Unknown Input Observer design for coupled PDE/ODE linear systems

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***Abstract*— The problem of unknown input observer design is considered for coupled PDE/ODE linear systems subject to unknown boundary inputs. Assuming available measurements at the boundary of the distributed domain, the synthesis of the observer is based on geometric conditions and Lyapunov methods. Numerical simulations support and validate the theoretical findings, illustrating the robust estimation performances of the proposed unknown input observer.**

I. INTRODUCTION

A. Background and motivation

The dynamics of several complex physical processes is described by partial differential equations, modelling the evolution of spacial distributed phenomena. Examples may be found in hydraulic networks [1], multiphase flow [2], transmission networks [3], road traffic networks [4] or gas flow in pipelines [5].

Actuators and sensors are typically placed at the boundary of the domain, while the state variables are neither directly controlled nor measured in the interior. Classical problems such as feedback control or state estimation become particularly interesting and challenging in this context, leading to the interest for design of boundary controllers and boundary observers. Among the wide range of systems governed by PDEs, hyperbolic systems encountered a major interest from the control community. In this regard, sufficient conditions for controllability and observability of first-order hyperbolic systems are discussed in [6]. The stability problem for boundary control in hyperbolic systems has been largely explored; see for instance [7] [8] [9] and the references therein. The boundary observability of infinite dimensional linear systems has been formally investigated in [10] using semigroups defined on Hilbert spaces. Observer design for linearized first-order hyperbolic systems based on Lyapunov methods has been addressed in [11], where exponential convergence is guaranteed using boundary injections. For quasilinear first-order hyperbolic systems, the tasks of boundary stabilization and state estimation have been considered in [12].

In some cases, e.g. in the presence of processes having both a distributed and a finite-dimensional behavior, the partial differential equations may be entangled with ordinary differential equations. Typically, such interconnection takes place at the boundary of the space domain, with the output of the ODEs providing dynamic boundary conditions for the PDEs. A LMI-based approach to boundary observer design for conservation laws with static and asymptotically stable dynamic boundary control is proposed in [13]. A design procedure for backstepping observers is proposed in [14] based on the solution of an auxiliary set of PDEs for computing a suitable change of coordinates. A thoughtful stability analysis is presented in [15] introducing cross-terms defined through supplementary integral states, while non-diagonal Lyapunov functionals are considered in [16] for coupled systems of scalar PDEs and ODEs and in [17] for more general systems.

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B. Contributions

Adopting a setting similar to [13], [17], the problem of observer design in the presence of unknown inputs is considered in this paper. Assuming that partial measurements are available at both boundary points of the domain, the goal is to design a full state, i.e. infinite-dimensional, observer with the property of providing an estimation error completely decoupled from the unknown inputs. In the literature, estimators with such property are referred to as *Unknown Input Observers* (UIO) and have been recognized as an excellent tool for robust estimation and fault diagnosis [18], [19], [20]. In the infinite-dimensional context, the problem of UIO design has been previously investigated in [21] from an abstract viewpoint. The construction proposed in the paper hinges upon the validity of some geometric conditions, and involves the combined use of left and right boundary outputs in order to guarantee an exponentially stable error dynamics. Unlike in the finite-dimensional case, Unknown Input Observers for coupled PDEs/ODEs shows an interesting new feature: the finite-dimensional part of the observer is allowed to be not internally stable, whereas stabilization is provided by the interconnection with the infinite-dimensional part, i.e. through the right boundary output injection.

An extensive numerical simulation study is provided to highlight this and other features of the proposed UIO, and show the robust estimation accuracy as compared to a Luenberger-like observer.

C. Notation

The sets $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{>0}$ represent the set of nonnegative and positive real scalars, respectively. The symbols S_+^n and D_+^n denote, respectively, the set of real $n \times n$ symmetric positive definite matrices and the set of diagonal positive definite matrices. For a matrix $A \in \mathbb{R}^{n\times m}$, $A^\top$ denotes the transpose of A and when $n = m$, $\mathrm{He}(A) = A + A^\top$. Given two matrices A and B, $A \oplus B$ denotes the block diagonal matrix with matrices A and B on its diagonal.

For a symmetric matrix A, positive definiteness (negative definiteness) and positive semidefiniteness (negative semidefiniteness) are denoted, respectively, by $A \succ 0$ ($A \prec 0$) and $A \succeq 0$ ($A \preceq 0$). Given $A, B \in \mathsf{S}_+^{n_x}$, we say that $A \preceq B$ ($A \succeq B$) if $A - B \preceq 0$ ($A - B \succeq 0$). Given $A \in \mathsf{S}_+^{n_x}$, $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ stand, respectively, for the largest and the smallest eigenvalue of A. In partitioned symmetric matrices, the symbol $\bullet$ stands for symmetric blocks. Given $x, y \in \mathbb{R}^n$, we denote by $\langle x, y\rangle_{\mathbb{R}^n}$ the standard Euclidean inner product. Let $U \subset \mathbb{R}$, $V \subset \mathbb{R}^n$, and $f\colon U \to V$, we denote by $\|f\|_{\mathcal{L}^2} = (\int_U |f(x)|^2 dx)^{\frac{1}{2}}$ the $\mathcal{L}^2$ norm of f. In particular, we say that $f \in \mathcal{L}^2(U;V)$ if $\|f\|_{\mathcal{L}^2}$ is finite. Given $f, g \in \mathcal{L}^2(U;V)$, $\langle f, g\rangle_{\mathcal{L}^2} := \int_U \langle f(x), g(x)\rangle_{\mathbb{R}^n} dx$. Let $U \subset \mathbb{R}$ be open and V be a linear normed space,

$$\mathcal{H}^1(U;V) := \Big\{f \in \mathcal{L}^2(U;V) : f \text{ is absolutely continuous on } U, \frac{d}{dz} f \in \mathcal{L}^2(U;V)\Big\}$$

where $\frac{d}{dz}$ stands for the weak derivative of f. The symbol $\mathcal{C}^k(U;V)$ denotes the set of class k functions $f\colon U \to V$. Let $I \subset \mathbb{R}$, $\phi\colon I \to \mathcal{H}^1(U;V)$, $t \in I$, and $z^\star \in U$. We denote by $(\phi(t))(z^\star) \in V$ the value of $\phi(t)$ taken at $z = z^\star$. Let X and Y be linear normed spaces, U be an open subset of X, $f\colon U \to Y$, and $x \in U$, we denote by $Df(x)$ the Fréchet derivative of f at x.

D. Preliminary results and definitions

In this paper, we consider linear abstract dynamical systems of the form:

$$\dot{x} = \mathscr{A} x \tag{1}$$

where $x \in \mathcal{Z}$ is the system state, $\mathcal{Z}$ is the state space that we assume to be a Hilbert space equipped with the inner product $\langle \cdot, \cdot\rangle_{\mathcal{Z}}$, and $\mathscr{A}$ is a linear operator on $\mathcal{Z}$. In particular, we consider the following notion of solution for (1).

Definition 1 *Let $\mathcal{I} \subset \mathbb{R}_{\geq 0}$ be an interval containing zero. A function $\psi \in \mathcal{C}^0(\mathcal{I}, \mathcal{Z})$ is a solution to* (1) *if for all $t \in \mathcal{I}$*

$$\int_0^t \psi(s)ds \in \operatorname{dom} \mathscr{A}, \quad \psi(t) = \psi(0) + \mathscr{A} \int_0^t \psi(s)ds$$

Moreover, we say that ψ is maximal if its domain cannot be extended and it is complete if $\sup \mathcal{I} = \infty$. $\circ$

We say that (1) is well-posed if for any $\xi \in \mathcal{Z}$, there exists a unique maximal solution φ to (1) such that $\varphi(0) = \xi$.

Definition 2 *We say that* (1) *is globally exponentially stable if there exist $\lambda, \kappa > 0$ such that any maximal solution φ to* (1) *is complete and satisfies the following bound:*

$$\|\varphi(t)\|_{\mathcal{Z}} \leq \kappa e^{-\lambda t} \|\varphi(0)\|_{\mathcal{Z}} \qquad \forall t \in \operatorname{dom} \varphi$$

Theorem 1 (Global Exponential Stability) *Let* (1) *be well-posed. Assume that there exists a Fréchet differentiable functional $V\colon \mathcal{Z} \to \mathbb{R}$ and positive scalars $\alpha_1, \alpha_2, \alpha_3$, and p such that the following items hold:*

(i) For all $x \in \mathcal{Z}$

$$\alpha_1 \|x\|_{\mathcal{Z}}^p \leq V(x) \leq \alpha_2 \|x\|_{\mathcal{Z}}^p$$

(ii) For all $x \in \operatorname{dom} \mathscr{A}$

$$DV(x)\mathscr{A} x \leq -\alpha_3 \|x\|_{\mathcal{Z}}^p$$

Then, (1) *is globally exponentially stable.*

II. Problem Statement and Solution Outline

A. Problem Setup

Let $\Omega := (0,1)$, we consider a system of n_x linear 1-D conservation laws with dynamic boundary conditions formally written as:

$$\begin{aligned}
&\partial_t x(t,z) + \Lambda \partial_z x(t,z) = 0 && (t,z) \in \mathbb{R}_{\geq 0} \times \Omega \\
&x(t,0) = M\chi(t) && \forall t \in \mathbb{R}_{\geq 0} \\
&\dot{\chi}(t) = A\chi(t) + Bu(t) + Ew(t) && \forall t \in \mathbb{R}_{\geq 0} \\
&y_1(t) = Cx(t,0) && \forall t \in \mathbb{R}_{\geq 0} \\
&y_2(t) = Nx(t,1) && \forall t \in \mathbb{R}_{\geq 0}
\end{aligned} \tag{2}$$

where $\partial_t x$ and $\partial_z x$ denote, respectively, the derivative of x with respect to "time" and the "spatial" variable z, $(z \mapsto x(\cdot, z), \chi) \in \mathcal{L}^2(0,1;\mathbb{R}^{n_x}) \times \mathbb{R}^{n_\chi}$ is the system state, $u \in \mathbb{R}^{n_u}$ is a known boundary input, $w \in \mathbb{R}^{n_w}$ is an unknown boundary input and $y = [y_1 \; y_2] \in \mathbb{R}^{n_{y_1}+n_{y_2}}$ is a measured output. We assume that the matrices $\Lambda \in \mathsf{D}_+^{n_x}$, $M \in \mathbb{R}^{n_x \times n_\chi}$, $A \in \mathbb{R}^{n_\chi \times n_\chi}$, $B \in \mathbb{R}^{n_\chi \times n_u}$, $E \in \mathbb{R}^{n_\chi \times n_w}$, $C \in \mathbb{R}^{n_{y_1} \times n_x}$ and $N \in \mathbb{R}^{n_{y_2} \times n_x}$ are given.

The goal is to design an observer providing an exponentially convergent estimate $(z \mapsto \hat{x}(\cdot, z), \hat{\chi})$ of the state $(z \mapsto x(\cdot, z), \chi)$, irrespectively of the unknown input w.

B. Outline of the Proposed Observer

We propose an unknown input observer of the following form

$$\begin{aligned}
&\partial_t \hat{x}(t,z) + \Lambda \partial_z \hat{x}(t,z) = 0 && (t,z) \in \mathbb{R}_{\geq 0} \times \Omega \\
&\hat{x}(t,0) = M\hat{\chi}(t) && \forall t \in \mathbb{R}_{\geq 0} \\
&\dot{\psi}(t) = F\psi(t) + RBu(t) + Ky_1(t) + L(y_2(t) - \hat{y}_2(t)) && \forall t \in \mathbb{R}_{\geq 0} \\
&\hat{\chi}(t) = \psi(t) + Hy_1(t) && \forall t \in \mathbb{R}_{\geq 0} \\
&\hat{y}_2(t) = N\hat{x}(t,1) && \forall t \in \mathbb{R}_{\geq 0}
\end{aligned}$$

where matrices F, R, K, H, L are to be designed. At this stage, define the following two estimation errors $\epsilon_x := x - \hat{x}$ and $\epsilon_\chi := \chi - \hat{\chi}$. The dynamics of these errors are as follows:

$$\begin{aligned}
&\partial_t \epsilon_x(t,z) + \Lambda \partial_z \epsilon_x(t,z) = 0 \\
&\epsilon_x(t,0) = M\epsilon_\chi(t) \\
&\dot{\epsilon}_\chi(t) = -F\psi(t) + G_u u(t) + G_w w(t) + G_\chi \chi(t) - LN\epsilon_x(t,1) \\
&(t,z) \in \mathbb{R}_{\geq 0} \times \Omega
\end{aligned} \tag{3a}$$

where:

$$\begin{aligned}
G_u &:= (I - R - HCM)B \\
G_w &:= (I - HCM)E \\
G_\chi &:= (A - HCMA - KCM)
\end{aligned} \tag{3b}$$

Paralleling the literature of unknown input observers [18], [22], in the result given next we propose a selection of the observer gains R, F, and K enabling to decouple the error dynamics (3a) from the input u and the state χ.

Proposition 1 *Let $K_1 \in \mathbb{R}^{n_\chi \times n_{y_1}}$ and select:*

$$R = (I - HCM) \quad (4a)$$

$$K = K_1 + K_2, \quad K_2 = FH \quad (4b)$$

$$F = A - HCMA - K_1CM \quad (4c)$$

Then, the error dynamics (3a) *turn into:*

$$\begin{aligned} &\partial_t \epsilon_x(t,z) + \Lambda \partial_z \epsilon_x(t,z) = 0 && (t,z) \in \mathbb{R}_{\geq 0} \times \Omega \\ &\epsilon_x(t,0) = M\epsilon_\chi(t) && \forall t \in \mathbb{R}_{\geq 0} \\ &\dot{\epsilon}_\chi(t) = F\epsilon_\chi(t) + REw(t) - LN\epsilon_x(t,1) && \forall t \in \mathbb{R}_{\geq 0} \end{aligned} \quad (5)$$

Proof: The result can be easily proven by noticing that the selection of the matrices R, K, and F in (4) implies that matrices G_u and G_χ in (3b) are zero. ■

As a second step, to decouple the error dynamics (5) from the unknown input w, we select R, or actually H, such that

$$RE = (I - HCM)E = 0$$

To ensure that a feasible solution for the above identity exists, we consider the following assumption:

Assumption 1 *The matrix CME is full column rank.*

Indeed, as long as the following assumption is in force, CME is left invertible and one can pick:

$$H = E((CME)^\top CME)^{-1}(CME)^\top \quad (6)$$

This selection ensures that the matrix G_w in (3b) vanishes. Namely, the error dynamics are totally decoupled from the unknown input w. In particular, under (4) and (6) the error dynamics read:

$$\begin{aligned} &\partial_t \epsilon_x(t,z) + \Lambda \partial_z \epsilon_x(t,z) = 0 && (t,z) \in \mathbb{R}_{\geq 0} \times \Omega \\ &\epsilon_x(t,0) = M\epsilon_\chi(t) && \forall t \in \mathbb{R}_{\geq 0} \\ &\dot{\epsilon}_\chi(t) = F\epsilon_\chi(t) - LN\epsilon_x(t,1) && \forall t \in \mathbb{R}_{\geq 0} \end{aligned} \quad (7)$$

C. Abstract Formulation of the Error Dynamics

To analyze the error dynamics (7), we reformulate those as an abstract differential equation on the Hilbert space

$$\mathcal{Z} := \mathcal{L}^2(0,1;\mathbb{R}^{n_x}) \times \mathbb{R}^{n_\chi}$$

endowed with the following inner product:

$$\langle (f_1, f_2), (g_1, g_2) \rangle_{\mathcal{Z}} := \langle f_1, g_1 \rangle_{\mathcal{L}^2} + \langle f_2, g_2 \rangle_{\mathbb{R}^{n_\chi}} \quad (8)$$

In particular, let $\mathcal{X} := \mathcal{H}^1(0,1;\mathbb{R}^{n_x}) \times \mathbb{R}^{n_\chi} \subset \mathcal{Z}$. Define $\mathcal{D} := \{(\epsilon_x, \varepsilon_\chi) \in \mathcal{X} \colon \epsilon_x(0) = M\epsilon_\chi\}$ and consider the following operator:

$$\begin{aligned} &\mathscr{A} \colon \operatorname{dom} \mathscr{A} \to \mathcal{Z} \\ &(\epsilon_x, \epsilon_\chi) \mapsto \begin{pmatrix} -\Lambda \frac{d}{dz} & 0 \\ 0 & F \end{pmatrix} \begin{pmatrix} \epsilon_x \\ \epsilon_\chi \end{pmatrix} + \begin{pmatrix} 0 \\ -LN\epsilon_x(1) \end{pmatrix} \end{aligned} \quad (9)$$

where $\operatorname{dom} \mathscr{A} := \mathcal{D}$. Then, the error dynamics can be formally written as the following abstract differential equation on the Hilbert space $\mathcal{Z}$

$$\begin{pmatrix} \dot{\epsilon}_x \\ \dot{\epsilon}_\chi \end{pmatrix} = \mathscr{A} \begin{pmatrix} \epsilon_x \\ \epsilon_\chi \end{pmatrix} \quad (10)$$

Invoking the results given in [17, Section III.A], the error system (7) can be proven to be well posed on the state space $\mathcal{Z}$. This statement is made more precise in the result given next.

Theorem 2 *[17] System* (10) *is well-posed and its maximal solutions are complete.* □

The following *detectability property* is instrumental to decide the approach for designing the observer gain matrices. In particular, while stability of the observer is essentially granted under the validity of such property, in the opposite case the observer synthesis requires an additional effort.

Property 1 *[Detectability] Let us assume that the expression of H in (6) is well defined, and set $R = (I - HCM)$ accordingly. The pair (RA, CM) is detectable.* ◇

III. Stability analysis of the Error Dynamics

The analysis of the stability of the error system depends on whether the Property 1 holds or not. Therefore, we analyze the two cases separately.

A. Stability with detectability

The following result provides sufficient conditions for the design of the proposed observer when Property 1 holds.

Theorem 3 *Assume that Property 1 is fulfilled. System* (10) *is globally exponentially stable if there exist $\mu \in \mathbb{R}_{>0}, P \in \mathsf{D}_+^{n_x}, Q \in \mathsf{S}_+^{n_\chi}, K_1 \in \mathbb{R}^{n_x \times n_x}$ and L such that the following matrix inequality is satisfied:*

$$\begin{bmatrix} -e^{-\mu}\Lambda P & -QLN \\ \bullet & \operatorname{He}(F^\top Q) + M^\top \Lambda P M \end{bmatrix} \prec 0 \quad (11)$$

with $F = RA - K_1CM$.

Proof: For all $(\epsilon_x, \epsilon_\chi) \in \mathcal{Z}$, let us introduce the candidate Lyapunov functional

$$V(\epsilon_x, \epsilon_\chi) := \int_0^1 e^{-\mu z} \epsilon_x^\top(z) P \epsilon_x(z) dz + \epsilon_\chi^\top Q \epsilon_\chi \quad (12)$$

In particular, observe that for all $(\epsilon_x, \epsilon_\chi) \in \mathcal{Z}$ one has

$$\alpha_1 \|(\epsilon_x, \epsilon_\chi)\|_{\mathcal{Z}}^2 \leq V(\epsilon_x, \epsilon_\chi) \leq \alpha_2 \|(\epsilon_x, \epsilon_\chi)\|_{\mathcal{Z}}^2 \quad (13)$$

where

$$\alpha_1 := \lambda_{\min}\left(\begin{bmatrix} Pe^{-\mu} & 0 \\ \bullet & Q \end{bmatrix}\right), \quad \alpha_2 := \lambda_{\max}\left(\begin{bmatrix} P & 0 \\ \bullet & Q \end{bmatrix}\right) \quad (14)$$

are strictly positive. Moreover, it can be easily shown that for all $(\epsilon_x, \epsilon_\chi) \in \mathcal{Z}$:

$$(h_x,h_\chi)\mapsto DV(\epsilon_x, \epsilon_\chi) \begin{pmatrix} h_x \\ h_\chi \end{pmatrix} = 2\left(\int_0^1 e^{-\mu z}\epsilon_x^\top(z) P h_x(z)dz + \epsilon_\chi^\top Q h_\chi\right)$$

Thus, for all $(\epsilon_x, \epsilon_\chi) \in \operatorname{dom}\mathscr{A}$:

$$\begin{aligned} DV(\epsilon_x, \epsilon_\chi)\mathscr{A}\begin{pmatrix} \epsilon_x \\ \epsilon_\chi \end{pmatrix} =& -2\int_0^1 e^{-\mu z}\epsilon_x^\top(z)\Lambda P \frac{d}{dz}\epsilon_x(z)dz \\ &+ \epsilon_\chi^\top \operatorname{He}(QF)\epsilon_\chi \\ &- 2\epsilon_\chi^\top QLN\epsilon_x(1) \end{aligned}$$

Integrating by parts, for all $(\epsilon_x, \epsilon_\chi) \in \operatorname{dom}\mathscr{A}$ one gets:

$$DV(\epsilon_x, \epsilon_\chi)\mathscr{A}\begin{pmatrix} \epsilon_x \\ \epsilon_\chi \end{pmatrix} = \int_0^1 \begin{bmatrix} \epsilon_x(z) \\ \epsilon_x(1) \\ \epsilon_\chi \end{bmatrix}^\top \Psi(z) \begin{bmatrix} \epsilon_x(z) \\ \epsilon_x(1) \\ \epsilon_\chi \end{bmatrix} dz$$

where for all $z \in [0,1]$

$$\Psi(z) := \begin{bmatrix} -\mu e^{-\mu z}\Lambda P & 0 & 0 \\ \bullet & -e^{-\mu}\Lambda P & -Q^\top LN \\ \bullet & \bullet & \operatorname{He}(QF) + M^\top \Lambda P M \end{bmatrix}$$

The latter, by using (11) shows that for all $(\epsilon_x, \epsilon_\chi) \in \operatorname{dom}\mathscr{A}$

$$DV(\epsilon_x, \epsilon_\chi)\mathscr{A}\begin{pmatrix} \epsilon_x \\ \epsilon_\chi \end{pmatrix} \leq -\alpha_3 ||(\epsilon_x, \epsilon_\chi)||^2_{\mathcal{Z}} \qquad (15)$$

where:

$$\alpha_3 := |\lambda_{\max}(\Psi(1))|$$

By the virtue of (13) and (15), Theorem 1 and Theorem 2 ensure that (10) is globally exponentially stable. This ends the proof. ■

Remark 1 *When μ is fixed, condition* (11) *can be cast into a linear matrix inequality (LMI) via the following invertible change of variables: $Y = QK_1$ and $J = QL$. In this sense, the observer gains L and K_1 can be efficiently designed via the solution to an LMI coupled with a line search on the scalar μ.*

Remark 2 *It can be noticed that a somewhat trivial solution to the previous matrix inequality can always be found setting $L = 0$. In other words, in the case of detectability of (RA, CM), there is no need to use the right boundary output y_2 in the observer design. On the other hand, by tuning the gain L, the observer performance can be improved.*

B. Stability without detectability

When Property 1 does not hold, we can directly set $K_1 = 0$, and consider the following more sophisticated Lyapunov functional proposed in [17]:

$$W(\epsilon_x, \epsilon_\chi) := \int_0^1 \begin{bmatrix} \epsilon_x(z) \\ \epsilon_\chi \end{bmatrix}^\top \begin{bmatrix} e^{-\mu z}P & T^\top \\ \bullet & Q \end{bmatrix} \begin{bmatrix} \epsilon_x(z) \\ \epsilon_\chi \end{bmatrix} dz \qquad (17)$$

Accordingly, we can derive a sufficient stability condition based on the following matrix inequalities:

$$\begin{bmatrix} e^{-\mu}P & T^\top \\ \bullet & Q \end{bmatrix} \succ 0 \qquad (18)$$

$$\Phi(1) \prec 0 \qquad (19)$$

where Φ is defined in (16) (at the top of the next page).

Theorem 4 *Let $F = RA$ and $R = I - HCM$, where H is selected as in* (6)*. The error dynamics are globally exponentially stable if there exist $\mu \in \mathbb{R}_{>0}, P \in \mathsf{D}_+^{n_x}, T \in \mathbb{R}^{n_\chi \times n_x}$, $Q \in \mathsf{S}_+^{n_\chi}$, and $L \in \mathbb{R}^{n_\chi \times n_{y_2}}$ can be found such that matrix inequalities (18) and (19) are satisfied.*

Proof: The proof follows the same lines as the proof of Thereom 3. In particular, let, for all $(\epsilon_x, \epsilon_\chi) \in \mathcal{Z}$, W be defined as in (17). Then, for for all $(\epsilon_x, \epsilon_\chi) \in \mathcal{Z}$ one has:

$$\beta_1 \|(\epsilon_x, \epsilon_\chi)\|^2_{\mathcal{Z}} \leq W(\epsilon_x, \epsilon_\chi) \leq \beta_2 \|(\epsilon_x, \epsilon_\chi)\|^2_{\mathcal{Z}} \qquad (20)$$

where

$$\beta_1 := \lambda_{\min}\left(\begin{bmatrix} Pe^{-\mu} & T^\top \\ \bullet & Q \end{bmatrix}\right), \quad \beta_2 := \lambda_{\max}\left(\begin{bmatrix} P & T^\top \\ \bullet & Q \end{bmatrix}\right) \qquad (21)$$

are strictly positive due to (18). Then, as shown in [17] for all $(\epsilon_x, \epsilon_\chi) \in \operatorname{dom}\mathscr{A}$:

$$DW(\epsilon_x, \epsilon_\chi)\mathscr{A}\begin{pmatrix} \epsilon_x \\ \epsilon_\chi \end{pmatrix} = \int_0^1 \begin{bmatrix} \epsilon_x(z) \\ \epsilon_x(1) \\ \epsilon_\chi \end{bmatrix}^\top \Phi(z) \begin{bmatrix} \epsilon_x(z) \\ \epsilon_x(1) \\ \epsilon_\chi \end{bmatrix} dz \qquad (22)$$

where, for all $z \in [0,1]$, $\Phi(z)$ is defined in (16). At this stage notice that from (19), the following chain of matrix inequalities holds:

$$\Phi(0) \preceq \Phi(z) \preceq \Phi(1) \prec 0 \quad \forall z \in [0,1].$$

Therefore, for all $z \in [0,1]$, $\Phi(z) \prec 0$, there exists $\beta_3 > 0$ such that:

$$\Phi(z) \preceq -\beta_3 I \qquad \forall z \in [0,1] \qquad (23)$$

in particular

$$\beta_3 = |\lambda_{\max}(\Phi(1))|$$

Combining (22) and (23) gives for all $(\epsilon_x, \epsilon_\chi) \in \operatorname{dom}\mathscr{A}$:

$$DW(\epsilon_x, \epsilon_\chi)\mathscr{A}\begin{pmatrix} \epsilon_x \\ \epsilon_\chi \end{pmatrix} \leq -\beta_3 \|(\epsilon_x, \epsilon_\chi)\|^2_{\mathcal{Z}} \qquad (24)$$

Thanks to (20) and (24), Theorem 1 ensures that (10) is globally exponentially stable. This concludes the proof. ■

Remark 3 *Differently from* (11)*, when μ is fixed* (18) *cannot be cast into a linear matrix inequality (LMI) via a simple change of variables. This is due to the presence of the cross term introduced in the Lyapunov functional* (17)*. A specific approach to get an LMI-based design algorithm from* (18) *has been proposed recently in [23].*

$$\Phi(z) = \begin{bmatrix} -\mu e^{-\mu z}\Lambda P & -T^\top LN & T^\top F \\ \bullet & -\Lambda P e^{-\mu z} & -\Lambda T^\top - N^\top L^\top Q \\ \bullet & \bullet & \mathrm{He}(QF + T\Lambda M) + M^\top \Lambda P M \end{bmatrix} \tag{16}$$

IV. Numerical Example

In this section, we showcase the application of the proposed observer in a numerical example. In particular, we select:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ B = 0, \ E = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \ \Lambda = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 2 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

and consider the scenario without detectability discussed in Section III-B. From (6) one gets $H = [0 \ \ 0 \ \ 1]^\top$. Hence using (4) with $K_1 = 0$ yields:

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix}, F = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix}, K_2 = 0$$

Notice that the pair (RA, CM) is not detectable. This can be checked by using the PBH-test of observability. In particular, if one denotes:

$$\mathcal{O}(s) := \left[\frac{RA - sI}{CM}\right] = \left[\begin{array}{ccc} -s & 1 & 0 \\ -1 & -s & 0 \\ 1 & -1 & -s \\ \hline 1 & 1 & 1 \end{array}\right] \qquad \forall s \in \mathbb{C}$$

Then, it follows that $\operatorname{rank} \mathcal{O}(\pm j) = 2$, showing that (RA, CM) is not detectable. This prevents one from using Theorem 3 to design an unknown input observer. To overcome this problem, we consider the additional boundary measurement y_2 defined in (2) with $N = I$ and make use of Theorem 4 to design the observer. By relying on the approach outlined in [23], the following feasible solution to (18) and (19) is obtained:

$$L = \begin{bmatrix} 0.02904 & -0.01437 \\ 0.002278 & 0.2589 \\ 0.3725 & -0.2736 \end{bmatrix}, \mu = 1, P = \begin{bmatrix} 157.5 & 0 \\ 0 & 104.1 \end{bmatrix},$$

$$T = \begin{bmatrix} -99.98 & 2.941 \\ -90.95 & -79.97 \\ -92.64 & 6.472 \end{bmatrix}, Q = \begin{bmatrix} 913.5 & 312.8 & 267.2 \\ 312.8 & 907.9 & 269 \\ 267.2 & 269 & 284.8 \end{bmatrix}$$

To validate our theoretical findings, next we present some simulations of the proposed observer[1]. In these simulations the unknown input $w(t) = \sin(2t)$ is considered and initial conditions are taken as follows:

$$\begin{aligned} &x_1(0, z) = 0.5(\sin(2\pi z) - 1) && \forall z \in [0, 1] \\ &x_2(0, z) = 0.5(\sin(4\pi z) - 1) && \forall z \in [0, 1] \\ &\chi(0) = (1, -1, 0.5) && \\ &\hat{x}(0, z) = 0 && \forall z \in [0, 1] \\ &\hat{\chi}(0) = 0 && \end{aligned} \tag{25}$$

[1]Numerical integration of hyperbolic PDEs is performed via the use of the Lax-Friedrichs (Shampine's two-step variant) scheme implemented in Matlab® by Shampine [24].

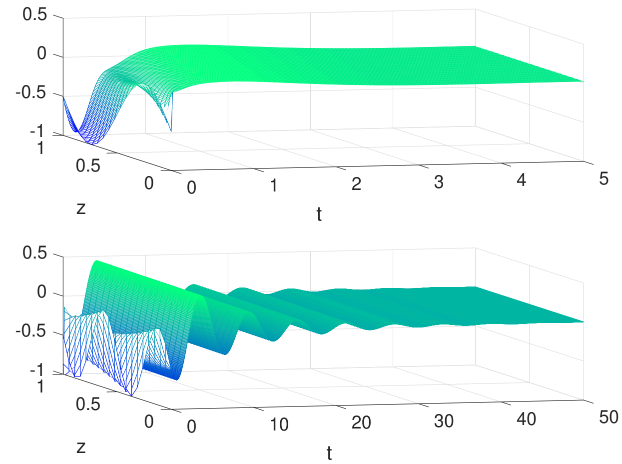

Fig. 1. Evolution of the estimation error ϵ_x from the initial condition in (25).

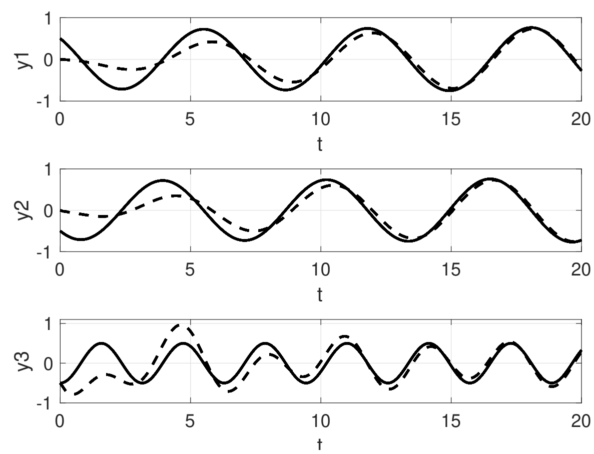

Fig. 2. Evolution of the state χ (solid line) and of its estimate $\hat{\chi}$ (dashed line) from the initial condition in (25).

Exponential state reconstruction is confirmed by Fig. 1, and Fig. 2 where the evolution of ϵ_x and of the states χ and $\hat{\chi}$, respectively, are reported. To emphasize the benefit of the proposed methodology in the presence of unknown inputs, in Fig. 3 we report the evolution of the squared norm of the estimation error $(\epsilon_x, \epsilon_\chi)$ for the proposed observer and for the standard Luenberger-like observer proposed in [23]. Fig. 2 clearly points out that due to the unknown input w, a Luenberger-like observer is not effective in this case.

V. Conclusions and Discussion

Design of unknown input observers (UIO) for coupled PDE/ODE linear systems subject to unknown boundary inputs has been addressed in this paper. The structure of

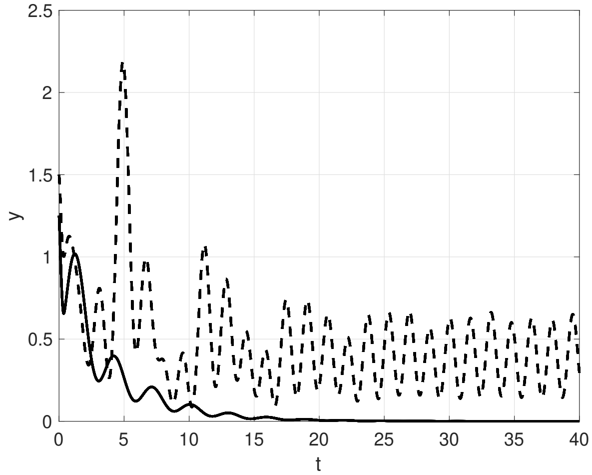

Fig. 3. Evolution of $\|(\epsilon_x, \epsilon_\chi)\|^2_{\mathcal{Z}}$ for the proposed observer (solid line) and for a standard Luenberger-like observer (dashed line).

the proposed UIOs is analogous to their finite-dimensional version, and the interconnection with the system to be estimated is made by means of injection of the boundary outputs. The synthesis of observer parameters and gains is based on geometric conditions and Lyapunov methods. An interesting feature of the considered infinite-dimensional UIO is that, even though the finite-dimensional part lacks of detectability in some cases, the complete error system may still be made exponentially stable. Properties and performances of the observer are exploited through extensive simulations.

The interest in considering UIOs lies on their robust estimation capabilities on the one side and, on the other side, on their suitability for generating fault detection/isolation filters. In this regard PDE systems with dynamic boundary conditions, which fall in the setup considered in this paper, are prone to actuator faults as many other controlled physical processes. In order to detect the faults, and identify the faulty actuators, a bank of unknown input observers can be designed with the aim of generating complementary residual signals giving information on the occurrence and the location of the faults. This can be done, for example, by decoupling the error from selected inputs only or by projecting the output of the error system on prescribed directions.

The application of the proposed infinite-dimensional UIO to fault diagnosis in coupled PDE/ODE linear systems is object of ongoing research.

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