otimes 2} & (I_2 \mp \sigma_2)^{\otimes 2} \end{pmatrix} \otimes I_{N/8} \\
&= \frac{1}{8}\begin{pmatrix}
1 & \mp i & \mp i & -1 & -1 & \mp i & \mp i & 1 \\
\pm i & 1 & 1 & \mp i & \mp i & 1 & 1 & \pm i \\
\pm i & 1 & 1 & \mp i & \mp i & 1 & 1 & \pm i \\
-1 & \pm i & \pm i & 1 & 1 & \pm i & \pm i & -1 \\
-1 & \pm i & \pm i & 1 & 1 & \pm i & \pm i & -1 \\
\pm i & 1 & 1 & \mp i & \mp i & 1 & 1 & \pm i \\
\pm i & 1 & 1 & \mp i & \mp i & 1 & 1 & \pm i \\
1 & \mp i & \mp i & -1 & -1 & \mp i & \mp i & 1
\end{pmatrix} \otimes I_{N/8} \\
&= B^*_\pm A_\pm
\end{aligned}$$

where

$$A_\pm = \frac{1}{8}\begin{pmatrix} (I_2 \pm \sigma_2)^{\otimes 2} & (\sigma_1 \mp i\sigma_3)^{\otimes 2} \\ (I_2 \pm \sigma_2)^{\otimes 2} & (\sigma_1 \mp i\sigma_3)^{\otimes 2} \end{pmatrix} \otimes I_{N/8} = B_\pm.$$

3.5.4.4 Case of RA(iv)

In dimension $n = 2$ there are no potentials, and this can be easily seen as in the above case of RA(ii). In even dimension $n \geq 4$ instead, recalling the definition of the Dirac matrices in 4-dimensions (3.5.4) and Lemma 3.6, it is easy to check that a couple of examples for our desired matrices is obtained choosing $V_\pm = B^*_\pm A_\pm$ with

$$A_\pm = \rho^2_\pm \otimes \rho^0_\pm \otimes I_{N/4}, \qquad B_\pm = \tau^2_\pm \otimes \tau^0_\pm \otimes I_{N/4},$$

hence videlicet

$$V_\pm = \frac{i}{4}\begin{pmatrix}
1 \pm 1 & -1 \mp 1 & 0 & 0 \\
1 \pm 1 & -1 \mp 1 & 0 & 0 \\
0 & 0 & -1 \pm 1 & -1 \pm 1 \\
0 & 0 & 1 \mp 1 & 1 \mp 1
\end{pmatrix} \otimes I_{N/4}.$$

Thus concludes the parade of examples for the matricial parts V of the potentials satisfying our Rigidity Assumptions (i)–(iv), and also the first part of this thesis.

Part II

Blow-up phenomena for wave-like models

Fun years for me, for a guy who used to like to blow up things.
We had lots of explosions, lots of blowups.

John Kobak, engineer at the NASA Propulsion Systems Laboratory

Chapter 4

Heat-like and wave-like lifespan estimates for solutions of semilinear damped wave equations via a Kato-type lemma

The aim of the present chapter is to study blow-up phenomena and lifespan estimates for solutions of Cauchy problems with small data for several semilinear damped wave models, especially the semilinear wave equations with power-nonlinearity and scale-invariant damping and mass terms. In particular, we are interested in exploring the competition between so-called "heat-like" and "wave-like" behavior of the solutions, which concerns not only critical exponents, but also lifespan estimates, in a way that we will clarify later.

This chapter contains the results proved in [S4], joint work with Ning-An Lai and Hiroyuki Takamura.

4.1 Preamble

The problem we mainly concern about is

$$\begin{cases} \Box u + \dfrac{\mu_1}{1+t} u_t + \dfrac{\mu_2}{(1+t)^2} u = |u|^p, & \text{in } \mathbb{R}^n \times (0,T), \\ u(x,0) = \varepsilon f(x), \quad u_t(x,0) = \varepsilon g(x), & x \in \mathbb{R}^n, \end{cases} \tag{4.1.1}$$

where $\Box := \partial_{tt} - \Delta$ is the d'Alembert operator, $\mu_1, \mu_2 \in \mathbb{R}$, $p > 1$, $n \in \mathbb{N}$, $T > 0$ and $\varepsilon > 0$ is a "small" parameter. First of all, let us denote the energy and weak solutions of our problem (4.1.1).

Definition 4.1. We say that u is an energy solution of (4.1.1) over $[0,T)$ if

$$u \in C([0,T), H^1(\mathbb{R}^n)) \cap C^1([0,T), L^2(\mathbb{R}^n)) \cap C((0,T), L^p_{\text{loc}}(\mathbb{R}^n))$$

satisfies $u(x,0)=\varepsilon f(x)$ in $H^1(\mathbb{R}^n)$, $u_t(x,0)=\varepsilon g(x)$ in $L^2(\mathbb{R}^n)$ and

$$\begin{aligned}
&\int_{\mathbb{R}^n} u_t(x,t)\phi(x,t)dx-\int_{\mathbb{R}^n}\varepsilon g(x)\phi(x,0)dx\\
&+\int_0^t ds\int_{\mathbb{R}^n}\{-u_t(x,s)\phi_t(x,s)+\nabla u(x,s)\cdot\nabla\phi(x,s)\}\,dx\\
&+\int_0^t ds\int_{\mathbb{R}^n}\frac{\mu_1}{1+s}u_t(x,s)\phi(x,s)dx+\int_0^t ds\int_{\mathbb{R}^n}\frac{\mu_2}{(1+s)^2}u(x,s)\phi(x,s)dx\\
=&\int_0^t ds\int_{\mathbb{R}^n}|u(x,s)|^p\phi(x,s)dx
\end{aligned}\tag{4.1.2}$$

for $t\in[0,T)$ and any test function $\phi\in C_0^\infty(\mathbb{R}^n\times[0,T))$.

Employing the integration by parts in the above equality and letting $t\to T$, we reach to the definition of the weak solution of (4.1.1), that is

$$\begin{aligned}
&\int_{\mathbb{R}^n\times[0,T)} u(x,s)\left\{\Box\phi(x,s)-\frac{\partial}{\partial s}\left(\frac{\mu_1}{1+s}\phi(x,s)\right)+\frac{\mu_2}{(1+s)^2}\phi(x,s)\right\}dxds\\
=&\,\varepsilon\int_{\mathbb{R}^n}\{\mu_1 f(x)\phi(x,0)+g(x)\phi(x,0)-f(x)\phi_t(x,0)\}\,dx\\
&+\int_{\mathbb{R}^n\times[0,T)}|u(x,s)|^p\phi(x,s)\,dxds.
\end{aligned}$$

We recall that the *critical exponent* p_{crit} of (4.1.1) is the smallest exponent greater than 1 such that, if $p>p_{\text{crit}}$, there exists a unique global-in-time energy solution to the problem, whereas if $1<p\le p_{\text{crit}}$ the solution blows up in finite time. In the latter case, one is also interested in finding estimates for the *lifespan* T_ε, which is the maximal existence time of the solution, depending on the parameter ε.

Our principal model is the one in (4.1.1), for which we obtain Theorem 4.2 and Theorem 4.4, according to the different conditions imposed on the initial data. As straightforward consequences, we also obtain Theorem 4.1 and Theorem 4.3 for the massless case, i.e. the model with $\mu_2=0$. The lifespan estimate in dimension $n=1$ in this case is improved, comparing to the known results. Moreover, we continue the study of semilinear wave equations with scattering damping, negative mass term and power nonlinearity, which we introduced together with Lai and Takamura in [LST19, LST20].

In the rest of the section, we compare the classical models for the heat and wave equations with power-nonlinearity in order to introduce the "heat-like" and "wave-like" terminology. In Section 4.2 we sketch the background of the problems under consideration and we exhibit our main results, which will be proved in Section 4.4, exploiting, as main tool, a Kato-type lemma in integral form presented in Section 4.3.

4.1.1 Heat versus wave

Let us consider the toy-models of the heat and wave equations, respectively given by

$$\begin{cases}u_t-\Delta u=|u|^p,\\ u(x,0)=\varepsilon f(x),\end{cases}\qquad\qquad \begin{cases}u_{tt}-\Delta u=|u|^p,\\ (u,u_t)(x,0)=\varepsilon(f,g)(x).\end{cases}$$

Nowadays the study of these two equations is almost classic: the well-known results include the lifespan estimates and the critical exponents, which are the so-called Fujita exponent $p_F(n)$ and the Strauss exponent $p_S(n)$, corresponding to the heat and the wave equation respectively. For our purposes, let us define these two exponents for all $\nu \in \mathbb{R}$:

$$p_F(\nu) := \begin{cases} 1 + \dfrac{2}{\nu} & \text{if } \nu > 0, \\ +\infty & \text{if } \nu \le 0, \end{cases} \qquad p_S(\nu) := \begin{cases} \dfrac{\nu + 1 + \sqrt{\nu^2 + 10\nu - 7}}{2(\nu - 1)} & \text{if } \nu > 1, \\ +\infty & \text{if } \nu \le 1. \end{cases}$$

We remark that

$$\begin{aligned} 1 < p < p_F(\nu) &\Longrightarrow \gamma_F(p, \nu) := 2 - \nu(p - 1) > 0, \\ 1 < p < p_S(\nu) &\Longrightarrow \gamma_S(p, \nu) := 2 + (\nu + 1)p - (\nu - 1)p^2 > 0. \end{aligned}$$

In particular, if $\nu > 0$, $p_F(\nu)$ is the solution of the linear equation $\gamma_F(p, \nu) = 0$, whereas if $\nu > 1$, $p_S(\nu)$ is the positive solution of the quadratic equation $\gamma_S(p, \nu) = 0$. Although the expression $\gamma_S(p, \nu)$ is well-known in the literature, the introduction of $\gamma_F(p, \nu)$ is justifyed from the fact that γ_F plays for the heat equation the same role that γ_S plays for the wave equation, as it emerges from the lifespan estimates.

Suppose for simplicity that f, g are non-negative, non-vanishing, compactly supported functions (for different conditions on the initial data, we can have different lifespan estimates, see Subsection 4.2.4). We have that the blow-up results are the ones collected in the following table.

Table 4.1: Heat versus wave blow-up results.

	Heat	Wave
Critical exponent p_{crit}	$p_F(n)$	$p_S(n)$
Subcritical lifespan T_ε for $1 < p < p_{\text{crit}}$	$\sim \varepsilon^{-2(p-1)/\gamma_F(p,n)}$	$\sim \varepsilon^{-(p-1)/\gamma_F(p,n-1)}$ if $n = 1$ or $n = 2$, $1 < p < 2$ $\sim a(\varepsilon)$ if $n = p = 2$, $\varepsilon^2 a^2 \log(1 + a) = 1$ $\sim \varepsilon^{-2p(p-1)/\gamma_S(p,n)}$ if $n = 2$, $2 < p < p_S(n)$ or $n \ge 3$
Critical lifespan T_ε for $p = p_{\text{crit}}$	$\sim \exp(C\varepsilon^{-(p-1)})$	$\sim \exp(C\varepsilon^{-p(p-1)})$ (in general, lower bound open for $n \ge 9$)

Here and in the following, we use the notation $F \lesssim G$ (respectively $F \gtrsim G$) if there exists a constant $C > 0$ independent of ε such that $F \le CG$ (respectively $F \ge CG$), and the notation $F \sim G$ if $F \lesssim G$ and $F \gtrsim G$.

For a more detailed story of these results, we refer to the book [ER18], the doctoral thesis [Wak14b], the introductions of [IKTW19, Tak15, TW11, TW14] and the references therein.

For the comparison between the heat and wave equations, let us introduce an informal but evocative notation to describe the behavior of the critical exponents and of the lifespan estimates in our models. We will call the critical exponent *heat-like* if it is related to the

Fujita exponent, i.e. $p_{\text{crit}} = p_F(\nu)$ for some $\nu \in \mathbb{R}$, whereas we will call it *wave-like* if it is related to the Strauss exponent, i.e. $p_{\text{crit}} = p_S(\nu)$ for some $\nu \in \mathbb{R}$.

Similarly, we will say that the lifespan estimate is *heat-like* if it is related in some way to the one of the heat equation, i.e. to the exponent $2(p-1)/\gamma_F(p,\nu)$ in the subcritical case and to $\exp(\varepsilon^{-(p-1)})$ in the critical one, whereas we will say it *wave-like* if related to the one of the wave equation, i.e. to the exponent $2p(p-1)/\gamma_S(p,\nu)$ in the subcritical case and to $\exp(\varepsilon^{-p(p-1)})$ in the critical one. However, we also define a *mixed-type* behavior when the lifespan estimate is related to $2p(p-1)/\gamma_F(p,\nu)$ in the subcritical case (as we will see in Theorem 4.3 and 4.4), to remark that the lifespan is longer than the heat-like one, due to the additional p in the exponent.

4.2 Problems and main results

This section is devoted to presenting the models under consideration and to stating our results. More precisely, we start to consider the damped wave equation by adding the damping term $\mu/(1+t)^\beta u_t$ to the wave equation, focusing then on the scale-invariant case, i.e. setting $\beta = 1$. Afterwards, we add also the mass term $\mu_2/(1+t)^2 u$. In Subsection 4.2.4, we observe that a special condition on the initial data can significantly change the blow-up results. Finally, in Subsection 4.2.5 we consider a special wave model with scattering damping and negative mass term, the study of which can be essentially reduced to that of the previous models.

4.2.1 Damped wave equation

Let us proceed by adding the damping term $\mu/(1+t)^\beta u_t$ to the wave equation, with $\mu \geq 0$ and $\beta \in \mathbb{R}$, hence we consider the Cauchy problem

$$\begin{cases} u_{tt} - \Delta u + \dfrac{\mu}{(1+t)^\beta} u_t = |u|^p, & \text{in } \mathbb{R}^n \times (0,T), \\ u(x,0) = \varepsilon f(x), \quad u_t(x,0) = \varepsilon g(x), & x \in \mathbb{R}^n. \end{cases} \tag{4.2.1}$$

According to the works by Wirth [Wir04, Wir06, Wir07], in the study of the associated homogeneous problem

$$\begin{cases} u^0_{tt} - \Delta u^0 + \dfrac{\mu}{(1+t)^\beta} u^0_t = 0, \\ u^0(x,0) = f(x), \quad u^0_t(x,0) = g(x), \end{cases} \tag{4.2.2}$$

we can classify the damping term into four cases, depending on the different values of β. When $\beta < 1$, the damping term is said to be *overdamping* and the solution does not decay to zero when $t \to \infty$. If $-1 \leq \beta < 1$, the solution behaves like that of the heat equation and we say that the damping term is *effective*. Hence, the term u^0_{tt} in (4.2.2) has no influence on the behavior of the solution and the $L^p - L^q$ decay estimates of the solution are almost the same as those of the heat equation. In contrast, when $\beta > 1$, it is known that the solution behaves like that of the wave equation, which means that the damping term in (4.2.2) has no influence on the behavior of the solution. In fact, in this case the solution scatters to that of the free wave equation when $t \to \infty$, and thus we say that we have *scattering*. Finally, when $\beta = 1$, the equation in (4.2.2) is invariant under the scaling

$$\widetilde{u^0}(x,t) := u^0(\lambda x, \lambda(1+t) - 1), \quad \lambda > 0,$$

and hence we say that the damping term is *scale-invariant*. In this case the behavior of the solution of (4.2.2) has been observed to be determined by the value of μ. We summarize all the classifications of the damping term in (4.2.2) in the next table.

Table 4.2: Classification of damped wave equations.

Range of β	Classification
$\beta \in (-\infty, -1)$	overdamping
$\beta \in [-1, 1)$	effective
$\beta = 1$	scale-invariant
$\beta \in (1, \infty)$	scattering

Let us return to problem (4.2.1), which inherits the above terminology and has very different behaviors from case to case. Indeed, in the overdamping case the solution exists globally for any $p > 1$. In the effective case, the problem is heat-like, both in the critical exponent and in the lifespan estimates, while in the scattering case the problem seems to be wave-like. Finally, the scale-invariant case has an intermediate behavior, and a competition between heat-like and wave-like arises. Before moving to the last case, let us collect in the following two tables some blow-up and global existence results for $\beta \neq 1$, at the best of our knowledge.

Table 4.3: Blow-up in finite time for $\beta \neq 1$.

Authors	Range of β	Exponent p	Lifespan T_ε
Fujiwara, Ikeda, Wakasugi [FIW19] Ikeda, Inui [II19]	$\beta = -1$	$1 < p < p_F(n)$ $p = p_F(n)$	$\sim \exp(C\varepsilon^{-\frac{2(p-1)}{\gamma_F(p,n)}})$ $\sim \exp\exp(C\varepsilon^{-(p-1)})$
Li, Zhou [LZ95] Zhang [Zha01] Todorova, Yordanov [TY01] Kirane, Qafsaoui [KQ02] Ikeda, Ogawa [IO16] Lai, Zhou [LZ] Ikeda, Wakasugi [IW15] Nishihara [Nis11] Fujiwara, Ikeda, Wakasugi [FIW19]	$\beta = 0$	$1 < p < p_F(n)$ $p = p_F(n)$	$\sim \varepsilon^{-\frac{2(p-1)}{\gamma_F(p,n)}}$ $\sim \exp(C\varepsilon^{-(p-1)})$
Fujiwara, Ikeda, Wakasugi [FIW19] Ikeda, Inui [II19] Ikeda, Ogawa [IO16] Ikeda, Wakasugi [IW15]	$-1 < \beta < 1$ $\beta \neq 0$	$1 < p < p_F(n)$ $p = p_F(n)$	$\sim \varepsilon^{-\frac{2(p-1)}{(1+\beta)\gamma_F(p,n)}}$ $\sim \exp(C\varepsilon^{-(p-1)})$
Lai, Takamura [LT18] Wakasa, Yordanov [WY19]	$\beta > 1$	$1 < p < p_S(n)$ $p = p_S(n)$	$\lesssim \varepsilon^{-\frac{2p(p-1)}{\gamma_S(p,n)}}$ $\lesssim \exp(C\varepsilon^{-p(p-1)})$

Table 4.4: Global-in-time existence for $\beta \neq 1$.

Authors	Range of β	Dimension n	Exponent p
Ikeda, Wakasugi [IW20]	$\beta < -1$	$n \geq 1$	$p > 1$
Wakasugi [Wak17]	$\beta = -1$	$n = 1, 2$ $n \geq 3$	$p > p_F(n)$ $p_F(n) < p < \frac{n}{n-2}$
Todorova, Yordanov [TY01]	$\beta = 0$	$n = 1, 2$ $n \geq 3$	$p > p_F(n)$ $p_F(n) < p \leq \frac{n}{n-2}$
D'Abbicco, Lucente, Reissig [DLR15] Nishihara [Nis11] Lin, Nishihara, Zhai [LNZ12]	$-1 < \beta < 1$ $\beta \neq 0$	$n = 1, 2$ $n \geq 3$	$p > p_F(n)$ $p_F(n) < p < \frac{n+2}{n-2}$
Liu, Wang [LW20]	$\beta > 1$	$n = 3, 4$	$p > p_S(n)$

4.2.2 Scale-invariant damped wave equation

We consider now (4.2.1) for $\beta = 1$, hence we consider the Cauchy problem

$$\begin{cases} u_{tt} - \Delta u + \dfrac{\mu}{1+t} u_t = |u|^p, & \text{in } \mathbb{R}^n \times (0,T), \\ u(x,0) = \varepsilon f(x), \quad u_t(x,0) = \varepsilon g(x), & x \in \mathbb{R}^n. \end{cases} \tag{4.2.3}$$

The scale-invariant problem has been studied intensively in the last years. This great interest is motivated by the fact that, differently from the damped wave equation with $\beta \neq 1$, in the scale-invariant case the results depend also on the damping coefficient μ, for determining both the critical exponent and the lifespan estimate. Hence, the situation is a bit more complicated, since the scale-invariant case shows results intermediate between the ones for the effective ($-1 \leq \beta < 1$) and non-effective ($\beta > 1$) damping cases, and then it seems to be a threshold between a heat-like and a wave-like behavior.

In the following two tables we collect, at the best of our knowledge, results concerning the existence and blow-up for the scale-invariant damping.

Table 4.5: Global-in-time existence for $\beta = 1$.

Authors	Dimension n	Coefficient μ	Exponent p
D'Abbicco [D'A15]	$n = 1$ $n = 2$ $n \geq 3$	$\mu \geq \frac{5}{3}$ $\mu \geq 3$ $\mu \geq n+2$	$p > p_F(1)$ $p > p_F(2)$ $p_F(n) < p \leq \frac{n}{n-2}$
D'Abbicco, Lucente, Reissig [DLR15] Kato, Sakuraba [KS19] Lai [Lai20]	$n = 2, 3$	$\mu = 2$	$p > p_S(n+2)$
D'Abbicco, Lucente [DL15]	$n \geq 5$ (odd dim., rad. symm.)	$\mu = 2$	$p_S(n+2) < p < \min\left\{2, \frac{n+1}{n-3}\right\}$
Palmieri [Pal19a]	$n \geq 4$ (even dim.)	$\mu = 2$	$p_S(n+2) < p < p_F(2)$

Table 4.6: Blow-up in finite time for $\beta = 1$.

Authors	Dim. n	Coefficient μ	Exponent p	Lifespan T_ε
Wakasugi [Wak14a, Wak14b]	$n \geq 1$	$\mu \geq 1$ $0 < \mu < 1$	$1 < p \leq p_F(n)$ $1 < p < 1 + \frac{2}{n+\mu-1}$	$\lesssim \varepsilon^{-(p-1)/\gamma_F(p,n)}$ $\lesssim \varepsilon^{-(p-1)/\gamma_F(p,n+\mu-1)}$
D'Abbicco, Lucente, Reissig [DLR15]	$n = 1$ $n = 2, 3$	$\mu = 2$	$1 < p \leq p_F(1)$ $1 < p \leq p_S(n+2)$	
Wakasa [Wak16] Kato, Takamura, Wakasa [KTW19]	$n = 1$	$\mu = 2$	$1 < p < p_F(1)$ $p = p_F(1)$	$\sim \varepsilon^{-(p-1)/\gamma_F(p,1)}$ $\sim \exp(C\varepsilon^{-(p-1)})$
Imai, Kato, Takamura, Wakasa [IKTW20]	$n = 2$	$\mu = 2$	$1 < p < p_F(2) = p_S(2)$ $p = p_F(2) = p_S(2)$	$\sim \varepsilon^{-(p-1)/\gamma_F(p,2)}$ $\sim \exp(C\varepsilon^{-1/2})$
Kato, Sakuraba [KS19]	$n = 3$	$\mu = 2$	$1 < p < p_S(5)$ $p = p_S(5)$	$\sim \varepsilon^{-2p(p-1)/\gamma_S(p,5)}$ $\sim \exp(C\varepsilon^{-p(p-1)})$
Lai, Takamura, Wakasa [LTW17]	$n \geq 2$	$0 < \mu < \frac{n^2+n+2}{2(n+2)}$	$p_F(n) \leq p < p_S(n+2\mu)$	$\lesssim \varepsilon^{-2p(p-1)/\gamma_S(p,n+2\mu)}$
Ikeda, Sobajima [IS18]	$n \geq 1$	$0 \leq \mu < \frac{n^2+n+2}{n+2}$ ($\mu \neq 0$ if $n = 1$)	$p_F(n) < p \leq p_S(n+\mu)$	$\lesssim \varepsilon^{-2p(p-1)/\gamma_S(p,n+\mu)-\delta}$ if $\begin{cases} n = 1, \frac{2}{3} \leq \mu < \frac{4}{3} \\ n = 1, 0 < \mu < \frac{2}{3}, p \geq \frac{2}{\mu} \\ n \geq 2, p > p_S(n+2+\mu) \end{cases}$ $\lesssim \varepsilon^{-\frac{2(p-1)}{\mu}-\delta}$ if $n = 1, 0 < \mu < \frac{2}{3}, p < \frac{2}{\mu}$ $\lesssim \varepsilon^{-1-\delta}$ if $n \geq 2, p < p_S(n+2+\mu)$ $\lesssim \exp(C\varepsilon^{-p(p-1)})$ if $p = p_S(n+\mu)$.
Tu, Lin [TL17, TL19]	$n \geq 2$	$\mu > 0$ $0 < \mu < \frac{n^2+n+2}{n+2}$	$1 < p < p_S(n+\mu)$ $p = p_S(n+\mu)$	$\lesssim \varepsilon^{-2p(p-1)/\gamma_S(p,n+\mu)}$ $\lesssim \exp(C\varepsilon^{-p(p-1)})$

Observe that the special case $\mu = 2$ was widely studied, starting from D'Abbicco, Lucente and Reissig [DLR15]. The reason is that, if we exploit the Liouville transform

$$v(x,t) := (1+t)^{\mu/2} u(x,t)$$

in problem (4.2.3), it turns out to be

$$\begin{cases} v_{tt} - \Delta v + \dfrac{\mu(2-\mu)}{4(1+t)^2} v = \dfrac{|v|^p}{(1+t)^{\mu(p-1)/2}}, & \text{in } \mathbb{R}^n \times (0,T), \\ v(x,0) = \varepsilon f(x), \quad v_t(x,0) = \varepsilon \left\{ \dfrac{\mu}{2} f(x) + g(x) \right\}, & x \in \mathbb{R}^n. \end{cases}$$

For $\mu = 2$ the damping term disappears, making the analysis more manageable and related to the undamped wave equation. From the works [DL15, DLR15, IS18, Pal19a, Wak14a] it is now clear that the critical exponent for $\mu = 2$ is $p_{\text{crit}} = \max\{p_F(n), p_S(n+2)\}$, with the lifespan estimates stated in low dimensions $n \leq 3$ by the works [IKTW20, KS19, KTW19, Wak16].

When $\mu \neq 2$, it was observed that for small μ the problem is wave-like in the critical exponent and in the lifespan estimates, whereas it is heat-like for larger μ. However, the exact threshold was still unclear. We conjecture, in accordance with Remarks 1.2 and 1.4 in [IS18], that the threshold value should be

$$\mu_* \equiv \mu_*(n) := \frac{n^2+n+2}{n+2},$$

and that the critical exponent should be

$$\begin{aligned} p_{\text{crit}} = p_\mu(n) &:= \max\{p_F(n-[\mu-1]_-), p_S(n+\mu)\} \\ &= \begin{cases} p_S(n+\mu) & \text{if } 0 \le \mu < \mu_*, \\ p_F(n) & \text{if } \mu \ge \mu_*. \end{cases} \end{aligned} \tag{4.2.4}$$

Here and in the following, $[x]_\pm = \frac{|x|\pm x}{2}$ indicates the positive and negative part functions respectively.

The blow-up part of this conjecture has already been proved, combining [Wak14a] and [IS18]. In our next theorem, which is a straightforward corollary of Theorem 4.2, we reconfirm the blow-up range and we give cleaner estimates for the lifespan in the subcritical case, obtaining improvements mainly in the 1-dimensional case (see Remark 4.2). We refer to Figure 4.1 for a graphic representation of the results below.

Theorem 4.1. *Let $\mu \ge 0$ and $1 < p < p_\mu(n)$, with $p_\mu(n)$ defined in (4.2.4). Assume that $f \in H^1(\mathbb{R}^n)$, $g \in L^2(\mathbb{R}^n)$ and*

$$f, h \ge 0, \quad h \not\equiv 0, \quad \text{where } h := [\mu-1]_+ f + g.$$

Suppose that u is an energy solution of (4.2.3) on $[0,T)$ that satisfies

$$\operatorname{supp} u \subset \{(x,t) \in \mathbb{R}^n \times [0,\infty) \colon |x| \le t+R\}$$

with some $R \ge 1$.

Then, there exists a constant $\varepsilon_1 = \varepsilon_1(f,g,\mu,p,R) > 0$ such that the blow-up time T_ε of problem (4.2.3), for $0 < \varepsilon \le \varepsilon_1$, has to satisfy:

- *if $0 \le \mu < \mu_*$, then*

$$T_\varepsilon \lesssim \begin{cases} \varepsilon^{-(p-1)/\gamma_F(p,n-[\mu-1]_-)} & \text{if } 1 < p \le \dfrac{2}{n-|\mu-1|}, \\ \varepsilon^{-2p(p-1)/\gamma_S(p,n+\mu)} & \text{if } \dfrac{2}{n-|\mu-1|} < p < p_\mu(n); \end{cases}$$

- *if $\mu \ge \mu_*$, then*

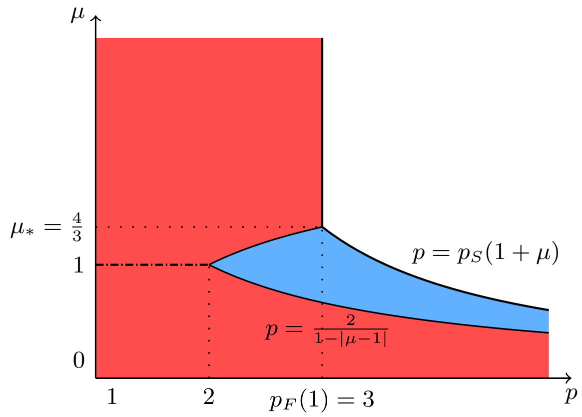
$$T_\varepsilon \lesssim \varepsilon^{-(p-1)/\gamma_F(p,n)} = \varepsilon^{-[2/(p-1)-n]^{-1}}.$$

Moreover, if $\mu = n = 1$ and $1 < p \le 2$ the estimate for T_ε is improved by

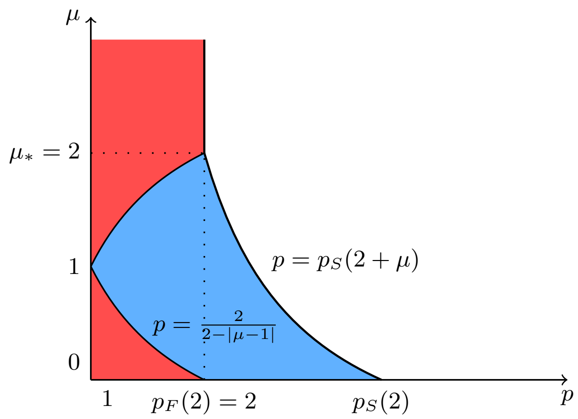
$$T_\varepsilon \lesssim \phi_0(\varepsilon)$$

where $\phi_0 \equiv \phi_0(\varepsilon)$ is the solution of

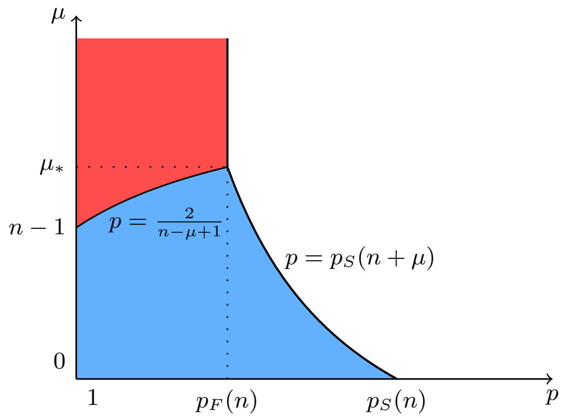
$$\varepsilon \phi_0^{\frac{2}{p-1}-1} \ln(1+\phi_0) = 1.$$



(a) Case $n = 1$.



(b) Case $n = 2$.



(c) Case $n \geq 3$.

Figure 4.1: In this figure we collect the results from Theorem 4.1. If (p, μ) is in the blue area, we have that $T_\varepsilon \lesssim \varepsilon^{-2p(p-1)/\gamma_S(p,n+\mu)}$ and hence the lifespan estimate is wave-like. Otherwise, if (p, μ) is in the red area, then $T_\varepsilon \lesssim \varepsilon^{-(p-1)/\gamma_F(p,n-[\mu-1]_-)}$ and the lifespan estimate is heat-like. In the case $n = 1$, the dash-dotted line given by $\mu = 1$, $1 < p \leq 2$ highlights the improvement $T_\varepsilon \lesssim \phi_0(\varepsilon)$.

Remark 4.1. Note that, if $n \geq 3$ and $0 \leq \mu < n-1$, we can write the lifespan estimates in Theorem 4.1 explicitly as

$$T_\varepsilon \lesssim \begin{cases} \varepsilon^{-2p(p-1)/\gamma_S(p,n+\mu)} & \begin{array}{l}\text{if } 0 \leq \mu \leq n-1 \text{ or} \\ \text{if } n-1 < \mu < \mu_* \text{ and } \dfrac{2}{n-\mu+1} < p < p_\mu(n),\end{array} \\ \varepsilon^{-(p-1)/\gamma_F(p,n)} & \text{if } n-1 < \mu < \mu_* \text{ and } 1 < p \leq \dfrac{2}{n-\mu+1}. \end{cases}$$

Remark 4.2. Comparing the lifespan estimates in Theorem 4.1 with the known results summarized in Table 4.6, we remark that the heat-like estimates for $n \geq 1$ were already proved by Wakasugi [Wak14b], whereas the wave-like ones for $n \geq 2$ by Tu and Lin [TL17]. The wave-like estimates for $n = 1$ were almost obtained by Ikeda and Sobajima [IS18] for $p_F(n) \leq p < p_S(n+\mu)$, with a loss in the exponent given by a constant $\delta > 0$.

Hence our improvements are given by the wave-like estimates if $n = 1$ and by the logarithmic gain $T_\varepsilon \lesssim \phi_0(\varepsilon)$ if $n = \mu = 1$ and $1 < p \leq 2$. Moreover, about the wave-like estimates for $n \geq 2$, in [TL17] the initial data are supposed to be non-negative, whereas our conditions on the initial data are less restrictive.

Anyway, our approach is different and based on an iteration argument rather than on a test function method.

Remark 4.3. We conjecture that the lifespan estimates in Theorem 4.1 are indeed optimal, except on the "*transition curve*" (in the (p,μ)-plane) from the wave-like to the heat-like zone, given by

$$p = \frac{2}{n - |\mu - 1|} \quad \text{for } 0 \leq \mu \leq \mu_* \text{ and } 1 < p \leq p_\mu(n).$$

On this curve, the identity

$$2p\,\gamma_F(p, n - [\mu-1]_-) = \gamma_S(p, n+\mu)$$

holds true and here we expect a logarithmic gain, as already obtained for the case $p = 2$, $\mu = n = 1$ in the previous theorem, and for the case $n = p = 2$, $\mu = 0$ for the wave equation (see Subsection 4.1.1). As we see from [IKTW20, KS19, KTW19, Wak16] the conjecture holds true if $\mu = 2$ and $n \leq 3$.

Remark 4.4. In the current analysis we do not treat the critical case, but, to conclude our prospectus, it is natural to conjecture that

$$T_\varepsilon \sim \begin{cases} \exp\left(C\varepsilon^{-p(p-1)}\right) & \text{if } 0 \leq \mu < \mu_* \text{ and } p = p_\mu(n) = p_S(n+\mu), \\ \exp\left(C\varepsilon^{-(p-1)}\right) & \text{if } \mu > \mu_* \text{ and } p = p_\mu(n) = p_F(n), \end{cases}$$

for some constant $C > 0$. We refer to [IS18, TL19] for the wave-like lifespan estimate from above in the critical case and to [IKTW20, KS19, KTW19, Wak16] for the proof of the conjecture if $\mu = 2$ and $n = 1, 3$.

However, we expect a different behavior if $\mu = \mu_*$ and $p = p_{\mu_*}(n)$, that is when the transition curve from Remark 4.3 intersects the blow-up curve. This expectation is motivated from [IKTW20], where the authors prove for $n = \mu = \mu_* = p_F(2) = p_S(4) = 2$ that $T_\varepsilon \sim \exp(C\varepsilon^{-1/2})$, which is neither a wave-like critical lifespan, nor a heat-like one.

4.2.3 Wave equation with scale-invariant damping and mass

Finally, we return to our main problem (4.1.1). The scale-invariant damped and massive wave equation was studied by A. Palmieri as object of his doctoral dissertation [Pal18b], under the supervision of M. Reissig. However, as far as we know, the research of the lifespan estimates in case of blow-up is still underdeveloped.

A key parameter for the study of this problem is

$$\delta \equiv \delta(\mu_1, \mu_2) := (\mu_1 - 1)^2 - 4\mu_2,$$

which, roughly speaking, quantifies the interaction between the damping and the mass term. Indeed, if $\delta \geq 0$, the damping term is predominant and we observe again a competition between the wave-like and heat-like behaviors. In particular, the critical exponent seems to be wave-like for small positive values of δ, while it is heat-like for large ones. If on the contrary $\delta < 0$, the mass term has more influence and the equation becomes of Klein-Gordon-type. To see this, apply again the Liouville transform $v(x,t) := (1+t)^{\mu_1/2} u(x,t)$ to problem (4.1.1), which therefore becomes

$$\begin{cases} v_{tt} - \Delta v + \dfrac{(1-\delta)/4}{(1+t)^2} v = \dfrac{|v|^p}{(1+t)^{\mu_1(p-1)/2}}, & \text{in } \mathbb{R}^n \times (0,T), \\ v(x,0) = \varepsilon f(x), \quad v_t(x,0) = \varepsilon \left\{ \dfrac{\mu_1}{2} f(x) + g(x) \right\}, & x \in \mathbb{R}^n. \end{cases} \tag{4.2.5}$$

In the following, we will consider only the case $\delta \geq 0$.

Let us start by collecting some known results. From [dNPR17, Pal18a, PR18], we know that for $\mu_1, \mu_2 > 0$ and $\delta \geq (n+1)^2$ the critical exponent for problem (4.1.1) is the shifted Fujita exponent

$$p_{\text{crit}} = p_F \left(n + \frac{\mu_1 - 1 - \sqrt{\delta}}{2} \right).$$

On the contrary, from [Pal19a, Pal19b], in the special case $\delta = 1$ and under radial symmetric assumptions for $n \geq 3$, Palmieri proved that the critical exponent is

$$p_{\text{crit}} = p_S(n + \mu_1).$$

The case $\delta = 1$ is clearly the analogous of the case $\mu = 2$ for the scale-invariant damped wave equation without mass: under this assumption we see from (4.2.5) that the equation can be transformed into a wave equation without damping and mass and with a suitable nonlinearity. In [PR19], Palmieri and Reissig proved, by using the Kato's lemma and Yagdjian integral transform, a blow-up result for $\delta \in (0,1]$, showing a competition between the shifted Fujita and Strauss exponents. Indeed, they obtained the blow-up result for

$$1 < p \leq \max \left\{ p_F \left(n + \frac{\mu_1 - 1 - \sqrt{\delta}}{2} \right), p_S(n + \mu_1) \right\}$$

except for the critical case $p = p_S(n+\mu_1)$ in dimension $n = 1$. Finally, Palmieri and Tu in [PT19], under suitable sign assumptions on the initial data and for μ_1, μ_2, δ non-negative, established a blow-up result for $1 < p \leq p_S(n+\mu_1)$ and furthermore the following lifespan estimates:

$$T_\varepsilon \lesssim \begin{cases} \varepsilon^{-2p(p-1)/\gamma_S(p, n+\mu_1)} & \text{if } 1 < p < p_S(n+\mu_1), \\ \exp(C\varepsilon^{-p(p-1)}) & \text{if } p = p_S(n+\mu_1) \text{ and } p > \dfrac{2}{n - \sqrt{\delta}}. \end{cases}$$

They used an iteration argument based on the technique of double multiplier for the subcritical case and a version of test function method developed by Ikeda and Sobajima [IS18] for the critical case. Of course, we refer to the works by Palmieri and to his doctoral thesis for a more detailed background. We also mention the recent work [IM21] by Inui and Mizutani for results on the scattering and asymptotic order for the wave equation with scale-invariant damping and mass terms and energy critical nonlinearity.

We present now our main result, concerning the blow-up of (4.1.1) for $\mu_1, \mu_2 \in \mathbb{R}$ and $\delta \geq 0$, and the upper bound for the lifespan estimates. Firstly, let us introduce the value

$$d_*(\nu) := \begin{cases} \frac{1}{2}\left(-1-\nu+\sqrt{\nu^2+10\nu-7}\right) & \text{if } \nu > 1, \\ 0 & \text{if } \nu \leq 1, \end{cases} \tag{4.2.6}$$

and set for simplicity

$$d_* := d_*(n+\mu_1) \in [0,2). \tag{4.2.7}$$

Observe that, if $n+\mu_1 > 1$, then

$$\begin{aligned} \sqrt{\delta} = n - d_* &\iff \gamma_S(p, n+\mu_1) = 2\,\gamma_F\left(p, n + \frac{\mu_1 - 1 - \sqrt{\delta}}{2}\right) = 0 \\ &\iff p_S(n+\mu_1) = p_F\left(n + \frac{\mu_1 - 1 - \sqrt{\delta}}{2}\right) = \frac{2}{n-\sqrt{\delta}}. \end{aligned} \tag{4.2.8}$$

The following result holds.

Theorem 4.2. *Let $\mu_1, \mu_2 \in \mathbb{R}$, $\delta \geq 0$ and $1 < p < p_{\mu_1,\delta}(n)$, with*

$$p_{\mu_1,\delta}(n) := \max\left\{ p_F\left(n + \frac{\mu_1 - 1 - \sqrt{\delta}}{2}\right), p_S\left(n+\mu_1\right)\right\}. \tag{4.2.9}$$

Assume that $f \in H^1(\mathbb{R}^n)$, $g \in L^2(\mathbb{R}^n)$ and

$$f, h \geq 0, \quad h \not\equiv 0, \quad \text{where} \quad h := \frac{\mu_1 - 1 + \sqrt{\delta}}{2} f + g. \tag{4.2.10}$$

Suppose that u is an energy solution of (4.1.1) on $[0,T)$ that satisfies

$$\operatorname{supp} u \subset \{(x,t) \in \mathbb{R}^n \times [0,\infty) \colon |x| \leq t + R\} \tag{4.2.11}$$

with some $R \geq 1$.

Then, there exists a constant $\varepsilon_2 = \varepsilon_2(f, g, \mu_1, \mu_2, n, p, R) > 0$ such that the blow-up time T_ε of problem (4.1.1), for $0 < \varepsilon \leq \varepsilon_2$, has to satisfy:

- *if $\sqrt{\delta} \leq n-2$, then*
$$T_\varepsilon \lesssim \varepsilon^{-2p(p-1)/\gamma_S(p,n+\mu_1)};$$
- *if $n-2 < \sqrt{\delta} < n - d_*(n+\mu_1)$, then*
$$T_\varepsilon \lesssim \begin{cases} \phi(\varepsilon) & \text{if } 1 < p \leq \dfrac{2}{n-\sqrt{\delta}}, \\ \varepsilon^{-2p(p-1)/\gamma_S(p,n+\mu_1)} & \text{if } \dfrac{2}{n-\sqrt{\delta}} < p < p_{\mu_1,\delta}(n), \end{cases}$$

where $\phi \equiv \phi(\varepsilon)$ *is the solution of*

$$\varepsilon\phi^{\frac{\gamma_F(p,n+(\mu_1-1-\sqrt{\delta})/2)}{p-1}} \ln(1+\phi)^{1-\operatorname{sgn}\delta} = 1;$$

- *if* $\sqrt{\delta} \geq n - d_*(n+\mu_1)$, *then*

$$T_\varepsilon \lesssim \phi(\varepsilon).$$

If in particular $\delta > 0$, *then*

$$\phi(\varepsilon) = \varepsilon^{-(p-1)/\gamma_F(p,n+(\mu_1-1-\sqrt{\delta})/2)} = \varepsilon^{-\left[2/(p-1)-n-(\mu_1-1-\sqrt{\delta})/2\right]^{-1}}.$$

Here and in the following, the sign function is defined as $\operatorname{sgn} x = \frac{|x|}{x}$ if $x \neq 0$, whereas $\operatorname{sgn} x = 0$ if $x = 0$.

Remark 4.5. We can write the exponent in (4.2.9) explicitly as

$$p_{\mu_1,\delta}(n) = \begin{cases} p_S\,(n+\mu_1) & \text{if } n+\mu_1>1,\ \sqrt{\delta}\leq n-d_*,\\ p_F\left(n+\dfrac{\mu_1-1-\sqrt{\delta}}{2}\right) & \text{if } n+\mu_1>1,\ n-d_*<\sqrt{\delta}<2n+\mu_1-1,\\ +\infty & \text{if } n+\mu_1>1,\ \sqrt{\delta}\geq 2n+\mu_1-1 \\ & \text{or if } n+\mu_1\leq 1. \end{cases}$$

Remark 4.6. Note that, setting the mass coefficient $\mu_2 = 0$ and the damping coefficient $\mu_1 = \mu > 0$, then $\sqrt{\delta} = |\mu - 1|$ and

$$\sqrt{\delta} \leq n - d_*(n+\mu) \iff 0 < \mu \leq \mu_*.$$

It is straightforward to check that, by imposing $\mu_2 = 0$, the results in Theorem 4.2 coincide with those in Theorem 4.1.

Remark 4.7. Analogously as in Remark 4.3, we conjecture that $p_{\mu_1,\delta}(n)$ defined in (4.2.9) is indeed the critical exponent and that the lifespan estimates presented in Theorem 4.2 are optimal, except on the "*transition surface*" (in the (p,μ_1,δ)-space) defined by

$$p = \frac{2}{n-\sqrt{\delta}} \quad \text{for } n-2<\sqrt{\delta}<n-d_*(n+\mu_1) \text{ and } 1<p\leq p_{\mu_1,\delta}(n), \tag{4.2.12}$$

on which we expect a logarithmic gain.

The exponent $p = \frac{2}{n-\sqrt{\delta}}$ already emerged in Palmieri and Tu [PT19], but as a technical condition. We underline that this exponent comes out to be the solution of the equation

$$2p\,\gamma_F\left(p, n+\frac{\mu_1-1-\sqrt{\delta}}{2}\right) = \gamma_S(p,n+\mu_1)$$

when $n-2<\sqrt{\delta}<n-d_*(n+\mu_1)$.

Remark 4.8. Similarly as in Remark 4.4, we expect that, if $p = p_{\mu_1,\delta}(n)$, then

$$T_\varepsilon \sim \begin{cases} \exp\left(C\varepsilon^{-p(p-1)}\right) & \text{if } n+\mu_1>1 \text{ and } \sqrt{\delta}<n-d_*,\\ \exp\left(C\varepsilon^{-(p-1)}\right) & \text{if } n+\mu_1>1 \text{ and } n-d_*<\sqrt{\delta}<2n+\mu_1-1, \end{cases}$$

for some constant $C > 0$. See [PT19] for the proof of the wave-like upper bound of the lifespan estimate in the critical case. Moroever, if $\sqrt{\delta} = n - d_*(n+\mu_1)$ and $p = p_{\mu_1,\delta}(n)$, we expect a different lifespan estimate, as in the massless case.

4.2.4 Different lifespans for different initial conditions

In Theorems 4.1 and 4.2 we impose on the initial data the condition

$$h = \frac{\mu_1 - 1 + \sqrt{\delta}}{2} f + g \not\equiv 0.$$

One could ask if this is only a technical condition, but it turns out that this is not the case: if we impose $h \equiv 0$, the lifespan estimates change drastically. This phenomenon was recently taken in consideration also in the works by Imai, Kato, Takamura and Wakasa [IKTW19, IKTW20, KTW19].

Let us return to the wave equation

$$\begin{cases} u_{tt} - \Delta u = |u|^p, & \text{in } \mathbb{R}^n \times (0,T), \\ u(x,0) = \varepsilon f(x), \quad u_t(x,0) = \varepsilon g(x), & x \in \mathbb{R}^n. \end{cases}$$

Since $\mu_1 = \mu_2 = 0$, in this case the condition $h \equiv 0$ is equivalent to $g \equiv 0$. Indeed, under the assumption

$$\int_{\mathbb{R}^n} g(x)dx = 0,$$

collecting the results from the works [IKTW19, LZ14, Lin90, LS96, Tak15, TW11, Zho92b, Zho92a, Zho93], we have that, for $n \geq 1$, the lifespan estimates

$$T_\varepsilon \sim \begin{cases} \varepsilon^{-2p(p-1)/\gamma_S(p,n)} & \text{if } 1 < p < p_S(n), \\ \exp\left(C\varepsilon^{-p(p-1)}\right) & \text{if } p = p_S(n), \end{cases}$$

hold, with the exclusion of the critical case $p = p_S(n)$ if $n \geq 9$ and there are not radial symmetry assumptions. We refer to the Introduction of [IKTW19] by Imai, Kato, Takamura and Wakasa for a detailed background on these results. What is interesting is the fact that now we observe always a wave-like lifespan. This is in contrast with the estimates presented in Subsection 4.1.1, where, under the assumption

$$\int_{\mathbb{R}^n} g(x)dx > 0,$$

we have heat-like lifespans in low dimensions, more precisely if $n = 1$ or if $n = 2$ and $1 < p \leq 2$, with a logarithmic gain if $n = p = 2$.

Let us consider now the Cauchy problem for the scale-invariant damped wave equation (4.2.1) with $\mu = 2$, that is

$$\begin{cases} u_{tt} - \Delta u + \dfrac{2}{1+t} u_t = |u|^p, & \text{in } \mathbb{R}^n \times (0,T), \\ u(x,0) = \varepsilon f(x), \quad u_t(x,0) = \varepsilon g(x), & x \in \mathbb{R}^n. \end{cases}$$

Since $\mu_1 = 2$ and $\mu_2 = 0$, the condition $h \equiv 0$ is equivalent to $f + g \equiv 0$. In low dimensions $n = 1$ and $n = 2$, Kato, Takamura and Wakasa [KTW19] and Imai, Kato, Takamura and Wakasa [IKTW20] proved that, if the initial data satisfy

$$\int_{\mathbb{R}^n} \{f(x) + g(x)\}dx = 0,$$

then the lifespan estimates in 1-dimension are

$$T_\varepsilon \sim \begin{cases} \varepsilon^{-2p(p-1)/\gamma_S(p,3)} & \text{if } 1 < p < 2, \\ b(\varepsilon) & \text{if } p = 2, \\ \varepsilon^{-p(p-1)/\gamma_F(p,1)} & \text{if } 2 < p < p_F(1), \\ \exp(C\varepsilon^{-p(p-1)}) & \text{if } p = p_F(1) = 3, \end{cases}$$

where $b \equiv b(\varepsilon)$ satisfies the equation $\varepsilon^2 b \log(1+b) = 1$, and in 2-dimensions are

$$T_\varepsilon \sim \begin{cases} \varepsilon^{-2p(p-1)/\gamma_S(p,4)} & \text{if } 1 < p < p_F(1) = p_S(4) = 2, \\ \exp(C\varepsilon^{-2/3}) & \text{if } p = p_F(2) = p_S(4) = 2. \end{cases}$$

These estimates are greatly different from the ones presented in Subsection 4.2.2, which hold under the assumption

$$\int_{\mathbb{R}^n} \{f(x) + g(x)\} dx \neq 0.$$

In dimension $n = 1$, we have no more a heat-like behavior, but a wave-like one appears for $p < 2$, whereas for $p > 2$ we have a mixed-like behavior, according to the notation introduced in Subsection 4.1.1. Indeed, in the latter case, even if the lifespan is related to the heat-like one, an additional p appears. In dimension $n = 2$, we have no more a heat-like behavior, but a wave-like one. The strange exponent in the critical lifespan can be explained by the same phenomenon underlined in Remark 4.4.

We are ready to exhibit our results, which give upper lifespan estimate in the subcritical case when $h \equiv 0$. It is easy to see that our estimates coincide with the ones just showed above in the respective cases. Going on with the exposition followed until now, we will present firstly the particular massless case, then the more general one with also the mass term. For simplicity, we will consider only non-negative damping coefficients.

Let us introduce the exponent

$$p_* \equiv p_*(n+\mu_1, n-\sqrt{\delta}) := \begin{cases} 1 + \dfrac{n-\sqrt{\delta}+2}{n+\mu_1-1}, & \text{if } n+\mu_1 \neq 1, \\ +\infty, & \text{if } n+\mu_1 = 1, \end{cases} \tag{4.2.13}$$

and note that, for $p > 1$ and $n + \mu_1 \neq 1$,

$$p = p_* \iff \gamma_S(p, n+\mu_1) = 2\,\gamma_F\left(p, n + \frac{\mu_1 - 1 - \sqrt{\delta}}{2}\right). \tag{4.2.14}$$

The following results hold. See Figure 4.2 for a graphic representation of the claim in Theorem 4.3.

Theorem 4.3. *Let $\mu \geq 0$ and $1 < p < p_\mu(n)$, with $p_\mu(n)$ as in Theorem 4.1. Assume that $f \in H^1(\mathbb{R}^n)$, $g \in L^2(\mathbb{R}^n)$ and*

$$f \geq 0, \quad f \not\equiv 0, \quad [\mu - 1]_+ f + g \equiv 0.$$

Suppose that u is an energy solution of (4.2.3) on $[0, T)$ that satisfies (4.2.11) for some $R \geq 1$.

Then there exists a constant $\varepsilon_3 = \varepsilon_3(f, g, \mu, p, R) > 0$ such that the blow-up time T_ε of problem (4.2.3), for $0 < \varepsilon \leq \varepsilon_3$, has to satisfy:

- *if* $0 \le \mu \le \mu_*$*, then*
$$T_\varepsilon \lesssim \varepsilon^{-2p(p-1)/\gamma_S(p,n+\mu)};$$

- *if* $\mu_* < \mu < n+3$*, then*
$$T_\varepsilon \lesssim \begin{cases} \varepsilon^{-2p(p-1)/\gamma_S(p,n+\mu)}, & \text{if } 1 < p < p_*, \\ \sigma_0(\varepsilon), & \text{if } p = p_*, \\ \varepsilon^{-p(p-1)/\gamma_F(p,n)}, & \text{if } p_* < p < p_\mu(n), \end{cases}$$
where $\sigma_0 \equiv \sigma_0(\varepsilon)$ *is the solution of*
$$\varepsilon^p \sigma_0^{\frac{2}{p-1}-n} \ln(1+\sigma_0) = 1$$
and
$$p_* = 1 + \frac{n-\mu+3}{n+\mu-1};$$

- *if* $\mu \ge n+3$*, then*
$$T_\varepsilon \lesssim \varepsilon^{-p(p-1)/\gamma_F(p,n)}.$$

Moreover, if $n = 1$, $0 < \mu < 2$ *and*
$$1 < p < \frac{2}{1+|\mu-1|},$$
then the estimate for the blow-up time T_ε *is improved by*
$$T_\varepsilon \lesssim \varepsilon^{-(p-1)/\gamma_F(p,1+[\mu-1]_+)}.$$

Theorem 4.4. *Let* $\mu_1 \ge 0$, $\mu_2 \in \mathbb{R}$, $\delta \ge 0$ *and* $1 < p < p_{\mu_1,\delta}(n)$*, with* $p_{\mu_1,\delta}(n)$ *defined in* (4.2.9)*. Assume that* $f \in H^1(\mathbb{R}^n)$, $g \in L^2(\mathbb{R}^n)$ *and* $f \ge 0$, $f \not\equiv 0$, $h \equiv 0$*, with* h *defined in* (4.2.10)*. Suppose that* u *is an energy solution of* (4.1.1) *on* $[0,T)$ *that satisfies* (4.2.11) *with some* $R \ge 1$.

Then, there exists a constant $\varepsilon_4 = \varepsilon_4(f,g,\mu_1,\mu_2,p,R) > 0$ *such that the blow-up time* T_ε *of problem* (4.1.1)*, for* $0 < \varepsilon \le \varepsilon_4$*, has to satisfy:*

- *if* $\sqrt{\delta} \le n - d_*(n+\mu_1)$*, then*
$$T_\varepsilon \lesssim \varepsilon^{-2p(p-1)/\gamma_S(p,n+\mu_1)};$$

- *if* $n - d_*(n+\mu_1) < \sqrt{\delta} < n+2$*, then*
$$T_\varepsilon \lesssim \begin{cases} \varepsilon^{-2p(p-1)/\gamma_S(p,n+\mu_1)}, & \text{if } 1 < p < p_*, \\ \sigma_*(\varepsilon) & \text{if } p = p_*, \\ \sigma(\varepsilon), & \text{if } p_* < p < p_{\mu_1,\delta}(n), \end{cases}$$
where $\sigma \equiv \sigma(\varepsilon)$ *and* $\sigma_* \equiv \sigma_*(\varepsilon)$ *are the solutions respectively of*
$$\varepsilon^p \sigma^{\frac{\gamma_F(p,n+(\mu_1-1-\sqrt{\delta})/2)}{p-1}} \ln(1+\sigma)^{1-\operatorname{sgn}\delta} = 1,$$
$$\varepsilon^p \sigma_*^{\frac{\gamma_F(p,n+(\mu_1-1-\sqrt{\delta})/2)}{p-1}} \ln(1+\sigma_*)^{2-\operatorname{sgn}\delta} = 1;$$

- *if* $\sqrt{\delta} \geq n+2$, *then*
$$T_\varepsilon \lesssim \sigma(\varepsilon).$$

Moreover, if $n = 1$, $0 \leq \delta < 1$ *and*

$$1 < p < r_*(\mu_1, \delta) := \begin{cases} 1 + 2\,\dfrac{2-\sqrt{\delta}}{1+\mu_1+\sqrt{\delta}}, & \text{if } \sqrt{\delta} < \theta, \\ 1 + 2\,\dfrac{2-\theta}{1+\mu_1+\theta} = \dfrac{2}{1+\theta}, & \text{if } \sqrt{\delta} = \theta, \\ \dfrac{2}{1+\sqrt{\delta}}, & \text{if } \sqrt{\delta} > \theta, \end{cases} \tag{4.2.15}$$

with

$$\theta \equiv \theta(\mu_1) := 1 + \frac{\mu_1}{2} - \frac{1}{2}\sqrt{\mu_1^2 + 16} \in (-1, 1), \tag{4.2.16}$$

then the estimate for the blow-up time T_ε *is improved by*

$$T_\varepsilon \lesssim \varepsilon^{-(p-1)/\gamma_F\left(p,(\mu_1+1+\sqrt{\delta})/2\right)}.$$

Remark 4.9. In the 1-dimensional case of Theorem 4.4, one can check that $r_* < p_{\mu_1,\delta}(1)$ holds always, except when $\mu_1 = 3$ and $\delta = 0$, since in this case $r_* = p_{3,0}(1) = p_S(4) = 2$. About the relation between p_* and r_*, we have that, for $0 \leq \delta < 1$, if $\sqrt{\delta} \lesseqgtr \theta$ then $p_* \lesseqgtr r_*$.

Remark 4.10. We conjecture that the estimates in the previous two theorems are indeed optimal, except in dimension $n = 1$ for Theorem 4.3 on the *transition curve* defined by

$$p = \frac{2}{1+|\mu-1|} \quad \text{for } 0 \leq \mu \leq 2,$$

and for Theorem 4.4 on the *transition surface*

$$p = r_*(\mu_1, \delta) \quad \text{for } 0 \leq \delta \leq 1.$$

Moreover, in the critical case we expect, due to the wave-like and mixed-type behaviors,

$$T_\varepsilon \sim \exp(C\varepsilon^{-p(p-1)}),$$

except for $\sqrt{\delta} = n - d_*(n+\mu_1)$ and $p = p_{\mu_1,\delta}(n)$, where the lifespan should be different.

Remark 4.11. The conditions (4.2.10) on the initial data in Theorem 4.1 and 4.2 can be replaced by the less strong conditions

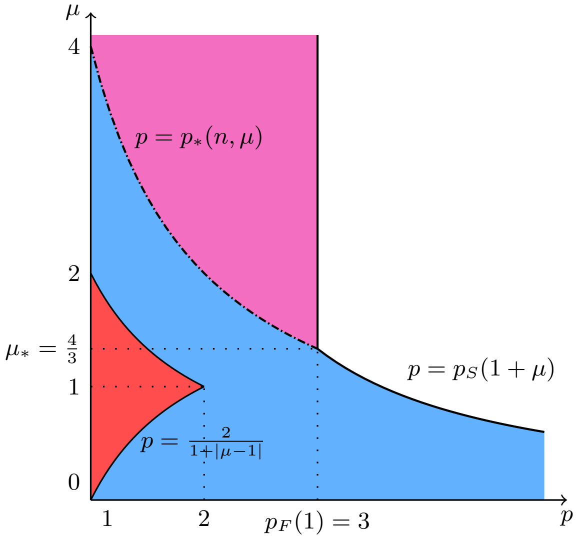
$$\int_{\mathbb{R}^n} f(x) \geq 0, \qquad \int_{\mathbb{R}^n} f(x)\phi_1(x) \geq 0, \qquad \int_{\mathbb{R}^n} h(x) > 0, \qquad \int_{\mathbb{R}^n} h(x)\phi_1(x) > 0,$$

where the positive function $\phi_1(x)$ is defined later in (4.4.9).

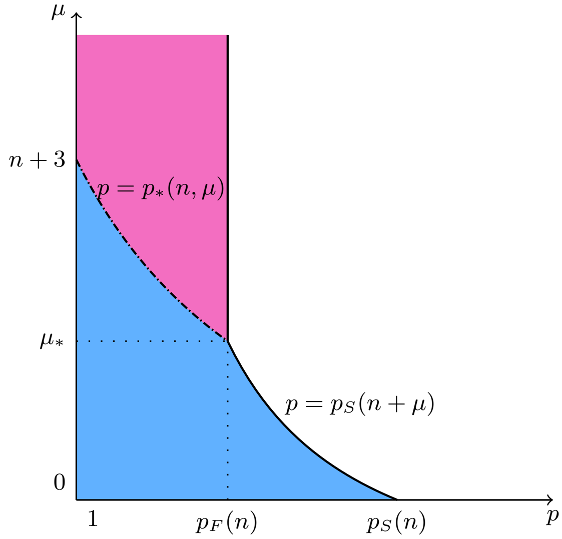
Similarity can be done for the initial conditions of Theorem 4.3 and 4.4, requiring

$$\int_{\mathbb{R}^n} f(x) > 0, \qquad \int_{\mathbb{R}^n} f(x)\phi_1(x) > 0, \qquad \int_{\mathbb{R}^n} h(x) = 0, \qquad \int_{\mathbb{R}^n} h(x)\phi_1(x) = 0.$$

It will be clear from the proof of our theorems that these weaker hypothesis are sufficient.



(a) Case $n = 1$.



(b) Case $n \geq 2$.

Figure 4.2: Here we collect the results from Theorem 4.3. If (p, μ) is in the blue area, then $T_\varepsilon \lesssim \varepsilon^{-2p(p-1)/\gamma_S(p,n+\mu)}$, hence the lifespan estimate is wave-like. If (p, μ) is in the purple area, then $T_\varepsilon \lesssim \varepsilon^{-p(p-1)/\gamma_F(p,n)}$ and the lifespan estimate is of mixed-type. The dash-dotted line given by $p = p_*(n, \mu)$ highlights the improvement $T_\varepsilon \lesssim \sigma_0(\varepsilon)$. In the case $n = 1$, if (p, μ) is in the red area, $T_\varepsilon \lesssim \varepsilon^{-(p-1)/\gamma_F(p,1+[\mu-1]_-)}$ and the lifespan estimate is heat-like.

4.2.5 Wave equation with scattering damping and negative mass

In the end, in this subsection we want to continue the study of a problem we examined in [LST19, LST20] together with Ning-An Lai and Hiroyuki Takamura. In these two works, we considered the Cauchy problem for the wave equation with scattering damping and negative mass term, viz.

$$\begin{cases} w_{tt} - \Delta w + \dfrac{\nu_1}{(1+t)^\beta} w_t + \dfrac{\nu_2}{(1+t)^{\alpha+1}} w = |w|^p, & \text{in } \mathbb{R}^n \times (0,T), \\ w(x,0) = \varepsilon f(x), \quad w_t(x,0) = \varepsilon g(x), & x \in \mathbb{R}^n, \end{cases} \tag{4.2.17}$$

where $\nu_1 \geq 0$, $\nu_2 < 0$, $\alpha \in \mathbb{R}$ and $\beta > 1$.

In Subsection 4.2.2 we already observed that, if the damping is of scattering type, the solution of the homogeneous damped wave equation "scatters" to the one of the wave equation. For the equation with power non-linearity, according to the results by Lai and Takamura [LT18] and Wakasa and Yordanov [WY19], the solution again seems to be wave-like both in the critical exponent and in the lifespan estimate.

In [LST19], we took in consideration (4.2.17) with $\alpha > 1$ and observed a double scattering phenomenon, in the sense that both the damping and the mass terms seem to be not effective. Hence, the solution behaves like that of the wave equation with power non-linearity $u_{tt} - \Delta u = |u|^p$. More precisely, supposing for simplicity that f, g are non-negative, non-vanishing, compactly supported functions, we established the blow-up for $1 < p < p_S(n)$ and the upper bound for the lifespan estimates

$$T_\varepsilon \lesssim \begin{cases} \varepsilon^{-\frac{p-1}{\gamma_F(p,n-1)}} & \text{if } n = 1 \text{ or } n = 2, 1 < p < 2, \\ a(\varepsilon) & \text{if } n = p = 2, \\ \varepsilon^{-\frac{2p(p-1)}{\gamma_S(p,n)}} & \text{if } n = 2, 2 < p < p_S(n) \text{ or if } n \geq 3, \end{cases}$$

where $a \equiv a(\varepsilon)$ satisfies $\varepsilon^2 a^2 \log(1+a) = 1$, although in the case $n = p = 2$ more technical conditions were required.

In [LST20], we studied the case $\alpha < 1$, discovering a new behavior in the lifespan estimate. Indeed, we proved that there is blow-up for every $p > 1$ and that the upper lifespan estimate

$$T_\varepsilon \lesssim \zeta(C\varepsilon)$$

hold, where $\zeta \equiv \zeta(\overline{\varepsilon})$ is the larger solution of the equation

$$\overline{\varepsilon} \zeta^{\frac{\gamma_F(p, n-(1+\alpha)/4)}{p-1}} \exp\left(K \zeta^{\frac{1-\alpha}{2}}\right) = 1,$$

with

$$K = \frac{2\sqrt{|\nu_2|}}{1-\alpha} \exp\left(\frac{\nu_1}{2(1-\beta)}\right).$$

As observed in Remark 2.1 of [LST20], a less sharp but more clear estimate for the lifespan in the case $\alpha < 1$ is

$$T_\varepsilon \lesssim \left[\log\left(\frac{1}{\varepsilon}\right)\right]^{\frac{2}{1-\alpha}}.$$

Hence, the negative mass term with $\alpha > 1$ seems to have no influence on the behavior of the solution; on the contrary, if $\alpha < 1$ the negative mass term becomes extremely relevant, implying the blow-up for all $p > 1$ and a lifespan estimate which is much shorter, compared to the ones introduced previously.

We now come to the case $\alpha = 1$. This is particular and was not deepened in our previous works. Indeed in Subsection 4.4.5, after introducing a multiplier to absorb the damping term, we will show that we can get blow-up results and lifespan estimates for this problem by reducing ourself to calculations similar to the ones we will perform to prove the results exhibited in the previous subsections. Roughly speaking, we will find out that (4.2.17) with $\alpha = 1$ has the same behavior as that of (4.1.1) with $\mu_1 = 0$ and $\mu_2 = \nu_2 e^{\nu_1/(1-\beta)}$.

Therefore, in the rest of this subsection we will consider the Cauchy problem

$$\begin{cases} w_{tt} - \Delta w + \dfrac{\nu_1}{(1+t)^\beta} w_t + \dfrac{\nu_2}{(1+t)^2} w = |w|^p, & \text{in } \mathbb{R}^n \times (0,T), \\ w(x,0) = \varepsilon f(x), \quad w_t(x,0) = \varepsilon g(x), & x \in \mathbb{R}^n, \end{cases} \tag{4.2.18}$$

where $\nu_1 \geq 0$, $\nu_2 < 0$ and $\beta > 1$.

Definition 4.2. We say that u is an energy solution of (4.2.18) over $[0,T)$ if

$$w \in C([0,T), H^1(\mathbb{R}^n)) \cap C^1([0,T), L^2(\mathbb{R}^n)) \cap C((0,T), L^p_{\text{loc}}(\mathbb{R}^n))$$

satisfies $w(x,0) = \varepsilon f(x)$ in $H^1(\mathbb{R}^n)$, $w_t(x,0) = \varepsilon g(x)$ in $L^2(\mathbb{R}^n)$ and

$$\begin{aligned} &\int_{\mathbb{R}^n} w_t(x,t)\phi(x,t)dx \\ &+ \int_0^t ds \int_{\mathbb{R}^n} \{-w_t(x,s)\phi_t(x,s) + \nabla w(x,s)\cdot\nabla\phi(x,s)\}\, dx \\ &+ \int_0^t ds \int_{\mathbb{R}^n} \frac{\nu_1}{(1+s)^\beta} w_t(x,s)\phi(x,s)dx \\ &+ \int_0^t ds \int_{\mathbb{R}^n} \frac{\nu_2}{(1+s)^2} w(x,s)\phi(x,s)dx \\ =& \int_{\mathbb{R}^n} \varepsilon g(x)\phi(x,0)dx \\ &+ \int_0^t ds \int_{\mathbb{R}^n} |w(x,s)|^p \phi(x,s)dx \end{aligned} \tag{4.2.19}$$

with any test function $\phi \in C_0^\infty(\mathbb{R}^n \times [0,T))$ for $t \in [0,T)$.

We have the following result, graphically pictured in Figure 4.3.

Theorem 4.5. *Fix $\nu_1 \geq 0$, $\nu_2 < 0$, $\beta > 1$. Define*

$$\delta := 1 - 4\nu_2 e^{\nu_1/(1-\beta)} > 1,$$

and the parameter

$$d_*(n) := \frac{1}{2}\left(-1 - n + \sqrt{n^2 + 10n - 7}\right) \in [0,2).$$

Let $1 < p < p_\delta(n)$*, with*

$$\begin{aligned} p_\delta(n) &= \max\left\{ p_F\left(n - \frac{1+\sqrt{\delta}}{2}\right), p_S(n) \right\} \\ &= \begin{cases} p_S(n) & \text{if } n \geq 2, \sqrt{\delta} \leq n - d_*(n), \\ p_F\left(n - \frac{1+\sqrt{\delta}}{2}\right) & \text{if } n \geq 2, n - d_*(n) < \sqrt{\delta} < 2n-1, \\ +\infty & \text{if } n = 1 \text{ or if } n \geq 2, \sqrt{\delta} \geq 2n-1. \end{cases} \end{aligned}$$

Assume that $f \in H^1(\mathbb{R}^n)$*,* $g \in L^2(\mathbb{R}^n)$ *are non-negative and not both vanishing. Suppose that* w *is an energy solution of* (4.2.18) *on* $[0,T)$ *that, for some* $R \geq 1$*, satisfies*

$$\operatorname{supp} w \subset \{(x,t) \in \mathbb{R}^n \times [0,\infty) \colon |x| \leq t + R\}.$$

Then, there exists a constant $\varepsilon_5 = \varepsilon_5(f,g,\beta,\nu_1,\nu_2,n,p,R) > 0$ *such that the blow-up time* T_ε *of problem* (4.2.18)*, for* $0 < \varepsilon \leq \varepsilon_5$*, has to satisfy:*

- *if* $\sqrt{\delta} \leq n-2$*, then*
$$T_\varepsilon \lesssim \varepsilon^{-2p(p-1)/\gamma_S(p,n)};$$
- *if* $n - 2 < \sqrt{\delta} < n - d_*(n)$*, then*
$$T_\varepsilon \lesssim \begin{cases} \varepsilon^{-(p-1)/\gamma_F(p,n-(1+\sqrt{\delta})/2)}, & \text{if } 1 < p \leq \frac{2}{n-\sqrt{\delta}}, \\ \varepsilon^{-2p(p-1)/\gamma_S(p,n)}, & \text{if } \frac{2}{n-\sqrt{\delta}} < p < p_\delta(n); \end{cases}$$
- *if* $\sqrt{\delta} \geq n - d_*(n)$*, then*
$$T_\varepsilon \lesssim \varepsilon^{-(p-1)/\gamma_F(p,n-(1+\sqrt{\delta})/2)} = \varepsilon^{-\left[2/(p-1)-n+(1+\sqrt{\delta})/2\right]^{-1}}.$$

Remark 4.12. As a direct consequence of Remark 4.7 and 4.8, we expect that $p_\delta(n)$ is the critical exponent and that the lifespan estimates presented in Theorem 4.5 are optimal, except on the *transition curve* (in the (p,δ)-plane) defined by

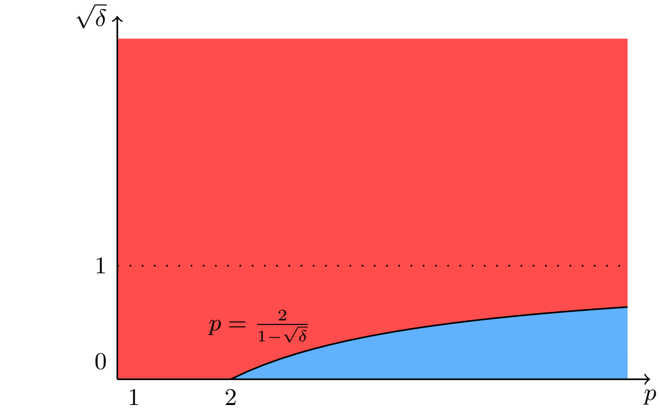
$$p = \frac{2}{n-\sqrt{\delta}} \quad \text{for } n-2 < \sqrt{\delta} < n - d_*(n) \text{ and } 1 < p \leq p_\delta(n),$$

on which we presume a logarithmic gain can appear.

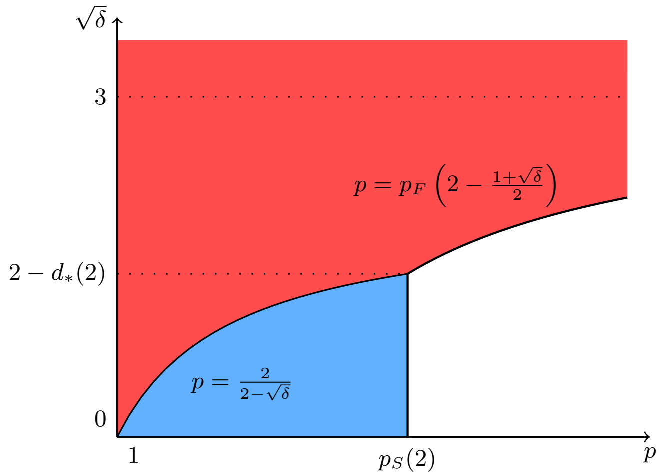
Moreover, we expect that, if $p = p_\delta(n)$, then

$$T_\varepsilon \sim \begin{cases} \exp\left(C\varepsilon^{-p(p-1)}\right) & \text{if } n \geq 2, \sqrt{\delta} < n - d_*(n), \\ \exp\left(C\varepsilon^{-(p-1)}\right) & \text{if } n \geq 2, n - d_*(n) < \sqrt{\delta} < 2n-1, \end{cases}$$

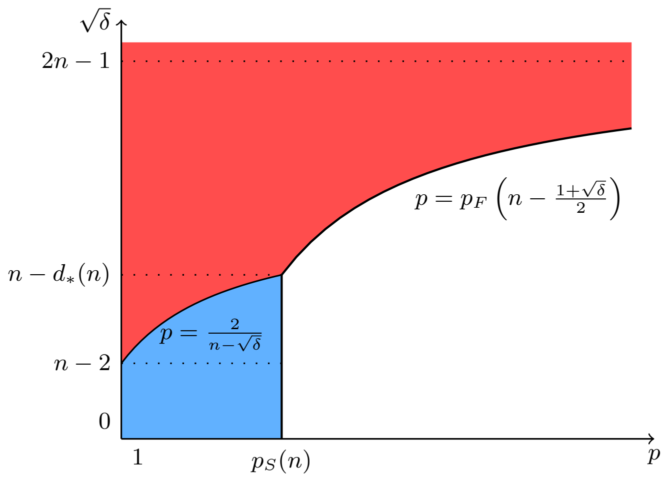
for some constant $C > 0$. If $\sqrt{\delta} = n - d_*(n)$ and $p = p_\delta(n)$, we presume a lifespan estimate of different kind.



(a) Case $n = 1$.



(b) Case $n = 2$.



(c) Case $n \geq 3$.

Figure 4.3: Here we collect the results from Theorem 4.5. If $(p, \sqrt{\delta})$ is in the blue area, then $T_\varepsilon \lesssim \varepsilon^{-2p(p-1)/\gamma_S(p,n)}$, hence the lifespan estimate is wave-like. Otherwise, if $(p, \sqrt{\delta})$ is in the red area, then $T_\varepsilon \lesssim \varepsilon^{-(p-1)/\gamma_F(p,n-(1+\sqrt{\delta})/2)}$ and the lifespan is heat-like. Note that this figure represents also the results of Theorem 4.2 for the case $\mu_1 = 0$, $\mu_2 \leq 1/4$.

4.3 The Kato-type lemma

The principal ingredient we will employ in the demonstration of our theorems is the following Kato-type lemma. Although this tool is well known and used in the literature, here we will reformulate it in such a way that, in the following sections, we can directly apply it to obtain not only the condition to find the possible critical exponent, but also the upper bound of the lifespan estimate. We will prove it using the so-called iteration argument.

Lemma 4.1. *Let $p > 1$, $a, b \in \mathbb{R}$ satisfy*

$$\gamma := 2[(p-1)a - b + 2] > 0.$$

Assume that $F \in C([0,T))$ satisfies, for $t \geq T_0$,

$$F(t) \geq EAt^a \left[\ln(1+t)\right]^c, \tag{4.3.1}$$

$$F(t) \geq B \int_{T_0}^t ds \int_{T_0}^s r^{-b} F(r)^p dr, \tag{4.3.2}$$

where $c, T_0 \geq 0$ and $E, A, B > 0$. Suppose that there exists $\widetilde{T} \geq T_0$ which solves

$$E\widetilde{T}^{\frac{\gamma}{2(p-1)}} \left[\ln(1+\widetilde{T})\right]^c = 1. \tag{4.3.3}$$

Then, we have that

$$T < C\widetilde{T}$$

for some positive constant C independent of E.

Proof. Let $\widetilde{T}$ be as in the statement of the lemma and start with the ansatz

$$F(t) \geq D_j \left[\ln(1+\widetilde{T})\right]^{c_j} t^{-b_j} (t-\widetilde{T})^{a_j} \quad \text{for } t \geq \widetilde{T}, \quad j = 1,2,3,\dots \tag{4.3.4}$$

where D_j, a_j, b_j, c_j are positive constants to be determined later. Due to hypothesis (4.3.1), note that (4.3.4) is true for $j = 1$ with

$$D_1 = EA, \quad a_1 = [a]_+, \quad b_1 = [a]_-, \quad c_1 = c, \tag{4.3.5}$$

where $[x]_\pm := (|x| \pm x)/2$. Plugging (4.3.4) into (4.3.2), we get

$$\begin{aligned} F(t) &\geq D_j^p B \int_{\widetilde{T}}^t ds \int_{\widetilde{T}}^s \left[\ln(1+\widetilde{T})\right]^{pc_j} r^{-b-pb_j} (r-\widetilde{T})^{pa_j} dr \\ &\geq \frac{D_j^p B}{(pa_j + [b]_- + 2)^2} \left[\ln(1+\widetilde{T})\right]^{pc_j} t^{-pb_j - [b]_+} (t-\widetilde{T})^{pa_j + [b]_- + 2} \end{aligned}$$

for $t \geq \widetilde{T}$, and then we can define the sequences $\{D_j\}_{j\in\mathbb{N}}$, $\{a_j\}_{j\in\mathbb{N}}$, $\{b_j\}_{j\in\mathbb{N}}$, $\{c_j\}_{j\in\mathbb{N}}$ by

$$a_{j+1} = pa_j + [b]_- + 2, \qquad b_{j+1} = pb_j + [b]_+,$$

$$c_{j+1} = pc_j, \qquad D_{j+1} = \frac{D_j^p B}{(pa_j + [b]_- + 2)^2},$$

to establish (4.3.4) with j replaced by $j+1$. Hence for any $j \in \mathbb{N}$, it follows from the previous relations and from (4.3.5) that

$$a_j = p^{j-1}\left([a]_+ + \frac{[b]_- + 2}{p-1}\right) - \frac{[b]_- + 2}{p-1}, \qquad b_j = p^{j-1}\left([a]_- + \frac{[b]_+}{p-1}\right) - \frac{[b]_+}{p-1},$$
$$c_j = p^{j-1}c.$$

In particular, we obtain that

$$pa_j + [b]_- + 2 = a_{j+1} \le p^j\left([a]_+ + \frac{[b]_- + 2}{p-1}\right) \Longrightarrow D_{j+1} \ge \widetilde{C}p^{-2j}D_j^p, \tag{4.3.6}$$

where $\widetilde{C} := B/\{[a]_+ + ([b]_- + 2)/(p-1)\}^2 > 0$. From (4.3.6) and $D_1 = EA$, by an inductive argument we infer, for $j \ge 2$, that

$$D_j \ge \exp\left\{p^{j-1}\left[\ln(EA) - S_j\right]\right\},$$

where

$$S_j := \sum_{k=1}^{j-1} \frac{2k\ln p - \ln \widetilde{C}}{p^k}.$$

Since $\sum_{k=0}^{\infty} x^k = 1/(1-x)$ and $\sum_{k=1}^{\infty} kx^k = x/(1-x)^2$ when $|x| < 1$, we obtain

$$S_\infty := \lim_{j\to+\infty} S_j = \ln\{\widetilde{C}^{p/(1-p)}p^{2p/(1-p)^2}\}.$$

Moreover, there exists $j_0 \ge 2$ such that the sequence S_j is increasing for $j \ge j_0$. Hence we obtain that

$$D_j \ge (EAe^{-S_\infty})^{p^{j-1}}$$

for j sufficiently large. Let us turning back to (4.3.4) and let $C > 1$ be a constant to be determined later. If we suppose $t \ge C\widetilde{T}$, so that in particular $t - \widetilde{T} \ge (1 - 1/C)t$, and considering (4.3.3), we have

$$\begin{aligned} F(t) &\ge t^{\frac{[b]_+}{p-1}}(t-\widetilde{T})^{-\frac{[b]_-+2}{p-1}}\left\{EAe^{-S_\infty}\left[\ln(1+\widetilde{T})\right]^c t^{-[a]_- - \frac{[b]_+}{p-1}}(t-\widetilde{T})^{[a]_+ + \frac{[b]_-+2}{p-1}}\right\}^{p^{j-1}} \\ &\ge t^{\frac{[b]_+}{p-1}}(t-\widetilde{T})^{-\frac{[b]_-+2}{p-1}}\left\{EAe^{-S_\infty}\left(1-\frac{1}{C}\right)^{[a]_+ + \frac{[b]_-+2}{p-1}}\left[\ln(1+\widetilde{T})\right]^c t^{\frac{\gamma}{2(p-1)}}\right\}^{p^{j-1}} \\ &\ge t^{\frac{[b]_+}{p-1}}(t-\widetilde{T})^{-\frac{[b]_-+2}{p-1}} J^{p^{j-1}} \end{aligned}$$

with

$$J := Ae^{-S_\infty}\left(1-\frac{1}{C}\right)^{[a]_+ + \frac{[b]_-+2}{p-1}} C^{\frac{\gamma}{2(p-1)}}.$$

Since $\gamma > 0$, we can choose $C > 1$ large enough, in such a way that $J > 1$. Letting $j \to +\infty$ in the above inequality, we get $F(t) \to +\infty$. Then, $T < C\widetilde{T}$ as claimed. □

Remark 4.13. We can observe that the previous lemma is still true if in (4.3.2) an arbitrary number of integrals appear, more precisely if we replace (4.3.2) with

$$F(t) \ge B\int_{T_0}^{t} dt_1 \int_{T_0}^{t_1} dt_2 \cdots \int_{T_0}^{t_{k-1}} t_k^{-b} F(t_k)^p dt_k \quad \text{for } t \ge T_0,$$

and γ with $\gamma_k := 2[(p-1)a - b + k]$, for any positive integer $k \in \mathbb{N}$.

4.4 Proof for the theorems

We come now to the demonstration of Theorems 4.2 and 4.4. In the next two subsections, we will prove some key inequalities which will be employed in the machinery of the Kato-type lemma. Applying the latter, we will find a couple of results, which will be compared in Subsection 4.4.4 to find the claimed ones. The proof of Theorems 4.1 and 4.3 are clearly omitted, since they are straightforward corollaries of Theorems 4.2 and 4.4 respectively, just set the mass equal to zero. In the end, we will sketch the proof of Theorem 4.5 in Subsection 4.4.5.

4.4.1 The key estimates

Let us define the functional

$$F_0(t) := \int_{\mathbb{R}^n} u(x,t)dx.$$

Choosing the test function $\phi = \phi(x,s)$ in (4.1.2) to satisfy

$$\phi \equiv 1 \text{ in } \{(x,s) \in \mathbb{R}^n \times [0,t] : |x| \leq s + R\}, \tag{4.4.1}$$

we get

$$\begin{aligned}
&\int_{\mathbb{R}^n} u_t(x,t)dx - \int_{\mathbb{R}^n} u_t(x,0)dx \\
&+ \int_0^t ds \int_{\mathbb{R}^n} \frac{\mu_1}{1+s} u_t(x,s)dx + \int_0^t ds \int_{\mathbb{R}^n} \frac{\mu_2}{(1+s)^2} u(x,s)dx \\
= &\int_0^t ds \int_{\mathbb{R}^n} |u(x,s)|^p dx,
\end{aligned}$$

which yields, by taking derivative with respect to t,

$$F_0''(t) + \frac{\mu_1}{1+t} F_0'(t) + \frac{\mu_2}{(1+t)^2} F_0(t) = \int_{\mathbb{R}^n} |u(x,t)|^p dx. \tag{4.4.2}$$

Setting

$$\lambda := 1 + \sqrt{\delta} > 0, \quad \kappa := \frac{\mu_1 - 1 - \sqrt{\delta}}{2}, \quad \delta := (\mu_1 - 1)^2 - 4\mu_2,$$

we obtain that (4.4.2) is equivalent to

$$\frac{d}{dt}\left\{(1+t)^\lambda \frac{d}{dt}\left[(1+t)^\kappa F_0(t)\right]\right\} = (1+t)^{\kappa+\lambda} \int_{\mathbb{R}^n} |u(x,t)|^p dx.$$

Integrating twice the above equality over $[0,t]$, we get

$$F_0(t) = L(t) + M(t), \tag{4.4.3}$$

where

$$\begin{aligned}
L(t) &:= F_0(0)(1+t)^{-\kappa} + [\kappa F_0(0) + F_0'(0)](1+t)^{-\kappa} \int_0^t (1+s)^{-\lambda} ds, \\
M(t) &:= (1+t)^{-\kappa} \int_0^t (1+s)^{-\lambda} ds \int_0^s (1+r)^{\kappa+\lambda} dr \int_{\mathbb{R}^n} |u(x,r)|^p dx \geq 0.
\end{aligned}$$

Consider now the functional

$$\mathcal{F}(t) := (1+t)^{\kappa+\lambda} F_0(t),$$

and observe that F_0 and $\mathcal{F}$ imply the same blow-up results, so it is sufficient to study the latter functional. Since

$$\int_{\mathbb{R}^n} f(x)dx \geq 0, \quad H_0 := \int_{\mathbb{R}^n} h(x)dx \geq 0,$$

and they are not both equal to zero, we want to prove that there exists a time $T_0 > 0$, independent of ε, such that, for $t \geq T_0$, the following estimates hold:

$$\mathcal{F}(t) \gtrsim \int_{T_0}^t ds \int_{T_0}^s r^{-(n+\kappa+\lambda)(p-1)} \mathcal{F}(r)^p dr, \tag{4.4.4}$$

$$\mathcal{F}(t) \gtrsim \varepsilon \begin{cases} t & \text{if } H_0 = 0, \\ t^\lambda \ln^{1-\operatorname{sgn}\delta}(1+t) & \text{if } H_0 > 0, \end{cases} \tag{4.4.5}$$

$$\mathcal{F}(t) \gtrsim \varepsilon^p \begin{cases} t^{\kappa+\lambda-(n+\mu_1-1)\frac{p}{2}+n+1} & \text{if } \kappa-(n+\mu_1-1)\frac{p}{2}+n+1>0, \\ t^\lambda \ln^{2-\operatorname{sgn}\delta}(1+t) & \text{if } \kappa-(n+\mu_1-1)\frac{p}{2}+n+1=0, \\ t^\lambda \ln^{1-\operatorname{sgn}\delta}(1+t) & \text{if } \kappa-(n+\mu_1-1)\frac{p}{2}+n+1<0. \end{cases} \tag{4.4.6}$$

Thanks to the Hölder inequality and using the compact support of the solution (4.2.11), we have

$$\int_{\mathbb{R}^n} |u(x,t)|^p dx \gtrsim t^{-n(p-1)} |F_0(t)|^p = (1+t)^{-n(p-1)-(\kappa+\lambda)p} \mathcal{F}(t)^p \tag{4.4.7}$$

for $t \gtrsim 1$. Considering L and recalling the definition (4.2.10) of H_0 we obtain

$$L(t) = \begin{cases} (1+t)^{-\kappa}[F_0(0)+\varepsilon H_0 \ln(1+t)] & \text{if } \delta = 0, \\ \dfrac{(1+t)^{-\kappa}}{\sqrt{\delta}} \left\{ \varepsilon H_0 + [\sqrt{\delta} F_0(0) - \varepsilon H_0](1+t)^{-\sqrt{\delta}} \right\} & \text{if } \delta > 0. \end{cases}$$

So, from the condition on the initial data we get, for $t \gtrsim 1$ sufficiently large, that

$$L(t) \gtrsim \varepsilon \begin{cases} t^{-\kappa-\sqrt{\delta}} & \text{if } H_0 = 0, \\ t^{-\kappa} & \text{if } H_0 > 0,\, \delta > 0, \\ t^{-\kappa}\ln(1+t) & \text{if } H_0 > 0,\, \delta = 0, \end{cases} \tag{4.4.8}$$

and in particular the positiveness of L for large time. Neglecting L from (4.4.3), inserting (4.4.7) and recalling that $\lambda > 0$, we get (4.4.4). Instead, inserting (4.4.8) in (4.4.3) and neglecting M, we reach (4.4.5).

Finally, we will prove (4.4.6) in the next section.

4.4.2 The weighted functional

Let us introduce

$$F_1(t) := \int_{\mathbb{R}^n} u(x,t)\psi_1(x,t)dx,$$

where ψ_1 is the test function presented by Yordanov and Zhang in [YZ06],

$$\psi_1(x,t) := e^{-t}\phi_1(x), \qquad \phi_1(x) := \begin{cases} \int_{\mathbb{S}^{n-1}} e^{x\cdot\omega} d\omega & \text{for } n \geq 2, \\ e^x + e^{-x} & \text{for } n = 1, \end{cases} \tag{4.4.9}$$

which satisfies the following inequality (equation (2.5) in [YZ06]):

$$\int_{|x|\leq t+R} \psi_1(x,t)^{\frac{p}{p-1}} dx \lesssim (1+t)^{(n-1)\left\{1-\frac{p}{2(p-1)}\right\}}. \tag{4.4.10}$$

We want to establish the lower bound for F_1. From the definition of energy solution (4.1.2), we have that

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^n} u_t(x,t)\phi(x,t)dx \\ &-\int_{\mathbb{R}^n} u_t(x,t)\phi_t(x,t)dx - \int_{\mathbb{R}^n} u(x,t)\Delta\phi(x,t)dx \\ &+\int_{\mathbb{R}^n} \frac{\mu_1}{1+t}u_t(x,t)\phi(x,t)dx + \int_{\mathbb{R}^n} \frac{\mu_2}{(1+t)^2}u(x,t)\phi(x,t)dx \\ =&\int_{\mathbb{R}^n} |u(x,t)|^p\phi(x,t)dx. \end{aligned}$$

Integrating the above inequality over $[0,t]$, and in particular using the integration by parts on the second term in the first line and on the first term in the second line, we infer

$$\begin{aligned} &\int_{\mathbb{R}^n} u_t(x,t)\phi(x,t)dx - \varepsilon\int_{\mathbb{R}^n} g(x)\phi(x,0)dx \\ &-\int_{\mathbb{R}^n} u(x,t)\phi_t(x,t)dx + \varepsilon\int_{\mathbb{R}^n} f(x)\phi_t(x,0)dx \\ &+\int_0^t ds\int_{\mathbb{R}^n} u(x,s)\phi_{tt}(x,s)dx - \int_0^t ds\int_{\mathbb{R}^n} u(x,s)\Delta\phi(x,s)dx \\ &+\int_{\mathbb{R}^n} \frac{\mu_1}{1+t}u(x,t)\phi(x,t)dx - \varepsilon\mu_1\int_{\mathbb{R}^n} f(x)\phi(x,0)dx \\ &+\int_0^t ds\int_{\mathbb{R}^n} u(x,s)\frac{\mu_1}{(1+s)^2}\phi(x,s)dx - \int_0^t ds\int_{\mathbb{R}^n} u(x,s)\frac{\mu_1}{1+s}\phi_t(x,s)dx \\ &+\int_0^t ds\int_{\mathbb{R}^n} \frac{\mu_2}{(1+s)^2}u(x,s)\phi(x,s)dx \\ =&\int_0^t ds\int_{\mathbb{R}^n} |u(x,s)|^p\phi(x,s)dx. \end{aligned} \tag{4.4.11}$$

Setting

$$\phi(x,t) = \psi_1(x,t) = e^{-t}\phi_1(x) \quad \text{on } \operatorname{supp} u,$$

then we have

$$\phi_t = -\phi, \quad \phi_{tt} = \Delta\phi \quad \text{on } \operatorname{supp} u.$$

Hence from (4.4.11) we obtain

$$F_1'(t)+2F_1(t)+\frac{\mu_1}{1+t}F_1(t)+\int_0^t\left\{\frac{\mu_1}{1+s}+\frac{\mu_1+\mu_2}{(1+s)^2}\right\}F_1(s)ds$$
$$=\varepsilon\int_{\mathbb{R}^n}\{(1+\mu_1)f(x)+g(x)\}\,\phi_1(x)dx+\int_0^t ds\int_{\mathbb{R}^n}|u(x,s)|^p\phi(x,s)dx,$$

from which, after a derivation,

$$F_1''(t)+\left(2+\frac{\mu_1}{1+t}\right)F_1'(t)+\left(\frac{\mu_1}{1+t}+\frac{\mu_2}{(1+t)^2}\right)F_1(t)$$
$$=\int_{\mathbb{R}^n}|u(x,t)|^p\phi(x,t)dx \quad (4.4.12)$$

Let us define the multiplier

$$m(t):=e^t(1+t)^{\frac{\mu_1-1}{2}}>0.$$

Multiplying equation (4.4.12) by $m(t)$, using for convenience the change of variables $z:=1+t$ and denoting

$$\mathcal{B}(z):=m(t)F_1(t), \quad (4.4.13)$$

we obtain that $\mathcal{B}$ satisfies the nonlinear modified Bessel equation

$$z^2\,\frac{d^2\mathcal{B}}{dz^2}(z)+z\,\frac{d\mathcal{B}}{dz}(z)-\left(z^2+\frac{\delta}{4}\right)\mathcal{B}(z)=N(z) \quad (4.4.14)$$

with initial data

$$\begin{aligned}\mathcal{B}(1)&=\varepsilon\int_{\mathbb{R}^n}f(x)\phi_1(x)dx,\\ \frac{d\mathcal{B}}{dz}(1)&=\varepsilon\int_{\mathbb{R}^n}\left\{\frac{\mu_1-1}{2}f(x)+g(x)\right\}\phi_1(x)dx,\end{aligned} \quad (4.4.15)$$

and where

$$N(z):=z^2m\,(z-1)\int_{\mathbb{R}^n}|u(x,z-1)|^p\phi(x,z-1)dx\geq 0.$$

Now, let us estimate the function $\mathcal{B}$.

4.4.2.1 Homogeneous problem

Let us firstly consider the homogeneous Cauchy problem

$$\begin{cases}z^2\,\dfrac{d^2\mathcal{B}_0}{dz^2}(z)+z\,\dfrac{d\mathcal{B}_0}{dz}(z)-\left(z^2+\dfrac{\delta}{4}\right)\mathcal{B}_0(z)=0, \quad z\geq 1,\\ \mathcal{B}_0(1)=\mathcal{B}(1), \quad \dfrac{d\mathcal{B}_0}{dz}(1)=\dfrac{d\mathcal{B}}{dz}(1).\end{cases}$$

The fundamental solutions are the modified Bessel's functions $B^+_{\sqrt{\delta}/2}(z):=I_{\sqrt{\delta}/2}(z)$ and $B^-_{\sqrt{\delta}/2}(z):=K_{\sqrt{\delta}/2}(z)$. Therefore

$$\mathcal{B}_0(z)=\varepsilon c_+B^+_{\sqrt{\delta}/2}(z)+\varepsilon c_-B^-_{\sqrt{\delta}/2}(z),$$

where, thanks to the relations (A.3), (A.4) and (A.5), it holds

$$\begin{aligned} c_{\pm} &= \pm\,\varepsilon^{-1}\left\{\frac{d\mathcal{B}_0}{dz}(1)-\frac{\sqrt{\delta}}{2}\mathcal{B}_0(1)\right\}B^{\mp}_{\sqrt{\delta}/2}(1)+\varepsilon^{-1}\mathcal{B}_0(1)B^{\mp}_{1+\sqrt{\delta}/2}(1)\\ &= \pm\,B^{\mp}_{\sqrt{\delta}/2}(1)\int_{\mathbb{R}^n}h(x)\phi_1(x)dx\\ &\quad+\left[\mp\sqrt{\delta}B^{\mp}_{\sqrt{\delta}/2}(1)+B^{\mp}_{1+\sqrt{\delta}/2}(1)\right]\int_{\mathbb{R}^n}f(x)\phi_1(x)dx\\ &= \begin{cases} \pm\,B^{\mp}_0(1)\displaystyle\int_{\mathbb{R}^n}h(x)\phi_1(x)dx+B^{\mp}_1(1)\int_{\mathbb{R}^n}f(x)\phi_1(x)dx & \text{if } \delta=0,\\ \pm\,B^{\mp}_{\sqrt{\delta}/2}(1)\displaystyle\int_{\mathbb{R}^n}h(x)\phi_1(x)dx+B^{\mp}_{-1+\sqrt{\delta}/2}(1)\int_{\mathbb{R}^n}f(x)\phi_1(x)dx & \text{if } \delta>0. \end{cases} \end{aligned}$$

Due to the assumptions on the initial data and recalling that $B^+_\nu(z), B^-_\nu(z) > 0$ when $\nu > -1$ and $z > 0$ (see for example 9.6.1 in [AS64]), we can deduce that $c_+ > 0$ (see also Remark 4.11). Exploiting the asymptotic expansions for the modified Bessel's functions (A.9) and (A.10), we have that

$$\mathcal{B}_0(z)=\varepsilon\left[c_+\frac{e^z}{\sqrt{2\pi z}}+c_-\sqrt{\frac{\pi}{2z}}e^{-z}\right]\left(1+O\left(\frac{1}{z}\right)\right)$$

where O is the Big O from the Bachmann-Landau notation. Then, there exist two constants $C > 0$ and $z_0 \geq 1$, both independent of ε, such that

$$\mathcal{B}_0(z)\geq C\varepsilon z^{-1/2}e^z \quad \text{for } z\geq z_0. \tag{4.4.16}$$

4.4.2.2 Inhomogeneous problem

Let us consider now the Cauchy problem

$$\begin{cases} z^2\dfrac{d^2\mathcal{B}_N}{dz^2}(z)+z\dfrac{d\mathcal{B}_N}{dz}(z)-\left(z^2+\dfrac{\delta}{4}\right)\mathcal{B}_N(z)=N(z), \quad z\geq 1,\\ \mathcal{B}_N(1)=\dfrac{d\mathcal{B}_N}{dz}(1)=0. \end{cases}$$

Exploiting the method of variation of parameters, we have that

$$\mathcal{B}_N(z)=B^+_{\sqrt{\delta}/2}(z)\int_1^z\xi B^-_{\sqrt{\delta}/2}(\xi)N(\xi)d\xi-B^-_{\sqrt{\delta}/2}(z)\int_1^z\xi B^+_{\sqrt{\delta}/2}(\xi)N(\xi)d\xi.$$

Recalling that $N(z) \geq 0$ and using the fact that $B^+_{\sqrt{\delta}/2}(z)$ is increasing and $B^-_{\sqrt{\delta}/2}(z)$ is decreasing respect to the argument for $z > 0$ (due to the relations (A.4) and (A.5)), we get that

$$\mathcal{B}_N(z)\geq 0 \quad \text{for } z\geq 1. \tag{4.4.17}$$

Since the solution $\mathcal{B}$ to the Cauchy problem (4.4.14)–(4.4.15) is the sum of $\mathcal{B}_0$ and $\mathcal{B}_N$, from estimates (4.4.16) and (4.4.17) we get

$$\mathcal{B}(z)=\mathcal{B}_0(z)+\mathcal{B}_N(z)\gtrsim\varepsilon z^{-1/2}e^z \quad \text{for } z\geq z_0.$$

At this point, recalling the definition (4.4.13) of $\mathcal{B}$ and changing again the variables, we reach

$$F_1(t) \gtrsim \varepsilon(1+t)^{-\mu_1/2} \quad \text{for } t \gtrsim 1. \tag{4.4.18}$$

By Hölder's inequality and using estimates (4.4.10) and (4.4.18), we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} |u(x,t)|^p dx &\geq \left(\int_{\mathbb{R}^n} |\psi_1(x,t)|^{p/(p-1)}\right)^{1-p} |F_1(t)|^p \\ &\gtrsim \varepsilon^p (1+t)^{-(n+\mu_1-1)\frac{p}{2}+n-1}\end{aligned}$$

for $t \gtrsim 1$, plugging which into (4.4.3) and recalling that $L(t)$ is positive for t large enough, we get

$$F_0(t) \gtrsim \varepsilon^p (1+t)^{-\kappa} \int_{T_1}^t (1+s)^{-\lambda} ds \int_{T_1}^s (1+r)^{q+\sqrt{\delta}-1} dr$$

for $t \geq T_1$ with a suitable $T_1 > 0$ independent of ε, and where we define

$$q \equiv q(p) := \kappa - (n+\mu_1-1)\frac{p}{2} + n + 1. \tag{4.4.19}$$

We obtain, for large time $t \gtrsim 1$, that:

- if $q > -\sqrt{\delta}$, then
$$F_0(t) \gtrsim \varepsilon^p t^{-\kappa} \begin{cases} t^q & \text{if } q > 0, \\ \ln(1+t) & \text{if } q = 0, \\ 1 & \text{if } q < 0; \end{cases}$$
- if $q = -\sqrt{\delta}$, then
$$F_0(t) \gtrsim \varepsilon^p t^{-\kappa} \begin{cases} 1 & \text{if } \delta > 0, \\ \ln^2(1+t) & \text{if } \delta = 0; \end{cases}$$
- if $q < -\sqrt{\delta}$, then
$$F_0(t) \gtrsim \varepsilon^p t^{-\kappa} \begin{cases} 1 & \text{if } \delta > 0, \\ \ln(1+t) & \text{if } \delta = 0. \end{cases}$$

Summing all up, we finally deduce the relations in (4.4.6).

4.4.3 Application of the Kato-type lemma

We are ready now to apply the Kato-type lemma, as presented in Section 4.3, twice to two different couples of inequalities, and subsequently we will infer which result is optimal. The calculations in this subsection are all elementary (and quite tedious), so we will only sketch them.

Apply Lemma 4.1 to the inequalities (4.4.4) and (4.4.5), with

$$E=\varepsilon,$$

$$a=\begin{cases}1 & \text{if } H_0=0,\\ \lambda & \text{if } H_0>0,\end{cases}\quad b=(n+\kappa+\lambda)(p-1),\quad c=\begin{cases}0 & \text{if } H_0=0,\\ 1-\operatorname{sgn}\delta & \text{if } H_0>0,\end{cases}$$

$$1<p<p_c:=\begin{cases}p_F(n+\kappa+\sqrt{\delta}) & \text{if } H_0=0,\\ p_F(n+\kappa) & \text{if } H_0>0,\end{cases}$$

$$\gamma=\begin{cases}2\gamma_F(p,n+\kappa+\sqrt{\delta}) & \text{if } H_0=0,\\ 2\gamma_F(p,n+\kappa) & \text{if } H_0>0.\end{cases}$$

We chose $p\in(1,p_c)$ since this is equivalent to $\gamma>0$ for $p>1$. Then, for every $p\in(1,p_c)$, we have $T_\varepsilon\lesssim\widetilde{T}\equiv\widetilde{T}(\varepsilon)$, with

$$\varepsilon^p\widetilde{T}^{\frac{p\gamma}{p-1}}\left[\ln(1+\widetilde{T})\right]^{pc}=1. \tag{4.4.20}$$

Apply Lemma 4.1 to the inequalities (4.4.4) and (4.4.6), with

$$\overline{E}=\varepsilon^p,$$

$$\overline{a}=\begin{cases}\lambda+q & \text{if } q>0,\\ \lambda & \text{if } q\le 0,\end{cases}\quad \overline{b}=(n+\kappa+\lambda)(p-1),\quad \overline{c}=\begin{cases}0 & \text{if } q>0,\\ 2-\operatorname{sgn}\delta & \text{if } q=0,\\ 1-\operatorname{sgn}\delta & \text{if } q<0,\end{cases}$$

$$1<p<\overline{p}_c,\qquad \overline{\gamma}=\begin{cases}\gamma_S(p,n+\mu_1) & \text{if } q>0,\\ 2\gamma_F(p,n+\kappa) & \text{if } q\le 0,\end{cases}$$

where q is the one in (4.4.19) and $\overline{p}_c\in(1,+\infty]$ is defined as the exponent such that $\overline{\gamma}>0$ for $1<p<\overline{p}_c$ (we will explicitly define this exponent later). Then, for every $p\in(1,\overline{p}_c)$, we have $T_\varepsilon\lesssim\widetilde{S}\equiv\widetilde{S}(\varepsilon)$, with

$$\varepsilon^p\widetilde{S}^{\frac{\overline{\gamma}}{p-1}}\left[\ln(1+\widetilde{S})\right]^{\overline{c}}=1. \tag{4.4.21}$$

In both cases, since (4.4.4), (4.4.6) and (4.4.5) are true for $t\ge T_0$ with some time T_0, and since we need to require $\widetilde{T},\widetilde{S}\ge T_0$ to apply the Kato-type lemma, we need to impose also that ε is sufficiently small. From these computations, we deduce the blow-up for $1<p<p_k:=\max\{p_c,\overline{p}_c\}$ and the upper bound of the lifespan estimate $T_\varepsilon\lesssim\min\{\widetilde{T},\widetilde{S}\}$. We will go further in the analysis to clarify these values.

Before moving forward, in order to understand the definition of $\widetilde{S}$ we need to write down more explicitly the definitions of $\overline{c}$, $\overline{p}_c$ and $\overline{\gamma}$, since they depend on q and therefore on the exponent p. Firstly, recall the definition (4.2.13) of $p_*=p_*(n+\mu_1,n-\sqrt{\delta})$ and that, by (4.2.14), for $p>1$ and $\mu_1+n\neq 1$, it holds

$$p=p_*\iff q(p)=0\iff \gamma_S(p,n+\mu_1)=2\gamma_F(p,n+\kappa).$$

We will consider several cases, due to the generality of the constants involved, but what lies beneath is the elementary comparison between the parabola γ_S (line in the case $\mu_1+n=1$) and the line $2\gamma_F$. Also, since we want to be in the hypothesis of the Kato-type lemma, our interest is directed to $\overline{\gamma}>0$, and so we explicit its definition only for the range $1<p<\overline{p}_c$.

4.4.3.1 Case $n + \mu_1 > 1$

Recalling the definition (4.2.6)–(4.2.7) of $d_* := d_*(n + \mu_1)$ and the relation (4.2.8), we have that the following hold true:

$$\begin{gathered} 0 < d_* < 2, \\ \sqrt{\delta} = n - d_* \iff p_* = p_S(n + \mu_1) = p_F(n + \kappa) = \frac{2}{d_*}. \end{gathered}$$

Taking also in account that

$$\begin{gathered} \sqrt{\delta} \le n - d_*(n + \mu_1) \iff p_* \ge p_S(n + \mu_1), \\ \sqrt{\delta} < n + 2 \iff p_* > 1, \\ q > 0 \iff p < p_*, \end{gathered}$$

we have:

- if $\sqrt{\delta} \le n - d_*$, then

$$\begin{gathered} \overline{p}_c = p_S(n + \mu_1), \\ \overline{\gamma} = \gamma_S(p, n + \mu_1), \quad \text{for } 1 < p < \overline{p}_c, \\ \overline{c} = 0; \end{gathered}$$

- if $n - d_* < \sqrt{\delta} < n + 2$, then

$$\begin{gathered} \overline{p}_c = p_F(n + \kappa), \\ \overline{\gamma} = \begin{cases} \gamma_S(p, n + \mu_1), & \text{for } 1 < p < p_*, \\ 2\gamma_F(p, n + \kappa), & \text{for } p_* \le p < \overline{p}_c, \end{cases} \\ \overline{c} = \begin{cases} 0, & \text{for } 1 < p < p_*, \\ 2 - \operatorname{sgn} \delta, & \text{for } p = p_*, \\ 1 - \operatorname{sgn} \delta, & \text{for } p_* < p < \overline{p}_c; \end{cases} \end{gathered}$$

- if $\sqrt{\delta} \ge n + 2$, then

$$\begin{gathered} \overline{p}_c = p_F(n + \kappa), \\ \overline{\gamma} = 2\gamma_F(p, n + \kappa) \quad \text{for } 1 < p < \overline{p}_c, \\ \overline{c} = 1 - \operatorname{sgn} \delta. \end{gathered}$$

4.4.3.2 Case $n + \mu_1 = 1$

Taking in account that

$$q > 0 \iff \sqrt{\delta} < n + 2$$

we have:

- if $\sqrt{\delta} < n + 2$, then

$$\begin{gathered} \overline{p}_c = p_S(n + \mu_1) = p_S(1) = +\infty, \\ \overline{\gamma} = \gamma_S(p, n + \mu_1) = \gamma_S(p, 1) = 2 + 2p, \quad \text{for } 1 < p < \overline{p}_c, \\ \overline{c} = 0; \end{gathered}$$

- if $\sqrt{\delta} = n+2$, then

$$\begin{gathered}\overline{p}_c = p_S(n+\mu_1) = p_F(n+\kappa) = +\infty,\\ \overline{\gamma} = \gamma_S(p, n+\mu_1) = 2\gamma_F(p, n+\kappa) = 2+2p, \quad \text{for } 1<p<\overline{p}_c,\\ \overline{c} = 2 - \operatorname{sgn}\delta;\end{gathered}$$

- if $\sqrt{\delta} > n+2$, then

$$\begin{gathered}\overline{p}_c = p_F(n+\kappa) = p_F\left((n-\sqrt{\delta})/2\right) = +\infty,\\ \overline{\gamma} = 2\gamma_F(p, n+\kappa) = 2\gamma_F\left(p, (n-\sqrt{\delta})/2\right), \quad \text{for } 1<p<\overline{p}_c,\\ \overline{c} = 1 - \operatorname{sgn}\delta.\end{gathered}$$

4.4.3.3 Case $n+\mu_1 < 1$

Taking in account that

$$\begin{aligned}p_* > 1 &\iff \sqrt{\delta} > n+2,\\ q > 0 &\iff p > p_*,\end{aligned}$$

we have:

- if $\sqrt{\delta} \le n+2$, then

$$\begin{gathered}\overline{p}_c = p_S(n+\mu_1) = +\infty,\\ \overline{\gamma} = \gamma_S(p, n+\mu_1) \quad \text{for } 1<p<\overline{p}_c,\\ \overline{c} = 0;\end{gathered}$$

- if $\sqrt{\delta} > n+2$, then

$$\begin{gathered}\overline{p}_c = p_S(n+\mu_1) = +\infty,\\ \overline{\gamma} = \begin{cases} 2\gamma_F(p, n+\kappa), & \text{for } 1<p\le p_*,\\ \gamma_S(p, n+\mu_1), & \text{for } p_*<p<\overline{p}_c,\end{cases}\\ \overline{c} = \begin{cases} 1-\operatorname{sgn}\delta, & \text{for } 1<p<p_*,\\ 2-\operatorname{sgn}\delta, & \text{for } p=p_*,\\ 0, & \text{for } p_*<p<\overline{p}_c.\end{cases}\end{gathered}$$

Now that the definitions of $p_c, \overline{p}_c$ and $\widetilde{T}, \widetilde{S}$ are clear, we can proceed further.

4.4.4 Proof for Theorem 4.2 and Theorem 4.4

As we said, from our computations we found the blow-up for $1<p<p_k=\max\{p_c,\overline{p}_c\}$ and the upper bound of the lifespan estimates $T_\varepsilon \lesssim \min\{\widetilde{T}, \widetilde{S}\}$. Observing that

$$\widetilde{T}(\varepsilon),\ \widetilde{S}(\varepsilon) \to +\infty \quad \text{for } \varepsilon \to 0^+$$

and comparing the relations (4.4.20) and (4.4.21), we get that

$$p\gamma \gtrless \overline{\gamma} \Longrightarrow \widetilde{T} \lessgtr \widetilde{S}.$$

If $p\gamma = \overline{\gamma}$, the exponent of the logarithm comes into play, indeed

$$pc \gtreqless \overline{c} \Longrightarrow \widetilde{T} \lesseqgtr \widetilde{S}.$$

Now, we need to consider two cases according to the fact that $H_0 = \int_{\mathbb{R}^n} h(x)dx$ is positive or null.

4.4.4.1 Case $H_0 > 0$

We can easily infer that $p_k = p_{\mu_1,\delta}(n)$ defined in (4.2.9). We establish the upper bound for the lifespan T_ε without making distinctions according to the value of $n + \mu_1$. Taking in account that, for $p > 1$,

$$2p\,\gamma_F(p, n+\kappa) > \gamma_S(p, n+\mu_1) \Longleftrightarrow \begin{cases} p > 1, & \text{if } \sqrt{\delta} \geq n, \\ 1 < p < \dfrac{2}{n-\sqrt{\delta}}, & \text{if } n-2 < \sqrt{\delta} < n, \end{cases}$$

$$n - d_* < \sqrt{\delta} < n \text{ and } n + \mu_1 > 1 \Longrightarrow p_F(n+\kappa) < \frac{2}{n-\sqrt{\delta}},$$

$$\sqrt{\delta} \leq n - d_* \text{ and } 1 < p < p_k \Longrightarrow q > 0,$$

we have:

- if $\sqrt{\delta} \leq n-2$ and $1 < p < p_k$, then $p\gamma < \overline{\gamma}$ and so $\widetilde{S} < \widetilde{T}$;
- if $n - 2 < \sqrt{\delta} < n - d_*$ and
  - if $1 < p < \frac{2}{n-\sqrt{\delta}}$, then $p\gamma > \overline{\gamma}$ and so $\widetilde{T} < \widetilde{S}$;
  - if $p = \frac{2}{n-\sqrt{\delta}}$, then $p\gamma = \overline{\gamma}$ and $pc \geq \overline{c}$, so that $\widetilde{T} \leq \widetilde{S}$;
  - if $\frac{2}{n-\sqrt{\delta}} < p < p_k$, then $p\gamma < \overline{\gamma}$, so that $\widetilde{S} < \widetilde{T}$;
- if $\sqrt{\delta} \geq n - d_*$ and if $1 < p < p_k$, then $p\gamma > \overline{\gamma}$ so that $\widetilde{T} < \widetilde{S}$.

4.4.4.2 Case $H_0 = 0$

From now on we will impose the additional hypothesis that $\mu_1 > 0$ (which however can be relaxed to $n + \mu_1 > 1$).

Obviously, $p_F(n + \kappa + \sqrt{\delta}) \leq p_F(n + \kappa)$, hence again $p_k = p_{\mu_1,\delta}(n)$ defined in (4.2.9). Consider that, for $p > 1$,

$$p\gamma_F(p, n+\kappa+\sqrt{\delta}) > \gamma_F(p, n+\kappa) \Longleftrightarrow \sqrt{\delta} < 2 \text{ and } 1 < p < 1 + \frac{2-\sqrt{\delta}}{n+\kappa+\sqrt{\delta}},$$

$$2p\,\gamma_F(p, n+\kappa+\sqrt{\delta}) > \gamma_S(p, n+\mu_1) \Longleftrightarrow n = 1 \text{ and } \sqrt{\delta} < 1 \text{ and } 1 < p < \frac{2}{1+\sqrt{\delta}}.$$

If $n \geq 2$, taking into account that

$$n - d_* < \sqrt{\delta} < n + 2 \Longrightarrow 1 + \frac{2 - \sqrt{\delta}}{n + \kappa + \sqrt{\delta}} < p_*,$$

we can prove that $p\gamma < \overline{\gamma}$ for $1 < p < p_k$, and so $\widetilde{S} < \widetilde{T}$.

Suppose now that $n = 1$. Recall the definition (4.2.16) of θ and note that it satisfies $\operatorname{sgn}\theta = \operatorname{sgn}(\mu_1 - 3)$. Moreover the following relations hold:

$$\begin{aligned}
\mu_1 > 0 &\Longrightarrow 1 - d_* < 1 \text{ and } 1 + \frac{2 - \sqrt{\delta}}{n + \kappa + \sqrt{\delta}} < p_S(1 + \mu_1),\\
0 < \mu_1 < 3 &\Longleftrightarrow 1 - d_* > 0,\\
0 < \mu_1 < 3 &\Longrightarrow |1 - d_*| > \theta,\\
\sqrt{\delta} > -1 + d_* &\Longrightarrow \frac{2}{1 + \sqrt{\delta}} < p_S(1 + \mu_1),\\
\theta < \sqrt{\delta} < 3 &\Longrightarrow 1 + \frac{2 - \sqrt{\delta}}{n + \kappa + \sqrt{\delta}} < p_* \text{ and } \frac{2}{1 + \sqrt{\delta}} < p_*.
\end{aligned}$$

Recall also the definition (4.2.15) of $r_* \equiv r_*(\mu_1, \delta)$ and Remark 4.9. Hence, we get that:

- if $\sqrt{\delta} = 0$, $\mu_1 = 3$ and if $1 < p < p_k$, then $p\gamma > \overline{\gamma}$ and so $\widetilde{T} < \widetilde{S}$;
- if $\sqrt{\delta} = 0$ and $\mu_1 \neq 3$, or if $0 < \sqrt{\delta} < 1$, we have:
  - if $1 < p < r_*$, then $p\gamma > \overline{\gamma}$ and so $\widetilde{T} < \widetilde{S}$;
  - if $p = r_*$, then $p\gamma = \overline{\gamma}$ and $pc \leq \overline{c}$, so that $\widetilde{S} \leq \widetilde{T}$;
  - if $r_* < p < p_k$, then $p\gamma < \overline{\gamma}$, so that $\widetilde{S} < \widetilde{T}$;
- if $\sqrt{\delta} \geq 1$ and if $1 < p < p_k$, then $p\gamma < \overline{\gamma}$ so that $\widetilde{S} < \widetilde{T}$.

In the end, recalling the definitions of $\gamma, \overline{\gamma}, c$ and $\overline{c}$ in the various cases and summing all up, we can conclude the proof for Theorem 4.2 and Theorem 4.4.

4.4.5 Proof for Theorem 4.5

We will only sketch the demonstration, since it is a variation of the previous one. Let us introduce the functional

$$G_0(t) = \int_{\mathbb{R}^n} w(x,t)dx$$

and, as in [LST19, LST20], the bounded multiplier

$$m(t) := \exp\left(\nu_1 \frac{(1+t)^{1-\beta}}{1-\beta}\right).$$

Choosing the test function $\phi = \phi(x,s)$ in (4.2.19) to satisfy (4.4.1), deriving respect to the time and multiplying by m, we get that

$$[m(t)G_0'(t)]' + \frac{\nu_2}{(1+t)^2}m(t)G_0(t) = m(t)\int_{\mathbb{R}^n}|w(x,t)|^p dx,$$

and hence

$$\begin{aligned}G_0(t) &= G_0(0) + m(0)G_0'(0)\int_0^t m^{-1}(s)ds \\ &\quad - \int_0^t m^{-1}(s)ds\int_0^s m(r)\frac{\nu_2}{(1+r)^2}G_0(r)dr \\ &\quad + \int_0^t m^{-1}(s)ds\int_0^s m(r)dr\int_{\mathbb{R}^n}|w(x,r)|^p dx.\end{aligned} \tag{4.4.22}$$

It is simple to see, by a comparison argument, that G_0 is positive. Indeed, by the hypothesis on initial data, we know that $G_0(0) = \int_{\mathbb{R}^n} f(x)dx$ and $G_0'(0) = \int_{\mathbb{R}^n} g(x)dx$ are non-negative and not both zero. If $G_0(0) > 0$, by continuity G_0 is positive for small time. If $G_0(0) = 0$ and $G'(0) > 0$, then G_0 is increasing and again positive for small time $t > 0$. If we suppose that there exists a time $t_0 > 0$ such that $G_0(t_0) = 0$, calculating (4.4.22) in $t = t_0$ we get a contradiction, since the left-hand term would be zero and the right-hand term would be strictly positive. Then, G_0 is positive for any time $t > 0$. Define now the functional $\overline{G}_0$ as the solution of the integral equation

$$\begin{aligned}\overline{G}_0(t) &= \frac{1}{2}G_0(0) + \frac{m(0)}{2}G_0'(0)t - m(0)\int_0^t ds\int_0^s \frac{\nu_2}{(1+r)^2}\overline{G}_0(r)dr \\ &\quad + m(0)\int_0^t ds\int_0^s dr\int_{\mathbb{R}^n}|w(x,r)|^p dx.\end{aligned} \tag{4.4.23}$$

Since $m(0) < m(t) < 1$ for any $t > 0$ and $\nu_2 < 0$, we have that

$$\begin{aligned}G_0(t) - \overline{G}_0(t) \geq \frac{1}{2}G_0(0) &+ \frac{m(0)}{2}G_0'(0)t \\ &- m(0)\int_0^t ds\int_0^s \frac{\nu_2}{(1+r)^2}[G_0(r) - \overline{G}_0(r)]dr,\end{aligned}$$

and, again by a comparison argument, we may infer that $G_0 \geq \overline{G}_0$. From (4.4.23) we get that $\overline{G}_0$ satisfies

$$\overline{G}_0''(t) + \frac{m(0)\nu_2}{(1+t)^2}\overline{G}(t) = m(0)\int_{\mathbb{R}^n}|w(x,t)|^p dx,$$

which has the same structure of (4.4.2) with $\mu_1 = 0$ and $\mu_2 = m(0)\nu_2$. Setting

$$\lambda := 1 + \sqrt{\delta}, \quad \kappa := -\lambda/2, \quad \mathcal{G}(t) := (1+t)^{\kappa+\lambda}\overline{G}_0(t),$$

similarly as in Subsection 4.4.1 we obtain

$$\begin{aligned}\overline{G}_0(t) &= \overline{G}_0(0)(1+t)^{-\kappa} + [\kappa\overline{G}_0(0) + \overline{G}_0'(0)](1+t)^{-\kappa}\int_0^t (1+s)^{-\lambda}ds \\ &\quad + (1+t)^{-\kappa}\int_0^t (1+s)^{-\lambda}ds\int_0^s (1+r)^{\kappa+\lambda}dr\int_{\mathbb{R}^n}|w(x,r)|^p dx\end{aligned} \tag{4.4.24}$$

and then

$$\mathcal{G}(t) \gtrsim \int_{T_0}^{t} ds \int_{T_0}^{s} r^{-(n+\kappa+\lambda)(p-1)} \mathcal{G}(r)^p dr, \tag{4.4.25}$$

$$\mathcal{G}(t) \gtrsim \varepsilon t^{\lambda}. \tag{4.4.26}$$

Now, to get the counterpart of (4.4.6), define the functional

$$G_1(t) := \int_{\mathbb{R}^n} w(x,t)\psi_1(x,t)dx,$$

with ψ_1 defined in (4.4.9). After taking a derivative respect to the time in the definition of energy solution (4.2.19) and multiplying both of its sides with $m(t)$, we have that

$$\begin{aligned}
&\frac{d}{dt}\left\{m(t)\int_{\mathbb{R}^n} w_t(x,t)\phi(x,t)dx\right\} \\
&+ m(t)\int_{\mathbb{R}^n}\{-w_t(x,t)\phi_t(x,t) - w(x,t)\Delta\phi(x,t)\}\,dx \\
=& -m(t)\int_{\mathbb{R}^n}\frac{\nu_2}{(1+t)^2}w(x,t)\phi(x,t)dx + m(t)\int_{\mathbb{R}^n}|w(x,t)|^p\phi(x,t)dx.
\end{aligned}$$

By integration on $[0,t]$ we get

$$\begin{aligned}
&m(t)\int_{\mathbb{R}^n} w_t(x,t)\phi(x,t)dx - m(0)\varepsilon\int_{\mathbb{R}^n} g(x)\phi(x,0)dx \\
&- m(t)\int_{\mathbb{R}^n} w(x,t)\phi_t(x,t)dx + m(0)\varepsilon\int_{\mathbb{R}^n} f(x)\phi_t(x,0)dx \\
&+ \int_0^t ds\int_{\mathbb{R}^n} m(s)\frac{\nu_1}{(1+s)^\beta}w(x,s)\phi_t(x,s)dx \\
&+ \int_0^t ds\int_{\mathbb{R}^n} m(s)w(x,s)\phi_{tt}(x,s)dx - \int_0^t ds\int_{\mathbb{R}^n} m(s)w(x,s)\Delta\phi(x,s)dx \\
=& -\int_0^t ds\int_{\mathbb{R}^n} m(s)\frac{\nu_2}{(1+s)^2}w(x,s)\phi(x,s)dx \\
&+ \int_0^t ds\int_{\mathbb{R}^n} m(s)|w(x,s)|^p\phi(x,s)dx.
\end{aligned}$$

Setting $\phi(x,t) = \psi_1(x,t) = e^{-t}\phi_1(x)$ on $\operatorname{supp} w$ and recalling the bounds on the multiplier $m(t)$, we obtain

$$\begin{aligned}
G_1'(t) + 2G_1(t) \geq\ & m(0)G_1'(0) + 2m(0)G_1(0) \\
&+ m(0)\int_0^t\left\{\frac{\nu_1}{(1+s)^\beta} - \frac{\nu_2}{(1+s)^2}\right\}G_1(s)ds \\
&+ m(0)\int_0^t ds\int_{\mathbb{R}^n}|w(x,s)|^p dx.
\end{aligned}$$

Integrating the above inequality over $[0,t]$ after a multiplication by e^{2t}, we get

$$\begin{aligned}G_1(t) &\geq G_1(0)e^{-2t} + m(0)\{G_1'(0) + 2G_1(0)\}\frac{1-e^{-2t}}{2}\\&+ m(0)e^{-2t}\int_0^t e^{2s}ds\int_0^s\left\{\frac{\nu_1}{(1+r)^\beta} - \frac{\nu_2}{(1+r)^2}\right\}G_1(r)dr\\&+ m(0)e^{-2t}\int_0^t e^{2s}ds\int_0^s dr\int_{\mathbb{R}^n}|w(x,r)|^p\phi(x,r)dx,\end{aligned}$$

from which, thanks again to a comparison argument, we infer that G_1 is non-negative, and so, neglecting the last two term in the above inequality, it is easy to reach

$$G_1(t) \gtrsim \varepsilon \quad \text{for } t \gtrsim 1.$$

Hence, we have also

$$\int_{\mathbb{R}^n}|w(x,t)|^p dx \gtrsim \varepsilon^p(1+t)^{-(n-1)\frac{p}{2}+n-1} \quad \text{for } t \gtrsim 1,$$

and so, taking into account (4.4.24), it holds

$$\overline{G}_0(t) \gtrsim \varepsilon^p(1+t)^{-\kappa}\int_{T_1}^t(1+s)^{-\lambda}ds\int_{T_1}^s(1+r)^{q+\sqrt{\delta}-1}dr \quad \text{for } t \geq T_1,$$

for some $T_1 > 0$, where

$$q \equiv q(p) := -\frac{1+\sqrt{\delta}}{2} - (n-1)\frac{p}{2} + n + 1.$$

Finally, we obtain the inequality analogous to (4.4.6), i.e.

$$\mathcal{G}(t) \gtrsim \varepsilon^p\begin{cases}t^{\lambda+q} & \text{if } q > 0,\\ t^\lambda\ln(1+t) & \text{if } q = 0,\\ t^\lambda & \text{if } q < 0.\end{cases} \tag{4.4.27}$$

Thanks to (4.4.25), (4.4.26) and (4.4.27) and applying the Kato-type lemma as in Subsection 4.4.3, we can conclude the proof of Theorem 4.5.

Chapter 5

Blow-up and lifespan estimate for generalized Tricomi equations related to Glassey conjecture

In this chapter, we consider the small data Cauchy problem for the semilinear generalized Tricomi equations with a power-nonlinearity of derivative type, suggesting the *papabili* candidates both for the critical exponent and for the lifespan estimates. Other than the blow-up phenomena, we prove also a local existence result.

The reference for this chapter is [S5], joint work with Ning-An Lai.

5.1 The generalized Tricomi model

The object of our investigation is the problem

$$\begin{cases} u_{tt} - t^{2m}\Delta u = |u_t|^p & \text{in } [0,T)\times\mathbb{R}^n, \\ u(x,0) = \varepsilon f(x), \quad u_t(x,0) = \varepsilon g(x), & x\in\mathbb{R}^n, \end{cases} \tag{5.1.1}$$

where $m \geq 0$ is a real constant, $n \geq 1$ is the dimension and $\varepsilon > 0$ is a "small" parameter. The initial data f, g are compactly supported functions from the energy spaces

$$f \in H^1(\mathbb{R}^n), \quad g \in H^{1-\frac{1}{m+1}}(\mathbb{R}^n),$$

and, without loss of generality, we may assume

$$\operatorname{supp} f, \operatorname{supp} g \subseteq \{x \in \mathbb{R}^n : |x| \leq 1\}. \tag{5.1.2}$$

The mathematical investigation of the semilinear generalized Tricomi equations and related models is motivated by the fact that such kind of equations apper in the study of gas dynamic problems, see e.g. [Ber58]. If we set ourselves in dimension $n = 1$, letting $m = 1/2$, the equation becomes

$$u_{tt} - t u_{xx} = 0,$$

namely the classical linear Tricomi equation, introduced by the Italian mathematician in [Tri23] apropos of boundary value problems for partial differential equations of mixed-type. Later, Frankl [Fra45] highlighted the connection between the study of gas flows with nearly sonic speed and the Tricomi equation, which indeed describes the transition from subsonic flow (for $t < 0$, when the Tricomi equation is elliptic) to supersonic flow (for $t > 0$, when it is hyperbolic). For more details and applications, we refer to the series of works by Yagdjian [Yag04, Yag06, Yag07a, Yag07b, Yag07c] and to the references therein, such as the already cited [Ber58] and moreover [CC86, Ger98, Mor82, Mor04, Noc86, Ras90].

For $k > 0$ and $n \geq 1$, the operator

$$\mathscr{T} := \partial_t^2 - t^{2k}\Delta \tag{5.1.3}$$

is also known as Gellerstedt operator. The first steps in the study of (generalized) Tricomi equations move in the direction of building the explicit fundamental solution. In their works [BNG99, BNG02, BNG05], Barros-Neto and Gelfand established the fundamental solution for

$$yu_{xx} + u_{yy} = 0$$

in the whole plane. Instead, for the Gellerstedt operator (5.1.3) with $2k \in \mathbb{N}$, Yagdjian constructed in [Yag04] a fundamental solution with support located in the "forward cone"

$$\mathcal{C}(t_0, x_0) := \left\{ (t,x) \in \mathbb{R}^{n+1} \colon |x - x_0| \leq \frac{t^{k+1} - t_0^{k+1}}{k+1} \right\}$$

and relative to any arbitrary point $(t_0, x_0) \in [0, +\infty) \times \mathbb{R}^n$.

Recently, the long time behavior of solutions for small data Cauchy problem of the semilinear generalized Tricomi equation

$$\begin{cases} u_{tt} - t^{2k}\Delta u = |u|^p, & \text{in } [0,T) \times \mathbb{R}^n \\ u(x,0) = \varepsilon f(x), \quad u_t(x,0) = \varepsilon g(x), & x \in \mathbb{R}^n, \end{cases} \tag{5.1.4}$$

has attracted scholarly attention. The main goal is determining the *critical power* $p_c(k,n)$, namely, as we know, the value such that if $1 < p \leq p_c(k,n)$ then the solution blows up in a finite time, whereas if $p > p_c(k,n)$ there exists a unique global-in-time solution. Yagdjian [Yag06] obtained some partial results, nevertheless in his work there was still a gap between the blow-up and global existence ranges. The critical power was finally established in a recent series of works by He, Witt and Yin [HWY17a, HWY17b, HWY16, HWY18] (see also the doctoral dissertation by He [He16]). For $k > 1/2$, $p_c(k,n)$ admits the following expression:

- if $n = 1$, then $p_c(k,1) = 1 + \dfrac{2}{k}$;
- if $n \geq 2$, then $p_c(k,n)$ is the positive root of the quadratic equation

$$2 + \left[n + 1 - 3\left(1 - \frac{1}{k+1}\right)\right] p - \left[n - 1 + \left(1 - \frac{1}{k+1}\right)\right] p^2 = 0.$$

Lately, Lin and Tu [LT19b] studied the upper bound of lifespan estimate for (5.1.4), and Ikeda, Lin and Tu [ILT21] established the blow-up and upper bound of lifespan estimate for the weakly coupled system of the generalized Tricomi equations with multiple propagation speed. The critical power above should be compared with the corresponding one for the semilinear wave equation $u_{tt} - \Delta u = |u|^p$. Indeed, letting $k = 0$ in the definition of $p_c(k, n)$, we infer $p_c(0, 1) = +\infty$ and, for $n \geq 2$, $p_c(0, n)$ becomes the Strauss exponent, which is the critical power for the small data Cauchy problem in (5.1.4) with $k = 0$ (see Chapter 4). Finally, we refer also to Ruan, Witt and Yin [RWY14, RWY15a, RWY15b, RWY18] for results about the local existence and local singularity structure of low regularity solutions for the equation $u_{tt} - t^k \Delta u = f(t, x, u)$.

In this chapter, we consider the semilinear generalized Tricomi equations with power-nonlinearity of derivative type, focusing on blow-up result and lifespan estimate from above for the small data Cauchy problem. Note that, setting $m = 0$ in (5.1.1), we come back to the semilinear wave equation

$$u_{tt} - \Delta u = |u_t|^p. \tag{5.1.5}$$

For this problem, Glassey [Gla] conjectured that the critical exponent is the power, now named after him, defined by

$$p_G(n) := \begin{cases} 1 + \dfrac{2}{n-1} & \text{if } n \geq 2, \\ +\infty & \text{if } n = 1. \end{cases} \tag{5.1.6}$$

The research on this problem was initiated by John [Joh81], where more general equations in dimension $n = 3$ are considered, proving the blow-up of solutions for $p = 2$. Then, the study of the blow-up was continued in the low dimensional case by Masuda [Mas], Schaeffer [Sch86], John [Joh85] and Agemi [Age91], whereas Rammaha [Ram87] treated the high dimensional case $n \geq 4$ under radial symmetric assumptions. Finally, Zhou [Zho01] proved the blow-up for $n \geq 1$ and $1 < p \leq p_G(n)$, furnishing the upper bound for the lifespan of the solutions, namely

$$T_\varepsilon \leq \begin{cases} C\varepsilon^{-\frac{2(p-1)}{2-(n-1)(p-1)}} & \text{if } 1 < p < p_G(n), \\ \exp(C\varepsilon^{-(p-1)}) & \text{if } p = p_G(n), \end{cases} \tag{5.1.7}$$

for some positive constant C independent of ε. We recall that the *lifespan* T_ε is defined as the maximal existence time of the solution, depending on the parameter ε. Regarding the global existence part, we refer to Sideris [Sid83], Hidano and Tsutaya [HT95] and Tzvetkov [Tzv98] for results in dimension $n = 2, 3$ and Hidano, Wang15 and Yokoyama [HWY12] for the high dimensional cases $n \geq 4$ under radially symmetric assumptions. For more details about the Glassey conjecture, one can see the references [LT19a] and [Wan15].

The study of problem (5.1.1) under consideration generalizes the Glassey conjecture. Therefore, it is interesting to find the critical exponent and lifespan estimate for (5.1.1), which will coincide with the Glassey exponent (5.1.6) and Zhou's lifespan estimate (5.1.7) respectively for $m = 0$. The main tool we are going to use is the test function method. In [HWY17b], the blow-up result for (5.1.4) is based on a test function given by the product of the harmonic function $\int_{\mathbb{S}^{n-1}} e^{x \cdot \omega} d\omega$ and the solution of the ordinary differential equation

$$\lambda''(t) - t^k \lambda'(t) = 0.$$

Inspired by the works [ISW19] and [LT20], we construct a nonnegative test function composed by a cut-off function, the harmonic function $\int_{\mathbb{S}^{n-1}} e^{x\cdot\omega} d\omega$ and the solution of the ODE (5.4.1) below. Since we consider the Tricomi-type equations with derivative nonlinear term, the first derivative with respect to the time variable and a factor t^{-2m} are included in the special test function. Before proving the blow-up result, we give also a local existence result following the approach of Yagdjian [Yag06], from which we can deduce the optimality of the lifespan estimates at least for $n = 1$.

When the paper [S5], the results of which this chapter refers to, was almost finished, we found the paper [LP21] by Lucente and Palmieri, where they independently studied the same problem with a different approach. However, the result we are going to present here seems to improve the blow-up range and lifespan estimates there found.

We cite also the very recent papers [CLP21] by Chen, Lucente and Palmieri and [HH20] by Hamouda and Hamza, where the blow-up phenomena for generalized Tricomi equations with combined nonlinearities, i.e. $u_{tt} - t^{2m}\Delta u = |u_t|^p + |u|^q$, is independently studied exploiting the iteration argument. In particular, the work [HH20] confirms the blow-up result presented in this chapter by giving an alternative proof. Conversely, we are confident that also our method can be adapted to study various blow-up problems involving generalized Tricomi equations, including the combined nonlinearity. This means that the test function method presented in the following and the iteration argument developed in [CLP21, HH20] furnish two different approaches for the study of blow-up phenomena for Tricomi-related problems.

5.2 Main result

Let us start stating the definition of energy solution for our problem (5.1.1), similarly as in [ISW19, LT18] and as in the previous Chapter 4.

Definition 5.1. We say that the function

$$u \in C([0,T), H^1(\mathbb{R}^n)) \cap C^1([0,T), H^{1-\frac{1}{m+1}}(\mathbb{R}^n)), \quad \text{with } u_t \in L^p_{\text{loc}}((0,T)\times\mathbb{R}^n),$$

is a weak solution of (5.1.1) on $[0,T)$ if

$$u(0,x) = \varepsilon f(x) \text{ in } H^1(\mathbb{R}^n), \qquad u_t(0,x) = \varepsilon g(x) \text{ in } H^{1-\frac{1}{m+1}}(\mathbb{R}^n)$$

and

$$\begin{aligned} &\varepsilon \int_{\mathbb{R}^n} g(x)\Psi(0,x)dx + \int_0^T \int_{\mathbb{R}^n} |u_t|^p \Psi(t,x)\,dxdt \\ &= \int_0^T \int_{\mathbb{R}^n} -u_t(t,x)\Psi_t(t,x)\,dxdt + \int_0^T \int_{\mathbb{R}^n} t^{2m}\nabla u(t,x)\cdot\nabla\Psi(t,x)\,dxdt, \end{aligned} \tag{5.2.1}$$

for any $\Psi(t,x) \in C_0^1([0,T)\times\mathbb{R}^n) \cap C^\infty((0,T)\times\mathbb{R}^n)$.

Remark 5.1. The choice of the functional spaces $H^1(\mathbb{R}^n)$ and $H^{1-\frac{1}{m+1}}(\mathbb{R}^n)$ for the initial data $u(0,x)$ and $u_t(0,x)$ respectively are suggested by [He16] and by Theorem 5.2 below. Of course, if $m = 0$ we have $H^0(\mathbb{R}^n) = L^2(\mathbb{R}^n)$.

In the same spirit of Chapter 4, let us define the exponent

$$p_T(n,m) := \begin{cases} 1+\dfrac{2}{(m+1)(n-1)-m} & \text{if } n\geq 2, \\ +\infty & \text{if } n=1, \end{cases}$$

as the root (when $n \geq 2$) of the expression $\gamma_T(n,m;p)=0$, where

$$\gamma_T(n,m;p) := 2-[(m+1)(n-1)-m](p-1),$$

and observe that $\gamma_T(n,m;p)>0$ for $1<p<p_T(n,m)$.

We state now our main result for (5.1.1).

Theorem 5.1. *Let $n \geq 1$, $m \geq 0$ and $1 < p \leq p_T(n,m)$. Assume that $f \in H^1(\mathbb{R}^n)$, $g \in H^{1-\frac{1}{m+1}}(\mathbb{R}^n)$ satisfy the compact support assumption (5.1.2) and that*

$$a(m)f+g, \quad a(m) := [2(m+1)]^{\frac{m}{m+1}} \frac{\Gamma\left(\frac{1}{2}+\frac{m}{2(m+1)}\right)}{\Gamma\left(\frac{1}{2}-\frac{m}{2(m+1)}\right)}, \tag{5.2.2}$$

is non-negative and not identically vanishing. Suppose that u is an energy solution of (5.1.1) with compact support in the "cone"

$$\operatorname{supp} u \in \left\{(t,x)\in[0,T)\times\mathbb{R}^n \colon |x| \leq \gamma(t) := 1+\frac{t^{m+1}}{m+1}\right\}. \tag{5.2.3}$$

Then, there exists a constant $\varepsilon_0=\varepsilon_0(f,g,m,n,p)>0$ such that the lifespan T_ε satisfies

$$T_\varepsilon \leq \begin{cases} C\varepsilon^{-\frac{2(p-1)}{\gamma_T(n,m;p)}} & \text{if } 1<p<p_T(n,m), \\ \exp\left(C\varepsilon^{-(p-1)}\right) & \text{if } p=p_T(n,m), \end{cases} \tag{5.2.4}$$

for $0<\varepsilon\leq\varepsilon_0$ and some positive constant C independent of ε.

Remark 5.2. For $m=0$ the exponent p_T becomes the Glassey exponent (5.1.6), namely $p_T(n,0)=p_G(n)$, and the lifespan estimate (5.2.4) is exactly the same as (5.1.7).

Remark 5.3. It is interesting to note that, if $n=2$, then the blow-up power $p_T(2,m)=3$ and the subcritical lifespan estimate $T_\varepsilon \leq C\varepsilon^{-\left(\frac{1}{p-1}-\frac{1}{2}\right)^{-1}}$ are independent of m.

Remark 5.4. We conjecture that $p_T(n,m)$ is indeed the critical exponent for problem (5.1.1) and the lifespan (5.2.4) are optimal. The next goal should be to verify this conjecture considering the global-in-time existence for solutions to (5.1.1).

5.3 Local existence result

Before to proceed with the demonstration of Theorem 5.1 in Section 5.4, we firstly want to present in this section a local existence result. As observed in [LP21], it is possible to prove a local-in-time existence result for problem (5.1.1), regardless the size of the Cauchy data, following the steps in Section 2.1 of [DDG01]. However, we believe that the following

Theorem 5.2 is interesting to justify the choice of the energy space for the solution and the initial data in Theorem 5.1. In addition, we can verify the optimality of the lifespan estimate in the 1-dimensional case.

Let us consider the integral equation

$$\begin{aligned} u(t,x) =& \varepsilon V_1(t,D_x)f(x) + \varepsilon V_2(t,D_x)g(x) \\ &+ \int_0^t [V_2(t,D_x)V_1(s,D_x) - V_1(t,D_x)V_2(s,D_x)]|u_t(s,x)|^p ds \end{aligned} \tag{5.3.1}$$

where $\varepsilon > 0$ is not necessarily small, $f \in H^1(\mathbb{R}^n)$, $g \in H^{1-\frac{1}{m+1}}(\mathbb{R}^n)$ and the Fourier multiplier $V_1(t,D_x)$ and $V_2(t,D_x)$ will be defined below. As remarked in [Yag06], any classical or distributional solution to our problem (5.1.1) solves also the integral equation (5.3.1). We have the following result.

Theorem 5.2. *Let* $0 \le m < 2$, $p > \max\{2, 1+\frac{n}{2}\}$, $\sigma \in \left(\frac{n}{2} - \frac{m}{2(m+1)}, p-1-\frac{m}{2(m+1)}\right)$ *and* $f \in H^{\sigma+1}(\mathbb{R}^n)$, $g \in H^{\sigma+1-\frac{1}{m+1}}(\mathbb{R}^n)$. *Then there exists a unique solution* $u = u(t,x)$ *to equation* (5.3.1) *satisfying*

$$u \in C\left((0,T); \dot{H}^{\sigma+\frac{m}{2(m+1)}+1}(\mathbb{R}^n)\right), \quad u_t \in C\left([0,T); H^{\sigma+\frac{m}{2(m+1)}}(\mathbb{R}^n)\right)$$

for some $T > 0$.

Moreover, if $\varepsilon > 0$ *is small enough, then* $T \gtrsim \varepsilon^{-\left(\frac{1}{p-1}+\frac{m}{2}\right)^{-1}}$.

As in Yagdjian [Yag07c] and Taniguchi and Tozaki [TT80], we introduce the differential operators $V_1(t,D_x)$ and $V_2(t,D_x)$ as follows. Set

$$z := 2i\phi(t)|\xi|, \quad \phi(t) := \frac{t^{m+1}}{m+1}, \quad \mu := \frac{m}{2(m+1)}.$$

Then $V_1(t,D_x)$ and $V_2(t,D_x)$ are the Fourier multiplier

$$\begin{aligned} V_1(t,D_x)\psi &= \mathscr{F}^{-1}[V_1(t,|\xi|)\mathscr{F}\psi], \\ V_2(t,D_x)\psi &= \mathscr{F}^{-1}[V_2(t,|\xi|)\mathscr{F}\psi], \end{aligned}$$

defined by the symbols

$$\begin{aligned} V_1(t,|\xi|) &:= e^{-z/2}\Phi(\mu, 2\mu; z), \\ V_2(t,|\xi|) &:= te^{-z/2}\Phi(1-\mu, 2(1-\mu); z), \end{aligned}$$

where $\mathscr{F}, \mathscr{F}^{-1}$ are the Fourier transform and its inverse respectively, and $\Phi(a,c;z)$ is the confluent hypergeometric function. Recall that $\Phi(a,c;z)$ is an entire analytic function of z such that

$$\Phi(a,c;z) = 1 + O(z) \quad \text{for } z \to 0 \tag{5.3.2}$$

and which satisfies the following differential relations (see e.g. [AS64, Section 13.4]):

$$\frac{d^n}{dz^n}\Phi(a,c;z) = \frac{(a)_n}{(c)_n}\Phi(a+n, c+n; z), \tag{5.3.3}$$

$$\frac{d}{dz}\Phi(a,c;z) = \frac{1-c}{z}\left[\Phi(a,c;z) - \Phi(a,c-1;z)\right], \tag{5.3.4}$$

where $(x)_n = x(x+1)\cdots(x+n-1)$ is the Pochhammer's symbol. Moreover $\Phi(a,c;z)$ satisfies the estimate

$$|\Phi(a,c;2i\phi(t)|\xi|)| \le C_{a,c,m}(\phi(t)|\xi|)^{\max\{a-c,-a\}} \quad \text{for } 2\phi(t)|\xi| \ge 1. \tag{5.3.5}$$

Remark 5.5. In the case of the wave equation, i.e. when $m=0$, the definitions of $V_1(t,D_x)$ and $V_2(t,D_x)$ should be understood taking the limit for $m \to 0$ in their formulas. Indeed, using the identities 10.2.14, 13.6.3 and 13.6.14 in [AS64], we get

$$\lim_{m\to 0} V_1(t,|\xi|) = \lim_{\mu\to 0}\Gamma\left(\mu+\frac{1}{2}\right)\left(\frac{z}{4}\right)^{1/2-\mu} I_{\mu-1/2}\left(\frac{z}{2}\right) = \cosh\left(\frac{z}{2}\right),$$
$$\lim_{m\to 0} V_2(t,|\xi|) = \frac{2t}{z}\sinh\left(\frac{z}{2}\right),$$

where $I_\nu(w)$ is the modified Bessel function of first kind. Thus for $m=0$ one recovers the well-known wave operators $V_1(t,D_x) = \cos(t\sqrt{-\Delta})$ and $V_2(t,D_x) = \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}$.

As Yagdjian observes, there are two different phase functions of two different waves hidden in $\Phi(a,c;z)$. More precisely, for $0 < \arg z < \pi$, we can write (see [Inu67])

$$e^{-z/2}\Phi(a,c;z) = \frac{\Gamma(c)}{\Gamma(a)}e^{z/2}H_+(a,c;z) + \frac{\Gamma(c)}{\Gamma(c-a)}e^{-z/2}H_-(a,c;z) \tag{5.3.6}$$

where

$$H_+(a,c;z) = \frac{e^{-i\pi(c-a)}}{e^{i\pi(c-a)}-e^{-i\pi(c-a)}}\frac{1}{\Gamma(c-a)}z^{a-c}\int_\infty^{(0+)} e^{-\omega}\omega^{c-a-1}\left(1-\frac{\omega}{z}\right)^{a-1}d\omega,$$
$$H_-(a,c;z) = \frac{1}{e^{i\pi a}-e^{-i\pi a}}\frac{1}{\Gamma(a)}z^{-a}\int_\infty^{(0+)} e^{-\omega}\omega^{a-1}\left(1+\frac{\omega}{z}\right)^{c-a-1}d\omega.$$

For $|z|\to\infty$ and $0<\arg z<\pi$, the following asymptotic estimates hold:

$$H_+(a,c;z) \sim z^{a-c}\left[1+\sum_{k=1}^{\infty}\frac{(c-a)_k(1-a)_k}{k!}z^{-k}\right],$$
$$H_-(a,c;z) \sim (e^{-i\pi}z)^{-a}\left[1+\sum_{k=1}^{\infty}(-1)^k\frac{(a)_k(1+a-c)_k}{k!}z^{-k}\right].$$

Combining the asymptotic estimates for $H_+(a,c;z)$ and $H_-(a,c;z)$ with their definitions, one can infer, for $2\phi(t)|\xi|\ge 1$, the relations

$$|\partial_t^k\partial_\xi^\beta H_+(a,c;2i\phi(t)|\xi|)| \le C_{a,c,m,k,\beta}(\phi(t)|\xi|)^{a-c}\langle\xi\rangle^{\frac{k}{m+1}-|\beta|}, \tag{5.3.7}$$
$$|\partial_t^k\partial_\xi^\beta H_-(a,c;2i\phi(t)|\xi|)| \le C_{a,c,m,k,\beta}(\phi(t)|\xi|)^{-a}\langle\xi\rangle^{\frac{k}{m+1}-|\beta|}, \tag{5.3.8}$$

where $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$ are the Japanese brackets.

Finally, let us introduce for simplicity of notation the operators

$$W_1(s,t,D_x) := V_1(t,D_x)V_2(s,D_x),$$
$$W_2(s,t,D_x) := V_2(t,D_x)V_1(s,D_x),$$

whose symbols, if we set $z := 2i\phi(t)|\xi|$ and $\zeta := 2i\phi(s)|\xi|$, are given by

$$W_1(s,t,|\xi|) = se^{-(z+\zeta)/2}\Phi(\mu, 2\mu; z)\Phi(1-\mu, 2(1-\mu); \zeta),$$
$$W_2(s,t,|\xi|) = te^{-(z+\zeta)/2}\Phi(\mu, 2\mu; \zeta)\Phi(1-\mu, 2(1-\mu); z).$$

The key estimates employed in the proof of Theorem 5.2 are given in Corollary 5.1, which comes straightforwardly from Theorem 5.3, and in Lemma 5.1. The estimates in the following Theorem 5.3, which are of independent interest, are obtained adapting the argument exploited by Yagdjian [Yag06] and Reissig [Rei97] for the case of the operators $V_1(t, D_x)$ and $V_2(t, D_x)$. In order to not weigh down the exposition, we postpone the proof of this theorem in Appendix 5.B.

Theorem 5.3. *Let $n \geq 1$, $m \geq 0$, $\mu := \frac{m}{2(m+1)}$ and $\psi \in C_0^\infty(\mathbb{R}^n)$. Then the following $L^q - L^{q'}$ estimates on the conjugate line, i.e. for $\frac{1}{q} + \frac{1}{q'} = 1$, hold for $0 < s \leq t$ and for all admissible $q \in (1, 2]$:*

(i) if $n\left(\frac{1}{q} - \frac{1}{q'}\right) - 1 \leq \sigma \leq -\mu + n\left(\frac{1}{q} - \frac{1}{q'}\right)$, then

$$\left\|(\sqrt{-\Delta})^{-\sigma} W_1(s,t,D_x)\psi\right\|_{L^{q'}} \lesssim (t/s)^{-m/2} s^{1+\left[\sigma - n\left(\frac{1}{q}-\frac{1}{q'}\right)\right](m+1)} \|\psi\|_{L^q};$$

(ii) if $n\left(\frac{1}{q} - \frac{1}{q'}\right) - 1 \leq \sigma \leq -1 + \mu + n\left(\frac{1}{q} - \frac{1}{q'}\right)$, then

$$\left\|(\sqrt{-\Delta})^{-\sigma} W_2(s,t,D_x)\psi\right\|_{L^{q'}} \lesssim (t/s)^{-m/2} s^{1+\left[\sigma - n\left(\frac{1}{q}-\frac{1}{q'}\right)\right](m+1)} \|\psi\|_{L^q};$$

(iii) if $n\left(\frac{1}{q} - \frac{1}{q'}\right) \leq \sigma \leq 1 - \mu + n\left(\frac{1}{q} - \frac{1}{q'}\right)$, then

$$\left\|(\sqrt{-\Delta})^{-\sigma} \partial_t W_1(s,t,D_x)\psi\right\|_{L^{q'}} \lesssim (t/s)^{m/2} s^{\left[\sigma - n\left(\frac{1}{q}-\frac{1}{q'}\right)\right](m+1)} \|\psi\|_{L^q};$$

(iv) if $\sigma = n\left(\frac{1}{q} - \frac{1}{q'}\right)$, then

$$\left\|(\sqrt{-\Delta})^{-\sigma} \partial_t W_2(s,t,D_x)\psi\right\|_{L^{q'}} \lesssim (t/s)^{m/2} \|\psi\|_{L^q}.$$

Remark 5.6. As in Yagdjian [Yag06], it is easy to obtain similar estimates for the (homogeneous) Besov spaces and then for the Sobolev-Slobodeckij spaces.

In the previous theorem, choosing $q = q' = 2$ and $\sigma = -1$ for $W_1(s,t,D_x)$ and $W_2(s,t,D_x)$, and $\sigma = 0$ for their derivatives, we immediately get the next corollary.

Corollary 5.1. *The following estimates hold*

$$\|W_j(s,t,D_x)\psi\|_{\dot{H}^{\gamma+1}} \lesssim (ts)^{-m/2} \|\psi\|_{\dot{H}^\gamma},$$
$$\|\partial_t W_j(s,t,D_x)\psi\|_{\dot{H}^\gamma} \lesssim (t/s)^{m/2} \|\psi\|_{\dot{H}^\gamma},$$
$$\|\partial_t W_j(s,t,D_x)\psi\|_{H^\gamma} \lesssim (t/s)^{m/2} \|\psi\|_{H^\gamma},$$

for $n \geq 1$, $m \geq 0$, $\gamma \in \mathbb{R}$ and $j \in \{1, 2\}$.

We furnish now estimates in the energy space $\dot{H}^\gamma(\mathbb{R}^n)$ and $H^\gamma(\mathbb{R}^n)$ also for $V_1(t, D_x)$, $V_2(t, D_x)$ and their derivatives with respect to time.

Lemma 5.1. *Let $\gamma \in \mathbb{R}$, $m \geq 0$ and $\mu := \frac{m}{2(m+1)}$. The following estimates hold:*

$$\begin{aligned}
\|V_1(t, D_x)\psi\|_{\dot{H}^{\gamma-\sigma}} &\lesssim t^{\sigma(m+1)} \|\psi\|_{\dot{H}^\gamma} && \text{for } -\mu \leq \sigma \leq 0; \\
\|V_2(t, D_x)\psi\|_{\dot{H}^{\gamma-\sigma}} &\lesssim t^{\sigma(m+1)+1} \|\psi\|_{\dot{H}^\gamma} && \text{for } -1+\mu \leq \sigma \leq 0; \\
\|\partial_t V_1(t, D_x)\psi\|_{H^{\gamma-\sigma}} &\lesssim t^{\sigma(m+1)-1} \|\psi\|_{H^\gamma} && \text{for } 1-\mu \leq \sigma \leq 1; \\
\|\partial_t V_2(t, D_x)\psi\|_{H^{\gamma-\sigma}} &\lesssim \langle t\rangle^{\sigma(m+1)} \|\psi\|_{H^\gamma} && \text{for } \mu \leq \sigma.
\end{aligned}$$

Proof. By estimates (5.3.5), for the range of σ in the hypothesis we have that

$$\begin{aligned}
||\xi|^{-\sigma} V_1(t, |\xi|)| &\lesssim \begin{cases} |\xi|^{-\sigma}(\phi(t)|\xi|)^{-\mu} & \text{if } \phi(t)|\xi| \geq 1, \\ |\xi|^{-\sigma} & \text{if } \phi(t)|\xi| \leq 1, \end{cases} \\
&\leq t^{\sigma(m+1)}; \\
||\xi|^{-\sigma} V_2(t, |\xi|)| &\lesssim \begin{cases} t|\xi|^{-\sigma}(\phi(t)|\xi|)^{\mu-1} & \text{if } \phi(t)|\xi| \geq 1, \\ t|\xi|^{-\sigma} & \text{if } \phi(t)|\xi| \leq 1, \end{cases} \\
&\leq t^{\sigma(m+1)+1}; \\
|\langle\xi\rangle^{-\sigma} \partial_t V_1(t, |\xi|)| &\lesssim \begin{cases} t^m|\xi|^{1-\sigma}(\phi(t)|\xi|)^{-\mu} & \text{if } \phi(t)|\xi| \geq 1, \\ t^m|\xi|^{1-\sigma} & \text{if } \phi(t)|\xi| \leq 1, \end{cases} \\
&\leq t^{\sigma(m+1)-1}; \\
|\langle\xi\rangle^{-\sigma} \partial_t V_2(t, |\xi|)| &\lesssim \begin{cases} \langle\xi\rangle^{-\sigma}(\phi(t)|\xi|)^{\mu} & \text{if } \phi(t)|\xi| \geq 1, \\ \langle\xi\rangle^{-\sigma}[1+\phi(t)|\xi|] & \text{if } \phi(t)|\xi| \leq 1, \end{cases} \\
&\leq \langle\xi\rangle^{-\sigma}\langle\phi(t)|\xi|\rangle^{\mu} \\
&\leq \langle\phi(t)\rangle^{\mu}\langle\xi\rangle^{\mu-\sigma} \\
&\lesssim \langle t\rangle^{\sigma(m+1)}.
\end{aligned}$$

Consequently

$$\begin{aligned}
\|V_1(t, D_x)\psi\|_{\dot{H}^{\gamma-\sigma}} &= \left\| |\xi|^{\gamma-\sigma} V_1(t, |\xi|)\widehat{\psi} \right\|_{L^2} \\
&\leq \left\| |\xi|^{-\sigma} V_1(t, |\xi|) \right\|_{L^\infty} \left\| |\xi|^{\gamma}\widehat{\psi} \right\|_{L^2} \\
&\lesssim t^{\sigma(m+1)} \|\psi\|_{\dot{H}^\gamma},
\end{aligned}$$

and similarly we can obtain the other estimates. □

Remark 5.7. The previous lemma should be compared with Lemma 3.2 in [RWY15a], where similar estimates are obtained under the restriction $0 < t \leq T$, for some fixed positive constant T.

Finally, let us recall also the following useful relations that come from an application of Theorems 4.6.4/2 and 5.4.3/1 in [RS96].

Lemma 5.2. *The following estimates hold:*

(i) if $\gamma > 0$ and $u, v \in H^\gamma(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, then

$$\|uv\|_{H^\gamma} \lesssim \|u\|_{L^\infty} \|v\|_{H^\gamma} + \|u\|_{H^\gamma} \|v\|_{L^\infty} ;$$

(ii) if $p > 1$, $\gamma \in \left(\frac{n}{2}, p\right)$ and $u \in H^\gamma(\mathbb{R}^n)$, then

$$\||u|^p\|_{H^\gamma} \lesssim \|u\|_{H^\gamma} \|u\|_{L^\infty}^{p-1} .$$

We can now start the proof of the local existence result.

Proof of Theorem 5.2. Let us consider the map

$$\begin{aligned}\Psi[v](t,x) =& \varepsilon V_1(t,D_x)f(x) + \varepsilon V_2(t,D_x)g(x)\\ &+ \int_0^t [V_2(t,D_x)V_1(s,D_x) - V_1(t,D_x)V_2(s,D_x)]|v_t(s,x)|^p ds\end{aligned}$$

and the complete metric space

$$X(a,T) := \left\{v\colon v \in C((0,T); \dot{H}^{\gamma+1}(\mathbb{R}^n)), v_t \in C([0,T); H^\gamma(\mathbb{R}^n)) \text{ and } \|v\|_X \le a\right\}$$

for some $a, T > 0$ to be chosen later, where $\gamma := \sigma + \mu$, $\mu := \frac{m}{2(m+1)}$ and

$$\|v\|_X := \sup_{0\le t\le T} \left[t^{m/2} \|v\|_{\dot{H}^{\gamma+1}} + \langle t\rangle^{-m/2} \|v_t\|_{H^\gamma}\right] .$$

Note that, since the operators $V_1(t, D_x)$ and $V_2(t, D_x)$ commute, we have

$$\begin{aligned}\partial_t\Psi[v](t,x) =& \varepsilon\partial_t V_1(t,D_x)f(x) + \varepsilon\partial_t V_2(t,D_x)g(x)\\ &+ \int_0^t [\partial_t V_2(t,D_x)V_1(s,D_x) - \partial_t V_1(t,D_x)V_2(s,D_x)]|v_t(s,x)|^p ds.\end{aligned}$$

We want to show that Ψ is a contraction mapping on $X(a, T)$.

By Lemma 5.1 and the immersion $H^s(\mathbb{R}^n) \hookrightarrow \dot{H}^s(\mathbb{R}^n)$ for $s > 0$, we get

$$\begin{aligned}\|V_1(t,D_x)f\|_{\dot{H}^{\gamma+1}} &\lesssim t^{-m/2} \|f\|_{H^{\gamma-\mu+1}} ,\\ \|V_2(t,D_x)g\|_{\dot{H}^{\gamma+1}} &\lesssim t^{-m/2} \|g\|_{H^{\gamma+\mu}} ,\\ \|\partial_t V_1(t,D_x)f\|_{H^\gamma} &\lesssim t^{m/2} \|f\|_{H^{\gamma-\mu+1}} ,\\ \|\partial_t V_2(t,D_x)g\|_{H^\gamma} &\lesssim \langle t\rangle^{m/2} \|g\|_{H^{\gamma+\mu}} .\end{aligned}$$

Moreover by Corollary 5.1 we infer

$$\begin{aligned}\|V_2(t,D_x)V_1(s,D_x)|v_t(s,x)|^p\|_{\dot{H}^{\gamma+1}} &\lesssim (st)^{-m/2} \|v_t(s,x)\|_{H^\gamma}^p ,\\ \|V_1(t,D_x)V_2(s,D_x)|v_t(s,x)|^p\|_{\dot{H}^{\gamma+1}} &\lesssim (st)^{-m/2} \|v_t(s,x)\|_{H^\gamma}^p ,\\ \|\partial_t V_2(t,D_x)V_1(s,D_x)|v_t(s,x)|^p\|_{H^\gamma} &\lesssim (s/t)^{-m/2} \|v_t(s,x)\|_{H^\gamma}^p ,\\ \|\partial_t V_1(t,D_x)V_2(s,D_x)|v_t(s,x)|^p\|_{H^\gamma} &\lesssim (s/t)^{-m/2} \|v_t(s,x)\|_{H^\gamma}^p ,\end{aligned}$$

where we used the estimates

$$\| |v_t(s,x)|^p \|_{\dot H^\gamma} \le \| |v_t(s,x)|^p \|_{H^\gamma} \lesssim \|v_t(s,x)\|_{H^\gamma}^p \,,$$

which come from Lemma 5.2 and the Sobolev embeddings.

From these estimates and from the fact that $t\langle t\rangle^{-1} < 1$ for any $t > 0$, we obtain

$$\begin{aligned} t^{m/2} \left\| \Psi[v](t,\cdot) \right\|_{\dot H^{\gamma+1}} + \langle t\rangle^{-m/2} \left\| \partial_t \Psi[v](t,\cdot) \right\|_{H^\gamma} \\ \lesssim \varepsilon \left[\|f\|_{H^{\gamma-\mu+1}} + \|g\|_{H^{\gamma+\mu}} \right] + \int_0^t s^{-m/2} \left\| v_t(s,\cdot) \right\|_{H^\gamma}^p ds \end{aligned}$$

and hence, since $m < 2$, we get

$$\left\| \Psi[v] \right\|_X \le C_0 \varepsilon \left[\|f\|_{H^{\gamma-\mu+1}} + \|g\|_{H^{\gamma+\mu}} \right] + C_0 T^{1-\frac{m}{2}} \langle T\rangle^{\frac{m}{2}p} \left\| v \right\|_X^p$$

for some constant $C_0 > 0$ independent of ε. Choosing a sufficiently large and T sufficiently small, namely $a \ge 2C_0\varepsilon \left[\|f\|_{H^{\gamma-\mu+1}} + \|g\|_{H^{\gamma+\mu}} \right]$ and $T^{1-\frac{m}{2}} \langle T\rangle^{\frac{m}{2}p} \le (2C_0 a^{p-1})^{-1}$, we infer that $\Psi[v] \in X(a,T)$.

Now we show that Ψ is a contraction. Fixed $v, \widetilde{v} \in X(a,T)$, we have similarly as above

$$\begin{aligned} t^{m/2} \left\| \Psi[v](t,\cdot) - \Psi[\widetilde{v}](t,\cdot) \right\|_{\dot H^{\gamma+1}} + \langle t\rangle^{-m/2} \left\| \partial_t \Psi[v](t,\cdot) - \partial_t \Psi[\widetilde{v}](t,\cdot) \right\|_{H^\gamma} \\ \lesssim \int_0^t s^{-m/2} \left\| |v_t(s,\cdot)|^p - |\widetilde{v}_t(s,\cdot)|^p \right\|_{H^\gamma} ds. \end{aligned} \tag{5.3.9}$$

Since we can write

$$|v_t|^p - |\widetilde{v}_t|^p = 2^{-p} p \int_{-1}^1 (v_t - \widetilde{v}_t)(v_t + \widetilde{v}_t + \lambda(v_t - \widetilde{v}_t)) |v_t + \widetilde{v}_t + \lambda(v_t - \widetilde{v}_t)|^{p-2} d\lambda$$

and recalling that $p > 2$ and $\gamma \in (n/2, p-1)$, an application of Lemma 5.2 combined with Sobolev embeddings give us

$$\begin{aligned} \left\| |v_t|^p - |\widetilde{v}_t|^p \right\|_{H^\gamma} &\lesssim \|v_t - \widetilde{v}_t\|_{L^\infty} \left\| (|v_t| + |\widetilde{v}_t|)^{p-1} \right\|_{H^\gamma} \\ &\quad + \|v_t - \widetilde{v}_t\|_{H^\gamma} \left\| (|v_t| + |\widetilde{v}_t|)^{p-1} \right\|_{L^\infty} \\ &\lesssim \|v_t - \widetilde{v}_t\|_{L^\infty} \left(\|v_t\|_{H^\gamma}^{p-1} + \|\widetilde{v}_t\|_{H^\gamma}^{p-1} \right) \\ &\quad + \|v_t - \widetilde{v}_t\|_{H^\gamma} \left(\|v_t\|_{L^\infty}^{p-1} + \|\widetilde{v}_t\|_{L^\infty}^{p-1} \right) \\ &\lesssim \|v_t - \widetilde{v}_t\|_{H^\gamma} \left(\|v_t\|_{H^\gamma}^{p-1} + \|\widetilde{v}_t\|_{H^\gamma}^{p-1} \right). \end{aligned}$$

Inserting this inequality into (5.3.9) we get

$$\left\| \Psi[v] - \Psi[\widetilde{v}] \right\|_X \le C_1 T^{1-\frac{m}{2}} \langle T\rangle^{\frac{m}{2}p} a^{p-1} \left\| v - \widetilde{v} \right\|_X$$

for some $C_1 > 0$, and so Ψ is a contraction for $T^{1-\frac{m}{2}} \langle T\rangle^{\frac{m}{2}p} \le (C_1 a^{p-1})^{-1}$. By the Banach fixed point theorem we conclude that there exists a unique $v \in X(a,T)$ such that $\Psi[v] = v$.

As a by-product of the computations, from the conditions on T and a we can choose the existence time such that $T^{1-\frac{m}{2}} \langle T\rangle^{\frac{m}{2}p} = C\varepsilon^{-(p-1)}$ for some $C > 0$ independent of ε, hence $T \gtrsim \varepsilon^{-\left[\frac{1}{p-1} + \frac{m}{2}\right]^{-1}}$ for ε small enough. □

5.4 Blow-up via a test function method

We come now to the proof of Theorem 5.1, which heavily relies on a special test function, closely related to a time dependent function satisfying the following ordinary differential equation:

$$\lambda''(t) - 2mt^{-1}\lambda'(t) - t^{2m}\lambda(t) = 0, \tag{5.4.1}$$

where $t > 0$ and $m \in \mathbb{R}$.

Lemma 5.3. *The fundamental solutions λ_-, λ_+ of (5.4.1) are the functions defined by:*

- *if $m = -1$:*
$$\lambda_-(t) = t^{-\frac{1+\sqrt{5}}{2}}, \qquad \lambda_+(t) = t^{-\frac{1-\sqrt{5}}{2}};$$
- *if $m \neq -1$:*
$$\lambda_-(t) = t^{m+\frac{1}{2}} K_{\frac{1}{2}+\frac{m}{2(m+1)}}\left(\frac{t^{m+1}}{|m+1|}\right), \quad \lambda_+(t) = t^{m+\frac{1}{2}} I_{\frac{1}{2}+\frac{m}{2(m+1)}}\left(\frac{t^{m+1}}{|m+1|}\right),$$

where $I_\nu(z), K_\nu(z)$ are the modified Bessel functions of the first and second kind respectively.

Proof. The result trivially follows from straightforward computations based on formulas for Bessel functions collected in Appendix A. Instead in Appendix 5.A we show a way to reach the expression of the solutions for $m \neq -1$.

If $m = -1$, it is immediate to check that λ_- and λ_+ are two independent solutions of (5.4.1). Suppose now $m \neq -1$ and set $z = t^{m+1}/|m+1|$ and $\sigma = \operatorname{sgn}(m+1)$ for simplicity. From (A.5) and (A.1), we get

$$\begin{aligned}
\lambda_-'(t) &= \left(m+\frac{1}{2}\right) t^{m-1/2} K_{\frac{m+1/2}{m+1}}(z) + \sigma t^{2m+1/2} K'_{\frac{m+1/2}{m+1}}(z) \\
&= \left(m+\frac{1}{2}\right) t^{m-1/2} K_{\frac{m+1/2}{m+1}}(z) - \sigma t^{2m+1/2} \\
&\quad \times \left[K_{-\frac{1}{2(m+1)}}(z) + \sigma\left(m+\frac{1}{2}\right) t^{-m-1} K_{\frac{m+1/2}{m+1}}(z)\right] \\
&= -\sigma t^{2m+1/2} K_{-\frac{1}{2(m+1)}}(z) \\
&= -\sigma t^{2m+1/2} K_{\frac{1}{2(m+1)}}(z),
\end{aligned}$$

and

$$\begin{aligned}
\lambda_-''(t) =& -\sigma\left(2m+\frac{1}{2}\right) t^{2m-1/2} K_{\frac{1}{2(m+1)}}(z) - t^{3m+1/2} K'_{\frac{1}{2(m+1)}}(z) \\
=& -\sigma\left(2m+\frac{1}{2}\right) t^{2m-1/2} K_{\frac{1}{2(m+1)}}(z) \\
&+ t^{3m+1/2}\left[K_{-\frac{m+1/2}{m+1}}(z) + \sigma\frac{t^{-m-1}}{2} K_{\frac{1}{2(m+1)}}(z)\right] \\
=& t^{3m+1/2} K_{\frac{m+1/2}{m+1}}(z) - 2m\sigma\, t^{2m-1/2} K_{\frac{1}{2(m+1)}}(z) \\
=& t^{2m}\lambda_-(t) + 2mt^{-1}\lambda_-'(t).
\end{aligned}$$

Analogously, using (A.4) and (A.5), we obtain

$$\begin{aligned}\lambda_+'(t) &= \sigma t^{2m+1/2} I_{-\frac{1}{2(m+1)}}(z),\\ \lambda_+''(t) &= t^{3m+1/2} I_{\frac{m+1/2}{m+1}}(z) + 2m\sigma\, t^{2m-1/2} I_{-\frac{1}{2(m+1)}}(z)\\ &= t^{2m}\lambda_+(t) + 2m\, t^{-1}\lambda_+(t).\end{aligned}$$

Then, it is clear that λ_- and λ_+ solve equation (5.4.1) and, from relation (A.3), we can check that the Wronskian $W(t) = \lambda_-(t)\lambda_+'(t) - \lambda_+(t)\lambda_-'(t)$ is

$$\begin{aligned}W(t) &= \sigma t^{3m+1}[I_{-\frac{1}{2(m+1)}} K_{-\frac{1}{2(m+1)}+1} + I_{-\frac{1}{2(m+1)}+1} K_{-\frac{1}{2(m+1)}}](z)\\ &= (m+1)t^{2m} > 0\end{aligned}$$

for $t > 0$, hence the two solutions are independent. □

Lemma 5.4. *Suppose $m > -1/2$. Define $\mu := \frac{m}{2(m+1)}$ and*

$$\lambda(t) := t^{m+1/2} K_{\mu+\frac{1}{2}}\left(\frac{t^{m+1}}{m+1}\right).$$

Then, $\lambda \in C^1([0,+\infty)) \cap C^\infty(0,+\infty)$ and satisfies the following properties:

(i) $\lambda(t) > 0$, $\lambda'(t) < 0$,

(ii) $\lim_{t\to 0^+} \lambda(t) = 2^{\mu-\frac{1}{2}}(m+1)^{\mu+\frac{1}{2}}\Gamma\left(\mu+\frac{1}{2}\right) =: c_0(\mu) > 0$,

(iii) $\lim_{t\to 0^+} \frac{\lambda'(t)}{t^{2m}} = -c_0(-\mu) < 0$,

(iv) $\lambda(t) = \sqrt{\frac{(m+1)\pi}{2}}\, t^{m/2} \exp\left(-\frac{t^{m+1}}{m+1}\right) \times (1 + O(t^{-(m+1)}))$, *for large* $t > 0$,

(v) $\lambda'(t) = -\sqrt{\frac{(m+1)\pi}{2}}\, t^{3m/2} \exp\left(-\frac{t^{m+1}}{m+1}\right) \times (1 + O(t^{-(m+1)}))$, *for large* $t > 0$,

where Γ is the Gamma function and O is the Big O from the Bachmann-Landau notation.

Proof. From (A.4) we know that λ is smooth for $t > 0$. Since $K_\nu(z)$ is real and positive for $\nu \in \mathbb{R}$ and $z > 0$, also λ is real and positive. Recall from the proof of Lemma 5.3 that

$$\lambda'(t) = -t^{2m+1/2} K_{-\mu+\frac{1}{2}}\left(\frac{t^{m+1}}{m+1}\right),$$

and hence λ' is negative. From (A.8) we have $\lambda(t) \sim c_0(\mu)$ and $\lambda'(t) \sim -c_0(-\mu)t^{2m}$ for $t \to 0^+$, so we can prove (ii) and (iii). Finally, from (A.10) we obtain (iv) and (v). □

We can start now the proof of our main theorem.

Proof of Theorem 5.1. As in [ISW19], let $\eta(t) \in C^\infty([0,+\infty))$ satisfying

$$\eta(r) := \begin{cases} 1 & \text{for } r \le \frac{1}{2}, \\ \text{decreasing} & \text{for } \frac{1}{2} < r < 1, \\ 0 & \text{for } r \ge 1, \end{cases}$$

and denote, for $M \in (1,T)$,

$$\eta_M(t) := \eta\left(\frac{t}{M}\right), \qquad \eta^0(t,x) := \eta\left(\frac{|x|}{2}\left(1+\frac{t^{m+1}}{m+1}\right)^{-1}\right).$$

We remark that one can assume $1 < T \le T_\varepsilon$, since otherwise our result holds obviously by choosing ε small enough. The last ingredient other than λ, η_M and η^0 to construct our test function is

$$\phi(x) := \begin{cases} \displaystyle\int_{\mathbb{S}^{n-1}} e^{x\cdot\omega} d\omega & \text{if } n \ge 2, \\ e^x + e^{-x} & \text{if } n = 1, \end{cases}$$

which satisfies

$$\Delta\phi = \phi, \quad 0 < \phi(x) \le C_0(1+|x|)^{-\frac{n-1}{2}} e^{|x|}, \tag{5.4.2}$$

for some $C_0 > 0$. We can finally introduce the test function

$$\begin{aligned} \Phi(t,x) :=& -t^{-2m}\partial_t\left(\eta_M^{2p'}(t)\lambda(t)\right)\phi(x)\eta^0(t,x) \\ =& -t^{-2m}\left(\partial_t\eta_M^{2p'}(t)\lambda(t) + \eta_M^{2p'}(t)\lambda'(t)\right)\phi(x)\eta^0(t,x), \end{aligned} \tag{5.4.3}$$

where $M \in (1,T)$ and $p' = p/(p-1)$ is the conjugate exponent of p. It is straightforward to check that $\Phi(t,x) \in C_0^1([0,+\infty)\times\mathbb{R}^n) \cap C_0^\infty((0,+\infty)\times\mathbb{R}^n)$ if we set

$$\Phi(0,x) := \lim_{t\to 0^+}\Phi(t,x) = c_0(-\mu)\phi(x)\eta\left(\frac{|x|}{2}\right) \ge 0,$$

where c_0 is defined in Lemma 5.4.(ii). Note also that

$$\Phi(t,x) = -t^{-2m}\partial_t\left(\eta_M^{2p'}(t)\lambda(t)\right)\phi(x)$$

in the cone defined in (5.2.3).

Taking Φ as the test function in the definition of weak solution (5.2.1), exploiting the compact support condition (5.2.3) on u and integrating by parts, we obtain

$$\begin{aligned} &\varepsilon c_0(-\mu)\int_{\mathbb{R}^n} g\phi dx + \int_0^T\int_{\mathbb{R}^n} |u_t|^p t^{-2m}\eta_M^{2p'}|\lambda'|\phi\, dxdt \\ &+ \int_0^T\int_{\mathbb{R}^n} |u_t|^p t^{-2m}|\partial_t\eta_M^{2p'}|\lambda\phi\, dxdt \\ =& \int_0^T\int_{\mathbb{R}^n} u_t t^{-2m}\left[-2mt^{-1}\left(\partial_t\eta_M^{2p'}\lambda + \eta_M^{2p'}\lambda'\right) + \partial_t^2\eta_M^{2p'}\lambda + 2\partial_t\eta_M^{2p'}\lambda' + \eta_M^{2p'}\lambda''\right]\phi\, dxdt \\ &- \int_0^T\int_{\mathbb{R}^n} \nabla u\cdot\nabla\phi\,\partial_t\left(\eta_M^{2p'}\lambda\right)\, dxdt \end{aligned}$$

and hence

$$\begin{aligned}
&\varepsilon c_0(-\mu)\int_{\mathbb{R}^n} g\phi dx + \int_0^T\int_{\mathbb{R}^n} |u_t|^p t^{-2m}\eta_M^{2p'}|\lambda'|\phi\,dxdt \\
&+\int_0^T\int_{\mathbb{R}^n} |u_t|^p t^{-2m}|\partial_t\eta_M^{2p'}|\lambda\phi\,dxdt \\
=&-\varepsilon c_0(\mu)\int_{\mathbb{R}^n} f\phi dx - 2m\int_0^T\int_{\mathbb{R}^n} u_t t^{-2m-1}\partial_t\eta_M^{2p'}\lambda\phi\,dxdt \\
&+\int_0^T\int_{\mathbb{R}^n} u_t t^{-2m}\left(\partial_t^2\eta_M^{2p'}\lambda + 2\partial_t\eta_M^{2p'}\lambda'\right)\phi\,dxdt \\
&\int_0^T\int_{\mathbb{R}^n} u_t t^{-2m}\eta_M^{2p'}\left(\lambda'' - 2mt^{-1}\lambda' - t^{2m}\lambda\right)\phi\,dxdt.
\end{aligned}$$

Neglecting the third term in the left hand-side and recalling that λ solve the ODE (5.4.1), it follows that

$$\begin{aligned}
&\varepsilon C_1 + \int_0^T\int_{\mathbb{R}^n} |u_t|^p t^{-2m}\eta_M^{2p'}|\lambda'|\phi\,dxdt, \\
\le& -2m\int_0^T\int_{\mathbb{R}^n} u_t t^{-2m-1}\partial_t\eta_M^{2p'}\lambda\phi\,dxdt \\
&+\int_0^T\int_{\mathbb{R}^n} u_t t^{-2m}\partial_t^2\eta_M^{2p'}\lambda\phi\,dxdt \\
&+2\int_0^T\int_{\mathbb{R}^n} u_t t^{-2m}\partial_t\eta_M^{2p'}\lambda'\phi\,dxdt \\
=:& \, I + II + III,
\end{aligned} \tag{5.4.4}$$

where

$$C_1 \equiv C_1(m,f,g) := c_0(\mu)\int_{\mathbb{R}^n} f\phi dx + c_0(-\mu)\int_{\mathbb{R}^n} g\phi dx > 0$$

is a positive constant thanks to (5.2.2).

Now we will estimate the three terms I, II, III by Hölder's inequality. Firstly let us define the functions

$$\theta(t) := \begin{cases} 0 & \text{for } t < \frac{1}{2}, \\ \eta(t) & \text{for } t \ge \frac{1}{2}, \end{cases} \qquad \theta_M(t) := \theta\left(\frac{t}{M}\right),$$

for which it is straightforward to check the following relations:

$$|\partial_t\eta_M^{2p'}| \le \frac{2p'}{M}\left\|\eta'\right\|_{L^\infty}\theta_M^{2p'/p}, \tag{5.4.5}$$

$$|\partial_t^2\eta_M^{2p'}| \le \frac{2p'}{M^2}\left[(2p'-1)\left\|\eta'\right\|_{L^\infty}^2 + \left\|\eta\eta''\right\|_{L^\infty}\right]\theta_M^{2p'/p}. \tag{5.4.6}$$

From now on, C will stand for a generic positive constant, independent of ε and M, which can change from line to line.

Exploiting the estimates (5.4.2) and (5.4.5), the asymptotic behaviors (iv)–(v) in Lemma 5.4 and the finite speed of propagation property (5.2.3), for I we obtain

$$\begin{aligned}
I =& -2m\int_0^T\int_{\mathbb{R}^n} u_t t^{-2m-1}\partial_t\eta_M^{2p'}\lambda\phi\,dxdt \\
\leq& CM^{-2}\left(\int_{\frac{M}{2}}^M\int_{|x|\leq\gamma(t)} t^{-2m}|\lambda'|^{-\frac{1}{p-1}}|\lambda|^{\frac{p}{p-1}}\phi\,dxdt\right)^{\frac{1}{p'}} \\
&\times\left(\int_0^T\int_{\mathbb{R}^n}|u_t|^p t^{-2m}\theta_M^{2p'}|\lambda'|\phi\,dxdt\right)^{\frac{1}{p}} \\
\leq& CM^{-2}\left(\int_{\frac{M}{2}}^M\int_0^{1+\frac{t^{m+1}}{m+1}} t^{-2m+\frac{mp-3m}{2(p-1)}}(1+r)^{\frac{n-1}{2}}e^{r-\frac{t^{m+1}}{m+1}}\,drdt\right)^{\frac{1}{p'}} \\
&\times\left(\int_0^T\int_{\mathbb{R}^n}|u_t|^p t^{-2m}\theta_M^{2p'}|\lambda'|\phi\,dxdt\right)^{\frac{1}{p}} \\
\leq& CM^{-2-\frac{3m}{2}+\frac{m}{2p}+\left[\frac{(m+1)(n-1)}{2}+1\right]\frac{p-1}{p}}\left(\int_0^T\int_{\mathbb{R}^n}|u_t|^p t^{-2m}\theta_M^{2p'}|\lambda'|\phi\,dxdt\right)^{\frac{1}{p}}.
\end{aligned}\tag{5.4.7}$$

Analogously, for II and III we have

$$\begin{aligned}
II =& \int_0^T\int_{\mathbb{R}^n} u_t t^{-2m}\partial_t^2\eta_M^{2p'}\lambda\phi\,dxdt \\
\leq& CM^{-2-\frac{3m}{2}+\frac{m}{2p}+\left[\frac{(m+1)(n-1)}{2}+1\right]\frac{p-1}{p}} \\
&\times\left(\int_0^T\int_{\mathbb{R}^n}|u_t|^p t^{-2m}\theta_M^{2p'}|\lambda'|\phi\,dxdt\right)^{\frac{1}{p}},
\end{aligned}\tag{5.4.8}$$

and

$$\begin{aligned}
III =& 2\int_0^T\int_{\mathbb{R}^n} u_t t^{-2m}\partial_t\eta_M^{2p'}\lambda'\phi\,dxdt \\
\leq& CM^{-1}\left(\int_{\frac{M}{2}}^M\int_{|x|\leq\gamma(t)} t^{-2m}|\lambda'|\phi\,dxdt\right)^{\frac{1}{p'}} \\
&\times\left(\int_0^T\int_{\mathbb{R}^n}|u_t|^p t^{-2m}\theta_M^{2p'}|\lambda'|\phi\,dxdt\right)^{\frac{1}{p}} \\
\leq& CM^{-1-\frac{m}{2}+\frac{m}{2p}+\left[\frac{(m+1)(n-1)}{2}+1\right]\frac{p-1}{p}} \\
&\times\left(\int_0^T\int_{\mathbb{R}^n}|u_t|^p t^{-2m}\theta_M^{2p'}|\lambda'|\phi\,dxdt\right)^{\frac{1}{p}}.
\end{aligned}\tag{5.4.9}$$

Since $m\geq -1$ is equivalent to

$$-1-\frac{m}{2}+\frac{m}{2p}\geq -2-\frac{3m}{2}+\frac{m}{2p},$$

we conclude, by plugging (5.4.7), (5.4.8) and (5.4.9) in (5.4.4), that

$$\begin{aligned} &C_1\varepsilon+\int_0^T\int_{\mathbb{R}^n}|u_t|^p t^{-2m}\eta_M^{2p'}|\lambda'|\phi\,dxdt \\ &\leq CM^{-1-\frac{m}{2}+\frac{m}{2p}+\left[\frac{(m+1)(n-1)}{2}+1\right]\frac{p-1}{p}} \\ &\quad\times\left(\int_0^T\int_{\mathbb{R}^n}|u_t|^p t^{-2m}\theta_M^{2p'}|\lambda'|\phi\,dxdt\right)^{\frac{1}{p}}. \end{aligned} \tag{5.4.10}$$

Define now the function

$$Y[w](M):=\int_1^M\left(\int_0^T\int_{\mathbb{R}^n}w(t,x)\theta_\sigma^{2p'}(t)\,dxdt\right)\sigma^{-1}d\sigma$$

and let us denote for simplicity

$$Y(M):=Y\left[|u_t|^p t^{-2m}|\lambda'(t)|\phi(x)\right](M).$$

From direct computations we see that

$$\begin{aligned} Y(M)&=\int_1^M\left(\int_0^T\int_{\mathbb{R}^n}|u_t|^p t^{-2m}|\lambda'(t)|\phi(x)\theta_\sigma^{2p'}(t)\,dxdt\right)\sigma^{-1}d\sigma \\ &=\int_0^T\int_{\mathbb{R}^n}|u_t|^p t^{-2m}|\lambda'(t)|\phi(x)\int_1^M\theta^{2p'}(t/\sigma)\sigma^{-1}d\sigma\,dxdt \\ &=\int_0^T\int_{\mathbb{R}^n}|u_t|^p t^{-2m}|\lambda'(t)|\phi(x)\int_{\frac{t}{M}}^t\theta^{2p'}(s)s^{-1}ds\,dxdt \\ &\leq\int_0^T\int_{\mathbb{R}^n}|u_t|^p t^{-2m}|\lambda'(t)|\phi(x)\eta^{2p'}\left(\frac{t}{M}\right)\int_{\frac12}^1 s^{-1}ds\,dxdt \\ &=\ln 2\int_0^T\int_{\mathbb{R}^n}|u_t|^p t^{-2m}\eta_M^{2p'}(t)|\lambda'(t)|\phi(x)\,dxdt, \end{aligned} \tag{5.4.11}$$

where we used the definition of $\theta(t)$. Moreover

$$Y'(M)=\frac{d}{dM}Y(M)=M^{-1}\int_0^T\int_{\mathbb{R}^n}|u_t|^p t^{-2m}\theta_M^{2p'}(t)|\lambda'(t)|\phi(x)\,dxdt. \tag{5.4.12}$$

Hence by combining (5.4.10), (5.4.11) and (5.4.12), we get

$$M^{\left[\frac{(m+1)(n-1)-m}{2}\right](p-1)}Y'(M)\geq\left[C_1\varepsilon+(\ln 2)^{-1}Y(M)\right]^p,$$

which leads to

$$M\leq\begin{cases} C\varepsilon^{-\left(\frac{1}{p-1}+\frac{m-(m+1)(n-1)}{2}\right)^{-1}} & \text{for } 1<p<p_T(n,m),\\ \exp\left(C\varepsilon^{-(p-1)}\right) & \text{for } p=p_T(n,m). \end{cases}$$

Since M is arbitrary in $(1,T)$, we finally obtain the blow-up for $1<p\leq p_T(n,m)$ and the lifespan estimates (5.2.4). □

5.A Solution formula for the ODE

We show in this section how to discover the formula of the solution for equation (5.4.1). Let us suppose $m \in \mathbb{N}$ and make the ansatz

$$\lambda(t) = \sum_{h=0}^{\infty} a_h t^h, \tag{5.A.1}$$

for some constants $\{a_h\}_{h\in\mathbb{N}}$. Hence,

$$\lambda'(t) = \sum_{h=1}^{\infty} h a_h t^{h-1}, \qquad \lambda''(t) = \sum_{h=2}^{\infty} h(h-1) a_h t^{h-2}.$$

Substituting in (5.4.1) and multiplying by t^2, we get

$$\begin{aligned}
0 &= \sum_{h=2}^{\infty} h(h-1)a_h t^h - 2m \sum_{h=1}^{\infty} h a_h t^h - \sum_{h=0}^{\infty} a_h t^{h+2m+2} \\
&= \sum_{h=2}^{\infty} h(h-1)a_h t^h - 2m \sum_{h=1}^{\infty} h a_h t^h - \sum_{h=2m+2}^{\infty} a_{h-2m-2} t^h \\
&= \sum_{h=1}^{2m} h(h-2m-1)a_h t^h + \sum_{2m+2}^{\infty} [h(h-2m-1)a_h - a_{h-2m-2}] t^h.
\end{aligned} \tag{5.A.2}$$

Let us fix the constant a_0 and a_{2m+1}. We will write the other constants in dependence of these ones. Indeed, we infer from (5.A.2) that

$$\begin{aligned}
a_h &= 0 && \text{for } h = 1, \dots, 2m, \\
a_h &= \frac{a_{h-2m-2}}{h(h-2m-1)} && \text{for } h \geq 2m+2.
\end{aligned}$$

Hence, by an inductive argument, we can prove that, for any $k \in \mathbb{N}$

$$\begin{aligned}
a_h &= \begin{cases}
\dfrac{a_0}{[2(m+1)]^k\, k!\, \prod_{j=1}^{k} [2(m+1)j - (2m+1)]} & \text{if } h = 2(m+1)k, \\[2ex]
\dfrac{a_{2m+1}}{[2(m+1)]^k\, k!\, \prod_{j=1}^{k} [2(m+1)j + (2m+1)]} & \text{if } h = 2(m+1)k + 2m + 1, \\[2ex]
0 & \text{otherwise,}
\end{cases} \\
&= \begin{cases}
\dfrac{[2(m+1)]^{-2k}\, \Gamma\left(1 - \frac{m+1/2}{m+1}\right)}{k!\, \Gamma\left(k + 1 - \frac{m+1/2}{m+1}\right)} a_0 & \text{if } h = 2(m+1)k, \\[2ex]
\dfrac{[2(m+1)]^{-2k}\, \Gamma\left(1 + \frac{m+1/2}{m+1}\right)}{k!\, \Gamma\left(k + 1 + \frac{m+1/2}{m+1}\right)} a_{2m+1} & \text{if } h = 2(m+1)k + 2m + 1, \\[2ex]
0 & \text{otherwise,}
\end{cases}
\end{aligned}$$

where we used the relations

$$\prod_{j=1}^{k} (cj \pm 1) = c^k \frac{\Gamma(k+1 \pm 1/c)}{\Gamma(1 \pm 1/c)}.$$

Substituting the values of a_h into (5.A.1), we have

$$\begin{aligned}\lambda(t) &= a_0\Gamma\left(1-\frac{m+1/2}{m+1}\right)\sum_{k=0}^{\infty}\frac{[2(m+1)]^{-2k}}{k!\,\Gamma\left(k+1-\frac{m+1/2}{m+1}\right)}t^{2(m+1)k}\\ &\quad + a_{2m+1}\Gamma\left(1+\frac{m+1/2}{m+1}\right)t^{2m+1}\sum_{k=0}^{\infty}\frac{[2(m+1)]^{-2k}}{k!\,\Gamma\left(k+1+\frac{m+1/2}{m+1}\right)}t^{2(m+1)k}\\ &= c_-a_0t^{m+1/2}\sum_{k=0}^{\infty}\frac{1}{k!\,\Gamma\left(k+1-\frac{m+1/2}{m+1}\right)}\left[\frac{t^{m+1}}{2(m+1)}\right]^{2k-\frac{m+1/2}{m+1}}\\ &\quad + c_+a_{2m+1}t^{m+1/2}\sum_{k=0}^{\infty}\frac{1}{k!\,\Gamma\left(k+1+\frac{m+1/2}{m+1}\right)}\left[\frac{t^{m+1}}{2(m+1)}\right]^{2k+\frac{m+1/2}{m+1}},\end{aligned}$$

with

$$c_\pm = \Gamma\left(1\pm\frac{m+1/2}{m+1}\right)[2(m+1)]^{\pm\frac{m+1/2}{m+1}}.$$

Taking into account the relations (A.2) and (A.1) we get

$$\begin{aligned}\lambda(t) &= c_-a_0t^{m+1/2}I_{-\frac{m+1/2}{m+1}}\left(\frac{t^{m+1}}{m+1}\right) + c_+a_{2m+1}t^{m+1/2}I_{\frac{m+1/2}{m+1}}\left(\frac{t^{m+1}}{m+1}\right)\\ &= k_1t^{m+1/2}I_{\frac{m+1/2}{m+1}}\left(\frac{t^{m+1}}{m+1}\right) + k_2t^{m+1/2}K_{\frac{m+1/2}{m+1}}\left(\frac{t^{m+1}}{m+1}\right)\end{aligned}$$

with

$$k_1 = c_-a_0 + c_+a_{2m+1}, \qquad k_2 = \frac{2}{\pi}c_-a_0\sin\left(\frac{m+1/2}{m+1}\pi\right).$$

In this way we can deduce the fundamental solutions of the equation (5.4.1) when $m \in \mathbb{N}$, and from Lemma 5.3 we know that the solution formula hold also for $m \in \mathbb{R}\setminus\{-1\}$.

5.B Proof of Theorem 5.3

In this appendix we prove the $L^p - L^q$ estimates on the conjugate line for $W_1(s,t,D_x)$, $W_2(s,t,D_x)$ and their derivatives respect to time collected in Theorem 5.3. The argument is adapted from the proof of Theorem 3.3 by Yagdjian [Yag06] (see also [Rei97] and [ER18, Chapter 16]), where similar estimates for $V_1(t,D_x)$ and $V_2(t,D_x)$ are presented. Note that in [Yag06] the additional hypothesis $\sigma \geq 0$ is supposed, but this can be dropped, as we will show.

Before to proceed, we recall the following key lemmata.

Definition 5.2. Denote by $L_p^q \equiv L_p^q(\mathbb{R}^n)$ the space of tempered distributions T such that

$$\|T * f\|_{L^q} \leq C\,\|f\|_{L^p}$$

for a suitable positive constant C independent on f and all Schwartz functions $f \in \mathscr{S}(\mathbb{R}^n)$.

Denote instead with $M_p^q \equiv M_p^q(\mathbb{R}^n)$ the set of multiplier of type (p,q), i.e. the set of Fourier transforms $\mathscr{F}(T)$ of distributions $T \in L_p^q$.

Lemma 5.5 ([Hör60], Theorem 1.11). *Let f be a measurable function such that for all positive λ, we have*

$$\operatorname{meas}\{\xi \in \mathbb{R}^n : |f(\xi)| \leq \lambda\} \leq C\lambda^{-b}$$

for some suitable $b \in (1, \infty)$ and positive C. Then, $f \in M_p^q$ if $1 < p \leq 2 \leq q < \infty$ and $1/p - 1/q = 1/b$.

Lemma 5.6 ([Bre75], Lemma 2). *Fix a nonnegative smooth function $\chi \in C_0^\infty([0, \infty))$ with compact support* $\operatorname{supp}\chi \subset \{x \in \mathbb{R}^n : 1/2 \leq |x| \leq 2\}$ *such that $\sum_{k=-\infty}^{\infty} \chi(2^{-k}x) = 1$ for $x \neq 0$. Set $\chi_k(x) := \chi(2^{-k}x)$ for $k \geq 1$ and $\chi_0(x) := 1 - \sum_{k=1}^{\infty} \chi_k(x)$, so that* $\operatorname{supp}\chi_0 \subset \{x \in \mathbb{R}^n : |x| \leq 2\}$.

Let $a \in L^\infty(\mathbb{R}^n)$, $1 < p \leq 2$ and assume that

$$\left\| \mathscr{F}^{-1}(a\chi_k \widehat{v}) \right\|_{L^{p'}} \leq C \|v\|_{L^p} \quad \textit{for } k \geq 0.$$

Then for some constant A independent of a we have

$$\left\| \mathscr{F}^{-1}(a\widehat{v}) \right\|_{L^{p'}} \leq AC \|v\|_{L^p}.$$

Lemma 5.7 (Littman-type lemma, see Lemma 4 in [Bre75]). *Let P be a real function, smooth in a neighbourhood of the support of $v \in C_0^\infty(\mathbb{R}^n)$. Assume that the rank of the Hessian matrix $(\partial^2_{\eta_j\eta_k} P(\eta))_{j,k\in\{1,\dots,n\}}$ is at least ρ on the support of v. Then for some integer N the following estimate holds:*

$$\left\| \mathscr{F}^{-1}(e^{itP(\eta)}v(\eta)) \right\|_{L^\infty} \leq C(1 + |t|)^{-\rho/2} \sum_{|\alpha|\leq N} \left\| \partial_\eta^\alpha v \right\|_{L^1}.$$

We will prove now only estimate (iii) of Theorem 5.3, since the computation for estimates (i) and (ii) are completely analogous; about estimate (iv), we will sketch the proof since it could be strange to the reader that this is the only case where the range of σ collapses to be only a value.

First of all, let us set $\tau := t/s \geq 1$, $z = 2i\phi(t)\xi$, $\zeta = 2i\phi(s)\xi$ and let us introduce the smooth functions $X_0, X_1, X_2 \in C^\infty(\mathbb{R}^n; [0, 1])$ satisfying

$$X_0(x) = \begin{cases} 1 & \text{for } |x| \leq 1/2, \\ 0 & \text{for } |x| \geq 3/4, \end{cases}$$
$$X_2(x) = \begin{cases} 1 & \text{for } |x| \geq 1, \\ 0 & \text{for } |x| \leq 3/4, \end{cases}$$
$$X_1(x) = 1 - X_0(\tau^{m+1}x) - X_2(x).$$

In particular, observe that

$$X_0(\phi(t)\xi) + X_1(\phi(s)\xi) + X_2(\phi(s)\xi) \equiv 1$$

for $0 < s \leq t$ and $\xi \in \mathbb{R}^n$.

By relations (5.3.3) and (5.3.4), it is straightforward to get

$$\partial_t V_1(t, |\xi|) = \frac{m+1}{2} t^{-1} z e^{-z/2} [\Phi(\mu + 1, 2\mu + 1; z) - \Phi(\mu, 2\mu; z)]$$
$$\partial_t V_2(t, |\xi|) = e^{-z/2} \left[\Phi(1 - \mu, 1 - 2\mu; z) - \frac{m+1}{2} z \Phi(1 - \mu, 2(1 - \mu); z) \right].$$

Thus one can check, using identity (5.3.6), that

$$\begin{aligned}
\partial_t W_1(s,t,|\xi|) =& ist^m e^{-(z+\zeta)/2}|\xi| \\
&\times [\Phi(\mu+1,2\mu+1;z)-\Phi(\mu,2\mu;z)]\Phi(1-\mu,2(1-\mu);\zeta) \\
=& ist^m e^{-\zeta/2}|\xi|[e^{z/2}H^0_+(z)+e^{-z/2}H^0_-(z)+]\Phi(1-\mu,2(1-\mu);\zeta) \\
=& ist^m|\xi| \Big[e^{[1+\tau^{m+1}]\zeta/2}H^0_+(z)H^1_+(\zeta)+e^{-[1-\tau^{m+1}]\zeta/2}H^0_+(z)H^1_-(\zeta) \\
&+e^{[1-\tau^{m+1}]\zeta/2}H^0_-(z)H^1_+(\zeta)+e^{-[1+\tau^{m+1}]\zeta/2}H^0_-(z)H^1_-(\zeta)\Big] \\
\partial_t W_2(s,t,|\xi|) =& e^{-(z+\zeta)/2} \\
&\times \Phi(\mu,2\mu;\zeta)\left[\Phi(1-\mu,1-2\mu;z)-it^{m+1}|\xi|\Phi(1-\mu,2(1-\mu);z)\right] \\
=& e^{-\zeta/2}\Phi(\mu,2\mu;\zeta)[e^{z/2}H^2_+(z)+e^{-z/2}H^2_-(z)] \\
=& e^{[1+\tau^{m+1}]\zeta/2}H^2_+(z)H^3_+(\zeta)+e^{-[1-\tau^{m+1}]\zeta/2}H^2_+(z)H^3_-(\zeta) \\
&+e^{[1-\tau^{m+1}]\zeta/2}H^2_-(z)H^3_+(\zeta)+e^{-[1+\tau^{m+1}]\zeta/2}H^2_-(z)H^3_-(\zeta)
\end{aligned}$$

where for the simplicity we set

$$\begin{aligned}
H^0_\pm(z) :=& \frac{\Gamma(2\mu+1)}{\Gamma\left(\mu+\frac{1}{2}\pm\frac{1}{2}\right)}H_\pm(\mu+1,2\mu+1;z)-\frac{\Gamma(2\mu)}{\Gamma(\mu)}H_\pm(\mu,2\mu;z), \\
H^1_\pm(\zeta) :=& \frac{\Gamma(2(1-\mu))}{\Gamma(1-\mu)}H_\pm(1-\mu,2(1-\mu);\zeta),
\end{aligned}$$

and

$$\begin{aligned}
H^2_\pm(z) :=& \frac{\Gamma(1-2\mu)}{\Gamma\left(\frac{1}{2}\pm\frac{1}{2}-\mu\right)}H_\pm(1-\mu,1-2\mu;z) \\
&-it^{m+1}|\xi|\frac{\Gamma(2(1-\mu))}{\Gamma(1-\mu)}H_\pm(1-\mu,2(1-\mu);z), \\
H^3_\pm(\zeta) :=& \frac{\Gamma(2\mu)}{\Gamma(\mu)}H_\pm(\mu,2\mu;\zeta).
\end{aligned}$$

Estimates at low frequencies for $\partial_t W_1(s,t,D_x)$

Let us consider the Fourier multiplier

$$\mathscr{F}^{-1}_{\xi\to x}\left(X_0(\phi(t)\xi)|\xi|^{-\sigma}\partial_t W_1(s,t,|\xi|)\widehat{\psi}\right).$$

By the change of variables $\eta := \phi(t)\xi$ and $x := \phi(t)y$ we get

$$\begin{aligned}
&\left\|\mathscr{F}^{-1}_{\xi\to x}\left(X_0(\phi(t)\xi)|\xi|^{-\sigma}\partial_t W_1(s,t,|\xi|)\widehat{\psi}\right)\right\|_{L^{q'}} \\
&\qquad\lesssim \tau^{-1}t^{(n/q'-n+\sigma)(m+1)}\left\|T_0*\mathscr{F}^{-1}_{\eta\to y}\left(\widehat{\psi}(\eta/\phi(t))\right)\right\|_{L^{q'}}
\end{aligned}$$

where

$$\begin{aligned}
T_0 :=& \mathscr{F}^{-1}_{\eta\to y}\left(X_0(\eta)|\eta|^{1-\sigma}e^{-i[1+1/\tau^{m+1}]|\eta|}\Phi_0(\mu;\tau;|\eta|)\right) \\
\Phi_0(\mu;\tau;|\eta|) :=& [\Phi(\mu+1,2\mu+1;2i|\eta|)-\Phi(\mu,2\mu;2i|\eta|)] \\
&\times\Phi(1-\mu,2(1-\mu);2i|\eta|/\tau^{m+1}) \\
=& O(|\eta|)[1+\tau^{-(m+1)}O(|\eta|)].
\end{aligned}$$

The last equality above is implied by (5.3.2), from which we deduce $|\Phi_0(\mu;\tau;|\eta|)| \lesssim 1$ if $|\eta| \le 3/4$. So, for any $\lambda > 0$, we obtain

$$\begin{aligned}
\operatorname{meas}\{\eta \in \mathbb{R}^n \colon |\mathscr{F}_{y\to\eta}(T_0)| \ge \lambda\} &\le \operatorname{meas}\{\eta \in \mathbb{R}^n \colon |\eta| \le 3/4 \text{ and } |\eta|^{1-\sigma} \gtrsim \lambda\} \\
&\lesssim \begin{cases} 1 & \text{if } 0 < \lambda \le 1, \\ 0 & \text{if } \lambda \ge 1 \text{ and } \sigma \le 1, \\ \lambda^{-\frac{n}{\sigma-1}} & \text{if } \lambda \ge 1 \text{ and } \sigma > 1, \end{cases} \\
&\lesssim \lambda^{-b},
\end{aligned}$$

where $1 < b < \infty$ if $\sigma \le 1$ and $1 < b \le \frac{n}{\sigma-1}$ if $\sigma > 1$. Hence by Lemma 5.5, we get $T_0 \in L_q^{q'}$ for $1 < q \le 2 \le q' < \infty$ and $\sigma \le 1 + n(\frac{1}{q} - \frac{1}{q'})$. Then we obtain the Hardy-Littlewood-type inequality

$$\left\| \mathscr{F}^{-1}_{\xi\to x}\left(X_0(\phi(t)\xi)|\xi|^{-\sigma}\partial_t W_1(s,t,|\xi|)\widehat{\psi}\right)\right\|_{L^{q'}} \lesssim \tau^{-1} t^{\left[\sigma - n\left(\frac{1}{q}-\frac{1}{q'}\right)\right](m+1)} \|\psi\|_{L^q}.$$

Observing that, by the assumption on the range of σ,

$$\begin{aligned}
\tau^{-1} t^{\left[\sigma - n\left(\frac{1}{q}-\frac{1}{q'}\right)\right](m+1)} &= \tau^{-\left[1-\mu+n\left(\frac{1}{q}-\frac{1}{q'}\right)-\sigma\right](m+1)} \tau^{m/2} s^{\left[\sigma - n\left(\frac{1}{q}-\frac{1}{q'}\right)\right](m+1)} \\
&\le \tau^{m/2} s^{\left[\sigma - n\left(\frac{1}{q}-\frac{1}{q'}\right)\right](m+1)},
\end{aligned}$$

we finally get

$$\left\| \mathscr{F}^{-1}_{\xi\to x}\left(X_0(\phi(t)\xi)|\xi|^{-\sigma}\partial_t W_1(s,t,|\xi|)\widehat{\psi}\right)\right\|_{L^{q'}} \lesssim \tau^{\frac{m}{2}} s^{\left[\sigma - n\left(\frac{1}{q}-\frac{1}{q'}\right)\right](m+1)} \|\psi\|_{L^q}. \tag{5.B.1}$$

Estimates at intermediate frequencies for $\partial_t W_1(s,t,D_x)$

We proceed similarly as before. Let us consider now the Fourier multiplier

$$\mathscr{F}^{-1}_{\xi\to x}\left(X_1(\phi(s)\xi)|\xi|^{-\sigma}\partial_t W_1(s,t,|\xi|)\widehat{\psi}\right).$$

Exploiting this time the change of variables $\eta := \phi(s)\xi$ and $x := \phi(s)y$, we get

$$\begin{aligned}
&\left\| F^{-1}_{\xi\to x}\left(X_1(\phi(s)\xi)|\xi|^{-\sigma}\partial_t W_1(s,t,|\xi|)\widehat{\psi}\right)\right\|_{L^{q'}} \\
&\qquad\lesssim \tau^{m/2} s^{(n/q'-n+\sigma)(m+1)} \left\| T_1 * \mathscr{F}^{-1}_{\eta\to y}\left(\widehat{\psi}(\eta/\phi(s))\right)\right\|_{L^{q'}}
\end{aligned}$$

where

$$\begin{aligned}
T_1 &:= \mathscr{F}^{-1}_{\eta\to y}\left(X_1(\eta)|\eta|^{1-\sigma} e^{-i[1+\tau^{m+1}]|\eta|}\Phi_1(\mu;\tau;|\eta|)\right) \\
\Phi_1(\mu;\tau;|\eta|) &:= \tau^{m/2}[\Phi(\mu+1,2\mu+1;2i\tau^{m+1}|\eta|) - \Phi(\mu,2\mu;2i\tau^{m+1}|\eta|)] \\
&\quad \times \Phi(1-\mu,2(1-\mu);2i|\eta|).
\end{aligned}$$

Taking in account (5.3.2) and (5.3.5), we infer that

$$|\Phi_1(\mu;\tau;|\eta|)| \lesssim \tau^{m/2}(\tau^{m+1}|\eta|)^{-\mu} = |\eta|^{-\mu} \quad \text{on } \operatorname{supp} X_1(\eta) \subseteq \left[(2\tau^{m+1})^{-1}, 1\right],$$

and thus, for any $\lambda > 0$, we obtain

$$\begin{aligned}\text{meas}\{\eta\in\mathbb{R}^n\colon |\mathscr{F}_{y\to\eta}(T_1)|\geq\lambda\} &\leq \text{meas}\{\eta\in\mathbb{R}^n\colon |\eta|\leq 1 \text{ and } |\eta|^{1-\mu-\sigma}\gtrsim\lambda\}\\ &\lesssim \begin{cases}1 & \text{if } 0<\lambda\leq 1,\\ 0 & \text{if } \lambda\geq 1 \text{ and } \sigma\leq 1-\mu,\\ \lambda^{-\frac{n}{\sigma-1+\mu}} & \text{if } \lambda\geq 1 \text{ and } \sigma>1-\mu,\end{cases}\\ &\lesssim \lambda^{-b},\end{aligned}$$

where $1<b<\infty$ if $\sigma\leq 1-\mu$ and $1<b\leq\frac{n}{\sigma-1+\mu}$ if $\sigma>1-\mu$. Hence by Lemma 5.5, we get $T\in L_q^{q'}$ for $1<q\leq 2\leq q'<\infty$ and $\sigma\leq 1-\mu+n(\frac{1}{q}-\frac{1}{q'})$. Then we reach

$$\left\|F^{-1}_{\xi\to x}\left(X_1(\phi(s)\xi)|\xi|^{-\sigma}\partial_t W_1(s,t,|\xi|)\widehat{\psi}\right)\right\|_{L^{q'}} \lesssim \tau^{\frac{m}{2}} s^{\left[\sigma-n\left(\frac{1}{q}-\frac{1}{q'}\right)\right](m+1)}\|\psi\|_{L^q}. \tag{5.B.2}$$

Estimates at high frequencies for $\partial_t W_1(s,t,D_x)$

Finally, we want to estimate the Fourier multiplier

$$\mathscr{F}^{-1}_{\xi\to x}\left(X_2(\phi(s)\xi)|\xi|^{-\sigma}\partial_t W_1(s,t,|\xi|)\widehat{\psi}\right).$$

We choose a set of functions $\{\chi_k\}_{k\geq 0}$ as in the statement of Lemma 5.6.

L^1-L^∞ *estimates.* We claim that, for $k\geq 0$,

$$\left\|\mathscr{F}^{-1}_{\xi\to x}\left(X_2(\phi(s)\xi)\,\chi_k(\phi(s)\xi)\,|\xi|^{-\sigma}\partial_t W_1(s,t,|\xi|)\right)\right\|_{L^\infty} \lesssim 2^{k(n-\sigma)}\tau^{\frac{m}{2}} s^{(\sigma-n)(m+1)}. \tag{5.B.3}$$

Exploiting the change of variables $\phi(s)\xi=2^k\eta$ and $2^k x=\phi(s)y$, by the expression of the symbol $\partial_t W_1(s,t,|\xi|)$, we obtain

$$\begin{aligned}&\left\|\mathscr{F}^{-1}_{\xi\to x}\left(X_2(\phi(s)\xi)\,\chi_k(\phi(s)\xi)\,|\xi|^{-\sigma}\partial_t W_1(s,t,|\xi|)\right)\right\|_{L^\infty}\\ &\qquad\lesssim 2^{k(n-\sigma+1)}\tau^m s^{(\sigma-n)(m+1)}[A_+^+ + A_-^+ + A_+^- + A_-^-]\end{aligned} \tag{5.B.4}$$

where

$$\begin{aligned}A_\pm^+ &:= \left\|\mathscr{F}^{-1}_{\eta\to y}\left(e^{i[\pm 1+\tau^{m+1}]2^k|\eta|}v_k^{+,\pm}(\eta)\right)\right\|_{L^\infty},\\ A_\pm^- &:= \left\|\mathscr{F}^{-1}_{\eta\to y}\left(e^{i[\pm 1-\tau^{m+1}]2^k|\eta|}v_k^{-,\pm}(\eta)\right)\right\|_{L^\infty},\end{aligned}$$

and

$$\begin{aligned}v_k^{+,\pm}(\eta) &:= X_2(2^k\eta)\chi(\eta)|\eta|^{1-\sigma}H_+^0(2i\tau^{m+1}2^k|\eta|)H_\pm^1(2i2^k|\eta|),\\ v_k^{-,\pm}(\eta) &:= X_2(2^k\eta)\chi(\eta)|\eta|^{1-\sigma}H_-^0(2i\tau^{m+1}2^k|\eta|)H_\pm^1(2i2^k|\eta|).\end{aligned}$$

The functions $v_k^{\pm,\pm}(\eta)$ are smooth and compactly supported on $\{\eta\in\mathbb{R}^n\colon 1/2\leq|\eta|\leq 2\}$. When $k=0$, it is easy to see by estimates (5.3.7) and (5.3.8) that

$$\begin{aligned}\left\|\mathscr{F}^{-1}_{\eta\to y}\left(e^{i[\pm 1+\tau^{m+1}]|\eta|}v_0^{+,\pm}(\eta)\right)\right\|_{L^\infty} &\leq \left\|v_0^{+,\pm}\right\|_{L^1}\\ &\lesssim \tau^{-(m+1)\mu}\left\|X_2(\eta)\chi(\eta)|\eta|^{-\sigma}\right\|_{L^1}\\ &\lesssim \tau^{-m/2}.\end{aligned}$$

For $k \geq 1$, by Lemma 5.7 we have, for some integer $N > 0$, that

$$\left\| \mathscr{F}^{-1}_{\eta \to y} \left(e^{i[\pm 1 + \tau^{m+1}]2^k|\eta|} v_k^{+,\pm}(\eta) \right) \right\|_{L^\infty} \lesssim (1 + [\pm 1 + \tau^{m+1}]2^k)^{-\frac{n-1}{2}} \sum_{|\alpha| \leq N} \left\| \partial_\eta^\alpha v_k^{+,\pm} \right\|_{L^1}. \quad (5.B.5)$$

Since $X_2(2^k\eta)\chi(\eta) = \chi(\eta)$ for $k \geq 1$, by estimates (5.3.7)–(5.3.8) and Leibniz rule we infer

$$\begin{aligned} &|\partial_\eta^\alpha v_k^{+,\pm}(\eta)| \\ &= \left| \sum_{\gamma \leq \beta \leq \alpha} \binom{\alpha}{\beta} \binom{\beta}{\gamma} \partial_\eta^{\alpha-\beta} \left(\chi(\eta)|\eta|^{1-\sigma} \right) \partial_\eta^{\beta-\gamma} H_+^0(2i\tau^{m+1}2^k|\eta|) \partial_\eta^\gamma H_\pm^1(2i2^k|\eta|) \right| \\ &\lesssim \tau^{-m/2} 2^{-k} \sum_{\beta \leq \alpha} C_{\mu,\alpha,\beta} \mathbb{1}_{[1/2,2]}(\eta)|\eta|^{-1-|\beta|} \end{aligned}$$

where $\mathbb{1}_{[1/2,2]}(\eta) = 1$ for $1/2 \leq |\eta| \leq 2$ and $\mathbb{1}_{[1/2,2]}(\eta) = 0$ otherwise. From the latter estimate and (5.B.5), we get

$$A_\pm^+ \lesssim \tau^{-m/2} 2^{-k} (1 + [\pm 1 + \tau^{m+1}])^{-\frac{n-1}{2}} \leq \tau^{-m/2} 2^{-k}.$$

Similarly we obtain also that $A_\pm^- \lesssim \tau^{-m/2} 2^{-k}$. Thus, inserting in (5.B.4) we obtain (5.B.3), which combined with the Young inequality give us the $L^1 - L^\infty$ estimate

$$\left\| \mathscr{F}^{-1}_{\xi \to x} \left(X_2(\phi(s)\xi)\, \chi_k(\phi(s)\xi)\, |\xi|^{-\sigma} \partial_t W_1(s,t,|\xi|) \widehat{\psi} \right) \right\|_{L^\infty} \lesssim 2^{k(n-\sigma)} \tau^{m/2} s^{(\sigma-n)(m+1)} \|\psi\|_{L^1}. \quad (5.B.6)$$

$L^2 - L^2$ estimates. By the Plancherel formula, Hölder inequality, estimate (5.3.5) and the substitution $\phi(s)\xi = 2^k\eta$, we obtain

$$\begin{aligned} &\left\| \mathscr{F}^{-1}_{\xi \to x} \left(X_2(\phi(s)\xi)\, \chi_k(\phi(s)\xi)\, |\xi|^{-\sigma} \partial_t W_1(s,t,|\xi|) \widehat{\psi} \right) \right\|_{L^2} \\ &\leq \left\| X_2(\phi(s)\xi)\, \chi_k(\phi(s)\xi)\, |\xi|^{-\sigma} \partial_t W_1(s,t,|\xi|) \right\|_{L^\infty} \|\psi\|_{L^2} \\ &\lesssim 2^{-k\sigma} \tau^{m/2} s^{\sigma(m+1)} \|\psi\|_{L^2}. \end{aligned} \quad (5.B.7)$$

$L^q - L^{q'}$ estimates. The interpolation between (5.B.6) and (5.B.7) give us the estimates on the conjugate line

$$\left\| \mathscr{F}^{-1}_{\xi \to x} \left(X_2(\phi(s)\xi)\, \chi_k(\phi(s)\xi)\, |\xi|^{-\sigma} \partial_t W_1(s,t,|\xi|) \widehat{\psi} \right) \right\|_{L^{q'}} \lesssim 2^{k\left[n\left(\frac{1}{q}-\frac{1}{q'}\right)-\sigma\right]} \tau^{m/2} s^{\left[\sigma - n\left(\frac{1}{q}-\frac{1}{q'}\right)\right](m+1)} \|\psi\|_{L^q}, \quad (5.B.8)$$

where $1 < q \leq 2$. Now, choosing $n\left(\frac{1}{q}-\frac{1}{q'}\right) \leq \sigma$, putting together (5.B.1), (5.B.2) and (5.B.8) with an application of Lemma 5.6, we finally obtain the $L^q - L^{q'}$ estimate for $\partial_t W_1(s,t,D_x)$.

Estimates for $\partial_t W_2(s,t,D_x)$

For the intermediate and high frequencies cases, proceeding as above we straightforwardly obtain, under the constrains $\sigma \le \mu + n(\frac{1}{q}-\frac{1}{q'})$ and $n\left(\frac{1}{q}-\frac{1}{q'}\right) \le \sigma$ respectively, that

$$\left\|\mathscr{F}^{-1}_{\xi\to x}\left(X_j(\phi(s)\xi)|\xi|^{-\sigma}\partial_t W_2(s,t,|\xi|)\widehat{\psi}\right)\right\|_{L^{q'}} \lesssim \tau^{m/2} s^{\left[\sigma-n\left(\frac{1}{q}-\frac{1}{q'}\right)\right](m+1)} \|\psi\|_{L^q}. \tag{5.B.9}$$

for $j\in\{1,2\}$ and $1<q\le 2\le q'<\infty$.

At low frequencies, by computations similar to that for $\partial_t W_1(s,t,D_x)$, we obtain

$$\begin{aligned}&\left\|\mathscr{F}^{-1}_{\xi\to x}\left(X_0(\phi(t)\xi)|\xi|^{-\sigma}\partial_t W_2(s,t,|\xi|)\widehat{\psi}\right)\right\|_{L^{q'}}\\ &\qquad\qquad\lesssim t^{(n/q'-n+\sigma)(m+1)}\left\|T_0 * \mathscr{F}^{-1}_{\eta\to y}\left(\widehat{\psi}(\eta/\phi(t))\right)\right\|_{L^{q'}}\end{aligned}$$

where this time

$$\begin{aligned}T_0 &:= \mathscr{F}^{-1}_{\eta\to y}\left(X_0(\eta)|\eta|^{-\sigma}e^{-i[1+1/\tau^{m+1}]|\eta|}\Phi_0(\mu;\tau;|\eta|)\right)\\ \Phi_0(\mu;\tau;|\eta|) &:= \Phi(\mu,2\mu;2i|\eta|/\tau^{m+1})\\ &\quad\times[\Phi(1-\mu,1-2\mu;2i|\eta|)-i(m+1)|\eta|\Phi(1-\mu,2(1-\mu);2i|\eta|)]\\ &= [1+\tau^{-(m+1)}O(|\eta|)][1+O(|\eta|)],\end{aligned}$$

and hence again $|\Phi_0(\mu;\tau;|\eta|)| \lesssim 1$ if $|\eta|\le 3/4$. For any $\lambda>0$, we get

$$\mathrm{meas}\{\eta\in\mathbb{R}^n\colon |\mathscr{F}_{y\to\eta}(T_0)|\ge\lambda\} \le \mathrm{meas}\{\eta\in\mathbb{R}^n\colon |\eta|\le 3/4 \text{ and } |\eta|^{-\sigma}\gtrsim\lambda\} \lesssim \lambda^{-b},$$

where $1<b<\infty$ if $\sigma\le 0$ and $1<b\le\frac{n}{\sigma}$ if $\sigma>0$. Another application of Lemma 5.5 tell us that $T_0\in L^{q'}_q$ for $1<q\le 2\le q'<\infty$ with the condition on σ given by $\sigma\le n(\frac{1}{q}-\frac{1}{q'})$. Finally, similarly as in the case of $\partial_t W_1(s,t,D_x)$ we conclude that (5.B.9) holds true also for $j=0$.

The proof of estimates (iv) in Theorem 5.3 is thus reached combining the inequality (5.B.9) for $j\in\{0,1,2\}$; putting together all the constrains on the range of σ, we are forced to choose $\sigma = n(\frac{1}{q}-\frac{1}{q'})$.

Appendix A

Some formulas for the modified Bessel functions

For the reader's convenience, here we gather some formulas, often employed in the thesis, from Section 9.6 and Section 9.7 of the handbook by Abramowitz and Stegun [AS64].

- The solutions to the differential equation

$$z^2 \frac{d^2}{dz^2} w(z) + z \frac{d}{dz} w(z) - \left(z^2 + \nu^2\right) w(z) = 0$$

  are the modified Bessel functions $I_{\pm\nu}(z)$ and $K_\nu(z)$. $I_\nu(z)$ and $K_\nu(z)$ are real and positive when $\nu > -1$ and $z > 0$.

- Relations between solutions:

$$K_\nu(z) = K_{-\nu}(z) = \frac{\pi}{2} \frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)}. \tag{A.1}$$

  When $\nu \in \mathbb{Z}$, the right hand-side of this equation is replaced by its limiting value.

- Ascending series:

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{2k+\nu}}{k!\,\Gamma(k+1+\nu)}, \tag{A.2}$$

  where Γ is the Gamma function.

- Wronskian:

$$I_\nu(z) K_{\nu+1}(z) + I_{\nu+1}(z) K_\nu(z) = \frac{1}{z}. \tag{A.3}$$

- Recurrence relations:

$$\partial_z I_\nu(z) = I_{\nu+1}(z) + \frac{\nu}{z} I_\nu(z), \qquad \partial_z K_\nu(z) = -K_{\nu+1}(z) + \frac{\nu}{z} K_\nu(z), \tag{A.4}$$

$$\partial_z I_\nu(z) = I_{\nu-1}(z) - \frac{\nu}{z} I_\nu(z), \qquad \partial_z K_\nu(z) = -K_{\nu-1}(z) - \frac{\nu}{z} K_\nu(z). \tag{A.5}$$

- Limiting forms for fixed ν and $z \to 0$:

$$I_\nu(z) \sim \frac{1}{\Gamma(\nu+1)}\left(\frac{z}{2}\right)^\nu \quad \text{for } \nu \neq -1, -2, \ldots \tag{A.6}$$

$$K_0(z) \sim -\ln(z), \tag{A.7}$$

$$K_\nu(z) \sim \frac{\Gamma(\nu)}{2}\left(\frac{z}{2}\right)^{-\nu}, \quad \text{for } \Re\nu > 0. \tag{A.8}$$

- Asymptotic expansions for fixed ν and large $|z|$:

$$I_\nu(z) = \frac{1}{\sqrt{2\pi}} z^{-1/2} e^z \times (1 + O(z^{-1})), \qquad \text{for } |\arg z| < \frac{\pi}{2}, \tag{A.9}$$

$$K_\nu(z) = \sqrt{\frac{\pi}{2}} z^{-1/2} e^{-z} \times (1 + O(z^{-1})), \qquad \text{for } |\arg z| < \frac{3}{2}\pi. \tag{A.10}$$

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