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Nonlinear parabolic stochastic evolution equations in critical spaces

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Cycle XXXIII

Mathematics is a dangerous profession;
an appreciable proportion of us go mad.

J. E. Littlewood

Abstract

In this thesis we develop a new approach to nonlinear stochastic partial differential equations (SPDEs) with Gaussian noise. Our aim is to provide an abstract framework which is applicable to a large class of SPDEs and includes many important cases of nonlinear parabolic problems which are of quasi- or semilinear type. One of the main contributions of this thesis is a new method to bootstrap Sobolev and Hölder regularity in time and space, which does not require smoothness of the initial data. This leads to new results even in the classical L^2-settings, which we illustrate for a parabolic SPDE and for the stochastic Navier-Stokes equations in two dimensions.

Our theory is formulated in an L^p-setting, and because of this we can deal with nonlinearities in a very efficient way. Applications to local-well posedness to several concrete problems and their quasilinear variants are given. This includes Stochastic Navier-Stokes equations, Burger's equation, the Allen-Cahn equation, the Cahn-Hilliard equation, reaction-diffusion equations, and the porous media equation. The interplay of the nonlinearities and the critical spaces of initial data leads to new results and insights for these SPDEs. Most of the previous equations will be considered with a gradient-noise term.

The thesis is divided into three parts. The first one concerns local well-posedness for stochastic evolution equations. Here, we study stochastic maximal L^p-regularity for semigroup generators, and in particular, we prove a sharp time-space regularity result for stochastic convolutions which will play a basic role for the nonlinear theory. Next, we show local existence of solutions to stochastic evolution equations with rough initial data which allows us to define 'critical spaces' in an abstract way. The proofs are based on weighted maximal regularity techniques for the linearized problem as well as on a combination of several sophisticated splitting and truncation arguments. The local-existence theory developed here can be seen as a stochastic version of the theory of critical spaces due to Prüss-Simonett-Wilke (2018). We conclude the first part by applying our main result to several SPDEs. In particular, we check that critical spaces defined abstractly coincide with the critical spaces from a PDEs perspective, i.e. spaces invariant under the natural scaling of the SPDE considered.

The second part is devoted to the study of blow-up criteria and instantaneous regularization. Here we prove several blow-up criteria for stochastic evolution equations. Some of them were not known even in the determinstic setting. For semilinear equations we obtain a Serrin type blow-up criterium, which extends a recent result of Prüss-Simonett-Wilke (2018) to the stochastic setting. Blow-up criteria can be used to prove global well-posedness for SPDEs. As in the first part, maximal regularity techniques and weights in time play a central role in the proofs. Next we present a new abstract bootstrapping method to show Sobolev and Hölder regularity in time and space, which does not require smoothness of the initial data. The blow-up criteria are at the basis of these new methods. Moreover, in applications the bootstrap results can be combined with our blow-up criteria, to obtain efficient ways to prove global existence. This fact will be illustrated for a concrete SPDE.

In the third part, we apply the previous results to study quasilinear reaction-diffusion equations and stochastic Navier-Stokes equations with gradient noise. As regards the former, we show global well-posedness and instantaneous regularization of solutions employing suitable dissipative conditions. Here we also prove a suitable stochastic version of the parabolic DeGiorgi-Nash-Moser estimates by employing a standard reduction method. The last chapter concerns stochastic Navier-Stokes equations and in the three dimensional case we prove local existence with data in the critical spaces L^3 and $B^{\frac{3}{q}-1}_{q,p}$. In addition, we prove a blow-up criterium for solutions with paths in $L^p(L^q)$ where $\frac{2}{p}+\frac{3}{q}=1$ which extends the usual Serrin blow-up criteria to the stochastic setting. Finally, we prove existence of global solutions in two dimensions under minimal assumptions on the noise term and on the initial data.

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Contents

Chapter 1

Introduction

Stochastic partial differential equations (SPDEs) of evolution type arise in many areas of mathematics, physics as well as in engeneering applications and have attracted a lot of attention in the past decades. SPDEs are typically derived as 'random' perturbation of partial differential equations (PDEs) and the stochastic terms can model thermal fluctations, uncertain determinations of the parameters and non-predictable forces acting on the system. For instance, stochastic perturbation of the Navier-Stokes equations are widely used in the study of turbulence in fluid flows.

SPDEs are usually modelled as an ordinary stochastic differential equations (SDEs) in an infinite dimensional state space. The latter are typically referred to as *stochastic evolution equations* and have been extensely studied in the literature. In this thesis we will deal with stochastic evolution equations of parabolic type. This means that the leading operators provide (some) smoothing effects. We will turn to this point when we will analyse maximal regularity techniques. There are two concepts of solutions to stochastic evolution equations: strong and martingale solutions. Roughly speaking, in the first case the probability space is fixed, while in the second one, it is variable and has to be constructed jointly with the solution. In this thesis we focus on strong solutions only. Let us stress that existence of strong solutions to stochastic evolution equations can be (sometimes) proven by analysing martingale solutions and applying the Yamada-Watanabe theorem (see [148, Appendix E]).

There are several fundamental approaches or methods to prove existence and uniqueness of strong solutions. One method is based on monotonicity of the operators under consideration (see [147]). It has been applied in the stochastic setting by Pardoux [171] and by Krylov and Rozovskiĭ in [135] (see also [148] for a textbook exposition). Another method is the semigroup approach and in this thesis we mainly follow its philosophy but we also borrow some ideas from the monotone approach. The semigroup method started with Curtain and Falb [48], Dawson [55] and has been continued by DaPrato and collaborators (see [50] and the refereces therein). The references given so far, only deal with spaces having a Hilbert space structure which is (sometimes) restrictive in applications. An important step in the recent developments of stochastic evolution equations in a non Hilbertian setting was made by Brzeźniak in [25, 26] where the author studied stochastic evolution equations in Banach spaces with martingale type 2 (for such spaces we refer to [107, Subsection 3.5.d] and the references therein). In [25] maximal regularity results have been considered and in [26] applications to (semilinear) stochastic evolution equations were given by using fixed point methods. In the semigroup approach, a breakthrough was made by van Neerven, Veraar and Weis in [163, 166]. In [163] the authors developed a stochastic integration theory for processes with values in UMD Banach spaces (see also [169]) and sharp two-side estimates for the stochastic integrals which generalize the Itô-isometry known for Hilbert valued processes. In [166], the authors showed optimal space-time regularity estimates for stochastic convolutions where the above mentioned two-side estimates played a crucial role. Finally, maximal regularity technique was used by van Neerven, Veraar and Weis in [165] to study semilinear stochastic evolution equations, and subsequently extended by Hornung [104] to quasilinear equations.

The aim of this thesis is to completely revise this theory and provide a new and systematic treatment of parabolic stochastic evolution equations by employing maximal regularity techniques. Here we are concerned with equations of the form

$$\begin{cases} du + A(u)udt = F(u)dt + (B(u)u + G(u))dW_H, & t \in (0,T), \\ u(0) = u_0. \end{cases} \tag{1.1}$$

The leading operator A is of quasilinear type which means that for each v in a suitable interpolation space

$$u \mapsto A(v)u,$$

defines a mapping from X_1 into X_0, where $X_1 \hookrightarrow X_0$ densely. The problem (1.1) includes the semilinear case where the pair $(A(u)u, B(u)u)$ is replaced by $(\widetilde{A}u, \widetilde{B}u)$. Here $(\widetilde{A}, \widetilde{B})$ are operators not depending on u. The noise term W_H is a cylindrical Brownian motion. The nonlinearities F and G are of semilinear type. As we will see, many SPDEs fit in this framework.

In this thesis we build an $L^p(L^q)$-theory for (1.1) in which the coercivity condition can be formulated for an abstract pair (A, B) and where we allow (t, ω)-dependence in the operators in an adapted way. Krylov's L^p-theory [129] is an important step in this direction, and recently an evolution equation approach has been found by Veraar and Portal in [174], which additionally gives optimal space-time-regularity. Here we will completely revise the general theory, and our approach has a lot of flexibility. In particular, we allow:

- a quasilinear couple (A, B);
- measurable dependence in (t, ω);
- (A, B) without smallness conditions on B;
- weights in the time variable $w_\kappa(t) = t^\kappa$ with $\kappa \in [0, \frac{p}{2} - 1)$;
- rough initial data: $u_0 \in (X_0, X_1)_{1-\frac{1+\kappa}{p},p}$;
- nonlinearities F and G defined on interpolation spaces $[X_0, X_1]_{1-\varepsilon}$ which are locally Lipschitz and have polynomial growth;
- $L^p(0,T; L^q)$-theory and $L^p(0,T; H^{s,q})$-theory for a range of $s \in \mathbb{R}$.

In the above $(X_0, X_1)_{\theta,p}$ and $[X_0, X_1]_\theta$ denote the real and complex interpolation spaces, respectively. In applications these can be identified with certain Besov spaces and Bessel potential spaces.

Using the weights w_κ we will introduce a stochastic version of the theory of *critical spaces*, which we will briefly discuss in the deterministic setting in the next section of the introduction. After that we will give a simplified introduction to stochastic maximal L^p-regularity which will play a basic role throughout the thesis. In Section 1.3 we give an informal overview of the abstract results proven and in Section 1.4 we discuss applications to SPDEs with particular emphasis to stochastic reaction diffusion equations and stochastic Navier-Stokes equations with gradient noise.

1.1 Criticality

In the literature *critical spaces* are often introduced as those spaces which satisfy a scaling invariance similar to the one of the PDE itself, or as those spaces in which the energy bound and nonlinearity are of the same order. More details on this can be found in [31, 121, 144, 190, 199], and references therein. For example for the Navier-Stokes equations on $\mathbb{R}^3$ one can obtain solutions in $L^p(0,T; L^q)$ for small initial data in the critical space $\dot{B}_{q,p}^{-1+\frac{3}{q}}$ provided the criticality condition $\frac{2}{p} + \frac{3}{q} = 1$ holds, and $q \in (3, \infty)$ (see [144, p. 182]).

Another way to introduce criticality would be to consider a specific nonlinearity, e.g. $F(u) = |u|^r$ in a given PDE. Typically, some exponent r turns out to be critical in the sense that the "usual" estimates are not powerful enough anymore. Below that value of r the problem is usually called *subcritical* and above that value it is called *supercritical*.

In a recent paper of Prüss-Simonett-Wilke [178] a new viewpoint on critical spaces has been discovered in the deterministic setting. Special cases have been considered before in [175, 180, 181]. The authors consider abstract evolution equations in spaces of the form $L^p((0,T), w_\kappa; X_0)$, where $p \in (1,\infty)$, $w_\kappa(t) = t^\kappa$ is a weight function with $\kappa \in [0, p-1)$ and typically X_0 is a Sobolev or an L^q-space. Assuming *maximal regularity* (see Section 1.2) for the leading term and several other conditions, the authors establish local well-posedness. The weight can be chosen in correspondence with the polynomial growth rate of the nonlinearity to obtain what they call a critical weight. After the weight exponent κ is fixed, the so-called *trace space* of initial values which one can consider becomes $(X_0, X_1)_{1-\frac{1+\kappa}{p},p}$, and this space they call critical.

A surprising feature is that in many concrete examples the latter trace space coincides with the critical space from a PDE point of view. In [178] this leads to several new results for classical PDEs of evolution type such as the Navier–Stokes equation, Cahn-Hilliard equations, convection-diffusion equations, and many more. A crucial point in their theory is that F does not have to be defined on the real interpolation spaces $(X_0, X_1)_{1-\frac{1}{p},p}$ and one can allow it to be defined on a much smaller space X_θ with $\theta > 1 - \frac{1}{p}$ at the cost of a growth condition on F.

In this thesis we will develop a stochastic version of the above theory. For this many additional difficulties have to be overcome. Some of them are connected to $L^p(\Omega)$-integrability issues for the nonlinearities, and others are connected with the fact that in stochastic maximal regularity (see next section) one needs to work with vector-valued spaces with *fractional* smoothness to obtain the right trace theory. Note that in the stochastic case the condition on κ becomes more restrictive $\kappa \in [0, \frac{p}{2} - 1)$ (in the deterministic case this was $\kappa \in [0, p-1)$). Another issue in the examples is that the stochastic version of maximal L^p-regularity theory is more complicated and less developed than the deterministic case. Fortunately, there is a lot of current research in this direction and some new progress is contained in the thesis as well.

We will show that our theory can be applied to several classes of parabolic SPDEs. With a hands on approach for each SPDE separately one can often obtain very detailed properties of solutions. Our theory can provide other information as one usually obtains new spaces in which the problem can be analyzed, and thus provides different regularity results which where often not available yet.

Before we continue our discussion, we will first introduce the reader to so-called *stochastic maximal regularity*, which is one of the main tools used to study (1.1).

1.2 Stochastic maximal regularity

Maximal regularity has many forms and has always played a fundamental role in modern PDE. Below we will try to explain some of the background in a nontechnical way. The precise definitions can be found in Chapter 3 and Subsection 4.2.

Arguably the most common form of maximal regularity for elliptic equations is: the solution u to $\lambda u - \Delta u = f$ with $f \in L^q(\mathbb{R}^d)$ and $\lambda > 0$ satisfies

$$\|u\|_{W^{2,q}(\mathbb{R}^d)} \leq C\|f\|_{L^q(\mathbb{R}^d)},$$

where $q \in (1,\infty)$ and C is a constant depending on λ and q. The result fails for the endpoints $q \in \{1,\infty\}$. For $q = 2$ this result is simple, and for general q one typically uses Calderón–Zygmund theory (see [94]).

For the heat equation a similar result holds: the solution u to $\partial_t u - \Delta u = f$, with initial condition $u_0 \in B^{2-2/p}_{q,p}(\mathbb{R}^d)$ (Besov space) and $f \in L^p(0,T; L^q(\mathbb{R}^d))$ satisfies

$$\|u\|_{W^{1,p}(0,T;L^q(\mathbb{R}^d))} + \|u\|_{L^p(0,T;W^{2,q}(\mathbb{R}^d))} \eqsim \|u_0\|_{B^{2-2/p}_{q,p}(\mathbb{R}^d)} + \|f\|_{L^p(0,T;L^q(\mathbb{R}^d))},$$

where $p, q \in (1,\infty)$. Again the result fails if p or q are in $\{1,\infty\}$. There are many ways how to deduce the latter results, and again Calderón–Zygmund theory plays a central role. The fact that the estimate is two-sided shows that the result is optimal.

An efficient reformulating of the last result is

$$\|u\|_{W^{1,p}(0,T;X_0)} + \|u\|_{L^p(\mathbb{R}_+;X_1)} \eqsim \|u_0\|_{(X_0,X_1)_{1-\frac{1}{p},p}} + \|f\|_{L^p(0,T;X_0)},$$

where $X_0 = L^q(\mathbb{R}^d)$ and $X_1 = W^{2,q}(\mathbb{R}^d)$. In this form the result can be extended to many other parabolic problems, and this has been an important field of research for decades:

- For a PDE perspective see [93, 133, 140];
- For an evolution equation perspective see [62, 138, 177].

This topic is still very active in various schools as it is evident from the many recent results (see e.g. [63, 66, 67, 75, 106, 172, 185, 196]). As we explained before sharp estimates for the linear setting can be used very effectively in the nonlinear case. In the quasilinear case for deterministic equations the standard reference for this is [46], and the recent monograph [177].

In the stochastic situation the above theory is much more recent. If u is a solution to the stochastic heat equation $du + \Delta u dt = f dt + g dW$, then for all $p \in (2,\infty)$ and $q \in [2,\infty)$

$$\begin{aligned}\|u\|_{L^p(\Omega;H^{\theta,p}(0,T;H^{2-2\theta,q}(\mathbb{R}^d)))} \lesssim \|u_0\|_{L^p(\Omega;B^{2-2/p}_{q,p}(\mathbb{R}^d))} &+ \|f\|_{L^p(\Omega;L^p(0,T;L^q(\mathbb{R}^d)))}\\ &+ \|g\|_{L^p(\Omega;L^p(0,T;W^{1,q}(\mathbb{R}^d;\ell^2)))},\end{aligned}$$

for any $\theta \in [0,1/2)$. Moreover, if $q = 2$, then $p = 2$ is also allowed. Here $H^{\theta,p}$ denotes the Bessel-potential space with smoothness θ. The above result was proved in [166] by van Neerven, Veraar and Weis. The case $\theta = 0$ and $p \geq q \geq 2$ was obtained before in [125, 128, 129, 130] with a slightly stronger assumption on u_0. Recently, a stochastic version of Calderón–Zygmund theory was developed by Lorist and Veraar [149]. The latter can be used to derive the full range $p \in (2,\infty)$ and $q \in (2,\infty)$ from the case $p = q$.

As before the evolution equation reformulation is of the form

$$\begin{aligned}\|u\|_{L^p(\Omega;H^{\theta,p}(0,T;X_{1-\theta}))} \lesssim \|u_0\|_{L^p(\Omega;(X_0,X_1)_{1-\frac{1}{p},p})} &+ \|f\|_{L^p(\Omega;L^p(0,T;X_0))}\\ &+ \|g\|_{L^p(\Omega;L^p(0,T;\gamma(\ell^2,X_{\frac{1}{2}})))},\end{aligned} \tag{1.2}$$

where $X_{1-\theta} = [X_0, X_1]_{1-\theta}$ denotes the complex interpolation spaces and coincides with $\mathsf{D}(A^{1-\theta})$ for $A = 1 - \Delta$ on X_0. This is the setting in which in [166] the stochastic maximal regularity was proved for a large class of SPDEs. An important difference with the deterministic case is that the estimate does not hold for the end-point $\theta = 1/2$. However, the half-open interval $\theta \in [0,1/2)$ is good enough for applications.

The abstract view-point in [166] will be pursued in Chapter 3 where we study stochastic maximal L^p-regularity by looking at estimates of the form (1.2) for solutions to $du + Au dt = f dt + g dW_H$ where A is a semigroup generator. Among others, we prove the independence on $\kappa \in [0, \frac{p}{2} - 1)$ of the *weighted* maximal L^p-regularity for power weights $w_\kappa = |t|^\kappa$. In particular, (1.2) is equivalent to the weighted estimate

$$\begin{aligned}\|u\|_{L^p(\Omega;H^{\theta,p}(0,T,w_\kappa;X_{1-\theta}))} \lesssim \|u_0\|_{L^p(\Omega;(X_0,X_1)_{1-\frac{1+\kappa}{p},p})} &+ \|f\|_{L^p(\Omega;L^p(0,T,w_\kappa;X_0))}\\ &+ \|g\|_{L^p(\Omega;L^p(0,T,w_\kappa;\gamma(\ell^2,X_{\frac{1}{2}})))},\end{aligned}$$

where $X_0 = L^q(\mathbb{R}^d)$ and $X_1 = W^{2,q}(\mathbb{R}^d)$ as above. A key observation is the following. If $\kappa \uparrow \frac{p}{2} - 1$, then $(X_0, X_1)_{1-\frac{1+\kappa}{p},p}$ becomes larger than $(X_0, X_1)_{1-\frac{1}{p},p}$ appearing in (1.2). Therefore the weights allow us to study stochatic evolution equations with rough initial data.

It is important to note that the natural formulation of the stochastic heat equations is actually $du + \Delta u dt = f dt + (g + Bu) dW$, where

$$Bu\, dW = \sum_{j=1}^{d} \sum_{n\geq 1} b_{jn} \partial_j u \, dw^n, \tag{1.3}$$

where $(b_{jn})_{n\geq 1} \in \ell^2$. Under the parabolicity/coercivity assumption

$$|\xi|^2 - \frac{1}{2}\sum_{i,j=1}^{d}\sum_{n\geq 1} b_{in}b_{jn}\xi_i\xi_j \geq \delta|\xi|^2, \tag{1.4}$$

the above estimates for the stochastic heat equation still hold (see [174, Theorem 5.3] and Section 5.1.1 for a more general formulation). A similar situation appears for the stochastic Navier-Stokes equations arising in the study of turbulent flows (see (1.11) and (1.14) below). To handle such situations, in Chapter 4 we introduce and study a (generalized) notion of stochastic maximal L^p-regularity where we consider couple of operators (A, B) rather than a single operator A. Several properties of this generalized maximal regularity will be given in Subsections 4.2, 6.2 and 9.1.2 where the analysis of the case $B = 0$ will play of basic importance to extrapolate the space-time regularity estimate (1.2) from the case $\theta = 0$ (see Proposition 4.2.8).

Although we will only use stochastic maximal regularity in the $L^p(L^q)$ scale, it is important to note that it can also be considered in different scales such as the Besov scale and Hölder scale. For details on this we refer to [25, 28, 49, 149] for the Besov scale and to [70, 71, 205]. The Hölder case can be seen as a stochastic analogue of Schauder theory (see [93, 127]).

1.3 Stochastic evolution equations in critical spaces

In this subsection we present our main results concerning (1.1) proven in Chapters 4, 6 and 7 in an informal way. The precise statements will be given in the main text of the thesis.

1.3.1 Local well-posedness and critical spaces

The main result of Chapter 4 is a local well-posedness result. Details can be found in Section 4.3, and in particular the precise statements can be found in Theorem 4.3.5, 4.3.7 and 4.3.8 below.

Theorem 1.3.1. *Let $p \in [2, \infty)$ and $w_\kappa(t) = t^\kappa$ with $\kappa \in [0, \frac{p}{2} - 1)$ (set $\kappa = 0$ if $p = 2$). Under maximal L^p-regularity assumptions on the pair (A, B), and local Lipschitz conditions and polynomial growth conditions on A, B, F and G, and assuming $u_0 \in X^{\mathsf{Tr}}_{\kappa,p}$ a.s., there exists a unique L^p_κ-maximal solution (u, σ) to (1.1), and the paths of u almost surely satisfy*

$$u \in L^p_{\text{loc}}([0, \sigma), w_\kappa; X_1) \cap C([0, \sigma); X^{\mathsf{Tr}}_{\kappa,p}) \cap C((s, \sigma); X^{\mathsf{Tr}}_p). \tag{1.5}$$

In the above $\sigma > 0$ a.s., and $X^{\mathsf{Tr}}_p = (X_0, X_1)_{1-\frac{1}{p},p}$ and $X^{\mathsf{Tr}}_{\kappa,p} := (X_0, X_1)_{1-\frac{1+\kappa}{p},p}$. By analyzing the precise polynomial growth conditions of F and G we obtain conditions on (p, κ) for criticality of the space $X^{\mathsf{Tr}}_{\kappa,p}$. Of course this condition also depends on the choice of the spaces X_0 and X_1. However, the corresponding 'trace space' $X^{\mathsf{Tr}}_{\kappa,p}$ in the critical case is usually independent of the choice of the scale (see [178, Section 2.4] and the applications in Chapters 5, 8 and 9).

1.3.2 Blow-up criteria

In applications to problems of mathematical physics and/or enginnering one expects that the corresponding solutions are global in time or, in case of it ceases to exist, one is willing to understand the mechanics behind the 'explosion' of the solutions. Theorem 1.3.1 does not provide any information on the life-span σ of solutions to (1.1) and to answer the questions above we need to investigate conditions that allows us to determinate whether $\sigma = \infty$ a.s. or not. Such requirements are (usually) referred to as *blow-up criteria*. For (deterministic) evolution equations such conditions are widely known and we cannot give a complete overview. We refer to [177, Chapter 5] and [178, Theorem 2.4] and for a PDE perspective we refer to [182]. On the contrary, little is known about explosion criteria for stochastic evolution equations, and to the best of our knowledge, in this thesis we give a first systematic treatment of such criteria for solutions to (1.1). Let us mention that in [26, 104, 164, 165] some abstract settings appear in which global existence is proved using an argument which resembles a blow-up criterium. Of these criteria, [104, Theorem 4.3] comes closest to ours, for a comparison see the text below Theorem 1.3.2.

Next we state our blow-up criteria for stochastic evolution equations which will be proven in Chapter 6. Let us begin by looking at the quasilinear case. More details and other criteria can be found in Theorem 6.3.6 and its proof. To formulate it we introduce the following notation:

$$\mathcal{N}^{\kappa}(u;t) := \|F(\cdot,u)\|_{L^p(0,t,w_\kappa;X_0)} + \|G(\cdot,u)\|_{L^p(0,t,w_\kappa;\gamma(H,X_{1/2}))}.$$

Theorem 1.3.2 (Quasi-linear case)**.** *Under suitable conditions, the maximal solution (u,σ) of Theorem 1.3.1 satisfies*

(1) $\mathbb{P}\Big(\sigma<\infty,\, \lim_{t\uparrow\sigma} u(t)$ *exists in* $X^{\mathsf{Tr}}_{\kappa,p},\, \mathcal{N}^{\kappa}(u;\sigma)<\infty\Big)=0$*;*

(2) $\mathbb{P}\Big(\sigma<\infty,\, \lim_{t\uparrow\sigma} u(t)$ *exists in* $X^{\mathsf{Tr}}_{\kappa,p}\Big)=0$ *if* $X^{\mathsf{Tr}}_{\kappa,p}$ *is not critical for* (1.1)*;*

(3) $\mathbb{P}\Big(\sigma<\infty,\, \lim_{t\uparrow\sigma} u(t)$ *exists in* $X^{\mathsf{Tr}}_{\kappa,p},\, \|u\|_{L^p(0,\sigma;X_{1-\frac{\kappa}{p}})}<\infty\Big)=0.$

The blow-up criteria of Theorem 1.3.2 can often be used to prove global existence. Indeed, for this one needs a suitable a priori bound for the solution which implies the existence of the limit $\lim_{t\uparrow\sigma} u(t)$ in the 'trace space' $X^{\mathsf{Tr}}_{\kappa,p}$ on the set $\{\sigma<\infty\}$. According to Theorem 1.3.2 this can only happen if $\mathbb{P}(\sigma<\infty)=0$, and thus $\sigma=\infty$ a.s. Of course to prove an a priori bound or energy estimate we need to use structural properties of a given SPDE. To obtain such bounds one can typically use Ito's formula, combine it with one-sided growth conditions of F, and subtle regularity results for linear SPDEs.

Theorem 1.3.2(3) in the case $\kappa=0$ can be compared with [104, Theorem 4.3]. However, in that case Theorem 1.3.2(2) is applicable and actually easier to check as we do not need to consider $\|u\|_{L^p(0,\sigma;X_1)}<\infty$ in the criterium. Moreover, there are many differences, and in particular the assumptions on the nonlinearities and initial data in [104] are much more restrictive, and only $\kappa=0$ is considered.

In the semilinear case much more can be said (see Theorems 6.3.7 and 6.3.8 for the precise statements).

Theorem 1.3.3 (Semi-linear case)**.** *Under suitable conditions, the maximal solution (u,σ) of Theorem 1.3.1 satisfies*

(1) $\mathbb{P}\Big(\sigma<\infty,\, \sup_{t\in[0,\sigma)}\|u(t)\|_{X^{\mathsf{Tr}}_{\kappa,p}}<\infty\Big)=0$ *if* $X^{\mathsf{Tr}}_{\kappa,p}$ *is non-critical for* (1.1)*;*

(2) $\mathbb{P}\Big(\sigma<\infty,\, \sup_{t\in[0,\sigma)}\|u(t)\|_{X^{\mathsf{Tr}}_{\kappa,p}}+\|u\|_{L^p(0,\sigma;X_{1-\frac{\kappa}{p}})}<\infty\Big)=0;$

(3) $\mathbb{P}\Big(\sigma<\infty,\, \|u\|_{L^p(0,\sigma;X_{1-\frac{\kappa}{p}})}<\infty\Big)=0$ *under extra conditions on* κ*.*

The above results extend the blow-up criteria in [178, Corollaries 2.2, 3.3 and Theorem 2.4] to the stochastic setting. The criterium (3) is a Serrin type blow-up condition, and probably the deepest of the criteria stated here. It seems that our result is the first systematic approach to blow-up criteria in the stochastic case. The global existence results for stochastic Navier-Stokes equations in two dimensions, and equations of reaction diffusion type in Chapters 8 and 9, will be based on these new criteria. Let us mention that some of the criteria we obtain are even new in the deterministic setting.

The advantage of our approach is that for a given concrete SPDE, the local well-posedness theory, and blow-up criteria can be used as a black box. So to prove global existence one only needs to prove energy estimates (which can be hard). However, the rest of the argument can be completed in a rather soft way. We summarize this in the following roadmap of which a more extensive version can be found in Roadmap 6.3.11. Applications of the results will be discussed in Chapter 7, 8 and 9.

Roadmap 1.3.4 (Proving global existence and regularity).

(a) Prove local well-posedness and regularity with Theorem 1.3.1.

(b) Prove an energy estimate.

(c) Combine the energy estimate with Theorem 1.3.2 or 1.3.3 to prove $\sigma = \infty$.

Moreover, instantaneous regularization (see Subsection 1.3.3 below) can help in the above scheme as often the extra regularity enable us to prove energy estimates.

1.3.3 Instantaneous regularization

In order to introduce the reader to instantaneous regularization in this section we let $X_1 \subsetneqq X_0$ which is usual in applications to SPDEs. From (1.5) one sees that if $\kappa > 0$, then the solution to (1.1) instantaneously regularizes 'in space' as the regularity of u for $t > 0$ is better than the one in $t = 0$ since $X_p^{\mathsf{Tr}} \subsetneqq X_{\kappa,p}^{\mathsf{Tr}}$. This simple but central result is the key behind our new bootstrapping method, which we will now explain in the special setting of Corollary 7.1.5 which requires the conditions $p > 2$ and $\kappa > 0$. The case $p = 2$ or $\kappa = 0$ can be studied as well, see Proposition 7.1.7 and the text below it.

Fix $s > 0$, $r \in (p, \infty)$. Since $\kappa > 0$, we can choose $\alpha \in [0, \frac{r}{2} - 1)$ such that $\frac{1}{p} < \frac{1+\alpha}{r} < \frac{1+\kappa}{p}$. By (1.5) we have $u(s) \in X_p^{\mathsf{Tr}} \hookrightarrow X_{\alpha,r}^{\mathsf{Tr}}$ a.s. and one can construct a maximal local solution to (1.1) starting at s with initial data $u(s) \in X_{\alpha,r}^{\mathsf{Tr}}$ by Theorem 1.3.1. This gives a maximal local solution (v, τ) on $[s, \infty)$ and by (1.5),

$$v \in L^r_{\mathrm{loc}}([s, \tau), w_\alpha; X_1) \cap C([s, \tau); X_{\alpha,r}^{\mathsf{Tr}}) \cap C((s, \tau); X_r^{\mathsf{Tr}}) \quad \text{a.s.} \tag{1.6}$$

Since $r > p$, $X_r^{\mathsf{Tr}} \subsetneqq X_p^{\mathsf{Tr}}$ and hence the regularity of v seems to be better than the one of u in (1.5). Now if we could show that $\tau = \sigma$ and $u = v$ on $[s, \sigma)$, then this would improve the regularity of u significantly. By choosing r large one can even obtain Hölder regularity in time (see Corollary 7.1.5 for details).

To prove $u = v$, first note that by using the regularity of v and the uniqueness of the maximal local solution (u, σ), one can obtain $\tau \leq \sigma$ a.s. and $v = u$ on $[\![s, \tau[\![$. This is not surprising since v is 'more regular' than u and therefore one expects that v blows-up before u. The key step is proving that $\sigma = \tau$. To prove it note that on the set $\{\tau < \sigma\}$, $v = u \in C((s, \tau]; X_p^{\mathsf{Tr}}) \subseteq C((s, \tau]; X_{\alpha,r}^{\mathsf{Tr}})$, and hence

$$\begin{aligned}
\mathbb{P}(\tau < \sigma) &= \mathbb{P}\Big(\{\tau < \sigma\} \cap \{\tau < T\} \cap \Big\{\lim_{t \uparrow \tau} v(t) \text{ exists in } X_{\alpha,r}^{\mathsf{Tr}}\Big\}\Big) \\
&\leq \mathbb{P}\Big(\tau < T, \lim_{t \uparrow \tau} v(t) \text{ exists in } X_{\alpha,r}^{\mathsf{Tr}}\Big) = 0,
\end{aligned}$$

which follows from the blow-up criterium of Theorem 1.3.2(2) applied to (v, τ).

The above gives an abstract bootstrap mechanism to obtain time regularization of solutions to (1.1). A variation of this strategy can be used to bootstrap regularity in space. This requires two Banach couples (Y_0, Y_1) and $(\widehat{Y}_0, \widehat{Y}_1)$ in which the equation (1.1) can be considered as well. Next we state this result. The precise assumptions are too technical to state here, but the conditions to be checked seem to be natural in all examples we have considered in Chapters 8 and 9. For the precise details we refer to Theorem 7.1.3.

Theorem 1.3.5. *Let (Y_0, Y_1) and $(\widehat{Y}_0, \widehat{Y}_1)$ be Banach spaces such that*

$$Y_1 \hookrightarrow Y_0, \qquad \widehat{Y}_1 \hookrightarrow \widehat{Y}_0, \qquad \text{and} \qquad \widehat{Y}_i \hookrightarrow Y_i \hookrightarrow X_i.$$

Let $\widehat{r} \geq r \geq p > 2$, $\alpha \in [0, \frac{r}{2} - 1)$, and $\widehat{\alpha} \in [0, \frac{\widehat{r}}{2} - 1)$. Let (u, σ) be the L^p_κ-maximal solution of Theorem 1.3.1. Under suitable conditions, the following implication holds:

$$u \in \bigcap_{\theta \in [0,1/2)} H^{\theta,r}_{\mathrm{loc}}(0, \sigma; Y_{1-\theta}) \quad a.s. \implies u \in \bigcap_{\theta \in [0,1/2)} H^{\theta,\widehat{r}}_{\mathrm{loc}}(0, \sigma; \widehat{Y}_{1-\theta}) \quad a.s.$$

We emphasize that we do not need additional regularity on the initial data u_0, since the arguments all take place on $[s, \infty)$ with $s > 0$. The main idea of the theorem is that regularity in the (Y_0, Y_1, r, α)-setting be transferred to the $(\widehat{Y}_0, \widehat{Y}_1, \widehat{r}, \widehat{\alpha})$-setting. Since we can choose the spaces Y_i and $\widehat{Y}_i$, we can iterate the above to gain regularity. The regularity class $\bigcap_{\theta\in[0,1/2)} H^{\theta,r}_{\mathrm{loc}}(0, \sigma; Y_{1-\theta})$ seems rather obscure at first sight. However, as we have seen in Subsection 1.2, it is the one that contains all information concerning stochastic maximal L^r-regularity.

The extra regularity obtained by bootstrapping, is of course interesting from a theoretical point of view, but it can also assist in proving global existence. Indeed, due to the extra smoothness and integrability, often one can prove energy estimates on an interval $[s, \sigma)$ with $s > 0$ by applying Itô's formula and integration by parts arguments.

In classical bootstrapping arguments one argues in a completely different way. Given the maximal solution (u, σ) one investigates what regularity $f := F(\cdot, u)$ and $g := G(\cdot, u)$ have, and combines this with regularity estimate for *linear equations* with inhomogeneities f and g to (hopefully) find more space and time regularity for u. With the new information on u, one can repeat this argument over and over again. This method is of course very important, but it also has some disadvantages. First of all it requires a smooth initial value. Moreover, in case of critical nonlinearities or unweighted situations it often not possible to use this argument as $F(\cdot, u)$ or $G(\cdot, u)$ does not have the right integrability/regularity properties. This will be discussed for the 2D Navier-Stokes equations (see (1.16) below) and in an 1D example see Subsection 7.2 and the introduction in Chapter 7.

In order to deal with the critical and unweighted case (in particular if $p = 2$) we proved a further variation of the bootstrapping result of Theorem 1.3.5 in Proposition 7.1.7. Here the idea is to exchange some of the space regularity to create a weighted setting out of unweighted setting. As soon as the weight is there, the loss of integrability and regularity can be recovered with Theorem 1.3.5.

The applications of Theorem 1.3.5 (and its variants) are 'universal' to some extent. Indeed, the implementation of such result depends only on X_0, X_1, p, κ and does not depend on the concrete SPDEs under investigation. We will see an instance of this fact in studying the regularization of solutions to the stochastic Navier-Stokes equations (see Theorem 1.4.3(1) below). The previous somewhat surprising fact could be understood by recalling that the quadruple (X_0, X_1, p, κ) encodes the scaling property of the underlined SPDE and this is the only information we need to prove instantaneous regularization results.

Finally, we mention that in deterministic theory one can often use the implicit function theorem to prove higher order regularity in time and space. This method is referred to as the *parameter trick* and usually attributed to [12, 11]. It can be used to prove differentiability and even real analyticity in time. For further details on this method we refer to [177, Chapter 5] and to the notes of that chapter for further historical accounts. Of course, differentiability in time is completely out of reach in the stochastic setting, since already Brownian itself is not differentiable. Therefore, it seems impossible to extend this method to the stochastic framework.

1.4 Applications to parabolic SPDEs

Chapters 5, 8 and 9 will be devoted to applications of the theory developed for (1.1) to SPDEs. Our aim is to show the flexibility and usefulness of the abstract results proven here. In all cases we obtain sharp results in an $L^p(L^q)$-scale and most results in this setting seem to be new. Specifically, in Chapter 5 we apply well-posedness result of Theorem 1.3.1 to several parabolic SPDEs including reaction-diffusion equations in conservative and non-conservative form, quasilinear SPDEs with or without spatial weights, the porous media equation, and 1D Burger's equation with white noise. Relying also on the results in Chapters 6 and 7, in Part III we study regularization and global well-posedness for quasilinear reaction diffusion equations and for the stochastic Navier-Stokes equations. In Chapters 8-9, we mainly consider equations on the d-dimensional torus $\mathbb{T}^d$ and/or bounded domains. This choice simplifies the proofs of energy estimates that are needed to check the blow-up criteria of Theorems 1.3.2 and 1.3.3. However, with some additional work and assumptions, we expect that our methods can be applied to study equations on unbounded domains.

To give a flavour of the results one can obtain with the theory developed in this thesis, in the next subsections we state some results which will be proven in this work. Here we pay particular attention to the stochastic Navier-Stokes equations with gradient noise which, to the author's opinion, seem the most important and explanatory application treated in this thesis. In all cases, we consider $H = \ell^2$ and the cylindrical Brownian motion W_{ℓ^2} is solely determinated by a sequence $(w^n)_{n\geq 1}$ of independent standard Brownian motions on a complete filtered probability space $(\Omega, \mathbb{F} = (\mathscr{F}_t)_{t\geq 0}, \mathscr{A}, \mathbb{P})$ (see Example 2.3.6). Other choices are possible, see Subsection 5.1.5 and 5.2.7 for the 1D Burger's equations with white and coloured noises, respectively. First we give a local well-posedness result for a parabolic SPDE on $\mathbb{R}^d$ in critical spaces, then we discuss a quasilinear Allen-Cahn equation and the stochastic Navier-Stokes equations.

1.4.1 Parabolic SPDEs with power-type nonlinearities

Consider the following stochastic reaction-diffusion equations on $\mathbb{R}^d$ with $d \geq 3$

$$\begin{cases} du - \Delta u dt = u|u|^2 dt + \sum_{j=1}^{d} \sum_{n\geq 1} \Big(b_{jn} \partial_j u(x) + g_n u|u| \Big) dw_t^n, & \text{on } \mathbb{R}^d, \\ u(0) = u_0, & \text{on } \mathbb{R}^d, \end{cases} \tag{1.7}$$

where $u : [0,\infty) \times \Omega \times \mathbb{R}^d \to \mathbb{R}$ is the unknown process, $(b_{jn})_{n\geq 1}$ and $(g_n)_{n\geq 1} \in \ell^2$ and the b_{jn} satisfy (1.4). The following is a special case of Theorem 5.1.10. Problem (1.7) can be recasted in the form (1.1). The definition of maximal local solution will be given in Section 4.3 (see the text below (5.23) and Definition 4.3.4).

Theorem 1.4.1. *Let $d \geq 3$. Assume that $q \in [2, d)$ and $q > d/2$. Let $p \in (2,\infty)$ be such that $\frac{1}{p} + \frac{d}{2q} \leq 1$ holds, and let $\kappa_{\mathsf{crit}} = p(1 - \frac{d}{2q}) - 1$. Then for each*

$$u_0 \in L^0_{\mathscr{F}_0}(\Omega; B^{\frac{d}{q}-1}_{q,p}(\mathbb{R}^d))$$

there exists a maximal local solution (u,σ) to (1.7) such that

$$u \in L^p_{\mathrm{loc}}([0,\sigma), w_{\kappa_{\mathsf{crit}}}; W^{1,q}(\mathbb{R}^d)) \cap C([0,\sigma); B^{\frac{d}{q}-1}_{q,p}(\mathbb{R}^d)) \cap C((0,\sigma); B^{1-\frac{2}{p}}_{q,p}(\mathbb{R}^d)) \ a.s.$$

Here $L^p(0,T,w_\kappa)$ denotes the weighted L^p-space with weight $w_\kappa(t) = t^\kappa$. One can check that (1.7) is invariant under the scaling (see Subsection 5.1.3)

$$u_\lambda(t,x) := \lambda^{1/2} u(\lambda t, \lambda^{1/2} x), \qquad \text{for } \lambda > 0,\ x \in \mathbb{R}^d. \tag{1.8}$$

Moreover, the space of initial data $B^{\frac{d}{q}-1}_{q,p}(\mathbb{R}^d)$ (or actually its homogeneous version) is invariant under this scaling as well. Employing the results in Theorem 1.3.3 one can obtain useful criteria to determinate whetever (u,σ) is a global solution to (1.7), i.e. $\sigma = \infty$. Global existence results for semilinear reaction-diffusion equations will be investigate in Chapter 8 under a dissipation condition. Let us point out some interesting features of Theorem 1.4.1:

- We can obtain $W^{1,q}$-solutions for any initial data with arbitrary low but positive smoothness. Using the results in Chapter 7 one can show that u instantaneously gains regularity in space and time;
- Only part of the structure of the nonlinearities $u|u|^2$ and $u|u|$ plays a role in the formulation of the result. In particular, if the nonlinearities have a different growth, then the above spaces need to be changed accordingly (see Theorem 5.1.10 for details);
- The deterministic and stochastic nonlinearities in (1.7) have the same scaling under the map (1.8) (see Subsection 5.1.3).

1.4.2 Stochatic quasilinear Allen-Cahn equation

Allen-Cahn equation is a reaction-diffusion equation of mathematical physics which describes the process of phase separation in multi-component alloy systems, including order-disorder transitions. Here we consider the following quasilinear stochastic Allen-Cahn equation on a C^1-bounded domain $\mathscr{O} \subseteq \mathbb{R}^d$ with $d \geq 2$

$$\begin{cases} du - \operatorname{div}(a(u)\cdot\nabla u)dt = (u-u^3)dt + \sum_{j=1}^d \sum_{n\geq 1} \Phi_n(u)\, dw_t^n, & \text{on } \mathscr{O}, \\ u = 0 & \text{on } \partial\mathscr{O}, \\ u(0) = u_0, & \text{on } \mathscr{O}, \end{cases} \tag{1.9}$$

where $u : [0,\infty)\times\Omega\times\mathscr{O} \to \mathbb{R}$ is the unknown process, $\Phi := (\Phi)_{n\geq 1} : \mathbb{R} \to \ell^2$ is measurable, $a : \mathbb{R} \to \mathbb{R}^{d\times d}$ is bounded, locally Lipschitz and

$$\sum_{i,j=1}^d a^{i,j}(y)\xi_j\xi_j \geq \nu|\xi|^2, \text{ for all } \xi\in\mathbb{R}^d \text{ and } \quad \|(\Phi_n(y)-\Phi(y'))_{n\geq 1}\|_{\ell^2} \leq C|y-y'|^2$$

for some $\nu > 0$ and $C \in (0, \frac{2}{d-1})$ independent of $y, y' \in \mathbb{R}$. To recast (1.9) in the form (1.1), we introduce spaces with Dirichlet boundary conditions:

$${}_DH^{1,q}(\mathscr{O}) := \{v \in W^{1,q}(\mathscr{O}) : v = 0 \text{ on } \partial\mathscr{O}\}, \text{ and } {}_DB^s_{q,p}(\mathscr{O}) := \{v \in B^s_{q,p}(\mathscr{O}) : v = 0 \text{ on } \partial\mathscr{O}\}$$

where $q,p \in (1,\infty)$ and $s \in (\frac{1}{q}, 1)$ and B stands for Besov spaces. Let us note that ${}_DC^{s'}(\mathscr{O}) \hookrightarrow {}_DB^s_{q,p}(\mathscr{O})$ for all $s' > s$ where ${}_DC^{s'}(\mathscr{O}) := \{v \in C^s(\mathscr{O}) : v = 0 \text{ on } \partial\mathscr{O}\}$ (see Remark 8.3.4). For each $\theta_1, \theta_2 > 0$ and open set $J \subseteq \mathbb{R}_+ = (0,\infty)$, we denote by $C^{\theta_1,\theta_2}(J\times\overline{\mathscr{O}})$ the set of all bounded maps $v : J\times\overline{\mathscr{O}} \to \mathbb{R}$ such that

$$|v(t,x) - v(t',x')| \lesssim |t-t'|^{\theta_1} + |x-x'|^{\theta_2}, \quad \text{for all } \ (t,x),(t',x') \in J\times\overline{\mathscr{O}} \tag{1.10}$$

and $C^{\theta_1,\theta_2}_{\rm loc}(\mathbb{R}_+\times\overline{\mathscr{O}})$ the set of all maps $v \in C^{\theta_1,\theta_2}(J\times\overline{\mathscr{O}})$ for all compact sets $J \subseteq \mathbb{R}_+$.

The following is a special case of Theorem 8.3.2. Weak solutions to (1.9) are defined in Subsection 8.3.1 and are intended as weak in the analytic sense (or strong in the probability sense).

Theorem 1.4.2. *Let the above assumptions be satisfied. Suppose that $q \in [2,\infty)$, $p \in (2,\infty)$ and $\kappa \in [0, \frac{p}{2}-1)$ satisfy $1 - 2\frac{1+\kappa}{p} > \frac{d}{q}$. Then for any $u_0 \in L^0_{\mathscr{F}_0}(\Omega; {}_DB^{1-2\frac{1+\kappa}{p}}_{q,p}(\mathscr{O}))$ there exists a* global *weak solution u to (8.66) such that*

$$u \in L^p_{\rm loc}([0,\infty), w_\kappa; {}_DH^{1,q}(\mathscr{O})) \cap C([0,\infty); {}_DB^{1-2\frac{1+\kappa}{p}}_{q,p}(\mathscr{O})), \qquad a.s.$$

Moreover, the global solution u instantaneously regularizes in time and space: For all $\theta_1 \in (0,1/2)$ and $\theta_2 \in (0,1)$ one has $u \in C^{\theta_1,\theta_2}_{\rm loc}(\mathbb{R}_+\times\overline{\mathscr{O}})$ a.s.

Some special features of Theorem 1.4.2 are the following:

- Theorem 1.4.2 is applicable for $u_0 \in {}_DC^\delta(\mathscr{O})$ with $\delta > 0$ choosing p, κ such that $1 - 2\frac{1+\kappa}{p} < \delta$;
- As in (1.7) the deterministic and the stochastic nonlinearities have the same scaling;
- Weak solutions to (1.9) instantaneously become 'almost' C^1 in space. As we will show in [6], it is possible also to improve such result. In particular, we will prove that u *instantaneously becomes a strong solution* to (1.9) and $u \in C^{\theta_1,\theta_2}(I_\sigma\times\overline{\mathscr{O}})$ for all $\theta_1 \in (0,1/2)$ and $\theta_2 \in (0,2)$;
- In Theorem 8.3.2 we allow $a^{i,j}$ to be VMO in space and measurable in time.

The proof of Theorem 1.4.2 relies on the blow-up criterium in Theorem 1.3.2(2) and the stochastic DeGiorgi-Nash-Moser estimates. In absence of gradient-noise term, such estimates can be obtained by reducing the deterministic problem to a deterministic PDEs (see Subsection 8.1.2). The argument breaks down if a gradient-noise term is present, and therefore we omitted in (1.9) the term $b_{j,n}\partial_j u$ which appears in (1.7).

1.4.3 Stochastic Navier-Stokes equations for turbulent flows

In Chapter 9 we are concerned with the study of existence and regularity of solutions to the stochastic Navier-Stokes equations with gradient noise:

$$\begin{cases} du - \Delta u\, dt = \big(-\nabla P - \operatorname{div}(u\otimes u)\big)dt + \displaystyle\sum_{n\geq 1} (\phi_n \cdot \nabla) u\, dw_t^n, & \text{on } \mathbb{T}^d, \\ \operatorname{div} u = 0, \qquad u(\cdot,0)=u_0, & \text{on } \mathbb{T}^d. \end{cases} \tag{1.11}$$

Here $u := (u^k)_{k=1}^d : I_T \times \Omega \times \mathbb{T}^d \to \mathbb{R}^d$ denotes the unknown velocity field, $P, Q_n : I_T \times \Omega \times \mathbb{T}^d \to \mathbb{R}$ the unknown pressures, $u \otimes u := (u^j u^k)_{j,k=1}^d$ and

$$(\phi_n \cdot \nabla) u := \Big(\sum_{j=1}^d \phi_n^j \partial_j u^k \Big)_{k=1}^d, \qquad \operatorname{div}(u\otimes u) := \Big(\sum_{j=1}^d \partial_j (u^j u^k) \Big)_{k=1}^d.$$

We also consider (1.11) with additional forcing terms and variable viscosity, see (9.48).

The relation between Navier-Stokes equations and hydrodynamic turbulence is one of the most challenging problems in fluid mechanics. It is believed that the onset of the turbulence is related to the randomness of background movement. The mathematical study of stochastic perturbations of Navier-Stokes equations began with the work of Bonsoussan and Temam in [19] and it was later extended in several directions, see e.g. [24, 27, 40, 32, 72, 83, 84, 109, 158, 206] and the references therein. In the mathematical literature, usually, the random perturbation of the Navier-Stokes equations is postulated and does not have a clear physical motivation. In [160, 161] a new approach was taken to derive stochastic perturbation of the Navier-Stokes equations. Indeed, it is assumed that the dynamics of the fluid particles is given by the *stochastic flow*

$$\begin{cases} d\mathcal{X}(t,x) = u(t,\mathcal{X}(t,x))dt + \phi(t,\mathcal{X}(t,x)) \circ dW, \\ \mathcal{X}(0,x) = x, \qquad x \in \mathscr{O}, \end{cases} \tag{1.12}$$

where u,ϕ are undetermined, W and $\circ$ denote a white-noise and the Stratonovich integration, respectively. Roughly speaking, (1.12) says that the velocity field splits into a sum of slow oscillating part $u\,dt$ and a fast oscillating part $\phi \circ dW$. As noticed in [161], such splitting can be traced back to the work of Reynolds in 1880. In more recent works, closely related models were proposed by Kraichnan [122, 123] in the study of turbulence transportation of passive scalars and then developed further by other authors, see e.g. [89, 90]. The corresponding theory of turbulence transportation is called Kraichnan's theory in which one typically has $\phi = ((\phi_n^j)_{j=1}^d)_{n\geq 1}$ and

$$\phi_n = (\phi_n^j)_{j=1}^d \in H^{\frac{d}{2}+\alpha} \cap \{\operatorname{div}\xi = 0\} \hookrightarrow C^\alpha, \text{ for all } n \geq 1 \tag{1.13}$$

where $\alpha > 0$ is small and $H^s := H^{s,2}$ are Bessel-potential spaces, cf. [162, Section 1]. We refer to [82, 160, 161, 162] for additional motivations and to [161] for a derivation of (1.11) from (1.12) *via* the second Newton's law or balance of forces. A similar stochastic perturbation was also introduced in the study of Cammassa-Holm equation in [47], and subsequentially analysed in [8].

The problem (1.11) fits into the setting (1.1). Reasoning as in Subsection 1.2 (see (1.3) and the text before it), B is *not* a lower-order term w.r.t. $-\Delta$ and therefore we need to use the full strenght of the theory of stochastic evolution equations developed here. To recast (1.11) in the form (1.1) we let $\mathbb{H}^{s,q}(\mathbb{T}^d)$ and $\mathbb{B}^s_{q,p}(\mathbb{T}^d)$ be the Bessel-potential and Besov spaces of $\mathbb{R}^d$-valued divergence free vector fields (see Subsection 9.2.1 for the precise definition), respectively. To ensure parabolicity of (1.11), we assume

$$|\xi|^2 - \frac{1}{2}\sum_{i,j=1}^d \sum_{n\geq 1} \phi_n^i(x)\phi_n^j(x)\xi_i\xi_j \geq \nu|\xi|^2, \quad \text{for all } \xi \in \mathbb{R}^d \text{ and } x \in \mathbb{T}^d. \tag{1.14}$$

Let us state the following special case of Theorems 9.3.2, 9.3.3, 9.3.5 and 9.3.6. Recall that the spaces $C^{\theta_1,\theta_2}_{\text{loc}}$ have been defined below (1.10). For the definition of δ-weak solution to (1.11) see the text before Theorem 9.3.2.

Theorem 1.4.3. *Let $d \in \{2,3\}$. Assume that (1.14) holds and $\phi^j \in C^\alpha(\mathbb{T}^3;\ell^2)$ for some $\alpha > 0$ and all $j \in \{1,\dots,d\}$. Let $\delta \in [-\frac{1}{2},0]$ be such that $|\delta| \geq \alpha$. Assume that either $p = q = d = 2$ and $\delta = 0$, or*

$$q \in [2,\infty), \quad p \in (2,\infty) \text{ satisfy } \quad \frac{d}{2+\delta} < q < \frac{d}{1+\delta} \quad \text{and} \quad \frac{2}{p} + \frac{d}{q} \leq 2+\delta.$$

Then for each $u_0 \in L^0_{\mathscr{F}_0}(\Omega;\mathbb{B}^{\frac{d}{q}-1}_{q,p}(\mathbb{T}^d))$ there exists a unique δ-weak solution (u,σ) to (1.11) with $\sigma > 0$ a.s. and

$$u \in L^p_{\mathrm{loc}}([0,\sigma),w_\kappa;\mathbb{H}^{1+\delta,q}(\mathbb{T}^d)) \cap C([0,\sigma);\mathbb{B}^{\frac{d}{q}-1}_{q,p}(\mathbb{T}^d)) \text{ a.s.} \tag{1.15}$$

where $\kappa = -1 + \frac{p}{2}(2+\delta-\frac{d}{q})$. Moreover the following assertions hold:

(1) (Almost C^1-regularization) *$u \in C^{\theta_1,\theta_2}_{\mathrm{loc}}(I_\sigma \times \mathbb{T}^d)$ a.s. for all $\theta_1 \in (0,1/2)$ and $\theta_2 \in (0,1)$;*

(2) (Global existence in 2D) *If $d = 2$, then $\sigma = \infty$ a.s.;*

(3) (Stochastic Serrin's criteria) *Let $d = 3$, $r \in (2,\infty)$, $\delta \in [-\frac{1}{2},0)$ and $\xi \in (3,\frac{3}{1+\delta})$ be such that $\frac{2}{r} + \frac{3}{\xi} = 1$. Then*

$$\mathbb{P}\Big(\varepsilon < \sigma < \infty,\ \|u(t)\|_{L^r(\varepsilon,\sigma;L^\xi(\mathbb{T}^3;\mathbb{R}^3))} < \infty\Big) = 0 \quad \text{for all } \varepsilon > 0.$$

The first statement in Theorem 1.4.3 yields a stochastic version of the well-known local existence result for the deterministic Navier-Stokes equations in the critical spaces $\mathbb{B}^{\frac{d}{q}-1}_{q,p}(\mathbb{T}^d)$ (see e.g. [31, 144, 181, 199]). However, some limitations on q appear due to the regularity of the noise term $(\phi_n \cdot \nabla)u\, dw^n_t$. Moreover, if ϕ_n is smooth, then setting $\delta = -\frac{1}{2}$ we get $q < 2d$ and Theorem 1.4.3 provides local existence for (1.11) with data in critical spaces with smoothness $\frac{d}{q} - 1 > -\frac{1}{2}$. The optimality of such threeshold goes beyond the scope of this thesis.

Since $\alpha > 0$, Theorem 1.4.3 is always applicable for some $\delta < 0$ and therefore we may choose $q \in (d, \frac{d}{1+\delta})$ so that $\mathbb{B}^{\frac{d}{q}-1}_{q,p}(\mathbb{T}^d)$ has negative smoothness. As we will see in Corollary 9.3.4, this allows one to prove local-well posedness for data in the critical space $\mathbb{L}^3(\mathbb{T}^3)$ (i.e. the subset of divergence free $\mathbb{R}^d$-valued vector fields in L^3). The content of (1) yields the following

$$u_0 \in \mathbb{B}^{\frac{d}{q}-1}_{q,p}(\mathbb{T}^d) \text{ a.s.} \quad \Rightarrow \quad u(t) \in C^\theta(\mathbb{T}^d) \text{ a.s. for all } t \in (0,\sigma)$$

for all $\theta \in (0,1)$. In other words, (1) shows that solutions to (1.11) *instantaneous regularize in space* regardless the regularity of the initial data.

Theorem 1.4.3 yields new results even in the widely studied 2D-case. Indeed, if $q > 2$, then (2) ensures global existence for (1.11) with data in $\mathbb{B}^{\frac{2}{q}-1}_{p,q}(\mathbb{T}^2)$ for some $q > 2$. Since $\mathbb{L}^2(\mathbb{T}^2) \hookrightarrow \mathbb{B}^{\frac{2}{q}-1}_{p,q}(\mathbb{T}^2)$, (2) improves the usual 2D-global existence result (see e.g. [162, Theorem 2.2] for the similar $\mathbb{R}^2$-case) including initial data with *infinite energy*. In addition, (1) is of particular interest in the case $d = q = p = 2$ and $\delta = 0$. Indeed, the usual bootstrapping argument sketched in Subsection 1.3.3 cannot be applied to show smoothing of u. Indeed, one is tempted to prove $|u| \in L^r_{\mathrm{loc}}([0,\sigma);L^\zeta(\mathbb{T}^2))$ (i.e. $u \otimes u \in L^{r/2}_{\mathrm{loc}}([0,\sigma);L^{\zeta/2}(\mathbb{T}^2))$) for $r,\zeta \geq 4$ where either $r > 4$ or $\zeta > 4$. However this is not possible since the following embeddings are sharp

$$C([0,t];L^2(\mathbb{T}^2)) \cap L^2(0,t;H^1(\mathbb{T}^2)) \hookrightarrow L^4(0,t;H^{1/2}(\mathbb{T}^2)) \hookrightarrow L^4(0,t;L^4(\mathbb{T}^2)), \quad t > 0. \tag{1.16}$$

Therefore using (1.15) with $p = q = 2$ and $\kappa = 0$ we merely obtain $\mathrm{div}(u\otimes u) \in L^2_{\mathrm{loc}}([0,\sigma);H^{-1}(\mathbb{T}^2))$ which is useless if we want to improve the regularity by standard methods.

The above is related to the fact that $\mathrm{div}(u\otimes u)$ is a *critical nonlinearity* for (1.11) in the 2D-case. Roughly speaking this means that the standard energy bound in $L^\infty(0,\sigma;L^2(\mathbb{T}^2))\cap L^2(0,\sigma;H^1(\mathbb{T}^2))$ has the same order as the nonlinearity.

Item (3) is the stochastic analogue of [144, Theorem 11.2] and will be derived from Theorem 1.3.3(3). Actually, we prove a more refined result where L^{ξ} is replaced by a Sobolev space with (possible) *negative* smoothness.

To conclude, let us mention that in [7] the abstract results proven here will be used to improve the results in Theorem 1.4.3. Indeed, we will show that (1) can be improved to $u \in C^{\theta_1,\theta_2}(I_\sigma \times \mathbb{T}^d)$ where $\theta_1 \in (0, 1/2)$, $\theta_2 \in (0, 1+\eta)$ with $\eta > 0$ and even to an optimal regularization in the Sobolev scale provided $\phi \in H^{d/2+\alpha}$ (cf. (1.13)). Moreover, we will prove that solution to (1.11) are global in probability if the initial data is small in the critical space $\mathbb{B}^{d/q-1}_{q,p}(\mathbb{T}^d)$.

1.5 Overview

This thesis is divided into three parts. Part I consists of Chapters 3-5 and is devoted to stochastic maximal L^p-regularity, local well-posedness for (1.1) and its application to parabolic SPDEs. Chapters 6 and 7 represent Part II where we study blow-up criteria and regularization results for solutions to (1.1). Part III consists of Chapters 8-9 and provides applications of the theory in Parts I-II to stochastic reaction diffusion and stochastic Navier-Stokes equations.

In Chapter 2 we explain some preliminary results on sectorial operators, H^∞-calculus, deterministic maximal L^p-regularity, fractional Sobolev spaces with power weights and stochastic integration in UMD Banach spaces. In Chapter 3 we study stochastic maximal L^p-regularity presenting a weighted sharp space-time regularity estimate for stochastic convolutions which is the main result in [5]. In Chapters 4 and 5 we prove Theorem 1.3.1 and we apply it to several SPDEs of parabolic type. The latter results are taken from [3].

In Chapters 6 and 7 we prove Theorems 1.3.2, 1.3.3 and 1.3.5. In Chapter 7 we also apply our bootstrapping method to a 1D SPDE with cubic nonlinearity. These results are taken from [4].

In Chapter 8 we study semilinear and quasilinear reaction-diffusion equations showing global well-posedness and regularization phenomena. The latter results will be presented in [6]. Chapter 9 is devoted to the study of stochastic Navier-Stokes where we prove Theorem 1.4.3 which will be treated in [7].

Resumé of the activities

The research activities of my PhD programme were supervised by my advisor Prof. M. Veraar (TU Delft) but I also collaborated with Prof. D. Andreucci, Prof. P. Loreti and Prof. D. Sforza (Sapienza University - SBAI Department). During my PhD I mainly studied functional anaytic methods for stochastic evolution equations and related topics. Moreover, I also worked on control theory and evolution equations with non-local terms.

During my first year of PhD I took the following exams:

- Large Deviations, course held by Prof. A. Faggionato at the Math Department - Sapienza University;
- Analisi Funzionale, course held by Prof. C. Pinzari at the Math Department - Sapienza University;
- Mean Field Limits in Classical and Quantum Mechanics, course held by Prof. F. Golse at the Math Department - Sapienza University;
- Nonlinear Diffusion in inhomogeneous Environments, course held by Prof. A. Tedeev and D. Andreucci at the SBAI Department - Sapienza University.

I spent three visiting periods of study at the technology University of Delft (TU Delft) during the following periods: 5 Nov 2018 - 7 Dec 2018, 3 Jun 2019 - 6 Jul 2019, 10 Nov 2019 - 7 Dec 2019.

Publications

During my PhD program I contributed to the following publications and preprints which are partially contained in this thesis.

- A. Agresti, M. Veraar, *Stability properties of Stochastic maximal L^p-regularity*, J. Math. Anal. Appl. 482 (2020), no. 2, 123553, 35 pp.
- A. Agresti, M. Veraar, *Nonlinear parabolic stochastic evolution equations in critical spaces Part I: Stochastic maximal regularity and local existence*, Accepted for publication in Nonlinearity, arXiv preprint arXiv:2001.00512, 2020, 106 pp.
- A. Agresti, M. Veraar, *Nonlinear parabolic stochastic evolution equations in critical spaces Part II: Blow-up criteria and instantaneous regularization*, arXiv preprint arXiv:2012.04448, 78 pp.

Besides the contents of this thesis I contributed to the following publications.

- A. Agresti, D. Andreucci, P. Loreti, *Variable Support Control for the Wave Equation - A Multiplier Approach*, Proceedings of the 15th International Conference on Informatics in Control, Automation and Robotics (ICINCO), 2, 33–42, 2018, 10 pp.
- A. Agresti, D. Andreucci, P. Loreti, *Observability for the Wave Equation with Variable Support in the Dirichlet and Neumann Cases*, Informatics in Control, Automation and Robotics, 2019, 25 pp.
- A. Agresti, D. Andreucci, P. Loreti, *Alternating and Variable Controls for the wave equation*, ESAIM Control Optim. Calc. Var. 26 (2020), Paper No. 38, 35 pp.
- A. Agresti, P. Loreti, D. Sforza, *Time memory effect in entropy decay of Ornstein-Uhlenbeck operators*, Accepted for publication in Minimax Theory and its Applications, arXiv preprint arXiv:1910.12931, 2020, 17 pp.

Conferences and Schools

During my PhD program I attended the following conferences and schools.

- Contributed talk at V Congreso de jovenes investigadores 2020, Castello (Spain), January 2020.
- Department Seminar at the TU Delft, Delft (Netherlands), November 2019.
- Summer School: New Frontiers in Singular SPDEs and Scaling Limits, Bonn (Germany), September 2019.
- Recent Trends in Stochastic Analysis and SPDEs, Pisa (Italy), July 2019.
- Equadiff 2019, Leiden (Netherlands), July 2019.
- Contributed talk at International Conference on Elliptic and Parabolic Problems, Gaeta (Italy), May 2019.
- Spring School: Random Interfaces, Augsburg (Germany), March 2019.
- Winter School: Stochastic PDEs and Mean-Field Games, Bologna (Italy), January 2019.
- Contributed talk at 15th International Conference on Informatics in Control, Automation and Robotics (ICINCO), Porto (Portugal), July 2018.
- Summer School: Developments in Stochastic Partial Differential Equations in honour of Giuseppe Da Prato, Varese (Italy), July 2018.
- Contributed talk at Harmonic Analysis for Stochastic PDEs, Delft (Netherlands), July 2018.
- Contributed talk at SIMAI 2018, Rome (Italy), July 2018.
- Department Seminar at the TU Delft, Delft (Netherlands), April 2018.
- Winter School: Mathematical Challenges in Quantum Mechanics, Roma (Italy), February 2018.

Chapter 2

Preliminaries

In this preliminary chapter we recall some definitions and results that we will frequently use. Further definitions and references will be given in the next chapters where needed. In Section 2.1 we discuss several operator theoretic concepts such as sectorial operators, H^∞-calculus and R-boundedness. In Subsection 2.1.2 we discuss some basic properties of deterministic maximal L^p-regularity and its connection with the R-boundedness. In Section 2.2 we introduce Sobolev spaces with fractional smoothness and power weights which will play a basic role throughout the thesis. In particular, in Subsection 2.2.1 we prove an 'optimal' trace embedding for anisotropic Sobolev spaces with power weights. Stochastic integration in UMD spaces will be the topic of Section 2.3. Here we only provide basic facts and references to the existing literature. In Subsections 2.3.1-2.3.2 we discuss also the concept of type 2 and the class of γ-radonifying operators.

This chapter also contains definitions and notations which will be used throughout the thesis. For the reader's convenience we provide a list here. Further variants of the following notation will be also employed in the later chapters. For a Banach space X, $\theta \in (0,1)$, $I = (a,b)$ an open interval with $-\infty \leq a < b \leq \infty$ and a sectorial operator A we let:

- $[\cdot,\cdot]_\theta$ and $(\cdot,\cdot)_{\theta,p}$ denote the complex and the real interpolation functors respectively, see e.g. [20, 151, 197];
- $\mathsf{D}(A^\theta)$ and $\mathsf{D}_A(\theta,p)$ denote the domain of its fractional power and the real interpolation space defined in (2.4), respectively;
- $w_\kappa(t) = |t|^\kappa$ for $\kappa, t \in \mathbb{R}$ is a power weight;
- $L^p(I, w_\kappa; X)$ or $L^p(a,b,w_\kappa;X)$ denote weighted Lebesgue spaces (see (2.6));
- $H^{\theta,p}(I,w_\kappa;X)$ or $H^{\theta,p}(a,b,w_\kappa;X)$ (resp. ${}_0H^{\theta,p}(a,b,w_\kappa;X)$ or ${}_0H^{\theta,p}(a,b,w_\kappa;X)$) denote the fractional Sobolev spaces introduced in Definition 2.2.1;
- W_H denotes a cylindrical Brownian motion, see Definition 2.3.5;
- For $I \in \{(a,b),(a,b],[a,b),[a,b]\}$, $C(I;X)$ denotes the set of all continuous functions $f : I \to X$. If $\max\{|a|,|b|\} < \infty$, then $C([a,b];X)$ is a Banach space when it is endowed with the norm
$$\|f\|_{C([a,b];X)} := \sup_{t\in I} \|f(t)\|_X. \tag{2.1}$$

Most of the results presented here are taken from the existing literature. For the extension operators in Proposition 2.2.4 and the optimal trace embedding in Proposition 2.2.5, we follow the presentation given in [3] and [5], respectively.

2.1 Operator theory

2.1.1 Sectorial operators and H^∞-calculus

Let $A : \mathsf{D}(A) \subseteq X \to X$ be a closed operator on a Banach space X. We say that A is sectorial if the domain and the range of A are dense in X and there exists $\phi \in (0, \pi)$ such that $\sigma(A) \subseteq \overline{\Sigma_\phi}$, where $\Sigma_\phi := \{z \in \mathbb{C} : |\arg z| < \phi\}$, and there exists $C > 0$ such that

$$|\lambda| \|(\lambda - A)^{-1}\|_{\mathscr{L}(X)} \leq C, \qquad \forall \lambda \in \mathbb{C} \setminus \overline{\Sigma_\phi}. \tag{2.2}$$

Moreover, $\omega(A) := \inf\{\phi \in (0, \pi) : (2.2) \text{ holds for some } C > 0\}$ is called the angle of sectoriality of A.

Next we define the H^∞-calculus for a sectorial operator A. Let $\phi \in (0, \pi)$ and let us denote by $H_0^\infty(\Sigma_\phi)$ the set of all holomorphic function $f : \Sigma_\phi \to \mathbb{C}$ such that $|f(z)| \lesssim \min\{|z|^\varepsilon, |z|^{-\varepsilon}\}$ for some $\varepsilon > 0$.

For $\phi > \omega(A)$ and $f \in H_0^\infty(\Sigma_\phi)$, we set

$$f(A) := \frac{1}{2\pi i} \int_\Gamma f(z)(z - A)^{-1} dz.$$

Note that $f(A)$ is well defined and $f(A) \in \mathscr{L}(X)$. We say that A has a bounded H^∞-calculus of angle ϕ if there exists $C > 0$ such that

$$\|f(A)\|_{\mathscr{L}(X)} \leq C \|f\|_{H^\infty(\Sigma_\phi)}, \qquad \forall f \in H_0^\infty(\Sigma_\phi). \tag{2.3}$$

Finally, we set $\omega_{H^\infty}(A) := \inf\{\phi \in (0, \pi) : (2.3) \text{ holds for some } C > 0\}$ is the angle of the H^∞-calculus of A.

For the reader's convenience, we list some operators with a bounded H^∞-calculus. However, this list is far from complete. Moreover, there are still many new developments on H^∞-calculus for differential operators.

Example 2.1.1.

(1) Positive self-adjoint operators on Hilbert spaces [108, Proposition 10.2.23];

(2) $-A$ generates an analytic contraction semigroups on a Hilbert space [108, Theorem 10.2.24 and Corollary 10.4.10];

(3) $-A$ generates a positive contraction semigroup on L^q which is analytic and bounded on a sector.

(4) Second order uniformly elliptic operators with Dirichlet or Neumann boundary conditions on $L^q(\mathscr{O})$, where $q \in (1, \infty)$ and $\mathscr{O} \in \{\mathbb{R}^d, \mathbb{R}^{d-1} \times \mathbb{R}_+\}$ or $\mathscr{O}$ is a C^2-domain with compact boundary [61] or [111];

(5) Second and high-order uniformly elliptic operators with Lopatinskii-Shapiro boundary conditions (see [177, Chapter 6]) on $L^q(\mathscr{O})$, where $q \in (1, \infty)$ and $\mathscr{O}$ is a sufficiently smooth domain with compact boundary [61];

(6) The Stokes operator on $\mathbb{L}^q(\mathscr{O})$ (i.e. divergence-free vector fields in $L^q(\mathscr{O}; \mathbb{R}^d)$), where $q \in (1, \infty)$ and $\mathscr{O}$ is a bounded $C^{2,\alpha}$-domain [111];

(7) The Stokes operator on $\mathbb{L}^q(\mathscr{O})$, where $|\frac{1}{q} - \frac{1}{2}| \leq \frac{1}{2d}$ and $\mathscr{O}$ is a bounded Lipschitz domain [139].

Some more examples can be found in [108, Chapter 10] and in particular the notes to that chapter. Moreover, by interpolation-extrapolation arguments one obtains similar results on other spaces (see [3, Appendix A] and the references therein).

Next, we introduce the class of operators with bounded imaginary powers (or briefly BIP). For details we refer to [97]. Let A be a sectorial operator on X. Let us note that, for a given sectorial operator A, the operator A^z for $z \in \mathbb{C}$ is defined through the extended functional calculus [177, Subsection 3.3.2]. We say that $A \in \mathrm{BIP}(X)$ if $A^{it} \in \mathscr{L}(X)$ for all $t \in \mathbb{R}$. In this case, one can check that $t \mapsto \|A^{it}\|_{\mathscr{L}(X)}$ has exponential growth and we denote by θ_A the *power-angle* of A, i.e.

$$\theta_A := \limsup_{t\uparrow\infty} \frac{1}{t} \log \|A^{it}\|_{\mathscr{L}(X)}.$$

For future convenience, let us recall the following properties:

- If $A \in \mathrm{BIP}(X)$, then $[X, \mathsf{D}(A)]_\theta = \mathsf{D}(A^\theta)$ for any $\theta \in (0,1)$, see e.g. [177, Theorem 3.3.7];
- If A has a bounded H^∞-calculus, then $A \in \mathrm{BIP}(X)$ and $\theta_A \leq \omega_{H^\infty}(A)$.

We conclude this section by definying additional spaces related to sectorial operators. Let A be a sectorial operator on a Banach spaces X and assume $0 \in \rho(A)$. As usual, for each $m \in \mathbb{N}$, we denote by $\mathsf{D}(A^m)$ the domain of A^m endowed with the norm $\|\cdot\|_{\mathsf{D}(A^m)} := \|A^m \cdot\|_X$. Then for each $\vartheta > 0$ and $p \in (1,\infty)$ we define

$$\mathsf{D}_A(\vartheta, p) := (X, \mathsf{D}(A^m))_{\vartheta/m,p}; \tag{2.4}$$

where $\vartheta < m \in \mathbb{N}$ and $(\cdot,\cdot)_{\vartheta/m,p}$ denotes the real interpolation functor. It follows from reiteration (see [197, Theorem 1.15.2]) that $\mathsf{D}_A(\mu, p)$ does not depend on the choice of $m > \vartheta$, moreover

$$(X, \mathsf{D}_A(\vartheta, p))_{\nu,q} = \mathsf{D}_A(\nu\vartheta, q),$$

for all $\nu > 0$ and $q \in (1,\infty)$. We refer to [197, Chapter 1], [151, Chapter 1] and [177, Chapter 3] for more on this topic.

2.1.2 Deterministic Maximal L^p-regularity and R-boundedness

Deterministic maximal L^p-regularity has been investigated by many authors and plays an important role in the modern treatment of parabolic equations, see e.g. [62, 138, 177, 178] and the references therein. The following is taken from [177, Definition 3.5.1].

Definition 2.1.2 (Deterministic maximal L^p-regularity)**.** *Let $T > 0$ and $p \in [1\infty]$. A closed linear operator A on a Banach space X is said to have* (deterministic) maximal L^p-regularity on $(0,T)$ *if for all $f \in L^p(0,T;X)$ there exists an unique $u \in W^{1,p}(0,T;X) \cap L^p(0,T;D(A))$ such that*

$$u' + Au = f, \qquad u(0) = 0.$$

In this case we write $A \in \mathrm{DMR}(p,T)$.

Several stability properties of the deterministic maximal L^p-regularity have been studied in [68] (see also the monograph [177]): For all $p \in [1,\infty]$ and $T \in (0,\infty]$

- the class $\mathrm{DMR}(p,T)$ is stable under appropriate translations and dilations;
- if $A \in \mathrm{DMR}(p,T)$, then $-A$ generates an analytic semigroup;
- if $A \in \mathrm{DMR}(p,\infty)$, then $S = (S(t))_{t\geq 0}$ is exponential stable;
- $\mathrm{DMR}(p,\infty) \subseteq \mathrm{DMR}(p,T) = \mathrm{DMR}(p,\widetilde{T})$ if $T, \widetilde{T} \in (0,\infty)$.
- if $A \in \mathrm{DMR}(p,T)$ and $S = (S(t))_{t\geq 0}$ is exponential stable, then $A \in \mathrm{DMR}(p,\infty)$;
- $\mathrm{DMR}(p,T) \subseteq \mathrm{DMR}(q,T)$ for all $q \in (1,\infty)$ with equality if $p \in (1,\infty)$.

Moreover let us recall that weighted versions of deterministic maximal L^p-regularity have been studied in [176] for power weights and in [37, 38] for weights of A_p-type. The aim of Chapter 3 is to provide similar results for the stochastic maximal L^p-regularity.

For the reader's convenience, we recall an important result due to L. Weis [207] which characterize deterministic maximal L^p-regularity in terms of R-boundedness of certian family of operators. To this end, we introduce the concept of R-boundedness for a family of operators bounded linear operators $\mathscr{J} \subseteq \mathscr{L}(X,Y)$ where X,Y are Banach spaces.

Let $(\widetilde{r}_n)_{n\geq 1}$ be a Rademacher sequence on $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$, i.e. a sequence of independent random variables with $\mathbb{P}(\widetilde{r}_n = 1) = \mathbb{P}(\widetilde{r}_n = -1) = \frac{1}{2}$ for all $n \geq 1$. A family of bounded linear operators $\mathscr{J} \subseteq \mathscr{L}(X,Y)$ is said to be *R-bounded* if there exists a constant $C > 0$ such that for all $x_1, \dots, x_N \in X$, $T_1, \dots, T_N \in \mathscr{J}$ one has

$$\Big\| \sum_{j=1}^{N} r_j T_j x_j \Big\|_{L^2(\Omega;X)} \leq C \Big\| \sum_{j=1}^{N} r_j x_j \Big\|_{L^2(\Omega;X)}.$$

The least constant in the above inequality will be denoted by $R(\mathscr{J})$. One can readily check that R-boundedness is stronger than uniform boundedness. More precisely, if $\mathscr{J}$ is R-bounded, then $\mathscr{J}$ is uniformly bounded and $\sup_{T\in\mathscr{J}} \|T\|_{\mathscr{L}(X,Y)} \leq R(\mathscr{J})$. For a detailed study of the R-boundedness, we refer to [108, Chapter 8].

A sectorial operator A is said to be R-sectorial if for some $\phi \in (0, \omega(A))$ the set

$$\{\lambda R(\lambda, A) \,:\, \lambda \in \mathbb{C} \setminus \Sigma_\phi\} \quad \text{is } R\text{-bounded}. \tag{2.5}$$

Finally, we set $\omega_R(A) := \inf\{\phi \in (0, \omega(A)) \,:\, (2.5) \text{ holds}\}$. For the notion of UMD Banach space we refer to Subsection 2.3.1 and the references therein.

Theorem 2.1.3 (L. Weis [207])**.** *Let A be a sectorial operator on X such that $0 \in \rho(A)$. Let X be UMD. Then $A \in \mathrm{DMR}(p,\infty)$ if and only A is R-sectorial with angle $< \pi/2$.*

Remark 2.1.4.

- Actually in [207] a more general version of Theorem 2.1.3 is proven. Indeed, the condition $0 \in \rho(A)$ can be removed at the expense of using an homogeneous version of the deterministic maximal L^p-regularity.
- For any UMD space X, one has $A \in \mathrm{BIP}(X)$ implies that A is R-sectorial on X and $\omega_R(A) \leq \theta_A$ (see [177, Theorem 4.4.5]).

2.2 Fractional Sobolev spaces with power weights

Here and in the rest of the thesis for $p \in (1,\infty)$ and $\kappa \in (-1, p-1)$ we set $w_\kappa(t) := |t|^\kappa$ for $t \in \mathbb{R}$. For p, κ as before and for an open interval I we denote by $L^p(I, w_\kappa; X)$ the Banach space of all strongly measurable functions $f : I \to X$ for which

$$\|f\|^p_{L^p(I,w_\kappa;X)} := \int_I \|f(t)\|^p_X w_\kappa(t) dt < \infty. \tag{2.6}$$

If $\kappa = 0$, then $w_\kappa = 1$ and we write $L^p(I;X)$ instead of $L^p(I, w_0; X)$. Moreover, we note that if $0 \notin \overline{I}$ and I is bounded, then $L^p(I, w_\kappa; X) = L^p(I;X)$ isomorphically. Moreover, for $I = (a,b)$ and p, κ as above, we set $L^p(a,b,w_\kappa;X) := L^p(I, w_\kappa; X)$. A similar convention will be used for the spaces introduced below.

To introduce Sobolev spaces we need to introduce the space of X-valued distributions. For an open subset $I \subseteq \mathbb{R}$, let $\mathscr{D}(I) := C_0^\infty(I)$ with the usual topology. Then we define the set of all X-valued distribution as $\mathscr{D}'(I;X) := \mathscr{L}(\mathscr{D}(I);X)$. Note that $L^1_{\mathrm{loc}}(I;X) \hookrightarrow \mathscr{D}'(I;X)$ and the distributional derivatives $f^{(j)} \in \mathscr{D}'(I;X)$ for all $j \geq 1$ and $f \in L^1_{\mathrm{loc}}(I;X)$ in the usual way.

For $n \geq 1$ and an open interval $I \subset \mathbb{R}$, we denote by $W^{n,p}(I, w_\kappa; X)$ the set of all $f \in L^p(I, w_\kappa; X)$ such that $f^{(j)} \in L^p(I, w_\kappa; X)$ for all $j \in \{1, \dots, n\}$, where $f^{(j)}$ denotes the j-th distributional derivative of f. We endow $W^{n,p}(I, w_\kappa; X)$ with the norm

$$\|f\|_{W^{n,p}(I,w_\kappa;X)} := \sum_{j=0}^{n} \|f^{(j)}\|_{L^p(I,w_\kappa;X)}.$$

If $\kappa \in (-1, p-1)$ and $0 \in \overline{I}$, then the trace map $f \mapsto f(0)$ is a bounded mapping from $W^{1,p}(I, w_\kappa; X)$ into X (see [146, Lemma 3.1]). Define a closed subspace of $W^{1,p}(I, w_\kappa; X)$ as

$$_0W^{1,p}(I, w_\kappa; X) = \{f \in W^{1,p}(I, w_\kappa; X) : f(0) = 0 \text{ if } 0 \in \overline{I}\}. \tag{2.7}$$

We define fractional Sobolev spaces by complex interpolation as in [153] and [177, Section 3.4.5].

Definition 2.2.1. *Let $-\infty \leq a < b \leq \infty$, $I = (a, b)$ $p \in (1, \infty)$, $\kappa \in (-1, p-1)$ and $\theta \in (0, 1)$. Let*

$$H^{\theta,p}(I, w_\kappa; X) := [L^p(I, w_\kappa; X), W^{1,p}(I, w_\kappa; X)]_\theta.$$

If $0 \in \overline{I}$ let

$$_0H^{\theta,p}(I, w_\kappa; X) := [L^p(I, w_\kappa; X), {_0W^{1,p}}(I, w_\kappa; X)]_\theta.$$

As before, $H^{\theta,p}(I, w_\kappa; X) = H^{\theta,p}(I; X)$ isomorphically if $0 \notin \overline{I}$ and I is bounded. Furthermore, by interpolation it is immediate that

$$_0H^{\theta,p}(I, w_\kappa; X) \hookrightarrow H^{\theta,p}(I, w_\kappa; X) \quad \text{contractively}. \tag{2.8}$$

Let us note some further properties of the above spaces.

Proposition 2.2.2. *Let X be a Banach space. Let $\theta \in (0, 1)$, $p \in (1, \infty)$, $\kappa \in (-1, p-1)$, $J \subseteq I \subseteq \mathbb{R}$ intervals, $I_T = (0, T)$ with $T \in (0, \infty]$, $\varepsilon > 0$, and $\mathcal{A} \in \{H, {_0H}\}$. Then for all $f \in \mathcal{A}^{\theta,p}(I_T, w_\kappa; X)$,*

$$\begin{aligned} \|f\|_{\mathcal{A}^{\theta,p}(J,w_\kappa;X)} &\leq \|f\|_{\mathcal{A}^{\theta,p}(I,w_\kappa;X)}, \\ \|f\|_{H^{\theta,p}(\varepsilon,T;X)} &\leq \varepsilon^{-\kappa} \|f\|_{\mathcal{A}^{\theta,p}(I_T,w_\kappa;X)}, \qquad \text{if } \kappa \in [0, p-1). \end{aligned}$$

Proof. For convenience of the reader we provide the details. The first estimate follows by interpolating the restriction operator mapping from $\mathcal{A}^{k,p}(I, w_\kappa; X)$ into $\mathcal{A}^{k,p}(J, w_\kappa; X)$ for $k \in \{0, 1\}$.

To prove the second estimate by (2.8) it suffices to consider the case $\mathcal{A} = H$. Let $r : f \mapsto f|_{(\varepsilon,T)}$ be the restriction operator on (ε, T). It is immediate to see that

$$\|r\|_{\mathscr{L}(W^{j,p}(I_T,w_\kappa;X)),W^{j,p}(\varepsilon,T;X))} \leq \varepsilon^{-\kappa}, \quad \text{for } j \in \{0, 1\}.$$

Thus, interpolation gives $r : H^{\theta,p}(I_T, w_\kappa; X) \to H^{\theta,p}(\varepsilon, T; X)$ with norm at most $\varepsilon^{-\kappa}$. □

Here we discuss extension operators for the spaces just introduced. Further properties of fractional Sobolev spaces will be investigated in Chapter 4. In [153], extension operators for the above spaces are already given. However, we found a different and (to our viewpoint) simpler approach to build extension operators. It will give some more information, which will be needed in the following. Let us begin with a definition.

Definition 2.2.3 (Extension operator). *Let $\mathcal{A} \in \{H^{s,p}, {_0H^{s,p}}\}$ for some $s \in [0, 1]$, $p \in (1, \infty)$ and let $\kappa \in (-1, p-1)$. Let $I_T = (0, T)$ for some $T \in (0, \infty)$. We say that a bounded linear operator*

$$\mathsf{E}_T : \mathcal{A}(I_T, w_\kappa; X) \to \mathcal{A}(\mathbb{R}, w_\kappa; X),$$

is an extension operator on $\mathcal{A}(I_T, w_\kappa; X)$ if $\mathsf{E}_T f = f$ on I_T.

Let E be the extension operator which maps,

$$\mathcal{A}(0,1,w_\kappa;X) \to \mathcal{A}(\mathbb{R},w_\kappa;X), \quad \text{where} \quad \mathcal{A} \in \{L^p, W^{1,p}\} \tag{2.9}$$

given by the classical reflection argument (see e.g. [1, Theorems 5.19 and 5.22]), which can be extended to the weighted setting. By construction it follows that

$$\|\mathsf{E}f\|_{L^p(\mathbb{R},w_\kappa;X)} \leq C_{p,\kappa}\|f\|_{L^p(0,1,w_\kappa;X)}, \tag{2.10}$$

$$\|(\mathsf{E}f)'\|_{L^p(\mathbb{R},w_\kappa;X)} \leq C_{p,\kappa}(\|f\|_{L^p(0,1,w_\kappa;X)} + \|f'\|_{L^p(0,1,w_\kappa;X)}), \tag{2.11}$$

where $C_{p,\kappa}$ is a constant which depends only on p,κ. The last ingredient needed in our approach is the Poincaré inequality (see [153, Lemma 2.12]): For all $T>0$

$$\|f\|_{L^p(0,T,w_\kappa;X)} \lesssim_{p,\kappa} T\|f'\|_{L^p(0,T,w_\kappa;X)}, \quad \forall f \in {}_0W^{1,p}(I,w_\kappa;X). \tag{2.12}$$

Proposition 2.2.4. *Let $s\in[0,1]$, $p\in(1,\infty)$, $\kappa\in(-1,p-1)$ and let $T\in(0,\infty)$. Let $\mathsf{E}_T : L^p(0,T,w_\kappa;X) \to L^p(\mathbb{R},w_\kappa;X)$ be the operator given by*

$$\mathsf{E}_T f(t) := \mathsf{E}(f(T\cdot))\Big(\frac{t}{T}\Big), \qquad t\in\mathbb{R},$$

where E is as above. Then the following assertion holds.

(1) *The restriction ${}_0\mathsf{E}_T$ of E_T to ${}_0H^{s,p}(I_T,w_\kappa;X)$ defines a bounded extension operator with values in ${}_0H^{s,p}(\mathbb{R},w_\kappa;X)$ with*

$$\|{}_0\mathsf{E}_T\|_{\mathscr{L}({}_0H^{s,p}(I_T,w_\kappa;X),\,{}_0H^{s,p}(\mathbb{R},w_\kappa;X))} \leq {}_0C,$$

where ${}_0C$ depends only on p,s,κ.

(2) *Let $\eta>0$ and $T\in(\eta,\infty]$. Then E_T induces an extension operator on $H^{s,p}(I_T,w_\kappa;X)$, which will be still denoted by E_T. Moreover,*

$$\|\mathsf{E}_T\|_{\mathscr{L}(H^{s,p}(I_T,w_\kappa;X),H^{s,p}(\mathbb{R},w_\kappa;X))} \leq C,$$

where C depends only on p,s,κ,η.

Proof. (1): By a change of variable and (2.10),

$$\|{}_0\mathsf{E}_Tf\|_{L^p(\mathbb{R},w_\kappa;X)} = \Big\|t\mapsto \mathsf{E}(f(T\cdot))\Big(\frac{t}{T}\Big)\Big\|_{L^p(\mathbb{R},w_\kappa;X)} \lesssim \|f\|_{L^p(I_T,w_\kappa;X)},$$

and

$$\begin{aligned}
\|({}_0\mathsf{E}_Tf)'\|_{L^p(\mathbb{R},w_\kappa;X)} &= T^{-1}\Big\|t\mapsto (\mathsf{E}(f(T\cdot)))'\Big(\frac{t}{T}\Big)\Big\|_{L^p(\mathbb{R},w_\kappa;X)} \\
&= T^{-1+\frac{1+\kappa}{p}}\|(\mathsf{E}(f(T\cdot)))'\|_{L^p(\mathbb{R},w_\kappa;X)} \\
&\overset{(i)}{\lesssim} T^{-1+\frac{1+\kappa}{p}}(\|f(T\cdot)\|_{L^p(0,1,w_\kappa;X)} + \|f'(T\cdot)T\|_{L^p(0,1,w_\kappa;X)}) \\
&\overset{(ii)}{\lesssim} T^{\frac{1+\kappa}{p}}\|f'(T\cdot)\|_{L^p(0,1,w_\kappa;X)} = \|f'\|_{L^p(I_T,w_\kappa;X)}
\end{aligned}$$

where in (i) we used (2.11) and in (ii) the weighted Poincaré inequality (2.12). We can conclude that also $\|{}_0\mathsf{E}_T\|_{\mathscr{L}({}_0W^{1,p}(0,T,w_\kappa;X),{}_0W^{1,p}(\mathbb{R},w_\kappa;X))} \leq C$ with C independent of T.

Now complex interpolation gives that ${}_0\mathsf{E}_T$ is a bounded linear operator from ${}_0H^{s,p}(I_T,w_\kappa;X)$ into ${}_0H^{s,p}(\mathbb{R},w_\kappa;X)$. Moreover, it has the extension property, i.e. ${}_0\mathsf{E}_Tf=f$ on I_T, which follows from the extension property of E.

(2): This follows in the same way, but since we cannot use Poincaré inequality, we obtain $\|\mathsf{E}_T\|_{\mathscr{L}(W^{1,p}(0,T,w_\kappa;X),W^{1,p}(\mathbb{R},w_\kappa;X))} \leq C(1+T^{-1})$. ∎

2.2.1 A trace embedding of weighted anisotropic space

The aim of this subsection is to prove an optimal trace embedding for anisotropic space with power weights which will play a basic role throghout this thesis. Recall that, for a given sectorial operator A, the space $\mathsf{D}_A(\theta,p)$ is defined in (2.4). Finally, for an interval I, $C_0(\overline{I};X)$ denotes the Banach space of all continuous functions on $\overline{I}$ with values in X which vanish at infinity.

Proposition 2.2.5 (Trace embedding)**.** *Let $p\in(1,\infty)$, $\kappa\in[0,p-1)$, $s\in(0,1)$ and let X be a UMD space and define $I_T=(0,T)$ where $T\in(0,\infty]$. Let A be an invertible sectorial operator on X. Then the following assertions hold:*

(1) *If $s>\frac{1+\kappa}{p}$, then*

$$H^{s,p}(I_T,w_\kappa;X)\cap L^p(I_T,w_\kappa;\mathsf{D}(A^s))\hookrightarrow C_0\Big(\overline{I},\mathsf{D}_A\Big(s-\frac{1+\kappa}{p},p\Big)\Big).$$

(2) *If $s>\frac{1}{p}$ and $\delta\in(0,T)$, then setting $J_{\delta,T}:=(\delta,T)$ we have*

$$H^{s,p}(I_T,w_\kappa;X)\cap L^p(I_T,w_\kappa;\mathsf{D}(A^s)))\hookrightarrow C_0\Big(\overline{J}_{\delta,T};\mathsf{D}_A\Big(s-\frac{1}{p},p\Big)\Big).$$

By Proposition 2.2.4 we can keep track of the constants in the embeddings (1)-(2). This will be exploited in Chapter 4.

The proof of Proposition 2.2.5 will be given at the end of this subsection. It will be useful to use a Fourier analytic description of the fractional Sobolev spaces $H^{\theta,p}$ introduced in Definition 2.2.1. Here we provide only basic facts and we refer to [145, 154] for details. Proposition 2.2.5 is stated only for smoothness $s\in(0,1)$. However, our arguments can be extended also to high-order regularity (see [5, Section 7]) but it will be not needed here.

Let $\mathscr{S}(\mathbb{R};X)$ be the space of X-valued Schwartz functions endowed with the usual topology, and $\mathscr{S}'(\mathbb{R};X):=\mathscr{L}(\mathscr{S}(\mathbb{R});X)$ denotes the space of X-valued tempered distributions. Let $\mathcal{J}_s$ be the *Bessel potential operator* of order $s\in\mathbb{R}$, i.e.

$$\mathcal{J}_s f=\mathcal{F}^{-1}((1+|\cdot|^2)^{s/2}\mathcal{F}(f))\,,\quad f\in\mathscr{S}(\mathbb{R};X);$$

where $\mathcal{F}$ denotes the Fourier transform. Thus, one also has $\mathcal{J}_s:\mathscr{S}'(\mathbb{R};X)\to\mathscr{S}'(\mathbb{R};X)$. For $s\in\mathbb{R}$, $p\in(1,\infty)$, $\kappa\in(-1,p-1)$, $\mathscr{H}^{s,p}(\mathbb{R},w_\kappa;X)\subseteq\mathscr{S}'(\mathbb{R};X)$ denote the *Bessel potential space*, i.e. the set of all $f\in\mathscr{S}'(\mathbb{R};X)$ for which $\mathcal{J}_s f\in L^p(\mathbb{R},w_\kappa;X)$ and we set

$$\|f\|_{\mathscr{H}^{s,p}(\mathbb{R},w_\kappa;X)}:=\|\mathcal{J}_s f\|_{L^p(\mathbb{R},w_\kappa;X)}.$$

To define vector valued weighted Bessel potential spaces on the half-line $\mathbb{R}_+=(0,\infty)$, we use a standard method. Recall that $\mathscr{D}'(\mathbb{R}_+;X)$ is the space all X-valued distributions.

Definition 2.2.6. *Let $p\in(1,\infty)$ and $\kappa\in(-1,p-1)$. Let*

$$\mathscr{H}^{s,p}(\mathbb{R}_+,w_\kappa;X)=\{f\in\mathscr{D}'(\mathbb{R}_+;X)\,:\,\exists g\in\mathscr{H}^{s,p}(\mathbb{R},w_\kappa;X);\ s.t.\ g|_{\mathbb{R}_+}=f\},$$

endowed with the quotient norm $\|f\|_{\mathscr{H}^{s,p}(\mathbb{R}_+,w_\kappa;X)}=\inf\{\|g\|_{\mathscr{H}^{s,p}(\mathbb{R},w_\kappa;X)}\,:\,g|_I=f\}$.

To handle Bessel potential space we need the following result, see [145, Propositions 5.5 and 5.6].

Proposition 2.2.7. *Let $p\in(1,\infty)$, $\kappa\in(-1,p-1)$, $I\in\{\mathbb{R},\mathbb{R}_+\}$ and let X be a UMD Banach space.*

(1) *There exists an extension operator $\mathscr{E}:\mathscr{H}^{s,p}(\mathbb{R}_+,w_\kappa;X)\to\mathscr{H}^{s,p}(\mathbb{R},w_\kappa;X)$ such that $\mathscr{E}f|_{\mathbb{R}_+}=f$ for all $f\in\mathscr{H}^{s,p}(\mathbb{R}_+,w_\kappa;X)$ and $\mathscr{E}:C^1([0,\infty);X)\to C^1(\mathbb{R};X)$.*

(2) *If* $k \in \mathbb{N}$, $p \in (1,\infty)$, *then* $\mathscr{H}^{k,p}(I, w_\kappa; X) = W^{k,p}(I, w_\kappa; X)$.

(3) *Let* $\theta \in (0,1)$ *and* $s_0, s_1 \in \mathbb{R}$ *and set* $s := s_0(1-\theta) + \theta s_1$. *Then*

$$[\mathscr{H}^{s_0,p}(I, w_\kappa; X), \mathscr{H}^{s_1,p}(I, w_\kappa; X)]_\theta = \mathscr{H}^{s,p}(I, w_\kappa; X).$$

By Definition 2.2.1 and Proposition 2.2.7 we have

$$\mathscr{H}^{s,p}(I, w_\kappa; X) = H^{s,p}(I, w_\kappa; X), \quad \text{for } I \in \{\mathbb{R}, \mathbb{R}_+\} \text{ provided } X \text{ is UMD and } s \in [0,1]. \tag{2.13}$$

Besides this identification, sometimes it will be useful to keep a different notation for $\mathscr{H}$ and H.

Next, we prove a density lemma. we $I \in \{\mathbb{R}, \mathbb{R}_+\}$, we denote by $C_c^1(\overline{I}; X)$ the space of X-valued functions $f : \overline{I} \to X$ such that f, f' are continuous and bounded with compact support.

Lemma 2.2.8. *Let* X *and* Y *be Banach spaces such that* $Y \hookrightarrow X$ *densely. Let* $k \in \mathbb{N}$, $s \in [0,1]$, $p \in (1,\infty)$, $\kappa \in (-1, p-1)$. *Let* $I \in \{\mathbb{R}, \mathbb{R}_+\}$. *Then* $C_c^1(\overline{I}) \otimes Y$ *is dense in* $\mathscr{H}^{s,p}(I, w_\kappa; X)$ *and in* $\mathscr{H}^{s,p}(I, w_\kappa; X) \cap L^p(I; w_\kappa; Y)$.

Proof. By Proposition 2.2.7 it suffices to prove the statements in the case $I = \mathbb{R}$. The density of $C_c^k(\mathbb{R}) \otimes X$ in $H^s(\mathbb{R}, w_\kappa; X)$ follows from [145, Lemma 3.4]. Now since Y is densely embedded in X the result follows.

To prove the density in $E := \mathscr{H}^{s,p}(\mathbb{R}, w_\kappa; X) \cap L^p(\mathbb{R}; w_\kappa; Y)$, let $f \in E$. Let $\varphi \in C_c^\infty(\mathbb{R})$ be such that $\varphi \geq 0$ and $\|\varphi\|_1 = 1$. Let $\varphi_n(x) = n^{-1}\varphi(nx)$. Then $\varphi_n * f \to f$ in E. Therefore, it suffices to approximate $g = \varphi_n * f$ for fixed n. Since $g \in \mathscr{H}^{s,p}(\mathbb{R}, w_\kappa; Y)$ and $\mathscr{H}^{s,p}(\mathbb{R}, w_\kappa; Y) \hookrightarrow E$ it suffices to approximate g in $\mathscr{H}^{s,p}(\mathbb{R}, w_\kappa; Y)$. This follows from the first statement of the lemma. □

The key ingredient in the proof of Proposition 2.2.5 is the following trace result due to [156, Theorem 1.1] where the result was stated on the full real line. The result on $\mathbb{R}_+$ is immediate from the boundedness of the extension operator of Proposition 2.2.7 and the density Lemma 2.2.8.

Theorem 2.2.9. *Let* A *be an invertible sectorial operator with dense domain. Let* $p \in (1,\infty)$, $\kappa \in (-1, p-1)$ *and* $s \in (\frac{1+\kappa}{p}, 1]$. *Then the trace operator* $(\mathsf{Tr} f) := f(0)$ *initially defined on* $C_c^1([0,\infty); \mathsf{D}(A))$, *extends to a bounded linear operator on* $\mathscr{H}^{s,p}(\mathbb{R}_+, w_\kappa; X) \cap L^p(\mathbb{R}_+, w_\kappa; \mathsf{D}(A^s))$. *Moreover,*

$$\mathsf{Tr} : \mathscr{H}^{s,p}(\mathbb{R}_+, w_\kappa; X) \cap L^p(\mathbb{R}_+, w_\kappa; \mathsf{D}(A^s)) \to \mathsf{D}_A\Big(s - \frac{1+\kappa}{p}, p\Big).$$

Proof of Proposition 2.2.5. By Proposition 2.2.4 it is enough to consider the case $T = \infty$. Thus by (2.13) we can also replace H by $\mathscr{H}$.

(1): For notational convenience let us set $E := \mathscr{H}^{s,p}(\mathbb{R}_+, w_\kappa; X) \cap L^p(\mathbb{R}_+, w_\kappa; \mathsf{D}(A^s))$. To prove the required embedding by the density Lemma 2.2.8 it suffices to check that $\sup_{t\geq 0} \|f(t)\|_{\mathsf{D}_A(\mu,p)} \leq C\|f\|_E$ for every $f \in C_c^1(\mathbb{R}_+; \mathsf{D}(A))$. To prove this we extend a standard translation argument to the weighted setting. Let $(T(t))_{t\geq 0}$ the left-translation semigroup, i.e. $(T(t)f)(s) := f(t+s)$ on $L^p(\mathbb{R}_+; X)$. Since $\kappa \geq 0$, $T(t)$ is contractive on $L^p(\mathbb{R}_+, w_\kappa; X)$ as well. Since $T(t)$ commutes with the first derivative ∂_s it is immediate that $(T(t))_{t\geq 0}$ defines a contraction on $W^{1,p}(\mathbb{R}_+, w_\kappa; X)$. By complex interpolation and Proposition 2.2.7 it follows that there exists a constant M such that $\|T(t)\|_{\mathscr{L}(\mathscr{H}^{s,p}(\mathbb{R}_+, w_\kappa; X))} \leq M$ for $t \in \mathbb{R}_+$, and consequently the same holds on E. Now by Theorem 2.2.9 we obtain

$$\|f(t)\|_{\mathsf{D}_A(\mu,p)} = \|(T(t)f)(0)\|_{\mathsf{D}_A(\mu,p)} \leq C\|T(t)f\|_E \leq CM\|f\|_E$$

as required.

(2): By Proposition 2.2.2 and (2.13) we get

$$\mathscr{H}^{s,p}(I_T, w_\kappa; X) \cap L^p(I_T, w_\kappa; \mathsf{D}(A^s)) \hookrightarrow \mathscr{H}^{s,p}(J_{\delta,T}; X) \cap L^p(J_{\delta,T}; \mathsf{D}(A^s)).$$

Therefore, since (1) extends to any half line $[\delta, \infty) \subseteq [0,\infty)$ the required result follows from (1) in the unweighted case. □

2.3 UMD spaces and stochastic integration

In this section we revise the theory of stochastic integration in UMD spaces [29, 163] (see also [200]). For the reader's convenience, we also provide some basic definition on UMD spaces and γ-radonifying operators that are needed to formulate such theory.

Throughout this section, $(\Omega, \mathbb{F} = (\mathscr{F}_t)_{t\geq 0}, \mathscr{A}, \mathbb{P})$ denotes a filtered probability space.

2.3.1 UMD spaces and Banach spaces with type 2

Let us begin with a standard definition. Let $(\widetilde{\Omega}, (\widetilde{\mathscr{F}_n})_{n\geq 1}, \widetilde{\mathscr{A}}, \widetilde{\mathbb{P}})$ be a filtered probability space. A sequence $(M_n)_{n\geq 1} \subset L^1(\widetilde{\Omega}; X)$ is said to be a martingale sequence provided $\mathbb{E}[M_n|\mathscr{F}_{n-1}] = M_{n-1}$ for all $n \geq 2$. To each martingale sequence, we can associate the *martingale difference sequence*

$$dM_1 = 0, \quad \text{and} \quad dM_n := M_n - M_{n-1}, \quad \text{for all } n \geq 2.$$

Next, we define the *unconditional martingale differences* (or briefly UMD) property.

Definition 2.3.1 (UMD property). *A Banach space X is said to have the UMD property if for there exists a constant $\beta > 0$ depending only on X such that the following holds. Whenever $(\widetilde{\Omega}, (\widetilde{\mathscr{F}_n})_{n\geq 1}, \widetilde{\mathscr{A}}, \widetilde{\mathbb{P}})$ is a filtered probability space and $(dM_n)_{n\geq 1}$ is a martingale difference sequence, then for any $N \geq 1$ and every finite sequence $(\varepsilon_n)_{n=1}^N \subseteq \{-1, 1\}^N$ we have*

$$\Big\| \sum_{n=1}^N \varepsilon_n dM_n \Big\|^2_{L^2(\widetilde{\Omega};X)} \leq \beta \Big\| \sum_{n=1}^N dM_n \Big\|^2_{L^2(\widetilde{\Omega};X)}.$$

UMD spaces will play a central role in this thesis. Let us list some properties and examples of UMD spaces. We refer to [107, Chapter 4-5] for further properties and references.

- If X is UMD, then every closed subspace $C \subseteq X$ is UMD;
- X is UMD if and only if X^* is UMD;
- UMD space are reflexive;
- If (X_0, X_1) is a compatible couple of UMD spaces, then $(X_0, X_1)_{\theta,p}$ and $[X_0, X_1]_\theta$ are UMD provided $\theta \in (0, 1)$ and $p \in (1, \infty)$.

Typical examples of UMD spaces are: L^p-spaces and Sobolev spaces $H^{s,p}$ and/or Besov spaces $B^s_{q,p}$ on either $\mathbb{R}^d, \mathbb{T}^d$ or domains with (sufficiently) regular boundary provided $s \in \mathbb{R}$ and $p, q \in (1, \infty)$. To apply stochastic maximal L^p-regularity techniques, we will use UMD space having type 2. For the reader's convenience, we recall the definition. Further properties and examples can be found in [108, Chapter 7] and the references therein. As above, we say that $(\widetilde{r}_n)_{n\geq 1}$ is a Rademacher sequence on $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ provided they are independent random variables and $\mathbb{P}(\widetilde{r}_n = 1) = \mathbb{P}(\widetilde{r}_n = -1) = \frac{1}{2}$ for all $n \geq 1$.

Definition 2.3.2 (Banach space with type 2). *Let $(\widetilde{r}_n)_{n\geq 1}$ be a Rademacher sequence on a probability space $(\widetilde{\Omega}, \widetilde{\mathscr{A}}, \widetilde{\mathbb{P}})$. A Banach space X is said to have* type 2 *if there exists a constant C such that for all finite subset $(x_i)_{i=1}^N \subseteq X$ one has*

$$\Big\| \sum_{n=1}^N \widetilde{r}_n x_n \Big\|^2_{L^2(\widetilde{\Omega};X)} \leq C \sum_{n=1}^N \|x_n\|^2_X.$$

Example of UMD Banach spaces with type 2 are L^p-spaces and Sobolev spaces $H^{s,p}$ and/or Besov spaces $B^s_{q,p}$ on either $\mathbb{R}^d, \mathbb{T}^d$ or domains with (sufficiently) regular boundary provided $s \in \mathbb{R}$ and $p, q \in [2, \infty)$ (cf. [108, Proposition 7.1.4]).

2.3.2 γ-radonifying operators

In this subsection we briefly review some basic facts regarding γ-radonifying operators; for further discussions see [108, Chapter 9]. Through this subsection $(\gamma_n)_{n\in\mathbb{N}}$ denotes a Gaussian sequence, i.e. a sequence of independent standard normal variables over a probability space $(\widetilde{\Omega}, \widetilde{\mathscr{A}}, \widetilde{\mathbb{P}})$.

Let $\mathcal{H}$ be a Hilbert space (with scalar product $(\cdot,\cdot)_{\mathcal{H}}$) and X be a Banach space with finite cotype. Recall that $\mathcal{H}\otimes X$ is the space of finite rank operators from $\mathcal{H}$ to X. In other words, each $T\in\mathcal{H}\otimes X$ has the form

$$T=\sum_{n=1}^{N} h_n\otimes x_n\,,$$

for $N\in\mathbb{N}$ and $(h_n)_{n=1}^N\subseteq\mathcal{H}$. Here $h\otimes x$ denotes the operator $g\mapsto (g,h)_{\mathcal{H}}x$.

For $T\in\mathcal{H}\otimes X$ define

$$\|T\|^2_{\gamma(\mathcal{H},X)}:=\sup\Big\|\sum_{n=1}^{N}\gamma_n T h_n\Big\|^2_{L^2(\widetilde{\Omega};X)}<\infty,$$

where the supremum is taken over all finite orthonormal systems $(h_n)_{n=1}^N$ in $\mathcal{H}$. Then $\|T\|\le\|T\|_{\gamma(\mathcal{H},X)}$. The closure of $\mathcal{H}\otimes X$ with respect to the above norm is called *the space of γ-radonifying operators* and is denoted by $\gamma(\mathcal{H},X)$.

For $X=L^p(S)$ with $p\in[1,\infty)$, where (S,Σ,μ) is a measure space one has (see [108, Proposition 9.3.2])

$$\gamma(\mathcal{H},X)=L^p(S;\mathcal{H}). \tag{2.14}$$

The previous identification show that γ-radonifying operators can be considered as a natural generalisation of 'squre function' widely used in harmonic analysis.

The following property will be used through the thesis.

Proposition 2.3.3 (Ideal Property)**.** *Let $T\in\gamma(\mathcal{H},X)$. If $\mathcal{G}$ is another Hilbert space and Y a Banach space, then for all $U\in\mathscr{L}(X,Y)$ and $V\in\mathscr{L}(\mathcal{G},\mathcal{H})$ we have $UTV\in\gamma(G,Y)$ and*

$$\|UTV\|_{\gamma(\mathcal{G},Y)}\le\|U\|_{\mathscr{L}(X,Y)}\|T\|_{\gamma(\mathcal{H},X)}\|V\|_{\mathscr{L}(\mathcal{G},\mathcal{H})}.$$

We will be mainly interested in the case that $\mathcal{H}=L^2(S;H)$ where $(S,\mathcal{A},\mu)$ is a measure space and H is another Hilbert space. In this situation we employ the following notation:

$$\gamma(S;H,X):=\gamma(L^2(S;H),X)$$

and $\gamma(a,b;H,X):=\gamma(L^2(a,b;H),X)$, if $S=(a,b)$, μ is the one dimensional Lebesgue measure and $\mathcal{A}$ is the natural σ-algebra. If $H=\mathbb{R}$ we simply write $\gamma(a,b;X):=\gamma(L^2(a,b),X)$.

An H-strongly measurable function $G:S\to\mathscr{L}(H,X)$ (i.e. for each $h\in H$ the map $s\mapsto f(s)h$ is strongly measurable) *belongs to $L^2(S;H)$ scalarly* if $G^*(s)x^*\in L^2(S;H)$ for each $x^*\in X^*$. Such a function *represents* an operator $R\in\gamma(S;H,X)$ if for all $f\in L^2(S;H)$ and $x^*\in X^*$ we have

$$\int_S\langle G(s)f(s),x^*\rangle\,ds=\langle R(f),x^*\rangle.$$

It can be shown that if R is represented by G_1 and G_2 then $G_1=G_2$ almost everywhere. It will be convenient to identify R with G and we will simply write $G\in\gamma(S;H,X)$ and $\|G\|_{\gamma(S;H,X)}:=\|R\|_{\gamma(S;H,X)}$. By the ideal property, if $S=S_1\cup S_2$ and S_1 and S_2 are disjoint, then

$$\|G\|_{\gamma(S;H,X)}\le\|G\|_{\gamma(S_1;H,X)}+\|G\|_{\gamma(S_2;H,X)}. \tag{2.15}$$

Another consequence of the ideal property is that for $G\in\gamma(S;H,X)$, $\phi\in L^\infty(S)$ and $S_0\subseteq S$, we have

$$\|\phi G\|_{\gamma(S;H,X)}\le\|\phi\|_\infty\|G\|_{\gamma(S;H,X)},\quad \|\mathbf{1}_{S_0}G\|_{\gamma(S;H,X)}=\|G\|_{\gamma(S_0;H,X)} \tag{2.16}$$

To conclude this section, we recall the following embedding:

Proposition 2.3.4. *Let X be a Banach space with type 2, then*

$$L^2(S;\gamma(H,X)) \hookrightarrow \gamma(L^2(S),\gamma(H,X)) \hookrightarrow \gamma(L^2(S;H),X).$$

Proof. Since X has type 2, also $\gamma(H,X)$ has type 2, because it is isomorphic to a closed subspace of $L^2(\widetilde{\Omega};X)$ (see [108, Proposition 7.1.4]). Now the first embedding follows from [108, Theorem 9.2.10]. The second embedding follows by considering finite rank operators and applying [108, Theorem 7.1.20] with orthonormal family $\{\widetilde{\gamma}_i\widehat{\gamma}_j : i,j\in\mathbb{N}\}$, where $\widetilde{\gamma}_i$ and $\widehat{\gamma}_j$ are defined on probability spaces $\widetilde{\Omega}$ and $\widehat{\Omega}$, respectively. □

2.3.3 Stochastic Integration in UMD Banach spaces

The aim of this section is to present basic results of the stochastic integration theory in UMD Banach spaces developed in [163]. Throughout this section $(\Omega,\mathbb{F}=(\mathscr{F}_t)_{t\geq 0},\mathscr{A},\mathbb{P})$ denotes a filtered probability space. An adapted step process is a linear combination of functions

$$(\mathbf{1}_{A\times(s,t]}\otimes(h\otimes x))(\omega,t) := \mathbf{1}_{A\times(s,t]}(\omega,t)(h\otimes x)\,,$$

where $0\leq s<t\leq T$ and $A\in\mathscr{F}_s$. Let $T>0$, we say that a stochastic process $G:[0,T]\times\Omega\to\mathscr{L}(H,X)$ *belongs to* $L^2(0,T;H)$ *scalarly almost surely* if for all $x^*\in X^*$ a.s. the $G^*x^*\in L^2(0,T;H)$. Such a process G is said to represents an $L^2(0,T;H)$-strongly measurable $R\in L^0(\Omega;\gamma(0,T;H,X))$ if for all $f\in L^2(0,T;H)$ and $x^*\in X^*$ we have

$$\langle R(\omega)f,x^*\rangle = \int_0^T \langle G(t,\omega)f(t),x^*\rangle\,dt.$$

As done in Subsection 2.3.2, we identify G and R in the case that R is represented by G. Moreover, we say that $G\in L^p(\Omega;\gamma(0,T;H,X))$ if $R\in L^p(\Omega;\gamma(0,T;H,X))$ for some $p\in[0,\infty)$. We say that $R:\Omega\to\gamma(0,T;H,X)$ is *elementary adapted* if it is represented by an adapted step process G. Finally,

$$L^p_{\mathscr{P}}(\Omega;\gamma(0,T;H,X))$$

denotes the closure of all elementary adapted $R\in L^p(\Omega;\gamma(0,T;H,X))$. Throughout the thesis we will consider cylindrical Gaussian noise.

Definition 2.3.5. *Let H be a separable Hilbert space. A bounded linear operator $W_H : L^2(\mathbb{R}_+;H)\to L^2(\Omega)$ is said to be an* cylindrical Brownian motion *in H if the following are satisfied:*

- *for all $f\in L^2(\mathbb{R}_+;H)$ the random variable $W_H(f)$ is centered Gaussian.*
- *for all $t\in\mathbb{R}_+$ and $f\in L^2(\mathbb{R}_+;H)$ with support in $[0,t]$, $W_H(f)$ is $\mathscr{F}_t$-measurable.*
- *for all $t\in\mathbb{R}_+$ and $f\in L^2(\mathbb{R}_+;H)$ with support in $[t,\infty]$, $W_H(f)$ is independent of $\mathscr{F}_t$.*
- *for all $f_1,f_2\in L^2(\mathbb{R}_+;H)$ we have $\mathbb{E}(W_H(f_1)W_H(f_2)) = (f_1,f_2)_{L^2(\mathbb{R}_+;H)}$.*

Given a cylindrical Brownian motion in H, the process $(W_H(t)h)_{t\geq 0}$, where

$$W_H(t)h := W_H(\mathbf{1}_{(0,t]}\otimes h)\,, \tag{2.17}$$

is a Brownian motion.

Example 2.3.6. Let $(w_n)_{n\geq 1}$ be independent standard Brownian motions on $(\Omega,\mathbb{F},\mathscr{A},\mathbb{P})$. Then $W_{\ell^2}(f)=\sum_{n\geq 1}\int_{\mathbb{R}_+}\langle f,e_n\rangle dw_n$ converges in $L^2(\Omega)$ and defines a cylindrical Brownian motion in ℓ^2, where $e_n=(\delta_{jn})_{n\geq 1}$ and δ_{jn} denotes the Kronecker's delta.

At this point, we can define the *stochastic integral with respect to a cylindrical Brownian* motion in H of the process $\mathbf{1}_{A\times(s,t]}\otimes(h\otimes x)$:

$$\int_0^\infty \mathbf{1}_{A\times(s,t]}\otimes(h\otimes x)(s)\,dW_H(s) := \mathbf{1}_A\otimes(W_H(t)h-W_H(s)h)\,x\,, \tag{2.18}$$

and we extend it to adapted step processes by linearity.

Theorem 2.3.7 (Itô-isomorphism)**.** *Let $T > 0$, $p \in (0, \infty)$ and let X be a UMD Banach space, then the mapping $G \to \int_0^T G\, dW_H$ admits a unique extension to a isomorphism from $L^p_{\mathscr{P}}(\Omega; \gamma(0, T; H, X))$ into $L^p(\Omega; X)$ and*

$$\mathbb{E} \sup_{0\leq t\leq T} \Big\| \int_0^t G(s)\, dW_H(s)\Big\|_X^p \eqsim_{p,X} \mathbb{E}\|G\|^p_{\gamma(0,T;H,X)}.$$

If G does not depend on Ω, then the above holds for every Banach space X and the norm equivalence only depends on $p \in (0, \infty)$.

To conclude, we make the following simple observation. To state this, we denote by $L^p_{\mathscr{P}}(\Omega \times (0, T); \gamma(H, X))$ the closure in $L^p(\Omega \times (0, T); \gamma(H, X))$ of all simple adapted stochastic process. As a consequence of Proposition 2.3.4 one easily obtains the following:

Corollary 2.3.8. *Let $T > 0$, $p \in (0, \infty)$ and let X be a UMD Banach space with type 2. Then the mapping $G \mapsto \int_0^T G\, dW_H$ extends to a bounded linear operator from $L^p_{\mathscr{F}}(\Omega \times (0, T); \gamma(H, X))$ into $L^p(\Omega; X)$. Moreover,*

$$\mathbb{E} \sup_{0\leq t\leq T} \Big\| \int_0^t G(s)\, dW_H(s)\Big\|_X^p \lesssim_{p,X,T} \mathbb{E}\|G\|^p_{L^2(0,T;\gamma(H,X))}.$$

Part I

Local well-posedness

Chapter 3

Stochastic maximal L^p-regularity for semigroup generators

In this chapter, X denotes a Banach space with UMD and type 2, H denotes an Hilbert space with dimension $\dim H \geq 1$, and $A : \mathsf{D}(A) \subseteq X \to X$ is an operator such that $-A$ generates a strongly continuous semigroup $(S(t))_{t\geq 0}$. In addition $(\Omega, \mathbb{F} = (\mathscr{F}_t)_{t\geq 0}, \mathscr{A}, \mathbb{P})$ and $\mathscr{P}$ denote an underlying filtered probability space and the progressive sigma algebra, respectively.

The aim of this chapter is to introduce and study stochastic maximal L^p-regularity for semigroup generators. The latter concerns optimal (space-time) regularity estimates for stochastic convolutions

$$S \diamond G(t) := \int_0^t S(t-s)G(s)dW_H(s), \quad t \in \mathbb{R}_+,$$

where W_H denotes a cylindrical Brownian motion in H and G a progressively measurable process. The study of stochastic convolutions is motivated by the fact that $S \diamond G$ satisfies $du + Audt = gdW_H$ and $u(0) = 0$. The set of all operators having stochastic maximal L^p-regularity will be denoted by $\mathrm{SMR}(p, T)$. Some variants of the latter will be also employed.

This chapter is organized as follows. In Section 3.1 we present some basic definitions and properties of the class $\mathrm{SMR}(p, T)$ such as a deterministic characterization and independence of on the noise dimension $\dim H$. In Section 3.2 by means of square-function estimates we prove that if $A \in \mathrm{SMR}(p, T)$, then $(S(t))_{t\geq 0}$ is an analytic semigroup. To this end we prove several additional results which are of independent interests. In Section 3.3 we prove that the class $\mathrm{SMR}(p, T)$ is independent of the lenght of the interval. As a by-product of the latter result we obtain that

$$A \in \mathrm{SMR}(p, T) \text{ for some } T < \infty \quad \Rightarrow \quad \text{there exists } \lambda \in \mathbb{R} \text{ such that } \lambda + A \in \mathrm{SMR}(p, \infty).$$

In particular, the latter shows that one can always reduce the study of maximal L^p-regularity to exponentially stable semigroup generators.

Section 3.4 is devoted to the study of the *weighted* stochastic maximal L^p-regularity. Here we prove that weighted stochastic maximal L^p-regularity is equivalent to the *un-weighted* one. Finally, we introduce the class $\mathrm{SMR}_\theta(p, \infty)$ consisting of generators of an exponetially stable semigroup satisfying suitable (weighted) estimates. Following [166], we use the DaPrato-Kwapień-Zabczyk factorization argument to show optimal space-time regularity for $S \diamond G$ in the case $A \in \mathrm{SMR}_\theta(p, \infty)$. The results in Section 3.4 will be of basic importance for the subsequent chapters where space-time regularity estimates will be used in fixed point arguments to prove existence for *nonlinear* problems.

The results in this chapter are taken from my work [5].

3.1 Definitions and basic facts

Let us begin by discussing the relation between abstract stochastic Cauchy problem and stochastic convolutions.

3.1.1 Solution concepts

For processes $F \in L^1_{\mathscr{P}}(\Omega \times (0,T); X)$ and $G \in L^2_{\mathscr{P}}(\Omega \times (0,T); \gamma(H,X))$ for every $T < \infty$, consider the following stochastic evolution equation

$$\begin{cases} dU + AU dt = F dt + G dW_H, & \text{on } \mathbb{R}_+, \\ U(0) = 0. \end{cases} \tag{3.1}$$

The *mild solution* to (3.1) is given by

$$U(t) = S * F(t) + S \diamond G(t) := \int_0^t S(t-s)F(s)ds + \int_0^t S(t-s)G(s)\, dW_H(s).$$

for $t \geq 0$. It is well-known that the mild solution is a so-called *weak solution* to (3.1): for all $x^* \in \mathsf{D}(A^*)$, for all $t \geq 0$, a.s.

$$\langle U(t), x\rangle + \int_0^t \langle U(s), A^* x^*\rangle ds = \int_0^t \langle F(s), x^*\rangle ds + \int_0^t G(s)^* x^* dW_H(s)$$

Conversely, if $U \in L^1_{\mathrm{loc}}(\mathbb{R}_+; X)$ a.s. is a weak solution to (3.1), then U is a mild solution. Moreover, if $U \in L^1_{\mathrm{loc}}(\mathbb{R}_+; \mathsf{D}(A))$ a.s., then additionally U is a *strong solution* to (3.1): for all $t \geq 0$ a.s.

$$U(t) + \int_0^t AU(s)ds = \int_0^t F(s)ds + \int_0^t G(s)dW_H(s).$$

For details we refer to [50] and [200].

3.1.2 Main definitions

Here and in the rest of this chapter, $\omega_0(-A)$ denotes the growth bound of S:

$$\omega_0(-A) := \inf\Big\{\omega \in \mathbb{R} \,:\, \sup_{t>0} e^{-\omega t}\|S(t)\| < \infty\Big\}.$$

In particular, $\omega_0(-A) < 0$ if and only if S is exponentially stable. Moreover, if A is a densely defined operator and $w > \omega_0(-A)$, then $w + A$ is a sectorial operator on X, and as noticed in Subsection 2.1.1, $(w+A)^{1/2}$ is a well-defined closed operator on X.

Let us begin by defining the class of operators having *stochastic maximal L^p-regularity.*

Definition 3.1.1 (Stochastic maximal L^p-regularity)**.** *Let X be a UMD space with type 2, let $p \in [2,\infty)$, $w > \omega_0(-A)$ and let $J = (0,T)$ with $T \in (0,\infty]$. The operator A is said to have* stochastic maximal L^p-regularity *on J if for each $G \in L^p_{\mathscr{P}}(\Omega \times J; \gamma(H,X))$ the stochastic convolution $S \diamond G$ takes values in $\mathsf{D}((w+A)^{1/2})$ $\mathbb{P} \times dt$-a.e., and satisfies*

$$\|S \diamond G\|_{L^p(\Omega\times J;\mathsf{D}((w+A)^{1/2}))} \leq C\|G\|_{L^p(\Omega\times J;\gamma(H,X))}, \tag{3.2}$$

for some $C > 0$ independent of G. In this case we write $A \in \mathrm{SMR}(p,T)$.

Note that, the class $\mathrm{SMR}(p,T)$ does not depend on $w > \omega_0(-A)$. Indeed, for any $w, w' > \omega_0(-A)$, $\mathsf{D}((w+A)^{1/2}) = \mathsf{D}((w'+A)^{1/2})$ isomorphically.

Some helpful remarks may be in order.

Remark 3.1.2. In Definition 3.1.1 it suffices to consider G in a dense class of a subset of $L^p_{\mathscr{P}}(\Omega \times J; \gamma(H,X))$ for which the stochastic convolution process $(w+A)^{1/2}S \diamond G(t)$ is well-defined for each $t \geq 0$. For example, the set of all adapted step processes with values in $\mathsf{D}(A)$ (or the space $L^p_{\mathscr{P}}(\Omega \times J; \gamma(H, \mathsf{D}(A))))$ can be used. Indeed, if $G \in L^p_{\mathscr{P}}(\Omega \times J; \gamma(H, \mathsf{D}(A)))$, then $s \mapsto (w+A)^{1/2}S(t-s)G(t)$ belongs to $L^p(\Omega \times J; \gamma(H,X))$ for each $t \in J$. Thus for $t \in J$,

$$\begin{aligned}\mathbb{E}\int_0^t \|(w+A)^{1/2}S(t-s)G(s)\|^p_{\gamma(H,X)}\,ds &\leq M^2\mathbb{E}\int_0^t \|(w+A)^{1/2}G(s)\|^p_{\gamma(H,X)}\,ds\\ &\leq c\,M^2\|G\|_{L^p(\Omega\times J;\gamma(H,\mathsf{D}(A)))},\end{aligned}$$

where $M := \sup_{s\leq t}\|S(t)\|$. Therefore, for each $t \in J$, the well-definedness of $(w+A)^{1/2}S \diamond G(t)$ follows from Corollary 2.3.8.

Remark 3.1.3. In the setting of Definition 3.1.1, for $\alpha \in [1/2, 1]$, one could ask for

$$\|S \diamond G\|_{L^p(\Omega\times J;\mathsf{D}((w+A)^\alpha))} \leq C\|G\|_{L^p(\Omega\times J;\gamma(H,\mathsf{D}((w+A)^{\alpha-1/2})))}, \tag{3.3}$$

for each $G \in L^p_{\mathscr{P}}(\Omega \times J; \gamma(H, \mathsf{D}((w+A)^{\alpha-\frac{1}{2}})))$. One can easily deduce that A satisfies (3.3) if and only if $A \in \mathrm{SMR}(p,T)$.

Before going further, we introduce an homogeneous version of stochastic maximal L^p-regularity:

Definition 3.1.4 (Homogeneous Stochastic Maximal L^p-regularity)**.** *Let X be a UMD space with type 2 and let $p \in [2,\infty)$. The operator A is said to have* homogeneous stochastic maximal L^p-regularity *if for each $G \in L^p_{\mathscr{P}}(\Omega \times \mathbb{R}_+; \gamma(H,X))$ the stochastic convolution $S \diamond G$ takes values in $\mathsf{D}(A^{1/2})$ $\mathbb{P} \times dt$-a.e. and*

$$\|A^{1/2}S \diamond G\|_{L^p(\Omega\times\mathbb{R}_+;X)} \leq C\|G\|_{L^p(\Omega\times\mathbb{R}_+;\gamma(H,X))}, \tag{3.4}$$

for some $C > 0$ independent of G. In this case we write $A \in \mathrm{SMR}^0(p,\infty)$.

There is no need for the homogeneous version of $\mathrm{SMR}(p,T)$ for $J = (0,T)$ with $T < \infty$, since in this situation by Corollary 2.3.8 we have

$$\|S \diamond G\|_{L^p(\Omega\times J;X)} \leq c_T\|G\|_{L^p(\Omega\times J;\gamma(H,X))}.$$

Moreover, it is clear that if $A \in \mathrm{SMR}^0(p,\infty)$ for some $p \in [2,\infty)$ and $0 \in \rho(A)$ (thus $0 \in \rho(A^{1/2})$) then $A \in \mathrm{SMR}(p,\infty)$. The converse is also true as Corollary 3.2.9 below shows.

We will mainly study the class $\mathrm{SMR}(p,T)$ (for $T \in (0,\infty]$). However, many results can be extended to the class $\mathrm{SMR}^0(p,\infty)$ without difficulty.

In order to state the following result we introduce the following condition:

Assumption 3.1.5. *Let X be a UMD Banach space with type 2 and let $p \in [2,\infty)$. Assume that the following family is R-bounded*

$$\{J_\delta\}_{\delta>0} \subseteq \mathscr{L}(L^p_{\mathscr{P}}(\Omega \times \mathbb{R}_+; \gamma(H,X)), L^p(\Omega \times \mathbb{R}_+; X)),$$

where $J_\delta f(t) := \frac{1}{\sqrt{\delta}}\int_{(t-\delta)\vee 0}^t f(s)dW_H(s)$.

The above holds for $p \in (2,\infty)$ if X is isomorphic to a closed subspace of an $L^q(S)$ space with $q \in [2,\infty)$. If $q = 2$, one can also allow $p = 2$. The following central result was proved in [166, 167, 168]; see also Remark 3.4.7.

Theorem 3.1.6. *Suppose that Assumption 3.1.5 is satisfied. If A has a bounded H^∞-calculus with $\omega_{H^\infty}(A) < \pi/2$, then $A \in \mathrm{SMR}^0(p,\infty)$.*

3.1.3 Deterministic characterization and immediate consequences

In the next proposition we make a first reduction to the case where G does not depend on Ω.

Proposition 3.1.7. *Let X be a UMD space with type 2, let $p \in [2,\infty)$, let $J = (0,T)$ with $T \in (0,\infty]$ and fix $w > \omega_0(-A)$. Then the following are equivalent:*

(1) $A \in \mathrm{SMR}(p,T)$.

(2) *There exists a constant C such that for all $G \in L^p(J;\gamma(H,\mathsf{D}(A)))$,*

$$\Big(\int_0^T \|s \mapsto (w+A)^{1/2}S(t-s)G(s)\|^p_{\gamma(0,t;H,X)}\,dt\Big)^{1/p} \le C\|G\|_{L^p(J;\gamma(H,X))}.$$

Proof. (1) $\Rightarrow$ (2): For $G \in L^p(J;\gamma(H,\mathsf{D}(A)))$, Theorem 2.3.7 provides the two-sides estimates

$$\|(w+A)^{1/2}S \diamond G(t)\|_{L^p(\Omega;X)} \eqsim_{p,X} \|s \mapsto (w+A)^{1/2}S(t-s)G(s)\|_{\gamma(0,t;H,X)}.$$

Now the claim follows by taking $L^p(J)$-norms in the previous inequalities.

(2) $\Rightarrow$ (1): As in the previous step, we employ Theorem 2.3.7. Indeed, for any $t \in J$ and G an adapted step process, we have

$$\|(w+A)^{1/2}S \diamond G(t)\|^p_{L^p(\Omega;X)} \eqsim_{p,X} \mathbb{E}\|s \mapsto (w+A)^{1/2}S(t-s)G(s)\|^p_{\gamma(0,t;H,X)}.$$

Integrating over $t \in J$, we get

$$\begin{aligned}\|(w+A)^{1/2}S \diamond G\|^p_{L^p(\Omega\times J;X)} &\eqsim_{X,p} \mathbb{E}\int_0^T \|s \mapsto (w+A)^{1/2}S(t-s)G(s)\|^p_{\gamma(0,t;H,X)}\,dt\\ &\le C^p\mathbb{E}\int_0^T \|G(t)\|^p_{\gamma(H,X)}\,dt = C^p\|G\|^p_{L^p(\Omega\times J;\gamma(H,X))},\end{aligned}$$

where in the last we have used the inequality in (2) pointwise in Ω. The claim follows by density of the adapted step process in $L^p_{\mathscr{P}}(\Omega\times J;\gamma(H,X))$. ☐

Proposition 3.1.8. *Let X be a UMD space with type 2, let $p \in [2,\infty)$. Let $J = (0,T)$ with $T \in (0,\infty]$ and assume $A \in \mathrm{SMR}(p,T)$. Then:*

(1) *If $T < \infty$ and $\lambda \in \mathbb{C}$, then $A + \lambda \in \mathrm{SMR}(p,T)$.*

(2) *If $T = \infty$ and $\lambda \in \mathbb{C}$ is such that $\Re\lambda \ge 0$, then $A+\lambda \in \mathrm{SMR}(p,\infty)$.*

(3) *If $T \in (0,\infty]$ and $\lambda > 0$, then $\lambda A \in \mathrm{SMR}(p,T/\lambda)$.*

Proof. (1): Note that $-A-\lambda$ generates $(e^{-\lambda t}S(t))_{t>0}$. Then, fix $w > \omega_0(-A-\lambda)$ (thus $w+\lambda > \omega_0(-A)$) and let $G \in L^p(J;\gamma(H,\mathsf{D}(A)))$. By (2.16) one has

$$\begin{aligned}&\|s \mapsto (w+\lambda+A)^{1/2}e^{-\lambda(t-s)}S(t-s)G(s)\|_{\gamma(0,t;H,X)}\\ &\qquad\le M_{T,\lambda}\|s \mapsto (w+\lambda+A)^{1/2}S(t-s)G(s)\|_{\gamma(0,t;H,X)},\end{aligned}$$

where $M_{T,\lambda} = \sup_{\{0<s<t<T\}} e^{-(\mathrm{Re}\lambda)(t-s)}$. Therefore, taking the $L^p(J)$-norms, Proposition 3.1.7 implies the required result.

(2): Follows by the same argument of (1) but in this case $M_{\infty,\lambda} = \sup_{\{0<s<t\}} e^{-(\mathrm{Re}\lambda)(t-s)}$ is finite if and only if $\Re\lambda \ge 0$.

(3): Note that $-\lambda A$ generates $(S(\lambda t))_{t>0}$. Fix $G \in L^p(0,T/\lambda;\gamma(H,\mathsf{D}(A)))$ and $w > \omega_0(-\lambda A)$ (thus $w/\lambda > \omega_0(-A)$), one has

$$\begin{aligned}&\|s \mapsto (w+\lambda A)^{1/2}S(\lambda(t-s))G(s)\|_{\gamma(0,t;H,X)}\\ &\qquad= \|s \mapsto (w+\lambda A)^{1/2}S(\lambda s)G(t-s)\|_{\gamma(0,t;H,X)}\end{aligned}$$

$$\eqsim_\lambda \|s \mapsto (\frac{w}{\lambda} + A)^{1/2} S(s) G(t - \frac{s}{\lambda})\|_{\gamma(0,\lambda t;H,X)}.$$

Then integrating over $0 < t < T/\lambda$, one has

$$\begin{aligned}\int_0^{\frac{T}{\lambda}} &\|s \mapsto (w+\lambda A)^{1/2} S(\lambda(t-s)) G(s)\|^p_{\gamma(0,t;H,X)}\, dt \\ &\eqsim_\lambda \int_0^{\frac{T}{\lambda}} \|s \mapsto (\frac{w}{\lambda} + A)^{1/2} S(s) G(t - \frac{s}{\lambda})\|^p_{\gamma(0,\lambda t;H,X)}\, dt \\ &\eqsim_\lambda \int_0^{T} \|s \mapsto (\frac{w}{\lambda} + A)^{1/2} S(s) G(\frac{\tau - s}{\lambda})\|^p_{\gamma(0,\tau;H,X)}\, d\tau \\ &\le C_{\lambda,p,A} \int_0^T \|G(\frac{s}{\lambda})\|^p_{\gamma(H,X)} ds = C_{\lambda,p,A} \int_0^{\frac{T}{\lambda}} \|G(s)\|^p_{\gamma(H,X)} ds;\end{aligned}$$

where in the last inequality we have used that $A \in \mathrm{SMR}(p,T)$. Thus Proposition 3.1.7 ensures that $\lambda A \in \mathrm{SMR}(p, T/\lambda)$. □

In Corollary 3.3.3 we will see a refinement of Proposition 3.1.8.

3.1.4 Independence of H

Theorem 3.1.9. *Let X be a UMD space with type 2, let $p \in [2,\infty)$ and let $J = (0,T)$ with $T \in (0,\infty]$. The following are equivalent:*

(1) *$A \in \mathrm{SMR}(p,T)$ for $H = \mathbb{R}$.*

(2) *$A \in \mathrm{SMR}(p,T)$ for any Hilbert space H.*

Proof. It suffices to prove (1)⇒(2), since the converse is trivial. Assume (1) holds. Without loss of generality we can assume H is separable (see [108, Proposition 9.1.7]). Let $\Gamma : \mathbb{R}_+ \to L^p(\widetilde{\Omega}; X)$ be defined by $\Gamma(s) = \sum_{n\ge 1} \gamma_n G(s) h_n$, where $(h_n)_{n\ge1}$ is an orthonormal basis for H and (γ_n) on $\widetilde{\Omega}$ is as in Section 2.3.2. Then by the Kahane–Khintchine inequalities and the definition of the γ-norm we have

$$\|G(s)\|_{\gamma(H,X)} = \|\Gamma(s)\|_{L^2(\widetilde{\Omega};X)} \eqsim_p \|\Gamma(s)\|_{L^p(\widetilde{\Omega};X)}. \tag{3.5}$$

By Proposition 2.3.4

$$\begin{aligned}\|s \mapsto (w + A)^{1/2} &S(t-s) G(s)\|_{\gamma(0,t;H,X)} \\ &\lesssim_X \|s \mapsto (w + A)^{1/2} S(t-s) G(s)\|_{\gamma(0,t;\gamma(H,X))} \\ &= \|s \mapsto (w + A)^{1/2} S(t-s) \Gamma(s)\|_{\gamma(0,t;L^2(\widetilde{\Omega};X))} \\ &\overset{(*)}{=} \|s \mapsto (w + A)^{1/2} S(t-s) \Gamma(s)\|_{L^2(\widetilde{\Omega};\gamma(0,t;X))} \\ &\le \|s \mapsto (w + A)^{1/2} S(t-s) \Gamma(s)\|_{L^p(\widetilde{\Omega};\gamma(0,t;X))},\end{aligned}$$

where we applied the γ-Fubini's theorem (see [108, Theorem 9.4.8]) in $(*)$. By Fubini's theorem and Proposition 3.1.7 we obtain

$$\begin{aligned}\int_J \|s \mapsto &(w + A)^{1/2} S(t-s) G(s)\|^p_{\gamma(0,t;H,X)} dt \\ &\le \widetilde{\mathbb{E}} \int_J \|s \mapsto (w + A)^{1/2} S(t-s) \Gamma(s)\|^p_{\gamma(0,t;X)} dt \\ &\le C^p \widetilde{\mathbb{E}} \|\Gamma\|^p_{L^p(J;X)} = C^p \|\Gamma\|^p_{L^p(J;L^p(\widetilde{\Omega};X))} \eqsim_p C^p \|G\|_{L^p(J;\gamma(H,X))}.\end{aligned}$$

where in "$\eqsim_p$" we used (3.5). Now the result follows from Proposition 3.1.7. □

3.2 Analyticity and exponential stability

The main result of this section is the following.

Theorem 3.2.1. *Let X be a Banach space with UMD and type 2 and let $p \in [2, \infty)$. Let $J = (0, T)$ with $T \in (0, \infty]$. If $A \in \mathrm{SMR}(p, T)$, then $-A$ generates an analytic semigroup.*

The proof consists of several steps and will be explained in the next subsections.

3.2.1 Square function estimates

Next we derive a simple square function estimates from $\mathrm{SMR}(p, T)$. In order to include the case $T = \infty$ we need a careful analysis of the constants.

Lemma 3.2.2. *Let X be a UMD space with type 2, let $p \in [2, \infty)$, let $J = (0, T)$ with $T \in (0, \infty]$ and let $w > \omega_0(-A)$. If $A \in \mathrm{SMR}(p, T)$, then there is a constant C such that for all $x \in X$,*

$$\|s \mapsto (w + A)^{1/2} S(s)x\|_{\gamma(J;X)} \leq C\|x\|. \tag{3.6}$$

Proof. First assume $T < \infty$ and fix $h \in H$ with $\|h\| = 1$. Let $G \in L^p(J; \gamma(H, X))$ be given by $G(t) = \mathbf{1}_J h \otimes x$. Then for $t \in [T/2, T]$ one can write

$$\begin{aligned}\|s \mapsto (w + A)^{1/2} S(s)x\|_{\gamma(0,T/2;X)} &\leq \|s \mapsto (w + A)^{1/2} S(s)x\|_{\gamma(0,t;X)} \\ &= \|s \mapsto (w + A)^{1/2} S(t - s)x\|_{\gamma(0,t;X)} \\ &= \|s \mapsto (w + A)^{1/2} S(t - s)G(s)\|_{\gamma(0,t;H,X)}.\end{aligned}$$

Therefore, taking p-th powers on both sides integration over $t \in J$, and applying Proposition 3.1.7 yields

$$\begin{aligned}T\|s \mapsto (w + A)^{1/2} S(s)x\|^p_{\gamma(0,T/2;X)} &\leq \int_0^T \|s \mapsto (w + A)^{1/2} S(t - s)G(s)\|^p_{\gamma(0,t;H,X)} \, dt \\ &\leq C^p \|G\|^p_{L^p(J;\gamma(H,X))} = C^p T\|x\|^p.\end{aligned}$$

Therefore,

$$\|s \mapsto (w + A)^{1/2} S(s)x\|_{\gamma(0,T/2;X)} \leq C\|x\|, \quad x \in X. \tag{3.7}$$

By the left-ideal property and (3.7) we see that

$$\begin{aligned}\|s \mapsto (w + A)^{1/2} S(s)x\|_{\gamma(T/2,T;X)} &= \|s \mapsto S(\tfrac{T}{2})(w + A)^{1/2} S(s - \tfrac{T}{2})x\|_{\gamma(T/2,T;X)} \\ &\leq \|S(\tfrac{T}{2})\| \, \|s \mapsto (w + A)^{1/2} S(s)x\|_{\gamma(0,T/2;X)} \\ &\leq C\|S(\tfrac{T}{2})\| \, \|x\|.\end{aligned}$$

Combining this with (3.7) and (2.15) yields

$$\begin{aligned}&\|(w + A)^{1/2} S(s)x\|_{\gamma(J;X)} \\ &\quad \leq \|(w + A)^{1/2} S(s)x\|_{\gamma(0,T/2;X)} + \|(w + A)^{1/2} S(s)x\|_{\gamma(T/2,T;X)} \\ &\quad \leq C_{S,T}\|x\|.\end{aligned}$$

Next we consider $T = \infty$. Applying Proposition 3.1.7 with $G\mathbf{1}_{[0,R]}$ with $R > 0$ fixed and (2.16) gives that

$$\Big(\int_0^R \|s \mapsto (w + A)^{1/2} S(t - s)G(s)\|^p_{\gamma(0,t;H,X)} \, dt\Big)^{1/p} \leq C\|G\|_{L^p(0,R;\gamma(H,X))},$$

where C is independent of R. Therefore, arguing as in (3.7) we obtain that for all $R < \infty$,

$$\|s \mapsto (w + A)^{1/2} S(s)x\|_{\gamma(0,R/2;X)} \leq C\|x\|.$$

The result now follows since (see [163, Proposition 2.4])

$$\|s \mapsto (w + A)^{1/2} S(s)x\|_{\gamma(\mathbb{R}_+;X)} = \sup_{R>0} \|s \mapsto (w + A)^{1/2} S(s)x\|_{\gamma(0,R/2;X)}.$$

□

3.2.2 Sufficient conditions for analyticity

To prove Theorem 3.2.1 we need several additional results which are of independent interest. The next result is a comparison result between γ-norms and L^p-norms of certain orbits for spaces with cotype p. Related estimates for general analytic functions can be found in [201, Theorem 4.2], but are not applicable here.

Lemma 3.2.3. *Let X be a Banach space with cotype p. Let $\omega_0(-A) < 0$. Then for all $q > p$ there exists a $C > 0$ such that for all $x \in \mathsf{D}(A^2)$,*

$$\|t \mapsto A^{1/q}S(t)x\|_{L^q(\mathbb{R}_+;X)} \leq C\|t \mapsto A^{1/2}S(t)x\|_{\gamma(\mathbb{R}_+;X)}.$$

Moreover, if $p = 2$, then one can take $q = 2$ in the above.

The right-hand side of the above estimate is finite. Indeed, for $x \in \mathsf{D}(A^2)$, we have $A^{1/2}S(\cdot)x = S(\cdot)A^{1/2}x \in C^1([0,T];X)$, thus it follows from [108, Proposition 9.7.1] that $A^{1/2}S(\cdot)x \in \gamma(0,T;X)$. Now since S is exponentially stable we can conclude from [170, Proposition 4.5] that $A^{1/2}S(\cdot)x \in \gamma(\mathbb{R}_+;X)$.

Proof. By an approximation argument we can assume $x \in \mathsf{D}(A^3)$. Let $(\phi_n)_{n\geq 0}$ be a Littlewood-Paley partition of unity as in [20, Section 6.1]. Let $f : \mathbb{R} \to X$ be given by $f(t) := A^{1/q}S(|t|)x$. Then $f'(t) = \text{sign}(t)Af(t)$ for $t \in \mathbb{R} \setminus \{0\}$. Let $f_n := \phi_n * f$ for $n \geq 0$. Let ψ be such that $\widehat{\psi} = 1$ on $\operatorname{supp} \widehat{\phi}_1$ and $\widehat{\psi} \in C_c^\infty(\mathbb{R} \setminus \{0\})$. Set $\widehat{\psi}_n(\xi) = \widehat{\psi}_1(2^{-(n-1)}\xi)$ for $n \geq 1$. Then $f_n = \psi_n * f_n$.

Step 1: We will first show that for all $\alpha \in (0,1)$, there is a constant C such that for all $n \geq 0$

$$\|f_n\|_p \leq C2^{-\alpha n}\|A^\alpha f_n\|_p, \tag{3.8}$$

where we write $\|\cdot\|_p := \|\cdot\|_{L^p(\mathbb{R};X)}$. As a consequence the estimate (3.8) holds for an arbitrary $\alpha > 0$ if one takes $x \in \mathsf{D}(A^{r+2})$ (where $\alpha < r \in \mathbb{N}$). For $n = 0$ the estimate is clear from $0 \in \rho(A^\alpha)$. To prove the estimate for $n \geq 1$ note that by the moment inequality (see [76, Theorem II.5.34]) and Hölder inequality,

$$\|Af_n\|_p \leq C\|A^\alpha f_n\|_p^{\frac{1}{2-\alpha}}\|A^2 f_n\|_p^{\frac{1-\alpha}{2-\alpha}}. \tag{3.9}$$

Using $f_n = \psi_n * f_n$ and the properties of S we obtain

$$\text{sign}(\cdot)Af_n = \frac{d}{dt}f_n = \psi_n' * f_n. \tag{3.10}$$

Therefore, by Young's inequality

$$\|A^2 f_n\|_p = \|\psi_n' * Af_n\|_p \leq \|\psi_n'\|_1\|Af_n\|_p \leq C_\psi 2^n\|Af_n\|_p.$$

Combining this with (3.9) we obtain

$$\|Af_n\|_p \leq C2^{n(1-\alpha)}\|A^\alpha f_n\|_p. \tag{3.11}$$

Next we prove an estimate for $\|f_n\|_p$. Let $d_t = \frac{d}{dt}$ and set $J_\beta = (1-d_t^2)^{\beta/2}$ for $\beta \in \mathbb{R}$. Then $J_{\beta_1}J_{\beta_2} = J_{\beta_1+\beta_2}$ for $\beta_1, \beta_2 \in \mathbb{R}$. Recall from the proof of [10, Theorem 6.1] that for any $g \in L^p(\mathbb{R};X)$ and $\beta \in \mathbb{R}$, we have

$$\|J_\beta\psi_n * g\|_p \leq C_{\beta,\psi}2^{\beta n}\|\psi_n * g\|_p.$$

Therefore,

$$\|f_n\|_p = \|\psi_n * \varphi_n * f\|_p = \|J_{-2}\psi_n * (J_2\varphi_n) * f\|_p \leq C_\psi 2^{-2n}\|\psi_n * (J_2\varphi_n) * f\|_p.$$

Now since $J_2 = 1 - d_t^2$ we can estimate

$$\begin{aligned}\|\psi_n * (J_2\varphi_n) * f\|_p &\leq \|\psi_n * \varphi_n * f\|_p + \|d_t^2(\psi_n * \varphi_n * f)\|_p \\ &\leq C_\psi\|f_n\|_p + \|\psi_n' * \varphi_n * f'\|_p.\end{aligned}$$

By Young's inequality

$$\|\psi_n' * \varphi_n * f'\|_p \leq \|\psi_n'\|_1 \|\varphi_n * f'\|_p \leq C_\psi 2^n \|(f_n)'\|_p = C_\psi 2^n \|Af_n\|_p,$$

where in the last equality we have used (3.10). Thus we can conclude

$$\|f_n\|_p \leq C_\psi 2^{-n}(\|f_n\|_p + \|Af_n\|_p) \leq C_{\psi,A} 2^{-n}\|Af_n\|_p, \tag{3.12}$$

where in the last step we used the fact that A is invertible.

Now (3.8) follows by combining (3.11) and (3.12).

Step 2: By Step 1 with $\alpha := \frac{1}{2} - \frac{1}{q}$ and [186, Lemma 4.1] we can estimate

$$\|f_n\|_p \leq C 2^{-n\alpha}\|A^\alpha f_n\|_p \leq C_{p,X} 2^{-n\alpha} 2^{\frac{n}{2}-\frac{n}{p}} \|A^\alpha f_n\|_{\gamma(\mathbb{R};X)}.$$

Multiplying by $2^{\frac{n}{p}-\frac{n}{q}}$ and taking ℓ^p-norms and applying [112, Lemma 2.2] in the same way as in [112, Theorem 1.1] gives

$$\begin{aligned}\|f\|_{B^{\frac{1}{p}-\frac{1}{q}}_{p,p}(\mathbb{R};X)} &\leq C_{p,X}\Big(\sum_{n\geq 0} \|A^\alpha f_n\|^p_{\gamma(\mathbb{R};X)}\Big)^{1/p} \\ &\leq C'_{p,X}\|A^\alpha f\|_{\gamma(\mathbb{R};X)} \leq 2C'_{p,X}\|t \mapsto A^{1/2}S(t)x\|_{\gamma(\mathbb{R}_+;X)},\end{aligned}$$

where in the last step we used (2.15).

It remains to note that $B^{\frac{1}{p}-\frac{1}{q}}_{p,p}(\mathbb{R};X) \hookrightarrow L^q(\mathbb{R};X)$ (see [154, Theorem 1.2 and Proposition 3.12]). The final assertion for $p = 2$ is immediate from Proposition 2.3.4. □

Next we show that certain L^p-estimates for orbits implies analyticity of the semigroup S.

Lemma 3.2.4. *Let X be a Banach space and let $w > \omega_0(-A)$. If for some $T \in (0,\infty]$, $C > 0$, $p \geq 2$, the operator A satisfies*

$$\|t \mapsto (w + A)^{1/p}S(t)x\|_{L^p(0,T;X)} \leq C\|x\|_X, \quad x \in \mathsf{D}(A), \tag{3.13}$$

then $-A$ generates an analytic semigroup.

It seems that the above result was first observed in [23, Proposition 2.7]. The proof below is different and was found independently.

Proof. Clearly, we can assume $T < \infty$. Moreover, without loss of generality, one can reduce to the case that S is exponentially stable and $w = 0$. Finally, we can also assume that $p \geq 2$ is an integer. Indeed, fix $n \in \mathbb{N}$ such that $n \geq p$. By the moment inequality (see [76, Theorem II.5.34]) for all $t \in [0,T]$, we have

$$\begin{aligned}\|(w + A)^{1/n}S(t)x\|^n &\lesssim_{n,p,A,w} \|S(t)x\|^{n-p}\|(w + A)^{1/p}S(t)x\|^p \\ &\lesssim_{n,p,A,T} \|x\|^{n-p}\|(w + A)^{1/p}S(t)x\|^p.\end{aligned}$$

Therefore,

$$\int_0^T \|(w + A)^{1/n}S(t)x\|^n dt \lesssim_{n,p,A,T,w} \|x\|^{n-p}\int_0^T \|(w + A)^{1/p}S(t)x\|^p dt \leq C^n\|x\|^n.$$

To prove that $(S(t))_{t\geq 0}$ is analytic, it suffices by [76, Theorem II.4.6] to show that $\{tAS(t) : t \in (0,T]\} \subseteq \mathscr{L}(X)$ is bounded. To prove this fix $x \in \mathsf{D}(A)$. Let $M = \sup_{t\geq 0}\|S(t)\|$. Let $t_n = \frac{T}{p2^n}$ for $n \geq 0$. Then for all $t \in [t_{n+1}, t_n]$ we have $\|A^{1/p}S(t_n)x\| \leq M\|A^{1/p}S(t)x\|$ and thus integration gives

$$\frac{1}{2}t_n\|A^{1/p}S(t_n)x\|^p = (t_n - t_{n+1})\|A^{1/p}S(t_n)x\|^p$$

$$\leq M^p \int_J \|A^{1/p}S(t)x\|^p dt \leq M^p C^p \|x\|^p.$$

Now fix $t \in (0, T/p]$. Choose $n \geq 0$ such that $t \in [t_{n+1}, t_n]$. Then we obtain

$$t\|A^{1/p}S(t)x\|^p \leq 2M^p t_{n+1}\|A^{1/p}S(t_{n+1})x\|^p \leq 4M^{2p}C^p\|x\|^p.$$

By density it follows that $S(t) : X \to \mathsf{D}(A^{1/p})$ is bounded and $t^{1/p}\|A^{1/p}S(t)\| \leq 4^{1/p}M^2C$ for each $t \in (0, T/p]$. We can conclude that for all $t \in (0, T]$,

$$\|tAS(t)\| = \|(t^{1/p}A^{1/p}S(t/p))^p\| \leq t\|A^{1/p}S(t/p)\|^p \leq 4pM^{2p}C^p.$$

□

Proposition 3.2.5. *Let X be a Banach space with finite cotype. Let $J = (0, T)$ with $T \in (0, \infty]$. Let $w > \omega_0(-A)$. If there exists a $c > 0$ such that*

$$\|t \mapsto (w + A)^{1/2}S(t)x\|_{\gamma(J;X)} \leq c\|x\|, \quad x \in X, \tag{3.14}$$

then $-A$ generates an analytic semigroup.

Proof. By rescaling we can assume that S is exponentially stable, thus we may take $w = 0$. Moreover, by [170, Proposition 4.5] we can assume $T = \infty$. Now the result follows by combining Lemmas 3.2.3 and 3.2.4. □

Proof of Theorem 3.2.1. By Lemma 3.2.2 the estimate (3.14) holds. Moreover, since X has type 2, it has finite cotype (see [108, Theorem 7.1.14]). Therefore, by Proposition 3.2.5, $-A$ generates an analytic semigroup. □

From the proof of Theorem 3.2.1 we obtain the following.

Remark 3.2.6. Assume $A \in \mathrm{SMR}(p, T)$, $\omega_0(-A) < 0$ and X has cotype p_0. Let $p > p_0$. Then there is a constant C such that for all $x \in X$,

$$\int_{\mathbb{R}_+} \|A^{1/p}S(t)x\|^p dt \leq C^p\|x\|^p.$$

This type of estimate gives the boundedness of some singular integrals.

3.2.3 Exponential stability

Proposition 3.2.7 (Stability)**.** *Let X be a UMD space with type 2, let $p \in [2, \infty)$. If $A \in \mathrm{SMR}(p, \infty)$, then $\omega_0(-A) < 0$.*

Proof. Let $w > \omega_0(-A)$. Let $y \in X$ be arbitrary. Taking $x = (w + A)^{-1/2}y$ in Lemma 3.2.2 one obtains

$$\|s \mapsto S(s)y\|_{\gamma(\mathbb{R}_+;X)} \leq C\|(w + A)^{-1/2}y\| \leq C'\|y\|.$$

Thus from [96, Theorem 3.2] it follows that there is an $\varepsilon > 0$ such that $\{(\lambda + A)^{-1} : \lambda > -\varepsilon\}$ is uniformly bounded. From Theorem 3.2.1 it follows that A generates an analytic semigroup, and hence $0 > s_0(-A) = \omega_0(-A)$ (see [76, Corollary IV.3.12]). □

By combining Theorem 3.2.1 and Proposition 3.2.7 we now obtain that every $A \in \mathrm{SMR}(p, \infty)$ is a sectorial operator. Therefore, choosing $w = 0$ in (3.2.2) in Lemma 3.2.2, we obtain the following:

Corollary 3.2.8. *Suppose that $A \in \mathrm{SMR}(p, \infty)$, $\omega_0(-A) < 0$ and set $\varphi(z) := z^{1/2}e^{-z}$, then there exists a constant $c > 0$ such that*

$$\|t \mapsto \varphi(tA)x\|_{\gamma(\mathbb{R}_+, \frac{dt}{t};X)} \leq c\|x\|,$$

for all $x \in X$.

As announced in Section 3.1 we now can prove the following:

Corollary 3.2.9. *Let $A \in \mathrm{SMR}^0(p,\infty)$. Then $A \in \mathrm{SMR}(p,\infty)$ if and only if $0 \in \rho(A)$.*

Proof. It remains to show that $A \in \mathrm{SMR}(p,\infty)$ implies $0 \in \rho(A)$ and this follows by Proposition 3.2.7. □

Remark 3.2.10. The assertion of Proposition 3.2.7 does not hold if instead we only assume $A \in \mathrm{SMR}^0(p,T)$. Indeed, $-\Delta$ satisfies $\mathrm{SMR}^0(p,T)$ on $L^q(\mathbb{R}^d)$ with $q \in [2,\infty)$ (see [166, Theorem 1.1 and Example 2.5]), but of course $\omega_0(\Delta) = 0$.

3.3 Independence of the time interval

3.3.1 Independence of T

It is well-known in deterministic theory of maximal L^p-regularity that maximal regularity on a finite interval J and exponential stability imply maximal regularity on $\mathbb{R}_+$. We start with a simple result which allows to pass from $\mathbb{R}_+$ to any interval $(0,T)$.

Proposition 3.3.1. *Let X be a UMD space with type 2, let $p \in [2,\infty)$ and let $J = (0,T)$ with $T \in (0,\infty)$. If $A \in \mathrm{SMR}(p,\infty)$, then $A \in \mathrm{SMR}(p,T)$.*

Proof. Let $w > \omega_0(-A)$. Let $G \in L^p_{\mathscr{P}}(\Omega \times J;\gamma(H,X))$ and extending G as 0 on (T,∞) it follows that

$$\begin{aligned}\|S\diamond G\|_{L^p(\Omega\times J;\mathsf{D}((w+A)^{1/2}))} &\leq \|S\diamond G\|_{L^p(\Omega\times\mathbb{R}_+;\mathsf{D}((w+A)^{1/2}))}\\ &\leq C\|G\|_{L^p(\Omega\times\mathbb{R}_+;\gamma(H,X))} = C\|G\|_{L^p(\Omega\times J;\gamma(H,X))}.\end{aligned}$$

□

Next we present a stochastic version of [68, Theorem 5.2] of which its tedious proof is due to T. Kato. Our proof is a variation of the latter one.

Theorem 3.3.2. *Let X be a UMD Banach space with type 2 and let $p \in [2,\infty)$. If $A \in \mathrm{SMR}(p,T)$ and $\omega_0(-A) < 0$, then $A \in \mathrm{SMR}(p,\infty)$.*

Proof. It suffices to check the estimate in Proposition 3.1.7(2) with $w = 0$. Let $J = (0,T)$ and for each $j \in \mathbb{N}$ set $T_j := jT/2$ and $G_j := \mathbf{1}_{[T_j,T_{j+1})}G$. In this proof, to shorten the notation below, we will write

$$\|G\|_{\gamma(a,b)} := \|G\|_{\gamma((a,b);H,X)}.$$

It follows from the triangle inequality and (2.15) that

$$\begin{aligned}&\Big(\int_0^\infty \|s\mapsto A^{1/2}S(t-s)G(s)\|^p_{\gamma(0,t)}\,dt\Big)^{\frac1p}\\
&\quad\leq \Big(\int_0^T \|s\mapsto A^{1/2}S(t-s)G(s)\|^p_{\gamma(0,t)}\,dt\Big)^{\frac1p}\\
&\qquad+\Big(\sum_{j\geq2}\int_{T_j}^{T_{j+1}} \|s\mapsto A^{1/2}S(t-s)G(s)\|^p_{\gamma(0,t)}\,dt\Big)^{\frac1p}\\
&\quad\leq \Big(\int_0^T \|s\mapsto A^{1/2}S(t-s)G(s)\|^p_{\gamma(0,t)}\,dt\Big)^{\frac1p}\\
&\qquad+\Big(\sum_{j\geq2}\int_{T_j}^{T_{j+1}} \|s\mapsto A^{1/2}S(t-s)G(s)\|^p_{\gamma(0,T_{j-1})}\,dt\Big)^{1/p}\\
&\qquad+\Big(\sum_{j\geq2}\int_{T_j}^{T_{j+1}} \|s\mapsto A^{1/2}S(t-s)(G_{j-1}(s)+G_j(s))\|^p_{\gamma(T_{j-1},t)}\,dt\Big)^{\frac1p}\end{aligned}$$

$$=: R_1 + R_2 + R_3.$$

By Proposition 3.1.7, to prove the claim, it is enough to estimate R_i for $i = 1, 2, 3$. By assumption, $A \in \mathrm{SMR}(p, T)$, then by Definition 3.1.1 one has

$$R_1 := \Big(\int_0^T \|s \mapsto A^{1/2} S(t-s) G(s)\|_{\gamma(0,t)}^p \, dt \Big)^{\frac{1}{p}} \leq C\|G\|_{L^p(J;X)} \leq C\|G\|_{L^p(\mathbb{R}_+;X)}.$$

Since $t - T/2 \geq T_{j-1}$ for $t \in [T_j, T_{j+1}]$, by (2.16) the second term can estimated as,

$$\begin{aligned} R_2 &= \Big(\sum_{j\geq 2} \int_{T_j}^{T_{j+1}} \|s \mapsto A^{1/2} S(t-s) G(s)\|_{\gamma(0,T_{j-1})}^p \, dt \Big)^{\frac{1}{p}} \\ &\leq \Big(\int_T^\infty \|s \mapsto A^{1/2} S(t-s) G(s)\|_{\gamma(0,t-\frac{T}{2})}^p \, dt \Big)^{\frac{1}{p}}. \end{aligned}$$

By Theorem 3.2.1, $(S(t))_{t\geq 0}$ is exponentially stable and analytic. Therefore, there are constants $a, M > 0$ such that for all $t \in \mathbb{R}_+$ one has $\|A^{1/2} S(t)\| \leq M t^{-1/2} e^{-at/2}$. By Proposition 2.3.4, for $t \geq T$ one has

$$\begin{aligned} \|s \mapsto &A^{1/2} S(t-s) G(s)\|_{\gamma(0,t-\frac{T}{2})} \\ &\leq \tau_{2,X} \|s \mapsto A^{1/2} S(t-s) G(s)\|_{L^2((0,t-\frac{T}{2});\gamma(H,X))} \\ &\leq \tau_{2,X} \|s \mapsto M(t-s)^{-1/2} e^{-a(t-s)/2} G(s)\|_{L^2((0,t-\frac{T}{2});\gamma(H,X))} \\ &\leq L \|s \mapsto e^{-a(t-s)/2} G(s)\|_{L^2((0,t-\frac{T}{2});\gamma(H,X))} \\ &\leq L \Big(\int_0^t e^{-a(t-s)} \|G(s)\|_{\gamma(H,X)}^2 \, ds \Big)^{1/2} \\ &= L(k * g)^{1/2}, \end{aligned}$$

where $L = \tau_{2,X} M (T/2)^{-1/2}$, $k(s) = \mathbf{1}_{\mathbb{R}_+}(s) e^{-as}$ and $g(s) = \mathbf{1}_{\mathbb{R}_+}(s)\|G(s)\|_{\gamma(H,X)}^2$. Taking $L^p(T,\infty)$-norms with respect to t, from Young's inequality we find that

$$R_2 \leq L\|(k * g)^{1/2}\|_{L^p(\mathbb{R})} \leq L\|k\|_{L^1(\mathbb{R})}^{1/2} \|g\|_{L^{p/2}(\mathbb{R})}^{1/2} = L a^{-1/2} \|G\|_{L^p(\mathbb{R}_+;\gamma(H,X))}.$$

To estimate R_3, writing $G_{j-1,j} = G_{j-1} + G_j$ for each $j \geq 2$ we can estimate

$$\begin{aligned} R_{3j}^p &:= \int_{T_j}^{T_{j+1}} \|s \mapsto A^{1/2} S(t-s) G_{j-1,j}(s)\|_{\gamma(T_{j-1},t)}^p \, dt \\ &= \int_{T_j}^{T_{j+1}} \|s \mapsto A^{1/2} S(t-s-T_{j-1}) G_{j-1,j}(s+T_{j-1})\|_{\gamma(0,t-T_{j-1})}^p \, dt \\ &\leq \int_{T/2}^{T} \|s \mapsto A^{1/2} S(t-s) G_{j-1,j}(s+T_{j-1}))\|_{\gamma(0,t)}^p \, dt \\ &\leq \int_0^T \|s \mapsto A^{1/2} S(t-s) G_{j-1,j}(s+T_{j-1}))\|_{\gamma(0,t)}^p \, dt \\ &\leq C^p \|G_{j-1,j}(\cdot + T_{j-1})\|_{L^p(J;\gamma(H,X))}^p, \end{aligned}$$

where in the last step we have used the assumption and Proposition 3.1.7. Thus, for the third term we write

$$\begin{aligned} R_3 &= \Big(\sum_{j\geq 2} R_{3j}^p \Big)^{\frac{1}{p}} \leq C \Big(\sum_{j\geq 2} \|G_{j-1,j}(\cdot + T_{j-1})\|_{L^p(J;\gamma(H,X))}^p \Big)^{\frac{1}{p}} \\ &\leq 2C \Big(\sum_{j\geq 1} \|G_j\|_{L^p(\mathbb{R}_+;\gamma(H,X))}^p \Big)^{\frac{1}{p}} \leq 2C\|G\|_{L^p(\mathbb{R}_+;\gamma(H,X))}, \end{aligned}$$

in the last step used that the G_j's have disjoint support. This concludes the proof. ∎

Now we can extend Proposition 3.1.8.

Corollary 3.3.3. *Let X be a UMD space with type 2, let $p \in [2,\infty)$. Let $T_1 < \infty$ and suppose that $A \in \mathrm{SMR}(p, T_1)$, then the following holds true:*

(1) *For any $\lambda > \omega_0(-A)$ one has $\lambda + A \in \mathrm{SMR}(p,\infty)$.*

(2) *For any $T_2 > 0$, $A \in \mathrm{SMR}(p, T_2)$.*

(3) *If $T \in (0,\infty]$ and $\lambda > 0$, then $\lambda A \in \mathrm{SMR}(p,T)$.*

Proof. (1): By Proposition 3.1.8(2) $\lambda + A \in \mathrm{SMR}(p, T_1)$ if $\lambda > \omega_0(-A)$. Since $\omega_0(-(A+\lambda)) < 0$ for $\lambda > \omega_0(A)$, by Theorem 3.3.2, we obtain that $A + \lambda \in \mathrm{SMR}(p,\infty)$.

(2): By (1) we know that there exists w such that $A + w \in \mathrm{SMR}(p,\infty)$. Now applying Proposition 3.3.1 we find $w + A \in \mathrm{SMR}(p, T_2)$, and thus the result follows from Proposition 3.1.8(1).

(3): Proposition 3.1.8(3) ensures that $\lambda A \in \mathrm{SMR}(p, T/\lambda)$. Now (2) implies $\lambda A \in \mathrm{SMR}(p,T)$. □

3.3.2 Counterexample

In this final section we give an example of an analytic semigroup generator $-A$ such that $A \notin \mathrm{SMR}(p,T)$.

Proposition 3.3.4. *Let X be an infinite dimensional Hilbert space. Then there exists an operator A such that $-A$ generates an analytic semigroup with $\omega_0(-A) < 0$, but $A \notin \mathrm{SMR}(p,T)$ for any $T \in (0,\infty]$ and $p \in [2,\infty)$.*

Proof. Let $(e_n)_{n\in\mathbb{N}}$ be a Schauder basis of H, for which there exists a $K > 0$ such that for each finite sequence $(\alpha_n)_{n=1}^N \subset \mathbb{C}$ and

$$\Big\| \sum_{1\le n\le N} \alpha_n e_n \Big\| \le K \Big(\sum_{1\le n\le N} |\alpha_n|^2 \Big)^{1/2},$$
$$\sup \Big\{ \sum_{n\ge1} |\alpha_n|^2 \, : \, \Big\| \sum_{n\ge1} \alpha_n e_n \Big\| \le 1 \Big\} = \infty;$$

for the existence of such basis see [188, Example II.11.2] and [108, Example 10.2.32]. Then, define the diagonal operator A by $Ae_n = 2^n e_n$ with its natural domain. By [108, Proposition 10.2.28] A is sectorial of angle zero and $0 \in \rho(A)$. This implies that $-A$ generates an exponentially stable and analytic semigroup S on X. In [142, Theorem 5.5] it was shown that for such operator A there exists no $C > 0$ such that for all $x \in \mathsf{D}(A)$,

$$\|t \mapsto A^{1/2} S(t) x\|_{L^2(\mathbb{R}_+;X)} \le C\|x\|, \quad x \in X.$$

If $A \in \mathrm{SMR}(p,\infty)$, for some $p \in [2,\infty)$, then Lemma 3.2.2 for $w = 0$ provides such estimate (recall that for Hilbert space X one has $\gamma(\mathbb{R}_+;X) = L^2(\mathbb{R}_+;X)$), this implies $A \notin \mathrm{SMR}(p,\infty)$ for all $p \in [2,\infty)$. Since $\omega_0(-A) < 0$, then Theorem 3.3.2 shows that $A \notin \mathrm{SMR}(p,T)$ for any $T \in (0,\infty]$. □

Remark 3.3.5. The adjoint of the example in Proposition 3.3.4 gives an example of an operator which has $\mathrm{SMR}(2,\infty)$, but which does not have a bounded H^∞-calculus (see [14, Section 4.5.2], [142, Theorems 5.1-5.2] and [108, Example 10.2.32]). Note that in the language of [142] for the Weiss conjecture, $A \in \mathrm{SMR}(2,\infty)$ if and only if $A^{1/2}$ is admissible for A. See [149] for more on this.

3.4 Weighted Stochastic Maximal L^p-regularity

As before, in this section X is a Banach space with UMD and type 2. For $p \in [2,\infty)$ and $\kappa \in \mathbb{R}$ and $T \in (0,\infty]$, let $L^p_{\mathscr{P}}(\Omega \times (0,T), w_\kappa; X)$ be the closure of the adapted step processes in $L^p(\Omega; L^p((0,T), w_\kappa; X)))$.

First we extend Definition 3.1.1 to the weighted setting:

Definition 3.4.1. *Let X be a UMD space with type 2, let $p \in [2,\infty)$, $w > \omega_0(-A)$, $T \in (0,\infty]$ and $\kappa \in \mathbb{R}$. We say that A belongs to* $\mathrm{SMR}(p,T,\kappa)$ *if there is a constant C such that for all $G \in L^p_{\mathscr{P}}(\Omega \times (0,T), w_\kappa; \gamma(H,X))$ one has*

$$\|S \diamond G\|_{L^p(\Omega\times(0,T),w_\kappa;\mathsf{D}((w+A)^{1/2}))} \leq C\|G\|_{L^p_{\mathscr{P}}(\Omega\times(0,T),w_\kappa;\gamma(H,X))}.$$

Remark 3.4.2. Note that for every $G \in L^p_{\mathscr{P}}(\Omega \times (0,T), w_\kappa; \gamma(H, \mathsf{D}(A)))$ the stochastic integral $(w+A)^{1/2} S \diamond G$ is well-defined in X. Indeed, since $\kappa < \frac{p}{2} - 1$ by Hölder's inequality one obtains that for all $T < \infty$

$$L^p(0,T,w_\kappa;X) \subseteq L^2(0,T;X);$$

and the claim follows as in Remark 3.1.2.

The main result of this subsection is a stochastic analogue of [176, Theorem 2.4].

Theorem 3.4.3. *Let X be a UMD space with type 2, let $p \in [2,\infty)$ and $\kappa \in (-1, \frac{p}{2}-1)$. Then the following assertions are equivalent:*

1. $A \in \mathrm{SMR}(p,\infty)$.
2. $A \in \mathrm{SMR}(p,\infty,\kappa)$.

As a consequence $\mathrm{SMR}(p,\infty,\kappa) = \mathrm{SMR}(p,\infty)$ for all $\kappa \in (-1, \frac{p}{2}-1)$.

To prove the result we will prove the following more general result, which can be viewed as a stochastic operator-valued analogue of [189].

Theorem 3.4.4. *Let $p \in [2,\infty)$, $\kappa \in (-\infty, \frac{p}{2}-1)$ and let X be a Banach space and let Y be a UMD Banach space with type 2. Let X_0 be a Banach space which densely embeds into X. Let $\Delta = \{(t,s) : 0 < s < t < \infty\}$ and let $K \in C(\Delta; \mathscr{L}(X,Y))$ be such that $\|K(t,s)\| \leq M/(t-s)^{1/2}$ and $\|K(t,s)x\| \leq M\|x\|_{X_0}$ for all $t > s > 0$. For adapted step processes G let $T_K G$ be defined by*

$$T_K G(t) = K \diamond G(t) = \int_0^t K(t,s)G(s)\,dW_H(s), \quad t \in \mathbb{R}_+.$$

Let $p \in [2,\infty)$ and $\kappa \in (-\infty, \frac{p}{2}-1)$. The following assertions are equivalent:

(1) *T_K is bounded from $L^p_{\mathscr{P}}(\Omega \times \mathbb{R}_+, w_\kappa; \gamma(H,X))$ into $L^p(\Omega \times \mathbb{R}_+, w_\kappa; Y)$.*

(2) *T_K is bounded from $L^p_{\mathscr{P}}(\Omega \times \mathbb{R}_+; \gamma(H,X))$ into $L^p(\Omega \times \mathbb{R}_+; Y)$.*

As a consequence the boundedness of T_K does not depend on $\kappa \in (-\infty, \frac{p}{2}-1)$.

To prove the theorem we prove a stochastic version of a standard lemma (see [189], [124] and [176, Proposition 2.3]).

Lemma 3.4.5. *Let X be a Banach space and let Y be a UMD Banach space with type 2. Let $p \in [2,\infty)$ and $\beta \in (-\infty, \frac{1}{2} - \frac{1}{p})$. Let $\Delta = \{(t,s) : 0 < s < t < \infty\}$. Let K be as in Theorem 3.4.3. Then the operator $T_{K,\beta} : L^p_{\mathscr{P}}(\Omega \times \mathbb{R}_+; \gamma(H,X)) \to L^p(\Omega \times \mathbb{R}_+; Y)$ defined by*

$$T_{K,\beta}G(t) = \int_0^t K(t,s)((t/s)^\beta - 1)G(s)\,dW_H(s)$$

is bounded and satisfies $\|T_{K,\beta}\| \leq C_{p,Y} C_\beta M$.

Proof. By density it suffices to bound $T_{K,\beta}G$ for adapted step processes G. Note that for all $t > s > 0$ one has

$$\|K(t,s)((t/s)^\beta - 1)\|^2 \leq M^2 k_\beta(t,s),$$

where $k_\beta : \{(s,t) \in (0,\infty)^2 : s < t\} \to \mathbb{R}_+$ is given by $k_\beta(t,s) = ((t/s)^\beta - 1)^2/(t-s)$.

By Corollary 2.3.8 we have

$$\begin{aligned}\mathbb{E}\|T_{K,\beta}G(t)\|^p &\leq C_{p,Y}^p \mathbb{E}\Big(\int_0^t \|K(t,s)((t/s)^\beta - 1)\|^2 \|G(s)\|^2_{\gamma(H,X)}\,ds\Big)^{p/2}\\ &\leq C_{p,Y}^p M^p \mathbb{E}\Big(\int_0^t k_\beta(t,s)\|G(s)\|^2_{\gamma(H,X)}\,ds\Big)^{p/2},\end{aligned}$$

To conclude, it suffices to prove that

$$\int_{\mathbb{R}_+}\Big(\int_0^t k_\beta(t,s)|f(s)|^2\,ds\Big)^{p/2}\,dt \leq C_\beta^p \|f\|^p_{L^p(\mathbb{R}_+)},$$

for any $f \in L^p(\mathbb{R}_+)$. Let us set $g(s) = |f(s)s^{1/p}|^2$ for $s > 0$, then

$$\int_0^t k_\beta(t,s)|f(s)|^2\,ds = \frac{1}{t^{2/p}}\int_0^\infty h_\beta(t/s)g(s)\,\frac{ds}{s} = \frac{h_\beta * g(t)}{t^{2/p}},$$

where the convolution is in the multiplicative group $(*, \mathbb{R}_+ \setminus \{0\})$ with Haar measure $d\mu(s) = \frac{ds}{s}$ and $h_\beta(x) := \mathbf{1}_{(1,\infty)}(x)\frac{(x^\beta-1)^2}{x-1}x^{2/p}$ for $x > 0$. Taking $\frac{p}{2}$-powers and integrating over $t \in \mathbb{R}_+$ and applying Young's inequality yields

$$\begin{aligned}\int_{\mathbb{R}_+}\Big(\int_0^t k_\beta(t,s)|f(s)|^2\,ds\Big)^{p/2}\,dt &= \|h_\beta * g\|^{p/2}_{L^{p/2}(\mathbb{R}_+,\mu)} \leq \|h_\beta\|^{p/2}_{L^1(\mathbb{R}_+,\mu)}\|g\|^{p/2}_{L^{p/2}(\mathbb{R}_+,\mu)}\\ &= \|h_\beta\|^{p/2}_{L^1(\mathbb{R}_+,\mu)}\|f\|^p_{L^p(\mathbb{R}_+)}.\end{aligned}$$

Finally, one easily checks that

$$\|h_\beta\|_{L^1(\mathbb{R}_+,\mu)} = \int_1^\infty \frac{(x^\beta - 1)^2}{x-1}x^{2/p}\,\frac{dx}{x}$$

is finite if and only if $\beta < \frac{1}{2} - \frac{1}{p}$. This concludes the proof. ☐

Proof of Theorem 3.4.4. By density it suffices to prove uniform estimates for $T_K G$ where G is a X_0-valued adapted step process.

(1)⇒(2): Set $G_\beta(s) := s^\beta G(s)$ where $\beta = \kappa/p$. Observe that

$$t^\beta T_K G(t) = T_K G_\beta(t) + T_{K,\beta}G_\beta(t), \tag{3.15}$$

where $T_{K,\beta}$ is as in Lemma 3.4.5. By (1) one has

$$\|T_K G_\beta\|_{L^p(\Omega\times\mathbb{R}_+;Y)} \leq C\|G_\beta\|_{L^p(\Omega\times\mathbb{R}_+;\gamma(H,X))} = C\|G\|_{L^p(\Omega\times\mathbb{R}_+,w_\kappa;\gamma(H,X))}.$$

Moreover, by Lemma 3.4.5 one has

$$\|T_{K,\beta}G_\beta\|_{L^p(\Omega\times\mathbb{R}_+;Y)} \leq C\|G_\beta\|_{L^p(\Omega\times\mathbb{R}_+;\gamma(H,X))} = C\|G\|_{L^p(\Omega\times\mathbb{R}_+,w_\kappa;\gamma(H,X))}.$$

Then by (3.15) and the previous estimates,

$$\begin{aligned}\|T_K G\|_{L^p(\Omega\times\mathbb{R}_+,w_\kappa;Y)} &= \|t \mapsto t^\beta T_K G(t)\|_{L^p(\Omega\times\mathbb{R}_+;Y)}\\ &\leq \|T_K G_\beta\|_{L^p(\Omega\times\mathbb{R}_+;Y)} + \|T_{K,\beta}G_\beta\|_{L^p(\Omega\times\mathbb{R}_+;Y)}\\ &\leq 2C\|G\|_{L^p(\Omega\times\mathbb{R}_+,w_\kappa;\gamma(H,X))}.\end{aligned}$$

(2)⇒(1): Let $F_{-\beta}(s) = s^{-\beta}G(s)$ where $\beta = \kappa/p$. Similarly to (3.15), one has

$$T_K F(t) = t^\beta T_K F_{-\beta}(t) - T_{K,\beta}F(t).$$

As before, applying the assumption to $F_{-\beta}$ and Lemma 3.4.5 gives that

$$\begin{aligned}\|T_K F\|_{L^p(\Omega\times\mathbb{R}_+;Y)} &\leq \|t \mapsto t^\beta T_K F_{-\beta}(t)\|_{L^p(\Omega\times\mathbb{R}_+;Y)} + \|T_{K,\beta}F\|_{L^p(\Omega\times\mathbb{R}_+;Y)}\\ &= \|T_K F_{-\beta}\|_{L^p(\Omega\times\mathbb{R}_+,w_\kappa;Y)} + \|T_{K,\beta}F\|_{L^p(\Omega\times\mathbb{R}_+;Y)}\\ &\leq C\|F_{-\beta}\|_{L^p(\Omega\times\mathbb{R}_+,w_\kappa;\gamma(H,X))} + C''\|F\|_{L^p(\Omega\times\mathbb{R}_+;\gamma(H,X))}\\ &= (C + C'')\|F\|_{L^p(\Omega\times\mathbb{R}_+;\gamma(H,X))},\end{aligned}$$

from which the result follows. □

Proof of Theorem 3.4.3. If (1) holds, then by Theorem 3.2.1 the semigroup S generated by A is analytic. To see that (2) also implies analyticity of S, note that the statement of Lemma 3.2.2 still holds if instead we assume $A \in \mathrm{SMR}(p,\infty,\kappa)$. To see this one can repeat the argument given there by using $\kappa > -1$. Therefore, if (2) holds, then Proposition 3.2.5 implies that S is analytic.

By the analyticity of S, the operator-valued family $K : \Delta \to \mathscr{L}(X)$ defined by

$$K(t,s) := A^{\frac{1}{2}}S(t-s)$$

satisfies $\|K(t,s)\| \leq C/(t-s)^{1/2}$ for $t > s > 0$. Therefore, the equivalence of (1) and (2) follows from Theorem 3.4.4 with $X_0 = \mathsf{D}(A)$. □

3.4.1 Space-time regularity results

To state the last results of this section, we introduce a further class of operators. From now on we will assume $(S(t))_{t\geq 0}$ is exponentially stable. For $\theta \in [0, 1/2)$ we set

$$S_\theta(t) := \frac{t^{-\theta}}{\Gamma(1-\theta)}S(t)\,, \qquad t \geq 0.$$

Definition 3.4.6. *Let X be a UMD space with type 2, let $p \in [2,\infty)$, and $\theta \in [0,1/2)$ and assume $\omega_0(-A) < 0$. We say that operator A belongs to $\mathrm{SMR}_\theta(p,\infty)$ if for each $G \in L^p_{\mathscr{P}}(\Omega\times\mathbb{R}_+;\gamma(H,X))$ the stochastic convolution process*

$$S_\theta \diamond G(t) := \int_0^t S_\theta(t-s)G(s)\,dW_H(s)\,,$$

is well-defined in X, takes values in $\mathsf{D}(A^{1/2-\theta})$ $\mathbb{P}\times dt$-a.e. and satisfies

$$\|S_\theta \diamond G\|_{L^p(\Omega\times\mathbb{R}_+;\mathsf{D}(A^{\frac{1}{2}-\theta}))} \leq C\|G\|_{L^p(\Omega\times\mathbb{R}_+;\gamma(H,X))}.$$

for some $C > 0$ independent of G.

By definition, we have $\mathrm{SMR}_0(p,\infty) = \mathrm{SMR}(p,\infty)$.

The following important remark gives sufficient conditions for $A \in \mathrm{SMR}_\theta(p,\infty)$ which reduces to Theorem 3.1.6 if $\theta = 0$.

Remark 3.4.7. It was shown in [166, 167, 168] that, if X satisfies Assumption 3.1.5, $0 \in \rho(A)$ and A has a bounded H^∞-calculus of angle $< \pi/2$ then $A \in \mathrm{SMR}_\theta(p,\infty)$ for any $\theta \in [0,1/2)$ and $p \in (2,\infty)$. In addition, if $q = 2$, then $A \in \mathrm{SMR}_\theta(p,\infty)$ for any $p \in [2,\infty)$. Lastly, the assumption $0 \in \rho(A)$ can be avoided using a homogeneous version of $\mathrm{SMR}_\theta(p,\infty)$ (see [166, Theorem 4.3]).

Before going further, we make the following observation:

Proposition 3.4.8. *Let X be a UMD space with type 2 and let $p \in [2,\infty)$. Let $A \in \mathrm{SMR}_\theta(p,\infty)$ be such that $\omega_0(-A) < 0$ and A is an R-sectorial operator of angle $\omega_R(A) < \pi/2$. Then, for any $0 \leq \psi < \theta < 1/2$, we have $A \in \mathrm{SMR}_\psi(p,\infty)$.*

Proof. First observe that an analogue of Proposition 3.1.7 for $\mathrm{SMR}_\theta(p,\infty)$ holds and we will use it in the proof below. By [111, Lemma 3.3] (or [108, Proposition 10.3.2]) the set $\{(sA)^{\theta-\psi}S(s/2) : s>0\}$ is R-bounded and hence γ-bounded (see [108, Theorem 8.1.3(2)]). Therefore, by the γ-multiplier theorem (see [108, Theorem 9.5.1]) we obtain

$$\|s\mapsto A^{1/2-\psi}S_\psi(t-s)G(s)\|_{\gamma(0,t;H,X)} \le C\|s\mapsto A^{1/2-\theta}S_\theta((t-s)/2)G(s)\|_{\gamma(0,t;H,X)}.$$

Taking L^p-norms on both sides we find that

$$\begin{aligned}
&\int_0^\infty \|s\mapsto A^{1/2-\psi}S_\psi(t-s)G(s)\|^p_{\gamma(0,t;H,X)}\,dt\\
&\qquad\le C^p\int_0^\infty \|s\mapsto A^{1/2-\theta}S_\theta((t-s)/2)G(s)\|^p_{\gamma(0,t;H,X)}dt\\
&\qquad= \frac{C^p}{2}\int_0^\infty \|s\mapsto A^{1/2-\theta}S_\theta((2\tau-s)/2)G(s)\|^p_{\gamma(0,2\tau;H,X)}d\tau\\
&\qquad\le 2^{\frac{p}{2}-1}C^p\int_0^\infty \|\sigma\mapsto A^{1/2-\theta}S_\theta(\tau-\sigma)G(2\sigma)\|^p_{\gamma(0,\tau;H,X)}d\tau\\
&\qquad\le 2^{\frac{p}{2}-1}C^pK^p\|G\|_{L^p(\Omega\times\mathbb{R}_+;\gamma(H,X))}.
\end{aligned}$$

where we only used elementary substitutions and in the last step we used the assumption applied to the function $G(2\cdot)$. ☐

The following proposition is the analogue of Theorem 3.4.3 for the class $\mathrm{SMR}_\theta(p,\infty)$.

Proposition 3.4.9. *Let X be a UMD space with type 2. Assume $\omega_0(-A)<0$ and S is an analytic semigroup. Let $p\in[2,\infty)$, $\kappa\in(-1,\frac{p}{2}-1)$ and $\theta\in[0,1/2)$. Then the following are equivalent:*

(1) $A\in \mathrm{SMR}_\theta(p,\infty)$.

(2) *There is a constant $C>0$ such that for all $G\in L^p_{\mathscr{P}}(\Omega\times\mathbb{R}_+,w_\kappa;\gamma(H,X))$ we have $S_\theta\diamond G(t)\in \mathsf{D}(A^{\frac12-\theta})$ $\mathbb{P}\times dt$-a.e. and*

$$\|S_\theta\diamond G\|_{L^p(\Omega\times\mathbb{R}_+,w_\kappa;\mathsf{D}(A^{\frac12-\theta}))}\le C\|G\|_{L^p(\Omega\times\mathbb{R}_+,w_\kappa;\gamma(H,X))}.$$

Proof. Let $K_\theta:\Delta\to\mathscr{L}(X)$ be defined by $K_\theta(t,s)=A^{\frac12-\theta}(t-s)^{-\theta}S(t-s)$. By analyticity of the semigroup $(S(t))_{t\ge0}$, one has $\|K_\theta(t,s)\|\le C/(t-s)^{1/2}$ for $t>s>0$, and thus the result follows from Theorem 3.4.4 in the same way as in Theorem 3.4.3. ☐

We are ready to prove the main result of this section. Recall from Remark 3.4.7 that all the conditions are satisfied if X is isomorphic to a closed subspace of L^q with $q\in[2,\infty)$, $0\in\rho(A)$ and A has a bounded H^∞-calculus of angle $<\pi/2$.

Theorem 3.4.10. *Let X be a UMD space with type 2. Assume $\omega_0(A)<0$, $A\in \mathrm{BIP}(X)$ with $\theta_A<\pi/2$. Let $p\in(2,\infty)$, let $\kappa\in(-1,\frac{p}{2}-1)$ (or $p=2$ and $\kappa=0$) and let $\theta\in[0,\frac12)$. Assume that $A\in\mathrm{SMR}_\theta(p,\infty)$.*

(1) (Space-time regularity) *Then*

$$\mathbb{E}\|S\diamond G\|^p_{H^{\theta,p}(\mathbb{R}_+,w_\kappa;\mathsf{D}(A^{\frac12-\theta}))}\le C^p\,\mathbb{E}\|G\|^p_{L^p(\mathbb{R}_+,w_\kappa;\gamma(H,X))}.$$

(2) (Maximal estimates) *If $\kappa\ge0$ and $\theta>\frac{1+\kappa}{p}$, then*

$$\mathbb{E}\sup_{t\in\mathbb{R}_+}\|S\diamond G(t)\|^p_{\mathsf{D}_A(\frac12-\frac{1+\kappa}{p},p)}\le C^p\,\mathbb{E}\|G\|^p_{L^p(\mathbb{R}_+,w_\kappa;\gamma(H,X))}.$$

(3) (Parabolic regularization) *If $\kappa \geq 0$ and $\theta > \frac{1}{p}$, then for any $\delta > 0$*

$$\mathbb{E} \sup_{t\in[\delta,\infty)} \|S \diamond G(t)\|^p_{\mathsf{D}_A\left(\frac{1}{2}-\frac{1}{p},p\right)} \leq C^p \, \mathbb{E}\|G\|^p_{L^p(\mathbb{R}_+,w_\kappa;\gamma(H,X))}.$$

In all cases the constant C is independent of G.

Proof. To prepare the proof, we collect some useful facts. Let $\mathscr{A}$ be the closed and densely defined operator on $L^p(\mathbb{R}_+, w_\kappa; X)$ with domain $\mathsf{D}(\mathscr{A}) := L^p(\mathbb{R}_+, w_\kappa; \mathsf{D}(A))$ defined by

$$(\mathscr{A} f)(t) := Af(t);$$

since $A \in \mathrm{BIP}(X)$ then also $\mathscr{A} \in \mathrm{BIP}(L^p(\mathbb{R}_+, w_\kappa; X))$ and $\theta_{\mathscr{A}} = \theta_A < \pi/2$. Moreover, $0 \in \rho(\mathscr{A})$ since $0 \in \rho(A)$. Let $\mathscr{B}$ be the closed and densely defined operator on $L^p(\mathbb{R}_+, w_\kappa; X)$ with domain $\mathsf{D}(\mathscr{B}) := {}_0W^{1,p}(\mathbb{R}_+, w_\kappa; X)$ given by (see (2.7))

$$\mathscr{B} f := f'.$$

By [145, Theorem 6.8], $\mathscr{B}$ has a bounded H^∞-calculus of angle $\omega_{H^\infty}(\mathscr{B}) = \pi/2$; in particular $\theta_{\mathscr{B}} \leq \pi/2$. Since $\theta_{\mathscr{A}} + \theta_{\mathscr{B}} < \pi$, by [179, Theorems 4 and 5] the operator

$$\mathscr{C} := \mathscr{A} + \mathscr{B}, \qquad \mathsf{D}(\mathscr{C}) := \mathsf{D}(\mathscr{A}) \cap \mathsf{D}(\mathscr{B}),$$

is an invertible sectorial on $L^p(\mathbb{R}_+, w_\kappa; X)$, moreover has bounded imaginary powers with $\theta_{\mathscr{C}} \leq \pi/2$. By [26, Proposition 3.1] one has

$$(\mathscr{C}^{-\gamma} f)(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} S(t-s) f(s)\, ds. \tag{3.16}$$

Moreover, for all $\gamma \in (0, 1]$ one has (see [77, Lemma 9.5(b)])

$$\begin{aligned} \mathsf{D}(\mathscr{C}^\gamma) &= [L^p(\mathbb{R}_+, w_\kappa; X), \mathsf{D}(\mathscr{B})]_\gamma \cap [L^p(\mathbb{R}_+, w_\kappa; X), \mathsf{D}(\mathscr{A})]_\gamma \\ &= H_0^{\gamma,p}(\mathbb{R}_+, w_\kappa; X) \cap L^p(\mathbb{R}_+, w_\kappa; \mathsf{D}(A^\gamma)), \end{aligned} \tag{3.17}$$

where in the last equality we have used Definition 2.2.1. To prove (1)-(3), by a density argument, it suffices to consider an adapted rank step process $G : [0, \infty) \times \Omega \to \gamma(H, \mathsf{D}(A))$.

(1): By the Da Prato–Kwapień–Zabczyk factorization argument (see [26] and [50, Section 5.3] and references therein), using (3.16) for $\gamma = \theta$, the stochastic Fubini theorem and the equality

$$\frac{1}{\Gamma(\theta)\Gamma(1-\theta)} \int_r^t (t-s)^{\theta-1}(s-r)^{-\theta}\, ds = 1$$

one obtains, for all $t \in \mathbb{R}_+$,

$$\mathscr{C}^{-\theta}(A^{\frac{1}{2}-\theta} S_\theta \diamond G)(t) = A^{\frac{1}{2}-\theta} S \diamond G(t) \ \text{ almost surely.} \tag{3.18}$$

Then,

$$\begin{aligned} \|A^{\frac{1}{2}-\theta} S \diamond G\|_{L^p(\Omega;H^{\theta,p}(\mathbb{R}_+,w_\kappa;X))} &\overset{(i)}{\leq} C \|\mathscr{C}^\theta A^{\frac{1}{2}-\theta} S \diamond G\|_{L^p(\Omega\times\mathbb{R}_+,w_\kappa;X)} \\ &\overset{(ii)}{=} C \|A^{\frac{1}{2}-\theta} S_\theta \diamond G\|_{L^p(\Omega\times\mathbb{R}_+,w_\kappa;X)} \\ &\overset{(iii)}{\leq} C' \|G\|_{L^p(\Omega\times\mathbb{R}_+,w_\kappa;\gamma(H,X))}, \end{aligned}$$

where in (i) we have used (3.17) and (2.8), in (ii) (3.18) and in (iii) we used Proposition 3.4.9.

(2): By Proposition 2.2.5(1), we have

$$H^{\theta,p}(\mathbb{R}_+, w_\kappa; X) \cap L^p(\mathbb{R}_+, w_\kappa; \mathsf{D}(A^\theta)) \hookrightarrow C_0\Big([0,\infty); \mathsf{D}_A\Big(\theta - \frac{1+\kappa}{p}, p\Big)\Big).$$

Moreover, since $A \in \mathrm{BIP}(X)$ with $\theta_B < \pi/2$ then $\omega_R(A) < \pi/2$ thus $-A$ generates an analytic semigroup on X (see Remark 2.1.4). Setting $\zeta_\lambda = A^\lambda S \diamond G$, by Proposition 3.4.9 and the fact that $0 \in \varrho(A)$, one has

$$\begin{aligned}
&\|\zeta_{\frac{1}{2}-\theta}\|_{L^p\left(\Omega;C_0\left([0,\infty);\mathsf{D}_A\left(\theta-\frac{1+\kappa}{p},p\right)\right)\right)} \\
&\qquad \leq K\|\zeta_{\frac{1}{2}-\theta}\|_{L^p(\Omega;H^{\theta,p}([0,\infty),w_\kappa;X))} + K\|\zeta_{\frac{1}{2}-\theta}\|_{L^p(\Omega;L^p(\mathbb{R}_+,w_\kappa;\mathsf{D}(A^\theta)))} \\
&\qquad = K\|\zeta_{\frac{1}{2}-\theta}\|_{L^p(\Omega;H^{\theta,p}(\mathbb{R}_+,w_\kappa;X))} + K\|\zeta_{\frac{1}{2}}\|_{L^p(\Omega\times\mathbb{R}_+,w_\kappa;X))} \\
&\qquad \leq CK\|G\|_{L^p(\Omega\times\mathbb{R}_+,w_\kappa;\gamma(H,X))}.
\end{aligned} \tag{3.19}$$

Since $A^{\frac{1}{2}-\theta} : \mathsf{D}_A\big(\frac{1}{2} - \frac{1+\kappa}{p}, p\big) \to \mathsf{D}_A\big(\theta - \frac{1+\kappa}{p}, p\big)$ is an isomorphism (see [197, Theorem 1.15.2 (e)]), we have

$$\begin{aligned}
\|S \diamond G\|_{L^p\left(\Omega;C_0\left([0,\infty);\mathsf{D}_A\left(\frac{1}{2}-\frac{1+\kappa}{p},p\right)\right)\right)} &\eqsim_{A,\theta,p} \|\zeta_{\frac{1}{2}-\theta}\|_{L^p\left(\Omega;C_0\left([0,\infty);\mathsf{D}_A\left(\theta-\frac{1+\kappa}{p},p\right)\right)\right)} \\
&\leq CK\|G\|_{L^p(\Omega\times\mathbb{R}_+,w_\kappa;\gamma(H,X))};
\end{aligned}$$

where in the last inequality we have used (3.19).

(3): This follows from the same argument as in (2) using Proposition 2.2.5(2) instead of Proposition 2.2.5(1). □

Remark 3.4.11. Similar to [166, Remark 5.1] (see also the references therein), Theorem 3.4.10 can be localized via a standard stopping time argument. For future references, we give the explicit formulation for Theorem 3.4.10(3):

Let $\theta > \frac{1}{p}$, $\kappa \in [0, \frac{p}{2} - 1)$, $A \in \mathrm{SMR}_\theta(p,\infty)$ and let $\tau > 0$ be a stopping time then for any $G \in L^0_{\mathscr{P}}(\Omega; L^p((\tau,\infty), w_\kappa; \gamma(H,X)))$,

$$S \diamond G \in L^0\Big(\Omega; C_0\Big((\tau;\infty); \mathsf{D}_A\Big(\frac{1}{2} - \frac{1}{p}, p\Big)\Big)\Big).$$

Chapter 4

Local existence for stochastic evolution equations in critical spaces

In this chapter X_0, X_1 denote Banach spaces with UMD and type 2 such that $X_1 \hookrightarrow X_0$ densely, H denotes an Hilbert space with dimension $\dim H \geq 1$, and $X_\theta := [X_0, X_1]_\theta$ and $X^{\mathsf{Tr}}_{\kappa,p} = (X_0, X_1)_{1-\frac{1+\kappa}{p},p}$ (see Assumption 4.2.1). Here $(\Omega, \mathbb{F} = (\mathscr{F}_t)_{t\geq 0}, \mathscr{A}, \mathbb{P})$ and $\mathscr{P}$ denote an underlying complete filtered probability space and $\mathscr{P}$ the progressive sigma algebra, respectively. Moreover we assume that $\mathscr{F}_0$ contains all the $\mathbb{P}$-null sets in $\mathscr{A}$.

The aim of this chapter is to give a new and systematic treatment of the well-posedness of semilinear and quasilinear parabolic evolution equations of the form

$$\begin{cases} du + A(t,u)u dt = F(t,u)dt + (B(t,u) + G(t,u))dW_H, \quad t \in \mathbb{R}_+, \\ u(0) = u_0. \end{cases} \tag{4.1}$$

The nonlinearities F and G decompose as $F = F_{\mathsf{Tr}} + F_c + F_L$ and $G = G_{\mathsf{Tr}} + G_c + G_L$ where $F_{\mathsf{Tr}}, G_{\mathsf{Tr}}$ are locally Lipschitz on the 'trace space' $X^{\mathsf{Tr}}_{\kappa,p}$, F_L, G_L are globally Lipschitz nonlinearities on X_1, and F_c and G_c are locally Lipschitz with polynomial growth maps defined on $[X_0, X_1]_\varphi$ with $\varphi \in (0,1)$. The growth ρ and the 'roughness in space' φ of the nonlinearities F_c, G_c have to satisfy the following relation (cf. (4.18) and (4.20))

$$\rho\Big(\varphi - 1 + \frac{1+\kappa}{p}\Big) + \varphi \leq 1. \tag{4.2}$$

If in (4.2) the equality holds, then we say that $X^{\mathsf{Tr}}_{\kappa,p}$ is *critical* for (4.1) and in applications to SPDEs such spaces turn out to enjoy the right (local) scaling of the equations under study.

This chapter is organised as follows. In Section 4.1 we provide additional preliminary results on anisotropic function spaces with power weights and we define suitable spaces of stopped processes. In Section 4.2 we introduce and present several results on a generalized notion of weighted stochastic maximal L^p-regularity. The set of all *couples* (A, B) having (the generalized) maximal L^p-regularity will be denoted by either $\mathcal{SMR}_{p,\kappa}(T)$ or $\mathcal{SMR}^{\bullet}_{p,\kappa}(T)$ in the case that sharp space-time estimates also hold. For the classes just introduced we prove the following transference result

$$\text{if } (A,B) \in \mathcal{SMR}_{p,\kappa}(T) \text{ and } \mathcal{SMR}^{\bullet}_{p,\kappa}(T) \neq \varnothing \quad \Rightarrow \quad (A,B) \in \mathcal{SMR}^{\bullet}_{p,\kappa}(T),$$

which will be employed several times in applications. In Section 4.3, after introducing the notion of L^p_κ-solutions to (4.1) we state and prove a local existence result for (4.1). For the reader's convenience, we give the basic idea behind the proof of Theorem 4.3.5 which motivates the proof of several lemmas proven in Section 4.3. The idea is to linearize the quasilinear part of the equation at the initial data, i.e. writing $A(t,u) = A(t,u_0) + (A(t,u) - A(t,u_0))$, and $B(t,u) =$

$B(t,u_0)+(B(t,u)-B(t,u_0))$ and to consider the following truncatation of (4.1)

$$\begin{cases} du + A(t,u_0)udt = \Lambda(u,u_0;t)(F(t,u)+A(t,u_0)-A(t,u))dt \\ \qquad\qquad +[B(t,u_0)+\Lambda(t,u_0,u)(G(t,u)+B(t,u)-B(t,u_0))]dW_H, \quad t\in\mathbb{R}_+, \\ u(0)=u_0, \end{cases} \tag{4.3}$$

where $\Lambda(t,u,u_0)=1$ provided u and $u-u_0$ are small (say less than 1) in $L^p(I_T\times\Omega,w_\kappa;X_1)\cap\mathfrak{X}(T)$- and $C([0,T];X^{\mathsf{Tr}}_{\kappa,p})$-norms, respectively. Here the space $\mathfrak{X}$ is a suitable intersection of Lebesgue spaces which is designed to control the nonlinearities F_c and G_c (see Subsection 4.3.3). Defining the stopping times $\sigma:=\inf\{t\in[0,T]\,:\,\|u\|_{L^p(I_T\times\Omega,w_\kappa;X_1)\cap\mathfrak{X}(T)}+\|u-u_0\|_{C([0,T];X^{\mathsf{Tr}}_{\kappa,p})}\geq 1\}$, we check that the stopped process $u|_{[0,\sigma)\times\Omega}$ is a local solution to (4.1). The above strategy requires the study of the truncated nonlinearity appearing in (4.3). For technical reasons, we use the decompositions $F=F_{\mathsf{Tr}}+F_c+F_L$, $G=G_{\mathsf{Tr}}+G_c+G_L$ and we estimate the truncations separately.

The results in this chapter are taken from Sections 2-4 of my work [3].

4.1 Preliminaries

4.1.1 Embedding results for Sobolev spaces with power weights

In this subsection we collect some basic embedding results for the spaces introduced in Subsection 2.2. To begin, let us introduce Sobolev embeddings and interpolation inequalities for $H^{s,p}$. Some of the following results might also hold for general Banach spaces, but since we will use the UMD property many times we prefer the presentation below. Note that the difficulty in the proofs below is that we want estimates with T-independent constants as this is required in fixed point arguments below.

The following result on vector-valued Sobolev spaces follows from [145, sections 5 and 6]. The scalar unweighted case is simpler, and in that case the result is a special case of [187].

Theorem 4.1.1. *Let X be a UMD space, $p\in(1,\infty)$, $\kappa\in(-1,p-1)$, $s\in(0,1)$, and $I\in\{\mathbb{R},\mathbb{R}_+\}$. If $s\neq\frac{1+\kappa}{p}$, then*

$${}_0H^{s,p}(I,w_\kappa;X)=\begin{cases}\{u\in H^{s,p}(I,w_\kappa;X):u(0)=0\}, & \text{if } s>\frac{1+\kappa}{p},\\ H^{s,p}(I,w_\kappa;X), & \text{if } s<\frac{1+\kappa}{p},\end{cases}$$

isomorphically.

By using the extension operator of Proposition 2.2.4 one can see that Theorem 4.1.1 extends to $I=(0,T)$ with $T\in(0,\infty)$. In particular, if $s\neq\frac{1+\kappa}{p}$, then ${}_0H^{s,p}(I,w_\kappa;X)$ is a closed subspace of $H^{s,p}(I,w_\kappa;X)$. As a consequence the estimate $\|u\|_{{}_0H^{s,p}(I,w_\kappa;X)}\eqsim\|u\|_{H^{s,p}(I,w_\kappa;X)}$ holds, where we need the condition $u(0)=0$ if $s>\frac{1+\kappa}{p}$. The theorem will usually be applied through the latter norm equivalence.

Proposition 4.1.2 (Sobolev embedding)**.** *Let X be a UMD Banach space. Let $T\in(0,\infty]$ and set $I_T=(0,T)$. Assume that $1<p_0\leq p_1<\infty$, $s_0,s_1\in(0,1)$ and $\kappa_i\in(-1,p_i-1)$ for $i\in\{0,1\}$. Assume $\frac{\kappa_1}{p_1}\leq\frac{\kappa_0}{p_0}$ and $s_0-\frac{1+\kappa_0}{p_0}\geq s_1-\frac{1+\kappa_1}{p_1}$. Then there is a constant C independent of T such that for all $f\in{}_0H^{s_0,p_0}(I_T,w_{\kappa_0};X)$,*

$$\|f\|_{{}_0H^{s_1,p_1}(I_T,w_{\kappa_1};X)}\leq C\|f\|_{{}_0H^{s_0,p_0}(I_T,w_{\kappa_0};X)}.$$

The same holds with ${}_0H^{s_i,p_i}(I_T,w_{\kappa_i};X)$ replaced by $H^{s_i,p_i}(I_T,w_{\kappa_i};X)$ with a constant C which depends on T.

Proof. First assume $s_1\neq\frac{1+\kappa_1}{p_1}$. Let ${}_0\mathsf{E}_T$ be as in Proposition 2.2.4(1). Then

$$\|f\|_{{}_0H^{s_1,p_1}(I_T,w_{\kappa_1};X)}\leq\|{}_0\mathsf{E}_Tf\|_{{}_0H^{s_1,p_1}(\mathbb{R},w_{\kappa_1};X)}.$$

where we used Proposition 2.2.2 for ${}_0\mathsf{E}_T f$. By Theorem 4.1.1 it remains to estimate the term $\|{}_0\mathsf{E}_T f\|_{H^{s_1,p_1}(\mathbb{R},w_{\kappa_1};X)}$. By [157, Propositions 3.2 and 3.7], $\|{}_0\mathsf{E}_T f\|_{H^{s_i,p_i}(\mathbb{R},w_{\kappa_i};X)}$ is equivalent to $\|{}_0\mathsf{E}_T f\|_{\mathscr{H}^{s_i,p_i}(\mathbb{R},w_{\kappa_i};X)}$, where $\mathscr{H}$ denotes the Bessel potential space. Therefore, by the weighted Sobolev embedding result [154, Corollary 1.4] we obtain

$$\|{}_0\mathsf{E}_T f\|_{H^{s_1,p_1}(\mathbb{R},w_{\kappa_1};X)} \lesssim \|{}_0\mathsf{E}_T f\|_{H^{s_0,p_0}(\mathbb{R},w_{\kappa_0};X)}.$$

By (2.8) and Proposition 2.2.4(1) we obtain

$$\|{}_0\mathsf{E}_T f\|_{H^{s_0,p_0}(\mathbb{R},w_{\kappa_0};X)} \leq \|{}_0\mathsf{E}_T f\|_{{}_0H^{s_0,p_0}(\mathbb{R},w_{\kappa_0};X)} \lesssim \|f\|_{{}_0H^{s_0,p_0}(I_T,w_{\kappa_0};X)},$$

and the result follows by combining the estimates.

In the case $s_1 - \frac{1+\kappa_1}{p_1} = 0$ we use an interpolation argument. Let $\varepsilon > 0$ be so small that $s_j^{\pm} := s_j \pm \varepsilon \in (0,1)$. Then by the previous considerations

$${}_0H^{s_0^{\pm},p_0}(I_T,w_{\kappa_0};X) \hookrightarrow {}_0H^{s_1^{\pm},p_1}(I_T,w_{\kappa_1};X),$$

where the embedding constants can be taken T-independent. Interpolating both embeddings gives the desired embedding in the remaining case.

The final assertion can be proved with the same method, but one can avoid Theorem 4.1.1. Moreover, one needs to use the extension operator on $H^{s,p}$ spaces provided by Proposition 2.2.4. ☐

Next we prove a version of the mixed derivative result [146, Theorem 3.18], but with T-independent estimates.

Proposition 4.1.3 (Mixed derivative inequality). *Let (X_0, X_1) be an interpolation couple such that both X_0 and X_1 are UMD spaces. Let $p_i \in (1,\infty)$, $\kappa_i \in (-1, p_i - 1)$, and $s_i \in (0,1)$ for $i \in \{0,1\}$. For $\theta \in (0,1)$ set*

$$s := s_0(1-\theta) + s_1\theta, \quad \frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \kappa = (1-\theta)\frac{p}{p_0}\kappa_0 + \theta\frac{p}{p_1}\kappa_1.$$

Assume $T \in (0,\infty]$ and $s \neq \frac{1+\kappa}{p}$. Then there exists a constant $C > 0$ independent of $T \in (0,\infty]$ such that for all $f \in {}_0H^{s_0,p_0}(I_T,w_{\kappa_0};X_0) \cap {}_0H^{s_1,p_1}(I_T,w_{\kappa_1};X_1)$,

$$\|f\|_{{}_0H^{s,p}(I_T,w_\kappa;[X_0,X_1]_\theta)} \leq C\|f\|^{1-\theta}_{{}_0H^{s_0,p_0}(I_T,w_{\kappa_0};X_0)}\|f\|^{\theta}_{{}_0H^{s_1,p_1}(I_T,w_{\kappa_1};X_1)}.$$

The same holds with ${}_0H^{s_i,p_i}(I_T,w_{\kappa_i};X_i)$ replaced by $H^{s_i,p_i}(I_T,w_{\kappa_i};X_i)$ with a constant C which depends on T in which case $s = \frac{1+\kappa}{p}$ is also allowed.

Proof. Let ${}_0\mathsf{E}_T$ be as in Proposition 2.2.4(1). By construction (see Proposition 2.2.4) ${}_0\mathsf{E}_T$ does not depend on p_i, κ_i, s_i, X_i. Therefore, Proposition 2.2.2 gives

$$\|f\|_{{}_0H^{s,p}(I_T,w_\kappa;[X_0,X_1]_\theta)} \leq \|{}_0\mathsf{E}_T f\|_{{}_0H^{s,p}(\mathbb{R},w_\kappa;[X_0,X_1]_\theta)}.$$

Since $s \neq \frac{1+\kappa}{p}$, by Theorem 4.1.1 it suffices to estimate $\|{}_0\mathsf{E}_T f\|_{H^{s,p}(\mathbb{R},w_\kappa;[X_0,X_1]_\theta)}$. The interpolation result [146, Theorem 3.18] implies

$$\|{}_0\mathsf{E}_T f\|_{H^{s,p}(\mathbb{R},w_\kappa;[X_0,X_1]_\theta)} \leq C\|{}_0\mathsf{E}_T f\|^{1-\theta}_{H^{s_0,p_0}(\mathbb{R},w_{\kappa_0};X_0)}\|{}_0\mathsf{E}_T f\|^{\theta}_{H^{s_1,p_1}(\mathbb{R},w_{\kappa_1};X_1)}.$$

As in the proof of Proposition 4.1.2 one can check that

$$\|{}_0\mathsf{E}_T f\|_{H^{s_i,p_i}(\mathbb{R},w_{\kappa_i};X_i)} \leq \|{}_0\mathsf{E}_T f\|_{{}_0H^{s_i,p_i}(\mathbb{R},w_{\kappa_i};X_i)} \lesssim \|f\|_{{}_0H^{s_i,p_i}(I_T,w_{\kappa_i};X_i)},$$

and we can conclude the required embedding holds.

The final assertion can be proved in a similar way. ☐

Remark 4.1.4. It is to be expected that combining the methods of [145] with [146, Theorem 3.18], Proposition 4.1.3 can be improved to

$$[{}_0H^{s_0,p_0}(\mathbb{R}_+,w_{\kappa_0};X_0),{}_0H^{s_1,p_1}(\mathbb{R}_+,w_{\kappa_1};X_1)]_\theta={}_0H^{s,p}(\mathbb{R}_+,w_\kappa;[X_0,X_1]_\theta) \tag{4.4}$$

under the condition $s\neq\frac{1+\kappa}{p}$. In the case that $s=\frac{1+\kappa}{p}$, we expect the embedding

$${}_0H^{s_0,p}(I_T,w_\kappa;X_0)\cap{}_0H^{s_1,p}(I_T,w_\kappa;X_1)\hookrightarrow{}_0H^{s,p}(I_T,w_\kappa;[X_0,X_1]_\theta)$$

to be valid with T-independent constants as well. This could be proved by a reiteration and interpolation argument using (4.4).

We conclude this section by recalling an optimal trace result for anisotropic spaces. This result is a special case of the trace embedding of [2]. In the case that $X_1=\mathsf{D}(A)$ where $A\in\mathrm{BIP}$ and $0\in\rho(A)$, the following is a consequence of Proposition 2.2.5. Moreover, the UMD condition can be avoided. As above, for an interval $J\subseteq\mathbb{R}_+$ and a Banach space X, we denote by $C_0(\overline{J};X)$ the set of all continuous functions $f:\overline{J}\to X$ vanishing at infinity endowed with the norm given by the right-hand side of (2.1).

Proposition 4.1.5. *Let (X_0,X_1) be a couple of Banach space such that $X_1\hookrightarrow X_0$. Set $X_{1-\theta}:=[X_0,X_1]_{1-\theta}$ or $X_{1-\theta}=(X_0,X_1)_{1-\theta,r}$ with $r\in[1,\infty]$. Assume that $p\in(1,\infty)$, $\kappa\in[0,p-1)$, $\theta\in(0,1)$ and $T\in(0,\infty]$. Then the following holds:*

(1) *If $\theta>\frac{1+\kappa}{p}$, then*

$$H^{\theta,p}(I_T,w_\kappa;X_{1-\theta})\cap L^p(I_T,w_\kappa;X_1)\hookrightarrow C_0\big(\overline{I}_T;(X_0,X_1)_{1-\frac{1+\kappa}{p},p}\big);$$

(2) *If $\theta>\frac{1}{p}$, then for any $0<\varepsilon<T$ and $J_{\varepsilon,T}=(\varepsilon,T)$*

$$H^{\theta,p}(I_T,w_\kappa;X_{1-\theta})\cap L^p(I_T,w_\kappa;X_1)\hookrightarrow C_0\big(\overline{J}_{\varepsilon,T};(X_0,X_1)_{1-\frac{1}{p},p}\big).$$

Moreover, the constants in (1) and (2) depend only on η if $T\in(\eta,\infty]$. Furthermore, if we replace $H^{\theta,p}$ by ${}_0H^{\theta,p}$ in (1) and (2) the constants in the embeddings can be chosen independent of $T>0$.

Here (1) follows from the above mentioned references and Proposition 2.2.4. To prove (2) one can reduce to (1) with $\kappa=0$ by Proposition 2.2.2 and a translation argument. To prove the embeddings (1) and (2) for ${}_0H^{\theta,p}$ by Proposition 2.2.4 it suffices to consider the case $T=\infty$ in which case the result follows from (1) for $H^{\theta,p}$.

4.1.2 Stochastic setting

A stopping time τ is a measurable map $\tau:\Omega\to[0,T]$ such that $\{\tau\leq t\}\in\mathscr{F}_t$ for all $t\in[0,T]$. We denote by $[\![0,\sigma]\!]$ the stochastic interval

$$[\![0,\sigma]\!]:=\{(t,\omega)\in I_T\times\Omega\,:\,0\leq t\leq\sigma(\omega)\}.$$

Analogously definitions hold for $[\![0,\sigma)\!)$, $(\!(0,\sigma)\!)$ etc.

In accordance with the previous notation, for $A\subseteq\Omega$ and τ,μ two stopping times such that $\tau\leq\mu$, we set

$$[0,T]\times\Omega\supseteq[\tau,\mu]\times A:=\{(t,\omega)\in[0,T]\times A\,:\,\tau(\omega)\leq t\leq\mu(\omega)\}.$$

In particular, $[\![0,\sigma]\!]=[0,\sigma]\times\Omega$.

Let X be a Banach space and let $A\in\mathcal{A}$. We say that $u:[0,\mu]\times A\to X$ is strongly measurable (resp. strongly progressively measurable) if the process

$$\mathbf{1}_{[0,\mu]\times A}u:=\begin{cases}u, & \text{on } [0,\mu]\times A,\\ 0, & \text{otherwise},\end{cases} \tag{4.5}$$

is strongly measurable (resp. strongly progressively measurable).

To each stopping time τ we can associate the σ-algebra of the τ-past,

$$\mathscr{F}_\tau := \{A \in \mathcal{A} : \{\tau \leq t\} \cap A \in \mathscr{F}_t, \ \forall t \in [0,T]\}.$$

The following well-known results will be used frequently in the thesis without further mentioning (see [110, Lemmas 7.1 and 7.5]).

Proposition 4.1.6. *Let τ be a stopping time. Then $\mathscr{F}_\tau$ is a σ-algebra and satisfies the following properties.*

- *If $\tau = t$ a.s. for some $t \in [0,T]$, then $\mathscr{F}_\tau = \mathscr{F}_t$.*
- *If $X : [0,T] \times \Omega \to X$ is a strongly progressively measurable process, then the random variable $X_\tau(\omega) := X(\tau(\omega), \omega)$ is strongly $\mathscr{F}_\tau$-measurable.*

We continue with another measurability lemma.

Lemma 4.1.7. *Let X be a Banach space. For each $t \in [0,T]$, let Y_t be a space of functions $f : [0,t] \to X$. Assume that for each $f \in Y_T$ and each $t \in [0,T]$,*

- *$f|_{[0,t]} \in Y_t$;*
- *$t \mapsto \|f|_{[0,t]}\|_{Y_t}$ is increasing;*

Let $u : \Omega \to Y_T$ be strongly measurable and τ be a stopping time. Then $\omega \mapsto \|u(\omega)|_{[0,\tau(\omega)]}\|_{Y_{\tau(\omega)}}$ is measurable.

Proof. Since u is strongly measurable, we may assume that Y_T is separable.

Let $\Psi : [0,T] \times Y_T \to [0,\infty)$ be given by $\Psi(t,f) = \|f|_{[0,t]}\|_{Y_t}$. Then since for $f \in Y_T$, $\Psi(\cdot, f)$ is increasing, it follows that $\Psi(\cdot, f)$ is measurable. For $t \in [0,T]$ and $f, g \in Y_T$,

$$|\Psi(t,f) - \Psi(t,g)| \leq \|(f-g)|_{[0,t]}\|_{Y_t} \leq \|f - g\|_{Y_T}.$$

Therefore, $\Psi(t, \cdot)$ is continuous. Since Y_T is separable this implies Ψ is measurable (see [9, Lemma 4.51]).

On the other hand, $\zeta : \Omega \to [0,T] \times Y_T$ defined by $\zeta(\omega) = (\tau(\omega), u(\omega))$ is measurable. Since $\|u(\omega)|_{[0,\tau(\omega)]}\|_{Y_{\tau(\omega)}} = \Psi(\zeta(\omega)) = (\Psi \circ \zeta)(\omega)$ the required measurability follows. ☐

The lemma will be applied to the spaces Y_t such as

$$C([0,t]; X), L^p(0,t,w_\kappa; X), H^{\theta,p}(I_t, w_\kappa; X), {}_0H^{\theta,p}(I_t, w_\kappa; X).$$

The first two examples are simple because the norm is actually a continuous function of $t \in [0,T]$. In the cases $H^{\theta,p}$ and ${}_0H^{\theta,p}$ it is not obvious whether the norms are continuous in $t \in [0,T]$, but fortunately, they are increasing by Proposition 2.2.4.

The above lemma implies that the following versions of stopped spaces with stopped norms are well-defined.

Definition 4.1.8. *Let X be a Banach space. Let $T > 0$, $p, q \in (1,\infty)$, $r \in \{0\} \cup [1,\infty)$ and $\theta \in [0,1]$. Assume that τ is a stopping time such that $\tau : \Omega \to [0,T]$. Let $(Y_t)_{t\in[0,T]}$ be as in Lemma 4.1.7. We say that $u \in L^r_{\mathscr{P}}(\Omega; Y_\tau)$ if there exists a strongly progressively measurable $\widetilde{u} \in L^r(\Omega; Y_T)$ such that $\widetilde{u}|_{[\![0,\tau]\!]} = u$. If in addition $r \in [1,\infty)$, we set*

$$\|u\|^r_{L^r(\Omega;Y_\tau)} := \mathbb{E}\big(\|\widetilde{u}|_{[0,\tau]}\|^r_{Y_\tau}\big). \tag{4.6}$$

Using Lemma 4.1.7 one can check that the expectation in (4.6) is well-defined. Moreover, one can check that the norm does not depend on the choice of $\widetilde{u}$.

4.2 Stochastic maximal L^p-regularity

The following assumptions will be made throughout Sections 4.2 and 4.3.

Assumption 4.2.1. *Let X_0, X_1 be UMD Banach spaces with type 2 and assume $X_1 \hookrightarrow X_0$ densely. Assume one of the following two settings is satisfied*

- $p \in (2, \infty)$ *and* $\kappa \in [0, \frac{p}{2} - 1)$;
- $p = 2$, $\kappa = 0$ *and* X_0, X_1 *are Hilbert spaces.*

For $\theta \in (0,1)$, and p, κ as above let

$$X_\theta := [X_0, X_1]_\theta, \qquad X^{\mathsf{Tr}}_{\kappa,p} := (X_0, X_1)_{1-\frac{1+\kappa}{p},p}, \qquad X^{\mathsf{Tr}}_p := X^{\mathsf{Tr}}_{0,p}.$$

The spaces X_θ have UMD and type 2 (see [107, Proposition 4.2.17] and [108, Proposition 7.1.3]). The same holds for X^{Tr}_p but this will not be needed.

Moreover, in the case $p = 2$ and $\kappa = 0$, by [107, Corollary C.4.2] we have $X_{\frac{1}{2}} = (X_0, X_1)_{\frac{1}{2},2} = X^{\mathsf{Tr}}_2$. This is the reason we only consider Hilbert spaces if $p = 2$ and it will be used without further mentioning it.

4.2.1 Stochastic maximal L^p-regularity

In this subsection we collect some basic definitions.

The next assumption is solely for Section 4.2, where the linear theory is treated.

Assumption 4.2.2. *Let $T \in (0, \infty]$ and set $I_T := (0, T)$. The maps $A : I_T \times \Omega \to \mathscr{L}(X_1, X_0)$ and $B : I_T \times \Omega \to \mathscr{L}(X_1, \gamma(H, X_{1/2}))$ are strongly progressively measurable. Moreover, we assume there exists $C_{A,B} > 0$ such that*

$$\|A(t,\omega)\|_{\mathscr{L}(X_1,X_0)} + \|B(t,\omega)\|_{\mathscr{L}(X_1,\gamma(H,X_{1/2}))} \le C_{A,B},$$

for a.a. $\omega \in \Omega$ and all $t \in I_T$.

Note that A is a family of unbounded operators on X_0 and $\mathsf{D}(A(t,\omega)) = X_1$, and B is a family of unbounded operators on $X_{1/2}$ with domain $\mathsf{D}(B(t,\omega)) = X_1$. The orders of both terms are comparable as the A-term is for the deterministic part, and the B-term for the stochastic part.

Stochastic maximal L^p-regularity is concerned with the optimal regularity estimate for the linear abstract stochastic Cauchy problem:

$$\begin{cases} du(t) + A(t)u(t)dt = f(t)dt + (B(t)u(t) + g(t))dW_H(t), & t \in [0,T], \\ u(0) = u_0. \end{cases} \tag{4.7}$$

Next we give the definition of a strong solution.

Definition 4.2.3. *Let τ be a stopping time which takes values in $[0, T]$. Let the Assumptions 4.2.1-4.2.2 be satisfied. Assume that*

$$u_0 \in L^0_{\mathscr{F}_0}(\Omega; X_0), \quad f \in L^0_{\mathscr{P}}(\Omega; L^1(I_\tau; X_0)), \quad g \in L^0_{\mathscr{P}}(\Omega; L^2(I_\tau; \gamma(H, X_0))).$$

A strongly progressive process $u : [\![0,\tau]\!] \to X_1$ is a strong solution to (4.7) on $[\![0,\tau]\!]$ if a.s. $u \in L^2(I_\tau; X_1)$, and a.s. for all $t \in I_\tau$,

$$u(t) - u_0 + \int_0^t A(s)u(s)ds = \int_0^t (B(s)u(s) + g(s))dW_H(s) + \int_0^t f(s)ds. \tag{4.8}$$

Note that a strong solution automatically satisfies $u \in L^0(\Omega; C([0,\tau]; X_0))$.

We are ready to define weighted stochastic maximal L^p-regularity in a similar way as in [174].

Definition 4.2.4 (Stochastic maximal L^p-regularity)**.** *Let the Assumptions 4.2.1-4.2.2 be satisfied. We write $(A,B)\in\mathcal{SMR}_{p,\kappa}(T)$ if for every*

$$f\in L^p_{\mathscr{P}}(\Omega;L^p(I_T,w_\kappa;X_0))\qquad\text{and}\qquad g\in L^p_{\mathscr{P}}(\Omega;L^p(I_T,w_\kappa;\gamma(H,X_{1/2})))$$

there exists a strong solution u to (4.7) on $[\![0,T]\!]$ with $u_0=0$ such that $u\in L^p(I_T\times\Omega,w_\kappa;X_1)$, and moreover for all stopping times $\tau:\Omega\to[0,T]$ and any strong solution $u\in L^p(I_\tau\times\Omega,w_\kappa;X_1)$ the following estimate holds

$$\|u\|_{L^p(I_\tau\times\Omega,w_\kappa;X_1)}\leq C\|f\|_{L^p(\Omega;L^p(I_\tau,w_\kappa;X_0))}+C\|g\|_{L^p(\Omega;L^p(I_\tau,w_\kappa;\gamma(H,X_{1/2})))},$$

where C is independent of f, g and τ.

In the unweighted case we set $\mathcal{SMR}_p(T):=\mathcal{SMR}_{p,0}(T)$. Finally, we write $A\in\mathcal{SMR}_{p,\kappa}(T)$ if $(A,0)\in\mathcal{SMR}_{p,\kappa}(T)$.

As a consequence of the estimate in the above definition, a strong solution $u\in L^p(I_\tau\times\Omega,w_\kappa;X_1)$ on $[\![0,\tau]\!]$ to (4.7) is unique.

Often we will need the following stronger form of stochastic maximal L^p-regularity, where additional time-regularity is required. For technical reasons the definitions for $p>2$ and $p=2$ are different.

Definition 4.2.5. *Let the Assumptions 4.2.1-4.2.2 be satisfied.*

(1) *For $p>2$, we write $(A,B)\in\mathcal{SMR}^\bullet_{p,\kappa}(T)$ if $(A,B)\in\mathcal{SMR}_{p,\kappa}(T)$ and for every $f\in L^p_{\mathscr{P}}(\Omega;L^p(I_T,w_\kappa;X_0))$ and $g\in L^p_{\mathscr{P}}(\Omega;L^p(I_T,w_\kappa;\gamma(H,X_{1/2})))$ the strong solution u to (4.7) on $[\![0,T]\!]$ with $u_0=0$ satisfies $u\in L^p(\Omega;H^{\theta,p}(I_T,w_\kappa;X_{1-\theta}))$ for every $\theta\in[0,1/2)$, and*

$$\|u\|_{L^p(\Omega;H^{\theta,p}(I_T,w_\kappa;X_{1-\theta}))}\leq C\|f\|_{L^p(\Omega;L^p(I_T,w_\kappa;X_0))}+C\|g\|_{L^p(\Omega;L^p(I_T,w_\kappa;\gamma(H,X_{1/2})))},$$

where C does not depend on f and g.

(2) *We write $(A,B)\in\mathcal{SMR}^\bullet_{2,0}(T)$ if $(A,B)\in\mathcal{SMR}_{2,0}(T)$ and for every $f\in L^2_{\mathscr{P}}(I_T\times\Omega;X_0)$ and $g\in L^2_{\mathscr{P}}(I_T\times\Omega;\gamma(H,X_{1/2}))$ the solution u to (4.7) with $u_0=0$ satisfies $u\in L^2(\Omega;C(\overline{I}_T;X_{\frac12}))$ and*

$$\|u\|_{L^2(\Omega;C(\overline{I}_T;X_{\frac12}))}\leq C\|f\|_{L^2(I_T\times\Omega;X_0)}+C\|g\|_{L^2(I_T\times\Omega;\gamma(H,X_{1/2}))},$$

where C does not depend on f and g.

In the unweighted case we set $\mathcal{SMR}^\bullet_p(T):=\mathcal{SMR}^\bullet_{p,0}(T)$. Furthermore, we write $A\in\mathcal{SMR}^\bullet_{p,\kappa}(T)$ if $(A,0)\in\mathcal{SMR}^\bullet_{p,\kappa}(T)$.

Although we allow $\theta=\frac{1+\kappa}{p}$ in the above definition, later on we will omit this case since some technical difficulties arise related to Theorem 4.1.1.

In the next section we give examples of pairs (A,B) which are in $\mathcal{SMR}^\bullet_{p,\kappa}(T)$.

4.2.2 Operators with stochastic maximal L^p-regularity

There exists an extensive list of examples on stochastic maximal L^p-regularity and in this section we review a selection. We will only consider maximal L^p-regularity in the Bessel-potential scale.

The case Hilbert space case for $\mathcal{SMR}_{p,\kappa}(T)$ was first studied by several different methods for $p=2$ and $\kappa=0$. We refer to the following papers for more detailed information.

- [50, Theorem 6.14] the semigroup approach under restrictions on the interpolation spaces.
- [148] the monotone operators approach, where A and B not even need to be linear.
- [126] $W^{k,2}$-theory on domains with weights.

In some cases one can even obtain that the operator is in $\mathcal{SMR}_2^\bullet(T)$. For instance this holds if A is the generator of a C_0-semigroup on $X_{\frac{1}{2}}$ which has a dilation to a C_0-group (see [98]). In particular, this holds if the semigroup is quasi-contractive $\|e^{-tA}\|_{\mathscr{L}(X_{\frac{1}{2}})} \leq e^{t\omega}$ or A has a bounded H^∞-calculus of angle $< \pi/2$ on X_0 (see [138, Theorem 11.13]).

In the setting $X_0 = H^{s,p}$ the stochastic maximal regularity of the form $\mathcal{SMR}_{p,\kappa}(T)$ has been obtained mostly for second order elliptic operators starting in [128, 129, 130] in the $\mathbb{R}^d$-case in what is usually called Krylov's L^p-theory for SPDEs. It was afterwards extended to domains:

Example 4.2.6.

- [45] and [149] heat equation on an angular domain with weights;
- [44] heat equation on polygonal domains with weights;
- [69] C^2-domains no weights;
- [116, 117, 118] C^1-domains with weights;
- [134] half space case with weights;

and second order systems:

- [120] second order systems with B of special form;
- [159] second order systems with B of special form.

The stronger form of stochastic maximal regularity $\mathcal{SMR}_p^\bullet(T)$ was proved in [166] for $B = 0$ and A independent of (t, ω) using the H^∞-calculus. Combined with a perturbation argument, the case $\kappa \in [0, \frac{p}{2} - 1)$ was obtained in Theorem 3.4.10.

Theorem 4.2.7. *Let Assumption 4.2.1 be satisfied. Let X_0 be isomorphic to a closed subspace of an L^q-space for some $q \in [2, \infty)$ on a σ-finite measure space. Let A be a closed operator on X_0 such that $\mathsf{D}(A) = X_1$. Assume that there exists a $\lambda \in \mathbb{R}$ such that $\lambda + A$ has a bounded H^∞-calculus of angle $< \pi/2$. Then $A \in \mathcal{SMR}_{p,\kappa}^\bullet(T)$ for all $T < \infty$. Furthermore, if A is invertible and $\lambda = 0$, then the result extends to $T = \infty$.*

In particular, this result can be combined with the examples listed in Example 2.1.1.

In [165] $\mathcal{SMR}_{p,\kappa}^\bullet(T)$ was obtained for regular time dependent A for small B using perturbation arguments. By combining ideas from Krylov's L^p-theory and the semigroup approach of [166] this was improved in [174] to a large class of abstract operators (A, B) as in Assumption 4.2.2 and where no time-regularity is assumed. In particular, it applies to second order systems with $B \neq 0$, and higher order systems with small $B \neq 0$ and in particular improves [128, 129, 130] and [120]. We will come back to those examples in later sections.

By definition $\mathcal{SMR}_{p,\kappa}^\bullet(T) \subseteq \mathcal{SMR}_{p,\kappa}(T)$. The following somewhat surprising result states that $\mathcal{SMR}_{p,\kappa}^\bullet(T) \neq \varnothing$ is a necessary and sufficient condition for the reverse inclusion to hold. Usually the non-emptyness can be checked with Theorem 4.2.7 by showing that there is some operator $\widetilde{A}$ on X_0 with $\mathsf{D}(\widetilde{A}) = X_1$ and which has a bounded H^∞-calculus of angle $< \pi/2$.

Proposition 4.2.8 (Transference of stochastic maximal regularity)**.** *Let the Assumptions 4.2.1-4.2.2 be satisfied. Let $(A, B) \in \mathcal{SMR}_{p,\kappa}(T)$ and assume the existence of a couple $(\widetilde{A}, \widetilde{B})$ which satisfies Assumption 4.2.2 and belongs to $\mathcal{SMR}_{p,\kappa}^\bullet(T)$. Then $(A, B) \in \mathcal{SMR}_{p,\kappa}^\bullet(T)$.*

Proof. Let us analyse the case $p > 2$. The other case follows in the same way. By Definition 4.2.5 we have to prove that for any $f \in L^p_{\mathscr{P}}(I_T \times \Omega, w_\kappa; X_0)$, $g \in L^p_{\mathscr{P}}(I_T \times \Omega, w_\kappa; \gamma(H, X_{1/2}))$ and $\theta \in [0, 1/2)$ the unique strong solution $u \in L^p_{\mathscr{P}}(I_T \times \Omega, w_\kappa; X_1)$ to (4.7) on $[\![0, T]\!]$ with $u_0 = 0$ verifies

$$u \in L^p(\Omega; H^{\theta,p}(I_T, w_\kappa; X_{1-\theta})).$$

To this end, note that

$$\begin{cases} du + \widetilde{A}udt = \widetilde{B}udW_H + ((\widetilde{A} - A)u + f)dt + ((B - \widetilde{B})u + g)dW_H, & t \in [0, T], \\ u(0) = 0. \end{cases}$$

Fix $\theta \in [0, 1/2)$. Since $u \in L^p_{\mathscr{P}}(I_T \times \Omega, w_\kappa; X_1)$ and $(\widetilde{A}, \widetilde{B}) \in \mathcal{SMR}^\bullet_{p,\kappa}(T)$, one has

$$\begin{aligned} &\|u\|_{L^p(\Omega;H^{\theta,p}(I_T,w_\kappa;X_{1-\theta}))} \\ &\lesssim \|(\widetilde{A} - A)u + f\|_{L^p(I_T\times\Omega,w_\kappa;X_0)} + \|(B - \widetilde{B})u + g\|_{L^p(I_T\times\Omega,w_\kappa;\gamma(H,X_{1/2}))} \\ &\overset{(i)}{\lesssim} \|u\|_{L^p(I_T\times\Omega,w_\kappa;X_1)} + \|f\|_{L^p(I_T\times\Omega,w_\kappa;X_0)} + \|g\|_{L^p(I_T\times\Omega,w_\kappa;\gamma(H,X_{1/2}))} \\ &\overset{(ii)}{\lesssim} \|f\|_{L^p(I_T\times\Omega,w_\kappa;X_0)} + \|g\|_{L^p(I_T\times\Omega,w_\kappa;\gamma(H,X_{1/2}))}, \end{aligned}$$

where in (i) we used Assumption 4.2.2 and in (ii) we used $(A, B) \in \mathcal{SMR}_{p,\kappa}(T)$. □

Remark 4.2.9.

(1) Proposition 4.2.8 is actually needed in the proof [174, Theorem 3.18] and it was overlooked. The result can be used to deduce the stronger form of stochastic maximal L^p-regularity $\mathcal{SMR}^\bullet_{p,\kappa}(T)$ also for some cases of the list in Example 4.2.6. In particular, this will play a role in later sections.

(2) [174, Theorem 3.9] contains another transference result which allows to deduce $A \in \mathcal{SMR}^\bullet_{p,\kappa}(T)$ from maximal L^p-regularity for the deterministic problem (i.e. $g = 0$, $B = 0$) and $\widetilde{A} \in \mathcal{SMR}_{p,\kappa}(T)$ for some family $\widetilde{A}$. Moreover, in special cases it is shown that one can reduce to $B = 0$ in [174, Theorem 3.18].

(3) Theorem 4.2.7 also holds for operators $A : \Omega \to \mathscr{L}(X_1, X_0)$ as long as the estimates for the H^∞-calculus are uniform in Ω.

To finish this subsection we mention that there are also perturbation results for $\mathcal{SMR}^\bullet_{p,\kappa}(T)$ (see [174, Theorem 3.15] and Theorem 9.1.4 below).

4.2.3 Initial values and the solution operator

The aim of this subsection is the study of the linear problem (4.7) with non-trivial initial data and to introduce some notations.

Proposition 4.2.10. *Suppose Assumptions 4.2.1, and 4.2.2 hold. Let $(A, B) \in \mathcal{SMR}_{p,\kappa}(T)$. Then for any $u_0 \in L^p_{\mathscr{F}_0}(\Omega; X^{\mathsf{Tr}}_{\kappa,p})$, $f \in L^p_{\mathscr{P}}(I_T \times \Omega, w_\kappa; X_0)$ and $g \in L^p_{\mathscr{P}}(I_T \times \Omega, w_\kappa; \gamma(H, X_{1/2}))$ there exists a unique strong solution $u \in L^p(I_T \times \Omega, w_\kappa; X_1)$ to (4.7) on $[\![0, T]\!]$ and*

$$\begin{aligned} \|u\|_{L^p(I_T\times\Omega,w_\kappa;X_1)} \leq{}& C\|f\|_{L^p(I_T\times\Omega,w_\kappa;X_0)} \\ &+ C\|g\|_{L^p(I_T\times\Omega,w_\kappa;\gamma(H,X_{1/2}))} + C\|u_0\|_{L^p(\Omega;X^{\mathsf{Tr}}_{\kappa,p})}, \end{aligned} \tag{4.9}$$

where C is independent of f, g and u_0.

If in addition $(A, B) \in \mathcal{SMR}^\bullet_{p,\kappa}(T)$, then for all $\theta \in [0, 1/2)$ the left-hand side of (4.9) can be replaced by $\|u\|_{L^p(\Omega;H^{\theta,p}(I_T,w_\kappa;X_{1-\theta}))}$ if $p > 2$ with C additionally depending on θ, and replaced by $\|u\|_{L^p(\Omega;C(\overline{I}_T;X_{1/2}))}$ if $p = 2$.

Proof. The proof is similar to [15, Lemma 2.2]. For the reader's convenience, we include the details. In steps 1-3, we assume only that $(A, B) \in \mathcal{SMR}_{p,\kappa}(T)$.

Step 1: Uniqueness. This follows from $(A, B) \in \mathcal{SMR}_{p,\kappa}(T)$ and Definition 4.2.4.

Step 2: u exists and (4.9) holds provided u_0 is simple. Recall that (see [20, Theorem 3.12.2] or [198, Theorem 1.8.2, p. 44]) the real interpolation space $X^{\mathsf{Tr}}_{\kappa,p}$ can be characterized as the set of all $x \in X_0 + X_1$ such that there exists $h \in W^{1,p}(\mathbb{R}_+, w_\kappa; X_0) \cap L^p(\mathbb{R}_+, w_\kappa; X_1)$ which satisfies $x = h(0)$. Moreover,

$$\|x\|_{X^{\mathsf{Tr}}_{\kappa,p}} \eqsim \inf\{\|h\|_{W^{1,p}(\mathbb{R}_+,w_\kappa;X_0)\cap L^p(\mathbb{R}_+,w_\kappa;X_1)} : h(0) = x\}. \tag{4.10}$$

Let $u_0 \in L^p_{\mathscr{F}_0}(\Omega; X^{\mathsf{Tr}}_{\kappa,p})$ be simple. By (4.10) applied pointwise w.r.t. $\omega \in \Omega$, one can check that there exists a simple map $h \in L^p_{\mathscr{F}_0}(\Omega; W^{1,p}(\mathbb{R}_+, w_\kappa; X_0) \cap L^p(\mathbb{R}_+, w_\kappa; X_1))$ such that

$$\|h\|_{L^p(\Omega;W^{1,p}(\mathbb{R}_+,w_\kappa;X_0)\cap L^p(\mathbb{R}_+,w_\kappa;X_1))} \lesssim \|u_0\|_{L^p(\Omega;X^{\mathsf{Tr}}_{\kappa,p})}, \tag{4.11}$$

where the implicit constant does not depend on u_0. Set $u := h + v$. Then u is a strong solution to (4.7) on $[\![0,T]\!]$ if and only if v is a strong solution on $[\![0,T]\!]$ to

$$\begin{cases} dv + A(t)vdt = (f + \dot{h} - A(t)h)dt + (B(t)v + B(t)h + g)dW_H, \ t \in I_T, \\ v(0) = 0. \end{cases} \tag{4.12}$$

By (4.11) and the fact that $(A,B) \in \mathcal{SMR}_{p,\kappa}(T)$, (4.9) follows.

Step 3: u exists and (4.9) holds for all $u_0 \in L^p_{\mathscr{F}_0}(\Omega; X^{\mathsf{Tr}}_{\kappa,p})$. By [107, Lemma 1.2.19], there exists a uniformly bounded sequence of simple maps $(u_{0,n})_{n\geq 1} \subseteq L^p_{\mathscr{F}_0}(\Omega; X^{\mathsf{Tr}}_{\kappa,p})$ such that $u_{0,n} \to u_0$ in $L^p_{\mathscr{F}_0}(\Omega; X^{\mathsf{Tr}}_{\kappa,p})$. Thus, the conclusion follows from Step 2 and the completeness of $L^p_{\mathscr{P}}(I_T \times \Omega, w_\kappa; X_1)$.

Step 4: The last claim holds. Similarly to Step 3, it is enough to consider u_0 simple. Thus, as in Step 2, there exists $h \in L^p_{\mathscr{F}_0}(\Omega; W^{1,p}(\mathbb{R}_+, w_\kappa; X_0) \cap L^p(\mathbb{R}_+, w_\kappa; X_1))$ such that (4.10) holds. Then by Proposition 4.1.3 and the fact that $(A,B) \in \mathcal{SMR}^\bullet_{p,\kappa}(T)$, the claim follows by writing $u = h + v$ where v solves (4.12). ☐

Remark 4.2.11. Under the assumption that $X_1 = \mathsf{D}(\widetilde{A})$, for a sectorial operator $\widetilde{A}$ on X_0 with angle $\omega(\widetilde{A}) < \pi/2$, the proof of Proposition 4.2.10 simplifies. See step 0 in [174, Theorem 3.15]. This type of assumption is satisfied in all the applications which will be presented in this thesis.

Next we will define certain solution operators which will be used in Section 4.3. Suppose $(A,B) \in \mathcal{SMR}^\bullet_{p,\kappa}(T)$ and that Assumptions 4.2.1-4.2.2 hold. Using Proposition 4.2.10 for $p > 2$ we can define $\mathscr{R}_{(A,B)}(u_0, f, g) = u$, where u is the strong solution to (4.7) as a mapping from

$$L^p_{\mathscr{F}_0}(\Omega; X^{\mathsf{Tr}}_{\kappa,p}) \times L^p_{\mathscr{P}}(I_T \times \Omega, w_\kappa; X_0) \times L^p_{\mathscr{P}}(I_T \times \Omega, w_\kappa; \gamma(H, X_{1/2}))$$

into

$$\bigcap_{\theta\in[0,1/2)} L^p(\Omega; H^{\theta,p}(I_T, w_\kappa; X_{1-\theta})).$$

By linearity, we can write

$$\mathscr{R}_{(A,B)}(u_0, f, g) = \mathscr{R}_{(A,B)}(u_0, 0, 0) + \mathscr{R}_{(A,B)}(0, f, 0) + \mathscr{R}_{(A,B)}(0, 0, g).$$

Note that $\mathscr{R}_{(A,B)}(0, \cdot, \cdot)$ actually maps into $L^p(\Omega; {}_0H^{\theta,p}(I_T, w_\kappa; X_{1-\theta})$ for any $\theta \in [0, \frac{1}{2}) \setminus \{\frac{1+\kappa}{p}\}$. Indeed, this follows from $u(0) = 0$ in X_0, Theorem 4.1.1 and the text below it.

For later use, in the case $p > 2$ and $\theta \in [0, \frac{1}{2}) \setminus \{\frac{1+\kappa}{p}\}$, we define

$$\begin{aligned} C^{\det,\theta}_{(A,B)} &= \|\mathscr{R}_{(A,B)}(0, \cdot, 0)\|_{L^p(I_T\times\Omega,w_\kappa;X_0))\to L^p(\Omega;{}_0H^{\theta,p}(I_T,w_\kappa;X_{1-\theta}))}, \\ C^{\mathsf{sto},\theta}_{(A,B)} &= \|\mathscr{R}_{(A,B)}(0, 0, \cdot)\|_{L^p_{\mathscr{P}}(I_T\times\Omega,w_\kappa;\gamma(H,X_{1/2}))\to L^p(\Omega;{}_0H^{\theta,p}(I_T,w_\kappa;X_{1-\theta}))}. \end{aligned} \tag{4.13}$$

In the case $p = 2$ and $\theta \in (0, 1/2)$, we replace the range space by $L^p(\Omega; C(\overline{I}_T; X_{1/2}))$ (which is constant in $\theta \in (0, 1/2)$). Moreover, for $\theta \in [0, \frac{1}{2}) \setminus \{\frac{1+\kappa}{p}\}$ we set

$$K^{\det,\theta}_{(A,B)} := C^{\det,\theta}_{(A,B)} + C^{\det,0}_{(A,B)}, \qquad K^{\mathsf{sto},\theta}_{(A,B)} := C^{\mathsf{sto},\theta}_{(A,B)} + C^{\mathsf{sto},0}_{(A,B)}. \tag{4.14}$$

In the next proposition we collect some simple properties of the solution operator $\mathscr{R}_{(A,B)}$.

Proposition 4.2.12. *Suppose Assumptions 4.2.1-4.2.2 hold. Let $(A,B)\in\mathcal{SMR}^{\bullet}_{p,\kappa}(T)$ and let $\mathscr{R}:=\mathscr{R}_{(A,B)}$. Let $u_0\in L^p_{\mathscr{F}_0}(\Omega;X^{\mathsf{Tr}}_{\kappa,p})$, $f\in L^p_{\mathscr{P}}(I_T\times\Omega,w_\kappa;X_0)$, $g\in L^p_{\mathscr{P}}(I_T\times\Omega,w_\kappa;\gamma(H,X_{1/2}))$ and set $u:=\mathscr{R}(u_0,f,g)$. Then the following assertions hold*

(1) *For each $F\in\mathscr{F}_0$,*

$$\mathbf{1}_F\mathscr{R}(u_0,f,g)=\mathscr{R}(\mathbf{1}_Fu_0,\mathbf{1}_Ff,\mathbf{1}_Fg)=\mathbf{1}_F\mathscr{R}(\mathbf{1}_Fu_0,\mathbf{1}_Ff,\mathbf{1}_Fg).$$

(2) *Assume that $v\in L^p_{\mathscr{P}}([\![0,\sigma]\!],w_\kappa;X_1)$ is a strong solution to (4.7) on $[\![0,\sigma]\!]$, where σ is a stopping time. Then*

$$v=u|_{[\![0,\sigma]\!]}=\mathscr{R}(u_0,\mathbf{1}_{[\![0,\sigma]\!]}f,\mathbf{1}_{[\![0,\sigma]\!]}g),\qquad \text{on } [\![0,\sigma]\!].$$

(3) *For all $T_1\le T$, the following estimates on the maximal regularity constants hold*

$$K^{\mathsf{det},\theta}_{(A|_{[\![0,T_1]\!]},B|_{[\![0,T_1]\!]})}\le K^{\mathsf{det},\theta}_{(A,B)}\quad\text{and}\quad K^{\mathsf{sto},\theta}_{(A|_{[\![0,T_1]\!]},B|_{[\![0,T_1]\!]})}\le K^{\mathsf{sto},\theta}_{(A,B)}.$$

Proof. (1): By Definition 4.2.3, u verifies (4.8). It follows that $v:=\mathbf{1}_Fu$ satisfies

$$v(t)-\mathbf{1}_Fu_0+\int_0^t A(s)(\mathbf{1}_Fu(s))ds=\int_0^t(B(s)(v(s))+\mathbf{1}_Fg(s))dW_H+\int_0^t\mathbf{1}_Ff(s)ds.$$

By uniqueness we obtain $v=\mathscr{R}(\mathbf{1}_Fu_0,\mathbf{1}_Ff,\mathbf{1}_Fg)$. This proves the first identity. The second identity follows from the first identity and $\mathbf{1}_F^2=\mathbf{1}_F$.

(2): From Definition 4.2.3 we immediately see that $u|_{[\![0,\sigma]\!]}$ is a strong solution on $[\![0,\sigma]\!]$. By uniqueness, this implies $v=u|_{[\![0,\sigma]\!]}$. Thus, a.s. for all $t\in[0,\sigma]$,

$$u(t)-u_0+\int_0^t A(s)u(s)ds=\int_0^t(B(s)u(s)+g(s))dW_H(s)+\int_0^t f(s)ds.$$

On the other hand, $\widetilde{u}:=\mathscr{R}(u_0,\mathbf{1}_{[\![0,\sigma]\!]}f,\mathbf{1}_{[\![0,\sigma]\!]}g)$ satisfies a.s. for all $t\in[0,\sigma]$,

$$\begin{aligned}\widetilde{u}(t)-u_0+\int_0^t A(s)\widetilde{u}(s)ds&=\int_0^t(B(s)\widetilde{u}(s)+\mathbf{1}_{[\![0,\sigma]\!]}g(s))dW_H(s)+\int_0^t\mathbf{1}_{[\![0,\sigma]\!]}f(s)ds\\&=\int_0^t B(s)\widetilde{u}(s)dW_H(s)+\int_0^{t\wedge\sigma}g(s)dW_H(s)+\int_0^{t\wedge\sigma}f(s)ds\\&=\int_0^t B(s)\widetilde{u}(s)dW_H(s)+\int_0^t g(s)dW_H(s)+\int_0^t f(s)ds\end{aligned}$$

Therefore, again by uniqueness $\widetilde{u}=v$.

(3): This is immediate from (2) and Proposition 2.2.2. ☐

We end this section with a lemma which can be extracted from the proof of [174, Theorem 3.15 Step 1]. For the reader's convenience we sketch the proof.

Lemma 4.2.13. *Let Assumption 4.2.1 be satisfied, and suppose that (A,B) satisfies Assumption 4.2.2. Then for each $s\in I_T$ there exists a constant $c_s>0$ such that $\lim_{s\downarrow0}c_s=0$ and for all τ stopping time satisfying $0\le\tau\le s$ a.s. and any $f\in L^p_{\mathscr{P}}(I_s\times\Omega,w_\kappa;X_0)$, $g\in L^p_{\mathscr{P}}(I_s\times\Omega,w_\kappa;\gamma(H,X_{1/2}))$ and any strong solution $u\in L^p_{\mathscr{P}}(I_\tau\times\Omega,w_\kappa;X_1)$ to (4.7) on $[\![0,\tau]\!]$ with $u_0=0$ one has*

$$\begin{aligned}\|u\|_{L^p(I_\tau\times\Omega,w_\kappa;X_0)}\le c_s\|u\|_{L^p(I_\tau\times\Omega,w_\kappa;X_1)}&+c_s\|f\|_{L^p(I_\tau\times\Omega,w_\kappa;X_0)}\\&+c_s\|g\|_{L^p(I_\tau\times\Omega,w_\kappa;\gamma(H,X_{1/2}))}.\end{aligned}$$

If additionally $(A,B)\in\mathcal{SMR}^{\bullet}_{p,\kappa}(T)$, then $u=\mathscr{R}_{(A,B)}(0,f,g)$ a.e. on $[\![0,\tau]\!]$ and

$$\|u\|_{L^p(I_\tau\times\Omega,w_\kappa;X_0)}\le c_s\|f\|_{L^p(I_\tau\times\Omega,w_\kappa;X_0)}+c_s\|g\|_{L^p(I_\tau\times\Omega,w_\kappa;\gamma(H,X_{1/2}))}.$$

Proof. Let us begin by proving the first claim. Recall that $u(t) = \int_0^t (-A(r)u(r) + f(r))dr + \int_0^t (B(r)u(r) + g(r))dW_H(r)$ a.s. for each $t \in I_\tau$. Let us set $v(t) := \int_0^t \mathbf{1}_{[\![0,\tau]\!]}(-A(r)u(r) + f(r))dr + \int_0^t \mathbf{1}_{[\![0,\tau]\!]}(B(r)u(r) + g(r))dW_H(r)$ a.s. for each $t \in [0, s]$. Note that $v = u$ a.e. on $[\![0, \tau]\!]$. By Corollary 2.3.8,

$$
\begin{aligned}
&\|v\|_{L^p(\Omega;C(\bar{I}_s;X_0))} \\
&\lesssim_{X,p} \|\mathbf{1}_{[\![0,\tau]\!]}(-Au + f)\|_{L^p(\Omega;L^1(I_s;X_0))} + \|\mathbf{1}_{[\![0,\tau]\!]}(Bu + g)\|_{L^p(\Omega;L^2(I_s;\gamma(H,X_0)))} \\
&\overset{(i)}{\lesssim}_{p,\kappa} k_s\big[\| - Au + f\|_{L^p(I_\tau\times\Omega,w_\kappa;X_0)} + \|Bu + g\|_{L^p(I_\tau\times\Omega,w_\kappa;\gamma(H,X_0))}\big] \\
&\overset{(ii)}{\lesssim}_{A,B} k_s\big[\|u\|_{L^p(I_\tau\times\Omega,w_\kappa;X_1)} + \|f\|_{L^p(I_\tau\times\Omega,w_\kappa;X_0)} + \|g\|_{L^p(I_\tau\times\Omega,w_\kappa;\gamma(H,X_{1/2}))}\big],
\end{aligned}
$$

where in (i) we used Hölder's inequality and $\kappa \in [0, \frac{p}{2} - 1)$, in (ii) we used Assumptions 4.2.1-4.2.2. The constant k_s satisfies $\lim_{s\downarrow 0} k_s =: k \in [0, \infty)$. Therefore,

$$
\begin{aligned}
\|v\|_{L^p(I_s\times\Omega,w_\kappa;X_0)} \leq k_s c_s\big[&\|u\|_{L^p(I_\tau\times\Omega,w_\kappa;X_1)} + \|f\|_{L^p(I_\tau\times\Omega,w_\kappa;X_0)} \\
&+ \|g\|_{L^p(I_\tau\times\Omega,w_\kappa;\gamma(H,X_{1/2}))}\big],
\end{aligned}
$$

where $c_s > 0$ satisfies $\lim_{s\downarrow 0} c_s = 0$. Since $v = u$ a.e. on $[\![0, \tau]\!]$ and $\tau \leq s$ a.s., one has $\|u\|_{L^p(I_\tau\times\Omega,w_\kappa;X_0)} \leq \|v\|_{L^p(I_s\times\Omega,w_\kappa;X_0)}$, and thus the first estimate follows.

If $(A, B) \in \mathcal{SMR}^{\bullet}_{p,\kappa}(T)$ and $u \in L^p_{\mathscr{P}}(I_\tau \times \Omega, w_\kappa; X_1)$ is a strong solution to (4.7) on $[\![0, \tau]\!]$, then by Proposition 4.2.12(2), $u = \mathscr{R}_{(A,B)}(0, f, g) = \mathscr{R}_{(A,B)}(0, \mathbf{1}_{[\![0,\tau]\!]}f, \mathbf{1}_{[\![0,\tau]\!]}g)$ a.e. on $[\![0, \tau]\!]$. Thus

$$
\begin{aligned}
\|u\|_{L^p(I_\tau\times\Omega,w_\kappa;X_1)} &\leq \|\mathscr{R}_{(A,B)}(0, \mathbf{1}_{[\![0,\tau]\!]}f, \mathbf{1}_{[\![0,\tau]\!]}g)\|_{L^p(I_T\times\Omega,w_\kappa,X_1)} \\
&\lesssim \|f\|_{L^p(I_\tau\times\Omega,w_\kappa;X_1)} + \|g\|_{L^p(I_\tau\times\Omega,w_\kappa;\gamma(H,X_{1/2}))},
\end{aligned}
\tag{4.15}
$$

and this implies the second estimate. □

4.3 Local existence results

In this section we consider the following nonlinear evolution equation

$$
\begin{cases}
du + A(\cdot, u)udt = (F(\cdot, u) + f)dt + (B(\cdot, u)u + G(\cdot, u) + g)dW_H, \\
u(0) = u_0;
\end{cases}
\tag{4.16}
$$

for $t \in [0, T]$ on a Banach space X_0 where $T < \infty$. Recall that Assumption 4.2.1 holds throughout this section.

The equation (4.16) covers both the case of quasilinear and semilinear equations. In the quasilinear case the reader should have in mind that for each fixed $x \in X^{\mathsf{Tr}}_{\kappa,p}$, the operators $A(t, x)$ and $B(t, x)$ satisfy the mapping properties of Assumption 4.2.2. We refer to (HA) below for the precise definitions. In the semilinear case $A(t, x)$ and $B(t, x)$ do not depend on x and therefore are precisely as in Assumption 4.2.2.

The structure of the nonlinearities F and G which will be assumed below is very flexible and extends many known results. Moreover, the structural conditions are satisfied by large classes of SPDE.

Compared to [103, 104, 165] there are several important differences:

- we assume a joint condition on (A, B) and therefore B is not assumed to be small as one sometimes needs with the semigroup approach to SPDEs (see e.g. [30, 80]);
- the operators A and B are allowed to be time and Ω-dependent in just a measurable way;
- we allow weights in time, so that our initial values can be very rough;
- we allow critical nonlinearities in the sense of [143, 178, 180].

4.3.1 Assumptions on the nonlinearities

In this section we discuss the assumptions and the main results regarding (4.16). Moreover, the definition of a strong solution to (4.16) is given in Definition 4.3.3-4.3.4 below.

Concerning the random operators A, B, the nonlinearities F, G, and the initial data, we make the following hypothesis.

Hypothesis $(\mathbf{H}_p)$.

(HA) Assumption 4.2.1 holds. Let $A : [0,T]\times\Omega\times X^{\mathsf{Tr}}_{\kappa,p} \to \mathscr{L}(X_1, X_0)$ and $B : [0,T]\times\Omega\times X^{\mathsf{Tr}}_{\kappa,p} \to \mathscr{L}(X_1, \gamma(H, X_{1/2}))$. Assume that for all $x \in X^{\mathsf{Tr}}_{\kappa,p}$ and $y \in X_1$, the maps $(t,\omega)\mapsto A(t,\omega,x)y$ and $(t,\omega)\mapsto B(t,\omega,x)y$ are strongly progressively measurable.

Moreover, for all $n \geq 1$, there exists $C_n, L_n \in \mathbb{R}_+$ such that for all $x, y \in X^{\mathsf{Tr}}_{\kappa,p}$ with $\|x\|_{X^{\mathsf{Tr}}_{\kappa,p}}, \|y\|_{X^{\mathsf{Tr}}_{\kappa,p}} \leq n$, $t \in [0,T]$, and a.a. $\omega\in\Omega$.

$$\begin{aligned}\|A(t,\omega,x)\|_{\mathscr{L}(X_1,X_0)} &\leq C_n(1+\|x\|_{X^{\mathsf{Tr}}_{\kappa,p}}),\\ \|B(t,\omega,x)\|_{\mathscr{L}(X_1,\gamma(H,X_{1/2}))} &\leq C_n(1+\|x\|_{X^{\mathsf{Tr}}_{\kappa,p}}),\\ \|A(t,\omega,x)-A(t,\omega,y)\|_{\mathscr{L}(X_1,X_0)} &\leq L_n\|x-y\|_{X^{\mathsf{Tr}}_{\kappa,p}},\\ \|B(t,\omega,x)-B(t,\omega,y)\|_{\mathscr{L}(X_1,\gamma(H,X_{1/2}))} &\leq L_n\|x-y\|_{X^{\mathsf{Tr}}_{\kappa,p}}.\end{aligned}$$

(HF) The map $F : [0,T]\times\Omega\times X_1 \to X_0$ decomposes as $F := F_L + F_c + F_{\mathsf{Tr}}$ where $F_L, F_c, F_{\mathsf{Tr}}$ are strongly measurable and for all $t\in[0,T]$, $x \in X_1$ the map $t \mapsto F_\ell(t,\omega,x)$ is strongly $\mathscr{F}_t$-measurable for $\ell \in \{L, c, \mathsf{Tr}\}$. Moreover, $F_L, F_c, F_{\mathsf{Tr}}$ satisfy the following estimates.

(i) There exist constants $L_F, \widetilde{L}_F, C_F > 0$, such that for all $x, y \in X_1$, $t\in[0,T]$ and a.a. $\omega\in\Omega$,

$$\begin{aligned}\|F_L(t,\omega,x)\|_{X_0} &\leq C_F(1+\|x\|_{X_1}),\\ \|F_L(t,\omega,x)-F_L(t,\omega,y)\|_{X_0} &\leq L_F\|x-y\|_{X_1} + \widetilde{L}_F\|x-y\|_{X_0}.\end{aligned}$$

(ii) There exist an $m_F \geq 1$, $\varphi_j \in (1-(1+\kappa)/p, 1)$, $\beta_j \in (1-(1+\kappa)/p, \varphi_j]$, $\rho_j \geq 0$ for $j\in\{1,\dots,m_F\}$ such that $F_c : [0,T]\times\Omega\times X_1 \to X_0$ and for each $n\geq 1$ there exist $C_{c,n}, L_{c,n}\in\mathbb{R}_+$ for which

$$\begin{aligned}\|F_c(t,\omega,x)\|_{X_0} &\leq C_{c,n}\sum_{j=1}^{m_F}(1+\|x\|^{\rho_j}_{X_{\varphi_j}})\|x\|_{X_{\beta_j}} + C_{c,n},\\ \|F_c(t,\omega,x)-F_c(t,\omega,y)\|_{X_0} &\leq L_{c,n}\sum_{j=1}^{m_F}(1+\|x\|^{\rho_j}_{X_{\varphi_j}}+\|y\|^{\rho_j}_{X_{\varphi_j}})\|x-y\|_{X_{\beta_j}},\end{aligned} \tag{4.17}$$

a.s. for all $x, y\in X_1$, $t\in[0,T]$ such that $\|x\|_{X^{\mathsf{Tr}}_{\kappa,p}}, \|y\|_{X^{\mathsf{Tr}}_{\kappa,p}} \leq n$. Moreover, $\rho_j, \varphi_j, \beta_j, \kappa$ satisfy

$$\rho_j\Big(\varphi_j - 1 + \frac{1+\kappa}{p}\Big) + \beta_j \leq 1, \qquad j\in\{1,\dots,m_F\}. \tag{4.18}$$

(iii) For each $n\geq 1$ there exist $L_{\mathsf{Tr},n}, C_{\mathsf{Tr},n}\in\mathbb{R}_+$ such that the mapping $F_{\mathsf{Tr}} : [0,T]\times\Omega\times X^{\mathsf{Tr}}_{\kappa,p} \to X_0$ satisfies

$$\begin{aligned}\|F_{\mathsf{Tr}}(t,\omega,x)\|_{X_0} &\leq C_{\mathsf{Tr},n}(1+\|x\|_{X^{\mathsf{Tr}}_{\kappa,p}}),\\ \|F_{\mathsf{Tr}}(t,\omega,x)-F_{\mathsf{Tr}}(t,\omega,y)\|_{X_0} &\leq L_{\mathsf{Tr},n}\|x-y\|_{X^{\mathsf{Tr}}_{\kappa,p}},\end{aligned}$$

for a.a. $\omega\in\Omega$, for all $t\in[0,T]$ and $\|x\|_{X^{\mathsf{Tr}}_{\kappa,p}}, \|y\|_{X^{\mathsf{Tr}}_{\kappa,p}} \leq n$.

(HG) The map $G : [0,T] \times \Omega \times X_1 \to \gamma(H, X_{1/2})$ decomposes as $G := G_L + G_c + G_{\mathsf{Tr}}$ where $G_L, G_c, G_{\mathsf{Tr}}$ are strongly measurable and for all $t \in [0,T]$, $x \in X_1$ the map $t \mapsto G_\ell(t,\omega,x)$ is strongly $\mathscr{F}_t$-measurable for $\ell \in \{L, c, \mathsf{Tr}\}$. Moreover, $G_L, G_c, G_{\mathsf{Tr}}$ satisfy the following estimates.

(i) There exist constants $L_G, \widetilde{L}_G, C_G$, such that for all $x, y \in X_1$, $t \in [0,T]$ and a.a. $\omega \in \Omega$,

$$\|G_L(t,\omega,x)\|_{\gamma(H,X_{1/2})} \le C_G(1 + \|x\|_{X_1}),$$
$$\|G_L(t,\omega,x) - G_L(t,\omega,y)\|_{\gamma(H,X_{1/2})} \le L_G\|x-y\|_{X_1} + \widetilde{L}_G\|x-y\|_{X_0}.$$

(ii) There exist an $m_G \ge 1$, $\varphi_j \in (1-(1+\kappa)/p, 1)$, $\beta_j \in (1-(1+\kappa)/p, \varphi_j]$, $\rho_j \ge 0$ for $j \in \{m_F+1, \dots, m_F+m_G\}$ such that $G_c : [0,T] \times \Omega \times X_1 \to X_0$ and for each $n \ge 1$ there exist $C_{c,n}, L_{L_c,n} \in \mathbb{R}_+$ for which

$$\|G_c(t,\omega,x)\|_{\gamma(H,X_{1/2})} \le C_{c,n} \sum_{j=m_F+1}^{m_F+m_G} (1 + \|x\|_{X_{\varphi_j}}^{\rho_j})\|x\|_{X_{\beta_j}} + C_{c,n},$$
$$\|G_c(t,\omega,x) - G_c(t,\omega,y)\|_{\gamma(H,X_{1/2})} \le L_{c,n} \sum_{j=m_F+1}^{m_F+m_G} (1 + \|x\|_{X_{\varphi_j}}^{\rho_j} + \|y\|_{X_{\varphi_j}}^{\rho_j})\|x-y\|_{X_{\beta_j}}, \tag{4.19}$$

a.s. for all $x, y \in X_1$, $t \in [0,T]$ such that $\|x\|_{X^{\mathsf{Tr}}_{\kappa,p}}, \|y\|_{X^{\mathsf{Tr}}_{\kappa,p}} \le n$. Moreover, $\varphi_j, \beta_j, \kappa$ satisfy

$$\rho_j\Big(\varphi_j - 1 + \frac{1+\kappa}{p}\Big) + \beta_j \le 1, \qquad j \in \{m_F+1, \dots, m_F+m_G\}. \tag{4.20}$$

(iii) For each $n \ge 1$ there exist constants $L_{\mathsf{Tr},n}, C_{\mathsf{Tr},n}$ such that mapping $G_{\mathsf{Tr}} : [0,T] \times \Omega \times X^{\mathsf{Tr}}_{\kappa,p} \to X_0$ satisfies

$$\|G_{\mathsf{Tr}}(t,\omega,x)\|_{\gamma(H,X_{1/2})} \le C_{\mathsf{Tr},n}(1 + \|x\|_{X^{\mathsf{Tr}}_{\kappa,p}}),$$
$$\|G_{\mathsf{Tr}}(t,\omega,x) - G_{\mathsf{Tr}}(t,\omega,y)\|_{\gamma(H,X_{1/2})} \le L_{\mathsf{Tr},n}\|x-y\|_{X^{\mathsf{Tr}}_{\kappa,p}},$$

for a.a. $\omega \in \Omega$, for all $t \in [0,T]$ and $\|x\|_{X^{\mathsf{Tr}}_{\kappa,p}}, \|y\|_{X^{\mathsf{Tr}}_{\kappa,p}} \le n$.

(Hf) $f \in L^p_{\mathscr{P}}(I_T \times \Omega, w_\kappa; X_0)$ and $g \in L^p_{\mathscr{P}}(I_T \times \Omega, w_\kappa; \gamma(H, X_{1/2}))$.

The relations (4.18)-(4.20) will play an important role in the analysis of (4.16). As announed in Subsection 1.1, following [178], we may give an abstract definition of critical space for (4.16).

The space $X^{\mathsf{Tr}}_{\kappa,p}$ will be called a *critical space* for (4.16) if for some $j \in \{1, \dots, m_F+m_G\}$ equality in (4.18) or (4.20) holds. Moreover, the value of κ for which equality in (4.18) or (4.20) holds, will be called the *critical weight* and it will be denoted by κ_{crit}. Some remarks may be in order.

Remark 4.3.1. Let us note that in Theorem 4.3.5 and 4.3.7 below, only the constants L_F, L_G are assumed to be small. The other constants are arbitrary. At first sight the splitting of the nonlinearities F and G in several parts seems quite complicated. Let us emphasise that the most important part is F_c and G_c as these will usually determine *critical spaces* as defined above. The flexibility in the form we choose the nonlinearities is quite important in application to SPDEs. It will allow us in many cases to find a broad class of initial value spaces for which the SPDE can be solved. Let us note that usually it is enough to take $m_F = m_G = 1$.

Remark 4.3.2. Below we collect some observations which will be used later on to check (HF) or (HG). We discuss this only for F since the same arguments apply to G.

(1) If $F : I_T \times \Omega \times X_\theta \to X_0$, for some $\theta < 1 - (1+\kappa)/p$, is locally Lipschitz uniformly on X_θ uniformly w.r.t. $(t, \omega) \in I_T \times \Omega$, then F verifies (HF). Indeed, it is enough to recall that

$$X_\theta \hookrightarrow (X_0, X_1)_{\theta,\infty} \hookrightarrow (X_0, X_1)_{1-\frac{1+\kappa}{p},p} = X^{\mathsf{Tr}}_{\kappa,p},$$

where in the first inclusion we used [20, Theorem 3.9.1] and in the last inclusion [20, Theorem 3.4.1] since $X_1 \hookrightarrow X_0$. Then the conclusion follows by setting $F_{\mathsf{Tr}} := F$, $F_c = F_L = 0$. As soon as θ is larger, then we need a nonzero F_c as the situation is more sophisticated.

(2) We can additionally allow the case $\beta_j = \varphi_j = 1 - (1+\kappa)/p$ for all $j \in \{1, \dots, m_F\}$ in (HF). Indeed, $\rho_j(\varphi_j + \varepsilon - 1 + \frac{1+\kappa}{p}) + \beta_j = \rho_j\varepsilon + \beta_j$. Thus, there exists $\varepsilon > 0$ such that $\rho_j\varepsilon + \beta_j < 1$ for all $j \in \{1, \dots, m_F + m_G\}$. Since $X_{1-(1+\kappa)/p+\varepsilon} \hookrightarrow X_{1-(1+\kappa)/p}$, we may replace $\varphi_j = \beta_j$ by $1 - (1+\kappa)/p + \varepsilon$ obtaining that F_c satisfies (HF) and (4.18) holds with strict inequality.

(3) Assume that $\beta_j = \varphi_j < 1$ for some $j \in \{1, \dots, m_F\}$ and that equality in (4.18) holds. Then $\rho_j > 0$ and thus $\varphi_j > 1-(1+\kappa)/p$ holds since $\varphi_j - 1 + (1+\kappa)/p = (1-\beta_j)/\rho_j > 0$. Therefore, in applications we do not need to check $\beta_j > 1 - (1+\kappa)/p$ if equality in (4.18) holds (e.g. in the critical case).

Next we define L^p_κ-strong solutions to (4.16). Here we add the prefix L^p_κ since the definition depends on (p, κ).

Definition 4.3.3 (L^p_κ-strong solutions)**.** *Let the Hypothesis* $(\mathbf{H}_p)$ *be satisfied and let σ be a stopping time with $0 \le \sigma \le T$. A strongly progressively measurable process u on $[\![0, \sigma]\!]$ satisfying*

$$u \in L^p(I_\sigma, w_\kappa; X_1) \cap C(\overline{I}_\sigma; X^{\mathsf{Tr}}_{\kappa,p}) \quad a.s.$$

is called an L^p_κ-strong solution to (4.16) on $[\![0, \sigma]\!]$ if $F(\cdot, u) \in L^p(I_\sigma, w_\kappa; X_0)$ and $G(\cdot, u) \in L^p(I_\sigma, w_\kappa; \gamma(H, X_{1/2}))$ a.s., and the following identity holds a.s. and for all $t \in [0, \sigma]$,

$$\begin{aligned} u(t) - u_0 + \int_0^t A(s, u(s))u(s)ds &= \int_0^t F(s, u(s)) + f(s)ds \\ &+ \int_0^t \mathbf{1}_{[\![0,\sigma]\!]}(B(s, u(s))u(s) + G(s, u(s)) + g(s))\, dW_H(s). \end{aligned} \tag{4.21}$$

Note that if u is an L^p_κ-*strong solution*, then the integrals appearing in (4.21) are well-defined. To see this, note that $s \mapsto A(s, u(s))u(s)$ and $s \mapsto B(s, u(s))u(s)$ are strongly progressively measurable by the conditions on u and (HA) (see [9, Lemma 4.51]). Moreover, pointwise in Ω we can take $\mathbb{N} \ni n \ge \|u\|_{C(\overline{I}_\sigma; X^{\mathsf{Tr}}_{\kappa,p})}$ and write

$$\|A(s, u(s))u(s)\|_{X_0} \le C_n(1 + \|u(s)\|_{X^{\mathsf{Tr}}_{\kappa,p}})\|u(s)\|_{X_1} \le C_n(1+n)\|u(s)\|_{X_1}.$$

Integrating over $s \in [0, \sigma]$ we obtain

$$\|s \mapsto A(s, u(s))u(s)\|_{L^1(0,\sigma;X_0)} \le C_n(1+n)\|u\|_{L^1(0,\sigma;X_1)},$$

and the latter is finite since $u \in L^p(I_\sigma, w_\kappa; X_1)$ a.s. Thus, the integral on the left-hand side of (4.21) is well-defined. In the same way one can check that $s \mapsto B(s, u(s))u(s)$ is in $L^2(0, \sigma; X_{1/2})$ and the first stochastic integral on the right-hand side of (4.21) is well-defined by Corollary 2.3.8. Using the above argument and that $f, F(\cdot, u) \in L^p(I_\sigma, w_\kappa; X_0)$ a.s. and $g, G(\cdot, u) \in L^p(I_\sigma, w_\kappa; \gamma(H, X_{1/2}))$ a.s., one can check that the remaining integrals are well-defined.

Next we define L^p_κ-local and L^p_κ-maximal local solutions to (4.16).

Definition 4.3.4 (L^p_κ-Local and L^p_κ-maximal solution)**.** *Let σ be a stopping time with $0 \le \sigma \le T$. Let $u : [\![0, \sigma)\!) \to X_1$ be strongly progressively measurable.*

- (u,σ) *is called an* L^p_κ-local solution *to* (4.16) *on* $[0,T]$*, if there exists an increasing sequence* $(\sigma_n)_{n\geq 1}$ *of stopping times such that* $\lim_{n\uparrow\infty}\sigma_n=\sigma$ *a.s. and* $u|_{[\![0,\sigma_n]\!]}$ *is an* L^p_κ*-strong solution to* (4.16) *on* $[\![0,\sigma_n]\!]$*. In this case,* $(\sigma_n)_{n\geq1}$ *is called a* localizing sequence *for the local solution* (u,σ)*.*
- *An* L^p_κ*-local solution* (u,σ) *to* (4.16) *on* $[0,T]$ *is called* unique*, if for every* L^p_κ*-local solution* (v,ν) *to* (4.16) *on* $[0,T]$ *for a.a.* $\omega\in\Omega$ *and for all* $t\in[0,\nu(\omega)\wedge\sigma(\omega))$ *one has* $v(t,\omega)=u(t,\omega)$*.*
- *An* L^p_κ*-local solution* (u,σ) *to* (4.16) *on* $[0,T]$ *is called an* L^p_κ-maximal local solution*, if for any other* L^p_κ*-local solution* (v,ϱ) *to* (4.16) *on* $[0,T]$*, we have a.s.* $\varrho\leq\sigma$ *and for a.a.* $\omega\in\Omega$ *and all* $t\in[0,\varrho(\omega))$*,* $u(t,\omega)=v(t,\omega)$*.*

Note that L^p_κ-maximal local solutions are unique by definition. In addition, an (unique) L^p_κ-strong solution u on $[\![0,\sigma]\!]$ gives an (unique) L^p_κ-local solution (u,σ) to (4.16). In the following, we omit the prefix L^p_κ and the "on $[0,T]$" if no confusion seems likely.

4.3.2 Statement of the main results

Our first result on (4.16) reads as follows.

Theorem 4.3.5 (Quasilinear I)**.** *Let Hypothesis* **(H$_p$)** *be satisfied. Assume that* $u_0\in L^\infty_{\mathscr{F}_0}(\Omega;X^{\mathsf{Tr}}_{\kappa,p})$ *and* $(A(\cdot,u_0),B(\cdot,u_0))\in\mathcal{SMR}^\bullet_{p,\kappa}(T)$*. Then there exists an* $\varepsilon>0$ *such that if*

$$\max\{L_F,L_G\}<\varepsilon, \tag{4.22}$$

then the following assertions hold:

(1) (Existence and uniqueness) *There exists an* L^p_κ*-maximal local solution* (u,σ) *to* (4.16) *such that* $\sigma>0$ *a.s.*

(2) (Regularity) *There exists a localizing sequence* $(\sigma_n)_{n\geq1}$ *for* (u,σ) *such that* $\sigma_n>0$ *a.s. and*

- *If* $p>2$ *and* $\kappa\in[0,\frac{p}{2}-1)$*, then for all* $n\geq1$*,* $\theta\in[0,\frac12)$*,*

$$u\in L^p(\Omega;H^{\theta,p}(I_{\sigma_n},w_\kappa;X_{1-\theta}))\cap L^p(\Omega;C(\overline{I}_{\sigma_n};X^{\mathsf{Tr}}_{\kappa,p})).$$

Moreover, (u,σ) *instantaneously regularizes to* $u\in C((0,\sigma);X^{\mathsf{Tr}}_p)$ *a.s.*

- *If* $p=2$ *and* $\kappa=0$*, then for all* $n\geq1$*,*

$$u\in L^2(\Omega;L^2(I_{\sigma_n};X_1))\cap L^2(\Omega;C(\overline{I}_{\sigma_n};X_{1/2})).$$

(3) (Continuous dependence on the initial data) *There exist* $\eta,C>0$ *depending on* u_0 *such that if* $v_0\in B_{L^\infty_{\mathscr{F}_0}(\Omega;X^{\mathsf{Tr}}_{\kappa,p})}(u_0,\eta)$*, then the following hold:*

- *there exists an* L^p_κ*-maximal local solution* (v,τ) *to* (4.16) *with* $\tau>0$ *a.s. and initial data* v_0*;*
- *For each stopping time* ν *with* $\nu\in(0,\tau\wedge\sigma]$ *a.s. one has*

$$\|u-v\|_{L^p(\Omega;E)}\leq C\|u_0-v_0\|_{L^p(\Omega;X^{\mathsf{Tr}}_{\kappa,p})},$$

where either $E\in\{H^{\theta,p}(I_\nu,w_\kappa;X_{1-\theta}),C(\overline{I}_\nu;X^{\mathsf{Tr}}_{\kappa,p})\}$ *with* $p>2$*,* $\kappa\in[0,\frac{p}{2}-1)$*,* $\theta\in[0,\frac12)$ *or* $E\in\{L^2(I_\nu;X_1),C(\overline{I}_\nu;X_{1/2})\}$ *and* $p=2$ *and* $\kappa=0$*.*

(4) (Localization) *If* (v,τ) *is an* L^p_κ*-maximal local solution to* (4.16) *with data* $v_0\in L^\infty_{\mathscr{F}_0}(\Omega;X^{\mathsf{Tr}}_{\kappa,p})$*, then setting* $\Gamma:=\{v_0=u_0\}$ *one has*

$$\tau|_\Gamma=\sigma|_\Gamma,\qquad v|_{\Gamma\times[0,\tau)}=u|_{\Gamma\times[0,\sigma)}.$$

A more explicit bound for the number ε in (4.22) will be provided in Remark 4.3.17.

In (2) we see that the paths of the solution are in $C([0,\sigma);X^{\mathsf{Tr}}_{\kappa,p})$. However, if $\kappa>0$, after $t=0$ the regularity immediately improves to $C((0,\sigma);X^{\mathsf{Tr}}_{p})$, where we recall $X^{\mathsf{Tr}}_{\kappa,p}=(X_0,X_1)_{1-\frac{1+\kappa}{p},p}$ and $X^{\mathsf{Tr}}_{p}=(X_0,X_1)_{1-\frac{1}{p},p}$ This phenomena will play a crucial role in Chapter 7. Note that the $L^p(\Omega)$-norms in (3) are well-defined due to Lemma 4.1.7 and the text below it. Furthermore, in Step 4 in the proof of Theorem 4.3.5 we show that the estimates in (3) also hold for the choice $E=\mathfrak{X}(\nu)$ where the space $\mathfrak{X}$ is defined in (4.31) below.

Remark 4.3.6. In applications to SPDEs, one does not always have $u_0\in L^{\infty}_{\mathscr{F}_0}(\Omega;X^{\mathsf{Tr}}_{\kappa,p})$. To weaken this condition we make a further extension of Theorem 4.3.5 at the expense of a stronger hypothesis on F_L,G_L, see Theorem 4.3.7 below.

On the other hand, if the filtration $\mathbb{F}=(\mathscr{F}_t)_{t\geq 0}$ is the augmented filtration generated by the cylindrical Brownian motion W_H, then $L^0_{\mathscr{F}_0}(\Omega;X^{\mathsf{Tr}}_{\kappa,p})=X^{\mathsf{Tr}}_{\kappa,p}$. Thus, Theorem 4.3.5 can be applied without any restriction.

We would like to present an additional result on the quasilinear case, where we weaken the integrability hypothesis on the initial data, at the cost of more restrictions on the nonlinearities F_L,G_L. More specifically, we need a local version of the assumptions (HF)-(HG) and (Hf).

(HF′) The map $F:[0,T]\times\Omega\times X_1\to X_0$ has the same measurability properties in (HF) and it can be decomposed as $F:=F_L+F_c+F_{\mathsf{Tr}}$, where F_c,F_{Tr} are as in (HF). Assume that for each $n\geq 1$ there exist constants $L_{F,n},\widetilde{L}_{F,n},C_{F,n}$, such that for all $x,y\in X_1$, $t\in[0,T]$ and a.a. $\omega\in\Omega$, and $\|x\|_{X^{\mathsf{Tr}}_{\kappa,p}},\|y\|_{X^{\mathsf{Tr}}_{\kappa,p}}\leq n$ one has

$$\|F_L(t,\omega,x)\|_{X_0}\leq C_{F,n}(1+\|x\|_{X_1}),$$
$$\|F_L(t,\omega,x)-F_L(t,\omega,y)\|_{X_0}\leq L_{F,n}\|x-y\|_{X_1}+\widetilde{L}_{F,n}\|x-y\|_{X_0}.$$

(HG′) The map $G:[0,T]\times\Omega\times X_1\to X_0$ has the same measurability properties in (HF) and it can be decomposed as $G:=G_L+G_c+G_{\mathsf{Tr}}$ where G_c,G_{Tr} are as in (HG). Assume that for each $n\geq 1$ there exist constants $L_{G,n},\widetilde{L}_{G,n},C_{G,n}$, such that for all $x,y\in X_1$, $t\in[0,T]$ and a.a. $\omega\in\Omega$, and $\|x\|_{X^{\mathsf{Tr}}_{\kappa,p}},\|y\|_{X^{\mathsf{Tr}}_{\kappa,p}}\leq n$ one has

$$\|G_L(t,\omega,x)\|_{\gamma(H,X_{1/2})}\leq C_{G,n}(1+\|x\|_{X_1}),$$
$$\|G_L(t,\omega,x)-G_L(t,\omega,y)\|_{\gamma(H,X_{1/2})}\leq L_{G,n}\|x-y\|_{X_1}+\widetilde{L}_{G,n}\|x-y\|_{X_0}.$$

(Hf′) $f\in L^0_{\mathscr{P}}(\Omega;L^p(I_T,w_\kappa;X_0))$ and $g\in L^0_{\mathscr{P}}(\Omega;L^p(I_T,w_\kappa;\gamma(H,X_{1/2})))$.

We say that the **Hypothesis (H$_0$)** holds if (HA), (HF′), (HG′), and (Hf′) are satisfied. Definitions 4.3.3 and 4.3.4 clearly extend to the setting of Hypothesis **(H$_0$)**.

To extend Theorem 4.3.5 in case of L^0-data we employ a cut-off argument. To this end, given $u_0\in L^0_{\mathscr{F}_0}(\Omega;X^{\mathsf{Tr}}_{\kappa,p})$ we denote by $(u_{0,n})_{n\geq 1}$ a sequence such that

$$u_{0,n}\in L^{\infty}_{\mathscr{F}_0}(\Omega;X^{\mathsf{Tr}}_{\kappa,p}),\qquad\text{and}\qquad u_{0,n}=u_0\ \text{ on }\ \{\|u_0\|_{X^{\mathsf{Tr}}_{\kappa,p}}\leq n\}.\tag{4.23}$$

For possible choices of $(u_{0,n})_{n\geq 1}$ see (4.26) and the text below it.

Theorem 4.3.7 (Quasilinear II)**.** *Let Hypothesis* **(H$_0$)** *be satisfied. Let $u_0\in L^0_{\mathscr{F}_0}(\Omega;X^{\mathsf{Tr}}_{\kappa,p})$. Assume that there exists $(u_{0,n})_{n\geq 1}$ satisfying* (4.23) *and for all $n\geq 1$*

$$(A(\cdot,u_{0,n}),B(\cdot,u_{0,n}))\in\mathcal{SMR}^{\bullet}_{p,\kappa}(T).\tag{4.24}$$

There exists a decreasing sequence $(\varepsilon_n)_{n\geq 1}$ in $(0,\infty)$ such that if

$$\max\{L_{F,n},L_{G,n}\}<\varepsilon_n,\qquad\textit{for all } n\geq 1,\tag{4.25}$$

then the following assertions hold:

(1) (Existence and uniqueness) *There exists an* L^p_κ*-maximal local solution* (u,σ) *to* (4.16) *such that* $\sigma>0$ *a.s.*

(2) (Regularity) *For each localizing sequence* $(\sigma_n)_{n\geq 1}$ *for* (u,σ)*, one has*

- *If* $p>2$ *and* $\kappa\in[0,\frac{p}{2}-1)$*, then for all* $n\geq 1$*,* $\theta\in[0,\frac12)$*,*

$$u\in H^{\theta,p}(I_{\sigma_n},w_\kappa;X_{1-\theta})\cap C(\overline{I}_{\sigma_n};X^{\mathsf{Tr}}_{\kappa,p}) \quad a.s.$$

Moreover, (u,σ) *instantaneously regularizes to* $u\in C((0,\sigma);X^{\mathsf{Tr}}_p)$ *a.s.*

- *If* $p=2$ *and* $\kappa=0$*, then for all* $n\geq 1$*,*

$$u\in L^2(I_{\sigma_n};X_1)\cap C(\overline{I}_{\sigma_n};X_{1/2}) \quad a.s.$$

(3) (Local existence and continuous dependence on the initial data) *For any* $n\geq 1$*, let* $\Gamma_n:=\{\|u_0\|_{X^{\mathsf{Tr}}_{\kappa,p}}\leq n\}$*. Then Theorem* 4.3.5(3) *holds with* u_0,v_0 *and* Ω *replaced by* $\mathbf{1}_{\Gamma_n}u_0,\mathbf{1}_{\Gamma_n}v_0$ *and* Γ_n *respectively.*

(4) (Localization) *Theorem* 4.3.5(4) *holds, where the assumptions on* u_0,v_0 *are replaced by* $u_0,v_0\in L^0_{\mathscr{F}_0}(\Omega;X^{\mathsf{Tr}}_{\kappa,p})$*.*

For the more precise estimates on the sequence $(\varepsilon_n)_{n\geq 1}$ we refer to Remark 4.3.19.

Let $u_0\in L^0_{\mathscr{F}_0}(\Omega;X^{\mathsf{Tr}}_{\kappa,p})$ and set $\Gamma_n:=\{\|u_0\|_{X^{\mathsf{Tr}}_{\kappa,p}}\leq n\}\in\mathscr{F}_0$. A typical choice of the sequence $(u_{0,n})_{n\geq 1}$ in Theorem 4.3.7 is given by

$$u_{0,n}:=\mathbf{1}_{\Gamma_n}u_0+\mathbf{1}_{\Omega\setminus\Gamma_n}\frac{nu_0}{\|u_0\|_{X^{\mathsf{Tr}}_{\kappa,p}}}. \tag{4.26}$$

However, the condition (4.23) allows us to choose $u_{0,n}|_{\Omega\setminus\Gamma_n}$ differently, and we will exploit this fact in applications. More precisely, instead of (4.26) one can use $u_{0,n}=\mathbf{1}_{\Gamma_n}u_0+\mathbf{1}_{\Omega\setminus\Gamma_n}x$ where $x\in X^{\mathsf{Tr}}_{\kappa,p}$ can be chosen such that (4.24) holds. Throughout Chapter 4 we will use the choice (4.26), but in Subsection 5.2.6 we need a different choice (see (5.81)).

If (4.16) is of semilinear type, (see Assumption 4.2.2), the condition $u_0\in L^\infty_{\mathscr{F}_0}(\Omega;X^{\mathsf{Tr}}_{\kappa,p})$ can be weakened and we still get L^p-integrability with respect to $\omega\in\Omega$. More precisely, we have the following.

Theorem 4.3.8 (Semilinear)**.** *Let the Hypothesis* **(H$_p$)** *be satisfied, where* A *and* B *are of semilinear type as in Assumption* 4.2.2*. There exists an* $\varepsilon>0$ *such that if*

$$\max\{L_G,L_F\}<\varepsilon, \tag{4.27}$$

then the following assertions hold:

(1) *If* $u_0\in L^\infty_{\mathscr{F}_0}(\Omega;X^{\mathsf{Tr}}_{\kappa,p})$*, then the statements in Theorem* 4.3.5(1)–(4) *hold.*

(2) *If* $u_0\in L^p_{\mathscr{F}_0}(\Omega;X^{\mathsf{Tr}}_{\kappa,p})$ *and the constants* $C_{c,n},L_{c,n},C_{\mathsf{Tr},n},L_{\mathsf{Tr},n}$ *in* (HF)-(HG) *do not depend on* $n\geq 1$*, then the statements in Theorem* 4.3.5(1)–(4) *hold.*

(3) *If* $u_0\in L^0_{\mathscr{F}_0}(\Omega;X^{\mathsf{Tr}}_{\kappa,p})$*, then the statements in Theorem* 4.3.7(1)–(4) *hold.*

Assertion (1) is immediate from Theorem 4.3.5. Under additional growth conditions one can often derive L^p-estimates as well. Assertion (2) shows that in the semilinear case the condition $u_0\in L^\infty_{\mathscr{F}_0}(\Omega;X^{\mathsf{Tr}}_{\kappa,p})$ in Theorem 4.3.5 can be weakened. Assertion (3) will be immediate from the proof of Theorem 4.3.7.

4.3.3 The role of the space $\mathfrak{X}(T)$

In the proofs of the results stated in Subsection 4.3.2 in the case $p > 2$ we need a family of function spaces $(\mathfrak{X}(t))_{t\in[0,T]}$ having the following three properties: the nonlinearities $F_c(\cdot, u), G_c(\cdot, u)$ can be controlled by $\|u\|_{\mathfrak{X}(t)}$, the map $[0, T] \ni t \mapsto \|f|_{(0,t)}\|_{\mathfrak{X}(t)}$ is continuous for all $f \in \mathfrak{X}(T)$, and

$$H^{\delta,p}(I_T; w_\kappa; X_{1-\delta}) \cap L^p(I_T, w_\kappa; X_1) \hookrightarrow \mathfrak{X}(T) \tag{4.28}$$

for some $\delta \in (\frac{1+\kappa}{p}, \frac{1}{2})$. Note that the left-hand side in (4.28) is part of our usual maximal regularity space (see Definition 4.2.5). As mentioned below Lemma 4.1.7, it is not obvious whether the $H^{\delta,p}(I_t, w_\kappa; X_{1-\delta})$-norm is continuous in t and therefore we do not define $\mathfrak{X}(T)$ as the left-hand side of (4.28).

Recall that the numbers $(\rho_j)_{j=1}^{m_F+m_G}$, $(\beta_j)_{j=1}^{m_F+m_G}$ and $(\varphi_j)_{j=1}^{m_F+m_G}$ are defined in (HF) and (HG). In the case that for some $j \in \{1, \dots, m_F + m_G\}$, (4.18) or (4.20) holds with *strict* inequality, we may increase ρ_j in order to obtain equality. More precisely, we set

$$\rho_j^\star := \frac{1-\beta_j}{\varphi_j - 1 + (1+\kappa)/p}, \qquad j \in \{1, \dots, m_F + m_G\}. \tag{4.29}$$

Since $\beta_j < 1$ and $\varphi_j > 1 - (1 + \kappa)/p$, one has $\rho_j^\star > 0$ for all $j \in \{1, \dots, m_F + m_G\}$.

To define a space $\mathfrak{X}(T)$ which satisfies the previous requirements, suppose (HF)-(HG) are satisfied and let $\rho_j^\star$ be as in (4.29). For $j \in \{1, \dots, m_F + m_G\}$ we let

$$\frac{1}{r_j'} := \frac{\rho_j^\star(\varphi_j - 1 + (1+\kappa)/p)}{(1+\kappa)/p} < 1, \qquad \frac{1}{r_j} := \frac{\beta_j - 1 + (1+\kappa)/p}{(1+\kappa)/p} < 1 \tag{4.30}$$

and, for $T > 0$,

$$\mathfrak{X}(T) := \Big(\bigcap_{j=1}^{m_F+m_G} L^{pr_j}(I_T, w_\kappa; X_{\beta_j})\Big) \cap \Big(\bigcap_{j=1}^{m_F+m_G} L^{\rho_j^\star pr_j'}(I_T, w_\kappa; X_{\varphi_j})\Big). \tag{4.31}$$

We will see that $\mathfrak{X}(T)$ is the natural space to control the non-linearities F_c, G_c; see Lemmas 4.3.11 and 4.3.13 below. Moreover, if (4.28) holds, then the solution paths will automatically be in (4.31) as soon as we have maximal regularity. Finally, we note that the continuity of the $\mathfrak{X}$-norm in $t \in [0, T]$ follows from the Lebesgue dominated convergence theorem.

Next we prove (4.28) with the above defined $\mathfrak{X}$ under a trace condition. The general case is discussed in Remark 4.3.10. Here we follow [180, Section 2].

Lemma 4.3.9. *Let (HF)-(HG) be satisfied. Let $T \in (0, \infty]$ and let r_j, r_j' and $\mathfrak{X}(T)$ be as in (4.30) and (4.31) respectively. If $p > 2$ and $\kappa \in [0, \frac{p}{2} - 1)$, then for any $\delta \in (\frac{1+\kappa}{p}, \frac{1}{2})$*

$${}_0H^{\delta,p}(0, T; w_\kappa; X_{1-\delta}) \cap L^p(0, T, w_\kappa; X_1) \hookrightarrow \mathfrak{X}(T),$$

where the embedding constant can be chosen to be independent of T.

Furthermore, if $p = 2$ and $\kappa = 0$, the same holds with ${}_0H^{\delta,p}(0, T; w_\kappa; X_{1-\delta})$ replaced by $C([0, T]; X_{1/2})$.

Proof. Recall that in (4.29) we have defined $\rho_j^\star$ such that (4.18)-(4.20) hold with equality for each $j \in \{1, \dots, m_F + m_G\}$. Due to (4.30), this implies that

$$\frac{1}{r_j} + \frac{1}{r_j'} = 1, \text{ for all } j \in \{1, \dots, m_F + m_G\}.$$

Step 1: Case $p = 2$, $\kappa = 0$. Let $\vartheta \in (0, 1)$ be arbitrary. By interpolation one has

$$\|x\|_{X_{\frac{1}{2}+\frac{\vartheta}{2}}} \leq \|x\|_{X_{\frac{1}{2}}}^{1-\vartheta} \|x\|_{X_1}^{\vartheta},$$

for $x \in X_1$. Thus, by Young's inequality

$$\begin{aligned}\|u\|_{L^{\frac{2}{\vartheta}}(0,T;X_{\frac12+\frac{\vartheta}{2}})} &\le \|u\|^{1-\vartheta}_{C([0,T];X_{1/2})}\|u\|^{\vartheta}_{L^2(I_T;X_1)}\\ &\le (1-\vartheta)\|u\|_{C([0,T];X_{1/2})} + \vartheta\|u\|_{L^2(I_T;X_1)}.\end{aligned}$$

Therefore, we have the following contractive embedding

$$C([0,T];X_{1/2}) \cap L^2(0,T;X_1) \hookrightarrow L^{\frac{2}{\vartheta}}(0,T;X_{\frac12+\frac{\vartheta}{2}}). \tag{4.32}$$

By (4.32) with $\vartheta = 1/r_j = 2(\beta_j - 1/2)$ and $\vartheta = 1/(\rho_j^\star r_j') = 2(\varphi_j - 1/2)$ one obtains

$$C([0,T];X_{1/2}) \cap L^2(0,T;X_1) \hookrightarrow L^{2r_j}(0,T;X_{\beta_j}) \cap L^{2\rho_j^\star r_j'}(0,T;X_{\varphi_j}).$$

Step 2: case $p>2$ and $\kappa \in [0,\frac{p}{2}-1)$. By Proposition 4.1.2 for each $j \in \{1,\dots,m_F+m_G\}$

$$ {}_0H^{1-\beta_j,p}(0,T,w_\kappa;X_\lambda) \hookrightarrow L^{pr_j}(0,T,w_\kappa;X_\lambda), \tag{4.33}$$

$$ {}_0H^{1-\varphi_j,p}(0,T,w_\kappa;X_\lambda) \hookrightarrow L^{\rho_j^\star pr_j'}(0,T,w_\kappa;X_\lambda), \tag{4.34}$$

for each $\lambda \in [0,1]$ and where the embedding constants do not depend on T.

Let $0<\eta<\zeta<1$ and assume $\eta \neq (1+\kappa)/p$. Using Proposition 4.1.3 with $\theta := \frac{\zeta-\eta}{\zeta} \in (0,1)$, one obtains

$$ {}_0H^{\zeta,p}(0,T;w_\kappa;X_{1-\zeta}) \cap L^p(0,T,w_\kappa;X_1) \hookrightarrow {}_0H^{\eta,p}(0,T,w_\kappa;X_{1-\eta}), \tag{4.35}$$

where we used that $[X_{1-\zeta},X_1]_\theta = X_\eta$, which follows immediately from the reiteration theorem for complex interpolation and Assumption 4.2.1.

Let $\delta \in (\frac{1+\kappa}{p},\frac12)$ be arbitrary. Since $\mathcal{B}_j \in (1-\frac{1+\kappa}{p},1)$ one has $\delta > 1-\beta_j \in (0,\frac{1+\kappa}{p})$ for each $j \in \{1,\dots,m_F+m_G\}$, and hence it follows that

$$\begin{aligned} {}_0H^{\delta,p}(0,T;w_\kappa;X_{1-\delta}) \cap L^p(0,T,w_\kappa;X_1) &\hookrightarrow {}_0H^{1-\beta_j,p}(0,T,w_\kappa;X_{\beta_j})\\ &\hookrightarrow L^{pr_j}(0,T,w_\kappa;X_{\beta_j}),\end{aligned}$$

where in the first embedding we used $\delta > 1-(1+\kappa)/p > 1-\beta_j$ and (4.35), and the second one follows from (4.33). Analogously, for $j \in \{1,\dots,m_F+m_G\}$, using $\delta > \frac{1+\kappa}{p} > 1-\varphi_j$, (4.34), and (4.35), one obtains

$$\begin{aligned} {}_0H^{\delta,p}(0,T;w_\kappa;X_{1-\delta}) \cap L^p(0,T,w_\kappa;X_1) &\hookrightarrow {}_0H^{1-\varphi_j,p}(0,T,w_\kappa;X_{\varphi_j})\\ &\hookrightarrow L^{\rho_j^\star pr_j'}(0,T,w_\kappa;X_{\varphi_j}).\end{aligned}$$

Putting together the above inclusions the result follows. □

Remark 4.3.10. Let $p>2$. The embedding in Lemma 4.3.9 also holds in the case where ${}_0H^{\delta,p}$ is replaced by $H^{\delta,p}$, but with an embedding constant which depends on $T>0$.

Let us show that the space $\mathfrak{X}(T)$ defined in (4.31) is well suited to bound the nonlinearities F_c, G_c. Actually, we prove a more refined result since this will be needed in Chapter 6.

Lemma 4.3.11. *Let the hypothesis (HF)-(HG) be satisfied. Let $0<T<\infty$ and $N \ge 1$ be fixed. Then there exists $C_T>0$ and $\zeta>1$ such that for all $u \in C(\overline{I}_T;X^{\mathsf{Tr}}_{\kappa,p}) \cap \mathfrak{X}(T)$ which verifies $\|u\|_{C(\overline{I}_T;X^{\mathsf{Tr}}_{\kappa,p})} \le N$, one has a.s.*

$$\begin{aligned}\|F_c(\cdot,u)-F_c(\cdot,0)\|_{L^p(I_T,w_\kappa;X_0)} + \|G_c(\cdot,u)-G_c(\cdot,0)\|_{L^p(I_T,w_\kappa;\gamma(H,X_{1/2}))}&\\ \le C_T(\|u\|_{\mathfrak{X}(T)} + \|u\|^{\zeta}_{\mathfrak{X}(T)}).&\end{aligned}$$

Moreover, if $X^{\mathsf{Tr}}_{\kappa,p}$ is not critical for (4.16), then $\lim_{T\downarrow 0} C_T = 0$.

Proof. For notational simplicity we only consider the case $m_F = 1$. Thus, we set $\rho^\star := \rho_1^\star$, $\rho := \rho_1$, $\varphi := \varphi_1$ and $\beta := \beta_1$. In this case,

$$\mathfrak{X}(T) = L^{pr}(I_T, w_\kappa; X_\beta) \cap L^{\rho p r'}(I_T, w_\kappa; X_\varphi), \tag{4.36}$$

where $r := r_1$ and $r' := r_1'$ are defined in (4.30). Thus, by (HF), for $x \in X_\varphi$,

$$\|F_c(t,x) - F_c(t,0)\|_{X_0} \leq L_{c,N}(1 + \|x\|_{X_\varphi}^\rho)\|x\|_{X_\beta}.$$

This implies

$$\begin{aligned} &\|F_c(\cdot,u) - F_c(\cdot,0)\|_{L^p(0,T,w_\kappa;X_0)} \\ &\leq L_{c,N}\|u\|_{L^p(0,T,w_\kappa;X_\beta)} + \big\|\|u\|_{X_\varphi}^\rho \|u\|_{X_\beta}\big\|_{L^p(0,T,w_\kappa)} \\ &\leq L_{c,N}(C_T\|u\|_{L^{pr}(0,T,w_\kappa;X_\beta)} + \|u\|^\rho_{L^{\rho p r'}(0,T,w_\kappa;X_\varphi)}\|u\|_{L^{pr}(0,T,w_\kappa;X_\beta)}); \end{aligned} \tag{4.37}$$

where $\lim_{T\downarrow 0} C_T = 0$. For simplicity, let us distinguish two cases:

Case $\rho^\star = \rho := \rho_1$. In other words $X^{\mathsf{Tr}}_{\kappa,p}$ is critical for (4.16). The claimed inequality follows from (4.36)-(4.37) by setting $\zeta = 1 + \rho$.

Case $\rho^\star > \rho := \rho_1$. By the Hölder inequality, one has

$$\|u\|^\rho_{L^{\rho p r'}(0,T,w_\kappa)} \leq C_T\|u\|^\rho_{L^{\rho^\star p r'}(0,T,w_\kappa;X_\varphi)} \leq C_T\|u\|^\rho_{\mathfrak{X}(T)},$$

where $\lim_{T\downarrow 0} C_T = 0$.

The assertion for G_c is proved in the same way. ☐

Remark 4.3.12. If the constants $L_{c,n}$ in (HF)(*ii*) and (HG)(*ii*) do not depend on $n \geq 1$, then the constant C_T can be chosen independent of N and the above proof extends to any $u \in \mathfrak{X}(T)$.

4.3.4 Truncation lemmas

In this subsection we collect several truncation lemmas which are needed in the proofs of Theorems 4.3.5, 4.3.7 and 4.3.8.

First we define suitable truncations of F_c, G_c. To this end let $\xi \in W^{1,\infty}([0,\infty))$ be such that $\xi = 1$ on $[0,1]$ and $\xi = 0$ outside $[2,\infty)$ and ξ is linear on $[1,2]$. For each $\lambda > 0$, set $\xi_\lambda(x) := \xi(x/\lambda)$ for $x \in \mathbb{R}_+$. Then $\operatorname{supp} \xi_\lambda \subseteq (0, 2\lambda)$, $\xi_\lambda|_{(0,\lambda)} = 1$ and $\|\xi_\lambda'\|_{L^\infty(\mathbb{R}_+)} \leq 1/\lambda$. For $t \in [0,T]$, $x \in X^{\mathsf{Tr}}_{\kappa,p}$, and $u \in \mathfrak{X}(T) \cap C(\overline{I}_T; X^{\mathsf{Tr}}_{\kappa,p})$ we set

$$\Theta_\lambda(t,x,u) := \xi_\lambda\Big(\|u\|_{\mathfrak{X}(t)} + \sup_{s\in[0,t]} \|u(s) - x\|_{X^{\mathsf{Tr}}_{\kappa,p}}\Big). \tag{4.38}$$

In the next lemma we fix $\omega \in \Omega$, but omit it from our notation.

Lemma 4.3.13. *Let (HF)-(HG) be satisfied. Let $T > 0$ and let $\sigma \in [0,T]$. Let Θ_λ be defined in (4.38). For any $\lambda \in (0,1)$, let the maps*

$$\begin{aligned} F_{c,\lambda} &: X^{\mathsf{Tr}}_{\kappa,p} \times \mathfrak{X}(\sigma) \cap C(\overline{I}_\sigma; X^{\mathsf{Tr}}_{\kappa,p}) \to L^p(I_\sigma, w_\kappa; X_0)), \\ G_{c,\lambda} &: X^{\mathsf{Tr}}_{\kappa,p} \times \mathfrak{X}(\sigma) \cap C(\overline{I}_\sigma; X^{\mathsf{Tr}}_{\kappa,p}) \to L^p(I_\sigma, w_\kappa; \gamma(H, X_{1/2})), \end{aligned}$$

be given by

$$\begin{aligned} F_{c,\lambda}(x,u) &:= \Theta_\lambda(\cdot,x,u)(F_c(\cdot,u) - F_c(\cdot,0)), \\ G_{c,\lambda}(x,u) &:= \Theta_{c,\lambda}(\cdot,x,u)(G_c(\cdot,u) - G_c(\cdot,0)). \end{aligned}$$

Then for any $N \geq 1$ there exist constants $C_\lambda, L_{T,\lambda}$ such that if $\|x\|_{X^{\mathsf{Tr}}_{\kappa,p}} \leq N$,

$$\|F_{c,\lambda}(x,u)\|_{L^p(I_\sigma,w_\kappa;X_0)} \leq C_\lambda$$

$$\|G_{c,\lambda}(x,u)\|_{L^p(I_\sigma,w_\kappa;\gamma(H,X_{1/2}))} \le C_\lambda$$
$$\|F_{c,\lambda}(x,u) - F_{c,\lambda}(\cdot,x,v)\|_{L^p(I_\sigma,w_\kappa;X_0)} \le L_{\lambda,T}(\|u-v\|_{\mathfrak{X}(\sigma)} + \|u-v\|_{C(\bar{I}_\sigma;X^{\mathsf{Tr}}_{\kappa,p})}),$$
$$\|G_{c,\lambda}(\cdot,x,u) - G_{c,\lambda}(\cdot,x,v)\|_{L^p(I_\sigma,w_\kappa;\gamma(H,X_{1/2}))} \le L_{\lambda,T}(\|u-v\|_{\mathfrak{X}(\sigma)} + \|u-v\|_{C(\bar{I}_\sigma;X^{\mathsf{Tr}}_{\kappa,p})});$$

a.s. Moreover, for each $\varepsilon > 0$ there exists $\bar{T} = \bar{T}(\varepsilon) > 0$ and $\bar{\lambda} = \bar{\lambda}(\varepsilon) > 0$ such that for all $T \in (0,\bar{T})$, $\lambda \in (0,\bar{\lambda})$ one has $L_{\lambda,T} < \varepsilon$.

Proof. We only consider the estimates for $F_{c,\lambda}$ since the other case is similar. Recall that in (4.29) we have defined $\rho_j^\star$ such that (4.18)-(4.20) hold with equality for each $j \in \{1,\dots,m_F+m_G\}$. For notational convenience, we assume that $m_F = 1$ and we set $\rho := \rho_1^\star$, $\varphi := \varphi_1$, $\beta := \beta_1$ and $r := r_1$, $r' := r_1'$ (see (4.30)). The general case can be proven with the same considerations. Moreover, it is enough to consider the case $\sigma = T$. Moreover, in the proof C_T denotes a suitable constant, which can be different from line to line, and verifies $\lim_{T\downarrow 0} C_T = 0$.

Set $\widetilde{F}_c(t,u) := F_c(t,u) - F_c(t,0)$. Thus, $\widetilde{F}_c(t,0) = 0$, and by (HF) it follows that for $u,v \in X_\varphi$,

$$\|\widetilde{F}_c(t,u) - \widetilde{F}_c(t,v)\|_{X_0} \le L_{c,N+2}(1 + \|u\|^\rho_{X_\varphi} + \|v\|^\rho_{X_\varphi})\|u-v\|_{X_\beta}, \tag{4.39}$$

provided $\|u\|_{X^{\mathsf{Tr}}_{\kappa,p}}, \|v\|_{X^{\mathsf{Tr}}_{\kappa,p}} \le N+2$. For convenience we set $L_{c,N+2} =: C_F$.

Let us set

$$\tau_u := \inf\Big\{t \in [0,T] \,:\, \|u\|_{\mathfrak{X}(t)} + \sup_{s\in[0,t]} \|u(s) - x\|_{X^{\mathsf{Tr}}_{\kappa,p}} \ge 2\lambda\Big\} \wedge T. \tag{4.40}$$

Then since $\Theta_\lambda(t,x,u) = 0$ if $t \ge \tau_u$ we can write

$$\begin{aligned}
\|F_{c,\lambda}(x,u)\|_{L^p(I_T,w_\kappa;X_0)} &= \|F_{c,\lambda}(x,u)\|_{L^p(0,\tau_u,w_\kappa;X_0)} \\
&\overset{(i)}{\le} C_F\Big(\int_0^{\tau_u} (1+\|u\|^\rho_{X_\varphi})^p \|u\|^p_{X_\beta} t^\kappa dt\Big)^{1/p} \\
&\overset{(ii)}{\le} C_F(\|u\|_{L^p(I_{\tau_u},w_\kappa;X_\beta)} + \|u\|^\rho_{L^{\rho p r'}(I_{\tau_u},w_\kappa;X_\varphi)} \|u\|_{L^{pr}(I_{\tau_u},w_\kappa;X_\beta)}) \\
&\overset{(iii)}{\le} C_F(C_T\|u\|_{L^{pr}(I_{\tau_u},w_\kappa;X_\beta)} + \|u\|^\rho_{L^{\rho p r'}(I_{\tau_u},w_\kappa;X_\varphi)} \|u\|_{L^{pr}(I_{\tau_u},w_\kappa;X_\beta)}) \\
&\overset{(iv)}{\le} C_F(2C_T\lambda + (2\lambda)^\rho 2\lambda) =: C_\lambda.
\end{aligned}$$

In (i) we used (4.39) and the fact that $\|u\|_{C(\bar{I}_{\tau_u};X^{\mathsf{Tr}}_{\kappa,p})} \le N+2$, $\|x\|_{L^\infty(\Omega;X^{\mathsf{Tr}}_{\kappa,p})} \le N$, and $\lambda \in (0,1)$. In (ii) and (iii) we used Hölder's inequality with exponent r, r' defined in (4.30). In (iv) we used (4.40).

Next we estimate $\Delta F := F_{c,\lambda}(x,u) - F_{c,\lambda}(x,v)$. Without loss of generality, we may assume that $\tau_u \le \tau_v$. Clearly, we can estimate

$$\begin{aligned}
\|\Delta F\|_{L^p(I_T,w_\kappa;X_0)} \le &\|\Theta_\lambda(\cdot,x,u)(\widetilde{F}_c(\cdot,u) - \widetilde{F}_c(\cdot,v))\|_{L^p(I_T,w_\kappa;X_0)} \\
&+ \|(\Theta_\lambda(\cdot,x,u) - \Theta_\lambda(\cdot,x,v))\widetilde{F}_c(\cdot,v)\|_{L^p(I_T,w_\kappa;X_0)} =: R_1 + R_2.
\end{aligned}$$

Since $\Theta(t,x,u) = 0$, the term R_1 can be estimated as

$$\begin{aligned}
R_1 &= \|\Theta_\lambda(\cdot,x,u)(\widetilde{F}_c(u) - \widetilde{F}_c(v))\|_{L^p(0,\tau_u,w_\kappa;X_0)} \\
&\overset{(i)}{\le} C_F\Big(\int_0^{\tau_u} (1+\|u(t)\|^\rho_{X_\varphi} + \|v(t)\|^\rho_{X_\varphi})^p \|u(t)-v(t)\|^p_{X_\beta} t^\kappa dt\Big)^{\frac{1}{p}} \\
&\overset{(ii)}{\le} C_F\Big(C_T + \|u\|^\rho_{L^{\rho p r'}(0,\tau_u,w_\kappa;X_\varphi)} + \|v\|^\rho_{L^{\rho p r'}(0,\tau_u,w_\kappa;X_\varphi)}\Big)\|u-v\|_{L^{pr}(I_T,w_\kappa;X_\beta)} \\
&\overset{(iii)}{\le} C_F\Big(C_T + 2^{1+\rho}\lambda^\rho\Big)\|u-v\|_{\mathfrak{X}(T)}.
\end{aligned}$$

In (i) we used (4.39), in (ii) we used Hölder's inequality with exponent r, r', and (iii) follows from $\tau_u \leq \tau_v$. For R_2 note that since $\|\xi_\lambda'\|_{L^\infty(\mathbb{R}_+)} \leq 1/\lambda$, for all $t \in [0, T]$, one has

$$\begin{aligned}
&|\Theta_\lambda(t, x, u) - \Theta_\lambda(t, x, v)| \\
&\quad \leq \frac{1}{\lambda}\big|\|u\|_{\mathfrak{X}(t)} - \|v\|_{\mathfrak{X}(t)} + \|u - x\|_{C(\bar{I}_t; X^{\mathsf{Tr}}_{\kappa,p})} - \|v - x\|_{C(\bar{I}_t; X^{\mathsf{Tr}}_{\kappa,p})}\big| \\
&\quad \leq \frac{1}{\lambda}\big[\|u - v\|_{\mathfrak{X}(T)} + \|u - v\|_{C(\bar{I}_T; X^{\mathsf{Tr}}_{\kappa,p})}\big].
\end{aligned}$$

Therefore, using that $\Theta_\lambda(t, x, u) = \Theta_\lambda(t, x, v) = 0$ if $t \geq \tau_v$, we obtain

$$\begin{aligned}
R_2 &= \|(\Theta_\lambda(\cdot, x, u) - \Theta_\lambda(\cdot, x, v))\widetilde{F}_c(\cdot, v)\|_{L^p(I_{\tau_v}, w_\kappa; X_0)} \\
&\leq \frac{1}{\lambda}\big[\|u - v\|_{\mathfrak{X}(T)} + \|u - v\|_{C(\bar{I}_T; X^{\mathsf{Tr}}_{\kappa,p})}\big]\|\widetilde{F}_c(\cdot, v)\|_{L^p(0, \tau_v, w_\kappa; X_0)}.
\end{aligned}$$

By Hölder's inequality, and $\|v\|_{\mathfrak{X}(\tau_v)} \leq 2\lambda$ (see (4.40)), we obtain

$$\begin{aligned}
\|\widetilde{F}_c(\cdot, v)\|_{L^p(0, \tau_v, w_\kappa; X_0)} &\leq C_F\Big(\int_0^{\tau_v} (1 + \|v\|^\rho_{X_\varphi})^p \|v\|^p_{X_\beta} t^\kappa dt\Big)^{\frac{1}{p}} \\
&\leq C_F[C_T + \|v\|^\rho_{L^{p\rho r'}(0, \tau_v, w_\kappa; X_\varphi)}]\|v\|_{L^{pr}(0, \tau_v, w_\kappa; X_\beta)} \\
&\leq 2C_F(C_T + (2\lambda)^\rho)\lambda
\end{aligned}$$

It follows that

$$R_2 \leq 2C_F(C_T + (2\lambda)^\rho)(\|u - v\|_{\mathfrak{X}(T)} + \|u - v\|_{C(\bar{I}_T; X^{\mathsf{Tr}}_{\kappa,p})}).$$

□

Remark 4.3.14. In the setting of Lemma 4.3.13, if the constants $L_{n,c}, C_{n,c}$ in (HF)(ii)-(HG)(ii) do not depend on $n \geq 1$, then Lemma 4.3.13 also holds with $\Theta_\lambda(t, u, x)$ replaced by $\widetilde{\Theta}_\lambda(t, x, u) := \xi_\lambda(\|u\|_{\mathfrak{X}(t)})$.

The last ingredient we need for the proof of Theorem 4.3.5 is a suitable truncation of the remaining non-linearities $A, B, F_{\mathsf{Tr}}, G_{\mathsf{Tr}}$. Here the proof in [104, Lemma 4.4] extends to our setting. Let ξ_λ be the truncation defined before Lemma 4.3.13. For $t \in [0, T]$, $x \in X^{\mathsf{Tr}}_{\kappa,p}$, and $u \in C(\bar{I}_T; X^{\mathsf{Tr}}_{\kappa,p}) \cap L^p(I_T, w_\kappa; X_1)$ we set

$$\Psi_\lambda(t, x, u) := \xi_\lambda\Big(\sup_{s \in [0,t]} \|u(s) - x\|_{X^{\mathsf{Tr}}_{\kappa,p}} + \|u\|_{L^p(I_t, w_\kappa; X_1)}\Big). \tag{4.41}$$

Similar to Lemma 4.3.13, we have the following.

Lemma 4.3.15. *Let (HA), (HF)-(HG) be satisfied. Let $T > 0$, $\lambda \in (0, 1)$, and let σ be a stopping time with values in $[0, T]$. Moreover, let the maps*

$$\begin{aligned}
F_{A,\lambda}(x, \cdot) &: X^{\mathsf{Tr}}_{\kappa,p} \times L^p(I_\sigma, w_\kappa; X_1) \cap C(\bar{I}_\sigma; X^{\mathsf{Tr}}_{\kappa,p}) \to L^p(I_\sigma, w_\kappa; X_0), \\
G_{B,\lambda}(x, \cdot) &: X^{\mathsf{Tr}}_{\kappa,p} \times L^p(I_\sigma, w_\kappa; X_1) \cap C(\bar{I}_\sigma; X^{\mathsf{Tr}}_{\kappa,p}) \to L^p(I_\sigma, w_\kappa; \gamma(H, X_{1/2})),
\end{aligned}$$

be given by

$$\begin{aligned}
F_{A,\lambda}(x, u) &:= \Psi_\lambda(\cdot, x, u)[(A(\cdot, x) - A(\cdot, u))u + F_{\mathsf{Tr}}(\cdot, u) - F_{\mathsf{Tr}}(\cdot, x)], \\
G_{B,\lambda}(x, u) &:= \Psi_\lambda(\cdot, x, u)[-(B(\cdot, x) - B(\cdot, u))u + G_{\mathsf{Tr}}(\cdot, u) - G_{\mathsf{Tr}}(\cdot, x)].
\end{aligned}$$

Then for any $N \geq 1$ there exist constants $\widetilde{C}_\lambda, L_{T,\lambda}$ such that for all $\|x\|_{X^{\mathsf{Tr}}_{\kappa,p}} < N$,

$$\|F_{A,\lambda}(x, u)\|_{L^p(I_\sigma, w_\kappa; X_0)} \leq \widetilde{C}_\lambda\,,$$

$$\|G_{B,\lambda}(x,u)\|_{L^p(I_\sigma,w_\kappa;\gamma(H,X_{1/2}))} \leq \widetilde{C}_\lambda,$$
$$\|F_{A,\lambda}(x,u)-F_{A,\lambda}(x,v)\|_{L^p(I_\sigma,w_\kappa;X_0)} \leq \widetilde{L}_{\lambda,T}(\|u-v\|_{L^p(I_\sigma,w_\kappa;X_1)}+\|u-v\|_{C(\overline{I}_\sigma;X^{\mathsf{Tr}}_{\kappa,p})}),$$
$$\|G_{B,\lambda}(x,u)-G_{B,\lambda}(x,v)\|_{L^p(I_\sigma,w_\kappa;\gamma(H,X_{1/2}))} \leq \widetilde{L}_{\lambda,T}(\|u-v\|_{L^p(I_\sigma,w_\kappa;X_1)}+\|u-v\|_{C(\overline{I}_\sigma;X^{\mathsf{Tr}}_{\kappa,p})}),$$

a.s. Moreover, for each $\varepsilon>0$ *there exist* $\bar{T}=\bar{T}(\varepsilon)>0$ *and* $\bar{\lambda}=\bar{\lambda}(\varepsilon)>0$ *such that*

$$\widetilde{L}_{\lambda,T}<\varepsilon,$$

for any $T<\bar{T}$, $\lambda<\bar{\lambda}$.

Proof. Recall that $L_{\mathsf{Tr},n}, L_{A,n}, L_F, \widetilde{L}_F$ are the constants defined in (HA), (HF)-(HG). For simplicity we set $L:=L_{N+2}:=\max\{L_{\mathsf{Tr},N+2}, L_{A,N+2}, \widetilde{L}_F\}$, where N is as in the statement. Moreover, as before $C_T>0$ denotes a constant which may change from line to line and satisfies $\lim_{T\downarrow 0} C_T=0$. We proof only the estimates for $F_{A,\lambda}$, since the other follows similarly. Again, as in Lemma 4.3.13, the above claimed estimates are pointwise with respect to $\omega\in\Omega$. Thus, it is enough to consider the case $\sigma=T$.

To begin, we set

$$\zeta_u := \inf\{t\in[0,T] \,:\, \|u\|_{L^p(I_t,w_\kappa;X_1)}+\sup_{s\in[0,t]}\|u(s)-x\|_{X^{\mathsf{Tr}}_{\kappa,p}}>2\lambda\}\wedge T. \tag{4.42}$$

Without loss of generality we can assume $\zeta_u\geq\zeta_v$. Firstly,

$$\begin{aligned}
&\|F_{A,\lambda}(x,u)\|_{L^p(I_T,w_\kappa;X_0)}\\
&\overset{(i)}{\leq} \|A(\cdot,x)u-A(\cdot,u)u\|_{L^p(I_{\zeta_u},w_\kappa;X_0)}+\|F_{\mathsf{Tr}}(\cdot,u)-F_{\mathsf{Tr}}(\cdot,x)\|_{L^p(I_{\zeta_u},w_\kappa;X_0)}\\
&\overset{(ii)}{\leq} (N+2)L\|u\|_{L^p(I_{\zeta_u},w_\kappa;X_1)}+L\|u-x\|_{L^p(I_{\zeta_u},w_\kappa;X^{\mathsf{Tr}}_{\kappa,p})}\\
&\overset{(iii)}{\leq} 2L\lambda(N+2+C_T)=:C_{\lambda,T},
\end{aligned}$$

where in (i) we used (4.42) and $\Psi_\lambda(t,x,u)=0$ if $t\geq\zeta_u$. In (ii) we used the assumption (HA), (HF) and $\sup_{t\in[0,\zeta_u]}\|u(t)-x\|_{X^{\mathsf{Tr}}_{\kappa,p}}\leq N+2$ by (4.42). In (iii) we used that $\|u\|_{L^p(I_{\zeta_u},w_\kappa;X_1)}+\sup_{s\in[0,\zeta_u]}\|u(s)-x\|_{X^{\mathsf{Tr}}_{\kappa,p}}\leq 2\lambda$.

To prove the Lipschitz estimate we split the proof into two steps.

Step 1: Lipschitz estimate for $t\mapsto\Psi(t,x,u)(F_{\mathsf{Tr}}(t,u)-F_{\mathsf{Tr}}(t,0))$. For simplicity, let us set $\widetilde{F}_{\mathsf{Tr}}(u):=F_{\mathsf{Tr}}(\cdot,u)-F_{\mathsf{Tr}}(\cdot,0)$. As in the proof of Lemma 4.3.13, one has

$$\begin{aligned}
&\|\Psi(\cdot,x,u)\widetilde{F}_{\mathsf{Tr}}(u)-\Psi(\cdot,x,v)\widetilde{F}_{\mathsf{Tr}}(v)\|_{L^p(I_T,w_\kappa;X_0)}\\
&\leq \|(\Psi(\cdot,x,u)-\Psi(\cdot,x,v))\widetilde{F}_{\mathsf{Tr}}(u)\|_{L^p(I_T,w_\kappa;X_0)}+\|\Psi(\cdot,x,v)(\widetilde{F}_{\mathsf{Tr}}(u)-\widetilde{F}_{\mathsf{Tr}}(v))\|_{L^p(I_T,w_\kappa;X_0)}\\
&\leq \|(\Psi(\cdot,x,u)-\Psi(\cdot,x,v))\widetilde{F}_{\mathsf{Tr}}(u)\|_{L^p(I_{\zeta_u},w_\kappa;X_0)}+\|\widetilde{F}_{\mathsf{Tr}}(u)-\widetilde{F}_{\mathsf{Tr}}(v)\|_{L^p(I_{\zeta_v},w_\kappa;X_0)}.
\end{aligned}$$

Note that

$$\begin{aligned}
\|\widetilde{F}_{\mathsf{Tr}}(u)-\widetilde{F}_{\mathsf{Tr}}(v)\|_{L^p(I_{\zeta_v},w_\kappa;X_0)} &\leq L\|u-v\|_{L^p(I_{\zeta_v},w_\kappa;X^{\mathsf{Tr}}_{\kappa,p})}\\
&\leq LC_T\|u-v\|_{C(\overline{I}_T;X^{\mathsf{Tr}}_{\kappa,p})},
\end{aligned}$$

and

$$\begin{aligned}
&\|(\Psi(\cdot,x,u)-\Psi(\cdot,x,v))\widetilde{F}_{\mathsf{Tr}}(u)\|_{L^p(I_{\zeta_u},w_\kappa;X_0)}\\
&\leq \sup_{t\in[0,\zeta_u]}|\Psi(t,x,u)-\Psi(t,x,v)|\|\widetilde{F}_{\mathsf{Tr}}(u)\|_{L^p(I_{\zeta_u},w_\kappa;X_0)}\\
&\leq L\frac{1}{\lambda}(\|u-v\|_{C(\overline{I}_T;X^{\mathsf{Tr}}_{\kappa,p})}+\|u-v\|_{L^p(I_{\zeta_u},w_\kappa;X_1)})\|u-x\|_{L^p(I_{\zeta_u},w_\kappa;X^{\mathsf{Tr}}_{\kappa,p})}
\end{aligned}$$

$$\leq 2C_T L(\|u-v\|_{C(\overline{I}_T;X^{\mathsf{Tr}}_{\kappa,p})} + \|u-v\|_{L^p(I_{\zeta_u},w_\kappa;X_1)});$$

where in the last inequality we used that $\|u-x\|_{L^p(I_{\zeta_u},w_\kappa;X^{\mathsf{Tr}}_{\kappa,p})} \leq 2C_T\lambda$ by (4.42).

Step 2: Lipschitz estimate for $t\mapsto \Psi_\lambda(t,x,u)(A(t,x)u-A(t,u)u)$. Writing

$$\begin{aligned}
&\|\Psi_\lambda(\cdot,x,u)(A(\cdot,x)u-A(\cdot,u)u)-\Psi(\cdot,x,v)(A(\cdot,x)v-A(\cdot,v)v)\|_{L^p(I_T,w_\kappa;X_0)}\\
&\leq \|(\Psi_\lambda(\cdot,x,u)-\Psi_\lambda(\cdot,x,v))((A(\cdot,x)-A(\cdot,u))u)\|_{L^p(I_T,w_\kappa;X_0)}\\
&\quad + \|\Psi_\lambda(\cdot,x,v)((A(\cdot,v)-A(\cdot,x))(u-v))\|_{L^p(I_T,w_\kappa;X_0)}\\
&\quad + \|\Psi(\cdot,x,v)((A(\cdot,v)-A(\cdot,u))u)\|_{L^p(I_T,w_\kappa;X_0)} =: R_1+R_2+R_3.
\end{aligned}$$

For R_1 note that

$$\begin{aligned}
R_1 &= \|(\Psi_\lambda(\cdot,x,u)-\Psi_\lambda(\cdot,x,v))((A(\cdot,x)-A(\cdot,u))u)\|_{L^p(I_{\zeta_u},w_\kappa;X_0)}\\
&\leq \sup_{t\in[0,\zeta_u]}|\Psi_\lambda(t,x,u)-\Psi_\lambda(t,x,v)|\|(A(\cdot,x)-A(\cdot,u))u\|_{L^p(I_{\zeta_u},w_\kappa;X_0)}
\end{aligned}$$

As before, for all $t\in[0,\zeta_u]$,

$$|\Psi_\lambda(t,x,u)-\Psi_\lambda(t,x,v)| \leq \frac{1}{\lambda}(\|u-v\|_{C(\overline{I}_T;X^{\mathsf{Tr}}_{\kappa,p})}+\|u-v\|_{L^p(I_T,w_\kappa;X_1)}).$$

Moreover,

$$\|(A(\cdot,x)-A(\cdot,u))u\|_{L^p(I_{\zeta_u},w_\kappa;X_0)} \leq L\Big(\int_0^{\zeta_u}\|u(t)-x\|^p_{X^{\mathsf{Tr}}_{\kappa,p}}\|u(t)\|^p_{X_1}t^\kappa dt\Big)^{\frac{1}{p}} \leq 4\lambda^2 L.$$

Therefore,

$$R_1 \leq 4CL\lambda(\|u-v\|_{C(\overline{I}_T;X^{\mathsf{Tr}}_{\kappa,p})}+\|u-v\|_{L^p(I_T,w_\kappa;X_1)}).$$

Similarly, one gets

$$R_2+R_3 \leq L\lambda\|u-v\|_{L^p(I_T,w_\kappa;X_1)} + L\lambda\|u-v\|_{C(\overline{I}_T;X^{\mathsf{Tr}}_{\kappa,p})}.$$

Putting together the estimates in Step 1-2 the conclusion follows. □

In the proof of Theorem 4.3.7 we need a further truncation. To this end, let ξ_λ be as above. Then for $u\in C(\overline{I}_T;X^{\mathsf{Tr}}_{\kappa,p})\cap L^p(I_T,w_\kappa;X_1)$, $n\geq 1$ and $t\in I_T$ we set

$$\Phi_n(t,u) := \xi_n\Big(\|u\|_{L^p(I_t,w_\kappa;X_1)} + \sup_{s\in[0,t]}\|u(s)\|_{X^{\mathsf{Tr}}_{\kappa,p}}\Big). \tag{4.43}$$

As before, we fix $\omega\in\Omega$, but we omit it from the notation.

Lemma 4.3.16. *Let (HF′)-(HG′) be satisfied. Let $T>0$ and let σ be a stopping time with value in $[0,T]$. Let Φ_n be as in (4.43). For any $n\geq 1$, let the maps*

$$\begin{aligned}
F_{L,n} &: L^p(I_\sigma,w_\kappa;X_1)\cap C(\overline{I}_\sigma;X^{\mathsf{Tr}}_{\kappa,p}) \to L^p(I_\sigma,w_\kappa;X_0),\\
G_{L,n} &: L^p(I_\sigma,w_\kappa;X_1)\cap C(\overline{I}_\sigma;X^{\mathsf{Tr}}_{\kappa,p}) \to L^p(I_\sigma,w_\kappa;\gamma(H,X_{1/2})),
\end{aligned}$$

be given by

$$\begin{aligned}
F_{L,n}(u) &:= \Phi_n(\cdot,u)(F_L(\cdot,u)-F_L(\cdot,0)),\\
G_{L,n}(u) &:= \Phi_n(\cdot,u)(G_L(\cdot,u)-G_L(\cdot,0)).
\end{aligned}$$

Then there exist constants $C_n, C_T>0$ such that a.s.

$$\|F_{L,n}(\cdot,u)\|_{L^p(I_\sigma,w_\kappa;X_0)} \leq C_n$$

$$\|G_{L,n}(\cdot,u)\|_{L^p(I_\sigma,w_\kappa;\gamma(H,X_{1/2}))} \leq C_n$$
$$\|F_{L,n}(\cdot,u)-F_{F,n}(\cdot,v)\|_{L^p(I_\sigma,w_\kappa;X_0)} \leq L'_{F,n}(\|u-v\|_{L^p(I_\sigma,w_\kappa;X_1)}+\|u-v\|_{C(\overline{I}_\sigma;X^{\mathsf{Tr}}_{\kappa,p})}),$$
$$\|G_{L,n}(\cdot,u)-G_{F,n}(\cdot,v)\|_{L^p(I_\sigma,w_\kappa;\gamma(H,X_{1/2}))} \leq L'_{G,n}(\|u-v\|_{L^p(I_\sigma,w_\kappa;X_1)}+\|u-v\|_{C(\overline{I}_\sigma;X^{\mathsf{Tr}}_{\kappa,p})});$$

where $L'_{F,n} := 3L_{F,2n}+C_T\widetilde{L}_{F,2n}$, $L'_{G,n} := 3L_{G,2n}+C_T\widetilde{L}_{F,2n}$ *and* $\lim_{T\downarrow 0} C_T = 0$.

Proof. The proof is similar to the one given in Lemmas 4.3.13 and 4.3.15. For the sake of completeness we sketch the proof of the Lipschitz continuity of $F_{L,n}$. Since the estimates are pointwise with respect to $\omega\in\Omega$, we may assume $\sigma = T$. Let $u,v\in C(\overline{I}_T;X^{\mathsf{Tr}}_{\kappa,p})\cap L^p(I_T,w_\kappa;X_1)$ and set

$$\lambda_u := \inf\Big\{t\in[0,T]\,:\,\|u\|_{L^p(I_t,w_\kappa;X_1)}+\sup_{s\in[0,t]}\|u(s)\|_{X^{\mathsf{Tr}}_{\kappa,p}}\geq 2n\Big\}\wedge T. \tag{4.44}$$

A similar definition holds for λ_v. As usual, we assume $\lambda_u\geq\lambda_v$. Therefore

$$\begin{aligned}\|F_{L,n}(\cdot,u)-F_{L,n}(\cdot,v)\|_{L^p(I_T,w_\kappa;X_0)} &= \|F_{L,n}(\cdot,u)-F_{L,n}(\cdot,v)\|_{L^p(I_{\lambda_u},w_\kappa;X_0)}\\ &\leq \|(\Phi_n(\cdot,u)-\Phi_n(\cdot,v))\widetilde{F}_L(\cdot,u)\|_{L^p(I_{\lambda_u},w_\kappa;X_0)}\\ &\quad+\|\Phi_n(\cdot,v)(F_L(\cdot,u)-F_L(\cdot,v))\|_{L^p(I_{\lambda_v},w_\kappa;X_0)};\end{aligned}$$

where we have set $\widetilde{F}_L(\cdot,u) := F_L(\cdot,u)-F_L(\cdot,0)$. Since $\|\xi'\|_{L^\infty(\mathbb{R}_+)}\leq 1$, one has

$$\begin{aligned}&\|(\Phi_n(\cdot,u)-\Phi_n(\cdot,v))\widetilde{F}_L(\cdot,u)\|_{L^p(I_{\lambda_u},w_\kappa;X_0)}\\ &\leq \frac{1}{n}(\|u-v\|_{C(\overline{I}_T;X^{\mathsf{Tr}}_{\kappa,p})}+\|u-v\|_{L^p(I_T,w_\kappa;X_1)})\|L_{F,2n}\|u\|_{X_1}+\widetilde{L}_{F,2n}\|u\|_{X_0}\|_{L^p(I_{\lambda_u},w_\kappa)}\\ &\leq 2(\|u-v\|_{C(\overline{I}_T;X^{\mathsf{Tr}}_{\kappa,p})}+\|u-v\|_{L^p(I_T,w_\kappa;X_1)})(L_{F,2n}+C_T\widetilde{L}_{F,2n});\end{aligned}$$

where in the last inequality we used (4.44). Finally, since $\lambda_v\leq\lambda_u$,

$$\begin{aligned}\|\Phi_n(\cdot,v)(F_L(\cdot,u)-F_L(\cdot,v))\|_{L^p(I_{\lambda_v},w_\kappa;X_0)} &\leq \|F_L(\cdot,u)-F_L(\cdot,v)\|_{L^p(I_{\lambda_v},w_\kappa;X_0)}\\ &\leq L_{F,2n}\|u-v\|_{L^p(I_{\lambda_v},w_\kappa;X_0)}+C_T\widetilde{L}_{F,2n}\|u-v\|_{C(\overline{I}_{\lambda_v};X^{\mathsf{Tr}}_{\kappa,p})}.\end{aligned}$$

The above estimates readily imply the claim. ☐

4.3.5 Proofs of Theorems 4.3.5 and 4.3.7-4.3.8

With this preparation, we are ready to prove our first result concerning (4.16).

Proof of Theorem 4.3.5. To begin, we look to a suitable modification of (4.16). More specifically, fix $w_0\in L^p_{\mathscr{F}_0}(\Omega;X^{\mathsf{Tr}}_{\kappa,p})$ and let us consider the following semilinear equation:

$$\begin{cases} du+A(\cdot,u_0)udt = (\widetilde{F}_\lambda(u)+\widetilde{f})dt+(B(\cdot,u_0)u+\widetilde{G}_\lambda(u)+\widetilde{g})dW_H,\\ u(0)=w_0;\end{cases} \tag{4.45}$$

on $[0,T]$, where

$$\begin{aligned}\widetilde{F}_\lambda(u) &:= F_{c,\lambda}(u_0,u)+F_{A,\lambda}(u_0,u)+F_L(\cdot,u),\\ \widetilde{G}_\lambda(u) &:= G_{c,\lambda}(u_0,u)+G_{A,\lambda}(u_0,u)+G_L(\cdot,u),\\ \widetilde{f} &:= f+F_c(\cdot,0)+F_{\mathsf{Tr}}(\cdot,u_0),\\ \widetilde{g} &:= g+G_c(\cdot,0)+G_{\mathsf{Tr}}(\cdot,u_0),\end{aligned} \tag{4.46}$$

where $F_{c,\lambda},G_{c,\lambda},F_{A,\lambda}$ and $G_{A,\lambda}$ are defined in Lemmas 4.3.13 and 4.3.15. By (HF)-(HG) and the fact that $T<\infty$, it follows that $\widetilde{f}\in L^p_{\mathscr{P}}(I_T\times\Omega,w_\kappa;X_0)$ and $\widetilde{g}\in L^p_{\mathscr{P}}(I_T\times\Omega,w_\kappa;\gamma(H,X_{1/2}))$.

Let $\mathscr{R} := \mathscr{R}_{(A(\cdot,u_0),B(\cdot,u_0))}$ be the solution operator associated to the couple $(A(\cdot,u_0),B(\cdot,u_0)) \in \mathcal{SMR}^{\bullet}_{p,\kappa}(T)$.

To study existence of strong solutions to (4.45) let σ be a stopping time with values in $[0,T]$ and consider

$$\mathcal{Z}_\sigma := L^p_{\mathscr{P}}(\Omega;\mathfrak{X}(\sigma)) \cap L^p_{\mathscr{P}}(I_\sigma\times\Omega, w_\kappa; X_1) \cap L^p_{\mathscr{P}}(\Omega; C(\overline{I}_\sigma; X^{\mathsf{Tr}}_{\kappa,p})), \tag{4.47}$$

equipped with the sum of the three norms. Note that the stopped space and norm were defined in Definition 4.1.8. Recall that $\mathfrak{X}(\sigma)$ was defined in (4.31). On $\mathcal{Z}_\sigma$ we define an equivalent norm by

$$|\!|\!|\cdot|\!|\!|_{\mathcal{Z}_\sigma} = \|\cdot\|_{\mathcal{Z}_\sigma} + M\|\cdot\|_{L^p(\Omega;L^p(I_\sigma,w_\kappa;X_0))},$$

here $M\geq 0$ will be specified below. We shall study the map Π_{w_0} defined on $\mathcal{Z}_\sigma$ by

$$\Pi_{w_0}(v) := \mathscr{R}(w_0, \widetilde{F}_\lambda(v)+\widetilde{f}, \widetilde{G}_\lambda(v)+\widetilde{g}). \tag{4.48}$$

For the sake of clarity, we divide the proof into several steps.

Step 1: There exist $M>0$, $\lambda^*>0$, $T^*\in(0,T]$, $\varepsilon>0$ and $\alpha<1$ such that if $\max\{L_F,L_G\}\leq\varepsilon$, *then for any stopping time $\sigma:\Omega\to[0,T^*]$ and any $w_0\in L^p_{\mathscr{F}_0}(\Omega;X^{\mathsf{Tr}}_{\kappa,p})$ one has $\Pi_{w_0}:\mathcal{Z}_\sigma\to\mathcal{Z}_\sigma$ and for all $v,w\in\mathcal{Z}_\sigma$,*

$$|\!|\!|\Pi_{w_0}(v)-\Pi_{w_0}(w)|\!|\!|_{\mathcal{Z}_\sigma} \leq \alpha|\!|\!|u-w|\!|\!|_{\mathcal{Z}_\sigma}. \tag{4.49}$$

In the following, we consider $p>2$, the case $p=2$ follows by replacing ${}_0H^{\delta,p}(I_\sigma,w_\kappa;X_{1-\delta})$ by $C(\overline{I}_\sigma,X_{1/2})$ below.

Let $p>2$, and fix a stopping time σ with values in $[0,T]$. Fix $\delta\in((1+\kappa)/p,1/2)$. Note that for $z\in L^p_{\mathscr{P}}(I_T\times\Omega,w_\kappa;X_1)\cap L^p_{\mathscr{P}}(\Omega;H^{\delta,p}(I_T,w_\kappa;X_{1-\delta}))$

$$\|z\|_{\mathcal{Z}_T} \leq k_T(\|z\|_{L^p(I_T\times\Omega,w_\kappa;X_1)} + \|z\|_{L^p(\Omega;H^{\delta,p}(I_T,w_\kappa;X_{1-\delta}))}), \tag{4.50}$$

where k_T is a constant which depends on T. Moreover

$$\|z\|_{\mathcal{Z}_\sigma} \leq C_1(\|z\|_{L^p(I_\sigma\times\Omega,w_\kappa;X_1)} + \|z\|_{L^p(\Omega;{}_0H^{\delta,p}(I_\sigma,w_\kappa;X_{1-\delta}))}), \tag{4.51}$$

for all $z\in L^p_{\mathscr{P}}(I_\sigma\times\Omega,w_\kappa;X_1)\cap L^p_{\mathscr{P}}(\Omega;{}_0H^{\delta,p}(I_\sigma,w_\kappa;X_{1-\delta}))$, where the constant C_1 is independent of T. Both estimates (4.50) and (4.51) follow from Proposition 4.1.5, Lemma 4.3.9 and Remark 4.3.10.

By Proposition 4.2.10 and (4.50) one has

$$\|\mathscr{R}(w_0,0,0)\|_{\mathcal{Z}_T} \leq k_T\|w_0\|_{L^p(\Omega;X^{\mathsf{Tr}}_{\kappa,p})}. \tag{4.52}$$

Since $(A(\cdot,u_0),B(\cdot,u_0))\in\mathcal{SMR}^{\bullet}_{p,\kappa}(T)$, Definition 4.2.5, (4.14), Proposition 4.2.12 and (4.51) give that for all $\phi\in L^p_{\mathscr{P}}(I_T\times\Omega,w_\kappa;X_0)$ and $\psi\in L^p_{\mathscr{P}}(I_T\times\Omega,w_\kappa;\gamma(H,X_{1/2}))$,

$$\begin{aligned}\|\mathscr{R}(0,\phi,\psi)\|_{\mathcal{Z}_\sigma} &\leq \|\mathscr{R}(0,\mathbf{1}_{[\![0,\sigma]\!]}\phi,\mathbf{1}_{[0,\sigma]}\psi)\|_{\mathcal{Z}_T}\\ &\leq C_1K^{\det,\delta}\|\phi\|_{L^p(\Omega;L^p(I_\sigma,w_\kappa;X_0))} + C_1K^{\mathsf{sto},\delta}\|\psi\|_{L^p(\Omega;L^p(\Omega;I_\sigma,w_\kappa;\gamma(H,X_{1/2})))},\end{aligned} \tag{4.53}$$

where $K^{\det,\delta} := K^{\det,\delta}_{(A(\cdot,u_0),B(\cdot,u_0))}$, $K^{\det,\delta} := K^{\mathsf{sto},\delta}_{(A(\cdot,u_0),B(\cdot,u_0))}$ and C_1 is as in (4.51).

Next we show that Π_{w_0} maps $\mathcal{Z}_\sigma$ into itself. Let $v\in\mathcal{Z}_\sigma$. By (4.52) and (4.53) we can write

$$\begin{aligned}\|\Pi_{w_0}(v)\|_{\mathcal{Z}_\sigma} &\leq \|\mathscr{R}(w_0,0,0)\|_{\mathcal{Z}_T} + \|\mathscr{R}(0,\widetilde{F}_\lambda(v)+\widetilde{f},\widetilde{G}_\lambda(v)+\widetilde{g}\|_{\mathcal{Z}_\sigma}\\ &\leq k_T\|w_0\|_{L^p(\Omega;X^{\mathsf{Tr}}_{\kappa,p})} + C_1K^{\det,\delta}\|\widetilde{F}_\lambda(v)+\widetilde{f}\|_{L^p(\Omega;L^p(I_\sigma,w_\kappa;X_0))}\\ &\quad + C_1K^{\mathsf{sto},\delta}\|\widetilde{G}_\lambda(v)+\widetilde{g}\|_{L^p(\Omega;L^p(I_\sigma,w_\kappa;\gamma(H,X_{1/2})))}\end{aligned}$$

and the latter is finite by Lemmas 4.3.13 and 4.3.15.

Moreover, for $v, w \in \mathcal{Z}_\sigma$ by Proposition 4.2.12 we can write

$$\Pi_{w_0}(v) - \Pi_{w_0}(w) = \mathscr{R}(0, \mathbf{1}_{[\![0,\sigma]\!]}(\widetilde{F}_\lambda(v) - \widetilde{F}_\lambda(w)), \mathbf{1}_{[\![0,\sigma]\!]}(\widetilde{G}_\lambda(v) - \widetilde{G}_\lambda(w))) \tag{4.54}$$

on $[\![0, \sigma]\!]$. The previous identity and (4.53) gives

$$\begin{aligned}
&\|\Pi_{w_0}(v) - \Pi_{w_0}(w)\|_{\mathcal{Z}_\sigma} \\
&= \|\mathscr{R}(0, \mathbf{1}_{[\![0,\sigma]\!]}(\widetilde{F}_\lambda(v) - \widetilde{F}_\lambda(w)), \mathbf{1}_{[\![0,\sigma]\!]}(\widetilde{G}_\lambda(v) - \widetilde{G}_\lambda(w)))\|_{\mathcal{Z}_\sigma} \\
&\leq C_1 K^{\det,\delta} \|\widetilde{F}_\lambda(v) - \widetilde{F}_\lambda(w)\|_{L^p(\Omega; L^p(I_\sigma, w_\kappa; X_0))} \\
&\qquad + C_1 K^{\mathsf{sto},\delta} \|\widetilde{G}_\lambda(v) - \widetilde{G}_\lambda(w)))\|_{L^p(\Omega; L^p(I_\sigma, w_\kappa; \gamma(H, X_{1/2})))} \\
&\leq C_1 [K^{\det,\delta}(L'_{\lambda,T} + L_F) + K^{\mathsf{sto}}(L'_{\lambda,T} + L_G)] \|v - w\|_{\mathcal{Z}_\sigma} \\
&\qquad + C_1 (K^{\det,\delta} \widetilde{L}_F + K^{\mathsf{sto},\delta} \widetilde{L}_G) \|v - w\|_{L^p(\Omega; L^p(I_\sigma, w_\kappa; X_0))},
\end{aligned} \tag{4.55}$$

where the last estimate follows from Lemmas 4.3.13 and 4.3.15 and where we have set $L'_{\lambda,T} = L_{\lambda,T} + \widetilde{L}_{\lambda,T}$.

Let $\varepsilon > 0$ be such that if (4.22) holds, then

$$C_1[K^{\det,\delta} L_F + K^{\mathsf{sto},\delta} L_G] < 1. \tag{4.56}$$

By Lemmas 4.3.13 and 4.3.15 one can find $\widetilde{T}$ and $\widetilde{\lambda}$ such that

$$C_1[K^{\det,\delta}(L_F + L'_{\lambda,T}) + K^{\mathsf{sto},\delta}(L_G + L'_{\lambda,T})] := \alpha' < 1; \tag{4.57}$$

for all $T \leq \widetilde{T}$ and $\lambda \leq \widetilde{\lambda}$. To complete the proof we extend the argument in [165, Theorem 4.5] to our setting. Set

$$M := \frac{K^{\det,\delta} \widetilde{L}_F + K^{\mathsf{sto},\delta} \widetilde{L}_G}{K^{\det,\delta} L_F + K^{\mathsf{sto},\delta} L_G}.$$

With such a choice the inequality (4.55) implies that

$$\|\Pi_{w_0}(v) - \Pi_{w_0}(w)\|_{\mathcal{Z}_\sigma} \leq \alpha' \|v - w\|_{\mathcal{Z}_\sigma}.$$

Applying Lemma 4.2.13 with u given by (4.54) we find

$$\begin{aligned}
&\|\Pi_{w_0}(v) - \Pi_{w_0}(w)\|_{L^p(\Omega; L^p(I_\sigma, w_\kappa; X_0))} \\
&\leq c_T \big[\|\widetilde{F}_\lambda(v) - \widetilde{F}_\lambda(w)\|_{L^p(I_\sigma \times \Omega, w_\kappa; X_0)} + \|\widetilde{G}_\lambda(v) - \widetilde{G}_\lambda(w)\|_{L^p(I_\sigma \times \Omega, w_\kappa; X_0)}\big] \\
&\leq \widetilde{c}_T \|v - w\|_{\mathcal{Z}_\sigma};
\end{aligned} \tag{4.58}$$

where the last step follows from Lemmas 4.3.13 and 4.3.15, and where $c_T, \widetilde{c}_T > 0$ and both tend to zero as $T \downarrow 0$. The claim follows from (4.55) and (4.58) by choosing $T^* > 0$ such that $M\widetilde{c}_{T^*} < 1 - \alpha'$, $\lambda^* = \widetilde{\lambda}$ and $\alpha := \alpha' + M c_{T^*} < 1$.

Step 2: Let λ^*, T^* be as in Step 1. Then for each $w_0 \in L^p_{\mathscr{F}_0}(\Omega; X^{\mathsf{Tr}}_{\kappa,p})$ the problem (4.45) has a unique strong solution $u_{w_0} \in \mathcal{Z}_{T^*}$ on $[\![0, T^*]\!]$. Moreover, there exists a constant $C = C(T^*, \lambda^*) > 0$ such that for all $w_0, w_1 \in L^p_{\mathscr{F}_0}(\Omega; X^{\mathsf{Tr}}_{\kappa,p})$, one has

$$\|u_{w_0} - u_{w_1}\|_{\mathcal{Z}_{T^*}} \leq C \|w_0 - w_1\|_{L^p_{\mathscr{F}_0}(\Omega; X^{\mathsf{Tr}}_{\kappa,p})}. \tag{4.59}$$

Applying Step 1 to $\sigma \equiv T^*$, we obtain that $\Pi_{w_0} : \mathcal{Z}_{T^*} \to \mathcal{Z}_{T^*}$ is a contraction. Therefore, by the Banach fixed point theorem there exists a unique $u_{w_0} \in \mathcal{Z}_{T^*}$ such that $\Pi_{w_0}(u_{w_0}) = u_{w_0}$. From this we can conclude that u_{w_0} is a strong solution to (4.45) on $[\![0, T^*]\!]$ (see Definition 4.3.3 and (4.48)).

It remains to prove (4.59). The linearity of $\mathscr{R}$ shows that

$$u_{w_0} - u_{w_1} = \Pi_{w_0}(u_{w_0}) - \Pi_{w_1}(u_{w_1}) = \mathscr{R}(w_0 - w_1, 0, 0) + \Pi_0(u_{w_0}) - \Pi_0(u_{w_1}).$$

Therefore, by (4.52) and (4.49),

$$
\begin{aligned}
\|u_{w_0}-u_{w_1}\|_{\mathcal{Z}_{T^*}} &\leq \|\mathscr{R}(w_0-w_1,0,0)\|_{\mathcal{Z}_{T^*}} + \|\Pi_0(u_{w_0})-\Pi_0(u_{w_1})\|_{\mathcal{Z}_{T^*}}\\
&\leq \widetilde{k}_{T^*}\|w_0-w_1\|_{L^p_{\mathscr{F}_0}(\Omega;X^{\mathsf{Tr}}_{\kappa,p})} + \alpha\|u_{w_0}-u_{w_1}\|_{\mathcal{Z}_{T^*}}.
\end{aligned}
$$

Since $\alpha<1$, the latter implies (4.59).

Step 3: Let (v,τ) be a local solution to (4.45) with initial data $w_0\in L^p_{\mathscr{F}_0}(\Omega;X^{\mathsf{Tr}}_{\kappa,p})$. Then $v=u_{w_0}$ on $[\![0,\tau\wedge T^*[\![$. Without loss of generality, we can assume that $\tau<T^*$. For $n\geq 1$ let

$$
\tau_n := \inf\{t\in[0,\tau) \,:\, \|v\|_{\mathfrak{X}(t)} + \|v-w_0\|_{C(\overline{I}_t;X^{\mathsf{Tr}}_{\kappa,p})} + \|v\|_{L^p(I_t,w_\kappa;X_1)}\geq n\}
$$

and $\tau_n:=\tau$ if the set is empty. Then $(\tau_n)_{n\geq 1}$ is a localizing sequence for (v,τ).

Fix $n\geq 1$. Lemmas 4.3.13 and 4.3.15 ensure that $\mathbf{1}_{[\![0,\tau_n]\!]}(\widetilde{F}_\lambda(v)+\widetilde{f})\in L^p_{\mathscr{P}}(I_T\times\Omega,w_\kappa;X_0)$ and $\mathbf{1}_{[\![0,\tau_n]\!]}(\widetilde{G}_\lambda(v)+\widetilde{f})\in L^p_{\mathscr{P}}(I_T\times\Omega,w_\kappa;g(H,X_{1/2}))$. Moreover, by Proposition 4.2.12 one obtains

$$
\begin{aligned}
v &= \mathscr{R}(w_0,\mathbf{1}_{[\![0,\tau_n]\!]}(\widetilde{f}+\widetilde{F}_\lambda(v)),\mathbf{1}_{[\![0,\tau_n]\!]}(\widetilde{G}_\lambda(v)+\widetilde{g})),\\
u_{w_0} &= \mathscr{R}(w_0,\mathbf{1}_{[\![0,\tau_n]\!]}(\widetilde{f}+\widetilde{F}_\lambda(u_{w_0})),\mathbf{1}_{[\![0,\tau_n]\!]}(\widetilde{G}_\lambda(u_{w_0})+\widetilde{g}));
\end{aligned}
$$

on $[\![0,\tau_n]\!]$. Using (4.54) this implies that

$$
\begin{aligned}
\|u_{w_0}-v\|_{\mathcal{Z}_{\tau_n}} &= \|\mathscr{R}(0,\mathbf{1}_{[\![0,\tau_n]\!]}(\widetilde{F}_\lambda(v)-\widetilde{F}_\lambda(u_{w_0})),\mathbf{1}_{[\![0,\tau_n]\!]}(\widetilde{G}_\lambda(v)-\widetilde{G}_\lambda(u_{w_0})))\|_{\mathcal{Z}_{\tau_n}}\\
&= \|\Pi_0(u_{w_0})-\Pi_0(v)\|_{\mathcal{Z}_{\tau_n}} \leq \alpha\|u_{w_0}-v\|_{\mathcal{Z}_{\tau_n}};
\end{aligned}
$$

where in the last step we used (4.49). Since $\alpha<1$, we obtain that $u_{w_0}=v$ on $[\![0,\tau_n\wedge T^*]\!]$. Since $n\geq 1$ was arbitrary, it follows that $u_{w_0}=v$ on $[\![0,\tau\wedge T[\![$.

Steps 1-3 complete our treatment of (4.45). Below we apply these results to study (4.16).

Step 4: Let $\eta:=\lambda^*/2$. Then (4.16) has a strong solution (v,τ) with initial data $v_0\in L^\infty(\Omega;X^{\mathsf{Tr}}_{\kappa,p})$ and $\tau>0$ a.s. provided $v_0\in B_{L^\infty_{\mathscr{F}_0}(\Omega;X^{\mathsf{Tr}}_{\kappa,p})}(u_0,\eta)$. In particular, this gives a strong solution (u,σ) to (4.16) with $\sigma>0$ a.s.

Step 1 ensures that (4.45) with initial data v_0 has a unique strongly progressively measurable solution u_{v_0} if $\lambda=\lambda^*$ and $T=T^*$. Set

$$
\tau := \inf\big\{t\in[0,T] \,:\, \|u_{v_0}\|_{\mathfrak{X}(t)} + \|u_{v_0}-u_0\|_{C(\overline{I}_t;X^{\mathsf{Tr}}_{\kappa,p})} + \|u_{v_0}\|_{L^p(I_t,w_\kappa;X_1)} > \lambda^*/2\big\}.
$$

Since the maps $t\mapsto\|u_{u_0}\|_{\mathfrak{X}(t)}$, $t\mapsto\sup_{s\in[0,t]}\|u_{u_0}(s)-v_0\|_{X^{\mathsf{Tr}}_{\kappa,p}}$ are continuous and adapted, τ is a stopping time. Note that if $v_0\in B_{L^\infty_{\mathscr{F}_0}(\Omega;X^{\mathsf{Tr}}_{\kappa,p})}(u_0,\eta)$, then $0<\tau$ a.s.

Setting $v:=u_{v_0}|_{[\![0,\tau]\!]}$, then a.s. for $t\in[0,\tau]$, one has

$$
\Theta_{\lambda^*}(t,u_0,v)=1, \qquad \Psi_{\lambda^*}(t,u_0,v)=1.
$$

Using the latter, by (4.46) a.s. on $[\![0,\tau]\!]$

$$
\begin{aligned}
\widetilde{F}_{\lambda^*}(v) &= A(\cdot,u_0)v - A(\cdot,v)v + F_c(\cdot,v) - F_c(\cdot,0) + F_{\mathsf{Tr}}(\cdot,v) - F_{\mathsf{Tr}}(\cdot,u_0) + F_L(\cdot,v),\\
\widetilde{G}_{\lambda^*}(v) &= B(\cdot,u_0)v - B(\cdot,v)v + G_c(\cdot,v) - G_c(\cdot,0) + G_{\mathsf{Tr}}(\cdot,v) - G_{\mathsf{Tr}}(\cdot,u_0) + G_L(\cdot,v).
\end{aligned}
$$

Using this and (4.45), it follows that v is a strong solution to (4.16) on $[\![0,\tau]\!]$ with initial data v_0.

Next, we prove the continuity estimate claimed in (3) for the solutions just constructed. Let (u,σ), (v,τ) be solutions of (4.16) constructed above with initial value u_0,v_0 respectively. Therefore, $u=u_{u_0}|_{[\![0,\sigma]\!]}$ and $v=u_{v_0}|_{[\![0,\tau]\!]}$.

Let $\nu:=\sigma\wedge\tau$, this implies that $u=u_{u_0}|_{[\![0,\nu]\!]}$, $v=u_{v_0}|_{[\![0,\nu]\!]}$ and

$$
\|u-v\|_{\mathcal{Z}_\nu} \leq \|u_{u_0}-u_{v_0}\|_{\mathcal{Z}_{T^*}} \leq C\|u_0-v_0\|_{L^p_{\mathscr{F}_0}(\Omega;X^{\mathsf{Tr}}_{\kappa,p})}; \tag{4.60}
$$

where in the last step we used (4.59).

Step 5: (1) and the first part of (3) hold. For the sake of clarity, we divide the proof of this step into two parts.

Step 5a: Uniqueness of the strong solution (v,τ) constructed in Step 4. Recall that (v,τ) is a strong solution to (4.16) with initial data v_0 and satisfies $v = u_{v_0}$ on $[\![0,\tau]\!]$. Let (w,μ) be a local solution to (4.16) with initial data v_0. By Definition 4.3.4, it is enough to prove that $v = w$ on $[0,\tau\wedge\mu)$. We claim that

$$w \in \mathfrak{X}(t) \quad \text{a.s. for all } t \in [0,\mu). \tag{4.61}$$

Let us first show that (4.61) implies the claim of Step 5a. Thus, suppose that (4.61) holds. Let $(\mu_n)_{n\geq 1}$ be a localizing sequence for (w,μ), and define the following stopping times

$$\mu_n^* := \inf\big\{t\in[0,\mu_n) \,:\, \|w\|_{\mathfrak{X}(t)} + \|w-u_0\|_{C(\overline{I}_t;X^{\mathsf{Tr}}_{\kappa,p})} + \|w\|_{L^p(I_t,w_\kappa;X_1)} > \lambda^*/2\big\},$$
$$\mu^* := \inf\big\{t\in[0,\mu) \,:\, \|w\|_{\mathfrak{X}(t)} + \|w-u_0\|_{C(\overline{I}_t;X^{\mathsf{Tr}}_{\kappa,p})} + \|w\|_{L^p(I_t,w_\kappa;X_1)} > \lambda^*/2\big\},$$

where $\lambda^*>0$ is as in Step 1 and where we set $\mu_n^* = \mu_n$ and $\mu^* = \mu$ if the set is empty. Let $n\geq 1$ be fixed. The argument used in Step 4 shows that (w,μ_n^*) is a local solution to (4.45) with initial data $v_0 \in L^\infty_{\mathscr{F}_0}(\Omega;X^{\mathsf{Tr}}_{\kappa,p})$. Therefore, by Step 3 $w = u_{v_0}$ on $[\![0,\mu_n^*\wedge\tau]\!]$. Letting $n\uparrow\infty$ we find $v = w$ on $[\![0,\mu^*\wedge\tau[\![$. From the latter equality, it follows that $\mu\wedge\tau = \mu^*\wedge\tau$ a.s. This proves the uniqueness of (v,τ).

Now we turn to the proof of (4.61). To this end we set, for a.a. $(\omega,t)\in[0,\mu)\times\Omega$,

$$\mathcal{N}_w(t,\omega) := \|F(\cdot,\omega,w(\cdot,\omega))\|_{L^p(0,t,w_\kappa;X_0)} + \|G(\cdot,\omega,w(\cdot,\omega))\|_{L^p(0,t,w_\kappa;\gamma(H,X_{1/2}))}.$$

By Definitions 4.3.3-4.3.4 we have $\mathcal{N}_w(t)<\infty$ a.s. for all $t\in[0,\mu)$. Define a sequence of stopping times by

$$\nu_n := \inf\big\{t\in[0,\mu) \,:\, \mathcal{N}_w(t) + \|w-u_0\|_{C(\overline{I}_t;X^{\mathsf{Tr}}_{\kappa,p})} + \|w\|_{L^p(I_t,w_\kappa;X_1)} > n\big\},$$

where $\inf\varnothing := \mu$. Then $\lim_{n\uparrow\infty}\nu_n = \mu$ a.s., and therefore to prove (4.61) it is enough to show $w\in\mathfrak{X}(\nu_n)$ a.s. for all $n\geq 1$. Note that for any $n\geq 1$,

$$w|_{[\![0,\nu_n]\!]} \in L^\infty(\Omega; C(\overline{I}_{\nu_n};X^{\mathsf{Tr}}_{\kappa,p}) \cap L^p(I_{\nu_n}\times\Omega,w_\kappa;X_1)), \tag{4.62}$$

where we used that $u_0\in L^\infty(\Omega;X^{\mathsf{Tr}}_{\kappa,p})$ by assumption, and thus

$$\begin{aligned} &\mathbf{1}_{[\![0,\nu_n]\!]}F(\cdot,w) \in L^p(I_T\times\Omega,w_\kappa;X_0),\\ &\mathbf{1}_{[\![0,\nu_n]\!]}G(\cdot,w) \in L^p(I_T\times\Omega,w_\kappa;\gamma(H,X_{1/2})). \end{aligned} \tag{4.63}$$

Since (w,μ) is a local solution to (4.16) and $\nu_n\leq\mu$ a.s. we have that $w|_{[\![0,\nu_n[\![}$ is a strong solution to (4.16) on $[\![0,\nu_n]\!]$. Writing $A(\cdot,w) = A(\cdot,u_0) + (A(\cdot,w)-A(\cdot,u_0))$ and $B(\cdot,w) = B(\cdot,u_0) + (B(\cdot,w)-B(\cdot,u_0))$, one sees that (w,ν_n) is a strong solution to (4.7) on $[\![0,\nu_n]\!]$ with (A,B) and (f,g) replaced by $(A(\cdot,u_0),B(\cdot,u_0))$ and (f_n^w,g_n^w), where

$$\begin{aligned} f_n^w &:= \mathbf{1}_{[\![0,\nu_n]\!]}[(A(\cdot,u_0)-A(\cdot,w)) + F(\cdot,w) + f],\\ g_n^w &:= \mathbf{1}_{[\![0,\nu_n]\!]}[(B(\cdot,w)-B(\cdot,u_0)) + G(\cdot,w) + g], \end{aligned}$$

respectively. By (4.62)-(4.63) and (HA),

$$f_n^w \in L^p_{\mathscr{P}}(I_T\times\Omega,w_\kappa;X_0) \quad \text{ and } g_n^w \in L^p_{\mathscr{P}}(I_T\times\Omega,w_\kappa;\gamma(H,X_{1/2})).$$

Since $(A,B)\in\mathcal{SMR}^\bullet_{p,\kappa}(T)$, $w = \mathscr{R}_{(A(\cdot,u_0),B(\cdot,u_0))}(u_0,f_n^w,g_n^w)$ on $[\![0,\nu_n]\!]$ by Proposition 4.2.12(2). Therefore, the last statement in Proposition 4.2.10 ensures that for all $\delta\in(\frac{1+\kappa}{p},\frac12)$ and $n\geq 1$,

$$w|_{[\![0,\nu_n]\!]} \in H^{\delta,p}(I_{\nu_n},w_\kappa;X_{1-\delta}) \cap L^p(I_{\nu_n},w_\kappa;X_1) \hookrightarrow \mathfrak{X}(\nu_n) \quad \text{a.s.},$$

where we used Lemma 4.3.9 for the embedding (see Remark 4.3.10).

Step 5b: Proof of the claim in Step 5. It remains to prove the existence of a maximal solution (v, τ) of (4.16) with initial data v_0 as in (3). Let Ξ be the set of all stopping time τ such that (4.16) admits a unique local solution on $[0, \tau)$ in the sense of Definitions 4.3.3-4.3.4 with initial value v_0. Then the above ensures that Ξ is not empty. We claim that Ξ is closed under pairwise maximization, i.e. if $\tau_0, \tau_1 \in \Xi$, then $\tau_0 \vee \tau_1 \in \Xi$. A similar argument appears in [104, Lemma 4.6], but our setting is different. Let (v_i, τ_i) be the unique local solution to (4.16) with the same initial data and localizing sequences $(\tau_i^n)_{n\geq 1}$ for $i = 0, 1$. The uniqueness ensures that $v_0 = v_1$ on $[\![0, \tau_0 \wedge \tau_1)\!)$. Define the process $u^n : [\![0, \tau_0^n \vee \tau_1^n]\!] \to X_0$ given by

$$u^n(t) = v_0(t \wedge \tau_0^n) + v_1(t \wedge \tau_1^n) - v_0(t \wedge \tau_0^n \wedge \tau_1^n).$$

Note that $u^n(t) = v_1(t)$ on $\{\tau_0^n \leq t \leq \tau_1^n\}$ and $u^n(t) = v_0(t) + v_1(\tau_1^n) - v_0(\tau_1^n) = v_0(t)$ on $\{\tau_1^n \leq t \leq \tau_0^n\}$. By definition u^n is strongly progressively measurable and has the same regularity properties of v_0 and v_1 on $[\![0, \tau_0^n \vee \tau_1^n]\!]$. Letting $n \uparrow \infty$ we obtain a unique local solution $(v, \tau_0 \vee \tau_1)$ and thus $\tau_0 \vee \tau_1 \in \Xi$.

By [113, Theorem A.3], $\sigma := \operatorname{ess\,sup} \Xi$ exists, and there exists a sequence of stopping times $(\tau_n)_{n\geq 1} \subseteq \Xi$ such that $\tau_n \leq \sigma$, $\lim_{n\uparrow\infty} \tau_n = \sigma$ a.s. and by the above uniqueness there exists a process $v : [0, \tau] \times \Omega \to X_0$ such that u is a local solution to (4.16) on $[\![0, \tau_n)\!)$. In addition, $\tau > 0$ a.s. by Step 4. This implies, the existence of a unique maximal local solution (v, τ) to (4.16) with initial value v_0 and localizing sequence $(\tau_n)_{n\geq 1}$. This finishes the proof of the first part of (3) and in particular (1).

Step 6: (2). Let (v, τ^v) be the maximal solution to (4.16) with initial value v_0, where v_0 is as in (3). Let $(\tau_n^v)_{n\geq 1}$ be a localizing sequence for (v, τ^v) with $\tau_n^v > 0$ a.s. For each $n \geq 1$, set

$$\widetilde{\tau}_n^v := \inf\{t \in [0, \tau_n^v) : \|v\|_{\mathfrak{X}(t)} + \|v - v_0\|_{C(\overline{I}_t; X^{\mathsf{Tr}}_{\kappa,p})} + \|v\|_{L^p(I_t, w_\kappa; X_1)} \geq n\}, \tag{4.64}$$

where we set $\widetilde{\tau}_n^v = \tau_n^v$ if the set is empty. Thus, each $\widetilde{\tau}_n^v$ is a stopping time and $\lim_{n\uparrow\infty} \widetilde{\tau}_n^v = \tau^v$. Moreover, $\tau_n^v > 0$ a.s. Let $\nu_n = \min\{\widetilde{\tau}_n^u, \widetilde{\tau}_n^v\}$.

Hypothesis (HA) and (HF)–(Hf) and Lemma 4.3.11 show that

$$\begin{aligned} f_n^v &:= \mathbf{1}_{[\![0,\nu_n]\!]}[(A(\cdot, u_0) - A(\cdot, v))v + F(\cdot, v) + f] \in L^p_{\mathscr{P}}(I_T \times \Omega, w_\kappa; X_0), \\ g_n^v &:= \mathbf{1}_{[\![0,\nu_n]\!]}[(B(\cdot, v) - B(\cdot, v_0))u + G(\cdot, v) + g] \in L^p_{\mathscr{P}}(I_T \times \Omega, w_\kappa; \gamma(H, X_{1/2})), \end{aligned}$$

for all $n \geq 1$. As in Step 5a, since u and v are strong solution to (4.16), by Proposition 4.2.12(2) we have

$$v = \mathscr{R}(v_0, f_n^v, g_n^v), \qquad \text{on } [\![0, \nu_n]\!],$$

where $\mathscr{R} := \mathscr{R}_{(A(\cdot,u_0), B(\cdot,u_0))}$. Since $(A(\cdot, u_0), B(\cdot, u_0)) \in \mathcal{SMR}^\bullet_{p,\kappa}(T)$, it follows from Proposition 4.2.10 that

$$v \in \bigcap_{\theta\in[0,1/2)} L^p_{\mathscr{P}}(\Omega; H^{\theta,p}(I_{\nu_n}, w_\kappa; X_{1-\theta})), \qquad \forall n \geq 1. \tag{4.65}$$

In particular, by Proposition 4.1.5(1)

$$v \in L^p(\Omega; C(\overline{I}_{\nu_n}; X^{\mathsf{Tr}}_{\kappa,p})).$$

It remains to prove the instantaneously regularization effect. Let $\kappa > 0$, by (4.65) and Definition 4.1.8, for each $n \geq 1$ there exists $\widetilde{v}_n \in L^p_{\mathscr{P}}(\Omega; H^{\delta,p}(I_{T^*}, w_\kappa; X_{1-\delta}) \cap L^p(I_{T^*}, w_\kappa; X_1))$ such that $v|_{[\![0,\widetilde{\sigma}_n]\!]} = \widetilde{v}_n|_{[\![0,\nu_n]\!]}$ and for any $\varepsilon > 0$,

$$\widetilde{v}_n \in L^p_{\mathscr{P}}(\Omega; H^{\delta,p}(I_{T^*}, w_\kappa; X_{1-\delta}) \cap L^p(I_{T^*}, w_\kappa; X_1)) \hookrightarrow L^p_{\mathscr{P}}(\Omega; C([\varepsilon, T^*]; X^{\mathsf{Tr}}_p)),$$

where in the last inclusion we used Proposition 4.1.5(2) and the fact that $\delta > \frac{1+\kappa}{p} \geq \frac{1}{p}$ since $\kappa \geq 0$. The claim follows from the arbitrariness of $n \geq 1$ and $\varepsilon > 0$. By taking $v = u$ this completes the proof of (2)

Step 7: the second part of (3). The cases $E \in \{L^p(I_\nu, w_\kappa; X_1), C(\overline{I}_\nu; X^{\mathsf{Tr}}_{\kappa,p}), \mathfrak{X}(\nu)\}$ have already been considered in (4.60). It remains to consider $E = H^{\theta,p}(I_\nu, w_\kappa; X_{1-\theta})$. Carefully checking the proofs of (4.49) and (4.52) one also obtains the latter case.

Step 8: (4) *holds.* Let (u,σ) and (v,τ) be as in the statement. Recall that $\Gamma := \{u_0 = v_0\}$. Without loss of generality we assume $\mathbb{P}(\Gamma) > 0$.

Set $\widetilde{\sigma} := \mathbf{1}_\Gamma \sigma + \mathbf{1}_{\Omega\setminus\Gamma}\tau$ and $\widetilde{u} := \mathbf{1}_{\Gamma\times[0,\tau)} v + \mathbf{1}_{(\Omega\setminus\Gamma)\times[0,\sigma)} u$. Then with the same argument used in the proof of Proposition 4.2.12, one can check that $(\widetilde{u}, \widetilde{\sigma})$ is a unique local solution to (4.16) since $u_0 = v_0$ on Γ.

The maximality of (u,σ) implies $\tau \le \sigma$ on Γ and

$$u = \widetilde{u} = v, \qquad \Gamma \times [0,\tau).$$

Exchanging the role of (u,σ) and (v,τ), one obtains also $\sigma \le \tau$ on Γ and $u = v$ on $\Gamma \times [0,\sigma)$. This implies the claim. ☐

Some remark may be in order.

Remark 4.3.17. Due to (4.56), the argument used in Step 1 in the proof of Theorem 4.3.5 ensures that instead of (4.22) we can assume

$$C_1(L_F K^{\mathsf{det},\delta}_{(A(\cdot,u_0),B(\cdot,u_0))} + L_B K^{\mathsf{sto},\delta}_{(A(\cdot,u_0),B(\cdot,u_0))}) < 1.$$

Here C_1 is the constant in (4.51) and $\delta \in ((1+\kappa)/p, 1/2)$. Typically the above constants are difficult to compute. See [165, Section 5] for examples in which explicit computations can be worked out.

Remark 4.3.18. By analysing the argument in the above proof one can readily check that Theorem 4.3.5 holds in case that the assumptions (HF)(i) and (HG)(i) are replaced by:

(1) For any stopping time $\mu : \Omega \to [0,T]$, one has

$$\begin{aligned} F_L &: L^0_{\mathscr{P}}(\Omega; L^p(I_\mu, w_\kappa; X_1) \cap C(\overline{I}_\mu; X^{\mathsf{Tr}}_{\kappa,p})) \to L^0_{\mathscr{P}}(\Omega; L^p(I_\mu, w_\kappa; X_0)),\\ G_L &: L^0_{\mathscr{P}}(\Omega; L^p(I_\mu, w_\kappa; X_1) \cap C(\overline{I}_\mu; X^{\mathsf{Tr}}_{\kappa,p})) \to L^0_{\mathscr{P}}(\Omega; L^p(I_\mu, w_\kappa; \gamma(H, X_{1/2}))).\end{aligned}$$

Moreover, there exist $\widetilde{C}, L_F, L_G, \widetilde{L}_F, \widetilde{L}_G > 0$ such that for a.a. $\omega \in \Omega$ and for all $u, v \in L^p(I_\mu, w_\kappa; X_1) \cap C(\overline{I}_\mu; X^{\mathsf{Tr}}_{\kappa,p})$

$$\begin{aligned} \|F_L(\cdot,\omega,u)\|_{L^p(I_\mu,w_\kappa,X_0)} &\le \widetilde{C}(1 + \|u\|_{L^p(I_\mu,w_\kappa;X_1)} + \|u\|_{C(\overline{I}_\mu;X^{\mathsf{Tr}}_{\kappa,p})}),\\ \|G_L(\cdot,\omega,u)\|_{L^p(I_\mu,w_\kappa,\gamma(H,X_{1/2}))} &\le \widetilde{C}(1 + \|u\|_{L^p(I_\mu,w_\kappa;X_1)} + \|u\|_{C(\overline{I}_\mu;X^{\mathsf{Tr}}_{\kappa,p})}),\\ \|F_L(\cdot,\omega,u) - F_L(\cdot,\omega,v)\|_{L^p(I_\mu,w_\kappa,X_0)} &\le L_F(\|u-v\|_{L^p(I_\mu,w_\kappa;X_1)} + \|u-v\|_{C(\overline{I}_\mu;X^{\mathsf{Tr}}_{\kappa,p})})\\ &\quad + \widetilde{L}_F\|u-v\|_{L^p(I_\mu,w_\kappa;X_0)},\\ \|G_L(\cdot,\omega,u) - G_L(\cdot,\omega,v)\|_{L^p(I_\mu,w_\kappa,\gamma(H,X_{1/2}))} &\le L_G(\|u-v\|_{L^p(I_\mu,w_\kappa;X_1)} + \|u-v\|_{C(\overline{I}_\mu;X^{\mathsf{Tr}}_{\kappa,p})})\\ &\quad + \widetilde{L}_G\|u-v\|_{L^p(I_\mu,w_\kappa;X_0)}.\end{aligned}$$

(2) For $\Gamma \in \{F_L, G_L\}$ and all stopping times $\nu \in [0,\mu]$ a.s., $\mathbf{1}_{[0,\nu]}\Gamma(\cdot,u) = \mathbf{1}_{[0,\nu]}\Gamma(\cdot,v)$ provided $\mathbf{1}_{[0,\nu]}u = \mathbf{1}_{[0,\nu]}v$ and $u, v \in L^0_{\mathscr{P}}(\Omega; L^p(I_\mu, w_\kappa; X_1) \cap C(\overline{I}_\mu; X^{\mathsf{Tr}}_{\kappa,p}))$.

To see that (1)-(2) are sufficient to prove Theorem 4.3.5 it is enough to note that only (1) and (2) are needed in Step 1 (resp. 5) to prove existence (resp. uniqueness). The other steps hold without any changes.

Next, we prove Theorem 4.3.7.

Proof of Theorem 4.3.7. We start by collecting some useful facts. To begin, let

$$\xi := \inf\{t \in [0,T] \,:\, \|f\|_{L^p(I_t,w_\kappa;X_0)} + \|g\|_{L^p(I_t,w_\kappa;\gamma(H,X_{1/2}))} \ge 1\}.$$

Then ξ is an stopping time, $\xi > 0$ a.s. and

$$\mathbf{1}_{[\![0,\xi]\!]} f \in L^p_{\mathscr{P}}(I_T \times \Omega, w_\kappa; X_0), \qquad \mathbf{1}_{[\![0,\xi]\!]} g \in L^p_{\mathscr{P}}(I_T \times \Omega, w_\kappa; \gamma(H, X_{1/2})).$$

Moreover, let $n \geq 1$ be fixed and define $\Gamma_n := \{\|u_0\|_{X^{\mathsf{Tr}}_{\kappa,p}} \leq n\} \in \mathscr{F}_0$. Recall that $(u_{0,n})_{n\geq 1}$ satisfies (4.23). Finally, let $F_{L,n}, G_{L,n}$ be as in Lemma 4.3.16. The same lemma implies that $F_{L,n}$ and $G_{L,n}$ verify the condition in Remark 4.3.18 for

$$L_F = 3L_{F,2n} + C_T \widetilde{L}_{F,2n}, \quad \widetilde{L}_F = 0, \quad L_G = 3L_{G,2n} + C_T \widetilde{L}_{G,2n}, \quad \widetilde{L}_G = 0,$$

where $\lim_{T\downarrow 0} C_T = 0$. For $n \geq 1$, set $F_n = F_{L,n} + F_c + F_{\mathsf{Tr}}$, $G_n = G_{L,n} + G_c + G_{\mathsf{Tr}}$. By (4.23), $\sup_\Omega \|u_{0,n}\|_{X^{\mathsf{Tr}}_{\kappa,p}} < \infty$. Let $R_n \geq 1$ be the smallest integer satisfying

$$R_n \geq \sup_\Omega \|u_{0,n}\|_{X^{\mathsf{Tr}}_{\kappa,p}}. \tag{4.66}$$

Theorem 4.3.5 and Remarks 4.3.17-4.3.18 ensure the existence of a maximal local solution (u_n, σ_n) to (4.16) with (u_0, f, g, F, G) replaced by

$$(u_{0,n}, \mathbf{1}_{[\![0,\xi]\!]} f + F_L(t,0), \mathbf{1}_{[\![0,\xi]\!]} g + F_L(t,0), F_{R_n}, G_{R_n})$$

provided

$$3C_1(L_{F,2R_n} K^{\mathsf{det},\delta}_{(A(\cdot,u_{0,n}),B(\cdot,u_{0,n}))} + L_{B,2R_n} K^{\mathsf{sto},\delta}_{(A(\cdot,u_{0,n}),B(\cdot,u_{0,n}))}) < 1, \ \forall n \geq 1, \tag{4.67}$$

where $C_1 > 0$ is the constant in the embedding of Lemma 4.3.9 and does not depend on $T > 0$. Note that choosing $\varepsilon_n > 0$ suitably we obtain (4.67). Recall that the constants $K^{\mathsf{det},\delta}_{(A(\cdot,u_{0,n}),B(\cdot,u_{0,n}))}$, $K^{\mathsf{sto},\delta}_{(A(\cdot,u_{0,n}),B(\cdot,u_{0,n}))}$ are defined in (4.14) and $\delta \in ((1+\kappa)/p, 1/2)$ is arbitrary.

For the sake of clarity, we split the proof into several steps.

Step 1: Existence of a local solution to (4.16) *if* $u_0 \in L^0_{\mathscr{F}_0}(\Omega; X^{\mathsf{Tr}}_{\kappa,p})$. Let (u_n, σ_n) as above. Then let us define the following stopping time

$$\tau_n := \inf\Big\{t \in [0, \sigma_n) \, : \, \|u_n\|_{L^p(I_t, w_\kappa; X_1)} + \sup_{s\in[0,t]} \|u_n\|_{X^{\mathsf{Tr}}_{\kappa,p}} \geq 2R_n\Big\}$$

and $\tau_n := \sigma_n$ if the set is empty. Then reasoning as in Step 4 in the proof of Theorem 4.3.5 one immediately sees that $(u_n, \sigma_n \wedge \tau_n)$ verifies (4.16) with initial data $u_{0,n}$. Note that $u_{0,n}$ has norm less than R_n (see (4.66)), and therefore $\tau_n > 0$ a.s. Thus, $\sigma_n \wedge \tau_n > 0$ a.s.

Set $\sigma_n' := \sigma_n \wedge \tau_n$. Let $(\Lambda_n)_{n\geq 1} \subseteq \mathscr{F}_0$ be defined as $\Lambda_1 := \Gamma_1$ and $\Lambda_n := \Gamma_{n+1} \setminus \Gamma_n$ for each $n > 1$. Define (u, σ) as $\sigma := \sigma_n'$ on Λ_n, and $u = u_n$ on $\Lambda_n \times [0, \sigma_n')$. Since (u_n, σ_n') is a local solution to (4.16) with initial data $u_{0,n}$, one can check that (u, σ) is a local solution to (4.16).

Step 2: Uniqueness of (u, σ). Let (v, μ) be another local solution to (4.16). Set

$$\mu_n := \inf\Big\{t \in [0, \mu) \, : \, \|v\|_{L^p(I_t, w_\kappa; X_1)} + \sup_{s\in[0,t]} \|v\|_{X^{\mathsf{Tr}}_{\kappa,p}} \geq 2R_n\Big\},$$

and $\tau_n = \mu$ if the set is empty. Then $(\mathbf{1}_{\Lambda_n} v, \mathbf{1}_{\Lambda_n} \mu_n)$ is a local solution to (4.16) with data $(\mathbf{1}_{\Lambda_n} u_{0,n}, \mathbf{1}_{\Lambda_n}(\mathbf{1}_{[\![0,\xi]\!]} f + F_L(t,0)), \mathbf{1}_{\Lambda_n}(\mathbf{1}_{[\![0,\xi]\!]} g + G_L(t,0)))$ and $F = F_{R_n}, G = G_{R_n}$. At this stage, the conclusion follows as in Step 5 in the proof of Theorem 4.3.5.

Step 2: Existence of a maximal unique local solution. Similarly as in Step 6 in the proof of Theorem 4.3.5, consider the set Ξ of all stopping time τ such that (4.16) admits a unique local solution. Steps 1-2 ensure that Ξ is not empty, and that there exists $\tau \in \Xi$ such that $\tau > 0$ a.s. The rest of the proof follows as Step 5 in the proof of Theorem 4.3.5.

Step 3: Regularity. The claimed regularity follows as in Step 6 in the proof of Theorem 4.3.5 by replacing $\widetilde{\tau}_n^v$ in (4.64) by $\mathbf{1}_{\Gamma_n} \widetilde{\tau}_n^v$. □

Remark 4.3.19. As in Remark 4.3.17 the proof of Theorem 4.3.7 shows that the condition (4.25) can be replaced by (4.67).

Proof of Theorem 4.3.8. (1): Follows by Theorem 4.3.5.

(2): The proof is similar to the one proposed for Theorem 4.3.5. Indeed, we may replace the truncations in Step 1 by

$$\begin{aligned}\widetilde{F}_\lambda(u) &:= F_{c,\lambda}(u_0,u) + F_L(\cdot,u) + F_{\mathsf{Tr}}(\cdot,u),\\ \widetilde{G}_\lambda(u) &:= G_{c,\lambda}(u_0,u) + G_L(\cdot,u) + G_{\mathsf{Tr}}(\cdot,u),\\ \widetilde{f} &:= f + F_c(\cdot,0) + F_{\mathsf{Tr}}(\cdot,u_0),\\ \widetilde{g} &:= g + G_c(\cdot,0) + G_{\mathsf{Tr}}(\cdot,u_0).\end{aligned}$$

Due to Remark 4.3.14 and the assumptions, the assertion of Lemma 4.3.13 still holds. Now one can repeat the proof of Theorem 4.3.5 literally.

(3): This follows from Theorem 4.3.5 and the fact that the constants ε_n do not depend on $n \geq 1$ (see Remark 4.3.19). □

Chapter 5

Applications to parabolic SPDEs

Let $(\Omega, \mathbb{F} = (\mathscr{F}_t)_{t\geq 0}, \mathscr{A}, \mathbb{P})$ and $\mathscr{P}$ be a filtered probability space and the progressive sigma algebra, respectively.

In this chapter we apply the abstract results of Chapter 4 to several concrete SPDEs of parabolic type. The general study of stochastic evolution equations will continue in Chapters 6 and 7. The applications include: reaction-diffusion equations in conservative and non-conservative form, Burger's equations with white and colored noise, quasilinear SPDEs in divergence and non-divergence form on $\mathbb{R}^d$ and/or domains and porous media equations. In all of the applications a gradient noise term is allowed.

The overall aim of this chapter is to show the flexibility and applicability of the results in Chapter 4 for some classical SPDEs. Indeed, our approach provides several new results in an $L^p(L^q)$-setting which seem not to be known before and might be difficult to reach with pure PDEs arguments. Moreover, if the SPDEs admits critical spaces, then we compute them explictly and in all cases they coincide with spaces having the right (local) scaling of the SPDEs considered. In particular, in the case of reaction-diffusion equations in Section 5.1.3, we show that the deterministic and the stochastic nonlinearity have the same scaling if and only if the 'corresponding critical equations' (i.e. (4.18) and (4.20)) coincide.

The results are taken from Sections 5-6 of my work [3]. For applications to stochastic Allen-Cahn and Cahn-Hilliard equations on domains we refer to [3, Section 7].

5.1 Applications to semilinear SPDEs with gradient noise

In this section we will consider semilinear SPDEs on $X_0 = H^{s,q}$ which can be written in the form

$$\begin{cases} du + A(\cdot)udt = F(\cdot, u)dt + (G(\cdot, u) + B(\cdot)u)dW_H, & t \in I_T, \\ u(0) = u_0, \end{cases} \tag{5.1}$$

which is a special case of the setting considered in Theorem 4.3.8. In Subsections 5.1.2-5.1.4 we take $H = \ell^2$ and in Subsection 5.1.5 $H = L^2(\mathbb{T})$.

In the next section we motivate this setting and explain which class of operator pairs (A, B) we will be considering.

5.1.1 Introduction and motivations

In this section we study a large class of nonlinear second order equations with gradient noise. Such equations are commonly known as stochastic–reaction diffusion equations, but they also include the filtering equation see [129, Section 8] and Allen-Cahn equations [21, 22, 87, 184].

Stochastic reaction–diffusion equations have been extensively studied in the last decades. Nonlinear reaction–diffusion models arise in many scientific areas such as chemical reactions, pattern-formation, population dynamics. Stochastic perturbations of such models can model thermal fluctuations, uncertain determinations of the parameters and non-predictable forces acting on the system. For the sake of completeness let us mention some works on the deterministic case [34, 86, 182, 208] and for the stochastic case one may consult [33, 35, 36, 58, 74, 78, 81, 88, 102, 202, 204] and the references therein.

To the best of our knowledge, the results presented below are new. The reader can compare our results with the results in [178, Section 3] in the deterministic framework.

In this section we analyse second order stochastic PDEs in non-divergence form with gradient noise:

$$\begin{cases} du + \mathcal{A}u dt = f(u, \nabla u)dt + \sum_{n\geq 1} \big(\mathcal{B}_n u + g_n(u)\big)dw_t^n, & \text{on } \mathscr{O}, \\ u(0) = u_0, & \text{on } \mathscr{O}. \end{cases} \tag{5.2}$$

here $(w_t^n : t \geq 0)_{n\geq 1}$ denotes a sequence of independent standard Brownian motions and $u : I_T \times \Omega \times \mathscr{O} \to \mathbb{R}$ is the unknown process. Moreover, the differential operators $\mathcal{A}, \mathcal{B}_n$ for each $x \in \mathscr{O}$, $\omega \in \Omega$, $t \in (0, T)$ are given by

$$\begin{aligned} (\mathcal{A}(t,\omega)u)(t,\omega,x) &:= -\sum_{i,j=1}^{d} a_{ij}(t,\omega)\partial_{ij}^2 u(x), \\ (\mathcal{B}_n(t,\omega)u)(t,\omega,x) &:= \sum_{j=1}^{d} b_{jn}(t,\omega)\partial_j u(x). \end{aligned} \tag{5.3}$$

Lower order terms in the previous differential operators can be added (see Subsection 5.1.6). The assumptions on f, g_n will be specified below.

In the applications of Theorem 4.3.8, the following splitting arises naturally:

- $\mathscr{O} = \mathbb{R}^d$ or $\mathscr{O} = \mathbb{T}^d$;
- $\mathscr{O}$ is a smooth domain in $\mathbb{R}^d$.

We will only consider $\mathbb{R}^d$ in detail since $\mathbb{T}^d$ can be treated by the same arguments. This will be done in Sections 5.1.2, 5.1.3, and 5.1.4 using the maximal regularity result of Lemma 5.1.2 below.

To avoid the need for too many subcases, we will only consider $d \geq 2$. However, under suitable conditions on the parameters the case $d = 1$ could also be included in most examples.

Next we introduce the function spaces which will be needed below. As usual, for $q \in (1, \infty)$ and $k \geq 1$, we denote by $W^{k,q}(\mathbb{R}^d)$ the set of all $f \in L^q(\mathbb{R}^d)$ such that $\partial^\alpha f \in L^q(\mathbb{R}^d)$ for any $\alpha \in \mathbb{N}_0^d$ such that $|\alpha| \leq k$ endowed with the natural norm. Let $\mathfrak{F}$ be the Fourier transform on $\mathbb{R}^d$. Then for any $s \in \mathbb{R}$ and $q \in (1, \infty)$ we set $H^{s,q}(\mathbb{R}^d) = \{f \in \mathscr{S}'(\mathbb{R}^d) : \mathfrak{F}^{-1}((1 + |\cdot|^2)^{s/2}\mathfrak{F}(f)) \in L^q(\mathbb{R}^d)\}$ with its natural norm. For $s \in \mathbb{R}$, $q \in (1, \infty)$ and $p \in [1, \infty]$, we define Besov spaces through real interpolation:

$$B_{q,p}^s(\mathbb{R}^d) = (H^{s_0,q}(\mathbb{R}^d), H^{s_1,q}(\mathbb{R}^d))_{\theta,p},$$

where $s_0 < s < s_1$ and $\theta \in (0, 1)$ are chosen in such a way that $s = s_0(1-\theta) + s_1\theta$. We refer to [20, Chapter 6] for alternative descriptions of the Besov spaces $B_{q,p}^s(\mathbb{R}^d)$. For $s \in \mathbb{R}$ and $q \in (1, \infty)$, we denote the Sobolev-Slobodeckij spaces by $W^{s,q}(\mathbb{R}^d) := B_{q,q}^s(\mathbb{R}^d)$.

Recall from [20, Theorem 6.4.5] that

$$[H^{s_0,q}(\mathbb{R}^d), H^{s_1,q}(\mathbb{R}^d)]_\theta = H^{s,q}(\mathbb{R}^d), \qquad s := (1-\theta)s_0 + \theta s_1. \tag{5.4}$$

For the sake of simplicity, sometimes, we write $H^{s,q}$ instead of $H^{s,q}(\mathbb{R}^d)$ (and analogously for other spaces) if no confusion seems possible.

The following will be a standing assumption in this section:

Assumption 5.1.1. *Suppose that one of the two conditions hold:*

- $q \in [2, \infty)$, $p \in (2, \infty)$ *and* $\kappa \in [0, \frac{p}{2} - 1)$;
- $q = p = 2$ *and* $\kappa = 0$.

Assume the following two conditions on a_{ij} and b_{in}:

(1) *The functions $a_{ij} : (0,T) \times \Omega \to \mathbb{R}$ and $b_{jn} : (0,T) \times \Omega \to \mathbb{R}$ are progressively measurable. Moreover, there exists $K > 0$ such that*

$$|a_{ij}(t,\omega)| + \|(b_{jn}(t,\omega))_{n\geq 1}\|_{\ell^2} \leq K, \qquad a.a.\ \omega \in \Omega,\ \text{for all } t \in I_T.$$

(2) *There exists $\epsilon > 0$ such that a.s. for all $\xi \in \mathbb{R}^d$, $t \in I_T$,*

$$\sum_{i,j=1}^{d} \Big(a_{ij}(t) - \frac{1}{2} \sum_{n\geq 1} b_{in}(t) b_{jn}(t) \Big) \xi_i \xi_j \geq \epsilon |\xi|^2.$$

The following result will be employed several times.

Lemma 5.1.2. *Let the Assumption 5.1.1 be satisfied. Let $X_0 = H^{s,q}(\mathbb{R}^d)$ and $X_1 = H^{s+2,q}(\mathbb{R}^d)$ with $s \in \mathbb{R}$. Let $A : I_T \times \Omega \to \mathscr{L}(X_1; X_0)$ and $B : I_T \times \Omega \to \mathscr{L}(X_1, \gamma(\ell^2, X_{\frac{1}{2}}))$ be given by*

$$A(t)u := \mathcal{A}(t)u, \quad (B(t)u)_n := \mathcal{B}_n(t)u, \qquad n \geq 1,$$

where $\mathcal{A}, \mathcal{B}_n$ are as in (5.3). Then $(A, B) \in \mathcal{SMR}^{\bullet}_{p,\kappa}(T)$ (see Definition 4.2.5).

Proof. Since the coefficients a_{ij}, b_{jn} are x-independent by applying $(1-\Delta)^{s/2}$ to the equation, one can reduce to the case $s = 0$. Now the result follows from [174, Theorem 5.3]. □

5.1.2 Conservative stochastic reaction diffusion equations

In this subsection we study the following differential problem for the unknown process $u : [0,T] \times \Omega \times \mathbb{R}^d \to \mathbb{R}$,

$$\begin{cases} du - \mathcal{A}u dt = \operatorname{div}(f(\cdot,u))dt + \sum_{n\geq 1} (\mathcal{B}_n u + g_n(\cdot,u)) dw_t^n, & \text{on } \mathbb{R}^d, \\ u(0) = u_0, & \text{on } \mathbb{R}^d; \end{cases} \tag{5.5}$$

for $t \in I_T$. Here $\mathcal{A}, \mathcal{B}_n$ are as in (5.3).

A formal integration of (5.5) shows that the system preserves mass under the flow, i.e.

$$\mathbb{E} \int_{\mathbb{R}^d} u(x,t)dx = \mathbb{E} \int_{\mathbb{R}^d} u_0(x)dx.$$

This feature is very important from a modelling point of view, since u (typically) represents the mass of chemical reactants. This motivates the name 'conservative reaction-diffusion equations'.

We study (5.5) under the following assumption:

Assumption 5.1.3. *The maps $f : I_T \times \Omega \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$, $g := (g_n)_{n\geq 1} : I_T \times \Omega \times \mathbb{R}^d \times \mathbb{R} \to \ell^2$ are $\mathscr{P} \otimes \mathscr{B}(\mathbb{R}^d) \otimes \mathscr{B}(\mathbb{R})$-measurable with $f(\cdot,0) = 0$ and $g(\cdot,0) = 0$. Moreover, there exist $h > 1$ and $C > 0$ such that a.s. for all $t \in I_T$, $z, z' \in \mathbb{R}$ and $x \in \mathbb{R}^d$,*

$$|f(t,x,z) - f(t,x,z')| + \|g(t,x,z) - g(t,x,z')\|_{\ell^2} \leq C(|z|^{h-1} + |z'|^{h-1})|z - z'|.$$

Typical examples of f and g which satisfies Assumption 5.1.3 are:

$$f(x,u) = \widetilde{f}(x)|u|^{h-1}u, \qquad g(x,u) = \widetilde{g}(x)|u|^{h-1}u, \qquad h \in (1,\infty), \tag{5.6}$$

where $\widetilde{f} \in L^\infty_{\mathscr{P}}((0,T) \times \Omega \times \mathbb{R}^d; \mathbb{R}^d)$ and $\widetilde{g} \in L^\infty_{\mathscr{P}}((0,T) \times \Omega \times \mathbb{R}^d; \ell^2)$. The condition $f(\cdot,0) = 0$ and $g(\cdot,0) = 0$ can be weakened to a decay condition in the x-variable.

We study (5.5) directly in 'the almost very weak setting', i.e. in $X_0 := H^{-1-s,q}$ with $s \in [0,1)$ (cf. [178, Subsection 4.5]). This will give us additional flexibility in the treatment of (5.5). The weak setting can be derived by setting $s = 0$.

Almost very weak setting

Let $s \in [0,1)$ and let $q \in [2,\infty)$. The differential problem (5.5) can be rephrased as a stochastic evolution equation of the form (5.1) with $X_0 := H^{-1-s,q}$ and $X_1 := H^{1-s,q}$. Here

$$\begin{aligned} A(t)u &= \mathcal{A}(t)u, & B(t)u &= (\mathcal{B}_n(t)u)_{n\geq 1}, \\ F(t,u) &= \operatorname{div}(f(t,\cdot,u)), & G(t,u) &= (g_n(t,\cdot,u))_{n\geq 1} \end{aligned}$$

for $u \in H^{1-s,q}$. We say that (u,σ) is a maximal local solution to (5.5) if (u,σ) is a maximal local solution to (5.1) in the sense of Definition 4.3.4.

To show local existence for (5.5) we employ Theorem 4.3.8. By Lemma 5.1.2 it is enough to look at suitable bounds for the non-linearities F, G. To this end, let us start by looking at F. By Assumption 5.1.3, it follows that

$$\begin{aligned} \|F(\cdot,u)-F(\cdot,v)\|_{H^{-1-s,q}} &\overset{(i)}{\lesssim} \|F(\cdot,u)-F(\cdot,v)\|_{H^{-1,r}} \\ &\lesssim \|f(\cdot,u)-f(\cdot,v)\|_{L^r} \\ &\lesssim \Big\|(|u|^{h-1}+|v|^{h-1})|u-v|\Big\|_{L^r} \\ &\overset{(ii)}{\lesssim} (\|u\|^{h-1}_{L^{hr}}+\|v\|^{h-1}_{L^{hr}})\|u-v\|_{L^{hr}} \\ &\overset{(iii)}{\lesssim} (\|u\|^{h-1}_{H^{\theta,q}}+\|v\|^{h-1}_{H^{\theta,q}})\|u-v\|_{H^{\theta,q}}; \end{aligned} \tag{5.7}$$

where in (i) we used the Sobolev embedding with r defined by $-1-\frac{d}{r} = -1-s-\frac{d}{q}$, in (ii) the Hölder inequality with exponent $h, \frac{h}{h-1}$ and in (iii) the Sobolev embedding and $\theta - \frac{d}{q} = -\frac{d}{hr}$. Note that $r \in (1,\infty)$ since $q \geq 2$, $d \geq 2$ and $s \in [0,1)$ by assumption. Note that θ has to satisfy $\theta \in (0, 1-s)$ in order to obtain a space in between X_0 and X_1. Combining the identities we obtain

$$\frac{d}{q} - \theta = \frac{d}{hr} = \frac{1}{h}\Big(\frac{d}{q}+s\Big) \Rightarrow \theta = \frac{d}{q}\Big(1-\frac{1}{h}\Big) - \frac{s}{h}.$$

Therefore, to ensure that $\theta \in (0,1-s)$ we assume[1]

$$\frac{d(h-1)}{h-s(h-1)} < q < \frac{d(h-1)}{s}. \tag{5.8}$$

Since $s \neq 1$ and $h > 1$ the set of q which satisfies (5.8) is not-empty. If (5.8) holds, due to (5.4) one has $H^{\theta,q} = [H^{-1-s,q}, H^{1-s,q}]_{\beta_1}$ where

$$\beta_1 = \frac{1+\theta+s}{2} = \frac{1}{2}\Big[\Big(\frac{d}{q}+s\Big)\Big(1-\frac{1}{h}\Big)+1\Big] \in (0,1). \tag{5.9}$$

To check the condition (HF) we may split the discussion into three cases:

(1) If $1-\frac{1+\kappa}{p} > \beta_1$, by Remark 4.3.2(1), (HF) follows by setting $F_{\mathsf{Tr}}(t,u) := \operatorname{div}(f(t,\cdot,u))$ and $F_L \equiv F_c \equiv 0$.

(2) If $1-\frac{1+\kappa}{p} = \beta_1$, by (5.7) and Remark 4.3.2(2), (HF) follows by setting $F_L \equiv F_{\mathsf{Tr}} \equiv 0$, $F_c(t,u) := \operatorname{div}(f(t,\cdot,u))$, $m_F = 1$, $\rho_1 = h-1$ and $\varphi_1 = \beta_1$.

(3) If $1-\frac{1+\kappa}{p} < \beta_1$ we set $F_c(t,u) := \operatorname{div}(f(t,\cdot,u))$ and $F_L \equiv F_{\mathsf{Tr}} \equiv 0$. As in the previous item we set $m_F = 1$, $\rho_1 = h-1$ and $\varphi_1 = \beta_1$. By (5.7) it remains to check the condition (4.18). In this situation, (4.18) becomes,

$$\frac{1+\kappa}{p} \leq \frac{\rho_1+1}{\rho_1}(1-\beta_1) = \frac{1}{2}\frac{h}{h-1} - \frac{1}{2}\Big(\frac{d}{q}+s\Big). \tag{5.10}$$

[1] Here we have set $1/0 := \infty$.

Note that the assumption $\kappa \geq 0$ implies

$$\frac{1}{p} + \frac{d}{2q} + \frac{s}{2} \leq \frac{h}{2(h-1)}. \tag{5.11}$$

Since $d/2q + s/2 < h/[2(h-1)]$ (thanks to the lower bound in (5.8)) the above inequality is always verified for p sufficiently large.

It remains to estimate G. To this end we can reasoning as in (5.7). First, note that $X_{1/2} = H^{-s,q}$ (see (5.4)) and let r, θ be as in (5.7). By Assumption 5.1.3 one has

$$\begin{aligned}\|G(\cdot,u) - G(\cdot,v)\|_{\gamma(\ell^2;H^{-s,q})} &\lesssim \|G(\cdot,u) - G(\cdot,v)\|_{\gamma(\ell^2;L^r)} \\ &\overset{(i)}{\eqsim} \|G(\cdot,u) - G(\cdot,v)\|_{L^r(\ell^2)} \\ &\lesssim \|(|u|^{h-1} + |v|^{h-1})|u-v|\|_{L^r} \\ &\lesssim (\|u\|^{h-1}_{H^{\theta,q}} + \|v\|^{h-1}_{H^{\theta,q}})\|u-v\|_{H^{\theta,q}};\end{aligned} \tag{5.12}$$

where in (i) we used the identification $\gamma(\ell^2, L^r) = L^r(\ell^2) := L^r(\mathbb{R}^d;\ell^2)$ (see (2.14)). The previous considerations show that G verifies (HG) under the same assumptions on F.

Therefore, Theorem 4.3.8 gives the following result.

Theorem 5.1.4. *Let Assumptions 5.1.1 and 5.1.3 be satisfied and $d \geq 2$. Let $s \in [0,1)$. Assume (5.8). Let β_1 be as in (5.9). Assume that one of the following conditions is satisfied*

- $1 - (1+\kappa)/p \geq \beta_1$*;*
- $1 - (1+\kappa)/p < \beta_1$ *and (5.10) holds.*

Then for each $u_0 \in L^0_{\mathscr{F}_0}(\Omega; B^{1-s-2(1+\kappa)/p}_{q,p}(\mathbb{R}^d))$ there exists a maximal local solution (u,σ) to (5.5). Moreover, there exists a localizing sequence $(\sigma_n)_{n\geq 1}$ such that a.s. for all $n \geq 1$

$$u \in L^p(I_{\sigma_n}, w_\kappa; H^{1-s,q}) \cap C(\overline{I}_{\sigma_n}; B^{1-s-2(1+\kappa)/p}_{q,p}) \cap C((0,\sigma_n]; B^{1-s-2/p}_{q,p}).$$

Critical spaces for (5.5)

In this subsection we study the existence of critical spaces for (5.5).

To motivate the setting let f, g_n be as in (5.6) with $\widetilde{f}, \widetilde{g} \in \ell^2$ constant w.r.t. It will turn out that our abstract notion of critical spaces as introduced in Remark 4.3.2 (3) is consistent with the natural scaling of (5.5)-(5.6). First consider the deterministic setting, i.e. $b_{jn} \equiv \widetilde{g}_n \equiv 0$. If u is a (local smooth) solution to (5.5)-(5.6) on $(0,T) \times \mathbb{R}^d$, then $u_\lambda(x,t) := \lambda^{1/[2(h-1)]}u(\lambda t, \lambda^{1/2}x)$ is a (local smooth) solution to (5.5) on $(0, T/\lambda) \times \mathbb{R}^d$ for each $\lambda > 0$. Note that the map $u \mapsto u_\lambda$ induces a mapping on the initial data u_0 given by $u_0 \mapsto u_{0,\lambda}$ where $u_{0,\lambda}(x) := \lambda^{1/[2(h-1)]}u_0(\lambda^{1/2}x)$ for $x \in \mathbb{R}^d$.

In the theory of PDEs a function space is called critical for (5.5)-(5.6) (in absence of noise) if it is invariant under the above mapping $u_0 \mapsto u_{0,\lambda}$. An example of a Besov spaces which is (locally) invariant under this scaling is $B^{d/q-1/(h-1)}_{q,p}$ for $q, p \in (1,\infty)$. This can be made precise by looking at the so-called homogeneous version of such spaces. Indeed, one has

$$\begin{aligned}\|u_{0,\lambda}\|_{\dot{B}^{d/q-1/(h-1)}_{q,p}} &\eqsim \lambda^{1/[2(h-1)]}(\lambda^{1/2})^{d/q-1/(h-1)-d/q}\|u_0\|_{\dot{B}^{d/q-1/(h-1)}_{q,p}} \\ &= \|u_0\|_{\dot{B}^{d/q-1/(h-1)}_{q,p}};\end{aligned} \tag{5.13}$$

where the implicit constants do not depend on $\lambda > 0$. It will turn out that this space appears naturally when equality in (5.10) is reached. This observation was made in [178, Sections 2.3 and 3-6] for many PDEs.

Next consider the stochastic problem. At least formally, we can show that if u is a (local smooth) solution to (5.5), then u_λ is a (local smooth) solution to (5.5) where the $(w^n_t : t \geq 0)_{n\geq 1}$ is replaced by the sequence of independent Brownian motions $(b^n_{t,\lambda}\,:\,t\geq 0)_{n\geq 1}:=(\lambda^{-1/2}w^n_{\lambda t}\,:\,t\geq 0)_{n\geq 1}$. To see this, let $t\in(0,T)$ and let us look at the strong formulation of (5.5) as in Definition 4.3.3. As we have seen before, under the map $u\mapsto u_\lambda$ all the deterministic integrals have all the same scaling, therefore it is enough to study one of them. For instance,

$$\int_0^{t/\lambda}\Delta u_\lambda(s,x)ds=\lambda^{\frac{1}{2(h-1)}}\int_0^t \Delta u(s',\lambda^{1/2}x)ds'.$$

Such scaling agrees with the scaling of the stochastic integrals,

$$\begin{aligned}\int_0^{t/\lambda}|u_\lambda(s,x)|^{h-1}u_\lambda(s,x)db^n_{s,\lambda}&=\int_0^{t/\lambda}\lambda^{\frac{1}{2(h-1)}}|u(\lambda s,\lambda^{1/2}x)|^{h-1}u(\lambda s,\lambda^{1/2}x)dw^n_{\lambda s}\\&=\lambda^{\frac{1}{2(h-1)}}\int_0^t|u(s,\lambda^{1/2}x)|^{h-1}u(s,\lambda^{1/2}x)dw^n_s,\end{aligned}\tag{5.14}$$

where $n\geq 1$ is fixed. The same holds for the stochastic integral for the b-term. Therefore, u_λ is a solution to (5.5) with a scaled noise.

After these formal calculations, let us turn to our setting. We will analyse when equality in (5.10) can be allowed. We begin by looking at the case $p\in(2,\infty)$. Note that $\kappa\in[0,\frac{p}{2}-1)$ if and only if $\frac{1+\kappa}{p}\in[\frac1p,\frac12)$ and due to (5.11) to ensure the existence of a weight κ which realizes equality in (5.10) we have to assume

$$\frac12\frac{h}{h-1}-\frac12\Big(\frac dq+s\Big)<\frac12.\tag{5.15}$$

Simple computations show that the previous is verified if and only if

$$h\geq\frac{1+s}{s}\qquad\text{or}\qquad\Big[h<\frac{1+s}{s}\ \text{ and }\ q<\frac{d(h-1)}{1-s(h-1)}\Big].\tag{5.16}$$

If (5.11) and (5.16) hold, then we set

$$\kappa_{\mathsf{crit}}=\frac p2\Big(\frac{h}{h-1}-\frac dq-s\Big)-1.\tag{5.17}$$

Then $\kappa_{\mathsf{crit}}\in[0,\frac p2-1)$ and the corresponding critical space is

$$X^{\mathsf{Tr}}_{\kappa_{\mathsf{crit}},p}=B^{1-s-2\frac{1+\kappa_{\mathsf{crit}}}{p}}_{q,p}(\mathbb{R}^d)=B^{\frac dq-\frac{1}{h-1}}_{q,p}(\mathbb{R}^d).\tag{5.18}$$

Note that the above space coincides with the one appearing in the above discussion. Moreover, the space does not depend on the parameter $s>0$, and depends on p only through the microscopic parameter. The independence on $s>0$ is in accordance with the independence of the scale founded in the deterministic case for (4.16) without noise and bilinear non-linearities, see [178, Section 2.4].

It remains to consider the case $p=q=2$ and $\kappa=0$. We expect that a similar space appears also in this case. Indeed, the condition (5.10) implies the identity

$$h=\frac{2+d+2s}{d+2s}>1.\tag{5.19}$$

Note that the lower bound in (5.8) is automatically verified and the upper bound in (5.8) is equivalent to $d>2s^2/(1-s)$. Therefore, in the case $p=q=2$, $\kappa=0$ and h as in (5.19), the trace space for (5.5) becomes

$$X^{\mathsf{Tr}}_{\kappa,p}=B^{-s}_{2,2}(\mathbb{R}^d)=B^{\frac d2-\frac{1}{h-1}}_{2,2}(\mathbb{R}^d)=H^{\frac d2-\frac{1}{h-1}}(\mathbb{R}^d).$$

In the case $s=0$ one has $h=(2+d)/d=2/d+1$ and condition (5.8) is satisfied.

Let us summarize what we have proved in the following:

Theorem 5.1.5. *Let Assumptions 5.1.1 and 5.1.3 be satisfied and $d \geq 2$. Let $s \in [0,1)$ and let one of the following conditions be satisfied:*

- *$p, q \in (2,\infty)$, (5.8), (5.11) and (5.16) hold;*
- *$p = q = 2$, $d > 2s^2/(1-s)$, and h is as in (5.19).*

Let κ_{crit} be as in (5.17). Then for each

$$u_0 \in L^0_{\mathscr{F}_0}(\Omega; B^{\frac{d}{q}-\frac{1}{h-1}}_{q,p}(\mathbb{R}^d))$$

there exists a maximal local solution (u,σ) to (5.5). Moreover, there exists a localizing sequence $(\sigma_n)_{n\geq 1}$ such that a.s. for all $n \geq 1$

$$u \in L^p(I_{\sigma_n}, w_{\kappa_{\mathsf{crit}}}; H^{1-s,q}(\mathbb{R}^d)) \cap C(\overline{I}_{\sigma_n}; B^{\frac{d}{q}-\frac{1}{h-1}}_{q,p}(\mathbb{R}^d)) \cap C((0,\sigma_n]; B^{1-s-\frac{2}{p}}_{q,p}(\mathbb{R}^d)).$$

Note that the space $B^{\frac{d}{q}-\frac{1}{h-1}}_{q,p}(\mathbb{R}^d)$ becomes larger as p tends to ∞. Therefore, for u_0 as above and any $\delta < 1-s$, there exists a maximal local solution (u,σ) to (5.5) such that $u \in C((0,\sigma_n]; B^{\delta}_{q,\infty}(\mathbb{R}^d))$ a.s. In particular, if $s = 0$, then for all $\delta < 1$ we find a maximal local solution to (5.5) such that $u \in C((0,\sigma_n]; B^{\delta}_{q,\infty}(\mathbb{R}^d))$ a.s. Bootstrapping arguments related to such regularization phenomena will be investigated in Chapter 7.

Let us conclude this section by giving an example which illustrates the usefulness of $s \in (0,1)$.

Example 5.1.6. Let $d = 3$ and $h = 2$. The restriction on $q \geq 2$ becomes

$$\frac{3}{2-s} < q < \min\Big\{\frac{3}{s}, \frac{3}{1-s}\Big\}, \qquad s \in [0,1). \tag{5.20}$$

Therefore, in the weak setting $s = 0$ one needs $q \in [2,3)$, and the critical space $B^{\frac{3}{q}-1}_{q,p}(\mathbb{R}^d)$ has strictly positive smoothness. To admit critical spaces with negative smoothness, we need $s > 0$. Indeed, note that the choice $s = 1/2$ optimizes the right hand-side of (5.20). Therefore, with $s = 1/2$ we can allow $q \in [2,6)$ and thus we have a larger class of critical spaces which goes down to smoothness $-\frac{1}{2}$.

Also the space $L^{d(h-1)}(\mathbb{R}^d)$ is invariant under the scaling $u_0 \mapsto u_{0,\lambda}$. From the previous result we obtain the following corollary.

Corollary 5.1.7. *Let Assumptions 5.1.1 and 5.1.3 be satisfied and $d \geq 2$. Let $h > 1 + \frac{2}{d}$, $q := d(h-1)$ and $p \in (q,\infty)$. Then there exists $\bar{s} > 0$ such that for all $s \in (0,\bar{s})$ and*

$$u_0 \in L^0_{\mathscr{F}_0}(\Omega; L^{d(h-1)}(\mathbb{R}^d))$$

there exists a maximal local solution to (5.5), and there exists a localizing sequence $(\sigma_n)_{n\geq 1}$ such that for any $n \geq 1$ and a.s.

$$u \in L^p(I_{\sigma_n}, w_{\kappa_{\mathsf{crit}}}; H^{1-s,q}(\mathbb{R}^d)) \cap C(\overline{I}_{\sigma_n}; B^0_{q,p}(\mathbb{R}^d)) \cap C((0,\sigma_n]; B^{1-s-\frac{2}{p}}_{q,p}(\mathbb{R}^d)),$$

where κ_{crit} is given by (5.17).

Recall that p in Theorem 5.1.5 can be chosen as large as one wants.

Proof. Since $h \geq 1 + \frac{2}{d}$, $q \geq 2$. One can check that there exists $s_1 > 0$ such that (5.8) and (5.16) hold for $q = d(h-1)$ and $s \in (0,s_1)$. Moreover, for $q = d(h-1)$, there exists $s_2 > 0$, such that (5.11) holds for all $p \in (2,\infty)$ and $s \in (0,s_2)$. Set $s := \min\{s_1,s_2\}$. Thus, Theorem 5.1.5 ensures the existence of a maximal local solution to (5.5) for any $s \in (0,\bar{s})$ and $u_0 \in L^0_{\mathscr{F}_0}(\Omega; B^0_{q,p}(\mathbb{R}^d))$ with the required regularity. To conclude, it remains to recall that $L^q(\mathbb{R}^d) \hookrightarrow B^0_{q,p}(\mathbb{R}^d)$, since $p \geq q$. ☐

By choosing s small enough such that $1-s-2/p > 0$, the solution u to (5.5) provided by Corollary 5.1.7, instantaneously regularizes in space, i.e. $u \in C(I_{\sigma_n}; B^{1-s-\frac{2}{p}}_{q,p}(\mathbb{R}^d)) \hookrightarrow C(I_{\sigma_n}; L^q(\mathbb{R}^d))$ a.s. for all $n \geq 1$.

5.1.3 Stochastic reaction diffusion equations

In this subsection we study local existence for the following non-conservative reaction-diffusion equation for the unknown $u : [0,T] \times \Omega \times \mathbb{R}^d \to \mathbb{R}$,

$$\begin{cases} du + \mathcal{A}udt = f(\cdot,u)dt + \sum_{n\geq 1}(\mathcal{B}_n u + g_n(\cdot,u))dw_t^n, & \text{on } \mathbb{R}^d, \\ u(0) = u_0, & \text{on } \mathbb{R}^d, \end{cases} \tag{5.21}$$

where $\mathcal{A}, \mathcal{B}_n$ are as in (5.3). In this subsection we assume that

Assumption 5.1.8. *The maps $f : I_T \times \Omega \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$, $g := (g_n)_{n\geq 1} : I_T \times \Omega \times \mathbb{R}^d \times \mathbb{R} \to \ell^2$ are $\mathscr{P} \otimes \mathscr{B}(\mathbb{R}^d) \otimes \mathscr{B}(\mathbb{R})$-measurable with $f(\cdot, 0) = 0$ and $g(\cdot, 0) = 0$. Moreover, there exist $m, h > 1$ and $C > 0$ such that a.s. for all $t \in I_T$, $x \in \mathbb{R}^d$ and $z, z' \in \mathbb{R}$*

$$\begin{aligned} |f(t,x,z) - f(t,x,z')| &\leq C(|z|^{m-1} + |z'|^{m-1})|z - z'|, \\ \|g(t,x,z) - g(t,x,z')\|_{\ell^2} &\leq C(|z|^{h-1} + |z'|^{h-1})|z - z'|. \end{aligned}$$

Typical choices for such non-linearities are:

$$f(u) = |u|^{m-1}u, \qquad g_n(\cdot,u) = \widetilde{g}_n|u|^{h-1}u, \quad n \geq 1, \tag{5.22}$$

for some $\widetilde{g} = (\widetilde{g}_n)_{n\geq 1} \in L^\infty_{\mathscr{P}}(I_T \times \Omega \times \mathbb{R}^d; \ell^2)$.

To make the results more readable we choose to analyse (5.21) only in the weak setting. The interested reader can adapt the argument below and the one given in Subsection 5.1.2, to study (5.21) in the almost weak setting. As we have seen before, the latter choice gives local existence in a wider set of critical spaces. For more on this we refer to [3, Subsection 7.1].

Again we will focus on the setting of critical spaces. Some noncritical cases could be included by simpler methods. Part of this is covered in the quasilinear setting in Subsection 5.2.5.

Weak setting

As in Subsection 5.1.2 we rewrite (5.5) in the form (5.1) by setting $X_0 := H^{-1,q}(\mathbb{R}^d)$, $X_1 := W^{1,q}(\mathbb{R}^d) = H^{1,q}(\mathbb{R}^d)$ and, for $u \in X_1$,

$$\begin{aligned} A(t)u &= \mathcal{A}(t)u, \quad & B(t)u &= (\mathcal{B}_n(t)u)_{n\geq 1}, \\ F(t,u) &= f(t,u), \quad & G(t,u) &= (g_n(t,u))_{n\geq 1}. \end{aligned} \tag{5.23}$$

As before (u, σ) is a maximal local solution to (5.21) if (u, σ) is a maximal local solution to (5.1) in the sense of Definition 4.3.4.

To prove local existence we apply Theorem 4.3.8. By Lemma 5.1.2, it is enough to estimate the nonlinearities F, G. We start by estimating F:

$$\begin{aligned} \|F(\cdot,u) - F(\cdot,v)\|_{H^{-1,q}} &\overset{(i)}{\lesssim} \|F(\cdot,u) - F(\cdot,v)\|_{L^t} \\ &\lesssim \|(|u|^{m-1} + |v|^{m-1})|u - v|\|_{L^t} \\ &\overset{(ii)}{\lesssim} (\|u\|_{L^{mt}}^{m-1} + \|v\|_{L^{mt}}^{m-1})\|u - v\|_{L^{mt}} \\ &\overset{(iii)}{\lesssim} (\|u\|_{H^{\theta,q}}^{m-1} + \|v\|_{H^{\theta,q}}^{m-1})\|u - v\|_{H^{\theta,q}}. \end{aligned} \tag{5.24}$$

where in (i) we used the Sobolev embedding with $\frac{d}{t} := 1 + \frac{d}{q}$, in (ii) we applied the Hölder inequality, and in (iii) we used Sobolev embedding with $\theta - \frac{d}{q} = -\frac{d}{mt}$. Note that to ensure that $t \in (1, \infty)$, it is enough to assume $q \neq 2$ if $d = 2$. Further, we need $\theta \in (0, 1)$ in order to obtain a space between X_0 and X_1. Combining the identities we obtain

$$\frac{1}{q} - \frac{\theta}{d} = \frac{1}{mt} = \frac{1}{m}\Big(\frac{1}{q} + \frac{1}{d}\Big) \Rightarrow \theta = \frac{d}{q}\Big(1 - \frac{1}{m}\Big) - \frac{1}{m}.$$

Therefore, $\theta \in (0,1)$ is equivalent to

$$d\Big(\frac{m-1}{m+1}\Big) < q < d(m-1). \tag{5.25}$$

Since $q \geq 2$, we also need $m > 1 + \frac{2}{d}$. Setting $\beta_1 = \varphi_1 = \frac{1+\theta}{2} < 1$ we obtain $H^{\theta,q} = [H^{-1,q}, H^{1,q}]_{\beta_1}$ by (5.4). More explicitly

$$\beta_1 = \frac{\theta+1}{2} = \frac{1}{2}\Big(\frac{d}{q}+1\Big)\Big(1-\frac{1}{m}\Big).$$

As in Subsection 5.1.2 to check (HF) we split into three subcases:

(1) If $1-(1+\kappa)/p > \beta_1$, then by Remark 4.3.2(1) (2), (HF) holds.

(2) If $1-(1+\kappa)/p = \beta_1$, then by Remark (2), (HF) holds.

(3) If $1-(1+\kappa)/p < \beta_1$, then (HF) holds with $m_F = 1$, $\beta_1 = \varphi_1$, $\rho_1 = m-1$ if the condition (4.18) holds:

$$\frac{1+\kappa}{p} \leq \frac{\rho_1+1}{\rho_1}(1-\beta_1) = \frac{m}{m-1} - \frac{1}{2}\Big(\frac{d}{q}+1\Big). \tag{5.26}$$

To ensure that $\kappa \geq 0$ we have to assume that

$$\frac{1}{p} + \frac{d}{2q} \leq \frac{m}{m-1} - \frac{1}{2} = \frac{m+1}{2(m-1)}. \tag{5.27}$$

From (5.25) one can check that (5.27) is solvable for p sufficiently large.

Next, we estimate G using the same strategy of (5.12). Indeed, since $X_{1/2} = L^q(\mathbb{R}^d)$ and $\gamma(\ell^2, L^q) = L^q(\mathbb{R}^d;\ell^2) =: L^q(\ell^2)$ (see (2.14)) one has

$$\begin{aligned}\|G(\cdot,u)-G(\cdot,v)\|_{L^q(\ell^2)} &\lesssim \|(|u|^{h-1}+|v|^{h-1})|u-v|\|_{L^q}\\ &\overset{(i)}{\lesssim} (\|u\|^{h-1}_{L^{hq}} + \|v\|^{h-1}_{L^{hq}})\|u-v\|_{L^{hq}}\\ &\overset{(ii)}{\lesssim} (\|u\|^{h-1}_{H^{\phi,q}} + \|v\|^{h-1}_{H^{\phi,q}})\|u-v\|_{H^{\phi,q}}.\end{aligned} \tag{5.28}$$

where in (i) we applied the Hölder inequality and in (ii) we used Sobolev embedding with $\phi - \frac{d}{q} = -\frac{d}{hq}$. Therefore, $\phi = \frac{d}{q}\frac{h-1}{h}$. Note that $\phi > 0$ and to ensure that $\phi < 1$ we have to assume

$$q > \frac{d(h-1)}{h}. \tag{5.29}$$

In addition, let us set

$$\beta_2 = \frac{\phi+1}{2} = \frac{1}{2} + \frac{d}{2q}\Big(1-\frac{1}{h}\Big), \qquad \varphi_2 = \beta_2.$$

As in the previous cases, the discussion splits in two cases:

(1) If $1-(1+\kappa)/p > \beta_2$, then (HG) holds by Remark 4.3.2(1).

(2) If $1-(1+\kappa)/p = \beta_2$, then (HG) holds by Remark 4.3.2(2).

(3) If $1-(1+\kappa)/p < \beta_2$, then (HG) holds with $m_G = 1$, $\rho_2 = h-1$, $\beta_2 = \varphi_2$ if the condition (4.20) holds:

$$\frac{1+\kappa}{p} \leq \frac{h}{h-1}(1-\beta_2) = \frac{h}{2(h-1)} - \frac{d}{2q}. \tag{5.30}$$

To ensure that $\kappa \geq 0$ we have to assume that

$$\frac{1}{p} + \frac{d}{2q} \leq \frac{h}{2(h-1)}. \tag{5.31}$$

These preparation give the following theorem.

Theorem 5.1.9. *Let Assumptions 5.1.1 and 5.1.8 be satisfied and $d \geq 2$. Let $m > 1 + \frac{2}{d}$ and $h > 1$. Moreover, assume that (5.25) and (5.29) hold. Assume one of the following conditions is satisfied*

- $1 - (1+\kappa)/p \geq \beta_1$ *and* $1 - (1+\kappa)/p \geq \beta_2$*;*
- $1 - (1+\kappa)/p < \beta_1$, $1 - (1+\kappa)/p \geq \beta_2$ *and (5.26) holds;*
- $1 - (1+\kappa)/p \geq \beta_1$, $1 - (1+\kappa)/p < \beta_2$ *and (5.30) holds;*
- $1 - (1+\kappa)/p < \beta_1$ *and* $1 - (1+\kappa)/p < \beta_2$ *and (5.26), (5.30) hold.*

If $d = 2$ we further assume further that $q \neq 2$. Then for each

$$u_0 \in L^0_{\mathscr{F}_0}(\Omega; B^{1-2\frac{1+\kappa}{p}}_{q,p}(\mathbb{R}^d)),$$

the problem (5.21) has a maximal local solution (u, σ). Moreover, there exists a localizing sequence $(\sigma_n)_{n\geq 1}$ such that a.s. for all $n \geq 1$

$$u \in L^p(I_{\sigma_n}, w_\kappa; W^{1,q}(\mathbb{R}^d)) \cap C(\overline{I}_{\sigma_n}; B^{1-2\frac{1+\kappa}{p}}_{q,p}(\mathbb{R}^d)) \cap C((0, \sigma_n]; B^{1-\frac{2}{p}}_{q,p}(\mathbb{R}^d)).$$

Critical spaces for (5.21)

As in Subsection 5.1.2 we study critical spaces for (5.21). Therefore, we need to study when equality in (5.26) and (5.30) can be reached.

As in Subsection 5.1.2, before embarking in this discussion let us analyse the scaling properties of the equation (5.21) in the case that (5.22) holds.

In the deterministic case, i.e. $b_{jn} \equiv \widetilde{g}_n \equiv 0$,the map $u \mapsto u_\lambda$ where

$$u_\lambda(x,t) := \lambda^{1/(m-1)} u(\lambda t, \lambda^{1/2} x)$$

for $\lambda > 0$ preserves the set of (smooth local) solutions to (5.21). More precisely, if u is a (smooth local) solution to (5.21) on $(0,T) \times \mathbb{R}^d$ then u_λ is a (smooth local) solution to (5.21) on $(0, T/\lambda) \times \mathbb{R}^d$. Reasoning as (5.13), one discovers that $B^{d/q-2/(m-1)}_{q,p}(\mathbb{R}^d)$ is 'locally' invariant under the induced map $u_0 \mapsto u_{0,\lambda} := \lambda^{1/(m-1)} u_0(\lambda^{1/2} \cdot)$.

Since (5.21)-(5.22) presents two non-linearities, it is not immediate to see whether there is scaling-invariance as in Subsection 5.1.2. To check this, we mimic the scaling argument performed in Subsection 5.1.2 to discover a relation between h and m. Indeed, using the strong formulation of solutions given in Definition 4.3.3, substituting $s' = \lambda s$ for the deterministic integral one obtains

$$\begin{aligned} \int_0^{t/\lambda} |u_\lambda|^{m-1} u_\lambda ds &= \int_0^t \lambda^{1+\frac{1}{m-1}} |u(s', \lambda x)|^{m-1} u(s', \lambda x) \frac{ds'}{\lambda} \\ &= \lambda^{\frac{1}{m-1}} \int_0^t |u(s', \lambda x)|^{m-1} u(s', \lambda x) ds'. \end{aligned}$$

where, u_λ is as above. For the stochastic term the same calculation as in (5.14) gives that the scalings coincide if $\frac{h}{m-1} - \frac{1}{2} = \frac{1}{m-1}$, or in other words $\frac{h-1}{m-1} = \frac{1}{2}$, thus $h = (m+1)/2$. This relation holds if and only if the right hand-sides of the inequalities (5.26) and (5.30) coincide. Moreover, if $h = (m+1)/2$ the lower bound in (5.25) coincides with (5.29).

For the sake of simplicity, let us continue the discussion on critical spaces for (5.21) under the assumption $h = (m+1)/2$. In this case, (5.26) and (5.30) coincide, and in order to have equality in the latter two we need to assume

$$\frac{m}{m-1} - \frac{1}{2}\Big(\frac{d}{q} + 1\Big) < \frac{1}{2} \Leftrightarrow q < \frac{d(m-1)}{2}.$$

Since $q > 2$, to avoid trivial situations we assume $m > 1 + \frac{4}{d}$. Under the above assumption we can set

$$\kappa_{\mathsf{crit}} := \frac{pm}{m-1} - \frac{p}{2}\Big(\frac{d}{q} + 1\Big) - 1 \tag{5.32}$$

and the trace space for the solution to (5.21)-(5.22) becomes

$$X^{\mathsf{Tr}}_{\kappa,p} = B^{1-\frac{2(1+\kappa_{\mathsf{crit}})}{p}}_{q,p}(\mathbb{R}^d) = B^{1-2\frac{m}{m-1}+\frac{d}{q}+1}_{q,p}(\mathbb{R}^d) = B^{\frac{d}{q}-\frac{2}{m-1}}_{q,p}(\mathbb{R}^d).$$

Note that the above space depends on p only through the microscopic parameter and it presents the same scaling as in the deterministic case, due to the choice $h = (m+1)/2$. Moreover, one can check that in the case $p = q = 2$ and $\kappa = 0$, no other critical space arises. Therefore, Theorem 5.1.9 implies the following result.

Theorem 5.1.10. *Let Assumptions 5.1.1 and 5.1.8 be satisfied and $d \geq 2$. Let $m > 1 + \frac{4}{d}$ and $h = \frac{m+1}{2}$. Assume that $q \in (\frac{d(m-1)}{m+1}, \frac{d(m-1)}{2})$, and if $d = 2$ we assume $q \neq 2$. Assume $\frac{1}{p} + \frac{d}{2q} \leq \frac{m+1}{2(m-1)}$, and let κ_{crit} be given by (5.32). Then for each*

$$u_0 \in L^0_{\mathscr{F}_0}(\Omega; B^{\frac{d}{q}-\frac{2}{m-1}}_{q,p}(\mathbb{R}^d))$$

there exists a maximal local solution (u, σ) to (5.21). Moreover, there exists a localizing sequence $(\sigma_n)_{n\geq 1}$ such that a.s. for all $n \geq 1$

$$u \in L^p(0, \sigma_n, w_{\kappa_{\mathsf{crit}}}; W^{1,q}(\mathbb{R}^d)) \cap C(\overline{I}_{\sigma_n}; B^{\frac{d}{q}-\frac{2}{m-1}}_{q,p}(\mathbb{R}^d)) \cap C((0, \sigma_n]; B^{1-\frac{2}{p}}_{q,p}(\mathbb{R}^d)).$$

5.1.4 Stochastic reaction-diffusion with gradient nonlinearities

In this section we study reaction-diffusion equations with gradient non-linearities:

$$\begin{cases} du + \mathcal{A}u dt = f(\cdot, u, \nabla u)dt + \sum_{n\geq 1}(\mathcal{B}_n u + g_n(\cdot, u))dw^n_t, & \text{on } \mathbb{R}^d, \\ u(0) = u_0, & \text{on } \mathbb{R}^d; \end{cases} \tag{5.33}$$

where $\mathcal{A}, \mathcal{B}_n$ are as in (5.3). We study (5.33) under the following assumption:

Assumption 5.1.11. *The maps $f : I_T \times \Omega \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, $g := (g_n)_{n\geq 1} : I_T \times \Omega \times \mathbb{R}^d \times \mathbb{R} \to \ell^2$ are $\mathscr{P} \otimes \mathscr{B}(\mathbb{R}^d) \otimes \mathscr{B}(\mathbb{R})$-measurable with $f(\cdot, 0, 0) = 0$ and $g(\cdot, 0) = \nabla_y g(\cdot, 0) = 0$. In addition there exist $m > 2$ and $\eta \in (0, 1)$ such that for each $R > 0$ there exists $C_R > 0$ for which one has*

$$\begin{aligned} |f(t, x, y, z) - f(t, x, y', z')| \leq & C_R(1 + |z|^{m-1} + |z'|^{m-1})|z - z'| \\ & + C_R(1 + |z|^{m-\eta} + |z'|^{m-\eta})|y - y'|, \\ \|g(t, x, y) - g(t, x, y')\|_{\ell^2} + \|\nabla_y g(t,& x, y) - \nabla_y g(t, x, y')\|_{\ell^2} \leq C_R|y - y'|, \end{aligned}$$

a.s. for all $t \in I_T$, $x \in \mathbb{R}^d$, $y, y' \in B_{\mathbb{R}}(R)$ and $z, z' \in \mathbb{R}^d$.

Typical choices for f are

$$f(u, \nabla u) = u^c|\nabla u|^r, \quad \text{or} \quad f(u, \nabla u) = |\nabla u|^r; \quad \text{where } c \in [1, \infty),\ r > 1; \tag{5.34}$$

see the monograph [182, Chapter 5, Section 34] for related problems and motivations. For the first example it is straightforward to check that the assumption on f holds for any $m > \max\{r, 2\}$. For $c = 1$ and $r = 2$ we obtain a non-linearity similar to the one appearing in the study of harmonic maps into the sphere, see e.g. [195, p. 225]. The second example in (5.34) satisfies the assumption for $m = r$ if $r > 2$ or for any $m > 2$ if $r \in (1, 2]$. The latter example covers the stochastic version of [182, eq. (34.5), p. 406] and it appears in stochastic control theory see e.g. [18, 173] with $\mathcal{B}_n = 0$

and $g_n = 0$. A further motivation for (5.33) comes from the analysis of high-order regularity of quasilinear equations in divergence form with gradient type nonlinearities (see e.g. [178, Section 3, Example 2]). In such a case, one may take

$$f(u, \nabla u) = a(u)|\nabla u|^2 + |\nabla u|^r,$$

where $r > 1$ and $a : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz. As above, Assumption 5.1.11 holds for $m = r$ if $r > 2$ or for any $m > 2$ if $r \in (1, 2]$.

As usual we consider (5.33) as (5.1) with $X_0 := L^q(\mathbb{R}^d)$, $X_1 := W^{2,q}(\mathbb{R}^d)$ and

$$\begin{aligned} A(t)u &= \mathcal{A}(t)u, & B(t)u &= (\mathcal{B}_n(t)u)_{n\geq 1}, \\ F(t, u) &= f(t, u, \nabla u), & G(t, u) &= (g_n(t, u))_{n\geq 1}, \end{aligned}$$

for $u \in W^{2,q}(\mathbb{R}^d)$. As before (u, σ) is a maximal local solution to (5.33) if (u, σ) is a maximal local solution to (5.1) in the sense of Definition 4.3.4.

The main result of this section reads as follows.

Theorem 5.1.12. *Let Assumptions 5.1.1 and 5.1.11 be satisfied, $d \geq 1$ and $q > \frac{d(m-1)}{m}$. Let $\beta = \frac{1}{2} + \frac{d}{2q}\frac{m-1}{m}$. Assume that one of the following holds:*

(1) $1 - \frac{1+\kappa}{p} \geq \beta$;

(2) $1 - \frac{1+\kappa}{p} < \beta$ *and* $\frac{1+\kappa}{p} \leq \frac{m}{2(m-1)} - \frac{d}{2q}$.

Then for each

$$u_0 \in L^0_{\mathscr{F}_0}(\Omega; B^{2-2\frac{1+\kappa}{p}}_{q,p})$$

there exists a maximal local solution (u, σ) to (5.33). Moreover, there exists a localizing sequence $(\sigma_n)_{n\geq 1}$ such that and a.s. for all $n \geq 1$

$$u \in L^p(I_{\sigma_n}, w_\kappa; W^{2,q}) \cap C(\overline{I}_{\sigma_n}; B^{2-2\frac{1+\kappa}{p}}_{q,p}) \cap C((0, \sigma_n]; B^{2-\frac{2}{p}}_{q,p}).$$

Proof. By Theorem 4.3.8 and Lemma 5.1.2, it remains to check that the nonlinearities satisfies the conditions (HF)-(HG).

First observe that $2 - 2\frac{(1+\kappa)}{p} > \frac{d}{q}$ in each case. Indeed, if (1) holds then the latter follows from $q > \frac{d(m-1)}{m} > \frac{d}{m}$. If (2) holds, then $2 - 2\frac{1+\kappa}{p} \geq 2 - \frac{m}{m-1} + \frac{d}{q} > \frac{d}{q}$ where in the last inequality we used that $m > 2$. The previous observation combined with Sobolev embedding gives that

$$X^{\mathsf{Tr}}_{\kappa,p} = B^{2-\frac{2(1+\kappa)}{p}}_{q,p} \hookrightarrow C^\epsilon(\mathbb{R}^d), \qquad \text{for some } \epsilon > 0. \tag{5.35}$$

Let $n \geq 1$ and let $u, v \in X_1$ be such that $u, v \in B_{X^{\mathsf{Tr}}_{\kappa,p}}(n)$. By the previous embedding $\|u\|_{L^\infty(\mathbb{R}^d)} \leq C\|u\|_{X^{\mathsf{Tr}}_{\kappa,p}} \leq Cn$ and the same for v. Let $\phi \in \big(2 - 2\frac{1+\kappa}{p}, 2\big)$ be arbitrary. Setting $R = Cn$, then by Assumption 5.1.11,

$$\begin{aligned} \|F(\cdot, u)-F(\cdot, v)\|_{L^q} & \\ &\leq C_R\|(1 + |\nabla u|^{m-1} + |\nabla v|^{m-1})|\nabla u - \nabla v|\|_{L^q} \\ &\quad + C_R\|(1 + |\nabla u|^{m-\eta} + |\nabla v|^{m-\eta})|u - v|\|_{L^q} \\ &\lesssim_R \|\nabla u - \nabla v\|_{L^q} + (\|\nabla u\|^{m-1}_{L^{qm}} + \|\nabla v\|^{m-1}_{L^{qm}})\|\nabla u - \nabla v\|_{L^{qm}} \\ &\quad + \|u - v\|_{X^{\mathsf{Tr}}_{\kappa,p}} + (\|\nabla u\|^{(m-\eta)}_{L^{q(m-\eta)}} + \|\nabla u\|^{(m-\eta)}_{L^{q(m-\eta)}})\|u - v\|_{C^\epsilon} \\ &\leq (1 + \|u\|^{m-1}_{H^{\theta,q}} + \|v\|^{m-1}_{H^{\theta,q}})\|u - v\|_{H^{\theta,q}} \\ &\quad + (1 + \|u\|^{m-\eta}_{H^{\theta,q}} + \|v\|^{m-\eta}_{H^{\theta,q}})\|u - v\|_{H^{\phi,q}}; \end{aligned} \tag{5.36}$$

where in the last line we used the Sobolev embedding with $\theta - \frac{d}{q} = 1 - \frac{d}{qm}$ and the fact that $H^{\phi,q} \hookrightarrow B^{2-2\frac{1+\kappa}{p}}_{q,p}$. Note that $\theta < 2$ since $q > \frac{d(m-1)}{m}$ and $\beta = \theta/2$. Moreover, by (5.4), $H^{\theta,q} = [L^q, W^{2,q}]_\beta$ and $H^{\phi,q} = [L^q, W^{2,q}]_{\frac{\phi}{2}}$. To check (HF) we split the argument in two cases:

1. If $1-(1+\kappa)/p \geq \beta$, then $\phi > 2(1-\frac{1+\kappa}{p}) \geq \theta$. Since $\eta<1$, (5.36) implies

$$\|F(\cdot,u)-F(\cdot,v)\|_{L^q} \lesssim (1+\|u\|^{m-\eta}_{H^{\phi,q}}+\|v\|^{m-\eta}_{H^{\phi,q}})\|u-v\|_{H^{\phi,q}}.$$

Set $m_F=1$, $\rho_1=m-\eta$ and $\varphi_1=\beta_1=\phi/2$. Choosing $\phi=2(1-\frac{1+\kappa}{p})+\varepsilon$, for some ε small, (4.18) is equivalent to

$$(m-\eta)\Big(\varphi_1-1+\frac{1+\kappa}{p}\Big)+\beta_1=(m-\eta+1)\frac{\varepsilon}{2}+1-\frac{1+\kappa}{p}\leq 1.$$

The latter inequality is satisfied if $\varepsilon>0$ is sufficiently small. In turn, (HF) is satisfied by setting $F_c=F$, $F_{\mathsf{Tr}}=F_L=0$.

2. If $1-(1+\kappa)/p<\beta$, then by (5.36), we may set $m_F=2$, $\rho_1=m-1$, $\rho_2=m-\eta$, $\varphi_1=\varphi_2=\theta/2$, $\beta_1=\varphi_1$ and $\beta_2=\phi/2$. It remains to verify (4.18), which is equivalent to the following

$$(m-1)\Big(\varphi_1-1+\frac{1+\kappa}{p}\Big)+\varphi_1\leq 1, \tag{5.37}$$

$$(m-\eta)\Big(\varphi_1-1+\frac{1+\kappa}{p}\Big)+\beta_2\leq 1. \tag{5.38}$$

Note that (5.37) implies (5.38). To see this, set $\phi=2-2\frac{1+\kappa}{p}+\varepsilon$ for $\varepsilon>0$ small. Then (5.38) holds provided $m\varphi_1-(m-1)(1-\frac{1+\kappa}{p})\leq 1+\eta'$ where $\eta'>0$. Now, standard considerations show that (5.37) implies the latter. Thus, (HF) is satisfied by setting $F_c=F$, $F_{\mathsf{Tr}}=F_L=0$.

Finally, we note that (5.37) is equivalent to

$$\frac{1+\kappa}{p}\leq\frac{\rho+1}{\rho}\Big(\frac{1}{2}-\frac{d}{2q}\frac{m-1}{m}\Big)=\frac{m}{2(m-1)}-\frac{d}{2q}. \tag{5.39}$$

A more simple argument applies to g. Indeed,

$$\begin{aligned}\|G(\cdot,u)-G(\cdot,v)\|_{W^{1,q}(\ell^2)} &\lesssim \|(g_n(\cdot,u)-g_n(\cdot,v))_{n\geq1}\|_{L^q(\ell^2)}\\ &\quad+\|(\nabla g_n(\cdot,u)(\nabla u-\nabla v))_{n\geq1}\|_{L^q(\ell^2)}\\ &\quad+\|(\nabla g_n(\cdot,u)-\nabla g_n(\cdot,v))\nabla v)_{n\geq1}\|_{L^q(\ell^2)}\\ &\leq C_R\|u-v\|_{W^{1,q}}\lesssim C_R\|u-v\|_{X^{\mathsf{Tr}}_{\kappa,p}};\end{aligned} \tag{5.40}$$

where we used that $X^{\mathsf{Tr}}_{\kappa,p}\hookrightarrow L^\infty\cap W^{1,q}$ by (5.35) and $2-2(1+\kappa)/p>1$. Therefore, G satisfies (HG) with $G_c=G_L=0$. □

Critical spaces for (5.33)

Analogously to Subsections 5.1.2, 5.1.3 let us first analyse the scaling property of the equation (5.33) under the assumption

$$f(u,\nabla u)=|\nabla u|^m, \qquad m>2, \tag{5.41}$$

cf. (5.34). In the deterministic case, i.e. $b_{jn}\equiv g_n\equiv 0$, the equation (5.33) with (5.41) is 'locally invariant' under the transformation $u\mapsto u_\lambda$ where

$$u_\lambda(t,x):=\lambda^{-\alpha/2}u(\lambda t,\lambda^{1/2}x), \qquad \text{for } \lambda>0,\ x\in\mathbb{R}^d,$$

and where we have set $\alpha:=\frac{m-2}{m-1}$. As in [178, Example 3, Section 3] one can see that the Besov space $B^{d/q+(m-2)/(m-1)}_{q,p}$ has the right 'local' scaling for the problem (5.33) with f as in (5.41), i.e. the homogeneous version of this space is invariant under the induced map $u_0 \mapsto u_{0,\lambda} := \lambda^{-\alpha} u_0(\lambda^{1/2}\cdot)$. More precisely, one has

$$\|u_{0,\lambda}\|_{\dot{B}^{\frac{d}{q}+\alpha}_{q,p}} \eqsim \lambda^{-\alpha/2}(\lambda^{1/2})^{\frac{d}{q}+\alpha-\frac{d}{q}}\|u_0\|_{\dot{B}^{\frac{d}{q}+\alpha}_{q,p}} = \|u_0\|_{\dot{B}^{\frac{d}{q}+\alpha}_{q,p}};$$

here $\dot{B}^{d/q+(m-2)/(m-1)}_{q,p}$ denotes the homogeneous Besov space and the implicit constant does not depend on $\lambda > 0$.

It turns out that the above spaces arise naturally as critical spaces for (5.33) in our abstract framework. Moreover, using our abstract theory we do not assume that f has the form in (5.41) but Assumption 5.1.11 is enough. To this end, as in Subsections 5.1.2, 5.1.3 we study when equality holds in (5.39) for some $\kappa := \kappa_{\mathsf{crit}}$.

Let us begin by analysing the case $p \in (2,\infty)$ and $\kappa \in [0, p/2-1)$. In this case, to ensure $\kappa \geq 0$, by (5.39) we need

$$\frac{1}{p} + \frac{d}{2q} \leq \frac{m}{2(m-1)}. \tag{5.42}$$

To ensure $\kappa < \frac{p}{2} - 1$ we assume

$$\frac{m}{2(m-1)} - \frac{d}{2q} < \frac{1}{2} \Leftrightarrow q < d(m-1).$$

Since $q \geq 2$, we assume $m > 1 + \frac{2}{d}$. Since $m > 2$ by Assumption 5.1.11, the latter is automatically satisfied in the case $d > 1$. Under the previous conditions, we set

$$\kappa_{\mathsf{crit}} = \frac{pm}{2(m-1)} - \frac{pd}{2q} - 1. \tag{5.43}$$

Then the trace space becomes

$$X^{\mathsf{Tr}}_{\kappa_{\mathsf{crit}},p} = B^{2-\frac{2(1+\kappa_{\mathsf{crit}})}{p}}_{q,p}(\mathbb{R}^d) = B^{\frac{d}{q}+\frac{m-2}{m-1}}_{q,p}(\mathbb{R}^d).$$

In the case $q = p = 2$ and $\kappa = 0$, if equality in (5.39) holds, then $m = 1 + 2/d$, and therefore $d = 1$ since $m > 2$. Thus, we can also allow $d = 1$, $m = 3$, $p = q = 2$, and $\kappa = 0$, and the corresponding critical space becomes $X^{\mathsf{Tr}}_{\kappa,p} = B^1_{2,2}(\mathbb{R}) = H^1(\mathbb{R})$. Now Theorem 5.1.12 implies the following result.

Theorem 5.1.13. *Let Assumptions 5.1.1 and 5.1.11 be satisfied. Let either $d \geq 2$, or $d = 1$ and $m > 3$. Assume that $\frac{d(m-1)}{m} < q < d(m-1)$ and that $p \in (2,\infty)$ verifies (5.42). Let κ_{crit} be given by (5.43). Then for each*

$$u_0 \in L^0_{\mathscr{F}_0}(\Omega; B^{\frac{d}{q}+\frac{m-2}{m-1}}_{q,p}(\mathbb{R}^d)),$$

there exists a maximal local solution to (5.33). Moreover, there exists a localizing sequence $(\sigma_n)_{n\geq 1}$ such that a.s. for all $n \geq 1$

$$u \in L^p(I_{\sigma_n}, w_{\kappa_{\mathsf{crit}}}; W^{2,q}(\mathbb{R}^d)) \cap C(\overline{I}_{\sigma_n}; B^{\frac{d}{q}+\frac{m-2}{m-1}}_{q,p}(\mathbb{R}^d)) \cap C((0,\sigma_n]; B^{2-\frac{2}{p}}_{q,p}(\mathbb{R}^d)).$$

Furthermore, the same is true if $d = 1$, $m = 3$, $p = q = 2$ and $\kappa_{\mathsf{crit}} = 0$.

5.1.5 Stochastic Burger's equation with white noise

In this section we consider a stochastic Burger's equation with space-time white noise on $\mathbb{T}$. The space-time white noise will be denoted by w_t. More precisely, we analyse the following problem for the unknown process $u : I_T \times \Omega \times \mathbb{T} \to \mathbb{R}$

$$\begin{cases} du + \mathcal{A}u dt = \partial_x(f(\cdot,u))dt + g(\cdot,u)dw_t, & \text{on } \mathbb{T}, \\ u(0) = u_0 & \text{on } \mathbb{T}. \end{cases} \tag{5.44}$$

Here $\mathcal{A}$ is as in (5.3) and for simplicity we took $\mathcal{B}=0$. For results with Dirichlet boundary conditions see [3, Subsection 5.6].

Compared with the previous sections, due to the space-time white noise we restrict ourselves to the one-dimensional torus, and require a suitable interpretation of the g-term. Indeed, the term $g(\cdot,u)dw_t$ in (5.44) will be interpreted as $M_{g(\cdot,u)}W_{L^2(\mathbb{T})}$ where $M_{g(\cdot,u)}$ denotes multiplication by $g(\cdot,u)$, and $W_{L^2(\mathbb{T})}$ is an $L^2(\mathbb{T})$-cylindrical Brownian motion induced by the space-time white noise w_t.

Assumption 5.1.14.

(1) *Assumption 5.1.1 is satisfied.*

(2) *The maps $f: I_T\times\Omega\times\mathbb{T}\times\mathbb{R}\to\mathbb{R}$, $g: I_T\times\Omega\times\mathbb{T}\times\mathbb{R}\to\mathbb{R}$ are $\mathscr{P}\otimes\mathscr{B}(\mathbb{T})\otimes\mathscr{B}(\mathbb{R})$-measurable with $f(\cdot,0)=g(\cdot,0)\in L^q(\mathbb{T})$. Moreover, there exist $h,m>1$ and $C>0$ such that such that for all $z,z'\in\mathbb{R}$*

$$|f(\cdot,z)-f(\cdot,z')|\le C(1+|z|^{h-1}+|z'|^{h-1})|z-z'|,$$
$$|g(\cdot,z)-g(\cdot,z')|\le C(1+|z|^{m-1}+|z'|^{m-1})|z-z'|.$$

The Burger's nonlinearity $f(u)=-u^2$ satisfies the above condition for any $h\ge 2$.

As above, to prove local existence for (5.44) we employ Theorem 4.3.8. Recall that the space-time white noise can be model as an $L^2(\mathbb{T})$-cylindrical Brownian motion. Therefore, we set $H=L^2(\mathbb{T})$. Fix $s\in(0,1)$ and $q\in[2,\infty)$. We rewrite (5.44) in the form (5.1) by setting $X_0:=H^{-1-s,q}(\mathbb{T})$, $X_1=H^{1-s,q}(\mathbb{T})$. Note that by (5.4),

$$X_{\frac12}=H^{-s,q}(\mathbb{T})\quad\text{and}\quad X^{\mathsf{Tr}}_{\kappa,p}=B^{1-s-2\frac{(1+\kappa)}{p}}_{q,p}(\mathbb{T}).$$

For $u\in X_1$ and $t\in I_T$ we set

$$A(t)u=\mathcal{A}(t)u,\qquad B(t)u=0,$$
$$F(t,u)=\partial_x(f(t,u)),\qquad G(t,u)=iM_{g(t,u)}.$$

Here $\mathcal{A}(t)$ is as (5.3) and for fixed $u\in L^\ell(\mathbb{T})$ measurable, $M_{g(t,u)}:L^2(\mathbb{T})\to L^r(\mathbb{T})$ is the multiplication operator

$$(M_{g(t,u)}h)(x)=g(t,u(x))h(x),$$

for $r\in(1,2)$ and $\ell\in(2,\infty)$ which satisfy $\frac1r=\frac12+\frac1\ell$ and we will need $s-\frac1r>0$ later for the G term. Moreover, $i:L^r(\mathbb{T})\to H^{-s,q}(\mathbb{T})=X_{\frac12}$ denotes the embedding which holds since $-s-\frac1q\le-\frac1r$. Since $s>\frac1r>\frac12$ we only will consider $s\in(\frac12,1)$ below.

As usual, we say that (u,σ) is a maximal local solution to (5.83) if (u,σ) is a maximal local solution to (5.1) in the sense of Definition 4.3.4 with the above choice of A,B,F,G,H. To estimate the nonlinearity we start by looking at F. As in (5.7), by Assumption 5.1.14(2) we get

$$\begin{aligned}\|F(\cdot,u)-F(\cdot,v)\|_{H^{-1-s,q}}&\overset{(i)}{\lesssim}\|F(\cdot,u)-F(\cdot,v)\|_{H^{-1,\xi}}\\&\lesssim(1+\|u\|^{h-1}_{L^{h\xi}}+\|v\|^{h-1}_{L^{h\xi}})\|u-v\|_{L^{h\xi}}\\&\overset{(ii)}{\lesssim}(1+\|u\|^{h-1}_{H^{\theta,q}}+\|v\|^{h-1}_{H^{\theta,q}})\|u-v\|_{H^{\theta,q}};\end{aligned}\tag{5.45}$$

where in (i) we used the Sobolev embedding with ξ defined by $-1-\frac1\xi=-1-s-\frac1q$ and in (iii) the Sobolev embedding with $\theta-\frac1q=-\frac1{h\xi}$. To ensure that $\xi\in(1,\infty)$ we have to assume $q>\frac1{1-s}$. Moreover,

$$\frac1q-\theta=\frac1{h\xi}=\frac1h\Big(\frac1q+s\Big)\quad\Rightarrow\quad\theta=\frac1q\Big(1-\frac1h\Big)-\frac sh.$$

Since θ has to satisfy $\theta \in (0, 1-s)$, we require $\frac{h-1}{h-s(h-1)} < q < \frac{h-1}{s}$. Since $\frac{h-1}{h-s(h-1)} < \frac{1}{1-s}$ for all $h \geq 1$ and $s \in (0,1)$, it is enough to assume

$$\frac{1}{1-s} < q < \frac{h-1}{s}. \tag{5.46}$$

Note that since $s \in (\frac{1}{2}, 1)$ then $\frac{1}{1-s} > 2$. Thus, if q verifies (5.46), then $q > 2$. Moreover, the condition (5.46) is not empty provided

$$h > \frac{1}{1-s}. \tag{5.47}$$

Since $s > \frac{1}{2}$, the previous implies $h > 2$. If (5.46) holds, then $H^{\theta,q} = [H^{-1-s,q}, H^{1-s,q}]_{\beta_1}$ where

$$\beta_1 = \frac{1+\theta+s}{2} = \frac{1}{2}\Big[\Big(\frac{1}{q}+s\Big)\Big(1-\frac{1}{h}\Big)+1\Big] \in (0,1). \tag{5.48}$$

To check the condition (HF) we may split the discussion into three cases:

(1) If $1 - \frac{1+\kappa}{p} > \beta_1$, by Remark 4.3.2(1), (HF) follows by setting $F_{\mathsf{Tr}}(t,u) = \partial_x(f(t,\cdot,u))$ and $F_L \equiv F_c \equiv 0$.

(2) If $1 - \frac{1+\kappa}{p} = \beta_1$, by (5.45) and Remark 4.3.2(2), (HF) follows by setting $F_L \equiv F_{\mathsf{Tr}} \equiv 0$, $F_c(t,u) = \partial_x(f(t,\cdot,u))$, $m_F = 1$, $\rho_1 = h-1$ and $\varphi_1 = \beta_1$.

(3) If $1 - \frac{1+\kappa}{p} < \beta_1$ we set $F_c(t,u) = \partial_x(f(t,\cdot,u))$ and $F_L \equiv F_{\mathsf{Tr}} \equiv 0$. As in the previous item we set $m_F = 1$, $\rho_1 = h-1$ and $\varphi_1 = \beta_1$. By (5.45) it remains to check the condition (4.18). In this situation, (4.18) becomes

$$\frac{1+\kappa}{p} \leq \frac{\rho_1+1}{\rho_1}(1-\beta_1) = \frac{1}{2}\frac{h}{h-1} - \frac{1}{2}\Big(\frac{1}{q}+s\Big). \tag{5.49}$$

Next we estimate G. By Assumption 5.1.14(2) it follows that

$$\begin{aligned}
\|G(\cdot,u)-G(\cdot,v)\|_{\gamma(L^2;H^{-s,q})} &\eqsim \|(I-\partial_x^2)^{-\frac{s}{2}}(M_{g(\cdot,u)}-M_{g(\cdot,v)})\|_{\gamma(L^2;L^q)}\\
&\overset{(i)}{\lesssim} \|(I-\partial_x^2)^{-\frac{s}{2}}(M_{g(\cdot,u)}-M_{g(\cdot,v)})\|_{\mathscr{L}(L^2;L^\infty)}\\
&\overset{(ii)}{\lesssim} \|M_{g(\cdot,u)}-M_{g(\cdot,v)}\|_{\mathscr{L}(L^2;L^r)}\\
&\overset{(iii)}{\lesssim} \|g(\cdot,u)-g(\cdot,v)\|_{L^\ell}\\
&\lesssim (1+\|u\|_{L^{m\ell}}^{m-1}+\|v\|_{L^{m\ell}}^{m-1})\|u-v\|_{L^{m\ell}}\\
&\overset{(iv)}{\lesssim} (1+\|u\|_{H^{\phi,q}}^{m-1}+\|v\|_{H^{\phi,q}}^{m-1})\|u-v\|_{H^{\phi,q}};
\end{aligned} \tag{5.50}$$

where in (i) we used [164, Lemma 2.1], in (ii) we used Sobolev embedding with $s - \frac{1}{r} > 0$. In (iii) we used Hölder's inequality with $\frac{1}{\ell} = \frac{1}{r} - \frac{1}{2}$, and in (iv) Sobolev embedding with $\phi - \frac{1}{q} = -\frac{1}{\ell m} = \frac{1}{m}(\frac{1}{2} - \frac{1}{r})$. Thus, to ensure that $\phi \in (0, 1-s)$ we require

$$\frac{m}{m(1-s)+\frac{1}{r}-\frac{1}{2}} < q < \frac{m}{\frac{1}{r}-\frac{1}{2}}. \tag{5.51}$$

The lower estimate in (5.51) is immediate from $q > 1/(1-s)$. The upper estimate gives a restriction, but we will take $r \in (1,2)$ large enough to avoid any additional restrictions coming from (5.51).

Due to (5.4) one has $H^{\phi,q} = [H^{-1-s,q}, H^{1-s,q}]_{\beta_2}$ where

$$\beta_2 = \frac{1+s+\phi}{2} = \frac{1+s}{2} + \frac{1}{2q} - \frac{1}{2m}\Big(\frac{1}{r}-\frac{1}{2}\Big) \in (0,1). \tag{5.52}$$

As usual, to check assumption (HG) we split the discussion in several cases. Since $r \in (1,2)$ will be chosen large, we will set

$$\widetilde{\beta}_2 = \frac{1+s}{2} + \frac{1}{2q} \in (0,1). \tag{5.53}$$

Then $\widetilde{\beta}_2 > \beta_2$.

(1) If $1 - \frac{1+\kappa}{p} \geq \widetilde{\beta}_2$, then since $\widetilde{\beta}_2 > \beta_2$, by Remark 4.3.2(1), (HG) follows by setting $G_{\mathsf{Tr}}(t,u) := g(\cdot,u)$ and $G_L \equiv G_c \equiv 0$.

(2) If $1 - \frac{1+\kappa}{p} < \widetilde{\beta}_2$, we can choose $r \in (1,2)$ so large that the same holds with β_2 instead of $\widetilde{\beta}_2$, and we set $G_c(t,u) := g(\cdot,u)$ and $G_L \equiv G_{\mathsf{Tr}} \equiv 0$. As in the previous item we set $m_G = 1$, $\rho_2 = m-1$ and $\varphi_2 = \beta_2$. By (5.50) it remains to check the condition (4.20). Now (4.20) becomes

$$\frac{1+\kappa}{p} \leq \frac{\rho_2+1}{\rho_2}(1-\beta_2) = \frac{m}{m-1}(1-\beta_2)$$

Choosing $r \in (1,2)$ large enough the latter holds if

$$\frac{1+\kappa}{p} < \frac{m}{(m-1)}(1-\widetilde{\beta}_2) = \frac{m}{2(m-1)}\Big(1 - s - \frac{1}{q}\Big). \tag{5.54}$$

Since $\kappa \in [0, \frac{p}{2}-1)$ and $\widetilde{\beta}_2 < 1$, then the above inequality is always verified for p sufficiently large and κ small.

Combining the above considerations with Theorem 4.3.8 and Lemma 5.1.2, we obtain the following:

Theorem 5.1.15. *Let $s \in (\frac{1}{2},1)$ and $h > 1/(1-s)$. Assume that Assumption 5.1.14 holds. Let (5.46) be satisfied. Let β_1 be as in (5.48) and $\widetilde{\beta}_2$ as in (5.53). Assume that one of the following conditions is satisfied:*

- $1-(1+\kappa)/p \geq \beta_1$ *and* $1-(1+\kappa)/p \geq \widetilde{\beta}_2$*;*
- $1-(1+\kappa)/p < \beta_1$, $1-(1+\kappa)/p \geq \widetilde{\beta}_2$ *and* (5.49) *holds;*
- $1-(1+\kappa)/p \geq \beta_1$, $1-(1+\kappa)/p < \widetilde{\beta}_2$ *and* (5.54) *holds;*
- $1-(1+\kappa)/p < \beta_1$ *and* $1-(1+\kappa)/p < \widetilde{\beta}_2$ *and* (5.49), (5.54) *hold.*

Then for each

$$u_0 \in L^0_{\mathscr{F}_0}(\Omega; B^{1-s-2\frac{1+\kappa}{p}}_{q,p}(\mathbb{T})),$$

the problem (5.44) has a maximal local solution (u,σ). Moreover, there exists a localizing sequence $(\sigma_n)_{n\geq 1}$ such that a.s. for all $n \geq 1$

$$u \in L^p(I_{\sigma_n}, w_\kappa; H^{-1-s,q}(\mathbb{T})) \cap C(\overline{I}_{\sigma_n}; B^{1-s-2\frac{1+\kappa}{p}}_{q,p}(\mathbb{T})) \cap C((0,\sigma_n]; B^{1-s-\frac{2}{p}}_{q,p}(\mathbb{T})).$$

Example 5.1.16. In the case of Burger's equation, i.e. $f(u) = -u^2$ and $h = 2$, Theorem 5.1.15 gives a sub-optimal result. To see this recall that $f(u) = -u^2$ verifies Assumption 5.1.14 for all $h \geq 2$. Fix $\varepsilon > 0$ and write $h = 2+\varepsilon$. Then (5.47) implies $s \in (\frac{1}{2}, \frac{1+\varepsilon}{2+\varepsilon})$. Since $s \in (\frac{1}{2}, \frac{1+\varepsilon}{2+\varepsilon})$ is arbitrary, choosing $s > \frac{1}{2}$ small enough, the limitation (5.46) gives $2 < q < 2(1+\varepsilon)$. Since $\beta_1, \widetilde{\beta}_2 < 1$, by choosing p large enough, Theorem 5.1.15 gives the existence of a maximal solution to (5.44) with $f(u) = -u^2$.

Critical spaces for (5.44)

Here we analyse the existence of critical spaces for (5.44). By definition, it means that (5.49) or (5.54) has to be satisfied with equality. Since in (5.54) equality is not allowed, we have to require that the right-hand side of (5.49) is smaller than the one in (5.54). A straightforward computation shows that this holds if and only if

$$m < h + (1-h)\Big(s + \frac{1}{q}\Big). \tag{5.55}$$

In particular, the latter implies $m < h$. Note that (5.55) is not empty since $h+(1-h)(s+\frac{1}{q}) > 1$, by (5.46). If (5.55) holds, then the critical spaces arise when equality in (5.49) is reached. Reasoning as in Subsection 5.1.2, for $p \in (2,\infty)$ equality in (5.49) holds for some $\kappa \in [0, \frac{p}{2}-1)$ if the following are satisfied

$$\frac{1}{p} + \frac{1}{2}\Big(\frac{1}{q} + s\Big) \leq \frac{1}{2}\frac{h}{h-1}, \tag{5.56}$$

$$h \geq \frac{1+s}{s} \qquad \text{or} \qquad \Big[h < \frac{1+s}{s} \ \text{ and } \ q < \frac{h-1}{1-s(h-1)}\Big]. \tag{5.57}$$

Note that if $h < \frac{1+s}{s}$ one always has $\frac{h-1}{1-s(h-1)} > \frac{h-1}{s}$ as follows from $s > \frac{1}{2}$. Therefore, by (5.46), condition (5.57) is always verified. Defining κ_{crit} as in (5.17), one obtains $X^{\mathsf{Tr}}_{\kappa_{\mathsf{crit}},p} = B^{\frac{1}{q}-\frac{1}{h-1}}_{q,p}(\mathbb{T})$. These considerations and Theorem 5.1.15 give the following.

Theorem 5.1.17. *Let $s \in (\frac{1}{2}, 1)$ and $h > 1/(1-s)$. Assume that Assumption 5.1.14 holds. Assume that (5.46), (5.55) and (5.56) hold. Let $\kappa_{\mathsf{crit}} := \frac{p}{2}(\frac{h}{h-1} - \frac{1}{q} - s) - 1$. Then for each*

$$u_0 \in L^0_{\mathscr{F}_0}(\Omega; B^{\frac{1}{q}-\frac{1}{h-1}}_{q,p}(\mathbb{T}))$$

there exists a maximal local solution (u,σ) to (5.5). Moreover, there exists a localizing sequence $(\sigma_n)_{n\geq 1}$ such that a.s. for all $n \geq 1$

$$u \in L^p(I_{\sigma_n}, w_{\kappa_{\mathsf{crit}}}; H^{1-s,q}(\mathbb{T})) \cap C(\overline{I}_{\sigma_n}; B^{\frac{1}{q}-\frac{1}{h-1}}_{q,p}(\mathbb{T})) \cap C((0,\sigma_n]; B^{1-s-\frac{2}{p}}_{q,p}(\mathbb{T})).$$

Example 5.1.18. Here we continue the study of (5.44) in the case of Burger's equation, i.e. (5.44) with $f(u) = -u^2$. As in Example 5.1.16, let $\varepsilon > 0$ and $h = 2+\varepsilon$. Thus, (5.47) and (5.46) gives $s \in (\frac{1}{2}, \frac{1+\varepsilon}{2+\varepsilon})$ and $q \in (\frac{1}{1-s}, \frac{1+\varepsilon}{s})$. In addition, (5.55) is equivalent to $m \in (1, 2+\varepsilon-(1+\varepsilon)(s+\frac{1}{q}))$. Therefore, if $p \in (2,\infty)$ verifies (5.56) and q, s, m, h are as above, then Theorem 5.1.17 ensure the existence of a maximal local solution to (5.44) for $u_0 \in L^0_{\mathscr{F}_0}(\Omega; B^{\frac{1}{q}-\frac{1}{1+\varepsilon}}_{q,p}(\mathbb{T}))$.

5.1.6 Discussion and further extensions

x-dependent coefficients

In the results of Sections 5.1.2-5.1.5 we only used the assertion $(A,B) \in \mathcal{SMR}^{\bullet}_{p,\kappa}(T)$ of Lemma 5.1.2. If (A,B) in Assumption 5.1.1 have x-dependent coefficients but still satisfies $(A,B) \in \mathcal{SMR}^{\bullet}_{p,\kappa}(T)$, then all local existence and regularity results extend to this setting. In the time-independent case (or time-continuous case) many of such results are available as follows from Theorem 4.2.7 (and [165, Section 5]). However, only under a smallness condition on b_{jn}.

In the case $p = q$ much more is known on $(A,B) \in \mathcal{SMR}^{\bullet}_{p,\kappa}(T)$ with x-dependent coefficients. In particular, from [129] and the discussion in Section 4.2.2 we see that stochastic maximal L^p-regularity holds in the case the coefficients a_{ij} and b_{jn} are smooth in space. Moreover, some results can be extended to systems as in [174, Section 5]. In our opinion the restriction $p = q$ seem quite unnatural for the x-dependent variant of the SPDEs considered in the previous sections. This motivates to extend the theory to $p \neq q$ as well. At the moment this seems out of reach if the coefficients a_{ij} and b_{jn} only have measurable dependence in $(t,\omega) \in [0,T] \times \Omega$ or if the b_{jn} are not small.

As an illustrations let us mention that for $s = 0$ and $p = q$, the conditions of Theorem 5.1.5 become

$$\frac{d(h-1)}{h} < p < d(h-1) \quad \text{and} \quad p \geq \frac{d+2}{h-1}.$$

One can check that this will create cases in which not all $h > 1$ can be treated. For instance for $d = 2$, $h \in (1,2]$ has to be excluded. Similar restrictions occur in Theorems 5.1.10 and 5.1.13. On the other hand, as explained before we can allow x-dependent coefficients a_{ij} and b_{jn} using the pointwise extension of Assumption 5.1.1 to the x-dependent setting under some smoothness conditions in x.

Lower order terms

The results of the previous subsections hold if we add lower order terms in the differential operators (5.3). For instance, one may substitute $\mathcal{A}$ by $\mathcal{A} + \mathcal{A}_\ell$ where $\mathcal{A}_\ell(t)u := \sum_{j=1}^d a_j(t,\cdot)\partial_j u + a_0(t,\cdot)u$. To see this, one can take $F_L(t,u) := \mathcal{A}_\ell(t)u$ and, under suitable assumptions on $a_0, \dots, a_d$, the assumption (HF′) is satisfied. Another possibility, to allow lower order terms in (5.3) is to use a perturbation theorem to check stochastic maximal L^p-regularity. Yet another possibility is to include the lower order terms in the nonlinearity f. It depends on each specific case what is the best solution.

Results on $\mathbb{T}^d$

The results of Subsections 5.1.2-5.1.4 hold if $\mathbb{R}^d$ is replaced by the torus $\mathbb{T}^d$. Moreover, in this case, the assumptions on the nonlinearities can be slightly weakened. Indeed, for instance in Section 5.1.2 the Lipschitz condition can be replaced by the following: there exist $h > 1$ and $C > 0$ such that a.s. for all $t \in I_T$, $z, z' \in \mathbb{R}$ and $x \in \mathbb{R}^d$,

$$|f(t,x,z) - f(t,x,z')| + \|g(t,x,z) - g(t,x,z')\|_{\ell^2} \leq C(1 + |z|^{h-1} + |z'|^{h-1})|z - z'|.$$

The only difference is that an additional constant C is added on the right-hand side. Since $\mathbb{T}^d$ has finite volume this does not lead to any problems. The same applies to Sections 5.1.3 and 5.1.4. Moreover, the conditions on f and g in Section 5.1.5 can be weakened in the same way.

5.2 Applications to quasilinear SPDEs with gradient noise

In this section we study quasilinear SPDEs which can be rewritten in the form (4.16) with $H = \ell^2$ (Subsections 5.2.2-5.2.6) or $H = H^{\delta,2}$ (Subsection 5.2.7). In the next subsection we motivate and explain the class of equations which will be considered.

5.2.1 Introduction and motivations

Quasilinear parabolic SPDEs have been intensively studied in literature. In the deterministic case the monograph [141] contains the classical theory. Quasilinear SPDEs arise in many areas of applied science since they model reaction-diffusion equations in which the diffusivity depends strongly on the property itself. For this and more physical motivations we refer to [56, 60, 65, 114, 152]. For a mathematical perspective one may consult [59, 101, 115, 137]. To the best of our knowledge, except for the paper [79], there is no other treatment in the literature for quasilinear stochastic PDEs where the coefficients b_{jnk} appearing in the gradient noise term (see (5.59) below) may depend on u. However, our approach and setting is quite different from the one used in [79] due to a different choice of the leading operators (in [79] they may be degenerate) and a different choice of the noise.

In this section we analyse quasilinear systems of second order stochastic PDEs in non-divergence form with nonlinear gradient noise on a domain $\mathscr{O} \subseteq \mathbb{R}^d$:

$$\begin{cases} du + \mathcal{A}(\cdot, u, \nabla u)u dt = f(\cdot, u, \nabla u)dt + \sum_{n\geq 1}(\mathcal{B}_n(\cdot, u)\cdot \nabla u + g_n(\cdot, u))dw_t^n, \\ u(0) = u_0. \end{cases} \tag{5.58}$$

Here $(w_t^n : t \geq 0)_{n\geq 1}$ denotes a sequence of independent Brownian motions and $u : [0, T] \times \Omega \times \mathscr{O} \to \mathbb{R}^N$ is the unknown process where $N \geq 1$. The differential operators $\mathcal{A}, \mathcal{B}_n$ for each $x \in \mathscr{O}$, $\omega \in \Omega$, $t \in (0, T)$, are given by

$$\begin{aligned} (\mathcal{A}(t, \omega, v, \nabla v)u)(t, \omega, x) &:= - \sum_{i,j=1}^{d} a_{ij}(t, \omega, x, v(x), \nabla v(x))\partial_{ij}^2 u(x), \\ (\mathcal{B}_n(t, \omega, v)u)(t, \omega, x) &:= \Big(\sum_{j=1}^{d} b_{jkn}(t, \omega, x, v(x))\partial_j u_k(x)\Big)_{k=1}^{N}. \end{aligned} \tag{5.59}$$

Note that $\mathcal{A}, \mathcal{B}_n$ generalize the differential operators in (5.3) studied in Section 5.1. As we saw in Subsection 5.1.6, lower order terms in (5.59) can be allowed here as well. Furthermore, as in Subsection 5.1.1, the following splitting arises naturally:

- $\mathscr{O} = \mathbb{R}^d$ or $\mathscr{O} = \mathbb{T}^d$;
- $\mathscr{O}$ is a sufficiently smooth domain in $\mathbb{R}^d$.

In Subsection 5.2.2 we will only consider $\mathbb{R}^d$ in detail since the case $\mathbb{T}^d$ is similar. Under additional assumptions, in Subsection 5.2.3 we study (5.58) with Dirichlet boundary condition. Subsection 5.2.5 is devoted to equations in divergence form. We remark that in the latter section, we can deal only with a small gradient noise term.

The following assumption will be in force in Subsections 5.2.2-5.2.3:

Assumption 5.2.1. *Suppose that one of the following two conditions hold:*

- $p \in (2, \infty)$ *and* $\kappa \in [0, \frac{p}{2} - 1)$.
- $p = 2$ *and* $\kappa = 0$.

Assume the following conditions on a_{ij}, b_{jkn}:

(1) *For each* $i, j \in \{1, \dots, d\}$ *and* $n \geq 1$, *the maps* $a_{ij} : (0, T) \times \Omega \times \mathscr{O} \times \mathbb{R}^N \times \mathbb{R}^{N\times d} \to \mathbb{R}^{N\times N}$ *and* $b_{jkn} : (0, T) \times \Omega \times \mathscr{O} \times \mathbb{R}^N \to \mathbb{R}$ *are* $\mathscr{P} \otimes \mathscr{B}(\mathscr{O}) \otimes \mathscr{B}(\mathbb{R}^N) \otimes \mathscr{B}(\mathbb{R}^{N\times d})$ *and* $\mathscr{P} \otimes \mathscr{B}(\mathscr{O}) \otimes \mathscr{B}(\mathbb{R}^N)$-*measurable, respectively.*

Moreover, for every $r > 0$ *there exist constants* $L_r, M_r > 0$ *and an increasing continuous function* $K_r : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $K_r(0) = 0$ *and for a.a.* $\omega \in \Omega$ *for all* $t \in [0, T]$, $i, j \in \{1, \dots, d\}$, $x, x' \in \mathscr{O}$, $y \in B_{\mathbb{R}^N}(r)$, $z \in B_{\mathbb{R}^{N\times d}}(r)$,

$$\begin{aligned} |a_{ij}(t, \omega, x, y, z)| + \|(b_{jkn}(t, \omega, \cdot, y))_{n\geq 1}\|_{W^{1,\infty}(\mathscr{O};\ell^2)} &\leq M_r, \\ |a_{ij}(t, \omega, x, y, z) - a_{ij}(t, \omega, x', y, z)| &\leq K_r(|x - x'|). \end{aligned}$$

(2) *For each* $r > 0$ *there exists* $C_r > 0$ *such that for all* $i, j \in \{1, \dots, d\}$, $x \in \mathscr{O}$, $y, y' \in B_{\mathbb{R}^N}(r)$, $z, z' \in B_{\mathbb{R}^{N\times d}}(r)$, $t \in [0, T]$, $k \in \{1, \dots, N\}$ *and a.a.* $\omega \in \Omega$,

$$\begin{aligned} |a_{ij}(t, \omega, x, y, z) - a_{ij}(t, \omega, x, y', z')| + \|b_{jkn}(t, \omega, x, y) - b_{jkn}(t, \omega, x', y')\|_{\ell^2\times\mathbb{R}^N} \\ \|\nabla_y b_{jkn}(t, \omega, x, y) - \nabla_y b_{jkn}(t, \omega, x', y')\|_{\ell^2\times\mathbb{R}^N} \leq C_r(|y - y'| + |z - z'|). \end{aligned}$$

(3) *For each* $r > 0$ *there exists* $\epsilon_r > 0$ *such that a.s. for all* $\xi \in \mathbb{R}^d$, $\theta \in \mathbb{R}^N$, $t \in [0,T]$, $x \in \mathscr{O}$, $y \in B_{\mathbb{R}^N}(r)$ *and* $z \in B_{\mathbb{R}^{N\times d}}(r)$ *one has*

$$\sum_{i,j=1}^{d} \xi_i\xi_j((a_{ij}(t,\omega,x,y,z) - \Sigma_{ij}(t,\omega,x,y))\theta,\theta)_{\mathbb{R}^N} \geq \epsilon_r|\xi|^2|\theta|^2.$$

Here for each fixed $i,j \in \{1,\dots,d\}$, $\Sigma_{ij}(t,\omega,x,y)$ *is the* $N \times N$ *matrix with the diagonal elements*

$$\Big(\frac{1}{2}\sum_{n\geq 1} b_{ikn}(t,\omega,x,y)b_{jkn}(t,\omega,x,y)\Big)_{k=1}^{N}.$$

In Subsection 5.2.2 we study (5.58) under the following assumption.

Assumption 5.2.2. *The maps* $f : I_T \times \Omega \times \mathscr{O} \times \mathbb{R}^N \times \mathbb{R}^{N\times d} \to \mathbb{R}^N$, $g := (g_n)_{n\geq 1} : I_T \times \Omega \times \mathscr{O} \times \mathbb{R}^N \times \mathbb{R}^{N\times d} \to \ell^2 \times \mathbb{R}^N$ *are* $\mathscr{P} \otimes \mathscr{B}(\mathscr{O}) \otimes \mathscr{B}(\mathbb{R}^N) \otimes \mathscr{B}(\mathbb{R}^{N\times d})$ *and* $\mathscr{P} \otimes \mathscr{B}(\mathscr{O}) \otimes \mathscr{B}(\mathbb{R}^N)$*-measurable respectively. Assume* $f(\cdot,0) = 0$ *and* $g(\cdot,0) = \nabla_y g(\cdot,0) = 0$*. Moreover, for each* $r > 0$ *there exists* $C_r > 0$ *such that a.a.* $\omega \in \Omega$*, for all* $t \in [0,T]$, $x \in \mathscr{O}$, $y,y' \in B_{\mathbb{R}^N}(r)$ *and* $z,z' \in B_{\mathbb{R}^{N\times d}}(r)$,

$$\begin{aligned}|f(t,x,y,z) - f(t,x,y',z')| &\leq C_r(|y-y'| + |z-z'|),\\ \|g(t,x,y) - g(t,x,y')\|_{\ell^2} + \|\nabla_y g(t,x,y) - \nabla_y g(t,x,y')\|_{\ell^2} &\leq C_r|y-y'|.\end{aligned}$$

In the next subsection, under additional assumption on f,g, we extend the results in Subsection 5.1.4 to suitable quasilinear equations; see Theorems 5.2.5-5.2.6 there.

Remark 5.2.3. The parabolicity condition in Assumption 5.2.1(3) extends the one we have seen in Assumption 5.1.1(2) to the case of x-dependent coefficients and systems. It was considered in the above form in [174], where complex matrix-valued a_{ij} were allowed as well. Some diagonal condition is assumed for the b-term, because otherwise the result does not hold in general (see [30, 71, 120] for further discussion on this topic).

Unlike in Sections 5.1.2, 5.1.3, and 5.1.4 we will be assuming $p = q$ in many of the results below. This is mainly because the quasilinear structure of the equation will imply that our operators will have coefficients depending on (t,ω,x). Unfortunately, no $L^p(L^q)$-theory is available for $p \neq q$ if only measurability in time is assumed. Of course in the case the coefficients are (ω,x)-dependent, there is a theory with $p \neq q$ as follows from Theorem 4.2.7. However, at the same time we would like the b-term to satisfy the right parabolicity condition, and almost no general $L^p(L^q)$-theory with $p \neq q$ is available in this case.

5.2.2 Quasilinear SPDEs in non-divergence form on $\mathbb{R}^d$

In this section we study (5.58) on $\mathbb{R}^d$. For the function spaces needed below, we employ the notation introduced in Subsection 5.1.1.

To begin, we recast (5.58) as a quasilinear evolution equations in the form (4.16) on $X_0 := L^p(\mathbb{R}^d;\mathbb{R}^N)$ and $X_1 := W^{2,p}(\mathbb{R}^d;\mathbb{R}^N)$ by setting, for $u \in C^1(\mathbb{R}^d;\mathbb{R}^N)$ and $v \in W^{2,p}(\mathbb{R}^d;\mathbb{R}^N)$

$$\begin{aligned}A(t,u)v &= \mathcal{A}(t,u,\nabla u)v, & B(t,u)v &= (\mathcal{B}_n(t,u)v)_{n\geq 1},\\ F(t,u) &= f(t,u,\nabla u), & G(t,u) &= (g_n(t,\cdot,u))_{n\geq 1}.\end{aligned}$$

By (5.59) and $u \in C^1(\mathbb{R}^d;\mathbb{R}^N)$ all the above maps are well-defined. As usual, we say that (u,σ) is a maximal local solution to (5.58) on $\mathbb{R}^d$ if (u,σ) is a maximal local solution to (4.16) in the sense of Definition 4.3.4.

The first result of this section is as follows:

Theorem 5.2.4. *Let the Assumptions 5.2.1-5.2.2 be satisfied for* $\mathscr{O} = \mathbb{R}^d$*. Assume that* $p > 2(1+\kappa) + d$*. Then for any*

$$u_0 \in L^0_{\mathscr{F}_0}(\Omega; W^{2-\frac{2(1+\kappa)}{p},p})$$

there exists a maximal local solution (u, σ) *to* (5.58). *Moreover, there exists a localizing sequence* $(\sigma_n)_{n\geq 1}$ *such that for all* $n \geq 1$ *and a.s.*

$$u \in L^p(I_{\sigma_n}, w_\kappa; W^{2,p}) \cap C(\overline{I}_{\sigma_n}; W^{2-2\frac{1+\kappa}{p},p}) \cap C((0, \sigma_n]; W^{2-\frac{2}{p},p}).$$

Proof. We apply Theorem 4.3.7 with $F_L \equiv F_c \equiv G_L \equiv G_c \equiv 0$, $F_{\mathsf{Tr}} := f$ and $G_{\mathsf{Tr}} := (g_n)_{n\geq 1}$. For this it remains to check (HA), (HF′), (HG′) and (4.24). For the sake of clarity we split the proof into several steps.

Step 1: (HA) *holds.* Since $p > 2(1+\kappa)+d$, by Sobolev embedding one has

$$X^{\mathsf{Tr}}_{\kappa,p} = W^{2-\frac{2(1+\kappa)}{p},p} \hookrightarrow C^{1+\epsilon}, \qquad \text{for some } \epsilon > 0. \tag{5.60}$$

Fix $r > 0$, and let $u_1, u_2 \in B_{X^{\mathsf{Tr}}_{\kappa,p}}(r)$. By (5.60) it follows that $\|u_1\|_{W^{1,\infty}}, \|u_2\|_{W^{1,\infty}} \leq Cr =: R$ where C depends only on p, d. Thus,

$$\|A(t, u_1)v - A(t, u_2)v\|_{L^q} \leq C_R \|u_1 - u_2\|_{W^{1,\infty}} \|v\|_{W^{2,q}} \leq C_R \|u_1 - u_2\|_{X^{\mathsf{Tr}}_{\kappa,p}} \|v\|_{W^{2,q}},$$

where C_R is as in Assumption 5.2.1 (2). The same argument holds for B.

Step 2: (4.24) *holds.* It is enough to prove that $(A(\cdot, w_0), B(\cdot, w_0)) \in \mathcal{SMR}^\bullet_{p,\kappa}(T)$ for all $w_0 \in L^\infty_{\mathscr{F}_0}(\Omega; X^{\mathsf{Tr}}_{\kappa,p})$. By (5.60), it follows that $w_0 \in L^\infty_{\mathscr{F}_0}(\Omega; C^{1+\epsilon})$. Now the claim follows from [174, Theorem 5.4] and Assumption 5.2.1.

Step 3: (HF′) *and* (HG′) *holds.* By (5.60) and the assumption on f, g_n it follows easily that for any $n \geq 1$ and any $u, v \in B_{X^{\mathsf{Tr}}_{\kappa,p}}(n)$ one has

$$\|f(\cdot, u, \nabla u) - f(\cdot, v, \nabla v)\|_{L^p} + \|g(\cdot, u) - g(\cdot, v)\|_{W^{1,p}(\ell^2)} \leq C_n \|u - v\|_{X^{\mathsf{Tr}}_{\kappa,p}};$$

where $C_n > 0$ may depends on $n \geq 1$. □

Theorem 5.2.4 gives local existence for (5.58) under quite general assumptions on f, g_n. The drawback in applying Theorem 5.2.4 is that the trace space in (5.60) is very regular and therefore the initial values have to be rather smooth. Under additional assumptions on a_{ij}, b_{jnk} we can admit rougher trace spaces $X^{\mathsf{Tr}}_{\kappa,p}$ for (5.58). To do so we will partially extend the results in Subsection 5.1.4. In particular, the following extends Theorem 5.1.12 in the case $q = p$.

Theorem 5.2.5. *Suppose that Assumptions* 5.1.11 *and* 5.2.1 *hold. Assume* $d \geq 1$. *Assume that* $a_{ij}(t, \omega, x, y, z)$ *does not depend on the* z*-variable and* $b_{jkn}(t, \omega, x, y)$ *does not depend on the* y *variable. Moreover, suppose that*

$$p \geq \frac{m-1}{m}(2(1+\kappa)+d). \tag{5.61}$$

Then for each

$$u_0 \in L^0_{\mathscr{F}_0}(\Omega; W^{2-\frac{2(1+\kappa)}{p},p})$$

there exists a maximal local solution (u, σ) *to* (5.58). *Moreover, there exists a localizing sequence* $(\sigma_n)_{n\geq 1}$ *such that for all* $n \geq 1$ *and a.s.*

$$u \in L^p(I_{\sigma_n}, w_\kappa; W^{2,p}) \cap C(\overline{I}_{\sigma_n}; W^{2-\frac{2(1+\kappa)}{p},p}) \cap C((0, \sigma_n]; W^{2-\frac{2}{p},p}).$$

Recall that typical examples of non-linearities which satisfies Assumptions 5.1.11 are $f(u, \nabla u) = |u|^c |\nabla u|^r$ with $c, r > 1$ and $f(\nabla u) = |\nabla u|^r$ with $r > 2$.

Proof. The proof is similar to the one proposed in Theorem 5.1.12 with $q = p$. Note that if $q = p$, the restrictions in Theorem 5.1.13 reduce to (5.61).

Additionally, we need to check that for $w_0 \in L^\infty_{\mathscr{F}_0}(\Omega; X^{\mathsf{Tr}}_{\kappa,p})$ and $q = p$, one has $(A(w_0), B(w_0)) \in \mathcal{SMR}^\bullet_{p,\kappa}(T)$. Since these operators have x-dependent coefficients, Lemma 5.1.2 is not applicable. By (5.61) it follows that $2 - 2(1+\kappa)/p > d/p$. Therefore, by Sobolev embedding

$$X^{\mathsf{Tr}}_{\kappa,p} = W^{2-2\frac{1+\kappa}{p},p} \hookrightarrow C^\eta, \qquad \text{for some } \eta > 0. \tag{5.62}$$

Thus, $(A(w_0), B(w_0)) \in \mathcal{SMR}^\bullet_{p,\kappa}(T)$ follows from (5.62), Assumption 5.2.1 and [174, Theorem 5.4]. □

As a consequence we obtain the following result in the critical case in the same way as in the proof of Theorem 5.1.13. However, since we need $p = q$, we need further restrictions on q. To avoid this, one needs further results on stochastic maximal $L^p(L^q)$-regularity with x-dependent coefficients.

Theorem 5.2.6. *Suppose that Assumptions 5.1.11 and 5.2.1 hold. Assume $d \geq 1$ and $m > 1 + \frac{2}{d}$. Assume that $a_{ij}(t, \omega, x, y, z)$ does not depend on the z-variable and $b_{jkn}(t, \omega, x, y)$ does not depend on the y variable. Suppose that*

$$\frac{(m-1)}{m}(2+d) < p < d(m-1). \tag{5.63}$$

Then for any

$$u_0 \in L^0_{\mathscr{F}_0}\Big(\Omega; W^{\frac{d}{p}+\frac{m-2}{m-1},p}\Big)$$

there exists a maximal local solution (u, σ) to (5.58). Moreover, there exists a localizing sequence $(\sigma_n)_{n\geq 1}$ such that a.s. for all $n \geq 1$

$$u \in L^p(I_{\sigma_n}, w_{\kappa_{\mathsf{crit}}}; W^{2,p}) \cap C(\overline{I}_{\sigma_n}; W^{\frac{d}{p}+\frac{m-2}{m-1},p}) \cap C((0,\sigma_n]; W^{2-\frac{2}{p},p}),$$

where $\kappa_{\mathsf{crit}} := \frac{pm}{2(m-1)} - \frac{d}{2} - 1$.

Note that since $m > 1 + \frac{2}{d}$, one has $d(m-1) > 2$. Therefore, the set of p which satisfies (5.63) is not empty.

5.2.3 Quasilinear SPDEs in non-divergence form on domains

In this subsection we investigate the quasilinear problem (5.58) with Dirichlet boundary conditions

$$u = 0 \qquad \text{on } \partial\mathscr{O}. \tag{5.64}$$

Here we assume $\mathscr{O}$ is a bounded domain with C^2-boundary. Moreover, we let $N = 1$ and write $b_{jn} := b_{j1n}$.

As usual, we recast (5.58) in the form (4.16). To this end, for $p \in (1, \infty)$ and $s \in (0, 1)$ we set

$$W^{1,p}_0(\mathscr{O}) = \{u \in W^{1,p}(\mathscr{O}) \,:\, u|_{\mathscr{O}} = 0\}$$
$${}_DH^{2,p}(\mathscr{O}) = W^{2,p}(\mathscr{O}) \cap W^{1,p}_0(\mathscr{O}).$$
$${}_DW^{2s,p}(\mathscr{O}) = (L^p(\mathscr{O}), {}_DW^{2,p}(\mathscr{O}))_{s,p}.$$

Recall that, by [187],

$$[L^q(\mathscr{O}), {}_DH^{2,p}(\mathscr{O})]_{1/2} = W^{1,p}_0(\mathscr{O}) \qquad \text{for all } p \in (1, \infty).$$

To proceed further, let $X_0 = L^p(\mathscr{O})$, $X_1 = {}_DH^{2,q}(\mathscr{O})$ and for $u \in X^{\mathsf{Tr}}_{\kappa,p} = {}_DB^{1-2\frac{1+\kappa}{p}}_{q,p}(\mathscr{O}) = \big\{z \in B^{1-2\frac{1+\kappa}{p}}_{q,p}(\mathscr{O}) \,:\, z = 0 \text{ on } \partial\mathscr{O}\big\}$ (see [95]), $v \in X_1$ we set

$$\begin{aligned} A(t,u)v &= \mathcal{A}(t,u,\nabla u)v, \qquad & B(t,u)v &= (\mathcal{B}_n(t,u)v)_{n\geq 1},\\ F(t,u) &= f(t,u,\nabla u), & G(t,u) &= (g_n(t,\cdot,u))_{n\geq 1}. \end{aligned} \tag{5.65}$$

where $\mathcal{A}, \mathcal{B}_n$ are as in (5.59). We say that (u, σ) is a maximal local solution to (5.58) with boundary condition (5.64) if (u, σ) is a maximal local solution to (4.16) with A, B, F, G in (5.65).

Below we will show that for $p > d + 2$

$$B(\cdot, u)v \in \gamma(\ell^2, W^{1,p}_0(\mathscr{O})), \text{ a.e. on } I_T \times \Omega \text{ for all } u \in X^{\mathsf{Tr}}_{\kappa,p},\ v \in X_1. \tag{5.66}$$

As remarked in [69] (see the text below Assumption 1.4), to check (5.66) it is sufficient to require an "orthogonality condition" for b at the boundary of $\mathscr{O}$. In the quasilinear setting, this condition reads as follows

$$\sum_{j=1}^{d} b_{jn}(t,\omega,x,0)\nu_j(x) = 0, \text{ for a.a. } (t,\omega,x)\in I_T\times\Omega\times\partial\mathscr{O} \text{ and all } n\geq 1, \tag{5.67}$$

where $\nu=(\nu_j)_{j=1}^d$ is the exterior normal field on $\partial\mathscr{O}$. To see that (5.67) implies (5.66), we argue as follows. Since $p>d+2$, one has

$$\begin{aligned} X^{\mathsf{Tr}}_{\kappa,p} = {}_D W^{2-\frac{2}{p},p}(\mathscr{O}) &= \big\{u\in W^{2-\frac{2}{p},p}(\mathscr{O})\,:\, u=0 \text{ a.e. on } \partial\mathscr{O}\big\}\\ &\hookrightarrow \big\{u\in C^{1+\varepsilon}(\mathscr{O})\,:\, u=0 \text{ a.e. on } \partial\mathscr{O}\big\}. \end{aligned} \tag{5.68}$$

Note that $\gamma(\ell^2, W^{1,p}_0(\mathscr{O})) = W^{1,p}_0(\mathscr{O};\ell^2)$ by (2.14). Thus, (5.66) holds thanks to $u=v=0$ a.e. on $\partial\mathscr{O}$ (thus ∇v is parallel to ν). Assumption (5.67) was first introduced in [69] (for $\kappa=0$) where the author showed that under suitable conditions on the coefficients, (5.67) yields stochastic maximal L^p-regularity estimates for the linear case of (5.58) on domains. Ellipticity and smoothness of the coefficients alone are not enough to show maximal regularity estimates for parabolic SPDEs on domains with the choice $X_0=L^q(\mathscr{O})$ and $X_1={}_D H^{2,q}(\mathscr{O})$, see [131, Theorem 5.3]. To reduce the conditions on the b-term one needs to use suitable weighted Sobolev spaces (see Subsection 5.2.4 below).

A similar argument shows that item (2) in the following assumption is sufficient to obtain (5.66) with $B(\cdot,u)v$ replaced by $G(\cdot,v)$ where G is as in (5.65).

Assumption 5.2.7. *Let Assumption 5.2.1 be satisfied. Assume that $N=1$. Suppose that (5.67) holds and that the following are satisfied.*

(1) *$\mathscr{O}$ is a bounded C^2-domain in $\mathbb{R}^d$;*

(2) *$g_n(t,\omega,0)=0$ for a.a. $\omega\in\Omega$ and for all $x\in\partial\mathscr{O}$.*

The main result of this subsection is an extension of Theorem 5.2.4 to domains in case $\kappa=0$. Using the results of [3, Example A.4], the reader can check that also Theorem 5.2.5 (resp. 5.2.6) extends to the problem (5.58) with boundary condition (5.64) provided $\kappa=0$ (resp. $p=\frac{m-1}{m}(d+2)$ i.e. $\kappa_{\mathsf{crit}}=0$). For the sake of brevity, we do not include any statement here.

Theorem 5.2.8. *Suppose Assumptions 5.2.2 and 5.2.7 hold. Let $p\in(d+2,\infty)$. Then for each $u_0\in L^0_{\mathscr{F}_0}(\Omega;{}_D W^{2-\frac{2}{p},p}(\mathscr{O}))$ there exists a maximal local solution to (5.58) with boundary condition (5.64), and a localizing sequence $(\sigma_n)_{n\geq1}$ such that a.s. for all $n\geq 1$*

$$u\in L^p(I_{\sigma_n}; W^{2,p}(\mathscr{O})\cap W^{1,p}_0(\mathscr{O}))\cap C(\overline{I}_{\sigma_n}; {}_D W^{2-\frac{2}{p},p}(\mathscr{O})).$$

Proof. Similar to the proof of Theorem 5.2.4, we set $F_L\equiv F_c\equiv G_L\equiv G_c\equiv 0$, $F_{\mathsf{Tr}}(t,u):=F(t,u)$ and $G_{\mathsf{Tr}}(u):=G(t,u)$. Here F,G are as in (5.65). As before one sees that (HF′) and (HA) hold. To check (HG′) recall that $\gamma(\ell^2,W^{1,p}_0(\mathscr{O}))=W^{1,p}_0(\mathscr{O};\ell^2)$. By Assumption 5.2.7(2) one has $g_n(\cdot,u)=g_n(\cdot,v)=0$ a.e. on $I_T\times\Omega\times\partial\mathscr{O}$ for all $u,v\in B_{X^{\mathsf{Tr}}_{\kappa,p}}(n)$. The latter considerations imply, for all $u,v\in B_{X^{\mathsf{Tr}}_{\kappa,p}}(n)$,

$$\|(g_n(\cdot,u)-g_n(\cdot,v))_{n\geq1}\|_{\gamma(\ell^2,W^{1,p}_0(\mathscr{O}))} \eqsim \|(g_n(\cdot,u)-g_n(\cdot,v))_{n\geq1}\|_{W^{1,p}(\mathscr{O};\ell^2)},$$

where the implicit constants are independent of u,v. By (5.68) and the former one can show that (HG′) holds.

To apply Theorem 4.3.7 it remains to check that the stochastic maximal L^p-regularity assumption. Fix $w_0\in L^\infty_{\mathscr{F}_0}(\Omega;X^{\mathsf{Tr}}_{\kappa,p})$. By (5.68), $w_0=0$ a.e. on $\Omega\times\partial\mathscr{O}$. The latter and (5.67) yield $\sum_{j=1}^d b_{jn}(\cdot,w_0)\nu_j=0$ a.e. on $I_T\times\Omega\times\partial\mathscr{O}$. Therefore, by [69, Theorem 2.5] one has $(A(\cdot,w_0),B(\cdot,w_0))\in\mathcal{SMR}_p(T)$. Moreover, by [3, Example A.4] and Assumption 5.2.7(1), the Dirichlet Laplacian $-{}_D\Delta_p : {}_D W^{2,p}(\mathscr{O})\subseteq L^q(\mathscr{O})\to L^q(\mathscr{O})$ has a bounded H^∞-calculus of angle $<\pi/2$. Thus, by Theorem 4.2.7 and the transference result Proposition 4.2.8 we also obtain $(A(\cdot,w_0),B(\cdot,w_0))\in\mathcal{SMR}^\bullet_p(T)$. ∎

5.2.4 Quasilinear SPDEs in non-divergence form on domains with weights

In a series of papers by Krylov and his collaborators stochastic maximal L^p-regularity is derived on weighted L^p spaces on bounded domains. For special choices of the weight no additional conditions on b and g arise. We consider exactly the same problem as in Section 5.2.3, but this time with weighted function spaces which are more complicated. For function spaces on $\mathbb{R}^d$ with weights we refer to [154, 155] and the references therein. In particular, to define Besov spaces on $\mathbb{R}^d$ with weights we employ the Definition 3.2 in [154].

Let $v_\alpha : \mathbb{R}^d \to (0,\infty)$ be given by $v_\alpha(x) = \mathrm{dist}(x, \partial\mathscr{O})^\alpha$ where $\alpha \in \mathbb{R}$. For an integer $n \geq 1$, let $W^{n,p}(\mathscr{O}, v_\alpha)$ be the space of all $u \in L^p(\mathscr{O}, v_\alpha)$ for which $\partial^\beta u \in L^p(\mathscr{O}, v_\alpha)$ for all $|\beta| \leq n$ endowed with its natural norm. Let

$$ {}_DW^{n,p}(\mathscr{O}, v_\alpha) = \{u \in W^{n,p}(\mathscr{O}, v_\alpha) : \mathsf{Tr}_{\partial\mathscr{O}} u = 0 \text{ if } n > (1+\alpha)/p\}. $$

The trace operator is a bounded operator into $L^p(\partial\mathscr{O})$ (see [146, Section 3.2]). We will only use the above space for $n \in \{1,2\}$ below. For $s \in (0,1)$ let

$$ \mathcal{V}^{2s,p}(\mathscr{O}, v_\alpha) := (L^p(\mathscr{O}, v_\alpha), \mathcal{V}^{2,p}(\mathscr{O}, v_\alpha))_{s,p}, \qquad \mathcal{V} \in \{{}_DW, W\}. \tag{5.69} $$

The latter definition requires some care. In the case $\alpha \in (-1, p-1)$ the space $W^{2s,p}(\mathscr{O}, v_\alpha)$ is equivalent to the Besov space $B^{2s}_{p,p}(\mathscr{O}, v_\alpha)$. Here $B^{2s}_{p,p}(\mathscr{O}, v_\alpha)$ is the restricted space to $\mathscr{O}$ of $B^{2s}_{p,p}(\mathbb{R}^d, v_\alpha)$, see e.g. [145, Definition 5.2]. This follows by combining [43] and [154, Proposition 6.1]. To see this, it is enough to note that by [43, Theorem 1.1] and (5.69), for each $s \in (0,1)$, $p \in (1,\infty)$ and $\alpha \in (-1, p-1)$ there exists an extension operator E (cf. Definition 2.2.3), i.e. a bounded linear operator

$$ E : W^{2s,p}(\mathscr{O}, v_\alpha) \to W^{2s,p}(\mathbb{R}^d, v_\alpha), \quad \text{such that} \quad Ef|_{\mathscr{O}} = f, \tag{5.70} $$

where $W^{2s,p}(\mathbb{R}^d, v_\alpha) = B^{2s}_{p,p}(\mathbb{R}^d, v_\alpha)$. In the case $\alpha \geq p-1$, the space $W^{2s,p}(\mathscr{O}, v_\alpha)$ does not coincide with a weighted Besov space. However, it densely contains the Besov space $B^{2s}_{p,p}(\mathscr{O}, v_\alpha)$ (see [146, Remark 7.14]).

The following is the main assumption of this subsection.

Assumption 5.2.9. *Suppose that Assumption 5.2.1 holds with $N = 1$, $\kappa = 0$ and write $b_{jn} = b_{j1n}$. Suppose that $a_{ij}(t,\omega,x,y,z)$ does not depend on the z-variable and $b_{jn}(t,\omega,x,y)$ does not depend on the y variable. Let $\mathscr{O}$ be a bounded C^2-domain. Moreover, let $\delta \in (0,1]$ and suppose that for each $r > 0$ there exists $\epsilon_r > 0$ such that a.s. for all $\xi \in \mathbb{R}^d$, $\theta \in \mathbb{R}^N$, $t \in [0,T]$, $x \in \mathscr{O}$, $y \in B_{\mathbb{R}^N}(r)$ one has*

$$ \sum_{i,j=1}^d \xi_i\xi_j\big(a_{ij}(t,\omega,x,y) - \Sigma_{ij}(t,\omega,x)\big) \geq \delta \sum_{i,j=1}^d \xi_i\xi_j a_{ij}(t,\omega,x,y) \geq \epsilon_r|\xi|^2. $$

Here for each fixed $i,j \in \{1,\dots,d\}$,

$$ \Sigma_{ij}(t,\omega,x) = \frac{1}{2}\sum_{n\geq 1} b_{in}(t,\omega,x) b_{jn}(t,\omega,x). $$

Finally, suppose that $p \in (d+2,\infty)$ and δ satisfy

$$ 2p - 1 - \frac{p}{p(1-\delta)+\delta} < \alpha < 2p - d - 2. $$

The above assumptions imply that $\alpha > p-1$. In the special case that $b_{jn} \equiv 0$, we can take $\delta = 1$, and thus $p - 1 < \alpha < 2p - d - 2$. The above parabolicity condition is introduced in [134] and also considered in [117].

In this subsection we let

$$ X_0 := L^p(\mathscr{O}, v_\alpha), \quad X_1 := {}_DW^{2,p}(\mathscr{O}, v_\alpha), \quad X^{\mathsf{Tr}}_p = {}_DW^{2-2/p}(\mathscr{O}, v_\alpha), \tag{5.71} $$

where the last equality follows from (5.69). Moreover, we define A, B, F, G be as in (5.65). Let us first analyse the linear problem.

Lemma 5.2.10. *Suppose that Assumption 5.2.9 holds. Then the following hold:*

(1) *There exists $\eta>0$ such that ${}_DW^{2-\frac{2}{p},p}(\mathscr{O},v_\alpha)\hookrightarrow C^\eta(\mathscr{O})$;*

(2) *For every*

$$w_0\in L^\infty_{\mathscr{F}_0}(\Omega;{}_DW^{2-\frac{2}{p},p}(\mathscr{O},v_\alpha))$$

one has $(A(\cdot,w_0),B(\cdot,w_0))\in \mathcal{SMR}^\bullet_p(T)$.

Proof. (1): By (5.69) and [20, Theorem 4.7.2],

$${}_DW^{2-\frac{2}{p},p}(\mathscr{O},v_\alpha)=X^{\mathsf{Tr}}_p=(X_0,X_1)_{1-\frac{1}{p},p}=(X_{1/2},X_1)_{1-\frac{2}{p},p}.$$

By [146, Proposition 3.16] one has

$$X_{\frac{1}{2}}=W^{1,p}(\mathscr{O},v_\alpha). \tag{5.72}$$

Therefore, by Hardy's inequality (see [146, Corollary 3.4]),

$$X^{\mathsf{Tr}}_p=(X_{\frac{1}{2}},X_1)_{1-\frac{2}{p}}\hookrightarrow (L^p(\mathscr{O},v_{\alpha-p}),W^{1,p}(\mathscr{O},v_{\alpha-p}))_{1-\frac{2}{p},p}=W^{1-\frac{2}{p},p}(\mathscr{O},v_{\alpha-p}),$$

where the last equality follows from (5.70). By Assumption 5.2.9 one has $\alpha-p\in(-1,p-1)$, therefore the considerations at the beginning of this section imply that $W^{1-\frac{2}{p},p}(\mathscr{O},v_{\alpha-p})=B^{1-\frac{2}{p}}_{p,p}(\mathscr{O},v_{\alpha-p})$. To complete the proof of (1) it is enough to show that $B^{1-\frac{2}{p}}_{p,p}(\mathscr{O},v_{\alpha-p})\hookrightarrow C^\eta(\mathscr{O})$ for some $\eta>0$. Since $\mathscr{O}$ is bounded, using a standard localization argument (see e.g. [146, Section 2.2] and the references therein) it is enough to prove

$$B^{1-\frac{2}{p}}_{p,p}(\mathbb{R}^d,g_{\alpha-p})\hookrightarrow B^\eta_{\infty,\infty}(\mathbb{R}^d); \tag{5.73}$$

where, for $x=(x_1,x_2,\dots,x_d)$, $g_\beta(x):=x_1^\beta$ on $|x|\le 1$ and $g_\beta(x):=1$ otherwise.

The embedding in (5.73) follows from [155, Proposition 4.2] and the fact that $1-\frac{2}{p}-\frac{\alpha-p+d}{p}>0$ and $1-\frac{2+d}{p}>0$. The latter are equivalent to $\alpha<2p-d-2$ and $p>d+2$, respectively, which hold by Assumption 5.2.9.

(2): Combining (1), Assumption 5.2.9 and [117, Theorem 2.9] one has $(A(\cdot,w_0),B(\cdot,w_0))\in \mathcal{SMR}_p(T)$. To see the latter note that by Hardy's inequality (see [146, Corollary 3.4]), and [150, Proposition 2.2] (also see [119, Remark 2.9]) the spaces in [117] coincide with the ones considered here.

Since by [146, Theorem 1.1], $-{}_D\Delta_p$ has a bounded H^∞-calculus of angle zero on $L^p(\mathscr{O},v_\alpha)$ (with domain ${}_DW^{2,p}(\mathscr{O},v_\alpha)$), by Theorem 4.2.7 and the transference result Proposition 4.2.8 we also obtain that $(A(\cdot,w_0),B(\cdot,w_0))\in\mathcal{SMR}^\bullet_p(T)$. ☐

In the next result, we say that (u,σ) is maximal local solution to (5.58) if (u,σ) is a maximal local solution to (4.16) with A,B,F,G and X_0,X_1 are as in (5.65) and (5.71) respectively.

Theorem 5.2.11. *Suppose Assumptions 5.2.2 and 5.2.9 hold, and that f does not dependent on the z-variable. Then for each*

$$u_0\in L^0_{\mathscr{F}_0}(\Omega;{}_DW^{2-\frac{2}{p},p}(\mathscr{O},w_\alpha)),$$

there exists a unique maximal local solution (u,σ) to (5.58). Moreover, there exists a localizing sequence $(\sigma_n)_{n\ge1}$ such that a.s. for all $n\ge 1$

$$u\in L^p(I_{\sigma_n};{}_DW^{2,p}(\mathscr{O},v_\alpha))\cap C(\overline{I}_{\sigma_n};{}_DW^{2-\frac{2}{p},p}(\mathscr{O},v_\alpha)).$$

Recall that the space ${}_DW^{2-\frac{2}{p},p}(\mathscr{O},v_\alpha)$ is defined as in (5.69) and does not coincide with a weighted Besov space if $\alpha\ge p-1$.

Proof. By Lemma 5.2.10 and the fact that $\mathscr{O}$ is bounded, one can argue in the same way as in Theorem 5.2.8. We remark that (HG′) is satisfied by setting $G_c = G_L = 0$ and $G_{\mathsf{Tr}} = G$. To see this one can argue as in (5.40) since $X^{\mathsf{Tr}}_{\kappa,p} \hookrightarrow W^{1,p}(\mathscr{O}, v_\alpha) \cap C^\eta(\mathscr{O})$ for some $\eta > 0$. The latter embedding follows from Lemma 5.2.10(1), (5.72) and $X^{\mathsf{Tr}}_{\kappa,p} \hookrightarrow X_{1/2}$ due to $1 - 2\frac{1+\kappa}{p} > \frac{1}{2}$. □

Remark 5.2.12.

(1) It would be interesting to extend the above to $\kappa \neq 0$ and $p \neq q$. However, at the moment almost no weighted theory is available in the case a_{ij} depend on (t, ω). Except in the case $\mathcal{A} = -\Delta$, one has a bounded H^∞-calculus on $L^q(\mathscr{O}, v_\alpha)$ by [146], and thus Theorem 4.2.7 implies stochastic maximal L^p-regularity in the full range. The latter can very likely be extended to elliptic second order operators in non-divergence form with smooth x-dependent coefficients by standard arguments. This would make it possible to do a variant of Theorem 5.2.11 with general (p, q, κ) as long as the coefficients a_{ij} are independent of time.

(2) In [115] a quasilinear SPDE is considered in weighted spaces as well. However, the results seem not comparable. For instance, they consider operators in divergence form and they do not allow a gradient type noise term.

5.2.5 Quasilinear SPDEs in divergence form on domains

Unlike in the previous sections we will consider an example where there is no time-dependence in the operator A and $B = 0$. In this way we can obtain a full $L^p(L^q)$-theory. We study the following differential problem for the unknown $u : I_T \times \Omega \times \mathscr{O} \to \mathbb{R}$:

$$\begin{cases} du - \operatorname{div}(a(u)\nabla u)dt = (\operatorname{div}(f_1(\cdot, u, \nabla u)) + f_2(\cdot, u, \nabla u))dt & \\ \qquad\qquad\qquad\qquad + \sum_{n\geq 1} g_n(\cdot, u, \nabla u)dw^n_t, & \text{on } \mathscr{O}, \\ u = 0, & \text{on } \partial\mathscr{O}; \\ u(0) = u_0, & \text{on } \mathscr{O}. \end{cases} \tag{5.74}$$

The problem (5.74) was already considered in [104, Secion 5.5]. The aim of this section is to partially extend [104, Theorem 5.6] and at the same time correct it (see the discussion in [103, p. 66] on this matter). Note that in [104, Section 5.5] equations in divergence form have been considered with Neumann and/or mixed type boundary conditions. Our framework also allows this setting, but we will only consider Dirichlet conditions here. The interested reader can adapt the proofs below with the functional analytic set-up proposed [104, Section 5.5] to correct [104, Theorem 4.11] with different boundary conditions.

We study (5.74) under the following assumption.

Assumption 5.2.13.

(1) *Let* $q \in [2, \infty)$, $p \in (2, \infty)$ *and* $\kappa \in [0, \frac{p}{2} - 1)$ *be such that* $1 - \frac{2(1+\kappa)}{p} > \frac{d}{q}$.

(2) $\mathscr{O} \subseteq \mathbb{R}^d$ *is a bounded* C^1*-domain.*

(3) *The map* $a : \Omega \times \mathscr{O} \times \mathbb{R} \to \mathbb{R}^{d\times d}$ *is* $\mathscr{F}_0 \otimes \mathscr{B}(\mathscr{O}) \otimes \mathscr{B}(\mathbb{R})$*-measurable. Assume that* $a(\cdot, 0) \in L^\infty(\Omega \times \mathscr{O})$ *and for each* $r > 0$ *there exists an increasing continuous function* $K_r : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $K_r(0) = 0$ *and for each* $i, j \in \{1, \dots, d\}$, $x, x' \in \mathscr{O}$ *and* $y \in B_{\mathbb{R}}(r)$,

$$|a(x, y) - a(x', y)| \leq K_r(|x - x'|).$$

Moreover, a *is locally Lipschitz w.r.t.* $y \in \mathbb{R}$ *uniformly in* (ω, x)*, i.e. for each* $r > 0$ *there exists* $C_r > 0$ *such that a.s. for all* $x \in \mathscr{O}$ *and* $y, y' \in B_{\mathbb{R}}(r)$ *one has*

$$|a(x, y) - a(x, y')|_{\mathbb{R}^{d\times d}} \leq C_r|y - y'|.$$

Furthermore, a is locally uniformly ellipticity, i.e. for each $r > 0$ there exists $\epsilon_r > 0$ such that a.s. for all $x \in \mathscr{O}$ and $y \in B_{\mathbb{R}}(r)$ one has

$$\sum_{i,j=1}^{d} \xi_i \xi_j a_{ij}(x, y) \geq \epsilon_r |\xi|^2.$$

(4) *Let $\varepsilon \geq 0$. The mappings $f_1 : I_T \times \Omega \times \mathscr{O} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$, $f_2 : I_T \times \Omega \times \mathscr{O} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ and $g := (g_n)_{n\geq 1} : I_T \times \Omega \times \mathscr{O} \times \mathbb{R} \times \mathbb{R}^d \to \ell^2$ are $\mathscr{P} \otimes \mathscr{B}(\mathscr{O}) \otimes \mathscr{B}(\mathbb{R}^d) \otimes \mathscr{B}(\mathbb{R})$-measurable. Assume that $f_1(\cdot, 0, 0) = 0$, $f_2(\cdot, 0) = g_n(\cdot, 0) = 0$ for all $n \geq 1$. Finally, we assume that for each $r > 0$ there exists $C_r > 0$ such that a.s. for all $x \in \mathscr{O}$, $y, y' \in B(r)$ and $z, z' \in \mathbb{R}$*

$$\sum_{i=1}^{2} |f_i(t, x, y, z) - f_i(t, x, y', z')| + \|g(t, x, y, z) - g(t, x, y', z')\|_{\ell^2} \leq C_{r,\varepsilon}|y - y'| + \varepsilon|z - z'|.$$

Typical examples of f_1, f_2, g are

$$\begin{aligned} f_1(x, u, \nabla u) &= \varepsilon \nabla u, & f_1(x, u) &= (\widetilde{f}_i(x)|u|^{h-1}u)_{i=1}^d, \\ f_2(t, u) &= |u|^{m-1}u, & (g_n(x, u))_{n\geq 1} &= (\widetilde{g}_n(x)|u|^{r-1}u)_{n\geq 1}. \end{aligned}$$

where $h, m, r > 1$, $\varepsilon > 0$, $(\widetilde{f}_i)_{i=1}^d \in L^\infty(\Omega \times \mathscr{O}; \mathbb{R}^d)$ and $(\widetilde{g}_n)_{n\geq 1} \in L^\infty(\Omega \times \mathscr{O}; \ell^2)$.

Let us briefly recall the function spaces which will be needed below. Let $s \in (-1, 1)$ and $q, p \in (1, \infty)$, we set

$$\begin{aligned} W_0^{1,q}(\mathscr{O}) &= \{u \in W^{1,q}(\mathscr{O}) : u|_{\partial\mathscr{O}} = 0\}, \\ W^{-1,q}(\mathscr{O}) &= (W_0^{1,q'}(\mathscr{O}))^* \\ {}_D B^s_{q,p}(\mathscr{O}) &= (W^{-1,q}(\mathscr{O}), W_0^{1,q}(\mathscr{O}))_{\frac{s+1}{2},p}. \end{aligned} \tag{5.75}$$

For further properties we refer to [3, Example A.4] and the references therein.

To recast the problem (5.74) in the form (4.16) let us set $X_0 := W^{-1,q}(\mathscr{O})$, $X_1 := W_0^{1,q}(\mathscr{O})$, and for $u \in C(\overline{\mathscr{O}})$ and $v \in X_1$

$$\begin{aligned} A(t, u)v &= -\mathrm{div}(a(u)\nabla v), & B(t, u)v &= 0, \\ F(t, u) &= \mathrm{div}(f_1(t, u, \nabla u)) + f_2(t, u, \nabla u), & G(t, u) &= (g_n(t, u, \nabla u))_{n\geq 1}. \end{aligned}$$

Here the divergence operator is defined as in (8.3), i.e. for $u \in C(\overline{\mathscr{O}})$ and $v \in X_1$,

$$\langle \phi, A(u)v \rangle = \int_{\mathscr{O}} (a(u) \cdot \nabla v) \cdot \nabla \phi \, dx, \qquad \phi \in W^{1,q'}(\mathscr{O}). \tag{5.76}$$

The same applies to $F(t, u)$. As usual we say that (u, σ) is a maximal local solution of (5.74) if (u, σ) is a maximal local solution of (4.16) with the above choice of A, B, F, G and $H = \ell^2$.

Before stating the main result of this subsection, let us note that a maximal local solution to (5.74) verifies the natural weak formulation of (5.74): a.s. for all $t \in [0, \sigma)$ and all $\phi \in C_c^1(\mathscr{O})$,

$$\begin{aligned} \int_{\mathscr{O}} (u(t) - u_0)\phi \, dx + \int_0^t \int_{\mathscr{O}} (a(u) \cdot \nabla u) \cdot \nabla \phi \, dx \, ds &= -\int_0^t \int_{\mathscr{O}} f_1(u, \nabla u) \cdot \nabla \phi \, dx \, ds \\ &+ \int_0^t \int_{\mathscr{O}} f_2(u, \nabla u)\phi \, dx \, ds + \sum_{n\geq 1} \int_0^t \int_{\mathscr{O}} g_n(u, \nabla u)\phi \, dx \, dw_s^n. \end{aligned}$$

To see this, use (4.21) and note that $\phi \in C_c^1(\mathscr{O}) \subseteq (W^{-1,q}(\mathscr{O}))^* = W_0^{1,q'}(\mathscr{O})$.

Theorem 5.2.14. *Suppose Assumption 5.2.13 holds. Then for each $N \geq 1$ there exists $\bar{\varepsilon}_N > 0$ such that if $\varepsilon \in (0, \bar{\varepsilon}_N)$ and*

$$u_0 \in L^\infty_{\mathscr{F}_0}(\Omega; {}_D B^{1-\frac{2(1+\kappa)}{p}}_{q,p}(\mathscr{O}))$$

has norm $\leq N$, then there exists a unique local solution (u, σ) to (5.74). Moreover, there exists a localizing sequence $(\sigma_n)_{n\geq 1}$ such that for all $n \geq 1$ and a.s.

$$u \in L^p(I_{\sigma_n}, w_\kappa; W^{1,q}_0(\mathscr{O})) \cap C(\bar{I}_{\sigma_n}; {}_D B^{1-\frac{2(1+\kappa)}{p}}_{q,p}(\mathscr{O})) \cap C((0, \sigma_n]; {}_D B^{1-\frac{2}{p}}_{q,p}(\mathscr{O})).$$

Proof. By Assumption 5.2.13(1), (5.75), and Sobolev embeddings one has

$$X^{\mathsf{Tr}}_{\kappa,p} = {}_D B^{1-\frac{2(1+\kappa)}{p}}_{q,p}(\mathscr{O}) \hookrightarrow B^{1-\frac{2(1+\kappa)}{p}}_{q,p}(\mathscr{O}) \hookrightarrow C^\eta(\mathscr{O}) \hookrightarrow L^\infty(\mathscr{O}); \tag{5.77}$$

for some $\eta > 0$. Therefore, $A(u)v := -\mathrm{div}(a(u) \cdot \nabla v)$ for $u \in X^{\mathsf{Tr}}_{\kappa,p}$ and $v \in X_1$ is well-defined. By [16, Remark 4.3(ii) and Theorem 11.5], [73, Remark 2.4(3)] and Assumption 5.2.13 $A(u_0)$ has a bounded H^∞-calculus of angle $< \pi/2$. Therefore, by Theorem 4.2.7 (see also Remark 4.2.9(3)) we find that $A(u_0) \in \mathcal{SMR}^\bullet_{p,\kappa}(T)$ and for each $\theta \in [0, 1/2)$

$$\max\left\{K^{\mathsf{det},\theta}_{A(u_0)}, K^{\mathsf{sto},\theta}_{A(u_0)}\right\} \leq C_N, \tag{5.78}$$

where C_N depends only on $N > 0$. To check (HA) let us fix $n \geq 1$ and $u_1, u_2 \in X^{\mathsf{Tr}}_{\kappa,p}$ of norm $\leq n$. Then by (5.77) it follows that $\|u_1\|_{L^\infty(\mathscr{O})}, \|u_2\|_{L^\infty(\mathscr{O})} \leq C_n =: R$, and for each $v \in X_1$

$$\begin{aligned}\|\mathrm{div}(a(u_1) \cdot \nabla v) - \mathrm{div}(a(u_2) \cdot \nabla v)\|_{W^{-1,q}(\mathscr{O})} &\lesssim \|(a(u_1) - a(u_2))\nabla v\|_{L^q(\mathscr{O})} \\ &\leq C_R \|u_1 - u_2\|_{L^\infty(\mathscr{O})} \|v\|_{W^{1,q}(\mathscr{O})} \\ &\lesssim_R \|u_1 - u_2\|_{X^{\mathsf{Tr}}_{\kappa,p}} \|v\|_{X_1};\end{aligned}$$

where we used (8.3) and Assumption 5.2.13(3).

Since $X_{1/2} = L^q(\mathscr{O})$ by [187], using the same argument as above combined with Assumption 5.2.13(4) one obtains

$$\begin{aligned}\|F(\cdot, u) - F(\cdot, v)\|_{X_0} &+ \|G(\cdot, u) - G(\cdot, v)\|_{\gamma(\ell^2, X_{1/2})} \\ &\leq C_R \|u - v\|_{X^{\mathsf{Tr}}_{\kappa,p}} + C\varepsilon \|u - v\|_{X_1};\end{aligned} \tag{5.79}$$

where $C > 0$ does not depend on $n \geq 1$. By setting $F_L = F$ and $G_L = G$ the assumptions (HF′)-(HG′) are verified. Moreover, the inequalities (5.78), (5.79) and Remark 4.3.19 show that the condition (4.25) holds. The result now follows from Theorem 4.3.7. ☐

Remark 5.2.15.

(1) The assumption $u_0 \in L^\infty_{\mathscr{F}_0}(\Omega; X^{\mathsf{Tr}}_{\kappa,p})$ is automatically satisfied if $\mathbb{F}$ is generated by W_{ℓ^2} (see Remark 4.3.6).

(2) In Chapter 7 we will see that the instantaneous regularization effect in Theorem 5.2.14 can be bootstrapped to prove further regularization of solutions to (5.74). In such situation weights in time play a basic role.

In the case $u_0 \notin L^\infty_{\mathscr{F}_0}(\Omega; X^{\mathsf{Tr}}_{\kappa,p})$, we do not have any control on the constants of maximal regularity of $A(u_{0,n})$ as n grows see [103, p. 66] (here $(u_{0,n})_{n\geq 1}$ is as in (4.26)). However, by choosing $\varepsilon_n \downarrow 0$ appropriately, the arguments used in the proof of Theorem 5.2.14 still lead to the following.

Theorem 5.2.16. *Let the Assumption 5.2.13 be satisfied for any $\varepsilon > 0$. Then for each*

$$u_0 \in L^0_{\mathscr{F}_0}(\Omega; {}_D B^{1-\frac{2(1+\kappa)}{p}}_{q,p}(\mathscr{O}))$$

there exists a unique local solution (u, σ) to (5.74). Moreover, there exists a localizing sequence $(\sigma_n)_{n\geq 1}$ such that for all $n \geq 1$ and a.s.

$$u \in L^p(I_{\sigma_n}, w_\kappa; W^{1,q}_0(\mathscr{O})) \cap C(\bar{I}_{\sigma_n}; {}_D B^{1-\frac{2(1+\kappa)}{p}}_{q,p}(\mathscr{O})) \cap C((0, \sigma_n]; {}_D B^{1-\frac{2}{p}}_{q,p}(\mathscr{O})).$$

5.2.6 Stochastic porous media equations with positive initial data

In this subsection we investigate porous media type equations on the d-dimensional torus $\mathbb{T}^d$ with uniformly positive initial data. More precisely, we investigate the following problem for the unknown $u : I_T \times \Omega \times \mathbb{T}^d \to \mathbb{R}$

$$\begin{cases} du - \big(\Delta(|u|^{r-1}u) - \sum_{i,j=1}^d \Xi_{ij}(\cdot,u)\partial^2_{ij}u\big)dt = f(u,\nabla u)dt \\ \qquad\qquad + \sum_{n\geq 1}\big(\sum_{j=1}^d b_{nj}(\cdot,u)\partial_j u + g_n(u)\big)w^n_t, & \text{on } \mathbb{T}^d, \\ u(0) = u_0, & \text{on } \mathbb{T}^d; \end{cases} \tag{5.80}$$

where $r \in [1,\infty)$, $u_0 \geq c > 0$ a.e. on $\mathbb{T}^d$ and $\Xi_{i,j}(\cdot,u) = \frac{1}{2}\sum_{n\geq 1} b_{jn}(\cdot,u)b_{jn}(\cdot,u)$. The problem (5.80) in the case $r = 1$ fits in the framework of Section 5.1, in such a case the condition $u_0 \geq c$ can be avoided. We will only consider the range $r \geq 3$ for technical reasons. The range $r \in (1,3)$ is more sophisticated and requires other solution concepts than to the one below. For physical motivations we refer to [17], [79, Subsection 1.1] and the references therein. To see the link with the works [54, 79], let us note that at least formally (see [54, Remark 2.1])

$$\sum_{n\geq 1}\sum_{j=1}^d b_{nj}\partial_j u \circ dw^n_t = \sum_{n\geq 1}\sum_{j=1}^d b_{nj}\partial_j u\, dw^n_t + \sum_{i,j=1}^d \Big(\Xi_{ij}(\cdot,u)\partial^2_{ij}u + \text{lower order terms}\Big)dt,$$

where $\circ$ denotes the Stratonovich integration. We refer to Subsection 5.2.6 for a comparison to the literature.

To study (5.80), we exploit the fact that in Theorem 4.3.7, stochastic maximal L^p-regularity is required on $(A(u_{0,n}), B(u_{0,n}))$ for appropriate A and B (see (4.24)). We mainly deal with the strong setting and we refer to Remark 5.2.19 for the weak one. To begin, let us note that at least formally,

$$\Delta(|u|^{r-1}u) = r|u|^{r-1}\Delta u + r(r-1)u|u|^{r-3}|\nabla u|^2.$$

Therefore, in the case $u \geq c > 0$, the porous media operators acts like Δ plus a lower order term. For notational convenience, we set $\mathcal{A}_r(t,u)v := -r|u|^{r-1}\Delta v$ and $f_r(u,\nabla v) := -r(r-1)u|u|^{r-3}|\nabla v|^2$. To recast (5.80) in the form (4.16), we set $X_0 = L^q(\mathbb{T}^d)$, $X_1 = W^{2,q}(\mathbb{T}^d)$ and for $v \in X_1$, $u \in C(\mathbb{T}^d) \cap W^{1,q}(\mathbb{T}^d)$

$$A(t,u)v = \mathcal{A}_r(t,u)v + \sum_{i,j=1}^d \Xi_{ij}(t,u)\partial^2_{ij}v, \qquad B(t,u)v = \Big(\sum_{j=1}^d b_{jn}(t,u)\partial_j v\Big)_{n\geq 1},$$
$$F(t,v) = f(t,v,\nabla v) - f_r(v,\nabla v), \qquad G(t,v) = (g_n(t,v))_{n\geq 1}.$$

Here f and g_n are as in Assumption 5.2.2. The following is the main result of this subsection.

Theorem 5.2.17. *Let $r \geq 3$. Let $p \in (2,\infty)$ and $\kappa \in [0,\frac{p}{2}-1)$ be such that $p > 2(1+\kappa)+d$. Assume that b_{jn} and f,g verifies Assumption 5.2.1(1)-(2) and Assumption 5.2.2, respectively. Then for each*

$$u_0 \in L^0_{\mathscr{F}_0}(\Omega; W^{2-2\frac{1+\kappa}{p},p}(\mathbb{T}^d)), \qquad u_0 \geq c > 0 \text{ a.e. on } \mathbb{T}^d \times \Omega,$$

there exists a maximal local solution (u,σ) to (5.80). Moreover, there exists a localizing sequence $(\sigma_n)_{n\geq 1}$ such that for all $n \geq 1$ and a.s.

$$u \in L^p(I_{\sigma_n}, w_\kappa; W^{2,q}(\mathbb{T}^d)) \cap C(\overline{I}_{\sigma_n}; W^{2-2\frac{1+\kappa}{p},p}(\mathbb{T}^d)) \cap C((0,\sigma_n]; W^{2-\frac{2}{p},p}(\mathbb{T}^d)).$$

Proof. The proof is similar to the one given for Theorem 5.2.4. As in the proof of the latter theorem, by Sobolev embedding $X^{\mathsf{Tr}}_{\kappa,p} = W^{2-2\frac{1+\kappa}{p},p}(\mathbb{T}^d) \hookrightarrow C^{1+\eta}(\mathbb{T}^d)$ for some $\eta > 0$. Thus, using $r \geq 3$ the estimates on the nonlinearities can be performed as in Theorem 5.2.4. The fact that (HA) holds follows from standard computations.

To check the stochastic maximal regularity condition (4.24), for all $n \geq 1$ we set

$$u_{0,n} := \mathbf{1}_{\Gamma_n}u_0 + \mathbf{1}_{\Omega\setminus\Gamma_n}(c\mathbf{1}_{\mathbb{T}^d}), \quad \text{where} \quad \Gamma_n := \{\|u_0\|_{X^{\mathsf{Tr}}_{\kappa,p}} \leq n\}. \tag{5.81}$$

Thus, $u_{0,n} \in L^\infty(\Omega; C^{1,\eta}(\mathbb{T}^d))$ verifies $u_{0,n} \geq c$. Reasoning as in the proof of Theorem 5.2.4, $(A(\cdot, u_{0,n}), B(\cdot, u_{0,n})) \in \mathcal{SMR}^\bullet_{p,\kappa}(T)$ by [174, Theorem 5.4] and $u_{0,n} \geq c_1$. We remark that the ellipticity condition in [174, Assumption 5.2(2)] is satisfied since $|u_{0,n}|^{r-1} \geq c^{r-1} > 0$ a.s. and a.e. on $\mathbb{T}^d$. ☐

In the above proof we used the choice (5.81) instead of (4.26). Indeed, if $u_{0,n}$ is as in (4.26), then $u_{0,n}$ are not uniformly bounded from below in general.

The proof of Theorem 5.2.17 shows that Theorems 5.2.5-5.2.6 extends to (5.80). To avoid repetitions, we only state the extension of Theorem 5.2.6 to (5.80).

Theorem 5.2.18. *Let $r \geq 3$. Assume that b_{jn} and f, g verifies Assumption 5.2.1(1)-(2) and Assumption 5.1.11, respectively. Moreover, assume that $m > 1 + \frac{2}{d}$ and $b_{jn}(t, \omega, x, y)$ does not depend on the y variable. Suppose that $p \in (2, \infty)$ verifies (5.63). Then for any*

$$u_0 \in L^0_{\mathscr{F}_0}\Big(\Omega; W^{\frac{d}{p}+\frac{m-2}{m-1},p}(\mathbb{T}^d)\Big), \quad \text{with} \quad u_0 \geq c > 0 \ \text{ a.e. on } \mathbb{T}^d \times \Omega,$$

there exists a maximal local solution (u, σ) to (5.58). Moreover, there exists a localizing sequence $(\sigma_n)_{n\geq 1}$ such that a.s. for all $n \geq 1$

$$u \in L^p(I_{\sigma_n}, w_{\kappa_{\mathsf{crit}}}; W^{2,p}(\mathbb{T}^d)) \cap C(\overline{I}_{\sigma_n}; W^{\frac{d}{p}+\frac{m-2}{m-1},p}(\mathbb{T}^d)) \cap C((0, \sigma_n]; W^{2-\frac{2}{p},p}(\mathbb{T}^d)),$$

where $\kappa_{\mathsf{crit}} := \frac{pm}{2(m-1)} - \frac{d}{2} - 1$.

Proof. Comparing the proof of Theorem 5.2.17 and Theorem 5.2.6, it remains to estimate f_r. To this end let us note that for each $R > 0$, $y, y' \in B_{\mathbb{R}}(R)$ and $z, z' \in \mathbb{R}^d$,

$$|f_r(y,z) - f_r(y',z')| \leq C_R\big[(1 + |z|^2 + |z'|^2)|y - y'| + (1 + |z| + |z'|)|z - z'|\big], \tag{5.82}$$

for some $C_R > 0$ independent of y, y', z, z'. Therefore, due to (5.82), if f verifies Assumption 5.1.11 for $m > 2$, then $f - f_r$ verifies Assumption 5.1.11 with the same m. Thus, reasoning as in the proof of Theorem 5.2.6, the conclusion follows. ☐

Remark 5.2.19. Equation (5.80) has a natural weak formulation. One can check that the arguments used in Theorems 5.2.17-5.2.18 can be adapted to prove local existence in the weak setting (see Subsection 5.2.5). In such a case, $r \in (2, 3)$ is also allowed.

Discussions

Under some structural assumptions on the nonlinearities b_{jn}, f, g, (5.80) (and its generalizations) has been extensively studied (see for instance [54, 79, 92, 91, 53] and the references therein). One of the first paper on the topic is [92] where only x-independent b_{nj} are considered. In the x-dependent case the situation is more complicated and one often needs the assumption $m \geq 2$, see [91, 79]. In [54, 53], the authors allow the more complicated range $r \in (1, 2)$ as well, in some cases they need to work with other type of solutions such as kinetic or entropy solutions. Our results appear weaker than the ones in [54]. For instance, the assumption $u_0 \geq c$ is unnatural. However, this case was also considered in the deterministic setting, see e.g. [185]. Moreover, our setting differs from the one in [54], and the main differences are:

- the functions spaces considered for the initial data are different;
- the nonlinearity f can be of arbitrary polynomial growth in u and $|\nabla u|$;
- less regularity is required for b_{jn}.

It seems to us that the theory developed here can be used to study (5.80) with general u_0, employing a standard approximation argument (see e.g. [79, eq. (3.2)]). Firstly, one replaces $\Delta(|u|^{r-1}u)$ by $\Delta(\varepsilon + |u|^{r-1}u)$ in (5.80). With such modification, we can apply Theorem 4.3.7

to (5.80), obtaining a family of maximal local solutions $(u_\varepsilon, \sigma_\varepsilon)_{\varepsilon>0}$ to the modified equations. Secondly, one provides a-priori bound (uniform in $\varepsilon > 0$) in C^α-norm for $(u_\varepsilon)_{\varepsilon>0}$ for some uniform $\alpha > 0$. Thus, by the blow-up criteria in Chapter 6, $\sigma_\varepsilon = T$ and one can study the behaviour of u_ε as $\varepsilon \downarrow 0$. We remark that a-priori estimates for the C^α-norm for the deterministic version of (5.80) are known, see the discussion in [64, p. vii-viii]. However, we are not aware of any contribution on this topic for (5.80). Note that the arguments used for (5.80) seem to be applicable to other degenerate parabolic equations.

5.2.7 Stochastic Burger's equation with coloured noise

Here, we consider a quasilinear version of the stochastic Burger's equation on $\mathbb{T}$ with space-time coloured noise, which can be seen as the quasilinear analogue of (5.44). However, for technical reasons, we cannot deal with white noise as in Subsection 5.1.5.

More precisely, we consider the following problem for $u : I_T \times \Omega \times \mathbb{T} \to \mathbb{R}$,

$$\begin{cases} du - \partial_x(a(\cdot,u)\partial_x u)dt = \big(\partial_x(f_1(\cdot,u)) + f_2(\cdot,u)\big)dt + g(\cdot,u)dw_t^c, & \text{on } \mathbb{T}, \\ u(0) = u_0, & \text{on } \mathbb{T}; \end{cases} \tag{5.83}$$

here w_t^c denotes a coloured space-time noise on $\mathbb{T}$. More precisely, for some $\delta > 0$, we assume that w_t^c induces an $H^{\delta,2}(\mathbb{T})$-cylindrical Brownian motion in the sense of Definition 2.3.5.

The noise in (5.83) is different than in Subsections 5.2.2-5.2.6. The setting in (5.83) is as in Subsection 5.1.5, but with a coloured noise.

Assumption 5.2.20.

(1) *$q \in [2,\infty)$, $p \in (2,\infty)$ and $\kappa \in [0, \frac{p}{2}-1)$ verifies $2\delta - 2\frac{1+\kappa}{p} > \frac{1}{q}$.*

(2) *The map $a : \Omega \times \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is $\mathscr{F}_0 \otimes \mathscr{B}(\mathbb{T}) \otimes \mathscr{B}(\mathbb{R})$-measurable and it verifies the Assumption 5.2.13(3) with $d = 1$ and $\mathscr{O}$ replaced by $\mathbb{T}$.*

(3) *The maps $f_1, f_2, g : I_T \times \Omega \times \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ are $\mathscr{P} \otimes \mathscr{B}(\mathbb{T}) \otimes \mathscr{B}(\mathbb{R})$-measurable. Assume that $f_1(\cdot,0), f_2(\cdot,0) \in L^\infty(I_T \times \Omega; L^q(\mathbb{T}))$ and $g(\cdot,0) \in L^\infty(I_T \times \Omega \times \mathbb{T})$. Moreover, for each $r > 0$ there exists $C_r > 0$ such that for all $t \in I_T$, $x \in \mathbb{T}$ and $y, y' \in B_R(r)$,*

$$\sum_{i\in\{1,2\}} |f_i(t,x,y) - f_i(t,x,y')| + |g(t,x,y) - g(t,x,y')| \le C_r|y - y'|.$$

Remark 5.2.21.

- For any $\delta > 0$, Assumption 5.2.20(1) is satisfied for p, q large and κ small.
- Assumption 5.2.20(3) includes the Burger's type nonlinearity $f(u) = -u^2$.

In what follows, we only consider the case $\delta \in (0, \frac{1}{2})$, the other cases being simpler. To begin, note that by Assumption 5.2.20(1), there exists $s > \frac{1}{2}$ such that $1-2s+2\delta-2\frac{(1+\kappa)}{p} > \frac{1}{q}$. With such a choice, we rewrite (5.83) in the form (4.16). To this end, set $H = H^{2\delta,2}(\mathbb{T})$, $X_0 := H^{-1-s+\delta,q}(\mathbb{T})$ and $X_1 = H^{1-s+\delta,q}(\mathbb{T})$. Then by (5.4),

$$X_{\frac{1}{2}} = H^{-s+\delta,q}(\mathbb{T}) \quad \text{and} \quad X^{\mathsf{Tr}}_{\kappa,p} = B_{q,p}^{1-s+\delta-\frac{2(1+\kappa)}{p}}(\mathbb{T}). \tag{5.84}$$

As in Subsection 5.2.3, by Sobolev embedding and Assumption 5.2.20(1), one has

$$B_{q,p}^{1-s+\delta-\frac{2(1+\kappa)}{p}}(\mathbb{T}) \hookrightarrow C^\eta(\mathbb{T}), \qquad \eta := 1 - s + \delta - 2\frac{1+\kappa}{p} - \frac{1}{q} > s - \delta. \tag{5.85}$$

For $v \in X^{\mathsf{Tr}}_{\kappa,p}$, $u \in X_1$ let

$$A(v)u = -\partial_x(a(v)\partial_x u), \qquad\qquad B(t)u = 0,$$

$$F(t,u)=\partial_x(f(t,u)), \qquad\qquad G(t,u)=iM_{g(t,u)}.$$

Similar to Subsection 5.1.5, for fixed $u\in C(\mathbb{T})$, $M_{g(t,u)}:L^\xi(\mathbb{T})\to L^\xi(\mathbb{T})$ is the multiplication operator $(M_{g(t,u)}h)(x)=g(t,u(x))h(x)$ where $\xi\in(2,\infty)$ verifies $\delta-\frac{1}{2}=-\frac{1}{\xi}$, which is needed below for the Sobolev embedding $H^{\delta,2}\hookrightarrow L^\xi$ (and here we need $\delta\in(0,\frac{1}{2})$). Moreover, $i:L^\xi(\mathbb{T})\to X_{\frac{1}{2}}$ denotes the embedding. As usual, we say that (u,σ) is a maximal local solution to (5.83) if (u,σ) is a maximal local solution to (5.1) in the sense of Definition 4.3.4 with the above choice of A,B,F,G,H.

To estimate F, similar to (5.79), one has

$$\|F(\cdot,u)-F(\cdot,v)\|_{H^{-s,q}}\lesssim\sum_{i\in\{1,2\}}\|f_i(\cdot,u)-f_i(\cdot,v)\|_{L^q}\lesssim_r\|u-v\|_{X^{\mathsf{Tr}}_{\kappa,p}},$$

where in the last inequality we used Assumption 5.2.20(3) and (5.85). Therefore, F verifies (HF′) by setting $F_{\mathsf{Tr}}=F$, $F_L=F_c=0$. To estimate G, we argue as in (5.50), (5.79). Then for $u,v\in X^{\mathsf{Tr}}_{\kappa,p}$ such that $\|u\|_{X^{\mathsf{Tr}}_{\kappa,p}},\|v\|_{X^{\mathsf{Tr}}_{\kappa,p}}\leq r$, one has

$$\begin{aligned}
&\|G(\cdot,u)-G(\cdot,v)\|_{\gamma(H^{\delta,2};H^{-s+\delta,q})}\\
&\qquad\eqsim\|(I-\partial_x^2)^{-\frac{s}{2}+\frac{\delta}{2}}(M_{g(\cdot,u)}-M_{g(\cdot,v)})(1-\partial_x^2)^{-\frac{\delta}{2}}\|_{\gamma(L^2,L^q)}\\
&\qquad\overset{(i)}{\lesssim}\|(I-\partial_x^2)^{-\frac{s}{2}+\frac{\delta}{2}}(M_{g(\cdot,u)}-M_{g(\cdot,v)})(1-\partial_x^2)^{-\frac{\delta}{2}}\|_{\mathscr{L}(L^2,L^\infty)}\\
&\qquad\overset{(ii)}{\lesssim}\|(I-\partial_x^2)^{-\frac{s}{2}+\frac{\delta}{2}}(M_{g(\cdot,u)}-M_{g(\cdot,v)})\|_{\mathscr{L}(L^\xi,L^\infty)}\\
&\qquad\overset{(iii)}{\lesssim}\|M_{g(\cdot,u)}-M_{g(\cdot,v)}\|_{\mathscr{L}(L^\xi,L^\xi)}\\
&\qquad\leq\|g(\cdot,u)-g(\cdot,v)\|_{L^\infty}\overset{(iv)}{\lesssim_r}\|u-v\|_{X^{\mathsf{Tr}}_{\kappa,p}};
\end{aligned}$$

where in (i) we used [108, Corollary 9.3.3], in (ii) we used that $(1-\partial_x^2)^{-\frac{\delta}{2}}:L^2(\mathbb{T})\to H^{\delta,2}(\mathbb{T})\hookrightarrow L^\xi(\mathbb{T})$ as mentioned before. In (iii) we used $(1-\partial_x^2)^{-\frac{s}{2}+\frac{\delta}{2}}:L^\xi(\mathbb{T})\to H^{s-\delta,\xi}(\mathbb{T})\hookrightarrow L^\infty(\mathbb{T})$ and Sobolev embedding with $s-\delta-\frac{1}{\xi}=s-\frac{1}{2}>0$. Finally, (iv) follows from Assumption 5.2.20(3) and (5.85). Thus, (HG′) is verified by setting $G_{\mathsf{Tr}}=G$, $G_L=G_c=0$.

The following is the main result of this subsection.

Theorem 5.2.22. *Assume that the Assumption 5.2.20 holds. Let $s>\frac{1}{2}$ be such that $1-2s+2\delta-2\frac{(1+\kappa)}{p}>\frac{1}{q}$. Set $s_\delta:=1-s+\delta$. Then for each*

$$u_0\in L^0_{\mathscr{F}_0}(\Omega;B^{s_\delta-2\frac{1+\kappa}{p}}_{q,p}(\mathbb{T})),$$

there exists a maximal local solution to (5.83). Moreover, there exists a localizing sequence $(\sigma_n)_{n\geq1}$ such that a.s. for all $n\geq1$

$$u\in L^p(I_{\sigma_n},w_\kappa;H^{s_\delta,q}(\mathbb{T}))\cap C(\overline{I}_{\sigma_n};B^{s_\delta-2\frac{1+\kappa}{p}}_{q,p}(\mathbb{T}))\cap C((0,\sigma_n];B^{s_\delta-\frac{2}{p}}_{q,p}(\mathbb{T})).$$

Proof. To apply Theorem 4.3.7 it remains to check the condition (HA) and (4.24).

To prove that A verify (HA), it is enough to note that for any $u\in X_1$, $r>0$ and $v_1,v_2\in X^{\mathsf{Tr}}_{\kappa,p}$ such that $\|v_1\|_{X^{\mathsf{Tr}}_{\kappa,p}},\|v_2\|_{X^{\mathsf{Tr}}_{\kappa,p}}<r$,

$$\begin{aligned}
\|A(v_1)u-A(v_2)u\|_{H^{-1-s+\delta,q}(\mathbb{T})}&\lesssim\|(a(v_1)-a(v_2))\partial_x u\|_{H^{-s+\delta,q}(\mathbb{T})}\\
&\overset{(i)}{\lesssim}\|a(v_1)-a(v_2)\|_{C^\eta(\mathbb{T})}\|\partial_x u\|_{H^{-s+\delta,q}(\mathbb{T})}\\
&\overset{(ii)}{\lesssim_r}\|v_1-v_2\|_{X^{\mathsf{Tr}}_{\kappa,p}}\|u\|_{H^{1+s-\delta,q}(\mathbb{T})},
\end{aligned}$$

where in (i) follows by combining $\eta > s - \delta$, by (5.85), and [195, Chapter 14, eq. (4.14)] (or [157, Proposition 3.8]) and (ii) by Assumption 5.2.20(2), (5.84) and (5.85).

It remains to check the stochastic maximal regularity assumption (4.24), where in this case $B = 0$. By Theorem 4.2.7 and Remark 4.2.9(3), it is enough to show that for any $N \geq 1$ there exists $\lambda_N > 0$ such that for any $w_0 \in L^\infty_{\mathscr{F}_0}(\Omega; X^{\mathsf{Tr}}_{\kappa,p})$ the operator $\lambda_N + A(w_0)$ has a bounded H^∞-calculus on $H^{-1-\varepsilon,q}(\mathbb{T})$ with angle $< \pi/2$ and the estimates of the H^∞-calculus are uniform in $\omega \in \Omega$. To see this, recall that by (5.85), $w_0 \in L^\infty(\Omega; C^\eta(\mathbb{T}))$. Let $s' > s$ such that $1 - 2s' + 2\delta - 2\frac{1+\kappa}{p} > \frac{1}{q}$ and $\eta > s' - \delta$. Combining the proof of [177, Theorem 6.4.3] and the multiplication property in [195, Chapter 14, eq. (4.14)] one can check that there exists $\lambda_N > 0$ such that $\lambda_N + A(w_0)$ is R-sectorial on $H^{-1-\rho+\delta,q}(\mathbb{T})$ with $\rho \in \{0, s'\}$ (see e.g. [177, Definition 4.4.1] or [108, Defintion 10.3.1]) with angle $< \pi/2$. As we have seen in the proof of Theorem 5.2.14, up to enlarging $\lambda_N > 0$, $\lambda_N + A(w_0)$ has a bounded H^∞-calculus on $H^{-1,q}(\mathbb{T})$. The claim follows by using the argument in [139, Theorem 5], choosing $A = \lambda_N + A(w_0)$, $B = 1 - \partial_x^2$ and replacing L^2, L^{p_0} by $H^{-1,q}(\mathbb{T}), H^{-1-s'+\delta,q}(\mathbb{T})$ respectively. □

Part II

Blow-up criteria and regularization

Chapter 6

Blow-up criteria for stochastic evolution equations

Let $(\Omega, \mathbb{F} = (\mathscr{F}_t)_{t\geq 0}, \mathscr{A}, \mathbb{P})$ and $\mathscr{P}$ be a filtered probability space and the progressive sigma algebra, respectively. Moreover, we denote by H and W_H a separable Hilbert space and a cylindrical Brownian motion in H, respectively.

In this chapter we pursue the analysis of quasilinear stochastic evolution equations initiated in Chapter 4. Here we provide sufficient conditions for $\sigma = T$ a.s. where σ is the 'explosion time' of the L^p_κ-maximal local solution (u, σ) to

$$\begin{cases} du + A(t,u)udt = F(t,u)dt + (B(t,u) + G(t,u))dW_H, \quad t \in \mathbb{R}_+, \\ u(0) = u_0. \end{cases} \tag{6.1}$$

Such conditions will be called blow-up criteria and will be of basic importance for the following chapters. In Subsection 6.3.2 we collect our blow-up criteria, and distinguish between the quasilinear and semilinear case. Additionally, we provide a decision tree for applying these criteria in both cases. In the case of semilinear equation, under an additional assumption on the nonlinearities, we prove a *Serrin*-type blow up criteria, which in applications to the stochastic Navier-Stokes equations, implies a stochastic version of the well-known Serrin criteria (see e.g. Theorem 1.4.3).

One of the main results of this chapter is the following blow-up criterium

$$\mathbb{P}\Big(\sigma < T, \lim_{t\uparrow\sigma} u(t) \text{ exists in } X^{\mathsf{Tr}}_{\kappa,p}\Big) = 0, \text{ provided } X^{\mathsf{Tr}}_{\kappa,p} \text{ is not critical for (6.1)}. \tag{6.2}$$

If $\kappa = 0$, then the condition '$X^{\mathsf{Tr}}_{\kappa,p}$ is not critical for (6.1)' in (6.2) can be removed. As a by-product of the latter case, we show that σ is a *predictable* stopping time (see Corollary 6.3.9). Due to (6.2), to obtain $\sigma = T$ a.s. it is enough to prove [$\lim_{t\uparrow\sigma} u(t)$ exists in $X^{\mathsf{Tr}}_{\kappa,p}$ a.s.] In applications to SPDEs, this typically requires the proof of a-priori estimates. A futher use of blow-up criteria will be given in the next chapter where we provide a new bootstrapping machinary for proving regularization of solutions to SPDEs which is even new in the deterministic setting.

This chapter is the most technical of the thesis. The proof of the blow-up criteria requires several reductions and splitting arguments. Moreover, our strategy also requires some new results on stochastic maximal L^p-regularity and the study of the class $\mathcal{SMR}^{\bullet}_{p,\kappa}(\sigma, T)$ where σ is a stopping time. All the proofs rely on a contradiction argument which in the case of (6.2) with $\kappa = 0$ reads as follows. Let $\mathcal{O} = \{\sigma < T, \lim_{t\uparrow\sigma} u(t) \text{ exists in } X^{\mathsf{Tr}}_p\}$. If $\mathbb{P}(\mathcal{O}) > 0$, then we prove that the maximal L^p_0-local solution (u, σ) can be extended to a L^p_0-local solution $(\widetilde{u}, \widetilde{\sigma})$ such that $\mathbb{P}(\widetilde{\sigma} > \sigma) > 0$. The latter contradicts the maximality of (u, σ) and therefore $\mathbb{P}(\mathcal{O}) = 0$. The remaining criteria will be demonstrated in a concatenated manner in the sense that each will be deduced by contradiction from the previous one.

We attempt to make this chapter as self-contained as possible w.r.t. Chapter 4 provided the local existence theory for (6.1) is taken for granted. This chapter is organised as follows. In Section

6.1 we present some preliminary results which will be useful in the main proofs. In Section 6.2 we revise the theory of stochastic maximal L^p-regularity introducing the class $\mathcal{SMR}^{\bullet}_{p,\kappa}(\sigma,T)$ where σ is a stopping time. Here we prove a general 'method of continuity' for such class and we prove an useful perturbation result. Sections 6.3 and 6.4 are devoted to the statement and the proof of the main blow-up criteria, respectively.

The content of this chapter are taken from Sections 3-5 of my work [4].

6.1 Preliminaries

Here we collect some notation and some known results. Here we also state some of the results preven in the previous chapters in order to make Part II as independent as possible of Part I.

6.1.1 Weighted function spaces

Let $p\in(1,\infty)$, $\kappa\in(-1,p-1)$ and for $a\geq 0$, we denote by w_κ^a the shifted power weight

$$w_\kappa^a(t):=|t-a|^\kappa,\quad t\in\mathbb{R},\qquad w_\kappa:=w_\kappa^0. \tag{6.3}$$

For $I=(a,b)$ where $0\leq a<b\leq\infty$ and $\theta\in(0,1)$, define the following spaces:

- $L^p(I,w_\kappa^a;X)$ is the set of all strongly measurable functions $f:I\to X$ such that
$$\|f\|_{L^p(I,w_\kappa^a;X)}:=\Big(\int_a^b\|f(t)\|_X^p w_\kappa^a(t)dt\Big)^{1/p}<\infty.$$
- $W^{1,p}(I,w_\kappa^a;X)$ is the subspace of $L^p(I,w_\kappa^a;X)$ such that the weak derivative satisfies $f'\in L^p(I,w_\kappa^a;X)$. This space is endowed with the norm:
$$\|f\|_{W^{1,p}(I,w_\kappa^a;X)}:=\|f\|_{L^p(I,w_\kappa^a;X)}+\|f'\|_{L^p(I,w_\kappa^a;X)}.$$
- ${}_0W^{1,p}(I,w_\kappa^a;X)=\{f\in W^{1,p}(I,w_\kappa^a;X):f(a)=0\}$.
- $H^{\theta,p}(I,w_\kappa^a;X)=[L^p(I,w_\kappa^a;X),W^{1,p}(I,w_\kappa^a;X)]_\theta$ (complex interpolation).
- ${}_0H^{\theta,p}(I,w_\kappa^a;X)=[L^p(I,w_\kappa^a;X_1),{}_0W^{1,p}(I,w_\kappa^a;X)]_\theta$.
- For intervals $J\subseteq\overline{I}$ and $\mathcal{A}\in\{L^p,H^{\theta,p},W^{1,p}\}$, we denote by $\mathcal{A}_{loc}(J,w_\kappa^a;X)$ the set of all strongly measurable maps $f:J\to X$ such that $f\in\mathcal{A}(J',w_\kappa^a;X)$ for all bounded intervals J' with $\overline{J'}\subseteq J$.

If $\kappa=0$, the weight will be omitted from the notation. For the definition of $H^{\theta,p}$ and ${}_0H^{\theta,p}$ we used complex interpolation. For details on interpolation theory we refer to [20, 197] and [107, Appendix C].

In the case $\theta<\frac{1+\kappa}{p}$, the main result of [145, Section 6.2] (see Theorem 4.1.1 and the text below it for the case of bounded intervals) states that for all $0\leq a<b\leq\infty$,

$${}_0H^{\theta,p}(a,b,w_\kappa^a;X_{1-\theta})=H^{\theta,p}(a,b,w_\kappa^a;X_{1-\theta}) \tag{6.4}$$

with equivalent norms. This already played an important role in Chapter 4. In the current chapter it will play a key role in Subsection 6.4.4.

The following will be used many times in the manuscript. For each $f\in L^p(a,b,w_\kappa^a;X)$ and $\widetilde{a}\in(a,b]$ and $c\in[a,b)$, f is integrable on (a,b) since $\kappa\in(-1,p-1)$. Moreover, in the case that $\kappa\geq 0$,

$$\|f\|_{L^p(c,b;X)}\leq|c-a|^{-\kappa}\|f\|_{L^p(a,b,w_\kappa^a;X)}.$$

Let us collect some useful results in the following proposition.

Proposition 6.1.1. *Let X be a Banach space and $p \in (1,\infty)$. Let $0 \leq a \leq c < d \leq b < \infty$, $\kappa \in (-1, p-1)$, $\theta \in (0,1)$ and $\mathcal{A} \in \{{}_0H, H\}$. The following assertions hold.*

(1) *If $\kappa \geq 0$, then for each $f \in \mathcal{A}^{\theta,p}(a,b,w_\kappa^a;X)$, the following estimates hold:*

$$\begin{aligned} \|f\|_{\mathcal{A}^{\theta,p}(c,d,w_\kappa^a;X)} &\leq \|f\|_{\mathcal{A}^{\theta,p}(a,b,w_\kappa^a;X)}, \\ \|f\|_{\mathcal{A}^{\theta,p}(c,b;X)} &\leq (c-a)^{-\kappa/p}\|f\|_{\mathcal{A}^{\theta,p}(c,b,w_\kappa^a;X)}, \\ \|f\|_{H^{\theta,p}(c,b;X)} &\leq (c-a)^{-\kappa/p}\|f\|_{\mathcal{A}^{\theta,p}(a,b,w_\kappa^a;X)}. \end{aligned}$$

In particular $H^{\theta,p}(a,b,w_\kappa^a;X) \hookrightarrow H^{\theta,p}_{\mathrm{loc}}(a,b;X)$.

(2) *Let $a > 0$. Let ${}_0\mathsf{E}_a$ be the zero-extension operator from ${}_0W^{1,p}(a,b,w_\kappa^a;X)$ to ${}_0W^{1,p}(0,b,w_\kappa^a;X)$. Then ${}_0\mathsf{E}_a$ induces a contractive mapping*

$${}_0\mathsf{E}_a : {}_0H^{\theta,p}(a,b,w_\kappa^a;X) \to {}_0H^{\theta,p}(0,b,w_\kappa^a;X).$$

(3) *Let $1 < p \leq q < \infty$ and $\eta \in (-1, q-1)$. Assume that $\frac{1+\kappa}{p} > \frac{1+\eta}{q}$. Then $\mathcal{A}^{\theta,q}(a,b,w_\eta^a;X) \hookrightarrow \mathcal{A}^{\theta,p}(a,b,w_\kappa^a;X)$, and for all $f \in \mathcal{A}^{\theta,p}(a,b,w_\eta^a;X)$,*

$$\|f\|_{\mathcal{A}^{\theta,p}(a,b,w_\kappa^a;X)} \lesssim |b-a|^{\left(\frac{\kappa+1}{p}-\frac{\eta+1}{q}\right)}\|f\|_{\mathcal{A}^{\theta,q}(a,b,w_\eta^a;X)},$$

where the implicit constant depends only on p,q,η,κ.

(4) *Let $1 < p_0 \leq p_1 < \infty$, $\theta_0, \theta_1 \in (0,1)$ and $\kappa_i \in (-1, p_i - 1)$ for $i \in \{0,1\}$. Assume $\frac{\kappa_1}{p_1} \leq \frac{\kappa_0}{p_0}$ and $\theta_0 - \frac{1+\kappa_0}{p_0} \geq \theta_1 - \frac{1+\kappa_1}{p_1}$. Then for all $f \in \mathcal{A}^{\theta_0,p_0}(a,b,w_{\kappa_0}^a;X)$,*

$$\|f\|_{\mathcal{A}^{\theta_1,p_1}(a,b,w_{\kappa_1}^a;X)} \lesssim \|f\|_{\mathcal{A}^{\theta_0,p_0}(a,b,w_{\kappa_0}^a;X)}.$$

Proof. (1): This follow as in Proposition 2.2.2.

(2): One can check that ${}_0\mathsf{E}_a : {}_0H^{j,p}(a,b,w_\kappa^a;X) \to {}_0H^{j,p}(0,b,w_\kappa^a;X)$ is contractive for $j \in \{0,1\}$. Therefore, the claim follows by interpolation.

(3): We may assume $a = 0$ and $T := b < \infty$. Then for $f \in L^q(I_T, w_\eta; X)$,

$$\begin{aligned} \|f\|^p_{L^p(I_T,w_\kappa;X)} &= \int_0^T \big(t^{\frac{\eta}{q}}\|f(t)\|_X\big)^p t^{\kappa-\eta\frac{p}{q}}\,dt \\ &\leq \Big(\int_0^T t^{\frac{\kappa q-\eta p}{q-p}}\,dt\Big)^{\frac{q-p}{q}} \Big(\int_0^T t^\eta\|f(t)\|_X^q\,dt\Big)^{\frac{p}{q}} \\ &= C_{p,q,\kappa,\eta}T^{p\left(\frac{1+\kappa}{p}-\frac{1+\eta}{q}\right)}\|f\|^p_{L^q(I_T,w_\eta;X)}; \end{aligned} \tag{6.5}$$

where we used Hölder's inequality. Clearly, the same estimate holds for the first order Sobolev space. The general case follows by interpolation.

(4): See Proposition 4.1.2. □

Remark 6.1.2. In the above (3) is false in the limiting case $(1+\kappa)/p = (1+\eta)/q$ if $p > q$. This can be seen by taking $f(t) = t^{-\alpha}$ for an appropriate α.

We state a simple consequence of the above result.

Corollary 6.1.3. *Let X be a Banach space and $1 \leq q \leq p < \infty$. Let $0 \leq a < b < \infty$. Let $\varepsilon > 0$ and suppose that $\kappa \in (-1, p-1)$, $\eta \in (-1, q-1)$ and $\frac{1+\kappa}{p} < \varepsilon + \frac{1+\eta}{q}$. Then for each $\mathcal{A} \in \{{}_0H, H\}$,*

$$\mathcal{A}^{\theta,p}(a,b,w_\kappa^a;X) \hookrightarrow \mathcal{A}^{\theta-\varepsilon,q}(a,b,w_\eta^a;X), \qquad \theta \in [\varepsilon, 1].$$

The case $\frac{1+\kappa}{p} = \varepsilon + \frac{1+\eta}{q}$ is allowed provided $p = q$.

Proof. It suffices to consider $a = 0$ and $b = T$. Let $\mathcal{A} \in \{{}_0H, H\}$ and let $\theta \in [\varepsilon, 1]$ be fixed. We distinguish two cases.

Case (i): $\varepsilon < \frac{1+\kappa}{p}$. Let $\widetilde{\kappa} := \kappa - \varepsilon p$. Note that $\widetilde{\kappa} \le \kappa < p-1$ and $\widetilde{\kappa} > -1$ since $\varepsilon < \frac{1+\kappa}{p}$. Therefore, Proposition 6.1.1(4) and (3) (using $\frac{1+\widetilde{\kappa}}{p} = \frac{1+\kappa}{p} - \varepsilon < \frac{1+\eta}{q}$) give

$$\mathcal{A}^{\theta,p}(I_T, w_\kappa; X) \hookrightarrow \mathcal{A}^{\theta-\varepsilon,p}(I_T, w_{\widetilde{\kappa}}; X) \hookrightarrow \mathcal{A}^{\theta-\varepsilon,q}(I_T, w_\eta; X). \tag{6.6}$$

Case (ii): $\varepsilon \ge \frac{1+\kappa}{p}$. Take $\widetilde{\varepsilon} \in (0, \frac{1+\kappa}{p})$ such that $\frac{1+\kappa}{p} < \frac{1+\eta}{q} + \widetilde{\varepsilon}$. By the previous case, we have

$$\mathcal{A}^{\theta,p}(I_T, w_\kappa; X) \hookrightarrow \mathcal{A}^{\theta-\widetilde{\varepsilon},q}(I_T, w_\eta; X) \hookrightarrow \mathcal{A}^{\theta-\varepsilon,q}(I_T, w_\eta; X)$$

where the last inclusion follows by Proposition 6.1.1(4) and $\varepsilon > \widetilde{\varepsilon}$.

To prove the last claim, note that $\varepsilon < \frac{1+\kappa}{p}$ due to the assumption. Now the claim follows as in Case (i) if we omit the last embedding of (6.6). □

Next we recall some useful interpolation estimates.

Lemma 6.1.4 (Mixed derivative inequality)**.** *Let (X_0, X_1) be an interpolation couple of UMD spaces. Let $p_i \in (1,\infty)$, $\kappa_i \in (-1, p-1)$ and $s_i \in [0,1]$ for $i \in \{0,1\}$. For $\theta \in (0,1)$ set $s := s_0(1-\theta) + s_1\theta$. Then there exists a constant $C > 0$ such that for all $f \in \mathcal{A}_0(I_T, w_{\kappa_0}; X_0) \cap \mathcal{A}_1(I_T, w_{\kappa_1}; X_1)$*

$$\|f\|_{\mathcal{A}(I_T, w_\kappa; [X_0, X_1]_\theta)} \le C \|f\|^{1-\theta}_{\mathcal{A}_0(I_T, w_{\kappa_0}; X_0)} \|f\|^{\theta}_{\mathcal{A}_1(I_T, w_{\kappa_1}; X_1)},$$

in each of the following cases:

(1) $\mathcal{A} = H^{s,p}$, $\mathcal{A}_i = H^{s_i,p_i}$ *and* $s_i \in (0,1)$ *for* $i \in \{0,1\}$

(2) $\mathcal{A} = {}_0H^{s,p}$, $\mathcal{A}_i = {}_0H^{s_i,p_i}$ *and* $s_i \in (0,1)$ *for* $i \in \{0,1\}$*, provided* $s \neq \frac{1+\kappa}{p}$.

(3) $\mathcal{A} = H^{s,p}$, $\mathcal{A}_0 = L^p$, $\mathcal{A}_1 = W^{1,p}$, $s_0 = 0$, $s_1 = 1$,

where in case (2) the constant C can be chosen independently of T.

Proof. (1): See Proposition 4.1.3. The other cases follow by the same argument if one uses the extension operator of Proposition 2.2.4. □

6.1.2 An approximation result for sequence of stopping times

We will need the following approximation result for a sequence of stopping times.

Lemma 6.1.5. *Let $(\sigma_n)_{n\ge1}$ and σ be stopping times such that $0 \le \sigma_n < \sigma \le T$ for all $n \ge 1$ and* $\lim_{n\to\infty} \sigma_n = \sigma$ *a.s. Then for each $\varepsilon > 0$, there exists a sequence of stopping times $(\widetilde{\sigma}_n)_{n\ge1}$ such that the following assertion holds for each $n \ge 1$:*

(1) *$\widetilde{\sigma}_n$ takes values in a finite subset of* $[0,T]$*;*

(2) *$\widetilde{\sigma}_{n-1} \le \widetilde{\sigma}_n$ and $\sigma_n' \ge \sigma_n$ a.s.;*

(3) *$\mathbb{P}(\widetilde{\sigma}_n \ge \sigma) \le \varepsilon$.*

Proof. Let $\varepsilon > 0$. We approximate each σ_n from above in a suitable way. Since for each $n \ge 1$, one has $0 = \mathbb{P}(\sigma_n \ge \sigma) = \lim_{j\to\infty} \mathbb{P}(\sigma_n + 1/j > \sigma)$ it follows that there exists a $j_n \ge 1$ (also depending on ε) such that

$$\mathbb{P}(\sigma_n + 1/j_n > \sigma) \le 2^{-n}\varepsilon. \tag{6.7}$$

Let $0 = t_0^n < t_1^n < \ldots < t_{N_n}^n = T$ be such that $|t_i^n - t_{i+1}^n| < 1/j_n$ for each $i = 0, \ldots, N_n - 1$. Set $\mathcal{U}_i^n := \{t_i^n \le \sigma_n < t_{i+1}^n\} \in \mathscr{F}_{t_{i+1}^n}$ for $i = 0, \ldots, N_n - 1$. Let τ_n be the stopping time defined by

$$\tau_n := \sum_{i=0}^{N_n-1} t_{i+1}^n \mathbf{1}_{\mathcal{U}_i^n}.$$

Thus, $\sigma_n \leq \tau_n < \sigma_n + 1/j_n$ a.s. and by (6.7), $\mathbb{P}(\tau_n \geq \sigma) \leq \mathbb{P}(\sigma_n + 1/j_n > \sigma) < 2^{-n}\varepsilon$. Now for each $n \geq 1$, set $\widetilde{\sigma}_n := \tau_n \vee \tau_{n-1} \vee \cdots \vee \tau_1$. Then each $\widetilde{\sigma}_n$ takes values in a finite set. Moreover, $\widetilde{\sigma}_n \geq \widetilde{\sigma}_{n-1}$ and $\widetilde{\sigma}_n \geq \tau_n \geq \sigma_n$ for all $n \geq 1$ a.s. Finally,

$$\mathbb{P}(\widetilde{\sigma}_n \geq \sigma) \leq \sum_{i=1}^{n} \mathbb{P}(\tau_i \geq \sigma) \leq \sum_{i=1}^{\infty} 2^{-i}\varepsilon = \varepsilon.$$

□

Next, we introduce some stopped versions of the spaces we introduced in Section 6.1.1. As these definitions sometimes lead to measurability problems, we need to be careful here. This issue already appears in Chapter 4 (see Lemma 4.1.7 and Definition 4.1.8) for processes with random endpoint. The latter definitions can be extended easily to the case of random initial time σ provided it takes values in a finite set. In particular, a natural norm can be defined for the space

$$L^p(\Omega; \mathcal{A}^{\theta,p}(\sigma,\tau,w_\kappa^\tau;X)) \quad \text{for} \quad \mathcal{A} \in \{{}_0H, H\}, \qquad \text{and} \qquad L^p(\Omega; C([\sigma,\tau];X))$$

where X is a Banach space. Additionally, we will need some spaces of processes starting at a more general random time and this is defined below. If the starting random time takes values in a finite set, the definitions coincide.

We say that $f \in L^p(\Omega; {}_0H^{\theta,p}(\sigma,T;X))$ if $f : [\sigma,T] \times \Omega \to X$ is strongly measurable, $f \in {}_0H^{\theta,p}(\sigma,T;X)$ a.s. and ${}_0\mathsf{E}_\sigma f \in L^p(\Omega; {}_0H^{\theta,p}(I_T;X))$, where ${}_0\mathsf{E}_\sigma$ is the 0-extension operator in Proposition 6.1.1(2). Moreover, we set $\|f\|_{L^p(\Omega;{}_0H^{\theta,p}(\sigma,T;X))} := \|{}_0\mathsf{E}_\sigma f\|_{L^p(\Omega;{}_0H^{\theta,p}(I_T;X))}$.

Let $(Y_t)_{t\in I_T}$ be a family of function spaces such that for any $f : I_T \to Y$ and any $t \in I_T$, $f|_{(t,T)} \in Y_t$ and $t \mapsto \|f|_{(t,T)}\|_{Y_t}$ is continuous, we say $f \in L^p(\Omega; Y_\sigma)$ if there exists a strongly measurable map $\widetilde{f} : [\![0,T]\!] \to X$ such that $\widetilde{f}|_{(\!(\sigma,T)\!)} = f$ and $\|f\|_{L^p(\Omega;Y_\sigma)} := (\mathbb{E}\|\widetilde{f}|_{(\sigma,T)}\|^p_{Y_\sigma})^{1/p}$. The main spaces we need this for are $Y_t = C(\overline{I}_{T-t};X)$ and $Y_t = L^r(I_{T-t};X)$. In the above, we add the subscript $\mathscr{P}$ if in addition the process f is strongly progressively measurable. One can check that the spaces $L^p(\Omega;Y_\sigma)$ with $(Y_t)_{t\in I_T}$ as above, are Banach spaces.

6.2 Stochastic maximal L^p-regularity

In this section we revise the theory of stochastic maximal L^p-regularity of Section 3.1. We refer to Subsection 4.2.2 for examples.

Assumption 4.2.1 will be in force throughout Sections 6.2-7.1 where the abstract theory is studied.

6.2.1 Definitions and foundational results

In this section we recall and extends some definition given in Subsection 3.1. Let us begin with the following assumption which will be in force throughout Section 6.2.

Assumption 6.2.1. *Let $T \in (0,\infty]$ and $\sigma : \Omega \to [0,T]$ be a stopping time. The maps $A : [\![\sigma,T]\!] \to \mathscr{L}(X_1,X_0)$, $B : [\![\sigma,T]\!] \to \mathscr{L}(X_1,\gamma(H,X_{1/2}))$ are strongly progressively measurable. Moreover, there exists a constant $C_{A,B} > 0$ such that for a.a. $\omega \in \Omega$ and for all $t \in (\sigma(\omega),T)$,*

$$\|A(t,\omega)\|_{\mathscr{L}(X_1,X_0)} + \|B(t,\omega)\|_{\mathscr{L}(X_1,\gamma(H,X_{1/2}))} \leq C_{A,B}.$$

Stochastic maximal L^p-regularity is concerned with the optimal regularity estimate for the linear abstract stochastic Cauchy problem:

$$\begin{cases} du(t) + A(t)u(t)dt = f(t)dt + (B(t)u(t) + g(t))dW_H(t), & t \in [\![\sigma,T]\!], \\ u(\sigma) = u_\sigma. \end{cases} \tag{6.8}$$

A new feature is that we consider the initial condition at a random time σ since this will be needed in some of the proofs below.

The definition of strong solution to (6.8) is as follows: Let Assumptions 4.2.1 and 6.2.1 be satisfied. Let τ be a stopping time such that $\sigma \leq \tau \leq T$ a.s. Let

$$u_\sigma \in L^0_{\mathscr{F}_\sigma}(\Omega; X_0), \; f \in L^0_{\mathscr{P}}(\Omega; L^1(\sigma, \tau; X_0)), \; g \in L^0_{\mathscr{P}}(\Omega; L^2(\sigma, \tau; \gamma(H, X_0))).$$

A strongly progressive measurable map $u : [\![\sigma, \tau]\!] \to X_1$ is called a *strong solution* to (6.8) on $[\![\sigma, \tau]\!]$ if $u \in L^2(\sigma, \tau; X_1)$ a.s. and a.s. for all $t \in [\sigma, \tau]$,

$$u(t) - u_\sigma + \int_\sigma^t A(s)u(s)ds = \int_\sigma^t f(s)ds + \int_0^t \mathbf{1}_{[\![\sigma,\tau]\!]}(B(s)u(s) + g(s))dW_H(s).$$

If σ, τ are constants, then we simply write u is a strong solution to (6.8) on $[\sigma, \tau]$ instead of $[\![\sigma, \tau]\!]$. As in Section 3.1, we follow [174] to define stochastic maximal L^p-regularity. Recall that the norms at random times used below defined in Subsection 6.1.2.

Definition 6.2.2 (Stochastic maximal L^p-regularity)**.** *Let Assumptions 4.2.1 and 6.2.1 be satisfied. We write $(A, B) \in \mathcal{SMR}_{p,\kappa}(\sigma, T)$ if for every*

$$f \in L^p_{\mathscr{P}}((\!|\sigma, T|\!)), w^\sigma_\kappa; X_0), \qquad g \in L^p_{\mathscr{P}}((\!|\sigma, T|\!)), w^\sigma_\kappa; \gamma(H, X_{1/2})) \tag{6.9}$$

there exists a strong solution u to (6.8) with $u_\sigma = 0$ such that $u \in L^p_{\mathscr{P}}((\!|\sigma, T|\!)), w^\sigma_\kappa; X_1)$, and moreover for all stopping time τ, such that $\sigma \leq \tau \leq T$ a.s., and each strong solution $u \in L^p_{\mathscr{P}}((\!|\sigma, T|\!)), w^\sigma_\kappa; X_1)$ the following estimate holds

$$\|u\|_{L^p((\!|\sigma,\tau|\!)), w^\sigma_\kappa; X_1)} \leq C(\|f\|_{L^p((\!|\sigma,\tau|\!)), w^\sigma_\kappa; X_0)} + \|g\|_{L^p((\!|\sigma,\tau|\!)), w^\sigma_\kappa; \gamma(H, X_{1/2}))}).$$

where $C > 0$ is independent of (f, g, τ). We set $\mathcal{SMR}_p(\sigma, T) := \mathcal{SMR}_{p,0}(\sigma, T)$. Moreover, we write $A \in \mathcal{SMR}_{p,\kappa}(\sigma, T)$ if $(A, 0) \in \mathcal{SMR}_{p,\kappa}(\sigma, T)$.

If $(A, B) \in \mathcal{SMR}_{p,\kappa}(\sigma, T)$, then by the stated estimate a strong solution to (6.8) in $L^p((\sigma, T) \times \Omega, w^\sigma_\kappa; X_1)$ is unique.

Definition 6.2.3. *Let Assumptions 4.2.1 and 6.2.1 be satisfied.*

(1) *Let $p \in (2, \infty)$. In case $\kappa > 0$, suppose that σ takes values in a finite set. We write $(A, B) \in \mathcal{SMR}^\bullet_{p,\kappa}(\sigma, T)$ if $(A, B) \in \mathcal{SMR}_{p,\kappa}(\sigma, T)$ and for each f, g as in (6.9) the strong solution u to (6.8) with $u_\sigma = 0$ satisfies*

$$\|u\|_{L^p(\Omega; {}_0H^{\theta,p}(\sigma,T,w^\sigma_\kappa; X_{1-\theta}))} \leq C_\theta(\|f\|_{L^p((\!|\sigma,T|\!)), w^\sigma_\kappa; X_0)} + \|g\|_{L^p((\!|\sigma,T|\!)), w^\sigma_\kappa; \gamma(H, X_{1/2}))}),$$

for each $\theta \in [0, \frac{1}{2}) \setminus \{\frac{1+\kappa}{p}\}$, where C_θ is independent of f, g, τ.

(2) *Let $p = 2$ and $\kappa = 0$. We write $(A, B) \in \mathcal{SMR}^\bullet_{2,0}(\sigma, T)$ if $(A, B) \in \mathcal{SMR}_{2,0}(\sigma, T)$ and for each f, g as in (6.9) the strong solution u to (6.8) with $u_\sigma = 0$ satisfies*

$$\|u\|_{L^2(\Omega; C([\sigma,T]; X_{1/2}))} \leq C(\|f\|_{L^2((\!|\sigma,T|\!)); X_0)} + \|g\|_{L^2((\!|\sigma,T|\!)); \gamma(H, X_{1/2}))}),$$

where C is independent of f, g, τ.

Furthermore, we say that $A \in \mathcal{SMR}^\bullet_{p,\kappa}(\sigma, T)$ if $(A, 0) \in \mathcal{SMR}^\bullet_{p,\kappa}(\sigma, T)$ and we set

$$\mathcal{SMR}^\bullet_p(\sigma, T) := \mathcal{SMR}^\bullet_{p,0}(\sigma, T).$$

In the above setting we consider the mapping

$$u = \mathscr{R}_{\sigma,(A,B)}(0, f, g). \tag{6.10}$$

In (6.17) below the mapping will be extended to nonzero initial data. For $p > 2$ and $\kappa \in [0, \frac{p}{2} - 1)$ (where σ takes values in a finite set in the case $\kappa > 0$), and $\theta \in [0, \frac{1}{2}) \setminus \{\frac{1+\kappa}{p}\}$, $\mathscr{R}_{\sigma,(A,B)}(0, \cdot, \cdot)$ defines a mapping

$$L^p_{\mathscr{P}}((\!|\sigma, T|\!), w^\sigma_\kappa; X_0) \times L^p_{\mathscr{P}}((\!|\sigma, T|\!), w^\sigma_\kappa; \gamma(H, X_{1/2})) \to L^p(\Omega; {}_0H^{\theta,p}(\sigma, T; X_{1-\theta})).$$

Moreover, we define the constants

$$C^{\det,\theta,p,\kappa}_{(A,B)}(\sigma, T) = \|\mathscr{R}_{\sigma,(A,B)}(0, \cdot, 0)\|_{L^p_{\mathscr{P}}((\!|\sigma,T|\!),w^\sigma_\kappa;X_0)) \to L^p(\Omega;{}_0H^{\theta,p}(\sigma,T,w^\sigma_\kappa;X_{1-\theta}))},$$
$$C^{\mathsf{sto},\theta,p,\kappa}_{(A,B)}(\sigma, T) = \|\mathscr{R}_{\sigma,(A,B)}(0, 0, \cdot)\|_{L^p_{\mathscr{P}}((\!|\sigma,T|\!),w^\sigma_\kappa;\gamma(H,X_{1/2})) \to L^p(\Omega;{}_0H^{\theta,p}(\sigma,T,w^\sigma_\kappa;X_{1-\theta}))},$$

where in the case $p = 2$, $\kappa = 0$, $\theta \in (0, \frac{1}{2})$, we replace the range space by $L^2(\Omega; C([\sigma, T]; X_{1/2}))$ (which is constant in $\theta \in (0, \frac{1}{2})$). Moreover, we set

$$\begin{aligned} K^{\det,\theta,p,\kappa}_{(A,B)}(\sigma, T) &= C^{\det,\theta,p,\kappa}_{(A,B)}(\sigma, T) + C^{\det,0,p,\kappa}_{(A,B)}(\sigma, T), \\ K^{\mathsf{sto},\theta,p,\kappa}_{(A,B)}(\sigma, T) &= C^{\mathsf{sto},\theta,p,\kappa}_{(A,B)} + C^{\mathsf{sto},0,p,\kappa}_{(A,B)}(\sigma, T). \end{aligned} \tag{6.11}$$

Remark 6.2.4.

(1) Definition 6.2.3 (1) reduces to the one in Section 3.1 in case $\sigma = 0$. The case $\theta = \frac{1+\kappa}{p}$ is not considered, since the concrete description of the interpolation space is more complicated and not considered in [145, Theorem 6.5]. In case $\sigma = 0$, this case was included in Section 3.1 as it can be obtained by complex interpolation. This becomes unclear for random times σ.

(2) The stopping time σ in Definition 6.2.3(1) is assumed to take only takes finitely many values and this is sufficient for our purposes. We avoid the general case due to nontrivial measurability problems.

The following basic result can be proved in a similar way as in Proposition 4.2.8. It allows us to focus on proving $(A, B) \in \mathcal{SMR}_{p,\kappa}(\sigma, T)$ and obtain the stronger result $(A, B) \in \mathcal{SMR}^\bullet_{p,\kappa}(\sigma, T)$ almost for free.

Proposition 6.2.5 (Transference of stochastic maximal regularity)**.** *Let Assumptions 4.2.1 and 6.2.1 be satisfied. Let $\sigma : \Omega \to [0, T]$ be a stopping time which takes values in a finite set if $\kappa > 0$. If $(A, B) \in \mathcal{SMR}_{p,\kappa}(\sigma, T)$ and $\mathcal{SMR}^\bullet_{p,\kappa}(\sigma, T)$ is nonempty, then $(A, B) \in \mathcal{SMR}^\bullet_{p,\kappa}(\sigma, T)$.*

In order to check that $\mathcal{SMR}^\bullet_{p,\kappa}(\sigma, T)$ is nonempty we can often use Theorem 4.2.7.

6.2.2 Random initial times

In this section we study the role of random initial times. We will start by considering linear problem (6.8) for non-trivial initial data at a random initial time. A similar result was proved in Proposition 4.2.10 for fixed times, but without taking care of the dependence on the length of the time interval, which turns out to be important here. Therefore, we have to provide a detailed proof. The construction in the proof below will be used later on.

Proposition 6.2.6 (Nonzero initial data)**.** *Suppose that Assumptions 4.2.1 and 6.2.1 be satisfied. Let $(A, B) \in \mathcal{SMR}_{p,\kappa}(\sigma, T)$. Then for any $u_\sigma \in L^p_{\mathscr{F}_\sigma}(\Omega; X^{\mathsf{Tr}}_{\kappa,p})$, $f \in L^p_{\mathscr{P}}((\!|\sigma, T|\!), w^\sigma_\kappa; X_0)$, and $g \in L^p_{\mathscr{P}}((\!|\sigma, T|\!), w^\sigma_\kappa; \gamma(H, X_{1/2}))$. Then there exists a unique strong solution $u \in L^p_{\mathscr{P}}((\sigma, T) \times \Omega, w^s_\kappa; X_1)$ to (6.8) on $[\![\sigma, T]\!]$, and*

$$\begin{aligned} \|u\|_{L^p((\sigma,T)\times\Omega,w^\sigma_\kappa;X_1)} &\leq C\|u_\sigma\|_{L^p(\Omega;X^{\mathsf{Tr}}_{\kappa,p})} \\ &\quad + C\|f\|_{L^p((\!|\sigma,T|\!),w^\sigma_\kappa;X_0)} + C\|g\|_{L^p((\!|\sigma,T|\!),w^\sigma_\kappa;\gamma(H,X_{1/2}))} \end{aligned} \tag{6.12}$$

where C only dependents on $\widetilde{A}, p, K^{j,\theta,p,\kappa}_{(A,B)}(\sigma, T)$.

If additionally $(A, B) \in \mathcal{SMR}^\bullet_{p,\kappa}(\sigma, T)$ and σ takes finitely many values if $\kappa > 0$, then the left-hand side of (6.12) can be replaced by

(1) $\|u\|_{L^p(\Omega;C([\sigma,T];X^{\mathsf{Tr}}_{\kappa,p}))}$;

(2) $\|u\|_{L^p(\Omega;C([\sigma+\delta,T];X^{\mathsf{Tr}}_{\kappa,p}))}$ *if* $\delta>0$;

(3) $\|u\|_{L^p(\Omega;H^{\theta,p}(\sigma,T,w_\kappa^\sigma;X_{1-\theta}))}$ *if* $\theta\in[0,\frac12)\setminus\frac{1+\kappa}{p}$ *and* $p\in(2,\infty)$;

(4) $\|u\|_{L^p(\Omega;{}_0H^{\theta,p}(\sigma,T,w_\kappa^\sigma;X_{1-\theta}))}$ *if* $\theta\in(0,\frac{1+\kappa}{p})$ *and* $p\in(2,\infty)$;

where C *also depends on* δ *(resp.* θ*) in* (2) *(resp.* (3) *and* (4)*).*

The norms on random intervals in the above result are defined as at the end of Subsection 6.1.2. Part (4) is an estimate in terms of the ${}_0H$-norms, and will only play a role in the proof of Theorem 6.3.7(4). Note that due to (6.4), no trace restrictions is needed in (4).

Proof. For the reader's convenience, we split the proof into several steps. We only consider the case $p>2$, since the case $p=2$ is simpler.

Below we will use the so-called trace method of interpolation. By [20, Theorem 3.12.2] or [197, Theorem 1.8.2, p. 44], $X^{\mathsf{Tr}}_{\kappa,p}$ is the set of all $x\in X_0+X_1$ such that $x=h(0)$ for some $h\in W^{1,p}(\mathbb{R}_+,w_\kappa;X_0)\cap L^p(\mathbb{R}_+,w_\kappa;X_1)$ and

$$C^{-1}_{p,\kappa}\|x\|_{X^{\mathsf{Tr}}_{\kappa,p}}\leq \inf\|h\|_{W^{1,p}(\mathbb{R}_+,w_\kappa;X_0)\cap L^p(\mathbb{R}_+,w_\kappa;X_1)}\leq C_{p,\kappa}\|x\|_{X^{\mathsf{Tr}}_{\kappa,p}}, \tag{6.13}$$

where the infimum is taken over all h as above and where C only depends on p and κ.

Step 1: the case $(A,B)\in\mathcal{SMR}_{p,\kappa}(\sigma,T)$. Uniqueness is clear from $(A,B)\in\mathcal{SMR}_{p,\kappa}(\sigma,T)$. By completeness and density (cf. Proposition 4.2.10), it is enough to prove the claim for $u_\sigma=\sum_{j=1}^N \mathbf{1}_{\mathcal{U}_j}x_j$ where $N\geq1$, $x_1,\dots,x_N\in X^{\mathsf{Tr}}_{\kappa,p}$, and $(\mathcal{U}_i)_{i=1}^N$ in $\mathscr{F}_\sigma$ is a partition of Ω. By (6.13) there exist $h_j\in W^{1,p}(\mathbb{R}_+,w_\kappa;X_0)\cap L^p(\mathbb{R}_+,w_\kappa;X_1)$ such that $h_j(0)=x_j$ and

$$\|h_j\|_{W^{1,p}(\mathbb{R}_+,w_\kappa;X_0)\cap L^p(\mathbb{R}_+,w_\kappa;X_1)}\leq 2C_{p,\kappa}\|x_j\|_{X^{\mathsf{Tr}}_{\kappa,p}},\quad j\in\{1,\dots,N\}.$$

Then $v_1:=\sum_{j=1}^N\mathbf{1}_{\mathcal{U}_j}h_j(\cdot-\sigma)$ on $[\![\sigma,\infty[\![$ is strongly progressively measurable. It follows that u is a strong solution to (6.8) if and only if $v_2:=u-v_1$ is a strong solution to

$$\begin{cases} dv_2+Av_2dt=[f-\dot v_1-Av_1]dt+(Bv_2+Bv_1+g)dW_H,\\ v_2(\sigma)=0.\end{cases} \tag{6.14}$$

Let $\mathcal{A}^{\theta,p}=H^{\theta,p}$, or $\mathcal{A}={}_0H^{\theta,p}$ if $\theta\in(0,(1+\kappa)/p)$. Note that on $\mathcal{U}_j$,

$$\begin{aligned}\|v_1\|_{\mathcal{A}^{\theta,p}(\sigma,T,w_\kappa^\sigma;X_{1-\theta})}&\overset{(i)}{\leq}\|t\mapsto h_j(t-\sigma)\|_{\mathcal{A}^{\theta,p}(\sigma,\infty,w_\kappa^\sigma;X_{1-\theta})}\\ &\overset{(ii)}{\leq}\|h_j\|_{\mathcal{A}^{\theta,p}(\mathbb{R}_+,w_\kappa;X_{1-\theta})}\\ &\overset{(iii)}{\leq}C_\theta\|h_j\|_{H^{\theta,p}(\mathbb{R}_+,w_\kappa;X_{1-\theta})}\\ &\overset{(iv)}{\leq}C_\theta\|h_j\|_{L^p(\mathbb{R}_+,w_\kappa;X_1)}+C_\theta\|h_j\|_{W^{1,p}(\mathbb{R}_+,w_\kappa;X_0)}\\ &\overset{(v)}{\leq}2C_\theta C_{p,\kappa}\|u_\sigma\|_{X^{\mathsf{Tr}}_{\kappa,p}},\end{aligned}$$

where in (i) we used Proposition 6.1.1(1), and in (ii) a translation argument. In (iii) we used (6.4) if $\mathcal{A}^{\theta,p}={}_0H^{\theta,p}$. Finally, in (v) we used the choice of h_j. Note that (iii) can be avoided if $\mathcal{A}^{\theta,p}=H^{\theta,p}$.

In the following we set $C'_\theta:=2C_\theta C_{p,\kappa}$. If $\theta\in\{0,1\}$, then taking $L^p(\Omega)$-norms we obtain

$$\|v_1\|_{L^p(\Omega;H^{\theta,p}(\sigma,T,w_\kappa^\sigma;X_{1-\theta}))}\leq C'_\theta\|u_\sigma\|_{L^p_{\mathscr{F}_\sigma}(\Omega;X^{\mathsf{Tr}}_{\kappa,p})}, \tag{6.15}$$

Since v_2 satisfies (6.14) and $(A,B) \in \mathcal{SMR}_{p,\kappa}(\sigma,T)$, setting $C_1^\theta = C_{(A,B)}^{\det,\theta,p,\kappa}(\sigma,T)$ and $C_2^\theta = C_{(A,B)}^{\mathsf{sto},\theta,p,\kappa}(\sigma,T)$ it follows that

$$\begin{aligned}
&\|v_2\|_{L^p((\!|\sigma,T|\!),w_\kappa^\sigma;X_1)} \\
&\quad\le C_1^0\|f+\dot v_1 - Av_1\|_{L^p((\!|\sigma,T|\!),w_\kappa^\sigma;X_0)} + C_2^0\|Bv_1+g\|_{L^p((\!|\sigma,T|\!),w_\kappa^\sigma;\gamma(H,X_{1/2}))}\\
&\quad\le \widetilde C_0 C_1^0\|u_\sigma\|_{L^p(\Omega;X^{\mathsf{Tr}}_{\kappa,p})} + C_1^0\|f\|_{L^p((\!|\sigma,T|\!),w_\kappa^\sigma;X_0)} + C_2^0\|g\|_{L^p((\!|\sigma,T|\!),w_\kappa^\sigma;\gamma(H,X_{1/2}))},
\end{aligned}$$

where in the last step we used (6.15). Combining the estimates for v_1 and v_2, we obtain (1).

Next suppose $(A,B) \in \mathcal{SMR}^\bullet_{p,\kappa}(T)$. In the same way as in Step 1, by (6.15) for each $\theta \in [0,\frac12)\setminus\{\frac{1+\kappa}{p}\}$,

$$\begin{aligned}
\|v_2\|_{L^p(\Omega;{}_0H^{\theta,p}(\sigma,T,w_\kappa^\sigma;X_{1-\theta}))} \le \widetilde C_0 C_1^0\|u_\sigma\|_{L^p(\Omega;X^{\mathsf{Tr}}_{\kappa,p})} + C_1^\theta\|f\|_{L^p((\!|\sigma,T|\!),w_\kappa^\sigma;X_0)}&\\
+ C_2^\theta\|g\|_{L^p((\!|\sigma,T|\!),w_\kappa^\sigma;\gamma(H,X_{1/2}))}.&
\end{aligned}\tag{6.16}$$

Combining the estimates, we obtain (3) and (4), where for (3) one additionally needs to use that ${}_0H^{\theta,p} \hookrightarrow H^{\theta,p}$, contractively.

The maximal estimate in (1) follows by considering v_1 and v_2 separately again. Indeed, by Proposition 4.1.5(1) applied to each h_j we obtain

$$\|v_1\|_{C([\sigma,T];X^{\mathsf{Tr}}_{\kappa,p})} \le C\|u_\sigma\|_{X^{\mathsf{Tr}}_{\kappa,p}}, \qquad \text{a.s.}$$

The estimate for v_2 follows by combining (6.16) for $\theta=0$ and $\theta\in((1+\kappa)/p,1/2)$ with Proposition 4.1.5. To prove (2), one can argue similarly using Proposition 4.1.5(2) instead of Proposition 4.1.5(1). ☐

By the above we can extend the solution operator, defined for $u_\sigma = 0$ in (6.10), to nonzero initial values by setting

$$\mathscr{R}_{\sigma,(A,B)}(u_\sigma,f,g) := u, \tag{6.17}$$

where u is the unique strong solution to (6.8) on $[\![\sigma,T]\!]$. This defines a mapping from

$$L^p_{\mathscr{F}_\sigma}(\Omega;X^{\mathsf{Tr}}_{\kappa,p}) \times L^p_{\mathscr{P}}((\!|\sigma,T|\!),w_\kappa^\sigma;X_0) \times L^p_{\mathscr{P}}((\!|\sigma,T|\!),w_\kappa^\sigma;\gamma(H,X_{1/2})) \tag{6.18}$$

into $L^p((\!|\sigma,T|\!),w_\kappa^\sigma;X_0)$.

The following result can be obtained as in Proposition 4.2.12.

Proposition 6.2.7 (Localization and causality)**.** *Let Assumptions 4.2.1 and 6.2.1 be satisfied. Let $(A,B)\in\mathcal{SMR}_{p,\kappa}(\sigma,T)$. Let σ,τ be stopping times such that $\sigma\le\tau\le T$ a.s. Assume that (u_σ,f,g) belongs to the space in (6.18) and $u := \mathscr{R}_{\sigma,(A,B)}(u_\sigma,f,g)$. Then the following holds:*

(1) *If σ is a stopping time with values in I_T, then for any $F\in\mathscr{F}_\sigma$ one has*

$$\mathbf{1}_F u = \mathbf{1}_F\mathscr{R}_{\sigma,(A,B)}(\mathbf{1}_F u_\sigma, \mathbf{1}_F f, \mathbf{1}_F g), \qquad \textit{a.s. on } [\![s,T]\!].$$

(2) *For any strong solution $v\in L^p((\!|\sigma,\tau|\!),w_\kappa^\sigma;X_1)$ to (6.8), one has*

$$v = u|_{[\![\sigma,\tau]\!]} = \mathscr{R}_{\sigma,(A,B)}(u_\sigma, \mathbf{1}_{(\!|\sigma,\tau|\!)}f, \mathbf{1}_{(\!|\sigma,\tau|\!)}g), \qquad \textit{a.s. on } [\![\sigma,\tau]\!],$$

where we can replace $(\!|\sigma,\tau|\!)$ by its half open versions or closed version as well.

Note that the last assertion follows from the fact that the symmetric difference of the different sorts of intervals has zero Lebesgue measure.

Next we show that one can combine operators (A_j,B_j) at discrete random times to obtain an operator in $\mathcal{SMR}^\bullet_{p,\kappa}(\sigma,T)$.

Proposition 6.2.8 (Sufficient conditions for SMR at random initial times)**.** *Let Assumption 4.2.1 be satisfied. Suppose that* $\sigma = \sum_{j=1}^N \mathbf{1}_{\mathcal{U}_j} s_j$, *for* $N \in \mathbb{N}$, *where* $(s_j)_{j=1}^N$ *is in* $(0,T)$, *and* $(U_j)_{j=1}^N$ *is a partition of* Ω *with* $\mathcal{U}_j \in \mathscr{F}_{s_j}$ *for* $j \in \{1,\dots,N\}$. *Given* $(A_j, B_j) \in \mathcal{SMR}^{\bullet}_{p,\kappa}(s_j,T)$ *for* $j \in \{1,\dots,N\}$ *satisfying Assumption 6.2.1, set*

$$A := \sum_{j=1}^N \mathbf{1}_{\mathcal{U}_j\times[s_j,T]} A_j, \quad \text{and} \quad B := \sum_{j=1}^N \mathbf{1}_{\mathcal{U}_j\times[s_j,T]} B_j$$

Then $(A,B) \in \mathcal{SMR}^{\bullet}_{p,\kappa}(\sigma,T)$ *and for each* $i \in \{\mathsf{det},\mathsf{sto}\}$ *and* $\theta \in [0,\frac{1}{2}) \setminus \{\frac{1+\kappa}{p}\}$

$$K^{\theta,i,p,\kappa}_{(A,B)}(\sigma,T) \leq \max_{j\in\{1,\dots,N\}} K^{\theta,i,p,\kappa}_{(A_j,B_j)}(\sigma,T).$$

Proof. We only consider $p>2$. Let $\mathscr{R}_j := \mathscr{R}_{s_j,(A_j,B_j)}$ be the solution operator associated to (A_j,B_j). Using Proposition 6.2.7, one can check that the unique solution to (6.8) with f,g as in (6.9) and $u_\sigma = 0$ is given by

$$u := \sum_{j=1}^N \mathbf{1}_{\mathcal{U}_j} \mathscr{R}_j(0,f,g) = \sum_{j=1}^N \mathbf{1}_{\mathcal{U}_j} \mathscr{R}_j(0,\mathbf{1}_{\mathcal{U}_j} f,\mathbf{1}_{\mathcal{U}_j} g), \tag{6.19}$$

Therefore, if $f=0$, setting $K = \max_{j\in\{1,\dots,N\}} K^{\theta,i,p,\kappa}_{(A_j,B_j)}(s_j,T)$ we obtain

$$\begin{aligned}
\|u\|^p_{L^p_{\mathscr{P}}(\Omega;{}_0H^{\theta,p}(\sigma,T,w^\sigma_\kappa;X_{1-\theta}))} &= \sum_{j=1}^N \mathbb{E}[\mathbf{1}_{\mathcal{U}_j}\|\mathscr{R}_j(0,0,\mathbf{1}_{\mathcal{U}_j} g)\|^p_{{}_0H^{\theta,p}(s_j,T,w^{s_j}_\kappa;X_{1-\theta})}] \\
&\leq K^p \Big(\sum_{j=1}^N \mathbb{E}[\mathbf{1}_{\mathcal{U}_j}\|g\|^p_{L^p(s_j,T,w^{s_j}_\kappa;\gamma(H,X_{1/2}))}]\Big) \\
&= K^p \|g\|^p_{L^p((\!(\sigma,T]\!],w^\sigma_\kappa;\gamma(H,X_{1/2}))}.
\end{aligned}$$

This proves the required estimate for $K^{\theta,\mathsf{sto},p,\kappa}_{(A,B)}$. The other case is similar. □

We end this subsection with a result which shows that weighted maximal regularity implies unweighted maximal regularity for a shifted problem. Although in applications it is usually obvious that the latter holds, from a theoretical perspective it has some interest that weighted maximal regularity can be sufficient. It can be used to check Assumption 6.3.2 for blow-up criteria.

Proposition 6.2.9. *Let Assumptions 4.2.1 and 6.2.1 be satisfied. Let* τ *be a stopping time such that* $\sigma \leq \tau \leq T$ *a.s. Assume that one of the following conditions holds:*

- $\kappa = 0$.
- $\kappa > 0$, *σ takes values in a finite set, where we suppose* $s := \inf\{\tau(\omega) - \sigma(\omega) : \omega \in \Omega\} > 0$ *and* $r := \sup\{T - \sigma(\omega) : \omega \in \Omega\}$.

If $(A,B) \in \mathcal{SMR}^{\bullet}_{p,\kappa}(\sigma,T)$, *then* $(A,B) \in \mathcal{SMR}^{\bullet}_{p}(\tau,T)$, *and*

$$K^{j,\theta,p,0}_{(A,B)}(\tau,T) \leq s^{-\kappa/p} r^{\kappa/p} K^{j,\theta,p,\kappa}_{(A,B)}(\sigma,T),$$

for $j \in \{\mathsf{det},\mathsf{sto}\}$ *and* $\theta \in [0,1/2)$.

Proof. To prove the proposition, we only consider the case $p>2$ as the other case is simpler. Let $u = \mathscr{R}_{\sigma,(A,B)}(0,\mathbf{1}_{[\![\tau,T]\!]} f, \mathbf{1}_{[\![\tau,T]\!]} g)$. Then Proposition 6.2.7 and the assumption on (A,B) imply that $u|_{[\![\tau,T]\!]}$ is the unique strong solution to

$$du + Audt = fdt + (Bu+g)dW_H, \qquad u(\tau) = 0,$$

and $u = 0$ on $[\![\sigma, \tau]\!]$.

If $\sigma < \tau$, then combining two estimates in Proposition 6.1.1(1) we obtain

$$\|u\|_{{}_0H^{\theta,p}(\tau,T;X_{1-\theta})} \leq s^{-\kappa/p}\|u\|_{{}_0H^{\theta,p}(\tau,T,w_\kappa^\sigma;X_{1-\theta})} \leq s^{-\kappa/p}\|u\|_{{}_0H^{\theta,p}(\sigma,T,w_\kappa^\sigma;X_{1-\theta})}.$$

Clearly, the latter still holds with constant one if $\sigma = \tau$ and $\kappa = 0$.

Therefore, by the assumption on (A, B) for each $\theta \in [0, \frac{1}{2}) \setminus \{\frac{1+\kappa}{p}\}$, we obtain

$$\begin{aligned}
&\|u\|_{L^p(\Omega;{}_0H^{\theta,p}(\tau,T;X_{1-\theta}))}\\
&\quad\leq s^{-\kappa/p}\|u\|_{L^p(\Omega;{}_0H^{\theta,p}(\sigma,T,w_\kappa^\sigma;X_{1-\theta}))}\\
&\quad\leq s^{-\kappa/p}(K_1\|\mathbf{1}_{[\![\tau,T]\!]}f\|_{L^p((\!|\sigma,T|\!),w_\kappa^\sigma;X_0)} + K_2\|\mathbf{1}_{[\![\tau,T]\!]}g\|_{L^p((\!|\sigma,T|\!),w_\kappa^\sigma;\gamma(H,X_{1/2}))})\\
&\quad\leq s^{-\kappa/p}r^{\kappa/p}(K_1\|f\|_{L^p((\!|\tau,T|\!);X_0)} + K_2\|g\|_{L^p((\!|\tau,T|\!);\gamma(H,X_{1/2}))}),
\end{aligned}$$

where $K_1 = K_{(A,B)}^{\mathsf{det},\theta,p,\kappa}(\sigma,T)$ and $K_2 = K_{(A,B)}^{\mathsf{sto},\theta,p,\kappa}(\sigma,T)$ using the notation of (6.11). □

6.2.3 Perturbations

In this subsection we will discuss a simple perturbation result which will be needed in Theorem 6.3.6 on blow-up criteria. It is based on a version of the method of continuity, which extends the result [174, Proposition 3.18] in several ways.

To simplify the notation for stopping times σ, τ such that $s \leq \tau \leq \sigma \leq T$, we set

$$E_\theta(\sigma,\tau) = L^p_{\mathscr{P}}((\!|\sigma,\tau|\!), w_\kappa^\sigma; X_\theta) \quad\text{and}\quad E_\theta^\gamma(\sigma,\tau) = L^p_{\mathscr{P}}((\!|\sigma,\tau|\!), w_\kappa^\sigma; \gamma(H, X_\theta)), \quad \theta \in [0,1].$$

Proposition 6.2.10 (Method of continuity)**.** *Let Assumptions 4.2.1 and 6.2.1 be satisfied. Suppose that $(A, B) \in \mathcal{SMR}_{p,\kappa}(\sigma, T)$, where σ is a stopping time with values in $[s, T]$. Let $\widehat{A} : [\![\sigma, T]\!] \to \mathscr{L}(X_1, X_0)$, $\widehat{B} : [\![\sigma, T]\!] \to \mathscr{L}(X_1, \gamma(H, X_{1/2}))$ be strongly progressively measurable and assume there exists a constant $\widehat{C}$ such that*

$$\|\widehat{A}(t,\omega)x\|_{X_0} + \|\widehat{B}(t,\omega)x\|_{\gamma(H,X_{1/2})} \leq \widehat{C}_{A,B}\|x\|_{X_1}, \quad (t,\omega) \in [\![\sigma,T]\!], x \in X_1.$$

Let

$$A_\lambda = (1-\lambda)A + \lambda\widehat{A} \quad\text{and}\quad B_\lambda = (1-\lambda)B + \lambda\widehat{B}, \quad \lambda \in [0,1].$$

Suppose that there exist constants $C_{\mathsf{det}}, C_{\mathsf{sto}} > 0$ such that for all $\lambda \in [0,1]$, for all stopping time τ such that $\sigma \leq \tau \leq T$, $f \in E_0(\sigma,T)$, $g \in E^\gamma_{1/2}(\sigma,T)$ and each $u \in E_1(\sigma,\tau)$ which is a strong solution on $[\![\sigma,\tau]\!]$ to

$$\begin{cases} du(t) + A_\lambda u dt = f dt + (B_\lambda u + g) dW_H, & \text{on } [\![\sigma,T]\!],\\ u(\sigma) = 0, \end{cases} \tag{6.20}$$

the following estimate holds

$$\|u\|_{E_1(\sigma,\tau)} \leq C_{\mathsf{det}}\|f\|_{E_0(\sigma,\tau)} + C_{\mathsf{sto}}\|g\|_{E^\gamma_{1/2}(\sigma,\tau)}. \tag{6.21}$$

Then $(\widehat{A}, \widehat{B}) \in \mathcal{SMR}_{p,\kappa}(\sigma,T)$ and $C^{j,0,p,\kappa}_{(A,B)}(\sigma,T) \leq C_j$ for $j \in \{\mathsf{det},\mathsf{sto}\}$.

Of course the above result can be combined with Proposition 6.2.5 to find a similar result for $\mathcal{SMR}^\bullet_{p,\kappa}(\sigma,T)$.

Proof. Uniqueness of the solution to (6.20) is clear from (6.21). It remains to show existence of strong solution on $[\![\sigma,T]\!]$. Let $\Lambda \subseteq [0,1]$ denote the set of all λ such that for all $f \in E_0(\sigma,T)$ and $g \in E^\gamma_{1/2}(\sigma,T)$, (6.20) has a strong solution $u \in E_1(\sigma,T)$. Since $(\widehat{A},\widehat{B}) \in \mathcal{SMR}_{p,\kappa}(\sigma,T)$, one has $0 \in \Lambda$ and it is enough to check that $1 \in \Lambda$. For this it is enough to show that there exists a $\varepsilon_0 > 0$ such that for any $\lambda \in \Lambda$ one has $[\lambda, \lambda+\varepsilon_0] \cap [0,1] \subseteq \Lambda$.

let $\varepsilon_0 = \frac{1}{2C(\widehat{C}_{A,B}+C_{A,B})}$. Let $\lambda \in \Lambda$ and $\varepsilon \in (0, \varepsilon_0]$ be such that $\lambda + \varepsilon \le 1$. It is enough to show $\lambda + \varepsilon \in \Gamma$. Given $v \in E_1(\sigma, T)$, let $L_\varepsilon(v) = u$, where u is the unique strong solution to (6.20) with (f, g) replaced by $(f + \varepsilon(Av - \widehat{A}v), g + \varepsilon(\widehat{B}v - Bv))$.

Since $\lambda \in \Lambda$, L_ε defines a mapping on $E_1(\sigma, T)$. By definition, for $v_1, v_2 \in E_1(\sigma, T)$ one has that $u_{1,2} := L_\varepsilon(v_1) - L_\varepsilon(v_2)$ satisfies (6.20) with (f, g) replaced by $(\varepsilon(Av - \widehat{A}v), g + \varepsilon(\widehat{B}v - Bv))$, where $v = v_1 - v_2$. Therefore, by (6.21)

$$\begin{aligned}\|L_\varepsilon(v_1) - L_\varepsilon(v_2)\|_{E_1(\sigma,T)} &= \|u_{1,2}\|_{E_1(\sigma,T)} \\ &\le C(\|\varepsilon(Av - \widehat{A}v)\|_{E_0(\sigma,T)} + \|\varepsilon(\widehat{B}v - Bv)\|_{E^\gamma_{1/2}(\sigma,T)}) \\ &\le C(\widehat{C}_{A,B} + C_{A,B})\varepsilon\|v_1 - v_2\|_{E_1(\sigma,T)} \\ &\le \frac{1}{2}\|v_1 - v_2\|_{E_1(\sigma,T)}.\end{aligned}$$

By the Banach contraction principle it follows that there exists a unique $u \in E_1(\sigma, T)$, such that $L_\varepsilon(u) = u$, and thus u is the unique strong solution of (6.20) with λ replaced by $\lambda + \varepsilon$. From this we can conclude that $\lambda + \varepsilon \in \Lambda$.

The final estimate is immediate from (6.21) for $\lambda = 1$. ☐

Now we are able to state and proof our perturbation result, where the main novelty is that we can allow initial random times. The perturbation is assumed to be small in terms of the maximal regularity constants $C^{\mathsf{det},0,p,\kappa}_{(A,B)}$ and $C^{\mathsf{sto},0,p,\kappa}_{(A,B)}$ introduced below (6.10), but this will be sufficient for the proof of the blow-up criteria of Theorem 6.3.6. Other perturbation results allowing lower order terms can be found in [174] and will be discussed in Chapter 9, see Theorem 9.1.4 there.

Corollary 6.2.11 (Perturbation)**.** *Let Assumptions 4.2.1 and 6.2.1 be satisfied. Let $\sigma : \Omega \to [0, T]$ be a stopping time which takes values in a finite set if $\kappa > 0$. Assume that $(A, B) \in \mathcal{SMR}^\bullet_{p,\kappa}(\sigma, T)$. Let $\widehat{A} : [\![\sigma, T]\!] \to \mathscr{L}(X_1, X_0)$, $\widehat{B} : [\![\sigma, T]\!] \to \mathscr{L}(X_1, \gamma(H, X_{1/2}))$ be strongly progressively measurable such that for some positive constants C_A, C_B, L_A, L_B and for all $x \in X_1$, a.s. for all $t \in (\sigma, T)$,*

$$\|A(t,\omega)x - \widehat{A}(t,\omega)x\|_{X_0} \le C_A\|x\|_{X_1}, \quad \|B(t,\omega)x - \widehat{B}(t,\omega)x\|_{\gamma(H,X_{1/2})} \le C_B\|x\|_{X_1}.$$

If $\delta_{A,B} := C^{\mathsf{det},0,p,\kappa}_{(A,B)}(\sigma, T)C_A + C^{\mathsf{sto},0,p,\kappa}_{(A,B)}(\sigma, T)C_B < 1$, then $(\widehat{A}, \widehat{B}) \in \mathcal{SMR}^\bullet_{p,\kappa}(\sigma, T)$.

Proof. By Proposition 6.2.5 it suffices to prove $(\widehat{A}, \widehat{B}) \in \mathcal{SMR}_{p,\kappa}(\sigma, T)$, and actually the proof shows that we only need $\mathcal{SMR}_{p,\kappa}(\sigma, T)$ for the latter. We will use the method of continuity of Proposition 6.2.10. In the notation introduced there, let $\lambda \in [0, 1]$, and let $u \in E_1(\sigma, \tau)$ be a strong solution to (6.20) on $[\![\sigma, \tau]\!]$. It suffices to prove the a priori estimate (6.21). Since $u(\sigma) = 0$,

$$du(t) + Audt = \big[f + \lambda(A - \widehat{A})u\big]dt + \big[Bu + g + \lambda(\widehat{B} - B)u\big]dW_H \quad \text{on } [\![\sigma, \tau]\!],$$

and $(A, B) \in \mathcal{SMR}_{p,\kappa}(\sigma, T)$, it follows from Proposition 6.2.7 that a.s. on $[\![\sigma, \tau]\!]$

$$u = \mathscr{R}_{\sigma,(A,B)}(0, \mathbf{1}_{[\![\sigma,\tau]\!]}f + \lambda(A - \widehat{A})\mathbf{1}_{[\![\sigma,\tau]\!]}u, \mathbf{1}_{[\![\sigma,\tau]\!]}g + \lambda(\widehat{B} - B)\mathbf{1}_{[\![\sigma,\tau]\!]}u).$$

Therefore, by the properties of $\mathscr{R}_{\sigma,(A,B)}$ we obtain

$$\begin{aligned}\|u\|_{E_1(\sigma,\tau)} \le\ & C^{\mathsf{det},0,p,\kappa}_{(A,B)}(\sigma, T)\|\mathbf{1}_{[\![\sigma,\tau]\!]}f + \lambda(A - \widehat{A})\mathbf{1}_{[\![\sigma,\tau]\!]}u\|_{E_0(\sigma,T)} \\ &+ C^{\mathsf{sto},0,p,\kappa}_{(A,B)}(\sigma, T)\|\mathbf{1}_{[\![\sigma,\tau]\!]}g + \lambda(\widehat{B} - B)\mathbf{1}_{[\![\sigma,\tau]\!]}u\|_{E^\gamma_{1/2}(\sigma,\tau)} \\ \le\ & C^{\mathsf{det},0,p,\kappa}_{(A,B)}(\sigma, T)\|f\|_{E_0(\sigma,\tau)} + C^{\mathsf{sto},0,p,\kappa}_{(A,B)}(\sigma, T)\|g\|_{E^\gamma_{1/2}(\sigma,\tau)} + \delta_{A,B}\|u\|_{E_1(\sigma,\tau)}.\end{aligned}$$

Therefore, (6.21) follows, and this completes the proof. ☐

6.3 Blow-up criteria for stochastic evolution equations

In this section we present blow-up criteria for parabolic stochastic evolution equations of the form:

$$\begin{cases} du + A(\cdot,u)udt = (F(\cdot,u)+f)dt + (B(\cdot,u)u + G(\cdot,u)+g)dW_H, \\ u(s) = u_s, \end{cases} \tag{6.22}$$

where $s \geq 0$. Moreover, in the main theorems below we will actually consider (6.22) on a finite time interval $[s,T]$ where $T \in (s,\infty)$ is fixed. Extensions to $T=\infty$ are straightforward consequences. Moreover, by using uniqueness and combining solutions and can always reduce to the finite interval case (see Subsection 6.3.3). Before stating our main results we first review local existence results for (6.22) proven in Chapter 4.

6.3.1 Nonlinear parabolic stochastic evolution equations in critical spaces

For the reader's convenience, below we state a local existence for (6.22) which follows from Theorem 4.3.7 proven in Chapter 4. For the sake of simplicity, here we proven the main results under a weaker assumption compared to the one used in Chapter 4. However, the latter covers all the applications in Chapter 5.

We say that **Hypothesis (H)** holds if Hypothesis **($\mathbf{H}_0$)** is satisfied with $[0,T]$ replaced by $[s,T]$ and $F_L = G_L = 0$.

As in Chapter 4, we say that $X^{\mathsf{Tr}}_{\kappa,p}$ is a *critical space* for (6.22) if for some $j \in \{1,\dots,m_F+m_G\}$ equality holds in (4.18) or (4.20). In this case the corresponding power of the weight $\kappa := \kappa_{\mathsf{crit}}$ will be called *critical*.

The concept of L^p_κ-local, unique and maximal local solution to (6.22) can be defined as Definition 4.3.3-4.3.4 replacing $[0,T]$ by $[s,T]$. In Subsection 6.3.3 we extend the previous definitions to the case $T=\infty$. Recall that L^p_κ-maximal local solutions are unique by definition. In addition, an (unique) L^p_κ-strong solution u on $[\![s,\sigma]\!]$ gives an (unique) L^p_κ-local solution (u,σ) to (6.22) on $[s,T]$. In the following, we omit the prefix L^p_κ if no confusion seems likely.

Given $u_s \in L^0_{\mathscr{F}_s}(\Omega; X^{\mathsf{Tr}}_{\kappa,p})$ we denote by $(u_{s,n})_{n\geq 1}$ a sequence such that

$$u_{s,n} \in L^\infty_{\mathscr{F}_s}(\Omega; X^{\mathsf{Tr}}_{\kappa,p}), \quad \text{and} \quad u_{s,n} = u_s \ \text{ on } \ \{\|u_s\|_{X^{\mathsf{Tr}}_{\kappa,p}} \leq n\}. \tag{6.23}$$

A possible choice would be to set $u_{s,n} = R_n(u_s)$ where

$$R_n(x) = x, \quad \text{if } \|x\|_{X^{\mathsf{Tr}}_{\kappa,p}} \leq n, \quad \text{otherwise} \quad R_n(x) := nx/\|x\|_{X^{\mathsf{Tr}}_{\kappa,p}}.$$

The following follows from Theorem 4.3.7 in Chapter 4.

Theorem 6.3.1 (Local well-posedness)**.** *Let Hypothesis* **(H)** *be satisfied. Let $u_s \in L^0_{\mathscr{F}_s}(\Omega; X^{\mathsf{Tr}}_{\kappa,p})$ and that* (6.23) *holds. Suppose that*

$$(A(\cdot,u_{s,n}), B(\cdot,u_{s,n})) \in \mathcal{SMR}^{\bullet}_{p,\kappa}(s,T), \quad n \in \mathbb{N}.$$

Then the following assertions hold:

(1) (Existence and regularity) *There exists an L^p_κ-maximal local solution (u,σ) to* (6.22) *such that $\sigma > s$ a.s. Moreover, for each localizing sequence $(\sigma_n)_{n\geq 1}$ for (u,σ) one has*

- *If $p>2$, $\kappa \in [0,\frac{p}{2}-1)$, then for all $\theta \in [0,\frac{1}{2})$ and $n \geq 1$,*

$$u \in H^{\theta,p}(s,\sigma_n, w^s_\kappa; X_{1-\theta}) \cap C([s,\sigma_n]; X^{\mathsf{Tr}}_{\kappa,p}) \quad a.s.$$

Moreover, u instantaneously regularizes to $u \in C((s,\sigma_n]; X^{\mathsf{Tr}}_p)$ a.s.

- *If $p=2$, $\kappa=0$, then for all $n \geq 1$,*

$$u \in L^2(s,\sigma_n; X_1) \cap C([s,\sigma_n]; X_{1/2}) \quad a.s.$$

(2) (Localization) *If (v,τ) is an L^p_κ-maximal local solution to* (6.22) *with data $v_s \in L^0_{\mathscr{F}_s}(\Omega; X^{\mathsf{Tr}}_{\kappa,p})$, then setting $\Gamma := \{v_s = u_s\}$, one has*

$$\tau|_\Gamma = \sigma|_\Gamma, \qquad v|_{\Gamma\times[s,\tau)} = u|_{\Gamma\times[s,\sigma)}.$$

6.3.2 Main results

In this subsection, we state our main results regarding blow-up criteria for (6.22). For this we will need the following assumption on the nonlinearity (A, B). Recall that by Assumption 4.2.1, either $\kappa \in [0, \frac{p}{2} - 1)$, $p > 2$ or $\kappa = 0$, $p = 2$.

Assumption 6.3.2. *Suppose that Assumption (HA) holds for (A, B). Let $\ell \in [0, \frac{p}{2} - 1)$ (where $\ell = 0$ if $p = 2$). Assume that for each $M, \eta > 0$, $t \in [s + \eta, T)$ and $v \in L^\infty_{\mathscr{F}_t}(\Omega; X^{\mathsf{Tr}}_{\ell,p})$ with $\|v\|_{X^{\mathsf{Tr}}_{\ell,p}} \leq M$ a.s., one has $(A(\cdot, v), B(\cdot, v)) \in \mathcal{SMR}^\bullet_{p,\ell}(t, T)$, and for each $\theta \in [0, \frac{1}{2}) \setminus \{\frac{1+\kappa}{p}\}$ there exists a constant $K^\theta_{M,\eta}$ s.t.*

$$\max\{K^{\mathsf{det},\theta,p,\ell}_{(A(\cdot,v),B(\cdot,v))}(t, T), K^{\mathsf{sto},\theta,p,\ell}_{(A(\cdot,v),B(\cdot,v))}(t, T)\} \leq K^\theta_{M,\eta}, \quad t \in [s + \eta, T).$$

where the constants are as defined in (6.11).

Assumption 6.3.2 ensures that the maximal regularity constants are uniform on balls in $X^{\mathsf{Tr}}_{\kappa,p}$. It is important that the assumption is only formulated for non-random initial times t. Random initial times can be obtained afterwards using Proposition 6.2.9. In applications to semilinear equations, i.e. in the case that $(A(t, x), B(t, x)) = (\bar{A}(t), \bar{B}(t))$, the condition $(A(t, x), B(t, x)) \in \mathcal{SMR}^\bullet_{p,\kappa}(s, T)$ already implies Assumption 6.3.2 for $\ell = 0$ by Proposition 6.2.9. Finally, we note that on most applications we know $\mathcal{SMR}^\bullet_{p,\kappa}(t, T) \neq \varnothing$ with uniform estimates in t. Therefore, by transference (see Proposition 6.2.5) it is enough to check $(A(\cdot, v), B(\cdot, v)) \in \mathcal{SMR}_{p,\ell}(t, T)$ together with the above estimate for $\theta = 0$.

In the quasilinear case, Assumption 6.3.2 can be weakened is some situations of interest. For future convenience, we formulate this in the following remark.

Remark 6.3.3. Let $\mathcal{C} \subseteq X^{\mathsf{Tr}}_{\ell,p}$ be a closed subset and assume that the maximal L^p_κ-local solution (u, σ) to (6.22) satisfies $u(t) \in \mathcal{C}$ a.s. for all $t \in (0, \sigma)$. If the previous holds, then the requirement $v \in \mathcal{C}$ a.s. can be added in Assumption 6.3.2. For instance, in the case $X^{\mathsf{Tr}}_{\ell,p}$ is a function space the choice $\mathcal{C} = \{v \in X^{\mathsf{Tr}}_{\ell,p} : v \geq 0\}$ can be useful in applications to quasilinear SPDEs where the flow is positive preserving. For more on this see Remark 6.4.4.

For our main blow-up result we need another condition which states that the conditions on F and G are also satisfied in the unweighted setting.

Assumption 6.3.4. *Suppose that Assumption 4.2.1 holds for X_0, X_1, κ, p. Let F and G be as in Assumptions (HF) and (HG). Suppose that Assumption (HF) and (HG) hold with κ replaced by 0 and a possibly different choice of the parameters $(\rho'_j, \varphi'_i, \beta'_j)$ for $j \in \{1, \dots, m'_F + m'_G\}$ for certain integers m'_F and m'_G.*

Remark 6.3.5. In the important case that for a given $\kappa \in [0, \frac{p}{2} - 1)$, $\varphi_j = \beta_j$ for all $j \in \{1, \dots, m_F + m_G\}$ in (HF) and (HG), Assumption 6.3.4 is always satisfied. Indeed, in case $\beta_j = \varphi_j > 1 - 1/p$ one can choose $\varphi'_j = \beta'_j = \varphi_j$. In case $\beta_j = \varphi_j \leq 1 - 1/p$, one can apply Remark 4.3.2(2).

The main result of this section is a blow-up criterium for the L^p_κ-maximal local solution of Theorem 6.3.1. Theorems 6.3.6-6.3.8 below show that σ is an *explosion time* of the L^p_κ-maximal local solution of (6.22) in a certain norm. For notational convenience, set for $s, t \in [0, T]$

$$\mathcal{N}^\kappa_c(u; s, t) := \|F_c(\cdot, u)\|_{L^p(s,t,w^s_\kappa;X_0)} + \|G_c(\cdot, u)\|_{L^p(s,t,w^s_\kappa;\gamma(H,X_{1/2}))}, \tag{6.24}$$

where we recall $F = F_{\mathsf{Tr}} + F_c$ and $G = G_{\mathsf{Tr}} + G_c$.

Theorem 6.3.6 (Blow up criteria for quasilinear SPDEs)**.** *Let the Hypothesis **(H)** be satisfied. Let $u_s \in L^0_{\mathscr{F}_s}(\Omega; X^{\mathsf{Tr}}_{\kappa,p})$ and suppose that (4.23) holds. Suppose that*

$$(A(\cdot, u_{s,n}), B(\cdot, u_{s,n})) \in \mathcal{SMR}^\bullet_{p,\kappa}(s, T), \quad n \in \mathbb{N}, \tag{6.25}$$

and that Assumption 6.3.2 holds for $\ell \in \{0, \kappa\}$ and Assumption 6.3.4 holds. Let (u, σ) be the L^p_κ-maximal local solution to (6.22). Then

(1) $\mathbb{P}\Big(\sigma < T, \lim_{t\uparrow\sigma} u(t) \textit{ exists in } X_p^{\mathsf{Tr}}\Big) = 0$;

(2) $\mathbb{P}\Big(\sigma < T, \lim_{t\uparrow\sigma} u(t) \textit{ exists in } X_{\kappa,p}^{\mathsf{Tr}}, \mathcal{N}_c^{\kappa}(u;\, s,\, \sigma) < \infty\Big) = 0$;

(3) $\mathbb{P}\Big(\sigma < T, \lim_{t\uparrow\sigma} u(t) \textit{ exists in } X_{\kappa,p}^{\mathsf{Tr}}\Big) = 0$ *provided* $X_{\kappa,p}^{\mathsf{Tr}}$ *is not critical for* (6.22);

(4) $\mathbb{P}\Big(\sigma < T, \lim_{t\uparrow\sigma} u(t) \textit{ exists in } X_{\kappa,p}^{\mathsf{Tr}}, \|u\|_{L^p(s,\sigma;X_{1-\frac{\kappa}{p}})} < \infty\Big) = 0.$

In Figure 6.1 we provide a decision tree for applying Theorem 6.3.6.

Some comments are in order. For part (1) we only need Assumption 6.3.2 for $\ell = 0$. Part (3) is the easiest to check in applications since $X_p^{\mathsf{Tr}} \hookrightarrow X_{\kappa,p}^{\mathsf{Tr}}$. Of course if $\kappa = 0$, then (1) and (3) coincide.

To apply (2), one only needs to control F_c and G_c. which can be done with Lemmas 6.4.2 and 6.4.5. The control of F_c and G_c is needed only far from $t = s$. Actually one can replace $\mathcal{N}_c^{\kappa}(u;\, s,\, \sigma)$ by $\mathcal{N}_c^{0}(u;\, \tau,\, \sigma)$ for any random time $\tau \in (s,\sigma)$. Indeed, this follows from Theorem 6.3.1, and Lemmas 6.4.2 and 6.4.5.

As we will show in Section 7.1, the solution u is typically smoother than its values near $t = s$ and this may simplify the proof of *energy estimates*. In applications to concrete SPDEs we always prove the following stronger version of Theorem 6.3.6(2):

$$\mathbb{P}\Big(s' < \sigma < T, \lim_{t\uparrow\sigma} u(t) \text{ exists in } X_{\kappa,p}^{\mathsf{Tr}}, \mathcal{N}_c^{0}(u; s', \sigma) < \infty\Big) = 0, \text{ for all } s' \in (s,T).$$

Similar considerations hold for Theorem 6.3.7(2) below.

In (4) it suffices to estimate the $L^p(\tau,\sigma;X_{1-\frac{\kappa}{p}})$-norm of u for some stopping time $\tau \in (s,\sigma)$. This already implies that u is in L^p near $t = s$ as a map with values in $X_{1-\frac{\kappa}{p}}$. Indeed, if $p > 2$ this follows from Theorem 6.3.1(1), and

$$u \in H^{\frac{\kappa}{p},p}(s,\sigma_n,w_\kappa^s;X_{1-\frac{\kappa}{p}}) \hookrightarrow L^p(s,\sigma_n;X_{1-\frac{\kappa}{p}}), \qquad \text{a.s. for all } n \geq 1, \tag{6.26}$$

where we used Proposition 6.1.1(4). The case $p = 2$ is immediate from the fact that (u,σ) is an L^2_0-maximal local solution (see Definitions 4.3.3-4.3.4). Part (4) plays a key role in proving instantaneous regularization of solutions to (6.22) in the unweighted setting (see Proposition 7.1.7).

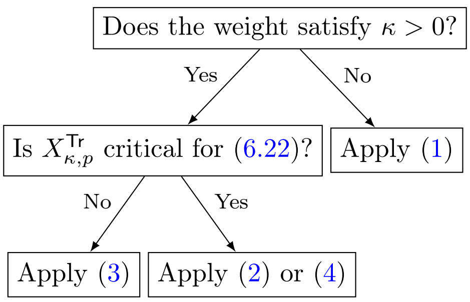

Figure 6.1. Decision tree for applying Theorem 6.3.6 to quasilinear SPDEs.

In applications to semilinear equations, the following improvement of Theorem 6.3.6 holds. For convenience, for $s,t \in [0,T]$ set

$$\mathcal{N}^{\kappa}(u;s,t) := \|F(\cdot,u)\|_{L^p(s,t,w_\kappa^s;X_0)} + \|G(\cdot,u)\|_{L^p(s,t,w_\kappa^s;\gamma(H,X_{1/2}))}. \tag{6.27}$$

Theorem 6.3.7 (Blow-up criteria for semilinear SPDEs)**.** *Let the Hypothesis* **(H)** *be satisfied, where we suppose that* $(A(t,x),B(t,x)) = (\bar{A}(t),\bar{B}(t))$ *does not depend on* x *and*

$$(\bar{A}(\cdot),\bar{B}(\cdot)) \in \mathcal{SMR}^{\bullet}_{p,\kappa}(s,T). \tag{6.28}$$

and that Assumption 6.3.2 holds for $\ell = \kappa$ and Assumption 6.3.4 holds. Let $u_s \in L^0_{\mathscr{F}_s}(\Omega; X^{\mathsf{Tr}}_{\kappa,p})$ and let (u,σ) be the L^p_κ-maximal local solution to (6.22). Then

(1) $\mathbb{P}\Big(\sigma < T, \lim_{t\uparrow\sigma} u(t)$ *exists in* $X^{\mathsf{Tr}}_p\Big) = 0$*;*

(2) $\mathbb{P}\Big(\sigma < T, \mathcal{N}^\kappa(u; s,\sigma) < \infty\Big) = 0$*;*

(3) $\mathbb{P}\Big(\sigma < T, \sup_{t\in[s,\sigma)} \|u(t)\|_{X^{\mathsf{Tr}}_{\kappa,p}} < \infty\Big) = 0$ *provided $X^{\mathsf{Tr}}_{\kappa,p}$ is not critical for (6.22);*

(4) $\mathbb{P}\Big(\sigma < T, \sup_{t\in[s,\sigma)} \|u(t)\|_{X^{\mathsf{Tr}}_{\kappa,p}} + \|u\|_{L^p(s,\sigma;X_{1-\frac{\kappa}{p}})} < \infty\Big) = 0.$

A decision tree is given in Figure 6.2, but this time one should always first try to apply (1), and only if this does not work, use the decision tree.

The proofs of Theorem 6.3.7(1)-(2) do not require Assumption 6.3.2 for $\ell = \kappa$. Moreover, Assumption 6.3.2 for $\ell = 0$ is not assumed since it follows from Assumption 6.3.2 for $\ell = \kappa$ and Proposition 6.2.9. Theorem 6.3.7(3)-(4) are slight improvements of Theorem 6.3.6(3)-(4) since only boundedness is required. As before in (2) we may replace $\mathcal{N}^\kappa(u; s,\sigma)$ by $\mathcal{N}^0(u;\tau,\sigma)$ any random time $\tau \in (s,\sigma)$. The same holds for (4) with $\|u\|_{L^p(s,\sigma;X_{1-\frac{\kappa}{p}})}$ replaced by $\|u\|_{L^p(\tau,\sigma;X_{1-\frac{\kappa}{p}})}$.

Below we will obtain a further improvement of Theorem 6.3.7(4) by removing the condition $\sup_{t\in[s,\sigma)} \|u(t)\|_{X^{\mathsf{Tr}}_{\kappa,p}} < \infty$ under suitable assumptions. In literature blow-up criteria which only require L^p-bounds are called of *Serrin type* due to the analogy with Serrin's blow up criteria for Navier-Stokes equations (se e.g. [144, Theorem 11.2]).

Theorem 6.3.8 (Serrin type blow-up criteria for semilinear SPDEs)**.** *Let Hypothesis* **(H)** *be satisfied, where we suppose that $(A(t,x), B(t,x)) = (\bar{A}(t), \bar{B}(t))$ does not depend on x and*

$$(\bar{A}(\cdot), \bar{B}(\cdot)) \in \mathcal{SMR}^\bullet_{p,\kappa}(s,T), \quad n \in \mathbb{N}, \tag{6.29}$$

$F_{\mathsf{Tr}} = 0$, $G_{\mathsf{Tr}} = 0$, the constants $C_{c,n}$ in (HF)-(HG) are independent of $n \geq 1$, and for each $j \in \{1,\dots,m_F+m_G\}$

$$\beta_j = \varphi_j \quad \textit{and} \quad [(\kappa > 0 \textit{ and } \rho_j < 1+\kappa) \textit{ or } (\kappa = 0 \textit{ and } \rho_j \leq 1)]. \tag{6.30}$$

Suppose that Assumption 6.3.2 holds for $\ell = \kappa$ and Assumption 6.3.4 holds. If $u_s \in L^0_{\mathscr{F}_s}(\Omega; X^{\mathsf{Tr}}_{\kappa,p})$ and (u,σ) is the L^p_κ-maximal local solution to (6.22), then

$$\mathbb{P}\Big(\sigma < T, \|u\|_{L^p(s,\sigma;X_{1-\frac{\kappa}{p}})} < \infty\Big) = 0. \tag{6.31}$$

To the best of our knowledge, Theorem 6.3.8 is new even in the case $p = 2$ and $\kappa = 0$. The second part of (6.30) holds if $\rho_j = 1$, and this covers the case of bilinear nonlinearities as considered in Chapter 9 and [178]. An extension of Theorem 6.3.8 allowing $\varphi_j \neq \beta_j$ can be found in Proposition 6.4.9.

Suppose that $m_F = m_G = 1$, $\beta := \beta_1 = \beta_2 = \varphi_1 = \varphi_2$, and $\rho := \rho_1 = \rho_2$ are fixed for a given problem (6.22). Let $S = \{(p,\kappa) : X^{\mathsf{Tr}}_{\kappa,p}$ is critical for (6.22)$\}$. Then for all $(p,\kappa) \in S$ the following identity holds $\frac{1+\kappa}{p} = \frac{\rho+1}{\rho}(1-\beta)$, which means that $\frac{1+\kappa}{p}$ is constant. Moreover, the second part of (6.30) holds if and only if $\frac{\rho}{p} < \frac{1+\kappa}{p} = \frac{\rho+1}{\rho}(1-\beta)$, which holds for p large enough. Moreover, $1-\frac{\kappa}{p} = \frac{\rho+1}{\rho}(\beta-1)+\frac{1}{p}$ which decreases in p. Therefore, Theorem 6.3.8 requires only a mild control of the regularity "in space" of u provided p and thus κ, are large.

Let us conclude by pointing out the following simple consequence of Theorem 6.3.6(1), which will be used to prove Theorems 6.3.6(2)–(4), 6.3.7(2)-(4) and 6.3.8.

Corollary 6.3.9 (Predictability of the explosion time σ)**.** *If the conditions of Theorem 6.3.6 hold, then any localizing sequence $(\sigma_n)_{n\geq 1}$ for (u,σ) satisfies, for all $n \geq 1$,*

$$\mathbb{P}(\sigma < T, \sigma_n = \sigma) = 0.$$

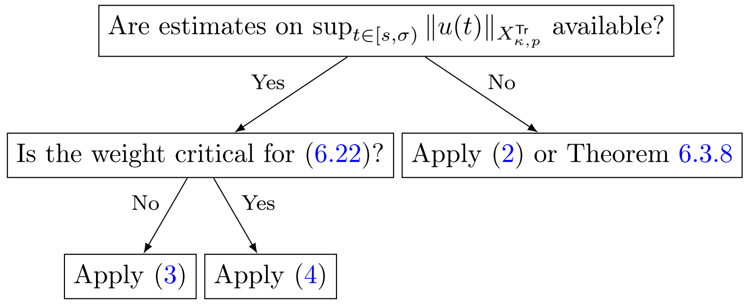

Figure 6.2. Decision tree for applying Theorems 6.3.7 and 6.3.8 to semilinear SPDEs in case Theorem 6.3.7(1) is not sufficient.

The above implies that σ is a so-called *predictable* stopping time. For the proof we only need Assumption 6.3.2 for $\ell = 0$.

Proof. Let $(\sigma_n)_{n\geq 1}$ be a localizing sequence. Suppose that there exists an $n \geq 1$ such that $\mathbb{P}(\sigma_n = \sigma < T) > 0$. Setting $\mathcal{V} := \{\sigma_n = \sigma < T\}$, by Theorem 6.3.1(1), one has $\sigma_n = \sigma > s$, $u \in C((s,\sigma_n]; X^{\mathsf{Tr}}_p) = C((s,\sigma]; X^{\mathsf{Tr}}_p)$ a.s. on $\mathcal{V}$. Therefore

$$\lim_{t\uparrow\sigma} u(t) \text{ exists in } X^{\mathsf{Tr}}_p, \quad \text{a.s. on } \mathcal{V},$$

where we used $s < \sigma_n = \sigma < T$ a.s. on $\mathcal{V}$. Thus,

$$\mathbb{P}(\mathcal{V}) = \mathbb{P}\Big(\mathcal{V} \cap \Big\{\lim_{t\uparrow\sigma} u(t) \text{ exists in } X^{\mathsf{Tr}}_p\Big\}\Big) \leq \mathbb{P}\Big(\sigma < T, \lim_{t\uparrow\sigma} u(t) \text{ exists in } X^{\mathsf{Tr}}_p\Big) = 0,$$

where in the last equality we used Theorem 6.3.6(1). This contradicts $\mathbb{P}(\mathcal{V}) > 0$ and therefore the result follows. $\square$

The next simple result will allow us to reduce to integrable or even bounded data. It will be used in the proofs of Theorems 6.3.6, 6.3.7 and 6.3.8, but it can also be a helpful reduction in proving global existence in concrete situations.

Proposition 6.3.10 (Reduction to uniformly bounded data)**.** *Let the Hypothesis* **(H)** *be satisfied. Let $u_s \in L^0_{\mathscr{F}_s}(\Omega; X^{\mathsf{Tr}}_{\kappa,p})$ and suppose that (4.23) holds. Let $f_n = f\mathbf{1}_{[0,\tau_n]}$ and $g = g\mathbf{1}_{[0,\tau_n]}$, where*

$$\tau_n = \inf\{t \in [0,T] : \|f\|_{L^p(0,t,w_\kappa;X_0)} \geq n, \ \|g\|_{L^p(0,t,w_\kappa;\gamma(H,X_{1/2}))} \geq n\}.$$

Let (u,σ) be the L^p_κ-maximal local solution to (6.22), and let (u_n,σ_n) be the L^p_κ-maximal local solution to (6.22) with (u_s,f,g) replaced by $(u_{s,n},f_n,g_n)$. For each of the statements in Theorems 6.3.6, 6.3.7 and 6.3.8 it suffices to prove that the corresponding probability is zero with u replaced by u_n for each $n \geq 1$.

Proof. By a translation argument we may assume that $s = 0$. We present the details in case of Theorem 6.3.7(2). The other cases can be obtained in the same way replacing the set (6.34) below by a suitable set in each case.

Set $\Gamma_n := \{\|u_0\|_{X^{\mathsf{Tr}}_{\kappa,p}} \leq n\} \in \mathscr{F}_0$. Observe that $\mathbb{P}(\{\tau_n = T\} \cap \Gamma_n) \to 1$, and for each $n \geq 1$, $(u, \sigma\wedge\tau_n)$ is an L^p_κ-local solution to (6.22) with (f,g) replaced by (f_n,g_n). Denoting by (v_n,μ_n) the L^p_κ-maximal solution to (6.22) with (u_0,f_n,g_n), we have $\tau_n\wedge\sigma \leq \mu_n$ and $u = v_n$ on $[\![0,\tau_n\wedge\sigma[\![$ by maximality (see Theorem 6.3.1). Similarly, since $(v_n,\mu_n\wedge\tau_n)$ is an L^p_κ-local solution to (6.22) with (u_0,f,g) we have $\mu_n\wedge\tau_n \leq \sigma$ and $u = v_n$ on $[\![0,\mu_n\wedge\tau_n[\![$. It follows that

$$\mu_n = \sigma \ \text{ on } \{\tau_n = T\}, \quad \text{and} \quad u = v_n \text{ on } [0,\sigma)\times\{\tau_n = T\}. \tag{6.32}$$

Moreover, by Theorem 6.3.7(2)

$$\mu_n = \sigma_n \ \text{ on } \Gamma_n, \quad \text{and} \quad v_n = u_n \text{ on } \Gamma_n. \tag{6.33}$$

For a stopping time ν such that $0 \leq \nu \leq \sigma$, and a process $(v(t))_{t\in(\tau,\nu)}$ set

$$\mathcal{O}(v,\nu) := \{\mathcal{N}^{\kappa}(v;0,\nu) < \infty\}. \tag{6.34}$$

Now if Theorem 6.3.7(2) holds with $(u_{0,n}, f_n, g_n)$, then by (6.32) and (6.33)

$$\begin{aligned}\mathbb{P}\big(\{\sigma < T\} \cap \mathcal{O}(u,\sigma)\big) &= \lim_{n\to\infty} \mathbb{P}\big(\{\sigma < T\} \cap \mathcal{O}(u,\sigma) \cap \{\tau_n = T\} \cap \Gamma_n\big) \\ &= \lim_{n\to\infty} \mathbb{P}\big(\{\sigma_n < T\} \cap \mathcal{O}(u_n,\sigma_n) \cap \{\tau_n = T\} \cap \Gamma_n\big) \\ &\leq \liminf_{n\to\infty} \mathbb{P}\big(\{\sigma_n < T\} \cap \mathcal{O}(u_n,\sigma_n)\big) = 0.\end{aligned}$$

□

The proofs of the blow-up criteria are given in Section 6.4:

- Subsection 6.4.2: Theorems 6.3.6(1) and 6.3.7(1)-(2);
- Subsection 6.4.3: Theorem 6.3.6(2)-(3) and Theorem 6.3.7(3);
- Subsection 6.4.4: Theorems 6.3.6(4), 6.3.7(4) and Theorem 6.3.8.

Blow-up criteria involving the space $\mathfrak{X}$ (see (6.41) below) will be given in Remarks 6.4.6 and 6.4.7 below.

6.3.3 Global existence

In this section we demonstrate how Theorem 6.3.7 can be used to prove global existence for an equation, where F and G satisfy a certain linear growth condition.

Definitions 4.3.3 and 4.3.4 can be extended to the half line case. Indeed, in Definition 4.3.3 one can just take $T = \infty$ and replace $[s,T]$, $L^p(s,\sigma)$ and $C([s,\sigma])$ by $[s,\infty)$, $L^p_{\text{loc}}([s,\sigma))$ and $C([s,\sigma] \cap [s,\infty))$, respectively. Definition 4.3.4 extends verbatim to $T = \infty$.

One can check that (u,σ) is an L^p_κ-(maximal) local solution to (6.22) on $[s,\infty)$ if for each $T < \infty$, $(u|_{[\![s,\sigma\wedge T)\!)}, \sigma\wedge T)$ is an L^p_κ-(maximal) local solution to (6.22) on $[s,T]$. As before an L^p_κ-maximal local solution on $[s,\infty)$ is unique. Conversely, one can construct L^p_κ-maximal local solutions on $[s,\infty)$ from the ones on finite time intervals. Indeed, suppose that an L^p_κ-maximal local solution (u^T,σ^T) exists on $[s,T]$ for every $T \in (s,\infty)$, Then by maximality (see Definition 4.3.4) $\sigma^T = \sigma^S \wedge T$ a.s. and $u^T = u^S$ a.e. on $[\![s,\sigma^T)\!)$ for $s < T \leq S$. Therefore, letting $u = u^T$ on $[\![s,\sigma_T)\!)$ and $\sigma := \lim_{T\to\infty}\sigma^T$, one has that u is an L^p_κ-maximal local solution on $[s,\infty)$. In particular, an L^p_κ-maximal local solution on $[s,\infty)$ exists if the conditions of Theorem 6.3.1 hold for all $T \in (s,\infty)$. Finally we mention that if for each $T \in (s,\infty)$, $(\sigma_n^T)_{n\geq 1}$ is a localizing sequence for (u^T,σ^T), then letting

$$\sigma_n = \sup_{m\in\{1,\dots,n\}} \sigma_n^m, \quad n \geq 1,$$

we obtain a localizing sequence $(\sigma_n)_{n\geq 1}$ for (u,σ).

The following roadmap can be used to prove global well-posedness and regularity.

Roadmap 6.3.11 (Proving global existence and regularity).

(a) Prove local well-posedness with Theorem 6.3.1.

(b) Obtain instantaneous regularization from Theorem 7.1.3 and Corollary 7.1.5 using as a first step Proposition 7.1.7 in the case $\kappa = 0$.

(c) Reduce the global existence proof to data u_0, f, g which is uniformly bounded in Ω (see Proposition 6.3.10).

(d) Prove an energy estimate for a certain norm $\|u\|_{Z(s,\sigma\wedge T)}$ by applying the equation and/or Itô's formula. In this part, the regularization proven in (b) can be used to simplify and/or obtain the estimate.

(e) Combine the energy estimate with Theorem 6.3.6, 6.3.7 or 6.3.8 to prove $\sigma \geq T$ a.s. possibly under restrictions on the integrability parameters and weights.

(f) Use the instantaneous regularization phenomena of Theorem 7.1.3 and Corollary 7.1.5 to reduce to the previous case.

Some steps of this roadmap can be skipped in certain situations. But the steps (a), (d), (e) seem essential in all cases. Furthermore, we mention that in (b) and (f) the use of weights is essential.

To illustrate the above roadmap concretely, we will now prove global existence of (6.22) in the semilinear setting under linear growth assumptions on F and G. Of course the linear growth assumptions fails to hold for many of the interesting SPDEs. So this result should only be seen as an illustration and test case.

Theorem 6.3.12 (Global well-posedness under linear growth conditions)**.** *Let Hypothesis* **(H)** *be satisfied for all $T \in (s,\infty)$, where we suppose that $(A(t,x), B(t,x)) = (\bar{A}(t), \bar{B}(t))$ does not depend on x and*

$$(\bar{A}(\cdot), \bar{B}(\cdot)) \in \mathcal{SMR}^{\bullet}_{p,\kappa}(s,T) \quad \text{for all } T \in (s,\infty). \tag{6.35}$$

and that Assumption 6.3.2 holds for $\ell = \kappa$ and all $T \in (s,\infty)$ and Assumption 6.3.4 holds for all $T \in (s,\infty)$. Suppose that for every $\varepsilon > 0$ there exist a constant $L_\varepsilon > 0$ such that for all $t \in (s,\infty)$, $\omega \in \Omega$ and $x \in X_1$,

$$\|F(t,\omega,x)\|_{X_0} + \|G(t,\omega,x)\|_{\gamma(H,X_{1/2})} \leq L_\varepsilon(1 + \|x\|_{X_0}) + \varepsilon\|x\|_{X_1}. \tag{6.36}$$

Then for each $u_s \in L^0_{\mathscr{F}_s}(\Omega; X^{\mathsf{Tr}}_{\kappa,p})$ there exists a unique L^p_κ-global solution u to (6.22) such that

- *If $p > 2$, and $\kappa \in [0, \frac{p}{2} - 1)$, then for all $\theta \in [0, \frac{1}{2})$,*

$$u \in H^{\theta,p}_{\text{loc}}([s,\infty), w^s_\kappa; X_{1-\theta}) \cap C([s,\infty); X^{\mathsf{Tr}}_{\kappa,p}) \quad a.s.$$

  Moreover, u instantaneously regularizes to $u \in C((s,\infty); X^{\mathsf{Tr}}_p)$ a.s.

- *If $p = 2$, and $\kappa = 0$, then*

$$u \in L^2_{\text{loc}}(s,\infty; X_1) \cap C([s,\infty); X_{1/2}) \quad a.s.$$

Moreover, if additionally $u_0 \in L^p_{\mathscr{F}_0}(\Omega; X^{\mathsf{Tr}}_{\kappa,p})$, $f \in L^p_{\mathscr{P}}((s,T) \times \Omega, w^s_\kappa; X_0)$ and $g \in L^p_{\mathscr{P}}((s,T) \times \Omega, w^s_\kappa; \gamma(H, X_{1/2}))$, then for every $T \in (s,\infty)$ and every $\theta \in [0,1/2)$ there exists a constant $C_{\theta,T}$ and C_T such that

$$\begin{aligned}\|v\|_{L^p(\Omega;E_{\theta,p})} \leq C_{\theta,T}(1 + \|u_0\|_{L^p(\Omega;X^{\mathsf{Tr}}_{\kappa,p})} \\ + \|f\|_{L^p((\!(s,T)\!), w^s_\kappa; X_0)} + \|g\|_{L^p((\!(s,T)\!), w^s_\kappa; \gamma(H,X_{1/2}))}),\end{aligned} \tag{6.37}$$

where we set for $s' > s$,

$$E_{\theta,p} \in \{H^{\theta,p}(s,T,w^s_\kappa; X_{1-\theta}), C([s,T]; X^{\mathsf{Tr}}_{\kappa,p}), C([s',T]; X^{\mathsf{Tr}}_p)\} \quad \text{if } \ p \in (2,\infty),$$
$$E_{\theta,2} \in \{L^2(s,T; X_1), C([s,T]; X_{1/2})\}.$$

By standard interpolation inequalities we can replace $\|x\|_{X_0}$ by $\|x\|_{X_{1-\delta}}$ with arbitrary $\delta \in (0,1)$ in (6.36). From the proof below one can actually see that it is enough to have (6.36) for some fixed small $\varepsilon > 0$.

Proof of Theorem 6.3.12. We may suppose that $s = 0$. We will only prove the result for $p > 2$ as the other case is simpler.

By Theorem 6.3.1 and the above discussion there exists local solution (u,σ) of (6.22) on $[0,\infty)$ with the required properties on $[0,\sigma)$ and thus we only need to show that $\sigma = \infty$ a.s. Replacing σ by $\sigma \wedge T$ it suffices to show that $\mathbb{P}(\sigma < T) = 0$. Moreover, by Proposition 6.3.10 it suffices to consider the case of $L^p(\Omega)$-integrable data u_0, f and g. To prove $\sigma = T$ a.s., we will apply Theorem 6.3.7(2). In order to do so we will first derive a suitable energy estimate.

Step 1: Energy estimate Let $(\sigma_n)_{n\geq 1}$ be a localizing sequence for (u, σ). Moreover, for each $n \geq 1$ define a stopping time by

$$\tau_n = \inf\{t \in [0, \sigma) : \|u\|_{L^p(0,t;X_1)} \geq n\} \wedge \sigma_n,$$

where we set $\inf \varnothing = \sigma$. Then $u|_{[\![0,\tau_n]\!]}$ is a strong solution of (6.22) on $[\![0, \tau_n]\!]$.

Set $\widetilde{f}_n = \mathbf{1}_{[0,\tau_n]}(f+F(\cdot, u))$ and $\widetilde{g}_n = \mathbf{1}_{[0,\tau_n]}(g+G(\cdot, u))$. Then by (6.36), $\widetilde{f}_n \in L^p((\!(0, T)\!), w_\kappa; X_0)$ and $\widetilde{g}_n \in L^p((\!(0, T)\!), w_\kappa; \gamma(H, X_{1/2}))$. By (6.35) for the strong solution v to

$$\begin{cases} dv(t) + A(t)v(t)dt = \widetilde{f}(t)dt + (B(t)v(t) + \widetilde{g}(t))dW_H(t), & t \in [\![0, T]\!], \\ u(0) = u_0, \end{cases}$$

we have $u = v$ on $[\![0, \tau_n]\!]$, and by Proposition 6.2.6,

$$\begin{aligned}\|v\|_{L^p((\!(0,T)\!),w_\kappa;X_1)} \leq C(&\|u_0\|_{L^p(\Omega;X^{\mathrm{Tr}}_{\kappa,p})} + \|\widetilde{f}\|_{L^p((\!(0,T)\!),w_\kappa;X_0)} \\ &+ \|\widetilde{g}\|_{L^p((\!(0,T)\!),w_\kappa;\gamma(H,X_{1/2}))}),\end{aligned}$$

By the linear growth assumption (6.36), and $\|\mathbf{1}_{[0,\tau_n]}u\|_{X_i} \leq \|v\|_{X_i}$ we obtain

$$\|\widetilde{f}\|_{X_0} + \|\widetilde{g}\|_{\gamma(H,X_{1/2})} \leq \|f\|_{X_0} + \|g\|_{\gamma(H,X_{1/2})} + L_\varepsilon(1 + \|v\|_{X_0}) + \varepsilon\|v\|_{X_1}.$$

Choose $\varepsilon = \frac{1}{2C}$ and set

$$K = \|u_0\|_{L^p(\Omega;X^{\mathrm{Tr}}_{\kappa,p})} + \|f\|_{L^p((\!(0,T)\!),w_\kappa;X_0)} + \|g\|_{L^p((\!(0,T)\!),w_\kappa;\gamma(H,X_{1/2}))} + L_\varepsilon,$$

Then combining the above we obtain

$$\|v\|_{L^p((\!(0,T)\!),w_\kappa;X_1)} \leq CK + CL_\varepsilon\|v\|_{L^p((\!(0,T)\!),w_\kappa;X_0)} + \frac{1}{2}\|v\|_{L^p((\!(0,T)\!),w_\kappa;X_1)},$$

and hence

$$\|v\|_{L^p((\!(0,T)\!),w_\kappa;X_1)} \leq 2CK + 2CL_\varepsilon\|v\|_{L^p((\!(0,T)\!),w_\kappa;X_0)}, \tag{6.38}$$

Similarly, by Proposition 6.2.6(1) there exists a $\widetilde{C} > 0$ independent of T such that

$$\begin{aligned}\|v\|_{L^p(\Omega;C([0,T];X^{\mathrm{Tr}}_{\kappa,p}))} &\leq \widetilde{C}(\|u_0\|_{L^p(\Omega;X^{\mathrm{Tr}}_{\kappa,p})} + \|\widetilde{f}\|_{L^p((\!(0,T)\!),w_\kappa;X_0)} \\ &\qquad + \|\widetilde{g}\|_{L^p((\!(0,T)\!),w_\kappa;\gamma(H,X_{1/2}))}), \\ &\leq \widetilde{C}K + \widetilde{C}L_\varepsilon\|v\|_{L^p((\!(0,T)\!),w_\kappa;X_0)} + \frac{1}{2}\widetilde{C}\|v\|_{L^p((\!(0,T)\!),w_\kappa;X_1)} \\ &\leq \widehat{C}K + \widehat{C}L_\varepsilon\|v\|_{L^p((\!(0,T)\!),w_\kappa;X_0)},\end{aligned}$$

where $\widehat{C} = \widetilde{C}(1 + C)$. Since $T > 0$ was arbitrary letting $y(t) = \|v\|^p_{L^p(\Omega;C([0,t];X^{\mathrm{Tr}}_{\kappa,p}))}$ it follows that for all $t \in (0, T]$,

$$y(t) \leq 2^{p-1}\widehat{C}^p K^p + 2^{p-1}\widehat{C}^p L^p_\varepsilon \int_0^t y(s)\, ds.$$

Thus, Gronwall's inequality implies $y(t) \leq 2^{p-1}\widehat{C}^p K^p e^{2^{p-1}\widehat{C}^p L^p_\varepsilon t}$. This gives

$$\|v\|_{L^p(\Omega;C([0,T];X^{\mathrm{Tr}}_{\kappa,p}))} \leq 2\widehat{C}Ke^{\frac{1}{p}2^{p-1}\widehat{C}^p L^p_\varepsilon t} := K\overline{C}_T.$$

Therefore, by $X^{\mathsf{Tr}}_{\kappa,p} \hookrightarrow X_0$ with embedding constant M, from (6.38) we obtain that

$$\|v\|_{L^p((0,T),w_\kappa;X_1))} \leq 2CK + 2CL_\varepsilon MK\overline{C}_T T^{1/p}$$

Since $u = v$ on $[\![0,\tau_n]\!]$ letting $n \to \infty$ the following energy estimate follows

$$\|u\|_{L^p((0,\sigma),w_\kappa;X_1))} \leq 2CK + 2CL_\varepsilon MK\overline{C}_T T^{1/p} \tag{6.39}$$

Step 3: From the estimate (6.39) and (6.36) we obtain that for a suitable $\widetilde{C}_T$

$$\Big\| \|F(\cdot,u)\|_{X_0} + \|G(\cdot,u)\|_{\gamma(H,X_{1/2})} \Big\|_{L^p((0,\sigma),w^s_\kappa))} \leq \widetilde{C}_T K < \infty.$$

Therefore, Theorem 6.3.7(2) implies $\sigma = T$ a.s. Furthermore, (6.37) follows from the latter estimate, (6.35) and Proposition 6.2.6. □

6.4 Proofs of Theorems 6.3.6, 6.3.7 and 6.3.8

In this section we have collected the proofs of the blow-up criteria stated in Section 6.3. The proofs are technical and require some preparations. In Section 6.4.1 we will first obtain a local existence result for (6.22) starting at a random initial time. It plays a key role in the proof of Theorem 6.3.6(1), which in turn is a central step in proving all the other blow-up criteria.

6.4.1 Local existence when starting at a random time

In this subsection, for a stopping time τ, we consider

$$\begin{cases} du + A(\cdot,u)dt = (F(\cdot,u)+f)dt + (B(\cdot,u)u + G(\cdot,u)+g)dW_H, \\ u(\tau) = u_\tau; \end{cases} \tag{6.40}$$

on $[\![\tau,T]\!]$. To define an L^p_κ-local solution to (6.40) on $[\![\tau,T]\!]$, one can just replace the initial time 0 by τ in Definition 4.3.4.

The following is the natural extension of the local existence part of Theorem 6.3.1 to the case of random initial times (6.40). It will be used to prove the blow-up results of Theorem 6.3.6.

Proposition 6.4.1 (Local existence starting at a random initial time)**.** *Let Assumption* **(H)** *be satisfied. Let τ be a stopping time with values in $[s,T]$, where we assume that τ takes values in a finite set if $\kappa > 0$. Assume that $u_\tau \in L^\infty_{\mathscr{F}_\tau}(\Omega;X^{\mathsf{Tr}}_{\kappa,p})$ and $(A(\cdot,u_\tau)|_{[\![\tau,T]\!]}, B(\cdot,u_\tau)|_{[\![\tau,T]\!]}) \in \mathcal{SMR}^\bullet_{p,\kappa}(\tau,T)$. Then there exists an L^p_κ-local solution (u,σ) to (6.40) on $[\![\tau,T]\!]$ such that $\sigma > \tau$ a.s. on the set $\{\tau < T\}$.*

The analogues assertions of Theorem 6.3.1 for (6.40) hold as well, but since these results will not be needed we do not consider this.

To prove of Proposition 6.4.1, we use a variation of the method in Theorem 4.3.5. As in Chapter 4, we introduce the space $\mathfrak{X}$ which allows to control the nonlinearity F_c, G_c. Recall that $\{\rho^\star_j, r_j, r'_j\}_{j\in\{1,\dots,m_F+m_G\}}$ have been defined in (4.29)-(4.30). Recall that $\frac{1}{r_j} + \frac{1}{r'_j} = 1$ for all $j \in \{1,\dots,m_F+m_G\}$. Finally, for each $0 \leq a < b \leq \infty$, we set

$$\mathfrak{X}(a,b) := \Big(\bigcap_{j=1}^{m_F+m_G} L^{pr_j}(a,b,w^a_\kappa;X_{\beta_j}) \Big) \cap \Big(\bigcap_{j=1}^{m_F+m_G} L^{\rho^\star_j pr'_j}(a,b,w^a_\kappa;X_{\varphi_j}) \Big). \tag{6.41}$$

Setting $\mathfrak{X}(T) := \mathfrak{X}(0,T)$, for all $T > 0$, our notation is consistent with (4.31). The following result was proven in Lemma 4.3.9.

Lemma 6.4.2. *Let Assumption 4.2.1, (HF) and (HG) be satisfied. Let $0 < a < b \leq T < \infty$ and let r_j, r'_j, $\mathfrak{X}(a,b)$ be as in (4.30) and (6.41) respectively. Then the following hold:*

(1) *If $p>2$, $\kappa\in[0,\frac{p}{2}-1)$, and $\mathcal{A}\in\{{}_0H,H\}$, then for any $\delta\in(\frac{1+\kappa}{p},\frac12)$,*

$$\mathcal{A}^{\delta,p}(a,b;w_\kappa^a;X_{1-\delta})\cap L^p(a,b,w_\kappa^a;X_1)\hookrightarrow\mathfrak{X}(a,b),$$

(2) *If $p=2$, and $\kappa=0$, then $C([a,b];X_{1/2})\cap L^2(a,b;X_1)\hookrightarrow\mathfrak{X}(a,b)$.*

Finally, if in (1) $\mathcal{A}={}_0H$, then the constants in the embeddings (1)-(2) can be chosen to be independent of $b-a>0$.

The following lemma contains the key estimate for the proof of Proposition 6.4.1. It is not immediate from Lemma 6.4.2, since we require uniformity in the constants if $|b-a|$ tends to zero.

Lemma 6.4.3. *Let Assumption 4.2.1, (HF) and (HG) be satisfied. Let $0\le a<b<T<\infty$ and let σ be a stopping time with values in $[a,b]$, where we assume that σ takes values in a finite set if $\kappa>0$. Let $(A,B)\in\mathcal{SMR}^\bullet_{p,\kappa}(\sigma,T)$. Let either $p>2$, $\kappa\in[0,\frac{p}{2}-1)$ and $\delta\in(\frac{1+\kappa}{p},\frac12)$ fixed or $p=2$, $\kappa=0$ and $\delta\in(0,\frac12)$. Set*

$$K_{(A,B)}:=\max\{K^{\det,\delta,p,\kappa}_{(A,B)},K^{\mathsf{sto},\delta,p,\kappa}_{(A,B)}\}.$$

Then there exists a constant $C>0$ independent of a,b,σ such that for each (u_σ,f,g) which belongs to (6.18),

$$\begin{aligned}\|\mathscr{R}_\sigma(u_\sigma,f,g)\|_{L^p(\Omega;\mathfrak{X}(\sigma,b)\cap L^p(\sigma,b,w_\kappa^\sigma;X_1)\cap C([\sigma,b];X^{\mathsf{Tr}}_{\kappa,p}))}\\ \le C(1+K_{(A,B)})(\|u_\sigma\|_{L^p(\Omega;X^{\mathsf{Tr}}_{\kappa,p})}+\|f\|_{L^p((\!|\sigma,b|\!),w^\sigma_\kappa;X_0)}\\ +\|g\|_{L^p((\!|\sigma,b|\!),w^\sigma_\kappa;\gamma(H,X_{1/2}))}),\end{aligned}\tag{6.42}$$

where $\mathscr{R}_\sigma:=\mathscr{R}_{\sigma,(A,B)}$ is the solution operator associated to (A,B).

Proof. We only consider the case $p>2$. From Proposition 6.2.6 we see that the constants in the estimates for the $L^p([\![\sigma,b]\!],w_\kappa^\sigma;X_1)$ and $L^p(\Omega;C([\sigma,b];X^{\mathsf{Tr}}_{\kappa,p}))$ norm of u do not depend on a,b,σ. It remains to estimate the $L^p(\Omega;\mathfrak{X}(\sigma,b))$-norm of $u:=\mathscr{R}_\sigma(u_\sigma,f,g)$, and for this we will reduce to the case with zero initial data. As in the proof of Proposition 6.2.6 we can assume that $u_\sigma=\sum_{j=1}^N\mathbf{1}_{\mathcal{U}_j}x_j$ is simple and set $v_1=\sum_{j=1}^N\mathbf{1}_{\mathcal{U}_j}h_j(\cdot-\sigma)$. The proof of Proposition 6.2.6 shows that $u=v_1+v_2$ and that on $\mathcal{U}_j$,

$$\begin{aligned}\|v_1\|_{\mathfrak{X}(\sigma,b)}&\le\|t\mapsto h_j(t-\sigma)\|_{\mathfrak{X}(\sigma,\infty)}\\&=\|h_j\|_{\mathfrak{X}(0,\infty)}\lesssim_{\widetilde{A}}\|h_j\|_{W^{1,p}(\mathbb{R}_+,w_\kappa;X_0)\cap L^p(\mathbb{R}_+,w_\kappa;X_1)}\lesssim_{\widetilde{A}}\|u_\sigma\|_{X^{\mathsf{Tr}}_{\kappa,p}};\end{aligned}$$

where we used Lemmas 6.1.4 and 4.3.9 and the same argument as the proof of Proposition 6.2.6. Taking $L^p(\Omega)$-norms we obtain $\|v_1\|_{L^p(\Omega;\mathfrak{X}(\sigma,b))}\lesssim\|u_\sigma\|_{L^p(\Omega;X^{\mathsf{Tr}}_{\kappa,p})}$, where the implicit constant does not depend on σ,a,b. To estimate v_2, choosing any $\delta\in(\frac{1+\kappa}{p},\frac12)$, by Lemma 4.3.9 we find

$$\begin{aligned}\|v_2\|_{L^p(\Omega;\mathfrak{X}(\sigma,b))}&\lesssim\|v_2\|_{L^p(\Omega;{}_0H^{\delta,p}(\sigma,b,w_\kappa^\sigma;X_{1-\delta})\cap L^p(\sigma,b,w^\sigma_\kappa;X_1))}\\&\lesssim2\widetilde{C}_0K_{(A,B)}\|u_\sigma\|_{L^p(\Omega;X^{\mathsf{Tr}}_{\kappa,p})}+K_{(A,B)}\|f\|_{L^p((\!|\sigma,b|\!),w^\sigma_\kappa;X_0)}\\&\quad+K_{(A,B)}\|g\|_{L^p((\!|\sigma,b|\!),w^\sigma_\kappa;\gamma(H,X_{1/2}))},\end{aligned}$$

where we used (6.16). Combining the estimates for v_1 and v_2, the result follows. ☐

After these preparations we can prove Proposition 6.4.1.

Proof of Proposition 6.4.1. The proof is a variation of the argument in Step 1 and 4 in the proof of Theorem 4.3.5. For each $\lambda\in(0,1)$, we look at the following truncation of (6.40) for $t\in[\tau,T]$ a.s.

$$\begin{cases}du+A(\cdot,u_\tau)udt=(\widetilde{F}_\lambda(u)+\widetilde{f})dt+(B(\cdot,u_\tau)u+\widetilde{G}_\lambda(u)+\widetilde{g})dW_H,\\ u(\tau)=u_\tau,\end{cases}$$

where

$$\begin{aligned}\widetilde{F}_\lambda(u) &:= \Theta_\lambda(\cdot,u_\tau,u)[F_c(\cdot,u)-F_c(\cdot,0)]\\ &\quad+\Psi_\lambda(\cdot,u_\tau,u)[(A(\cdot,u_\tau)u-A(\cdot,u)u)+F_{\mathsf{Tr}}(\cdot,u)-F_{\mathsf{Tr}}(\cdot,u_\tau)],\\ \widetilde{G}_\lambda(u) &:= \Theta_\lambda(\cdot,u_\tau,u)[G_c(\cdot,u)-G_c(\cdot,0)]\\ &\quad+\Psi_\lambda(\cdot,u_\tau,u)[(B(\cdot,u_\tau)u-B(\cdot,u)u)+G_{\mathsf{Tr}}(\cdot,u)-G_{\mathsf{Tr}}(\cdot,u_\tau)],\\ \widetilde{f} &:= f+F_c(\cdot,0)+F_{\mathsf{Tr}}(\cdot,u_\tau),\\ \widetilde{g} &:= g+G_c(\cdot,0)+G_{\mathsf{Tr}}(\cdot,u_\tau).\end{aligned} \tag{6.43}$$

Here, for $\xi_\lambda := \xi(\cdot/\lambda)$, $\xi\in W^{1,\infty}([0,\infty))$, $\xi=1$ on $[0,1]$, $\xi=0$ on $[1,\infty)$ and linear on $[1,2]$, we have set, a.s. for all $t\in[\tau,T]$,

$$\begin{aligned}\Theta_\lambda(t,u_\tau,u) &:= \xi_\lambda\Big(\|u\|_{\mathfrak{X}(\tau,t)}+\sup_{s\in[\tau,t]}\|u(s)-u_\tau\|_{X^{\mathsf{Tr}}_{\kappa,p}}\Big),\\ \Psi_\lambda(t,u_\tau,u) &:= \xi_\lambda\Big(\|u\|_{L^p(\tau,t,w^\tau_\kappa;X_1)}+\sup_{s\in[\tau,t]}\|u(s)-u_\tau\|_{X^{\mathsf{Tr}}_{\kappa,p}}\Big).\end{aligned} \tag{6.44}$$

Let $\mathscr{R}_\tau := \mathscr{R}_{(A(\cdot,u_\tau),B(\cdot,u_\tau))}$ be the solution operator associated to the couple $(A(\cdot,u_\tau),B(\cdot,u_\tau))$, see (6.17). For any $T'\in(0,T]$ we introduce the Banach space

$$\mathscr{Z}_{T'} := L^p_{\mathscr{P}}(\Omega;C([\tau,\mu_{T'}];X^{\mathsf{Tr}}_{\kappa,p})\cap\mathfrak{X}(\tau,\mu_{T'})\cap L^p(\tau,\mu_{T'},w^\tau_\kappa;X_1)),$$

where $\mu_{T'}:=T\wedge(\tau+T')$. In the following, for notational simplicity, we write μ instead of $\mu_{T'}$ if no confusion seems likely. Let us consider the map Υ defined as

$$\Upsilon(u)=\mathscr{R}_\tau(u_\tau,\mathbf{1}_{[\![\tau,\mu]\!]}(\widetilde{F}_\lambda(u)+\widetilde{f}),\mathbf{1}_{[\![\tau,\mu]\!]}(\widetilde{G}_\lambda(u)+\widetilde{g})). \tag{6.45}$$

For the sake of clarity, we split the proof into two steps.

Step 1: Υ maps $\mathscr{Z}_{T'}$ into itself for all $T'\in(0,T]$. Moreover, there exists a $T^*\in(0,T]$ and $\lambda^*\in(0,1)$ such that

$$\|\Upsilon(v)-\Upsilon(v')\|_{\mathscr{Z}_{T^*}}\le\frac{1}{2}\|v-v'\|_{\mathscr{Z}_{T^*}},\qquad\text{for all } v,v'\in\mathscr{Z}_{T^*}. \tag{6.46}$$

Let us note that by a translation argument and the pointwise estimates w.r.t. $\omega\in\Omega$ in Lemma 4.3.13 and 4.3.15, one can check that for all $v,v'\in\mathscr{Z}_{T'}$,

$$\|\widetilde{F}_\lambda(v)\|_{L^p((\!|\tau,\mu|\!),w^\tau_\kappa;X_0)}+\|\widetilde{G}_\lambda(v)\|_{L^p((\!|\tau,\mu|\!),w^\tau_\kappa;\gamma(H,X_{1/2}))}\le C_\lambda,$$

$$\begin{aligned}\|\widetilde{F}_\lambda(v)-\widetilde{F}_\lambda(v')\|_{L^p((\!|\tau,\mu|\!),w^\tau_\kappa;X_0)}+\|\widetilde{G}_\lambda(v)-\widetilde{G}_\lambda(v')\|_{L^p((\!|\tau,\mu|\!),w^\tau_\kappa;\gamma(H,X_{1/2}))}&\\ \le L_{\lambda,T}\|v-v'\|_{\mathscr{Z}_{T'}}.&\end{aligned}$$

In addition, for each $\varepsilon>0$ there exists a $\bar{\lambda}(\varepsilon)>0$ and $\bar{T}(\varepsilon)\in(0,T]$ such that

$$L_{\lambda,T}<\varepsilon,\qquad\text{for all }\lambda\in(0,\bar{\lambda}],\text{ and }T\in(0,\bar{T}].$$

We will only prove (6.46). The fact that Υ maps $\mathscr{Z}_{T'}$ into itself can be proved in a similar way. Let K be the least constant in (6.42) with (A,B), σ replaced by $(A(\cdot,u_\tau),B(\cdot,u_\tau))$, τ. Choose $\varepsilon^*>0$ such that $4KL_{\lambda^*,T^*}\le 1$ where $\lambda^*:=\bar{\lambda}(\varepsilon^*)$ and $T^*:=\bar{T}(\varepsilon^*)$. Thus, Lemma 6.4.3 and the previous choices yield

$$\begin{aligned}&\|\Upsilon(v)-\Upsilon(v')\|_{\mathscr{Z}_{T^*}}\\ &\quad=\|\mathscr{R}_\tau(0,\mathbf{1}_{[\![\tau,\mu]\!]}(\widetilde{F}_\lambda(v)-\widetilde{F}_\lambda(v')),\mathbf{1}_{[\![\tau,\mu]\!]}(\widetilde{G}_\lambda(v)-\widetilde{G}_\lambda(v')))\|_{\mathscr{Z}_{T^*}}\\ &\quad\le K\big(\|\widetilde{F}_\lambda(v)-\widetilde{F}_\lambda(v')\|_{L^p((\!|\tau,\mu|\!),w^\tau_\kappa;X_0)}+\|\widetilde{G}_\lambda(v)-\widetilde{G}_\lambda(v')\|_{L^p((\!|\tau,\mu|\!),w^\tau_\kappa;\gamma(H,X_{1/2}))}\big)\end{aligned}$$

$$\leq \frac{1}{2}\|v - v'\|_{\mathcal{Z}_{T^*}}.$$

Step 2: Conclusion. Let λ^*, T^* be as in Step 1. The conclusion of step 1 ensures that Υ is a contraction on $\mathcal{Z}_{T^*}$, and thus there exists a fixed point of the map Υ on $\mathcal{Z}_{T^*}$ which will be denoted by U. Setting

$$\nu := \inf\Big\{t \in [\tau, T] \,:\, \|U\|_{L^p(\tau,t,w_\kappa^\tau;X_1)\cap\mathfrak{X}(\tau,t)} + \sup_{s\in[\tau,t)}\|U(t) - u_\tau\| > \lambda^*\Big\}.$$

Then ν is a stopping time and $\nu > \tau$ a.s. on $\{\tau < T\}$. Moreover, as in Step 4 in the proof of Theorem 4.3.5, one obtains $u := U|_{[\![\tau,\nu]\!]}$ is an L^p_κ-local solution to (6.40). This follows since by (6.44), a.s. for all $t \in [\tau, \nu]$,

$$\Theta_{\lambda^*}(t, u_\tau, U) = 1, \qquad \Psi_{\lambda^*}(t, u_\tau, U) = 1.$$

By (6.43) the latter implies, a.s. for all $t \in [\tau, \nu]$,

$$\widetilde{F}_{\lambda^*}(U) = A(\cdot, u_\tau)U - A(\cdot, U)U + F_c(\cdot, U) - F_c(\cdot, 0) + F_{\mathsf{Tr}}(\cdot, U) - F_{\mathsf{Tr}}(\cdot, u_\tau),$$
$$\widetilde{G}_{\lambda^*}(U) = B(\cdot, u_\tau)U - B(\cdot, U)U + G_c(\cdot, U) - G_c(\cdot, 0) + G_{\mathsf{Tr}}(\cdot, U) - G_{\mathsf{Tr}}(\cdot, u_\tau).$$

Thus, (6.45) and Proposition 6.2.6 ensure that u is an L^p_κ-local solution to (6.22). □

6.4.2 Proofs of Theorems 6.3.6(1) and 6.3.7(1)-(2)

To prove Theorem 6.3.6(1), we argue by contradiction. If the probability would be nonzero, we will obtain a new equation on a set of positive probability at a random initial time. Using Proposition 6.4.1 we extend the solution which gives a contradiction with maximality. Note that Theorem 6.3.7(1) is just a special case of Theorem 6.3.6(1), and thus does not require further proof.

Proof of Theorem 6.3.6(1). By a translation argument we may assume that $s = 0$. Moreover, we will only consider $p > 2$, since the other case is simpler.

Step 1: Let $M \in \mathbb{N}$, $\eta > 0$ and $r \in [\eta, T]$. If μ is a stopping time with values in $[r, T]$, $u_\mu \in L^\infty_{\mathscr{F}_\mu}(\Omega; X_p^{\mathsf{Tr}})$ and $u_r \in L^\infty_{\mathscr{F}_r}(\Omega; X_p^{\mathsf{Tr}})$ are such that $u_\mu, u_r \in B_{L^\infty(\Omega;X^{\mathsf{Tr}}_{\kappa,p})}(M)$ and

$$\|u_\mu - u_r\|_{L^\infty(\Omega;X^{\mathsf{Tr}}_{\kappa,p})} \leq \frac{1}{4K_{M,\eta}L_M},$$

then one has $(A(\cdot, u_\mu), B(\cdot, u_\mu)) \in \mathcal{SMR}_p^\bullet(\mu, T)$, where L_M and $K_{M,\eta} = K^0_{M,\eta}$ are as in (HA) and Assumption 6.3.2 with $\ell = 0$, respectively. To prove the result, we use the perturbation result of Corollary 6.2.11. Note that by (HA), for $x \in X_1$, for all $t \in (r, T)$ and a.s.

$$\|(A(t, u_r) - A(t, u_\mu))x\|_{X_0} \leq L_M\|u_\mu - u_r\|_{X^{\mathsf{Tr}}_{\kappa,p}}\|x\|_{X_1} \leq \frac{1}{4K_{M,\eta}}\|x\|_{X_1}.$$

Similarly, $\|(B(t, u_r) - B(t, u_\mu))x\|_{\gamma(H,X_{1/2})} \leq 1/(4K_{M,\eta})\|x\|_{X_1}$ for all $t \in (r, T)$ a.s. Assumption 6.3.2 for $\ell = 0$ and Proposition 6.2.9 imply that $(A(\cdot, u_r)|_{[\![\mu,T]\!]}, B(\cdot, u_r)|_{[\![\mu,T]\!]})$ is in $\mathcal{SMR}_p^\bullet(\mu, T)$ with

$$\max\{K^{\mathsf{det},0,p,0}_{(A(\cdot,u_r)|_{[\![\mu,T]\!]},B(\cdot,u_r)|_{[\![\mu,T]\!]})}, K^{\mathsf{sto},0,p,0}_{(A(\cdot,u_r)|_{[\![\mu,T]\!]},B(\cdot,u_r)|_{[\![\mu,T]\!]})}\} \leq K_{M,\eta},$$

The claim follows from the previous estimates and Corollary 6.2.11 with

$$\delta_{A,B} \leq K_{M,\eta}\frac{1}{4K_{M,\eta}} + K_{M,\eta}\frac{1}{4K_{M,\eta}} = \frac{1}{2}.$$

Step 2: Conclusion. By contradiction assume that $\mathbb{P}(\mathcal{O}) > 0$, where

$$\mathcal{O} := \Big\{\sigma < T, \lim_{t\uparrow\sigma} u(t) \text{ exists in } X_p^{\mathsf{Tr}}\Big\} \in \mathscr{F}_\sigma.$$

Since $\sigma > 0$ a.s. there exists an $\eta > 0$ such that $\mathbb{P}(\mathcal{O} \cap \{\sigma > \eta\}) > 0$. Moreover, by Egorov's theorem, there exist $\mathcal{V} \subseteq \mathcal{O} \cap \{\sigma > \eta\}$ and $M \in \mathbb{N}$ such that $\mathcal{V} \in \mathscr{F}_\sigma$, $\mathbb{P}(\mathcal{V}) > 0$ and

$$\|u\|_{C([\eta,\sigma];X_p^{\mathsf{Tr}})} \leq MC_{p,\kappa}^{-1}, \quad \text{and} \quad \lim_{n\to\infty} \sup_{\mathcal{V}} \sup_{s\in[\sigma-\frac{1}{n},\sigma]} \|u(s) - u(\sigma)\|_{X_p^{\mathsf{Tr}}} = 0, \tag{6.47}$$

where we set $u(\sigma) := \lim_{t\uparrow\sigma} u(t)$ on $\mathcal{O}$ and $C_{\kappa,p}$ denotes constant in the embedding $X_p^{\mathsf{Tr}} \hookrightarrow X_{\kappa,p}^{\mathsf{Tr}}$. Let $N \in \mathbb{N}$ be such that $\eta > \frac{1}{N}$ and

$$\sup_{s\in[\sigma-\frac{1}{N},\sigma]} \|u(s) - u(\sigma)\|_{X_p^{\mathsf{Tr}}} \leq \frac{1}{4K_{M,\eta}L_M C_{\kappa,p}} \qquad \text{on } \mathcal{V}. \tag{6.48}$$

Set $\Xi := \operatorname{ess\,sup}_{\mathcal{V}} \sigma$ and fix $r \in (\Xi - \frac{1}{N}, \Xi)$ such that $r \geq \eta$, and set $\mathcal{U} := \mathcal{V} \cap \{\sigma > r\}$. Then $\mathcal{U} \in \mathscr{F}_\sigma \cap \mathscr{F}_r$, and by definition of essential supremum one has $\mathbb{P}(\mathcal{U}) > 0$.

Set $\mu = \sigma \mathbf{1}_{\mathcal{U}} + r\mathbf{1}_{\Omega\setminus\mathcal{U}}$ and $v_\mu := \mathbf{1}_{\mathcal{U}} u(\sigma)$. Then $\mu \in [r, T]$ and $v_\mu \in L^\infty_{\mathscr{F}_\mu}(\Omega; X_p^{\mathsf{Tr}})$. By (6.48) one has

$$\|u(r) - v_\mu\|_{X_{\kappa,p}^{\mathsf{Tr}}} \leq C_{\kappa,p}\|u(r) - v_\sigma\|_{X_p^{\mathsf{Tr}}} \leq \frac{1}{4K_{M,\eta}L_M} \quad \text{on } \mathcal{U},$$

and by the first estimate in (6.47), $\|\mathbf{1}_{\mathcal{U}} u(r)\|_{X_p^{\mathsf{Tr}}} \leq M$ and $\|v_\mu\|_{X_p^{\mathsf{Tr}}} \leq M$. Applying Step 1 to $(\mathbf{1}_{\mathcal{U}} u(r), v_\mu)$, we obtain that $(A(\cdot, v_\mu), B(\cdot, v_\mu)) \in \mathcal{SMR}_p^\bullet(\mu, T)$. Thus, Proposition 6.4.1 ensures that existence of an L^p_κ-local solution (v, ξ) to

$$\begin{cases} dv + A(\cdot, v)vdt = (F(\cdot, v) + f)dt + (B(\cdot, v)v + G(\cdot, v) + g)dW_H(t), \\ v(\mu) = v_\mu, \end{cases} \tag{6.49}$$

on $[\![\mu, T]\!]$, and where $\xi > \mu$ a.s. Note that $v_\mu = u(\sigma)$ on $\mathcal{U}$. Set

$$\widetilde{u} = u\mathbf{1}_{[\![0,\sigma]\!]} + v\mathbf{1}_{\mathcal{U}\times[\sigma,\xi)} \quad \text{and} \quad \widetilde{\sigma} := \mathbf{1}_{\Omega\setminus\mathcal{U}}\sigma + \mathbf{1}_{\mathcal{U}}\xi$$

Then $(\widetilde{u}, \widetilde{\sigma})$ is an L^p_κ-local solution to (6.22) which extends (u, σ) on $\mathcal{U}$. Since $\mathbb{P}(\mathcal{U}) > 0$, this contradicts the maximality of (u, σ) and this gives the desired contradiction. ☐

Remark 6.4.4. Suppose that we know that the maximal solution satisfies

$$u(t) \in \mathcal{C} \text{ a.s. for all } t \in (0, \sigma) \text{ where } \mathcal{C} \subseteq X_p^{\mathsf{Tr}} \text{ is closed subset.} \tag{6.50}$$

Then in Assumption 6.3.2 we only need to consider $v \in \mathcal{C}$ a.s. In this way Theorem 6.3.6(1) remains true. Indeed, one can repeat Step 1 for u_r, u_μ satisfying $u_r, u_\mu \in \mathcal{C}$ a.s. and replacing v_μ in (6.49) by $\mathbf{1}_{\mathcal{U}} u(\sigma) + \mathbf{1}_{\Omega\setminus\mathcal{U}} x$, where $x \in \mathcal{C}$. The same extension holds for the other assertions in Theorem 6.3.6.

Next we will prove Theorem 6.3.7(2). For this we only need Theorem 6.3.6(1) and the following elementary lemma (see Lemma 4.3.11 for a similar lemma).

Lemma 6.4.5. *Let the hypothesis (HF)-(HG) be satisfied. Let $s \leq a < b \leq T < \infty$ and $N \in \mathbb{N}$ be fixed. Let $\zeta := 1 + \max\{\rho_j : j \in \{1, \dots, m_F + m_G\}\}$. Then for all $h \in C([a, b]; X_{\kappa,p}^{\mathsf{Tr}}) \cap \mathfrak{X}(a, b)$ which satisfy a.s. $\|h\|_{C([a,b];X_{\kappa,p}^{\mathsf{Tr}})} \leq N$, one has a.s.*

$$\|F_c(\cdot, h)\|_{L^p(a,b,w_\kappa^a;X_0)} + \|G_c(\cdot, h)\|_{L^p(a,b,w_\kappa^a;\gamma(H,X_{1/2}))} \leq c_{a,b}(1 + \|h\|_{\mathfrak{X}(a,b)} + \|h\|^\zeta_{\mathfrak{X}(a,b)}),$$

where $c_{a,b} = c(|a - b|, N) > 0$ is independent of f and satisfies $c(\delta_1, N) \leq c(\delta_2, N)$ for all $0 \leq \delta_1 \leq \delta_2$. Moreover, if (4.17) and (4.19) are satisfied with constants $C_{c,n}$ independent of $n \geq 1$, then $c(b - a, N)$ can be chosen independent of N.

Proof. We prove the estimate for F_c in the case $m_F = 1$. The other cases are similar. Let N, h be as in the statement. Hypothesis (HF) ensures that for a.a. $\omega \in \Omega$ and all $t \in [a, b]$,

$$\|F_c(t, \omega, h(t, \omega))\|_{X_0} \leq C_{c,N}(1 + \|h(t, \omega)\|^{\rho}_{X_\varphi})\|h(t, \omega)\|_{X_\beta} + C_{c,N}.$$

where $\beta := \beta_1$, $\varphi := \varphi_1$ and $\rho := \rho_1$. By Hölder's inequality with exponent $r' := r_1', r := r_1$ (see (4.30) and the text below it) we get

$$\begin{aligned}\|F_c(\cdot, h)\|_{L^p(a,b,w^a_\kappa;X_0)} \leq C_{c,N}\big(&\|h\|_{L^p(a,b,w^a_\kappa;X_\beta)} + \\ &+ \|h\|^{\rho}_{L^{\rho p r'}(a,b,w^a_\kappa;X_\varphi)}\|h\|_{L^{pr}(a,b,w^a_\kappa;X_\beta)} + |b-a|^{1/p}\big).\end{aligned}$$

Since $\kappa \geq 0$, $r > 1$ and $\rho^\star := \rho_1^\star \geq \rho$ (see (4.29)), Hölder's inequality ensures that

$$\begin{aligned}\|h\|_{L^p(a,b,w^a_\kappa;X_\beta)} &\leq C_{b-a}\|h\|_{L^{pr}(a,b,w^a_\kappa;X_\beta)}\\ \|h\|_{L^{\rho p r'}(a,b,w^a_\kappa;X_\varphi)} &\leq C_{b-a}\|h\|_{L^{\rho^\star p r'}(a,b,w^a_\kappa;X_\varphi)},\end{aligned}$$

where $C_{\delta_1} \leq C_{\delta_2}$ for $0 \leq \delta_1 < \delta_2 < \infty$ and $\sup_{\delta \in (0,T)} C_\delta < \infty$ due to the assumption $T < \infty$. By (6.41) the previous inequalities imply the claimed estimate. ☐

Proof of Theorem 6.3.7(2). As before we assume $s = 0$, and we only consider $p > 2$. By (6.28) and Proposition 6.2.9, (A, B) satisfies Assumption 6.3.2 for $\ell = 0$. Therefore, Theorem 6.3.6(1) is applicable. For the reader's convenience, we divide the remaining part of the proof into several steps.

Step 1: Proof of Theorem 6.3.7(2). Let us argue by contradiction. Thus, we assume that $\mathbb{P}(\mathcal{O}) > 0$ where

$$\mathcal{O} := \{\sigma < T, \mathcal{N}^\kappa(u; 0, \sigma) < \infty\}.$$

Since $\sigma > 0$ a.s. by Theorem 6.3.1, it follows that there exist $\eta, M > 0$ such that $\mathbb{P}(\mathcal{V}) > 0$ where

$$\mathcal{V} := \{\eta < \sigma < T, \mathcal{N}^\kappa(u; 0, \sigma) < M\}.$$

Define a stopping time by

$$\nu := \inf\{t \in [0, \sigma) : \mathcal{N}^\kappa(u; 0, t) \geq M\},$$

where we take $\inf \varnothing := \sigma$. Note that

$$\|\mathbf{1}_{[\![0,\nu]\!]}F(\cdot, u)\|_{L^p(I_T,w_\kappa;X_0)} + \|\mathbf{1}_{[\![0,\nu]\!]}G(\cdot, u)\|_{L^p(I_T,w_\kappa;\gamma(H,X_{1/2}))} \leq M. \tag{6.51}$$

Since u is an L^p_κ-maximal local solution to (6.22), $u|_{[\![0,\nu]\!]}$ is an L^p_κ-local solution to (6.8) on $[\![0, \nu]\!]$. Proposition 6.2.7 and (6.51) gives

$$u = \mathscr{R}_{0,(A,B)}(u_0, \mathbf{1}_{[\![0,\nu]\!]}F(\cdot, u) + f, \mathbf{1}_{[\![0,\nu]\!]}G(\cdot, u) + g), \qquad \text{a.e. on } [\![0, \nu]\!], \tag{6.52}$$

On the other hand, by Proposition 6.2.6(2) and (6.51), the RHS of (6.52) satisfies

$$\mathscr{R}_{0,(A,B)}(u_0, \mathbf{1}_{[\![0,\nu]\!]}F(\cdot, u) + f, \mathbf{1}_{[\![0,\nu]\!]}G(\cdot, u) + g) \in L^p(\Omega; C([\eta, T]; X^{\mathsf{Tr}}_p)).$$

and therefore $\lim_{t\uparrow\sigma} u(t)$ exists in X^{Tr}_p a.s. on $\mathcal{V}$. Therefore,

$$\begin{aligned}0 < \mathbb{P}(\mathcal{V}) &= \mathbb{P}\big(\mathcal{V} \cap \{\sigma < T\} \cap \{\lim_{t\uparrow\sigma} u(t) \text{ exists in } X^{\mathsf{Tr}}_p\}\big)\\ &\leq \mathbb{P}\big(\sigma < T, \lim_{t\uparrow\sigma} u(t) \text{ exists in } X^{\mathsf{Tr}}_p\big) = 0,\end{aligned}$$

where in the last step we used Theorem 6.3.6(1). This contradiction completes the proof of (2). ☐

Remark 6.4.6. Let the assumptions of Theorem 6.3.7 be satisfied. If $F_{\mathsf{Tr}} = G_{\mathsf{Tr}} = 0$ and $C_{c,n}$ in (HF)-(HG) does not depend on $n \geq 1$, then Lemma 6.4.5, we get

$$\mathbb{P}(\sigma < T, \|u\|_{\mathfrak{X}(s,\sigma)} < \infty) = \mathbb{P}(\sigma < T, \mathcal{N}^\kappa(u; s, \sigma) < \infty) = 0,$$

where in the last step we applied Theorem 6.3.7(2)

6.4.3 Proofs of Theorems 6.3.6(2)-(3) and 6.3.7(3)

Using Theorem 6.3.6(1) we can now prove Theorem 6.3.6(2). Unfortunately, the proof is rather technical. It requires several reduction arguments and one key step is to approximate localizing sequences by stopping times taking finitely many values which in turn allow to apply the maximal regularity estimates of Proposition 6.2.8.

Proof of Theorem 6.3.6(2). By a translation argument we may assume $s = 0$. We will only consider $p > 2$ as the other case is similar. By Proposition 6.3.10 it is enough to consider

$$u_0 \in L^\infty_{\mathscr{F}_0}(\Omega; X^{\mathsf{Tr}}_{\kappa,p}),\ f \in L^p(I_T \times \Omega, w_\kappa; X_0), \quad \text{and} \quad g \in L^p(I_T \times \Omega, w_\kappa; \gamma(H, X_{1/2})).$$

Step 1: Setting up the proof by contradiction. We will prove (2) by a contradiction argument. So suppose that $\mathbb{P}(\mathcal{O}) > 0$ where

$$\mathcal{O} := \Big\{\sigma < T,\ \lim_{t\uparrow\sigma} u(t) \text{ exists in } X^{\mathsf{Tr}}_{\kappa,p},\ \mathcal{N}^\kappa_c(u; 0, \sigma) < \infty\Big\} \in \mathscr{F}_\sigma.$$

see (6.24) for the definition of $\mathcal{N}^\kappa_c$. For each $n \geq 1$, let

$$\sigma_n = \inf\Big\{t \in [0, \sigma) \,:\, \|u\|_{L^p(I_t, w_\kappa; X_0) \cap C([0,t]; X^{\mathsf{Tr}}_{\kappa,p})} + \mathcal{N}^\kappa_c(u; 0, t) \geq n\Big\} \wedge \frac{nT}{n+1}, \tag{6.53}$$

and $\inf \varnothing := \sigma$. Then $(\sigma_n)_{n\geq 1}$ is a localizing sequence for (u, σ). By Egorov's theorem and the fact that $\sigma > 0$ a.s., there exist $\eta > 0$, $\mathscr{F}_\sigma \ni \mathcal{V} \subseteq \mathcal{O}$, $M \in \mathbb{N}$ such that $\mathbb{P}(\mathcal{V}) > 0$, $\sigma \geq \eta$ a.s. on $\mathcal{V}$, and

$$\begin{aligned} \|u\|_{C([0,\sigma]; X^{\mathsf{Tr}}_{\kappa,p})} + \mathcal{N}^\kappa_c(u; 0, \sigma) &\leq M \text{ on } \mathcal{V}, \\ \lim_{n\to\infty} \sup_{\mathcal{V}} \sup_{s\in[\sigma_n,\sigma]} \|u(s) - u(\sigma)\|_{X^{\mathsf{Tr}}_{\kappa,p}} &= 0, \end{aligned} \tag{6.54}$$

where we have set $u(\sigma) := \lim_{t\uparrow\sigma} u(t)$ on $\mathcal{O}$. By decreasing η if necessary, we may suppose $\mathbb{P}(\sigma \leq \eta) \leq \frac{1}{4}\mathbb{P}(\mathcal{V})$.

By Corollary 6.3.9, one has $\sigma_n < \sigma$ on $\{\sigma < T\}$ for all $n \geq 1$. Moreover, the definition of σ_n implies $\sigma_n < \sigma$ on the set $\{\sigma = T\}$. Therefore, Lemma 6.1.5 implies there exists a sequence of stopping times $(\widetilde{\sigma}_n)_{n\geq 1}$ such that for each $n \geq 1$, $\widetilde{\sigma}_n$ takes values in a finite subset of $[0, T]$, $\widetilde{\sigma}_n \leq \widetilde{\sigma}_{n+1}$, $\widetilde{\sigma}_n \geq \sigma_n$ and $\mathbb{P}(\widetilde{\sigma}_n \geq \sigma) \leq \frac{1}{4}\mathbb{P}(\mathcal{V})$. Set

$$\sigma'_n = \widetilde{\sigma}_n \vee \eta \quad \text{ for } n \geq 1 \text{ and } \quad \mathcal{V}' := \mathcal{V} \cap (\cap_{n\geq 1}\{\sigma'_n < \sigma\}). \tag{6.55}$$

Then by Proposition 4.1.6, $\mathcal{V}' \in \mathscr{F}_\sigma$, and

$$\mathbb{P}(\mathcal{V}') = \lim_{n\to\infty} \mathbb{P}(\mathcal{V} \cap \{\sigma'_n < \sigma\}) \geq \lim_{n\to\infty} \mathbb{P}(\mathcal{V}) - \mathbb{P}(\sigma'_n \geq \sigma) \geq \frac{1}{2}\mathbb{P}(\mathcal{V}) > 0, \tag{6.56}$$

where in the last step we used

$$\mathbb{P}(\sigma'_n \geq \sigma) \leq \mathbb{P}(\sigma'_n \geq \sigma, \sigma > \eta) + \mathbb{P}(\sigma \leq \eta) \leq \mathbb{P}(\widetilde{\sigma}_n \geq \sigma) + \mathbb{P}(\sigma \leq \eta) \leq \frac{1}{2}\mathbb{P}(\mathcal{V}).$$

Step 2: In this step we will prove that $\mathbb{P}(\mathcal{O}) > 0$ implies

$$\mathbb{P}\Big(\sigma < T,\ \lim_{t\uparrow\sigma} u(t) \text{ exists in } X^{\mathsf{Tr}}_{\kappa,p},\ \|u\|_{L^p(I_\sigma, w_\kappa; X_1) \cap \mathfrak{X}(\sigma)} < \infty\Big) > 0. \tag{6.57}$$

To prove the above, we need some preliminary observations. By (6.54), for each $\varepsilon > 0$ there exists an $N(\varepsilon) \in \mathbb{N}$ such that

$$\sup_{s\in[\sigma_{N(\varepsilon)},\sigma]} \|u(s) - u(\sigma)\|_{X^{\mathsf{Tr}}_{\kappa,p}} < \varepsilon \text{ on } \mathcal{V}. \tag{6.58}$$

For each $\varepsilon > 0$ set $\lambda_\varepsilon = \sigma_{N(\varepsilon)}$, $\lambda'_\varepsilon = \sigma'_{N(\varepsilon)}$ and define the stopping time τ_ε by

$$\begin{aligned}\tau_\varepsilon := \inf\Big\{t \in [\lambda_\varepsilon, \sigma) : &\|u(t) - u(\lambda_\varepsilon)\|_{X^{\mathsf{Tr}}_{\kappa,p}} \geq 2\varepsilon, \\ &\|u\|_{C([0,t];X^{\mathsf{Tr}}_{\kappa,p})} + \mathcal{N}^\kappa_c(u;0,t) \geq M\Big\}\end{aligned} \tag{6.59}$$

where we set $\inf\varnothing := \sigma$. Note that $\tau_\varepsilon = \sigma$ on $\mathcal{V} \supseteq \mathcal{V}'$. Therefore,

$$\mathcal{V}' \subseteq \mathcal{U}_\varepsilon := \{\tau_\varepsilon > \lambda'_\varepsilon\} \in \mathscr{F}_{\lambda'_\varepsilon}.$$

For each $\varepsilon > 0$ we set

$$u_\varepsilon := \mathbf{1}_{\mathcal{U}_\varepsilon} u(\lambda'_\varepsilon) \in L^\infty_{\mathscr{F}_{\lambda'_\varepsilon}}(\Omega; X^{\mathsf{Tr}}_{\kappa,p}).$$

The latter random variable is well defined since $\sigma \geq \tau_\varepsilon > \lambda'_\varepsilon$ on $\mathcal{U}_\varepsilon$, and by (6.59), we have $\|u_\varepsilon\|_{X^{\mathsf{Tr}}_{\kappa,p}} \leq M$. Since $\lambda'_\varepsilon \geq \eta$ (see (6.55)), combining Assumption 6.3.2 with Proposition 6.2.8, we obtain that $(A(\cdot,u_\varepsilon), B(\cdot,u_\varepsilon)) \in \mathcal{SMR}^\bullet_{p,\kappa}(\lambda'_\varepsilon, T)$, and for each $\theta \in [0,\frac{1}{2}) \setminus \{\frac{1+\kappa}{p}\}$,

$$\max\{K^{\mathsf{det},\theta,p,\kappa}_{(A(\cdot,u_\varepsilon),B(\cdot,u_\varepsilon))}, K^{\mathsf{sto},\theta,p,\kappa}_{(A(\cdot,u_\varepsilon),B(\cdot,u_\varepsilon))}\} \leq K^\theta_{M,\eta}, \tag{6.60}$$

where $K^\theta_{M,\eta}$ is as in Assumption 6.3.2 for $\ell = \kappa$. Let us stress that $K^\theta_{M,\eta}$ does not depend on ε. Fix any $\theta \in \{\frac{1+\kappa}{p}, \frac{1}{2})$ and set $K_{M,\eta} = K^0_{M,\eta} + K^\theta_{M,\eta}$. For notational convenience, set $\mathscr{R}_\varepsilon = \mathscr{R}_{\lambda'_\varepsilon,(A(\cdot,u_\varepsilon),B(\cdot,u_\varepsilon))}$.

Lemma 6.4.3 ensures that (6.42) holds with $\mathscr{R}_\sigma$ replaced by $\mathscr{R}_\varepsilon$ and constant $K_{M,\eta} := C(1 + K_{M,\eta})$ which is independent of ε (see (6.60)). Thus, for L_n as in (HA), we set $\varepsilon = 1/(16K_{M,\eta}L_M)$. Let

$$\psi := \mathbf{1}_{\Omega\setminus\mathcal{U}_\varepsilon}\lambda'_\varepsilon + \mathbf{1}_{\mathcal{U}_\varepsilon}\tau_\varepsilon \quad \text{and} \quad \psi_n := \mathbf{1}_{\Omega\setminus\mathcal{U}_\varepsilon}\lambda'_\varepsilon + \mathbf{1}_{\mathcal{U}_\varepsilon}\min\{\max\{\sigma_n,\lambda'_\varepsilon\},\tau_\varepsilon\}. \tag{6.61}$$

Note that $\psi_n \uparrow \psi$ and for each $n \geq 1$, $[\![\lambda'_\varepsilon,\psi_n)\!) \subseteq [\![\lambda'_\varepsilon,\sigma_n)\!)$. Since (u,σ) is an L^p_κ-maximal local solution to (6.22), $(u|_{[\![\lambda'_\varepsilon,\psi)\!)},\psi)$ is an L^p_κ-local solution to

$$\begin{cases} dv + A(\cdot,v)vdt = (F(\cdot,v) + f)dt + (B(\cdot,v)v + G(\cdot,v) + g)dW_H(t), \\ v(\lambda'_\varepsilon) = u_\varepsilon, \end{cases} \tag{6.62}$$

with localizing sequence $(\psi_n)_{n\geq1}$. Here we used that $\sigma > \lambda'_\varepsilon$ on $\mathcal{U}_\varepsilon$ which follows from the definition of $\mathcal{U}_\varepsilon$. Finally, we set

$$\begin{aligned}\Lambda_\varepsilon &:= [\![\lambda'_\varepsilon,\psi)\!) = [\lambda'_\varepsilon,\tau_\varepsilon) \times \mathcal{U}_\varepsilon, \\ \Lambda^n_\varepsilon &:= [\![\lambda'_\varepsilon,\psi_n)\!) = [\lambda'_\varepsilon,\sigma_n \wedge \tau_\varepsilon) \times \mathcal{U}_\varepsilon.\end{aligned}$$

Since $A(\cdot,u) = A(\cdot,u_\varepsilon) + (A(\cdot,u) - A(\cdot,u_\varepsilon))$, $B(\cdot,u) = B(\cdot,u_\varepsilon) + (B(\cdot,u) - B(\cdot,u_\varepsilon))$, by (6.62) and Proposition 6.2.7 one has a.s. on Λ_ε

$$\begin{aligned}\mathbf{1}_{\mathcal{U}_\varepsilon}u &= \mathscr{R}_\varepsilon(u_\varepsilon, \mathbf{1}_{\Lambda^n_\varepsilon}f, \mathbf{1}_{\Lambda^n_\varepsilon}g) \\ &\quad + \mathscr{R}_\varepsilon(0, \mathbf{1}_{\Lambda^n_\varepsilon}(A(\cdot,u_\varepsilon) - A(\cdot,u))u, \mathbf{1}_{\Lambda^n_\varepsilon}(B(\cdot,u_\varepsilon) - B(\cdot,u))u) \\ &\quad + \mathscr{R}_\varepsilon(0, \mathbf{1}_{\Lambda^n_\varepsilon}F_{\mathsf{Tr}}(\cdot,u), \mathbf{1}_{\Lambda^n_\varepsilon}G_{\mathsf{Tr}}(\cdot,u)) \\ &\quad + \mathscr{R}_\varepsilon(0, \mathbf{1}_{\Lambda^n_\varepsilon}F_c(\cdot,u), \mathbf{1}_{\Lambda^n_\varepsilon}G_c(\cdot,u)) \\ &=: I + II + III + IV.\end{aligned} \tag{6.63}$$

Next, we estimate each of the above summands. To make the formulas more readable, in this step, we denote by $\mathcal{Z}$ the space $L^p(\Omega; L^p(\lambda'_\varepsilon,T,w^{\lambda'_\varepsilon}_\kappa;X_1) \cap \mathfrak{X}(\lambda'_\varepsilon,T))$. To begin, by Lemma 6.4.3,

$$\begin{aligned}\|I\|_{\mathcal{Z}} &\leq K_{M,\eta}(\|u_\varepsilon\|_{L^p(\mathcal{U}_\varepsilon;X^{\mathsf{Tr}}_{\kappa,p})} + \|f\|_{L^p(\Lambda^n_\varepsilon,w^{\lambda'_\varepsilon}_\kappa;X_0)} + \|g\|_{L^p(\Lambda^n_\varepsilon,w^{\lambda'_\varepsilon}_\kappa;\gamma(H,X_{1/2}))}) \\ &\leq CK_{M,\eta}(M + \|f\|_{L^p(I_T\times\Omega,w_\kappa;X_0)} + \|g\|_{L^p(I_T\times\Omega,w_\kappa;\gamma(H,X_{1/2}))}).\end{aligned}$$

Again, by Lemma 6.4.3,

$$\begin{aligned}\|II\|_{\mathcal{Z}} \leq K_{M,\eta}(&\|(A(\cdot,u_\varepsilon)-A(\cdot,u))u\|_{L^p(\Lambda^n_\varepsilon,w_\kappa^{\lambda'_\varepsilon};X_0)}\\ &+\|(B(\cdot,u_\varepsilon)-B(\cdot,u))u\|_{L^p(\Lambda^n_\varepsilon,w_\kappa^{\lambda'_\varepsilon};\gamma(H,X_{1/2}))}) \leq \frac{1}{2}\|u\|_{L^p(\Lambda^n_\varepsilon,w_\kappa^{\lambda'_\varepsilon};X_1)},\end{aligned} \tag{6.64}$$

where in the last inequality we used the choice of ε and the fact that

$$\sup_{s\in[\lambda'_\varepsilon,\tau_\varepsilon)}\|u(s)-u(\lambda'_\varepsilon)\|_{X^{\mathsf{Tr}}_{\kappa,p}} \leq 2\sup_{s\in[\lambda_\varepsilon,\tau_\varepsilon)}\|u(s)-u(\lambda_\varepsilon)\|_{X^{\mathsf{Tr}}_{\kappa,p}} \leq 4\varepsilon, \text{ a.s. on } \mathcal{U}_\varepsilon,$$

since $\lambda_\varepsilon \leq \lambda'_\varepsilon$. Similarly, one obtains

$$\begin{aligned}\|III\|_{\mathcal{Z}} &\leq K_{M,\eta}(\|F_{\mathsf{Tr}}(\cdot,u)\|_{L^p(\Lambda^n_\varepsilon,w_\kappa^{\lambda'_\varepsilon};X_0)}+\|G_{\mathsf{Tr}}(\cdot,u)\|_{L^p(\Lambda^n_\varepsilon,w_\kappa^{\lambda'_\varepsilon};X_0)})\\ &\leq 2K_{M,\eta}(1+C_{\mathsf{Tr},M}M),\end{aligned}$$

where in the last estimate we used (HF)-(HG) and (6.54). Finally,

$$\begin{aligned}\|IV\|_{\mathcal{Z}} &\leq K_{M,\eta}(\|F_c(\cdot,u)\|_{L^p(\Lambda^n_\varepsilon,w_\kappa^{\lambda'_\varepsilon};X_0)}+\|G_c(\cdot,u)\|_{L^p(\Lambda^n_\varepsilon,w_\kappa^{\lambda'_\varepsilon};X_0)})\\ &\leq K_{M,\eta}CM,\end{aligned} \tag{6.65}$$

in the last inequality we used (6.24) and the bound in (6.54). By (6.63), and the previous estimates, one obtains that for some $C_1 > 0$ for all $n \geq 1$,

$$\|u\|_{L^p(\mathcal{U}_\varepsilon;L^p(\lambda'_\varepsilon,\sigma_n\wedge\tau_\varepsilon,w_\kappa^{\lambda'_\varepsilon};X_1)\cap\mathfrak{X}(\lambda'_\varepsilon,\sigma_n\wedge\tau_\varepsilon))} \leq C_1, \tag{6.66}$$

and by Fatou's lemma, (6.66) also holds with $\sigma_n \wedge \tau_\varepsilon$ replaced by τ_ε.

Recall that $\tau_\varepsilon|_{\mathcal{V}'} = \sigma|_{\mathcal{V}'}$. Since $\lambda'_\varepsilon < \tau_\varepsilon = \sigma$ on $\mathcal{V}'$, (6.66) (with $n \to \infty$) implies

$$\begin{aligned}&\mathcal{V}'\cap\Big\{\sigma<T,\ \lim_{t\uparrow\sigma}u(t) \text{ exists in } X^{\mathsf{Tr}}_{\kappa,p},\ \|u\|_{L^p(I_\sigma,w_\kappa;X_1)\cap\mathfrak{X}(\sigma)}<\infty\Big\}\\ &=\mathcal{V}'\cap\Big\{\sigma<T,\ \lim_{t\uparrow\sigma}u(t) \text{ exists in } X^{\mathsf{Tr}}_{\kappa,p},\ \|u\|_{L^p(\lambda'_\varepsilon,\sigma,w_\kappa^{\lambda'_\varepsilon};X_1)\cap\mathfrak{X}(\lambda'_\varepsilon,\sigma)}<\infty\Big\}=\mathcal{V}',\end{aligned}$$

By (6.56), (6.57) follows, and this completes the proof of the claim in step 2.

Step 3: In this step we will prove that $\mathbb{P}(\mathcal{O}) > 0$ *implies*

$$\mathbb{P}\Big(\sigma<T, \lim_{t\uparrow\sigma}u(t) \,\text{exists in}\, X^{\mathsf{Tr}}_p\Big) > 0. \tag{6.67}$$

Clearly, this will contradict Theorem 6.3.6(1) and therefore, $\mathbb{P}(\mathcal{O}) = 0$.

By Step 2, we can find $\widetilde{M} \geq M$ (see (6.54)), such that $\mathbb{P}(\mathcal{I}) > 0$, where

$$\mathcal{I} := \big\{\sigma<T,\ \|u\|_{L^p(I_\sigma,w_\kappa;X_1)\cap\mathfrak{X}(\sigma)\cap C(\overline{I}_\sigma;X^{\mathsf{Tr}}_{\kappa,p})}<\widetilde{M}\big\}.$$

Let μ be the stopping time given by

$$\mu := \inf\{t\in[0,\sigma)\,:\,\|u\|_{L^p(I_t,w_\kappa;X_1)\cap\mathfrak{X}(t)\cap C([0,t];X^{\mathsf{Tr}}_{\kappa,p})}\geq M\}, \quad \inf\varnothing := \sigma. \tag{6.68}$$

By construction and (6.54), $\{\mu=\sigma\}\supseteq\mathcal{I}$ and $\mu > 0$ a.s. Since we already reduced to bounded initial values, we have $(A(\cdot,u_0),B(\cdot,u_0))\in\mathcal{SMR}^\bullet_{p,\kappa}(T)$ by (6.25). Set $v := \mathscr{R}_{0,(A(\cdot,u_0),B(\cdot,u_0))}(u_0,f_A,g_B)$ on $[\![0,T]\!]$. Here f_A and g_B are defined by

$$\begin{aligned}f_A &:= \mathbf{1}_{[\![0,\mu]\!]}((A(\cdot,u_0)-A(\cdot,u))u+F(\cdot,u))+f\in L^p_{\mathscr{P}}(I_T\times\Omega,w_\kappa;X_0),\\ g_B &:= \mathbf{1}_{[\![0,\mu]\!]}((B(\cdot,u_0)-B(\cdot,u))u+G(\cdot,u))+g\in L^p_{\mathscr{P}}(I_T\times\Omega,w_\kappa;\gamma(H,X_{1/2})),\end{aligned}$$

where we used Lemma 6.4.5, (6.68) and (HA) to check the required L^p-integrability. Since (u,σ) is an L^p_κ-maximal local solution to (6.22) with $s=0$ it follows from Proposition 6.2.7 that $u=v$ on $[\![0,\mu]\!]$. Since $\sigma>0$ a.s., there exists an $\widetilde{\eta}>0$ such that $\mathbb{P}(\{\sigma>\widetilde{\eta}\}\cap\mathcal{I})>0$. Set $\mathcal{U}:=\{\sigma>\widetilde{\eta}\}\cap\mathcal{I}$. Using the regularity estimate of Proposition 6.2.6(2) (and (6.17)) we obtain

$$\begin{aligned}\|u\|_{L^p(\mathcal{U};C([\widetilde{\eta},\mu];X^{\mathsf{Tr}}_p))} &\leq \|v\|_{L^p(\mathcal{U};C([\widetilde{\eta},T];X^{\mathsf{Tr}}_p))}\\ &\lesssim_{\widetilde{\eta}} \|u_0\|_{L^p(\Omega;X^{\mathsf{Tr}}_{\kappa,p})}+\|f_A\|_{L^p(I_T\times\Omega,w_\kappa;X_0)}+\|g_B\|_{L^p(I_T\times\Omega,w_\kappa;\gamma(H,X_{1/2}))},\end{aligned}$$

which is finite. Since $\{\mu=\sigma\}\supseteq\mathcal{U}$ and $\mathbb{P}(\mathcal{U})>0$, the above estimate gives (6.67). This finishes the proof of Step 3, and therefore the proof of Theorem 6.3.6(2). □

Remark 6.4.7.

- The arguments in the proof of Theorem 6.3.6(2) can be extended to prove (3) in the important case $F_c=G_c=0$. The only difference is in Step 2, where $IV=0$ by assumption. Of course the latter situation is also covered by Theorem 6.3.6(3) which is proved below.
- Similar to Remark 6.4.6, under the assumptions of Theorem 6.3.6,

$$\mathbb{P}\big(\sigma<T,\ \lim_{t\uparrow\sigma}u(t)\text{ exists in }X^{\mathsf{Tr}}_{\kappa,p},\ \|u\|_{\mathfrak{X}(s,\sigma)}<\infty\big)=0 \tag{6.69}$$

  To obtain (6.69) one can repeat the argument of Theorem 6.3.6(2) using Lemma 6.4.5 to get the estimate of IV in (6.65)

To derive the remaining part (3) of Theorem 6.3.6, we will exploit the non-criticality of $X^{\mathsf{Tr}}_{\kappa,p}$ by using Lemma 4.3.11. Also Theorem 6.3.6(2) is applied in a technical but essential step in the proof below.

Proof of Theorem 6.3.6(3). As before in part (2) we assume $s=0$ and $p>2$. By Proposition 6.3.10 we may assume u_0 is bounded, and f and g are L^p-integrable. Set

$$\mathcal{O}:=\Big\{\sigma<T,\ \lim_{t\uparrow\sigma}u(t)\text{ exists in }X^{\mathsf{Tr}}_{\kappa,p}\Big\} \tag{6.70}$$

and suppose that $\mathbb{P}(\mathcal{O})>0$. We will show that this leads to a contradiction.

Let $(\sigma_n)_{n\geq1}$ be the localizing sequence for (u,σ) defined in (6.53). By Egorov's theorem and the fact that $\sigma>0$ a.s., there exist $\eta>0$ and $\mathscr{F}_\sigma\ni\mathcal{V}\subseteq\mathcal{O}$, $M\in\mathbb{N}$ such that $\mathbb{P}(\mathcal{V})>0$, $\sigma>\eta$ a.s. on $\mathcal{V}$ and

$$\begin{aligned}&\sup_{\mathcal{V}}\|u\|_{C(\overline{I}_\sigma;X^{\mathsf{Tr}}_{\kappa,p})}\leq M,\quad\text{on }\mathcal{V}\\ &\lim_{n\to\infty}\sup_{\mathcal{V}}|\sigma_n-\sigma|=0,\qquad \lim_{n\to\infty}\sup_{\mathcal{V}}\sup_{s\in[\sigma_n,\sigma]}\|u(s)-u(\sigma)\|_{X^{\mathsf{Tr}}_{\kappa,p}}=0.\end{aligned}$$

Here, as usual, we have set $u(\sigma):=\lim_{t\uparrow\sigma}u(t)\in X^{\mathsf{Tr}}_{\kappa,p}$ on $\mathcal{V}$. Moreover, we may suppose that $\mathbb{P}(\sigma\leq\eta)\leq\frac14\mathbb{P}(\mathcal{V})$.

As in Step 1 of the proof of Theorem 6.3.6(2), there exists a sequence of stopping times $(\widetilde{\sigma}_n)_{n\geq1}$ such that for each $n\geq1$, $\widetilde{\sigma}_n$ takes values in a finite subset of $[0,T]$, $\widetilde{\sigma}_n\leq\widetilde{\sigma}_{n+1}$, $\widetilde{\sigma}_n\geq\sigma_n$ and $\mathbb{P}(\widetilde{\sigma}_n\geq\sigma)\leq\frac14\mathbb{P}(\mathcal{V})$. Defining σ_n' and $\mathcal{V}'$ as in (6.55), we have $\mathcal{V}'\in\mathscr{F}_\sigma$, and $\mathbb{P}(\mathcal{V}')>0$ as before. Moreover, for each $\varepsilon>0$ there exists an $N(\varepsilon)$ such that on the set $\mathcal{V}$, we have

$$\|u\|_{C(\overline{I}_\sigma;X^{\mathsf{Tr}}_{\kappa,p})}<M,\quad |\sigma_{N(\varepsilon)}-\sigma|<\varepsilon,\quad \sup_{s\in[\sigma_{N(\varepsilon)},\sigma]}\|u(s)-u(\sigma)\|_{X^{\mathsf{Tr}}_{\kappa,p}}<\varepsilon. \tag{6.71}$$

For each $\varepsilon>0$ set $\lambda_\varepsilon=\sigma_{N(\varepsilon)}$, $\lambda_\varepsilon'=\sigma'_{N(\varepsilon)}$ and define the stopping time τ_ε by

$$\tau_\varepsilon:=\inf\Big\{t\in[\lambda_\varepsilon,\sigma):\ \sup_{s\in[\lambda_\varepsilon,t]}\|u(s)-u(\lambda_\varepsilon)\|_{X^{\mathsf{Tr}}_{\kappa,p}}\geq2\varepsilon,\|u\|_{C(\overline{I}_t;X^{\mathsf{Tr}}_{\kappa,p})}\geq M\Big\},$$

where $\inf \varnothing := \sigma$. As in the proof of Theorem 6.3.6(2), $\tau_\varepsilon = \sigma$ a.s. on $\mathcal{V} \supseteq \mathcal{V}'$ for each $\varepsilon > 0$. Moreover, setting $\mathcal{U}_\varepsilon := \{\tau_\varepsilon > \lambda'_\varepsilon\} \in \mathscr{F}_{\lambda'_\varepsilon} \cap \mathscr{F}_{\tau_\varepsilon}$ and $u_\varepsilon := \mathbf{1}_{\mathcal{U}_\varepsilon} u(\lambda'_\varepsilon)$, one has $\mathcal{U}_\varepsilon \supseteq \mathcal{V}'$, u_ε is $\mathscr{F}_{\lambda'_\varepsilon}$-measurable and

$$u_\varepsilon \in B_{L^\infty(\Omega;X^{\mathsf{Tr}}_{\kappa,p})}(M), \qquad \text{for all } \varepsilon > 0, \quad N = N(\varepsilon). \tag{6.72}$$

Again, as in the proof of Theorem 6.3.6(2), combining Assumption 6.3.2 for $\ell = \kappa$, (6.72) and Proposition 6.2.8, one obtains $(A(\cdot, u_\varepsilon), B(\cdot, u_\varepsilon)) \in \mathcal{SMR}^\bullet_{p,\kappa}(\lambda'_\varepsilon, T)$, and for each $\theta \in [0, \frac{1}{2}) \setminus \{\frac{1+\kappa}{p}\}$,

$$\max\{K^{\mathsf{det},\theta,p,\kappa}_{(A(\cdot,u_\varepsilon),B(\cdot,u_\varepsilon))}, K^{\mathsf{sto},\theta,p,\kappa}_{(A(\cdot,u_\varepsilon),B(\cdot,u_\varepsilon))}\} \leq K^\theta_{M,\eta}. \tag{6.73}$$

Here $K^\theta_{M,\eta}$ does not depend on ε. Fix $\theta \in (\frac{1+\kappa}{p}, \frac{1}{2})$ and set $K_{M,\eta} = K^0_{M,\eta} + K^\theta_{M,\eta}$. Let $\mathscr{R}_\varepsilon := \mathscr{R}_{\lambda_\varepsilon,(A(\cdot,u_\varepsilon),B(\cdot,u_\varepsilon))}$ be the solution operator associated with $(A(\cdot, u_\varepsilon), B(\cdot, u_\varepsilon))$. By Lemma 6.4.3 and (6.73), the estimate (6.42) holds with $\mathscr{R}_\sigma$ replaced by $\mathscr{R}_\varepsilon$ and $K_{M,\eta}$ independent of $\varepsilon > 0$.

Step 1: There exist R, $\bar{\varepsilon} > 0$, $\zeta > 1$ such that for all $\varepsilon \in (0, \bar{\varepsilon})$ and all stopping time τ which satisfies

$$\lambda'_\varepsilon \leq \tau \leq \lambda'_\varepsilon + \varepsilon, \quad \text{ and } \quad \lambda'_\varepsilon \leq \tau \leq \tau_\varepsilon, \qquad \text{a.s. on } \mathcal{U}_\varepsilon, \tag{6.74}$$

one has

$$\mathbb{E}\big[\mathbf{1}_{\mathcal{U}_\varepsilon} \|u\|^p_{\mathfrak{X}(\lambda'_\varepsilon,\tau)}\big] \leq R + C_\varepsilon \mathbb{E}\big[\mathbf{1}_{\mathcal{U}_\varepsilon} \|u\|^{p\zeta}_{\mathfrak{X}(\lambda'_\varepsilon,\tau)}\big], \tag{6.75}$$

for some $C_\varepsilon > 0$ independent of u, τ such that $\lim_{\varepsilon\downarrow 0} C_\varepsilon = 0$.

It suffices to prove the result with τ replaced by $\lambda'_\varepsilon \vee (\tau \wedge \sigma_n)$ for each $n \geq 1$. This has the advantage that all norms which appear here will be finite.

Set $\varepsilon_1 := 1/(32\, K_{M,\eta} L_M)$. Let $\varepsilon_2 > 0$ be such that $C(\varepsilon_2, M) \leq 1/(4K_{M,\eta})$, where $C(\varepsilon_2, M)$ is as in Lemma 4.3.11. Here we used the fact that since F_c and G_c are noncritical $\lim_{\varepsilon\downarrow 0} C(\varepsilon, M) = 0$. Let $\bar{\varepsilon} := \varepsilon_1 \wedge \varepsilon_2$ and fix $\varepsilon \in (0, \bar{\varepsilon})$. Set $\psi := \mathbf{1}_{\Omega\setminus\mathcal{U}_\varepsilon} \lambda'_\varepsilon + \mathbf{1}_{\mathcal{U}_\varepsilon} \tau$. Since $\mathcal{U}_\varepsilon \in \mathscr{F}_{\tau_\varepsilon} \cap \mathscr{F}_{\lambda'_\varepsilon}$ and $\tau_\varepsilon \geq \tau$ a.s. on $\mathcal{U}_\varepsilon$, ψ is a stopping time. Let $\Lambda_\varepsilon := [\![\lambda'_\varepsilon, \tau]\!) = [\lambda'_\varepsilon, \tau) \times \mathcal{U}_\varepsilon$. Reasoning as in the proof of Theorem 6.3.6 (see (6.62)-(6.63)), by Proposition 6.2.7 and the linearity of $\mathscr{R}_\varepsilon$, a.s. on Λ_ε, one has

$$\begin{aligned}\mathbf{1}_{\mathcal{U}_\varepsilon} u &= \mathscr{R}_\varepsilon(u_\varepsilon, \mathbf{1}_{\Lambda_\varepsilon} F_c(\cdot, 0) + f, \mathbf{1}_{\Lambda_\varepsilon} G_c(\cdot, 0) + g) \\ &\quad + \mathscr{R}_\varepsilon(0, \mathbf{1}_{\Lambda_\varepsilon}(A(\cdot, u_\varepsilon) - A(\cdot, u))u, \mathbf{1}_{\Lambda_\varepsilon}(B(\cdot, u_\varepsilon) - B(\cdot, u))u) \\ &\quad + \mathscr{R}_\varepsilon(0, \mathbf{1}_{\Lambda_\varepsilon} F_{\mathsf{Tr}}(\cdot, u), \mathbf{1}_{\Lambda_\varepsilon} G_{\mathsf{Tr}}(\cdot, u)) \\ &\quad + \mathscr{R}_\varepsilon(0, \mathbf{1}_{\Lambda_\varepsilon}(F_c(\cdot, u) - F_c(\cdot, 0)), \mathbf{1}_{\Lambda_\varepsilon}(G_c(\cdot, u) - G_c(\cdot, 0)) \\ &:= I + II + III + IV.\end{aligned} \tag{6.76}$$

It remains to estimate each part separately. For notational convenience, we set

$$\mathcal{Z} := L^p(\Omega; \mathfrak{X}(\lambda'_\varepsilon, T) \cap L^p(\lambda'_\varepsilon, T), w^{\lambda'_\varepsilon}_\kappa; X_1).$$

By Lemma 6.4.3,

$$\begin{aligned}\|I\|_{\mathcal{Z}} &\leq K_{M,\eta}(\|u_\varepsilon\|_{L^p(\Omega;X^{\mathsf{Tr}}_{\kappa,p})} + \|F_c(\cdot, 0)\|_{L^p(\Lambda_\varepsilon, w^{\lambda'_\varepsilon}_\kappa; X_0)} + \|f\|_{L^p(\Lambda_\varepsilon, w^{\lambda'_\varepsilon}_\kappa; X_0)} \\ &\qquad + \|G_c(\cdot, 0)\|_{L^p(\Lambda_\varepsilon, w^{\lambda'_\varepsilon}_\kappa; \gamma(H, X_{1/2}))} + \|g\|_{L^p(\Lambda_\varepsilon, w^{\lambda'_\varepsilon}_\kappa; \gamma(H, X_{1/2}))}) \\ &\leq K_{M,\eta} C_\eta (C + M),\end{aligned}$$

where we used that (6.72). Moreover, as in (6.64) and (6.65) in the proof of Theorem 6.3.6(2) and using that $\tau \leq \tau_\varepsilon$ on $\mathcal{U}_\varepsilon$, one easily obtains

$$\|II\|_{\mathcal{Z}} \leq \frac{1}{4} \|u\|_{L^p(\Lambda_\varepsilon, w^{\lambda'_\varepsilon}_\kappa; X_1)}, \qquad \|III\|_{\mathcal{Z}} \leq K_{M,\eta} C(1 + M).$$

To estimate IV we employ Lemma 4.3.11.

For $\delta > 0$, set $C_\delta := C(\delta, M)$, where $C(\cdot,\cdot)$ is as in Lemma 4.3.11. By the choice of ε_2 we know that $C_{|b-a|}K_{M,\eta} \leq \frac{1}{4}$ for each $a < b$ with $|b-a| \leq \varepsilon_2$. By Lemma 6.4.3,

$$\begin{aligned}\|IV\|_{\mathcal{Z}} &\leq K_{M,\eta}(\|F_c(\cdot,u) - F_c(\cdot,0)\|_{L^p(\Lambda_\varepsilon, w_\kappa^{\lambda'_\varepsilon}; X_0)} \\ &\qquad\qquad + \|G_c(\cdot,u) - G_c(\cdot,0)\|_{L^p(\Lambda_\varepsilon, w_\kappa^{\lambda'_\varepsilon}; \gamma(H,X_{1/2}))}) \\ &\leq \frac{1}{4}\|u\|_{L^p(\mathcal{U}_\varepsilon; \mathfrak{X}(\lambda'_\varepsilon,\tau))} + C_\varepsilon(\mathbb{E}[\mathbf{1}_{\mathcal{U}_\varepsilon}\|u\|^{p\zeta}_{\mathfrak{X}(\lambda'_\varepsilon,\tau)}])^{1/p},\end{aligned}$$

where the last estimate follows from Lemma 4.3.11 and $|\tau - \lambda'_\varepsilon| \leq \varepsilon$ (see (6.74)).

Combining the estimates we obtain the claim of Step 1.

Step 2: Conclusion. Fix $\varepsilon > 0$ and set

$$\mathcal{I}_\varepsilon := \mathcal{V}' \cap \big\{\sigma < T,\, \|u\|_{\mathfrak{X}(\lambda'_\varepsilon,\sigma)} < \infty\big\},$$

Recall that $\lambda'_\varepsilon < \sigma$ on $\mathcal{V}'$. We claim that $\mathbb{P}(\mathcal{I}_\varepsilon) = 0$. Indeed, by (6.70), one has $\lim_{t\uparrow\sigma} u(t)$ exists in $X^{\mathsf{Tr}}_{\kappa,p}$ a.s. on $\mathcal{V}' \subseteq \mathcal{O}$. Therefore,

$$\begin{aligned}\mathbb{P}(\mathcal{I}_\varepsilon) &= \mathbb{P}\Big(\mathcal{V}' \cap \Big\{\sigma < T,\, \lim_{t\uparrow\sigma} u(t) \text{ exists in } X^{\mathsf{Tr}}_{\kappa,p},\, \|u\|_{\mathfrak{X}(\lambda'_\varepsilon,\sigma)} < \infty\Big\}\Big) \\ &= \mathbb{P}\Big(\mathcal{V}' \cap \Big\{\sigma < T,\, \lim_{t\uparrow\sigma} u(t) \text{ exists in } X^{\mathsf{Tr}}_{\kappa,p},\, \|u\|_{\mathfrak{X}(\sigma)} < \infty\Big\}\Big) \\ &\leq \mathbb{P}\Big(\sigma < T,\, \lim_{t\uparrow\sigma} u(t) \text{ exists in } X^{\mathsf{Tr}}_{\kappa,p},\, \|u\|_{\mathfrak{X}(\sigma)} < \infty\Big) = 0,\end{aligned}$$

where in the last step we used Theorem 6.3.6(2) and Lemma 6.4.5 (or (6.69)).

Next let $\bar\varepsilon, (C_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}, R, \zeta$ be as in Step 1. For each $\varepsilon \in (0,\bar\varepsilon)$ and $x \in \mathbb{R}_+$, set $\varphi_\varepsilon(x) := x - C_\varepsilon x^\zeta$. Standard considerations show that φ_ε has a unique maximum on $\mathbb{R}_+$ and $\lim_{\varepsilon\downarrow 0} M_\varepsilon = \infty$ where $\max_{\mathbb{R}_+}\varphi_\varepsilon =: M_\varepsilon$. Let $m_\varepsilon > 0$ be the unique number such that $\varphi_\varepsilon(m_\varepsilon) = M_\varepsilon$ and note that $\varphi_\varepsilon \geq 0$ on $[0, m_\varepsilon]$. Since $\lim_{\varepsilon\downarrow 0} M_\varepsilon = \infty$ and $\mathbb{P}(\mathcal{V}') > 0$, we can choose $\varepsilon \in (0,\bar\varepsilon)$ such that $M_\varepsilon\mathbb{P}(\mathcal{V}') > R$.

Since $\mathbb{P}(\mathcal{I}_\varepsilon) = 0$, for a.a. $\omega \in \mathcal{V}'$ there exists a $t < \sigma(\omega)$ such that

$$\|u(\cdot,\omega)\|_{\mathfrak{X}(\lambda'_\varepsilon,t)} > m_\varepsilon^{1/p}. \tag{6.77}$$

Define the stopping time μ by

$$\mu := \begin{cases} \inf\Big\{t \in [\lambda'_\varepsilon, \tau_\varepsilon) \,:\, \|u\|_{\mathfrak{X}(\lambda'_\varepsilon,t)} \geq m_\varepsilon^{1/p}\Big\}, & \text{on } \mathcal{U}_\varepsilon; \\ \lambda'_\varepsilon, & \text{on } \Omega \setminus \mathcal{U}_\varepsilon. \end{cases}$$

Here we set $\inf\varnothing := \tau_\varepsilon$. In this way $\|u\|_{\mathfrak{X}(\lambda'_\varepsilon,\mu)} = m_\varepsilon$ a.s. on $\mathcal{U}_\varepsilon$.

By the definition of τ_ε and (6.77), one has $\mu < \tau_\varepsilon = \sigma$ on $\mathcal{V}'$. Set $\mu_\varepsilon := \mu \wedge (\lambda'_\varepsilon + \varepsilon)$. By (6.71), $\sigma - \lambda'_\varepsilon \leq \sigma - \lambda_\varepsilon \leq \varepsilon$ a.s. on $\mathcal{V}'$, and therefore $\mu_\varepsilon = \mu$ on $\mathcal{V}'$. The latter combined with $\mathcal{V}' \subseteq \mathcal{U}_\varepsilon$ gives $\|u\|_{\mathfrak{X}(\lambda'_\varepsilon,\mu_\varepsilon)} = m_\varepsilon^{1/p}$ a.s. on $\mathcal{V}'$. Since $\varphi_\varepsilon|_{[0,m_\varepsilon]} \geq 0$ and $\mathcal{U}_\varepsilon \supseteq \mathcal{V}'$, we find

$$\mathbb{E}[\mathbf{1}_{\mathcal{U}_\varepsilon}\varphi_\varepsilon(\|u\|^p_{\mathfrak{X}(\lambda'_\varepsilon,\mu_\varepsilon)})] \geq \mathbb{E}[\mathbf{1}_{\mathcal{V}'}\varphi_\varepsilon(\|u\|^p_{\mathfrak{X}(\lambda'_\varepsilon,\mu_\varepsilon)})] = \mathbb{E}[\mathbf{1}_{\mathcal{V}'}\varphi_\varepsilon(m_\varepsilon)] = M_\varepsilon\mathbb{P}(\mathcal{V}').$$

Next observe that $\tau = \mu_\varepsilon$ satisfies condition (6.74), and the quantities appearing in (6.75) are finite. Therefore, Step 1 implies the following converse estimate

$$\mathbb{E}[\mathbf{1}_{\mathcal{U}_\varepsilon}\varphi_\varepsilon(\|u\|^p_{\mathfrak{X}(\lambda'_\varepsilon,\mu_\varepsilon)})] \leq R. \tag{6.78}$$

This leads to a contradiction since $R < M_\varepsilon\mathbb{P}(\mathcal{V}')$. Therefore, $\mathbb{P}(\mathcal{O}) = 0$ and this completes the proof of Theorem 6.3.6(3). □

Proof of Theorem 6.3.7(3). We use the same method as in the proof of Theorem 6.3.6(3). Suppose that $\mathbb{P}(\mathcal{O}) > 0$ where

$$\mathcal{O} := \Big\{\sigma < T,\, \sup_{t\in[0,\sigma)} \|u(t)\|_{X^{\mathsf{Tr}}_{\kappa,p}} < \infty\Big\}.$$

As before (see (6.71)) one can check that there exists a set $\mathcal{V}$ with positive probability such that for all $\varepsilon > 0$ there exists an $N(\varepsilon) \in \mathbb{N}$ for which

$$\|u\|_{C(\overline{I}_\sigma; X^{\mathsf{Tr}}_{\kappa,p})} < M, \quad \text{and} \quad |\sigma_{N(\varepsilon)} - \sigma| < \varepsilon. \tag{6.79}$$

Now the estimate (6.75) holds again. Indeed, in the proof the fact that $\lim_{t\uparrow\sigma} u(t)$ exists in $X^{\mathsf{Tr}}_{\kappa,p}$ was only used to estimate II. In the semilinear case, $II = 0$, and the bound in (6.79) can be used to reproduce the estimates for I, III, IV. The proof of Step 2 of Theorem 6.3.6(3) can be used to complete the proof. □

6.4.4 Proofs of Theorems 6.3.6(4), 6.3.7(4) and 6.3.8

In this subsection, we prove the remaining results stated in Subsection 6.3.2. We begin with the proof of Theorem 6.3.7(4) which will guide us into the remaining ones. The advantage is that in the semilinear case the argument used to control the nonlinearities is more transparent. Additional changes are needed to get Theorems 6.3.6(4) and 6.3.8.

Before starting with the proofs we introduce some notation which will be used only in this subsection and allows us to give an extension of Theorem 6.3.8, i.e. Proposition 6.4.9 below. Let (HF)-(HG) be satisfied and fix $j \in \{1, \dots, m_F + m_G\}$. If $\rho_j > 0$, then we define $\beta^\star_j, \varphi^\star_j \in (1 - \frac{1+\kappa}{p}, 1)$ as follows:

- If $\rho_j(\varphi_j - 1 + \frac{1+\kappa}{p}) + \varphi_j \geq 1$, then $\varphi^\star_j = \varphi_j$, and $\beta^\star_j = 1 - \rho_j(\varphi_j - 1 + \frac{1+\kappa}{p}) \in [\beta_j, \varphi_j]$;
- If $\rho_j(\varphi_j - 1 + \frac{1+\kappa}{p}) + \varphi_j < 1$, then $\beta^\star_j = \varphi^\star_j = 1 - \frac{\rho_j}{\rho_j+1}\frac{1+\kappa}{p} \in (\varphi_j, 1)$.

The previous definition implies that

$$\rho_j\Big(\varphi^\star_j - 1 + \frac{1+\kappa}{p}\Big) + \beta^\star_j = 1 \quad \text{for all } \ j \in \{1, \dots, m_F + m_G\}. \tag{6.80}$$

If $\rho_j = 0$, then with a slight abuse of notation we replace ρ_j by $\varepsilon_j > 0$ such that $\varepsilon_j(\varphi_j - 1 + \frac{1+\kappa}{p}) + \varphi_j < 1$, and define $\varphi^\star_j = \beta^\star_j = 1 - \frac{\varepsilon_j}{\varepsilon_j+1}\frac{1+\kappa}{p}$. By choosing ε_j small enough (e.g. $\varepsilon_j < \kappa + 1$) one always has $\beta^\star_j = \varphi^\star_j > 1 - \frac{1+\kappa}{p}\frac{1+\kappa}{2+\kappa}$, which is needed in Lemma 6.4.8(1) below.

In all cases we have $\varphi^\star_j \geq \varphi_j$ and $\beta^\star_j \geq \beta_j$. Therefore, by $X_\Phi \hookrightarrow X_\phi$ for $0 < \phi \leq \Phi < 1$, and by (HF)-(HG) for all $n \geq 1$ a.s. for all $x \in X_1$ s.t. $\|x\|_{X^{\mathsf{Tr}}_{\kappa,p}} \leq n$,

$$\|F_c(\cdot, x)\|_{X_0} + \|G_c(\cdot, x)\|_{\gamma(H, X_{1/2})} \leq C'_{c,n} \sum_{j=1}^{m_F+m_G} (1 + \|x\|^{\rho_j}_{X_{\varphi^\star_j}})\|x\|_{X_{\beta^\star_j}} + C'_{c,n}. \tag{6.81}$$

where $C'_{c,n} = C' C_{c,n}$ with C' depending only on $X_0, X_1, \beta_j, \beta^\star_j, \varphi_j, \varphi^\star_j$.

Next, we partially repeat the construction of the $\mathfrak{X}$-space (see (6.41)) using $(\rho_j, \beta^\star_j, \varphi^\star_j)$ instead of $(\rho_j, \beta_j, \varphi_j)$. As in Lemma 6.4.5 this will be needed to estimate the nonlinearities F_c, G_c. Similar to (4.30), for all $j \in \{1, \dots, m_F + m_G\}$ we set

$$\frac{1}{\xi'_j} := \frac{\rho_j(\varphi^\star_j - 1 + (1+\kappa)/p)}{(1+\kappa)/p}, \quad \text{and} \quad \frac{1}{\xi_j} := \frac{\beta^\star_j - 1 + (1+\kappa)/p}{(1+\kappa)/p}. \tag{6.82}$$

Since $\varphi^\star_j, \beta^\star_j \in (1 - \frac{1+\kappa}{p}, 1)$, we get $\frac{1}{\xi'_j}, \frac{1}{\xi_j} > 0$. Moreover, (6.80) yields $\frac{1}{\xi'_j} + \frac{1}{\xi_j} = 1$ and therefore $\frac{1}{\xi'_j}, \frac{1}{\xi_j} < 1$. Parallel to (6.41), for all $0 \leq a < b \leq T$ we define

$$\mathfrak{X}^\star(a, b) := \Big(\bigcap_{j=1}^{m_F+m_G} L^{p\xi_j}(a, b, w^a_\kappa; X_{\beta^\star_j})\Big) \cap \Big(\bigcap_{j=1}^{m_F+m_G} L^{\rho_j p \xi'_j}(a, b, w^a_\kappa; X_{\varphi^\star_j})\Big). \tag{6.83}$$

By (6.81) and Hölder's inequality we obtain that for all $M\geq 1$ and all $h\in\mathfrak{X}^\star(a,b)\cap C([a,b];X^{\mathsf{Tr}}_{\kappa,p})$ which satisfy $\|h\|_{C([a,b];X^{\mathsf{Tr}}_{\kappa,p})}\leq M$,

$$\begin{aligned}&\|F_c(\cdot,h)\|_{L^p(a,b,w_\kappa^a;X_0)}+\|G_c(\cdot,h)\|_{L^p(a,b,w_\kappa^a;\gamma(H,X_{1/2}))}\\&\leq C''_{c,M}\Big[\sum_{j=1}^{m_F+m_G}(1+\|h\|^{\rho_j}_{L^{\rho_j p\xi_j'}(a,b,w_\kappa^a;X_{\varphi_j^\star})})\|h\|_{L^{p\xi_j}(a,b,w_\kappa^a;X_{\beta_j^\star})}+1\Big],\end{aligned}\tag{6.84}$$

where $C''_{c,M}=C'_{c,M}(1\vee\|1\|_{L^p(0,T,w_\kappa)}\vee\max_j\|1\|_{L^{p\xi_j'}(0,T,w_\kappa)})$.

The key to the proofs of the blow-up criteria are interpolation inequalities. In order to simplify the notation for $\theta\in[0,1]$ and $0\leq a<b\leq T$, we set

$$ {}_0\mathsf{MR}^{\theta,\kappa}_X(a,b):={}_0H^{\theta,p}(a,b,w_\kappa^a;X_{1-\theta})\cap L^p(a,b,w_\kappa^a;X_1).\tag{6.85}$$

The reason for using the space ${}_0H^{\theta,p}$ instead of $H^{\theta,p}$ is that Proposition 6.2.6(4) allows to obtain uniformity of the estimates in T in the proof of Theorem 6.3.7(4) below. By (6.4) there are no trace restrictions when $\theta<\frac{1+\kappa}{p}$.

Lemma 6.4.8 (Interpolation inequality)**.** *Let $p\in(1,\infty)$, $\kappa\in[0,p-1)$, $\psi\in(1-\frac{1+\kappa}{p},1)$, and set $\zeta=(1+\kappa)/\big(\psi-1+\frac{1+\kappa}{p}\big)$. Then there exists a $\theta_0\in[0,\frac{1+\kappa}{p})$ such that for all $\theta\in[\theta_0,1)$ there is a constant $C>0$ such that the following estimate holds for all $0\leq a<b\leq T$ and all $f\in{}_0\mathsf{MR}^{\theta,\kappa}_X(a,b)\cap L^\infty(a,b;X^{\mathsf{Tr}}_{\kappa,p})$,*

$$\|f\|_{L^\zeta(a,b,w_\kappa;X_\psi)}\leq C\|f\|^{1-\phi}_{L^\infty(a,b;X^{\mathsf{Tr}}_{\kappa,p})}\|f\|^{(1-\delta)\phi}_{{}_0\mathsf{MR}^{\theta,\kappa}_X(a,b)}\|f\|^{\delta\phi}_{L^p(a,b;X_{1-\frac{\kappa}{p}})},\tag{6.86}$$

where we can take $\delta,\phi\in[0,1]$ as follows:

(1) $\delta=1-\frac{p}{1+\kappa}\Big(\psi-1+\frac{1+\kappa}{p}\Big)$ *and* $\phi=1$ *if* $\psi\in(1-\frac{1+\kappa}{p}\frac{1+\kappa}{2+\kappa},1)$ *and* $\kappa>0$*;*

(2) $\delta=\frac{\kappa}{\kappa+1}$ *and* $\phi=p\Big(\psi-1+\frac{1+\kappa}{p}\Big)$ *if* $\psi\in(1-\frac{1+\kappa}{p},1-\frac{\kappa}{p}]$ *and* $\kappa>0$*;*

(3) $\delta=1$ *and* $\phi=p\Big(\psi-1+\frac{1}{p}\Big)$ *if* $\kappa=0$.

In particular, in each of the above cases

$$(1-\delta)\phi\leq\frac{p}{1+\kappa}\Big(\psi-1+\frac{1+\kappa}{p}\Big)\tag{6.87}$$

Note that (1) and (2) have a nontrivial overlap since $1-\frac{\kappa}{p}>1-\frac{1+\kappa}{p}\frac{1+\kappa}{2+\kappa}$.

Proof. By a translation argument we can suppose that $a=0$ and we write t instead of b below. Let us begin by making some reductions. By Lemma 6.1.4(2) for all $\theta\in[\theta_0,1)$ we have

$$ {}_0H^{\theta,p}(I_t,w_\kappa;X_{1-\theta})\cap L^p(I_t;w_\kappa;X_1)\hookrightarrow{}_0H^{\theta_0,p}(I_t,w_\kappa;X_{1-\theta_0})\cap L^p(I_t;w_\kappa;X_1).$$

Therefore, it suffices to consider $\theta=\theta_0$ in all cases.

(1): First consider $\psi\in(1-\frac{\kappa}{p},1)$. For $\theta\in(0,1-\psi)$ one has $\theta<\kappa/p<(\kappa+1)/p$. Setting $\delta:=(1-\theta-\psi)/(\frac{\kappa}{p}-\theta)\in(0,1]$, we find

$$\|f\|_{L^\zeta(I_t,w_\kappa;X_\psi)}\overset{(i)}{\lesssim}\|f\|_{{}_0H^{\theta(1-\delta),p}(I_t,w_{\kappa(1-\delta)};X_\psi)}\overset{(ii)}{\lesssim}\|f\|^{1-\delta}_{{}_0H^{\theta,p}(I_t,w_\kappa;X_{1-\theta})}\|f\|^{\delta}_{L^p(I_t;X_{1-\frac{\kappa}{p}})}.\tag{6.88}$$

In (ii) we used Lemma 6.1.4(2). In (i) we used Proposition 6.1.1(4) with Sobolev exponents

$$\theta(1-\delta)-\frac{\kappa(1-\delta)+1}{p}=-\Big(\frac{\kappa}{p}-\theta\Big)(1-\delta)-\frac{1}{p}\overset{(a)}{=}-\Big(\psi-1+\frac{\kappa+1}{p}\Big)\overset{(b)}{=}-\frac{\kappa+1}{\zeta},$$

where we used $1-\delta = (\psi+\frac{\kappa}{p}-1)/(\frac{\kappa}{p}-\theta)$ in (a) and the definition of ζ in (b). The condition $\frac{\kappa}{\zeta} \leq \frac{\kappa}{p}(1-\delta)$ of Proposition 6.1.1(4) gives the following restriction on the parameter θ:

$$\frac{1}{1+\kappa}\Big(\psi - 1 + \frac{1+\kappa}{p}\Big) \leq \frac{1}{p}\frac{\psi - 1 + \frac{\kappa}{p}}{\frac{\kappa}{p}-\theta}. \tag{6.89}$$

One can check that (6.89) is satisfied with strict inequality for $\theta = 1-\psi$ (since $\psi<1$), and (6.89) is not satisfied for $\theta = 0$. Therefore, by continuity and linearity there is a unique $\theta \in (0, 1-\psi)$ such that equality holds in (6.89). Now (6.88) implies (6.86) with $\phi = 1$, and

$$1-\delta = \frac{(\psi - 1 + \frac{\kappa}{p})}{\frac{\kappa}{p}-\theta} \overset{(6.89)}{=} \frac{p}{1+\kappa}\Big(\psi - 1 + \frac{1+\kappa}{p}\Big)$$

which coincides with the choice of δ in (1).

Next consider $\psi \in (1-\frac{1+\kappa}{p}\frac{1+\kappa}{2+\kappa}, 1-\frac{\kappa}{p})$. Then $1-\psi > \kappa/p$ and we can apply the same argument starting with $\theta \in (1-\psi, (1+\kappa)/p)$ and setting $\delta := (\theta+\psi-1)/(\theta-\frac{\kappa}{p})$. Now the following variant of (6.89) has to be considered

$$\frac{1}{1+\kappa}\Big(\psi - 1 + \frac{1+\kappa}{p}\Big) \leq \frac{1}{p}\frac{1-\psi-\frac{\kappa}{p}}{\theta - \frac{\kappa}{p}}. \tag{6.90}$$

This time (6.90) is satisfied with strict inequality for $\theta = 1-\psi$ (since $\psi < 1$), and (6.90) is not satisfied for $\theta = \frac{1+\kappa}{p}$ (since $\psi > 1 - \frac{1+\kappa}{p}\frac{1+\kappa}{2+\kappa}$). Therefore, there is a unique $\theta \in (1-\psi, \frac{1+\kappa}{p})$ such that equality holds in (6.90). The rest of the proof is identical to (1).

The case $\psi = 1-\frac{1+\kappa}{p}$ is contained in (2) and will be proved below.

(2): First consider $\psi \in (1-\frac{1+\kappa}{p}, 1-\frac{\kappa}{p})$. Here we use a two step interpolation. By real interpolation [20, Theorems 3.5.3 and 4.7.1] and the definition of ϕ we obtain

$$\begin{aligned}(X^{\mathsf{Tr}}_{\kappa,p}, X_{1-\frac{\kappa}{p}})_{\phi,1} &\hookrightarrow \big((X_0,X_1)_{1-\frac{1+\kappa}{p},p}, (X_0,X_1)_{1-\frac{\kappa}{p}}\big)_{\phi,1}\\ &= (X_0,X_1)_{\psi,1} \hookrightarrow X_\psi,\end{aligned} \tag{6.91}$$

and hence $\|x\|_{X_\psi} \lesssim C\|x\|^{1-\phi}_{X^{\mathsf{Tr}}_{\kappa,p}}\|x\|^{\phi}_{X_{1-\frac{\kappa}{p}}}$ for all $x \in X_{1-\frac{\kappa}{p}}$. Since $\phi\zeta = p(1+\kappa)$, the latter estimate implies

$$\|f\|_{L^\zeta(I_t;w_\kappa;X_\psi)} \leq \|f\|^{1-\phi}_{L^\infty(I_t;X^{\mathsf{Tr}}_{\kappa,p})}\|f\|^{\phi}_{L^{p(1+\kappa)}(I_t,w_\kappa;X_{1-\frac{\kappa}{p}})}. \tag{6.92}$$

Reasoning as in (6.88) we get

$$\begin{aligned}\|f\|_{L^{p(\kappa+1)}(I_t,w_\kappa;X_{1-\frac{\kappa}{p}})} &\lesssim \|f\|_{{}_0H^{\frac{\kappa}{p}(1-\delta),p}(I_t,w_{\kappa(1-\delta)};X_{1-\frac{\kappa}{p}})}\\ &\lesssim \|f\|^{1-\delta}_{{}_0H^{\frac{\kappa}{p},p}(I_t,w_\kappa;X_{1-\frac{\kappa}{p}})}\|f\|^{\delta}_{L^p(I_t;X_{1-\frac{\kappa}{p}})},\end{aligned} \tag{6.93}$$

where for the Sobolev embedding of Proposition 6.1.1(4) we used

$$\frac{\kappa}{p}(1-\delta) - \frac{\kappa(1-\delta)+1}{p} = -\frac{1}{p} = -\frac{\kappa+1}{p(\kappa+1)},$$

and $\frac{\kappa}{p(\kappa+1)} = \frac{\kappa}{p}(1-\delta)$ (since $\delta = \frac{\kappa}{\kappa+1}$). Combining (6.92) and (6.93) gives (6.86).

Finally, the case $\psi = 1-\frac{\kappa}{p}$ of (2) follows from (6.93) with $\phi = 1$ and δ as before.

(3): This follows in a similar way as in (6.91) and (6.92). ☐

Proof of Theorem 6.3.7(4). As usual, we set $s = 0$ and we mainly focus on the case $p > 2$ since the case $p = 2$ is simpler. For the reader's convenience we split the proof into several steps. In Step 1 we apply Lemma 6.4.8 to obtain interpolation inequalities. In Step 2 we set-up the proof by contradiction and in Step 3 we derive the contradiction. Recall that from Theorem 6.3.1 we obtain that a.s. for all $\theta \in [0, \frac{1}{2})$ and $0 \le a < b < \sigma$,

$$u \in H^{\theta,p}(a, b, w_\kappa^a; X_{1-\theta}) \cap C([a, b]; X_{\kappa,p}^{\mathsf{Tr}}). \tag{6.94}$$

Step 1: Interpolation inequalities. Since $\varphi_j^\star, \beta_j^\star \in (1 - \frac{1+\kappa}{p}, 1)$ and

$$\rho_j p \xi_j' = \frac{1+\kappa}{(\varphi_j^\star - 1 + \frac{1+\kappa}{p})} \quad \text{and} \quad p\xi_j = \frac{1+\kappa}{(\beta_j^\star - 1 + \frac{1+\kappa}{p})}, \tag{6.95}$$

the exponents $\rho_j p \xi_j'$ and $p\xi_j$ satisfy the conditions of Lemma 6.4.8. Therefore, there exist $\theta \in [0, \frac{1+\kappa}{p})$ and $C > 0$ such that for all $j \in \{1, \dots, m_F + m_G\}$ there are $\phi_{1,j}, \phi_{2,j}, \delta_{1,j}, \delta_{2,j} \in (0, 1]$ such that a.s. for all $0 \le a < b < \sigma$

$$\|u\|_{L^{\rho_j p \xi_j'}(a,b,w_\kappa^a; X_{\varphi_j^\star})} \le C \|u\|^{1-\phi_{1,j}}_{L^\infty(a,b;X^{\mathsf{Tr}}_{\kappa,p})} \|u\|^{(1-\delta_{1,j})\phi_{1,j}}_{{}_0\mathsf{MR}^{\theta,\kappa}_X(a,b)} \|u\|^{\delta_{1,j}\phi_{1,j}}_{L^p(a,b;X_{1-\frac{\kappa}{p}})}, \tag{6.96}$$

$$\|u\|_{L^{p\xi_j}(a,b,w_\kappa^a; X_{\beta_j^\star})} \le C \|u\|^{1-\phi_{2,j}}_{L^\infty(a,b;X^{\mathsf{Tr}}_{\kappa,p})} \|u\|^{(1-\delta_{2,j})\phi_{2,j}}_{{}_0\mathsf{MR}^{\theta,\kappa}_X(a,b)} \|u\|^{\delta_{2,j}\phi_{2,j}}_{L^p(a,b;X_{1-\frac{\kappa}{p}})}, \tag{6.97}$$

Moreover, by (6.80) and (6.87), $\rho_j \phi_{1,j}(1 - \delta_{1,j}) + \phi_{2,j}(1 - \delta_{2,j}) \le 1$. Note that in (6.96) and (6.97) we used (6.94) and (6.4).

Step 2: Setting up the proof by contradiction. By contradiction we assume that $\mathbb{P}(\mathcal{O}) > 0$ where

$$\mathcal{O} := \Big\{\sigma < T, \sup_{t\in[0,\sigma)} \|u(t)\|_{X^{\mathsf{Tr}}_{\kappa,p}} < \infty, \|u\|_{L^p(I_\sigma; X_{1-\frac{\kappa}{p}})} < \infty\Big\}. \tag{6.98}$$

By Egorov's theorem and the fact that $\sigma > 0$ a.s., there exist $\eta > 0$, $M \ge 1$, $\mathscr{F}_\sigma \ni \mathcal{V} \subseteq \mathcal{O}$ such that $\mathbb{P}(\mathcal{V}) > 0$, $\sigma > \eta$ a.s. on $\mathcal{V}$,

$$\sup_{\mathcal{V}} \sup_{t\in[0,\sigma)} \|u(t)\|_{X^{\mathsf{Tr}}_{\kappa,p}} \le M, \quad \text{and} \quad \lim_{n\to\infty} \sup_{\mathcal{V}} \|u\|_{L^p(\sigma_n,\sigma; X_{1-\frac{\kappa}{p}})} = 0. \tag{6.99}$$

Reasoning as in the proof of Theorem 6.3.6(2), employing Corollary 6.3.9, there exist a sequence of stopping times $(\sigma_n')_{n\ge1}$ and a set $\mathscr{F}_\sigma \ni \mathcal{V}' \subseteq \mathcal{V}$ with positive measure such that $\sigma_n \le \sigma_n'$, $\sigma_n' \ge \eta$ a.s., and $\sigma_n' < \sigma$ a.s. on $\mathcal{V}'$ for all $n \ge 1$. Finally, by (6.99) and the fact that $\sigma_n' \ge \sigma_n$, for all $\varepsilon > 0$ there exists an $N(\varepsilon) > 0$ such that

$$\sup_{t\in[0,\sigma)} \|u(t)\|_{X^{\mathsf{Tr}}_{\kappa,p}} \le M, \quad \text{and} \quad \|u\|_{L^p(\sigma'_{N(\varepsilon)},\sigma; X_{1-\frac{\kappa}{p}})} \le \varepsilon, \quad \text{a.s. on } \mathcal{V}'. \tag{6.100}$$

Step 3: In this step we prove the desired contradiction. We begin by partially repeating the argument used in Step 2 in the proof of Theorem 6.3.6(2). Let θ and M, η be as in Steps 1 and 2, respectively. By Assumption 6.3.2 there exists a $K_{M,\eta} > 0$ such that for all $t \in (\eta, T)$ one has

$$\max\{K^{\mathsf{det},\theta,p,\kappa}_{(A,B)}(t, T), K^{\mathsf{sto},\theta,p,\kappa}_{(A,B)}(t, T)\} \le K_{M,\eta}. \tag{6.101}$$

Since σ_n' takes values in a finite set, (6.101) and Proposition 6.2.8 imply $(A, B) \in \mathcal{SMR}^\bullet_{p,\kappa}(\sigma_n', T)$ and

$$\max\{K^{\mathsf{det},\theta,p,\kappa}_{(A,B)}(\sigma_n', T), K^{\mathsf{sto},\theta,p,\kappa}_{(A,B)}(\sigma_n', T)\} \le K_{M,\eta}, \quad \text{for all } n \ge 1. \tag{6.102}$$

For notational convenience, for each $\varepsilon > 0$ we set $\lambda_\varepsilon := \sigma_{N(\varepsilon)}$ and $\lambda_\varepsilon' := \sigma'_{N(\varepsilon)}$. For each $\ell \ge 1$, let us define the following stopping time

$$\begin{aligned} \tau_{\varepsilon,\ell} := \inf\big\{t \in [\lambda_\varepsilon', \sigma) \,:\, &\|u\|_{L^p(\lambda_\varepsilon',t; X_{1-\frac{\kappa}{p}})} \ge \varepsilon, \\ &\|u(t)\|_{X^{\mathsf{Tr}}_{\kappa,p}} \ge M, \ \|u\|_{\mathfrak{X}(\lambda_\varepsilon',t)} \ge \ell\big\}, \end{aligned} \tag{6.103}$$

on $\{\lambda_\varepsilon' < \sigma\}$ and $\tau_{\varepsilon,\ell} = \lambda_\varepsilon'$ on $\{\lambda_\varepsilon' \ge \sigma\}$. Due to (6.100), $\lim_{\ell\to\infty} \tau_{\varepsilon,\ell} = \sigma$ a.s. on $\mathcal{V}'$. Next, fix $L \ge 1$ so large enough that

$$\mathbb{P}(\mathcal{V}'') > 0, \text{ where } \mathcal{V}'' := \mathcal{V}' \cap \mathcal{U}, \ \mathcal{U} := \{\sigma > \lambda_\varepsilon', \|u(\lambda_\varepsilon')\|_{X^{\mathsf{Tr}}_{\kappa,p}} \le L\} \in \mathscr{F}_{\lambda_\varepsilon'}. \tag{6.104}$$

By (6.103), for all $\varepsilon > 0$ and $\ell \geq 1$, we have a.s.

$$\|u\|_{L^p(\lambda'_\varepsilon,\tau_{\varepsilon,\ell};X_{1-\frac{\kappa}{p}})} \leq \varepsilon, \quad \|u\|_{\mathfrak{X}(\lambda'_\varepsilon,\tau_{\varepsilon,\ell})} \leq \ell, \quad \mathbf{1}_{\lambda'_\varepsilon<\tau_{\varepsilon,\ell}}\|u\|_{L^\infty(\lambda'_\varepsilon,\tau_{\varepsilon,\ell};X^{\mathsf{Tr}}_{\kappa,p})} \leq M. \tag{6.105}$$

Combining the last inequality in (6.105) and (HF)-(HG), we get

$$\|F_{\mathsf{Tr}}(\cdot,u)\|_{X_0} + \|G_{\mathsf{Tr}}(\cdot,u)\|_{\gamma(H,X_{1/2})} \leq C_{\mathsf{Tr},M}(1+M) \quad \text{a.e. on } [\![\lambda'_\varepsilon,\tau_{\varepsilon,\ell}[\![.$$

In particular, for some $R_{\mathsf{Tr},M} > 0$ independent of $\ell \geq 1$, we have a.s.

$$\|F_{\mathsf{Tr}}(\cdot,u)\|_{L^p(\lambda'_\varepsilon,\tau_{\varepsilon,\ell},w_\kappa^{\lambda'_\varepsilon};X_0)} + \|G_{\mathsf{Tr}}(\cdot,u)\|_{L^p(\lambda'_\varepsilon,\tau_{\varepsilon,\ell},w_\kappa^{\lambda'_\varepsilon};\gamma(H,X_{1/2}))} \leq R_{\mathsf{Tr},M}. \tag{6.106}$$

Finally, we estimate F_c, G_c. By (6.84) we get, for all $\varepsilon > 0$, $\ell \geq 1$ and a.s.

$$\begin{aligned}
&\|F_c(\cdot,u)\|_{L^p(\lambda'_\varepsilon,\tau_{\varepsilon,\ell},w_\kappa^{\lambda'_\varepsilon};X_0)} + \|G_c(\cdot,u)\|_{L^p(\lambda'_\varepsilon,\tau_{\varepsilon,\ell},w_\kappa^{\lambda'_\varepsilon};\gamma(H,X_{1/2}))}\\
&\leq C''_{c,M}\Big[\sum_{j=1}^{m_F+m_G}(1+\|u\|^{\rho_j}_{L^{\rho_j p\xi'_j}(\lambda'_\varepsilon,\tau_{\varepsilon,\ell},w_\kappa^{\lambda'_\varepsilon};X_{\varphi^\star_j})})\|u\|_{L^{p\xi_j}(\lambda'_\varepsilon,\tau_{\varepsilon,\ell},w_\kappa^{\lambda'_\varepsilon};X_{\beta^\star_j})}+1\Big].
\end{aligned} \tag{6.107}$$

Fix $j \in \{1,\dots,m_F+m_G\}$. From (6.96) and (6.97) we get a.s.

$$\begin{aligned}
&\|u\|^{\rho_j}_{L^{\rho_j p\xi'_j}(\lambda'_\varepsilon,\tau_{\varepsilon,\ell},w_\kappa^{\lambda'_\varepsilon};X_{\varphi^\star_j})}\|u\|_{L^{p\xi_j}(\lambda'_\varepsilon,\tau_{\varepsilon,\ell},w_\kappa^{\lambda'_\varepsilon};X_{\beta^\star_j})}\\
&\leq C\mathbf{1}_{\{\lambda'_\varepsilon<\tau_{\varepsilon,\ell}\}}\|u\|^{\rho_j(1-\phi_{1,j})+(1-\phi_{2,j})}_{L^\infty(\lambda'_\varepsilon,\tau_{\varepsilon,\ell};X^{\mathsf{Tr}}_{\kappa,p})}\|u\|^{\rho_j\phi_{1,j}(1-\delta_{1,j})+\phi_{2,j}(1-\delta_{2,j})}_{{}_0\mathsf{MR}^{\theta,\kappa}_X(\lambda'_\varepsilon,\tau_{\varepsilon,\ell})}\|u\|^{\rho_j\delta_{1,j}\phi_{1,j}+\delta_{2,j}\phi_{2,j}}_{L^p(\lambda'_\varepsilon,\tau_{\varepsilon,\ell};X_{1-\frac{\kappa}{p}})}\\
&\overset{(i)}{\leq} C_{M,j}\Upsilon_j(\varepsilon)\|u\|^{\rho_j\phi_{1,j}(1-\delta_{1,j})+\phi_{2,j}(1-\delta_{2,j})}_{{}_0\mathsf{MR}^{\theta,\kappa}_X(\lambda'_\varepsilon,\tau_{\varepsilon,\ell})}\\
&\overset{(ii)}{\leq} R_{M,j} + C_{M,j}\|u\|_{{}_0\mathsf{MR}^{\theta,\kappa}_X(\lambda'_\varepsilon,\tau_{\varepsilon,\ell})}\Upsilon_j(\varepsilon),
\end{aligned} \tag{6.108}$$

where $R_{M,j} > 0$, $\Upsilon_j \in C([0,\infty))$ are independent of $\ell \geq 1$ and $\lim_{\varepsilon\downarrow 0}\Upsilon_j(\varepsilon) = 0$ (since $\delta_{k,j}\phi_{k,j} > 0$) and in (i) we used the first and the last inequality in (6.105) and in (ii) the Young's inequality and the fact that $\rho_j\phi_{1,j}(1-\delta_{1,j}) + \phi_{2,j}(1-\delta_{2,j}) \leq 1$. Using the same argument one can provide a similar estimate for $\|u\|_{L^{p\xi_j}(\lambda'_\varepsilon,\tau_{\varepsilon,\ell},w_\kappa^{\lambda'_\varepsilon};X_{\beta^\star_j})}$. Thus, the latter and (6.107)-(6.108) yield, a.s. for all $\ell \geq 1$,

$$\begin{aligned}
\|F_c(\cdot,u)\|_{L^p(\lambda'_\varepsilon,\tau_{\varepsilon,\ell},w_\kappa^{\lambda'_\varepsilon};X_0)} &+ \|G_c(\cdot,u)\|_{L^p(\lambda'_\varepsilon,\tau_{\varepsilon,\ell},w_\kappa^{\lambda'_\varepsilon};\gamma(H,X_{1/2}))}\\
&\leq R + \Upsilon(\varepsilon)\|u\|_{{}_0\mathsf{MR}^{\theta,\kappa}_X(\lambda'_\varepsilon,\tau_{\varepsilon,\ell})}
\end{aligned} \tag{6.109}$$

where $R > 0$ and $\Upsilon \in C([0,\infty))$ are independent of $\ell \geq 1$ and $\lim_{\varepsilon\downarrow 0}\Upsilon(\varepsilon) = 0$.

To proceed further, note that by (6.101) there exists a constant $\widetilde{K}_{M,\eta} > 0$ independent of $\varepsilon > 0$ such that (6.12) and Proposition 6.2.6(4) holds with C, (A,B) replaced by $\widetilde{K}_{M,\eta}$, $(A|_{[\![\sigma'_n,T]\!]}, B|_{[\![\sigma'_n,T]\!]})$, respectively. Let $\varepsilon^* > 0$ be such that

$$\widetilde{K}_{M,\eta}\Upsilon(\varepsilon^*) \leq \frac{1}{4} \tag{6.110}$$

and set $\tau_{*,\ell} := \tau_{\varepsilon^*,\ell}$, $\lambda'_* := \lambda'_{\varepsilon^*}$ and $\mathscr{R}_* := \mathscr{R}_{\lambda'_{\varepsilon^*},(A,B)}$ (see (6.17)). Fix $\ell \geq 1$ an recall that $F_c = F_{\mathsf{Tr}} + F_c$, $G = G_{\mathsf{Tr}} + G_c$. Thus, by (6.106) and combining the second and the third estimate of (6.105) with Lemma 6.4.5 we get

$$\begin{aligned}
&\mathbf{1}_{[\lambda'_*,\tau_{*,\ell}]\times\mathcal{U}}F(\cdot,u) \in L^p_{\mathscr{P}}(\Omega;L^p(\lambda'_*,\tau_{*,\ell},w_\kappa^{\lambda'_*};X_0)),\\
&\mathbf{1}_{[\lambda'_*,\tau_{*,\ell}]\times\mathcal{U}}G(\cdot,u) \in L^p_{\mathscr{P}}(\Omega;L^p(\lambda'_*,\tau_{*,\ell},w_\kappa^{\lambda'_*};\gamma(H,X_{1/2}))),
\end{aligned} \tag{6.111}$$

with norms depending possibly on $\ell \geq 1$. To get the desired contradiction we need to prove an estimate which is uniform in $\ell \geq 1$.

To this end, recall that $\|u(\lambda'_*)\|_{X^{\mathsf{Tr}}_{\kappa,p}} \leq L$ a.s. on $\mathcal{U}$ by (6.104). Reasoning as in the proof of Theorem 6.3.6 (see (6.62)-(6.63)), by Proposition 6.2.7 and (6.111) one has, a.e. on $[\lambda'_*, \tau_{*,\ell}] \times \mathcal{U}$,

$$u = \mathscr{R}_*(\mathbf{1}_{\mathcal{U}} u(\lambda'_*), \mathbf{1}_{[\lambda'_*,\tau_{*,\ell}]\times\mathcal{U}} F(\cdot, u), \mathbf{1}_{[\lambda'_*,\tau_{*,\ell}]\times\mathcal{U}} G(\cdot, u)). \tag{6.112}$$

Thus, by (6.106), (6.109) with $\varepsilon = \varepsilon^*$ and Proposition 6.2.6 applied with θ as in Step 1, for all $\ell \geq 1$ we obtain

$$\begin{aligned}
&\|u\|_{L^p(\mathcal{U};{}_0\mathsf{MR}^{\theta;\kappa}_X(\lambda'_*,\tau_{*,\ell}))} \\
&\quad \leq \|\mathscr{R}_*(\mathbf{1}_{\mathcal{U}} u(\lambda'_*), \mathbf{1}_{[\lambda'_*,\tau_{*,\ell}]\times\mathcal{U}} F(\cdot, u), \mathbf{1}_{[\lambda'_*,\tau_{*,\ell}]\times\mathcal{U}} G(\cdot, u))\|_{L^p(\Omega;{}_0\mathsf{MR}^{\theta;\kappa}_X(\lambda'_*,T))} \\
&\quad \leq 2\widetilde{K}_{M,\eta}\big(L + R_{\mathsf{Tr},M} + \|F_c(\cdot,u)\|_{L^p((\lambda'_*,\tau_{*,\ell})\times\mathcal{U}, w_\kappa^{\lambda'_*};X_0)} \\
&\qquad\qquad\qquad + \|G_c(\cdot,u)\|_{L^p((\lambda'_*,\tau_{*,\ell})\times\mathcal{U}, w_\kappa^{\lambda'_*};\gamma(H,X_{1/2}))}\big) \\
&\quad \leq 2K_{M,\eta,L} + \frac{1}{2}\|u\|_{L^p(\mathcal{U};{}_0\mathsf{MR}^{\theta;\kappa}_X(\lambda'_*,\tau_{*,\ell}))},
\end{aligned}$$

where $K_{M,\eta,L}$ does not depend on $\ell \geq 1$ and in the last estimate we used the choice of ε^* in (6.110). Let us stress that $L^p(\mathcal{U})$-norms in the previous inequality are well-defined due to the measurability result Lemma 4.1.7 and the fact that λ'_ε takes values in a finite set.

Therefore, $\|u\|_{L^p(\mathcal{U};{}_0\mathsf{MR}^{\theta;\kappa}_X(\lambda'_*,\tau_{*,\ell}))} \leq C$ for $C > 0$ independent of $\ell \geq 1$. Since $\lim_{\ell\to\infty} \tau_{*,\ell} = \sigma$ a.s. on $\mathcal{V}'$, by (6.109) with $\varepsilon = \varepsilon^*$, by Fatou's lemma we get a.s. on $\mathcal{V}'' = \mathcal{V}' \cap \mathcal{U}$,

$$\begin{aligned}
\sup_{\ell\geq 1}\Big[&\|F_c(\cdot,u)\|_{L^p(\lambda'_*,\tau_{*,\ell},w_\kappa^{\lambda'_*};X_0)} + \|G_c(\cdot,u)\|_{L^p(\lambda'_*,\tau_{*,\ell},w_\kappa^{\lambda'_*};\gamma(H,X_{1/2}))}\Big] \\
&= \|F_c(\cdot,u)\|_{L^p(\lambda'_*,\sigma,w_\kappa^{\lambda'_*};X_0)} + \|G_c(\cdot,u)\|_{L^p(\lambda'_*,\sigma,w_\kappa^{\lambda'_*};\gamma(H,X_{1/2}))} \leq C''
\end{aligned}$$

for some $C'' > 0$. Thus $\mathcal{N}^\kappa(u;\lambda'_*,\sigma) \leq C'' + R_{\mathsf{Tr},M}$ a.s. on $\mathcal{V}''$ (see (6.27) for $\mathcal{N}^\kappa$). The former implies

$$\begin{aligned}
\mathbb{P}(\mathcal{V}'') &= \mathbb{P}\Big(\mathcal{V}'' \cap \big\{\sigma < T,\, \mathcal{N}^\kappa(u;\lambda'_*,\sigma) < \infty\big\}\Big) \\
&= \mathbb{P}\Big(\mathcal{V}'' \cap \big\{\sigma < T,\, \mathcal{N}^\kappa(u;0,\sigma) < \infty\big\}\Big) \leq \mathbb{P}\Big(\sigma < T,\, \mathcal{N}^\kappa(u;0,\sigma) < \infty\Big) = 0,
\end{aligned}$$

where in the last equality we used Theorem 6.3.7(2). The previous yields the desired contradiction with (6.104). ☐

The proof of Theorem 6.3.6(4) combines the argument used above and the one used in the Step 2 in the proof of Theorem 6.3.6(2).

Proof of Theorem 6.3.6(4). As usual $s = 0$ and we prove the claim by contradiction. Assume that $\mathbb{P}(\mathcal{O}) > 0$ where

$$\mathcal{O} := \Big\{\sigma < T,\, \lim_{t\uparrow\sigma} u(t) \text{ exists in } X^{\mathsf{Tr}}_{\kappa,p},\, \|u\|_{L^p(0,\sigma;X_{1-\frac{\kappa}{p}})} < \infty\Big\}.$$

Reasoning as in the proof of Theorem 6.3.6(2) there exist $\eta, M > 0$, a sequence of stopping times $(\sigma'_n)_{n\geq 1}$ taking values in a finite set and $\mathcal{V}' \in \mathscr{F}_\sigma$ such that $\eta \leq \sigma'_n < \sigma$ a.s. on $\mathcal{V}'$ and for each $\varepsilon > 0$ there exists an $N(\varepsilon) \geq 1$

$$\|u\|_{C(\overline{I}_\sigma;X^{\mathsf{Tr}}_{\kappa,p})} \leq M, \quad \sup_{t\in[\sigma'_{N(\varepsilon)},\sigma]} \|u(t) - u(\sigma)\|_{X^{\mathsf{Tr}}_{\kappa,p}} < \varepsilon \quad \text{and} \quad \|u\|_{L^p(\sigma'_{N(\varepsilon)},\sigma;X_{1-\frac{\kappa}{p}})} < \varepsilon.$$

where $u(\sigma) := \lim_{t\uparrow\sigma} u(t)$ a.s. on $\mathcal{O}$.

For each $\varepsilon > 0$, $\ell \geq 1$ define $\tau_{\varepsilon,\ell} := \sigma'_{N(\varepsilon)}$ on $\{\sigma'_{N(\varepsilon)} \geq \sigma\}$ and on $\{\sigma'_{N(\varepsilon)} < \sigma\}$ as

$$\tau_{\varepsilon,\ell} := \inf\Big\{t \in [\sigma'_{N(\varepsilon)}, \sigma) \,: \|u(t) - u(\sigma'_{N(\varepsilon)})\|_{X^{\mathsf{Tr}}_{\kappa,p}} \geq 2\varepsilon,\ \|u\|_{\mathfrak{X}^\star(\sigma'_{N(\varepsilon)},t)} \geq \ell, \\ \|u\|_{L^p(\sigma'_{N(\varepsilon)},t;X_{1-\frac{\kappa}{p}})} \geq \varepsilon,\ \|u\|_{C([0,t];X^{\mathsf{Tr}}_{\kappa,p})} \geq M\Big\}.$$

Choose $\varepsilon^* > 0$ such that the condition in the proof of Theorem 6.3.6(2) (see the text before (6.61)) and (6.110) both hold. At this point, one can repeat the estimates of u using the splitting in (6.63) with τ_ε replaced by $\tau_{\varepsilon^*,\ell}$. Note that $I - III$ (see (6.63)) can be estimated as in Theorem 6.3.6(2) and the remaining terms as in the proof of Theorem 6.3.7(4) above. The claim follows similar to Theorem 6.3.7(4) by contradiction with Theorem 6.3.6(2). □

It remains to prove Theorem 6.3.8. As announced in Subsection 6.3.2, we prove a generalization of Theorem 6.3.8, where we do not require $\varphi_j = \beta_j$. An example of such a situation is provided by stochastic reaction-diffusion equations with gradient nonlinearities, see Subsection 5.1.4.

Recall that $\varphi_j^\star$ and $\beta_j^\star$ are defined at the beginning of Subsection 6.4.4.

Proposition 6.4.9 (Serrin type blow-up criteria for semilinear SPDEs - revised)**.** *Let the assumptions of Theorem 6.3.8 be satisfied replacing (6.30) by the following condition: For each $j \in \{1, \dots, m_F + m_G\}$ such that $\rho_j > 0$ one of the following is satisfied*

- *$\kappa > 0$ and $\beta_j^\star, \varphi_j^\star > 1 - \frac{1+\kappa}{p}\frac{1+\kappa}{2+\kappa}$;*
- *$\kappa = 0$ and $\rho_j \leq 1$.*

Then the L^p_κ-maximal local solution (u, σ) to (6.22) satisfies

$$\mathbb{P}\big(\sigma < T,\ \|u\|_{L^p(s,\sigma;X_{1-\frac{\kappa}{p}})} < \infty\big) = 0.$$

Before we prove Proposition 6.4.9, we first show that it implies Theorem 6.3.8.

Proof of Theorem 6.3.8. It is enough to check the assumptions of Proposition 6.4.9. In case $\kappa = 0$ the assumptions coincide and hence this case is clear.

Next we assume $\kappa > 0$ and fix $j \in \{1, \dots, m_F + m_G\}$. If $\rho_j = 0$, then as agreed at the beginning of Subsection 6.4.4 we replaced ρ_j by ε_j, and the corresponding $\varphi_j^\star, \beta_j^\star$ satisfy $\varphi_j^\star, \beta_j^\star > 1 - \frac{1+\kappa}{p}\frac{1+\kappa}{2+\kappa}$ as assumed in Proposition 6.4.9. If $\rho_j > 0$, then note that the definition of $\varphi_j^\star, \beta_j^\star$ and the fact that $\varphi_j = \beta_j$ imply $\varphi_j^\star = \beta_j^\star = 1 - \frac{\rho_j}{\rho_j+1}\frac{1+\kappa}{p}$. Since $\rho_j < 1 + \kappa$ is equivalent to $\varphi_j^\star, \beta_j^\star > 1 - \frac{1+\kappa}{p}\frac{1+\kappa}{2+\kappa}$, the assumptions of Proposition 6.4.9 are satisfied also in this case. □

Proof of Proposition 6.4.9. As usual, we consider $s = 0$ and we split the proof into several cases. The proof follows a similar argument as Theorem 6.3.7(4).

Suppose that $\mathbb{P}(\mathcal{O}) > 0$ where

$$\mathcal{O} := \{\sigma < T, \|u\|_{L^p(I_\sigma;X_{1-\frac{\kappa}{p}})} < \infty\}. \tag{6.113}$$

Then there exist $\eta, M > 0$ and $\mathscr{F}_\sigma \ni \mathcal{V} \subseteq \mathcal{O}$ such that $\mathbb{P}(\mathcal{V}) > 0$ and

$$\sigma > \eta \quad \text{ and } \quad \|u\|_{L^p(I_\sigma;X_{1-\frac{\kappa}{p}})} \leq M \quad \text{ a.s. on } \mathcal{V}.$$

Note that in contrast to the proof of Theorem 6.3.7(4) we do not have an L^∞-bound for u in the trace space $X^{\mathsf{Tr}}_{\kappa,p}$. However, the assumption of Theorem 6.3.8 is that $F_{\mathsf{Tr}} = G_{\mathsf{Tr}} = 0$ and $C'_{c,n}$ in (6.84) are independent of $n \in \mathbb{N}$.

As in Step 1 of the proof of Theorem 6.3.7(4), from Lemma 6.4.8(1) and (3), it follows that for all $j \in \{1, \dots, m_F + m_G\}$, (6.96)-(6.97) hold with $\phi_{1,j}, \phi_{2,j}, \delta_{1,j}, \delta_{2,j} \in (0, 1]$ such that

$$\rho_j[(1 - \phi_{1,j}) + \phi_{1,j}(1 - \delta_{1,j})] + [(1 - \phi_{2,j}) + \phi_{2,j}(1 - \delta_{2,j})] \leq 1. \tag{6.114}$$

Therefore, we can extend the proof of Theorem 6.3.7(4) to the present case. Indeed, by (6.114) we can repeat the estimate (6.109) replacing the term $\|u\|_{{}_0\mathsf{MR}_X^{\theta,\kappa}(\lambda_\varepsilon,\tau_{\varepsilon,\ell})}$ by

$$\|u\|_{{}_0\mathsf{MR}_X^{\theta,\kappa}(\lambda_\varepsilon,\tau_{\varepsilon,\ell})\cap L^\infty(\lambda_\varepsilon,\tau_{\varepsilon,\ell};X^{\mathsf{Tr}}_{\kappa,p})}.$$

After this modification, one can repeat the argument of Step 3 of the proof of Theorem 6.3.8. $\square$

Chapter 7

Instantaneous regularization for SPDEs with illustration

Let $(\Omega, \mathbb{F} = (\mathscr{F}_t)_{t\geq 0}, \mathscr{A}, \mathbb{P})$ and $\mathscr{P}$ be a filtered probability space and the progressive sigma algebra, respectively. Moreover, we denote by H and W_H a separable Hilbert space and a cylindrical Brownian motion in H, respectively.

In this chapter we present instantaneous regularization results for solution to

$$\begin{cases} du + A(t,u)udt = F(t,u)dt + (B(t,u) + G(t,u))dW_H, & t \in \mathbb{R}_+, \\ u(0) = u_0. \end{cases} \tag{7.1}$$

Here we prove a general and abstract framework to prove that L^p_κ-maximal local solution (u,σ) to (7.1) satisfies

$$u \in \bigcap_{\theta\in[0,1/2)} H^{\theta,r}_{\mathrm{loc}}(0,\sigma; Y_{1-\theta}) \text{ a.s.}$$

where $(Y_{1-\theta})_{\theta\in[0,1/2)}$ is a family of suitable spaces incorporating regularization in 'space'. More precisely, the above inclusion yields a regularization in time if $r > p$ or in space if the inclusions $Y_{1-\theta} \hookrightarrow X_{1-\theta}$ for $\theta \in [0,1/2)$ are strict. Actually, we will provide sufficient conditions to ensure

$$u \in \bigcap_{\theta\in[0,1/2)} H^{\theta,r}_{\mathrm{loc}}(0,\sigma; Y_{1-\theta}) \text{ a.s.} \implies u \in \bigcap_{\theta\in[0,1/2)} H^{\theta,\widehat{r}}_{\mathrm{loc}}(0,\sigma; \widehat{Y}_{1-\theta}) \text{ a.s.}$$

and we can use the previous iteratively to prove instantaneous regularization of solutions. Here $(\widehat{Y}_{1-\theta})_{\theta\in[0,1/2)}$ is another family of spaces such that $\widehat{Y}_{1-\theta} \hookrightarrow Y_{1-\theta}$ for $\theta \in [0,1/2)$. The idea behind the bootstrap argument was sketched in Subsection 1.3.3 and we will not give additional details here. As an illustration of the above results, we prove regularization results for solutions to the following SPDE:

$$\begin{cases} du - \partial_x^2 u\, dt = \partial_x(u^3)dt + |u|^h dw_t^c, & \text{on } \mathbb{T}, \\ u(0) = u_0, & \text{on } \mathbb{T}, \end{cases} \tag{7.2}$$

where $u : [0,\infty) \times \Omega \times \mathbb{T}^d \to \mathbb{R}$ is the unknown process, $h \in [1,3)$ and w^c denotes a coloured noise. The choice of the cubic nonlinearity $\partial_x(u^3)$ is not accidental. Indeed $L^2(L^2)$-solutions to (7.2) present the same behaviour of the $L^2(L^2)$-solutions to the 2D stochastic Navier-Stokes (1.11) and one cannot use the standard argument to bootstrap regularity. Indeed, reasoning as in Subsection 1.4.3, if $u \in C([0,t]; L^2(\mathbb{T})) \cap L^2(0,\sigma; H^1(\mathbb{T}))$ for all $t < \sigma$, then for all $t < \sigma$

$$u \in C([0,t]; L^2(\mathbb{T}^2)) \cap L^2(0,t; H^1(\mathbb{T}^2)) \hookrightarrow L^6(0,t; H^{1/3}(\mathbb{T}^2)) \hookrightarrow L^6(0,t; L^6(\mathbb{T}^2)), \tag{7.3}$$

where the above embeddings are sharp. Thus, $u \in C([0,\sigma); L^2(\mathbb{T})) \cap L^2_{\mathrm{loc}}([0,\sigma); H^1(\mathbb{T}))$ implies $\partial_x(u^3) \in L^2_{\mathrm{loc}}([0,\sigma); H^{-1}(\mathbb{T}))$ and this is useless for proving regularization. Therefore to prove regularization of solutions to (7.2) we need to exploit the full strength of our theory.

This chapter is organized as follows. Section 7.1 contains three main results. In Theorem 7.1.3 we present a general iteration scheme to bootstrap regularity in time and space. In Corollary 7.1.5 we specialize to time regularity. In both results weights play an essential role. Finally, in Proposition 7.1.7 we present a result which allows to introduce weights after starting from an unweighted situation. The latter is important in several interesting situations. An example where this occurs will be given in Section 7.2 (see Roadmap 7.2.5 for a simple explanation).

7.1 Instantaneous regularization

As in Chapter 6 here we analyse the stochastic evolution equations (6.22). Recall that Assumption 4.2.1 is in force throughout this section where the abstract theory is analysed.

7.1.1 Assumptions

Below we state abstract conditions which can be used for bootstrapping arguments. We first present our main assumptions on the spaces $Y_1 \hookrightarrow Y_0$ in which we bootstrap regularity.

Assumption 7.1.1. *Suppose that Hypothesis* **(H)** *is satisfied. Let Y_0 and Y_1 be UMD Banach spaces with type 2, and such that $Y_1 \hookrightarrow Y_0$ densely. Let $r \in (2,\infty)$ and $\alpha \in [0, \frac{r}{2}-1)$. We say that hypothesis $\mathbf{H}(Y_0, Y_1, r, \alpha)$ holds if*

(1) *X_0 and Y_0 are compatible, and $Y_1 \cap X_1 \hookrightarrow Y_1$ is dense;*

(2) *There exist maps $A_Y : [\![s,T]\!] \times Y^{\mathsf{Tr}}_{\alpha,r} \to \mathscr{L}(Y_1, Y_0)$, $F_Y : [\![s,T]\!] \times Y_1 \to Y_0$, $B_Y : [\![s,T]\!] \times Y^{\mathsf{Tr}}_{\alpha,r} \to \mathscr{L}(Y_1, \gamma(H, Y_{1/2}))$ and $G_Y : [\![s,T]\!] \times Y_1 \to \gamma(H, Y_{1/2})$ such that a.s. for all $t \in (s,T)$,*

$$\begin{aligned} A_Y(t,z)v &= A(t,z)v, \qquad & B_Y(t,z)v &= B(t,z)v, \\ F_Y(t,v) &= F(t,v), & G_Y(t,v) &= G(t,v), \end{aligned}$$

for all $z, v \in X_1 \cap Y_1$. Moreover, the following hold:

- *A_Y, B_Y verify (HA) with (X_0, X_1, p, κ) replaced by (Y_0, Y_1, r, α);*
- *F_Y, G_Y satisfy (HF) with (X_0, X_1, p, κ) replaced by (Y_0, Y_1, r, α) and (possibly) different parameters $(\widetilde{\rho}_j, \widetilde{\varphi}_j, \widetilde{\beta}_j, \widetilde{m}_F, \widetilde{m}_G)$.*

(3) *There exists an invertible sectorial operator $\widetilde{A}_Y$ on Y_0 of angle $< \pi/2$ such that $\mathsf{D}(\widetilde{A}_Y) = Y_1$;*

(4) *$f \in L^0_{\mathscr{P}}(\Omega; L^r(s+\varepsilon, T; Y_0))$, $g \in L^0_{\mathscr{P}}(\Omega; L^r(s+\varepsilon, T; \gamma(H, Y_{1/2})))$ for all $\varepsilon > 0$.*

As before in case (6.22) is semilinear, we write $(A(\cdot,x), B(\cdot,x)) = (\bar{A}(\cdot), \bar{B}(\cdot))$ and $(\bar{A}_Y(\cdot), \bar{B}_Y(\cdot))$ instead of $(A_Y(\cdot,x), B_Y(\cdot,x))$. If it is necessary to make the dependency on (α, r) explicit as well, then we will write $(A_{Y,\alpha,r}, B_{Y,\alpha,r}, F_{Y,\alpha,r}, G_{Y,\alpha,r})$ instead of (A_Y, B_Y, F_Y, G_Y).

Let (u,σ) be the maximal L^p_κ-solution given by Theorem 6.3.1. Now the idea is as follows. The above setting allows to consider (6.22) in *the (Y_0, Y_1, r, α)-setting*, i.e.

$$\text{replace } \ (X_0, X_1, p, \kappa, A, B, F, G) \ \text{ by } \ (Y_0, Y_1, r, \alpha, A_Y, B_Y, F_Y, G_Y) \ \text{ in (6.22).}$$

Now if Assumption 6.3.2 holds in the (Y_0, Y_1, r, α)-setting for $\ell = \alpha$, it follows that all conditions of Theorem 6.3.1 also hold on $[s+\varepsilon, T]$ for $\varepsilon > 0$ arbitrary. Therefore, if $u(s+\varepsilon) \in Y^{\mathsf{Tr}}_{\alpha,r}$ a.s. there exists an L^r_α-maximal local solution (v,τ) to (6.22) in the (Y_0, Y_1, r, α)-setting with (s, u_s) replaced by $(s+\varepsilon, u(s+\varepsilon))$ and $\tau : \Omega \to (s+\varepsilon, T]$. Now one would expect that $\tau = \sigma$ and $u = v$ on $[\varepsilon, \sigma]$, and this typically improves the space-time regularity of u. In order to make the above bootstrap argument precise we need to be able to connect the (Y_0, Y_1, r, α)-setting to the (X_0, X_1, p, κ)-setting to assure:

(a) $X^{\mathsf{Tr}}_p \subseteq Y^{\mathsf{Tr}}_{\alpha,r}$;

(b) $v = u$ on $[\varepsilon, \tau]$ and $\tau \leq \sigma$;

(c) $\tau \geq \sigma$ via a blow-up criterium in the (Y_0, Y_1, r, α)-setting.

Below we will actually use an abstract (Y_0, Y_1, r, α)-setting and $(\widehat{Y}_0, \widehat{Y}_1, \widehat{r}, \widehat{\alpha})$-setting to be able to iteration the bootstrap argument. One important ingredients in the proof will be to show uniqueness (see (b) in the above), and this will be done by presuming the following inclusion:

$$\bigcap_{\theta \in [0,1/2)} H^{\theta,\widehat{r}}(s, T, w^s_{\widehat{\alpha}}; \widehat{Y}_{1-\theta}) \subseteq L^r(s, T, w^s_\alpha; Y_1) \cap \mathcal{Y}(s, T) \cap C([s, T]; Y^{\mathsf{Tr}}_{\alpha,r}). \tag{7.4}$$

Here $\mathcal{Y}(s, T)$ is defined as in (6.41) with the above new parameters (r, α) and X_θ replaced by Y_θ.

By translation and scaling (7.4) extends to all other bounded intervals. If u is in the space on RHS(7.4), then u has the required regularity for being an L^r_α-strong solution to (6.22) in the (Y_0, Y_1, r, α)-setting by Definition 4.3.3. This follows from Lemma 6.4.5 in the (Y_0, Y_1, r, α)-setting.

The following lemma gives sufficient conditions for (7.4), and is strong enough to cover all applications which we have studied so far. In particular, we never need to consider $\mathcal{Y}$ explicitly in applications. We only consider the case $\widehat{Y}_i \hookrightarrow Y_i$ for $i \in \{0, 1\}$, but there are also variations which avoid this condition.

Lemma 7.1.2. *Suppose that Hypothesis* **H**(Y_0, Y_1, r, α) *holds (see Assumption 7.1.1) and that*

- $\widehat{Y}_0$ *and* $\widehat{Y}_1$ *are Banach spaces such that* $\widehat{Y}_1 \hookrightarrow \widehat{Y}_0$*;*
- $\widehat{Y}_i \hookrightarrow Y_i$ *for* $i \in \{0, 1\}$*,* $\widehat{r} \in [r, \infty)$*, and* $\widehat{\alpha} \in [0, \frac{\widehat{r}}{2} - 1)$*.*

Then (7.4) holds in each of the following cases:

(1) $r = \widehat{r}$ *and* $\alpha = \widehat{\alpha}$*;*

(2) $\frac{1+\widehat{\alpha}}{\widehat{r}} < \frac{1+\alpha}{r}$*;*

(3) $\frac{1+\widehat{\alpha}}{\widehat{r}} < \frac{1+\alpha}{r} + \varepsilon$ *provided* $\widehat{Y}_{1-\varepsilon} \hookrightarrow Y_1$ *and* $\widehat{Y}_0 \hookrightarrow Y_\varepsilon$*, for some* $\varepsilon \in (0, \frac{1}{2} - \frac{1+\alpha}{r})$*;*

(4) $r = \widehat{r}$ *and* $\frac{1+\widehat{\alpha}}{r} = \frac{1+\alpha}{r} + \varepsilon$ *provided* $\widehat{Y}_{1-\varepsilon} \hookrightarrow Y_1$ *and* $\widehat{Y}_0 \hookrightarrow Y_\varepsilon$*, for some* $\varepsilon \in (0, \frac{1}{2} - \frac{1+\alpha}{r})$*.*

Proof. In all cases, it is enough to consider the case $s = 0$.

(1)-(2): By Proposition 6.1.1(3) and the fact that $\widehat{Y}_{1-\theta} \hookrightarrow Y_{1-\theta}$, for all $\theta \in [0, \frac{1}{2})$,

$$H^{\theta,\widehat{r}}(I_T, w_{\widehat{\alpha}}; \widehat{Y}_{1-\theta}) \hookrightarrow H^{\theta,r}(I_T, w_\alpha; \widehat{Y}_{1-\theta}) \hookrightarrow H^{\theta,r}(I_T, w_\alpha; Y_{1-\theta}).$$

Therefore, the inclusion follows by the former, Lemma 4.3.9 and Proposition 4.1.5(1).

(3): Due to Lemma 4.3.9, it is enough to show that for some $\theta_1, \theta_2 \in [0, \frac{1}{2})$, $\nu \in (\frac{1+\alpha}{r}, \frac{1}{2})$,

$$\bigcap_{\theta \in \{\theta_1, \theta_2\}} H^{\theta,\widehat{r}}(I_T, w_{\widehat{\alpha}}; \widehat{Y}_{1-\theta}) \subseteq H^{\nu,r}(I_t, w_\alpha; Y_{1-\nu}) \cap L^r(I_T, w_\alpha; Y_1). \tag{7.5}$$

The reiteration theorem for the complex interpolation (see e.g. [20, Theorem 4.6.1]) ensures that $\widehat{Y}_{\theta(1-\varepsilon)} \hookrightarrow Y_{\varepsilon+\theta(1-\varepsilon)}$, for each $\theta \in (0, 1)$. Therefore

$$\widehat{Y}_{1-\theta} \hookrightarrow Y_{1-\theta+\varepsilon}, \qquad \text{for all } \theta \in [\varepsilon, 1). \tag{7.6}$$

Since $\varepsilon \in (0, \frac{1}{2} - \frac{1+\alpha}{r})$, there exists a $\delta \in (0, \frac{1}{2})$ such that $\delta > \varepsilon + \frac{1+\alpha}{r} > \frac{1+\widehat{\alpha}}{\widehat{r}}$, where the last inequality follows by assumption. By (7.6) and the fact that $\delta > \varepsilon$ we obtain

$$H^{\delta,\widehat{r}}(I_T, w_{\widehat{\alpha}}; \widehat{Y}_{1-\delta}) \hookrightarrow H^{\delta,\widehat{r}}(I_T, w_{\widehat{\alpha}}; Y_{1-(\delta-\varepsilon)}) \hookrightarrow H^{\delta-\varepsilon,r}(I_T, w_\alpha; Y_{1-(\delta-\varepsilon)})$$

where the last embedding follows from Corollary 6.1.3. Similarly,

$$H^{\varepsilon,\widehat{r}}(I_T, w_{\widehat{\alpha}}; \widehat{Y}_{1-\varepsilon}) \hookrightarrow H^{\varepsilon,\widehat{r}}(I_T, w_{\widehat{\alpha}}; Y_1) \hookrightarrow L^r(I_T, w_\alpha; Y_1).$$

The above embeddings imply (7.5) with $\theta_1 = \varepsilon$, $\theta_2 = \delta$ and $\nu = \delta - \varepsilon$.

(4): The proof is similar to (3) using the last claim in Corollary 6.1.3 in the case that $\frac{1+\widehat{\alpha}}{r} = \varepsilon + \frac{1+\alpha}{r}$. ☐

7.1.2 Bootstrapping using weights

In this section we state our main result. The statement below is quite technical because the list of conditions is rather long. The strength of the result will be demonstrated in Section 7.2 and in Chapters 8-9 where we use the results below to show

- Hölder regularity results with rough initial data;
- weak solutions immediately become strong solutions.

To obtain this type of regularization, we build a general scheme which only depends on the structure of the SPDE through the parameters p, κ, X_0, X_1 in which the scaling properties of the underlined SPDEs is encoded. An example will be given in Section 7.2 (see Roadmap 7.2.5 for a short overview).

The assumptions below have a considerable overlap with Theorem 6.3.6, which plays a key role in the proof. Let us remind that critical spaces for (6.22) are defined below Hypothesis **(H)**. The picture one should have in mind is that Y-regularity and L^r-integrability is given, and $\widehat{Y}$-regularity and $L^{\widehat{r}}$-integrability are deduced as a consequence.

Theorem 7.1.3 (Bootstrapping regularity)**.** *Let Hypothesis* **(H)** *be satisfied. Let $u_s \in L^0_{\mathscr{F}_s}(\Omega; X^{\mathsf{Tr}}_{\kappa,p})$ and suppose that* (6.23) *holds for a sequence $(u_{s,n})_{n\geq 1}$. Suppose that*

$$(A(\cdot, u_{s,n}), B(\cdot, u_{s,n})) \in \mathcal{SMR}^{\bullet}_{p,\kappa}(s,T), \quad n \geq 1,$$

and let (u, σ) be the L^p_κ-maximal local solution to (6.22) *given by Theorem* 6.3.1*. Suppose that Assumption* 6.3.2 *holds for $\ell \in \{0, \kappa\}$ and Assumption* 6.3.4 *holds. Further suppose the following:*

(1) *Hypothesis* $\mathbf{H}(Y_0, Y_1, r, \alpha)$ *holds, Assumption* 6.3.2 *holds in the (Y_0, Y_1, r, α)-setting for $\ell = \alpha$, and*
   - $Y^{\mathsf{Tr}}_r \hookrightarrow X^{\mathsf{Tr}}_p$;
   - $u : (\!|s, \sigma|\!) \to Y_1$ *is strongly progressively measurable and*

$$u \in \bigcap_{\theta\in[0,1/2)} H^{\theta,r}_{\mathrm{loc}}(s, \sigma; Y_{1-\theta}) \quad a.s.$$

(2) *Hypothesis* $\mathbf{H}(\widehat{Y}_0, \widehat{Y}_1, \widehat{\alpha}, \widehat{r})$ *holds with $\widehat{r} \in [r, \infty)$ and $\widehat{\alpha} \in [0, \frac{\widehat{r}}{2} - 1)$, and the space $\widehat{Y}^{\mathsf{Tr}}_{\widehat{\alpha},\widehat{r}}$ is not critical for* (6.22)*, Assumption* 6.3.2 *for $\ell \in \{0, \widehat{\alpha}\}$ and* 6.3.4 *both hold in the $(\widehat{Y}_0, \widehat{Y}_1, \widehat{r}, \widehat{\alpha})$-setting.*

(3) $\widehat{Y}_i \hookrightarrow Y_i$ *for $i \in \{0, 1\}$, $Y^{\mathsf{Tr}}_r \hookrightarrow \widehat{Y}^{\mathsf{Tr}}_{\widehat{\alpha},\widehat{r}}$ and* (7.4) *holds.*

Then (u, σ) instantaneously regularizes in spaces and time in the sense that $u : (\!|s, \sigma|\!) \to \widehat{Y}_1$ is strongly progressively measurable and

$$u \in \bigcap_{\theta\in[0,1/2)} H^{\theta,\widehat{r}}_{\mathrm{loc}}(s, \sigma; \widehat{Y}_{1-\theta}) \subseteq C((s, \sigma); \widehat{Y}^{\mathsf{Tr}}_{\widehat{r}}) \quad a.s. \tag{7.7}$$

Observe that if Hypothesis **(H)** and Assumption 6.3.2 for $\ell = \kappa$ hold, then by Theorem 6.3.1, condition (1) is always satisfied for $(Y_0, Y_1, r, \alpha) = (X_0, X_1, p, \kappa)$. Also note that $Y^{\mathsf{Tr}}_{\alpha,r}$ can be critical for (6.22) in the (Y_0, Y_1, r, α)-setting.

In the above we did not assume $Y_i \hookrightarrow X_i$, but in most of the applications to SPDEs this holds, and therefore, $Y^{\mathsf{Tr}}_r \hookrightarrow X^{\mathsf{Tr}}_p$ is fulfilled provided $r \geq p$.

Remark 7.1.4.

(1) Theorem 7.1.3 yields an improvement in regularity in time if $\widehat{r} > r$ and in space if $\widehat{Y}_i \hookrightarrow Y_i$ is strict for some $i \in \{0, 1\}$.

(2) Often Theorem 7.1.3 can be applied iteratively where $(\widehat{Y}_0,\widehat{Y}_1,\widehat{r},\widehat{\alpha})$ takes over the role of (Y_0,Y_1,r,α), and another quadruple takes over the role of $(\widehat{Y}_0,\widehat{Y}_1,\widehat{r},\widehat{\alpha})$. In this way for concrete SPDEs, in finitely many steps one can often derive $(\frac{1}{2}-\varepsilon)$-Hölder regularity in time and higher order Hölder regularity in space for rough initial data.

Proof of Theorem 7.1.3. To prepare the proof, we collect some useful facts. It suffices to consider $s=0$. Fix $\varepsilon>0$ and set

$$\mathcal{V}:=\{\sigma>\varepsilon\}\in\mathscr{F}_\varepsilon. \tag{7.8}$$

By (1), Proposition 4.1.5, and (3),

$$\mathbf{1}_{\mathcal{V}}u(\varepsilon)\in L^0_{\mathscr{F}_\varepsilon}(\Omega;Y_r^{\mathsf{Tr}})\subseteq L^0_{\mathscr{F}_\varepsilon}(\Omega;\widehat{Y}^{\mathsf{Tr}}_{\widehat{\alpha},\widehat{r}}). \tag{7.9}$$

As explained below Assumption 7.1.1, by (1) and Theorem 6.3.1 applied in the (Y_0,Y_1,r,α)-setting we find that there exists an L^r_α-maximal local solution to (v,τ) to (6.22) in the (Y_0,Y_1,r,α)-setting with $s=\varepsilon$ and initial data $\mathbf{1}_{\mathcal{V}}u(\varepsilon)$, with localizing sequence $(\tau_k)_{k\ge1}$. Similarly, arguing in $(\widehat{Y}_0,\widehat{Y}_1,\widehat{r},\widehat{\alpha})$-setting (thus using (7.9) and (2)), we obtain an $L^{\widehat{r}}_{\widehat{\alpha}}$-maximal local solution $(\widehat{v},\widehat{\tau})$ to (6.22) in the $(\widehat{Y}_0,\widehat{Y}_1,\widehat{r},\widehat{\alpha})$-setting with $s=\varepsilon$, initial data $\mathbf{1}_{\mathcal{V}}u(\varepsilon)$ and localizing sequence $(\widehat{\tau}_k)_{k\ge1}$.

Step 1: $\tau=\sigma$ on $\mathcal{V}$ and $v=u$ a.e. on $[\varepsilon,\tau)\times\mathcal{V}$. By Theorem 6.3.1 and Proposition 4.1.5(2), for all $k\ge1$,

$$v\in\bigcap_{\theta\in[0,1/2)}H^{\theta,r}(\varepsilon,\tau_k,w^\varepsilon_\alpha;Y_{1-\theta})\subseteq C((\varepsilon,\tau_k];Y_r^{\mathsf{Tr}}). \tag{7.10}$$

By condition (1) and Theorem 6.3.1(1), $u\in Y_1\cap X_1$ a.e. on $[\![\varepsilon,\sigma)\!)$. Thus, $A_Y(\cdot,u)u=A(\cdot,u)u$, $B_Y(\cdot,u)u=B(\cdot,u)u$, $F_Y(\cdot,u)=F(\cdot,u)$, $G_Y(\cdot,u)=G(\cdot,u)$ a.e. on $[\![\varepsilon,\sigma)\!)$. This implies that $(\mathbf{1}_{\mathcal{V}}u|_{[\varepsilon,\sigma)},\mathbf{1}_{\mathcal{V}}\sigma+\mathbf{1}_{\Omega\setminus\mathcal{V}}\varepsilon)$ is an L^r_α-local solution to (6.22) in the (Y_0,Y_1,r,α)-setting with $s=\varepsilon$ and initial data $\mathbf{1}_{\mathcal{V}}u(\varepsilon)$. Therefore, maximality of (v,τ) implies

$$\sigma\le\tau,\quad\text{a.e. on }\mathcal{V},\qquad u=v,\quad\text{a.e. on }[\varepsilon,\sigma)\times\mathcal{V}. \tag{7.11}$$

It remains to prove that $\sigma\ge\tau$ a.e. on $\mathcal{V}$. For this it is enough to show

$$\mathbb{P}(\mathcal{V}\cap\{\sigma<\tau\})=0. \tag{7.12}$$

Since $\lim_{k\to\infty}\tau_k=\tau$ and $\tau>\varepsilon$ a.s., by (7.10) and (7.11), we get

$$u=v\in C((\varepsilon,\sigma];Y_r^{\mathsf{Tr}}),\qquad\text{a.e. on }\mathcal{V}\cap\{\tau<\sigma\}.$$

By (1), $Y_r^{\mathsf{Tr}}\hookrightarrow X_p^{\mathsf{Tr}}$ and therefore $\lim_{t\uparrow\sigma}u(t)$ exists in X_p^{Tr} a.e. on $\mathcal{V}\cap\{\sigma<\tau\}$. Combining this with $\sigma<\tau\le T$ a.e. on $\mathcal{V}\cap\{\sigma<\tau\}$, we obtain

$$\begin{aligned}\mathbb{P}(\mathcal{V}\cap\{\sigma<\tau\})&=\mathbb{P}\Big(\mathcal{V}\cap\{\sigma<\tau\}\cap\{\sigma<T\}\cap\big\{\lim_{t\uparrow\sigma}u(t)\text{ exists in }X_p^{\mathsf{Tr}}\big\}\Big)\\&\le\mathbb{P}\Big(\sigma<T,\ \lim_{t\uparrow\sigma}u(t)\text{ exists in }X_p^{\mathsf{Tr}}\Big)=0,\end{aligned}$$

where the last equality follows from Theorem 6.3.6(1) (here we used Assumption 6.3.2 for $\ell\in\{0,\kappa\}$ and Assumption 6.3.4). This implies (7.12) and completes this step.

Step 2: $\tau=\widehat{\tau}$ a.s. and $v=\widehat{v}$ on $[\![\varepsilon,\tau)\!)$. Since $(\widehat{\tau}_k)_{k\ge1}$ is a localizing sequence, for all $k\ge1$, one has

$$\widehat{v}\in\bigcap_{\theta\in[0,1/2)}H^{\theta,\widehat{r}}(\varepsilon,\widehat{\tau}_k,w^\varepsilon_{\widehat{\alpha}};\widehat{Y}_{1-\theta})\ \text{ a.s.} \tag{7.13}$$

Next, as in Step 1, we show that $(\widehat{v},\widehat{\tau})$ is an L^r_α-local solution to (6.22) in the (Y_0,Y_1,α,r)-setting. To this end, note that thanks to Hypothesis $\mathbf{H}(Y_0,Y_1,\alpha,r)$ and $\mathbf{H}(\widehat{Y}_0,\widehat{Y}_1,\widehat{\alpha},\widehat{r})$, and by density,

$$A_{\widehat{Y},\widehat{\alpha},\widehat{r}}(\cdot,\widehat{v})\widehat{v}=A_{Y,\alpha,r}(\cdot,\widehat{v})\widehat{v},\qquad F_{\widehat{Y},\widehat{\alpha},\widehat{r}}(\cdot,\widehat{v})=F_{Y,\alpha,r}(\cdot,\widehat{v}),$$

$$B_{\widehat{Y},\widehat{\alpha},\widehat{r}}(\cdot,\widehat{v})\widehat{v} = B_{Y,\alpha,r}(\cdot,\widehat{v})\widehat{v}, \qquad G_{\widehat{Y},\widehat{\alpha},\widehat{r}}(\cdot,\widehat{v}) = G_{Y,\alpha,r}(\cdot,\widehat{v})$$

a.e. on $[\![\varepsilon,\widehat{\tau})\!)$. The latter, (7.4), and (7.13), ensure that $(\widehat{v},\widehat{\tau})$ is also an L^r_α-local solution to (6.22) in the (Y_0,Y_1,r,α)-setting with $s=\varepsilon$ and initial data $\mathbf{1}_{\mathcal{V}}u(\varepsilon)$. The maximality of (v,τ) gives

$$\widehat{\tau} \le \tau, \qquad \text{a.s.,} \qquad v = \widehat{v}, \qquad \text{a.e. on } [\![\varepsilon,\widehat{\tau})\!). \tag{7.14}$$

It remains to prove $\tau \le \widehat{\tau}$ a.s. By (3), $Y^{\mathsf{Tr}}_r \hookrightarrow \widehat{Y}^{\mathsf{Tr}}_{\widehat{\alpha},\widehat{r}}$ and thus by (7.10) and (7.14),

$$v = \widehat{v} \in C((\varepsilon,\widehat{\tau}];Y^{\mathsf{Tr}}_r) \subseteq C((\varepsilon,\widehat{\tau}];\widehat{Y}^{\mathsf{Tr}}_{\widehat{\alpha},\widehat{r}}), \qquad \text{a.s. on } \{\widehat{\tau}<\tau\}.$$

Therefore, $\lim_{t\uparrow\widehat{\tau}}\widehat{v}(t)$ exists in $\widehat{Y}^{\mathsf{Tr}}_{\widehat{\alpha},\widehat{r}}$ a.e. on $\{\widehat{\tau}<\tau\}$. Since $\widehat{\tau}<\tau\le T$ on $\{\widehat{\tau}<\tau\}$,

$$\begin{aligned}\mathbb{P}(\widehat{\tau}<\tau) &= \mathbb{P}\Big(\{\widehat{\tau}<\tau\}\cap\{\widehat{\tau}<T\}\cap\big\{\lim_{t\uparrow\widehat{\tau}}\widehat{v}(t)\text{ exists in }\widehat{Y}^{\mathsf{Tr}}_{\widehat{\alpha},\widehat{r}}\big\}\Big)\\ &\le \mathbb{P}\Big(\widehat{\tau}<T,\ \lim_{t\uparrow\widehat{\tau}}\widehat{v}(t)\text{ exists in }\widehat{Y}^{\mathsf{Tr}}_{\widehat{\alpha},\widehat{r}}\Big) = 0,\end{aligned}$$

where in the last step we used condition (2) in order to apply Theorem 6.3.6(3) in the $(\widehat{Y}_0,\widehat{Y}_1,\widehat{r},\widehat{\alpha})$-setting.

Step 3: Conclusion. By Steps 1-2, $\sigma=\tau=\widehat{\tau}$ a.s. on $\mathcal{V}$ and $u=v=\widehat{v}$ on $\mathcal{V}\times[\varepsilon,\sigma)=[\![\varepsilon,\sigma)\!)$. Let $(\sigma_n)_{n\ge1}$ be the localizing sequence for u defined in (6.53). Then we have already seen that one has $\sigma_n<\sigma$ for all $n\ge1$. Thus, by (7.13) and the previous consideration, for all $n\ge1$ and $\delta>0$,

$$\mathbf{1}_{\mathcal{V}}u \in \bigcap_{\theta\in[0,1/2)} H^{\theta,\widehat{r}}(\varepsilon,\sigma_n,w^{\varepsilon}_{\widehat{\alpha}};\widehat{Y}_{1-\theta}) \subseteq \bigcap_{\theta\in[0,1/2)} H^{\theta,\widehat{r}}(\varepsilon+\delta,\sigma_n;\widehat{Y}_{1-\theta}) \ \text{ a.s.} \tag{7.15}$$

where we used that $\sigma_n<\sigma=\lim_{k\to\infty}\widehat{\tau}_k$ a.s., Proposition 6.1.1(1).

Now let $\varepsilon_k=\delta_k=\frac{1}{2k}$, $\mathcal{V}_k=\{\sigma>\frac{1}{2k}\}$ and set $\Omega_0=\bigcup_{k\ge1}\mathcal{V}_k$. Let $(\widehat{v}_k)_{k\ge1}$ denote the corresponding $\widehat{Y}_1$-valued solutions defined on $[\![1/k,\sigma)\!)$. Since $\sigma>0$ a.s., $\mathbb{P}(\Omega_0)=1$, and therefore, a.s. for all $k\ge1$ and all $n\ge1$,

$$u \in \bigcap_{\theta\in[0,1/2)} H^{\theta,\widehat{r}}(\tfrac{1}{k},\sigma_n;\widehat{Y}_{1-\theta}).$$

This implies the first part of (7.7). The final part of (7.7) follows from Proposition 4.1.5(1) in the unweighted case.

Finally, to check the progressive measurability of u as a $\widehat{Y}_1$-valued function, note that $u|_{[\![1/k,\sigma)\!)}=\widehat{v}_k$ on $\mathcal{V}_k$ a.s. Since $\mathbf{1}_{[1/k,\sigma)\times\mathcal{V}_k}\widehat{v}_k$ is strongly progressively measurable as a $\widehat{Y}_1$-valued process, and converges pointwise to u a.s. we find that u has the same property. ☐

In the special case $\widehat{Y}_i=Y_i$, the above result simplifies and can be used to derive time-regularity.

Corollary 7.1.5 (Bootstrapping time regularity). *Let Hypothesis* **(H)** *be satisfied. Let $u_s\in L^0_{\mathscr{F}_s}(\Omega;X^{\mathsf{Tr}}_{\kappa,p})$ and that (6.23) holds. Suppose that*

$$(A(\cdot,u_{s,n}),B(\cdot,u_{s,n}))\in\mathcal{SMR}^\bullet_{p,\kappa}(s,T), \qquad n\ge1,$$

and let (u,σ) be the L^p_κ-maximal local solution to (6.22) given by Theorem 6.3.1. Suppose that Assumption 6.3.2 holds for $\ell\in\{0,\kappa\}$ and Assumption 6.3.4 holds.

(1) *Suppose that Hypothesis* $\mathbf{H}(Y_0,Y_1,r,\alpha)$ *holds for some $\alpha\in(0,\frac{r}{2}-1)$, Assumption 6.3.2 holds in the (Y_0,Y_1,r,α)-setting for $\ell=\alpha$, and*

- $Y^{\mathsf{Tr}}_r \hookrightarrow X^{\mathsf{Tr}}_p$*;*

- *$u : (\!|s,\sigma|\!) \to Y_1$ is strongly progressively measurable and*

$$u \in \bigcap_{\theta\in[0,1/2)} H^{\theta,r}_{\rm loc}(s,\sigma;Y_{1-\theta}) \quad a.s.$$

(2) *Let $\widehat{r} \in [r,\infty)$ and suppose Assumption 6.3.2 for $\ell \in \{0,\widehat{\alpha}\}$ and Assumption 6.3.4 both hold in the $(Y_0,Y_1,\widehat{r},\widehat{\alpha})$-setting for all $\widehat{\alpha} \in [0,\frac{\widehat{r}}{2}-1)$ satisfying $\frac{1+\widehat{\alpha}}{\widehat{r}} < \frac{1+\alpha}{r}$.*

Then

$$u \in \bigcap_{\theta\in[0,1/2)} H^{\theta,\widehat{r}}_{\rm loc}(s,\sigma;Y_{1-\theta}) \subseteq C(s,\sigma;Y^{\mathsf{Tr}}_{\widehat{r}}) \quad a.s. \tag{7.16}$$

Note that if $f = 0$ and $g = 0$, then the above result may be applied with $\widehat{r}$ arbitrary. Recall that by Theorem 6.3.1, (1) is satisfied in the case $X_i = Y_i$ (for $i \in \{0,1\}$), $r = p$ and $\alpha = \kappa > 0$. In the proof of the Corollary 7.1.5 we will see that it is enough to assume (2) for a particular value $\widehat{\alpha}$ such that $\frac{1+\widehat{\alpha}}{\widehat{r}} < \frac{1+\alpha}{r}$, and the corresponding trace space $\widehat{Y}^{\mathsf{Tr}}_{\widehat{\alpha},\widehat{r}}$ is not critical for (6.22) in the $(\widehat{Y}_0,\widehat{Y}_1,\widehat{r},\widehat{\alpha})$-setting.

Proof of Corollary 7.1.5. The idea is to apply Theorem 7.1.3 with $\widehat{Y}_0 = Y_0, \widehat{Y}_1 = Y_1$, $\widehat{r} > r$ and $\widehat{\alpha}$ such that $\frac{1+\widehat{\alpha}}{\widehat{r}} < \frac{1+\alpha}{r}$ which will be chosen below. It remains to check Theorem 7.1.3(1)-(3). Note that (1) holds by assumption. Next, we check (2). Since hypothesis $\mathbf{H}(Y_0,Y_1,\alpha,r)$ holds, there exist $\widetilde{m}_F, \widetilde{m}_G$, $(\widetilde{\varphi}_j)_{j=1}^{\widetilde{m}_F+\widetilde{m}_G} \subseteq (1-\frac{1+\alpha}{r},1)$, $(\widetilde{\beta}_j)_{j=1}^{\widetilde{m}_F+\widetilde{m}_G}$ such that $\widetilde{\beta}_j \in (1-\frac{1+\alpha}{r},\widetilde{\varphi}_j]$ and (HF)-(HG) hold with (p,κ) replaced by (r,α). Set

$$2\varepsilon := \min_{j\in\{1,\dots,\widetilde{m}_F+\widetilde{m}_G\}} \Big\{\widetilde{\beta}_j - 1 + \frac{1+\alpha}{r}, \frac{\alpha}{r}\Big\} > 0,$$

where we used that $\alpha > 0$ by (1). In particular,

$$\min_{j\in\{1,\dots,\widetilde{m}_F+\widetilde{m}_G\}} \{\widetilde{\beta}_j,\widetilde{\varphi}_j\} > 1 - \frac{1+\alpha}{r} + \varepsilon, \quad \text{and} \quad \frac{1}{r} < \frac{1+\alpha}{r} - \varepsilon. \tag{7.17}$$

Since $\widehat{\alpha} \in [0,\frac{\widehat{r}}{2}-1)$ if and only if $\frac{1+\widehat{\alpha}}{\widehat{r}} \in [\frac{1}{\widehat{r}},1) \supseteq [\frac{1}{r},1)$ (where we used that $\widehat{r} \geq r$), there exists an $\widehat{\alpha} \in [0,\frac{\widehat{r}}{2}-1)$ such that

$$\frac{1+\alpha}{r} - \varepsilon < \frac{1+\widehat{\alpha}}{\widehat{r}} < \frac{1+\alpha}{r}. \tag{7.18}$$

Note that the above choice of $\widehat{\alpha}$ yields

$$Y_r \hookrightarrow Y^{\mathsf{Tr}}_{\widehat{\alpha},\widehat{r}} = \widehat{Y}^{\mathsf{Tr}}_{\widehat{\alpha},\widehat{r}} \quad \text{and} \quad Y^{\mathsf{Tr}}_{\widehat{\alpha},\widehat{r}} = \widehat{Y}^{\mathsf{Tr}}_{\widehat{\alpha},\widehat{r}} \hookrightarrow Y^{\mathsf{Tr}}_{\alpha,r}. \tag{7.19}$$

We claim that F_Y, G_Y satisfy (HF)-(HG) with (p,κ) replaced by $(\widehat{r},\widehat{\alpha})$. To see this, note that by (7.17)-(7.19),

$$\widetilde{\varphi}_j,\widetilde{\beta}_j > 1 - \frac{1+\alpha}{r} + \varepsilon > 1 - \frac{1+\widehat{\alpha}}{\widehat{r}},$$

and thus (4.18), (4.20) hold with $(p,\kappa,\rho_j,\varphi_i,\beta_j,m_F,m_G)$ replaced by $(\widehat{r},\widehat{\alpha},\widetilde{\rho}_j,\widetilde{\varphi}_i,\widetilde{\beta}_j,\widetilde{m}_F,\widetilde{m}_G)$. Moreover, due to the fact that $\frac{1+\widehat{\alpha}}{\widehat{r}} < \frac{1+\alpha}{r}$, (4.18) and (4.20) hold with the *strict* inequality and thus $\widehat{Y}^{\mathsf{Tr}}_{\widehat{\alpha},\widehat{r}}$ is not critical for (6.22) in the $(Y_0,Y_1,\widehat{r},\widehat{\alpha})$-setting. Finally, (HA) holds by the second inclusion in (7.19).

Due to the first inclusion in (7.19), to check Theorem 7.1.3(3) it remains to note that (7.4) with the above choice of $(\widehat{Y}_0,\widehat{Y}_1,\widehat{\alpha},\widehat{r})$ follows from Lemma 7.1.2(2) and the upper bound in (7.18). □

Remark 7.1.6. In the deterministic case, part of the arguments used in Theorem 7.1.3 appears in [175, 181, 100]. Our systematic treatment appears to be new. Let us note that an essential step in the proof is to use blow-up criteria to show the invariance of the explosion time σ in the different settings.

7.1.3 The emergence of weights

In Theorem 7.1.3(2), there are two difficulties:

- it is not applicable in the critical case;
- it is often not applicable in the unweighted setting.

In this subsection we show how to create a *weighted* situations from an unweighted one, which also allows criticality. For simplicity we only consider the case where we add a weight near $t = s$ as this is what is needed to start a bootstrapping argument. Moreover, we only consider the semilinear setting, as the extension to the quasilinear setting is quite cumbersome and harder to state. Unlike in Theorem 7.1.3(2) the case $p = 2$ is allowed, which is central in many applications.

Recall that $(\bar{A}_Y, \bar{B}_Y)$ is as below Assumption 7.1.1. We need an additional condition on F and G. Fix $r \in [1, \infty]$. Suppose that for each $n \geq 1$ there is a constant C_n such that for a.a. $\omega \in \Omega$, for all $t \in [s, T]$ and $\|x\|_{(X_0,X_1)_{1-\frac{1}{p},r}}, \|y\|_{(X_0,X_1)_{1-\frac{1}{p},r}} \leq n$,

$$\begin{aligned} \|F_c(t,\omega,x)\|_{X_0} &\leq C_n \sum_{j=1}^{m_F} (1 + \|x\|_{X_{\varphi_j}}^{\rho_j}) \|x\|_{X_{\beta_j}} + C_n \\ \|G_c(t,\omega,x)\|_{\gamma(H,X_{1/2})} &\leq C_n \sum_{j=m_F+1}^{m_F+m_G} (1 + \|x\|_{X_{\varphi_j}}^{\rho_j}) \|x\|_{X_{\beta_j}} + C_n, \\ \|F_{\mathsf{Tr}}(t,\omega,x)\|_{X_0} + \|G_{\mathsf{Tr}}(t,\omega,x)\|_{\gamma(H,X_{1/2})} &\leq C_n (1 + \|x\|_{(X_0,X_1)_{1-\frac{1}{p},r}}). \end{aligned} \tag{7.20}$$

This coincides with the growth condition in (HF) and (HG) if $r = p$ and $\kappa = 0$.

The main result of this section is the following:

Proposition 7.1.7 (Adding weights at the initial time)**.** *Let Hypothesis* **(H)** *be satisfied with* $\kappa = 0$*. Let* $r \in [p, \infty)$*,* $r > 2$*,* $\alpha \in [0, \frac{r}{2} - 1)$*, and suppose that* (7.20) *holds. Let* $u_s \in L^0_{\mathscr{F}_s}(\Omega; X^{\mathsf{Tr}}_p)$ *and suppose that*

$$(A(\cdot, x), B(\cdot, x)) \equiv (\bar{A}(\cdot), \bar{B}(\cdot)) \in \mathcal{SMR}^{\bullet}_p(s, T), \quad \textit{for all } x \in X_1. \tag{7.21}$$

Let (u, σ) *be the* L^p_0*-maximal local solution to* (6.22) *of Theorem* 6.3.1*. Suppose that* $\delta \in [0, 1 - \max_j \varphi_j)$*, where* $p > 2$ *in case* $\delta = 0$*, and the following are satisfied:*

(1) *Hypothesis* $\mathbf{H}(Y_0, Y_1, \alpha, r)$*, and Assumption* 6.3.4 *hold in the* (Y_0, Y_1, α, r)*-setting,*

$$Y_\delta = X_0, \quad Y_1 = X_{1-\delta}, \quad \frac{1}{p} = \frac{1+\alpha}{r} + \delta, \quad \textit{and} \quad \frac{1}{r} \geq \max_j \varphi_j - 1 + \frac{1}{p};$$

(2) $(\bar{A}_Y, \bar{B}_Y) \in \mathcal{SMR}_{q,\beta}(t, T)$ *for all* $t \in (s, T)$*,* $q \in (2, r]$ *and* $\beta \in [0, \frac{q}{2} - 1)$*.*

Then

$$u \in \bigcap_{\theta \in [0,1/2)} H^{\theta,r}_{\mathrm{loc}}(s, \sigma; X_{1-\delta-\theta}) \subseteq C(s, \sigma; (X_0, X_1)_{1-\delta-\frac{1}{r},r}) \quad \textit{a.s.} \tag{7.22}$$

Proposition 7.1.7 allows to bootstrap regularity in time provided $r > p$ is not too big at the expense of reducing the regularity 'in space' in the case that $\delta > 0$. Moreover, if $\alpha > 0$, then $(X_0, X_1)_{1-\delta-\frac{1}{r},r} = (X_0, X_1)_{1-\frac{1}{p}+\frac{\alpha}{r},r} \hookrightarrow X^{\mathsf{Tr}}_p$ and therefore Proposition 7.1.7 also yields a regularization in space. One of the interesting features of Proposition 7.1.7 is that in applications to SPDEs one can fix r and choose δ small enough so that $\frac{\alpha}{r} = \frac{1}{p} - \frac{1}{r} - \delta > 0$. Then by (7.22) and the fact that $Y^{\mathsf{Tr}}_r = (X_0, X_1)_{1-\delta-\frac{1}{r},r} \hookrightarrow X^{\mathsf{Tr}}_p$ we can apply Corollary 7.1.5 to obtain high integrability in time. After that one can bootstrap further regularity via Theorem 7.1.3.

In the case $p > 2$, one usually takes $\delta = 0$. This is not allowed if $p = 2$, since $\frac{1}{p} = \frac{1}{2} > \frac{1+\alpha}{r}$ for all $r \in [2, \infty)$, $\alpha \in [0, \frac{r}{2} - 1)$ and therefore (1) can hold if and only if $\delta > 0$. In this case, Y_0 can be thought as "$X_{-\delta}$" and (typically) can be defined as $X_{-\delta,\widetilde{A}}$, i.e. the extrapolated space (see e.g. [3, Appendix A]) constructed via $\widetilde{A}$ (see Assumption 4.2.1).

Actually, Proposition 7.1.7 holds under more general assumptions, and it has a version for a quasilinear equations. However, we prefer to state Proposition 7.1.7 in its current simple form as this is enough for many of the applications we have in mind and is less technical.

Proof of Proposition 7.1.7. As usual, we set $s = 0$. Due to [20, Theorems 4.6.1 and 4.7.2] and the fact that $Y_\delta = X_0$, $Y_1 = X_{1-\delta}$ we have

$$Y_\theta = X_{\theta-\delta}, \text{ and } (Y_0, Y_1)_{\theta,\zeta} = (X_0, X_1)_{\theta-\delta,\zeta} \text{ for all } \theta \in (\delta, 1),\ \zeta \in [1, \infty]. \tag{7.23}$$

The former, the fact that $r \geq p$ and $\frac{1+\alpha}{r} = \frac{1}{p} - \delta$ imply that for all $\varepsilon > 0$

$$\mathbf{1}_{\mathcal{V}} u(\varepsilon) \in L^0_{\mathscr{F}_\varepsilon}(\Omega; X^{\mathsf{Tr}}_p) \subseteq L^0_{\mathscr{F}_\varepsilon}(\Omega; Y^{\mathsf{Tr}}_{\alpha,r}) \quad \text{where} \quad \mathcal{V} := \{\sigma > \varepsilon\}. \tag{7.24}$$

As explained below Assumption 7.1.1, by (7.24) and Hypothesis $\mathbf{H}(Y_0, Y_1, \alpha, r)$, Theorem 6.3.1 gives existence of an L^r_α-maximal local solution (v, τ) to (6.22) on $[\varepsilon, T]$ with initial data $\mathbf{1}_{\mathcal{V}} u(\varepsilon)$ in the (Y_0, Y_1, α, r)-setting. Since $r > 2$ (see (1)), Theorem 6.3.1(1) ensures that a.s.

$$v \in \bigcap_{\theta\in[0,1/2)} H^{\theta,r}_{\mathrm{loc}}([\varepsilon, \tau), w^\varepsilon_\alpha; Y_{1-\theta}) \subseteq C([\varepsilon, \tau); Y^{\mathsf{Tr}}_{\alpha,r}) = C([\varepsilon, \tau); (X_0, X_1)_{1-\frac{1}{p},r}), \tag{7.25}$$

where the latter is not the "right" trace space. As in the proof of Theorem 7.1.3, to prove (7.22) it remains to show that

$$\tau = \sigma \ \text{ a.s.} \quad \text{and} \quad u = v \ \text{ a.e. on } [\![\varepsilon, \sigma)\!). \tag{7.26}$$

Indeed, if (7.26) holds, then (7.22) follows from (7.25), the arbitrariness of $\varepsilon > 0$, and the argument in Step 3 of the proof of Theorem 7.1.3.

For the reader's convenience, we split the proof of (7.26) into several steps. In Step 1 we prove that $\tau \leq \sigma$ a.s. and $u = v$ a.e. on $\mathcal{V} \times [\varepsilon, \tau)$ assuming that

$$\bigcap_{\theta\in[0,1/2)} H^{\theta,r}(a, b, w^a_\alpha; Y_{1-\theta}) \subseteq \mathfrak{X}(a, b), \qquad \text{for all } 0 \leq a < b < \infty, \tag{7.27}$$

in Step 2 we prove (7.26), and in Step 3 we prove (7.27).

Step 1: $\tau \leq \sigma$ a.s. and $u = v$ a.e. on $\mathcal{V} \times [\varepsilon, \tau)$. By uniqueness of the L^p_0-maximal local solution (u, σ), to prove the claim of this step it remains to check that (v, τ) is an L^p_0-local solution to (6.22) on $[\varepsilon, T]$ in the $(X_0, X_1, p, 0)$-setting with initial data $\mathbf{1}_{\mathcal{V}} u(\varepsilon)$. Since Hypothesis $\mathbf{H}(Y_0, Y_1, \alpha, r)$ holds, it is enough to check that the process v has the required regularity for being an L^p_0-local solution to (6.22) on $[\varepsilon, T]$ in the $(X_0, X_1, p, 0)$-setting (see Definitions 4.3.3-4.3.4 and Lemma 6.4.5), i.e.

$$v \in L^p(\varepsilon, \tau_k; X_1) \cap C([\varepsilon, \tau_k]; X^{\mathsf{Tr}}_p) \cap \mathfrak{X}(\varepsilon, \tau_k) \quad \text{a.s. for all } k \geq 1, \tag{7.28}$$

for a suitable localizing sequence $(\tau_k)_{k\geq1}$ for (v, τ). By (7.25) and (7.27), $v \in \mathfrak{X}(\varepsilon, \tau_k)$ a.s., and thus it remains to prove the first two parts of (7.28).

To proceed, we need a localization argument. For $j \geq 1$ set

$$\mathcal{V}_j := \mathcal{V} \cap \{\|u(\varepsilon)\|_{X^{\mathsf{Tr}}_p} \leq j\} \in \mathscr{F}_\varepsilon. \tag{7.29}$$

By (7.25) and (7.27), we can define a localizing sequence by

$$\tau_j := \inf\{t \in [\varepsilon, \tau) \,:\, \|v\|_{L^r(\varepsilon,t,w^\varepsilon_\kappa;Y_1)} + \|v(t)\|_{(X_0,X_1)_{1-\frac{1}{p},r}} + \|v\|_{\mathfrak{X}(\varepsilon,t)} \geq j\}, \tag{7.30}$$

where $\inf \varnothing := \tau$, and moreover, $(\tau_j)_{j\geq1}$ is a localizing sequence for (v, τ). Due to (7.20) one can check that Lemma 6.4.5 is also valid if $X^{\mathsf{Tr}}_{\kappa,p}$ is replaced by $(X_0, X_1)_{1-\frac{1}{p},r}$ everywhere. Therefore, by (7.30), we obtain that for all $j \geq 1$,

$$\begin{aligned} F_j &:= \mathbf{1}_{[\varepsilon,\tau_j]\times\mathcal{V}_j} F(\cdot, v) \in L^\infty(\Omega; L^p(\varepsilon, T; X_0)), \\ G_j &:= \mathbf{1}_{[\varepsilon,\tau_j]\times\mathcal{V}_j} G(\cdot, v) \in L^\infty(\Omega; L^p(\varepsilon, T; \gamma(H, X_{1/2}))). \end{aligned} \tag{7.31}$$

Due to (7.21), (7.29) and (7.31), for each $j \geq 1$ there exists a strong solution

$$z_j \in L^p_{\mathscr{P}}(\Omega; L^p(\varepsilon, T; X_1) \cap C([\varepsilon, T]; X_p^{\mathsf{Tr}}))$$

to the following (see Definition 4.2.3)

$$\begin{cases} dz_j + \bar{A}(\cdot)z_j dt &= F_j dt + (\bar{B}(\cdot)z_j + G_j)dW_H, \quad \text{on } [\varepsilon, T], \\ z_j(\varepsilon) &= \mathbf{1}_{\mathcal{V}_j} u(\varepsilon). \end{cases} \tag{7.32}$$

Recall that (v, τ) is an L^r_α-maximal local solution to (6.22) on $[\varepsilon, T]$ in the (Y_0, Y_1, α, r)-setting with initial data $\mathbf{1}_{\mathcal{V}} u(\varepsilon)$. By (7.30), $v \in L^\infty(\Omega; L^r(\varepsilon, \tau_j, w^\varepsilon_\alpha; Y_1))$. Set

$$v_j := \mathbf{1}_{\mathcal{V}_j}(v - z_j) \in L^p(\Omega; L^p(\varepsilon, \tau_j; X_1) + L^r(\varepsilon, \tau_j, w^\varepsilon_\alpha; Y_1)). \tag{7.33}$$

Then v_j is a strong solution to the following problem on $[\varepsilon, \tau_j] \times \mathcal{V}_j$

$$\begin{cases} dv_j + \bar{A}_Y(\cdot)v_j dt = \bar{B}_Y(\cdot)v_j dW_H, \quad \text{on } [\![\varepsilon, T]\!], \\ v_j(\varepsilon) = 0, \end{cases} \tag{7.34}$$

where $(\bar{A}_Y, \bar{B}_Y)$ are as in Assumption 7.1.1. Due to the regularity of z_j, it remains to prove that $v_j = 0$ a.e. on $[\![\varepsilon, \tau_j]\!]$. To this end, we apply assumption (2). For the sake of clarity we divide the argument into two cases.

(1) *Case $\delta = 0$, $p > 2$.* Recall that $r \geq p$ and $\frac{1+\alpha}{r} = \frac{1}{p}$ by (1). Fix $q \in (2, p)$. Then $\frac{1+\alpha}{r} < \frac{1}{q}$ and Proposition 6.1.1(3) yields $L^r(I_t, w_\alpha; Y_1) \hookrightarrow L^q(I_t; Y_1)$ for all $t > 0$. Recalling that $X_1 = Y_1$ (due to $\delta = 0$), we have

$$L^p(I_t; X_1) + L^r(I_t, w_\alpha; Y_1) \hookrightarrow L^q(I_t; X_1) \quad \text{for all } t > 0.$$

The former and (7.33) ensure $v_j \in L^q(\Omega; L^q(\varepsilon, \tau_j; X_1))$. Therefore, $v_j \equiv 0$ by (7.34), $(\bar{A}_Y, \bar{B}_Y) = (\bar{A}, \bar{B}) \in \mathcal{SMR}_q(\varepsilon, T)$, and Proposition 6.2.7.

(2) *Case $\delta > 0$.* Since $\frac{1+\alpha}{r} = \frac{1}{p} - \delta < \frac{1}{p}$ we have $L^r(I_t, w_\alpha; Y_1) \hookrightarrow L^p(I_t; Y_1)$ for all $t > 0$ by Proposition 6.1.1(3). Recalling that $X_1 \hookrightarrow Y_1$, we have

$$L^p(I_t; X_1) + L^r(I_t, w_\alpha; Y_1) \hookrightarrow L^p(I_t; Y_1) \quad \text{for all } t > 0.$$

As above, the former and (7.33) imply that $v_j \in L^p(\Omega; L^p(\varepsilon, \tau_j; Y_1))$. Therefore, $v_j \equiv 0$ by (7.34), $(\bar{A}_Y, \bar{B}_Y) \in \mathcal{SMR}_p(\varepsilon, T)$ and Proposition 6.2.7.

Step 2: (7.26) holds. By Step 1 and (7.25) it is enough to show that $\tau \geq \sigma$ a.s. on $\mathcal{V}$ and this will be done via Theorem 6.3.7(4). To this end, we claim that it suffices to show that

$$v \in L^r(\varepsilon, \tau; Y_{1-\frac{\alpha}{r}}) \cap C([\varepsilon, \tau]; Y^{\mathsf{Tr}}_{\alpha,r}) \ \text{ a.s. on } \mathcal{V} \cap \{\tau < \sigma\}. \tag{7.35}$$

Indeed, if (7.35) holds, then

$$\begin{aligned} \mathbb{P}(\mathcal{V} \cap \{\tau < \sigma\}) &\overset{(i)}{=} \mathbb{P}\Big(\mathcal{V} \cap \{\tau < \sigma\} \cap \Big\{ \sup_{t \in [\varepsilon, \tau)} \|v(t)\|_{Y^{\mathsf{Tr}}_{\alpha,r}} + \|v\|_{L^r(\varepsilon,\tau;Y_{1-\frac{\alpha}{r}})} < \infty \Big\}\Big) \\ &\overset{(ii)}{\leq} \mathbb{P}\Big(\tau < T, \sup_{t \in [\varepsilon, \tau)} \|v(t)\|_{Y^{\mathsf{Tr}}_{\alpha,r}} + \|v\|_{L^r(\varepsilon,\tau;Y_{1-\frac{\alpha}{r}})} < \infty \Big) = 0, \end{aligned}$$

where in (i) we used (7.35), and in (ii) we used Theorem 6.3.7(4).

To prove (7.35), recall that $\tau \leq \sigma$ a.s. on $\mathcal{V}$ and $u = v$ a.e. on $[\varepsilon, \tau) \times \mathcal{V}$ by Step 1. The latter, (7.23) and the fact that $r \geq p$ ensure

$$v = u \in C([\varepsilon, \tau]; X_p^{\mathsf{Tr}}) \subseteq C([\varepsilon, \tau]; Y^{\mathsf{Tr}}_{\alpha,r}), \quad \text{a.s. on } \mathcal{V} \cap \{\tau < \sigma\}. \tag{7.36}$$

To complete the proof of (7.35), we need to prove $v \in L^r(\varepsilon,\tau;Y_{1-\frac{\alpha}{r}})$. To this end, we consider $p>2$ and $p=2$ separately.

If $p>2$, then by Step 1 and Theorem 6.3.1(1) applied with $\theta=\frac{\alpha}{r}+\delta=\frac{1}{p}-\frac{1}{r}<\frac{1}{2}$ we have, a.s. on $\mathcal{V}\cap\{\tau<\sigma\}$,

$$v=u\in H^{\frac{\alpha}{r}+\delta,p}(\varepsilon,\tau;X_{1-\frac{\alpha}{r}-\delta})\overset{(7.23)}{=}H^{\frac{\alpha}{r}+\delta,p}(\varepsilon,\tau;Y_{1-\frac{\alpha}{r}})\overset{(*)}{\hookrightarrow}L^r(\varepsilon,\tau;Y_{1-\frac{\alpha}{r}})$$

where we used that $\frac{\alpha}{r}+\delta=\frac{1}{p}-\frac{1}{r}<\frac{1}{2}$, and $(*)$ follows from Proposition 6.1.1(4).

If $p=2$, then instead of Sobolev embedding, we can use the following standard interpolation inequality for $0\le a<b<\infty$ and $\theta\in(0,1)$:

$$C([a,b];X_{1/2})\cap L^2(a,b;X_1)\hookrightarrow L^{2/\theta}(a,b;X_{(1+\theta)/2}). \tag{7.37}$$

By Theorem 6.3.1(1) with $p=2$ and Step 1 we have, a.s. on $\mathcal{V}\cap\{\tau<\sigma\}$,

$$v=u\in C([\varepsilon,\tau];X_{1/2})\cap L^2(\varepsilon,\tau;X_1)\overset{(7.37)}{\hookrightarrow}L^r(\varepsilon,\tau;X_{1-\frac{\alpha}{r}-\delta})\overset{(7.23)}{=}L^r(\varepsilon,\tau;Y_{1-\frac{\alpha}{r}}),$$

where we used $\theta=1-2(\frac{\alpha}{r}+\delta)=\frac{2}{r}\in(0,1)$.

Step 3: (7.27) holds. By translation and scaling, it is enough to prove (7.27) for $a=0$ and $b=T$. Fix $k\in\{1,\dots,m_F+m_G\}$. Recall that $\kappa=0$ and by (4.29)-(4.30)

$$\frac{1}{\rho_k^\star pr_k'}=\varphi_k-1+\frac{1}{p}\quad\text{and}\quad\frac{1}{pr_k}=\beta_k-1+\frac{1}{p}. \tag{7.38}$$

By Hypothesis **(H)** for $\phi\in\{\beta_k,\varphi_k\}$ we have $\delta<1-\phi$ (since $\delta<1-\varphi_k$ and $\varphi_k\ge\beta_k$) and $1-\phi-\delta<\frac{1}{2}$ (since $\phi>1-\frac{1}{p}>\frac{1}{2}$). Thus, to prove (7.27) note that

$$\begin{aligned}\bigcap_{\theta\in[0,1/2)}H^{\theta,r}(I_T,w_\alpha;Y_{1-\theta})&\subseteq\bigcap_{\phi\in\{\varphi_k,\beta_k\}}H^{1-\phi-\delta,r}(I_T,w_\alpha;Y_{\phi+\delta})\\&\overset{(i)}{=}\bigcap_{\phi\in\{\varphi_k,\beta_k\}}H^{1-\phi-\delta,r}(I_T,w_\alpha;X_\phi)\\&\overset{(ii)}{\hookrightarrow}L^{\rho_k^\star pr_k'}(I_T;X_{\varphi_k})\cap L^{pr_k}(I_T;X_{\beta_k})\end{aligned} \tag{7.39}$$

where in (i) we used (7.23) and $\delta<1-\varphi_k$, and in (ii) we used (7.38), Proposition 6.1.1(4) and

$$r\le\min\{pr_k,\rho_k^\star pr_k'\},\quad 1-\varphi_k-\delta-\frac{1+\alpha}{r}=-\frac{1}{\rho_k^\star pr_k'},\quad 1-\beta_k-\delta-\frac{1+\alpha}{r}=-\frac{1}{pr_k}.$$

Note that $r\le\min\{pr_k,\rho_k^\star pr_k'\}$ is equivalent to $\frac{1}{r}\ge\frac{1}{\rho_k^\star pr_k'}=\varphi_k-1+\frac{1}{p}$ and $\frac{1}{r}\ge\frac{1}{pr_k}=\beta_k-1+\frac{1}{p}$ (see (7.38)), which hold by (2) and the fact that $\beta_k\le\varphi_k$. Since k was arbitrary (7.27) follows from (6.41) and (7.39). ☐

Remark 7.1.8. If additionally in Proposition 7.1.7, $(\bar{A}_Y,\bar{B}_Y)\in\mathcal{SMR}^\bullet_{r,\alpha}(s,T)$,

$$f\in L^0_{\mathscr{P}}(\Omega;L^r(s,T,w_\alpha;Y_0))\quad\text{and}\quad g\in L^0_{\mathscr{P}}(\Omega;L^r(s,T,w_\alpha;\gamma(H,Y_{1/2}))),$$

then

$$u\in\bigcap_{\theta\in[0,1/2)}H^{\theta,r}_{\rm loc}([s,\sigma),w_\alpha^s;X_{1-\delta-\theta})\ \text{ a.s.} \tag{7.40}$$

Indeed, this follows by taking $\varepsilon=0$ in (7.24), and using $X^{\mathsf{Tr}}_p\hookrightarrow Y^{\mathsf{Tr}}_{\alpha,r}$.

The previous regularization results allow us to prove instantaneous regularization for L^p_κ-maximal local solutions to (6.22). In applications to SPDEs, one can employ the following extrapolation result to transfer the regularity and life-span of solutions for a given setting to another one.

Lemma 7.1.9 (Extrapolating regularity and life-span)**.** *Let Hypothesis* **(H)** *be satisfied. Let* $u_s \in L^0_{\mathscr{F}_s}(\Omega; X^{\mathsf{Tr}}_{\kappa,p})$ *and suppose that* (u,σ) *the* L^p_κ*-maximal local solution to* (6.22) *exists. Suppose that Assumption* 6.3.2 *for* $\ell = 0$ *and* 6.3.4 *are satisfied, and that the following conditions hold for a given* $\varepsilon \in (0, T - s)$*:*

(1) *Hypothesis* $\mathbf{H}(Y_0, Y_1, r, \alpha)$ *and Assumption* 6.3.2 *for* $\ell = \alpha$ *hold in the* (Y_0, Y_1, r, α)*-setting, and one of the following holds:*
   - $u \in C((s,\sigma); Y_{1/2}) \cap L^2(s,\sigma; Y_1)$ *a.s. and* $r = 2$*;*
   - $u \in \bigcap_{\theta\in[0,1/2)} H^{\theta,r}_{\mathrm{loc}}(s,\sigma; Y_{1-\theta})$ *a.s. and* $r > 2$*.*

(2) *Hypothesis* $\mathbf{H}(\widehat{Y}_0, \widehat{Y}_1, \widehat{r}, \widehat{\alpha})$ *holds,* $\widehat{Y}_i \hookrightarrow Y_i$*,* $\widehat{r} \in [r, \infty)$*,* $\widehat{Y}^{\mathsf{Tr}}_{\widehat{r}} \hookrightarrow X^{\mathsf{Tr}}_p$*, and the* L^r_α*-maximal local solution* (v,τ) *to* (6.22) *on* $[s+\varepsilon, T]$ *in the* (Y_0, Y_1, r, α)*-setting with initial value* $v_{s+\varepsilon} = \mathbf{1}_{\sigma>\varepsilon} u(s+\varepsilon)$ *satisfies* $v \in \bigcap_{\theta\in[0,1/2)} H^{\theta,\widehat{r}}_{\mathrm{loc}}(s+\varepsilon,\tau; \widehat{Y}_{1-\theta})$*.*

Then $\sigma = \tau$ *and* $u = v$ *on* $[s+\varepsilon,\sigma)$ *a.s. on the set* $\{\sigma > \varepsilon\}$*.*

Conditions (1) and (2) can be checked using the results in Subsections 7.1.2-7.1.3. Typically the lemma can be applied for every $\varepsilon \in (0, T-s)$, and in this case we obtain that $u \in \bigcap_{\theta\in[0,1/2)} H^{\theta,\widehat{r}}_{\mathrm{loc}}(s,\sigma; \widehat{Y}_{1-\theta})$.

Remark 7.1.10.

(1) In applications to SPDEs, Lemma 7.1.9 allows to extrapolate *global existence* result from a given (Y_0, Y_1, r, α)-setting where $\tau = T$. Typically, this yields an improvement in the choice of the initial data (see Theorem 7.2.4 and the text below it).

(2) In the case that $X^{\mathsf{Tr}}_{\kappa,p}$ is critical, Theorems 6.3.6(3) and 6.3.7(3) are not applicable. Using Lemma 7.1.9 we can change into a (Y_0, Y_1, r, α)-setting, and in the case $Y^{\mathsf{Tr}}_{\alpha,r}$ is not critical, then one can often apply those result to find $\tau = T$, and therefore $\sigma = T$. See Chapter 8 for an application to reaction-diffusion equations.

Proof of Lemma 7.1.9. As usual, we set $s = 0$. We use the arguments used in Step 1 and 3 in the proof of Theorem 7.1.3 with minor modifications. Note that, due to (1) and Proposition 4.1.5, $v_\varepsilon := \mathbf{1}_{\sigma>\varepsilon} u(\varepsilon) \in L^0_{\mathscr{F}_\varepsilon}(\Omega; Y^{\mathsf{Tr}}_{\alpha,r})$. By (2) there exists a L^r_α-maximal local solution (v,τ) to (6.22) in the (Y_0, Y_1, α, r)-setting.

Reasoning as in Step 1 in Theorem 7.1.3, by (1) and Lemma 6.4.5 applied in the (Y_0, Y_1, α, r)-setting, one can check that $(u|_{[\![\varepsilon,\sigma)\!)}, \sigma\mathbf{1}_{\mathcal{V}} + \varepsilon\mathbf{1}_{\Omega\setminus\mathcal{V}})$ is an L^r_α-local solution to (6.22) with initial data v_ε in the (Y_0, Y_1, α, r)-setting. The maximality of (v,τ) ensures that $\sigma \le \tau$, and $u = v$ on $[\varepsilon,\sigma)$ a.s. on $\{\sigma > \varepsilon\}$. It remains to prove $\mathbb{P}(\{\varepsilon < \sigma < \tau\}) = 0$.

By Proposition 4.1.5, and (2), we have

$$u = v \in C((\varepsilon,\sigma]; \widehat{Y}^{\mathsf{Tr}}_{\widehat{r}}) \subseteq C((\varepsilon,\sigma]; X^{\mathsf{Tr}}_p) \ \text{ a.s. on } \{\varepsilon < \sigma < \tau\}.$$

Since $\tau \le T$ a.s. we have

$$\begin{aligned}\mathbb{P}(\{\varepsilon < \sigma < \tau\}) &= \mathbb{P}\Big(\{\varepsilon < \sigma < \tau\} \cap \big\{\sigma < T, \lim_{t\uparrow\sigma} u(t) \text{ exists in } X^{\mathsf{Tr}}_p\big\}\Big) \\ &\le \mathbb{P}\Big(\big\{\sigma < T, \lim_{t\uparrow\sigma} u(t) \text{ exists in } X^{\mathsf{Tr}}_p\big\}\Big) = 0,\end{aligned}$$

where we used Theorem 6.3.6(1). This completes the proof. □

7.2 A 1D problem with cubic nonlinearities and colored noise

The aim of this subsection is to demonstrate our main results in a fairly simple situation. In particular, we created this section to illustrate how Sections 6.3 and 7.1 can be used to transfer results in an $L^2(L^2)$-setting to $L^p(L^q)$. The arguments used in this simple 1D case can be extended to other situations, and this will be done in Chapters 8 and 9.

Below we study the existence and regularity of global solutions to

$$\begin{cases} du - \partial_x^2 u\, dt = \partial_x(f(\cdot,u))dt + g(\cdot,u)dw_t^c, & \text{on } \mathbb{T},\\ u(0) = u_0, & \text{on } \mathbb{T}, \end{cases} \tag{7.41}$$

where $u : [0,\infty)\times\Omega\times\mathbb{T}\to\mathbb{R}$ is the unknown process and w_t^c is a colored noise on $\mathbb{T}$, i.e. an $H^\lambda(\mathbb{T})$-cylindrical Brownian motion (see Definition 2.3.5). Here, for the sake of simplicity we will assume $\lambda\in(\frac{1}{2},1)$. Throughout this section we write $H^s(\mathbb{T}) := H^{s,2}(\mathbb{T})$ for $s\in\mathbb{R}$.

7.2.1 Statement of the main results

Let us begin by listing our assumptions.

Assumption 7.2.1. $\lambda\in(\frac{1}{2},1)$.

(1) $f:\mathbb{R}_+\times\Omega\times\mathbb{T}\times\mathbb{R}\to\mathbb{R}$ *and* $g:\mathbb{R}_+\times\Omega\times\mathbb{T}\times\mathbb{R}\to\mathbb{R}$ *are* $\mathscr{P}\otimes\mathscr{B}(\mathbb{T})\otimes\mathscr{B}(\mathbb{R})$*-measurable.*

(2) $f(\cdot,0)\in L^\infty(\mathbb{R}_+\times\Omega\times\mathbb{T})$ *and* $g(\cdot,0)\in L^\infty(\mathbb{R}_+\times\Omega\times\mathbb{T})$. *Moreover, there exists a* $\nu\in(0,2]$ *such that a.s. for all* $t\in\mathbb{R}_+$, $x\in\mathbb{T}$ *and* $y,y'\in\mathbb{R}$,

$$|f(\cdot,y)-f(\cdot,y')| \lesssim (1+|y|^2+|y'|^2)|y-y'|,$$
$$|g(\cdot,y)-g(\cdot,y')| \lesssim (1+|y|^{2-\nu}+|y'|^{2-\nu})|y-y'|.$$

Next, we define *weak solutions to* (7.41) *on* $\overline{I}_T$ where $T\in(0,\infty]$. To this end, we suitably interpret the term $g(\cdot,u)dw_t^c$ in (7.41). The operator $M_{g(\cdot,u)}$ denotes multiplication by $g(\cdot,u)$. Since $\lambda>\frac{1}{2}$, by Sobolev embeddings $\iota: H^\lambda(\mathbb{T})\to L^\zeta(\mathbb{T})$ for all $\zeta\in(1,\infty)$ and therefore, by Hölder's inequality, we may consider $M_{g(\cdot,u)}$ as a multiplication operator from $L^\zeta(\mathbb{T})$ into $L^2(\mathbb{T})$, where $u\in H^1(\mathbb{T})$. Typical examples of nonlinearities f and g which satisfy Assumption 7.2.1 are given by

$$f(y)=ay^3, \quad \text{and} \quad g(y)=by^{3-\nu}, \quad a,b\in\mathbb{R}.$$

For $T\in(0,\infty]$. We say that (u,σ) is a *weak solution to* (7.41) *on* $\overline{I}_T$ if (u,σ) is an L^2_0-maximal local solution to (6.22) on $\overline{I}_T$ (see Definitions 4.3.3, 4.3.4, and Subsection 6.3.3 for the extension to $[0,\infty)$) with $p=2$, $\kappa=0$, $H=H^\lambda(\mathbb{T})$, $X_0=H^{-1}(\mathbb{T})$, $X_1=H^1(\mathbb{T})$, and for $v\in X_1$,

$$\begin{aligned} A(\cdot)v &= -\partial_x^2 v, & B(\cdot)v &= 0,\\ F(\cdot,v) &= \partial_x(f(\cdot,v)), & G(\cdot,v) &= M_{g(\cdot,v)}. \end{aligned} \tag{7.42}$$

Weak solutions are unique by maximality. We say that (u,σ) (or simply u) is a *global weak solution to* (7.41) provided (u,σ) is a weak solution to (7.41) on $[0,\infty)$ with $\sigma=\infty$ a.s. Note that in the above the term *weak* is meant in the analytic sense and is motivated by the choice $X_0=H^{-1}(\mathbb{T})$.

For $s_1,s_2\in(0,1)$, $C^{s_1,s_2}([a,b]\times\mathbb{T})$ denotes the space continuous functions on $u:[a,b]\times\mathbb{T}\to\mathbb{R}$ for which there exists a $C\geq 0$ such that

$$|u(t_1,x_1)-u(t_2,x_2)|\leq C(|t_1-t_2|^{s_1}+|x_1-x_2|^{s_2}),\quad t_1,t_2\in[a,b],\ x_1,x_2\in\mathbb{T}.$$

Theorem 7.2.2 (Local existence and regularity)**.** *Let Assumption* 7.2.1 *be satisfied. Then for any* $u_0\in L^0_{\mathscr{F}_0}(\Omega;L^2(\mathbb{T}))$, (7.41) *has a weak solution on* $[0,\infty)$ *such that*

$$u\in L^2_{\rm loc}([0,\sigma);H^1(\mathbb{T}))\cap C([0,\sigma);L^2(\mathbb{T}))\quad a.s. \tag{7.43}$$

Moreover, u *instantaneously regularizes in time and space:*

$$u\in\bigcap_{\theta\in[0,1/2)} H^{\theta,r}_{\rm loc}(I_\sigma;H^{1-2\theta,\zeta}(\mathbb{T}))\quad a.s.\ \ \text{for all } r,\zeta\in(2,\infty). \tag{7.44}$$

In particular,

$$u \in \bigcap_{\theta\in(0,1/2)} C^{\theta}_{\rm loc}(I_\sigma;C^{1-2\theta}(\mathbb{T})) \subseteq \bigcap_{\theta_1\in(0,1/2),\,\theta_2\in(0,1)} C^{\theta_1,\theta_2}_{\rm loc}(I_\sigma\times\mathbb{T}) \quad a.s. \tag{7.45}$$

Under additional assumptions on the nonlinearities f and g but keeping still keeping $u_0 \in L^0_{\mathscr{F}_0}(\Omega;L^2(\mathbb{T}))$, one can prove higher order regularity result by using the bootstrap argument of Section 7.1 or by using Schauder theory. We emphasize that the main difficulty is to pass from (7.43) to (7.44). The regularization effect in (7.44)-(7.45) is also non-trivial if $g\equiv 0$, and even in that case it appears to be new (see the discussion related to (7.3) for details).

Next we will prove a global existence result under a sublinearity assumption on g (but without further growth conditions on f).

Theorem 7.2.3 (Global existence and regularity)**.** *Let Assumption 7.2.1 be satisfied. Assume that $f(t,x,y)$ does not depend on x, and there exists a $C_g>0$ such that*

$$|g(t,x,y)|\leq C_g(1+|y|) \quad a.s.\ \text{for all } t\in\mathbb{R}_+,\ x\in\mathbb{T} \text{ and } y\in\mathbb{R}. \tag{7.46}$$

Then for any $u_0\in L^0_{\mathscr{F}_0}(\Omega;L^2(\mathbb{T}))$, (7.41) has a global *weak solution u. In particular, u satisfies (7.43)-(7.45) with $\sigma=\infty$ a.s. Moreover, if $u_0\in L^2(\Omega;L^2(\mathbb{T}))$, then for each $T\in\mathbb{R}_+$ there exists a $C>0$ independent of u_0 such that*

$$\mathbb{E}\Big[\sup_{s\in I_T}\|u(s)\|^2_{L^2(\mathbb{T})}\Big]+\mathbb{E}\|u\|^2_{L^2(I_T;H^1(\mathbb{T}))}\leq C(1+\mathbb{E}\|u_0\|^2_{L^2(\mathbb{T})}). \tag{7.47}$$

By Lemma 7.1.9 and Theorem 7.2.3 we can extrapolate global existence of solutions to (7.41) with rough initial data. To this end we need to introduce (s,q,p,κ)-weak solutions to (7.41). Let $T\in(0,\infty]$. We say that (u,σ) is a (unique) *(s,q,p,κ)-weak solution* to (7.41) on $\overline{I}_T$ if (u,σ) is an L^p_κ-maximal local solution to (6.22) with the choice (7.42), $X_0=H^{-1-s,q}(\mathbb{T})$, $X_0=H^{1-s,q}(\mathbb{T})$ and $H=H^\lambda(\mathbb{T})$. As above, (u,σ) (or simply u) is a *global (s,q,p,κ)-weak solution* to (7.41) if (u,σ) is a (s,q,p,κ)-weak solution to (7.41) on $[0,\infty)$ with $\sigma=\infty$ a.s.

Theorem 7.2.4 (Global existence and regularity with rough initial data)**.** *Suppose that Assumption 7.2.1 and (7.46) hold. Let $s\in(0,\frac{1}{3})$, $p,q\in(2,\infty)$ be such that*

$$q\in\Big(2,\frac{2}{1-2s}\Big) \quad \text{and} \quad \frac{1}{p}+\frac{1}{2q}\leq\frac{3-2s}{4}.$$

Set $\kappa_{\mathsf{crit}}=-1+\frac{p}{2}(\frac{3}{2}-s-\frac{1}{q})$. Then for any $u_0\in L^0_{\mathscr{F}_0}(\Omega;B^{\frac{1}{q}-\frac{1}{2}}_{q,p}(\mathbb{T}))$, (7.41) has a global $(s,q,p,\kappa_{\mathsf{crit}})$-weak solution u on $[0,\infty)$ such that

$$u\in L^p_{\rm loc}([0,\infty),w_{\kappa_{\mathsf{crit}}};H^{1-s,q}(\mathbb{T}))\cap C([0,\infty);B^{\frac{1}{q}-\frac{1}{2}}_{q,p}(\mathbb{T})) \quad a.s. \tag{7.48}$$

and u satisfies (7.44)-(7.45) with $\sigma=\infty$.

Letting $s\in(0,\frac{1}{3})$ and $q\in(0,6)$ be large, Theorem 7.2.4 ensures global existence for initial data in critical spaces with negative smoothness up to $-\frac{1}{3}$. Since $L^2(\mathbb{T})\hookrightarrow B^{\frac{1}{q}-\frac{1}{2}}_{q,p}(\mathbb{T})$ this improves Theorem 7.2.3. The proof of Theorem 7.2.4 also yields instantaneous regularization results for $(s,q,p,\kappa_{\mathsf{crit}})$-weak solutions to (7.41) without condition (7.46).

7.2.2 Proofs of Theorems 7.2.2-7.2.4

Throughout this subsection, to abbreviate the notation, we often write L^q, $H^{s,q}$, $B^s_{q,p}$ etc. instead of $L^q(\mathbb{T})$, $H^{s,q}(\mathbb{T})$, $B^s_{q,p}(\mathbb{T})$. We begin by proving Theorem 7.2.2. In Roadmap 7.2.5 we summarized the strategy to obtain (7.44)-(7.45) using the results in Section 7.1.

Roadmap 7.2.5 (Instantaneous regularity for critical problems with $p=q=2$).

(a) Consider (7.41) in the $(H^{-1-\varepsilon}, H^{1-\varepsilon}, r, \alpha)$-setting with $\varepsilon \geq 0$ small, and for $\varepsilon = 0$ obtain a local solution from Theorem 6.3.1 (see Step 1 below).

(b) Exploit the case $\varepsilon > 0$ to bootstrap time regularity via Proposition 7.1.7 (Step 2a below) and Corollary 7.1.5 (Step 2b below). Here we create a weighted setting in time, but we lose some regularity in space. We recover space regularity via Theorem 7.1.3 still in case of an L^2-setting in space (Step 2c below).

(c) Apply Theorem 7.1.3 once more to bootstrap regularity in space by considering (7.41) in the $(H^{-1,\zeta}, H^{1,\zeta}, r, \alpha)$-setting where $\zeta > 2$ (Steps 3 and 4 below).

After this brief overview we will now actually start the proof.

Proof of Theorem 7.2.2. The proof will be divided into several steps. Recall that the operator $-\Delta_{s,q} : H^{2+s,q}(\mathbb{T}) \subseteq H^{s,q}(\mathbb{T}) \to H^{s,q}(\mathbb{T})$ has a bounded H^∞-calculus of angle 0 for all $s \in \mathbb{R}$ and $q \in (1,\infty)$ (see [108, Theorem 10.2.25] for the case of $\mathbb{R}^d$). Thus, by Theorem 4.2.7, for all $r \in (2,\infty)$, $q \in [2,\infty)$ and $\alpha \in [0, \frac{r}{2} - 1)$ (allowing $r = q = 2$ and $\alpha = 0$ as well) we have

$$-\Delta_{s,q} \in \mathcal{SMR}^{\bullet}_{r,\alpha}(s,T) \qquad \text{for all } 0 \leq s < T < \infty \tag{7.49}$$

and that Assumption 6.3.2 holds for $\ell = \alpha$. The complex and real interpolation spaces below will be obtained via [20, Theorem 6.4.5].

Step 1: For each $u_0 \in L^0_{\mathscr{F}_0}(\Omega; L^2)$ there exists a weak solution (u,σ) to (7.41) on $[0,\infty)$. Moreover, for all $r \in (2,\infty)$, $\alpha \in [0, \frac{r}{2} - 1)$ and $\varepsilon \in [0, \frac{1}{2})$ Hypothesis **H**$(H^{-1-\varepsilon}, H^{1-\varepsilon}, \alpha, r)$ *holds, and Assumption 6.3.4 holds in the $(H^{-1-\varepsilon}, H^{1-\varepsilon}, r, \alpha)$-setting provided*

$$\frac{1+\alpha}{r} \leq \frac{1}{2} - \frac{\varepsilon}{2}, \tag{7.50}$$

where the corresponding trace space $B^{1-\varepsilon-2\frac{1+\alpha}{r}}_{2,r}$ is critical for (7.41) if and only if (7.50) holds with equality. In this step we set $X_0 = H^{-1-\varepsilon}$ and $X_1 = H^{1-\varepsilon}$. Thus, $X_{1/2} = H^{-\varepsilon}$ and $X^{\mathsf{Tr}}_{\alpha,r} = B^{1-\varepsilon-2\frac{1+\alpha}{r}}_{2,r}$. To estimate F, note that by Assumption 7.2.1, for all $v, v' \in H^1$,

$$\begin{aligned} \|\partial_x(f(\cdot,v)) - \partial_x(f(\cdot,v'))\|_{H^{-1-\varepsilon}} &\lesssim \|f(\cdot,v) - f(\cdot,v')\|_{H^{-\varepsilon}} \\ &\overset{(i)}{\lesssim} \|f(\cdot,v) - f(\cdot,v')\|_{L^\xi} \\ &\overset{(ii)}{\lesssim} (1 + \|v\|^2_{L^{3\xi}} + \|v'\|^2_{L^{3\xi}})\|v - v'\|_{L^{3\xi}} \\ &\overset{(iii)}{\lesssim} (1 + \|v\|^2_{H^\theta} + \|v'\|^2_{H^\theta})\|v - v'\|_{H^\theta}, \end{aligned} \tag{7.51}$$

where $\xi = \frac{2}{1+2\varepsilon} \in (1,2)$, $\theta = \frac{1}{3} - \frac{\varepsilon}{3}$ and in (i), (iii) we used the Sobolev embeddings and in (ii) Hölder's inequality with exponent $(3, \frac{3}{2})$. Since $[H^{-1-\varepsilon}, H^{1-\varepsilon}]_\theta = H^{-1-\varepsilon+2\theta}$ for all $\theta \in (0,1)$, setting $p = r$, $\kappa = \alpha$, $m_F = 1$, $\rho_1 = 2$, and $\beta_1 = \varphi_1 = \frac{1+\varepsilon+\theta}{2} = \frac{2}{3} + \frac{\varepsilon}{3}$, the condition (4.18) becomes

$$\frac{1+\alpha}{r} \leq \frac{3}{2}(1 - \varphi_1) = \frac{1}{2} - \frac{\varepsilon}{2},$$

which coincides with (7.50).

Next we estimate G. Since $\lambda > \frac{1}{2}$ by Assumption 7.2.1, it follows from [108, Example 9.3.4] that $\iota : H^\lambda \to L^\zeta$ belongs to $\gamma(H^\lambda, L^\zeta)$ for all $\zeta \in [1,\infty)$. By the ideal-property of γ-radonifying operators (see e.g. [108, Theorem 9.1.10]), for all $v, v' \in H^{1-\varepsilon}$,

$$\begin{aligned} \|G(\cdot,v) - G(\cdot,v')\|_{\gamma(H^\lambda, H^{-\varepsilon})} &\lesssim \|G(\cdot,v) - G(\cdot,v')\|_{\gamma(H^\lambda, L^\xi)} \\ &\leq \|\iota\|_{\gamma(H^\lambda, L^\zeta)} \|M_{g(\cdot,v)} - M_{g(\cdot,v')}\|_{\mathscr{L}(L^\zeta, L^\xi)} \\ &\leq C_{\lambda,\zeta} \|g(\cdot,v) - g(\cdot,v')\|_{L^\varrho}, \end{aligned} \tag{7.52}$$

where $\xi = \frac{2}{1+2\varepsilon}$ is as above, and we applied Hölder's inequality with $\frac{1}{\varrho} + \frac{1}{\zeta} = \frac{1}{\xi}$. Therefore, by Assumption 7.2.1 and Hölder's inequality with exponents $(3-\nu, \frac{3-\nu}{2-\nu})$,

$$\begin{aligned}\|G(\cdot,v) - G(\cdot,v')\|_{\gamma(H^\lambda, H^{-\varepsilon})} &\lesssim \big\|(1+|v|^{2-\nu} + |v'|^{2-\nu})|v-v'|\big\|_{L^\varrho}\\ &\leq (1+\|v\|^{2-\nu}_{L^{(3-\nu)\varrho}} + \|v'\|^{2-\nu}_{L^{(3-\nu)\varrho}})\|v-v'\|_{L^{(3-\nu)\varrho}}\end{aligned}$$

Setting $\varrho = 3\xi/(3-\nu)$, by (7.51), the latter and $X_{1/2} = H^{-\varepsilon}$, it follows that (HG) holds with $m_G = 1$, $\rho_2 = 2-\nu$, $\varphi_2 = \beta_2 = \varphi_1$ and (4.20) holds with strict inequality.

Therefore, if $\varepsilon = 0$, Theorem 6.3.1 with $p = 2$, $\kappa = 0$, $F_c = F$ and $G_c = G$ implies existence and uniqueness of a weak solution to (7.41). The other assertions of Step 1 follow from the above considerations for general $\varepsilon \in [0, \frac{1}{2})$.

Step 2: The weak solution (u, σ) provided by Step 1 verifies

$$u \in \bigcap_{\theta\in[0,1/2)} H^r_{\rm loc}(I_\sigma; H^{1-2\theta}(\mathbb{T})), \text{ a.s. for all } r \in (2,\infty). \tag{7.53}$$

To prove this regularization effect in time, we will first use Proposition 7.1.7 to create a weighted setting with a slight increase in integrability. After that we will apply Corollary 7.1.5 to extend the integrability to arbitrary order. In the above procedure we lose some space regularity, and this will be recovered by applying Theorem 7.1.3. Observe that it suffices to consider r large. The proof is split into several sub-steps.

Step 2a: For each $\varepsilon \in (0, \frac{1}{2})$, (7.53) holds with $H^r_{\rm loc}(I_\sigma; H^{1-2\theta}(\mathbb{T}))$ replaced by $H^6_{\rm loc}(I_\sigma; H^{1-2\theta-\varepsilon}(\mathbb{T}))$. It suffices to apply Proposition 7.1.7, with $Y_0 = H^{-1-\varepsilon}$, $Y_1 = H^{1-\varepsilon}$, $X_0 = H^{-1}$, $X_1 = H^1$, $\delta = \frac{\varepsilon}{2}$, $p = 2$, and $\alpha > 0$ such that $\frac{1}{2} = \frac{1+\alpha}{6} + \delta$. Since $\varepsilon \in (0, 1/2)$, we have $\alpha \in (0,2)$. Recall from Step 1 that $\varphi_1 = \varphi_2 = \frac{2}{3}$. The requirements of Proposition 7.1.7 are now clear from the above choices and Step 1.

Step 2b: For each $\varepsilon \in (0, \frac{1}{2})$ and $\widehat{r} \in [6,\infty)$, (7.53) holds with $H^r_{\rm loc}(I_\sigma; H^{1-2\theta}(\mathbb{T}))$ replaced by $H^{\widehat{r}}_{\rm loc}(I_\sigma; H^{1-2\theta-\varepsilon}(\mathbb{T}))$. This is immediate from Step 1, Step 2a, and Corollary 7.1.5 applied with $X_i = H^{-1+i}$, $p = 2$, $\kappa = 0$, $Y_i = H^{-1+2i-\varepsilon}$, $r = 6$, $\alpha = 2 - 3\varepsilon$ and $\widehat{r} \in [6,\infty)$ arbitrary. Note that assumption (2) is satisfied due to Step 1. Note that the condition $L^2 = X^{\mathsf{Tr}}_p \hookrightarrow Y^{\mathsf{Tr}}_r = B^{1-\frac{2}{r}-\varepsilon}_{2,r}$ is satisfied since $1 - \frac{2}{r} - \varepsilon > 0$.

Step 2c: Proof of (7.53). Set $X_i = \widehat{Y}_i = H^{-1+2i}$, $Y_i = H^{-1+2i-\varepsilon}$,

$$\alpha = 0,\ \widehat{\alpha} > 0,\ r = \widehat{r} \geq 6,\ \varepsilon \in (0, \tfrac{1}{2}), \text{ such that } \frac{1+\widehat{\alpha}}{r} = \frac{1}{r} + \frac{\varepsilon}{2}. \tag{7.54}$$

To gain space regularity, it suffices to check the conditions (1)-(3) of Theorem 7.1.3.

(1): By Step 1 and (7.49), Hypothesis $\mathbf{H}(H^{-1-\varepsilon}, H^{-1-\varepsilon}, \alpha, r)$ and Assumption 6.3.2 hold provided (7.50) is satisfied for (r, α, ε). As in Step 2b, $Y^{\mathsf{Tr}}_r \hookrightarrow X^{\mathsf{Tr}}_p$, and the required regularity of u also follows from Step 2b.

(2): This follows from Step 1 with $\varepsilon = 0$ and the fact that $\frac{1+\widehat{\alpha}}{r} < \frac{1}{2}$. Moreover, by (7.50) with $\varepsilon = 0$ the space $Y^{\mathsf{Tr}}_{\widehat{\alpha},\widehat{r}} = B^{1-2\frac{1+\widehat{\alpha}}{r}}_{2,r}$ is not critical for (7.41) in the $(\widehat{Y}_0, \widehat{Y}_1, \widehat{r}, \widehat{\alpha})$-setting.

(3): By (7.54), one has $Y^{\mathsf{Tr}}_r = B^{1-\varepsilon-\frac{2}{r}}_{2,r} = B^{1-2\frac{1+\widehat{\alpha}}{r}}_{2,r} = \widehat{Y}^{\mathsf{Tr}}_{\widehat{\alpha},\widehat{r}}$. Also by (7.54) and Lemma 7.1.2(4) applied with ε replaced by $\varepsilon/2$, the embedding condition (7.4) holds.

Step 3: For all $\zeta, r \in (2,\infty)$ and $\alpha \in [0, \frac{r}{2} - 1)$, Hypothesis $\mathbf{H}(H^{-1,\zeta}, H^{1,\zeta}, \alpha, r)$ holds, Assumption 6.3.4 holds in the $(H^{-1,q}, H^{1,q}, \alpha, r)$-setting and the corresponding trace space $B^{1-2\frac{1+\alpha}{r}}_{\zeta,r}$ is not critical for (7.41). Let us begin by estimating F. Note that by Assumption 7.2.1, for all $v, v' \in H^{1,\zeta}$,

$$\begin{aligned}\|\partial_x(f(\cdot,v)) - \partial_x(f(\cdot,v'))\|_{H^{-1,\zeta}} &\lesssim \|f(\cdot,v) - f(\cdot,v')\|_{L^\zeta}\\ &\overset{(i)}{\lesssim} (1+\|v\|^2_{L^{3\zeta}} + \|v'\|^2_{L^{3\zeta}})\|v-v'\|_{L^{3\zeta}}\end{aligned}$$

$$\overset{(ii)}{\lesssim} (1 + \|v\|^2_{H^{\frac{2}{3}\zeta,\zeta}} + \|v'\|^2_{H^{\frac{2}{3}\zeta,\zeta}})\|v - v'\|_{H^{\frac{2}{3}\zeta,v}},$$

where in (i) we used Hölder's inequality with exponent $(3, \frac{3}{2})$ and in (ii) the Sobolev embedding $H^{\frac{2}{3}\zeta,\zeta} \hookrightarrow L^{3\zeta}$. Since $[H^{-1,\zeta}, H^{1,\zeta}]_\theta = H^{-1+2\theta,\zeta}$ for all $\theta \in (0,1)$, setting $m_F = 1$, $\rho_1 = 2$, $\beta_1 = \varphi_1 = \frac{1}{2} + \frac{1}{3q}$, condition (4.18) becomes

$$\frac{1+\alpha}{r} \leq \frac{3}{2}(1 - \varphi_1) = \frac{3}{4} - \frac{1}{2\zeta}.$$

Since $\frac{3}{4} - \frac{1}{2\zeta} > \frac{1}{2}$ due to $\zeta > 2$ and $\frac{1+\alpha}{r} < \frac{1}{2}$ for all $\alpha \in [0, \frac{r}{2} - 1)$, the above estimate is always strict and hence noncriticallity follows. As in (7.52) with L^ξ and $H^{-\varepsilon}$ replaced by L^ζ, one can estimate G to show that (HG) holds in the $(H^{-1,\zeta}, H^{1,\zeta}, \alpha, r)$-setting with $m_G = 1$, $\rho_2 = 2 - \nu$, $\varphi_2 = \beta_2 = \varphi_1$. Moreover, since $\frac{1+\alpha}{r} < \frac{1}{2}$, one can check that (4.20) holds with the strict inequality.

Step 4: u satisfies (7.44) *and* (7.45). Note that (7.45) follows from (7.44), Sobolev embedding, and standard considerations. To prove (7.44), we apply Theorem 7.1.3 with $Y_0 = H^{-1+2i}$, $\widehat{Y}_i = H^{-1+2i,\zeta}$,

$$r = \widehat{r} > 4, \;\; \widehat{\alpha} = \widehat{r}/4, \quad \text{and} \quad \alpha \in (\widehat{\alpha}, \frac{r}{2} - 1) \text{ arbitrary}.$$

Note that $\widehat{\alpha} = \frac{r}{4} < \frac{r}{2} - 1$. It remains to check Theorem 7.1.3(1)-(3).

(1): All conditions are clear from Steps 1 and 2, and $Y_r^{\mathsf{Tr}} = B^{1-\frac{2}{r}}_{2,r} \hookrightarrow L^2 = X_p^{\mathsf{Tr}}$.

(2): All conditions follow from Step 3.

(3): To check $Y_r^{\mathsf{Tr}} = B^{1-\frac{2}{r}}_{2,r} \hookrightarrow B^{1-2\frac{1+\widehat{\alpha}}{\widehat{r}}}_{\zeta,r} = \widehat{Y}^{\mathsf{Tr}}_{\widehat{r},\widehat{\alpha}}$, by Sobolev embeddings we need to show that

$$1 - \frac{2}{r} - \frac{1}{2} \geq 1 - 2\frac{1+\widehat{\alpha}}{r} - \frac{1}{\zeta} \quad \Leftrightarrow \quad 2\frac{\widehat{\alpha}}{r} + \frac{1}{\zeta} \geq \frac{1}{2}. \tag{7.55}$$

Since $r > 4$ and $\widehat{\alpha} = \frac{r}{4}$, (7.55) holds for all $\zeta \in (2, \infty)$. Finally, (7.4) follows from $\widehat{Y}_i \hookrightarrow Y_i$ for $i \in \{0, 1\}$, Lemma 7.1.2(2) and the choice $\alpha \in (\widehat{\alpha}, \frac{r}{2} - 1)$. □

To prove global existence for (7.41) under the assumptions of Theorem 7.2.3, we follow the roadmap provided in Subsection 6.3.3. Note that (a)-(b) are contained in the proof of Theorem 7.2.2. Our next step is to provide energy estimates under integrability assumptions on u_0 (see (c)-(d) and Proposition 6.3.10). The proof is based on an integration by parts argument. As noticed in (d), we can take advantage of the regularization results in Theorem 6.3.1 in the proof below. Indeed, due to (7.44)-(7.45), if we stay away from $t = 0$, then we have integrability in time and space of arbitrary order (see (7.57) and (7.61) below).

Lemma 7.2.6 (Energy estimates). *Let Assumption 7.2.1 be satisfied and suppose that $u_0 \in L^2_{\mathscr{F}_0}(\Omega; L^2)$. Let (u, σ) be the weak solution to (7.41) on $[0, \infty)$ provided by Theorem 7.2.2. . If (7.46) holds, then for each $T > 0$ there exists a $C > 0$ independent of u, u_0 such that*

$$\mathbb{E}\Big[\sup_{s\in[0,\sigma\wedge T)} \|u(s)\|^2_{L^2}\Big] + \mathbb{E}\|\nabla u\|^2_{L^2(0,\sigma\wedge T; L^2)} \leq C(1 + \mathbb{E}\|u_0\|^2_{L^2}).$$

Proof. Let $T > 0$ be fixed. By replacing σ by $\sigma \wedge T$, we may assume that σ takes values in $[0, T]$. Recall that (u, σ) is the unique weak solution to (7.41), and

$$u \in L^2_{\mathrm{loc}}([0, \sigma); H^1) \cap C([0, \sigma); L^2) \;\; \text{a.s.} \tag{7.56}$$

Let $s > 0$ and $n \geq 1$ be arbitrary (later on we let $s \downarrow 0$ and $n \to \infty$). By (7.45), the following stopping time is well-defined

$$\tau_n := \inf \big\{ t \in [s, \sigma) \, : \, \|u\|_{L^2(s,t;H^1)} + \|u(t) - u(s)\|_{C(\mathbb{T})} \geq n \big\}$$

if $\sigma > s$ and $\tau_n = s$ if $\sigma \leq s$. Here $\inf \varnothing := \sigma$. Note that $\lim_{n\to\infty} \tau_n = \sigma$ a.s. on $\{\sigma > s\}$. Moreover, we set

$$\Gamma_{s,n} := \{\tau_n > s,\ \|u(s)\|_{C(\mathbb{T})} \leq n\} \in \mathscr{F}_s. \tag{7.57}$$

Let

$$y(t) := \sup_{r\in[s,t\wedge\tau_n)} \mathbf{1}_{\Gamma_{s,n}} \|u(r)\|_{L^2}^2 + \int_s^t \int_{\mathbb{T}} \mathbf{1}_{[s,t\wedge\tau_n)\times\Gamma_{s,n}} |\nabla u(r)|^2\, dxdr, \quad t \in [s,T].$$

It is enough to prove the existence of $C > 0$ independent of u_0, s, n such that

$$\mathbb{E}y(t) \leq C(1 + t - s + \mathbb{E}y(s)) + C\int_s^t \mathbb{E}y(r)dr, \quad t \in [s,T]. \tag{7.58}$$

Indeed, by Grownall's inequality (7.58) implies that for all $t \in [s,T]$,

$$\mathbb{E}y(t) \leq Ce^{C(t-s)}(1 + t - s + \mathbb{E}[\mathbf{1}_{\Gamma_n}\|u(s)\|_{L^2}^2]). \tag{7.59}$$

The required a priori estimate, follows by letting $s \downarrow 0$ and $n \to \infty$ in (7.59).

For the reader's convenience, we split the remaining argument into several steps and we simply write Γ, τ instead of $\Gamma_{s,n}, \tau_n$, since s and n will be fixed.

Step 1: We apply Itô*'s formula to obtain the identity* (7.62) *below.* To this end, we extend u to a process v on $[s,T]\times\Omega$ in the following way. Let $v \in L^2((s,T)\times\Omega; H^1) \cap L^2(I_T; C([s,T]; L^2))$ be the strong solution to the problem

$$dv = \Delta v dt + f^u dt + g^u dW_{H^\lambda} \quad \text{and} \quad v(s) = \mathbf{1}_\Gamma u(s) \tag{7.60}$$

where by (7.44)-(7.45), and the definition of τ and Γ, for all $q \in (1,\infty)$ one has

$$\begin{aligned} f^u(t) &:= \mathbf{1}_{\Gamma\times[s,\tau)} \partial_x(f(\cdot,u)) \in L^2((s,T)\times\Omega; H^{-1,q}(\mathbb{T})),\\ g^u(t) &:= \mathbf{1}_{\Gamma\times[s,\tau)} g(\cdot,u) \in L^2((s,T)\times\Omega; L^q(\mathbb{T})). \end{aligned} \tag{7.61}$$

The existence of v is ensured by (7.49), (7.61), and Proposition 6.2.6. Note that since (u,σ) is a weak solution to (7.41) and v satisfies (7.60), by maximality of (u,σ) we get $v = u$ a.e. on $\Gamma\times[s,\tau)$.

Applying Itô's formula to $\|v\|_{L^2}^2$ (see [148, Theorem 4.2.5]), we obtain, a.s. for all $n \geq 1$ and $t \in [s,T]$,

$$\begin{aligned} &\|v(t)\|_{L^2}^2 - \mathbf{1}_\Gamma\|u(s)\|_{L^2}^2 + 2\int_s^t \|\nabla v(r)\|_{L^2}^2 dr\\ &= -2\int_s^t\int_{\mathbb{T}} \mathbf{1}_{\Gamma\times[s,\tau)} f(r,u(r))\partial_x u(r)\, dxdr\\ &\quad + \int_s^t \mathbf{1}_{\Gamma\times[s,\tau)} \|M_{g(r,u(r))}\|_{\gamma(H^\lambda,L^2)}^2\, dr\\ &\quad + 2\int_s^t \mathbf{1}_{\Gamma\times[s,\tau)} (u(r), M_{g(r,u(r))}(\cdot))_{L^2}\, dW_{H^\lambda}(r) =: I_t + II_t + III_t. \end{aligned} \tag{7.62}$$

Step 2: There exists C independent of u, u_0, s, n such that

$$\mathbb{E}\int_s^t \|\nabla v(r)\|_{L^2}^2\, dr \leq \mathbf{1}_\Gamma\|u(s)\|_{L^2}^2 + C\Big(1 + t - s + \mathbb{E}\int_s^t \mathbf{1}_{\Gamma\times[s,\tau)}\|u(r)\|_{L^2}^2\, dr\Big).$$

The idea is to take expectations in (7.62). Clearly, $\mathbb{E}[III_t] = 0$ for all $t \in [s,T]$. We claim that $I_t = 0$. To see this, it is enough to show that $\int_{\mathbb{T}} f(t,\phi)\partial_x\phi\, dx \equiv 0$ for any $\phi \in H^{1,q}$ with $q \geq 6$ suitably large. Here we used that u is smooth (see (7.44)).

Let $(\phi_k)_{k\geq1}$ in $C^\infty(\mathbb{T})$ be such that $\phi_k \to \phi$ in $H^{1,q}$. By Assumption 7.2.1 and $f(\cdot,\phi_k) \to f(\cdot,\phi)$ in $L^{q/3}$. Therefore, $f(\cdot,\phi_k)\partial_x\phi_k \to f(\cdot,\phi)\partial_x\phi$ in L^1, and hence

$$\int_{\mathbb{T}} f(\cdot,\phi)\partial_x\phi dx = \lim_{k\to\infty}\int_{\mathbb{T}} f(\cdot,\phi_k)\partial_x\phi_k dx = \lim_{k\to\infty}\int_{\mathbb{T}} \partial_x[F(\cdot,\phi_k)]dx \equiv 0,$$

where F is such $\partial_z F(\cdot,z)=f(\cdot,z)$. Thus, $I_t\equiv 0$.

It remains to estimate II. Here we argue as in (7.52). Fix $\xi>2$ and let $\zeta<\infty$ be such that $\frac{1}{\zeta}+\frac{1}{\xi}=\frac{1}{2}$. Using (7.46), a.s. for all $r\in[s,\tau)$,

$$\begin{aligned}
\|M_{g(r,u(r))}\|_{\gamma(H^\lambda,L^2)} &\le C\|M_{g(r,u(r))}\|_{\mathscr{L}(L^\zeta,L^2)}\\
&\le C\|g(r,u(r))\|_{L^\xi}\\
&\le CC_g(1+\|u(r)\|_{L^\xi})\\
&\le CC_g(1+\|u(r)\|_{H^1}^{\frac{2}{\xi}}\|u(r)\|_{L^2}^{1-\frac{2}{\xi}})\\
&\le \|u(r)\|_{H^1}+C'C_g(1+\|u(r)\|_{L^2})\\
&\le \|\nabla u(r)\|_{L^2}+(C'C_g+1)(1+\|u(r)\|_{L^2})
\end{aligned}\tag{7.63}$$

where C,C' only depend on z,ξ and we used $H^1\hookrightarrow L^\infty$. Thus, for all $t\in[s,T]$,

$$|II_t|\le \int_s^t \mathbf{1}_{\Gamma\times[s,\tau)}\|\nabla u\|_{L^2}^2\,dr + c\Big(t-s+\int_s^t \mathbf{1}_{\Gamma\times[s,\tau)}\|u\|_{L^2}^2\,dr\Big)\tag{7.64}$$

where c depends only on C,C_g, and where we used the definition of y. Therefore, taking expectations in (7.62) and (7.64), and using $u=v$ on $\Gamma\times[s,\tau)$, we obtain the required estimate by comparison with LHS(7.62).

Step 3: Proof of (7.58). We take absolute values and the supremum over time in (7.62), and then expectations. We already saw that $I\equiv 0$ on $[s,T]$. Moreover, $\mathbb{E}\big[\sup_{r\in[s,t]}|II_r|\big]\le\mathbb{E}\big[|II_t|\big]$ which can be estimate by the expectation of RHS(7.64). To conclude, it remains to estimate III. By the scalar Burkholder-Davis-Gundy inequality, we get

$$\begin{aligned}
\mathbb{E}\Big[\sup_{r\in[s,t]}|III_r|\Big] &\le C\mathbb{E}\Big[\int_s^t \mathbf{1}_{\Gamma\times[s,\tau)}(r)\Big\|(u(r),M_{g(r,u(r))}(\cdot))_{L^2}\Big\|_{\gamma(H^\lambda,\mathbb{R})}^2\,dr\Big]^{1/2}\\
&\le C\mathbb{E}\Big[\int_s^t \mathbf{1}_{\Gamma\times[s,\tau)}(r)\|u(r)\|_{L^2}^2\|M_{g(r,u(r))}\|_{\gamma(H^\lambda,L^2)}^2\,dr\Big]^{1/2}\\
&\le C\mathbb{E}\Big[\Big(\sup_{r\in[s,t\wedge\tau)}\mathbf{1}_\Gamma\|u(r)\|_{L^2}^2\Big)^{1/2}\Big(\int_s^t \mathbf{1}_{\Gamma\times[s,\tau)}(r)\|M_{g(r,u(r))}\|_{\gamma(H^\lambda,L^2)}^2\,dr\Big)^{1/2}\Big]\\
&\le \frac{1}{2}\mathbb{E}\Big(\sup_{r\in[s,t\wedge\tau)}\mathbf{1}_\Gamma\|u(r)\|_{L^2}^2\Big)+C'\mathbb{E}\Big[\int_s^t \mathbf{1}_{[s,\tau)}(r)\|M_{g(r,u(r))}\|_{\gamma(H^\lambda,L^2)}^2\,dr\Big].
\end{aligned}$$

where the last term coincides with $\mathbb{E}|II_t|$. Thus, by (7.64) and Step 2,

$$\mathbb{E}\Big[\sup_{r\in[s,t]}|III_r|\Big]\le\frac{1}{2}\mathbb{E}\Big(\sup_{r\in[s,t\wedge\tau)}\mathbf{1}_\Gamma\|u(r)\|_{L^2}^2\Big)+c''\Big(1+t-s+\mathbb{E}\int_s^t\|u(r)\|_{L^2}^2\,dr\Big),$$

where c'' is independent of u_0,s,n. Combining the estimates with (7.62), using $u=v$ on $\Gamma\times[s,\tau)$, and using the definition of y, we obtain (7.58). ☐

By Lemma 7.2.6 we can prove the global well-posedness to (7.41) following Roadmap 6.3.11(e) in Section 6.3.3. Here the criticality of the L^2-setting (see (7.50) with $r=2$ and $\varepsilon=\alpha=0$), forces us to use Theorem 6.3.7(4) in the proof below.

Proof of Theorem 7.2.3. Let (u,σ) be the weak solution to (7.41) on $[0,\infty)$. By Theorem 7.2.2 and Lemma 7.2.6 it remains to prove that $\sigma=\infty$ a.s.

Let $T\in(0,\infty)$. Replacing (u,σ) by $(u|_{[\![0,\sigma\wedge T[\![},\sigma\wedge T)$ it suffices to consider weak solutions to (7.41) on $[0,T]$ and to show that $\sigma=T$ a.s. For this we will use Theorem 6.3.7(4) with $p=2,\kappa=0,X_0=H^{-1},X_1=H^1$, and therefore $X^{\mathsf{Tr}}_{\kappa,p}=L^2$. Note that (7.49) holds, and that

Assumption 6.3.4 is satisfied by Step 1 of the proof of Theorem 7.2.2. By Proposition 6.3.10 we may assume that $u_0 \in L^2(\Omega; L^2)$. Thus, Lemma 7.2.6 yields

$$\sup_{s\in[0,\sigma)} \|u(s)\|_{L^2}^2 + \int_0^\sigma \|u(s)\|_{H^1}^2 ds < \infty \qquad \text{a.s.}$$

Therefore, applying Theorem 6.3.7(4), we obtain

$$\mathbb{P}(\sigma < T) = \mathbb{P}\Big(\sigma < T, \sup_{s\in[0,\sigma)} \|u(s)\|_{X^{\mathsf{Tr}}_{\kappa,p}} + \|u(s)\|_{L^2(I_\sigma;X_1)} < \infty\Big) = 0.$$

Finally, Lemma 7.2.6 also implies (7.47). □

It remains to prove Theorem 7.2.4. The idea of the proof is similar as in Roadmap 7.2.5, but since $p > 2$ we can use Proposition 7.1.7 with $\delta = 0$. Moreover, we will use the extrapolation technique of Lemma 7.1.9.

Proof of Theorem 7.2.4. Let $T \in (0,\infty)$. To prove global well-posedness we apply Lemma 7.1.9 (see also Remark 7.1.10(1)) with $X_i = H^{-1-s+2i,\zeta}$, $Y_i = H^{-1+2j}$, $r = 2$, $\alpha = 0$, $\widehat{Y}_j = H^{-1+2j,\zeta}$, $\zeta, \widehat{r}$ large, and $\widehat{\alpha} = \frac{\widehat{r}}{4}$ as in Step 4 of Theorem 7.2.2.

First we check Lemma 7.1.9 (2) with $\tau = T$. As in the proof of Theorem 7.2.2 one can check that $\mathbf{H}(\widehat{Y}_0, \widehat{Y}_1, \widehat{r}, \widehat{\alpha})$ holds. The global well-posednes in the $(Y_0, Y_1, 2, 0)$-setting follows from Theorem 7.2.3 and a translation argument, and the remaining conditions are clear.

It remains to check the local well-posedness and regularity assertions of Lemma 7.1.9(1), which is the existence of a $(s, q, p, \kappa_{\mathsf{crit}})$-weak solution to (7.41) and the instantaneous regularization requirement in (1). It is enough to show that for any $u_0 \in L^0_{\mathscr{F}_0}(\Omega; B^{\frac{1}{q}-\frac{1}{2}}_{q,p}(\mathbb{T}))$ there exists a $(s, q, p, \kappa_{\mathsf{crit}})$-weak solution (u, σ) to (7.41) on $[0, T]$ such that

$$u \in C(I_\sigma; L^2(\mathbb{T})) \cap L^2_{\mathrm{loc}}(I_\sigma; H^1(\mathbb{T})) \quad \text{a.s.} \tag{7.65}$$

Step 1: Assume that $s \in (0, \frac{1}{3})$, $r \in [2,\infty)$, $\alpha \in [0, \frac{r}{2} - 1)$ and $\zeta \in (2,\infty)$. Then, Hypothesis $\mathbf{H}(H^{-1-s,\zeta}, H^{1-s,\zeta}, r, \alpha)$, and Assumption 6.3.4 hold in the $(H^{-1-s,\zeta}, H^{1-s,\zeta}, r, \alpha)$-setting provided

$$\zeta < \frac{2}{s}, \quad \text{and} \quad \frac{1+\alpha}{r} + \frac{1}{2\zeta} \leq \frac{3-2s}{4}. \tag{7.66}$$

The corresponding trace space $B^{1-s-2\frac{1+\alpha}{r}}_{\zeta,r}$ is critical for (7.41) if and only if (7.66) holds with equality. In particular, for $u_0 \in L^0_{\mathscr{F}_0}(\Omega; B^{\frac{1}{q}-\frac{1}{2}}_{q,p})$ there exists a $(s, q, p, \kappa_{\mathsf{crit}})$-weak solution (u, σ) to (7.41) on $\overline{I}_T$. To prove local well-posedness we use Theorem 6.3.1 with $X_i = H^{-1-s+2i,\zeta}$. We first check (HF) in the $(H^{-1-s,\zeta}, H^{1-s,\zeta}, r, \alpha)$-setting. Fix $v, v' \in H^{1-s,\zeta}$ and note that

$$\begin{aligned}
\|\partial_x(f(\cdot, v)) - \partial_x(f(\cdot, v'))\|_{H^{-1-s,\zeta}} &\lesssim \|f(\cdot, v)) - f(\cdot, v')\|_{H^{-s,\zeta}} \\
&\overset{(i)}{\lesssim} \|f(\cdot, v)) - f(\cdot, v')\|_{L^\psi} \\
&\lesssim (1 + \|v\|^2_{L^{3\psi}} + \|v'\|^2_{L^{3\psi}})\|v - v'\|_{L^{3\psi}} \\
&\overset{(ii)}{\lesssim} (1 + \|v\|^2_{H^{\theta,q}} + \|v'\|^2_{H^{\theta,\zeta}})\|v - v'\|_{H^{\theta,\zeta}}
\end{aligned}$$

where in (i)-(ii) we used the Sobolev embedding with $-s-\frac{1}{\zeta} = -\frac{1}{\psi}$ and $\theta - \frac{1}{\zeta} = -\frac{1}{3\psi}$, where $\theta > 0$. To ensure that $\psi \in (1,\infty)$ one needs $\zeta > \frac{1}{1-s}$ which holds since $\zeta > 2$ and $s < \frac{1}{3}$. Combining the above identities we have $\theta = \frac{2}{3\zeta} - \frac{s}{3}$. To ensure $\theta > 0$ we need $\zeta < \frac{2}{s}$. Since $H^{\theta,\zeta} = X_\beta$, we obtain $\beta = \frac{1}{3}(\frac{1}{\zeta} + s) + \frac{1}{2}$, and one can check that $\beta \in (1/2, 1)$. Setting $p = r$, $m_F = 1$, $\rho_1 = 2$, and $\beta_1 = \varphi_1 = \beta$ the condition (4.18) becomes

$$\frac{1+\alpha}{r} \leq \frac{3}{2}(1-\beta) = \frac{3-2s}{4} - \frac{1}{2\zeta}.$$

which coincides with the second condition in (7.66). As in the proof of Theorem 7.2.2, one can check that condition (HG) holds in the $(H^{-1-s,\zeta}, H^{1-s,\zeta}, r, \alpha)$-setting with $m_G = 1$, $\rho_2 = 2-\nu$ $\varphi_2 = \beta_2 = \varphi_1$.

By the above, we can apply Theorem 6.3.1. It only remains to investigate criticality. By (7.66), criticality occurs if and only if

$$\frac{1+\kappa}{p} + \frac{1}{2q} = \frac{3-2s}{4}. \tag{7.67}$$

Since $\frac{1+\kappa}{p} \in [\frac{1}{p}, \frac{1}{2})$, (7.67) is admissible if and only if

$$\frac{1}{p} + \frac{1}{2q} \leq \frac{3-2s}{4}, \qquad \text{and} \qquad \frac{3-2s}{4} - \frac{1}{2q} < \frac{1}{2}.$$

The second inequality in the previous yields the limitation $q < \frac{2}{1-2s}$. Since $\frac{2}{1-2s} < \frac{2}{s}$ due to $s < \frac{1}{3}$, the first condition in (7.66) with $q = \zeta$ holds. By (7.67), $\kappa_{\mathsf{crit}} = -1 + \frac{p}{2}(\frac{3}{2} - s - \frac{1}{q})$ and the corresponding trace space becomes

$$X^{\mathsf{Tr}}_{\kappa_{\mathsf{crit}},p} = B^{1-s-2\frac{1+\kappa_{\mathsf{crit}}}{p}}_{q,p} = B^{1-s-\frac{3}{2}+s+\frac{1}{q}}_{q,p} = B^{\frac{1}{q}-\frac{1}{2}}_{q,p},$$

which finished this step.

Step 2: The $(s, q, p, \kappa_{\mathsf{crit}})$-weak solution provided by Step 1 verifies

$$u \in \bigcap_{\theta\in[0,1/2)} H^{\theta,r}_{\mathrm{loc}}(I_\sigma; H^{1-s-2\theta,q}(\mathbb{T})), \quad \text{a.s. for all } r \in (2,\infty). \tag{7.68}$$

As in Step 2 in the proof of Theorem 7.2.2 we use Proposition 7.1.7 and after that Corollary 7.1.5. In case $\kappa_{\mathsf{crit}} > 0$, Step 2a is not needed.

Step 2a: There exists an $r > p$ such that (7.68) holds. Let $\varphi_j = \beta$ where $\beta \in (1/2, 1)$ is as in Step 1. Let $r > p$ be such that $\frac{1}{r} \geq \max_j \varphi_j - 1 + \frac{1}{p}$. Then the claim follows by applying Proposition 7.1.7 with $\delta = 0$.

Step 2b: (7.68) holds. If $\kappa_{\mathsf{crit}} > 0$, then the claim follows from Corollary 7.1.5 applied to $Y_i = X_i = H^{-1-s+2i,q}$, $r = p$ and $\alpha = \kappa_{\mathsf{crit}}$. Next we consider the case $\kappa_{\mathsf{crit}} = 0$. Let r be as in Step 2a and let $\alpha \in (0, \frac{r}{2} - 1)$ be such that $\frac{1}{p} = \frac{1+\alpha}{r}$. By Step 2a, the assumptions of Corollary 7.1.5 are satisfies and this concludes the required regularity.

Step 3: The $(s, q, p, \kappa_{\mathsf{crit}})$-weak solution provided by Step 1 verifies

$$u \in \bigcap_{\theta\in[0,1/2)} H^{\theta,r}_{\mathrm{loc}}(I_\sigma; H^{1-2\theta,q}(\mathbb{T})), \quad \text{a.s. for all } r \in (2,\infty).$$

In particular, (7.65) holds. To conclude, it is enough to apply Theorem 7.1.3 to $Y_i = H^{1-s-2i,q}$, $\widehat{Y}_i = H^{-1+2i,q}$,

$$\alpha = 0,\ \widehat{\alpha} > 0,\ r = \widehat{r} > p \text{ large enough and } \frac{1+\widehat{\alpha}}{r} = \frac{1}{r} + \frac{s}{2}.$$

To check the assumptions of Theorem 7.1.3 one can use Step 2 and argue in a similar way as in Theorem 7.2.2 Step 2c. ☐

Part III

Applications

Chapter 8

Stochastic reaction diffusion equations: Global existence and regularity

Let $(w^n)_{n\geq 1}$ be a sequence of independent standard Brownian motions on a filtered probability space $(\Omega, \mathbb{F} = (\mathscr{F}_t)_{t\geq 0}, \mathscr{A}, \mathbb{P})$ and let $\mathscr{P}$ be the progressive sigma algebra.

In this chapter we study existence and regularity of global solutions reaction-diffusion equations of the form

$$\begin{cases} du - \operatorname{div}(a(\cdot)\nabla u)\,dt = (\operatorname{div}(\Psi(\cdot,u)) + \psi(\cdot,u))\,dt \\ \qquad\qquad\qquad\qquad\qquad + \sum_{n\geq 1}(b_n(\cdot)\cdot\nabla u + \Phi_n(\cdot,u))\,dw_t^n, & \text{on } \mathbb{T}^d, \\ u(0) = u_0, & \text{on } \mathbb{T}^d, \end{cases} \tag{8.1}$$

where $u : [0,\infty)\times\Omega\times\mathbb{T}^d \to \mathbb{R}$ is the unknown process and $\mathbb{T}^d$ denotes the d-dimensional torus. Here we will be mainly interested in studying (8.1) for initial data in critical spaces. Extensions to quasilinear equations will be considered on $\mathbb{T}^d$ as well as on bounded domains $\mathscr{O}$ with Dirichlet boundary conditions. However, in the latter situation we cannot allow a gradient-noise term, i.e. $b_n \equiv 0$. The aim of this chapter is to provide a nontrivial example that demonstrates how to use the abstract results of Chapters 4, 6 and 7. Extensions to systems and other boundary conditions are possible, but some additional complications have to be dealt with (see Subsection 8.2.4 for comments). To prove global well-posedness for (8.1) and its quasilinear case we introduce a suitable dissipation relation which relates Ψ, ψ and Φ and allow us to prove suitable a priori estimates. A protoype example of SPDEs which fits in our framework is the Allen-Cahn equation where

$$\psi(\cdot,u) = u - u^3, \quad |\Psi(\cdot,u)| \leq c + c_\psi |u|^2, \quad \text{and} \quad |\Phi(\cdot,u)| \leq c + c_\Phi |u|^2, \tag{8.2}$$

where c can be large and c_Ψ, c_ψ can be determinated explicitly using the ellipticity constant ν, the dimension d and the size of the gradient noise term $\operatorname{ess\,sup} \|b\|_{\ell^2}$. Let us mention that the growth in (8.2) is optimal also w.r.t. a scaling argument. The main problem here is to prove a priori estimates. Indeed, (8.1) admits solutions with very midly behaviour near $t = 0$ (see the beginning of Subsection 8.2.3). In particular, it could happen that $\nabla u \notin L^1_{\mathrm{loc}}([0,\sigma); L^2(\mathbb{T}^d))$ a.s. Thus, to prove energy estimates for (8.1) with integration by parts arguments, we exploit the instantaneous regularization that ensures $\nabla u \in L^r_{\mathrm{loc}}([s,\sigma); L^\zeta(\mathbb{T}^d))$ a.s. for all $r, \zeta \in (2,\infty)$, $s > 0$.

This chapter is organized as follows. In Section 8.1 we recall a generalized Itô's formula and we provide a short proof of the DeGiorgi-Nash-Moser estimates for parabolic SPDEs in absence of gradient noise. In Section 8.2 we deal with semilinear reaction-diffusion equations. Subsection 8.2.1 we state our main results and the proofs are collected in 8.2.2-8.2.3. In Section 8.3 we (partially) extend the result to the quasilinear case. Here, at the expense of removing the gradient noise term we may allow a to be merely measurable in time and VMO in space. To check the blow-up criteria we combine energy estimates the stochastic parabolic DeGiorgi-Nash-Moser estimates.

The results in this chapter will be presented in [6].

8.1 Preliminaries

In this section we have gathered some useful facts which will be employed in the later subsections. Subsection 8.1.1 we recall a generalized Itô's formula proven [57, Appendix] on the d-dimensional torus which will be used to prove a priori estimates for (8.1). In Subsection 8.1.2 we provide a short proof of the DeGiorgi-Nash-Moser estimates for stochastic parabolic SPDEs which will be of basic importance in Subsection 8.3.

8.1.1 A Generalized Itô's formula

In this subsection we state and sketch the proof of a generalized Itô's formula which will be employed in Lemma 8.2.12. In the following W_{ℓ^2} is as in Example 2.3.6, and for either $\mathscr{O} \in \{\mathbb{T}^d, \mathbb{R}^d\}$ or $\mathscr{O} \subseteq \mathbb{R}^d$ a bounded doman, we define the weak divergence operator $\mathrm{div} : L^q(\mathscr{O}; \mathbb{R}^d) \to {}_DH^{-1,q}(\mathscr{O}) := (\{v \in H^{1,q}(\mathscr{O}) : v = 0 \text{ on } \partial\mathscr{O}\})^*$ as

$$\langle g, \mathrm{div} f\rangle_{H^{1,q}(\mathscr{O}),H^{-1,q}(\mathscr{O})} = \int_{\mathscr{O}} f \cdot \nabla g \, dx \tag{8.3}$$

for all $f \in L^q(\mathscr{O}; \mathbb{R}^d)$ and $g \in \{v \in W^{1,q'}(\mathscr{O}) : v = 0 \text{ on } \partial\mathscr{O}\}$ where $\frac{1}{q} + \frac{1}{q'} = 1$. For notational convenience, we set ${}_DH^{1,q}(\mathscr{O}) := \{v \in W^{1,q'}(\mathscr{O}) : v = 0 \text{ on } \partial\mathscr{O}\}$ and ${}_DH^1(\mathscr{O}) = {}_DH^{1,2}(\mathscr{O})$.

Lemma 8.1.1. *Let $T \in (0,\infty)$ and let $\tau : \Omega \to [0,T]$ be a stopping time. Let Θ be a C^2 map with bounded second order derivatives. Suppose either $\mathscr{O} \in \{\mathbb{T}^d, \mathbb{R}^d\}$ or $\mathscr{O}$ is a bounded C^1-domain in $\mathbb{R}^d$. Assume that, for all $i \in \{1, \dots, d\}$,*

$$u_0 \in L^0_{\mathscr{F}_0}(\Omega; L^2(\mathscr{O})),\ G_i, F \in L^0_{\mathscr{P}}(\Omega; L^2(I_\tau \times \mathscr{O})),\ H \in L^0_{\mathscr{P}}(\Omega; L^2(I_\tau \times \mathscr{O}; \ell^2))).$$

Assume that the process

$$u \in L^0(\Omega; L^2(I_\tau; {}_DH^1(\mathscr{O}))) \cap L^0(\Omega; C([0,\tau]; L^2(\mathscr{O}))) \tag{8.4}$$

satisfies, a.s. for all $t \in [0,\tau]$, the following equality in ${}_DH^{-1}(\mathscr{O})$:

$$u(t) - u_0 = \int_0^t (\mathrm{div}(G(s)) + F(s))ds + \int_0^t H(s) dW_{\ell^2}(s). \tag{8.5}$$

Then, a.s. for all $t \in I_\tau$,

$$\begin{aligned}\int_{\mathscr{O}} \Theta(u(t))dx = &\int_{\mathscr{O}} \Theta(u_0)dx + \int_0^t \int_{\mathscr{O}} \Theta'(u(s))F(s)dx\, ds \\ &- \int_0^t \int_{\mathscr{O}} \Theta''(u(s))\nabla u(s) \cdot G(s) dx\, ds \\ &+ \int_0^t \int_{\mathscr{O}} \Theta'(u(s))H(s)dx\, dW_{\ell^2}(s) \\ &+ \frac{1}{2}\sum_{n\geq 1} \int_0^t \int_{\mathscr{O}} \Theta''(u(s))H_n^2(s)dx\, ds.\end{aligned} \tag{8.6}$$

Proof. The case $\mathscr{O} = \mathbb{T}^d$ follows by taking $\phi = \Theta$ and $\psi = 1$ in [57, Proposition A.1]. In [103, Lemma 3.2.13], the case $\mathscr{O} = \mathbb{R}^d$ was analised by using the argument in [57] using an approximation of the identity on $\mathbb{R}^d$. Here, we give some comments in the case $\mathscr{O}$ is a bounded C^1-domain.

Denoting by $\mathbf{1}_{\mathscr{O}}$ the extension by 0 outside $\mathscr{O}$ operator, let us set

$$v(t) := \mathbf{1}_{\mathscr{O}} u_0 + \int_0^t (\mathrm{div}_{\mathbb{R}^d}(\mathbf{1}_{\mathscr{O}} G(s)) + \mathbf{1}_{\mathscr{O}} F(s))ds + \int_0^t \mathbf{1}_{\mathscr{O}} H(s) dW_{\ell^2}(s). \tag{8.7}$$

By Corollary 2.3.8, v is well-defined with paths in $C(\overline{I}_\tau; H^{-1}(\mathbb{R}^d))$.

The idea is to apply (8.6) to v. We claim that v belong to the right regularity class and that $v = \mathbf{1}_{\mathscr{O}} u$. If the latter holds, then one has $v \in L^0(\Omega; L^2(I_\tau; H^1(\mathbb{R}^d))) \cap L^0(\Omega; C(\overline{I}_\tau; L^2(\mathbb{R}^d))$ due to (8.4). Therefore, (8.6) applies to v. Since $v = \mathbf{1}_{\mathscr{O}} u$, (8.6) shows the claimed equality for u.

Since $\mathscr{O}$ is a C^1-bounded domain and $u|_{\partial\mathscr{O}} = 0$ a.s. for all $t \in I_\tau$, to prove that $v = \mathbf{1}_{\mathscr{O}} u$ it remains to show that $v|_{\mathscr{O}} = u$ and $v|_{\mathbb{R}^d \setminus \mathscr{O}} = 0$ a.s. for all $t \in I_\tau$. To see the former, applying a linear functional $\mathbf{1}_{\mathscr{O}} \phi \in C^1_c(\mathscr{O}) \subseteq H^1(\mathbb{R}^d) = (H^{-1}(\mathbb{R}^d))^*$ to (8.7) we have, a.s. for all $t \in I_\tau$,

$$\begin{aligned}
\langle v(t), \mathbf{1}_{\mathscr{O}} \phi \rangle_{H^{-1}(\mathbb{R}^d), H^1(\mathbb{R}^d)} &= \langle \mathbf{1}_{\mathscr{O}} u_0, \mathbf{1}_{\mathscr{O}} \phi \rangle_{H^{-1}(\mathbb{R}^d), H^1(\mathbb{R}^d)} \\
&\quad + \int_0^t -\langle \mathbf{1}_{\mathscr{O}} G(s), \nabla(\mathbf{1}_{\mathscr{O}} \phi) \rangle_{H^{-1}(\mathbb{R}^d), H^1(\mathbb{R}^d)} ds \\
&\quad + \int_0^t \langle \mathbf{1}_{\mathscr{O}} F(s), \mathbf{1}_{\mathscr{O}} \phi \rangle_{H^{-1}(\mathbb{R}^d), H^1(\mathbb{R}^d)} ds \\
&\quad + \int_0^t \langle \mathbf{1}_{\mathscr{O}} H(s), \mathbf{1}_{\mathscr{O}} \phi \rangle_{H^{-1}(\mathbb{R}^d), H^1(\mathbb{R}^d)} dW_{\ell^2}(s).
\end{aligned}$$

Using that $\langle \mathbf{1}_{\mathscr{O}} z, \mathbf{1}_{\mathscr{O}} \varphi \rangle_{H^{-1}(\mathbb{R}^d), H^1(\mathbb{R}^d)} = \int_{\mathscr{O}} z\varphi \, dx = \langle z, \varphi \rangle_{{}_D H^{-1}(\mathscr{O}), {}_D H^1(\mathscr{O})}$ for all $z \in L^2(\mathscr{O}; \mathbb{R}^m)$ and $\varphi \in C^1_c(\mathscr{O}; \mathbb{R}^d)$ for all $m \geq 1$, one gets

$$\begin{aligned}
\langle v(t), \mathbf{1}_{\mathscr{O}} \phi \rangle_{H^{-1}(\mathbb{R}^d), H^1(\mathbb{R}^d)} &= \langle u_0, \phi \rangle_{{}_D H^{-1}(\mathscr{O}), {}_D H^1(\mathscr{O})} \\
&\quad + \int_0^t -\langle G(s), \nabla \phi \rangle_{{}_D H^{-1}(\mathscr{O}), {}_D H^1(\mathscr{O})} ds \\
&\quad + \langle F(s), \phi \rangle_{{}_D H^{-1}(\mathscr{O}), {}_D H^1(\mathscr{O})} ds \\
&\quad + \int_0^t \langle H(s), \phi \rangle_{{}_D H^{-1}(\mathscr{O}), {}_D H^1(\mathscr{O})} dW_{\ell^2}(s) \\
&= \langle u(t), \phi \rangle_{{}_D H^{-1}(\mathscr{O}), {}_D H^1(\mathscr{O})}
\end{aligned}$$

a.s. for all $t \in I_\tau$. Therefore, $v|_{\mathscr{O}} = u$ a.s. for all $t \in I_\tau$.

Similar considerations show that $v|_{\mathbb{R}^d \setminus \mathscr{O}} = 0$ a.s. for all $t \in I_\tau$. This yields the claim. □

8.1.2 Stochastic parabolic DeGiorgi-Nash-Moser estimates

In the papers [51, 52], the De Giorgi-Nash-Moser theory was extended to second order stochastic PDEs with gradient-noise under a stochastic parabolicity condition. Their main result yields estimates in $L^\infty(I_T \times \mathscr{O})$ only assuming a uniform ellipticity condition and measurability of the coefficients of the second order operator in divergence form. Unfortunately, no results on Hölder regularity of the solution seem to be know if one only assumed measurability of the coefficients. In the case there is no gradient noise, much more can be said. In this case it is possible to reduce to the deterministic DeGiorgi-Nash-Moser theory by a standard trick. This was done in [105, Theorem 4.2] by relying on Krylov's L^p-theory. Below, we give a derivation which relies on stochastic maximal L^p-regularity of Definition 4.2.4 and has the advantage that the setting to which it can be applied is more flexible (arbitrary domains, mixed integrability). We only consider the linear case, and in Subsection 8.3 we use it to cover the quasilinear setting.

In this appendix we prove Hölder estimates for the following SPDE:

$$\begin{cases} du - \mathrm{div}(a(\cdot)\nabla u)dt = f dt + \sum_{n\geq 1} g_n dw^n_t, & \text{on } \mathscr{O}, \\ u = 0, & \text{on } \partial\mathscr{O}, \\ u(0) = u_0, & \text{on } \mathscr{O}. \end{cases} \tag{8.8}$$

Here $(w^n)_{n\geq 1}$ are independent standard Brownian motions and div is as in (8.3). Moreover, we assume the following conditions.

Assumption 8.1.2. *Let $T \in (0,\infty)$ and let τ be a stopping time with values in $[0,T]$. Consider the following conditions:*

(1) *$\mathscr{O} \subset \mathbb{R}^d$ is a bounded C^1-domain.*

(2) *$a := (a^{i,j})_{i,j=1}^d : I_\tau \times \Omega \times \mathscr{O} \to \mathbb{R}$ are $\mathscr{P} \otimes \mathscr{B}(\mathscr{O})$-measurable and uniformly bounded by M and there exists $\nu > 0$ such that for a.a. $\omega \in \Omega$ and all $t \in I_\tau$, $x \in \mathscr{O}$, $\xi \in \mathbb{R}^d$*

$$\sum_{i,j=1}^{d} a^{i,j}(t,\omega,x)\xi_i\xi_j \geq \nu|\xi|^2.$$

It is also possible to consider lower order terms in (8.8) (see Remark 8.1.4). The main result of this subsection reads as follows. As in the previous subsection, ${}_D H^{1,q}(\mathscr{O}) = \{v \in H^{1,q}(\mathscr{O}) : v = 0 \text{ on } \partial\mathscr{O}\}$ and ${}_D H^{-1,q}(\mathscr{O}) = ({}_D H^{1,q'}(\mathscr{O}))$ where $\frac{1}{q} + \frac{1}{q'} = 1$.

Theorem 8.1.3. *Suppose Assumption 8.1.2 holds. Let $p \in (2,\infty)$, $q \in [2,\infty)$ be such that $\frac{2}{p} + \frac{d}{q} < 1$, and let $r \in (0,\infty)$. Suppose that*

$$u_0 \in L^r_{\mathscr{F}_0}(\Omega; L^2(\mathscr{O})),\ f \in L^r_{\mathscr{P}}(\Omega; L^p(I_\tau; W^{-1,q}(\mathscr{O}))),\ g \in L^r_{\mathscr{P}}(\Omega; L^p(I_\tau; L^q(\mathscr{O}; \ell^2))).$$

Then (8.8) has a unique strong solution u is on $[\![0,\tau]\!]$ in the sense of Definition 4.3.3 with $X_0 = {}_D H^{-1,2}(\mathscr{O})$, $X_1 = {}_D H^{1,2}(\mathscr{O})$, and

$$A(t,\omega)v = -\operatorname{div}(a(t,\omega,\cdot)\cdot\nabla v), \quad B(t,\omega)v \;= 0, \quad t \in [0,\tau], \omega \in \Omega, v \in X_1,$$

and the following a priori estimate holds with constant $K = K(M,\nu,d,r)$

$$\begin{aligned}\|u\|_{L^r(\Omega;L^\infty(I_\tau;L^2(\mathscr{O})))} + \|u\|_{L^r(\Omega;L^2(I_\tau;{}_D H^{1,2}(\mathscr{O})))} &\leq K(\|u_0\|_{L^r(\Omega;L^2(\mathscr{O}))} \\ &\quad + \|f\|_{L^r(\Omega;L^2(I_\tau;{}_D H^{-1,2}(\mathscr{O})))} + \|g\|_{L^r(\Omega;L^2(I_\tau;L^2(\mathscr{O};\ell^2)))}).\end{aligned} \tag{8.9}$$

Moreover, the following hold with

$$K_{f,g} := \|f\|_{L^r(\Omega;L^p(I_\tau;{}_D H^{-1,q}(\mathscr{O})))} + \|g\|_{L^r(\Omega;L^p(I_\tau;L^q(\mathscr{O};\ell^2)))}:$$

(1) *There exists $\eta = \eta(M,\nu,p,q,d) > 0$ such that for all $s > 0$ we can find a constant $C = C(M,\nu,p,q,r,s,\mathscr{O},T)$ such that $u \in L^r(\Omega; C^\eta([s,\tau] \times \mathscr{O}))$,*

$$\|u\|_{L^r(\Omega;C^\eta([s,\tau]\times\mathscr{O}))} \leq C(\|u_0\|_{L^r(\Omega;L^2(\mathscr{O}))} + K_{f,g}) \tag{8.10}$$

(2) *If $u_0 \in L^r(\Omega; L^\infty(\mathscr{O}))$, then $u \in L^r(\Omega; L^\infty(I_\tau \times \mathscr{O}))$, and there exists a constant $C = C(M,\nu,p,q,r,\mathscr{O},T) > 0$ such that*

$$\|u\|_{L^r(\Omega;L^\infty(I_\tau\times\mathscr{O}))} \leq C(\|u_0\|_{L^r(\Omega;L^\infty(\mathscr{O}))} + K_{f,g}).$$

(3) *If $u_0 \in L^r(\Omega; C^\delta(\mathscr{O}))$ for some $\delta > 0$, then there exist $\eta = \eta(M,\nu,p,q,d,\delta) > 0$ and $C = C(M,\nu,p,q,r,s,\mathscr{O},T,\delta) > 0$ such that $u \in L^r(\Omega; C^\eta(I_\tau \times \mathscr{O}))$ and*

$$\|u\|_{L^r(\Omega;C^\eta(I_\tau\times\mathscr{O}))} \leq C(\|u_0\|_{L^r(\Omega;C^\delta(\mathscr{O}))} + K_{f,g}).$$

In the above the convention is that $\|u\|_{C^\eta([s,t)\times\mathscr{O})} = 0$ if $t \leq s$.

Proof. By extending a, b, f, g suitably on (τ, ∞) we may assume $\tau = T$ for some $T \in (0, \infty)$. By classical theory(see [148, Theorem 5.1.3]) there exists a unique strong solution such that $u \in L^2(\Omega; L^2(0, T; {}_DH^{1,2}(\mathscr{O})) \cap C([0, T]; L^2(\mathscr{O})))$.

To obtain the estimate (8.9) first consider $r \in [2, \infty)$. By a localization argument we may assume that the LHS(8.9) is finite. From [148, Theorem 4.2.5] we obtain a.s. for all $t \in [0, \tau]$,

$$\|u(t)\|^2_{L^2(\mathscr{O})} + 2\nu \int_0^t \|\nabla u(s)\|^2_{L^2(\mathscr{O})} ds \leq \|u_0\|^2_{L^2(\mathscr{O})} + 2 \int_0^t \langle f(s), u(s)\rangle ds + \|g\|^2_{L^2(0,t;\gamma(H,L^2(\mathscr{O})))} + 2 \int_0^t g(s)^* u(s) dW(s).$$

Therefore, standard arguments imply

$$\|u(t)\|^2_{L^2(\mathscr{O})} + \int_0^t \|u(s)\|^2_{{}_DH^{1,2}(\mathscr{O})} ds \lesssim_\theta \|u_0\|^2_{L^2(\mathscr{O})} + \|f\|^2_{L^2(0,t;{}_DH^{-1,2}(\mathscr{O}))} + \|g\|^2_{L^2(0,t;\gamma(H,L^2(\mathscr{O})))} + \int_0^t g(s)^* u(s) dW(s).$$

Taking the essential supremum over $[0, \tau]$ and $L^{r/2}(\Omega)$-norms, we obtain

$$\begin{aligned} &\|u\|^r_{L^r(\Omega;L^\infty(0,\tau,L^2(\mathscr{O})))} + \|u\|^r_{L^r(\Omega;L^2(0,\tau;{}_DH^{1,2}(\mathscr{O})))} \\ &\lesssim_{r,\theta} \|u_0\|^r_{L^r(\Omega;L^2(\mathscr{O}))} + \|f\|^r_{L^r(\Omega;L^2(0,t;{}_DH^{-1,2}(\mathscr{O})))} \\ &\quad + \|g\|^r_{L^r(\Omega;L^2(0,t;L^2(\mathscr{O})))} + \|g^* u\|^{r/2}_{L^{r/2}(\Omega;L^2(0,\tau;\ell^2))} \end{aligned}$$

Since $\|g^* u\|_{L^{r/2}(\Omega;L^2(0,\tau;H))} \leq \|u\|_{L^r(\Omega;L^\infty(0,\tau;L^2(\mathscr{O})))} \|g\|_{L^r(\Omega;L^2(0,t;L^2(\mathscr{O})))}$, standard considerations imply (8.9). The case $r \in (0, 2)$ can be obtained from Lenglart's inequality (see [183, Proposition IV.4.7]).

Next we show that u has a Hölder continuous version by splitting $u = v_1 + v_2$ where v_1 satisfies the stochastic heat equation and v_2 satisfies a deterministic PDE for which we can obtain estimates in Hölder norms.

Let v_1 be the strong solution to

$$\begin{cases} dv_1 - \Delta v_1 dt = \sum_{n\geq 1} \mathbf{1}_{[\![0,\tau]\!]} g_n dw^n_t, & \text{on } \mathscr{O}, \\ v_1 = 0, & \text{on } \partial\mathscr{O}, \\ v_1(0) = 0, & \text{on } \mathscr{O}. \end{cases}$$

Recall that, by [16, Theorem 11.5], the weak Dirichlet Laplacian

$${}_D\Delta_{-1,q} : {}_DH^{1,q}(\mathscr{O}) \subseteq {}_DH^{-1,q}(\mathscr{O}) \to {}_DH^{-1,q}(\mathscr{O}),$$

defined as

$$\langle {}_D\Delta_{-1,q} f, g\rangle = \int_{\mathscr{O}} \nabla f \cdot \nabla g \, dx \tag{8.11}$$

for all $f \in {}_DH^{1,q}(\mathscr{O})$ and $g \in {}_DH^{1,q'}(\mathscr{O})$, has a bounded H^∞-calculus of angle $\omega_{H^\infty}(\Delta_{-1,D}) < \pi/2$ and $0 \in \rho({}_D\Delta_{-1,q}) = \rho({}_D\Delta_q)$. By [168, Corollary 7.4], one has $v_1 \in L^r(\Omega; H^{\theta,p}(\mathbb{R}_+; {}_DH^{1-2\theta,q}(\mathscr{O})))$ for all $\theta \in [0, 1/2)$ and

$$\|v_1\|_{L^r(\Omega;H^{\theta,p}(\mathbb{R}_+;{}_DH^{1-2\theta,q}(\mathscr{O})))} \leq C\|g\|_{L^r(\Omega;L^p(I_\tau;L^q(\mathscr{O};\ell^2)))} \tag{8.12}$$

In particular, setting $\theta = \frac{1}{p} + \epsilon$ for $\epsilon > 0$ so small that $\frac{1}{p} + \epsilon < \frac{1}{2}$ and $\frac{2}{p} + \frac{d}{q} < 1 - 2\epsilon$, by Sobolev embedding we obtain $v_1 \in L^r(\Omega; C^\eta(\mathbb{R}_+; C^\delta(\overline{\mathscr{O}}))) \hookrightarrow L^r(\Omega; C^\delta(\mathbb{R}_+ \times \overline{\mathscr{O}}))$ where $\gamma := \min\{\epsilon, 1 - \frac{2}{p} - 2\epsilon - \frac{d}{q}\}$.

Let $F = \mathbf{1}_{[\![0,\tau]\!]} f + \mathbf{1}_{[\![0,\tau]\!]}\operatorname{div}(a \cdot v_1) - \mathbf{1}_{[\![0,\tau]\!]}\Delta v_1$ and note that by (8.12)

$$\begin{aligned}\|F\|_{L^r(\Omega;L^p(\mathbb{R}_+;{}_D H^{-1,q}(\mathscr{O})))} \leq C\|f\|_{L^r(\Omega;L^p(I_\tau;{}_D H^{-1,q}(\mathscr{O})))}\\ + C\|g\|_{L^r(\Omega;L^p(I_\tau;L^q(\mathscr{O};\ell^2)))}.\end{aligned} \tag{8.13}$$

For each $\omega \in \Omega$ fixed, by the deterministic case of [148, Theorem 4.2.4] (or see [140, Theorem 4.1]) we can find a unique strong solution $v_2 \in L^2(0,\tau;{}_D H^{1,2}(\mathscr{O}))$ to the PDE

$$\begin{cases}\partial_t v_2 - \operatorname{div}(a \cdot \nabla v_2) = F, & \text{on } \mathscr{O},\\ v_2 = 0, & \text{on } \partial\mathscr{O},\\ v_2(0) = u_0, & \text{on } \mathscr{O},\end{cases}$$

As in [174, Theorem 3.9 step 1] one sees that v_2 is progressively measurable.

By (8.13) and the De Giorgi-Nash-Moser estimates [140, Theorems 7.1 and 10.1, Chapter III], applied pointwise in Ω, one obtains $v_2 \in L^r(\Omega; C^\eta((s,\tau)\times\mathscr{O}))$ for some $\eta = \eta(M,\nu,p,q,d) > 0$, and the estimate (8.10) holds with u replaced by v_2 with a constant only dependent on M,ν,p,q,d,s. Moreover, if $u_0 \in C^\delta(\mathscr{O})$ the latter holds with $s = 0$. Since $u = v_1 + v_2$, (1) and (3) follow with Hölder exponent $\min\{\eta,\gamma\}$. The proof of (2) is similar. □

Remark 8.1.4. The result of Theorem 8.1.3 also holds in case one adds lower order terms in the equation. However, in that case the constant K in (8.9) becomes T-dependent.

8.2 Semilinear stochastic reaction-diffusion equations

In this section we study the existence and regularity of global solutions to (8.1). The structure of this section is as follows. In Subsection 8.2.1 we state the assumptions and main results on global well-posedness and regularity. To obtain these results we first prove local well-posedness and regularity in Subsection 8.2.2 by applying Theorems 4.3.7 and 7.1.3. In Section 8.2.3 we prove an energy estimate (see Lemma 8.2.12) which allows us to derive global well-posedness using the blow-up criteria of Theorems 6.3.6 and 6.3.7.

8.2.1 Main results

In this subsection we state our main results concerning (8.1). Below, we study (8.1) on the state space $X_0 = H^{-\delta,q}$ with $\delta \in [1,2)$. In order to do this some smoothness requirements on a, b is required needed. Note that, if the following is satisfied with $\delta = 1$, then it also holds for some $\delta > 1$.

Assumption 8.2.1. *Suppose $d \geq 2$ and that the following hold:*

(1) $q \in [2,\infty)$, $p \in (2,\infty)$ *and* $\kappa \in [0, \frac{p}{2} - 1)$.

(2) *For each* $i,j \in \{1,\dots,d\}$, $a^{i,j} : \mathbb{R}_+ \times \Omega \times \mathbb{T}^d \to \mathbb{R}$, $(b_n^j)_{n\geq 1} : \mathbb{R}_+ \times \Omega \times \mathbb{T}^d \to \ell^2$ *are* $\mathscr{P} \otimes \mathscr{B}(\mathbb{T}^d)$*-measurable. In addition, there exist* $\alpha > |1-\delta|$ *and* $C_{a,b} > 0$*, such that a.s. for all* $t \in \mathbb{R}_+$, $i,j \in \{1,\dots,d\}$,

$$\|a^{i,j}(t,\cdot)\|_{C^\alpha(\mathbb{T}^d)} + \|(b_n^j(t,\cdot))_{n\geq 1}\|_{C^\alpha(\mathbb{T}^d;\ell^2)} \leq C_{a,b}.$$

(3) *There exists* $\nu > 0$ *such that, a.s. for all* $t \in \mathbb{R}_+$, $x \in \mathbb{T}^d$, $\xi \in \mathbb{R}^d$,

$$\sum_{i,j=1}^d \Big(a^{i,j}(t,x) - \frac{1}{2}\sum_{n\geq 1} b_n^j(t,x) b_n^i(t,x)\Big)\xi_i\xi_j \geq \nu|\xi|^2.$$

(4) *For all* $j \in \{1, \dots, d\}$*,* Ψ^j*,* $\psi : \mathbb{R}_+ \times \Omega \times \mathbb{T}^d \times \mathbb{R} \to \mathbb{R}$*,* $\Phi = (\Phi_n) : \mathbb{R}_+ \times \Omega \times \mathbb{T}^d \times \mathbb{R} \to \ell^2$ *are* $\mathscr{P} \otimes \mathscr{B}(\mathbb{T}^d) \otimes \mathscr{B}(\mathbb{R})$*-measurable. Set* $\Psi := (\Psi^j)_{j=1}^d$*. Assume that* $\Psi^j(\cdot, 0)$*,* $\psi(\cdot, 0) \in L^\infty(\mathbb{R}_+ \times \Omega \times \mathbb{T}^d)$*,* $\Phi(\cdot, 0) \in L^\infty(\mathbb{R}_+ \times \Omega \times \mathbb{T}^d; \ell^2)$*, and there exists* $h \geq 1$*, such that a.s. for all* $t \in \mathbb{R}_+$*,* $x \in \mathbb{T}^d$*,* $y \in \mathbb{R}$*,*

$$|\psi(t,x,y) - \psi(t,x,y')| \lesssim (1 + |y|^{h-1} + |y'|^{h-1})|y - y'|,$$
$$|\Psi(t,x,y) - \Psi(t,x,y')| + \|\Phi(t,x,y) - \Phi(t,x,y')\|_{\ell^2} \lesssim (1 + |y|^{\frac{h-1}{2}} + |y'|^{\frac{h-1}{2}})|y - y'|.$$

(5) *Set* $C_b := \operatorname{ess\,sup}_{(t,\omega,x)} \|((b_n^j(t,x))_{j=1}^d)_{n\geq 1}\|_{\ell^2(\mathbb{N};\mathbb{R}^d)}$*. Suppose that there exist* $\gamma \in (0, \nu)$*,* $M > 0$ *and* $\Lambda > \max\{\frac{d}{2}(h-1), 2\}$ *s.t. a.s. for all* $t \in \mathbb{R}_+$*,* $x \in \mathbb{T}^d$*,* $y \in \mathbb{R}$*,*

$$\frac{1}{4\gamma}|\Psi(t,x,y)|^2 + \frac{1}{2}\Big(1 + \frac{C_b^2}{4(\nu - \gamma)}\Big)\|\Phi(t,x,y)\|_{\ell^2}^2 \leq M(1 + |y|^2) - \frac{\psi(t,x,y)y}{\Lambda - 1}.$$

Furthermore, if $b \equiv 0$ *(resp.* $\Psi \equiv 0$*), then one can take* $\gamma = \nu$ *(resp.* $\gamma = 0$*) and* $\frac{\Lambda}{4\gamma}|\Psi(t,x,y)|^2$ *(resp.* $\frac{C_b^2}{4(\nu-\gamma)}\|\Phi(t,x,y)\|_{\ell^2}^2$*) can be omitted in the previous estimate.*

Let us discuss a prototype example of nonlinearities which satisfy Assumption 8.2.1(5) and appears often in the literature.

Example 8.2.2. In the study of reaction-diffusion equations, the following assumption on ψ arises naturally (see e.g. [203, formula (1.3)]): For some $c_1, c_2 > 0$ and a.s. for all $t \in \mathbb{R}_+$, $x \in \mathbb{T}^d$, $y \in \mathbb{R}$,

$$\psi(t,x,y)y \leq -c_1|y|^{h+1} + c_2(1 + |y|^2). \tag{8.14}$$

For instance, the above condition is satisfied in the case $\psi(\cdot, y) = -y|y|^{h-1}$. Moreover, (8.14) covers another well-known example (see e.g. [35, Remark 5.1(2)]): For some $k \geq 1$, $c > 0$ and a.s. for all $t \in \mathbb{R}_+$, $x \in \mathbb{T}^d$, $y \in \mathbb{R}$,

$$\psi(t,x,y) = -d_{2k+1}(t,x)y^{2k+1} + \sum_{j=0}^{2k} d_j(t,x)y^j, \quad \text{with} \quad d_{2k+1}(t,x) \geq c > 0. \tag{8.15}$$

where, $d_j : \mathbb{R}_+ \times \Omega \times \mathbb{T}^d \to \mathbb{R}$ is $\mathscr{P} \otimes \mathscr{B}(\mathbb{T}^d)$-measurable and uniformly bounded for $j \in \{1, \dots, 2k+1\}$. If ψ is as in (8.15), then it satisfies (8.14) with $h = 2k+1$. Note that, (8.15) covers (8.2) and the *Allen-Cahn nonlinearity* $\psi(\cdot, y) = y - y^3$.

If ψ satisfies (8.14), then Assumption 8.2.1(5) holds in the following cases:

- For some $M_1, M_2, \varepsilon > 0$, a.s. for all $t \in \mathbb{R}_+$, $x \in \mathbb{T}^d$, $y \in \mathbb{R}$,

$$|\Psi(t,x,y)| + \|\Phi(t,x,y)\|_{\ell^2} \leq M_1 + M_2|y|^{\frac{h+1}{2} - \varepsilon}. \tag{8.16}$$

  Indeed, using Young's inequality one obtain the last inequality in Assumption 8.2.1(5). Moreover, one can check that the case $\varepsilon = 0$, is admissible if M_2 is small enough.

- If $b \equiv 0$, the condition can be formulated as: there exist $M > 0$ and $\Lambda > \max\{\frac{d}{2}(h-1), 2\}$ such that a.s. for all $t \in \mathbb{R}_+$, $x \in \mathbb{T}^d$, $y \in \mathbb{R}$

$$\frac{|\Psi(t,x,y)|^2}{4\nu} + \frac{1}{2}\|\Phi(t,x,y)\|_{\ell^2}^2 \leq M(1 + |y|^2) + \frac{c_1}{\Lambda - 1}|y|^{h+1}.$$

  For instance if $\Psi(\cdot, y) = (b_j y|y|^{(h-1)/2})_{j=1}^d$ and $\Phi(\cdot, y) = (g_n y|y|^{(h-1)/2})_{n\geq 1}$ for $b \in \mathbb{R}^d$ and $g \in \ell^2$ small enough, then the assumption is satisfied. In this case as in Subsection 5.1.3 one can see that locally all terms in (8.1) have the same scaling.

Let us collect some comments on Assumption 8.2.1 in the following remark.

Remark 8.2.3.

- The condition (5) (typically) reflects a dissipation in the underlined physical model. The last condition in (5) says that the dissipation given by $\psi(\cdot, u)u$ is stronger than the forcing terms $\operatorname{div}(\Psi(\cdot, u))$, $\Phi(\cdot, u)$ and $b \cdot \nabla u$.
- In contrast to a large part of the literature, Example 8.2.2 shows that Ψ, Φ *are not* necessarily globally Lipschitz in y.
- The study of the optimality of the conditions in (5) goes beyond the scope of this thesis. However, since it may be useful in applications, we do make the constants as explicit as possible.

Next, we state the main results of this section. For $T \in (0, \infty]$, we say that (u, σ) is a *(unique L^p_κ-)weak solution to* (8.1) *on* $\overline{I}_T$ if (u, σ) is an L^p_κ-maximal local solution to (4.16) on $\overline{I}_T$ (see Definitions 4.3.3 and 4.3.4 and Subsection 6.3.3 for the extension to $[0, \infty)$) with $H = \ell^2$, W_ℓ as in Example 2.3.6, $X_0 = H^{-\delta,q}(\mathbb{T}^d)$, $X_1 = H^{2-\delta,q}(\mathbb{T}^d)$, $f = g = 0$ and for $v \in X_1$,

$$\begin{aligned} A(\cdot)v &= -\operatorname{div}(a(\cdot) \cdot \nabla v), & B(\cdot)v &= (b_n(\cdot)\nabla v)_{n\geq 1}, \\ F(\cdot, v) &= \operatorname{div}(\Psi(\cdot, v)) + \psi(\cdot, v), & G(\cdot, v) &= (\Phi_n(\cdot, v))_{n\geq 1}. \end{aligned} \tag{8.17}$$

Weak solutions are unique by maximality. We say that (u, σ) (or simply u) is a *global (L^p_κ-)weak solution to* (8.1) provided (u, σ) is a unique L^p_κ-weak solution to (8.1) on $[0, \infty)$ with $\sigma = \infty$ a.s.

Theorem 8.2.4 (Global existence and regularity)**.** *Let Assumption* 8.2.1 *be satisfied for some* $\delta \in (1, 2]$. *Suppose that* $q > \max\{\frac{d}{d-\delta}, \frac{d(h-1)}{2h-\delta(h-1)}\}$ *and that one of the following conditions holds:*

- $q < \frac{d(h-1)}{\delta}$ *and* $\frac{1+\kappa}{p} + \frac{1}{2}(\delta + \frac{d}{q}) \leq \frac{h}{h-1}$;
- $q \geq \frac{d(h-1)}{\delta}$ *and* $\frac{1+\kappa}{p} \leq \frac{h}{h-1}(1 - \frac{\delta}{2})$.

Then for any $u_0 \in L^0_{\mathscr{F}_0}(\Omega; B^{2-\delta-2\frac{1+\kappa}{p}}_{q,p}(\mathbb{T}^d))$, (8.1) *has a unique global weak solution*

$$u \in L^p_{\text{loc}}([0, \infty), w_\kappa; H^{2-\delta,q}(\mathbb{T}^d)) \cap C([0, \infty); B^{2-\delta-2\frac{1+\kappa}{p}}_{q,p}(\mathbb{T}^d)).$$

Moreover, u *instantaneously regularizes in time and space:*

$$u \in \bigcap_{\theta\in[0,1/2)} H^{\theta,r}_{\text{loc}}(\mathbb{R}_+; H^{1-2\theta,\zeta}(\mathbb{T}^d)) \text{ a.s.} \qquad \text{for all } r, \zeta \in (2, \infty). \tag{8.18}$$

In particular,

$$u \in \bigcap_{\theta\in(0,\frac{1}{2})} C^{\theta}_{\text{loc}}(\mathbb{R}_+; C^{1-2\theta}(\mathbb{T}^d)) \subseteq \bigcap_{\theta_1\in(0,\frac{1}{2}),\, \theta_2\in(0,1)} C^{\theta_1,\theta_2}_{\text{loc}}(\mathbb{R}_+ \times \mathbb{T}^d) \text{ a.s.} \tag{8.19}$$

As a by-product of Theorem 8.2.4, we obtain the global existence for (8.1) in critical space of Besov-type for (8.1). Set

$$\mathtt{h}_2 := 3, \qquad \text{and} \qquad \mathtt{h}_d := \frac{1}{2} + \frac{1}{d} + \sqrt{\Big(\frac{1}{2} + \frac{1}{d}\Big)^2 + \frac{2}{d}} \quad \text{for} \quad d \geq 3. \tag{8.20}$$

Motivation for this definition will be given in Remark 8.2.7 and Subsection 8.2.4. We will now specialize Theorem 8.2.4 to those parameters which lead to critical spaces.

Theorem 8.2.5 (Global existence in critical spaces)**.** *Let Assumption 8.2.1 be satisfied for some* $\delta \in [1, \frac{h+1}{h}]$. *Assume* $h > \mathtt{h}_d$, $\frac{1}{p} + \frac{1}{2}(\delta + \frac{d}{q}) \leq \frac{h}{h-1}$ *and*

$$\max\left\{\frac{d}{d-\delta}, \frac{d(h-1)}{2h-\delta(h-1)}\right\} < q < \frac{d(h-1)}{h+1-\delta(h-1)}. \tag{8.21}$$

Set $\kappa_{\mathsf{crit}} := p(\frac{h}{h-1} - \frac{1}{2}(\delta + \frac{d}{q})) - 1$. *Then for any* $u_0 \in L^0_{\mathscr{F}_0}(\Omega; B^{\frac{d}{q}-\frac{2}{h-1}}_{q,p}(\mathbb{T}^d))$, (8.1) *has a unique global solution*

$$u \in L^p_{\mathrm{loc}}([0,\infty), w_{\kappa_{\mathsf{crit}}}; H^{2-\delta,q}(\mathbb{T}^d)) \cap C([0,\infty); B^{\frac{d}{q}-\frac{2}{h-1}}_{q,p}(\mathbb{T}^d)). \tag{8.22}$$

Moreover u instantaneously regularizes in space and time, i.e. (8.18)-(8.19) *hold.*

It is important to note that by using the parameter δ one can allow much large parameter q. This has the advantage that the admissible smoothness of the initial value $s := \frac{d}{q} - \frac{2}{h-1}$ is lower. Indeed, letting $q \uparrow$RHS(8.21) it follows that we can deal with all smoothness $s > 1-\delta$. Thus using $\delta \in (1, \frac{h+1}{h})$ we obtain a wider class of critical spaces including ones with negative smoothness up to $-\frac{1}{h}$.

For a special choice of q we obtain the following L^q-version of the previous result.

Corollary 8.2.6 (Global existence in critical L^ξ-spaces)**.** *Let Assumption 8.2.1 be satisfied for some* $\delta \in (1,2]$. *Set* $\xi := \frac{d}{2}(h-1)$ *and assume that*

$$\text{either} \quad \Big[h > 1 + \frac{4}{d}, \quad \text{and} \quad p \geq \xi\Big] \quad \text{or} \quad \Big[h = 1 + \frac{4}{d}, \quad \text{and} \quad p, d > 2\Big]. \tag{8.23}$$

Then there exists $\bar{\delta}(h,d) \in (1, \frac{h+1}{h}]$ *such that for all* $\delta \in (1, \bar{\delta}]$ *and* $u_0 \in L^0_{\mathscr{F}_0}(\Omega; L^\xi(\mathbb{T}^d))$, (8.1) *has a unique global solution as in* (8.22) *with with* p, δ *as above,* $q = \xi$ *and* $\kappa_{\mathsf{crit}} = p(1 - \frac{\delta}{2}) - 1$. *Moreover, u instantaneously regularizes in spaces and time i.e.* (8.18)-(8.19) *hold.*

As in Subsection 5.1.3, one can see that solutions to (8.1) are (locally) invariant under the mapping

$$u \mapsto \lambda^{1/(h-1)} u(\lambda \cdot, \lambda^{1/2} \cdot), \quad \lambda > 0, \tag{8.24}$$

and that the spaces $B^{\frac{d}{q}-\frac{2}{h-1}}_{q,p}(\mathbb{T}^d)$ and $L^{\frac{d}{2}(h-1)}(\mathbb{T}^d)$ are (locally) invariant under the induced mapping on the initial data $u_0 \mapsto \lambda^{1/(h-1)} u_0(\lambda^{1/2} \cdot)$. Because of this they are called usually called critical in the literature. In Subsection 8.2.2 we will see that the spaces are also critical in the sense of Section 6.3 (see the text before (4.29)).

Remark 8.2.7.

- The assumption $h > \mathtt{h}_d$ ensures the existence of $q > 2$, $p \in (2, \infty)$ and $\delta \in [1, \frac{h+1}{h}]$ for which Theorem 8.2.5 applies. To see this, choosing p large enough, it is enough to find $q > 2$, $\delta \in [1, \frac{h+1}{h}]$ which satisfy (8.21). To begin, note that LHS(8.21)< RHS(8.21) leads to the condition $h > \frac{d+1}{d-1}$. Therefore, in order to find an admissible $q \geq 2$ for (8.21) we need $\phi(\delta) := \frac{d(h-1)}{h+1-\delta(h-1)} > 2$ for some $\delta \in [1, \frac{h+1}{h}]$. One can check that this holds if and only if $h > \frac{1}{2} + \frac{1}{d} + \left(\left(\frac{1}{2} + \frac{1}{d}\right)^2 + \frac{2}{d}\right)^{1/2}$. If $d \geq 3$, the latter condition coincides with (8.20) and is more restrictive than the previously obtained condition $h > \frac{d+1}{d-1}$. On the other hand, if $d = 2$, then this turns around and we require $h > 2$. This explains the two cases in (8.20).
- If Assumption 8.2.1 is satisfied with $h = \mathtt{h}_d$, then Theorem 8.2.5 can still applied with $\widetilde{h} > \mathtt{h}_d$, and gives a sub-optimal result in the sense that the local scaling in the space is not the right one for this nonlinearity. This observation applies to the standard Allen-Cahn equation with $d = 2$ (see Example 8.2.2).

To prove Theorem 8.2.4 and the stated consequences we will first consider *local* well-posedness in Subsection 8.2.2. In Subsection 8.2.3, we derive an a priori estimate which we use to obtain *global* well-posedness.

8.2.2 Local existence results for (8.1)

Throughout the remaining subsections, to abbreviate the notation, we often write L^q, $H^{s,q}$, $B^s_{q,p}$, etc. instead of $L^q(\mathbb{T}^d)$, $H^{s,q}(\mathbb{T}^d)$, $B^s_{q,p}(\mathbb{T}^d)$. Let us begin with a lemma. Recall that $C^{\det,\theta,p,\kappa}_{(A,B)}(s,T)$ and $C^{\mathsf{sto},\theta,p,\kappa}_{(A,B)}(s,T)$ are defined in Subsection 6.2.1.

Lemma 8.2.8. *Let Assumption 8.2.1(1)-(3) be satisfied. Then for each $\delta\in[1,2)$ there exists C for which the following holds. For any $s\in(0,T)$ the couple $(A,B)=(-\mathrm{div}(a\cdot\nabla),(b_n\cdot\nabla)_{n\geq 1})$ satisfies $(A,B)\in\mathcal{SMR}^\bullet_{p,\kappa}(s,T)$ with $X_0=H^{-\delta}(\mathbb{T}^d)$, $X_1=H^{2-\delta}(\mathbb{T}^d)$ and*

$$\max\big\{C^{\det,\theta,p,\kappa}_{(A,B)}(s,T),C^{\mathsf{sto},\theta,p,\kappa}_{(A,B)}(s,T)\big\}\leq C.$$

The proof of the above result follows from the argument employed to prove Theorem 9.2.2 below with some simplifications. To avoid repetitions we omit the details.

Our first result deals with the local well-posedness and smoothness of solutions to (8.1). For this we do not need Assumption 8.2.1(5). Recall that $I_\sigma=(0,\sigma)$.

Proposition 8.2.9 (Local existence and regularization)**.** *Let Assumption 8.2.1(1)-(4) be satisfied for some $\delta\in[1,2)$. Let p,q,κ be as in Theorem 8.2.4. Then for any $u_0\in L^0_{\mathscr{F}_0}(\Omega;B^{2-\delta-2\frac{1+\kappa}{p}}_{q,p}(\mathbb{T}^d))$, there exists a unique L^p_κ-weak solution (u,σ) to (8.1) on $[0,\infty)$ with $\sigma>0$ a.s., and for each localizing sequence $(\sigma_n)_{n\geq1}$, a.s. for all $n\geq 1$*

$$u\in L^p_{\mathrm{loc}}(\overline{I}_{\sigma_n},w_\kappa;H^{1-\delta,q}(\mathbb{T}^d))\cap C(\overline{I}_{\sigma_n};B^{2-\delta-2\frac{1+\kappa}{p}}_{q,p}(\mathbb{T}^d)). \tag{8.25}$$

Moreover, (u,σ) instantaneously regularizes in time and space

$$u\in\bigcap_{\theta\in[0,1/2)}H^{\theta,r}_{\mathrm{loc}}(I_\sigma;H^{1-2\theta,\zeta}(\mathbb{T}^d))\qquad a.s.\ for\ all\ r,\zeta\in(2,\infty). \tag{8.26}$$

In particular,

$$u\in\bigcap_{\theta\in(0,\frac12)}C^{\theta}_{\mathrm{loc}}(I_\sigma;C^{1-2\theta}(\mathbb{T}^d))\subseteq\bigcap_{\theta_1\in(0,\frac12),\,\theta_2\in(0,1)}C^{\theta_1,\theta_2}_{\mathrm{loc}}(I_\sigma\times\mathbb{T}^d)\qquad a.s. \tag{8.27}$$

Proof. By the argument given in the beginning of Subsection 6.3.3 it suffices to consider the equation on $[0,T]$ with $T\in(0,\infty)$ arbitrary. For the sake of clarity, we divide the proof into several steps. In Steps 1-3 we prove the local well-posedness and (8.25), and the argument is very similar to Subsection 5.1.3 where the case $\delta=1$ is considered. The most important part of the proof can be found in Steps 4-6 where we show the regularization via the results of Section 7.1.

Step 1: F satisfies (HF). Since F consist of two parts (see (8.17)), we check (4.18) for $j\in\{1,2\}$. By Assumption 8.2.1(4), a.s. for all $t\in I_T$ and $v,v'\in H^{2-\delta,q}$,

$$\begin{aligned}\|\psi(t,\cdot,v)-\psi(t,\cdot,v')\|_{H^{-\delta,q}}&\overset{(i)}{\lesssim}\|\psi(t,\cdot,v)-\psi(t,\cdot,v')\|_{L^\xi}\\&\lesssim\|(1+|v|^{h-1}+|v'|^{h-1})|v-v'|\|_{L^\xi}\\&\overset{(ii)}{\lesssim}(1+\|v\|^{h-1}_{L^{h\xi}}+\|v'\|^{h-1}_{L^{h\xi}})\|v-v'\|_{L^{h\xi}}\\&\overset{(iii)}{\lesssim}(1+\|v\|^{h-1}_{H^{\theta,q}}+\|v'\|^{h-1}_{H^{\theta,q}})\|v-v'\|_{H^{\theta,q}}\\&\eqsim(1+\|v\|^{h-1}_{X_\beta}+\|v'\|^{h-1}_{X_\beta})\|v-v'\|_{X_\beta},\end{aligned} \tag{8.28}$$

where $\beta:=\frac{\delta+\theta}{2}$. In (i) we used Sobolev embedding with $-\frac{d}{\xi}=-\delta-\frac{d}{q}$, where we used $q>\frac{d}{d-\delta}$ to ensure $\xi\in(1,\infty)$. In (ii) we used Hölder's inequality, and in (iii) Sobolev embedding where $\theta\in[0,2-\delta)$ is chosen (see below) so that $\theta-\frac{d}{q}\geq-\frac{d}{h\xi}$. To see that an admissible θ can be found such that (4.18) holds, we split into two cases:

(1) *Case* $q < \frac{d(h-1)}{\delta}$. Here, we may set $\theta = \frac{d}{q} - \frac{d}{h\xi} > 0$. To ensure that $\theta < 2 - \delta$ (i.e. ψ is a lower-order nonlinearity) we have to impose $q > \frac{d(h-1)}{2h-\delta(h-1)}$. To check (4.18) for $j = 1$, we split the discussion into two cases:

- If $1 - \frac{1+\kappa}{p} \geq \beta$ (i.e. $\frac{1+\kappa}{p} \leq 1 - \frac{1}{2}(\delta + \frac{d}{q})(1 - \frac{1}{h})$), then (4.18) for $j = 1$ follows by (8.28) and Remark 4.3.2(2). Moreover, the corresponding trace space is not critical for (8.1).
- If $1 - \frac{1+\kappa}{p} < \beta$ (i.e. $\frac{1+\kappa}{p} > 1 - \frac{1}{2}(\delta + \frac{d}{q})(1 - \frac{1}{h})$), then setting $\rho_1 := h - 1$, $\beta_1 := \varphi_1 := \beta = \frac{\delta+\theta}{2}$, by (8.28), (4.18) for $j = 1$ is equivalent to

$$\frac{1+\kappa}{p} \leq \frac{\rho_1 + 1}{\rho_1}(1 - \beta) = \frac{h}{h-1} - \frac{1}{2}\Big(\delta + \frac{d}{q}\Big). \tag{8.29}$$

  Note that, in this case, the corresponding trace space is critical for (8.1) if and only if the equality in (8.29) holds.

(2) *Case* $q \geq \frac{d(h-1)}{\delta}$. Here, we may set $\theta = 0$, and thus $\beta = \delta/2$. Since $\delta < 2$ (i.e. $\theta = 0 < 2 - \delta$), ψ is a lower order nonlinearity. As in the previous case, to check (4.18) for $j = 1$, we split the discussion into two cases:

- If $1 - \frac{1+\kappa}{p} \geq \frac{\delta}{2}$, then (4.18) for $j = 1$ follows by (8.28) and Remark 4.3.2(2). Moreover, the corresponding trace space is not critical for (8.1).
- If $1 - \frac{1+\kappa}{p} < \frac{\delta}{2}$, then setting $\rho_1 := h - 1$, $\beta_1 := \varphi_1 := \beta = \delta/2$, by (8.28), (4.18) for $j = 1$ is equivalent to

$$\frac{1+\kappa}{p} \leq \frac{\rho_1 + 1}{\rho_1}(1 - \beta) = \frac{h}{h-1}\Big(1 - \frac{\delta}{2}\Big). \tag{8.30}$$

  Note that, in this case, the corresponding trace space is critical for (8.1) if and only if the equality in (8.30) holds.

Next, we look at suitable bounds for $\mathrm{div}\Psi$. By Assumption 8.2.1(4), a.s. for all $t \in I_T$ and $v, v' \in H^{2-\delta,q}(\mathbb{T}^d)$,

$$\begin{aligned} &\|\mathrm{div}(\Psi(t,\cdot,v)) - \mathrm{div}(\Psi(t,\cdot,v'))\|_{H^{-\delta,q}} \\ &\overset{(iv)}{\lesssim} \|\Psi(t,\cdot,v) - \Psi(t,\cdot,v')\|_{L^\eta} \\ &\lesssim \|(1 + |v|^{\frac{h-1}{2}} + |v'|^{\frac{h-1}{2}})|v - v'|\|_{L^\eta} \\ &\overset{(v)}{\lesssim} (1 + \|v\|^{\frac{h-1}{2}}_{L^{\frac{h+1}{2}\eta}} + \|v'\|^{\frac{h-1}{2}}_{L^{\frac{h+1}{2}\eta}})\|v - v'\|_{L^{(\frac{h+1}{2})\eta}} \\ &\overset{(vi)}{\lesssim} (1 + \|v\|^{\frac{h-1}{2}}_{H^{\phi,q}} + \|v'\|^{\frac{h-1}{2}}_{H^{\phi,q}})\|v - v'\|_{H^{\phi,q}} \\ &\eqsim (1 + \|v\|^{\frac{h-1}{2}}_{X_\zeta} + \|v'\|^{\frac{h-1}{2}}_{X_\zeta})\|v - v'\|_{X_\zeta}, \end{aligned} \tag{8.31}$$

where $\zeta := \frac{\delta+\phi}{2}$. In (iv) we used $\mathrm{div} : H^{1-\delta,q} \to H^{-\delta,q}$ boundedly, and Sobolev embedding with $-\frac{d}{\eta} = 1 - \delta - \frac{d}{q}$, where $\eta \in (1, q)$ follows from $q > \frac{d}{d-\delta}$. In (v) we used Hölder's inequality, and in (vi) the Sobolev embedding with $\phi \in [0, 2 - \delta)$ and $\phi - \frac{d}{q} \geq -\frac{2d}{\eta(h+1)}$ (see below). To see that an admissible θ can be found such that (4.18) holds for $j = 2$, we split into two cases:

(3) *Case* $q < \frac{d(h-1)}{2(\delta-1)}$. Here, we may set $\phi := \frac{d}{q} - \frac{2d}{\eta(h+1)} = \frac{d}{q}\frac{h-1}{h+1} + 2\frac{1-\delta}{h+1} > 0$. Let us note that $\phi < 2 - \delta$, follows by $q > \frac{d(h-1)}{2h-\delta(h-1)}$ as assumed. To check (4.18) for $j = 2$, we split the discussion into two cases:

- $1 - \frac{1+\kappa}{p} \geq \zeta$ (i.e. $\frac{1+\kappa}{p} \leq \frac{h}{h+1} + \frac{1}{2}(\frac{d}{q} + \delta)\frac{h-1}{h+1}$), (4.18) for $j = 2$ follows by (8.28) and Remark 4.3.2(2). Moreover, the corresponding trace space is not critical for (8.1).

- $1-\frac{1+\kappa}{p}<\zeta$ (i.e. $\frac{1+\kappa}{p}>\frac{h}{h+1}+\frac{1}{2}(\frac{d}{q}+\delta)\frac{h-1}{h+1}$), then setting $\rho_2:=\frac{h-1}{2}$, $\beta_2:=\varphi_2:=\zeta$, by (8.31), (4.18) for $j=2$ is equivalent to

$$\frac{1+\kappa}{p}\leq\frac{\rho_2+1}{\rho_2}(1-\zeta)=\frac{h}{h-1}-\frac{1}{2}\Big(\delta+\frac{d}{q}\Big). \tag{8.32}$$

  As expected, (8.29) coincides with (8.32), and as above, the corresponding trace space is critical for (8.1) if and only if the equality in (8.32) holds.

(4) *Case* $q\geq\frac{d(h-1)}{2(\delta-1)}$. Here, we set $\phi=0$ and thus $\zeta=\delta/2$. Splitting the discussion into two cases, we get:

- If $1-\frac{1+\kappa}{p}\geq\zeta$, then (4.18) for $j=2$ follows by (8.28) and Remark 4.3.2(2). Moreover, the corresponding trace space is not critical for (8.1).
- If $1-\frac{1+\kappa}{p}<\zeta$, then setting $\rho_2:=\frac{h-1}{2}$, $\beta_2:=\varphi_2:=\zeta$, by (8.31), (4.18) for $j=2$ is equivalent to

$$\frac{1+\kappa}{p}\leq\frac{\rho_2+1}{\rho_2}(1-\zeta)=\frac{h+1}{h-1}\Big(1-\frac{\delta}{2}\Big). \tag{8.33}$$

  Note that, the corresponding trace space is critical for (8.1) if and only if the equality in (8.33) holds.

Since $F(\cdot,v)=\operatorname{div}(\Psi(\cdot,v))+\psi(\cdot,v)$, the above conditions have to be satisfied simultaneously. Therefore, we may argue as follows:

(5) Case $q<\frac{d(h-1)}{\delta}$. Thus, $q<\frac{d(h-1)}{2(\delta-1)}$ and by the results in (1) and (3), to check (HF) it is enough to assume that (8.29) holds. Moreover, the corresponding trace space is critical for (8.1) if and only if (8.29) holds with the equality.

(6) Case $\frac{d(h-1)}{\delta}\leq q<\frac{d(h-1)}{2(\delta-1)}$. By the results in (2), (3) and the fact that the RHS of (8.30) is less or equal than the one of (8.32), to check (HF) it is enough to assume that (8.30) holds. Moreover, the corresponding trace space is critical for (8.1) if and only if (8.30) holds with the equality.

(7) Case $q\geq\frac{d(h-1)}{2(\delta-1)}$. By (2), (4) and the fact that the RHS of (8.30) is less or equal than the one of (8.33), to check (HF) it is enough to assume that (8.30) holds. Moreover, the corresponding trace space is critical for (8.1) if and only if (8.30) holds with the equality.

To prove the claim of this step, it remains to note that the conditions in (5)-(7) are equivalent to the requirements in Theorem 8.2.4.

Step 2: G satisfies (HG). To check that G satisfies (HG), it is enough to prove that G satisfies the same bound of $\operatorname{div}(\Psi)$ in (8.31). Indeed, using that $X_{1/2}=H^{1-\delta,q}$ we get, a.s. for all $t\in I_T$ and $v,v'\in X_1$,

$$\begin{aligned}\|\Phi(t,\cdot,v)-\Phi(t,\cdot,v')\|_{\gamma(\ell^2,H^{1-\delta,q})}&\overset{(i)}{\lesssim}\|\Phi(t,\cdot,v)-\Phi(t,\cdot,v')\|_{\gamma(\ell^2,L^\eta)}\\&\overset{(ii)}{\eqsim}\|\Phi(t,\cdot,v)-\Phi(t,\cdot,v')\|_{L^\eta(\ell^2)}\\&\overset{(iii)}{\lesssim}\|(1+|v|^{\frac{h-1}{2}}+|v'|^{\frac{h-1}{2}})|v-v'|\|_{L^\eta},\end{aligned} \tag{8.34}$$

where in (i) we have used the left-ideal property of γ-spaces and the Sobolev embedding with $-\frac{d}{\eta}=1-\delta-\frac{d}{q}$, in (ii) (2.14) and in (iii) Assumption 8.2.1(4). Thus, to check (HG) with $\rho_3=(h-1)/2$, $\varphi_3=\beta_3=\zeta$ it is enough to repeat the estimates in (8.31) and the subsequent argument.

Step 3: If $u_0 \in L^0_{\mathscr{F}_0}(\Omega; B^{2-\delta-2\frac{1+\kappa}{p}}_{q,p}(\mathbb{T}^d))$, then there exists a unique weak solution u to (8.1), and

$$u \in \bigcap_{\theta\in[0,1/2)} H^{\theta,p}_{loc}(I_\sigma; H^{2\theta-\delta,q}), \qquad a.s. \tag{8.35}$$

*Condition **H**$(H^{-\delta,q}(\mathbb{T}^d), H^{2-\delta,q}(\mathbb{T}^d), p, \kappa)$ is satisfied whenever p, q, κ, δ satisfies the assumption of Theorem 8.2.4. Moreover, the corresponding trace space $B^{2-\delta-2\frac{1+\kappa}{p}}_{q,p}(\mathbb{T}^d)$ is critical for (8.1) if and only if one of the following holds:*

- *$q < \frac{d(h-1)}{\delta}$ and (8.29) holds with the equality;*
- *$q \geq \frac{d(h-1)}{\delta}$ and (8.30) holds with the equality.*

Finally, Assumption 6.3.2 in the $(H^{-\delta,q}(\mathbb{T}^d), H^{2-\delta,q}(\mathbb{T}^d), p, \kappa)$-setting holds for $\ell \in \{0, \kappa\}$ and Assumption 6.3.4 holds.

First of all, by Lemma 8.2.8, $(A, B) \in \mathcal{SMR}^\bullet_{p,\kappa}(s, T)$ for all $s \in I_T$, and (A, B) satisfies Assumption 6.3.2 in the $(H^{-\delta,q}(\mathbb{T}^d), H^{2-\delta,q}(\mathbb{T}^d), p, \kappa)$-setting (here A, B are as in (8.17)). Thus, the existence of a weak solution to (8.1) follows from Theorem 6.3.1, as well as the regularity (8.35). The claimed characterization of critical spaces follows from (5)-(7) in Step 1 and Step 2. Finally, Assumption 6.3.4 follows from the fact that $\varphi_j = \beta_j$ for all $j \in \{1, 2, 3\}$ as remarked before Assumption 6.3.4.

Step 4: For any $r_1 \in (2, \infty)$ we have

$$u \in \bigcap_{\theta\in[0,1/2)} H^{\theta,r_1}_{loc}(I_\sigma; H^{1-2\theta,q}), \qquad \text{a.s.} \tag{8.36}$$

By Step 3 and the fact that $f = g = 0$, in the case that $\kappa > 0$ we can apply Corollary 7.1.5 to $Y_0 = H^{-\delta,q}$, $Y_1 = H^{2-\delta,q}$, $\alpha = \kappa$, $r = p$ and $\widehat{r} = r_1 \in (2, \infty)$ arbitrary to get

$$u \in \bigcap_{\theta\in[0,1/2)} H^{\theta,r_1}_{loc}(I_\sigma; H^{2-2\theta-\delta,q}), \qquad \text{a.s.} \tag{8.37}$$

As explained after Proposition 7.1.7, in the case that $\kappa = 0$, using Step 3 one apply Proposition 7.1.7 $Y_0 = H^{-\delta,q}$, $Y_1 = H^{2-\delta,q}$, $\alpha = \kappa$, $r = p$ and $\widehat{r}$ such that $\frac{1}{\widehat{r}} = \max_j \varphi_j - 1 - \frac{1}{p}$ and then one can apply Corollary 7.1.5 as explained above, to get (8.37) also in the case $\kappa = 0$.

In the case $\delta = 1$, (8.37) implies (8.36). Thus, in the remaining part of this step we assume $\delta \in (1, 2)$ and we fix $r_1 \in (2, \infty)$. To complete the proof, it remains to check the applicability of Theorem 7.1.3 to $Y_0 = H^{-\delta,q}$, $Y_1 = H^{2-\delta,q}$, $\widehat{Y}_0 = H^{-1,q}$, $\widehat{Y}_1 = H^{1,q}$, $r = \widehat{r}, \alpha, \widehat{\alpha}$ to be chosen later and such that $\widehat{r} \geq r_1$. For the reader's convenience, we check the conditions in Theorem 7.1.3 separately.

Step 4a: If $\frac{1}{r} < 1 - \frac{\delta}{2}$, $\alpha > 0$, $\frac{1+\alpha}{r} < 1 - \frac{\delta}{2}$ and $\widehat{\alpha} = r(\frac{\delta-1}{2} + \frac{1}{r}) - 1$, then Theorem 7.1.3(3) holds. Since $\frac{1+\widehat{\alpha}}{r} = \frac{\delta-1}{2} + \frac{1}{r}$, one has

$$Y^{\mathsf{Tr}}_r = B^{2-\delta-\frac{2}{r}}_{q,r} = B^{1-2\frac{1+\widehat{\alpha}}{r}}_{q,r} = Y^{\mathsf{Tr}}_{\widehat{\alpha},\widehat{r}}.$$

Obviously, $\widehat{\alpha} > 0$ and $\widehat{\alpha} < \frac{r}{2} - 1$ due to $\frac{1}{r} < 1 - \frac{\delta}{2}$. To check (7.4), we apply Lemma 7.1.2(3). To this end, note that

$$\widehat{Y}_0 = Y_{\frac{\delta-1}{2}}, \qquad \text{and} \qquad \widehat{Y}_{1-\frac{\delta-1}{2}} = Y_1.$$

Set $\varepsilon := \frac{\delta-1}{2}$ and note that $\frac{1+\widehat{\alpha}}{\widehat{r}} = \varepsilon + \frac{1}{r} < \varepsilon + \frac{1+\alpha}{r}$. Thus, to apply Lemma 7.1.2(3) it remains to note that $\varepsilon < \frac{1}{2} - \frac{1+\alpha}{r}$ since $\frac{1+\alpha}{r} < 1 - \frac{\delta}{2}$.

Step 4b: If r is large enough and $\alpha > 0$ is small enough, then Theorem 7.1.3(1) holds. The claimed regularity in (1) follows from (8.37). Moreover, Assumption 6.3.2 holds in the (Y_0, Y_1, α, r)-setting for $\ell = \alpha$ by Step 3. It remains to note that, up to enlarging the above choice of $r = \widehat{r}$ and taking $\alpha > 0$ small, we can assume that either

$$\Big[q < \frac{d(h-1)}{\delta}, \qquad \text{and} \qquad \frac{1+\alpha}{r} \leq \frac{h}{h-1} - \frac{1}{2}\Big(\delta + \frac{d}{q}\Big)\Big] \tag{8.38}$$

or

$$\Big[q \geq \frac{d(h-1)}{\delta}, \quad \text{and} \quad \frac{1+\alpha}{r} \leq \frac{h}{h-1}\Big(1-\frac{\delta}{2}\Big)\Big]. \tag{8.39}$$

The above choice of r, α and Step 3 ensure that $\mathbf{H}(Y_0, Y_1, r, \alpha)$ holds.

Step 4c: If $\widehat{\alpha}, \widehat{r}$ are as in Step 4ab, then Theorem 7.1.3(2) holds. By Step 3, Assumption 6.3.2 holds in the $(\widehat{Y}_0, \widehat{Y}_1, \widehat{r}, \widehat{\alpha})$-setting for $\ell \in \{0, \widehat{\alpha}\}$ and Assumption 6.3.4 holds. Moreover, if $q \geq d(h-1)$, then $\mathbf{H}(\widehat{Y}_0, \widehat{Y}_1, \widehat{r}, \widehat{\alpha})$ follows by Step 3 with $\delta = 1$ by noticing that $\frac{1+\widehat{\alpha}}{\widehat{r}} < \frac{1}{2} < \frac{h}{2(h-1)}$. It remains to discuss the case $q < d(h-1)$. Recall that $\frac{1+\widehat{\alpha}}{\widehat{r}} = \frac{\delta-1}{2} + \frac{1}{r}$. The discussion splits into two cases:

- If $q < \frac{d(h-1)}{\delta}$, then by (8.38) we have
$$\frac{1+\widehat{\alpha}}{\widehat{r}} = \frac{\delta-1}{2} + \frac{1}{r} < \frac{\delta-1}{2} + \frac{h}{h-1} - \frac{1}{2}\Big(\delta + \frac{d}{q}\Big) = \frac{h}{h-1} - \frac{1}{2}\Big(1+\frac{d}{q}\Big). \tag{8.40}$$
Thus, $\mathbf{H}(\widehat{Y}_0, \widehat{Y}_1, \widehat{r}, \widehat{\alpha})$ follows by Step 3.

- If $q \in [\frac{d(h-1)}{\delta}, d(h-1))$, then by (8.39) and $q \geq \frac{d(h-1)}{\delta}$, we have
$$\begin{aligned}\frac{1+\widehat{\kappa}}{\widehat{r}} &= \frac{\delta-1}{2} + \frac{1}{r} < \frac{\delta-1}{2} + \frac{h}{h-1}\Big(1-\frac{\delta}{2}\Big)\\ &= \frac{h}{h-1} - \frac{1}{2} - \frac{\delta}{2}\frac{1}{h-1} \leq \frac{h}{h-1} - \frac{1}{2}\Big(1+\frac{d}{q}\Big).\end{aligned} \tag{8.41}$$
Thus, $\mathbf{H}(\widehat{Y}_0, \widehat{Y}_1, \widehat{r}, \widehat{\alpha})$ follows by Step 3.

Since (8.40)-(8.41) holds with strict inequality, $\widehat{Y}^{\mathsf{Tr}}_{\widehat{\alpha},\widehat{r}}$ is not critical for (8.1) in the $(\widehat{Y}_0, \widehat{Y}_1, \widehat{r}, \widehat{\alpha})$-setting. Therefore, Theorem 7.1.3 is applicable and yields the claim of this step.

Step 5: (8.26) holds. By Step 4, it remains to show regularization in space for (u, σ). To this end. It is enough to show the existence of $\mathsf{G} > 0$ and $r \in (2, \infty)$ such that for any $\xi \geq q$ and $r \in [r_0, \infty)$ the following implication holds:

$$u \in \bigcap_{\theta\in[0,1/2)} H^{\theta,r}_{loc}(I_\sigma; H^{1-2\theta,\xi}), \text{ a.s. } \Rightarrow u \in \bigcap_{\theta\in[0,1/2)} H^{\theta,r}_{loc}(I_\sigma; H^{1-2\theta,\xi+\mathsf{G}}), \text{ a.s.}, \tag{8.42}$$

To prove the above implication, Let $\alpha = \alpha(h,d,q) > 0$ and $r_0 = r_0(h,d,q) \in (2,\infty)$ such that

$$\frac{\alpha+1}{r_0} < \frac{h}{h-1} - \frac{1}{2}\Big(1+\frac{d}{q}\Big) \leq \frac{h}{h-1} - \frac{1}{2}\Big(1+\frac{d}{\xi}\Big).$$

By enlarging r_0 if necessary, we may assume that $\frac{\alpha+1}{r_0} < \frac{1}{2}$. Fix $r \geq r_0$ and $\xi \geq q$. As above, we apply Theorem 7.1.3 with $Y_0 = H^{-1,\xi}$, $Y_1 = H^{1,\xi}$, $\widehat{Y}_0 = H^{-1,\xi+\mathsf{G}}$, $\widehat{Y}_1 = H^{1,\xi+\mathsf{G}}$, where G will be chosen later. As in the previous step, we first check condition of Theorem 7.1.3(3). Let us set $\widehat{r} = r$, $\widehat{\alpha} = \alpha$. Thus

$$Y^{\mathsf{Tr}}_r = B^{1-\frac{2}{r}}_{\xi,r} \overset{(i)}{\hookrightarrow} B^{1-2\frac{1+\alpha}{r}}_{\xi+\mathsf{G},r} = Y^{\mathsf{Tr}}_{\widehat{\alpha},\widehat{r}} \tag{8.43}$$

where by Sobolev embedding (i) holds if and only if

$$1 - \frac{2}{r} - \frac{d}{\xi} \geq 1 - 2\frac{1+\alpha}{r} - \frac{d}{\xi+\mathsf{G}} \quad \Leftrightarrow \quad \frac{1}{\xi} - \frac{1}{\xi+\mathsf{G}} \leq \frac{2\alpha}{dr}.$$

Therefore, $\mathsf{G} = \frac{2\alpha}{dr}$, we find that with

$$\frac{1}{\xi} - \frac{1}{\xi+\mathsf{G}} = \frac{\mathsf{G}}{\xi(\xi+\mathsf{G})} \leq \mathsf{G} := \frac{2\alpha}{dr},$$

the embedding (8.43) holds. Finally, note that by Lemma 7.1.2(1) also (7.4) holds. Reasoning as in the previous step, it follows from Step 3 that the above choice of the parameters implies that also the conditions (1)-(2) of Theorem 7.1.3 holds and therefore, (8.42) follows.

Step 6: Proof of the last statement. This follows from (8.26) and Sobolev embeddings. ☐

We complete this subsection by deriving from Proposition 8.2.9 a local existence result for (8.1) in critical spaces.

Corollary 8.2.10 (Local existence in critical spaces I)**.** *Let Assumption 8.2.1(1)-(4) be satisfied for some* $\delta \in [1, \frac{h+1}{h}]$. *Assume that* $h > \mathtt{h}_d$ *(see (8.20)),* $\frac{1}{p} + \frac{1}{2}(\delta + \frac{d}{q}) < \frac{h}{h-1}$. *Suppose that* q *satisfies (8.21) and set*

$$\kappa_{\mathsf{crit}} := p\Big(\frac{h}{h-1} - \frac{1}{2}\Big(\delta + \frac{d}{q}\Big)\Big) - 1. \tag{8.44}$$

Then for any $u_0 \in L^0_{\mathscr{F}_0}(\Omega; B^{\frac{d}{q}-\frac{2}{h-1}}_{q,p}(\mathbb{T}^d))$, *there exists a weak solution* u *to (8.1), and for each localizing sequence* $(\sigma_n)_{n\in\mathbb{N}}$, *a.s. for all* $n \geq 1$,

$$u \in L^p_{\mathrm{loc}}(\overline{I}_{\sigma_n}, w_{\kappa_{\mathsf{crit}}}; H^{2-\delta,q}) \cap C(\overline{I}_{\sigma_n}; B^{\frac{d}{q}-\frac{2}{h-1}}_{q,p}).$$

Moreover u *instantaneously regularizes in space and time, i.e. (8.26)-(8.27) holds.*

Proof. The first claim follows by Proposition 8.2.9 once we checked that we can apply Proposition 8.2.9 for suitable p, q, κ. By $\delta < \frac{h+1}{h}$ and the upper bound in (8.21), one has $q < \frac{d(h-1)}{\delta}$. Therefore, Step 1(5) of the proof of Proposition 8.2.9 applies and κ_{crit} as defined in (8.44) gives equality in (8.29). Note that $\kappa_{\mathsf{crit}} \in (0, \frac{p}{2} - 1)$ holds, since the assumptions imply

$$\frac{1}{p} + \frac{1}{2}\Big(\delta + \frac{d}{q}\Big) < \frac{h}{h-1}, \quad \text{and} \quad \frac{h}{h-1} - \frac{1}{2}\Big(\delta + \frac{d}{q}\Big) < \frac{1}{2}.$$

Moreover, the corresponding trace space becomes

$$X^{\mathsf{Tr}}_{\kappa_{\mathsf{crit}},p} = B^{2-\delta-2\frac{1+\kappa_{\mathsf{crit}}}{p}}_{q,p}(\mathbb{T}^d) = B^{2-\delta-\frac{2h}{h-1}+\delta+\frac{d}{q}}_{q,p}(\mathbb{T}^d) = B^{\frac{d}{q}-\frac{2}{h-1}}_{q,p}(\mathbb{T}^d).$$

□

Part (2) of the next result plays an important role in the results on global existence.

Corollary 8.2.11 (Local existence in critical spaces II)**.** *Let Assumption 8.2.1(1)-(4) be satisfied for some* $\delta_0 \in (1, \frac{h+1}{h}]$. *Assume that (8.23) holds. Then there exists* $\overline{\delta}(h,d) \in (1, \delta_0]$ *such that if* $\delta \in (1, \overline{\delta})$, $\xi := \frac{d}{2}(h-1)$, *and* $\kappa_{p,\delta} := p(1 - \frac{\delta}{2}) - 1$, *then the following hold:*

(1) *For all* $u_0 \in L^0_{\mathscr{F}_0}(\Omega; L^{\xi}(\mathbb{T}^d))$, *there exists a weak solution* (u, σ) *to (8.1) which satisfies (8.25) and (8.26) with* $q = \xi$ *and* $\kappa = \kappa_{p,\delta}$*;*

(2) *If* $h > 1 + \frac{4}{d}$, *then there exists* $\overline{\lambda} > \xi$ *depending only on* $q, h, \overline{\delta}$ *such that for any* $\lambda \in (\xi, \overline{\lambda})$ *the space* $X^{\mathsf{Tr}}_{\kappa_{\lambda,\delta},p} = B^0_{\lambda,\lambda}(\mathbb{T}^d)$ *is not critical for (8.1) in the* $(H^{-\delta,\lambda}(\mathbb{T}^d), H^{2-\delta,\lambda}(\mathbb{T}^d), \lambda, \kappa_{\lambda,\delta})$*-setting.*

Proof. To prove (1)-(2), first note that by (8.23), there exists $\overline{\delta} \in (1, (h+1)/h]$ depending only on d, h, p such that

$$\max\Big\{\frac{d}{d-\overline{\delta}}, \frac{d(h-1)}{2h - \overline{\delta}(h-1)}\Big\} < \xi < \frac{d(h-1)}{h+1-\overline{\delta}(h-1)}, \quad \text{and} \quad p \geq 2/(2-\overline{\delta}). \tag{8.45}$$

(1): Fix $\delta \in (1, \overline{\delta}]$. By the second part of (8.45), $\frac{1}{p} + \frac{1}{2}(\delta + \frac{d}{\xi}) < \frac{h}{h-1}$. The conclusion follows from (8.45) and Corollary 8.2.10 where $\kappa_{\mathsf{crit}} = p(\frac{h}{h-1} - \frac{1}{2}(\delta + \frac{d}{\xi})) - 1 = p(1 - \frac{\delta}{2}) - 1$ (see (8.44)), and since $p \geq \frac{d}{2}(h-1)$),

$$L^{\frac{d}{2}(h-1)}(\mathbb{T}^d) \hookrightarrow B^0_{\frac{d}{2}(h-1),p}(\mathbb{T}^d).$$

(2): Let $\overline{\lambda} > \xi$ be such that (8.45) holds with ξ replaced by $\overline{\lambda}$. If $\delta \in (1, \overline{\delta}]$, then

$$\frac{1 + \kappa_{\lambda,\delta}}{\lambda} = 1 - \frac{\delta}{2} = \frac{h}{h-1} - \frac{1}{2}\Big(\frac{d}{\xi} + \delta\Big) < \frac{h}{h-1} - \frac{1}{2}\Big(\frac{d}{\lambda} + \delta\Big).$$

Therefore, the proof of Corollary 8.2.10 shows that the corresponding trace space

$$X^{\mathsf{Tr}}_{\kappa_{\lambda,\delta},\lambda} = B^{2-\delta-2(1+\kappa_{\lambda,\delta})/\lambda}_{\lambda,\lambda} = B^0_{\lambda,\lambda}$$

is not critical for (8.1). □

8.2.3 Proofs of Theorems 8.2.4, 8.2.5 and Corollary 8.2.6

Energy estimates for (8.1)

In this subsection, we establish energy estimates for solutions (8.1). If $\delta > 1$, then the regularity of u is insufficient at $t = 0$. To overcome this difficulty, we exploit the regularization of weak solutions to (8.1) and therefore giving estimates away from $t = 0$.

Let (u, σ) be the weak solution to (8.1) provided by Proposition 8.2.9. By (8.27), one has $\mathbf{1}_{\{\sigma>s\}} u(s) \in L^0_{\mathscr{F}_0}(\Omega; C^\eta(\mathbb{T}^d))$ for any $s > 0$ and $\eta \in (0, 1)$. Thus, for any $s > 0$ and $k \geq 1$, the following set is well-defined

$$\Gamma_{s,k} := \{\sigma > s \,,\, \|u(s)\|_{C(\mathbb{T}^d)} \leq k\} \in \mathscr{F}_s \tag{8.46}$$

and

$$\mathbb{P}\big(\{\sigma > s\} \setminus (\cup_{k\geq 1} \Gamma_{s,k})\big) = 0. \tag{8.47}$$

The last ingredient in the proof of Theorem 8.2.4 is the following result.

Lemma 8.2.12 (Energy estimates for (8.1))**.** *Let Assumption 8.2.1 be satisfied. Assume that $u_0 \in L^0_{\mathscr{F}_0}(\Omega; B^{2-\delta-2\frac{1+\kappa}{p}}_{q,p}(\mathbb{T}^d))$, and let (u, σ) be the weak solution to (8.1) provided by Proposition 8.2.9. Let $\Lambda > \max\{\frac{d}{2}(h-1), 2\}$ be as in Assumption 8.2.1(5). Fix $s > 0$, $k \geq 1$ and let $\Gamma_{s,k}$ be as in (8.46). Then, for each $\lambda \in [2, \Lambda)$ there exists C depending only on $c, \nu, \gamma, C_b, M, h, \Lambda, \lambda$ (see Assumption 8.2.1(5)) such that for all $t \in [s, \infty)$,*

$$\mathbb{E}\Big[\mathbf{1}_{\Gamma_{s,k}} \sup_{r\in[s,\sigma\wedge t)} \|u(r)\|^\lambda_{L^\lambda(\mathbb{T}^d)}\Big] \leq Ce^{C(t-s)}\Big(1 + t - s + \mathbb{E}[\mathbf{1}_{\Gamma_{s,k}} \|u(s)\|^\lambda_{L^\lambda(\mathbb{T}^d)}]\Big). \tag{8.48}$$

Proof. Since u is regular by (8.27), we can define

$$\tau_n = \inf\{t \in [s, \sigma) \,:\, \|u(t) - u(s)\|_{C(\mathbb{T}^d)} + \|u\|_{L^2(s,t;H^{1,2}(\mathbb{T}^d))} \geq n\} \wedge n, \tag{8.49}$$

if $\sigma > s$, and $\tau_n = s$ if $\sigma \leq s$. Here, as usual, we set $\inf \varnothing := \sigma$. Then each τ_n is a stopping time and by (8.27), we have $\lim_{n\uparrow\infty} \tau_n = \sigma$ a.s. on $\{\sigma > s\}$.

By Assumption 8.2.1(5), for each $\lambda \in (\max\{\frac{d}{2}(h-1), 2\}, \Lambda)$, there exist $\theta \in (0, 1)$, $M' > 0$, $\varepsilon_i \in (0, \nu)$ (for $i \in \{0, 1\}$), a.s. for all $t \in [0, \infty)$, $x \in \mathbb{T}^d$, $y \in \mathbb{R}$,

$$\varepsilon_1 + \varepsilon_2 < \theta, \tag{8.50}$$

$$\frac{|\Psi(t, x, y)|^2}{4\varepsilon_1} + \frac{1}{2}\Big(1 + \frac{C_b^2}{4\varepsilon_2}\Big)\|\Phi(t, x, y)\|^2_{\ell^2} \leq M'(1 + |y|^2) - \frac{\theta}{\lambda - 1}\psi(t, x, y)y. \tag{8.51}$$

All the constants in the proof below will depend on $\nu, \theta, \varepsilon_1, \varepsilon_2, C_b, M, \Lambda, \lambda, h$.

It suffices to prove (8.49) with $\Gamma_{s,k}$ replaced by

$$\Gamma^n_{s,k} := \{\tau_n > s \,,\, \|u(s)\|_{C(\mathbb{T}^d)} \leq k\} \in \mathscr{F}_s. \tag{8.52}$$

To abbreviate the notation and since k and n are fixed we will write $\Gamma := \Gamma^n_{s,k}$ and $\tau = \tau_n$ and denote

$$y(r) = \mathbf{1}_\Gamma \sup_{\eta\in[s,r\wedge\tau)} \|u(\eta)\|^\lambda_{L^\lambda(\mathbb{T}^d)}$$

By Gronwall's lemma and since $\tau \leq n$, it is enough to show, that there is a constant $C > 0$ such that

$$\mathbb{E}y(t) \leq C(1 + t - s + \mathbb{E}y(s)) + C\int_s^t \mathbb{E}y(r)\, dr, \quad t \in [s, n]. \tag{8.53}$$

The proof of (8.53) will be given in the next steps.

Step 1: We apply the Itô formula of Lemma 8.1.1 and the stochastic parabolicity to derive the a priori estimate (8.57) below.

By Proposition 6.2.7 and the fact that $(u|_{[s,\sigma)\times\Gamma}, \sigma\mathbf{1}_\Gamma + s\mathbf{1}_{\Omega\setminus\Gamma})$ is a weak solution to (8.1) on $[s,n]$, we obtain $\mathbf{1}_\Gamma u = v$ on $[s,\tau)$, where $v:[s,n]\times\Omega\to H^{1,2}(\mathbb{T})$ is a weak solution to

$$dv - Av\,dr = f^u\,dr + (Bv+g^u)\,dW_{\ell^2},\quad v(s)=\mathbf{1}_\Gamma u(s).$$

In the above A and B are given by (8.17), and by the definitions of τ and Γ,

$$\begin{aligned} f^u &= \mathbf{1}_{\Gamma\times[s,\tau)}\mathrm{div}(\Psi(\cdot,u)) + \mathbf{1}_{\Gamma\times[0,\tau)}\psi(\cdot,u)\in L^2_{\mathscr{P}}((s,n)\times\Omega;H^{-1,r}(\mathbb{T}^d))\\ g^u &= (\mathbf{1}_{\Gamma\times[s,\tau)}\Phi_n(\cdot,u))_{n\geq 1}\in L^2_{\mathscr{P}}((s,n)\times\Omega;L^r(\mathbb{T}^d;\ell^2)),\end{aligned}$$

for all $r\in(2,\infty)$.

Moreover, by Proposition 8.2.9 $\mathbf{1}_\Gamma u(s)\in L^2(\Omega;C(\mathbb{T}^d))\hookrightarrow L^2(\Omega;L^2(\mathbb{T}^d))$. Therefore, since it is classical that $(A,B)\in\mathcal{SMR}_2^\bullet(s,n)$ on the space $\widetilde{X}_k = H^{-1+2k,2}(\mathbb{T}^d)$ for $k\in\{0,1\}$ (see [148, Theorem 5.1.3]) we obtain that

$$v\in L^2_{\mathscr{P}}((s,n)\times\Omega;H^{1,2}(\mathbb{T}^d))\cap L^2(\Omega;C([s,n];L^2(\mathbb{T}^d))).$$

Therefore, it follows that (defining $u(\tau)=v(\tau)$ on Γ),

$$\mathbf{1}_\Gamma u\in L^2_{\mathscr{P}}([\![s,\tau)\!);H^{1,2}(\mathbb{T}^d))\cap L^2(\Omega;C([s,\tau];L^2(\mathbb{T}^d)))\tag{8.54}$$

and for $t\in[s,\tau]$,

$$\begin{aligned}\mathbf{1}_\Gamma u(t)-\mathbf{1}_\Gamma u(s) &= \int_s^t \mathbf{1}_{[s,\mu)\times\Gamma}\big[\mathrm{div}(a\cdot\nabla u)+\mathrm{div}(\Psi(\cdot,u))+\psi(\cdot,u)\big]\,dr\\ &\quad+\sum_{n\geq 1}\int_s^t \mathbf{1}_{[t,\mu)\times\Gamma}(b_n\cdot\nabla u+\Phi_n(\cdot,u))\,dw_r^n.\end{aligned}\tag{8.55}$$

Moreover, by (8.27), $\mathbf{1}_\Gamma u\in L^2(\Omega;C([s,\tau]\times\mathbb{T}^d))$ and by (8.49) and (8.52),

$$|u(t,x)|\leq n+k,\qquad \text{a.s. on }\Gamma,\ t\in[s,\tau),x\in\mathbb{T}^d.\tag{8.56}$$

After these preparation, we will apply Itô formula of Lemma 8.1.1 to rewrite $\phi_\lambda(\mathbf{1}_\Gamma u(t))$, where $\phi_\lambda(x)=\frac{1}{\lambda}|x|^\lambda$. Note, ϕ_λ does not have bounded second order derivatives as required. However, by (8.56), it is enough to apply the above mentioned Itô formula to a modification of ϕ_λ outside the interval $[-n-k-1,n+k+1]$. For instance, it is enough to take $\phi:=f_{n+k}(\phi_\lambda)$ where $f_{n+k}\in C^2(\mathbb{R})$, $f_{n+k}|_{[-n-k,n+k]}=\mathrm{Id}$ (identity on $\mathbb{R}$) and it is constant outside $[-n-k-1,n+k+1]$. In this form we can apply Lemma 8.1.1 (using (8.54) and (8.55)) to obtain a.s. for all $t\in[s,\tau]$

$$\frac{1}{\lambda}\|u(t)\|^\lambda_{L^\lambda}+\nu(\lambda-1)\mathscr{J}_t+\mathscr{K}_t\leq I_t+(\lambda-1)II_t+III_t,\tag{8.57}$$

where we have used Assumption 8.2.1(3), and we have set

$$\begin{aligned}\mathscr{J}_t &:= \int_s^t\int_{\mathbb{T}^d}\mathbf{1}_{[t,\tau)\times\Gamma}|u|^{\lambda-2}|\nabla u|^2\,dx\,dr\geq 0,\\ \mathscr{K}_t &:= \int_s^t\int_{\mathbb{T}^d}\mathbf{1}_{[t,\tau)\times\Gamma}|u|^{\lambda-2}\Big(-\psi(\cdot,u)u+c(1+|u|^2)\Big)\,dx\,dr\geq 0.\\ I_t &:= \mathbf{1}_\Gamma\frac{1}{\lambda}\|u(s)\|^\lambda_{L^\lambda}+c\int_s^t\int_{\mathbb{T}^d}\mathbf{1}_{[t,\tau)\times\Gamma}|u|^{\lambda-2}(1+|u|^2)\,dx\,dr,\\ II_t &:= \int_s^t\int_{\mathbb{T}^d}\mathbf{1}_{[t,\tau)\times\Gamma}|u|^{\lambda-2}\Big[-\Psi(\cdot,u)\nabla u\\ &\qquad+\frac{1}{2}\sum_{n\geq1}\Big(|\Phi_n(\cdot,u)|^2+2(b_n\cdot\nabla u)\Phi_n(\cdot,u)\Big)\Big]\,dx\,dr,\end{aligned}$$

$$III_t := \sum_{n\geq 1}\int_s^t \int_{\mathbb{T}^d} \mathbf{1}_{[t,\tau)\times\Gamma}|u|^{\lambda-2}(b_n\cdot\nabla u + \Phi_n(\cdot,u))u\,dx\,dw_r^n,$$

where $c := M(\Lambda-1)$ and the positivity of $\mathscr{K}$ follows by Assumption 8.2.1(5).

Step 2: There exists a constant $M_1>0$ such that for all $t\in(s,n]$

$$\mathbb{E}\mathscr{J}_t + \mathbb{E}\mathscr{K}_t \leq M_1\Big(t-s+\mathbb{E}\mathbf{1}_\Gamma\|u(s)\|^\lambda_{L^\lambda} + \mathbb{E}\int_s^t\int_{\mathbb{T}^d}\mathbf{1}_{[t,\tau)\times\Gamma}(r)|u(r)|^\lambda dx\,dr\Big). \tag{8.58}$$

To prove this result we will estimate the right-hand side of (8.57) and take expectations. For term I_η using $|u|^{\lambda-2}+|u|^\lambda\leq 1+2|u|^\lambda$, we can estimate

$$\begin{aligned}|I_t| &\leq \frac{1}{\lambda}\mathbf{1}_\Gamma\|u(s)\|^\lambda_{L^\lambda} + c\int_s^t\int_{\mathbb{T}^d}\mathbf{1}_{[s,\tau)\times\Gamma}(|u|^{\lambda-2}+|u|^\lambda)\,dx\,dr\\ &\leq \frac{1}{\lambda}\mathbf{1}_\Gamma\|u(s)\|^\lambda_{L^\lambda} + c(t-s) + 2c\int_s^t\int_{\mathbb{T}^d}\mathbf{1}_{[s,\tau)\times\Gamma}|u|^\lambda\,dx\,dr\end{aligned} \tag{8.59}$$

To estimate II note that

$$\begin{aligned}&\Big|-\Psi(\cdot,u)\nabla u + \frac12\sum_{n\geq1}\Big(|\Phi_n(\cdot,u)|^2 + 2(b_n\cdot\nabla u)\Phi_n(\cdot,u)\Big)\Big|\\ &\qquad\leq |\Psi(\cdot,u)||\nabla u| + \frac12\|\Phi(\cdot,u)\|^2_{\ell^2} + C_b|\nabla u|\|\Phi(\cdot,u)\|_{\ell^2}\\ &\qquad\leq (\varepsilon_1+\varepsilon_2)|\nabla u|^2 + \frac{1}{4\varepsilon_1}|\Psi(\cdot,u)|^2 + \frac12\Big(1+\frac{C_b^2}{4\varepsilon_2}\Big)\|\Phi(\cdot,u)\|^2_{\ell^2}\\ &\qquad\leq (\varepsilon_1+\varepsilon_2)|\nabla u|^2 + M'(1+|u|^2) - \frac{\theta}{\lambda-1}\psi(t,x,u)u =: Q(u)\end{aligned}$$

where the last estimate follows form (8.51).

Therefore, a.s. for all $0<s<t$,

$$\begin{aligned}|II_t| &\leq \int_s^t\int_{\mathbb{T}^d}\mathbf{1}_{[t,\tau)\times\Gamma}|u|^{\lambda-2}Q(u)dx\,dr\\ &\leq (\varepsilon_1+\varepsilon_2)\mathscr{J}_t + \frac{\nu}{\lambda-1}\mathscr{K}_t + M'\Big(t-s+\int_s^t\int_{\mathbb{T}^d}\mathbf{1}_{[s,\tau)\times\Gamma}|u|^\lambda\,dx\,dr\Big),\end{aligned} \tag{8.60}$$

Omitting the term $\frac{1}{\lambda}\|u(t)\|^\lambda_{L^\lambda}$, (8.57) holds for all $t\in[s,n]$, because all terms are constant on $[\tau,n]$. Therefore, taking expectations in (8.57) and using (8.59), (8.60) and $\mathbb{E}[III_t]=0$, we obtain

$$\begin{aligned}\nu(\lambda-1)\mathbb{E}\mathscr{J}_t + \mathbb{E}\mathscr{K}_t &\leq \frac{1}{\lambda}\mathbb{E}\mathbf{1}_\Gamma\|u(s)\|^\lambda_{L^\lambda} + (\varepsilon_1+\varepsilon_2)(\lambda-1)\mathbb{E}\mathscr{J}_t\\ &\qquad + \nu\,\mathbb{E}\mathscr{K}_t + \widetilde{M}\Big(t-s+\mathbb{E}\int_s^t\int_{\mathbb{T}^d}\mathbf{1}_{[s,\tau)\times\Gamma}|u|^\lambda dx\,dr\Big).\end{aligned}$$

Using (8.50) and the fact that $\nu<1$, one obtains (8.58).

Step 3: Proof of the estimate (8.53).

By (8.58), (8.59), and (8.60), we can find $\widehat{M}$ such that

$$\begin{aligned}&\mathbb{E}\Big[\sup_{\eta\in[s,t]}|I_\eta|\Big] + \mathbb{E}\Big[\sup_{\eta\in[s,t]}|II_\eta|\Big]\\ &\qquad\leq \widehat{M}\Big(t-s+\mathbb{E}\mathbf{1}_\Gamma\|u(s)\|^\lambda_{L^\lambda} + \mathbb{E}\int_s^t\int_{\mathbb{T}^d}\mathbf{1}_{[s,\tau)\times\Gamma}|u|^\lambda\,dx\,dr\Big)\\ &\qquad\leq \widehat{M}\Big(t-s+\mathbb{E}y(s) + \mathbb{E}\int_s^t y(r)\,dr\Big).\end{aligned} \tag{8.61}$$

To estimate III, by the scalar-valued Burkholder-Davis-Gundy inequality,

$$\mathbb{E}\Big[\sup_{\eta\in[s,t]}|III_\eta|\Big] \leq C_1\mathbb{E}\Big[\Big(\int_s^t \zeta(r)\,dr\Big)^{1/2}\Big],$$

where ζ satisfies (by the Cauchy-Schwarz inequality applied with $|u|^{\lambda-1} = |u|^{\frac{\lambda}{2}}|u|^{\frac{\lambda-2}{2}}$)

$$\begin{aligned}
\zeta(r) &:= \sum_{n\geq 1}\Big|\int_{\mathbb{T}^d} \mathbf{1}_{[s,\tau)\times\Gamma}(r)|u(r)|^{\lambda-1}(|b_n(r)\cdot\nabla u(r)| + |\Phi_n(r,u(r))|)\,dx\Big|^2\\
&\leq \|\mathbf{1}_{[s,\tau)\times\Gamma}(r)u(r)\|_{L^\lambda}^{\lambda}\int_{\mathbb{T}^d}\mathbf{1}_{[s,\tau)\times\Gamma}(r)|u(r)|^{\lambda-2}\sum_{n\geq 1}(|b_n(r)\cdot\nabla u(r)| + |\Phi_n(r,u(r))|)^2\,dx\\
&\leq y(t)\int_{\mathbb{T}^d}\mathbf{1}_{[s,\tau)\times\Gamma}(r)|u(r)|^{\lambda-2}2\sum_{n\geq 1}|b_n(r)|^2|\nabla u(r)|^2 + |\Phi_n(r,u(r))|^2\,dx.\\
&\leq 2y(t)\int_{\mathbb{T}^d}\mathbf{1}_{[s,\tau)\times\Gamma}(r)|u(r)|^{\lambda-2}(C_b^2|\nabla u(r)|^2 + \|\Phi(r,u(r))\|_{\ell^2}^2)\,dx,
\end{aligned}$$

for $r\in[s,t]$. Therefore, setting $\varepsilon_3 := \frac{1}{2\sqrt{2}C_1\lambda}$ we find

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{\eta\in[s,t]}|III_\eta|\Big]\\
&\leq \sqrt{2}C_1\,\mathbb{E}\Big[y^{1/2}(t)\int_s^t\int_{\mathbb{T}^d}\mathbf{1}_{[s,\tau)\times\Gamma}|u|^{\lambda-2}(C_b^2|\nabla u|^2 + \|\Phi(\cdot,u)\|_{\ell^2}^2)\,dxdr\Big)^{1/2}\Big]\\
&\overset{(i)}{\leq} \sqrt{2}C_1\Big(\varepsilon_3\mathbb{E}y(t) + \frac{1}{4\varepsilon_3}\mathbb{E}\Big[\int_s^t\int_{\mathbb{T}^d}\mathbf{1}_{[s,\tau)\times\Gamma}|u|^{\lambda-2}(C_b^2|\nabla u|^2 + \|\Phi(\cdot,u)\|_{\ell^2}^2)\,dxdr\Big]\Big)\\
&\overset{(ii)}{\leq} \frac{1}{2\lambda}\mathbb{E}y(t) + M_3\Big(t-s+\mathbb{E}\mathscr{J}_t + \mathbb{E}\mathscr{K}_t + \mathbb{E}\int_s^t\int_{\mathbb{T}^d}\mathbf{1}_{[s,\tau)\times\Gamma}|u|^\lambda dx\,dr\Big)\\
&\overset{(iii)}{\leq} \frac{1}{2\lambda}\mathbb{E}y(t) + M_4\Big(t-s+\mathbb{E}y(s) + \int_s^t\mathbb{E}y(r)\,dr\Big)
\end{aligned}$$

where M_3 and M_4 are constants. In (i) we used that $|ab|\leq\varepsilon_3 a^2 + \frac{1}{4\varepsilon_3}b^2$, in (ii) we used (8.51), and in (iv) we used (8.58).

Taking the supremum in (8.57) and using the estimates for I-III, we obtain

$$\frac{1}{\lambda}\mathbb{E}y(t) = \frac{1}{\lambda}\mathbb{E}\sup_{\eta\in[s,t\wedge\tau)}\|u(\eta)\|_{L^\lambda}^\lambda \leq M_5\Big(t-s+\mathbb{E}y(s) + \mathbb{E}\int_s^t y(r)\,dr\Big) + \frac{1}{2\lambda}\mathbb{E}y(t),$$

where M_5 is a constant. The latter implies which implies (8.53). ☐

Remark 8.2.13. The proof of Lemma 8.2.12 uses only four ingredients: the Itô formula, the uniform parabolicity (i.e. Assumption 8.2.1(3)), the dissipative conditions Assumption 8.2.1(5) and the instantaneous regularization (8.26).

The Itô formula also holds for equations on domains with Dirichlet boundary (see Lemma 8.1.1). Therefore, by the previous consideration the estimate (8.48) holds for weak solutions to quasilinear SPDEs in divergence form as (8.66) provided an uniform parabolicity is assumed and Assumption 8.2.1(5) hold for (8.66). This will be used in Section 8.3 in the case $b = 0$. Note that in case $b\neq 0$ and Dirichlet boundary conditions, local existence does not hold under the stated stochastic parabolicity condition of Assumption 8.2.1(3).

Proof of the main results

As the proofs of Corollaries 8.2.10-8.2.11 show, Theorem 8.2.5 and Corollary 8.2.6 follow from Theorem 8.2.4 with an appropriate choice of the parameters p,κ,q. Therefore, it suffices to prove Theorem 8.2.4.

Proof of Theorem 8.2.4. Let (u,σ) be the weak solution provided by Proposition 8.2.9.

For the reader's convenience, we split the proof as follows: In Step 1, we prove that (u,σ) is global in time (i.e. $\sigma=\infty$ a.s.) under additional restrictions on q,p,κ,δ,h. In Step 2 we remove these restrictions by using the regularization results of Section 7.1. As soon as we know that $\sigma=\infty$, (8.18)-(8.19) are immediate from Proposition 8.2.9.

Let $T\in(0,\infty)$. Replacing (u,σ) by $(u|_{[\![0,\sigma\wedge T]\!)},\sigma\wedge T)$ it suffices to consider weak solutions on $[0,T]$ and to show $\sigma=T$ a.s. below.

Step 1: Assume that $h>1+\frac{4}{d}$ and let Λ be as in Assumption 8.2.1(5). Let $\bar{\delta}>1$, $\bar{\lambda}>\xi$ be as in Corollary 8.2.11(2). Fix $\lambda\in(\xi,\bar{\lambda})$ where $\xi:=\frac{d}{2}(h-1)>2$ and assume that $q=\lambda$, $p=\lambda$, $\delta\in(1,\bar{\delta}]$ and $\kappa=\lambda(1-\frac{\delta}{2})-1$. Then $\sigma=T$.

In order to prove the result we will use the blow-up criterium of Theorem 6.3.7(3). To this end, we collect some useful facts. To begin, recall that, by Corollary 8.2.11(2) the space $B^0_{\lambda,\lambda}$ is *not* critical for (8.1) in the $(H^{-\delta,\lambda}(\mathbb{T}^d),H^{2-\delta,\lambda}(\mathbb{T}^d),\lambda,\kappa_{\lambda,\delta})$-setting. By Proposition 8.2.9, for any localizing sequence $(\sigma_n)_{n\geq1}$, a.s. for all $n\in\mathbb{N}$ (see (8.25))

$$u\in C(\overline{I}_{\sigma_n};B^0_{\lambda,\lambda}(\mathbb{T}^d)). \tag{8.62}$$

Since $\lambda>2$, one has $L^\lambda(\mathbb{T}^d)\hookrightarrow B^0_{\lambda,\lambda}(\mathbb{T}^d)$ and thus Lemma 8.2.12 implies

$$\|u\|_{L^\infty(s,\sigma;B^0_{\lambda,\lambda}(\mathbb{T}^d))}<\infty \qquad \text{a.s. on } \Gamma_{s,k}, \text{ for all } s>0 \text{ and } k\geq1, \tag{8.63}$$

where $\Gamma_{s,k}=\{\sigma>s\,,\,\|u(s)\|_{C(\mathbb{T}^d)}\leq k\}$ (see (8.46)). Combining (8.62) and (8.63) we obtain

$$\|u\|_{L^\infty(0,\sigma;B^0_{\lambda,\lambda}(\mathbb{T}^d))}<\infty \qquad \text{a.s. on } \Gamma_{s,k}, \text{ for all } s>0 \text{ and } k\geq1, \tag{8.64}$$

Therefore, for any $s>0$,

$$\begin{aligned}\mathbb{P}(s<\sigma<T) &\overset{(8.47)}{=} \lim_{k\uparrow\infty}\mathbb{P}\Big(\{s<\sigma<T\}\cap\Gamma_{s,k}\cap\{\|u\|_{L^\infty(s,\sigma;B^0_{\lambda,\lambda})}<\infty\}\Big)\\ &\overset{(8.64)}{=} \lim_{k\uparrow\infty}\mathbb{P}\Big(\{s<\sigma<T\}\cap\Gamma_{s,k}\cap\{\|u\|_{L^\infty(I_\sigma;B^0_{\lambda,\lambda})}<\infty\}\Big)\\ &\leq \mathbb{P}(\sigma<T\,,\,\|u\|_{L^\infty(I_\sigma;B^0_{\lambda,\lambda})}<\infty)=0,\end{aligned}$$

where the last identity follows from Theorem 6.3.7(3) and the fact that $B^0_{\lambda,\lambda}$ is *not* critical for (8.1). Since $\sigma>0$ a.s. by Proposition 8.2.9, the previous yields

$$\mathbb{P}(\sigma<T)=\lim_{s\downarrow0}\mathbb{P}(s<\sigma<T)=0. \tag{8.65}$$

Therefore $\sigma=T$ a.s., as desired.

Step 2: Next we show that $\sigma=T$ a.s. in the general case. We reduce to Step 1 using the regularization effect of Proposition 8.2.9 and Lemma 7.1.9.

By Assumption 8.2.1(5) we can choose (h^*,Λ^*) such that such that $h^*>\max\{h,1+\frac{4}{d}\}$ and $\Lambda^*\in(\frac{d}{2}(h^*-1),\Lambda)$. Then Assumption 8.2.1(5) holds with (h,Λ) replaced by (h^*,Λ^*). Moreover, since $h^*>h$ also Assumption 8.2.1(4) is satisfied. Moreover, set $\xi^*=\frac{d}{2}(h^*-1)>2$ and let $\bar{\delta}^*,\bar{\lambda}^*$ be as in Corollary 8.2.11(2) with h replaced by h^*. Fix $\lambda^*\in(\xi^*,\bar{\lambda}^*)$, $\delta^*\in(1,\bar{\delta}^*]$ and set $\kappa^*:=\lambda^*(1-\frac{\delta^*}{2})-1$. By Step 1 applied with $(\Lambda^*,\lambda^*,h^*,\delta^*)$ as before, the assumptions of Lemma 7.1.9 hold and the conclusion follows. □

8.2.4 Discussion and further extensions

Comparison with the literature

Compared to the existing literature on reaction-diffusion equations our setting is more flexible:

- local/global well-posedness in critical spaces with negative smoothness (see Remark 8.2.7),

- instantaneous regularization of solutions (see (8.18)-(8.19))
- gradient noise term $b \cdot \nabla u$

Further regularity can be obtained by the same methods (if the coefficients and nonlinearities have additional smoothness) or by standard bootstrapping techniques. The most important step is to go from Sobolev regularity to Hölder regularity. Gradient noise in (8.1) appears for instance in Allen-Cahn equations [99, 184] (see also Example 8.2.2).

Systems of reaction diffusion equations

Although we have only considered a single SPDE, all our results have a suitable extension to systems of reaction diffusion equations, where in case $b \neq 0$ one has to assume a certain diagonal structure (see [174, Section 5]), and where the condition Assumption 8.2.1(5) becomes dependent on the number of coupled equations in the system (the suboptimal case in (8.16) is still admissible). Reaction diffusion systems have been treated in many papers (e.g. [35, 36, 136] and references therein) under more restrictive conditions on the initial data, on the nonlinearities, and with $b = 0$. Let us note that in the case of non-periodic boundary conditions one needs additional conditions on b.

Conservative reaction diffusion equations

The problem (8.1) does not preserve the 'mass', i.e. $\mathbb{E}\int_{\mathbb{T}^d} u(t,x)dx$ unless $\psi \equiv 0$. In several situations (see e.g. [13, 87]) the term $\psi(\cdot,u)$ in (8.1) is replaced by $\psi(\cdot,u) - \int_{\mathbb{T}^d} \psi(\cdot,u(\cdot,x))dx$ in order to preserve the mass. We expect that our arguments can be adjusted to also cover this case. It is clear that Proposition 8.2.9 extends to such situation. The energy estimates in Lemma 8.2.12 require additional work, which will not be pursued here.

On the number $\mathtt{h}_d$

If $d \geq 3$, then the definition of $\mathtt{h}_d$ in (8.20) reminds us the 'Fujita' exponent $1 + \frac{2}{d} < \mathtt{h}_d$ introduced in [86] in the study of *blowing-up* of positive (smooth) solutions to the PDE: $\partial_t u - \Delta u = u^{1+h}$. This similarity is not only formal. Indeed, in [6] we will show global existence (in probability) of solutions to (8.1) for small initial data without the sign-conditions of Assumption 8.2.1(5). Therefore one can allow nonlinearities as in [86] for $h > \mathtt{h}_d$. Such a threshold $\mathtt{h}_d$ seems optimal for these results to hold in presence of a non-trivial gradient noise term, i.e. $b \cdot \nabla u$. Recently, there has been an increasing attention in extending [86] to the stochastic framework (see e.g. [41, 42, 85] and the references therein). However, unlike in our setting, global solutions are required to be integrable in Ω in the previous mentioned references. Moreover, gradient-noise is not considered in these works.

8.3 Quasilinear SPDEs in divergence form

In this section we study global well-posedness of a class of stochastic quasilinear SPDEs in divergence form

$$\begin{cases} du - \operatorname{div}(a(\cdot,u)\cdot\nabla u)\,dt = (\operatorname{div}(\Psi(\cdot,u)) + \psi(\cdot,u))\,dt \\ \qquad\qquad\qquad\qquad + \sum_{n\geq 1} \Phi_n(\cdot,u)\,dw_t^n, & \text{on } \mathscr{O}, \\ u = 0, & \text{on } \partial\mathscr{O}, \\ u(0) = u_0, & \text{on } \mathscr{O}, \end{cases} \tag{8.66}$$

where $u : [0,\infty) \times \Omega \times \mathscr{O} \to \mathbb{R}$ is the unknown process.

The results presented in this section also holds in the case that $\mathscr{O}$ is replaced by $\mathbb{T}^d$ ignoring the boundary condition in (8.66). For the sake of brevity, we do not give any explicit statement.

8.3.1 Main results

In this section and the following ones, for any Banach space E and $T \in (0,\infty]$, we say that a map $\Theta : I_T \times \Omega \times \mathscr{O} \times \mathbb{R} \to E$ is *locally Lipschitz in y uniformly w.r.t. (t,ω,x)* if the following holds: $\Theta(\cdot,0) \in L^\infty(I_T \times \Omega \times \mathscr{O}; E)$, and for each $n > 0$ there exists C_n such that, a.s. for all $t \in I_T$, $x \in \mathscr{O}$ and $|y|, |y'| < n$,

$$\|\Theta(t,x,y) - \Theta(t,x,y')\|_E \le C_n|y - y'|. \tag{8.67}$$

For a measurable and bounded function $\phi : \mathbb{R}_+ \times \mathscr{O} \times \mathbb{R} \to \mathbb{R}$ we set

$$\mathrm{osc}^{r,n}_{t,x}(\phi) := \fint_{t-r^2}^{t} \fint_{B_{\mathbb{R}^d}(x,r)\cap\mathscr{O}} \Big(\sup_{|y|\le n} \Big|\phi(s,\xi,y) - \fint_{B_{\mathbb{R}^d}(x,r)\cap\mathscr{O}} \phi(s,\xi',y)d\xi'\Big|\Big) d\xi\, ds.$$

The following is in force throughout this section.

Assumption 8.3.1. *Suppose $d \ge 1$ and that the following hold:*

(1) *$q \in [2,\infty)$, $p \in (2,\infty)$ and $\kappa \in [0, \frac{p}{2} - 1)$ satisfy $1 - 2\frac{1+\kappa}{p} > \frac{d}{q}$.*

(2) *$\mathscr{O} \subset \mathbb{R}^d$ is a bounded C^1-domain.*

(3) *$a := (a^{i,j})^d_{i,j=1}$, $\Psi := (\Psi^j)^d_{j=1}$ and for any $i,j \in \{1,\dots,d\}$, the maps $a^{i,j}, \Psi^j, \psi : \mathbb{R}_+ \times \Omega \times \mathscr{O} \times \mathbb{R} \to \mathbb{R}$, $\Phi = (\Phi_n) : \mathbb{R}_+ \times \Omega \times \mathscr{O} \times \mathbb{R} \to \ell^2$ are $\mathscr{P} \otimes \mathscr{B}(\mathscr{O}) \otimes \mathscr{B}(\mathbb{R})$-measurable and locally Lipschitz in y uniformly w.r.t. (t,ω,x).*

(4) *$a \in L^\infty(\mathbb{R}_+ \times \Omega \times \mathscr{O} \times \mathbb{R}; \mathbb{R}^{d\times d})$, and for each $n \ge 1$, $i,j \in \{1,\dots,d\}$,*

$$\lim_{R\downarrow 0} \Big(\operatorname*{ess\,sup}_{(t,\omega,x)\in\mathbb{R}_+\times\Omega\times\mathscr{O}} \sup_{r\le R} \mathrm{osc}^{r,n}_{t,x}(a^{i,j}(\cdot,\omega,\cdot)) \Big) = 0$$

(5) *a is* uniformly *elliptic, i.e. there exist $K, \nu > 0$ such that*

$$\sum_{i,j=1}^{d} a^{i,j}(t,x,y)\xi_i\xi_j \ge \nu|\xi|^2, \qquad \text{a.s. for all } t \in \mathbb{R}_+,\ x \in \mathscr{O},\ y \in \mathbb{R} \text{ and } \xi \in \mathbb{R}^d.$$

(6) *There exist $h \in (1,\infty)$, $C > 0$ such that a.s. for all $t \in \mathbb{R}_+$, $x \in \mathbb{T}^d$ and $y \in \mathbb{R}$,*

$$|\psi(t,x,y)| \le C(1 + |y|^h),$$
$$|\Psi(t,x,y)| + \|(\Phi_n(t,x,y))_{n\ge1}\|_{\ell^2} \le C(1 + |y|^{\frac{h+1}{2}}).$$

(7) *There exist $\Lambda > \max\{\frac{d}{2}(h+1), 2\}$ and $M > 0$ such that, a.s. for all $t \in \mathbb{R}_+$, $x \in \mathbb{T}^d$ and $y \in \mathbb{R}$,*

$$\frac{1}{4\nu}|\Psi(t,x,y)|^2 + \frac{1}{2}\|\Phi(t,x,y)\|^2_{\ell^2} \le M(1 + |y|^2) - \frac{1}{\Lambda - 1}\psi(t,x,y)y.$$

In Example 8.2.2, we have seen that Assumption 8.3.1(3), (6),(7) hold for a wide class of nonlinearities including the usual Allen-Cahn one $\psi(\cdot,y) = y - y^3$ for $y \in \mathbb{R}$. Note that we do not allow a gradient noise term in the above. As noticed in Remark 8.2.3, from a physically point of view, Assumption 8.3.1(7) say that the term $\psi(\cdot,u)u$ represents a dissipation in the underlined model represented and that such dissipation is stronger than the action of $\Psi(\cdot,u)$ and $\Phi(\cdot,u)$. For (8.66) we need stronger assumptions on the initial data than the one used for (8.1).

Assumption 8.3.1(4) allow the coefficients $a^{i,j}$ to be merely measurable in time and VMO in space (see e.g. [132]). Note that, if $a^{i,j}$ satisfies

$$|a^{i,j}(t,x,y) - a^{i,j}(t,x',y)| \le \Theta_n(|x - x'|) \tag{8.68}$$

a.s. for all $n \geq 1$, $t \in I_T$, $x, x' \in \mathscr{O}$, $|y| \leq n$ where $\Theta_n \in C([0,\infty))$ with $\lim_{\delta \downarrow 0} \Theta_n(\delta) = 0$, then $a^{i,j}$ satisfies Assumption 8.3.1(4). However, the condition in Assumption 8.3.1(4) does not imply continuity in the x-variable.

Before stating the main result of this subsection, we introduce the needed function spaces. As in Subsection 8.1, for any $q \in (1,\infty)$ we set

$$ {}_D H^{1,q}(\mathscr{O}) := \{v \in W^{1,q}(\mathscr{O}) \,:\, v = 0 \text{ on } \partial\mathscr{O}\}, \qquad {}_D H^{-1,q}(\mathscr{O}) := ({}_D H^{1,q'}(\mathscr{O}))^* \tag{8.69} $$

where q' satisfies $\frac{1}{q} + \frac{1}{q'} = 1$. The prescript D reminds the Dirichlet boundary conditions. In addition, for all $s \in (-1,1)$ and $p \in (1,\infty)$, we set

$$ \begin{aligned} {}_D H^{s,q}(\mathscr{O}) &:= [{}_D H^{-1,q}(\mathscr{O}), {}_D H^{1,q}(\mathscr{O})]_{\frac{1+s}{2}}, \\ {}_D B^{s}_{q,p}(\mathscr{O}) &:= ({}_D H^{-1,q}(\mathscr{O}), {}_D H^{1,q}(\mathscr{O}))_{\frac{1+s}{2},p}. \end{aligned} \tag{8.70} $$

Moreover, for p, q, s as above and $\mathcal{A}^{s,q} \in \{H^{s,q}, B^s_{q,p}\}$ the following identification holds

$$ {}_D\mathcal{A}^{s,q}(\mathscr{O}) = \begin{cases} \mathcal{A}^{s,q}(\mathscr{O}) & \text{if } s \in (0, \frac{1}{q}), \\ \{\mathcal{A}^{s,q}(\mathscr{O}) \,:\, u = 0 \text{ on } \partial\mathscr{O}\} & \text{if } s \in (\frac{1}{q}, 1). \end{cases} \tag{8.71} $$

Indeed, in the case $\mathcal{A}^{s,q} = H^{s,q}$ (resp. $\mathcal{A} = B^s_{q,p}$) the identification (8.71) is due to [187] (resp. [95]). Moreover, (8.71) for $\mathcal{A}^{s,q} = H^{s,q}$ ensures ${}_D H^{0,q}(\mathscr{O}) = L^q(\mathscr{O})$.

Fix $T \in (0,\infty]$. We say that (u,σ) is a *(unique L^p_κ-)weak solution to* (8.66) *on* $\overline{I}_T$ if (u,σ) is an L^p_κ-maximal local solution to (4.16) on $\overline{I}_T$ (see Definition 4.3.3-4.3.4 and Subsection 6.3.3 for the case $T = \infty$) with $H = \ell^2$, W_ℓ as in Example 2.3.6, $X_0 = {}_D H^{-1,q}(\mathscr{O})$, $X_1 = {}_D H^{1,q}(\mathscr{O})$, $f = g = 0$, and for $v \in X_1$, $v \in X^{\mathsf{Tr}}_{\kappa,p} = {}_D B^{1-2\frac{1+\kappa}{p}}_{q,p}(\mathscr{O})$,

$$ \begin{aligned} A(\cdot, v)v &= -\mathrm{div}(a(\cdot,v)\cdot\nabla v), & B(\cdot)v &= 0, \\ F(\cdot, v) &= \mathrm{div}(\Psi(\cdot,v)) + \psi(\cdot,v), & G(\cdot,v) &= (\Phi_n(\cdot,v))_{n\geq 1}. \end{aligned} \tag{8.72} $$

where div acts as in (8.3). Finally, we say that (u,σ) (or simply u) is a *global (L^p_κ)-weak solution to* (8.66) provided (u,σ) is a weak solution to (8.66) on $[0,\infty)$ with $\sigma = \infty$ a.s. By Assumption 8.3.1(1) and Sobolev embedding,

$$ X^{\mathsf{Tr}}_{\kappa,p} = {}_D B^{1-2\frac{1+\kappa}{p}}_{q,p}(\mathscr{O}) \hookrightarrow C^\eta(\mathscr{O}), \qquad \text{where} \qquad \eta = 1 - 2\frac{1+\kappa}{p} - \frac{d}{q} > 0, \tag{8.73} $$

and therefore the operator A in (8.72) is well-defined. Finally, recall a weak solution to (8.66) verifies the natural weak formulation of (8.66) obtained by integration by parts (see Subsection 5.2.5).

Theorem 8.3.2 (Global existence and regularity)**.** *Let Assumption 8.3.1 be satisfied. Then for any $u_0 \in L^0_{\mathscr{F}_0}(\Omega; {}_D B^{1-2\frac{1+\kappa}{p}}_{q,p}(\mathscr{O}))$ there exists a* global *weak solution u to* (8.66) *such that*

$$ u \in L^p_{\mathrm{loc}}([0,\infty), w_\kappa; {}_D H^{1,q}(\mathscr{O})) \cap C([0,\infty); {}_D B^{1-2\frac{1+\kappa}{p}}_{q,p}(\mathscr{O})), \qquad a.s. $$

Moreover, the global solution u instantaneously regularizes in time and space:

$$ u \in \bigcap_{\theta \in [0,1/2)} H^{\theta,r}_{\mathrm{loc}}(\mathbb{R}_+; {}_D H^{1-2\theta,\zeta}(\mathscr{O})), \quad \textit{a.s. for all } r,\zeta \in (2,\infty). \tag{8.74} $$

In particular,

$$ u \in \bigcap_{\theta \in [0,1/2)} C^\theta_{\mathrm{loc}}(\mathbb{R}_+; C^{1-2\theta}(\mathscr{O})) \subseteq \bigcap_{\theta_1 \in (0,\frac{1}{2}),\, \theta_2 \in (0,1)} C^{\theta_1,\theta_2}_{\mathrm{loc}}(\mathbb{R}_+ \times \overline{\mathscr{O}}) \ \textit{a.s.} \tag{8.75} $$

Additionally, if $u_0 \in L^\lambda(\Omega; L^\lambda(\mathscr{O}))$ with $\lambda \in [2,\Lambda)$ (see Assumption 8.3.1(7)), then for all $T < \infty$ there exists $C > 0$ independent of u_0 such that

$$ \mathbb{E}\Big[\sup_{t \in I_T} \|u(t)\|^\lambda_{L^\lambda(\mathscr{O})}\Big] \leq C\big(1 + \mathbb{E}\|u_0\|^\lambda_{L^\lambda(\mathscr{O})}\big). \tag{8.76} $$

Remark 8.3.3.

- Global existence for (8.66) holds if $\mathscr{O}$ is replaced by $\mathbb{T}^d$ ignoring the boundary condition in (8.66) and replacing $\mathscr{O}, {}_DB, {}_DH$ by $\mathbb{T}^d, B, H$ in Theorem 8.3.2.
- In Proposition 8.3.7 the a priori bound (8.76) will be further improved provided Ψ, ψ, Φ satisfy linear growth assumptions.

Note that, by (8.71) and (8.73), if u_0 is as in Theorem 8.3.2, then $u_0 \in C^\eta(\mathscr{O})$ a.s. and $u_0 = 0$ on $\partial\mathscr{O}$ a.s. A partial converse of the latter observation holds and for future convenience, we formulate this in the following

Remark 8.3.4. Let $\delta \in (0,1)$, $p \in (2,\infty)$, $q \in [2,\infty)$ and $\kappa \in [0, \frac{p}{2}-1)$ be such that $1 - 2\frac{1+\kappa}{p} < \delta$. Then the condition $u_0 \in L^0_{\mathscr{F}_0}(\Omega; {}_DC^\delta(\mathscr{O}))$ in Theorem 8.3.2 can be replaced by $u_0 \in L^0_{\mathscr{F}_0}(\Omega; {}_DC^\delta(\mathscr{O}))$, where ${}_DC^\delta(\mathscr{O}) := \{v \in C^\delta(\mathscr{O}) : v = 0 \text{ on } \partial\mathscr{O}\}$. To see this, it is enough to show

$$ {}_DC^\delta(\mathscr{O}) \hookrightarrow {}_DB^{1-2\frac{1+\kappa}{p}}_{q,p}(\mathscr{O}). \tag{8.77}$$

The former follows from (8.71) applied to $B^{1-2\frac{1+\kappa}{p}}_{q,p}$ and

$$C^\delta(\mathscr{O}) \overset{(i)}{=} (C(\overline{\mathscr{O}}), C^1(\overline{\mathscr{O}}))_{\delta,\infty} \overset{(ii)}{\hookrightarrow} (L^q(\mathscr{O}), W^{1,q}(\mathscr{O}))_{1-2\frac{1+\kappa}{p},p} = B^{1-2\frac{1+\kappa}{p}}_{q,p}(\mathscr{O})$$

where we have used that $\mathscr{O}$ is a bounded C^1-domain and in (i) [151, Example 1.8 and 1.9], in (ii) [151, Proposition 1.4] and $C^j(\overline{\mathscr{O}}) \hookrightarrow W^{j,q}(\mathscr{O})$ for $j \in \{0,1\}$.

8.3.2 Proof of Theorem 8.3.2

The proof of Theorem 8.3.2 follows the strategy of Theorem 8.2.4. Let us begin by showing local well-posedness and smoothness of solutions to result for (8.66). In the following result the Assumption 8.3.1(6)-(7) are not used. Recall that $I_t = (0,t)$ for all $t > 0$.

Proposition 8.3.5 (Local existence and regularity)**.** *Let Assumption 8.3.1(1)-(5) be satisfied. Then for any $u_0 \in L^0_{\mathscr{F}_0}(\Omega; {}_DB^{1-2\frac{1+\kappa}{p}}_{q,p}(\mathscr{O}))$ there exists a weak solution (u,σ) to (8.66) on $[0,\infty)$, and for each localizing sequence $(\sigma_n)_{n\geq 1}$, a.s. for all $n \geq 1$*

$$u \in L^p_{\rm loc}(\overline{I}_{\sigma_n}, w_\kappa; {}_DH^{1,q}(\mathscr{O})) \cap C(\overline{I}_{\sigma_n}; {}_DB^{1-2\frac{1+\kappa}{p}}_{q,p}(\mathscr{O})).$$

Moreover, (u,σ) instantaneously regularizes in time and space

$$u \in \bigcap_{\theta\in[0,1/2)} H^{\theta,r}_{loc}(I_\sigma; {}_DH^{1-2\theta,\zeta}(\mathscr{O})), \qquad \text{a.s. for all } r,\zeta \in (2,\infty). \tag{8.78}$$

In particular,

$$u \in \bigcap_{\theta\in[0,1/2)} C^\theta_{\rm loc}(I_\sigma; C^{1-2\theta}(\mathscr{O})) \subseteq \bigcap_{\theta_1\in(0,\frac{1}{2}),\,\theta_2\in(0,1)} C^{\theta_1,\theta_2}_{\rm loc}(I_\sigma \times \overline{\mathscr{O}}) \text{ a.s.} \tag{8.79}$$

Proof. As in the proof of Proposition 8.2.9, it is sufficient to consider (8.66) on $\overline{I}_T$ with $T < \infty$. For the reader's convenience, we split the proof into two steps.

Step 1: Existence of a weak solution to (8.66) As usual, we apply Theorem 6.3.1. Ss in Theorem 5.2.14 one sees that (8.73) and Assumption 8.3.1(3) imply that (HA) and (HF)-(HG) are satisfied with $F_c = G_c = 0$. It remains to discuss the validity of (4.24). Reasoning as in the proof of Theorem 8.1.3, $-{}_D\Delta_{-1,q}$ has a bounded H^∞-calculus of angle $\omega_{H^\infty}(-{}_D\Delta_{-1,q}) < \pi/2$ and by Theorem 4.2.7,

$$-{}_D\Delta_{-1,q} \in \mathcal{SMR}^\bullet_{p,\kappa}(s,T), \text{ for all } 0 \leq s < T < \infty. \tag{8.80}$$

Therefore, by (8.80) and [174, Theorem 3.9], (4.24) would follows if we can show that for each $v \in X^{\mathsf{Tr}}_{\kappa,p}$ $\|v\|_{X^{\mathsf{Tr}}_{\kappa,p}} \leq n$, $A(\cdot, v)$ has *deterministic* maximal L^p-regularity (see e.g. [174, Definition 3.2]) with constant independent of v but possibly on $n \in \mathbb{N}$. To show this, we apply the initial value version of [39, Theorem 3.3]. To check [39, Assumption 3.1] note that by (8.73) and the fact that $a^{i,j}$'s are locally Lipschitz in y uniformly w.r.t. (t,x) (see (8.67)), a.s. for all $t \geq 0$, $x, \xi, \xi' \in \mathscr{O}$,

$$|a^{i,j}(t,x,z(\xi)) - a^{i,j}(t,x,z(\xi'))| \leq C_N |\xi - \xi'|^{\eta}, \tag{8.81}$$

where $N(p,q,\kappa,n) > 0$. Hence, for each $i,j \in \{1,\dots,d\}$, $r \leq R$, $x \in \mathscr{O}$ and $t \in I_T$,

$$\begin{aligned}
&\fint_{t-r^2}^{t} \fint_{B_{\mathbb{R}^d}(x,r)\cap\mathscr{O}} \Big| a^{i,j}(s,\xi,v(\xi)) - \fint_{B_{\mathbb{R}^d}(x,r)\cap\mathscr{O}} a^{i,j}(s,\xi',v(\xi'))d\xi' \Big| \, d\xi \, ds \\
&\leq \fint_{t-r^2}^{t} \fint_{B_{\mathbb{R}^d}(x,r)\cap\mathscr{O}} \Big| a^{i,j}(s,\xi,v(\xi)) - \fint_{B_{\mathbb{R}^d}(x,r)\cap\mathscr{O}} a^{i,j}(s,\xi',v(\xi))d\xi' \Big| \, d\xi \, ds + C_N r^{\eta} \\
&\leq \operatorname*{ess\,sup}_{(t,x)\in I_T\times\mathscr{O}} \Big(\sup_{r\leq R} \operatorname{osc}^{r,N}_{t,x}(a^{i,j}) \Big) + C_N R^{\eta},
\end{aligned}$$

where we have used (8.81) with $x = \xi'$. The latter combined with Assumption 8.3.1(4), implies [39, Assumption 3.1], and hence $A(\cdot, v)$ has deterministic maximal L^p-regularity with estimates depending only on $n \in \mathbb{N}$.

Step 2: Proof of (8.78)-(8.79). Note that (8.79) follows by (8.78) and the Sobolev embeddings. To prove (8.78), employing Sobolev embedding for ${}_DH$ and ${}_DB$-spaces (see e.g. [3, (A.11)]), one can follow the argument used in Step 5 in the proof of Proposition 8.2.9. It remains to check hypothesis $\mathbf{H}({}_DH^{-1,\xi}(\mathscr{O}), {}_DH^{1,\xi}(\mathscr{O}), r, \alpha)$ (see Assumption 7.1.1), Assumption 6.3.2 for $\ell \in \{0, \alpha\}$, and Assumption 6.3.4 in the $({}_DH^{-1,\xi}(\mathscr{O}), {}_DH^{1,\xi}(\mathscr{O}), r, \alpha)$-setting for all $\xi \in [2,\infty)$, $r \in (2,\infty)$ and $\alpha \in [0, \frac{r}{2} - 1)$ such that $1 - 2\frac{1+\alpha}{r} > \frac{d}{\xi}$. Hypothesis $\mathbf{H}({}_DH^{-1,\xi}(\mathscr{O}), {}_DH^{1,\xi}(\mathscr{O}), \alpha, r)$ and Assumption 6.3.4 follows from (8.73) using the argument in Theorem 5.2.14. To check Assumption 6.3.2 for $\ell \in \{0,\alpha\}$, we can repeat the argument in Step 1 by applying again (8.80), [39, Theorem 3.3] and [174, Theorem 3.9]. □

Lemma 8.3.6. *Let $\mathscr{O} \subseteq \mathbb{R}^d$ be a bounded. Let $T < \infty$ and $0 < \delta' < \delta \leq 1$. Then $C^{\delta}(I_T \times \mathscr{O}) \hookrightarrow C(\overline{I}_T; C^{\delta'}(\mathscr{O}))$ with embedding constant ≤ 2. In particular, $C^{\delta}(\mathbb{R}_+ \times \mathscr{O}) \hookrightarrow \mathrm{BC}([0,\infty); C^{\delta'}(\mathscr{O}))$, where* BC *stands for bounded and continuous.*

Proof. Note that for any $u \in C^{\delta}(I_T \times \mathscr{O})$,

$$\sup_{t\in I_T} \|u(t,\cdot)\|_{C^{\delta'}(\mathscr{O})} \leq 2 \sup_{t\in I_T} \|u(t,\cdot)\|_{C^{\delta}(\mathscr{O})} \leq \|u\|_{C^{\delta}(I_T\times\mathscr{O})},$$

where the first estimate follows by using $|u(t,x) - u(t,y)|/|x-y|^{\delta'} \leq 2\|u(t)\|_{\infty}$ if $|x - y| \geq 1$. The above estimate shows that it remains to prove that the map

$$\overline{I_T} \ni t \mapsto u(t,\cdot) \in C^{\delta'}(\mathscr{O}) \quad \text{is continuous.} \tag{8.82}$$

Fix $(t_n)_{n\geq1}$ in $[0,T]$ such that $t_n \to t$. Then by the above $\{u(t_n,\cdot) : n \geq 1\}$ is bounded in $C^{\delta}(\mathscr{O})$. Therefore, by compactness of $C^{\delta}(\mathscr{O}) \hookrightarrow C^{\delta'}(\mathscr{O})$ (see [93, Lemma 6.33]) it follows that $(u(t_n,\cdot))_{n\geq1}$ is convergent to some v in $C^{\delta'}(\mathscr{O})$. Since $u(t_n,x) \to u(t,x)$ for all $x \in \mathscr{O}$, it follows that $v = u(t,\cdot)$. Therefore, $u(t_n,\cdot) \to u(t,\cdot)$ in $C^{\delta'}(\mathscr{O})$, which proves the required continuity. □

Proof of Theorem 8.3.2. We only consider $d \geq 2$. The case $d = 1$ only requires minor modifications. Let (u,σ) be the weak solution to (5.74) on $\mathbb{R}_+$ provided by Proposition 8.3.5. Note that (8.74)-(8.75) follows by Proposition 8.3.5 if we prove that (u,σ) is a global weak solution to (8.66), i.e. $\sigma = \infty$ a.s.

Let $T \in \mathbb{R}_+$ be arbitrary. Replacing (u, σ) by $(u|_{[\![0,\sigma\wedge T)\!)}, \sigma \wedge T)$, to prove that (u, σ) is a global weak solution it is sufficient to consider weak solution on I_T and to show $\sigma = T$ a.s. To prove the latter, since $\sigma > 0$ a.s. it remains to show that

$$\mathbb{P}(s < \sigma < T) = 0, \qquad \text{for all } s \in I_T. \tag{8.83}$$

To prove (8.83), let us collect some useful facts. By Proposition 6.3.10, we may assume

$$u_0 \in L^\infty(\Omega; {}_D B^{1-2\frac{1+\kappa}{p}}_{q,p}(\mathscr{O})). \tag{8.84}$$

Let $\Lambda > \max\{\frac{d}{2}(h+1), 2\} = \frac{d}{2}(h+1)$ be as in Assumption 8.3.1(7). Recall that, by Remark 8.2.13, the estimate (8.48) holds for (8.66), i.e. for any $\lambda \in [2, \Lambda)$ there exists $C > 0$ depending only on $\nu, d, h, M, \lambda, \Lambda, T$ (see Assumption 8.3.1(7)) such that

$$\mathbb{E}\Big[\mathbf{1}_{\Gamma_{s,k}} \sup_{t\in[s,\sigma)} \|u(t)\|^\lambda_{L^\lambda(\mathscr{O})}\Big] \le C\Big(1 + \mathbb{E}[\mathbf{1}_{\Gamma_{s,k}}\|u(s)\|^\lambda_{L^\lambda(\mathscr{O})}]\Big),$$

for all $k \ge 1$, $s \in (0, T) = I_T$ and $\Gamma_{s,k} = \{\sigma > s\,,\, \|u(s)\|_{C(\overline{\mathscr{O}})} \le k\}$. By (8.73), (8.84) and the fact that $u_0 \in C^\eta(\mathscr{O})$ a.s. (see (8.73)), we can take the limit as $s \downarrow 0$ and $k \ge 1$ large enough (see (8.84)) to obtain

$$\mathbb{E}\Big[\sup_{t\in[0,\sigma)} \|u(t)\|^\lambda_{L^\lambda(\mathscr{O}))}\Big] \le C\big(1 + \mathbb{E}[\|u_0\|^\lambda_{L^\lambda(\mathscr{O})}]\big). \tag{8.85}$$

In the remaining part of the proof, we fix $\lambda \in (\frac{d}{2}(h+1), \Lambda)$ and set $\varrho = \frac{2\lambda}{h+1} \in (d, \infty)$ and choose $\phi \in (2, \infty)$ such that $\frac{2}{\phi} + \frac{d}{\varrho} < 1$. The goal is to apply Theorem 8.1.3 with exponents ϕ and ϱ to find Hölder regularity of the solution. For each $n \ge 1$, define the stopping time τ_n by

$$\tau_n := \inf\{t \in [0, \sigma) \,:\, \|u\|_{L^{\phi h}(0,t;L^\lambda(\mathscr{O}))} \ge n\}, \qquad \text{where} \qquad \inf \varnothing := \sigma. \tag{8.86}$$

Then by (8.85), $\lim_{n\uparrow\infty} \mathbb{P}(\tau_n = \sigma) = 1$. The rest of the proof of (8.83) for fixed $s \in (0, T)$ is divided into three steps. In Step 1 we prove that solutions to (8.66) are continuous on $(0, \sigma]$ with values in $X^{\mathsf{Tr}}_{\kappa,p}$, in Step 2 we prove (8.83) under additional assumption on κ, in Step 3 we get (8.83) in the general case by employing instantaneous regularization and Step 1. Finally, in Step 4 we complete the proof of Theorem 8.3.2. An alternative approach to Steps 2 and 3 will be given afterwards.

Step 1: There exists $\delta(\|a^{i,j}\|_{L^\infty}, \nu, \phi, h, \lambda, d) > 0$ (independent of s) such that

$$u \in C([s, \sigma]; C^\delta(\mathscr{O})), \qquad \text{a.s. on } \{\sigma > s\}. \tag{8.87}$$

By $\mathbb{P}(\tau_n = \sigma) \to 1$ and Lemma 8.3.6, it is enough to prove that for each $n \ge 1$,

$$u \in C^\gamma((s, \tau_n) \times \mathscr{O}), \qquad \text{a.s. on } \{\tau_n > s\}, \tag{8.88}$$

where $\gamma(\|a^{i,j}\|_{L^\infty}, \nu, \phi, h, \lambda, d) > 0$. Since $\|u\|_{L^{\phi h}(I_{\tau_n};L^\lambda(\mathscr{O}))} \le n$ (see (8.86)), a.s.

$$\begin{aligned}\|\psi(\cdot, u)\|_{L^\phi(I_{\tau_n};{}_D H^{-1,\varrho}(\mathscr{O}))} &\overset{(i)}{\lesssim} \|\psi(\cdot, u)\|_{L^\phi(I_{\tau_n};L^{\lambda/h}(\mathscr{O}))} \\ &\overset{(ii)}{\lesssim} 1 + \|u\|^h_{L^{\phi h}(I_{\tau_n};L^\lambda(\mathscr{O}))} \le 1 + n^h.\end{aligned} \tag{8.89}$$

Here (i) follows from Sobolev embedding $L^{\lambda/h}(\mathscr{O}) \hookrightarrow {}_D H^{-1,\varrho}(\mathscr{O})$ (see (8.69) and [3, Eq. (A.11)]) where $-\frac{dh}{\lambda} \ge -1 - \frac{d}{\varrho}$ follows from $\lambda > \frac{d}{2}(h+1) > \frac{d}{2}(h-1)$. In (ii) we used Assumption 8.3.1(6). The remaining terms can be estimated analogously:

$$\begin{aligned}&\|\mathrm{div}(\Psi(\cdot, u))\|_{L^\phi(I_{\tau_n};{}_D H^{-1,\varrho}(\mathscr{O})} + \|\Phi(\cdot, u)\|_{L^\phi(I_{\tau_n};L^\varrho(\mathscr{O};\ell^2))} \\ &\qquad\lesssim 1 + \|u\|^{\frac{h+1}{2}}_{L^{\phi\frac{h+1}{2}}(I_{\tau_n};L^\lambda(\mathscr{O}))} \le 1 + n^{\frac{1+h}{2}}.\end{aligned} \tag{8.90}$$

By (8.84) and (8.89)-(8.90), Theorem 8.1.3(1) implies $u \in L^r(\Omega; C^\gamma((s,\tau_n)\times \mathscr{O}))$ and for some $\gamma(\|a^{i,j}\|_{L^\infty}, \nu, h, \lambda, d) > 0$ independent of r, s, n. Thus (8.88) follows and this yields the claim of this step.

Step 2: Let δ be as in Step 1 and assume that $1 - 2\frac{1+\kappa}{p} \in (0,\delta)$. Then (8.83) *holds.* Note that, the requirement $1 - 2\frac{1+\kappa}{p} \in (0,\delta)$ does not contradict Assumption 8.3.1(1) but forces $q > \frac{d}{\delta}$ and κ is close to $\frac{p}{2} - 1$. To prove (8.83), note that, by (8.87) and the fact that $u|_{\partial\mathscr{O}} = 0$ a.s. for all $t \in [s,\sigma)$ (see (8.78)), the embedding (8.77) and (8.87) imply

$$u \in C([s,\sigma]; {}_D B^{1-2\frac{1+\kappa}{p}}_{q,p}(\mathscr{O})), \qquad \text{a.s. on } \{\sigma > s\}.$$

Therefore,

$$\begin{aligned}\mathbb{P}(s<\sigma<T) &= \mathbb{P}\Big(s<\sigma<T\,,\ \lim_{t\uparrow\sigma} u(t) \text{ exists in } {}_D B^{1-2\frac{1+\kappa}{p}}_{q,p}(\mathscr{O})\Big)\\ &\leq \mathbb{P}\Big(\sigma<T\,,\ \lim_{t\uparrow\sigma} u(t) \text{ exists in } {}_D B^{1-2\frac{1+\kappa}{p}}_{q,p}(\mathscr{O})\Big) = 0,\end{aligned} \tag{8.91}$$

where the last equality follows by Theorem 6.3.6(3). Here, we have used that Assumption 6.3.2 for $\ell \in \{0,\kappa\}$ and Assumption 6.3.4 hold in the $({}_D H^{-1,q}(\mathscr{O}), {}_D H^{1,q}(\mathscr{O}), p, \kappa)$-setting (see Step 2 in the proof of Proposition 8.3.5) and that $F_c = G_c = 0$ which ensures that $X^{\mathsf{Tr}}_{\kappa,p} = {}_D B^{1-2\frac{1+\kappa}{p}}_{q,p}(\mathscr{O})$ is not critical for (8.66). Thus, (8.91) gives (8.83) as desired.

Step 3: (8.83) *holds.* As in the proof of Theorem 8.2.4, the idea is to reduce the general case to the one analysed in Step 2 using the extrapolation result in Lemma 7.1.9. Let $\delta > 0$ be as in Step 1. Fix $q^* > \frac{d}{\delta}$, $p^* \in (2,\infty)$ and $\kappa^* \in [0, \frac{p^*}{2} - 1)$ such that $\delta > 1 - 2\frac{1+\kappa^*}{p^*} > \frac{d}{q^*}$. Let (u,σ) be as the beginning of the proof. By Step 2 applied with (q^*, p^*, κ^*) on $[s,T]$ and Proposition 8.3.5, Lemma 7.1.9 and (8.78) yield the claim in the general case. ☐

In case the coefficients a^{ij} are independent of (t,ω,x), the above proof does not use the full strength of [39, Theorem 3.3], but merely the fact that second order operators with continuous coefficients in space, have deterministic maximal regularity. Next we give an alternative prove of (8.83) which avoids the use of Lemma 7.1.9.

Global well-posedness for (8.66)*: alternative proof.* As in the previous proof, we may assume that (8.84) holds. Repeating the estimates (8.89)-(8.90) and using the last statement in Theorem 8.1.3, there exists $\eta'(\|a^{i,j}\|_{L^\infty}, \nu, r, \lambda, d, p, \kappa, q) > 0$ such that

$$u \in C(\overline{I}_\sigma; C^{\eta'}(\mathscr{O})), \qquad \text{a.s.} \tag{8.92}$$

More precisely, by Theorem 8.1.3, (8.92) holds with $\eta' := \min\{\delta,\eta\} > 0$ where δ, η are as in Step 1 in the previous proof and in (8.73), respectively. By (8.92), for each $n \geq 1$, the following stopping time is well-defined

$$\mu_n := \inf\big\{t \in [0,\sigma) \,:\, \|u(t) - u_0\|_{C^{\eta'}(\mathscr{O})} > n\big\}, \quad \inf\varnothing := \sigma. \tag{8.93}$$

Note that $\mu_n > 0$ a.s. and $\mathbb{P}(\mu_n = \sigma) \to 1$ as $n \to \infty$. Let us fix $n \geq 1$. By (8.93) and (8.84), for some constant $C(n,u_0) > 0$,

$$\|u(t)\|_{L^\infty(\mathscr{O})} \leq n + \|u_0\|_{L^\infty(\mathscr{O})} \leq C(n,u_0), \qquad \text{a.s. for all } t \in [0,\mu_n). \tag{8.94}$$

Therefore, by Assumption 8.3.1(6) one has, a.s. for all $t \in [0,\mu_n)$ and $x \in \mathscr{O}$,

$$|\Psi(t,x,u(t,x))| + |\psi(t,x,u(t,x))| + \|\Phi(t,x,u(t,x))\|_{\ell^2} \leq \widetilde{C} \tag{8.95}$$

for some $\widetilde{C}(h, C(n,u_0)) > 0$. Set

$$\mathsf{a}^{i,j}_n := a^{i,j}(\cdot,u)\mathbf{1}_{[\![0,\mu_n)\!)} + \delta^{i,j}\mathbf{1}_{[\![\mu_n,T]\!]},$$

where $\delta^{i,j}$ is the Kronecker's delta. Reasoning as in the proof of Proposition 8.3.5, by (8.80), [39, Theorem 3.3] and [174, Theorem 3.9], for all $n \geq 1$, one has

$$A_n := \mathrm{div}(\mathsf{a}_n \cdot \nabla) \in \mathcal{SMR}^{\bullet}_{p,\kappa}(T), \text{ with } X_0 = {}_D H^{-1,q}(\mathscr{O}) \text{ and } X_1 = {}_D H^{1,q}(\mathscr{O}).$$

Let $\mathscr{R}_n := \mathscr{R}_{A_n,0}$ be the solution operator associated to A_n. By Proposition 6.2.7, (8.84) and (8.95), one has

$$u = \mathscr{R}_{0,A_n}\big(u_0, \mathbf{1}_{[\![0,\mu_n]\!]}(\mathrm{div}(\Psi(\cdot,u)) + \psi(\cdot,u)), \mathbf{1}_{[\![0,\mu_n]\!]}\Phi(\cdot,u)\big), \quad \text{a.e. on } [\![0,\mu_n[\![.$$

Since $\mu_n > 0$ a.s., the former, (8.84), (8.95) and Proposition 4.1.5(2) imply

$$u \in C((0,\mu_n]; {}_D B^{1-\frac{2}{p}}_{q,p}(\mathscr{O})), \qquad \text{a.s.}$$

Since $\mathbb{P}(\mu_n = \sigma) \to 1$ we get

$$\begin{aligned}\mathbb{P}(\sigma < T) &= \lim_{n\uparrow\infty} \mathbb{P}(\{\sigma < T\} \cap \{\mu_n = \sigma\}) \\ &= \lim_{n\uparrow\infty} \mathbb{P}\Big(\{\sigma < T\} \cap \{\mu_n = \sigma\} \cap \Big\{\lim_{t\uparrow\sigma} u(t) \text{ exists in } {}_D B^{1-\frac{2}{p}}_{q,p}(\mathscr{O})\Big\}\Big) \\ &\leq \mathbb{P}\Big(\sigma < T\,,\, \lim_{t\uparrow\sigma} u(t) \text{ exists in } {}_D B^{1-\frac{2}{p}}_{q,p}(\mathscr{O})\Big) = 0,\end{aligned}$$

where in the last equality follows by Theorem 6.3.6(1). Thus $\sigma = T$ a.s. as desired. □

8.3.3 Discussion

As far as we know Theorem 8.3.2 appears to be new. It is one of the few results on well-posedness of quasilinear SPDEs with nonlinearities which are not assumed to be globally Lipschitz. Among the existing ones, ours result appears highly flexible in many ways. For instance our results should be compared with the results in [59, Theorem 2.6], where all nonlinearities are assumed to be globally Lipschitz. In the latter work the authors also have a stronger version of the regularity estimate (8.76) under these global Lipschitz assumptions. In our setting it is enough to assume linear growth assumptions on ψ, Ψ, Φ to obtain such a result and we do not assume any prior existence and regularity of solutions. Moreover, we consider coefficients $a^{i,j}$ which can be measurable in time and Ω, and VMO in x, and only locally Lipschitz in u.

Next we explain how to extend the L^r-estimate of [59, Theorem 2.6] to our setting.

Proposition 8.3.7 ($L^r(\Omega)$-estimates in case sublinear growth)**.** *Let Assumption 8.3.1 be satisfied and fix $T \in (0,\infty)$ and $r \in [2,\infty)$. Suppose there exist $C_1, C_2 \geq 0$ such that a.s.*

$$|\Psi(t,x,y)| + |\psi(t,x,y)| + \|\Phi(t,x,y)\|_{\ell^2} \leq C_1 + C_2|y|, \quad t \in I_T, x \in \mathscr{O}, y \in \mathbb{R}. \tag{8.96}$$

Then there exists a $\delta \in (0,\eta)$ (see (8.73)) and $C > 0$ such that for all $u_0 \in L^0_{\mathscr{F}_0}(\Omega; {}_D B^{1-2\frac{1+\kappa}{p}}_{q,p}(\mathscr{O}))$,

$$\mathbb{E}\Big[\sup_{t\in I_T} \|u(t)\|^r_{C^\delta(\mathscr{O})}\Big] + \mathbb{E}\|u\|^r_{L^2(I_T;{}_D H^1(\mathscr{O}))} \leq C(1 + \mathbb{E}\|u_0\|^r_{C^\delta(\mathscr{O})}),$$

where u is the global weak solution to (8.66) provided by Theorem 8.3.2.

One can check that (8.96) actually implies Assumption 8.3.1(6)-(7).

Proof. First we claim that there exists $C = C(r,T) > 0$ independent of u_0 such that

$$\Big(\mathbb{E}\sup_{t\in I_T} \|u(t)\|^r_{L^2(\mathscr{O})}\Big)^{1/r} + \|u\|_{L^r(\Omega;L^2(I_T;{}_D H^1(\mathscr{O})))} \leq C(1 + \|u_0\|_{L^r(\Omega;L^2(\mathscr{O}))}). \tag{8.97}$$

Indeed, let $v(t) = \|u\|_{L^\infty(I_t;L^2(\mathscr{O}))}$. Then by (8.9) and (8.96) the following estimates hold uniformly in $t \in (0,T]$,

$$\begin{aligned}
&\|v(t)\|_{L^r(\Omega)} + \|u\|_{L^r(\Omega;L^2(I_t;{}_DH^{1,2}(\mathscr{O})))}\\
&\lesssim \|u_0\|_{L^r(\Omega;L^2(\mathscr{O}))} + \|\Psi(\cdot,u)\|_{L^r(\Omega;L^2(I_t;L^2(\mathscr{O})))}\\
&\qquad + \|\psi(\cdot,u)\|_{L^r(\Omega;L^2(I_t;{}_DH^{-1,2}(\mathscr{O})))} + \|\Phi(\cdot,u)\|_{L^r(\Omega;L^2(I_t;L^2(\mathscr{O};\ell^2)))}\\
&\leq 3C_1 t^{1/2} + \|u_0\|_{L^r(\Omega;L^2(\mathscr{O}))} + 3C_2\|u\|_{L^r(\Omega;L^2(I_t;L^2(\mathscr{O})))}\\
&\leq 3C_1 + \|u_0\|_{L^r(\Omega;L^2(\mathscr{O}))} + 3C_2\|v\|_{L^2(0,t;L^r(\Omega))}
\end{aligned}$$

where in the last step we used Minkowski's inequality and $r \in [2,\infty)$. Taking squares and using $(a+b)^2 \leq 2a^2 + 2b^2$ on the RHS, it follows from Grownwall's inequality applied to $\|v(\cdot)\|^2_{L^r(\Omega)}$ that

$$\Big(\mathbb{E}\sup_{t\in I_T}\|u(t)\|^r_{L^2(\mathscr{O})}\Big)^{1/r} = \|v(T)\|_{L^r(\Omega)} \leq C_T(1 + \|u_0\|_{L^r(\Omega;L^2(\mathscr{O}))}).$$

Combining both estimates we also find that (8.97) holds.

By Theorem 8.3.2 and (8.73) for $n \in \mathbb{N}$, we define the stopping time σ_n by

$$\sigma_n := \inf\big\{t \in I_T \,:\, \|u(t) - u_0\|_{L^\infty(\mathscr{O})} \geq n\big\}, \quad \text{where} \quad \inf\varnothing := T. \tag{8.98}$$

Fix $\varrho > d+2$. From Theorem 8.1.3(3) with $q = p = \varrho > d+2$, we find that there exists $\eta' \in (0,\eta)$ (here η is as in (8.73)) and $K > 0$ independent of $n \in \mathbb{N}$ such that

$$\begin{aligned}
\|u\|_{L^r(\Omega;C^{\eta'}(I_{\sigma_n}\times\mathscr{O}))} &\leq K\big(\|u_0\|_{L^r(\Omega;C^{\eta'}(\mathscr{O}))} + \|\Psi(\cdot,u)\|_{L^r(\Omega;L^\varrho(I_{\sigma_n}\times\mathscr{O}))}\\
&\qquad + \|\psi(\cdot,u)\|_{L^r(\Omega;L^\varrho(I_{\sigma_n}\times\mathscr{O}))} + \|\Phi(\cdot,u)\|_{L^r(\Omega;L^\varrho(I_{\sigma_n}\times\mathscr{O};\ell^2))}\big)\\
&\overset{(i)}{\leq} KC'\big(1 + \|u_0\|_{L^r(\Omega;C^{\eta'}(\mathscr{O}))} + \|u\|_{L^r(\Omega;L^\varrho(I_{\sigma_n}\times\mathscr{O}))}\big)\\
&\overset{(ii)}{\leq} KC''\big(1 + \|u_0\|_{L^r(\Omega;C^{\eta'}(\mathscr{O}))} + \|u\|_{L^r(\Omega;L^2(I_{\sigma_n}\times\mathscr{O}))}\big) + \frac{1}{2}\|u\|_{L^r(\Omega;L^\infty(I_{\sigma_n}\times\mathscr{O}))}.
\end{aligned}$$

where in (i) we have used (8.96), and in (ii) we used

$$\|u\|_{L^\rho} \leq \|u\|^{2/\rho}_{L^2}\|u\|^{2-2/\rho}_{L^\infty} \leq C_\varepsilon\|u\|_{L^2} + \varepsilon\|u\|_{L^\infty}, \quad \forall\varepsilon > 0.$$

Combining the estimate for $\|u\|_{L^r(\Omega;C^{\eta'})}$ with (8.97), we obtain

$$\mathbb{E}\Big[\sup_{t\in I_{\sigma_n}}\|u(t)\|^r_{C^{\eta'}(\mathscr{O})}\Big] \leq 2KC''(1 + \mathbb{E}\|u_0\|^r_{C^{\eta'}(\mathscr{O})}.$$

The result follows by letting $n \uparrow \infty$. ☐

Chapter 9

Stochastic Navier-Stokes for turbulent flows in critical spaces

Let $(w^n)_{n\geq 1}$ be a sequence of independent standard Brownian motions on a filtered probability space $(\Omega, \mathbb{F} = (\mathscr{F}_t)_{t\geq 0}, \mathscr{A}, \mathbb{P})$ and let $\mathscr{P}$ be the progressive sigma algebra.

In this chapter we study the following stochastic Navier-Stokes problem arising in the study of turbulent flows

$$\begin{cases} du - \Delta u\, dt = \big(-\nabla P + \mathcal{F}(\cdot, u) - \operatorname{div}(u \otimes u)\big)dt \\ \qquad\qquad\qquad\qquad + \sum_{n\geq 1} \big((\phi_n \cdot \nabla)u + \mathcal{G}(\cdot, u)\big)dw_t^n, & \text{on } \mathscr{O}, \\ \operatorname{div} u = 0, \qquad u(\cdot, 0) = u_0, & \text{on } \mathscr{O}, \end{cases} \tag{9.1}$$

where $u : [0, \infty) \times \Omega \times \mathscr{O} \to \mathbb{R}^d$ denotes the unknown velocity field, $P, Q_n : [0, \infty) \times \Omega \times \mathscr{O} \to \mathbb{R}$ denote the unknown pressures and

$$(\phi_n \cdot \nabla)u := \Big(\sum_{j=1}^d \phi_n^j \partial_j u^k\Big)_{k=1}^d, \qquad \operatorname{div}(u \otimes u) := \Big(\sum_{j=1}^d \partial_j (u^j u^k)\Big)_{k=1}^d.$$

Here we will mainly be concerned with the case $\mathscr{O} = \mathbb{T}^d$ for $d \geq 2$. However, we also consider the 2D Navier-Stokes on Lipschitz domain with no-slip boundary conditions (i.e. $u = 0$ on $\partial\mathscr{O}$) see Subsection 9.4. Due to physical motivations behind (9.1), we focus on the case $\phi^j : I_T \times \Omega \times \mathscr{O} \to \ell^2$ is α-Hölder continuos for some $\alpha > 0$. The main results of this section are: local well-posedness in the critical spaces $B^{d/q-1}_{q,p} \cap \{\operatorname{div} u = 0\}$ and $L^d \cap \{\operatorname{div} u = 0\}$, instantaneous regularization of solutions, Serrin-type blow-up criteria and global existence for solutions in two dimensions.

The problem fits well into the abstract framework analysed in Chapters 4, 6 and 7 and the proofs are based on the results proven there. To recast (9.1) in the form of a stochastic evolution equation, one can apply the Helmholtz projection $\mathcal{P}$ to the first equation in (9.1). Using this standard strategy, one can reduce the problem (9.1) to a problem in the unknown velocity field u.

The main problem here is to show stochastic maximal regularity for the 'turbulent Stokes couple' $(-\mathcal{P}\Delta, (\mathcal{P}(\phi_n^j \cdot \nabla))_{n\geq 1})$. Moreover, it will be of particular interest to prove stochastic maximal L^p-regularity on Sobolev spaces $H^{1+\delta,q} \cap \{\operatorname{div} u = 0\}$ where $\delta \in (-1, 0]$. To handle the case $\delta < 0$, we employ a multiplication result in Sobolev spaces with negative smoothness. For the reader's covenience, we provide a short proof of the latter result in Subsection 9.1.1.

This chapter is organized as follows. In Section 9.1 we provide some preliminary results. In particular, we study a partition and perturbation result for the class $\mathcal{SMR}^\bullet_{p,\kappa}(T)$ which will be of basic importance in Section 9.2 where we prove stochastic maximal L^p-regularity for the turbulent Stokes couple. In Section 9.3 we study (9.1) on $\mathbb{T}^d$ and we prove existence regularity, global existence in 2D for data in $B^{2/q-1}_{q,p} \cap \{\operatorname{div} u = 0\}$ (with $q \in [2, \frac{2}{1+\delta})$ and $|\delta| < \alpha$) and blow-up criteria for (9.1). In Section 9.4 we look at the 2D-case of (9.1) and we prove global existence on bounded Lipschitz domain when ϕ^j is merely bounded and measurable.

The results of this chapter will be presented in [7].

9.1 Preliminaries

In this subsection we have gathered some preliminary results which will be useful below. In Subsection 9.1.1 we prove a product estimate in negative Sobolev spaces and in Subsection 9.1.2 we prove additional results on tha class $\mathcal{SMR}^{\bullet}_{p,\kappa}(T)$.

9.1.1 Estimates for products in negative Sobolev spaces

Let us begin this section by recalling the following product estimate in Sobolev spaces (see e.g. [195, Chapter 13]): Let $\mathcal{H}\in\{\mathbb{R},\ell^2\}$ and $\mathscr{O}\in\{\mathbb{T}^d,\mathbb{R}^d\}$. Then for each $s>0$, $\alpha>s$ and $q\in(1,\infty)$ we have

$$\|fg\|_{H^{s,q}(\mathscr{O};\mathcal{H})}\lesssim\|f\|_{H^{s,q}(\mathscr{O})}\|g\|_{L^\infty(\mathscr{O};\mathcal{H})}+\|g\|_{C^{s+\varepsilon}(\mathscr{O};\mathcal{H})}\|f\|_{L^q(\mathscr{O})}. \tag{9.2}$$

Here we prove the following result.

Proposition 9.1.1. *Let $\mathscr{O}\in\{\mathbb{T}^d,\mathbb{R}^d\}$ and $\mathcal{H}\in\{\mathbb{R},\ell^2\}$. Let $s\in\mathbb{R}$ and $q\in(1,\infty)$. Then for all $\gamma>0$ and $\varepsilon\in(0,\gamma)$*

$$\|fg\|_{H^{-s,q}(\mathscr{O};\mathcal{H})}\lesssim\|f\|_{H^{-s,q}(\mathscr{O})}\|g\|_{L^\infty(\mathscr{O};\mathcal{H})}+\|f\|_{H^{-s-\varepsilon,q}(\mathscr{O})}\|g\|_{C^{s+\gamma}(\mathscr{O};\mathcal{H})}$$

where $fg:=(fg_n)_{n\in\mathbb{N}}$ if $\mathcal{H}=\ell^2$.

Proof. By a standard localization argument it is enough to prove the claim in the case $\mathscr{O}=\mathbb{R}^d$. Moreover we only consider the case $\mathcal{H}=\mathbb{R}$. The general case follows similarly.

The proof relies on Bony's paraproducts as used in [192]. Let $(\psi_j)_{j\in\mathbb{N}}$ be a Littlewood-Paley partition of the unity [192, p. 4] and, for any $k\in\mathbb{N}$, set $\Psi_k:=\sum_{j\leq k}\psi_j$. For any $f\in\mathscr{S}'(\mathbb{R}^d)$, set $\psi_j(D)f:=\mathfrak{F}^{-1}(\psi_j(\cdot)\mathfrak{F}(f))$ where $\mathfrak{F}$ denotes the Fourier transform on $\mathbb{R}^d$. Then, for any $f,g\in\mathscr{S}'(\mathbb{R}^d)$, the Bony's decomposition of the product is given by $fg:=T_fg+R(f,g)+T_gf$, where

$$T_fg:=\sum_{k\geq5}\Psi_{k-5}(D)f\psi_{k+1}(D)g,\qquad R(f,g)=\sum_{|j-k|\leq4}\psi_j(D)f\psi_k(D)g.$$

The operator T_fg is called the Bony's paraproduct. To prove the claimed estimate let us collect some useful facts. By (1.6)-(1.7) in [192, Chapter 2],

$$\|R(f,g)\|_{H^{s,q}(\mathbb{R}^d)}+\|T_fg\|_{H^{s,q}(\mathbb{R}^d)}\lesssim\|f\|_{L^\infty(\mathbb{R}^d)}\|g\|_{H^{s,q}(\mathbb{R}^d)}. \tag{9.3}$$

Moreover, using that $\|g\|_{C^{s+\gamma}(\mathbb{R}^d)}=\sup_{j\in\mathbb{N}}2^{j(s+\gamma)}\|\psi_j(D)g\|_{L^\infty(\mathbb{R}^d)}$ and, reasoning as in [192, (1.6), Chapter 2], one obtain for all $\varepsilon\in(0,\gamma)$,

$$\begin{aligned}\|T_fg\|_{H^{s+\varepsilon,q}(\mathbb{R}^d)}&\eqsim_{q,s,d}\Big\|\Big(\sum_{k\geq5}2^{2k(s+\varepsilon)}|\Psi_{k-5}(D)f|^2|\psi_k(D)g|^2\Big)^{1/2}\Big\|_{L^q(\mathbb{R}^d)}\\&\lesssim_{q,s,d}\|g\|_{C^{s+\gamma}(\mathbb{R}^d)}\Big\|\Big(\sum_{k\geq5}2^{-2k(\gamma-\varepsilon)}|\Psi_{k-5}(D)f|^2\Big)^{1/2}\Big\|_{L^q(\mathbb{R}^d)}\\&\lesssim_{q,s,d,\gamma,\varepsilon}\|f\|_{L^q(\mathbb{R}^d)}\|g\|_{C^{s+\gamma}(\mathbb{R}^d)},\end{aligned} \tag{9.4}$$

where in the last inequality we have used that $|\Psi_{k-5}f(x)|\leq CMf(x)$ for all $x\in\mathbb{R}^d$ (here M is the maximal operator) and C is independent of k,x,f.

To complete the proof it remains to combine (9.3)-(9.4) with a duality argument. Recall that $(H^{-s,q}(\mathbb{R}^d))^*=H^{s,q'}(\mathbb{R}^d)$. Let $h\in H^{s,q'}(\mathbb{R}^d)$, $f\in H^{-s,q}(\mathbb{R}^d)$, $g\in C^{s+\gamma}(\mathbb{R}^d)$ and note that

$$|\langle fg,h\rangle|=|\langle f,gh\rangle|\leq|\langle f,T_gh\rangle|+|\langle f,R(g,h)\rangle|+|\langle f,T_hg\rangle|.$$

By applying (9.3) we have

$$\begin{aligned}|\langle f, T_g h\rangle| + |\langle f, R(g,h)\rangle| &\leq \|f\|_{H^{-s,q}(\mathbb{R}^d)}\big(\|T_g h\|_{H^{s,q'}(\mathbb{R}^d)} + \|R(g,h)\|_{H^{s,q'}(\mathbb{R}^d)}\big)\\ &\lesssim \|f\|_{H^{-s,q}(\mathbb{R}^d)}\|g\|_{L^\infty(\mathbb{R}^d)}\|h\|_{H^{s,q'}(\mathbb{R}^d)}.\end{aligned}$$

and by (9.4), for each $\varepsilon\in(0,\gamma)$,

$$\begin{aligned}|\langle f, T_h g\rangle| &\leq \|f\|_{H^{-s-\varepsilon,q}(\mathbb{R}^d)}\|T_h g\|_{H^{s+\gamma,q'}(\mathbb{R}^d)}\\ &\lesssim \|f\|_{H^{-s-\varepsilon,q}(\mathbb{R}^d)}\|g\|_{C^{s+\gamma}(\mathbb{R}^d)}\|h\|_{L^{q'}(\mathbb{R}^d)}.\end{aligned}$$

Putting together the previous estimates and taking the supremum over all $\|h\|_{H^{s',q}(\mathbb{R}^d)}\leq 1$, one can readily obtain the claim. ☐

9.1.2 Partion and perturbation for $\mathcal{SMR}^\bullet_{p,\kappa}(s,T)$

In this section we assume prove additional results on stochastic maximal L^p-regularity. Here we employ the notation introduced in Section 6.2. In this section, we fix $T\in(0,\infty)$ and $s\in(0,T)$. Let us recall that $\mathcal{SMR}^\bullet_{p,\kappa}(s,T)$ denotes the set of all couple having stochastic maximal L^p-regularity (see Definitions 6.2.2-6.2.3), $\mathscr{R}_{s,(A,B)}$ denotes the associated solution operator and $K^{\mathsf{det},\theta,p,\kappa}_{(A,B)}$, $K^{\mathsf{det},\theta,p,\kappa}_{(A,B)}$ the constant of stochastic maximal L^p-regularity, see (6.17) and (6.11) respectively. Finally, with a slight abuse of notation, for any $t,t'\in[s,T]$, we write $(A,B)\in\mathcal{SMR}^\bullet_{p,\kappa}(t,t')$ in the case that $(A|_{[\![t,t']\!]},B|_{[\![t,t']\!]})\in\mathcal{SMR}^\bullet_{p,\kappa}(t,t')$ if no confusion seems likely.

For future convenience, let us state the following quantitative version of Proposition 4.2.8 which follows from the argument given there.

Proposition 9.1.2 (Transference)**.** *Let Assumption 4.2.1 be satisfied. Let $(A,B)\in\mathcal{SMR}_{p,\kappa}(s,T)$ and assume that there exists $(\widehat{A},\widehat{B})\in\mathcal{SMR}^\bullet_{p,\kappa}(s,T)$. Then $(A,B)\in\mathcal{SMR}^\bullet_{p,\kappa}(s,T)$ and*

$$C^{\ell,\theta,p,\kappa}_{(A,B)}(s,T)\leq C\big(C^{\ell,\theta,p,\kappa}_{(\widehat{A},\widehat{B})}(s,T), C^{\ell,0,p,\kappa}_{(A,B)}(s,T), C_{A,B}, C_{\widehat{A},\widehat{B}}\big)$$

for all $\ell\in\{\mathsf{det},\mathsf{sto}\}$ and $\theta\in[0,\frac12)\setminus\{\frac{1+\kappa}{p}\}$.

For notational convenience, in this subsection, for any $\theta\in[0,1]$ and any stopping time $\tau:\Omega\to[s,T]$ we set

$$E_\theta(\tau) = L^p_{\mathscr{P}}([\![\tau,T]\!], w^s_\kappa; X_\theta) \quad\text{and}\quad E^\gamma_\theta(\tau) = L^p_{\mathscr{P}}([\![\tau,T]\!], w^s_\kappa;\gamma(H,X_\theta)). \tag{9.5}$$

Finally, we say that $(s_j)_{j=1}^N$ is a partition of $[s,T]$ if $s=s_0<s_1<\dots<s_{N-1}<s_N=T$.

Proposition 9.1.3 (Partitions)**.** *Let Assumption 4.2.1 be satisfied. Suppose that Assumption 6.2.1 holds with $\sigma=s$. Assume that $(A,B)\in\mathcal{SMR}^\bullet_{p,\kappa}(s,T)$. Fix $\delta\in(\frac{1+\kappa}{p},\frac12)$ if $p>2$, or $\delta\in[0,\frac12)$ if $p=2$. Assume that there exists $(\widehat{A},\widehat{B})\in\mathcal{SMR}^\bullet_{p,\kappa}(s,T)$ and let $(s_j)_{j=0}^N$ be a partition of $[s,T]$. Assume that, for all $j\in\{1,\dots,N-1\}$,*

$$(A,B)\in\mathcal{SMR}_{p,\kappa}(s,s_1), \qquad (A,B)\in\mathcal{SMR}_p(s_j,s_{j+1}), \tag{9.6}$$

and let $M>0$ be such that

$$\max\big\{C^{\mathsf{det},0,p,\kappa_j}_{(A,B)}(s_j,s_{j+1}), C^{\mathsf{sto},0,p,\kappa_j}_{(A,B)}(s_j,s_{j+1})\big\}\leq M$$

where $\kappa_0:=\kappa$ and $\kappa_j:=0$ if $j\geq 1$. Then $(A,B)\in\mathcal{SMR}^\bullet_{p,\kappa}(s,T)$ and

$$C^{\ell,0,p,\kappa}_{(A,B)}(s,T)\leq C(N,M,(s_j)_{j=1}^N, C^{\ell,\delta,p,\kappa}_{(\widehat{A},\widehat{B})}(s,T), C_{\widehat{A},\widehat{B}}), \tag{9.7}$$

for $\ell\in\{\mathsf{det},\mathsf{sto}\}$.

Proof. We provide the proof in the case $p > 2$, the other case is simpler. As usual, we set $s = 0$. Let us recall that, by Proposition 6.2.9,

$$(\widehat{A}, \widehat{B}) \in \mathcal{SMR}^{\bullet}_{p,\kappa}(T) \subseteq \mathcal{SMR}^{\bullet}_{p,\kappa}(t,T) \cap \mathcal{SMR}^{\bullet}_{p}(t,T) \tag{9.8}$$

for all $t \in (0,T)$. Thus, by Proposition 9.1.2, it is enough to show that if $(A,B) \in \mathcal{SMR}^{\bullet}_{p,\kappa}(s_n)$ for a given $n \leq N-1$, then $(A,B) \in \mathcal{SMR}_{p,\kappa}(s_{n+1})$ and that (9.7) holds with $s = 0$ and $T = s_n$. We content ourself to construct a unique strong solution u_n to (6.8) on $[\![0, s_n]\!]$ with a corresponding estimate. The fact that all strong solutions to (6.8) on $[\![0,\tau]\!]$ where τ is a stopping time such that $0 \leq \tau \leq s_n$ satisfies $v = u_n|_{[\![0,\tau]\!]}$ follows analogously. For the sake of simplicity, let us set

$$E_{\theta,n} := E_{\theta}(s_n), \quad E^{\gamma}_{\theta,n} := E^{\gamma}_{\theta}(s_n), \quad \text{for all } \theta \in [0,1],\, n \in \{1,\dots,N\},$$

see (9.5). Consider the problem (6.8) on $[\![0, s_n]\!]$ with

$$u_s = 0, \qquad f \in E_{0,n}, \qquad \text{and} \qquad g \in E^{\gamma}_{1/2,n}.$$

Fix $\varepsilon \in (0, s_1)$. Since $(A,B) \in \mathcal{SMR}^{\bullet}_{p,\kappa}(s_{n-1})$, by Proposition 6.2.6 there exists a unique strong solution u_{n-1} to (6.8) on $[\![0, s_{n-1}]\!]$ such that

$$\|u_{n-1}\|_{E_{1,n-1}} + \|u_{n-1}(s_{n-1})\|_{L^p(\Omega;X^{\mathsf{Tr}}_p)} \lesssim \|f\|_{E_{0,n-1}} + \|g\|_{E^{\gamma}_{1/2,n-1}}, \tag{9.9}$$

where the implicit constant depends on $\varepsilon, n, (\widehat{A},\widehat{B}), C^{\mathsf{det},0,p,\kappa}_{(A,B)}(s_{n-1}), C^{\mathsf{sto},0,p,\kappa}_{(A,B)}(s_{n-1})$. By (9.6), Proposition 9.1.2 and (9.8), one has $(A,B) \in \mathcal{SMR}^{\bullet}_{p}(s_j, s_{j+1})$. Thus, by (9.9) exists a unique strong solution U_n to the problem

$$\begin{cases} dU_n(t) + A(t)U_n(t)dt = f(t)dt + (B(t)U_n(t) + g(t))dW_H(t), \quad t \in [\![s, s_n]\!], \\ U_n(s_{n-1}) = u_{n-1}(s_{n-1}) \end{cases}$$

which satisfies

$$\begin{aligned} \|U_n\|_{L^p((s_{n-1},s_n)\times\Omega;X_1)} &\lesssim \|u_{n-1}(s_{n-1})\|_{L^p(\Omega;X^{\mathsf{Tr}}_p)} + \|f\|_{E_{0,n}} + \|g\|_{E^{\gamma}_{1/2,n}} \\ &\lesssim \|f\|_{E_{0,n}} + \|g\|_{E^{\gamma}_{1/2,n}} \end{aligned} \tag{9.10}$$

where in the last inequality we have used (9.9). Setting $u_n := u_{n-1}$ on $I_{s_{n-1}} \times \Omega$ and $u_n := U_n$ on $(s_{n-1}, s_n) \times \Omega$, one can readily check that u_n is a strong solution to (6.8). Moreover, by (9.9)-(9.10) one has $u_n \in E_{1,n}$. Keeping track on the constants in (9.9)-(9.10), one can check that (9.7) holds. □

Let us conclude with a perturbation result which extends Corollary 6.2.11 in case of a deterministic starting time.

Theorem 9.1.4 (Perturbation). *Let Assumption 4.2.1 be satisfied. Suppose that Assumption 6.2.1 holds with $\sigma = s$. Assume that $(A,B) \in \mathcal{SMR}^{\bullet}_{p,\kappa}(s,T)$. Let $A_0 : [s,T] \times \Omega \to \mathscr{L}(X_1, X_0)$, $B_0 : [s,T] \times \Omega \to \mathscr{L}(X_1, \gamma(H, X_{1/2}))$ be strongly progressively measurable such that for some positive constants C_A, C_B, L_A, L_B and for all $x \in X_1$, a.s. for all $t \in (s,T)$,*

$$\begin{aligned} \|A_0(t,\omega)x\|_{X_0} &\leq C_A\|x\|_{X_1} + L_A\|x\|_{X_0}, \\ \|B_0(t,\omega)x\|_{\gamma(H,X_{1/2})} &\leq C_B\|x\|_{X_1} + L_B\|x\|_{X_0}. \end{aligned}$$

Fix $\delta \in (\frac{1+\kappa}{p}, \frac{1}{2})$ if $p > 2$, or $\delta \in [0, \frac{1}{2})$ if $p = 2$. Then there exists

$$\varepsilon\big(p, \kappa, X_0, C_A, C_B, L_A, L_B, K^{\mathsf{det},\delta,p,\kappa}_{(A,B)}(s,T), K^{\mathsf{sto},\delta,p,\kappa}_{(A,B)}(s,T)\big) \in (0,1)$$

such that if

$$C^{\mathsf{det},0,p,\kappa}_{(A,B)} C_A + C^{\mathsf{sto},0,p,\kappa}_{(A,B)} C_B < \varepsilon,$$

then $(A + A_0, B + B_0) \in \mathcal{SMR}^{\bullet}_{p,\kappa}(s,T)$ *and, for* $\ell \in \{\mathsf{sto}, \mathsf{det}\}$,

$$C^{\ell,0,p,\kappa}_{(A+A_0,B+B_0)}(s,T) \le C(p,\kappa,X_0,C_A,C_B,L_A,L_B,K^{\mathsf{det},\delta,p,\kappa}_{(A,B)}(s,T),K^{\mathsf{sto},\delta,p,\kappa}_{(A,B)}(s,T)).$$

Finally, if $\kappa = 0$*, then one can choose* $\varepsilon = 1$.

Proof. As usual, we set $s = 0$. The claim will be proven by using Proposition 9.1.3. By Proposition 9.1.2 it is enough to show that $(A + A_0, B + B_0) \in \mathcal{SMR}_{p,\kappa}(T)$. For the sake of convenience, we divide the proof into three steps.

Step 1: There exists $s_1, C_1 > 0$ *depending only on* p, κ, X_0, C_A, C_B, L_A, L_B, $C^{\mathsf{det},0,p,\kappa}_{(A,B)}(T)$ $C^{\mathsf{sto},0,p,\kappa}_{(A,B)}(T)$ *such that* $(A,B) \in \mathcal{SMR}_{p,\kappa}(s_1)$ *and*

$$C^{\mathsf{det},0,p,\kappa}_{(A,B)}(s_1) + C^{\mathsf{sto},0,p,\kappa}_{(A,B)}(s_1) \le C_1.$$

Let $t \in [0,T]$ and let $\tau : \Omega \to [0,t]$ be a stopping time. The claim of this step will be proven by employing Proposition 6.2.10. To this end, let $E_\theta(\tau), E^{\gamma}_{\theta}(\tau)$ be as in (9.5). For any $\lambda \in [0,1]$, set $A_\lambda := A + \lambda A_0$, $B_\lambda := B + \lambda B_0$. Since $\varepsilon < 1$, one has

$$\eta := 1 - C^{\mathsf{det},0,p,\kappa}_{(A,B)}(T) C_A + C^{\mathsf{sto},0,p,\kappa}_{(A,B)}(T) C_B > 0.$$

Let $\mathscr{R} := \mathscr{R}_{0,(A,B)}$ be the solution operator associated to (A,B). With the above choice of A_λ, B_λ and Proposition 6.2.7, any strong solution u to (6.20) on $[\![0,\tau]\!]$ satisfies

$$u = \mathscr{R}(0, \mathbf{1}_{[\![0,\tau]\!]}(f - \lambda A_0 u), \mathbf{1}_{[\![0,\tau]\!]}(g + \lambda B_0 u)), \qquad \text{on } [\![0,\tau]\!].$$

For notational convenience, we set $v := \mathscr{R}(0, \mathbf{1}_{[\![0,\tau]\!]}(f - \lambda A_0 u), \mathbf{1}_{[\![0,\tau]\!]}(g + \lambda B_0 u))$ on $[\![0,t]\!]$. Thus, the former implies $v|_{[\![0,\tau]\!]} = u$. Using that $C^{\ell,0,p,\kappa}_{(A,B)}(t) \le C^{\ell,0,p,\kappa}_{(A,B)}(T)$ for $t \le T$ and $\ell \in \{\mathsf{det}, \mathsf{sto}\}$ (see Proposition 6.2.9), the former yields

$$\begin{aligned}
\|u\|_{E_1(\tau)} &\le \|v\|_{E_1(t)} \\
&\le C^{\mathsf{det},0,p,\kappa}_{(A,B)}(T)\|f + \lambda A_0 u\|_{E_0(\tau)} + C^{\mathsf{det},0,p,\kappa}_{(A,B)}(T)\|g + \lambda B_0 u\|_{E^{\gamma}_{1/2}(\tau)} \\
&\le (1-\eta)\|u\|_{E_1(\tau)} + \big[C^{\mathsf{det},0,p,\kappa}_{(A,B)}(T) L_A + C^{\mathsf{sto},0,p,\kappa}_{(A,B)}(T) L_B\big]\|u\|_{E_0(\tau)} \\
&\quad + C^{\mathsf{det},0,p,\kappa}_{(A,B)}(T)\|f\|_{E_0(\tau)} + C^{\mathsf{sto},0,p,\kappa}_{(A,B)}(T)\|g\|_{E^{\gamma}_{1/2}(\tau)}
\end{aligned} \tag{9.11}$$

for all $\lambda \in [0,1]$. Since $\eta < 1$, it remains to estimate $\|u\|_{E_0(\tau)}$. By Lemma 4.2.13

$$\begin{aligned}
\|u\|_{E_0(\tau)} &\le c(t)\big(\|f - \lambda A_0 u\|_{E_0(\tau)} + \|g + \lambda B_0 u\|_{E^{\gamma}_{1/2}(\tau)}\big) \\
&\le C(t)\big(\|u\|_{E_1(\tau)} + \|f\|_{E_0(\tau)} + \|g\|_{E^{\gamma}_{1/2}(\tau)}\big)
\end{aligned} \tag{9.12}$$

where $C(t)$ depend only on $p, X_0, \kappa, C_A, C_B, L_A, L_B, C_{A,B}$ and verifies $\lim_{t\downarrow 0} C(t) = 0$. Choose $s_1 > 0$ such that

$$\big[C^{\mathsf{det},0,p,\kappa}_{(A,B)}(T) L_A + C^{\mathsf{sto},0,p,\kappa}_{(A,B)}(T) L_B\big] C(s_1) < \frac{\eta}{2}.$$

Then (9.11)-(9.12) readily uniform estimate in $\lambda \in [0,1]$ for $\|u\|_{E_1(s_1)}$. Finally, Proposition 6.2.10 yields the claim of this step.

Step 2: Let s_1 *be as in Step 1. There exist* $s', C' > 0$ *depending only on* p, κ, X_0, C_A, C_B, L_A, L_B, $C^{\mathsf{det},0,p,\kappa}_{(A,B)}(s,T)$ $C^{\mathsf{sto},0,p,\kappa}_{(A,B)}(s,T)$ *such that for each* $t \in [s_1, T)$ *one has* $(A,B) \in \mathcal{SMR}_p(t,t')$ *with* $t' := \min\{t + s', T\}$ *and*

$$\max\big\{C^{\mathsf{det},0,p,0}_{(A,B)}(t,t'), C^{\mathsf{sto},0,p,0}_{(A,B)}(t,t')\big\} \le C'.$$

By Proposition 6.2.9, for all $\ell \in \{\mathsf{det}, \mathsf{sto}\}$ and $t' > t$,

$$C^{\ell,0,p,0}_{(A,B)}(t,t') \le s_1^{-\kappa/p} C^{\ell,0,p,\kappa}_{(A,B)}(T).$$

Setting $\varepsilon := \min\{s_1^{\kappa/p}, 1\}$, the assumption and the previous yield

$$C^{\mathsf{det},0,p,0}_{(A,B)}(t,t')C_A + C^{\mathsf{sto},0,p,0}_{(A,B)}(t,t')C_B < 1. \tag{9.13}$$

Up to a translation argument, the claim of this step follows by repeating the argument in Step 1 with $\kappa = 0$ and (9.13).

Step 3: Conclusion. Let s_1 and $s' > 0$ be as in Step 1 and 2, respectively. The claim follows from Steps 1-2 and Proposition 9.1.3 by setting $(\widehat{A}, \widehat{B}) = (A, B)$, $s_j := \min\{s_1 + js', T\}$ for $j \geq 2$ and choosing $N \in \mathbb{N}$ so that $s_1 + Ns' > T$. ☐

9.2 Maximal L^p-regularity for the turbulent Stokes couple

In this subsection we study maximal regularity estimates for the *turbulent stochastic Stokes system* on $\mathbb{T}^d$:

$$\begin{cases} du - \mathcal{P}\mathcal{A}u\, dt = f dt + \sum_{n\geq 1} \big(\mathcal{P}\mathcal{B}_n u + g_n\big) dw_t^n, & \text{on } \mathbb{T}^d, \\ u(0) = u_0, & \text{on } \mathbb{T}^d. \end{cases} \tag{9.14}$$

Here $\mathcal{P}$ denotes the Helmholtz projection, and for all $t \in I_T$ and sufficiently smooth maps $v := (v^k)_{k=1}^d$,

$$\mathcal{A}(t)v = \mathrm{div}(a(t) \cdot \nabla v), \qquad \text{and} \qquad \mathcal{B}_n(t)v = (\phi_n \cdot \nabla)v. \tag{9.15}$$

More explicitly, the above differential operators are given by

$$\mathcal{A}(t)v = \Big(\sum_{i,j=1}^d \partial_i(a^{i,j}(t,\cdot)\partial_j v^k)\Big)_{k=1}^d, \qquad \mathcal{B}_n(t)v = \Big(\sum_{j=1}^d \phi_n^j(t,\cdot)\partial_j v^k\Big)_{k=1}^d.$$

We prove sharp L^p-estimates for (9.14) under the following assumption. In the following $\delta \in (-1, 0]$ is fixed. Note that, if the following assumption is satisfied $\delta = 0$, then it is also satisfied for $\delta \in (-\varepsilon, 0)$ where $\varepsilon > 0$.

Assumption 9.2.1. *Assume that $0 \leq s < T < \infty$ and let $\delta \in (-1, 0]$ be fixed.*

(1) *Let one of the following be satisfied:*

- *$q \in [2, \infty)$, $p \in (2, \infty)$ and $\kappa \in [0, \frac{p}{2} - 1)$;*
- *$q = p = 2$ and $\kappa = 0$.*

(2) *$a^{i,j} : [\![s, T]\!] \times \mathbb{T}^d \to \mathbb{R}$ and $(\phi_n^j)_{n\geq 1} : [\![s, T]\!] \times \mathbb{T}^d \to \ell^2$ are $\mathscr{P} \otimes \mathscr{B}(\mathbb{T}^d)$-measurable.*

(3) *There exist $\alpha > |\delta|$ and a constant C_a such that, a.s. for all $t \in I_T$,*

$$\|a^{i,j}(t,\cdot)\|_{C^\alpha(\mathbb{T}^d)} \leq C_a, \qquad \textit{for all } i, j \in \{1, \dots, d\},$$
$$\|(\phi_n^j(t,\cdot))_{n\geq 1}\|_{C^\alpha(\mathbb{T}^d;\ell^2)} \leq C_\phi, \qquad \textit{for all } j \in \{1, \dots, d\}.$$

(4) *There exists $\vartheta > 0$ such that, a.s. for all $t \in (s, T)$, $\xi \in \mathbb{R}^d$ and $x \in \mathbb{T}^d$,*

$$\sum_{i,j=1}^d \Big(a^{i,j}(t,x) - \frac{1}{2}\Big(\sum_{n\geq 1} \phi_n^i(t,x)\phi_n^j(t,x)\Big)\Big)\xi_i\xi_j \geq \vartheta|\xi|^2.$$

The Hölder regularity of ϕ_n^j fits the physical motivation (1.13). Under the previous assumption we introduce the *turbulent Stokes couple*:

$$\begin{aligned} &(A^{\mathcal{S}}_{\delta,q}, B^{\mathcal{S}}_{\delta,q}) : I_T \times \Omega \to \mathscr{L}(\mathbb{H}^{1+\delta,q}, \mathbb{H}^{-1+\delta,q} \times \gamma(\ell^2, \mathbb{H}^{\delta,q})) \\ &(A^{\mathcal{S}}_{\delta,q}u, B^{\mathcal{S}}_{\delta,q}u) := (-\mathcal{P}\mathcal{A}(\cdot)u, (\mathcal{P}\mathcal{B}_n(\cdot)u)_{n\geq 1}), \qquad u \in \mathbb{H}^{1+\delta,q}, \end{aligned} \tag{9.16}$$

where $\mathbb{H}^{s,q}$ is the space of divergence-free fields in $H^{s,q}(\mathbb{T}^d;\mathbb{R}^d)$ (see (9.21)). Note that $\delta = 0$ correspond with the usual weak setting. However, in applications to (9.1) it will be very important also to allow $\delta < 0$.

The main result of this section reads as follows.

Theorem 9.2.2. *Let Assumption 9.2.1 be satisfied for some $\delta \in (-1, 0]$. Then the problem (9.14) has stochastic maximal L^p-regularity i.e.*

$$(A^{\mathcal{S}}_{\delta,q}, B^{\mathcal{S}}_{\delta,q}) \in \mathcal{SMR}^{\bullet}_{p,\kappa}(s,T), \quad \text{with } X_0 = \mathbb{H}^{-1+\delta,q}(\mathbb{T}^d),\ X_1 = \mathbb{H}^{1+\delta,q}(\mathbb{T}^d).$$

Moreover, for each $\theta \in [0, \frac{1}{2})$, one has

$$\max\left\{K^{\mathsf{det},\theta,p,\kappa}_{(A^{\mathcal{S}}_{\delta,q},B^{\mathcal{S}}_{\delta,q})}, K^{\mathsf{sto},\theta,p,\kappa}_{(A^{\mathcal{S}}_{\delta,q},B^{\mathcal{S}}_{\delta,q})}\right\} \leq C(q,p,\kappa,d,\delta,\alpha,C_{a,\phi},\theta). \tag{9.17}$$

We conclude by mentioning that combining Theorem 9.1.4 and 9.2.2 we could add to the $\mathcal{B}_n$-term in (9.15) a gradient-noise term of the form $(\sum_{i,j=1}^d \Phi^{j,k}_{n,i}\partial_j u^i)_{k=1}^d$ provided Φ is *small* in a suitable norm. To this end one can use the estimates in Proposition 9.1.1 (cf. Step 0 and 7 in the proof of Lemma 9.2.3).

9.2.1 The functional analytic set-up

For any $s \in \mathbb{R}$ and $q \in (1,\infty)$, we denote by $L^q := L^q(\mathbb{T}^d;\mathbb{R}^d)$, $H^{s,q} := H^{s,q}(\mathbb{T}^d;\mathbb{R}^d)$ and $B^s_{q,p} := B^s_{q,p}(\mathbb{T}^d;\mathbb{R}^d)$ the Lebesgue, Bessel-potential and Besov spaces on $\mathbb{T}^d$ with values in $\mathbb{R}^d$, respectively. Moreover, we denote by $\mathcal{P}$ the Helmholtz projection which, for $f \in C^\infty(\mathbb{T}^d;\mathbb{R}^d)$ and $n = 1,\dots,d$, is given by

$$(\widehat{\mathcal{P}f})_n(k) := \widehat{f}_n(k) - \sum_{j=1}^d \frac{k_j k_n}{|k|^2}\widehat{f}_j(k), \quad k \in \mathbb{Z}^d \setminus \{0\}, \quad (\widehat{\mathcal{P}f})_n(0) := \widehat{f}_n(0).$$

Here, $\widehat{f}(k)$ denotes the k-th Fourier coefficient of f. We also recall another possible construction of $\mathcal{P}$ which will be useful later on. Note that, for each $f \in C^\infty(\mathbb{T}^d;\mathbb{R}^d)$, there exists a unique $\phi \in C^\infty(\mathbb{T}^d;\mathbb{R}^d)$ such that

$$\begin{cases} \Delta\phi = \operatorname{div} f, & \text{on } \mathbb{T}^d, \\ \int_{\mathbb{T}^d} \phi\, dx = 0. \end{cases} \tag{9.18}$$

If we set $\mathcal{Q}f := \phi$, then the Helmholtz projection is given by

$$\mathcal{P}f := f - \nabla\mathcal{Q}f. \tag{9.19}$$

Fourier multiplier techniques ensure that, for s,q as above,

$$\mathcal{P} : H^{s,q} \to H^{s,q}, \qquad \mathcal{Q} : H^{s,q} \to H^{s+1,q}. \tag{9.20}$$

In addition, $\mathcal{P}, \nabla\mathcal{Q}$ are projections (i.e. $\mathcal{P} = \mathcal{P}^2$, $\nabla\mathcal{Q} = (\nabla\mathcal{Q})^2$). Lastly, we define $\mathbb{L}^q := \mathbb{L}^q(\mathbb{T}^d)$, $\mathbb{H}^{s,q} := \mathbb{H}^{s,q}(\mathbb{T}^d)$, $\mathbb{B}^s_{q,p} := \mathbb{B}^s_{q,p}(\mathbb{T}^d)$ be the set of all divergence free vector-field on $\mathbb{T}^d$ which belongs to L^q, $H^{s,q}$ and $B^s_{q,p}$, respectively:

$$\mathsf{B} = \mathcal{P}(\mathsf{A}), \qquad (\mathsf{A},\mathsf{B}) \in \{(L^q,\mathbb{L}^q), (H^{s,q},\mathbb{H}^{s,q}), (B^s_{q,p},\mathbb{B}^s_{q,p})\}. \tag{9.21}$$

cf. [63, Definition 1.48]. Combining [198, Thereom 1.2.4] and [20, Theorem 6.4.5], for each $s_0, s_1 \in \mathbb{R}$, $p,q \in (1,\infty)$ and $\theta \in (0,1)$, one has

$$\mathbb{H}^{s_0\theta+(1-\theta)s_1,q} = [\mathbb{H}^{s_0,q},\mathbb{H}^{s_1,q}]_\theta, \qquad \mathbb{B}^{s_0\theta+(1-\theta)s_1}_{q,p} = (\mathbb{H}^{s_0,q},\mathbb{H}^{s_1,q})_{\theta,p}. \tag{9.22}$$

Let us denote by $\mathcal{S}_{s,q}$ the Stokes operator on $\mathbb{H}^{s,q}$, i.e. $\mathcal{S}_{s,q} : \mathbb{H}^{s+2,q} \subset \mathbb{H}^{s,q} \to \mathbb{H}^{s,q}$, with $\mathcal{S}_{s,q}f := -\mathcal{P}\Delta f$. Since $\mathcal{P}$ and Δ commutes, by [108, Proposition 10.2.18 and Theorem 10.2.25], it follows that $\mathcal{S}_{s,q}$ has a bounded H^∞-calculus with angle 0. By Theorem 4.2.7,

$$\mathcal{S}_{\delta,q} \in \mathcal{SMR}^{\bullet}_{p,\kappa}(s,T), \qquad \text{for all } s,T \text{ such that } 0 \leq s < T < \infty. \tag{9.23}$$

9.2.2 Proof of Theorem 9.2.2

The key result in the proof of Theorem 9.2.2 is the following a priori estimate for solutions to (9.14) on small time intervals.

Lemma 9.2.3 (A priori estimate on small intervals)**.** *Let Assumption 9.2.1 be satisfied. Then there exist $T^\star, C>0$ depending only on $q,p,\kappa,d,\delta,\vartheta,\alpha,\beta,r,C_{a,\phi}$ such that for any $t\in[s,T)$, any stopping time τ with values in $[t,t^\star]$ where $t^\star=(t+T^\star)\wedge T$,*

$$f\in L^p((t,t^\star)\times\Omega,w_\kappa^t;\mathbb{H}^{-\delta,q}),\qquad g\in L^p((t,t^\star)\times\Omega,w_\kappa^t;\gamma(\ell^2,\mathbb{H}^{1-\delta,q})) \tag{9.24}$$

and any strong solution $u\in L^p([\![0,\tau]\!],w_\kappa;\mathbb{H}^{2-\delta,q})$ to (9.14) on $[\![0,\tau]\!]$ one has

$$\|u\|_{L^p([\![0,\tau]\!],w_\kappa;\mathbb{H}^{2-\delta,q})}\le C\big(\|f\|_{L^p([\![0,\tau]\!],w_\kappa;\mathbb{H}^{-\delta,q})}+\|g\|_{L^p([\![0,\tau]\!],w_\kappa;\gamma(\ell^2,\mathbb{H}^{1-\delta,q}))}\big).$$

Before proving Lemma 9.2.3 we show that it implies Theorem 9.2.2.

Proof of Theorem 9.2.2. By a translation argument we may set $s=0$. To begin, note that for any $\lambda\in[0,1]$ and any $v\in\mathbb{H}^{2-\delta,q}$

$$\begin{aligned}A^{\mathcal{S}}_{\delta,q,\lambda}v&:=\lambda\mathcal{S}_{\delta,q}v+(1-\lambda)A^{\mathcal{S}}_{\delta,q}v=-\mathcal{P}\big(\operatorname{div}(a_\lambda(\cdot)\cdot\nabla v)\big),\\ B^{\mathcal{S}}_{\delta,q,\lambda}v&:=\lambda B^{\mathcal{S}}_{\delta,q}v=\big(\mathcal{P}(\lambda(\phi_n\cdot\nabla)v)\big)_{n\ge1}\end{aligned}$$

where we set $a^{i,j}_\lambda=\lambda\delta^{i,j}+(1-\lambda)a^{i,j}$ for all $i,j\in\{1,\dots,d\}$. One can check that the couple $(A^{\mathcal{S}}_{\delta,q,\lambda},B^{\mathcal{S}}_{\delta,q,\lambda})$ satisfies the Assumption 9.2.1 uniformly w.r.t. $\lambda\in[0,1]$. Applying Proposition 6.2.10 and Lemma 9.2.3 twice for κ as in Assumption 9.2.1 and $\kappa=0$ there exist $\gamma,C>0$ depending only on $q,p,\kappa,d,\delta,\vartheta,\alpha,\beta,r,C_{a,\phi}$ such that

$$(A^{\mathcal{S}}_{\delta,q},B^{\mathcal{S}}_{\delta,q})\in\mathcal{SMR}_{p,\ell}(t,(t+\gamma)\wedge T)\quad\text{ for all } t\in[0,T) \text{ and } \ell\in\{0,\kappa\} \tag{9.25}$$

and the constants of maximal L^p-regularity are bounded by C. Set $N:=\lceil T/\gamma\rceil$ and $s_j:=j\gamma$. The claimed result follows by applying Proposition 9.1.3 whose hypotheses are satisfied due to (9.23) and (9.25). □

To prove Lemma 9.2.3 we need two lemmas. The first shows the existence of suitable extension operators and the second one concerns an operator appearing in the proof of Lemma 9.2.3.

Lemma 9.2.4. *Let $\alpha\in(0,N]$, $\mathscr{O}\in\{\mathbb{R}^d,\mathbb{T}^d\}$ and $\mathcal{H}\in\{\mathbb{R},\ell^2\}$. Then, for any $y\in\mathscr{O}$ and any $r\in(0,\frac12)$ there exists an extension operator $\mathcal{E}^{\mathscr{O}}_{y,r}:C^\alpha(B_{\mathscr{O}}(x,r);\mathcal{H})\to C^\alpha(\mathscr{O};\mathcal{H})$ which satisfies the following properties:*

(1) *$\mathcal{E}^{\mathscr{O}}_{y,r}f|_{B_{\mathscr{O}}(y,r)}=f$, $\mathcal{E}^{\mathscr{O}}_{x,r}c\equiv c$ for any $f\in C^\alpha(B_{\mathscr{O}}(y,r);\mathcal{H})$ and $c\in\mathcal{H}$;*

(2) *$\|\mathcal{E}^{\mathscr{O}}_{y,r}\|_{\mathscr{L}(C^\alpha(B_{\mathscr{O}}(y,r);\mathcal{H}),C^\alpha(\mathscr{O};\mathcal{H}))}\le C_r$ for some C_r independent of y;*

(3) *$\|\mathcal{E}^{\mathscr{O}}_{y,r}\|_{\mathscr{L}(\mathrm{BUC}(B_{\mathscr{O}}(y,r);\mathcal{H}),\mathrm{BUC}(\mathscr{O};\mathcal{H}))}\le C$ for some C independent of y,r.*

Proof. We prove only the case $\mathcal{H}=\mathbb{R}$, the other follows similarly.

Step 1: The case $\mathscr{O}=\mathbb{R}^d$. Note that, by localization and the well-known extension operator [193, Chapter 4, (4.2)] and [151, Example 1.9 and discussion below it], one can check that there exists a bounded linear operator $\mathcal{E}:C^\alpha(B_{\mathbb{R}^d}(1))\to C^\alpha(\mathbb{R}^d)$ such that $\|\mathcal{E}f\|_{L^\infty(\mathbb{R}^d)}\le C\|f\|_{L^\infty(B(1))}$, $\mathcal{E}c=c$ for any $c\in\mathbb{R}$ and it satisfies the extension property, i.e. $\mathcal{E}f|_{B_{\mathbb{R}^d}(1)}=f$ for any $f\in C^\alpha(B_{\mathbb{R}^d}(1))$. Let $\varphi\in C^\infty_0(\mathbb{R}^d)$ be such that $0\le\varphi\le1$, $\varphi|_{B_{\mathbb{R}^d}(1)}=1$ and $\varphi|_{\mathbb{R}^d\setminus B_{\mathbb{R}^d}(2)}=0$. For any $f\in C(B_{\mathbb{R}^d}(1))$, we set

$$(\mathcal{E}^{\mathbb{R}^d}_{0,1}f)(x):=\varphi(x)\mathcal{E}f(x)+(1-\varphi(x))f(0),\qquad x\in\mathbb{R}^d.$$

One can readily check that $\mathcal{E}^{\mathbb{R}^d}_{0,1}$ inherit the property of $\mathcal{E}$ and, in addition, $\mathcal{E}_{0,1}f = f(0)$ on $\mathbb{R}^d \setminus B_{\mathbb{R}^d}(2)$ is constant. Setting

$$\mathcal{E}^{\mathbb{R}^d}_{y,r}f(x) = \mathcal{E}^{\mathbb{R}^d}_{0,1}[f(y+r\cdot)]\Big(\frac{x-y}{r}\Big), \qquad x\in\mathbb{R}^d.$$

One can readily check that $\mathcal{E}^{\mathbb{R}^d}_{y,r}$ has the desired property.

Step 2: The case $\mathscr{O}=\mathbb{T}^d$. Let $\iota:\mathbb{R}^d\to\mathbb{R}^d/\mathbb{Z}^d=\mathbb{T}^d$ be the quotient map. For any $B_{\mathbb{T}^d}(y,r)\subset\mathbb{T}^d$ let $y\in[0,1)^d$ be such that $\iota(B_{\mathbb{R}^d}(y,r))=B_{\mathbb{T}^d}(y,r)$. For any $x\in y+[0,1)^d=:Q_y$, we set

$$\mathcal{E}^{\mathbb{T}^d}_{x,r}(f)(\iota x) := \mathcal{E}^{\mathbb{R}^d}_{y,r}f(x). \tag{9.26}$$

Since $\iota(Q_y)=\mathbb{T}^d$, the above formula completely determinates $\mathcal{E}^{\mathbb{T}^d}_{y,r}$ on $\mathbb{T}^d$. It remains to check that $\mathcal{E}^{\mathbb{T}^d}_{y,r}f$ is well-defined on $\mathbb{T}^d$. To see this, note that $\mathrm{dist}(\partial Q_y, B_{\mathbb{R}^d}(y,2r))>0$ since $r<\frac12$. Moreover, by construction $\mathcal{E}^{\mathbb{R}^d}_{y,r}f|_{Q_y\setminus B_{\mathbb{R}^d}(y,2r)}\equiv f(y)$ and thus $\mathcal{E}^{\mathbb{R}^d}_{y,r}f$ glues well on the periodic torus. Thus, $\mathcal{E}^{\mathbb{T}^d}_{y,r}$ inherits from $\mathcal{E}^{\mathbb{R}^d}_{y,r}$ the required properties. ☐

Lemma 9.2.5. *Let $\phi\in C^\infty(\mathbb{T}^d)$. For any $f\in C^\infty(\mathbb{T}^d;\mathbb{R}^d)$, we set $\mathcal{J}_\phi f=:\psi$, where ψ is the unique solution to the following elliptic problem*

$$\begin{cases}\Delta\psi = \nabla\phi\cdot f - \int_{\mathbb{T}^d}\nabla\phi\cdot f\,dx, & \text{on } \mathbb{T}^d,\\ \int_{\mathbb{T}^d}\psi(x)dx=0.\end{cases} \tag{9.27}$$

Then, for each $s\in\mathbb{R}$, $q\in(1,\infty)$, there exists $C_{s,q}>0$ such that $\|\mathcal{J}_\phi f\|_{H^{s+2,q}}\le C_{s,q}\|f\|_{H^{s,q}}$.

Proof. To begin, let us denote by Δ_R^{-1} the operator defined as $(\widehat{\Delta_R^{-1}f})(0)=0$ and $(\widehat{\Delta_R^{-1}f})(k)=\frac{1}{|k|^2}\widehat{f}(k)$ for $\mathbb{Z}^d\ni k\neq 0$. By standard Fourier techniques, one can show that, for each $s\in\mathbb{R}$ and $q\in(1,\infty)$,

$$\Delta_R^{-1}:\{f\in H^{s,q}:\widehat{f}(0)=0\}\to\{f\in H^{s+2,q}:\widehat{f}(0)=0\}. \tag{9.28}$$

The claim follows by noticing that

$$\mathcal{J}_\phi f = \Delta_R^{-1}(\nabla\phi\cdot f - \langle\nabla\phi, f\rangle)$$

where $\int_{\mathbb{T}^d}\nabla\phi\cdot f\,dx := \langle\nabla\phi,f\rangle_{\mathscr{D}(\mathbb{T}^d),\mathscr{D}'(\mathbb{T}^d)}$ for any $f\in\mathscr{D}'(\mathbb{T}^d)$. ☐

We are ready to prove Lemma 9.2.3. The proof of the remaining cases follow the same strategy and will be proven at the end of this subsection.

Proof of Lemma 9.2.3. The proof will be divided into several steps. Let us begin by collecting some useful facts. Let α be as in Assumption 9.2.1(3). Let $t\in[s,T)$ and let τ be a stopping time such that $t\le\tau\le(t+T^\star)\wedge T=t^\star$ a.s. where $T^\star>0$ will be fixed in Step 6. As we will see the contents of Steps 0-5 hold for $T^\star=T-t$ and a fortiori for all $T^\star\in(0,T-t]$.

Let f,g be as in (9.24). For notational convenience we will set

$$N_{f,g}(t,\tau) := \|f\|_{L^p([\![t,\tau]\!],w^t_\kappa;X_0)} + \|g\|_{L^p([\![t,\tau]\!],w^t_\kappa;\gamma(H,X_{1/2}))}. \tag{9.29}$$

Moreover, for any $y\in\mathbb{T}^d$, $r\in(0,\frac12)$, we set $B(y,r):=B_{\mathbb{T}^d}(y,r)$ and $v\in\mathbb{H}^{2-\delta,q}$

$$\begin{aligned}\mathcal{A}_y(t)v &:= \mathrm{div}(a(t,y)\cdot\nabla v), & \mathcal{B}_y(t) &:= ((\phi_n(t,y)\cdot\nabla)v)_{n\ge1},\\ \mathcal{A}^{\mathcal{E}}_{y,r}(t) &:= \mathrm{div}(a^{\mathcal{E}}_{y,r}(t,\cdot)\cdot\nabla v), & \mathcal{B}^{\mathcal{E}}_{y,r,\ell}(t) &:= ((\phi^{\mathcal{E}}_{n,y,r}(t,\cdot)\cdot\nabla)v)_{n\ge1},\end{aligned} \tag{9.30}$$

where

$$a^{\mathcal{E}}_{y,r} := \Big(\mathcal{E}^{\mathbb{T}^d}_{y,r}(a^{i,j}(t,\cdot))\Big)^d_{i,j=1}, \quad \phi^{\mathcal{E}}_{n,y,r} := \Big(\mathcal{E}^{\mathbb{T}^d}_{y,r}(\phi^j_n(t,\cdot))\Big)^d_{j=1}, \quad n\ge1$$

and $\mathcal{E}^{\mathbb{T}^d}_{y,r}$ is the extension operator provided by Lemma 9.2.4. Comparing the previous definitions with (9.15), one sees that $\mathcal{A}_y, \mathcal{B}_y$ are the operators with "frozen coefficient at $y \in \mathbb{T}^d$" and $\mathcal{A}^{\mathcal{E}}_{y,r}$, $\mathcal{B}^{\mathcal{E}}_{y,r}$ are the operators whose coefficients are the extensions of $a^{i,j}|_{B(y,r)}$, $\phi^j_n|_{B(y,r)}$. Lastly, we set $\mathcal{A}_y := \mathcal{A}_y(t)$, $\mathcal{B}_{y,r} := \mathcal{B}_{y,r}(t) := (\mathcal{B}_{n,y,r}(t))_{n\in\mathbb{N}}$ and similar for $\mathcal{A}^{\mathcal{E}}_{y,r}, \mathcal{B}^{\mathcal{E}}_{y,r}$.

With a slight abuse of notations, in this proof, we also regards $\mathcal{A}, \mathcal{A}_y, \mathcal{A}^{\mathcal{E}}_{y,r}$ (resp. $\mathcal{B}, \mathcal{B}_y, \mathcal{B}^{\mathcal{E}}_{y,r}$) as mapping $I_T \times \Omega \to \mathscr{L}(H^{2-\delta,q}, H^{-\delta,q})$ (resp. $I_T \times \Omega \to \mathscr{L}(H^{2-\delta,q}, \gamma(\ell^2, H^{1-\delta,q}))$) where $H^{2j-\delta,q} := H^{2j-\delta,q}(\mathbb{T}^d;\mathbb{R}^d)$ for $j \in \{0,1\}$. The constants appearing appearing below will depend only on quantities in $\mathscr{Q} := \{q,p,\kappa,d,\delta,\vartheta,\alpha,r,C_{a,\phi}\}$. To stress this we write $C(\mathscr{Q})$ instead of C and similar.

Step 0: The couple $(\mathcal{P}\mathcal{A}, \mathcal{P}\mathcal{B})$ satisfies Assumption 4.2.2 with $C_{(\mathcal{A},\mathcal{B})} \leq C_0(\mathscr{Q})$. Let us begin by noticing that, due to (9.21) we have

$$\|v\|_{\mathbb{H}^{s,q}} = \|v\|_{H^{s,q}}, \qquad \text{for all } v \in \mathbb{H}^{s,q} \text{ and } s \in \mathbb{R}. \tag{9.31}$$

Let us first analyse the $\mathcal{A}$-term. Fix $v \in \mathbb{H}^{2-\delta,q}$. Since $\eta > \delta - 1$, by Proposition 9.1.1, a.e. on $[\![t,T]\!]$

$$\begin{aligned}\|\mathcal{P}\mathcal{A}(\cdot)v\|_{\mathbb{H}^{2-\delta,q}} &\lesssim \|\mathrm{div}(a(\cdot)\cdot\nabla v)\|_{H^{2-\delta,q}}\\ &\leq \|a(\cdot)\cdot\nabla v\|_{H^{1-\delta,q}} \lesssim \Big(\max_{i,j}\|a^{i,j}(\cdot)\|_{C^\alpha}\Big)\|\nabla v\|_{H^{1-\delta,q}}.\end{aligned}$$

By Assumption 9.2.1(3) the previous and (9.31) readily yields

$$\|\mathcal{P}\mathcal{A}(\cdot)v\|_{\mathbb{H}^{2-\delta,q}} \leq C_{\delta,q,d}C_a\|v\|_{\mathbb{H}^{2-\delta,q}}, \qquad \text{a.s. for all } r \in (t,T). \tag{9.32}$$

Next, we prove that $\mathcal{P}\mathcal{A} : [\![t,T]\!] \to \mathscr{L}(\mathbb{H}^{2-\delta,q}, \mathbb{H}^{-\delta})$ is strongly progressively measurable. Since $\mathbb{H}^{s,\zeta} \overset{d}{\hookrightarrow} \mathbb{H}^{\delta-1,\zeta}$ for all $s \geq \delta - 1$ and $\zeta \in (1,\infty)$, by (9.32) and the Pettis measurability theorem it is enough to show that $(t,\omega) \mapsto \langle \mathcal{P}\mathcal{A}(t,\omega)v, v'\rangle_{\mathbb{H}^{-\delta,q},\mathbb{H}^{\delta,q'}}$ is strongly progressively measurable where $v, v' \in C^1(\mathbb{T}^d) \cap \mathbb{H}^{2-\delta,q}$. The latter claim follows by Assumption 9.2.1(2).

The reader can easily check that the same arguments applied to the $\mathcal{B}$-term. This completes Step 0.

Step 1: There exists $C_1(\mathscr{Q}) > 0$ such that for each $y \in \mathbb{T}^d$, one has $(\mathcal{P}\mathcal{A}_y, \mathcal{P}\mathcal{B}_y) \in \mathcal{SMR}_{p,\kappa}(t,T)$ and

$$\max\{K^{\mathsf{sto},0,p,\kappa}_{(\mathcal{P}\mathcal{A}_y,\mathcal{P}\mathcal{B}_y)}(t,T), K^{\mathsf{det},0,p,\kappa}_{(\mathcal{P}\mathcal{A}_y,\mathcal{P}\mathcal{B}_y)}(t,T)\} \leq C_1 \tag{9.33}$$

By (9.18)-(9.19), one can check that

$$\mathcal{P}\mathcal{A}_y f = \mathcal{A}_y\mathcal{P}f, \quad \mathcal{P}\mathcal{B}_y f = \mathcal{B}_y\mathcal{P}f, \qquad \text{on } [\![0,T]\!], \qquad \text{for all } f \in H^{2-\delta,q}.$$

Thanks to the previous and (9.21), it is enough to prove stochastic maximal L^p-regularity for the following couple

$$\begin{aligned}(A_{\delta,q}, B_{\delta,q}) &: I_T \times \Omega \to \mathscr{L}(H^{2-\delta,q}, H^{-\delta,q} \times \gamma(\ell^2, H^{1-\delta,q})),\\ (A_{\delta,q}, B_{\delta,q})u &:= (\mathcal{A}_y u, \mathcal{B}_y u), \qquad\qquad u \in H^{2-\delta,q},\end{aligned}$$

with $\max\{K^{\mathsf{sto},0,p,\kappa}_{(\mathcal{A}_y,\mathcal{B}_y)}(t,T), K^{\mathsf{det},0,p,\kappa}_{(\mathcal{P}\mathcal{A}_y,\mathcal{P}\mathcal{B}_y)}(t,T)\} \leq C_1'(\mathscr{Q})$. Since the coefficients of $\mathcal{A}_y, \mathcal{B}_y$ are x-independent, the last claim follows by [174, Theorem 5.3 and Remark 4.6] (see also Lemma 5.1.2). To see that the constant C_1 does not depend on $t \in (0,T)$ note that the proof of [174, Theorem 5.3] consists in a reduction to the case $\mathcal{B}_y = 0$ via [174, Theorem 3.18] and in such a case the independent of the constants w.r.t. to $t \in (0,T)$ can be obtained using [174, Theorem 3.9] and the deterministic characterization of stochastic maximal L^p-regularity for semigroup generator Proposition 3.1.7.

Step 2: There exists $\eta(\mathscr{Q}) > 0$ for which the following holds:

If $y \in \mathbb{T}^d$ and $r \in (0, \frac{1}{2})$ satisfy, a.s. for all $t \in I_T$, $i,j \in \{1,\dots,d\}$,

$$\|a^{i,j}(t,\cdot) - a^{i,j}(t,y)\|_{L^\infty(B(y,r))} + \|(\phi^j_n(t,\cdot) - \phi^j_n(t,y))_{n\in\mathbb{N}}\|_{L^\infty(B(y,r);\ell^2)} \leq \eta, \tag{9.34}$$

then $(\mathcal{P}\mathcal{A}^{\mathcal{E}}_{y,r},\mathcal{P}\mathcal{B}^{\mathcal{E}}_{y,r})\in\mathcal{SMR}_{p,\kappa}(t,T)$ *and*

$$\max\{K^{\mathsf{sto},0,p,\kappa}_{(\mathcal{P}\mathcal{A}^{\mathcal{E}}_{y,r},\mathcal{P}\mathcal{B}^{\mathcal{E}}_{y,r})}(t,T),K^{\mathsf{det},0,p,\kappa}_{(\mathcal{P}\mathcal{A}^{\mathcal{E}}_{y,r},\mathcal{P}\mathcal{B}^{\mathcal{E}}_{y,r})}(t,T)\}\leq C_2(\mathscr{Q}).$$

Recall that $(\mathcal{P}\mathcal{A}_y,\mathcal{P}\mathcal{B}_y)\in\mathcal{SMR}_{p,\kappa}(t,T)$ and that (9.33) holds for C_1 independent of $y\in\mathbb{T}^d$. To idea is to apply Theorem 9.1.4. To this end, let us write

$$\mathcal{P}\mathcal{A}^{\mathcal{E}}_{y,r}=\mathcal{P}\mathcal{A}_y+\mathcal{P}(\mathcal{A}^{\mathcal{E}}_{y,r}-\mathcal{A}_y),\qquad \mathcal{P}\mathcal{B}^{\mathcal{E}}_{y,r}=\mathcal{P}\mathcal{B}_y+\mathcal{P}(\mathcal{B}^{\mathcal{E}}_{y,r}-\mathcal{B}_y). \tag{9.35}$$

Note that, for each $f\in\mathbb{H}^{2-\delta,q}$ and some $\varepsilon>0$ independent of f,

$$\begin{aligned}
&\|\mathcal{P}(\mathcal{A}^{\mathcal{E}}_{y,r}-\mathcal{A}_y)f\|_{\mathbb{H}^{-\delta,q}}\\
&\quad\leq C_{d,q}\sup_{i,j}\big\|(a^{i,j}(t,y)-\mathcal{E}^{\mathbb{T}^d}_{y,r}(a^{i,j}(t,\cdot))\partial_j f\big\|_{H^{1-\delta,q}}\\
&\quad\overset{(i)}{=} C_{d,q}\sup_{i,j}\big\|\mathcal{E}^{\mathbb{T}^d}_{y,r}[a^{i,j}(t,y)-(a^{i,j}(t,\cdot)]\partial_j f\big\|_{H^{1-\delta,q}}\\
&\quad\overset{(ii)}{\leq} C_{d,\delta,q}\sup_{i,j}\Big(\big\|\mathcal{E}^{\mathbb{T}^d}_{y,r}[a^{i,j}(t,y)-a^{i,j}(t,\cdot)]\big\|_{L^\infty}\|\partial_j f\|_{H^{1-\delta,q}}\\
&\qquad\qquad+\big\|\mathcal{E}^{\mathbb{T}^d}_{y,r}[a^{i,j}(t,y)-a^{i,j}(t,\cdot)]\big\|_{C^\beta}\|\partial_j f\|_{H^{1-\delta-\varepsilon,q}}\Big)\\
&\quad\overset{(iii)}{\leq} C_{d,\delta,q}\Big(\eta\|f\|_{H^{2-\delta,q}}+C_rC_{a,b}\|f\|_{H^{2-\delta-\varepsilon,q}}\Big),
\end{aligned}\tag{9.36}$$

where in (i), (iii) we have used Lemma 9.2.4 and in (ii) Proposition 9.1.1.

Employing Proposition 9.1.1 for $\mathcal{H}=\ell^2$, one can check that that

$$\|\mathcal{P}(\mathcal{B}^{\mathcal{E}}_{y,r}-\mathcal{B}_y)f\|_{\gamma(\ell^2,\mathbb{H}^{1-\delta,q})}\leq C_{d,\delta,q}\Big(\eta\|f\|_{H^{2-\delta,q}}+C_rC_{a,b}\|f\|_{H^{2-\delta-\varepsilon,q}}\Big). \tag{9.37}$$

Therefore, the claim of Step 2 follows by combining Step 1, Theorem 9.1.4 and the above estimates.

Step 3: Let η be as in step 2. There exist $N\geq 1$, $(y_h)_{h=1}^N\subset\mathbb{T}^d$, $(r_h)_{h=1}^N\subset(0,\frac12)$, which depends only on the quantities in $\mathscr{Q}$, such that $\mathbb{T}^d\subset\cup_{h=1}^N B_h$, where $B_h:=B(y_h,r_h)$, and a.s. for all $t\in I_T$, $i,j\in\{1,\dots,d\}$,

$$\|a_{i,j}(t,y_h)-a_{i,j}(t,\cdot)\|_{L^\infty(B_h)}+\|(\phi^j_n(t,y_h)-\phi^j_n(t,\cdot))_{n\in\mathbb{N}}\|_{L^\infty(B_h;\ell^2)}<\eta.$$

In particular, $(\mathcal{P}\mathcal{A}^{\mathcal{E}}_h,\mathcal{P}\mathcal{B}^{\mathcal{E}}_h):=(\mathcal{P}\mathcal{A}^{\mathcal{E}}_{x_h,r_h},\mathcal{P}\mathcal{B}^{\mathcal{E}}_{x_h,r_h})\in\mathcal{SMR}_{p,\kappa}(t,T)$ *and for each* $h\{1,\dots,N\}$

$$\max\{K^{\mathsf{det},0,p,\kappa}_{(\mathcal{P}\mathcal{A}^{\mathcal{E}}_h,\mathcal{P}\mathcal{B}^{\mathcal{E}}_h)}(t,T),K^{\mathsf{sto},0,p,\kappa}_{(\mathcal{P}\mathcal{A}^{\mathcal{E}}_h,\mathcal{P}\mathcal{B}^{\mathcal{E}}_h)}(t,T)\}\leq C_3(\mathscr{Q}). \tag{9.38}$$

The last claim follows by the first one and Step 2. To prove the first claim let us fix $y\in\mathbb{T}^d$ and note that

$$\begin{aligned}
\|a^{i,j}(t,y)-a^{i,j}(t,\cdot)\|_{L^\infty(B(y,r))}&+\|\phi^j(t,y)-\phi^j(t,\cdot)\|_{L^\infty(B(y,r);\ell^2)}\\
&\leq[a^{i,j}(t,\cdot)]_{C^\alpha(B(y,r))}+[\phi^j(t,\cdot)]_{C^\alpha(B(y,r);\ell^2)}\\
&\leq C_{\delta,q}C_{a,\phi}\,r^\alpha<\eta,
\end{aligned}$$

where the last inequality follows by choosing $r\in(0,\frac12)$ small. Note that, r does not depend on y but only on quantities in $\mathscr{Q}$. Since $\mathbb{T}^d$ has finite volume, then the claim of this step follows.

Step 4: Let $(\pi_h)_{h=1}^N$ be a smooth partition of the unity subordinate to the covering $(B_h)_{h=1}^N$ (see Step 3). Recall that f,g are as in (9.24) and τ is a stopping time with values in $[t,(t+T^\star)\wedge T]$ and that $u\in L^p([\![t,\tau]\!],w_\kappa;\mathbb{H}^{2-\delta,q})$ is a strong solution to (9.14) on $[\![0,\tau]\!]$. Then for any $h\in\{1,\dots,N\}$ the following holds

$$\mathcal{P}(\pi_h u)=\mathscr{R}_h(0,\mathsf{F}_h u,\mathsf{G}_h u)+\mathscr{R}_h(0,\mathcal{P}f_h,\mathcal{P}g_h),\qquad\text{a.e. on }[\![t,\tau]\!], \tag{9.39}$$

where $\mathscr{R}_h := \mathscr{R}_{t,(\mathcal{P}\mathcal{A}^{\mathcal{E}}_h,\mathcal{P}\mathcal{B}^{\mathcal{E}}_h)}$ *is the solution operator associated to* $(\mathcal{P}\mathcal{A}^{\mathcal{E}}_h,\mathcal{P}\mathcal{B}^{\mathcal{E}}_h)\in\mathcal{SMR}_{p,\kappa}(t,T)$ *(see* (6.17)*),* $\mathcal{J}_h := \mathcal{J}_{\pi_h}$ *(see Lemma* 9.2.5*),* $g_{n,h} := g_n\pi_h$*,* $f_h := \pi_h f$ *and*

$$\begin{aligned} \mathsf{F}_h u &:= \mathcal{P}\mathcal{A}_h\nabla\mathcal{J}_h u + \mathcal{P}[\pi_h,\mathcal{A}]u + \mathcal{P}[(\nabla\pi_h)\mathcal{Q}(\mathcal{A}u)],\\ \mathsf{G}_h u &:= (\mathsf{G}_{n,h}u)_{n\in\mathbb{N}}\\ \mathsf{G}_{n,h}u &:= \mathcal{P}[(\nabla\pi_h)\mathcal{Q}(\mathcal{B}_n u)] + \mathcal{P}\mathcal{B}_n\nabla\mathcal{J}_h u + \mathcal{P}[\pi_h,\mathcal{B}_n]u \end{aligned}\tag{9.40}$$

and, as usual, $[\cdot,\cdot]$ *denotes the commutator.*

To begin, let us write $\mathcal{P}(\mathcal{A}u)=\mathcal{A}u-\nabla P$ and $\mathcal{P}(\mathcal{B}_n u)=\mathcal{B}_n u-\nabla Q_n$, where we set $P=\mathcal{Q}(\mathcal{A}_h u)$, $Q_n=\mathcal{Q}(\mathcal{B}_n u)$ and $\mathcal{Q}$ is as in Subsection 9.2.1. Finally $z_h := \pi_h u$. Using the previous identities and multiplying (9.14) with $s=t$ by π_h, one obtains

$$\begin{cases} dz_h - \mathcal{A}\,z_h\,dt = ([\pi_h,\mathcal{A}]u - \nabla(\pi_h P) + (\nabla\pi_h)P + f_h)dt\\ \qquad + \sum_{n\geq 1}(\mathcal{B}_n z_h + [\pi_h,\mathcal{B}_n]u - \nabla(\pi_h Q_n) + (\nabla\pi_h)Q_n + g_{n,h})dw^n_t,\\ z_h(t)=0, \end{cases}\tag{9.41}$$

on $\mathbb{T}^d$. Since $\operatorname{supp}(z_h)\subseteq B_h$, for each $h\in\{1,\dots,N\}$ and $n\geq 1$ one has (cf. (9.30))

$$\mathcal{A}\,z_h = \mathcal{A}^{\mathcal{E}}_{x_h,r_h}z_h = \mathcal{A}^{\mathcal{E}}_h z_h,\qquad \mathcal{B}_n z_h = \mathcal{B}^{\mathcal{E}}_{n,x_h,r_h}z_h = \mathcal{B}^{\mathcal{E}}_{n,h}z_h.$$

To conclude, note that the the Helmholtz decomposition gives

$$z_h = \nabla\mathcal{Q}u_h + \mathcal{P}z_h = \nabla\mathcal{J}_h u + \mathcal{P}z_h.$$

To see the last equality, recall that $\operatorname{div} u = 0$ in $\mathscr{D}'(\mathbb{T}^d)$. Thus the previous identity follows by (9.19), (9.27) and

$$\operatorname{div} z_h = \operatorname{div} z_h - \int_{\mathbb{T}^d}\operatorname{div} z_h dx = \nabla\phi_h\cdot u - \int_{\mathbb{T}^d}\nabla\phi_h\cdot u\,dx.$$

Using that $z_h = \nabla\mathcal{J}_h u + v_h$ with $v_h := \mathcal{P}z_h$, and applying the operator $\mathcal{P}$ to (9.41), we get

$$\begin{cases} dv_h - \mathcal{P}\mathcal{A}^{\mathcal{E}}_h v_h\,dt = (\mathcal{P}\mathcal{A}\nabla\mathcal{J}_h u + \mathcal{P}[\pi_h,\mathcal{A}]u + \mathcal{P}[(\nabla\pi_h)P] + \mathcal{P}f_h)dt\\ \quad + \sum_{n\geq 1}(\mathcal{P}\mathcal{B}^{\mathcal{E}}_{n,h}v_h + \mathcal{P}\mathcal{B}_n\nabla\mathcal{J}_h u\\ \qquad\qquad + \mathcal{P}[\pi_h,\mathcal{B}_n]u + \mathcal{P}[(\nabla\pi_h)Q_n] + \mathcal{P}g_{n,h})dw^n_t, & \text{on } \mathbb{T}^d,\\ v_h(t)=0, & \text{on } \mathbb{T}^d. \end{cases}$$

Recall that, by Step 3, one has $(\mathcal{P}\mathcal{A}^{\mathcal{E}}_h,(\mathcal{P}\mathcal{B}^{\mathcal{E}}_{n,h})_{n\in\mathbb{N}})\in\mathcal{SMR}_{p,\kappa}(T)$. Thus the claim follows by Proposition 6.2.7 and the previous system.

Step 5: There exists $\varepsilon(\mathscr{Q}),C(\mathscr{Q})>0$ *such that for each* $v\in\mathbb{H}^{2-\delta,q}$ *one has*

$$\|\mathsf{F}\,v\|_{\mathbb{H}^{-\delta,q}} + \|(\mathsf{G}_n v)_{n\in\mathbb{N}}\|_{\gamma(\ell^2,\mathbb{H}^{1-\delta,q})} \leq C\|v\|_{H^{2-\delta-\varepsilon,q}},\qquad \text{a.e. on } [\![t,T]\!].$$

Let us begin by looking at F. By Step 3, $N<\infty$, thus it is enough to prove suitable estimates for F_h and $h\in\mathbb{N}$ fixed. Let us write $\mathsf{F}_h := \mathsf{F}_{\mathcal{J}} + \mathsf{F}_{\mathcal{A}} + \mathsf{F}_{\mathcal{Q}}$, where for $\pi := \pi_h$ and $æ := æ_h$,

$$\mathsf{F}_{\mathcal{J}}v := \mathcal{P}(\mathcal{A}\nabla\mathcal{J}v),\quad \mathsf{F}_{\mathcal{A}}v := \mathcal{P}[\mathcal{A},\phi]v,\quad \mathsf{F}_{\mathcal{Q}}v := \mathcal{P}[(\nabla\pi)\mathcal{Q}(\mathcal{A}v)].$$

First, we estimate $\mathsf{F}_{\mathcal{J}}$:

$$\begin{aligned} \|\mathsf{F}_{\mathcal{J}}v\|_{\mathbb{H}^{-\delta,q}} &\leq C\max_i\|a^{i,j}(t,\cdot)\partial_j\nabla\mathcal{J}v\|_{H^{1-\delta,q}}\\ &\overset{(i)}{\leq} CC_{a,\phi}\max_j\|\partial_j\nabla\mathcal{J}v\|_{H^{1-\delta,q}} \overset{(ii)}{\leq} CC_{a,\phi}\|v\|_{H^{1-\delta,q}}, \end{aligned}$$

where in (i) we have used Assumption 9.2.1 and that $\alpha > |1-\delta|$, in (ii) the estimate in Lemma 9.2.5. To estimate $\mathsf{F}_{\mathcal{A}}$, let us note that

$$[\mathcal{A},\pi]v = \partial_i(a^{i,j}v)\partial_j\pi + v\,a^{i,j}\partial^2_{i,j}\pi + \partial_j v\,a^{i,j}\partial_i\pi, \quad \text{in } \mathscr{D}'(\mathbb{T}^d).$$

Let us begin by noticing that, for each $i,j = 1,\dots,d$,

$$\begin{aligned}\|\mathcal{P}(\partial_i(a^{i,j}v)\partial_j\phi)\|_{\mathbb{H}^{-\delta,q}} &\leq C\max_j \|\partial_i(a^{i,j}v)\|_{H^{-\delta,q}}\\ &\leq C\max_{i,j}\|a^{i,j}v\|_{H^{1-\delta,q}} \leq CC_{a,\phi}\|v\|_{H^{1-\delta,q}},\end{aligned}$$

where we have used that $\alpha > |1-\delta|$ and Proposition 9.1.1. Choosing $\varepsilon \in (0,\frac{1}{2})$ such that $\alpha \geq |1-\delta| + 2\varepsilon$, one has

$$\begin{aligned}\|\mathcal{P}(\partial_j v\,a^{i,j}\partial_i\pi)\|_{\mathbb{H}^{-\delta,q}} &\leq C\max_i \|a^{i,j}\partial_j v\|_{H^{-\delta,q}}\\ &\leq C\max_i \|a^{i,j}\partial_j v\|_{H^{1-\delta-\varepsilon,q}}\\ &\leq CC_{a,\phi}\max_j\|\partial_j v\|_{H^{1-\delta-\varepsilon,q}} \leq CC_{a,\phi}\|v\|_{H^{2-\delta-\varepsilon,q}}.\end{aligned} \tag{9.42}$$

A similar estimate holds for $\mathcal{P}(v\,a^{i,j}\partial^2_{i,j}\pi)$. This proves the desired estimate for $\mathsf{F}_{\mathcal{A}}$. Lastly, for ε as above and (9.20), $\mathsf{F}_{\mathcal{Q}}$ can be estimates as follows:

$$\|\mathsf{F}_{\mathcal{Q}}v\|_{\mathbb{H}^{-\delta,q}} \lesssim \|\mathcal{Q}(\mathcal{A}v)\|_{H^{-\delta,q}} \lesssim \max_i\|a^{i,j}\partial_j v\|_{H^{-\delta,q}} \leq \|v\|_{H^{2-\delta-\varepsilon,q}},$$

where in the last inequality we have the same argument in (9.42). Putting together the previous estimates one obtains the claim for F.

Next, we look at G. As above, we fix $h\in\{1,\dots,N\}$ and we remove it from the notation. Thus, by (9.40) we have $\mathsf{G}_n := \mathsf{G}_{\mathcal{Q},n} + \mathsf{G}_{\mathcal{J},n} + \mathsf{G}_{\mathcal{B},n}$ where

$$\mathsf{G}_{\mathcal{Q},n}v := \mathcal{P}[(\nabla\pi)\mathcal{Q}(\mathcal{B}_n v)], \quad \mathsf{G}_{\mathcal{J},n}v := \mathcal{P}\mathcal{B}_n\nabla\mathcal{J}v, \quad \mathsf{G}_{\mathcal{B},n}v := \mathcal{P}[(\partial_j\pi)\phi^j_n v] \tag{9.43}$$

where we have used that $[\pi,\mathcal{B}_n]v = (\partial_j\pi)\phi^j_n v$. Note that $\mathsf{G}_{\mathcal{Q},n}, \mathsf{G}_{\mathcal{J},n}$ can be estimated as $\mathsf{F}_{\mathcal{Q}}, \mathsf{F}_{\mathcal{J}}$ above. The last term can be estimated as follows:

$$\begin{aligned}\|(\mathcal{P}([\pi_h,\mathcal{B}_n]v))_{n\geq 1}\|_{\mathbb{H}^{1-\delta,q}(\ell^2)} &\lesssim \max_j\|((\partial_j\pi_h)\phi^j_n v)_{n\geq 1}\|_{L^q(\ell^2)}\\ &\lesssim \max_j\|(\phi^j_n)_{n\geq 1}\|_{L^\infty(\ell^2)}\|v\|_{L^q},\end{aligned}$$

where we have used that $\delta\geq 1$. Since $2-\delta>0$, the previous estimate shows that $\mathcal{P}[\pi_h,\mathcal{B}_n]v$ is a lower-order term.

Step 6: Conclusion. Let (u,f,g,τ,t) be at the beginning of the proof. By Step 3 we know that $(\mathcal{P}\mathcal{A}^{\mathcal{E}}_h, \mathcal{P}\mathcal{B}^{\mathcal{E}}_h)\in\mathcal{SMR}_{p,\kappa}(t,T)$ and that (9.38) holds. Combining the latter and the content of Steps 4-5, one can see that there exist $\varepsilon(\mathscr{Q}), C(\mathscr{Q})>0$ such that for each $h\in\{1,\dots,N\}$

$$\|\mathcal{P}(\pi_h u)\|_{L^p([\![t,\tau]\!],w_\kappa;\mathbb{H}^{2-\delta,q})} \leq CN_{f,g}(t,\tau) + C\|u\|_{L^p([\![t,\tau]\!],w_\kappa;H^{2-\delta-\varepsilon,q})} \tag{9.44}$$

where $N_{f,g}(t,\tau)$ is as in (9.29). As in Step 5, since $\operatorname{div} u = 0$ in $\mathscr{D}'(\mathbb{T}^d)$ a.e. on $[\![t,\tau]\!]$, one has $\pi_h u = \mathcal{P}(\pi_h u) + \nabla\mathcal{J}_{\pi_h}u$. Recall that $u = \sum_{h=1}^N(\pi_h u)$, since $(\pi_h)_{h=1}^N$ is a partition of the unity. The previous considerations yield, for $\varepsilon>0$ as in (9.44),

$$\begin{aligned}&\|u\|_{L^p([\![0,\tau]\!],w_\kappa;H^{2-\delta,q})}\\ &\leq \sum_{h=1}^N\big(\|\mathcal{P}(\pi_h u)\|_{L^p([\![0,\tau]\!],w_\kappa;H^{2-\delta,q})} + \|\nabla\mathcal{J}_{\pi_h}u\|_{L^p([\![0,\tau]\!],w_\kappa;H^{2-\delta,q})}\big)\\ &\overset{(ii)}{\leq} CN_{f,g}(t,\tau) + C\|u\|_{L^p([\![0,\tau]\!],w_\kappa;H^{2-\delta-\varepsilon,q})},\\ &\overset{(iii)}{\leq} CN_{f,g}(t,\tau) + \frac{1}{2}\|u\|_{L^p([\![0,\tau]\!],w_\kappa;\mathbb{H}^{2-\delta,q})} + \widehat{C}\|u\|_{L^p([\![0,\tau]\!],w_\kappa;\mathbb{H}^{-\delta,q})},\end{aligned} \tag{9.45}$$

where in (i) we used (9.44) and Lemma 9.2.5 and in (iii) a standard interpolation inequality and (9.21). Since u is a strong solution to (9.14) on $[\![t,\tau]\!]$, by Step 0 and Lemma 4.2.13 there exists $c_T(\mathscr{Q}) > 0$ such that $\lim_{T\downarrow 0} c_T = 0$ and

$$\|u\|_{L^p([\![t,\tau]\!],w_\kappa;\mathbb{H}^{-\delta,q})} \leq c_T \|u\|_{L^p([\![t,\tau]\!],w_\kappa;\mathbb{H}^{2-\delta,q})} + c_T\, N_{f,g}(t,\tau). \tag{9.46}$$

Combining (9.45)-(9.46), one obtains

$$\|u\|_{L^p([\![t,\tau]\!],w_\kappa;\mathbb{H}^{2-\delta,q})} \leq C N_{f,g}(t,\tau) + \Big(\frac{1}{2} + C_T\Big)\|u\|_{L^p([\![t,\tau]\!],w_\kappa;\mathbb{H}^{2-\delta,q})}, \tag{9.47}$$

where $C_T(\mathscr{Q}) > 0$ satisfies $\lim_{T\downarrow 0} C_T = 0$. Let $T^\star(\mathscr{Q}) > 0$ be such that $C_{T^\star} < \frac{1}{2}$. Then the previous formula implies the claimed a priori estimate in Lemma 9.2.3. $\square$

9.3 Stochastic Navier-Stokes equations for turbulent flows

In this section we take advantage of the work done in the previous sections to study the stochastic Navier-Stokes equations

$$\begin{cases} du - \operatorname{div}(a(\cdot)\cdot\nabla u)\,dt = (-\nabla P + \operatorname{div}(u\otimes u))dt \\ \qquad\qquad\qquad + \sum_{n\geq 1}(-\nabla Q_n + (\phi_n\cdot\nabla)u + \mathcal{G}_n(\cdot,u))dw_t^n, & \text{on } \mathbb{T}^d, \\ \operatorname{div} u = 0, \qquad u(\cdot,0) = u_0, & \text{on } \mathbb{T}^d \end{cases} \tag{9.48}$$

where $u := (u^k)_{k=1}^d : I_T \times \Omega \times \mathscr{O} \to \mathbb{R}^d$ is unknown velocity field and $P, Q_n : I_T \times \Omega \times \mathscr{O} \to \mathbb{R}$ the unknown pressures.

Note that (9.48) generalizes (9.1) since a may depend on (t,ω,x)-dependent which (physically speaking) takes into account the variability in space and time of the viscosity of the fluid. We collect our main results in Subsections 9.3.1-9.3.2 and we provide the proofs in Subsection 9.3.3.

The following assumptions will be in force throughout this section.

Assumption 9.3.1. *Let $d \geq 2$ be an integer and $T \in (0,\infty)$.*

(1) *Assume that one of the following conditions is satisfied:*
- $p = q = 2$ *and* $\kappa = 0$;
- $q \in [2,\infty)$, $p \in (2,\infty)$ *and* $\kappa \in [0, \frac{p}{2} - 1)$.

(2) $\mathcal{F} = (\mathcal{F}_k)_{k=1}^d : I_T \times \Omega \times \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}^d$ *and* $\mathcal{G} = ((\mathcal{G}_n^k)_{k=1}^d)_{n\in\mathbb{N}} : I_T \times \Omega \times \mathbb{T}^d \times \mathbb{R}^d \to \ell^2 \times \mathbb{R}^d$ *are* $\mathscr{P} \otimes \mathscr{B}(\mathbb{T}^d) \otimes \mathscr{B}(\mathbb{R}^d)$*-measurable.*

(3) *For all* $k \in \{1,\dots,d\}$, $\mathcal{F}_k(\cdot,0) \in L^\infty(I_T \times \Omega \times \mathbb{T}^d)$, $\mathcal{G}_k(\cdot,0) \in L^\infty(I_T \times \Omega \times \mathbb{T}^d;\ell^2)$ *and there exists* $C_1, C_2 \geq 0$ *such that, a.s. for all* $t \in I_T$, $x \in \mathbb{T}^d$ *and* $y, y' \in \mathbb{R}^d$,

$$|\mathcal{F}_k(t,x,y) - \mathcal{F}_k(t,x,y')| + \|\mathcal{G}_k(t,x,y) - \mathcal{G}_k(t,x,y')\|_{\ell^2} \lesssim (1 + |y| + |y'|)|y - y'|.$$

9.3.1 Existence, regularization and global 2D-solutions

Let us begin by introducing suitable a notion of solutions to (9.48). Recall that $\mathbb{H}, \mathbb{B}$ denotes Bessel potential and Besov spaces of divergence free vector fields (see Subsection 9.2.1 for the precise definition). Let $\delta \in [1,2)$. We say that (u,σ) is *a (unique almost very) δ-weak solution to* (9.48) *on* $\overline{I}_T$ if (u,σ) is an L^p_κ-maximal local solution to (4.16) with $X_0 := \mathbb{H}^{-1+\delta,q}(\mathbb{T}^d)$, $X_1 := \mathbb{H}^{1+\delta,q}(\mathbb{T}^d)$, $H = \ell^2$, W_{ℓ^2} as in Example 2.3.6 and, for $v \in X_1$,

$$\begin{aligned} A(\cdot)v &= A^{\mathcal{S}}_{\delta,q} v, & B(\cdot)v &= B^{\mathcal{S}}_{\delta,q} v, \\ F(\cdot,v) &= \mathcal{P}\mathcal{F}(\cdot,v) - \mathcal{P}(\operatorname{div}(v\otimes v)), & G(\cdot,v) &= (\mathcal{P}(\mathcal{G}_n(\cdot,v)))_{n\geq 1}. \end{aligned} \tag{9.49}$$

where $A^{\mathcal{S}}_{\delta,q}, B^{\mathcal{S}}_{\delta,q}$ are as in (9.16). By Definitions 4.3.3 and 4.3.4, δ-weak solutions to (4.16) are unique. To motivate our definition of δ-weak solutions to (9.48) let us remark that any 0-weak solutions satisfies the natural weak formulation of (9.48): a.s. for all $\psi = (\psi_k)_{k=1}^d \in \mathbb{H}^{1,q'}(\mathbb{T}^d)$ and $t \in I_T$,

$$\begin{aligned}\int_{\mathbb{T}^d} (u_k(t,\cdot) - u_{0,k})\psi_k\, dx &+ \int_0^t \int_{\mathbb{T}^d} a^{i,j}\partial_j u_k \partial_i \psi_k\, dx\, ds \\ = \int_0^t \int_{\mathbb{T}^d} u_j u_i \partial_j \psi_i\, dx\, ds &+ \int_0^t \int_{\mathbb{T}^d} \mathcal{F}^j(\cdot, u)\psi_j\, dx\, ds \\ + \sum_{n\geq 1} \int_0^t \int_{\mathbb{T}^d} \Big(\phi^j_n \partial_j u_k &+ \mathcal{G}_{n,k}(\cdot, u)\Big)\psi_k\, dx\, dw^n_s\end{aligned} \tag{9.50}$$

where we have used the Einstein summation convention. The previous follows by Definition 4.3.3, an integration by parts and $\mathbb{H}^{-1,q}(\mathbb{T}^d) = (\mathbb{H}^{1,q'}(\mathbb{T}^d))^*$.

Theorem 9.3.2 (Local existence in the critical spaces $\mathbb{B}^{\frac{d}{q}-1}_{q,p}$)**.** *Let Assumption 9.2.1 with $\delta \in [-\frac{1}{2}, 0]$. Let Assumption 9.3.1 be satisfied. Assume that either $p = q = d = 2$ or*

$$q \in [2, \infty),\ p \in (2, \infty) \ \ \textit{satisfy} \ \ \frac{d}{2+\delta} < q < \frac{d}{1+\delta}, \quad \textit{and} \quad \frac{2}{p} + \frac{d}{q} \leq 2 + \delta. \tag{9.51}$$

Set $\kappa_{\mathsf{crit}} := -1 + \frac{p}{2}(2 + \delta - \frac{d}{q})$. Then for any $u_0 \in L^0_{\mathscr{F}_0}(\Omega; \mathbb{B}^{\frac{d}{q}-1}_{q,p}(\mathbb{T}^d))$, (9.48) has a δ-weak solution (u, σ) such that

$$u \in L^p_{\rm loc}([0, \sigma), w_{\kappa_{\mathsf{crit}}}; \mathbb{H}^{1+\delta,q}(\mathbb{T}^d)) \cap C([0, \sigma); \mathbb{B}^{\frac{d}{q}-1}_{q,p}(\mathbb{T}^d)), \quad a.s.$$

Moreover, (u, σ) instantaneously regularizes in time and space:

$$u \in \bigcap_{\theta\in[0,1/2)} H^{\theta,r}_{\rm loc}(I_\sigma; \mathbb{H}^{1-2\theta,\zeta}(\mathbb{T}^d)) \ a.s. \qquad \textit{for all } r, \zeta \in (2, \infty). \tag{9.52}$$

In particular,

$$u \in \bigcap_{\theta\in[0,1/2)} C^{\theta}_{\rm loc}(I_\sigma; C^{1-2\theta}(\mathbb{T}^d)) \subseteq \bigcap_{\theta_1,\theta_2\in[0,1/2)} C^{\theta_1,\theta_2}_{\rm loc}(I_\sigma \times \mathbb{T}^d) \ a.s. \tag{9.53}$$

Since $\alpha > 0$, it is always possible to apply Theorem 9.3.2 for some $\delta < 0$. Note that, if δ increases, then the upper bound in (9.51) becomes less restrictive. In the limiting case $\delta = -\frac{1}{2}$, one has $q < 2d$ and therefore Theorem 9.3.2 yields local existence and smoothness for initial data in Besov spaces with smoothness up to $-\frac{1}{2}$. The study of the optimality of such threeshold goes beyond the scope of this thesis.

(9.52)-(9.53) yield instantaneous regularization results for (9.48). As showed in the introduction, (9.53) is *not* trivial even in the 2D-case. In the latter case, under a sublinearity assumption, one can even prove the solution to (9.48) are global in time.

Theorem 9.3.3 (Global existence in 2D-case)**.** *Let the assumptions of Theorem 9.3.2 be satisfied with $d = 2$. Assume that, a.s. for all $j \in \{1, 2\}$, $t \in I_T$ and $x \in \mathbb{T}^d$,*

$$\|(\mathcal{G}^j_n(t, x, y))_{n\geq 1}\|_{\ell^2} + |\mathcal{F}^j(t, x, y)| \lesssim (1 + |y|).$$

Then (u, σ) is global in time, i.e. $\sigma = \infty$ a.s.

Note that Theorem 9.3.3 holds for $u_0 \in L^0_{\mathscr{F}_0}(\Omega; \mathbb{B}^{\frac{2}{q}-1}_{q,p}(\mathbb{T}^2))$ with $q \in [2, \frac{d}{1+\delta})$. Since one can always choose $\delta < 0$ and $\mathbb{L}^2(\mathbb{T}^2) \hookrightarrow \mathbb{B}^{\frac{2}{q}-1}_{q,p}(\mathbb{T}^2)$ for all $p \geq 2$, Theorem 9.3.3 extends the usual global existence for L^2-data to (9.48).

Theorem 9.3.3 follows immediately from the extrapolation Lemma 7.1.9, the regularization and global existence for solutions with L^2-data given by Theorem 9.3.2 with $q = p = d = 2$, $\kappa = 0$ and Theorem 9.4.3 below. In Subsection 9.4 we will give additional result for the 2D stochastic Navier-Stokes with L^2-data. In particular, we remove any smoothness condition on $a^{i,j}, \phi_n^j$ and we also study stochastic Navier-Stokes equations on Lipschitz domains with no-slip boundary conditions.

Next we show how derive from Theorem 9.3.2 a local existence result for data in the scaling invariant space $\mathbb{L}^d(\mathbb{T}^d)$. Since the case $d = 2$ was already treated in Theorem 9.3.2 we will assume $d \geq 3$.

Corollary 9.3.4 (Local existence in the critical space $\mathbb{L}^d$)**.** *Let Assumption 9.2.1 for some $\bar{\delta} \in [-\frac{1}{3}, 0)$. Let Assumption 9.3.1 be satisfied. Assume that $d \geq 3$ and $p \in [d, \infty)$. Then for each $\delta \in [\bar{\delta}, 0)$ and $u_0 \in L^0_{\mathscr{F}_0}(\Omega; \mathbb{L}^d(\mathbb{T}^d))$, (9.48) has a δ-weak solution (u, σ) such that*

$$u \in L^p_{\mathrm{loc}}([0,\sigma), w_\ell; \mathbb{H}^{1+\delta,d}(\mathbb{T}^d)) \cap C([0,\sigma); \mathbb{B}^0_{d,p}(\mathbb{T}^d)), \quad a.s.$$

where $\ell = -1 + \frac{p}{2}(1+\delta)$. Moreover, (u, σ) satisfies (9.52)-(9.53).

Recall that, if Assumption 9.2.1 holds for $\delta = 0$, then it also holds for some $\delta < 0$. In particular, Corollary 9.3.4 can be applied provided Assumption 9.2.1 holds for $\delta = 0$.

Proof of Corollary 9.3.4. Let us begin by noticing that, Theorem 9.3.2 is applicable with $q = d$, $p \geq d$ and $\delta \in [\bar{\delta}, 0)$ since $\bar{\delta} \geq -\frac{1}{3}$. Recall that

$$L^d(\mathbb{T}^d; \mathbb{R}^d) \hookrightarrow B^0_{d,p}(\mathbb{T}^d; \mathbb{R}^d), \qquad \text{for all } p \in [d, \infty).$$

Thus $\mathbb{L}^d(\mathbb{T}^d) \hookrightarrow \mathbb{B}^0_{d,p}(\mathbb{T}^d)$ by (9.21) and the claim follows by Theorem 9.3.2 with $q = d$, $p \geq d$ and δ as above. ☐

9.3.2 Blow-up criteria

In this subsection we collect several blow-up criteria for (9.48). The following result extends the *Serrin blow-up* criteria to the stochastic setting (see e.g. [144, Theorem 11.2]) and might be used in combination with a priori estimates to prove the that solutions to (9.48) are global.

Theorem 9.3.5 (Stochastic Serrin's blow-up criteria)**.** *Let Assumption 9.2.1 with $\delta \in [-\frac{1}{2}, 0)$. Let Assumption 9.3.1 be satisfied. Assume that (9.51) holds and $u_0 \in L^0_{\mathscr{F}_0}(\Omega; \mathbb{B}^{\frac{d}{q}-1}_{q,p}(\mathbb{T}^d))$. Let (u, σ) be the δ-weak solution to (9.48) provided by Theorem 9.3.2. Suppose that*

$$\zeta \in [2, \infty), \; r \in (2, \infty) \;\; \textit{satisfy} \;\; \frac{d}{2+\delta} < \zeta < \frac{d}{1+\delta}, \quad \textit{and} \quad \frac{2}{r} + \frac{d}{\zeta} \leq 2 + \delta.$$

Set $\gamma = \frac{2}{r} + \frac{d}{\zeta} - 1$. Then

$$\mathbb{P}\Big(\varepsilon < \sigma < T, \, \|u\|_{L^r(\varepsilon,\sigma; H^{\gamma,\zeta}(\mathbb{T}^d;\mathbb{R}^d))} < \infty\Big) = 0, \quad \textit{for all } \varepsilon \in (0, T). \tag{9.54}$$

In particular, for all $\xi \in (d, \frac{d}{1+\delta})$ and $\eta \in (2, \infty)$ such that $\frac{2}{\eta} + \frac{d}{\xi} = 1$,

$$\mathbb{P}\Big(\varepsilon < \sigma < T, \, \|u\|_{L^\eta(\varepsilon,\sigma; L^\xi(\mathbb{T}^d;\mathbb{R}^d))} < \infty\Big) = 0, \quad \textit{for all } \varepsilon \in (0, T). \tag{9.55}$$

Note that the choice $(\zeta, r) \neq (q, p)$ is allowed and since u is smooth far from $t = 0$ (see (9.52)-(9.53)). If $\zeta > d$ and r is large enough so that $\frac{2}{r} + \frac{d}{\zeta} < 1$, then $\gamma < 0$ and (9.54) provides a blow-up criteria in space of *negative* smoothness.

The blow-up criteria (9.55) follows from (9.54). To see this, let ξ, η as in Theorem 9.3.5, $r = \eta$ and fix $\zeta \in (\xi, \frac{d}{1+\delta})$. Since $\frac{2}{\eta} + \frac{d}{\xi} = 1$ and $\zeta > \xi$, we have $\frac{2}{r} + \frac{d}{\zeta} < 1 < 2 + \delta$ and therefore $\gamma < 0$. By Sobolev embeddings we have $L^\xi(\mathbb{T}^d; \mathbb{R}^d) \hookrightarrow H^{\gamma,\zeta}(\mathbb{T}^d; \mathbb{R}^d)$ and thus (9.54) implies (9.55). Let us note that (9.55) is the natural extension of the classical Serrin's blow-up criteria for the Navier-Stokes equations (resp. [144, Theorem 11.2]). Meanwhile (9.54) can be though as a stochastic version of [144, Theorem 11.3] since it allows also spaces with negative smoothness. The condition $\xi,\zeta < \frac{d}{1+\delta}$ is not present in the deterministic Serrin's criteria and such restriction is due to the effect of the gradient-noise (i.e. $(\phi^n\cdot\nabla)u\,dw_t^n$).

Finally, reasoning as in Subsection 6.3.2, for any stopping time $\tau\in(0,\sigma)$, the blow-up criteria (9.54) is equivalent to

$$\mathbb{P}\Big(\sigma<T,\, u(t)\in L^r(\tau,\sigma;H^{\gamma,\zeta}(\mathbb{T}^d;\mathbb{R}^d))\Big)=0. \tag{9.56}$$

This follows by $\sigma>0$ a.s. and (9.52). A similar reformulation also holds for (9.55).

Next we analyse the end-point case of Theorem 9.3.5, i.e. when the integrability in time equals to ∞.

Theorem 9.3.6 (Blow-up criteria in critical space). *Let Assumption 9.2.1 with $\delta\in[-\frac12,0)$. Let Assumption 9.3.1 be satisfied. Assume that (9.51) holds and $u_0\in L^0_{\mathscr{F}_0}(\Omega;\mathbb{B}^{\frac{d}{q}-1}_{q,p}(\mathbb{T}^d))$. Let (u,σ) be the δ-weak solution to (9.48) provided by Theorem 9.3.2. Suppose that $\zeta\in(2,\infty)$ and $r\in(2,\infty)$ satisfy*

$$\frac{d}{2+\delta}<\zeta<\frac{d}{1+\delta},\qquad \text{and}\qquad \frac{2}{r}+\frac{d}{\zeta}=2+\delta. \tag{9.57}$$

Then

$$\mathbb{P}\Big(\sigma<T,\,\lim_{t\uparrow\sigma}u(t)\ \text{exists in}\ B^{\frac{d}{\zeta}-1}_{\zeta,r}(\mathbb{T}^d;\mathbb{R}^d)\Big)=0. \tag{9.58}$$

In particular

$$\mathbb{P}\Big(\sigma<T,\,\lim_{t\uparrow\sigma}u(t)\ \text{exists in}\ L^d(\mathbb{T}^d;\mathbb{R}^d)\Big)=0. \tag{9.59}$$

As above, the choice $(\zeta,r)\neq(q,p)$ is allowed. Moreover, since $\sigma>0$ a.s. and u is smooth far from $t=0$, the requirements (9.58)-(9.59) makes sense. As noticed below Theorem 9.3.2, if $\delta=-\frac12$ and $\zeta\sim 2d$, then (9.58) provides a blow-up criteria in critical spaces with smoothness up to $-\frac12$. Let us note that (9.59) follows from (9.58). In the case $p=q=d=2$ is trivial. In the case $d\geq 3$, reasoning as in the proof of Corollary 9.3.4, (9.59) follows from (9.58) by choosing δ small, $r\geq d$ as in (9.57) and $L^d(\mathbb{T}^d;\mathbb{R}^d)\hookrightarrow\mathbb{B}^0_{d,r}(\mathbb{T}^d;\mathbb{R}^d)$.

To the best of our knowledge Theorems 9.3.5-9.3.6 are new. Let us conclude by pointing out that, in the deterministic setting, blow-up criteria in the critical space L^3 (here $d=3$ for simplicity) is know to be valid even with quantitative growth assumption in the $L^\infty((\varepsilon,\sigma);L^3)$-norm, see [191, Theorem 1.4]. We expect that Theorems 9.3.5-9.3.6 can be improved by exploiting the structure of the equation (9.48) and that the above blow-up criteria can be an useful tool in this direction. However, this goes beyond the scope of this thesis.

Throughout the next subsections, to abbreviate the notation, we often write $\mathbb{L}^q$, $\mathbb{H}^{s,q}$, $\mathbb{B}^s_{q,p}$ etc. instead of $\mathbb{L}^q(\mathbb{T}^d)$, $\mathbb{H}^{s,q}(\mathbb{T}^d)$, $\mathbb{B}^s_{q,p}(\mathbb{T}^d)$.

9.3.3 Proof of Theorems 9.3.2 and 9.3.5-9.3.6

We begin by proving Theorems 9.3.2. To prove the local existence and regularization for (9.48) we employ the main results in Chapter 4, 6 and 7. Below, Hypothesis $\mathbf{H}(\mathbb{H}^{-1+s,\zeta},\mathbb{H}^{-1+s,\zeta},\alpha,r)$ is defined as in Assumption 7.1.1. Let us begin by proving the following lemma.

Lemma 9.3.7. *Let Assumptions 9.2.1 and 9.3.1 be satisfied for $\delta\in(-1,0]$. Let either $[\zeta\in[2,\infty)$, $r\in(2,\infty)$ and $\alpha\in[0,\frac{r}{2}-1)]$ or $[\zeta=r=2$ and $\alpha=0]$. Assume that*[1]

$$\frac{d}{2+\delta}<\zeta<-\frac{d}{\delta}\qquad\text{and}\qquad 2\frac{1+\alpha}{r}+\frac{d}{\zeta}\leq 2+\delta. \tag{9.60}$$

[1] Here we have set $1/0=\infty$.

Then Hypothesis $\mathbf{H}(\mathbb{H}^{-1+s,\zeta},\mathbb{H}^{-1+s,\zeta},\alpha,r)$ *holds and the corresponding trace space* $\mathbb{B}^{1+\delta-2\frac{1+\alpha}{r}}_{\zeta,r}$ *is critical for* (9.48) *if and only if* (9.60) *holds with the equality.*

Proof. By Theorems 6.3.1 and 9.2.2, to conclude it remains to check (HF)-(HG). For notational convenience, we sometimes write Y_0, Y_1 instead of $\mathbb{H}^{-1+\delta,\zeta},\mathbb{H}^{1+\delta,\zeta}$. Thus $Y_\theta=\mathbb{H}^{-1+\delta+2\theta,\zeta}$ for $\theta\in(0,1)$.

Step 1: F verifies (HF) *in the* $(\mathbb{H}^{-1+\delta,\zeta},\mathbb{H}^{1+\delta,\zeta},r,\alpha)$*-setting.* Let $F=F_1+F_2$ where $F_1(u)=\mathcal{P}\mathrm{div}(u\otimes u)$ and $F_2=\mathcal{P}\mathcal{F}(u)$. We begin by estimating F_1: For all $v,v'\in Y_1$,

$$\begin{aligned}\|F_1(v)-F_1(v')\|_{\mathbb{H}^{1+\delta,q}}&\lesssim\|(v\otimes v)-(v'\otimes v')\|_{H^{\delta,q}}\\&\overset{(i)}{\lesssim}\|(v\otimes v)-(v'\otimes v')\|_{L^\lambda}\\&\lesssim\|v\otimes(v-v')\|_{L^\lambda}+\|(v-v')\otimes v\|_{L^\lambda}\\&\leq(\|v\|_{L^{2\lambda}}+\|v'\|_{L^{2\lambda}})\|v-v'\|_{L^{2\lambda}}\\&\overset{(ii)}{\lesssim}(\|v\|_{\mathbb{H}^{\theta,\zeta}}+\|v'\|_{\mathbb{H}^{\theta,\zeta}})\|v-v'\|_{\mathbb{H}^{\theta,\zeta}}\end{aligned}\tag{9.61}$$

where in (i) we have used the Sobolev embedding with $-\frac{d}{\lambda}=\delta-\frac{d}{\zeta}$ and in (ii) the Sobolev embedding with

$$\theta-\frac{d}{\zeta}=-\frac{d}{2\lambda}=-\frac{1}{2}\Big(\frac{d}{\zeta}-\delta\Big)\quad\Rightarrow\quad\theta=\frac{d}{2\zeta}+\frac{\delta}{2}.\tag{9.62}$$

To ensure that the Sobolev embeddings in (i)-(ii) are applied correctly we check that $\lambda>1$, $\theta>0$. One can readily check that $\lambda>1$ is always satisfied since $d\geq 2$, $\delta>-1$ and $\zeta\geq 2$. To check $\theta>0$, one has to impose $\zeta<-\frac{d}{\delta}$. Lastly, to ensure that F is a lower-order non linearity, we have to require $\theta<1+\delta$, which is equivalent to $\frac{d}{2+\delta}<\zeta$. Note that the above requirements follows by (9.60). Letting $\beta=\frac{1-\delta+\theta}{2}=\frac{1}{2}(1-\frac{\delta}{2}+\frac{d}{2\zeta})$, we have $Y_\beta=\mathbb{H}^{\theta,\zeta}$. Therefore the above estimate can be rewritten as follows

$$\|F_1(v)-F_1(v')\|_{Y_0}\lesssim(1+\|v\|_{Y_\beta}+\|v'\|_{Y_\beta})\|v-v'\|_{Y_\beta},\ \text{ for all } v,v'\in Y_1.\tag{9.63}$$

To check (HF) in the (Y_0,Y_1,r,α)-setting for F_1 we split the discussion into two cases.

(1) If $1-\frac{1+\alpha}{r}\geq\beta$, then by using that $X_{1-\frac{1+\kappa}{p}+\varepsilon}\hookrightarrow X_\beta$ for each $\varepsilon>0$ by (9.63) we get that (9.63) holds with β replaced by $1+\frac{1+\alpha}{r}+\varepsilon$. Letting $\rho_1=1,\beta_1=\varphi_1=1+\frac{1+\alpha}{r}+\varepsilon$ where $\varepsilon>0$ is such that $\rho_j\varepsilon+\beta<1$. Then by the previous choice of ε, F_1 is a *non*-critical part of the nonlinearity F in the $(\mathbb{H}^{-1+\delta,\zeta},\mathbb{H}^{1+\delta,\zeta},r,\alpha)$-setting (see (4.18)).

(2) If $1-\frac{1+\alpha}{r}<\beta$, then we set $\rho_1=1,\beta_1=\varphi_1=\beta$ and by (9.63) the condition (4.18) for $j=1$ becomes

$$\frac{1+\alpha}{r}\leq\frac{\rho_1+1}{\rho_1}(1-\beta)=1-\frac{d}{2\zeta}+\frac{\delta}{2}.\tag{9.64}$$

Note that the former is equivalent to (9.60). Finally, the space corresponding trace space $\mathbb{B}^{-1+\delta-2\frac{1+\alpha}{r}}_{\zeta,r}$ is critical for (9.48) in the $(\mathbb{H}^{-1+\delta,\zeta},\mathbb{H}^{1+\delta,\zeta},r,\alpha)$-setting if and only if the equality in (9.64) holds.

Let us discuss $F_2(u):=\mathcal{P}\mathcal{F}(u)$. By Assumption 9.3.1(3) we get, for all $v,v'\in Y_1$

$$\begin{aligned}\|F_2(v)-F_2(v')\|_{\mathbb{H}^{-1+\delta,q}}&\lesssim\|F_2(v)-F_2(v')\|_{\mathbb{H}^{\delta,q}}\\&\leq(1+\|v\|_{L^{2\lambda}}+\|v'\|_{L^{2\lambda}})\|v-v'\|_{L^{2\xi}}.\end{aligned}\tag{9.65}$$

Comparing the latter estimate with (9.61), one sees that the argument in (1)-(2) carries over the F_2 part. In turn, $F=F_1+F_2$ satisfies (HF) in the $(\mathbb{H}^{-1+\delta,\zeta},\mathbb{H}^{1+\delta,\zeta},r,\alpha)$-setting if and only if (9.64) holds and the corresponding trace space $\mathbb{B}^{-1+\delta-2\frac{1+\alpha}{r}}_{\zeta,r}$ is critical for (9.48) if and only if (9.64) holds with the equality.

Step 3: G verifies (HG) in the $(\mathbb{H}^{-1+\delta,\zeta},\mathbb{H}^{1+\delta,\zeta},r,\alpha)$-setting. Recall that $Y_{1/2}=\mathbb{H}^{\delta,\zeta}$ by (9.22). Let β,λ be as in Step 1. Then, Assumption 9.3.1 and (2.14) yield, for all $v,v'\in Y_1$,

$$\begin{aligned}\|G(\cdot,v)-G(\cdot,v')\|_{\gamma(\ell^2,\mathbb{H}^{\delta,\zeta})}&\lesssim \|\mathcal{G}(\cdot,v)-\mathcal{G}(\cdot,v')\|_{\gamma(\ell^2,L^\lambda)}\\ &\lesssim (1+\|v\|_{L^{2\lambda}}+\|v'\|_{L^{2\lambda}})\|v-v'\|_{L^{2\lambda}}\\ &\lesssim (1+\|v\|_{Y_\beta}+\|v'\|_{Y_\beta})\|v-v'\|_{Y_\beta}.\end{aligned} \tag{9.66}$$

The previous readily implies that (HG) is satisfied with $\beta_2=\varphi_2=\beta$. Therefore the conclusion follows as in Step 1. □

We are ready to prove Theorem 9.3.2.

Proof of Theorem 9.3.2. The instantaneous regularization result (9.52)-(9.53) follows from the results in Section 7.1. For the proof in the case $p=q=d=2$ and $\kappa=0$ one can argue in Theorem 7.2.2. For the remaining case one can argue as in Proposition 8.2.9. Therefore, we content ourself to prove the local well-posedness in the critical space $\mathbb{B}^{d/q-1}_{q,p}$.

Due to Lemma 9.3.7 for $\zeta=q$, $r=p$ and $\alpha=\kappa$, Theorem 6.3.1 implies the existence of a δ-maximal local solution to (9.48) provided $q<-\frac{d}{\delta}$ and

$$2\frac{1+\kappa}{p}+\frac{d}{q}\le 2+\delta. \tag{9.67}$$

It remains to show that the corresponding trace $\mathbb{B}^{1+\delta-2\frac{1+\kappa}{p}}_{q,p}$ coincide with $\mathbb{B}^{\frac{d}{q}-1}_{q,p}$ provided (9.51) holds. To show this it remains to investigate when the equality in (9.67) with can be reached. If $p=q=2$, $\kappa=0$ and the equality in (9.67) holds, then $d=2$ due to $\delta\le 0$. It remains to investigate the case $p>2$. In the latter case, since $\kappa\in[0,\frac{p}{2}-1)$ if and only if $\frac{1+\kappa}{p}\in[\frac{1}{p},\frac{1}{2})$, the equality in (9.67) can be reached provided

$$2+\delta-\frac{d}{q}<1 \qquad \text{and} \qquad \frac{2}{p}+\frac{d}{q}<2+\delta.$$

The first inequality in the former is equivalent to $q<\frac{d}{1+\delta}$. Combined with the above requirement on q we obtain $q<\min\{\frac{d}{1+\delta},-\frac{d}{\delta}\}$. Optimizing the right hand side in the former, one gets $\delta\in[-\frac{1}{2},0]$ and $q<\frac{d}{1+\delta}$ as assumed in Theorem 9.3.2. Under the previous assumptions, we may realise the equality in (9.67) by choosing $\kappa=\kappa_{\mathsf{crit}}=-1+\frac{p}{2}(2+\delta-\frac{d}{q})$. To conclude it remains to note that $1+\delta-2\frac{1+\kappa_{\mathsf{crit}}}{p}=\frac{d}{q}-1$ and therefore

$$X^{\mathsf{Tr}}_{\kappa_{\mathsf{crit}},p}=\mathbb{B}^{1+\delta-2\frac{1+\kappa_{\mathsf{crit}}}{p}}_{q,p}(\mathbb{T}^d)=\mathbb{B}^{\frac{d}{q}-1}_{q,p}(\mathbb{T}^d)$$

as desired. □

It remains to prove Theorems 9.3.5 and 9.3.6.

Proof of Theorem 9.3.5. Let (u,σ) be as in Theorem 9.3.2 and let ζ,r be as in Theorem 9.3.5. As showed below Thoerem 9.3.5, (9.55) follows from (9.54). To conclude it remains to note that (9.54) follows from the extrapolation Lemma 7.1.9 and Theorem 6.3.8 applied with $X_0=\mathbb{H}^{-1+\delta,\zeta}$, $X_1=\mathbb{H}^{1+\delta,\zeta}$, $p=r$ and the critical weight $\kappa=-1+\frac{r}{2}(2+\delta-\frac{d}{\zeta})$ (cf. Theorem 9.3.2). □

Proof of Theorem 9.3.6. The proof similar to the one of Theorem 9.3.5. It is enough to apply the extrapolation Lemma 7.1.9 and the blow-up criteria Theorem 6.3.7(1) in the $(\mathbb{H}^{-1+\delta,\zeta},\mathbb{H}^{1+\delta,\zeta},r,0)$-setting. □

9.4 Global smooth solutions for the 2D turbulent Navier-Stokes

In this subsection we study the forced stochastic Navier-Stokes equations for turbulent flows in two dimensions. More precisely we establish existence of global solutions to

$$\begin{cases} du - \operatorname{div}(a(\cdot)\cdot\nabla u)\,dt = (-\nabla P + \mathcal{F}(\cdot,u) + \operatorname{div}(u\otimes u) + g)dt \\ \qquad\qquad + \sum_{n\geq 1}(-\nabla Q_n + (\phi_n\cdot\nabla)u + \mathcal{G}_n(\cdot,u) + g)dw_t^n, & \text{on } \mathscr{O}, \\ \operatorname{div} u = 0, \qquad u(\cdot,0) = u_0, & \text{on } \mathscr{O}, \end{cases} \tag{9.68}$$

where $u : [0,\infty)\times\Omega\times\mathscr{O}\to\mathbb{R}^2$ is the unknown process and $\mathscr{O}$ is either $\mathbb{R}^2, \mathbb{T}^2$ or $\mathscr{O}\subset\mathbb{R}^2$. If $\mathscr{O}\subseteq\mathbb{R}^2$, then we complement (9.68) with no-slip boundary conditions:

$$u = 0, \quad \text{on } \partial\mathscr{O}. \tag{9.69}$$

Note that (9.68) coincides with (9.48) if $f = g = 0$. Let us list the assumptions needed below. Note that, $a^{i,j}, \phi_n^j$ are not even assumed to be continuous.

Assumption 9.4.1. *Let Assumption 9.3.1(2)-(3) be satisfied with $\mathbb{T}^d$ replaced by $\mathscr{O}$. Suppose that the following holds.*

(1) *One of the following is satisfied:*
- *$\mathscr{O}\subseteq\mathbb{R}^2$ is a bounded Lipschitz domain;*
- *$\mathscr{O}=\mathbb{T}^2$.*

(2) *$a_{i,j}, \phi_n^j : I_T\times\Omega\times\mathscr{O}\to\mathbb{R}$ are $\mathscr{P}\otimes\mathscr{B}(\mathscr{O})$-measurable. Moreover, there exists $C_{a,b}>0$ such that, a.s. for all $t\in I_T$, $i,j\in\{1,2\}$,*

$$\|a^{i,j}(t,\cdot)\|_{L^\infty(\mathscr{O})} + \|(\phi_n^j(t,\cdot))_{n\in\mathbb{N}}\|_{L^\infty(\mathscr{O};\ell^2)} \leq C_{a,\phi}.$$

Assume that there exists $\nu>0$ such that a.s., for all $t\in I_T$, $x\in\mathscr{O}$, $\xi\in\mathbb{R}^d$,

$$\sum_{i,j=1}^d \Big(a^{i,j}(t,x) - \frac{1}{2}\Big(\sum_{n\geq 1}\phi_n^j(t,x)\phi_n^i(t,x)\Big)\Big)\xi_i\xi_j \geq \nu|\xi|^2.$$

(3) *$f\in L^0_{\mathscr{P}}(\Omega;L^2(I_T\times\Omega;\mathbb{L}^2(\mathscr{O}))$ and $g\in L^0_{\mathscr{P}}(\Omega;L^2(I_T\times\Omega;\gamma(\ell^2,\mathbb{L}^2(\mathscr{O})))$.*

The arguments below could be adapted also the cover the case either $\mathscr{O}\subseteq\mathbb{R}^d$ is an unbounded domain with compact boundary. For the sake of simplicity, we do not pursue this here. In the following $T\in(0,\infty]$ is fixed.

In the case $\mathscr{O}=\mathbb{T}^2$, weak solutions can be defined as in Subsection 9.3.1. More precisely, we say that (u,σ) is a *weak solution to* (9.68) *on* $\overline{I}_T$ if (u,σ) is an L^2_0-maximal local solution to (4.16) on $\overline{I}_T$ with the choice $X_0 = \mathbb{H}^{-1}(\mathbb{T}^2)$, $X_0=\mathbb{H}^{-1}(\mathbb{T}^2)$, $H=\ell^2$ and A,B,F,G as in (9.49). Here and in the following we employ the usual abbreviation $\mathbb{H}^s = \mathbb{H}^{s,2}$ and similar.

In the case $\mathscr{O}\subseteq\mathbb{R}^d$, due to the boundary condition (9.69) we need to argue differently. To this end, set ${}_DH^1(\mathscr{O}) := \{f\in W^{1,2}(\mathscr{O}) : f = 0 \text{ on } \partial\mathscr{O}\}$ is well defined (here the prescript D reminds the Dirichlet boundary conditions). To construct the Helmholtz projection we argue as in Subsection 9.2.1. By standard elliptic regularity, for each $f\in L^2(\mathscr{O};\mathbb{R}^d)$ there exists a unique $\Upsilon\in H^1(\mathscr{O})$ such that

$$\begin{cases} \Delta\Upsilon = \operatorname{div} f, & \text{on } \mathscr{O}, \\ \partial_\nu\Upsilon = 0, & \text{on } \partial\mathscr{O}, \end{cases} \tag{9.70}$$

where ν denotes the exterior normal field on $\partial\mathscr{O}$. Here the above problem has to be understood in its natural weak formulation:

$$\Upsilon\in H^1(\mathscr{O}) \text{ satisfies (9.70)} \Leftrightarrow \Big[\int_{\mathscr{O}}\nabla\Upsilon\cdot\nabla\psi\,dx = \int_{\mathscr{O}} f\cdot\nabla\psi\,dx, \quad \forall\psi\in C^1(\overline{\mathscr{O}})\Big].$$

Then one can check that the operator $\mathcal{P}: L^2(\mathscr{O};\mathbb{R}^d) \to L^2(\mathscr{O};\mathbb{R}^d)$ given by (cf. (9.19)) $\mathcal{P}f := f - \nabla\phi$ is a projection. Moreover we set

$$\mathbb{L}^2(\mathscr{O}) := \mathcal{P}(L^2(\mathscr{O};\mathbb{R}^d)) \qquad {}_D\mathbb{H}^1(\mathscr{O}) := ({}_DH^1(\mathscr{O}))^d \cap \mathbb{L}^2(\mathscr{O}), \tag{9.71}$$

and ${}_D\mathbb{H}^{-1}(\mathscr{O}) := ({}_D\mathbb{H}^1(\mathscr{O}))^*$. Finally, we define the maps $v \mapsto \operatorname{div}(a(\cdot)\nabla v)$ and $v \mapsto \operatorname{div}(v \otimes v)$ on above spaces as follows. For each $v, v' \in {}_D\mathbb{H}^1(\mathscr{O})$ we let

$$\begin{aligned} \langle v', A^{\mathcal{S}}_{\mathscr{O}}(\cdot)v\rangle_{{}_D\mathbb{H}^1(\mathscr{O}),{}_D\mathbb{H}^{-1}(\mathscr{O})} &:= -\int_{\mathscr{O}} a(\cdot)\nabla v \cdot \nabla v'\, dx, \\ \langle v', F^{\mathcal{S}}_{\mathscr{O}}(\cdot, v)\rangle_{{}_D\mathbb{H}^1(\mathscr{O}),{}_D\mathbb{H}^{-1}(\mathscr{O})} &:= -\int_{\mathscr{O}} (v \otimes v) \cdot \nabla v'\, dx. \end{aligned} \tag{9.72}$$

Thus in the case $\mathscr{O} \subseteq \mathbb{R}^d$, we say that (u,σ) is a *weak solution to* (9.68)-(9.69) *on* $\overline{I}_T$ if (u,σ) is a L^2_0-maximal local solution to (4.16) on $\overline{I}_T$ with $X_0 = {}_D\mathbb{H}^{-1}(\mathscr{O})$, $X_1 = {}_D\mathbb{H}^1(\mathscr{O})$, $H = \ell^2$, W_{ℓ^2} as in Example 2.3.6 and, for $v \in X_1$,

$$\begin{aligned} A(\cdot)v &= -A^{\mathcal{S}}_{\mathscr{O}}(\cdot)v, & B(\cdot)v &= \big(\mathcal{P}\big[(\phi_n(\cdot)\cdot\nabla)v\big]\big)_{n\geq 1}, \\ F(\cdot, v) &= \iota\mathcal{P}\mathcal{F}(\cdot, v) + F^{\mathcal{S}}_{\mathscr{O}}(\cdot, v), & G(\cdot, v) &= (\mathcal{P}(g_n(\cdot, v)))_{n\geq 1}, \end{aligned} \tag{9.73}$$

where $\iota : \mathbb{L}^2(\mathscr{O}) \to \mathbb{H}^{-1}(\mathscr{O})$ denotes the natural embeddings.

Proposition 9.4.2 (Local existence in 2D)**.** *Let Assumption 9.4.1 be satisfied. For each* $u_0 \in L^0_{\mathscr{F}_0}(\Omega;\mathbb{L}^2(\mathscr{O}))$, *there exists a weak solution* (u,σ) *to* (9.68) *with boundary conditions* (9.69) *if* $\mathscr{O} \neq \mathbb{T}^2$ *such that*

$$u \in L^2_{\mathrm{loc}}([0,\sigma); {}_D\mathbb{H}^1(\mathscr{O})) \cap C([0,\sigma);\mathbb{L}^2(\mathscr{O})) \quad a.s. \tag{9.74}$$

Under a sublinear assumption on $\mathcal{F}, g$ the above solution is global.

Theorem 9.4.3 (Global existence in 2D)**.** *Let Assumption 9.4.1 be satisfied. Assume that, a.s. for all* $t \in \mathbb{R}_+, x \in \mathscr{O}$ *and* $y \in \mathbb{R}^2$

$$\|g(t,x,y)\|_{\ell^2} + |\mathcal{F}(t,x,y)| \leq C_{\mathcal{F},\mathcal{G}}(1+|y|). \tag{9.75}$$

Then for each $u_0 \in L^0_{\mathscr{F}_0}(\Omega;\mathbb{L}^2(\mathscr{O}))$, *the weak solution provided by Theorem 9.3.2 is global in time, i.e.* $\sigma = \infty$. *Moreover, if* $u_0 \in L^2_{\mathscr{F}_0}(\Omega;\mathbb{L}^2(\mathscr{O}))$, *then for each* $T > 0$ *there exists* $C_T > 0$ *such that*

$$\mathbb{E}\Big[\sup_{t\in I_T} \|u(t)\|^2_{L^2(\mathscr{O})}\Big] + \int_0^T \|\nabla u(t)\|^2_{L^2(\mathscr{O})}\, dt \leq C_T\Big(1 + \mathbb{E}\|u_0\|^2_{L^2(\mathscr{O})}\Big).$$

As mentioned below Assumption 9.4.1, our strategy to prove Theorem 9.4.3 can be also used in the case $\mathscr{O} \subseteq \mathbb{R}^2$ with compact boundary. Therefore, comparing the above result in the case $\mathscr{O} = \mathbb{R}^2$ (thus $\partial\mathbb{R}^2 = \varnothing$) with [162, Theorem 2.2], one notes that we do not require any condition on $\operatorname{div}\phi$. Finally, Theorem 9.4.3 does not seem to be follows from classical existence result (see e.g. [148, Theorem 5.1.3]).

The proof of the previous results will be given in the next section.

9.4.1 Proof of Proposition 9.4.2 and Theorem 9.4.3

In the remaining part of this subsection we mainly consider the case $\mathscr{O}$ is a bounded Lipschitz domain. The case $\mathscr{O} = \mathbb{T}^2$ is simpler. To abbreviate the notation, we sometimes write ${}_D\mathbb{H}^s$ instead of ${}_D\mathbb{H}^s(\mathscr{O})$ etc. Let us recall that the Stokes operator $\mathcal{S}_{\mathscr{O}}$ can be defined using the Friederich extension method (see [193, Appendix A]) and it is uniquely defined by the formula

$$\langle v', \mathcal{S}_{\mathscr{O}} v\rangle = -\int_{\mathscr{O}} \nabla v \cdot \nabla v'\, dx \text{ for all } v, v' \in {}_D\mathbb{H}^1.$$

By [193, Appendix A] the domain of $\mathcal{S}_{\mathscr{O}}$ is a self-adjoint operator on $\mathbb{L}^2$ and its domain is given by

$$\mathsf{D}(\mathcal{S}_{\mathscr{O}}) := \Big\{ v \in {}_D\mathbb{H}^1(\mathscr{O}) \, : \, \text{such that } \; {}_D\mathbb{H}^1(\mathscr{O}) \ni v' \mapsto -\int_{\mathscr{O}} \nabla v' \cdot \nabla v dx \\ \text{extends to a linear functional on } \mathbb{L}^2(\mathscr{O}) \Big\}.$$

Finally, $\mathsf{D}((-\mathcal{S}_{\mathscr{O}})^{1/2}) = {}_D\mathbb{H}^1(\mathscr{O})$ and $-\mathcal{S}_{\mathscr{O}}$ has a bounded H^∞-calculus with angle 0 by [194, Chapter 10, Proposition 1.10] and [108, Proposition 10.2.23]. By extrapolation-interpolation arguments (see e.g. [3, Appendix A]), there exists an extrapolated operator $(-\mathcal{S}_{\mathscr{O}})_{-\frac{1}{2}} : {}_D\mathbb{H}^1(\mathscr{O}) \subseteq {}_D\mathbb{H}^{-1}(\mathscr{O}) \to {}_D\mathbb{H}^{-1}(\mathscr{O})$ with a bounded H^∞-calculus with angle 0.

To Proposition 9.4.2 we need suitable embeddings. The following result is well-known to experts. For the reader's convenience, we include some details.

Lemma 9.4.4 (Sobolev embeddings for ${}_D\mathbb{H}^s$-scale)**.** *Let $\mathscr{O} \subseteq \mathbb{R}^2$ be a bounded Lipschitz domain. Let $s \in [0,1]$ and set ${}_D\mathbb{H}^s(\mathscr{O}) := [{}_D\mathbb{H}^{-1}(\mathscr{O}), {}_D\mathbb{H}^1(\mathscr{O})]_{\frac{1}{2}+\frac{s}{2}}$. If $p \in (2,\infty)$ satisfies $s - \frac{d}{2} \geq \frac{d}{p}$, then ${}_D\mathbb{H}^2(\mathscr{O}) \hookrightarrow L^p(\mathscr{O})$.*

Proof. By [3, Proposition A.2] applied to $-\mathcal{S}_{\mathscr{O}}$ it follows that ${}_D\mathbb{H}^s(\mathscr{O}) = [\mathbb{L}^2(\mathscr{O}), {}_D\mathbb{H}^1(\mathscr{O})]_s$. Therefore

$${}_D\mathbb{H}^s(\mathscr{O}) \hookrightarrow \big[(L^s(\mathscr{O}))^2, ({}_DH^1(\mathscr{O}))^2\big]_s \hookrightarrow L^p(\mathscr{O})$$

where the last equality embedding follows by an extension by 0-argument and the Sobolev embeddings on $\mathbb{R}^2$. ☐

Proof of Proposition 9.4.2. For the sake of clarity, we divide the proof into several steps.

Step 1: $(A,B) \in \mathcal{SMR}_2^\bullet(s,T)$ for each $s \in I_T$. The claim follows by [148, Theorem 4.2.4]. Indeed, assumptions (H1) and (H4) in [148, Chapter 4] are straightforward to check. Since A, B are linear, then it is enough to check the coercivity assumption (H3) in [148]. To see this, employing the Einstein notation, note that, for any $u \in X_1$,

$$\begin{aligned} &\langle u, \mathcal{P}(\operatorname{div}(a(\cdot)\nabla u)\rangle_{\mathbb{H}^1,\mathbb{H}^{-1}} + \|(\mathcal{P}((\phi_n \cdot \nabla)u)_{n\geq 1}\|^2_{\gamma(\ell^2,\mathbb{L}^2)} \\ &\leq -2\int_{\mathscr{O}} a(\cdot)\nabla u \cdot \nabla u \, dx + \sum_{n\geq 1} \|((\phi_n \cdot \nabla)u)_{n\geq 1}\|^2_{L^2} \leq -2\vartheta \|\nabla u\|_{L^2}, \end{aligned}$$

where we have used that $\mathcal{P} : L^2 \to L^2$ is a projection and thus $\|\mathcal{P}\|_{\mathscr{L}(L^2)} = 1$.

Step 2: Existence of a weak solution (u,σ) to (9.48) on $\overline{I}_T$. For any $v \in X_1$, let $F_1(\cdot, v) = \mathcal{P}(\operatorname{div}(v \otimes v))$ and $F_2(\cdot, v) := \mathcal{P}\mathcal{F}(\cdot, v)$. By repeating the estimates in the proof of Theorem 9.3.2 for $\delta = 0$ one easily obtain, a.s. for all $u, v \in X_1$,

$$\begin{aligned} \|F_1(\cdot,u) - F_1(\cdot,v)\|_{\mathbb{H}^{-1}} &+ \|G(\cdot,u) - G(\cdot,v)\|_{\gamma(\ell^2,\mathbb{L}^2)} \\ &\lesssim \|(1+|u|+|v|)|u-v|\|_{\mathbb{L}^2} \\ &\lesssim \|u-v\|_{\mathbb{L}^2} + (\|u\|_{\mathbb{L}^4} + \|v\|_{\mathbb{L}^4})\|u-v\|_{\mathbb{L}^4}, \\ &\lesssim (1 + \|u\|_{X_\beta} + \|v\|_{X_\beta})\|u-v\|_{X_\beta}, \end{aligned} \tag{9.76}$$

where $\beta = \frac{3}{4}$ and we have used that $X_\beta = {}_D\mathbb{H}^{-1+2\beta}(\mathscr{O}) \hookrightarrow \mathbb{L}^4 \cap \mathbb{L}^2$ by Lemma 9.4.4. Similarly, for all $v, v' \in X_1$ we have

$$\begin{aligned} \|F_2(\cdot,v) - F_2(\cdot,v)\|_{\mathbb{H}^{-1}} &\lesssim \|F_2(\cdot,v) - F_2(\cdot,v)\|_{\mathbb{L}^2} \\ &\lesssim \|u-v\|_{\mathbb{L}^2} + (\|u\|_{\mathbb{L}^4} + \|v\|_{\mathbb{L}^4})\|u-v\|_{\mathbb{L}^4}, \\ &\lesssim (1 + \|u\|_{X_\beta} + \|v\|_{X_\beta})\|u-v\|_{X_\beta}. \end{aligned}$$

where $\beta = \frac{3}{4}$. In particular, F, G satisfies (HF)-(HG) with $\rho_1 = \rho_2 = 1$, $m_F = 1$, $m_G = 1$, $\varphi_1 = \varphi_2 = \beta_1 = \beta_2 = \beta$, $p = 2$ and $\kappa = 0$. Thus, Theorem 6.3.1 gives the claim of this step and $\mathbb{L}^2$ is critical for (9.48) on $\mathscr{O} \subset \mathbb{R}^2$. ☐

It remains to prove Theorem 9.4.3. To this end we need the following

Lemma 9.4.5 (Energy estimates). *Let Assumption 9.4.1 be satisfied. Assume that* $u_0 \in L^2_{\mathscr{F}_0}(\Omega;\mathbb{L}^2)$ *and that* (9.75) *holds. Let* (u,σ) *be the weak solution provided by Proposition 9.4.2. Then for each* $T>0$ *there exists* $C_T>0$ *independent of* u_0, u *such that*

$$\mathbb{E}\Big[\sup_{t\in[0,T\wedge\sigma)}\|u(t)\|^2_{L^2(\mathscr{O})}\Big] + \int_0^{T\wedge\sigma}\|\nabla u(t)\|^2_{L^2(\mathscr{O})}\,dt \le C_T\Big(1+\mathbb{E}\|u_0\|^2_{L^2(\mathscr{O})}\Big).$$

Proof. The proof of the above energy inequality is based on a standard application of the Itô's formula. We sketch the proof. For notational convenience, we set $f=g=0$. The general case follows with minor modifications.

For each $j\ge 1$, let σ_j be the stopping time given by

$$\sigma_j := \inf\{t\in[0,\sigma\wedge T)\,:\,\|\nabla u\|_{L^2(0,t;L^2)} + \|u(t)\|_{\mathbb{L}^2}\ge n\},$$

where we have set $\inf\varnothing := \sigma$. By (9.74) it follows that $\lim_{j\to\infty}\sigma_j=\sigma$ a.s.

By Grownall and Fatou's lemmas, it is enough to prove the existence of a constant $C>0$ independent of j,u,u_0 such that

$$\mathbb{E}y(t) + \int_0^t y(s)ds \le C(1+t+\mathbb{E}\|u_0\|^2_{L^2(\mathscr{O})}) + \int_0^t \mathbb{E}y(s)ds,\quad t\in[0,T], \tag{9.77}$$

where

$$y(t) = \sup_{r\in[0,t\wedge\sigma_j)}\|u(t)\|^2_{L^2} + \int_0^{t\wedge\sigma_j}\int_{\mathscr{O}}\|\nabla u(s)\|^2_{L^2}ds. \tag{9.78}$$

For the reader's convenience, we split the proof into several steps.

Step 1: We apply the Itô*'s formula to obtain the identity* (9.81) *below.* The idea is to apply the Itô's formula [148, Theorem 4.2.5] and the usual cancellation $\langle \mathrm{div}(u\otimes u), u\rangle_{({}_DH^{-1})^2,({}_DH^1)^2}=0$ for $u\in X_1$. To begin, let us extend u to a process v on $[0,T]\times\Omega$ as follows. Let $v\in L^2(I_T\times\Omega;\mathbb{L}^2)$ be the strong solution to the following linear problem

$$dv - \mathrm{div}(a(\cdot)\nabla v)dt = f^u dt + ((\phi(\cdot,v)\cdot\nabla)v + g^u)dW_{\ell^2},\quad v(0)=u_0, \tag{9.79}$$

where (cf. (9.73))

$$\begin{aligned} f^u &:= \mathbf{1}_{[\![0,\sigma_j)\!)}(F^{\mathcal{S}}_{\mathscr{O}}(u) + \iota\mathcal{P}F(\cdot,u)) \in L^2(I_T\times\Omega;{}_D\mathbb{H}^{-1,2}),\\ g^u &:= \mathbf{1}_{[\![0,\sigma_j)\!)}(\mathcal{P}g_n(\cdot,u))_{n\ge1}\in L^2(I_T\times\Omega;\gamma(\ell^2,\mathbb{L}^2)). \end{aligned} \tag{9.80}$$

To see the claimed integrability of f^u, g^u, one can use the quadratic growth of the nonlinearities (see (9.72) and (9.75)), the fact that $\|\nabla u\|_{L^2(0,\sigma_j;L^2)} + \sup_{t\in[0,\sigma_j)}\|u(t)\|_{L^2}\le j$ and

$$L^2(0,t;H^1)\cap L^\infty(0,t;L^2(\mathscr{O}))\hookrightarrow L^4(0,t;H^{\frac12}(\mathscr{O}))\hookrightarrow L^4(0,t;L^4(\mathscr{O})).$$

Thus, the existence of such v follows by $(-\mathrm{div}(a(\cdot)\nabla),(\phi_n\cdot\nabla)_{n\ge1})\in\mathcal{SMR}^{\bullet}_{p,\kappa}(T)$ (see Step 1 in the proof of Proposition 9.4.2) and (9.80). Since (u,σ) is an L^2_0-maximal local solution to (9.68), we have $u=v$ a.e. on $[\![0,\sigma_j]\!]$.

Applying the Itô formula (see [148, Theorem 4.2.5]) to v, the ellipticity condition and the above mentioned cancellation imply that, a.s. for all $t\in I_T$,

$$\begin{aligned} &\|v(t)\|^2_{L^2} - \|u_0\|^2_{L^2} + 2\nu\int_0^t \mathbf{1}_{[0,\tau)}\|\nabla v\|^2_{L^2}ds\\ &\le \int_0^t \mathbf{1}_{[0,\sigma_j)}\Big[2\big|(\mathcal{F}(\cdot,u),u)_{L^2}\big| + \|\mathcal{G}(\cdot,u)\|^2_{\gamma(\ell^2,L^2)} + \big((\phi_n(s)\cdot\nabla)u,\mathcal{G}(\cdot,u)\big)_{\gamma(\ell^2,L^2)}\Big]ds\\ &2\sum_{n\ge1}\int_0^t \mathbf{1}_{[0,\sigma_j)}(\mathcal{G}_n(\cdot,u),u)_{L^2}dw^n_s + 2\sum_{n\ge1}\int_0^t\mathbf{1}_{[0,\sigma_j)}\big((\phi_n(s)\cdot\nabla)u,u\big)_{L^2}dw^n_s\\ &=: I_t + II_t + III_t + IV_t + V_t, \end{aligned} \tag{9.81}$$

where $(f,g)_{L^2} = \sum_{k\in\{1,2\}} \int_{\mathscr{O}} f^k g^k \, dx$ for $f,g \in L^2(\mathscr{O};\mathbb{R}^2)$.

For the reader's convenience, we split the remaining proof into two steps.

Step 2: There exists C independent of u, u_0, j such that

$$\mathbb{E}\int_0^{t\wedge\sigma_j} \|\nabla u(s)\|_{L^2} ds \leq C\Big(1+t-s+\mathbb{E}\int_0^{t\wedge\sigma_j} \|u(s)\|_{L^2} ds\Big).$$

The idea is to take expectations in (9.81) and using that $\mathbb{E}[IV_t] = \mathbb{E}[V_t] = 0$ for each $t \in [0,T]$. By (9.75) one can readily check that, for some constant $C > 0$ independent of j, u, u_0 the following holds

$$\mathbb{E}\big[|I_t| + |II_t|\big] \leq C\Big(1+t+\int_0^t \mathbf{1}_{[0,\sigma_j)}\|u(s)\|_{L^2} ds\Big) \quad \text{for all } t\in[0,T]. \tag{9.82}$$

To estimate III_t, by Assumption 9.4.1(2) we have

$$\begin{aligned} E|III_t| &\leq \int_0^t \mathbf{1}_{[0,\sigma_j)} \|(\phi_n(s)\cdot\nabla)u(s)\|_{L^2(\ell^2)} \|u(s)\|_{L^2} ds \\ &\leq C_{a,\phi}\int_0^t \mathbf{1}_{[0,\sigma_j)} \|\nabla u(s)\|_{L^2}\|u(s)\|_{L^2} ds \\ &\leq \nu \int_0^t \mathbf{1}_{[0,\sigma_j)} \|\nabla u(s)\|^2_{L^2} + \widehat{C}\int_0^t \mathbf{1}_{[0,\sigma_j)}\|u(s)\|^2_{L^2} ds \end{aligned} \tag{9.83}$$

where $\widehat{C}$ depends only on $C_{a,\phi}, \nu$ and we have used $\gamma(\ell^2, L^2) = L^2(\ell^2)$ by (2.14).

Therefore by taking expectations in (9.81) and using that $u = v$ a.e. on $[\![0,\sigma_j)\!)$ and (9.82)-(9.83), one obtains the claimed estimate in Step 2.

Step 2: Conclusion. The idea is to take absolute value and the supremum over $s \in [0,t]$ in (9.81). Since $\mathbb{E}[\sup_{t\in[0,T]} |\Lambda_t|] \leq \mathbb{E}[|\Lambda_T|]$ for $\Lambda \in \{I, II, III\}$, by (9.82)-(9.83) it remains to estimate IV and V. By (9.75) and the Burkholder-Davis-Gundy inequality, one can readily check that (see (9.78) for y)

$$\begin{aligned} \mathbb{E}\Big[\sup_{r\in[0,t]} |IV_s|\Big] &\leq \mathbb{E}\Big[\int_0^t \mathbf{1}_{[0,\sigma_j)} \|\mathcal{G}(s,u(s))\|^2_{\gamma(\ell^2,L^2)} \|u(s)\|^2_{L^2} ds\Big]^{1/2} \\ &\leq \mathbb{E}\Big[\Big(\sup_{s\in[0,\sigma_j\wedge t)} \|u(s)\|^2_{L^2}\Big)^{1/2}\Big(\int_0^t \mathbf{1}_{[0,\sigma_j)}\|\mathcal{G}(s,u(s))\|^2_{\gamma(\ell^2,L^2)} ds\Big)^{1/2}\Big] \\ &\leq \frac{1}{4}\mathbb{E}y(t) + \widehat{C}_{\mathcal{F},\mathcal{G}}\Big(1+t+\mathbb{E}\int_0^t \mathbf{1}_{[0,\sigma_j)}\|u(s)\|^2_{L^2} ds\Big). \end{aligned}$$

where $\widehat{C}_{\mathcal{F},\mathcal{G}}$ depends only on $C_{\mathcal{F},\mathcal{G}}$ in (9.75). Analogously, we can estimate V:

$$\begin{aligned} \mathbb{E}\Big[\sup_{r\in[0,t]} |V_s|\Big] &\leq \mathbb{E}\Big[\int_0^t \mathbf{1}_{[0,\sigma_j)} \|(\phi_n(s)\cdot\nabla)u(s)\|^2_{\gamma(\ell^2,L^2)} \|u(s)\|^2_{L^2} ds\Big]^{1/2} \\ &\leq C_{a,\phi}\mathbb{E}\Big[\Big(\sup_{r\in[0,\sigma_j\wedge t)} \|u(r)\|_{L^2}\Big)^{1/2}\Big(\int_0^t \mathbf{1}_{[0,\sigma_j)}\|\nabla u(s)\|^2_{L^2} ds\Big)^{1/2}\Big] \\ &\overset{(i)}{\leq} \frac{1}{4}\mathbb{E}y(t) + \widehat{C}_{a,\phi}\int_0^t \mathbf{1}_{[0,\sigma_j)}\|\nabla u(s)\|^2_{L^2} ds \\ &\overset{(ii)}{\leq} \frac{1}{4}\mathbb{E}y(t) + \widehat{C}_{a,\phi}C\Big(1+t+\int_0^t \mathbf{1}_{[0,\sigma_j)}\|u(s)\|^2_{L^2} ds\Big) \end{aligned}$$

where in (i) we have used Assumption 9.4.1(2) and $\widehat{C}_{a,\phi}$ depends only on $C_{a,\phi}$ and in (ii) we have used Step 2. By collecting the previous estimates and using (9.78) as well as $u = v$ a.e. on $[\![0,\sigma_j)\!)$ we get (9.77). This concludes the proof. □

Proof of Theorem 9.4.3. By replacing (u, σ) by $(u|_{[\![0,\sigma\wedge T[\![}, \sigma \wedge T)$ for $T \in (0, \infty)$, it is enough to show that weak solution to (9.68) on $[0, T]$ with $T \in (0, \infty)$ satisfies $\sigma = T$ a.s. By Proposition 6.3.10, it is enough to consider $u_0 \in L^2(\Omega; \mathbb{L}^2)$. By Lemma 9.4.5 and (9.71) we get

$$\sup_{s\in[0,\sigma)} \|u(s)\|^2_{\mathbb{L}^2} + \int_0^\sigma \|u(s)\|^2_{{}_D\mathbb{H}^1} ds < \infty \quad \text{a.s.} \tag{9.84}$$

Recall that weak solution to (9.68) are L^2_0-maximal local solution to (4.16) with $p = 2$, $\kappa = 0$, $H = \ell^2$, $X_0 = {}_D\mathbb{H}^{-1}$, $X_1 = {}_D\mathbb{H}^1$ (therefore $X^{\mathsf{Tr}}_{\kappa,p} = X_{1/2} = \mathbb{L}^2$) and A, B, F, G as in (9.73). Therefore

$$\mathbb{P}(\sigma < T) \overset{(9.84)}{=} \mathbb{P}\Big(\sigma < T, \sup_{t\in[0,\sigma)} \|u(s)\|_{X_{1/2}} + \|u\|_{L^2(0,\sigma;X_1)} < \infty\Big) = 0$$

where in the last inequality we used Theorem 6.3.7(4) whose assumption follows by Step 2 in the proof of Theorem 6.3.1. □

List of symbols

Weights

w_κ^a	power weight $\lvert t-a\rvert^\kappa$ for $a \in \mathbb{R}$.
$w_\kappa = w_\kappa^0$	power weight centered at the origin.

Operations

$(A, \mathsf{D}(A))$	closed operator.
$(A^*, \mathsf{D}(A^*))$	adjoint operator.
$(A^\theta, \mathsf{D}(A^\theta))$	fractional operator.
$(S(t))_{t\geq 0}$	semigroup generated by A on X.
$\omega_0(-A)$	growth bound of S, see p. 30.
$S \diamond G$	stochastic convolutions, see p. 30.

Interpolation and related notation

$(\cdot,\cdot)_{\theta,p}$	real interpolation space.
$[\cdot,\cdot]_\theta$	complex interpolation space.
$\mathsf{D}_A(\theta,p)$	$(X, \mathsf{D}(A^m))_{\theta/m,p}$ with $m > \theta$, see p. 18.
$X^{\mathsf{Tr}}_{\kappa,p}$	$(X_0, X_1)_{1-\frac{1+\kappa}{p},p}$.
X^{Tr}_p	$X^{\mathsf{Tr}}_{0,p}$.
X_θ	$[X_0, X_1]_\theta$.

Maximal regularity spaces

$\mathrm{SMR}(p,T)$	see p. 30.
$\mathrm{SMR}(p,T,\kappa)$	see p. 41.
$\mathrm{SMR}_\theta(p,\infty)$	see p. 43.
$\mathcal{SMR}_{p,\kappa}(T)$ and $\mathcal{SMR}_p(T)$	see p. 53.
$\mathcal{SMR}^\bullet_{p,\kappa}(T)$ and $\mathcal{SMR}^\bullet_p(T)$	see p. 53.
$\mathcal{SMR}_{p,\kappa}(\sigma,T)$ and $\mathcal{SMR}_p(\sigma,T)$	see p. 121.
$\mathcal{SMR}^\bullet_{p,\kappa}(\sigma,T)$ and $\mathcal{SMR}^\bullet_p(\sigma,T)$	see p. 121.

Spaces

$H^{s,q}(I, w_\kappa^a; X)$ or $H^{s,q}(a,b,w_a^\kappa;X)$	weighted Sobolev spaces on $I=(a,b)$ with values in X.
${}_0H^{s,q}(I, w_\kappa^a; X)$ or ${}_0H^{s,q}(a,b,w_a^\kappa;X)$	see p. 20.
$C(I;X)$	continuous maps on I with values in X.
$H^{s,q}(\mathscr{O})$	Sobolev spaces on $\mathscr{O}$.
$B^s_{q,p}(\mathscr{O})$	Besov spaces on $\mathscr{O}$.
${}_DH^{s,q}(\mathscr{O})$ and ${}_DB^s_{q,p}(\mathscr{O})$	spaces Dirichlet boundary conditions, see 200.
$\mathbb{H}^{s,q}(\mathbb{T}^d)$ and $\mathbb{B}^s_{q,p}(\mathbb{T}^d)$	spaces of divergence free vector fields on $\mathbb{T}^d$, see 213.
$\mathscr{L}(X,Y)$	bounded linear operators from X into Y.

$\gamma(H, X)$	γ-radonifying operators, see p. 25.
$\gamma(S; H, X)$	$\gamma(L^2(S; H), X)$, see p. 25.
$\gamma(a, b; H, X)$	$\gamma(L^2((a, b); H), X)$, see p. 25.

Probability notation

a.a.	almost all.
a.e.	almost everywhere.
a.s.	almost surely.
$(\Omega, \mathbb{F}, \mathscr{A}, \mathbb{P})$	underlined filtered probability space with $\mathbb{F} = (\mathscr{F}_t)_{t\geq 0}$.
$\mathbb{E}$	expectation with respect to $\mathbb{P}$.
$\widetilde{\mathbb{E}}$	expectation with respect to $\widetilde{\mathbb{P}}$.
W_H	cylindrical Brownian motion in H, see p. 26.

Miscellaneous

$a \lesssim_Q b$	$a \leq C_Q b$.
$a \gtrsim_Q b$	$a \geq C_Q b$.
$a \eqsim_Q b$	$a \lesssim_Q b$ and $a \gtrsim_Q b$.
$a \vee b$	$\max\{a, b\}$.
$a \wedge b$	$\min\{a, b\}$.
$\hookrightarrow$	continuous embedding.

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