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Packing conditions in metric spaces with curvature bounded above and applications

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Abstract

General metric spaces satisfying weak and synthetic notions of upper and lower curvature bounds will be studied. The relations between upper and lower bounds will be pointed out, especially the interactions between a packing condition and different forms of convexity of the metric. The main tools will be a new and flexible definition of entropy on metric spaces and a version of the Tits Alternative for groups of isometries of the metric spaces under consideration. The applications can be divided into classical and new results: the former consist in generalizations to a wider context of the theory of negatively curved Riemannian manifolds, while the latter include several compactness and continuity results.

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Chapter 1

Introduction

The purpose of this thesis is to study geometric properties of metric spaces satisfying weak and synthetic notions of upper and lower curvature bounds. The introduction is divided into sections according to the different chapters: we will present the setting, we will describe the main results comparing them to the literature and we will try to explain why the packing condition is the right one to consider in order to obtain a good control on the geometry of metric spaces with upper curvature bounds.

1.1 Upper and lower curvature bounds

Usually limits of Riemannian manifolds are no more Riemannian manifolds, even when the geometry of the manifolds is uniformly bounded. Several notions of upper and lower curvature bounds on arbitrary metric spaces have been developed during the years in order to overcome this problem and to describe the limits of sequences of Riemannian manifolds. The hope is to find large enough classes of metric spaces to contain every limit of sequences of Riemannian manifolds with bounded geometry (for instance: bounds on the sectional curvature, the diameter, the injectivity radius, etc.) but still with good local and global geometric properties.

1.1.1 Upper bounds

Locally CAT(κ) metric spaces, where κ is a real number, are currently one of the main topics in metric geometry. They have been studied from various points of view during the last decades. We will recall the CAT(κ) condition in Chapter 2. In general these metric spaces can be very wild and the local geometry can be difficult to understand. Under basic additional assumptions (local compactness and local geodesic completeness) it is possible to control much better the local and asymptotic properties of these spaces, as proved by Kleiner and Lytchak-Nagano. In particular, under these assumptions, the topological dimension coincides with the Hausdorff dimension, the local dimension can be detected from the tangent cones and there exist a decomposition of X in k-dimensional subspaces X^k (containing dense open subsets locally bilipschitz equivalent to $\mathbb{R}^k$ and admitting a regular Riemannian metric), and a canonical measure μ_X, coinciding with the restriction of the k-dimensional Hausdorff measure on each X^k, which is positive and finite on any open relatively

compact subset (cp. the foundational works [Kle99] , [LN19], [LN18]). Following [LN19] we will call for short *GCBA-spaces* the locally geodesically complete, locally compact and separable metric spaces satisfying some curvature upper bound, i.e. which are locally CAT(κ) for some κ. When we want to emphasize in a statement the role of κ we will write GCBA$^\kappa$.
It is a standard fact that a complete Riemannian manifold is GCBA$^\kappa$ if and only if its sectional curvature is bounded from above by κ. In particular GCBA-spaces arise in a natural way as generalizations and limits of Riemannian manifolds with sectional curvature bounded from above.

Usually the geometry of Riemannian manifolds is studied at level of universal covers, especially when the sectional curvature is non-positive. It is classical that universal covers of GCBA^0 spaces exist and are complete, geodesically complete, CAT(0) metric spaces. A second notion of upper bound on the curvature, that is implied by the CAT(0) assumption, is the convexity (or Busemann) condition. Convex metric spaces are simply connected, so they should be thought as generalization of universal covers of Riemannian manifolds with non-positive sectional curvature: indeed again the universal cover of a complete Riemannian manifold is a complete, convex and geodesically complete metric space if and only if its sectional curvature is non-positive. The main aspects of convex metric spaces will be discussed in Chapter 2.

The third kind of synthetic notion of upper bound on the curvature is the so-called Gromov-hyperbolicity condition, see Chapter 2. This is a widely studied property due to its relations with geometric group theory, dynamics, number theory, etc. (cp. [DSU17] for an overview).
There is a qualitative difference between the Gromov-hyperbolic condition and the other two: indeed the former is a sort of large scale version of negative curvature but it does not give information on the local structure at scales smaller than the hyperbolicity constant δ, while both CAT(κ) and convexity control the local geometry of the space. That is why it is difficult to use local estimates (or estimates at a fixed scale) to study the geometry of Gromov-hyperbolic spaces.
In large parts of the thesis we will consider metric spaces that are *both* Gromov-hyperbolic and convex: combining these two assumptions we will be able to link the local estimates given by the convexity to the large scale properties implied by the Gromov-hyperbolicity. Convex, Gromov-hyperbolic metric spaces should be thought as generalizations of Riemannian manifolds with negative sectional curvature.

1.1.2 Lower bounds

We recall here some notions of synthetic lower bound on the curvature for a metric space:
– curvature bounded from below in the sense of Alexandrov (cp. for instance [BGP92]). It is the analogue of the CAT(κ) condition, but the geometry of an Alexandrov metric space is simpler (for example the dimension is constant). A complete Riemannian manifold has curvature bounded from below by κ in the sense of Alexandrov if and only if its sectional curvature is at least κ;

– the RCD(K, N) condition, that is defined for metric measure spaces (X, d, μ) (cp. for instance [LV09]). Here N should be thought as an upper bound on the dimension of X, while K is a synthetic version of lower bound on the Ricci curvature. Indeed a Riemannian manifold (M, g, vol_g) of dimension n satisfies the RCD(K, n) condition if and only if its Ricci curvature is at least K;
– doubling condition on a metric measure space (X, d, μ): X satisfies a D_0*-doubling condition up to scale* $r_0 > 0$ if for any $0 < r \leq r_0$ and for all $x \in X$ it holds

$$\frac{\mu(B(x,2r))}{\mu(B(x,r))} \leq D_0.$$

– upper bound to the entropy: in [BCGS17] it is considered the case of δ-hyperbolic metric spaces X admitting a discrete group of isometries Γ with compact quotient. It is natural to consider the entropy h_Γ of the counting measure of an orbit (that is the critical exponent of Γ, see Section 1.6.2). Their version of lower bound on the curvature is then given by the condition $h_\Gamma \leq H$ for some fixed $H \geq 0$. We will see in a minute why this is a lower bound on the curvature.

The synthetic notion of lower bound we will use throughout the thesis, that is the packing condition, is extremely simple and natural. We say that a metric space (X, d) satisfies the P_0*-packing condition at scale* $r_0 > 0$ if all balls of radius $3r_0$ contain at most P_0 points that are $2r_0$-separated from each other (this can be equivalently expressed in terms of coverings with balls, see Chapter 2).
We observe that by Bishop-Gromov's Theorem every Riemannian manifold (M, g) with Ricci curvature bounded below by κ satisfies the P_0-packing condition, where P_0 depends only on κ and the dimension of M, at least at scales smaller than the injectivity radius of M. Notice however that for Riemannian manifolds a doubling or a packing condition at some scale $r_0 > 0$ are much weaker assumptions than a lower bound of the Ricci curvature (see [BCGS17], Sec.3.3, for different examples and a comparison of Ricci, packing and doubling conditions).
From a metric-geometry perspective the original interest in studying metric spaces satisfying a packing condition at *arbitrarily small* scales is Gromov's famous Precompactness Theorem [Gro81]. Another major outcome involving packing is Gromov's celebrated result on groups with polynomial growth, as extended by Breuillard-Green-Tao [BGT11] (cp. also the previous results [Kle10] and [ST10]), which shows that a uniform bound of the packing or doubling constant for X at *arbitrarily large* scale (or even at fixed, sufficiently large scale with respect to the diameter) yields an even stronger limitation on the complexity of the fundamental group of X, that is almost-nilpotency.
We can now come back to the bounded entropy condition given in [BCGS17]. Indeed one of the main results of that paper is that if (X, d) is a geodesically complete, δ-hyperbolic space admitting a discrete group of isometries Γ with compact quotient then X is P_0-packed at scale r_0, where P_0, r_0 depend only on δ and an upper bound on the diameter of the quotient and the critical exponent of Γ. That is why the upper bound on the entropy can be thought as a lower bound on the curvature condition.

1.1.3 Relations between upper and lower bounds

The packing condition alone is not enough: it is true that the packing at the fixed scale r_0 gives a control of the packing function at bigger scales (Lemma 2.4.2), but it does not give geometric information at scales smaller than r_0 (see Example 2.4.3). The fundamental lemma of the thesis affirms that if in addition we have an upper bound on the curvature (either convexity or the GCBA condition) and the locally geodesically completeness assumption then it is possible to propagate the packing estimate at scales smaller than r_0: this is the statement of Lemma 2.4.5, whose proof is almost trivial. However it is the starting point for all the results of the thesis. The idea is that a control of the packing function of X *at a fixed scale* gives a control of the local geometry of the metric space, provided X has an upper curvature bound and is geodesically complete. We report here this fact in case of convex metric space for simplicity.

Theorem A (Proposition 2.4.4, Packing Propagation Theorem)**.** *Let X be a convex and geodesically complete metric space that is P_0-packed at scale r_0. Then for every $0 < r \leq R$ it holds:*

$$\mathrm{Pack}(R, r) \leq P_0(1 + P_0)^{\frac{R}{r}-1}, \text{ if } r \leq r_0;$$

$$\mathrm{Pack}(R, r) \leq P_0(1 + P_0)^{\frac{R}{r_0}-1}, \text{ if } r > r_0.$$

Moreover if X is complete then it is proper.

The quantity $\mathrm{Pack}(R, r)$ is the packing function of X at scale $0 < r \leq R$, that is defined as the supremum among $x \in X$ of the maximal cardinality of a $2r$-separated subset inside the closed ball $\overline{B}(x, R)$. A related quantity is the covering function $\mathrm{Cov}(R, r)$ of X, which is defined as the supremum among $x \in X$ of the minimal cardinality of a r-dense subset of $\overline{B}(x, R)$. We notice that the packing condition can be stated as $\mathrm{Pack}(3r_0, r_0) \leq P_0$.

1.2 Packing in locally CAT(κ) spaces

In Chapter 3 we will study the case of GCBA spaces, where more local structure is known as explained in Section 1.1.1.
The first key-result of Chapter 3 is a Croke-type local volume estimate for GCBA-spaces of dimension bounded above for balls of radius smaller than the *almost-convexity radius*:

Theorem B (Theorem 3.1.1)**.** *For any complete* GCBA *space X of dimension $\leq n_0$ and any ball of radius $r < \min\{\rho_{\mathrm{ac}}(X), 1\}$ it holds:*

$$\mu_X(B(x, r)) \geq c_{n_0} \cdot r^{n_0} \tag{1}$$

where c_{n_0} is a constant depending only on the dimension n_0.

The almost-convexity radius $\rho_{\mathrm{ac}}(x)$ of a geodesic space X at a point x is defined as the supremum of the radii r such that for all $y, z \in B(x, r)$ and all $t \in [0, 1]$ it holds:

$$d(y_t, z_t) \leq 2t \cdot d(y, z),$$

where y_t, z_t denote points along geodesics $[x, y]$ and $[x, z]$ at distance $t \cdot d(x, y)$ and $t \cdot d(x, z)$ respectively from x. The almost-convexity radius of X is correspondingly defined as $\rho_{\text{ac}}(X) = \inf_{x \in X} \rho_{\text{ac}}(x)$. It is not difficult to show that every GCBA-space X always has positive almost-convexity radius at every point: namely if X is locally CAT(κ) and $x \in X$ then $\rho_{\text{ac}}(x)$ is always greater than or equal to the CAT(κ)-*radius* $\rho_{\text{cat}}(x)$ (see Section 2.2 for all details and the relation with the contraction and the logarithmic maps). However the almost-convexity radius is a more flexible geometric invariant than the CAT(κ)-radius, much alike the injectivity radius for Riemannian maniolds, since a space X might have a large curvature κ concentrated in a very small region around x, so that it may happen that $\rho_{\text{ac}}(x)$ is much larger than the CAT(κ)-radius at x.
We stress the fact that no explicit upper bound on the curvature is assumed for the estimate (1); the condition GCBA is only needed to ensure sufficient regularity of the space (and the existence of a natural measure to compute volumes). Notice that the above theorem is a generalization of the original Croke's result on manifolds ([Cro80], Proposition 14) at least for radii smaller than the almost-convexity radius: indeed on a manifold the curvature is bounded above on any compact ball. The differences are in the proofs: Croke's original proof is based on differential analysis, while our methods are purely metric.

For all subsequent results of this section we will consider, as standing assumption, GCBA-spaces with a uniform upper bound on the packing constant at some fixed scale r_0 smaller than the almost-convexity radius, or a doubling condition up to an arbitrary small scale. These classes of metric spaces are large enough to contain many interesting examples besides Riemannian manifolds and small enough to be, as we will see in Section 1.7, *compact* in the Gromov-Hausdorff sense.
There are a lot of non-manifolds examples in these classes of metric spaces. The simplest ones are simplicial complexes with locally constant curvature (also called M^κ-*complexes*, cp. [BH13]) and "bounded geometry" in an appropriate sense: they will be studied in detail in Section 3.4.1. Another interesting class of spaces satisfying a uniform packing condition at fixed scale is the class of (universal coverings of) compact, non-positively curved manifolds with bounded entropy admitting acylindrical splittings (see [CS]).

The first application of the Packing Propagation Theorem A is the characterization of the packing condition on GCBA spaces in terms of volume and dimension.

Theorem C (Extract from Theorem 3.2.1)**.** *Let X be a complete, geodesic,* GCBA-*space with almost-convexity radius $\rho_{\text{ac}}(X) \geq \rho_0 > 0$. The following conditions are equivalent:*

- *(a) there exist P_0 and $r_0 \leq \rho_0/3$ such that X satisfies the P_0-packing condition at scale r_0;*
- *(b) there exist n_0 and $V_0, R_0 > 0$ such that X has dimension $\leq n_0$ and $\mu_X(B(x, R_0)) \leq V_0$ for all $x \in X$;*

For Riemannian manifolds of dimension n the measure μ_X coincides with the n-dimensional Hausdorff measure, so (b) corresponds simply to a uniform upper bound on the Riemannian volume of balls of some fixed radius R_0, a condition that it is sometimes easier to verify than the bounded packing.
The proof of this theorem is essentially based on universal estimates from below and from above of the volume of small balls of X in terms of dimension and of the packing constants: the estimate from below is exactly given by (1). We want to remark out that many of the ideas behind these results are already implicitely present in [LN19].

In Section 3.3 we investigate the relation between the local doubling condition[1] with respect to the natural measure μ_X and the local structure of GCBA-spaces. It is easy to show that a local doubling condition implies the packing. However it turns out that the doubling property is much stronger and characterizes GCBA-spaces which are *purely dimensional spaces*, i.e. those whose points have all the same dimension. Indeed we prove:

Theorem D (Extract from Corollary 3.3.5 & Theorem 3.3.2)**.** *Let X be a complete, geodesic,* GCBA*-space with almost-convexity radius $\rho_{\text{ac}}(X) \geq \rho_0 > 0$. The following conditions are equivalent:*

(a) there exists $D_0 > 0$ such that the natural measure μ_X is D_0-doubling up to some scale $r_0 > 0$;

(b) X is purely n-dimensional for some n and there exist constants P_0 and $r_0 \leq \rho_0/3$ such that X satisfies the P_0-packing condition at scale r_0.

Finally in Section 3.4.1 of Chapter 3 we specialize our results to M^κ-complexes with bounded geometry. We will first establish some basic relations relating the injectivity radius to the size and valency of the complexes. Recall that the *valency* of a M^κ-complex X is the maximum number of simplices having a same vertex in common and the *size* of the simplices of X is defined as the smallest radius $R > 0$ such that any simplex contains a ball of radius $\frac{1}{R}$ and is contained in a ball of radius R; we refer to Section 3.4.1 for further definitions and details. Then we prove:

Theorem E (Proposition 3.4.12)**.** *Let X be a M^κ-complex whose simplices have size bounded by R, with valency at most N and no free faces. Then the following conditions are equivalent:*

(a) X is a complete GCBA*-space with curvature $\leq \kappa$;*

(b) X satisfies the link condition at all vertices;

(c) X is locally uniquely geodesic;

(d) X has positive injectivity radius;

(e) X has injectivity radius $\geq \iota_0$, for some ι_0 depending only on R and N.

[1]Beware that the *doubling constant* which is used in [LN19] is a different notion, which is purely metric and does not depend on the measure.

The equivalence of the first four conditions is well-known for M^κ-complexes with *finite shapes* (that is whose geometric simplices, up to isometry, vary in a finite set), see [BH13], while the last condition is new and we will use it to exhibit other examples of compact families of GCBA-spaces as explained in Section 1.7. We remark that the class of complexes satisfying the assumptions of Theorem E is made of locally CAT(κ) metric spaces with uniform lower bound on the almost-convexity radius and satisfying a uniform packing condition at the same scale (see Proposition 3.4.13).

1.3 Entropies

Chapter 4 is devoted to the investigation of different asymptotic quantities associated to a metric space, some of them classical and widely studied. The attention will be held on convex (and sometimes Gromov-hyperbolic) metric spaces satisfying a uniform covering (or packing) condition. We remark that convex metric spaces are the natural setting for the study of the geodesic flow, see Chapter 2. We are going to present the different notions of entropies we are interested in, starting from the Lipschitz-topological entropy.

1.3.1 Lipschitz-topological entropy of the geodesic flow

The topological entropy of the geodesic flow has been intensively studied in case of Riemannian manifolds, especially in the negatively curved setting. If such a manifold is denoted by $\bar{M} = M/\Gamma$, where M is its universal cover and Γ is its fundamental group, then the set of parametrized geodesic lines is identified with the unit tangent bundle $S\bar{M}$ and the non-wandering set of the geodesic flow is the set of unit tangent vectors whose lift to M generate a geodesic with endpoints in the limit set of Γ: we denote it by $S\bar{M}^{\text{nw}}$. Two cornerstones of the theory of the geodesic flow are the works of Eberlein ([Ebe72]), who proved that the geodesic flow restricted to $S\bar{M}^{\text{nw}}$ is topologically transitive, and Sullivan [Sul84], who proved the ergodicity of the geodesic flow when M is the 3-dimensional hyperbolic space and Γ is geometrically finite.

Probably the most important invariant associated to the geodesic flow is the topological entropy of its restriction to the non-wandering set, denoted $h_{\text{top}}^{\text{nw}}(\bar{M})$. It equals the Hausdorff dimension of the limit set of Γ and the critical exponent of Γ (see [Sul84], [OP04]). Moreover if $\bar{M}$ is compact then it coincides also with the volume entropy of M ([Man79]), while this is no more true in general, even when $\bar{M}$ has finite volume (cp. [DPPS09]): we will come back to these examples later. The topological entropy of the non-wandering set of the geodesic flow characterizes the hyperbolic metrics among Riemannian manifolds with pinched, negative curvature and with finite volume ([PS19]).

When X is a convex metric space the topological entropy of the geodesic flow is defined as the topological entropy (in the sense of Bowen, cp. [Bow73], [HKR95]) of the dynamical system $(\text{Geod}(X), \Phi_t)$, where $\text{Geod}(X)$ is the space of parametrized geodesic lines, endowed with the topology of uniform convergence on compact subsets, and Φ_t is the reparametrization flow. It is:

$$h_{\text{top}}(\text{Geod}(X)) = \inf_{\mathfrak{d}} \sup_{K \subseteq \text{Geod}(X)} \lim_{r \to 0} \lim_{T \to +\infty} \frac{1}{T} \log \text{Cov}_{\mathfrak{d}^T}(K, r),$$

where the infimum is taken *among all metrics on* Geod(X) inducing its topology, the supremum is taken among all compact subsets of Geod(X), $\mathfrak{d}^T$ is the distance $\mathfrak{d}^T(\gamma, \gamma') = \max_{t \in [0,T]} \mathfrak{d}(\Phi_t(\gamma), \Phi_t(\gamma'))$ and $\text{Cov}_{\mathfrak{d}^T}(K, r)$ is the minimal number of balls (with respect to the metric $\mathfrak{d}^T$) of radius r needed to cover K.
For this flow the non-wandering set is empty and applying the variational principle (cp. [HKR95]) it is straightforward to conclude that its topological entropy is zero since there are no flow-invariant probability measures (Lemma 4.2.1). Looking carefully at the proof of the variational principle it turns out that the metrics on Geod(X) almost realizing the infimum in the definition of the topological entropy are restriction to Geod(X) of metrics on its one-point compactification. In particular they are no the natural ones to consider: indeed the general idea behind the topological entropy is to compute the number of geodesic lines needed to stay at small distance r from any other geodesic line for a long time T. But a general metric $\mathfrak{d}$ on Geod(X) does not take into account this information. That is why, in Section 4.2, we will restrict the attention to the class of *geometric metrics* $\mathfrak{d}$: those with the property that the evaluation map $E\colon (\text{Geod}(X), \mathfrak{d}) \to (X, d)$ defined as $E(\gamma) = \gamma(0)$ is Lipschitz. Notice that for a geometric metric two geodesic lines are not close if they are distant at time 0. Accordingly the *Lipschitz-topological entropy* of the geodesic flow is defined as

$$h_{\text{Lip-top}}(\text{Geod}(X)) = \inf_{\mathfrak{d}} \sup_{K \subseteq \text{Geod}(X)} \lim_{r \to 0} \lim_{T \to +\infty} \frac{1}{T} \log \text{Cov}_{\mathfrak{d}^T}(K, r),$$

where now *the infimum is taken only among the geometric metrics of* Geod(X). Although the definition of the Lipschitz-topological entropy is quite complicated, its computation can be remarkably simplified. Indeed one of the most used metric on Geod(X) (see for instance [BL12]) is:

$$d_{\text{Geod}}(\gamma, \gamma') = \int_{-\infty}^{+\infty} d(\gamma(s), \gamma'(s)) \frac{1}{2e^{|s|}} ds$$

that induces the topology of Geod(X) and is geometric, and it turns out that it realizes the infimum in the definition of the Lipschitz-topological entropy.

Theorem F (Extract from Theorem 4.2.2 & Proposition 4.2.3)**.** *Let X be a complete, convex, geodesically complete metric space that is P_0-packed at scale r_0. Then*

$$h_{\text{Lip-top}}(\text{Geod}(X)) = \lim_{T \to +\infty} \frac{1}{T} \log \text{Cov}_{d^T_{\text{Geod}}}(\text{Geod}(x), r_0),$$

where Geod(x) *is the set of geodesic lines passing through x at time* 0.

Therefore the infimum in the definition of the Lipschitz topological entropy is actually realized by the metric d_{Geod} and the supremum among the compact sets can be replaced by a fixed (relative small) compact set. Moreover also the scale r can be fixed to be r_0 (or any other positive real number).

1.3.2 Volume and Covering entropy

The second definition of entropy we consider (see Section 4.1.2) is the volume entropy. If X is a metric space equipped with a measure μ it is classical to consider the exponential growth rate of the volume of balls, namely:

$$h_\mu(X) := \lim_{T\to+\infty} \frac{1}{T}\log \mu(\overline{B}(x,T)).$$

It is called the *volume entropy* of X with respect to the measure μ and it does not depend on the choice of the basepoint $x \in X$ by triangular inequality. This invariant has been studied intensively in case of complete Riemannian manifolds with non positive sectional curvature, where μ is the Riemannian volume on the universal cover. It is related to other interesting invariants as the simplicial volume of the manifold (see [Gro82]), [BS20]), a macroscopical condition on the scalar curvature (cp. [Sab17]) and the systole in case of compact, non-geometric 3-manifolds (cp. [CS19]). Moreover the infimum of the volume entropy among all the possible Riemannian metrics of volume 1 on a fixed closed manifold is a subtle homotopic invariant (see [Bab93], [Bru08] for general considerations and [BCG95], [Pie19] for the computation of the minimal volume entropy in case of, respectively, closed n-dimensional manifolds supporting a locally symmetric metric of negative curvature and 3-manifolds).

Another example, besides the Riemannian setting, is the counting measure of the orbit of a discrete, cocompact group of isometries of a convex, geodesically complete, Gromov-hyperbolic metric space (as studied in [BCGS17]). Both this case and the Riemannian one (at least when the curvature of the Riemannian manifold considered is pinched) share the following property: the measure μ under consideration satisfies

$$\frac{1}{H} \leq \mu(\overline{B}(x,r)) \leq H$$

for every $x \in X$, where r, H are positive real numbers. A measure with this property is called H-homogeneous at scale r. Among homogeneous measure there is a remarkable example: the volume measure μ_X of a complete, geodesically complete, CAT(0) metric space X that is P_0-packed at scale r_0 (see Section 1.2). We recall once again that if X is a Riemannian manifold of non-positive sectional curvature then μ_X coincides with the Riemannian volume, up to a universal multiplicative constant.

A more combinatoric and intrinsic version of the volume entropy of a generic metric space is the *covering entropy*, defined as:

$$h_{\text{Cov}}(X) := \lim_{T\to+\infty} \frac{1}{T}\log \text{Cov}(\overline{B}(x,T), r),$$

where x is a point of X and $\text{Cov}(\overline{B}(x,T), r)$ is the minimal number of balls of radius r needed to cover $\overline{B}(x,T)$. It does not depend on x but it can depend on the choice of r. This is not the case when X is a geodesically complete, convex metric space that is P_0-packed at scale r_0, as proved in Proposition 4.1.1. Moreover it is always finite (cp.Lemma 4.1.3). These results are again direct applications of the Packing Propagation Theorem A.

1.3.3 Shadow and Minkowski dimension of the boundary

The explicit computation of the Lipschitz-topological entropy suggests the possibility to relate that invariant to some property of the boundary at infinity of X. Once fixed a basepoint $x \in X$ we define the *shadow dimension* of the boundary ∂X of X as

$$\text{Shad-D}(\partial X) = \lim_{T\to+\infty} \frac{1}{T} \log \text{Shad-Cov}_r(\partial X, e^{-T}),$$

where $\text{Shad-Cov}_r(\partial X, e^{-T})$ is the minimal number of points $y_1, \ldots, y_N$ at distance T from x such that every geodesic ray issuing from x passes through one of the balls of radius r and center y_i. The limit above does not depend neither on x nor on r. It describes the asymptotic behaviour of the number of shadows casted by points at distance T from x needed to cover ∂X, when T goes to $+\infty$. The shadows, and especially their relations with other properties of the boundary at infinity, have been intensively studied during the years (starting from [Sul79]). In particular if X is Gromov-hyperbolic the boundary at infinity can be equipped with a metric and it turns out that the metric balls are approximately equivalent to the shadows. This equivalence remains true when we consider the generalized visual balls. If we denote by $(\cdot,\cdot)_x$ the Gromov product based on x then the generalized visual ball of center $z \in \partial X$ and radius ρ is $B(z,\rho) = \{z' \in \partial X \text{ s.t. } (z,z')_x > \log \frac{1}{\rho}\}$. The *visual Minkowski dimension* of ∂X is:

$$\text{MD}(\partial X) = \lim_{T\to+\infty} \frac{1}{T} \log \text{Cov}(\partial X, e^{-T}),$$

where $\text{Cov}(\partial X, e^{-T})$ is the minimal number of generalized visual balls of radius e^{-T} needed to cover ∂X. If the generalized visual balls are metric balls for some visual metric $D_{x,a}$ then we refind the usual definition of Minkowski dimension of the metric space $(\partial X, D_{x,a})$, once the obvious change of variable $\rho = e^{-T}$ is made. These invariants are presented in Section 4.3.

1.3.4 Equality of the entropies

One of the main results of this thesis is:

Theorem G. *Let X be a complete, convex, geodesically complete metric space that is P_0-packed at scale r_0. Then*

$$h_{\text{Lip-top}}(\text{Geod}(X)) = h_\mu(X) = h_{\text{Cov}}(X) = \text{Shad-D}(\partial X),$$

where μ is every homogeneous measure on X. Moreover if X is also δ-hyperbolic then they coincide also with $\text{MD}(\partial X)$.

Actually something more is true but in order to state it we need to introduce the notion of equivalent asymptotic behaviour of two functions.
Given $f, g\colon [0,+\infty) \to \mathbb{R}$ we say that f *and g have the same asymptotic behaviour*, and we write $f \asymp g$, if for all $\varepsilon > 0$ there exists $T_\varepsilon \geq 0$ such that if $T \geq T_\varepsilon$ then $|f(T) - g(T)| \leq \varepsilon$. The function T_ε is called the *threshold function*.
Usually we will write $f \underset{P_0,r_0,\delta,\ldots}{\asymp} g$ meaning that the threshold function can be

expressed only in terms of ε and $P_0, r_0, \delta, \ldots$ In particular if g is constantly equal to g_0 and $f \underset{P_0,r_0,\delta,\ldots}{\asymp} g_0$ then the function f tends to g_0 when T goes to $+\infty$ and moreover the rate of convergence to the limit can be expressed only in terms of $P_0, r_0, \delta, \ldots$

Theorem H. *Let X be a complete, convex, geodesically complete metric space that is P_0-packed at scale r_0. Then the functions defining the quantities of Theorem G have the same asymptotic behaviour and the threshold functions depend only on P_0, r_0, δ and the homogeneous constants of μ.*

Therefore not only all the introduced quantities define the same number, but all of them also have the same asymptotic behaviour. This means that if one can control the rate of convergence to the limit of one of these quantities then also the rate of convergence of all the other quantities is bounded. We remark that, differently from many of the papers in the literature, we do not require any group action on our metric spaces for the moment. The case of group actions will be studied in Chapter 7.

1.4 The Tits Alternative

The classical *Tits alternative*, proved by J. Tits [Tit72] says that any finitely generated linear group Γ over a commutative field $\mathbb{K}$ either is virtually solvable or contains a non-abelian free subgroup. This result has been extended to other classes of groups during the years. For instance, without intending to be exhaustive, it is now known that all the following classes of groups satisfy the Tits Alternative:

- any discrete, non-elementary group of isometries of a Gromov-hyperbolic space,
- any discrete group of isometries a proper CAT(0)-space containing a rank one element,
- any group acting properly and freely by isometries on a finite dimensional CAT(0)-cube complex,
- any acylindrical hyperbolic group,
- any subgroup of the mapping class group

(see respectively [Gro87], [Ham09], [SW05], [Osi16] [McC85]; also notice that the second class is a particular case of the fourth one, by [Sis18]).

Also, the original result has been improved by quantifying the depth in Γ of the free subgroup with respect to some fixed generating set S of Γ. E. Breuillard proved that there exists a universal function $N(d)$ such that for any finite subset $S \subset GL(d, \mathbb{K})$ either the group Γ generated by S is virtually solvable or there exist two words a, b on S of S-length less than $N(d)$ which generate a non-abelian free group (see [Bre08], and previous quantifications in this direction [BG08], [EMO05]). Similar forms of quantification of the Tits Alternative for Gromov hyperbolic groups were proved by T. Delzant [Del96], by M. Koubi [Kou98] (for Gromov hyperbolic groups with torsion) and by G. Arzantsheva and I. Lysenok [AL06] (for subgroup of a given hyperbolic group), for a constant N depending however always on the group Γ under consideration. A quantititative Tits Alternative for finitely generated subgroups of the mapping class group $\mathrm{Mod}(S)$ of any compact, orientable surface S was proved by J. Mangahas in [Man10].

A weaker form of the alternative, generally easier to establish, which we will call *weak Tits Alternative*, asks for the existence of free *semigroups* in Γ instead of free subgroups, provided that Γ is not virtually solvable[2]. For instance, it is known that there exists a universal constant N such that for any finitely generated group Γ acting properly without global fixed points on a CAT(0) square complex and for any generating set S of Γ, there always exist two elements $a, b \in \Gamma$ with S-norm smaller than N, generating a free semigroup, provided that Γ is not virtually abelian (which is the same as non-virtually solvable, for groups acting properly on finite-dimensional CAT(0)-cube complexes), cp. [KS19], [GJN20].
The same is true for finitely generated torsionless groups acting on d-dimensional cube complexes with isolated flats admitting a geometric group action, for a function $N(d)$ only depending on the dimension of the complex, see [GJN20], Theorem B).

Within the realm of negatively curved spaces very general quantifications of the weak Tits Alternative for discrete groups of isometries of Gromov-hyperbolic spaces were recently proved independently by Breuillard-Fujiwara [BF18], and by Besson-Courtois-Gallot-Sambusetti [BCGS17]. Namely they show that there exists a universal function $N(C)$ such that for any finite, symmetric subset S of isometries of a Gromov hyperbolic space X, either the group G generated by S is elementary or S^N contains two elements a, b which generate a free semigroup, *provided that* X satisfies a packing condition at scale δ with constant C or, better, if the counting measure of G satisfies a doubling condition at some scale r_0 with constant C. Less general forms of quantifications of the weak Tits Alternative for Gromov hyperbolic spaces were previuosly proved by C. Champetier and V. Guirardel [CG00], for δ-hyperbolic groups Γ (for some N depending on the hyperbolicity constant and on the cardinality of the generating set S) and by Besson-Courtois-Gallot [BCG03], for fundamental groups of pinched, negatively curved manifolds X (for some N depending on the injectivity radius of X and on a lower bound of the sectional curvature).

For discrete isometry groups of pinched, negatively curved manifolds S. Dey, M. Kapovich and B. Liu [DKL18] recently improved [BCG03], proving a quantitative, true Tits Alternative: there exists $N = N(k, d)$ such that for any couple of isometries a, b of a complete, simply connected, d-dimensional Riemannian manifold X with pinched, negative sectional curvature $-k^2 \leq K_X \leq -1$ which generate a discrete [3] non-elementary group, with a not elliptic, one can find an isometry w, which can be written as a word $w = w(a, b)$ on $\{a, b\}$ of length less than N, such that $\{a^N, w\}$ generate a non-abelian free subgroup. Noticably the authors not only find a true

[2]Notice that, in this weaker form, the Tits Alternative is no longer a *dichotomy* for linear groups, since it is well known that there exist solvable groups of $GL(n, \mathbb{R})$ which also contain free semigroups (and actually, *any* finitely generated solvable group which is not virtually nilpotent contains a free semigroup on two generators [Ros74]). It remains a dichotomy for those classes of groups for which “virtually solvable” implies sub-exponential growth, e.g. hyperbolic groups, groups acting geometrically on CAT(0)-spaces etc.

[3]We are not able to understand the proof in [DKL18] without the torsionless assumption, which seems to be used at page 14, below Lemma 4.5.

free subgroup with quantification, but they also specify that one of the generators of the free group can be prescribed a-priori (provided it is chosen not elliptic). This motivates the following definition:

Definition. We say that a discrete group Γ satisfies the quantitative Tits Alternative *with specification* $\mathcal{T}_{ws}(N)$ (respectively, the quantitative weak Tits Alternative with specification $\mathcal{T}^+_{ws}(N)$) if for every couple $a, b \in \Gamma$ which generates a non-virtually solvable group, with a of infinite order, there exists $w \in \Gamma$ which can be written as a word $w = w(a, b)$ in $\{a, b\}$ of length at most N, such that the subgroup $\langle a^N, w\rangle$ (resp. the semigroup $\langle a^N, w\rangle^+$) generated by $\{a^N, w\}$ is free.

The aim of Chapter 5 is two-fold:
(a) to generalise Dey-Kapovich-Liu's result, proving the property $\mathcal{T}_{ws}(N)$ for a large class of sufficiently mild, negatively curved *metric spaces*, and
(b) to extend the range of applications of the quantitative free sub-(semi)group theorem proved in [BCGS17] to *non-cocompact* actions [4].

With this in mind we recall that the lower sectional curvature bound can be replaced, in the metric setting, by the condition of bounded packing at some scale $r_0 > 0$. By Breuillard-Green-Tao's work, the packing condition provides sufficient information to deduce an analogue for metric spaces of the celebrated *Margulis Lemma* for Riemannian manifolds: *there exists a constant* ε_0*, only depending on* (P_0, r_0)*, such that for every discrete group of isometries* Γ *of a space* X *which is* P_0*-packed at scale* r_0*, the* ε_0*-almost stabilizer* $\Gamma_{\varepsilon_0}(x)$ *of any point* $x \in X$ *is virtually nilpotent* (cp. [BGT11], Corollary 11.17); that is, the elements of Γ which displace x less than ε_0 generate a virtually nilpotent group. This result allows us to mimick most of the arguments used in [DKL18].

On the other hand Gromov-hyperbolicity is the most natural metric replacement for the classical hypothesis of negative curvature. However we were not able to recover Dey-Kapovich-Liu's result without the additional assumption of the convexity of the space. This additional assumption gives much more regularity at scales smaller than the hyperbolicity constant, in particular it allows, using the Packing Propagation Theorem A, to obtain precise estimates of the ε-Margulis' domains of isometries for values of ε smaller than δ:

Theorem I (Proposition 2.5.6)**.** *Let* X *be a complete, convex, geodesically complete,* δ*-hyperbolic metric space that is* P_0*-packed at scale* r_0 *and let* $0 < \varepsilon_1 \leq \varepsilon_2$*. Then there exists* K_0*, only depending on* $P_0, r_0, \delta, \varepsilon_1$ *and* ε_2*, such that for every non-elliptic isometry* g *of* X *with* $\ell(g) \leq \varepsilon_1$ *it holds:*

$$\sup_{x \in \mathcal{M}_{\varepsilon_2}(g)} d(x, \mathcal{M}_{\varepsilon_1}(g)) \leq K_0.$$

Here $\ell(g)$ denotes the translation length of g and $\mathcal{M}_\varepsilon(g)$ is the generalized Margulis domain of g of displacement ε, see Chapter 2 for the definitions. This result is a key-tool for the proof of the following:

[4] *Non-cocompact* actions of a group Γ on general metric spaces are also considered in [BCGS17], especially in Chapter 6, but always with the underlying assumption that the group Γ also admits a cocompact action or a "well-behaved" action on some Gromov-hyperbolic space X_0, e.g. with some prescribed, lower bound of the minimal asymptotic displacement $\|g\|$ on X_0, for all $g \in \Gamma$.

Theorem J (Quantitative free subgroup theorem with specification).
Let P_0, r_0 and δ be fixed positive constants. Then there exists an integer $N(P_0, r_0, \delta)$, only depending on P_0, r_0, δ, satisfying the following properties. Let X be any complete, convex, geodesically complete, δ-hyperbolic metric space that is P_0-packed at scale r_0:

(a) for any couple of isometries $S = \{a, b\}$ of X, where a is non elliptic, such that the group $\langle a, b\rangle$ is discrete and non-elementary, there exists a word $w(a, b)$ in a, b of length $\leq N$ such that one of the semigroups $\langle a^N, w(a, b)\rangle^+$, $\langle a^{-N}, w(a, b)\rangle^+$ is free;

(b) for any couple of isometries $S = \{a, b\}$ of X such that the group $\langle a, b\rangle$ is discrete, non-elementary and torsion-free, there exists a word $w(a, b)$ in a, b of length $\leq N$ such that the group $\langle a^N, w(a, b)\rangle$ is free.

Therefore any discrete group of isometries Γ of a complete, convex, geodesically complete, packed δ-hyperbolic space satisfies property $\mathcal{T}^+_{ws}(N)$, and also property $\mathcal{T}_{ws}(N)$ if Γ is torsionless, for N depending only on δ and on the packing constants (P_0, r_0).

The first part of the theorem precises Proposition 5.18 in [BCGS17] and Theorem 5.11 of [BF18], showing the specification property under the additional hypothesis of convexity. The difficulty here is that there is no a apriori bounded power N such that $\ell(a^N)$ is greater than a specified constant (for instance, when a is parabolic this is false for every N). To avoid changing a with a bounded word in $\{a, b\}$ (as in [BCGS17] and [BF18]) we follow the strategy of [DKL18], which however requires the convexity property.
For the second part of Theorem J, it should be remarked that the torsionless assumption is actually necessary, as shown by some examples of groups acting on simplicial trees (with bounded valency, hence packed) produced in Proposition 12.2 of [BF18].
Finally notice that, in our setting, *elementary* is the same as *virtually nilpotent* because of the bounded packing assumption (see Section 2.6.2 for a proof of this fact; notice however that, without the packing assumption, there exist elementary groups of negatively curved manifolds which are even free non-abelian, cp.[Bow93]).

We want also to stress that the proof we give of this theorem heavily draws from techniques developed in [DKL18] and [BCGS17]. However some ideas are new, such as the bound of the distance between different levels of the Margulis domains of a single isometry, as well as the use of the free subgroup for some of the applications (estimate of the diastole and structure group of the thin subsets), which we will discuss in the next section.

1.5 Applications of the Tits Alternative

The motivation for us behind the quantification and the specification property, as well as for looking for true free subgroups (not only semigroups), is *geometric*. Actually the aim of Chapter 6 is to develop a geometric analysis for actions of discrete groups on packed, convex, Gromov hyperbolic spaces, and the Tits Alternative (in the sharp, quantitative form stated in Section 1.4) is a key-tool which is essential for many of the applications we will present. For instance a weak Tits alternative is not enough to properly describe the group structure of the connected components of the *thin subsets* for the action of Γ on a hyperbolic space X, as we will do in Section 6.4 (which is more precise than in [BCGS17]). On the other hand the specification property will be used to bound from below the *systole* of the action in terms of the *upper nilradius* thus yielding a new version of the classical Margulis' Lemma in our context. Some applications we will describe do not need the specification property or the quantitative Tits Alternative, but the packing assumption remains almost everywhere essential, in particular for Breuillard-Green-Tao's generalized Margulis Lemma.

A first, direct consequence of a quantitative free group or semigroup theorem is a uniform estimate from below of the *algebraic entropy* of the groups under consideration and of the entropy of the spaces they act on (from Theorem G we know that it can be computed in different ways, for simplicity we will refer to the covering entropy). On the other hand given any group Γ with a finite system of generators S one defines $\mathrm{Ent}(\Gamma, S)$ as the volume entropy of the Cayley graph of the couple (Γ, S). The algebraic entropy of a finitely generated group, denoted $\mathrm{EntAlg}(\Gamma)$, is accordingly defined as the infimum, over all possible finite generating sets S, of the exponential growth rates $\mathrm{Ent}(\Gamma, S)$. We record here two estimates for the entropy of the groups and the spaces under consideration, which stem from the simple weak quantified Tits Alternative (as previously proved in [BF18], Theorem 13.9, or in [BCGS17], Proposition 5.18(i)), combined with the packing assumption:

Theorem K. *Let X be a complete, convex, geodesically complete, δ-hyperbolic metric space that is P_0-packed at scale r_0. Assume that X admits a non-elementary, discrete group of isometries Γ. Then:*

(a) $\mathrm{EntAlg}(\Gamma) \geq C_0$*,*

(b) $h_{\mathrm{Cov}}(X) \geq C_0 \cdot \mathrm{nilrad}(\Gamma, X)^{-1}$*,*

where $C_0 = C_0(P_0, r_0, \delta)$ is a constant depending only on P_0, r_0 and δ.

The invariant $\mathrm{nilrad}(\Gamma, X)$ appearing here is the *nilradius* of the action of Γ on X, defined as the infimum over all $x \in X$ of the largest radius r such that the r-almost stabilizer $\Gamma_r(x)$ of x is virtually nilpotent. By definition it is always bounded below by Breuillard-Green-Tao's generalized Margulis constant ε_0. On the other hand the nilradius can be arbitrarily large, if the orbits of Γ are very sparse in X. However if Γ is non-elementary (which in our case means non-virtually nilpotent), it is always a finite number.

A second geometric consequence of Theorem J is a lower bound of the *systole* of the action of Γ on X, that is the smallest non-trivial displacement of points of X under the action of the group. Namely let $X_{\varepsilon_0} \subseteq X$ be the subset of points which are displaced less than the generalized Margulis constant ε_0 by some nontrivial element of the group Γ: this is classically called the ε_0-*thin subset* of X (we will come back to it later). We define the *upper nilradius* of Γ, as opposite to the nilradius, as the *supremum* over $x \in X_{\varepsilon_0}$ of the largest r such that $\Gamma_r(x)$ is virtually nilpotent. In other words the upper nilradius, denoted $\text{nilrad}^+(\Gamma, X)$, measures how far we need to travel from any x of X_{ε_0} to find two points g_1x, g_2x of the orbit such that the subgroup $\langle g_1, g_2 \rangle$ is non-elementary. A natural bound of the upper nilradius is given, for cocompact actions, by the diameter of the quotient Γ/X. However the upper nilradius can well be finite even for non-compact actions, for instance when Γ is a quasiconvex-cocompact group, or a subgroup of infinite index of a cocompact group of X (see Examples 6.2.3, 6.2.4 and Section 1.6 for details on quasiconvex-cocompact groups). The specification property in the quantitative Tits Alternative yields a lower bound of the systole of the action in terms of the geometric parameters P_0, r_0, δ and of an upper bound of the upper nilradius:

Theorem L. *Let X be a complete, convex, geodesically complete, δ-hyperbolic metric space that is P_0-packed at scale r_0. Then for any torsionless, non-elementary discrete group of isometries Γ of X it holds:*

$$\text{sys}(\Gamma, X) \geq \min \left\{ \varepsilon_0, \ \frac{1}{H_0} e^{-H_0 \cdot \text{nilrad}^+(\Gamma, X)} \right\}$$

where $H_0 = H_0(P_0, r_0, \delta)$ is a constant depending only on P_0, r_0, δ and $\varepsilon_0 = \varepsilon_0(P_0, r_0)$ is the generalized Margulis constant introduced before.

Remark that without any bound of the upper nilradius of Γ there is no hope of estimating $\text{sys}(\Gamma, X)$ from below in terms of δ and the packing parameters. This is clear for groups acting with parabolics (cp. Example 6.2.2), but also fails for groups without parabolics. It is enough to consider compact hyperbolic manifolds X possessing very small periodic geodesics γ of length ε, much smaller than the Margulis' constant: by the classical theory of Kleinian groups, γ has a very long tubular neighbourhood and, consequently, Γ has arbitrarily large upper nilradius. This is a general fact for actions of discrete groups on complete, convex, geodesically complete, Gromov-hyperbolic, packed metric spaces, as we will see in a minute.

Even without any a-priori bound of the upper nilradius of the action of Γ, one can always find a point x where the minimal displacement is bounded below by a universal function of the geometric parameters P_0, r_0 of X. Recall that the *diastole* of Γ acting on X, denoted $\text{dias}(\Gamma, X)$, is defined as the supremum over all $x \in X$ of the minimal displacement of x under all non trivial elements of the group. When X is a nonpositively curved manifold it corresponds to (twice) the greatest value of the cut radius for points of the quotient manifold $\Gamma \backslash X$. The next result generalizes one of the classical versions of the Margulis Lemma for negatively curved, pinched, Riemannian manifolds:

Theorem M. *Let X be a complete, convex, geodesically complete, δ-hyperbolic metric space that is P_0-packed at scale r_0. Then for any torsionless, discrete, non-elementary group of isometries Γ of X we have:*

$$\text{dias}(\Gamma, X) = \sup_{x \in X} \inf_{g \in \Gamma^*} d(x, gx) \geq \varepsilon_0$$

(where $\varepsilon_0 = \varepsilon_0(P_0, r_0)$ is the generalized Margulis constant).

Notice that the estimate, which holds also for cocompact groups, does not depend on the diameter (in contrast with Proposition 5.25 of [BCGS17]; also notice that our groups do not belong to any of the classes considered in [BCGS17], as they do not have a-priori a cocompact action on a convex, Gromov-hyperbolic space or an action on a Gromov-hyperbolic space with asymptotic displacement uniformly bounded below).

A consequence of the above estimates is an analogue of the classical thick-thin decomposition for Kleinian groups or isometry groups of pinched, negatively curved Riemannian manifolds (see [Thu97], [Bow95]) for discrete torsionless groups Γ acting on any complete, CAT(0), geodesically complete, Gromov-hyperbolic, packed metric space. Namely we show that for any $\varepsilon > 0$ smaller than the generalized Margulis constant ε_0 the connected components X^i_ε of the ε-thin subset of X are precisely invariant subsets whose stabilizer in Γ is a maximal, elementary subgroup Γ^i_ε (the subgroup generated by all the ε-almost stabilizers of points in X^i_ε): see Proposition 6.4.1 for a precise statement. This allows us to talk of *hyperbolic* and *parabolic components* of the *ε-thin subset of the quotient space* $\bar{X} = \Gamma\backslash X$, according to the type of the elementary subgroups Γ^i_ε, and opens the road to the notion of geometrical finiteness in our setting (whose study will be pursued elsewhere).
Propositions 6.4.2 & 6.4.4 in Section 6.4 resume the geometric picture of these components. Namely each component X^i_ε contains:
– in the hyperbolic case, a tubular neighbourhood $C_\varepsilon(\gamma)$ of any axis γ of the cyclic subgroup Γ^i_ε, which projects into a neighbourhood $C_\varepsilon(\bar{\gamma})$ of a periodic geodesic $\bar{\gamma}$ in $\bar{X}$;
– in the parabolic case, a connected, geodesic cone $C_\varepsilon(\gamma)$ for any geodesic ray γ with endpoint in the parabolic fixed point z of Γ^i_ε, which projects into a cone-neighbourhood $C_\varepsilon(\bar{\gamma})$ of the quotient ray $\bar{\gamma}$ in $\bar{X}$ containing definitely all rays asymptotic to $\bar{\gamma}$.
Moreover there exist universal functions $L_\varepsilon(r), R_\varepsilon(r)$ (only depending on the geometric parameters P_0, r_0, δ), tending to ∞ as $r \to 0$, such that if the above axis / rays γ are included in $X^i_r \subseteq X^i_\varepsilon$, for some $r < \varepsilon$, then the components X^i_r are:

- *the thinner, the longer*: the $L_\varepsilon(r)$-neighbourhood of $C_r(\gamma)$ is entirely contained in X^i_ε (thus X^i_ε contains a long tube around γ in the hyperbolic case and a large cone around any ray γ with endpoint z in the parabolic one);

- *have simple topology*: the $R_\varepsilon(r)$-neighbourhood in $\bar{X}$ of the periodic geodesic $\bar{\gamma}$ or of the geodesic cone $C_r(\bar{\gamma})$ are, respectively, isometric to the $R_\varepsilon(r)$-neighbourhood of γ in X modulo the cyclic group Γ^i_ε (in the hyperbolic case), and to the $R_\varepsilon(r)$-neighbourhood of the cone $C_r(\gamma)$ in X modulo the virtually nilpotent group Γ^i_ε (in the parabolic case).

The CAT(0) assumptions is needed to find a true invariant cone $C_\varepsilon(\gamma)$ in the parabolic case, since in this case the horospheres are preserved by the isometries of the group Γ^i_ε. This good description of the parabolic components opens the possibility to the study of geometrically finiteness in this setting. This topic will be investigated in further works.

1.6 Entropies and groups

In Chapter 7 we will introduce a variant of the different notions of entropies presented in Chapter 4 in order to study the critical exponent of groups acting on complete, convex, geodesically complete, δ-hyperbolic, packed metric spaces.

1.6.1 Entropies of the closed subsets of the boundary

In case X is a complete, convex, geodesically complete, δ-hyperbolic metric space that is P_0-packed at scale r_0 it is possible to define the version of all the different notions of entropies of Section 1.3 relative to subsets of the boundary at infinity ∂X (notice that in Section 1.3 the hyperbolicity assumption was not needed).
For every subset $C \subseteq \partial X$ we denote by $\text{Geod}(C)$ the set of parametrized geodesic lines with endpoints belonging to C and with $\text{QC-Hull}(C)$ the union of the points belonging to the geodesics of $\text{Geod}(C)$. Actually the hyperbolicity assumption (or at least a visibility assumption on ∂X) is necessary since otherwise the sets $\text{Geod}(C)$ and $\text{QC-Hull}(C)$ could be empty. The numbers

$$\begin{aligned} h_{\text{Cov}}(C) &= \lim_{T\to+\infty} \frac{1}{T} \log \text{Cov}(\overline{B}(x,T) \cap \text{QC-Hull}(C), r_0) \\ h_{\text{Lip-top}}(\text{Geod}(C)) &= \inf_{\mathrm{d}} \sup_{K\subseteq \text{Geod}(C)} \lim_{r\to 0} \lim_{T\to+\infty} \frac{1}{T} \log \text{Cov}_{\mathrm{d}^T}(K,r) \\ \text{Shad-D}(C) &= \lim_{T\to+\infty} \frac{1}{T} \log \text{Shad-Cov}_{r_0}(C, e^{-T}) \\ \text{MD}(C) &= \lim_{T\to+\infty} \frac{1}{T} \log \text{Cov}(C, e^{-T}) \end{aligned}$$

are called, respectively, covering entropy of C, Lipschitz-topological entropy of $\text{Geod}(C)$, shadow dimension of C and visual Minkowski dimension of C. The volume entropy of C with respect to a measure μ is

$$h_\mu(C) = \sup_{\sigma\geq 0} \lim_{T\to+\infty} \frac{1}{T} \log \mu(\overline{B}(x,T) \cap \overline{B}(\text{QC-Hull}(C), \sigma)),$$

where $\overline{B}(Y,\sigma)$ means the closed σ-neighbourhood of $Y \subseteq X$. If μ is H-homogeneous at scale r then the volume entropy can be computed putting $\sigma = r$ instead of the supremum over $\sigma \geq 0$ (Proposition 7.1.6). For instance when M is a Riemannian manifold with pinched negative curvature and μ_M is the Riemannian volume then it is $H(r)$-homogeneous at every scale $r > 0$, so the definition does not depend on σ at all. Most of the relations of Theorem G remain true for subsets of the boundary, but the asymptotic behaviour of the different functions involved in the definitions of the entropies depend also on the choice of the basepoint $x \in X$. The best possible choice, $x \in \text{QC-Hull}(C)$, allows us to give again uniform asymptotic estimates.

Theorem N. *Let X be a complete, convex, geodesically complete, δ-hyperbolic metric space that is P_0-packed at scale r_0 and let $C \subseteq \partial X$. Then*

$$h_{\text{Cov}}(C) = \text{Shad-D}(C) = \text{MD}(C) = h_\mu(C)$$

for every homogeneous measure μ on X. All the functions defining the quantities above have the same asymptotic behaviour and the threshold functions can be expressed only in terms of P_0, r_0, δ and the homogeneous constants of μ, if the basepoint x belongs to QC-Hull(C)*.*

The proof of this result does not follow by the same arguments of Theorem G, indeed it will be based heavily on the Gromov-hyperbolicity of X. The relation between the Lipschitz-topological entropy of $\mathrm{Geod}(C)$ and the other definitions of entropy is more complicated. We have:

Theorem O. *Let X be a complete, convex, geodesically complete, δ-hyperbolic metric space that is P_0-packed at scale r_0 and let $C \subseteq \partial X$. Then*

(a) if C is closed then $h_{\text{Lip-top}}(\mathrm{Geod}(C)) = h_{\mathrm{Cov}}(C)$ and the functions defining these two quantities have the same asymptotic behaviour with thresholds function depending only on P_0, r_0, δ.
Moreover if $x \in$ QC-Hull(C) then

$$h_{\text{Lip-top}}(\mathrm{Geod}(C)) = \lim_{T\to+\infty} \frac{1}{T} \log \mathrm{Cov}_{d^T_{\mathrm{Geod}}}(\mathrm{Geod}(\overline{B}(x,L), C), r_0),$$

where $\mathrm{Geod}(\overline{B}(x,L), C)$ is the set of geodesic lines with endpoints in C and passing through $\overline{B}(x,L)$ at time 0 and L is a constant depending only on δ.

(b) if C is not closed then

$$h_{\text{Lip-top}}(\mathrm{Geod}(C)) = \sup_{C' \subseteq C} h_{\text{Lip-top}}(\mathrm{Geod}(C')) \leq h_{\mathrm{Cov}}(C),$$

where the supremum is taken among the closed subsets of C.

In the forthcoming paper [Cav21] we will see how the inequality in (b) can be strict.

1.6.2 Critical exponent of discrete groups of isometries

When C is the limit set $\Lambda(\Gamma)$ of a discrete group of isometries Γ of X there is another largely studied invariant: the Γ-*entropy* of X defined as

$$h_\Gamma(X) = \lim_{T\to+\infty} \frac{1}{T} \log \#\Gamma x \cap \overline{B}(x,T),$$

which is precisely, when $x \in$ QC-Hull$(\Lambda(\Gamma))$, the volume entropy of the set $\Lambda(\Gamma)$ with respect to the counting measure μ_x^Γ of the orbit Γx. It equals the critical exponent of Γ, whose definition will be recalled in Section 7.2. The critical exponent of Γ has a dynamical and a measure-theoretical counterpart. Indeed when $\bar{M} = M/\Gamma$ is a complete Riemannian manifold with pinched negative sectional curvature then it equals the topological entropy of the non-wandering set of the geodesic flow, $h^{\mathrm{nw}}_{\mathrm{top}}(\bar{M})$, as showned in [OP04]. The same result will be generalized in case of discrete and torsion-free group of isometries of convex, geodesically complete, Gromov-hyperbolic, packed metric spaces in [Cav21]. On the other hand, by Bishop-Jones' Theorem, it equals the Hausdorff dimension of the radial limit set of Γ ([BJ97], [DSU17]).

We recall that a group Γ is said quasiconvex-cocompact if the action of Γ on $\text{QC-Hull}(\Lambda(\Gamma))$ is cocompact. In this case the codiameter of the action is the diameter of the compact space $\text{QC-Hull}(\Lambda(\Gamma))/\Gamma$. Quasiconvex-cocompact groups have been extensively studied, due to their regularity. For instance in [Coo93] is shown that the limit set of a quasiconvex cocompact group of isometries of a proper Gromov-hyperbolic metric space is Ahlfors-regular. Our main result of Chapter 7 is a refinement of this result, in terms of quantification of the Ahlfors-regularity constants. In the following the Γ-entropy of X will be denoted simply by h_Γ.

Theorem P. *Let X be a complete, convex, geodesically complete, δ-hyperbolic metric space that is P_0-packed at scale r_0 and let Γ be a discrete, quasiconvex-cocompact group of isometries of X with codiameter $\leq D$. Then the Patterson-Sullivan measure μ_{PS} on $\Lambda(\Gamma)$ is (A, h_Γ)-Ahlfors regular, i.e. for every $z \in \Lambda(\Gamma)$ and every $0 < \rho \leq 1$ it holds*

$$\frac{1}{A}\rho^{h_\Gamma} \leq \mu_{\text{PS}}(B(z,\rho)) \leq A\rho^{h_\Gamma},$$

where A is a constant depending only on P_0, r_0, δ and D. Moreover

$$\frac{1}{T}\log \text{Cov}(\Lambda(\Gamma), e^{-T}) \underset{P_0,r_0,\delta,D}{\asymp} h_\Gamma.$$

If Γ is elementary the proof is trivial, while in the non-elementary case it depends heavily on a uniform estimate of the critical exponent. Indeed by Theorem L in this case we have $0 < h^- \leq h_\Gamma \leq h^+$, where h^- and h^+ depend only on P_0, r_0, δ and D (see Remark 7.2.3 and Example 6.2.3).
Theorem P, together with its proof, has two main consequences: the first one is a quantified equidistribution of the orbit (see again [Coo93] for a non quantified version).

Theorem Q. *Let X be a complete, convex, geodesically complete, δ-hyperbolic metric space that is P_0-packed at scale r_0, let Γ be a discrete, quasiconvex-cocompact group of isometries of X with codiameter $\leq D$ and let $x \in \text{QC-Hull}(\Lambda(\Gamma))$. Then there exists $K > 0$ depending only on P_0, r_0, δ and D such that for all $T \geq 0$ it holds*

$$\frac{1}{K} \cdot e^{T\cdot h_\Gamma} \leq \Gamma x \cap \overline{B}(x,T) \leq K \cdot e^{T\cdot h_\Gamma}.$$

The second one is the continuity of the critical exponent as we will see in the last part of the introduction.

1.6.3 Differences of the invariants for geometrically finite groups

Theorem P affirms that for quasiconvex-cocompact groups the critical exponent equals the Minkowski dimension of the limit set (and so all the other relative notions of entropy, by Theorem N and Theorem O). We present here the situtation for geometrically finite groups in case of Riemannian manifolds, trying to give several interpretations of the possible difference between the critical exponent and the entropy of the limit set.
So we restrict the attention to the case of Riemannian manifolds $\bar{M} = M/\Gamma$ with pinched negative sectional curvature. If Γ is geometrically finite then the limit set

of Γ is the union of the radial limit set $\Lambda_{\mathrm{r}}(\Gamma)$ and the bounded parabolic points. The latter is a countable set, therefore the Hausdorff dimension of the limit set coincides with the Hausdorff dimension of the radial limit set and so by Bishop-Jones' Theorem it holds:

$$\mathrm{HD}(\Lambda(\Gamma)) = \mathrm{HD}(\Lambda_r(\Gamma)) = h_\Gamma. \tag{2}$$

We remark that this is not true if Γ is not geometrically finite, even when M is the hyperbolic space.

Example 1.6.1. In general it can happen $\mathrm{HD}(\Lambda_{\mathrm{r}}(\Gamma))) < \mathrm{HD}(\Lambda(\Gamma))$. Indeed let Γ be a cocompact group of $\mathbb{H}^2$ and let Γ' be a normal subgroup of Γ such that Γ/Γ' is non amenable. Let $F \subseteq \Lambda(\Gamma')$ be the subsets of points z that are fixed by some $g \in \Gamma'$. For every $z \in F$ and every $h \in \Gamma$ we have $hz = hgz = g'hz$ for some $g' \in \Gamma'$ since Γ' is normal. Then hz is fixed by g' and so it belongs to F, i.e. F is Γ-invariant. By minimality of $\Lambda(\Gamma)$ we get $\Lambda(\Gamma') = \Lambda(\Gamma)$, so $\mathrm{HD}(\Lambda(\Gamma')) = \mathrm{HD}(\Lambda(\Gamma))$. But by the growth tightness of Γ (cp. [Sam02]) we have

$$\mathrm{HD}(\Lambda_{\mathrm{r}}(\Gamma')) = h_{\Gamma'} < h_\Gamma = \mathrm{HD}(\Lambda(\Gamma)) = \mathrm{HD}(\Lambda(\Gamma')).$$

However if M is the hyperbolic space and Γ is geometrically finite then even something more is true, indeed by [SU96]:

$$h_\Gamma = \mathrm{HD}(\Lambda(\Gamma)) = \mathrm{MD}(\Lambda(\Gamma)). \tag{3}$$

This equality fails to be true for geometrically finite (actually of finite covolume) groups of manifolds with pinched, but variable, negative curvature. Indeed we have:

Example 1.6.2. In [DPPS09] it is presented an example of a smooth Riemannian manifold M with pinched negative sectional curvature admitting a (non-uniform) lattice (i.e. a group of isometries Γ with $\mathrm{Vol}(M/\Gamma) < +\infty$) such that $h_\Gamma < h_{\mu_M}(X)$. We observe that since Γ is a lattice then $\Lambda(\Gamma) = \partial M$, so $h_{\mu_M}(M) = \mathrm{MD}(\Lambda(\Gamma))$ by Theorem G, while $h_\Gamma = \mathrm{HD}(\Lambda_{\mathrm{r}}(\Gamma)) = \mathrm{HD}(\Lambda(\Gamma))$ by (2).

The example above is due to a relevant variation of the curvature of M. Indeed in [DPPS19] is shown that for non-uniform lattices Γ of asymptotically 1/4-pinched manifolds with negative curvature M it holds $h_{\mu_M}(M) = h_\Gamma$. The general situation in the geometrically finite case is:

$$\begin{gathered}\mathrm{HD}(\Lambda(\Gamma)) = h_\Gamma = h^{\mathrm{nw}}_{\mathrm{top}}(M/\Gamma)\\ h_{\text{Lip-top}}(\mathrm{Geod}(\Lambda(\Gamma))) = h_{\mathrm{Cov}}(\Lambda(\Gamma)) = h_{\mu_M}(\Lambda(\Gamma)) = \mathrm{MD}(\Lambda(\Gamma)),\end{gathered} \tag{4}$$

where the equalities follow by Theorem N, Theorem O and (2). Moreover it is clear that the first line is always less than or equal to the second one, since the Hausdorff dimension is always smaller than the Minkowski dimension.

The relations in (4) allow us to give new interpretations of the phenomena occurring in Example 1.6.2, i.e. the possible difference between the critical exponent of the group and the volume entropy of $\Lambda(\Gamma)$:

- *measure-theoretic interpretation:* it can be seen as the difference between the Hausdorff and the Minkowski dimension of the limit set $\Lambda(\Gamma)$, so it is related to the fractal structure of the limit set;

- *dynamical interpretation:* it can be seen as the difference between the topological entropy of the non-wandering set of the geodesic flow and the Lipschitz-topological entropy of $\mathrm{Geod}(\Lambda(\Gamma))$.
- *combinatoric interpretation:* it can be seen as the difference between h_Γ and $h_{\mathrm{Cov}}(\Lambda(\Gamma))$, where the former counts the exponential growth rate of an orbit while the latter counts the exponential growth rate of the cardinality of r-nets, for some (any) $r > 0$. Here the difference arises in terms of sparsity of the orbit.

1.7 Compactness and continuity

The last part of the thesis, Chapter 8, is focused on stability results under (pointed) Gromov Hausdorff convergence and ultralimits. The main properties of ultralimits will be recalled in Chapter 2, as well as their relation with Gromov-Hausdorff limit. For the purpose of the introduction we will expose the results in terms of Gromov-Hausdorff convergence when possible. In this chapter will be considered only CAT(κ) (or locally CAT(κ)) metric spaces since this condition is stable under limit, while the convexity assumption is not. In particular all the results of the previous section should be thought in terms of CAT(0) spaces instead of convex ones.

1.7.1 Compact classes

The families of spaces with uniformly bounded diameter, satisfying a packing condition for some universal function $P = P(r)$ and all $0 < r \leq r_0$, are classically called *uniformly compact*; actually one can always extract from them convergent subsequences for the Gromov-Hausdorff distance (see [Gro81]). Moreover it is classical that an upper bound on the curvature is stable under Gromov-Hausdorff convergence, provided that the corresponding CAT(κ)-radius is uniformly bounded below (see also Proposition 2.7.9). Starting from the Packing Propagation Theorem A it is possible to decline Gromov's Precompactness Theorem for GCBA-spaces as follows. Consider the classes [5]

$$\mathrm{GCBA}^{\kappa}_{\mathrm{pack}}(P_0, r_0; \rho_0), \qquad \mathrm{GCBA}^{\kappa}_{\mathrm{vol}}(V_0, R_0; \rho_0, n_0)$$

of complete, geodesic, GCBA-spaces with curvature $\leq \kappa$, almost-convexity radius $\rho_{\mathrm{ac}}(X) \geq \rho_0 > 0$ and satisfying, respectively, condition (a) or (b) of Theorem C. Let also denote by

$$\mathrm{GCBA}^{\kappa}_{\mathrm{vol}}(V_0; \rho_0, n_0^{=})$$

the class of complete, geodesic, GCBA-spaces with curvature $\leq \kappa$, total measure $\mu_X(X) \leq V_0$, almost-convexity radius $\rho_{\mathrm{ac}}(X)\geq\rho_0>0$ and dimension *precisely equal to* n_0. Then:

[5] Mnemonically we write before the semicolon the parameters which are relative to the packing condition or to the condition on the natural measure μ_X

Theorem R (Theorem 8.1.1, Corollary 8.1.9 & 8.1.7)**.**

(a) The classes $\mathrm{GCBA}^{\kappa}_{\mathrm{pack}}(P_0, r_0; \rho_0)$ *and* $\mathrm{GCBA}^{\kappa}_{\mathrm{vol}}(V_0, r_0; \rho_0, n_0)$ *are compact with respect to the pointed Gromov-Hausdorff convergence;*

(b) the class $\mathrm{GCBA}^{\kappa}_{\mathrm{vol}}(V_0; \rho_0, n_0^{=})$ *is compact with respect to the Gromov-Hausdorff convergence and contains only finitely many homotopy types.*

As our spaces are locally CAT(κ) with CAT(κ)-radius uniformly bounded below (see inequality (6) in Sec. 2.2), it is not surprising that the limit space is again locally CAT(κ). Less trivially, as a part of the proof of the compactness, we need to show that the conditions on the measure, on the almost-convexity radius and on the dimension are stable under Gromov-Hausdorff limits. So let us highlight the following results which are consequence of the estimates in Theorems B and C and are part of the compactness theorem.

Theorem S (Proposition 8.1.2 & Proposition 8.1.5)**.** *Let* (X_n, x_n) *be* GCBA^{κ}*-spaces converging to* (X, x) *with respect to the pointed Gromov-Hausdorff topology. Then:*

(a) $\rho_{\mathrm{ac}}(X) \geq \limsup_{n\to\infty} \rho_{\mathrm{ac}}(X_n)$*;*

(b) if $\rho_{\mathrm{ac}}(X_n) \geq \rho_0 > 0$ *for all* n *then* $\dim(X) \leq \lim_{n\to+\infty} \dim(X_n)$ *and the equality holds if and only if the distance from* x_n *to the maximal dimensional subspace* $X_n^{\max}$ *of* X_n *stays uniformly bounded when* $n \to \infty$*.*

(The second assertion refines Lemma 2.1 of [Nag18], holding for CAT(κ)*-spaces).*

Therefore GCBA spaces with curvature uniformly bounded from above and almost convexity radius uniformly bounded below can collapse only if the maximal dimensional subspaces go to infinity. We will see such an example in Section 8.1.1.

On the other hand the lower-semicontinuity of the natural measure of balls and of the total volume will follow from [LN19], where it is proved that if $(X_n)_{n\geq 0}$ is a sequence of GCBA-spaces converging to X then the natural measures μ_{X_n} converge weakly to the natural measure μ_X (see Lemma 2.2.7 and the proof of Corollary 3.3.7 for details). We will see in Section 3.3 that, under the stronger assumptions that the natural measure is doubling up to some arbitrarily small scale, the volume of balls is actually *continuous* (cp. Corollary 3.3.7).
Once proved that the bound on the total volume is stable under Gromov-Hausdorff convergence and that this implies the uniform boundedness of the spaces in our class, the homotopy finiteness stated in (b) is a particular case of Petersen's finiteness theorem [Pet90]; actually, as the CAT(κ)-radius is uniformly bounded below, these spaces have a common local geometric contractibility function $\mathrm{LGC}(r) = r$ for $r \leq \rho_0$.

It is not difficult (see Section 8.1.1) to check that also the doubling property is stable under pointed Gromov-Hausdorff convergence and so is the property of being pure dimensional. Namely let us also consider the classes (with the same conventions as before)

$$\mathrm{GCBA}^{\kappa}_{\mathrm{doub}}(D_0, r_0; \rho_0), \qquad \mathrm{GCBA}^{\kappa}_{\mathrm{vol}}(V_0; \rho_0, n_0^{\mathrm{pure}})$$

of complete, geodesic, GCBA-spaces X with curvature $\leq \kappa$, almost-convexity radius $\rho_{\text{ac}}(X) \geq \rho_0 > 0$ and which are, respectively, either D_0-doubling up to scale r_0 or purely n_0-dimensional with total measure $\mu_X(X) \leq V_0$.
We then deduce the following additional compactness results:

Theorem T (Extract from Corollaries 8.1.9 & 8.1.7)**.**
The classes $\text{GCBA}^\kappa_{\text{doub}}(D_0, r_0; \rho_0)$*,* $\text{GCBA}^\kappa_{\text{vol}}(V_0; \rho_0, n_0^{\text{pure}})$ *are compact with respect to pointed and unpointed Gromov-Hausdorff convergence respectively. Moreover* $\text{GCBA}^\kappa_{\text{vol}}(V_0; \rho_0, n_0^{\text{pure}})$ *contains only finitely many homotopy types.*

These theorems can be specialized to the case of M^κ-complexes with bounded geometry. Namely let

$$M^\kappa(R_0, N_0), \qquad M^\kappa(R_0; V_0, n_0)$$

be the class of M^κ-complexes K without free faces, with *positive* injectivity radius (but nor a-priori uniformly bounded below), simplices of size bounded by R_0 and, respectively, valency bounded by N_0 or total volume bounded by V_0 and $\dim(K) \leq n_0$. It is immediate to check that, for suitable $N_0 = N_0(R_0, V_0, n_0)$, the class $M^\kappa(R_0; V_0, n_0)$ is a subclass of $M^\kappa(R_0, N_0)$, made of *compact* M^κ-complexes, namely with a uniformly bounded number of simplices (cp. proof of Theorem 8.1.12); hence it contains only finitely many M^κ-complexes *up to simplicial homeomorphism.* On the other hand we prove:

Theorem U (Extract from Theorem 8.1.10 & Corollary 8.1.12)**.** *The classes* $M^\kappa(R_0, N_0)$ *and* $M^\kappa(R_0; V_0, n_0)$ *are compact, respectively, under pointed and unpointed Gromov-Hausdorff convergence. Moreover there are only finitely many* M^κ*-complexes of diameter* $\leq \Delta$ *in* $M^\kappa(R_0, N_0)$*, up to simplicial homeomorphisms.*

The proof is based on Theorem E and Proposition 3.4.13 that imply that the class $M^\kappa(R_0, N_0)$ is contained in $\text{GCBA}^\kappa_{\text{pack}}(P_0, r_0; \rho_0)$ for suitable P_0, r_0, ρ_0. The remaining part is to show that the limit of a sequence of spaces in $M^\kappa(R_0, N_0)$ has again the structure of a M^κ-complex with bounded geometry.
All the assumptions in this result are necessary. Indeed we will see how dropping the bounds on the valency or on the size of the simplices we do not have neither finiteness nor compactness (see Example 8.1.13).

1.7.2 Ultralimit groups

In Section 8.2 we introduce the notion of ultralimit groups: given a sequence of triples (X_n, x_n, Γ_n), where (X_n, x_n) is a pointed metric space and Γ_n is a group of isometries of X_n, and a non-principal ultrafilter ω we set Γ_ω as the set of admissible sequences (g_n), with $g_n \in \Gamma_n$ for every n. It acts naturally by isometries on the ultralimit space (X_ω, x_ω). By Theorem R (and by the classical stability of the δ-hyperbolicity condition) the class $\text{CAT}^0_{\text{pack}}(P_0, r_0, \delta)$ of complete, geodesically complete, CAT(0), δ-hyperbolic metric spaces that are P_0-packed at scale r_0 is closed under ultralimits. Moreover for all $P_0, r_0, \delta, \Delta > 0$ we denote by $\text{CAT}^0_{\text{nil}}(P_0, r_0, \delta; \Delta)$ the class of triples (X, x, Γ) where $(X, x) \in \text{CAT}^0_{\text{pack}}(P_0, r_0, \delta)$ and Γ is a discrete and torsion-free group of isometries of X satisfying $\text{nilrad}^+(\Gamma, X) \leq \Delta$. We have:

Theorem V (Theorem 8.2.4, Theorem 8.2.6 & Corollary 8.2.7)**.**
Let $(X_n, x_n) \subseteq \mathrm{CAT}^0_{\mathrm{pack}}(P_0, r_0, \delta)$. *Let* Γ_n *be a sequence of torsion-free, discrete groups of isometries of* X_n. *Let* ω *be a non-principal ultrafilter and* Γ_ω *be the ultralimit group of isometries of* X_ω. *Then:*

(a) Γ_ω *is either discrete and torsion-free, or elementary;*

(b) Γ_ω *is not elementary if and only if there exist admissible sequences* $(g_n), (h_n)$ *such that* $\langle g_n, h_n \rangle$ *is not elementary for* ω*-a.e.*(n)*;*

(c) the class $\mathrm{CAT}^0_{\mathrm{nil}}(P_0, r_0, \delta; \Delta)$ *is compact under pointed Gromov-Hausdorff convergence.*

Notice that the two conditions in (a) are not mutually exclusive: the group Γ_ω can be discrete, torsion-free *and* elementary. But if it is not discrete then it is elementary.

1.7.3 Continuity of the entropy

In general the entropy is not continuous under Gromov-Hausdorff convergence or ultralimits. However a control of the asymptotic behaviours of the function defining the different notions of entropies gives continuity, indeed:

Theorem W. *Let* (X_n, x_n) *be a sequence of complete, geodesically complete,* CAT(0) *metric spaces that are* P_0*-packed at scale* r_0 *converging in the pointed Gromov-Hausdorff sense to* (X_∞, x_∞). *If for every* n *it holds*

$$\frac{1}{T} \log \mathrm{Cov}(\overline{B}(x_n, T), r_0) \asymp h_n$$

and the threshold functions do not depend on n, *then* $h_{\mathrm{Cov}}(X_\infty) = \lim_{n\to+\infty} h_n$.

Under the assumptions of Theorem W we have that X_∞ is a proper, geodesically complete, CAT(0) metric space and moreover for every n the covering entropy of X_n is exactly h_n, so it states exactly the continuity of the covering entropy. Clearly by Theorem H, the assumption on the asymptotic behaviour of the covering entropy can be replaced by an equivalent assumption on the asymptotic behaviour of any other notion of entropy. This continuity result can be compared with [Rev05], where a continuity of the volume entropy is established in case of special classes of compact metric spaces. A similar result holds for the relative versions of the entropies, so for subsets of the boundaries (see Theorem 8.4.4). Theorem W and its relative version, are interesting if there is a family of metric spaces whose asymptotic behaviour of the covering entropy (or any other notion of entropy) is uniformly controlled. This is exactly the statement of Theorem P: namely if $\mathrm{CAT}^0_{\mathrm{qc}}(P_0, r_0, \delta; D)$ denotes the class of triples (X, x, Γ) where $(X, x) \in \mathrm{CAT}^0_{\mathrm{pack}}(P_0, r_0, \delta)$, Γ is a discrete, non-elementary, quasiconvex-cocompact group of isometries of X with codiameter $\leq D$ and $x \in \text{QC-Hull}(\Lambda(\Gamma))$ then we have a universal bounded asymptotic behaviour of the function defining the Minkowski dimension of $\Lambda(\Gamma)$.

Theorem X (Theorem 8.3.2). *The class* $\text{CAT}^0_{\text{qc}}(P_0, r_0, \delta; D)$ *is compact with respect to the pointed Gromov-Hausdorff convergence and with respect to this convergence the critical exponent is continuous, i.e. if* $(X_n, x_n, \Gamma_n) \subseteq \text{CAT}^0_{\text{qc}}(P_0, r_0, \delta; D)$ *converges to* $(X_\infty, x_\infty, \Gamma_\infty)$ *then* $h_{\Gamma_\infty} = \lim_{n\to+\infty} h_{\Gamma_n}$.

Remark that the lower semicontinuity of the critical exponent is known in some cases (see [BJ97] and [Pau97]) but several restrictions on the class of groups are made.
In the proof of the compactness part we will show the interesting fact that under the assumptions of the theorem the boundary at infinity of the limit space is homeomorphic (and actually isometric for a suitable choice of a metric) to the limit space of the boundaries (Theorem 8.3.1). Moreover in the quasiconvex-cocompact case with bounded codiameter we will see that the limit sets $\Lambda(\Gamma_n)$ converge to the limit set $\Lambda(\Gamma_\infty)$ (Theorem 8.3.2).

Chapter 2

Preliminaries on metric spaces

We fix the notation. The open and the closed ball of radius R centered at x in a metric space X will be denoted by $B_X(x,R)$ and $\overline{B}_X(x,R)$ respectively; if the metric space is clear from the context we will simply write $B(x,R)$ and $\overline{B}(x,R)$. The closed annulus with center at x and radii $r_1 < r_2$ will be denoted by $A(x,r_1,r_2)$. If (X,d) is a metric space and λ is a positive real number we denote by λX the metric space $(X,\lambda d)$, where $(\lambda d)(x,y) = \lambda d(x,y)$ for any $x,y \in X$, i.e. the rescaled metric space. We denote with $B_{\lambda X}(x,r)$ the ball of center x and radius r with respect to the metric λd. The identity map from (X,d) to $(X,\lambda d)$ is denoted by dil_λ.

A geodesic is a curve $\gamma\colon I \to X$, where I is an interval of $\mathbb{R}$, such that for any $t,s \in I$ it holds $d(\gamma(t),\gamma(s)) = |t-s|$. If $I = [a,b]$ we say that γ is a geodesic joining $x = \gamma(a)$ to $y = \gamma(b)$. A generic geodesic joining two points $x,y \in X$ will be denoted by $[x,y]$, even if there are more geodesics joining x and y. A curve is a local geodesic if it is a geodesic around any point in its interval of definition.
A *geodesic ray* is an isometric embedding from $[0,+\infty)$ to X, while a *geodesic line* (or, simply, a *geodesic*) is an isometric embedding from $\mathbb{R}$ to X. The space X is called *geodesic* if for all $x,y \in X$ there exists a geodesic segment joining x to y.

A metric space is *complete* if every Cauchy sequence has a limit point, while it is *proper* if every closed and bounded set is compact. Every proper metric space is complete.

X is said *locally geodesically complete* if any local geodesic in X defined on an interval $[a,b]$ can be extended, as a local geodesic, to a bigger interval $[a-\varepsilon, b+\varepsilon]$. It is said *geodesically complete* if any local geodesic can be extended, as a local geodesic, to the whole $\mathbb{R}$.
A complete, locally geodesically complete metric space is geodesically complete.

Let X be a metric space and $\gamma\colon [a,b] \to X$ be a curve. The length of γ is

$$\ell(\gamma) = \sup_{a=t_0<t_1,\ldots,t_n<t_{n+1}=b} \sum_{j=0}^{n} d(\gamma(t_j),\gamma(t_{j+1})),$$

where the supremum is among every possible finite partitions of $[a,b]$. A metric space X is a *length space* if for all $x, y \in X$ it holds

$$d(x, y) = \inf_{\gamma} \ell(\gamma),$$

where the infimum is taken among all the curves joining x to y. If X is a proper length space then it is geodesic.

Finally we stress the fact that we consider pointed Gromov-Hausdorff convergence *only for complete metric spaces*: so every time we write $(X_n, x_n) \to (X, x)$ in the pointed Gromov-Hausdorff sense we mean that X_n and X are complete. This condition is not restrictive; indeed if (X_n, x_n) converges to (X, x) then it converges also to the completion $(\hat{X}, \hat{x})$. As a consequence if (X_n, x_n) is a sequence of proper metric spaces converging to (X, x) then X is proper (see Corollary 3.10 of [Her16]).

2.1 Convex metric spaces

A metric space X is *convex* (or Busemann) if it is geodesic and for every couple of geodesic segments γ, γ' such that $\gamma(0) = \gamma'(0) = x$ the function $t \mapsto d(\gamma(t), \gamma'(t))$ is convex from the common maximal interval of definition to $\mathbb{R}$. Every convex metric space X is uniquely geodesic, i.e. for every two points $x, y \in X$ there exists exactly one geodesic segment joining them ([Pap05], Corollary 8.2.2). Moreover any local geodesic is a global geodesic ([Pap05], Corollary 8.2.3). Furthermore:

Lemma 2.1.1. *A complete, convex metric space is locally geodesically complete if and only if every geodesic segment can be extended to a geodesic line.*

The *radius* of a bounded subset $Y \subseteq X$, denoted by r_Y, is the infimum of the positive numbers r such that $Y \subseteq B(x, r)$ for some $x \in X$. The following fact is well-known for CAT(0)-spaces and will be used later to characterize elliptic isometries:

Lemma 2.1.2. *For any bounded subset Y of a proper, convex metric space X there exists a unique point $x \in X$ such that $Y \subseteq \overline{B}(x, r_Y)$. Such a point is called the* center *of Y.*

Proof. The existence of such a point is easy: just take a sequence of points x_n almost realizing the infimum in the definition of r_Y, i.e. $B(x_n, r_Y + \frac{1}{n}) \supseteq Y$. Since Y is bounded then the sequence x_n is bounded. We may suppose, up to taking a subsequence, that the sequence x_n converges to a point x_∞. We then have $d(x_\infty, y) = \lim_{n\to+\infty} d(x_n, y) \leq r_Y$ for any $y \in Y$.
Assume now that there exist two points $x \neq x'$ satisfying the thesis, that is $Y \subseteq \overline{B}(x, r_Y) \cap \overline{B}(x', r_Y)$. We take the midpoint m between x and x' and we claim that there exists $\varepsilon > 0$ such that $\overline{B}(m, r_Y - \varepsilon) \supseteq \overline{B}(x, r_Y) \cap \overline{B}(x', r_Y)$. Otherwise there exist a sequence of points $x_n \in \overline{B}(x, r_Y) \cap \overline{B}(x', r_Y)$ with $x_n \notin \overline{B}(m, r_Y - \frac{1}{n})$, that is $d(x_n, x) \leq r_Y$, $d(x_n, x') \leq r_Y$ and $d(x_n, m) > r_Y - \frac{1}{n}$. Then again we may assume that the x_n's converge to a point x_∞ satisfying

$$d(x_\infty, x) \leq r_Y, \quad d(x_\infty, x') \leq r_Y, \quad d(x_\infty, m) \geq r_Y.$$

So, by convexity (see Example 8.4.7.(iii) of [Pap05]), the distance from x_∞ to any point of $[x, x']$ is constant; this is impossible as the projection of x_∞ on $[x, x']$ is unique ([Pap05], Corollary 8.2.6). We have therefore proved that there exists $\varepsilon > 0$ such that

$$\overline{B}(m, r_Y - \varepsilon) \supseteq \overline{B}(x, r_Y) \cap \overline{B}(x', r_Y).$$

which contradicts with the definition of r_Y. Then $x = x'$. □

When X is a convex, geodesically complete metric space then for all $x \in X$ and $0 < r \leq R$ it is well defined the *contraction map*:

$$\varphi_r^R \colon \overline{B}(x, R) \to \overline{B}(x, r)$$

by sending a point $y \in \overline{B}(x, R)$ to the unique point y' along the geodesic $[x, y]$ satisfying $d(x, y')/r = d(x, y)/R$. By the geodesically completeness of X we conclude that the map φ_r^R is surjective. Moreover it is $\frac{r}{R}$-Lipschitz as follows directly from the definition of convexity of X.

2.1.1 Space of geodesic lines

The *space of parametrized geodesic lines* of a proper, convex, geodesically complete metric space is

$$\mathrm{Geod}(X) = \{\gamma \colon \mathbb{R} \to X \text{ isometry}\},$$

endowed with the topology of uniform convergence on compact subsets of $\mathbb{R}$. There is a natural action of $\mathbb{R}$ on $\mathrm{Geod}(X)$ defined by reparametrization:

$$\Phi_t \gamma(\cdot) = \gamma(\cdot + t)$$

for every $t \in \mathbb{R}$. It is easy to see it is a continuous action, i.e. $\Phi_t \circ \Phi_s = \Phi_{t+s}$ for all $t, s \in \mathbb{R}$ and for every $t \in \mathbb{R}$ the map Φ_t is a homeomorphism of $\mathrm{Geod}(X)$. This action is called the *geodesic flow* on X. The *evaluation map* $E \colon \mathrm{Geod}(X) \to X$, which is defined as $E(\gamma) = \gamma(0)$, is continuous and proper ([BL12], Lemma 1.10). Moreover it is surjective since X is assumed geodesically complete. The topology on $\mathrm{Geod}(X)$ is metrizable. Indeed we can construct a family of metrics on $\mathrm{Geod}(X)$ with the following method.
Let $\mathcal{F}$ be the class of continuous functions $f \colon \mathbb{R} \to \mathbb{R}$ satisfying

(a) $f(s) > 0$ for all $s \in \mathbb{R}$;

(b) $f(s) = f(-s)$ for all $s \in \mathbb{R}$;

(c) $\int_{-\infty}^{+\infty} f(s) ds = 1$;

(d) $\int_{-\infty}^{+\infty} 2|s| f(s) ds = C(f) < +\infty$.

For every $f \in \mathcal{F}$ we define the distance on $\mathrm{Geod}(X)$:

$$f(\gamma, \gamma') = \int_{-\infty}^{+\infty} d(\gamma(s), \gamma'(s)) f(s) ds. \tag{5}$$

We remark that the choice of $f = \frac{1}{2e^{|s|}}$ gives exactly the distance d_{Geod} of the introduction.

Lemma 2.1.3. *The expression defined in* (5) *satisfies these properties:*

(a) it is a well defined distance on $\mathrm{Geod}(X)$*;*

(b) for all $\gamma, \gamma' \in \mathrm{Geod}(X)$ *it holds* $f(\gamma, \gamma') \leq d(\gamma(0), \gamma(0)) + C(f)$*;*

(c) for all $\gamma, \gamma' \in \mathrm{Geod}(X)$ *it holds* $d(\gamma(0), \gamma'(0)) \leq f(\gamma, \gamma')$*;*

(d) it induces the topology of $\mathrm{Geod}(X)$*.*

Proof. For all $\gamma, \gamma' \in \mathrm{Geod}(X)$ we have

$$\begin{aligned} d(\gamma(s), \gamma'(s)) &\leq d(\gamma(s), \gamma(0)) + d(\gamma(0), \gamma'(0)) + d(\gamma'(0), \gamma'(s)) \\ &\leq 2|s| + d(\gamma(0), \gamma'(0)), \end{aligned}$$

so

$$\int_{-\infty}^{+\infty} d(\gamma(s), \gamma'(s)) f(s) ds \leq d(\gamma(0), \gamma'(0)) + \int_{-\infty}^{+\infty} 2|s| f(s) dt < +\infty.$$

This shows (b) and that the integral in (5) is finite. From the properties of the integral and the positiveness of f it is easy to prove that f is actually a distance. The proof of (c) follows from the convexity of the metric on X and the symmetry of f. Indeed for all $\gamma, \gamma' \in \mathrm{Geod}(X)$ the function $g(s) = d(\gamma(s), \gamma'(s))$ is convex. This means that for all $S, S' \in \mathbb{R}$ and for all $\lambda \in [0, 1]$ it holds

$$g(\lambda S + (1 - \lambda) S') \leq \lambda g(S) + (1 - \lambda) g(S').$$

We take $s \geq 0$ and we use the inequality above with $S = s, S' = -s$ and $\lambda = \frac{1}{2}$, obtaining

$$d(\gamma(0), \gamma'(0)) = g(0) \leq \frac{1}{2} g(-s) + \frac{1}{2} g(s) = \frac{d(\gamma(s), \gamma'(s)) + d(\gamma(-s), \gamma'(-s))}{2}.$$

We can now estimate the distance between γ and γ' as

$$\begin{aligned} f(\gamma, \gamma') &= \int_{-\infty}^{0} d(\gamma(s), \gamma'(s)) f(s) ds + \int_{0}^{+\infty} d(\gamma(s), \gamma'(s)) f(s) ds \\ &= \int_{0}^{+\infty} \left(d(\gamma(-s), \gamma'(-s)) + d(\gamma(s), \gamma'(s)) \right) f(s) ds \geq d(\gamma(0), \gamma'(0)), \end{aligned}$$

where we used the symmetry of f. This concludes the proof of (c).
If a sequence γ_n converges to γ_∞ uniformly on compact subsets then it is clear that for every $T \geq 0$ it holds

$$\lim_{n \to +\infty} \int_{-T}^{+T} d(\gamma_n(s), \gamma_\infty(s)) f(s) ds = 0.$$

For every $\varepsilon > 0$ we pick $T_\varepsilon \geq 0$ such that $\int_{T_\varepsilon}^{+\infty} 2|s| f(s) < \varepsilon$. Then it is easy to conclude, using the properties of f, that

$$\lim_{n \to +\infty} \int_{-\infty}^{+\infty} d(\gamma_n(s), \gamma_\infty(s)) f(s) ds \leq 2\varepsilon.$$

By the arbitrariness of ε we conclude that the sequence γ_n converges to γ_∞ with respect to the metric f.
Now suppose the sequence γ_n converges to γ_∞ with respect to f and suppose it does not converge uniformly on compact subsets to γ_∞. Therefore there exists $T \geq 0$, $\varepsilon_0 > 0$ and a subsequence γ_{n_j} such that $d(\gamma_{n_j}(t_j), \gamma_\infty(t_j)) > 6\varepsilon_0$ for every j, where $t_j \in [-T, T]$. We can suppose $t_j \to t_\infty$ and so $d(\gamma_{n_j}(t_\infty), \gamma_\infty(t_\infty)) > 4\varepsilon_0$ for every j. For all $t \in [t_\infty - \varepsilon_0, t_\infty + \varepsilon_0]$ we get $d(\gamma_{n_j}(t), \gamma_\infty(t)) > 2\varepsilon_0$. Therefore, if we set $m = \min_{t\in[t_\infty-\varepsilon_0, t_\infty+\varepsilon_0]} f(s) > 0$, we obtain

$$\int_{-\infty}^{+\infty} d(\gamma_{n_j}(s), \gamma_\infty(s)) f(s) ds > 4\varepsilon_0^2 m$$

for every j, which is a contradiction. □

A metric $\mathfrak{d}$ on $\mathrm{Geod}(X)$ inducing the topology of uniform convergence on compact subsets is said *geometric* if the evaluation map E is Lipschitz with respect to this metric. Any metric induced by $f \in \mathcal{F}$ is geometric by Lemma 2.1.3.(c).

Similar definitions can be given for the space of geodesic rays, $\mathrm{Ray}(X)$, which is

$$\mathrm{Ray}(X) = \{\xi \colon [0, +\infty) \to X \text{ isometry}\},$$

endowed with the topology of uniform convergence on compact subsets of $[0, +\infty)$. Any $f \in \mathcal{F}$ defines a distance on $\mathrm{Ray}(X)$ by

$$f(\xi, \xi') = \int_0^{+\infty} d(\xi(s), \xi'(s)) f(s) ds$$

that induces the topology of $\mathrm{Ray}(X)$. The evaluation map $E \colon \mathrm{Ray}(X) \to X$ that sends ξ to $\xi(0)$ is again continuous, surjective and proper. A metric $\mathfrak{d}$ on $\mathrm{Ray}(X)$ inducing its topology is *geometric* if the evaluation map is Lipschitz with respect to $\mathfrak{d}$. The reparametrization flow on $\mathrm{Ray}(X)$ is defined only for positive times and therefore it is a semi-flow called the *geodesic semi-flow*.

2.1.2 Boundary at infinity

The *boundary at infinity* of X is defined as the set $\mathrm{Ray}(X)$ modulo the equivalence relation $\xi \sim \xi'$ if and only if

$$\sup_{t\in[0,+\infty)} d(\xi(t), \xi'(t)) < +\infty.$$

Since X is proper then for every geodesic ray ξ and every point $x \in X$ there exists a unique geodesic ray ξ' which is equivalent to ξ and with $\xi'(0) = x$. A point of ∂X is denoted by z and the unique geodesic ray ξ in the class of z with $\xi(0) = x$ is denoted by $\xi_z = [x, z]$. There exists a topology on $X \cup \partial X$ that induces the original metric topology on X and the quotient topology on ∂X (as quotient of $\mathrm{Ray}(X)$). With this topology the space $X \cup \partial X$ is compact.

2.2 GCBA metric spaces

We recall the definition of locally CAT(κ) metric space. We fix $\kappa \in \mathbb{R}$. We denote by M^κ_2 the unique simply connected, complete, 2-dimensional Riemannian manifold of constant sectional curvature equal to κ and by D_κ the diameter of M^κ_2. So $D_\kappa = +\infty$ if $\kappa \leq 0$ and $D_\kappa = \frac{\pi}{\sqrt{\kappa}}$ if $\kappa > 0$.

A metric space X is CAT(κ) if any two points at distance less than D_κ can be connected by a geodesic and if the geodesic triangles with perimeter less than $2D_\kappa$ are thinner than their comparison triangles in the model space M^κ_2. This means the following. For any three points $x, y, z \in X$ such that $d(x, y)+d(y, z)+d(z, x) < 2D_\kappa$ a geodesic triangle with vertices x, y, z is the choice of three geodesics $[x, y]$, $[y, z]$ and $[x, z]$, denoted by $\Delta(x, y, z)$. For any such triangle there exists a unique triangle $\overline{\Delta}^\kappa(\bar{x}, \bar{y}, \bar{z})$ in M^κ_2, up to isometry, with vertices $\bar{x}$, $\bar{y}$ and $\bar{z}$ satisfying $d(\bar{x}, \bar{y}) = d(x, y)$, $d(\bar{y}, \bar{z}) = d(y, z)$ and $d(\bar{x}, \bar{z}) = d(x, z)$; such a triangle is called the *κ-comparison triangle* of $\Delta(x, y, z)$. The comparison point of $p \in [x, y]$ is the point $\bar{p} \in [\bar{x}, \bar{y}]$ such that $d(x, p) = d(\bar{x}, \bar{p})$. The triangle $\Delta(x, y, z)$ is thinner than $\overline{\Delta}^\kappa(\bar{x}, \bar{y}, \bar{z})$ if for any couple of points $p \in [x, y]$ and $q \in [x, z]$ we have $d(p, q) \leq d(\bar{p}, \bar{q})$.

A metric space X is called *locally* CAT(κ) if for any $x \in X$ there exists $r > 0$ such that $B(x, r)$ is a CAT(κ) metric space. The supremum among the radii $r < \frac{D_\kappa}{2}$ satisfying this property is called the CAT(κ)-radius at x and it is denoted by $\rho_{\text{cat}}(x)$. The infimum of $\rho_{\text{cat}}(x)$ among the points $x \in X$ is called the CAT(κ)-radius of X and it is denoted by $\rho_{\text{cat}}(X)$; therefore, by definition, $\rho_{\text{cat}}(X) \leq \frac{D_\kappa}{2}$.

A metric space X is *GCBA if there exists a κ such that X is locally CAT(κ), locally compact, separable and locally geodesically complete.* In some case we will write GCBA$^\kappa$, if we want to emphasize the role of κ. This class of metric spaces is the one studied in [LN19].

A *tiny ball*, according to [LN19], is a metric ball $B(x, r)$ such that $r < \min\{1, \frac{D_\kappa}{100}\}$ and $\overline{B}(x, 10r)$ is compact.

2.2.1 Contraction maps and almost-convexity radius

We suppose X is a complete, locally geodesically complete, locally CAT(κ), geodesic metric space. If $x, y \in X$ satisfy $d(x, y) < \rho_{\text{cat}}(x)$ then there exists a unique geodesic joining them. Hence for any $x \in X$ and $0 < r \leq R < \rho_{\text{cat}}(x)$ it is well defined the *contraction map*:

$$\varphi^R_r \colon \overline{B}(x, R) \to \overline{B}(x, r)$$

by sending a point $y \in \overline{B}(x, R)$ to the unique point y' along the geodesic $[x, y]$ satisfying $d(x, y')/r = d(x, y)/R$. Moreover any local geodesic starting at x which is contained in $B(x, \rho_{\text{cat}}(x))$ is a geodesic. This fact, together with the locally geodesically completeness and the completeness of X, shows that the map φ^R_r is surjective also in this setting. It is also $\frac{2r}{R}$-Lipschitz as stated in [LN19]. We skecth here the computation.

Lemma 2.2.1. *Any contraction map is $\frac{2r}{R}$-Lipschitz.*

Proof. By the CAT(κ) condition it is enough to prove the thesis on the model space M_2^κ. The result is clearly true when $\kappa \leq 0$, so we can assume $\kappa = 1$. In this case M_2^κ is the standard sphere $\mathbb{S}^2$.

Step 1. For any $x \in \mathbb{S}^2$ and for any $0 \leq R \leq \frac{\pi}{2}$ the inverse of the exponential map, the logarithmic map $\log_x \colon B(x,R) \to B_{T_x\mathbb{S}^2}(O,R)$, is $\frac{R}{\sin R}$-Lipschitz. So for any R in our range we have that the logarithmic map is 2-Lipschitz. Thus we can conclude that, for any $y,z \in B(x,\frac{\pi}{2})$,

$$d(y,z) \leq d(\log_x(y),\log_x(z)) \leq 2d(y,z)$$

where the first inequality follows by standard comparison results.

Step 2. We fix $0 < r \leq R \leq \frac{\pi}{2}$ and $y,z \in B(x,R)$. Let y' and z' be the contractions of y and z. We observe that the contraction of $\log_x(y)$, on the tangent space, from the radius R to r coincides with the point $\log_x(y')$ and the same holds for z; this contraction map is a dilation of factor $\frac{r}{R}$. Therefore

$$d(y',z') \leq d(\log_x(y'),\log_x(z')) = \frac{r}{R} d(\log_x(y),\log_x(z)) \leq \frac{2r}{R} d(y,z).$$

□

The natural set of scales where the contraction map is defined is not bounded from above by the CAT(κ)-radius but rather from the almost-convexity radius. The *almost-convexity radius at a point* $x \in X$ is defined as the supremum of the radii r such that for any two geodesics $[x,y],[x,z]$ of length at most r and any $t \in [0,1]$ it holds:

$$d(y_t,z_t) \leq 2td(y,z),$$

where y_t, z_t are respectively the points along $[x,y]$ and $[x,z]$ satisfying $d(x,y_t) = t{\cdot}d(x,y)$ and $d(x,z_t) = t{\cdot}d(x,z)$. The almost-convexity radius at x does not depend on κ and is denoted by $\rho_{\mathrm{ac}}(x)$. Then, by definition, for any point $y \in B(x,\rho_{\mathrm{ac}}(x))$ there exists a unique geodesic joining x to y (the existence follows from the assumptions on X), so the contraction map is well defined for any $0 < r \leq R < \rho_{\mathrm{ac}}(x)$. A straightforward modification of Corollary 8.2.3 of [Pap05] shows that any local geodesic joining x to a point y at distance $d(x,y) < \rho_{\mathrm{ac}}(x)$ is actually a geodesic. This fact and the geodesic completeness of X imply again that any contraction map within the almost-convexity radius is surjective and $\frac{2r}{R}$-Lipschitz, by definition.
The *(global) almost-convexity radius* of the space X, denoted by $\rho_{\mathrm{ac}}(X)$, is correspondingly defined as the infimum over x of the almost-convexity radius at x. Clearly we always have $\rho_{\mathrm{ac}}(X) \geq \rho_{\mathrm{cat}}(X)$. The inequality can be partially reversed when X is proper: indeed in this case it holds

$$\rho_{\mathrm{cat}}(X) \geq \min\left\{\frac{D_\kappa}{2}, \rho_{\mathrm{ac}}(X)\right\}, \tag{6}$$

therefore a lower bound on the almost-convexity radius and the knowledge of the upper bound κ yield a lower bound on the CAT(κ)-radius. The proof of (6) follows directly from Corollary II.4.12 of [BH13] once observed that any two points of X at distance less than $\rho_{\mathrm{ac}}(X)$ are joined by a unique geodesic.

2.2.2 Tangent cone and the logarithmic map

We fix a complete, geodesic, GCBA-space X.
Given two local geodesics γ, γ' starting at the same point $x \in X$ we can consider the geodesic triangle $\Delta(x, \gamma(t), \gamma'(t))$ for any small enough $t > 0$. The comparison triangle $\overline{\Delta}^\kappa(\bar{x}, \overline{\gamma(t)}, \overline{\gamma'(t)})$ has an angle α_t at $\bar{x}$. By the CAT(κ) condition the angle α_t is decreasing when $t \to 0$, see [BH13]. Hence it is possible to define the angle between γ and γ' at x as $\lim_{t\to 0} \alpha_t$: it is denoted by $\angle_x(\gamma, \gamma')$ and it takes values in $[0, \pi]$.
For any $x \in X$ the *space of directions of* X *at* x is defined as

$$\Sigma_x X = \{\gamma \text{ local geodesic s.t. } \gamma(0) = x\}/_\sim$$

where $\sim$ is the equivalence relation $\gamma \sim \gamma'$ if and only if $\angle_x(\gamma, \gamma') = 0$. The function $\angle_x(\cdot, \cdot)$ defines a distance which makes of $\Sigma_x X$ a compact, geodesically complete, CAT(1) metric space with diameter π (see [LN19]). The *tangent cone of* X *at the point* x is the metric space

$$T_x X = \Sigma_x X \times [0, +\infty)$$

up to the equivalence relation $(v, 0) \sim (w, 0)$ for every $v, w \in \Sigma_x X$. The point corrisponding to $t = 0$ is called the vertex of the tangent cone, denoted by O. The metric on $T_x X$ is given by the following formula: given two points $V = (v, t)$ and $W = (w, s)$ of $T_x X$ we define $d_T(V, W)$ as the unique positive real number satisfying:

$$d_T(V, W)^2 = t^2 + s^2 - 2ts \cos(\angle_x(v, w)). \qquad (7)$$

In other words $T_x X$ is the euclidean cone over $\Sigma_x X$. With this metric $T_x X$ is a proper, geodesically complete, CAT(0) metric space ([LN19]).

Remark 2.2.2. *Let* $Y = \mathbb{S}^{n-1}$ *be the euclidean standard sphere of radius* 1*. Then the euclidean cone over* Y *is isometric to* $\mathbb{R}^n$*.*

For any point $x \in X$ the *logarithmic map* at x is defined as:

$$\log_x \colon B(x, \rho_{\mathrm{ac}}(x)) \to T_x X, \qquad y \mapsto ([x, y], d(x, y)),$$

where $[x, y]$ is the unique geodesic from x to y (uniqueness is due to the definition of almost-convexity radius).

The logarithmic map can be recovered by the contraction maps as follows. First notice that if X is a GCBA-space and $\lambda > 0$ then the space λX is GCBA. Now, let the logarithmic map on the space λX at $\mathrm{dil}_\lambda(x)$ be denoted by

$$\log_{\mathrm{dil}_\lambda(x)} \colon B_{\lambda X}(\mathrm{dil}_\lambda(x), \lambda\rho_{\mathrm{ac}}(x)) \to T_{\mathrm{dil}_\lambda(x)}(\lambda X).$$

The spaces $T_{\mathrm{dil}_\lambda(x)}(\lambda X)$ and $T_x X$ are canonically isometric since the respective space of directions are canonically isometric. Let $R < \rho_{\mathrm{ac}}(x)$: we consider a sequence of real numbers $r_n \to 0$, we set $\lambda_n = \frac{R}{r_n}$ and we define the maps

$$g_n = \log_{\mathrm{dil}_{\lambda_n}(x)} \circ \mathrm{dil}_{\lambda_n} \circ \varphi^R_{r_n} \colon \overline{B}_X(x, R) \to T_x X$$

where we are using the natural identification $T_{\mathrm{dil}_{\lambda_n}(x)}(\lambda_n X) \cong T_x X$. By the CAT($\kappa$) condition the map $\log_{\mathrm{dil}_{\lambda_n}(x)}$ is $(1 + \varepsilon_n)$-Lipschitz with $\varepsilon_n \to 0$ for $r_n \to 0$. So, by

Lemma 2.2.1, the map g_n is $2(1+\varepsilon_n)$-Lipschitz and for any non-principal ultrafilter ω this sequence defines a ultralimit map g_ω between the ultralimit spaces (cp. Proposition 2.7.5). Since T_xX is proper we can apply Proposition 2.7.3 and find that the target space of g_ω is T_xX, i.e.

$$g_\omega : \omega\text{-}\lim \overline{B}_X(x,R) \to T_xX.$$

Using the definition of the logarithmic map and the natural identification between the tangent cones $T_{\mathrm{dil}_{\lambda_n}(x)}(\lambda_n X) \cong T_xX$ as metric spaces, it is straightforward to check that g_ω, restricted to the standard isometric copy of $\overline{B}_X(x,R)$ in $\omega\text{-}\lim \overline{B}_X(x,R)$ given by Proposition 2.7.3, coincides with $\log_x$.

In general the logarithmic map of a GCBA space is not injective, due to the possible branching of geodesics. We summarize its properties in the following lemma:

Lemma 2.2.3. *Let $x \in X$ be a point of a complete, geodesic,* GCBA *space. Then the logarithmic map* $\log_x$ *has the following properties:*

(a) $\log_x(\overline{B}(x,r)) = \overline{B}(O,r)$ *for any* $r < \rho_{\mathrm{ac}}(x)$*;*

(b) $d(O,\log_x(y)) = d(x,y)$ *for any* $y \in B(x,\rho_{\mathrm{ac}}(x))$*;*

(c) it is 2*-Lipschitz on* $B(x,\rho_{\mathrm{ac}}(x))$*.*

Proof. Let $y \in B(x,\rho_{\mathrm{ac}}(x))$. By definition we have $\log_x(y) = ([x,y], d(x,y))$, where $[x,y]$ is the unique geodesic from x to y. From (7) we immediately infer that $d_T(\log_x(y),O) = d(y,x)$. This proves (b) and that $\log_x(\overline{B}(x,r))$ is included in $\overline{B}(O,r)$ for any $r < \rho_{\mathrm{ac}}(x)$. Now let $V = (v,t) \in \overline{B}(O,r)$, for $r < \rho_{\mathrm{ac}}(x)$. We take a geodesic γ in the class of v. Since X is locally geodesically complete, there exists an extension of γ as a geodesic to the interval $[0,r]$ (this follows from the completeness of X and the fact that any local geodesic is a geodesic if it is contained in a ball of radius smaller than the almost-convexity radius). Then, using the definition of the logarithmic map, we deduce that $\log_x(\gamma(r)) = V$. Now $d(x,\gamma(r)) = r$, which concludes the proof of (a). Finally we have seen that the logarithmic map is obtained as the restriction of the limit map $g_\omega : \omega\text{-}\lim \overline{B}_X(x,R) \to T_xX$ to $\overline{B}_X(x,R)$. It is 2-Lipschitz for all $R < \rho_{\mathrm{ac}}$, therefore it is 2-Lipschitz on $B(x,\rho_{\mathrm{ac}}(x))$. □

The logarithmic map gives a good local approximation of X by the tangent cone, as expressed in the following result.

Lemma 2.2.4 ([LN19], Lemma 5.5)**.** *Let $x \in X$ be a point of a complete, geodesic,* GCBA *space. For any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $r < \delta$ and for every $y_1, y_2 \in B(x,r)$ it holds*

$$|d(y_1,y_2) - d_T(\log_x(y_1),\log_x(y_2))| \le \varepsilon r.$$

As a consequence of this fact, Lytchak and Nagano proved that the tangent cone at x can be seen as the Gromov-Hausdorff limit of a rescaled tiny ball around x. We explicit the proof of this fact because in the following we will need to write who are the maps realizing the Gromov-Hausdorff approximations.

Lemma 2.2.5 ([LN19], Corollary 5.7)**.** *Let $x \in X$ be a point of a complete, geodesic,* GCBA *space. For any sequence $\lambda_n \to \infty$ consider the sequence of* CAT(κ)*, pointed spaces $Y_n = (\lambda_n B(x,r), x)$, for any $r < \rho_{\text{cat}}(x)$. Then:*

(a) $Y_n \to (T_xX, d_T, O)$ in the pointed Gromov-Hausdorff convergence;

(b) the approximating maps $f_n \colon Y_n \to T_xX$ are given by $f_n = \log_{\text{dil}_{\lambda_n}(x)}$ (using again the natural identification $T_{\text{dil}_{\lambda_n}(x)}(\lambda_n X) \cong T_xX$)

Proof. Fix $R > 0$ and any $\varepsilon > 0$. Let δ be as in Lemma 2.2.4 and set $r_n = 1/\lambda_n$. We may assume that $r_n \cdot R < \delta$. Then for all $y_1, y_2 \in B_{Y_n}(x, R)$ we have $y_1, y_2 \in B_X(x, r_n R)$ and we can apply the Lemma 2.2.4, which yields

$$|d(y_1, y_2) - d_T(\log_x(y_1), \log_x(y_2))| \leq \varepsilon r_n R.$$

We have $d_{Y_n}(y_1, y_2) = \frac{d(y_1,y_2)}{r_n}$ and, by (7) and by the definition of the logarithmic map,

$$d_T(f_n(y_1), f_n(y_2)) = \frac{1}{r_n} d_T(\log_x(y_1), \log_x(y_2)).$$

In conclusion we get

$$|d_{Y_n}(y_1, y_2) - d_T(f_n(y_1), f_n(y_2))| \leq \varepsilon R.$$

Since this is true for any $\varepsilon > 0$ the thesis follows from Lemma 2.2.3. □

Finally we observe that this characterization of T_xX has another consequence. Fix any $v \in \Sigma_x X$, which can be naturally seen as an element of T_xX, and take any geodesic γ starting at x defining v: then for any sequence $r_n \to 0$ we have that the sequence $\gamma(r_n) \in Y_n$ defines v in the limit (indeed, $f_n(\gamma(r_n)) = v$ for any n).

2.2.3 Dimension and natural measure

We recall some fundamental properties of GCBA-spaces proved in [LN19]. For any point $x \in X$ there exists an integer number $k \in \mathbb{N}$ such that any sufficiently small ball around x has Hausdorff dimension k. This number is called *the dimension of X at the point x* and it is denoted by $\dim(x)$. It is possible to show that $\dim(x)$ is equal to the geometric dimension of the tangent cone to X at x as defined in [Kle99]. The *dimension of* X is the (possibly infinite) quantity $\dim(X) = \sup_{x\in X} \dim(x) \in [0, +\infty]$.

There exists a natural stratification of X into disjoint subsets X^k, where X^k is the set of points of dimension k, for $k \in \mathbb{N}$. In other words $X = \bigsqcup_{k\in\mathbb{N}} X^k$. Moreover the k-dimensional Hausdorff measure $\mathcal{H}^k$ is locally positive and locally finite on X^k. Hence it is defined a measure on X as

$$\mu_X = \sum_{k\in\mathbb{N}} \mathcal{H}^k \llcorner X^k.$$

The measure μ_X is locally positive and locally finite: we call it the *natural measure of X*.

Example 2.2.6. If X is a n-dimensional Riemannian manifold with sectional curvature $\leq \kappa$ then X is a locally geodesically complete, locally compact, separable, locally CAT(κ) metric space. In this case μ_X is the n-dimensional Hausdorff measure and it coincides with the Riemannian volume measure, up to a multiplicative constant.

This stratification of X has good local properties, as shown in [LN19]. For any $k \in \mathbb{N}$ it is possible to define the set of *regular points* $\mathrm{Reg}^k(X)$ of the k-dimensional part X^k of X. We do not present here the definition of regular points (they are those points that are (k, δ)-strained for a suitable small δ, according to [LN19], Sec. 11.4). Instead we recall the main properties of the set of k-dimensional and regular k-dimensional points we will need. For every $S \subseteq X$ we will denote $S^k = S \cap X^k$ and $\mathrm{Reg}^k(S) = S^k \cap \mathrm{Reg}^k(X)$.
Then:

- the set $\mathrm{Reg}^k(X)$ is open in X and dense in X^k (Cor. 11.8 of [LN19]);
- for any tiny ball $B(x, r)$ there exists k such that $B(x, r)$ does not contain points of dimension $> k$ (Corollary 5.4 of [LN19]);
- for any tiny ball $B(x, r)$ there exists a constant C, *only depending on the maximal number of r-separated points in $\overline{B}(x, 10r)$*, such that:

$$\mathcal{H}^k\left(B(x,r)^k\right) \leq C \cdot r^k \tag{8}$$

$$\mathcal{H}^{k-1}\left(\bar{B}(x,r)^k \setminus \mathrm{Reg}^k(B(x,r))\right) \leq C \cdot r^{k-1} \tag{9}$$

(Corollary 11.8 of [LN19]; see Sec.2.4 for the definition of r-separated points).

2.2.4 Gromov-Hausdorff convergence

We recall here some facts about the behaviour of the natural measures and the dimension under pointed Gromov-Hausdorff convergence.
Consider a proper GCBA-space X and its natural measure $\mu_X = \sum_{k=0}^{n} \mathcal{H}^k \llcorner X^k$, where $n = \dim(X)$ is assumed to be finite. The k-dimensional Hausdorff measure $\mathcal{H}^k$ restricted to the k-dimensional part is a Radon measure (indeed it is Borel regular and locally finite on the proper metric space X), so it is μ_X. In particular for any open subset $U \subset X$ it holds:

$$\mu_X(U) = \sup\{\mu_X(K) \text{ s.t. } K \text{ is a compact subset of } U\}.$$

Now suppose to have a sequence of proper GCBA-spaces X_n converging in the pointed Gromov-Hausdorff sense to some (proper) GCBA-space X. Arguing as in the first part of the proof of Theorem 1.5 of [LN19] we deduce that the natural measures μ_{X_n} converge in the weak sense to the natural measure of the limit, μ_X. This means that for any compact subsets $K_n \subset X_n$ converging to a compact subset $K \subset X$ it holds:

$$\lim_{\varepsilon \to 0} \liminf_{n \to +\infty} \mu_{X_n}(B(K_n, \varepsilon)) = \lim_{\varepsilon \to 0} \limsup_{n \to +\infty} \mu_{X_n}(B(K_n, \varepsilon)) = \mu_X(K) \tag{10}$$

where we denote by $B(K_n, \varepsilon)$ the ε-neighbourhood of K_n. As a consequence:

Lemma 2.2.7. *Let X_n be a sequence of proper,* GCBA-*spaces converging in the pointed Gromov-Hausdorff sense to a proper,* GCBA-*space X. Let $x_n \in X_n$ be a sequence of points converging to $x \in X$. Then for any $R > 0$ it holds:*

$$\mu_X(B(x, R)) \leq \limsup_{n \to +\infty} \mu_{X_n}(B(x_n, R)). \tag{11}$$

Proof. The natural measure μ_X is Radon and any compact subset contained in $B(x,R)$ is contained in $\overline{B}(x,R-2\eta)$ for some $\eta > 0$, therefore

$$\mu_X(B(x,R)) = \sup_{\eta>0} \mu_X(\overline{B}(x,R-2\eta)).$$

On the other hand for any $\eta > 0$ we have by (10)

$$\mu_X(\overline{B}(x,R-2\eta)) \leq \limsup_{n\to+\infty} \mu_{X_n}(\overline{B}(x_n,R-\eta)) \leq \limsup_{n\to+\infty} \mu_{X_n}(B(x_n,R)).$$

□

The equality in (11) would follow from a uniform estimate on the volumes of the annulii of a given thickness. Indeed this is the case when the metric spaces satisfy a uniform doubling condition, as we will see in Section 3.3.

We end the preliminaries about GCBA spaces recalling some facts about the stability of the dimension under Gromov-Hausdorff convergence. In [LN19] (Def. 5.12), Lytchak and Nagano introduce the notion of *standard setting of convergence.* This means considering a sequence of tiny balls

$$B(x_n, r_0) \subset \overline{B}(x_n, 10r_0)$$

in a sequence of GCBA-spaces X_n satisfying the following assumptions:

- the closed balls $\overline{B}(x_n, 10r_0)$ have uniformly bounded $\frac{r_0}{2}$-covering number (i.e. $\exists\, C_0$ such that the ball $\overline{B}(x_n, 10r_0)$ can be covered by C_0 closed balls of radius $\frac{r_0}{2}$ with centers in $\overline{B}(x_n, 10r_0)$ for all n , cp. Sec.2.4)
- the balls $\overline{B}(x_n, 10r_0)$ converge to a compact ball $\overline{B}(x, 10r_0)$ of a GCBA-space X in the Gromov-Hausdorff sense;
- the closures $\overline{B}(x_n, r_0)$ converge to the closure $\overline{B}(x, r_0)$ of a tiny ball in X.

We then have:

Lemma 2.2.8 (Lemma 11.5 & Lemma 11.7 of [LN19])**.**
Let $B(x_n, r_0)$ be a sequence of tiny balls in the standard setting of convergence. Let $y_n \in B(x_n, r_0)$ be a sequence converging to $y \in B(x, r_0)$. Then:

(a) $\dim(y) \geq \limsup_{n\to+\infty} \dim(y_n)$*;*

(b) if y is k-regular then $\dim(y) = \dim(y_n)$ *for all n large enough.*

For non-compact spaces the following general result is known:

Lemma 2.2.9 (Lemma 2.1 of [Nag18])**.**
Let (X_n, x_n) be a sequence of pointed, proper, geodesically complete, CAT*(κ) spaces converging to some (X, x) in the pointed Gromov-Hausdorff sense. Then* $\dim(X) \leq \liminf_{n\to+\infty} \dim(X_n)$.

2.3 Gromov-hyperbolic metric spaces

Let X be a geodesic space. Given three points $x, y, z \in X$, the *Gromov product* of y and z with respect to x is defined as

$$(y, z)_x = \frac{1}{2}(d(x, y) + d(x, z) - d(y, z)).$$

The space X is said *δ-hyperbolic* if for every four points $x, y, z, w \in X$ the following *4-points condition* hold:

$$(x, z)_w \geq \min\{(x, y)_w, (y, z)_w\} - \delta \tag{12}$$

or, equivalently,

$$d(x, y) + d(z, w) \leq \max\{d(x, z) + d(y, w), d(x, w) + d(y, z)\} + 2\delta. \tag{13}$$

The space X is *Gromov hyperbolic* if it is δ-hyperbolic for some $\delta \geq 0$.
The above formulations of δ-hyperbolicity are convenient when interested in taking limits (since they are preserved under ultralimits). However we will also make use of other classical characterizations of δ-hyperbolicity, depending on which one is more useful in the context.
Recall that a *geodesic triangle* in X is the union of three geodesic segments $[x, y], [y, z], [z, x]$ and is denoted by $\Delta(x, y, z)$. For every geodesic triangle there exists a unique *tripod* $\overline{\Delta}$ with vertices $\bar{x}, \bar{y}, \bar{z}$ such that the lengths of $[\bar{x}, \bar{y}], [\bar{y}, \bar{z}], [\bar{z}, \bar{x}]$ equal the lengths of $[x, y], [y, z], [z, x]$ respectively. There exists a unique map $f_{\bar{\Delta}}$ from $\Delta(x, y, z)$ to the tripod $\overline{\Delta}$ that identifies isometrically the corresponding edges, and there are exactly three points $c_x \in [y, z], c_y \in [x, z], c_z \in [x, y]$ such that $f_{\bar{\Delta}}(c_x) = f_{\bar{\Delta}}(c_y) = f_{\bar{\Delta}}(c_z) = c$, where c is the center of the tripod $\overline{\Delta}$. By definition of $f_{\bar{\Delta}}$ it holds:

$$d(x, c_z) = d(x, c_y), \qquad d(y, c_x) = d(y, c_z), \qquad d(z, c_x) = d(z, c_y).$$

The triangle $\Delta(x, y, z)$ is called *δ-thin* if for every $u, v \in \Delta(x, y, z)$ such that $f_{\bar{\Delta}}(u) = f_{\bar{\Delta}}(v)$ it holds $d(u, v) \leq \delta$; in particular the mutual distances between c_x, c_y and c_z are at most δ. It is well-known that every geodesic triangle in a geodesic *δ-hyperbolic* metric space (as defined above) is 4δ-thin, and moreover satisfies the *Rips' condition*:

$$[y, z] \subset \overline{B}([x, y] \cup [x, z], 4\delta). \tag{14}$$

Furthermore these last conditions are equivalent to the above definition of hyperbolicity, up to slightly increasing the hyperbolicity constant δ in (12). As a consequence of the δ-thinness of triangles we have the following: let $x, y, z \in X$, $f_{\bar{\Delta}} : \Delta(x, y, z) \to \bar{\Delta}$ the tripod approximation and c_x, c_y, c_z as before. Then

$$(y, z)_x = d(x, c_z) = d(x, c_y) \qquad \text{and} \qquad d(x, c_x) \leq d(x, c_y) + 4\delta \tag{15}$$

All Gromov-hyperbolic spaces, in this thesis, will be supposed proper; we will however stress this assumption in the statements where it is needed.

2.3.1 Gromov boundary

We fix a δ-hyperbolic metric space X and a base point x of X.
The *Gromov boundary* of X is defined as the quotient

$$\partial_G X = \{(y_n)_{n\in\mathbb{N}} \subseteq X \mid \lim_{n,m\to+\infty} (y_n, y_m)_x = +\infty\} /_\sim,$$

where $(y_n)_{n\in\mathbb{N}}$ is any sequence of points in X and $\sim$ is the equivalence relation defined by $(y_n)_{n\in\mathbb{N}} \sim (z_n)_{n\in\mathbb{N}}$ if and only if $\lim_{n,m\to+\infty}(y_n, z_m)_x = +\infty$. We will write $y = [(y_n)] \in \partial_G X$ for short, and we say that (y_n) *converges* to y. Clearly this definition does not depend on the basepoint x.
There is a natural topology on $X \cup \partial_G X$ that extends the metric topology of X. The Gromov product can be extended to points $y, z \in \partial_G X$ by

$$(y, z)_x = \sup_{(y_n),(z_n)} \liminf_{n,m\to+\infty} (y_n, z_m)_x$$

where the supremum is over all sequences such that $(y_n) \sim y$ and $(z_n) \sim z$. For any $x, y, z \in \partial_G X$ it continues to hold

$$(x, y)_x \geq \min\{(x, z)_x, (y, z)_x\} - \delta. \tag{16}$$

Moreover, for all sequences $(y_n), (z_n)$ converging to y, z respectively it holds

$$(y, z)_x - \delta \leq \liminf_{n,m\to+\infty} (y_n, z_m)_x \leq (y, z)_x. \tag{17}$$

In a similar way is defined the Gromov product between a point $y \in X$ and a point $z \in \partial_G X$. This product satisfies conditions analogue of (16) and (17).
Any geodesic ray ξ defines a point $\xi^+ = [(\xi(n))_{n\in\mathbb{N}}]$ of the Gromov boundary $\partial_G X$: we say that ξ *joins* $\xi(0) = y$ *to* $\xi^+ = z$, and we denote it by $[y, z]$. Notice that any point $y \in X$ can be joined to any point $z = [(z_n)] \in \partial_G X$: in fact the sequence (z_n) must be unbounded (as $(z_n, z_n)_x$ is unbounded), so the geodesic segments $[y, z_n]$ converge uniformly on compact sets, by properness of X, to a geodesic ray $\xi = [y, z]$. A geodesic ray connecting the basepoint x to z is denoted by $\xi_z = [x, z]$, even if it is not unique. If X is also convex then ξ_z is unique for all $z \in \partial_G X$. This fact defines a natural identification between ∂X and $\partial_G X$: indeed, a base point $x \in X$ being fixed, for every geodesic ray ξ starting at x one defines a point in the Gromov boundary as $\xi^+ = [(\xi(n)_{n\in\mathbb{N}}]$. This formula provides a well defined homeomorphism between ∂X and $\partial_G X$.

Analogously, given different points $z = [(z_n)], z' = [(z'_n)] \in \partial_G X$ there always exists a geodesic line γ joining z to z', i.e. such that $\gamma|_{[0,+\infty)}$ and $\gamma|_{(-\infty,0]}$ join $\gamma(0)$ to z, z' respectively (just consider the limit γ of the segments $[z_n, z'_n]$; notice that all these segments intersect a ball of fixed radius centered at x, since $(z_n, z'_m)_x$ is uniformly bounded above). We call z and z' the *positive* and *negative endpoints* of γ, respectively, denoted $\gamma^\pm$. We will also write, for short, $\partial\gamma := \{\gamma^+, \gamma^-\}$.

The *Busemann function* at $z \in \partial_G X$ can be defined as

$$\mathcal{B}_z(x, y) = \sup_{\xi} \lim_{t\to+\infty} d(x, \xi(t)) - d(\xi(t), y) \qquad \text{for } x, y \in X,$$

where the supremum is over all geodesic rays ξ with $\xi^+ = z$. It is clear from the definition that the Busemann function $\mathcal{B}_z(x,y)$ satisfies

$$(y,z)_x - (x,z)_y - \delta \leq \mathcal{B}_z(x,y) \leq (y,z)_x - (x,z)_y.$$

We remark that the formula $(y,z)_x - (x,z)_y$ is used in [DSU17] to define Busemann functions in arbitrary Gromov hyperbolic metric spaces.
The level set $\mathcal{B}_z(x,y) = 0$ is called the *horosphere of X centered at z passing through x*, while the subset $\mathcal{B}_z(x,y) \geq 0$ is the *horoball* through x; they are denoted, respectively, $H_z(x)$ and $H_z^+(x)$. Since $\mathcal{B}_z(x,y) = \mathcal{B}_z(x',y)$ for x, x' lying on the same horosphere H_z centered at z, we will also often write $\mathcal{B}_z(H_z, y)$, which should be thought of as a signed distance from H_z.

Lemma 2.3.1. *For all $x \in X$ and $z \in \partial_G X$ the function $\mathcal{B}_z(x,\cdot)\colon X \to \mathbb{R}$ is 1-Lipschitz.*

Proof. We fix $y, y' \in X$. For all $\varepsilon > 0$ we take a geodesic ray $\bar{\xi}$ that ε-almost realizes the supremum in the definition of $\mathcal{B}_z(x,y)$. Then

$$\mathcal{B}_z(x,y) - \mathcal{B}_z(x,y') \leq \lim_{t\to+\infty} -d(\bar{\xi}(t),y) + d(\bar{\xi}(t),y') + \varepsilon \leq d(y,y') + \varepsilon.$$

In the same way $\mathcal{B}_z(x,y') - \mathcal{B}_z(x,y) \leq d(y,y') + \varepsilon$. By the arbitrariness of ε we achieve the result. □

As X is not uniquely geodesic it may happen that there are several geodesic rays joining a point of X to some point $z \in \partial_G X$, or several geodesic lines joining two points of the boundary. However the following standard uniform estimates hold:

Lemma 2.3.2 (Prop. 8.10 of [BCGS17])**.** *Let X be a δ-hyperbolic space.*

(a) Let ξ_1, ξ_2 be two geodesic rays with $\xi_1^+ = \xi_2^+$ and $\xi_1(0) = \xi_2(0)$: then we have $d(\xi_1(t), \xi_2(t)) \leq 8\delta$, $\forall t \geq 0$;

(b) let ξ_1, ξ_2 be two geodesic rays with $\xi_1^+ = \xi_2^+$: then there exist $t_1, t_2 \geq 0$ with $t_1 + t_2 = d(\xi_1(0), \xi_2(0))$ such that $d(\xi_1(t+t_1), \xi_2(t+t_2)) \leq 8\delta$, $\forall t \geq 0$; moreover, if $\xi_1(0)$ and $\xi_2(0)$ lie on the same horosphere centered at the common endpoint, then $|t_1 - t_2| \leq 8\delta$;

(c) let γ_1, γ_2 be two geodesic lines with $\gamma_1^+ = \gamma_2^+$ and $\gamma_1^- = \gamma_2^-$: then for all $t \in \mathbb{R}$ there exists $s \in \mathbb{R}$ such that $d(\gamma_1(t), \gamma_2(s)) \leq 8\delta$.

2.3.2 Visual metrics

When X is a proper, δ-hyperbolic metric space it is known that the boundary $\partial_G X$ is metrizable. A metric $D_{x,a}$ on $\partial_G X$ is called a *visual metric* of parameter $a \in \left(0, \frac{1}{2\delta\cdot\log_2 e}\right)$ and center $x \in X$ if there exists $V > 0$ such that for all $z, z' \in \partial_G X$ it holds

$$\frac{1}{V} e^{-a(z,z')_x} \leq D_{x,a}(z,z') \leq V e^{-a(z,z')_x}. \tag{18}$$

A visual metric is said *standard* if for all $z, z' \in \partial_G X$ it holds

$$(3 - 2e^{a\delta})e^{-a(z,z')_x} \leq D_{x,a}(z, z') \leq e^{-a(z,z')_x}.$$

For all a as before and $x \in X$ there exists always a standard visual metric of parameter a and center x, see [Pau96]. We remark that the constants involved in the definition of a standard visual metric depend only on δ. As in [Pau96] we define the *generalized visual ball* of center $z \in \partial_G X$ and radius $\rho \geq 0$ as

$$B(z, \rho) = \left\{ z' \in \partial_G X \text{ s.t. } (z, z')_x > \log \frac{1}{\rho} \right\}.$$

It is comparable to the metric balls of the visual metrics on $\partial_G X$.

Lemma 2.3.3. *Let $D_{x,a}$ be a visual distance of center x and parameter a on $\partial_G X$. Then for all $z \in \partial_G X$ and for all $\rho > 0$ it holds*

$$B_{D_{x,a}}\left(z, \frac{1}{V}\rho^a\right) \subseteq B(z, \rho) \subseteq B_{D_{x,a}}(z, V\rho^a).$$

Proof. For all $z' \in B(z, \rho)$ by definition it holds $(z, z')_x > \log \frac{1}{\rho}$ and therefore $D_{x,a}(z, z') \leq Ve^{-a(z,z')_x} < V\rho^a$.
If $z' \in B_{D_{x,a}}(z, \frac{1}{V}\rho^a)$ then $\frac{1}{V}e^{-a(z,z')_x} \leq D_{x_0,a}(z, z') < \frac{1}{V}\rho^a$. This easily implies $z' \in B(z, \rho)$. □

A compact metric space Z is (A, s)-*Ahlfors regular* if there exists a probability measure μ on Z such that

$$\frac{1}{A}\rho^s \leq \mu(B(z, \rho)) \leq A\rho^s$$

for all $z \in Z$ and all $0 \leq \rho \leq \mathrm{Diam}(Z)$, where $\mathrm{Diam}(Z)$ is the diameter of Z. In case $Z = \partial_G X$ we say that Z is *visual* (A, s)-*Ahlfors regular* if there exists a probability measure μ on $\partial_G X$ such that

$$\frac{1}{A}\rho^s \leq \mu(B(z, \rho)) \leq A\rho^s$$

for all $z \in Z$ and all $0 \leq \rho \leq 1$, where $B(z, \rho)$ is the generalized visual ball of center z and radius ρ. From Lemma 2.3.3 it follows immediately the following.

Lemma 2.3.4. *If $\partial_G X$ is (A, s)-Ahlfors regular with respect to a visual metric of center x and parameter a, then it is visual (AV^s, as)-Ahlfors regular, where V is the constant of* (18).

2.3.3 Projections

Recall that a subset $C \subseteq X \cup \partial_G X$ is said *convex* if for every $x, y \in C$ there exists at least one geodesic (segment, ray, line) joining x to y that is included in C. Given any closed, convex subset C of X and a point $x \in X$, a *projection* of x to C is a point $c \in C$ such that $d(x, C) = d(x, c)$. Since C is closed and X is proper, it is clear that there exists at least a projection.
A fundamental tool in the study of projections in δ-hyperbolic spaces is the following:

Lemma 2.3.5 (Projection Lemma, cp. Lemma 3.2.7 of [CDP90])**.**
Let X be a δ-hyperbolic space, and let $x, y, z \in X$. For any geodesic segment $[y, z]$ we have:

$$(y, z)_x \geq d(x, [y, z]) - 4\delta.$$

Therefore if C is a convex subset and x_0 is a projection of x on C then $(x_0, c)_x \geq d(x, x_0) - 4\delta$ for all $c \in C$. This easily implies that the projection x_0 satisfies, for all $c \in C$:

$$(x, c)_{x_0} \leq 4\delta \tag{19}$$

One can then extend the definition of projection to boundary points, using this relation, as follows: we say that x_0 *is a projection of* $x \in \partial_G X$ *on* C if

$$(x, c)_{x_0} \leq 5\delta \quad \text{for all } c \in C.$$

In the next lemma we summarize the properties of projections we need. Recall that since C is convex and closed then it is naturally a geodesic, δ-hyperbolic, proper metric space; furthermore the Gromov boundary $\partial_G C$ of C canonically embeds into $\partial_G X$.

Lemma 2.3.6. *Let X be a proper, δ-hyperbolic metric space and C be a closed, convex subset of X. Let $x, x' \in X \cup \partial_G X \setminus \partial_G C$. The following facts hold:*

(a) there exists at least one projection of x on C;

(b) if x_1, x_2 are two projections of x on C then $d(x_1, x_2) \leq 10\delta$;

(c) if x_0 and x_0' are respectively projections of x and x' on C, then $d(x_0, x_0') \leq d(x, x') + 12\delta$.

Proof. We first show the existence of a projection for points $x \in \partial_G X \setminus \partial_G C$. Let (x_n) be a sequence converging to x and let c_n be a projection of x_n on C. First of all we claim that the sequence (c_n) is bounded. As the sequence (c_n) is in C and $x \notin \partial_G C$, then (c_n) is not equivalent to (x_n). In particular $(x_n, c_n)_{c_0} \leq D$ for some $0 \leq D < +\infty$ and some $c_0 \in C$. This means

$$d(c_0, x_n) + d(c_0, c_n) - d(c_n, x_n) \leq 2D.$$

As $x_0 \in C$ and c_n is a projection of x_n on C, we have $d(c_0, x_n) \geq d(c_n, x_n)$ and therefore $d(c_0, c_n) \leq 2D$ for all n. Therefore the sequence c_n converges, up to a subsequence, to a point $c \in C$. Notice that for any n and any $c' \in C$ we have $(x_n, c')_{c_n} \leq 4\delta$. Applying (17) we get for all $c' \in C$

$$(x, c')_c \leq \limsup_{n \to +\infty} (x_n, c')_c + \delta \leq \limsup_{n \to +\infty} (x_n, c')_{c_n} + d(c_n, c) + \delta \leq 5\delta.$$

This proves (a). Assertion (b) is an easy consequence of the definition, as

$$(x, x_1)_{x_2} \leq 5\delta, \qquad (x, x_2)_{x_1} \leq 5\delta,$$

so $d(x_1, x_2) = (x, x_1)_{x_2} + (x, x_2)_{x_1} \leq 10\delta$.
Finally the proof of (c) can be found in [CDP90], Corollary 10.2.2. □

Remark 2.3.7. We record here a consequence of the proof above: if (x_n) is a sequence of points converging to a point $x \in \partial_G X \setminus \partial_G C$ and c_n is a projection of x_n on C for all n, then, up to a subsequence, the limit point of the sequence c_n is a projection of x on C.

We now recall the Morse property of geodesic segments in a Gromov-hyperbolic space. A map $\alpha\colon [0,l] \to X$ is a $(1,\nu)$-*quasigeodesic segment* if for any $t, t' \in [0,l]$ it holds

$$|t-t'| - \nu \leq d(\alpha(t), \alpha(t')) \leq |t-t'| + \nu.$$

The points $\alpha(0)$ and $\alpha(l)$ are called the endpoints of α.

Proposition 2.3.8 (Morse Property)**.** *Let X be a δ-hyperbolic space and let α be a $(1,\nu)$-quasigeodesic segment. The following facts hold:*

(a) for any geodesic segment β joining the endpoints of α we have $d_H(\alpha,\beta) \leq \nu + 12\delta$, where d_H is the Hausdorff distance;

(b) for any $(1,\nu)$-quasigeodesic segment β with the same endpoints of α and for any time t where both α and β are defined it holds $d(\alpha(t), \beta(t)) \leq 6\nu + 48\delta$.

The proof of the first part can be found in [Bow06], while the second part is classical and follows from a straightforward computation.

As an immediate consequence of Lemma 2.3.5 and the previous proposition we get:

Lemma 2.3.9. *Let X be a δ-hyperbolic metric space, $x \in X$ and ξ be a geodesic ray such that $\xi(0)$ is a projection of x on ξ. Then*

(a) for all $T \geq 0$ the curve $\alpha = [x, \xi(0)] \cup [\xi(0), \xi(T)]$ is a $(1, 4\delta)$-quasigeodesic and $d(\alpha(t), \gamma(t)) \leq 72\delta$ for all possible t, where $\gamma = [x, \xi(T)]$;

(b) the curve $\alpha = [x, \xi(0)] \cup [\xi(0), \xi^+]$ is a $(1, 4\delta)$-quasigeodesic and $d(\alpha(t), \xi'(t)) \leq 72\delta$ for all $t \geq 0$, where $\xi' = [x, \xi^+]$;

Now we state the contracting property of projections:

Proposition 2.3.10 (Contracting Projections)**.** *Let X be a proper, δ-hyperbolic space, $C \subseteq X$ any closed convex subset and $Y \subseteq X$ another convex subset. The following facts hold:*

(a) suppose that the projections c, c' on C of, respectively, $y, y' \in Y$ satisfy $d(c,c') > 9\delta$: then, $[y,c] \cup [c,c'] \cup [c',y']$ is a $(1, 18\delta)$-quasigeodesic segment;

(b) if $d(Y,C) > 30\delta$ then any two projections on C of points of Y are at distance at most 9δ;

(c) if α, β are geodesic with $\partial\alpha \cap \partial\beta = \emptyset$, then any projection b_+ of β^+ on α satisfies $d(b_+, \beta) \leq \max\{49\delta,\ d(\alpha,\beta) + 19\delta\}$.

Proof. By assumption we have $(y, c')_c \leq 4\delta$ and $(y', c)_{c'} \leq 4\delta$, i.e.

$$d(y, c') \geq d(y, c) + d(c, c') - 8\delta, \qquad d(y', c) \geq d(y', c') + d(c, c') - 8\delta. \tag{20}$$

We apply the four-points condition (13) to (y, c', y', c) obtaining

$$d(y, c') + d(y', c) \leq \max\{d(y, y') + d(c, c'), d(y, c) + d(y', c')\} + 2\delta.$$

Assuming $d(c, c') > 9\delta$ we get by (20)

$$d(y, c') + d(y', c) \geq d(y, c) + d(c, c') + d(y', c') + d(c, c') - 16\delta > d(y, c) + d(y', c') + 2\delta.$$

Therefore the four-point condition becomes

$$d(y, c') + d(y', c) \leq d(y, y') + d(c, c') + 2\delta.$$

Using again (20) we get

$$d(y, c) + d(c, c') + d(y', c') + d(c, c') - 16\delta \leq d(y, y') + d(c, c') + 2\delta$$

which proves (a). We suppose now $d(Y, C) > 30\delta$ and that there are two points $y, y' \in Y$ with projections c, c' on C such that $d(c, c') > 9\delta$. Then the path $[y, c] \cup [c, c'] \cup [c', y']$ is a $(1, 18\delta)$-quasigeodesic segment by (a) and it is at Hausdorff distance at most 30δ from any geodesic segment $[y, y']$, by Lemma 2.3.8. As Y is convex, one of these geodesic segments is included in Y, so c is at distance at most 30δ from Y. This contradiction proves (b).
In order to prove (c) we observe that α and β are two closed, convex subsets of X. We divide the proof in two cases.
Case 1: $d(\alpha, \beta) > 30\delta$. Then let $x_0 \in \alpha$ and $y_0 \in \beta$ be points minimizing the distance between α and β; in particular x_0 is a projection of y_0 on α. By Remark 2.3.7 and by (b) there exists a projection b_+ of β^+ on α that falls at distance at most 9δ from x_0. Therefore we have $d(b_+, \beta) \leq d(b_+, x_0) + d(x_0, \beta) \leq 9\delta + d(\alpha, \beta)$. The thesis for all possible projections of β^+ on α follows from Lemma 2.3.6.(b).
Case 2: $d(\alpha, \beta) \leq 30\delta$. In this case we parameterize β in such a way that $\beta(0)$ is at distance at most 30δ from α. Then let

$$t_0 = \max\{t \in [0, +\infty) \ \text{ s.t. } \ d(\beta(t), \alpha) \leq 30\delta\},$$

let $y_0 = \beta(t_0)$ and let x_0 be any projection of y_0 on α. The convex subset $[\beta(t_0), \beta^+]$ of β is at distance $> 30\delta$ from α, so arguing as before we have that any projection b_+ of β^+ on α is at distance at most 19δ from x_0. Then, again $d(b_+, \beta) \leq d(b_+, x_0) + d(x_0, y_0) \leq 49\delta$. □

2.3.4 Helly's Theorem

A subset $C \subseteq X \cup \partial_G X$ is said λ-*quasiconvex*, where $\lambda \geq 0$, if for every $x, y \in C$ there exists at least one geodesic (segment, ray or line) joining x to y that is included in $\overline{B}(C, \lambda)$. The subset C is called *starlike with respect to a point* $x_0 \in C$ if for all $x \in C$ there exists at least one geodesic (segment, ray or line) $[x_0, x]$ entirely included in C. For instance a convex set is starlike with respect to all of its points. The proof of the following lemma can be found in [DKL18] and [CDP90]:

Lemma 2.3.11 (Lemma 3.3 of [DKL18] and Proposition 10.1.2 of [CDP90])**.** *Let X be δ-hyperbolic and let $C \subseteq X \cup \partial_G X$ be starlike with respect to x_0. Then C is 12δ-quasiconvex and $\overline{B}(C, \lambda)$ is 20δ-quasiconvex for all $\lambda \geq 0$.*

We state now the version of Helly's Theorem which we will need. The proof we give here follows the one given by [BF18], with the minor modifications needed to deal with *quasiconvex* subsets instead of convex ones.

Proposition 2.3.12 (Helly's Theorem)**.** *Let X be a δ-hyperbolic space and let $(C_i)_{i\in I}$ be a family of λ-quasiconvex subsets of X such that $C_i \cap C_j \neq \emptyset$ $\forall i, j$. Then:*

$$\bigcap_{i\in I} \overline{B}(C_i, 119\delta + 15\lambda) \neq \emptyset.$$

The proof is a direct consequence of the following lemma.

Lemma 2.3.13. *Let X be a δ-hyperbolic space, let $C_1, C_2 \subseteq X$ be two λ-quasiconvex subsets with non-empty intersection and let $x_0 \in X$ be fixed. Assume that we have points $x_1 \in C_1$ and $x_2 \in C_2$ which satisfy:*

$$d(x_0, x_i) \leq d(x_0, C_i) + \delta \text{ for } i = 1, 2$$

$$d(x_0, x_1) \geq d(x_0, x_2) - \delta$$

Then, $d(x_1, C_2) \leq 119\delta + 15\lambda$.

Proof. Let $u \in C_1 \cap C_2$. By the Projection Lemma 2.3.5 we have

$$(u, x_1)_{x_0} \geq d(x_0, [u, x_1]) - 4\delta.$$

Moreover

$$d(x_0, [u, x_1]) \geq d(x_0, \overline{B}(C_1, \lambda)) \geq d(x_0, C_1) - \lambda \geq d(x_0, x_1) - \lambda - \delta.$$

So $(u, x_1)_{x_0} \geq d(x_0, x_1) - \lambda - 5\delta$. Computing the Gromov product we get

$$d(u, z) + 10\delta + 2\lambda \geq d(x_1, z) + d(x_1, u).$$

The same conclusion holds for x_2. Hence the two paths $\alpha = [u, x_1] \cup [x_1, x_0]$ and $\beta = [u, x_2] \cup [x_2, x_0]$ are $(1, 10\delta + 2\lambda)$-quasigeodesic segments with same endpoints. Applying Proposition 2.3.8 we conclude that for any t where the two paths are defined it holds:

$$d(\alpha(t), \beta(t)) \leq 108\delta + 12\lambda.$$

We estimate now the distance between x_1 and the geodesic segment $[u, x_2]$. Let t_1, t_2 be such that $\alpha(t_1) = x_1$ and $\beta(t_2) = x_2$. If $t_1 \leq t_2$ then $\beta(t_1) \in [u, x_2]$ and in particular t_1 is a common time for both geodesic segments. Therefore we can conclude that $d(x_1, [u, x_2]) \leq 108\delta + 12\lambda$. We consider now the case $t_1 \geq t_2$. Since $d(x_0, x_1) \geq d(x_0, x_2) - \delta$ we know that

$$\begin{aligned} t_1 = d(u, x_1) &\leq d(x_0, u) - d(x_0, x_1) + 10\delta + 2\lambda \\ &\leq d(x_0, u) - d(x_0, x_2) + 11\delta + 2\lambda \\ &\leq d(x_0, x_2) + d(x_2, u) - d(x_0, x_2) + 11\delta + 2\lambda \\ &= t_2 + 11\delta + 2\lambda. \end{aligned}$$

Therefore we get

$$\begin{aligned} d(x_1, x_2) = d(\alpha(t_1), \beta(t_2)) &\leq d(\alpha(t_1), \alpha(t_2)) + d(\alpha(t_2), \beta(t_2)) \\ &\leq 11\delta + 2\lambda + 108\delta + 12\lambda = 119\delta + 14\lambda. \end{aligned}$$

In any case we have $d(x_1, [u, x_2]) \leq 119\delta + 14\lambda$. In conclusion

$$d(x_1, C_2) \leq d(x_1, B(C_2, \lambda)) + \lambda \leq d(x_1, [u, x_2]) + \lambda \leq 119\delta + 15\lambda.$$

□

Proof of Proposition 2.3.12. We choose a point $x_0 \in X$. Let $x_i \in C_i$ be as in the previous lemma, say with $d(x_0, x_1) \geq d(x_0, x_i) - \delta$ for all $i \in I$. Applying the lemma to any couple C_1, C_i we find that the point x_1 belongs to the intersection of all the desired neighbouroods of C_i. □

Finally we recall the definition of *quasiconvex hull* of a subset C of $\partial_G X$: it is the union of all the geodesic lines joining two points of C and it is denoted by QC-Hull(C).

2.4 Packing and covering

Let $Y \subset X$ be any subset of a metric space:
– a subset S of Y is called *r-dense* if $\forall y \in Y\ \exists z \in S$ such that $d(y, z) \leq r$;
– a subset S of Y is called *r-separated* if $\forall y, z \in S$ it holds $d(y, z) > r$.
The *r-packing number of* Y is the maximal cardinality of a $2r$-separated subset of Y and is denoted by $\text{Pack}(Y, r)$. The *r-covering number of* Y is the minimal cardinality of a r-dense subset of Y and is denoted by $\text{Cov}(Y, r)$. These two quantities are classically related by the following relations:

$$\text{Pack}(Y, 2r) \leq \text{Cov}(Y, 2r) \leq \text{Pack}(Y, r). \tag{21}$$

On a given space X the numbers $\text{Pack}(\overline{B}(x, R), r)$ and $\text{Cov}(\overline{B}(x, R), r)$, for $0 < r \leq R$, depend in general on the chosen point x. We are interested in the case where these numbers can be bounded independently of $x \in X$. Therefore consider the functions

$$\text{Pack}(R, r) = \sup_{x \in X} \text{Pack}(\overline{B}(x, R), r), \qquad \text{Cov}(R, r) = \sup_{x \in X} \text{Cov}(\overline{B}(x, R), r)$$

called, respectively, the *packing and covering functions* of X. They take values on $[0, +\infty]$; moreover, as an immediate consequence of (21), we have

$$\text{Pack}(R, 2r) \leq \text{Cov}(R, 2r) \leq \text{Pack}(R, r). \tag{22}$$

Definition 2.4.1. Let X be a metric space and let $C_0, P_0, r_0 > 0$.
We say that X *is P_0-packed at scale* r_0 if $\text{Pack}(3r_0, r_0) \leq P_0$, that is every ball of radius $3r_0$ contains no more than P_0 points that are $2r_0$-separated.
Analogously we say that X is *C_0-covered at scale* r_0 if $\text{Cov}(3r_0, r_0) \leq C_0$, i.e. every ball of radius $3r_0$ can be covered by at most C_0 balls of radius r_0.

The packing property can always be propagated at larger scales.

Lemma 2.4.2. *Let X be a geodesic metric space that is P_0-packed at scale r_0. Then for any $R \geq 3r_0$ it holds:*

$$\mathrm{Pack}(R, r_0) \leq P_0(1 + P_0)^{\frac{R}{r_0}-1}.$$

Proof. We prove the thesis by induction on k, where k is the smallest integer such that $R \leq 3r_0 + kr_0$. The case $k = 0$ clearly holds as for $R = 3r_0$ we have $\mathrm{Pack}(R, r_0) \leq P_0 \leq P_0(1+P_0)^2$. Let now $k \geq 1$ and $R \geq 3r_0$ such that $R \leq 3r_0 + kr_0$. We consider the sphere $S(x, R - r_0)$ of points at distance exactly $R - r_0$ from x. We observe that $R - r_0 \leq 3r_0 + (k - 1)r_0$, so by induction we can find a $2r_0$-separated subset $y_1, \ldots, y_n$ of $S(x, R - r_0)$ of maximal cardinality, where $n \leq P_0(1+P_0)^{\frac{R-r_0}{r_0}-1}$. Moreover

$$\bigcup_{i=1}^{n} \overline{B}(y_i, 3r_0) \supseteq A(x, R - r_0, R).$$

Indeed for any $y \in A(x, R - r_0, R)$ we take a geodesic $[x, y]$ and we call y' the point on the geodesic $[x, y]$ at distance $R - r_0$ from x. Then $y \in \overline{B}(y', r_0)$. Moreover there exists y_i such that $d(y', y_i) \leq 2r_0$, because of the maximality of the set $\{y_1, \ldots, y_n\}$. Hence $d(y, y_i) \leq 3r_0$. Therefore we get:

$$\begin{aligned}\mathrm{Pack}(\overline{B}(x, R), r_0) &\leq \mathrm{Pack}(\overline{B}(x, R - r_0), r_0) + \mathrm{Pack}(A(x, R - r_0, R), r_0) \\ &\leq \mathrm{Pack}(\overline{B}(x, R - r_0), r_0) + \sum_{i=1}^{n} \mathrm{Pack}(\overline{B}(y_i, 3r_0), r_0).\end{aligned}$$

Since $\mathrm{Pack}(\overline{B}(y_i, 3r_0), r_0) \leq P_0$, we obtain

$$\begin{aligned}\mathrm{Pack}(\overline{B}(x, R), r_0) &\leq \mathrm{Pack}(\overline{B}(x, R - r_0), r_0) + P_0 \cdot n \\ &\leq \mathrm{Pack}(\overline{B}(x, R - r_0), r_0) + P_0 \cdot \mathrm{Pack}(\overline{B}(x, R - r_0), r_0) \\ &\leq (1 + P_0)P_0(1 + P_0)^{\frac{R-r_0}{r_0}-1} = P_0(1 + P_0)^{\frac{R}{r_0}-1}.\end{aligned}$$

□

In general a control of the packing function at some fixed scale does not imply any control at smaller scales, as shown in the following example.

Example 2.4.3. Let $\mathbb{D}^n \subset \mathbb{R}^n$ be the closed Euclidean disk of radius 1. Let X_n be the space obtained gluing a Euclidean ray $[0, +\infty)$ to a point of the boundary of $\mathbb{D}^n$. Fix $r_0 = 1$. Any $2r_0$-separated subset S of X_n contains at most one point of $\mathbb{D}^n$. Hence $\mathrm{Pack}(3r_0, r_0) \leq 2$, in other words X_n is 2-packed at scale 1 for every n. However at smaller scales, for example at scale $r = \frac{1}{4}$, we can easily show that $\mathrm{Pack}(3r, r) \to +\infty$ when $n \to +\infty$. Notice that the spaces X_n in this example are complete and CAT(0) but they fail to be *geodesically complete*.

2.4.1 Packing in convex spaces

The next result states that in a *convex* and *geodesically complete* metric space, the packing function $\mathrm{Pack}(R, r)$ is well controlled by the packing condition at any fixed scale r_0:

Proposition 2.4.4. *Let X be a convex and geodesically complete metric space that is P_0-packed at scale r_0. Then:*

(a) for all $r \leq r_0$, the space X is P_0-packed at scale r;

(b) for every $0 < r \leq R$ it holds:

$$\mathrm{Pack}(R, r) \leq P_0(1 + P_0)^{\frac{R}{r} - 1}, \text{ if } r \leq r_0;$$

$$\mathrm{Pack}(R, r) \leq P_0(1 + P_0)^{\frac{R}{r_0} - 1}, \text{ if } r > r_0.$$

Moreover if X is complete then it is proper.

The proof follows by the next easy but fundamental lemma.

Lemma 2.4.5 (Packing propagation)**.** *Let X be a convex, geodesically complete metric space that is P_0-packed at scale r_0. Then X is P_0-packed at scale r for any $r \leq r_0$.*

Proof. We fix $x \in X$ and $r \leq r_0$. We take a $2r$-separated subset $\{x_1, \ldots, x_N\}$ of $\overline{B}(x, 3r)$. We consider the contraction map $\varphi_{3r}^{3r_0}$ which is surjective and $\frac{r}{r_0}$-Lipschitz. For any i we fix a preimage y_i of x_i under $\varphi_{3r}^{3r_0}$. We have $2r < d(x_i, x_j) \leq \frac{r}{r_0} d(y_i, y_j)$ for any $i \neq j$. This means that the set $\{y_1, \ldots, y_N\}$ is $2r_0$-separated in $\overline{B}(x, 3r_0)$, hence $N \leq P_0$. □

Proof of Proposition 2.4.4. By Lemma 2.4.5 we know that X is P_0-packed at scale r for all $0 < r \leq r_0$. Therefore, for these values of r, Lemma 2.4.2 yields

$$\mathrm{Pack}(R, r) \leq P_0(1 + P_0)^{\frac{R}{r} - 1}$$

$\forall R \geq 3r$; but this also holds for $R \leq 3r$ since then $\mathrm{Pack}(R, r) \leq \mathrm{Pack}(3r, r)$. On the other hand if $r \geq r_0$ the thesis follows directly from Lemma 2.4.2. Indeed when $R \geq 3r_0$ then $\mathrm{Pack}(R, r) \leq \mathrm{Pack}(R, r_0)$ and Lemma 2.4.2 concludes. If $R < 3r_0$ we get

$$\mathrm{Pack}(R, r) \leq \mathrm{Pack}(R, r_0) \leq \mathrm{Pack}(3r_0, r_0) \leq P_0$$

and $P_0(1 + P_0)^{\frac{R}{r} - 1} \geq P_0$.
Finally we assume that X is also complete. For all $x \in X$ and $R \geq 0$ the ball $\overline{B}(x, R)$ is complete since it is closed and X is complete. Moreover for all $\varepsilon > 0$ the maximal cardinality of a ε-separated subset of $\overline{B}(x, R)$ is finite, hence this ball is totally bounded and so compact. □

We can read this result in terms of the covering functions instead of the packing functions using (22).

Corollary 2.4.6. *Let X be a convex, geodesically complete metric space that is P_0-packed at scale r_0. Then for every $0 < r \leq R$ it holds:*

$$\mathrm{Cov}(R, r) \leq P_0(1 + P_0)^{\frac{2R}{r} - 1}, \text{ if } r \leq 2r_0;$$

$$\mathrm{Cov}(R, r) \leq P_0(1 + P_0)^{\frac{2R}{r_0} - 1}, \text{ if } r > 2r_0.$$

Proof. We have

$$\mathrm{Cov}(R,r) \leq \mathrm{Pack}\left(R, \frac{r}{2}\right) \leq P_0(1+P_0)^{\frac{2R}{r}-1}, \qquad \text{if } \frac{r}{2} \leq r_0$$
$$\mathrm{Cov}(R,r) \leq P_0(1+P_0)^{\frac{2R}{r_0}-1}, \qquad \text{if } \frac{r}{2} > r_0.$$ □

2.4.2 Packing in GCBA spaces

The proof of the packing propagation lemma is based on two properties of the contraction maps φ_r^R: they are surjective and $\frac{r}{R}$-Lipschitz. In case of GCBA spaces the contraction maps are again surjective but $\frac{2r}{R}$-Lipschitz. However we can mimic the proofs above in this setting, up to a factor 2. Indeed the next proposition affirms that the packing functions can be well controlled for complete, locally CAT(κ)-spaces which are locally geodesically complete (notice that no local compactness is assumed, since it will *follow* from the packing condition):

Proposition 2.4.7. *Let X be a complete, locally* CAT(κ)*, locally geodesically complete, geodesic metric space with $\rho_{\mathrm{ac}}(X) > 0$. Suppose that X satisfies*

$$\mathrm{Pack}\left(3r_0, \frac{r_0}{2}\right) \leq P_0 \qquad \textit{for } 0 < r_0 < \rho_{\mathrm{ac}}(X)/3.$$

Then X is proper and geodesically complete; so it is a GCBA *metric space. Moreover for any $0 < r \leq R$ it holds:*

$$\mathrm{Pack}(R,r) \leq P_0(1+P_0)^{\frac{R}{r}-1}, \textit{ if } r \leq r_0;$$

$$\mathrm{Pack}(R,r) \leq P_0(1+P_0)^{\frac{R}{r_0}-1}, \textit{ if } r > r_0.$$

We remark that a packing condition to scales bigger than the almost-convexity radius does not propagate to smaller scales:

Example 2.4.8. Let X_n be the graph with one vertex and n loops of length 1. For any n we glue an half-line to the vertex obtaining a complete, GCBA, geodesic metric space Y_n. As in Example 2.4.3 it is easy to show that at big scales the spaces Y_n satisfy a uniform packing condition, while at small scales they do not.

The proof of Proposition 2.4.7 is based again on the adapted version of Lemma 2.4.5

Lemma 2.4.9. *Let X be a space satisfying the assumptions of Proposition 2.4.7. Then X is P_0-packed at scale r for any $r \leq r_0$.*

The proof is exactly the same given for Lemma 2.4.5. The difference is the factor 2 in the Lipschitz constant of the contraction maps: the packing assumption on X is made especially to overcome this fact.

Proof of Proposition 2.4.7. The estimates on the packing functions can be made exactly as in the proof of Proposition 2.4.4, as well as the proof of the properness of X. Moreover a complete, locally geodesically complete metric space is geodesically complete. □

Also in this case we have:

Corollary 2.4.10. *Let X be a complete, locally* $\mathrm{CAT}(\kappa)$*, locally geodesically complete, geodesic metric space with* $\rho_{\mathrm{ac}}(X) > 0$*. Suppose that X satisfies*

$$\mathrm{Pack}\left(3r_0, \frac{r_0}{2}\right) \leq P_0 \qquad \text{for } r_0 < \rho_{\mathrm{ac}}(X)/3.$$

Then for any $0 < r \leq R$ it holds:

$$\mathrm{Cov}(R,r) \leq P_0(1+P_0)^{\frac{2R}{r}-1}, \text{ if } r \leq 2r_0;$$

$$\mathrm{Cov}(R,r) \leq P_0(1+P_0)^{\frac{2R}{r_0}-1}, \text{ if } r > 2r_0.$$

2.5 Isometries of Gromov-hyperbolic spaces

When X is a Gromov-hyperbolic space then its isometries are classified into three types according to the behaviour of their orbits (cp. for instance [CDP90]):

- an isometry g is *elliptic* if it has bounded orbits; when g acts discretely this is the same as asking that it is a torsion element, cp. [BCGS17];
- an isometry g is *parabolic* if there exists a point $g^\infty \in \partial_G X$ such that for all $x \in X$ the sequences $(g^k x)_{k \geq 0}$ and $(g^k x)_{k \leq 0}$ converge to g^∞;
- an isometry g is *hyperbolic* if the map $k \mapsto g^k x$ is a quasi-isometry $\forall x \in X$, i.e. there exist $L, C > 0$ such that for any $k, k' \in \mathbb{Z}$ it holds

  $$\frac{1}{L}|k-k'| - C \leq d(g^k x, g^{k'} x) \leq L|k-k'| + C.$$

  In this case there exist two points $g^- \neq g^+$ in $\partial_G X$ such that for any $x \in X$ the sequence $(g^k x)_{k \geq 0}$ converges to g^+ and the sequence $(g^k x)_{k \leq 0}$ converges to g^-.

Also recall that the *asymptotic displacement* of an isometry g is defined as the limit (which exists and does not depend on the choice of $x \in X$):

$$\|g\| = \lim_{n \to +\infty} \frac{d(x, g^n x)}{n}.$$

It is well known that for any isometry g of X and for any $k \in \mathbb{Z}^*$ it holds $\|g^k\| = k\|g\|$ and that g is hyperbolic if and only if $\|g\| > 0$ (see [CDP90]). The following lemma is well known and will be used to bound the displacement of an isometry by its asymptotic displacement:

Lemma 2.5.1 ([BCGS17], [CDP90])**.** *For any isometry g and every $x \in X$ we have: $d(x, g^n x) \leq d(x, gx) + (n-1) \cdot \|g\| + 4\delta \cdot \log_2 n$. for all $n > 0$.*

Any isometry of X acts naturally by homeomorphisms on $\partial_G X$. If g is parabolic then g^∞ is the unique fixed point of the action of g on $X \cup \partial_G X$, while if g is hyperbolic then g^-, g^+ are the only fixed points of the action of g on $X \cup \partial_G X$. The set of fixed points of an isometry g on the Gromov boundary is denoted by $\mathrm{Fix}_\partial(g)$. For any $k \in \mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$ we have that an isometry g is elliptic (resp.parabolic, hyperbolic) if and only if g^k is elliptic (resp.parabolic, hyperbolic); moreover if g is parabolic or hyperbolic it holds $\mathrm{Fix}_\partial(g^k) = \mathrm{Fix}_\partial(g)$.

2.5.1 Isometries of Gromov-hyperbolic and convex spaces

The *displacement function* of an isometry g is defined as $d_g(x) = d(x, gx)$, and the *minimal displacement* of g is

$$\ell(g) = \inf_{x \in X} d(x, gx).$$

In general the asymptotic displacement always satisfies $\|g\| \leq \ell(g)$. However on a convex metric space we always have $\|g\| = \ell(g)$ (cp. [BGS13]). In particular if X is convex and Gromov-hyperbolic all parabolic isometries have zero translation length (notice that this is false for arbitrary convex spaces, cp. [Wu18]) and $\ell(g) > 0$ if and only if g is of hyperbolic type. Moreover by Lemma 2.1.2 every elliptic isometry g of X has a fixed point (the center of any bounded orbit $g^n x$ of g, which is clearly invariant by g); reciprocally every isometry with a fixed point is clearly elliptic.
We can therefore restate the classification of isometries of a proper, convex, Gromov-hyperbolic space as follows:

- an isometry g is elliptic if and only if $\ell(g) = 0$ and the value of minimal displacement is attained for some $x \in X$;

- an isometry g is parabolic if and only if $\ell(g) = 0$ and the minimal displacement is not attained;

- an isometry g is hyperbolic if and only if $\ell(g) > 0$ and the minimal displacement is attained.

In the last case there exists a geodesic line joining g^- to g^+ on which g acts as a translation by $\ell(g)$; any such geodesic line is called an *axis of g* (see [Pap05]).

It is classical that in CAT(0), Gromov-hyperbolic spaces the Busemann function $\mathcal{B}_z(x, y)$ can be computed using any geodesic ray ξ with endpoint $\xi^+ = z$, cp. [BH13], [DPS12] (observe that this is false in convex metric spaces, see [And08]). As a consequence, under these assumptions, every parabolic isometry preserves the horospheres centered at its unique fixed point:

Lemma 2.5.2. *Let g be a parabolic isometry of a* CAT(0)*, Gromov-hyperbolic metric space X with fixed point z. Then $\mathcal{B}_z(x, gx) = 0$ for every $x \in X$. In particular g preserves the horospheres centered at z and passing through x.*

Proof. The metric derivative of a parabolic isometry g is equal to $g'(z) = 1$ (cp. [DSU17]), then by Proposition 4.2.16 of [DSU17] we conclude there exists $C > 0$ such that $|\mathcal{B}_z(x, g^n x)| \leq C$ for all $n \in \mathbb{Z}$. However by the cocycle condition satisfied by the Busemann function on CAT(0) metric spaces we have

$$\mathcal{B}_z(x, g^n x) = \mathcal{B}_z(x, gx) + \mathcal{B}_z(gx, g^n x) = n \cdot \mathcal{B}_z(x, gx)$$

which is larger than C for $n \gg 0$, unless $\mathcal{B}_z(x, gx) = 0$. □

2.5.2 The Margulis domain of an isometry

In this part X will be a proper, *convex*, Gromov-hyperbolic space.
From the convexity of the metric it follows that the displacement function d_g of an isometry g of X is convex. We are interested in the sublevel sets of d_g. Given $\varepsilon > 0$, the subset

$$M_\varepsilon(g) = \{x \in X \text{ s.t. } d(x, gx) \leq \varepsilon\}$$

is called the *Margulis domain* of g with displacement ε. As d_g is convex, the Margulis domain is a closed and convex subset of X. Finally we denote by

$$\text{Min}(g) = M_{\ell(g)}(g)$$

the subset of points of X where d_g attains its minimum (which is empty for a parabolic isometry g).

Lemma 2.5.3. *The Margulis domain $M_\varepsilon(g)$ of any isometry g of X, if non-empty, is starlike with respect to any point $z \in \text{Fix}_\partial(g) \cup \text{Min}(g)$.*

Proof. Fix a point $x \in M_\varepsilon(g)$ and $z \in \text{Fix}_\partial(g)$. Assume that the geodesic ray $[x, z]$ is not contained in $M_\varepsilon(g)$. Then there exists a point $y \in [x, z]$ such that $d(y, gy) \geq \varepsilon + \eta$ for some $\eta > 0$. Let $L = d(x, y)$ and consider the points y_n along $[x, z]$ at distance nL from x. By convexity of the displacement function, we have $d(y_n, gy_n) \geq \varepsilon + n\eta$. We observe that the points gy_n belong to the geodesic ray $[gx, gz]$, defining the point gz of the boundary. The two rays $[x, z]$ and $[gx, gz]$ are not parallel, hence $gz \neq z$ which is a contradiction since $z \in \text{Fix}_\partial(g)$. The case where $z \in \text{Min}(g)$ follows directly from the convexity of the displacement function and the minimality of z. □

The *generalized Margulis domain* of g at level ε is the set

$$\mathcal{M}_\varepsilon(g) = \bigcup_{i \in \mathbb{Z}^*} M_\varepsilon(g^i).$$

It clearly is a g-invariant subset of X. We remark that for all $\varepsilon > 0$ the union is finite when g is elliptic (and $\langle g \rangle$ is discrete) or hyperbolic, while it is infinite when g is of parabolic type.

Lemma 2.5.4. *The generalized Margulis domain $\mathcal{M}_\varepsilon(g)$ is 12δ-quasiconvex and connected.*

Proof. As a consequence of Lemma 2.5.3, the domain $\mathcal{M}_\varepsilon(g)$ is starlike with respect to any $x \in \text{Min}(g) \cup \text{Fix}_\partial(g)$. So, by Lemma 2.3.11, it is 12δ-quasiconvex.
The last assertion is trivial if g is elliptic or hyperbolic: in that case $\mathcal{M}_\varepsilon(g)$ is a finite union of connected sets with a common point. If g is parabolic we fix a point $x \in M_\varepsilon(g)$ and any $y \in \mathcal{M}_\varepsilon(g)$, so $y \in M_\varepsilon(g^i)$ for some $i \neq 0$. Since $\ell(g) = 0$ we can take a point $x' \in M_{\varepsilon/|i|}(g)$. By convexity we have that the geodesic segment $[x, x']$ is entirely contained in $M_\varepsilon(g)$. Moreover $d(x', g^i x') \leq \varepsilon$, so the geodesic segment $[y, x']$ is contained in $M_\varepsilon(g^i)$. As a consequence the curve $[x, x'] \cup [x', y]$ is contained in $\mathcal{M}_\varepsilon(g)$. We conclude that $\mathcal{M}_\varepsilon(g)$ is connected since every of its points can be connected to the fixed point x. □

One of the key ingredients in the proof of Theorem J and in the applications (notably, the existence of long tubes around small closed geodesics) are the following lower and upper uniform estimates of the distance between the boundaries of two different generalized Margulis domains. Hyperbolicity in used only in the upper estimate, while the packing condition is essential to both.

Proposition 2.5.5. *Let X be a complete, convex, geodesically complete, δ-hyperbolic metric space that is P_0-packed at scale r_0, and let $0 < \varepsilon_1 \leq \varepsilon_2$. Let g be any non-elliptic isometry, $x \in X \setminus \mathcal{M}_{\varepsilon_2}(g)$ and assume $\mathcal{M}_{\varepsilon_1}(g) \neq \emptyset$. Then:*

(a) $d(x, \mathcal{M}_{\varepsilon_1}(g)) \geq \frac{1}{2}(\varepsilon_2 - \varepsilon_1)$;

(b) if $\varepsilon_2 \leq r_0$ *then* $d(x, \mathcal{M}_{\varepsilon_1}(g)) > L_{\varepsilon_2}(\varepsilon_1) = \dfrac{\log\left(\frac{2}{\varepsilon_1} - 1\right)}{2\log(1+P_0)} \cdot \varepsilon_2 - \dfrac{1}{2}$.

Notice that the estimate (b) is significative only for ε_2 small enough, and has a different geometrical meaning from (a): it says that the $\mathcal{M}_{\varepsilon_2}(g)$ contains a large ball around any point $x \in \mathcal{M}_{\varepsilon_1}(g)$, of radius which is larger and larger as ε_1 tends to zero.

Proof. The first estimate is simple and does not need any additional condition on the metric space X. Let $\bar{x}$ be a projection of x on the closure $\overline{\mathcal{M}_{\varepsilon_1}(g)}$ of the generalized Margulis domain. By definition for all $\eta > 0$ there exists some nontrivial power g_η of g such that $d(\bar{x}, g_\eta \bar{x}) \leq \varepsilon_1 + \eta$. So:

$$\varepsilon_2 \leq d(x, g_\eta x) \leq d(x, \bar{x}) + d(\bar{x}, g_\eta \bar{x}) + d(g_\eta \bar{x}, g_\eta x) \leq 2d(x, \mathcal{M}_{\varepsilon_1}(g)) + \varepsilon_1 + \eta.$$

The estimate follows from the arbitrariness of η.
Let us now prove (b). Let again $\bar{x} \in \overline{\mathcal{M}_{\varepsilon_1}(g)}$ with $d(x,\bar{x}){=}d(x, \mathcal{M}_{\varepsilon_1}(g)){=}R$. For any $\eta > 0$ let g_η be some nontrivial power of g satisfying $d(\bar{x}, g_\eta \bar{x}) \leq \varepsilon_1 + \eta$. Then again

$$d(x, g_\eta^k x) \leq 2R + d(\bar{x}, g_\eta^k \bar{x}) \leq 2R + |k|(\varepsilon_1 + \eta)$$

so $d(x, g_\eta^k x) \leq 2R + 1$ for all k such that $|k| \leq 1/(\varepsilon_1 + \eta)$. Therefore we have at least $n(\varepsilon_1, \eta) = 1 + 2\lfloor 1/(\varepsilon_1 + \eta) \rfloor$ points in the orbit Γx inside the ball $\overline{B}(x, 2R+1)$. We deduce that if $n(\varepsilon_1, \eta) > \text{Pack}(2R+1, \varepsilon_2)$ two of these points stay at distance less than ε_2 one from the other, which implies that $x \in \mathcal{M}_{\varepsilon_2}(g)$, a contradiction. Therefore,

$$n(\varepsilon_1, \eta) = 1 + 2\lfloor 1/(\varepsilon_1 + \eta) \rfloor \leq \text{Pack}(2R+1, \varepsilon_2) \leq P_0(1+P_0)^{\frac{2R+1}{\varepsilon_2} - 1}$$

by Proposition 2.4.4 (since $\varepsilon_2 \leq r_0$), which implies that $R = d(x, \bar{x})$ is greater than the function $L_{\varepsilon_2}(\varepsilon_1)$ in (b) by the arbitrariness of η. □

The upper bound is more tricky:

Proposition 2.5.6. *Let X be a complete, convex, geodesically complete, δ-hyperbolic metric space that is P_0-packed at scale r_0 and let $0 < \varepsilon_1 \leq \varepsilon_2$. Then there exists K_0, only depending on $P_0, r_0, \delta, \varepsilon_1$ and ε_2, such that for every non-elliptic isometry g of X with $\ell(g) \leq \varepsilon_1$ it holds:*

$$\sup_{x \in \mathcal{M}_{\varepsilon_2}(g)} d(x, \mathcal{M}_{\varepsilon_1}(g)) \leq K_0.$$

The condition $\ell(g) \leq \varepsilon_1$ is necessary to guarantee that the set $\mathcal{M}_{\varepsilon_1}(g)$ is non-empty.

Proof of Proposition 2.5.6. Let $x \in \mathcal{M}_{\varepsilon_2}(g)$, so by definition there exists i_0 such that $d(x, g^{i_0}x) \leq \varepsilon_2$. If $x \in \overline{\mathcal{M}_{\varepsilon_1}(g)}$ there is nothing to prove. Otherwise we can find a point $\bar{x}$ of $\partial\mathcal{M}_{\varepsilon_1}(g)$ such that $d(x,\bar{x}) = d(x, \mathcal{M}_{\varepsilon_1}(g))$. We set $\tau = \max\{\varepsilon_1, \delta\}$ and $N_0 = \text{Pack}\left(42\tau, \frac{\varepsilon_1}{2}\right)$, which is a number depending only on P_0, r_0, δ and ε_1, by Proposition 2.4.4.

Step 1: we prove that there exists an integer $k \leq 2^{N_0+1}$ such that

$$d(\bar{x}, g^{k\cdot i_0}\bar{x}) > 42\tau \text{ and } d(x, g^{k\cdot i_0}x) \leq K := \varepsilon_2 + 84\tau + 2(N_0+1)\delta \tag{23}$$

If this was not true, then for all $k \leq 2^{N_0+1}$ such that $d(x, g^{k\cdot i_0}x) \leq K$ we would have $d(\bar{x}, g^{k\cdot i_0}\bar{x}) \leq 42\tau$. Let p_0 be the largest integer such that $d(\bar{x}, g^{2^p\cdot i_0}\bar{x}) \leq 42\tau$ for all $0 \leq p \leq p_0$. We affirm that $p_0 \geq N_0 + 1$.
Actually $p_0 \geq 0$ because $d(x, g^{i_0}x) \leq \varepsilon_2 \leq K$, hence $d(\bar{x}, g^{i_0}\bar{x}) \leq 42\tau$ by assumption. Also, by Lemma 2.5.1, we get

$$d(x, g^{2^i\cdot i_0}x) \leq d(x, g^{2^{i-1}\cdot i_0}x) + \ell(g^{2^{i-1}\cdot i_0}) + 2\delta = d(x, g^{2^{i-1}\cdot i_0}x) + 2^{i-1}i_0\ell(g) + 2\delta.$$

and, iterating,

$$d(x, g^{2^p\cdot i_0}x) \leq d(x, g^{i_0}x) + (2^p - 1)i_0\ell(g) + 2p\delta \leq (2^p - 1)i_0\ell(g) + 2p\delta + \varepsilon_2$$

for every $0 \leq p \leq N_0 + 1$. So, if $p_0 \leq N_0$ we would have:

$$\begin{aligned}
d(x, g^{2^{(p_0+1)}\cdot i_0}x) &\leq d(x, g^{2^{p_0}\cdot i_0}x) + 2^{p_0}\cdot i_0\ell(g) + 2\delta \\
&\leq d(x, g^{2^{p_0}\cdot i_0}x) + d(\bar{x}, g^{2^{p_0}\cdot i_0}\bar{x}) + 2\delta \\
&\leq (2^{p_0} - 1)i_0\ell(g) + 2p_0\delta + \varepsilon_2 + 42\tau + 2\delta \\
&\leq 84\tau + 2(p_0+1)\delta + \varepsilon_2 < K
\end{aligned}$$

since $2^{p_0}i_0\ell(g) \leq d(\bar{x}, g^{2^{p_0}\cdot i_0}\bar{x}) \leq 42\tau$ by definition. Hence by assumption $d(\bar{x}, g^{2^{(p_0+1)}\cdot i_0}\bar{x}) \leq 42\tau$ and p_0 would not be maximal.
Moreover since $\bar{x}$ is in the boundary of $\partial\mathcal{M}_{\varepsilon_1}(g)$ then

$$\inf_{i\in\mathbb{Z}^*} d(\bar{x}, g^i\bar{x}) \geq \varepsilon_1.$$

Indeed if $d(\bar{x}, g^i\bar{x}) = \varepsilon_1 - \eta$ for some $i \in \mathbb{Z}^*$ and some $\eta > 0$ then it is easy to show that for any $y \in B(\bar{x}, \eta/2)$ we would have $d(y, g^iy) < \varepsilon_1$; hence $B(\bar{x}, \eta/2) \subseteq \mathcal{M}_{\varepsilon_1}(g)$ and $\bar{x}$ would not belong to $\partial\mathcal{M}_{\varepsilon_1}(g)$.
Then the points $g^{2^p\cdot i_0}\bar{x}$, for $p = 1, \ldots, N_0+1$, are ε_1-separated. But, as they belong all to the ball $B(\bar{x}, 42\tau)$, they should be at most N_0 and this is a contradiction. This proves the first step.

Step 2: for any $k_0 \leq 2^{N_0+1}$ satisfying the conditions (23), we have:

$$d(x, g^{k_0\cdot i_0}x) \geq d(x,\bar{x}) + d(g^{k_0\cdot i_0}x, g^{k_0\cdot i_0}\bar{x}) \tag{24}$$

Indeed let us write $y = g^{k_0\cdot i_0}x$ and $\bar{y} = g^{k_0\cdot i_0}\bar{x}$. By definition the point $\bar{x}$ satisfies $d(x,\bar{x}) = d(x, \mathcal{M}_{\varepsilon_1}(g))$; so, from the 12δ-quasiconvexity of $\mathcal{M}_{\varepsilon_1}(g)$ (Remark 2.5.4), we deduce that

$$d(x, [\bar{x}, \bar{y}]) \geq d(x, \overline{B}(\mathcal{M}_{\varepsilon_1}(g), 12\delta)) = d(x,\bar{x}) - 12\delta.$$

Moreover from the Projection Lemma 2.3.5 we have

$$d(x, [\bar{x}, \bar{y}]) \leq (\bar{x}, \bar{y})_x + 4\delta.$$

Combining these estimates and expanding the Gromov product we obtain

$$d(x, \bar{y}) \geq d(x, \bar{x}) + d(\bar{x}, \bar{y}) - 20\delta. \tag{25}$$

Similarly, using that $d(y, \bar{y}) = d(y, \mathcal{M}_{\varepsilon_1}(g))$ (as $\mathcal{M}_{\varepsilon_1}(g)$ is g-invariant), we obtain

$$d(y, \bar{x}) \geq d(y, \bar{y}) + d(\bar{y}, \bar{x}) - 20\delta. \tag{26}$$

Adding these last two inequalities and using that $d(\bar{x}, \bar{y}) > 42\tau \geq 42\delta$ we deduce

$$d(x, \bar{y}) + d(y, \bar{x}) > d(x, \bar{x}) + d(y, \bar{y}) + 2\delta.$$

Therefore applying the four-points condition (13) to $x, \bar{y}, \bar{x}, y$ we find

$$\begin{aligned} d(x, \bar{y}) + d(\bar{x}, y) &\leq \max\{d(x, \bar{x}) + d(y, \bar{y}); d(x, y) + d(\bar{x}, \bar{y})\} + 2\delta \\ &= d(x, y) + d(\bar{x}, \bar{y}) + 2\delta \end{aligned}$$

It follows:

$$\begin{aligned} d(x, y) &\geq d(x, \bar{y}) + d(\bar{x}, y) - d(\bar{x}, \bar{y}) - 2\delta \\ &\geq d(x, \bar{x}) + d(\bar{x}, \bar{y}) - 20\delta + d(y, \bar{y}) + d(\bar{y}, \bar{x}) - 20\delta - 2\delta \\ &\geq d(x, \bar{x}) + d(y, \bar{y}), \end{aligned}$$

where we have used again (25), (26) and that $d(\bar{x}, \bar{y}) > 42\tau \geq 42\delta$ (the first condition in (23)). Moreover the second condition in (23) now yields

$$d(x, \bar{x}) + d(y, \bar{y}) \leq K$$

The conclusion follows observing that $d(y, \bar{y}) = d(x, \bar{x})$, so that

$$d(x, \mathcal{M}_{\varepsilon_1}(g)) \leq d(x, \bar{x}) \leq K/2$$

which is the announced bound, depending only on $P_0, r_0, \delta, \varepsilon_1$ and ε_2. □

The distance between two Margulis domains (not generalized) of a non-elliptic isometry can be bounded uniformly in CAT(0), δ-hyperbolic metric spaces, but the implied constant is not explicit. We will use this estimate in order to study the limit of sequences of isometries in Chapter 8.

Proposition 2.5.7. *Let $\varepsilon, \delta > 0$. There exists a constant $c(\varepsilon, \delta)$ that satisfies the following. Let X be a* CAT(0)*, δ-hyperbolic space and g be a non-elliptic isometry of X with $M_\varepsilon(g) \neq \emptyset$. Then for all $x \in X$ it holds*

$$d(x, gx) \geq c(\varepsilon, \delta) \cdot d(x, M_\varepsilon(g)).$$

In particular for all $0 < \varepsilon_1 \leq \varepsilon_2$ we get

$$\sup_{x \in M_{\varepsilon_2}(g)} d(x, M_{\varepsilon_1}(g)) \leq \frac{\varepsilon_2}{c(\varepsilon_1, \delta)} =: K_1(\varepsilon_1, \varepsilon_2, \delta).$$

Proof. Suppose by contradiction that this is not true. Then for every $n \in \mathbb{N}$ there exist a CAT(0), δ-hyperbolic space X_n, a non-elliptic isometry g_n of X_n such that $M_\varepsilon(g_n) \neq \emptyset$ and a point $x_n \in X_n$ such that

$$0 < d(x_n, g_n x_n) \leq \frac{1}{n} d(x_n, M_\varepsilon(g_n)),$$

where the first inequality follows from the hypothesis that g_n is not elliptic.
All the translation lengths $\ell_n = \ell(g_n)$ belong to the interval $[0, \varepsilon]$. We fix a non-principal ultrafilter ω and we put $\ell = \omega\text{-lim}\, \ell_n$. We divide the proof into two cases: $\ell > 0$ and $\ell = 0$.

Case $\ell > 0$. Clearly for ω-a.e.(n) we have $\ell(g_n) > \frac{\ell}{2} > 0$. As a consequence g_n is hyperbolic for ω-a.e.(n). Let y_n be the projection of x_n on the minimal set of g_n. Let X_ω be the ultralimit of the sequence of spaces (X_n, y_n). X_ω is a CAT(0), δ-hyperbolic space (the stability of the δ-hyperbolicity condition follows from (12), while the stability of the CAT(0) condition is classical, Proposition 2.7.9. The sequence of isometries (g_n) is admissible, i.e. for every n it holds $d(g_n y_n, y_n) = \ell(g_n) \leq \varepsilon$. Then it defines a limit isometry $g_\omega = \omega\text{-lim}\, g_n$ of X_ω, see Proposition 2.7.5.
First of all we show that g_ω is hyperbolic by proving that $\ell(g_\omega) > 0$. Indeed for all $w_\omega = \omega\text{-lim}\, w_n \in X_\omega$ it holds $d(g_\omega w_\omega, w_\omega) = \omega\text{-lim}\, d(g_n w_n, w_n) \geq \omega\text{-lim}\, \ell(g_n) = \ell > 0$, so g_ω is hyperbolic with $\ell(g_\omega) \geq \ell$.
Now we consider the sequence of geodesic segments $[y_n, x_n]$. The claim is that this sequence converge to a geodesic ray of X_ω. By Proposition 2.7.5 and Lemma 2.7.6 it is enough to show that the ω-limit of the sequence $d(y_n, x_n)$ is $+\infty$. For ω-a.e.(n) we have

$$d(y_n, x_n) \geq n \cdot d(x_n, g_n x_n) \geq n \cdot \ell(g_n) \geq n \cdot \frac{\ell}{2},$$

where the first inequality follows from the fact that $y_n \in M_\varepsilon(g_n)$ since y_n is in the minimal set of g_n. So the ultralimit of the geodesics $\gamma_n = [y_n, x_n]$ is a geodesic ray γ_ω. The next step is to show that g_ω acts on γ_ω by translation of length ℓ, i.e. for every $T \geq 0$ it holds $d(g_\omega \gamma_\omega(T), \gamma_\omega(T)) = \ell$. We fix $T \geq 0$ and, as shown before, for ω-a.e. n there exists a point along $\gamma_n = [y_n, x_n]$ at distance T from y_n. We denote this point by w_n^T. Clearly the sequence of points (w_n^T) defines the point $\gamma_\omega(T)$ of X_ω. We fix $\eta > 0$ and we claim that if $n \geq n_\eta$ then $d(w_n^T, g_n w_n^T) \leq \ell_n + \eta$. This would imply that $d(\gamma_\omega(T), g_\omega \gamma_\omega(T)) \leq \ell + \eta$ for every $\eta > 0$, hence that the point $\gamma_\omega(T)$ is translated exactly by ℓ since $\ell(g_\omega) \geq \ell$.
We fix an integer n and we suppose that $d(w_n^T, g_n w_n^T) > \ell_n + \eta$. We want to find an upper bound n_η for n. Let $m \in \mathbb{N}$ be the smallest integer such that $mT \geq d(x_n, y_n)$. For every integer $k \in \{0, \ldots, m-1\}$ let w_n^{kT} be the point along γ_n at distance kT from y_n. From the convexity of the metric it holds

$$d(w_n^{kT}, g_n w_n^{kT}) > \ell_n + k\eta.$$

In particular, again by convexity of the displacement function, we get

$$\begin{aligned} mT \geq d(x_n, y_n) \geq n \cdot d(x_n, g_n x_n) &\geq n \cdot d(w_n^{(m-1)T}, g_n w_n^{(m-1)T}) \\ &\geq n(\ell_n + (m-1)\eta). \end{aligned}$$

We deduce $n \leq \frac{m}{m-1}\frac{T}{\eta} \leq \frac{2T}{\eta} =: n_\eta$.
Finally we consider the geodesic rays γ_ω and $g_\omega\gamma_\omega$ of X_ω. They are two geodesic rays whose distance is constant, hence applying the Flat Quadrilateral Theorem (see [BH13]) it is possible to find a strip in X_ω isometric to $[0,+\infty)\times[0,\ell]$.
Notice that $\ell(g_\omega^N) = N\ell$ for every $N \in \mathbb{N}$. Moreover for every $x \in \gamma_\omega$ it holds $d(g_\omega^N x, x) \leq N\ell$, so $d(g_\omega^N\gamma_\omega(T), \gamma_\omega(T)) = N\ell$ for every $T > 0$ and for every $N \in \mathbb{N}$. Arguing as before one can find a strip in X_ω isometric to $[0,+\infty)\times[0,N\ell]$. If N is big enough we get a contradiction since X_ω is δ-hyperbolic.

Case $\ell = 0$. This case is similar to the previous one, but we cannot project on the minimal set of g_n. The projection will be done on a suitable Margulis domain. For ω-a.e. n it holds $\ell(g_n) < \frac{\varepsilon}{2}$. For ω-a.e.(n) let y_n' be the projection of x_n on the nonempty level set $M_{\frac{\varepsilon}{2}}(g_n)$ and let y_n be the point along $[y_n', x_n]$ whose displacement by g_n is exactly ε.
Once again we consider the sequence of spaces (X_n, y_n) and its ultralimit X_ω, which is CAT(0) and δ-hyperbolic. Moreover also in this case the sequence of isometries (g_n) is admissible since $d(g_n y_n, y_n) = \varepsilon$ for every n. Then it defines a limit isometry g_ω of X_ω. We will show that the isometry g_ω is hyperbolic by proving $\ell(g_\omega) > \frac{\varepsilon}{3} > 0$.
We start studying how the isometries g_n act on the geodesic segments $[y_n, x_n]$. Arguing exactly as in the previous case, replacing ℓ_n with ε (that is the displacement of y_n in this case), we find that ω-$\lim d(x_n, y_n) = +\infty$ and that for every $T > 0$ and for every $\eta > 0$ it is possible to find n_η such that for $n \geq n_\eta$ it holds

$$d(g_n w_n^T, w_n^T) < \varepsilon + \eta.$$

Here we are using the notation w_n^T with the same meaning of the previous case. We apply this for any fixed $T > 0$ and $\eta = \frac{\varepsilon}{2}$. Then for n big enough we have $d(y_n, w_n^T) = T$ and $d(g_n w_n^T, w_n^T) < \varepsilon + \frac{\varepsilon}{2}$. From the convexity of the displacement function of g_n along the geodesic $[y_n', x_n]$ we conclude necessarily $d(y_n', y_n) \geq T$ for all such n's and so for ω-a.e.(n).
Suppose now that there exists a point $p_\omega = \omega$-$\lim p_n \in X_\omega$ whose displacement under g_ω is less than $\frac{\varepsilon}{3}$. Since p_ω is a point of X_ω then by definition there exists $L \geq 0$ such that $d(p_n, y_n) \leq L$ for ω-a.e.(n). Moreover for ω-a.e. n we have $d(g_n p_n, p_n) < \frac{\varepsilon}{2}$ and since y_n' is the projection of y_n on the level set $M_{\frac{\varepsilon}{2}}(g_n)$ we obtain, for all fixed $T \geq 0$, $d(p_n, y_n) \geq d(y_n', y_n) \geq T$ for ω-a.e.(n). Choosing $T > L$ we find a contradiction, so $\ell(g) > \frac{\varepsilon}{3}$.
Now we observe that the geodesic segments $\gamma_n = [y_n', x_n]$ defined on the intervals $[-d(y_n', y_n), d(y_n, x_n)]$ form a sequence of admissible geodesics and then define a limit geodesic γ_ω, that is actually defined on the whole $\mathbb{R}$. Moreover every point of this geodesic line is displaced by g_ω of exactly ε. Indeed arguing as in the first case we conclude that for every $T \in \mathbb{R}$ it holds $\frac{\varepsilon}{3} < d(\gamma_\omega(T), g_\omega\gamma_\omega(T)) \leq \varepsilon$. By convexity of the displacement function we conclude that $d(\gamma_\omega(T), g_\omega\gamma_\omega(T))$ is constant and equals ε, since this is its value at $T = 0$ by construction. Arguing as in the previous case we conclude again that for all $N \in \mathbb{N}$ we have $d(\gamma_\omega(T), g_\omega^N\gamma_\omega(T)) = N\varepsilon$ and we find the contradiction applying once again the Flat Strip Theorem (see [BH13]). □

2.6 Discrete groups of isometries

If X is any proper metric space we denote by $\mathrm{Isom}(X)$ its group of isometries, endowed with the uniform convergence on compact subsets of X. A subgroup Γ of $\mathrm{Isom}(X)$ is called *discrete* if the following (equivalent) conditions (see [BCGS17]) hold:

(a) Γ is discrete as a subspace of $\mathrm{Isom}(X)$;

(b) $\forall x \in X$ and $R \geq 0$ the set $\Sigma_R(x) = \{g \in \Gamma \mid gx \in \overline{B}(x,R)\}$ is finite.

We will denote by Γ^* the subset of nontrivial elements of Γ, while $\Gamma^\diamond \subseteq \Gamma$ will denote the subset of elements with finite order.

We register here a number of invariants associated to the action of Γ on X we will be interested in later. For every $r > 0$ and every point $x \in X$ the *r-almost stabilizer* of x in Γ is the subgroup

$$\Gamma_r(x) = \langle \Sigma_r(x) \rangle.$$

The *r-thin subset* of X (with respect to the action of Γ) is the subset

$$X_r = \{x \in X \mid \exists g \in \Gamma^* \ \textit{s.t.} \ d(x,gx) < r\}$$

and the *free r-thin subset* $X_r^\diamond$ is obtained by replacing Γ^* in the definition above with $\Gamma \setminus \Gamma^\diamond$. Some numerical invariants associated to the action of Γ we will be interested in are:

- the *minimal displacement* of $g \in \Gamma$, defined as $\ell(g) = \inf_{x \in X} d(x,gx)$;
- the *asymptotic displacement* of $g \in \Gamma$, defined as $\|g\| = \lim_{n \to +\infty} \frac{d(x,g^n x)}{n}$;
- the *minimal displacement* of Γ at x, defined as $\mathrm{sys}(\Gamma,x) = \inf_{g \in \Gamma^*} d(x,gx)$;
- the *minimal free displacement* of Γ at x, defined as $\mathrm{sys}^\diamond(\Gamma,x) = \inf_{g \in \Gamma \setminus \Gamma^\diamond} d(x,gx)$;
- the *nilpotence radius* of Γ at x, defined as

$$\mathrm{nilrad}(\Gamma,x) = \sup\{r \geq 0 \ \text{ s.t. } \ \Gamma_r(x) \text{ is virtually nilpotent}\};$$

and their corresponding global versions:

- the *systole* and the *free systole*, defined respectively as:

$$\mathrm{sys}(\Gamma,X) = \inf_{x \in X} \mathrm{sys}(\Gamma,x), \qquad \mathrm{sys}^\diamond(\Gamma,X) = \inf_{x \in X} \mathrm{sys}^\diamond(\Gamma,x)$$

- the *diastole* and the *free diastole*, defined as:

$$\mathrm{dias}(\Gamma,X) = \sup_{x \in X} \mathrm{sys}(\Gamma,x), \qquad \mathrm{dias}^\diamond(\Gamma,X) = \sup_{x \in X} \mathrm{sys}^\diamond(\Gamma,x)$$

- the *nilradius*: $\mathrm{nilrad}(\Gamma,X) = \inf_{x \in X} \mathrm{nilrad}(\Gamma,x)$.

Moreover, for convex, packed metric spaces we can define an upper analogue of the nilradius (like the diastole for the systole):

- the *upper nilradius*: $\mathrm{nilrad}^+(\Gamma,X) = \sup_{x \in X_{\varepsilon_0}} \mathrm{nilrad}(\Gamma,x)$

where the constant ε_0 appearing in the definition is the *generalized Margulis constant*, which will be introduced in the next section.

2.6.1 Margulis constant

Consider a proper metric space X and a discrete group of isometries Γ of X. In general the nilradius of the action can well be zero. However, under the packing assumption it is always bounded away from zero:

Theorem 2.6.1 (Corollary 11.17 of [BGT11]).
Let X be a proper metric space such that $\mathrm{Cov}(\overline{B}(x,4),1) \leq C_0$ for all $x \in X$. Then there exists a constant $\varepsilon_M = \varepsilon_M(C_0) > 0$, only depending on C_0, such that for every discrete group of isometries Γ of X one has $\mathrm{nilrad}(\Gamma, X) \geq \varepsilon_M$.

In analogy with the case of Riemannian manifolds, the constant ε_M is called the *(generalized) Margulis constant* of X. Combining with Proposition 2.4.4, we immediately get a Margulis contant for the class of complete, convex, geodesically complete, metric spaces which are P_0-packed at some scale r_0:

Corollary 2.6.2. *Given $P_0, r_0 > 0$ there exists $\varepsilon_0 = \varepsilon_0(P_0, r_0) > 0$ such that for any complete, convex, geodesically complete metric space X which is P_0-packed at scale r_0 and for any discrete group of isometries Γ of X one has $\mathrm{nilrad}(\Gamma, X) \geq \varepsilon_0$.*

Proof. By Proposition 2.4.4 X is proper. We rescale the metric by a factor $\frac{1}{2r_0}$. As the packing property is invariant under rescaling, we have $\mathrm{Pack}(\frac{3}{2}, \frac{1}{2}) \leq P_0$ by assumption. Hence $\mathrm{Cov}(4,1) \leq \mathrm{Pack}(4, \frac{1}{2}) \leq P_0(1+P_0)^7$, as follows from (22) and from Proposition 2.4.4. Applying Theorem 2.6.1 we find a Margulis constant ε_M for the space $\frac{1}{2r_0}X$, only depending on P_0; then the constant $\varepsilon_0(P_0, r_0) = 2r_0 \cdot \varepsilon_M$ satisfies the thesis. □

2.6.2 Limit set and elementary groups

When X is a proper and δ-hyperbolic metric space then the *limit set* $\Lambda(\Gamma)$ of a discrete group of isometries Γ is the set of accumulation points of the orbit Γx on $\partial_G X$, where x is any point of X; it is the smallest Γ-invariant closed set of the Gromov boundary (cp. [Coo93], Theorem 5.1).

The set $\Lambda(\Gamma)$ is Γ-invariant, so is its quasiconvex hull. A discrete group of isometries Γ is called *quasiconvex-cocompact* if its action on QC-Hull$(\Lambda(\Gamma))$ is cocompact, i.e. if there exists $D \geq 0$ such that for all $x, y \in$ QC-Hull$(\Lambda(\Gamma))$ it holds $d(gx, y) \leq D$ for some $g \in \Gamma$. The smallest D satisfying this property is called the *codiameter* of Γ. A quasiconvex-cocompact group Γ is said *cocompact* if $\Lambda(\Gamma) = \partial X$ or equivalently QC-Hull$(\Lambda(\Gamma)) = X$.
The radial limit set of a discrete group of isometries Γ is defined as follows. Once fixed $x \in X$ a point $z \in \partial_G X$ is said *σ-radial* if there exists a sequence $\{g_n\}$ of elements of Γ such that $\{g_n x\}$ is unbounded and $d(g_n x, [x,z]) \leq \sigma$ for some geodesic ray $[x,z]$ and for all $n \in \mathbb{N}$. We denote the set of σ-radial points by $\Lambda_{\mathrm{r},\sigma}(\Gamma)$ and the set of radial points by
$$\Lambda_{\mathrm{r}}(\Gamma) = \bigcup_{\sigma \geq 0} \Lambda_{\mathrm{r},\sigma}(\Gamma).$$
The set $\Lambda_{\mathrm{r}}(\Gamma)$ is Γ-invariant, so its closure is $\Lambda(\Gamma)$.

The group Γ is called *elementary* if $\#\Lambda(\Gamma) \leq 2$. For an elementary discrete group Γ there are three possibilities (cp. [Gro87], [CDP90], [DSU17], [BCGS17]):

- Γ is *elliptic*, i.e. $\#\Lambda(\Gamma) = 0$; then $d(x, \Gamma x) < \infty$ for all $x \in X$, so the orbit of Γ is finite by discreteness;
- Γ is *parabolic*, i.e. $\#\Lambda(\Gamma) = 1$; then in this case Γ contains only parabolic or elliptic elements and all the parabolic elements have the same fixed point at infinity;
- Γ is *lineal*, i.e. $\#\Lambda(\Gamma) = 2$; in this case Γ contains only hyperbolic or elliptic elements and all the hyperbolic elements have the same fixed points at infinity.

So if two non-elliptic isometries a, b generate a discrete elementary group then they are either both parabolic or both hyperbolic and they have the same set of fixed points in $\partial_G X$; conversely if a, b are two non-elliptic isometries of X generating a discrete group $\langle a, b\rangle$ such that $\mathrm{Fix}_\partial(a) = \mathrm{Fix}_\partial(b)$, then $\langle a, b\rangle$ is elementary. We also recall the following property of elementary subgroups of general Gromov-hyperbolic spaces:

Lemma 2.6.3. *Let X be a Gromov-hyperbolic space and let Γ be a discrete group of isometries of X. Then for any non-elliptic $g \in \Gamma$ there exists a unique maximal, elementary subgroup of Γ containing g.*

Proof. The maximal, elementary subgroup of Γ containing g is:

$$\{g' \in \Gamma \mid g' \cdot \mathrm{Fix}_\partial(g) = \mathrm{Fix}_\partial(g)\}.$$

□

It is well known that any virtually nilpotent group of isometries Γ of X is elementary (since any non-elementary group Γ contains a free subgroup). Conversely if Γ is elliptic it is virtually nilpotent since it is finite. Also any lineal group is virtually cyclic (cp. Proposition 3.29 of [Cou16]), hence virtually nilpotent. On the other hand there are examples of non-virtually nilpotent, even free non abelian, parabolic groups acting on simply connected Riemannian manifolds with curvature ≤ -1 (see [Bow93], Sec. 6). However under a mild packing assumption it is possible to conclude that any parabolic group is virtually nilpotent. Some version of this fact is probably known to the experts and we present here the proof for completeness; we thank S. Gallot for explaining it to us.

Proposition 2.6.4. *Let X be a proper, geodesic, Gromov hyperbolic space that is P_0-packed at some scale r_0. Then any finitely generated, discrete, parabolic group of isometries Γ of X is virtually nilpotent.*

Hence, we deduce:

Corollary 2.6.5 (Elementary groups are virtually nilpotent)**.**
Let X be a proper, geodesic, Gromov hyperbolic space, P_0-packed at scale r_0. Then a discrete, finitely generated group of isometries of X is elementary if and only if it is virtually nilpotent.

We remark that the scale of the packing is not important: it plays the role of an asymptotic bound on the complexity of the space. The proof is based on the following fundamental result proved in [BGT11]:

Theorem 2.6.6 (Corollary 11.2 of [BGT11]). *For every $p \in \mathbb{N}$ there exists $N(p) \in \mathbb{N}$ such that the following holds for every group Γ and every finite, symmetric generating set S of Γ: if there exists some $A \subseteq \Gamma$ such that $S^{N(p)} \subseteq A$ and $\#(A \cdot A) \leq p \cdot \#A$ then Γ is virtually nilpotent.*

Proof of Proposition 2.6.4. Let S be a finite, symmetric generating set of Γ. Moreover let $\Lambda(\Gamma) = \{z\}$ and let γ be any geodesic ray such that $\gamma^+ = z$. Finally set $\Sigma_R(x) := \{g \in \Gamma \text{ s.t. } d(gx, x) \leq R\}$.

Step 1. Setting $R_0 = \max\{2r_0, 30\delta\}$ and $p = P_0(1+P_0)^{\frac{9R_0}{r_0}-1}$, we have for any $x \in X$:

$$\#(\Sigma_{R_0}(x) \cdot \Sigma_{R_0}(x)) \leq p \cdot \#\Sigma_{R_0}(x)$$

Actually by Lemma 2.4.2 we have

$$\text{Pack}\left(9R_0, \frac{R_0}{2}\right) \leq \text{Pack}(9R_0, r_0) \leq P_0(1+P_0)^{\frac{9R_0}{r_0}-1}.$$

Then it easily follows (cp. Lemma 3.12 of [BCGS17]) that

$$\frac{\#(\Sigma_{R_0}(x) \cdot \Sigma_{R_0}(x))}{\#\Sigma_{R_0}(x)} \leq \frac{\#\Sigma_{2R_0}(x)}{\#\Sigma_{R_0}(x)} \leq P_0(1+P_0)^{\frac{9R_0}{r_0}-1}$$

which is our claim. Remark that p does not depend on the point x.

Step 2. There exists $T = T(S, \gamma, p)$ such that $d(\gamma(T), g\gamma(T)) \leq 30\delta$ for all $g \in S^{N(p)}$ (where $N(p)$ is the value associated to p given by Theorem 2.6.6).

Let $\rho_0 = \max_{s \in S} d(\gamma(0), s\gamma(0))$. So we have $d(\gamma(0), g\gamma(0)) \leq N(p)\rho_0$ for all $g \in S^{N(p)}$. Let $g \in S^{N(p)}$. By definition we have $gz = z$, so $(g\gamma)^+ = z$. Then by Lemma 2.3.2 there exist $t_1, t_2 \geq 0$ such that $t_1 + t_2 = d(\gamma(0), g\gamma(0))$ and $d(\gamma(t+t_1), g\gamma(t+t_2)) \leq 8\delta$ for all $t \geq 0$. Therefore,

$$d(\gamma(s + t_1 - t_2), g\gamma(s)) \leq 8\delta \tag{27}$$

for all $s \geq T := \max\{t_1, t_2\} \leq N(p)\rho_0$. In the following we may assume $t_1 \geq t_2$ and call $\Delta = t_1 - t_2$. If $\Delta \leq 9\delta$, we apply the previous estimate to $s = T$ and we get $d(\gamma(T), g\gamma(T)) \leq 17\delta$, so the claim is true. Otherwise, we have $\Delta > 9\delta$, and we consider the triangle with vertices $A = \gamma(T + \Delta)$, $B = \gamma(T + 2\Delta)$ and $C = gA$. Let $(\bar{A}, \bar{B}, \bar{C})$ be the corresponding tripod with center $\bar{o}$ with edge lengths $\rho = \ell([\bar{A}, \bar{o}])$, $\sigma = \ell([\bar{C}, \bar{o}])$ and $\tau = \ell([\bar{B}, \bar{o}])$. We therefore have:

$$\rho + \tau = \Delta, \quad \sigma + \tau \leq 8\delta \text{ and } \sigma + \rho = d(A, gA).$$

This implies that $\rho - \sigma = (\rho + \tau) - (\sigma + \tau) \geq \Delta - 8\delta \geq 0$. In particular if m is the midpoint of $[A, gA]$ and $\bar{m}$ the corresponding point on the tripod we have $d(\bar{m}, \bar{A}) \leq d(\bar{o}, \bar{A})$, so there exists a point $m' \in [A, B]$ such that $d(m, m') \leq 4\delta$. Applying Lemma 8.21 of [BCGS17] we deduce

$$d(m', gm') \leq d(m, gm) + 8\delta \leq 14\delta$$

(as g is elliptic or parabolic). Moreover, as $m' = \gamma(s + \Delta)$ for some $s \geq T$ (since $\Delta \geq 0$), we have by (27) that $d(m', g\gamma(s)) \leq 8\delta$. By the triangle inequality we deduce

$$\Delta = d(g\gamma(s), gm') \leq 22\delta.$$

Therefore also in this case we get $d(\gamma(T), g\gamma(T)) \leq 30\delta$.

Conclusion. We have $S^{N(p)} \subset \Sigma_{R_0}(\gamma(T))$, where T is the constant of step 2. So we apply Theorem 2.6.6 and conclude that Γ is virtually nilpotent. □

2.7 Ultralimits

An *ultrafilter* on $\mathbb{N}$ is a subset ω of $\mathcal{P}(\mathbb{N})$ such that:

(a) $\emptyset \notin \omega$;

(b) if $A, B \in \omega$ then $A \cap B \in \omega$;

(c) if $A \in \omega$ and $A \subseteq B$ then $B \in \omega$;

(d) for any $A \subseteq \mathbb{N}$ then either $A \in \omega$ or $A^c \in \omega$.

We recall that there is a one-to-one correspondence between the ultrafilters ω on $\mathbb{N}$ and the finitely-additive measures defined on the whole $\mathcal{P}(\mathbb{N})$ with values on $\{0, 1\}$ such that $\omega(\mathbb{N}) = 1$. Indeed given an ultrafilter ω we define the measure $\omega(A) = 1$ if and only if $A \in \omega$; conversely given a measure ω as before we define the ultrafilter as the set $\omega = \{A \subseteq \mathbb{N} \text{ s.t. } \omega(A) = 1\}$ (it is easy to show it actually is an ultrafilter). In the following ω will denote both an ultrafilter and the measure that it defines. Therefore we will write that a property $P(n)$ holds ω-a.s. when the set $\{n \in \mathbb{N} \text{ s.t. } P(n)\} \in \omega$.

There is an easy example of ultrafilter: fix $n \in \mathbb{N}$ and consider the set ω of subsets of $\mathbb{N}$ containing n. An ultrafilter of this type is called *principal*. The interesting ultrafilters are the non-principal ones; it turns out that an ultrafilter is non-principal if and only if it does not contain any finite set. The existence of non-principal ultrafilters follows from Zorn's lemma. The interest on non-principal ultrafilters is due to the fact that they can define a notion of limit of a bounded sequence of real numbers:

Lemma 2.7.1. *Let $(a_n) \subseteq [a, b]$ be a bounded sequence of real numbers. Let ω be a non-principal ultrafilter. Then there exists a unique point x in $[a, b]$ such that for all $\eta > 0$ the set $\{n \in \mathbb{N} \text{ s.t. } |a_n - x| < \eta\}$ belongs to ω. The real number x is said the ω-limit of the sequence (a_n) and it is denoted by $x = \omega\text{-}\lim a_n$. Moreover if a_n and b_n are two bounded sequence of real numbers, it holds:*

(a) $\omega\text{-}\lim(a_n + b_n) = \omega\text{-}\lim a_n + \omega\text{-}\lim b_n$;

(b) if $\lambda \in \mathbb{R}$ *then* $\omega\text{-}\lim(\lambda a_n) = \lambda \cdot \omega\text{-}\lim a_n$;

(c) if $a_n \leq b_n$ *then* $\omega\text{-}\lim a_n \leq \omega\text{-}\lim b_n$;

(d) if $a = \omega\text{-}\lim a_n$ *and* f *is continuous at* a *then* $\omega\text{-}\lim f(a_n) = f(\omega\text{-}\lim a_n)$.

(The proof of the main part can be found in [DK18], Lemma 7.23, while properties (a)-(d) are trivial.)

The ultralimit of unbounded sequences of real numbers can be defined in the following way. Given an unbounded sequence of real numbers a_n the following mutually exclusive situations can occur:

- there exists $L > 0$ such that $a_n \in [-L, L]$ for ω-a.e.(n).
  In this case the ultralimit of (a_n) can be defined using Lemma 2.7.1.
- for any $L > 0$ the set $\{n \in \mathbb{N} \text{ s.t. } a_n \geq L\}$ belongs to ω.
  In this case we set $\omega\text{-}\lim a_n = +\infty$.
- for any $L < 0$ the set $\{n \in \mathbb{N} \text{ s.t. } a_n \leq -L\}$ belongs to ω.
  In this case we set $\omega\text{-}\lim a_n = -\infty$.

We remark that the limit depends strongly on the non-principal ultrafilter ω. The ultralimit of a sequence of metric spaces is defined as follows.

Definition 2.7.2. Let (X_n, x_n) be a sequence of pointed metric spaces and ω be a non-principal ultrafilter. We set:

$$X = \{(y_n) \,:\, y_n \in X_n \text{ and } \exists L > 0 \text{ s.t. } d(y_n, x_n) \leq L \text{ for every } n\}.$$

and, for $(y_n), (z_n) \in X$, we define the distance as:

$$d((y_n), (z_n)) = \omega\text{-}\lim d(y_n, z_n).$$

The space $X_\omega = (X, d)/_{d=0}$ is a metric space and it is called the *ω-limit of the sequence of spaces* (X_n, x_n). The fact that (X, d) is a metric space follows immediately from the properties of the ultralimit of a sequence of real numbers and from the fact that d_n is a distance for any n. In general the limit depends on the non-principal ultrafilter ω and on the basepoints.

A basic example is provided by the ultralimit of a constant sequence.

Proposition 2.7.3. *Let (X, x) be a metric space and ω a non-principal ultrafilter. Consider the constant sequence (X, x) and the corresponding ultralimit (X_ω, x_ω), where x_ω is the constant sequence of points (x). Then*

(a) The map $\iota\colon (X, x) \to (X_\omega, x_\omega)$ that sends y to the constant sequence $(y_n = y)$ is an isometric embedding;

(b) if X is proper then ι is surjective and (X_ω, x_ω) is isometric to (X, x).

Proof. The first part is obvious by the definitions. If X is proper and (y_n) is an admissible sequence defining a point of X_ω then it is contained in a closed ball of X, that is compact. By Lemma 7.23 of [DK18] we find $y \in X$ such that for all $\varepsilon > 0$ the set

$$\{n \in \mathbb{N} \text{ s.t. } d(y, y_n) < \varepsilon\}$$

belongs to ω. Therefore it is clear that the constant sequence $(y_n = y)$ defines the same point as the sequence (y_n) in X_ω, which proves (b). □

An interesting consequence of the definition is that the ultralimit of pointed metric spaces is always complete (the proof is given in [DK18], Proposition 7.44):

Proposition 2.7.4. *Let (X_n, x_n) be a sequence of pointed metric spaces and let ω be a non-principal ultrafilter. Then X_ω is a complete metric space.*

Once defined the limit of pointed metric spaces it is useful to define limit of maps. We take two sequences of pointed metric spaces (X_n, x_n) and (Y_n, y_n). A sequence of maps $f_n\colon X_n \to Y_n$ is said *admissible* if there exists $M \in \mathbb{R}$ such that $d(f_n(x_n), y_n) \leq M$ for any $n \in \mathbb{N}$. In general an admissible sequence of maps does not define a limit map, but it is the case if the maps are equi-Lipschitz. A sequence of maps $f_n\colon X_n \to Y_n$ is equi-Lipschitz if there exists $\lambda \geq 0$ such that f_n is λ-Lipschitz for any n.

Proposition 2.7.5. *Let (X_n, x_n), (Y_n, y_n) be two sequences of pointed metric spaces. Let $f_n\colon X_n \to Y_n$ be an admissible sequence of equi-Lipschitz maps. Let ω be a non-principal ultrafilter. Let X_ω and Y_ω be the ω-limits of (X_n, x_n) and (Y_n, y_n) respectively. Define $f = f_\omega\colon X_\omega \to Y_\omega$ as $f((z_n)) = (f_n(z_n))$. Then:*

(a) f is well defined;

(b) f is Lipschitz with the same constant of the sequence f_n.

In particular if for any n the map f_n is an isometry then f is an isometry, while if f_n is an isometric embedding for any n then f is again an isometric embedding.

The map $f = f_\omega$ is called the *ω-limit of the sequence of maps f_n* and we denote it by $f_\omega = \omega\text{-}\lim f_n$. The proof in case of isometric embeddings is given in [DK18], Lemma 7.47; the general case is analogous.

This result can be applied to the special case of geodesic segments since they are isometric embeddings of an interval into a metric space X. However we first need to explain what is the ultralimit of a sequence of intervals:

Lemma 2.7.6. *Let $I_n = [a_n, b_n] \subset \mathbb{R}$ be a sequence of intervals containing 0 (possibly with $a_n = -\infty$ or $b_n = +\infty$). Let ω be a non-principal ultrafilter. Then $\omega\text{-}\lim(I_n, 0)$ is isometric to I, where $I = [\omega\text{-}\lim a_n, \omega\text{-}\lim b_n] = [a, b]$ (possibly with $a = -\infty$ or $b = +\infty$) contains 0.*

Proof. We define a map from I_ω to I as follows. Given an admissible sequence (x_n) such that $x_n \in I_n$ then x_n is ω-a.s. bounded, so it is defined $\omega\text{-}\lim x_n$ by Lemma 2.7.1. We define the map as $(x_n) \mapsto \omega\text{-}\lim x_n$. It is easy to check it is surjective. Moreover it is an isometry, indeed:

$$|\omega\text{-}\lim x_n - \omega\text{-}\lim y_n| = \omega\text{-}\lim |x_n - y_n| = d((x_n), (y_n)).$$

□

In particular the limit of geodesic segments is a geodesic segment.

Lemma 2.7.7. *Let (X_n, x_n) be a sequence of pointed metric spaces and let ω be a non-principal ultrafilter. Let X_ω be the ultralimit of (X_n, x_n) and let $z = \omega\text{-}\lim z_n$, $w = \omega\text{-}\lim w_n \in X_\omega$. Suppose that for all n there exists a geodesic $\gamma_n\colon [0, d(z_n, w_n)] \to X_n$ joining z_n and w_n: then there exists a geodesic joining z and w in X_ω. In particular if X_n is a geodesic space for all n then the ultralimit X_ω is a geodesic space.*

Proof. We denote by I_n the interval $[0, d(z_n, w_n)]$. Since z and w belongs to X_ω then the distance between them is uniformly bounded. Hence from the previous lemma it follows that the ultralimit of the spaces $(I_n, 0)$ is $I_\omega = [0, \omega\text{-}\lim d(z_n, w_n)] = [0, d(z, w)]$. The maps γ_n define an admissible sequence of isometric embedding, so in particular they define a limit isometric embedding $\gamma_\omega \colon I_\omega \to X$. So γ_ω is a geodesic and clearly $\gamma_\omega(0) = \omega\text{-}\lim \gamma_n(0) = \omega\text{-}\lim z_n = z$ and $\gamma_\omega(d(z, w)) = w$. □

In order to prove stability results for classes of metric spaces we also need to establish the convergence of balls under ultralimits:

Lemma 2.7.8. *Let (X_n, x_n) be a sequence of geodesic metric spaces and ω be a non-principal ultrafilter. Let X_ω be the ultralimit of the sequence (X_n, x_n). Let $y = \omega\text{-}\lim y_n$ be a point of X_ω. Then for any $R \geq 0$ it holds*

$$\overline{B}(y, R) = \omega\text{-}\lim \overline{B}(y_n, R).$$

Proof. First of all $\omega\text{-}\lim \overline{B}(y_n, R) \subseteq \overline{B}(y, R)$. Indeed $z = \omega\text{-}\lim z_n$ belongs to $\omega\text{-}\lim \overline{B}(y_n, R)$ if and only if $d(z_n, y_n) \leq R$ for ω-a.e.(n). Then $d(z, y) \leq R$, i.e. $z \in \overline{B}(y, R)$. The next step is to show that the set $\omega\text{-}\lim \overline{B}(y_n, R)$ is closed. We take a sequence $z^k = \omega\text{-}\lim z_n^k$ of points of $\omega\text{-}\lim \overline{B}(y_n, R)$ that converges to some point $z = \omega\text{-}\lim z_n$ of X_ω. This implies that $d(y, z) \leq R$. We consider a geodesic segment of X_n between y_n and z_n and we denote by w_n the point along this geodesic at distance exactly R from y_n, if it exists. Otherwise $z_n \in \overline{B}(y_n, R)$ and in this case we set $w_n = z_n$. We observe that $w = \omega\text{-}\lim w_n \in \omega\text{-}\lim \overline{B}(y_n, R)$ by definition. We claim that $w = z$. In order to prove that we fix $\varepsilon > 0$. Then for ω-a.e.(n) we have $d(y_n, z_n) < R + \varepsilon$. This implies that $d(y_n, w_n) < \varepsilon$ and so $d(w, z) < \varepsilon$. From the arbitrariness of ε the claim is proved. The last step is to show that the open ball $B(y, R)$ is contained in $\omega\text{-}\lim \overline{B}(y_n, R)$. Indeed given $z = \omega\text{-}\lim z_n \in B(y, R)$ then there exists $\varepsilon > 0$ such that $d(z, y) < R - \varepsilon$. The set of indices n such that $d(z_n, y_n) < d(z, y) + \varepsilon < R$ belongs to ω, hence $z \in \omega\text{-}\lim \overline{B}(y_n, R)$. Since X_ω is geodesic and in any length space the closed ball is the closure of the open ball the proof is concluded. □

In general, even if every space X_n is uniquely geodesic, the ultralimit X_ω may be not uniquely geodesic. This is because, in general, it is not true that all the geodesics of X_ω are limit of sequences of geodesics of X_n. The fact that the geodesics of X_ω are actually limit of geodesics of the spaces X_n is true when all the X_n are $\mathrm{CAT}(\kappa)$. We recall the following fact which is well known (see [BH13] or [DK18] for instance):

Proposition 2.7.9. *Let (X_n, x_n) be a sequence of $\mathrm{CAT}(\kappa)$ pointed metric spaces and ω be a non-principal ultrafilter. Then any geodesic of length $< D_\kappa$ in X_ω is limit of a sequence of geodesics of X_n. As a consequence X_ω is a $\mathrm{CAT}(\kappa)$ metric space.*

The main result of this section is the following stability property for the $\mathrm{CAT}(\kappa)$-radius:

Corollary 2.7.10. *Let (X_n, x_n) be a sequence of complete, locally geodesically complete, locally $\mathrm{CAT}(\kappa)$, geodesic metric spaces with $\rho_{\mathrm{cat}}(X_n) \geq \rho_0 > 0$. Let ω be a non-principal ultrafilter. Then X_ω is a complete, locally geodesically complete, locally $\mathrm{CAT}(\kappa)$, geodesic metric space with $\rho_{\mathrm{cat}}(X_\omega) \geq \rho_0$.*

Proof. Let $y = \omega\text{-}\lim y_n$ be a point of X_ω. For any $r < \rho_0$ and for any n the ball $\overline{B}(y_n, r)$ is a CAT(κ) metric space. Moreover by Lemma 2.7.8 we have that $\overline{B}(y, r)$ is the ultralimit of a sequence of CAT(κ) metric spaces, hence it is CAT(κ) by Proposition 2.7.9. This shows that X_ω is locally CAT(κ) and $\rho_{\text{cat}}(X_\omega) \geq \rho_0$ by the arbitrariness of r. Moreover X_ω is geodesic by Corollary 2.7.7. We fix now a geodesic segment γ of X_ω defined on $[a, b]$. We look at the ball $B(\gamma(a), \rho_0)$, which is CAT(κ), and we take a sequence of points z_n such that $\omega\text{-}\lim z_n = \gamma(a)$. The subsegment of γ inside this ball, defined on $[a, a + \rho_0)$ is the limit of a sequence of geodesics γ_n inside the corresponding balls $B(z_n, \rho_0)$, by Proposition 2.7.9. Each γ_n can be extended to a geodesic segment $\tilde{\gamma}_n$ on the interval $(a - \rho_0, a + \rho_0)$ since each X_n is locally geodesically complete and complete. The ultralimit of the maps $\tilde{\gamma}_n$ is a geodesic segment defined on $[a - \rho_0, a + \rho_0]$ which extends γ. We can do the same around $\gamma(b)$. This proves that X_ω is locally geodesically complete. □

We conclude this section recalling, in the next two propositions, the relations between ultralimits and pointed Gromov-Hausdorff convergence.

Proposition 2.7.11 (see [Jan17])**.** *Let (X_n, x_n) be a sequence of proper, length metric spaces and ω be a non-principal ultrafilter. Then:*

(a) if the ultralimit (X_ω, x_ω) is proper then it is the limit of a convergent subsequence in the pointed Gromov-Hausdorff sense;

(b) reciprocally, if (X_n, x_n) converges to (X, x) in the pointed Gromov-Hausdorff sense then for any non-principal ultrafilter ω the ultralimit X_ω is isometric to (X, x) (we recall that, in this case, (X, x) is proper by definition of Gromov-Hausdorff convergence).

We now explicit the fact that continuity under ultralimits implies continuity under pointed Gromov-Hausdorff convergence.

Proposition 2.7.12. *Let $\mathcal{C}$ be a class of pointed, proper metric spaces and $h\colon \mathcal{C} \to \mathbb{R}$ be a function. Suppose that $\mathcal{C}$ is closed under ultralimits and h is continuous under ultralimits, i.e. for every non-principal ultrafilter ω and every sequence $(X_n, x_n) \in \mathcal{C}$ it holds $h(X_\omega) = \omega\text{-}\lim h(X_n)$. Suppose that $(X_n, x_n) \subseteq \mathcal{C}$ converges in the pointed Gromov-Hausdorff sense to (X_∞, x_∞). Then $X_\infty \in \mathcal{C}$ and $h(X_\infty) = \lim_{n\to+\infty} h(X_n)$.*

We need the following lemma.

Lemma 2.7.13. *Let a_n be a bounded sequence of real numbers. Let a_{n_j} be a subsequence converging to $\tilde{a}$. Then there exists a non-principal ultrafilter ω such that $\omega\text{-}\lim a_n = \tilde{a}$.*

Proof. The set $\{n_j\}_j$ is infinite, then there exists a non-principal ultrafilter ω containing $\{n_j\}_j$ (cp. [Jan17], Lemma 3.2). Moreover for every $\varepsilon > 0$ there exists j_ε such that for all $j \geq j_\varepsilon$ it holds $|a_{n_j} - \tilde{a}| < \varepsilon$. The set of indices where the inequality is true belongs to ω since the complementary is finite. This implies exactly that $\tilde{a} = \omega\text{-}\lim a_n$. □

Proof of Proposition 2.7.12. We fix every non-principal ultrafilter ω. Since the class $\mathcal{C}$ is made of proper metric spaces then X_ω is isometric to X_∞ (Proposition 2.7.11). Therefore $h(X_\infty) = h(X_\omega) = \omega\text{-}\lim h(X_n)$. This implies that $\omega\text{-}\lim h(X_n)$ does not depend on the ultrafilter ω. By the previous lemma we conclude that every converging subsequence of $h(X_n)$ has $h(X_\infty)$ as a limit, i.e. $\liminf_{n\to+\infty} h(X_n) = \limsup_{n\to+\infty} h(X_n) = h(X_\infty)$. □

Chapter 3

Packing and doubling on locally CAT(κ)-spaces

3.1 Uniform lower bounds on volume of balls

We fix a complete, geodesic, GCBA-space X. From (8) & (9) it follows that there exists an upper bound for the measure of any tiny ball $B(x,r)$; moreover one can find a uniform upper bound of the measure of *all* balls, independently of the center x, provided that X satisfies a uniform packing condition at some scale (see Theorem 3.2.1 for a precise statement). It is less clear if there exists a lower bound on the measure, and in particular if this lower bound depends only on some universal constant. Indeed in general the μ_X-volume of balls of a given radius is not uniformly bounded below independently of the space X. For instance consider the balls of radius $\frac{1}{2}$ inside $\mathbb{R}^n$: when n grows the measure of these balls tends to 0. The next theorem shows that if the dimension is bounded from above then there is a uniform bound from below to the measure of balls of a given (sufficiently small) radius:

Theorem 3.1.1. *Let X be a complete, geodesic,* GCBA *metric space. If* $\dim(X) \leq n_0$ *then for any $x \in X$ and any $r < \min\{1, \rho_{\text{ac}}(x)\}$ it holds*

$$\mu_X(\overline{B}(x,r)) \geq c_{n_0} \cdot r^{n_0},$$

where c_{n_0} is a constant only depending on n_0.

The proof of this fact is based on ideas most of which are already present in [LN19]. First of all we have:

Proposition 3.1.2. *Let X be a complete, geodesic,* GCBA *metric space and $x \in X$ be a point of dimension n. Then there exists a 1-Lipschitz, surjective map $P\colon T_xX \to \mathbb{R}^n$ such that:*

(a) $P(O) = 0$;

(b) $P(\overline{B}(O,r)) = \overline{B}(0,r)$ *for any $r > 0$;*

(c) $d_T(V,O) = d_{\mathbb{R}^n}(P(V),0)$ *for any $V \in T_xX$.*

Proof. As the point x has dimension n then the geometric dimension of T_xX is n. This implies that Σ_xX is a space of dimension $n-1$ satisfying the assumptions of Proposition 11.3 of [LN19]. So there exists a 1-Lipschitz surjective map $P'\colon \Sigma_xX \to \mathbb{S}^{n-1}$. We extend the map P' to a map P over the tangent cones by sending the point $V=(v,t)$ to the point $(P'(v),t)$. It is immediate to check that P is surjective and that $P(0)=0$.
Moreover the tangent cone over $\mathbb{S}^{n-1}$ is $\mathbb{R}^n$, as said in Example 2.2.2; therefore the equality $P(\overline{B}(O,R)) = \overline{B}(0,R)$ follows directly from (7). Always by (7) we have $d_T(V,O) = d_{\mathbb{R}^n}(P(V),0)$ for any $V \in C_xX$. Finally the 1-Lipschitz property of P follows from the same property of P' and from the properties of the cosine function. □

Combining this result with the properties of the logarithmic map explained in Section 2.2.2 we deduce the following:

Proposition 3.1.3. *Let X be a complete, geodesic,* GCBA *metric space and $x \in X$ be a point of dimension n. Then there exists a 2-Lipschitz map $\Psi_x\colon B(x,\rho_{\mathrm{ac}}(x)) \to \mathbb{R}^n$ such that*

(a) $\Psi_x(x) = 0$;

(b) $\Psi_x(\overline{B}(x,r)) = \overline{B}(0,r)$ for any $0<r<\rho_{\mathrm{ac}}(x)$;

(c) $d(x,y) = d(0,\Psi_x(y))$ for any $y \in B(x,\rho_{\mathrm{ac}}(x))$.

Proof. Define $\Psi_x = P \circ \log_x$, where P is the map of the previous proposition and $\log_x$ is the logarithmic map at x. Then Ψ satisfies the thesis. □

Using the map Ψ_x we can transport metric and measure properties from $\mathbb{R}^n$ to X. We denote by ω_n the $\mathcal{H}^n$-volume of the ball of radius 1 of $\mathbb{R}^n$.

Corollary 3.1.4. *Let X be a complete, geodesic,* GCBA *metric space and $x \in X$ be a point of dimension n. Then*

$$\mathcal{H}^n(B(x,r)) \geq \frac{1}{2^n}\omega_n r^n$$

for any $0<r<\rho_{\mathrm{ac}}(x)$.

Proof. It follows directly from the properties of the map Ψ_x and the behaviour of the Hausdorff measure under Lipschitz maps. □

Proof of Theorem 3.1.1. We fix $x \in X$, $0 < r < \min\{1, \rho_{\mathrm{ac}}(x)\}$ and $\varepsilon = \frac{r}{2n_0}$. We call d_0 the dimension of x. We look for the biggest ball around x of Hausdorff dimension exactly d_0. In order to do that we define

$$r_1 = \sup\{\rho > 0 \text{ s.t. } \mathrm{HD}(B(x,\rho)) = d_0\}.$$

(where HD denotes the Hausdorff dimension). Notice that $\mathrm{HD}(B(x,\rho))$ is monotone increasing in ρ. If $r_1 \geq r$ we stop and we redefine $r_1 = r$. Otherwise there exists a point x_1 such that $d(x,x_1) \leq r_1 + \varepsilon$ and the dimension of x_1 is $d_1 > d_0$, by definition

of r_1. Now we look for the biggest ball around x_1 of Hausdorff dimension d_1. We define
$$r_2 = \sup\{\rho > 0 \text{ s.t. } \mathrm{HD}(B(x_1, \rho)) = d_1\}.$$
Arguing as before, if $r_1 + \varepsilon + r_2 \geq r$ we stop the algorithm and we redefine r_2 as $r = r_1 + \varepsilon + r_2$. Otherwise we can find again a point x_2 such that $d(x_2, x_1) \leq r_2 + \varepsilon$ and whose dimension is $d_2 > d_1$. We continue the algorithm until $r_1+\varepsilon+\ldots+r_k = r$. It happens in at most n_0 steps. At the end we have points $x = x_0, x_1, \ldots x_k$ with $k \leq n_0$ such that $d(x_i, x_j) \leq r_j + \varepsilon$, $r_1 + \varepsilon + \ldots + r_k = r$ and such that the dimension of x_j is d_j, with $d_i > d_j$ if $i > j$. We observe that the d_j-dimensional parts of the balls $B(x_j, r_j)$, denoted by $B^{d_j}(x_j, r_j)$, are disjoint and contained in $\overline{B}(x, r)$, by construction. Moreover the open ball $B(x_j, r_j)$ has no point of dimension greater than d_j. So

$$\mu_X(\overline{B}(x, r)) = \sum_{k=0}^{n_0} \mathcal{H}^k \llcorner \overline{B}^k(x, r) \geq \sum_j \mathcal{H}^{d_j}(B^{d_j}(x_j, r_j)).$$

The last step is to estimate the last term of the sum. Since $k \leq n_0$ and $r_1+\varepsilon+\ldots+r_k = r$ then $r_1 + \ldots + r_k = r - (k-1)\varepsilon \geq \frac{r}{2}$. Hence there exists an index j such that $r_j \geq \frac{r}{2n_0}$. By definition any point of the ball $B(x_j, r_j)$ is of dimension $\leq d_j$. Hence by the properties of the Hausdorff measure we get

$$\mathcal{H}^{d_j}(B^{d_j}(x_j, r_j)) = \mathcal{H}^{d_j}(B(x_j, r_j)) \geq \frac{1}{2^{d_j}} \omega_{d_j} r_j^{d_j} \geq c_{n_0} r^{n_0},$$

where the first inequality follows directly from the previous corollary and the last one holds since $r \leq 1$. So we can choose

$$c_{n_0} = \left(\frac{1}{4n_0}\right)^{n_0} \min_{k=0,\ldots,n_0} \omega_k$$

that is a constant depending only on n_0. This concludes the proof. □

3.2 Characterization of the packing condition

We are ready to characterize the packing condition in terms of dimension and measure of a GCBA metric space.

Theorem 3.2.1. *Let X be a complete, geodesic,* GCBA$^\kappa$ *metric space with $\rho_{\mathrm{ac}}(X) \geq \rho_0 > 0$. The following facts are equivalent.*

(a) There exist $P_0 > 0$ and $0 < r_0 < \frac{\rho_0}{3}$ such that $\mathrm{Pack}(3r_0, \frac{r_0}{2}) \leq P_0$*;*

(b) There exist $n_0, V_0, R_0 > 0$ such that $\dim(X) \leq n_0$ and $\mu_X(B(x, R_0)) \leq V_0$ for any $x \in X$;

(c) There exists a measure μ on X and there exist two functions $c(r), C(r)$ such that for any $x \in X$ and for any $0 < r < \rho_0$:

$$0 < c(r) \leq \mu(B(x, r)) \leq C(r) < +\infty.$$

Moreover the set of constants $(n_0, V_0, R_0, \rho_0, \kappa)$ can be expressed only in terms of the set of constants $(P_0, r_0, \rho_0, \kappa)$ and viceversa.
Finally if any of the above conditions holds then the natural measure μ_X satisfies condition (c) and X is proper and geodesically complete.

Proof. Assume first that X satisfies $\text{Pack}(3r_0, \frac{r_0}{2}) \leq P_0$. First of all it follows that the dimension of X is bounded. Indeed we fix any point $x \in X$ and we denote by n its dimension. We consider the map $\Psi_x \colon B(x, 2r_0) \to \mathbb{R}^n$ given by Proposition 3.1.3. Let $x_1, \ldots, x_k$ be a $2r_0$-separated subset of $\overline{B}_{\mathbb{R}^n}(0, 2r_0)$. Since Ψ_x is surjective on this ball we can take preimages y_i of x_i under Ψ_x. Moreover $d(y_i, x) = d(\Psi_x(y_i), 0)$, hence $y_i \in \overline{B}(x, 2r_0)$. As Ψ_x is 2-Lipschitz the set $\{y_1, \ldots, y_k\}$ is a r_0-separated subset of $\overline{B}(x, 2r_0)$. Then

$$k \leq \text{Pack}\left(2r_0, \frac{r_0}{2}\right) \leq \text{Pack}\left(3r_0, \frac{r_0}{2}\right) \leq P_0$$

by Theorem 2.4.7. But it is easy to show that $k \geq 2n$. Therefore $2n \leq P_0$ is the bound on the dimension we were looking for. We observe that this bound is expressed only in terms of P_0. We fix now $x \in X$ and any $R > 0$. Let $r = \min\{1, R, \frac{1}{10}r_0, \frac{1}{100}D_\kappa\}$. We take a covering of $\overline{B}(x, R)$ with balls of radius r. By Theorem 2.4.7 it is possible to do that with k balls, where k can be estimated in the following way:

$$k = \text{Cov}(\overline{B}(x, R), r) \leq \text{Pack}\left(\overline{B}(x, R), \frac{r}{2}\right) \leq P_0(1 + P_0)^{\frac{2R}{r} - 1}.$$

We call $y_1, \ldots, y_k$ the centers of these balls. By Theorem 2.4.7 the space X is proper, then from the choice of r we get that $B(y_i, r)$ is a tiny ball for any i, as follows from (6). Moreover the maximal number of r-separated points inside $\overline{B}(y_i, 10r)$ is bounded by $\text{Pack}(10r, \frac{r}{2}) \leq P_0(1 + P_0)^{19}$, as follows again by Theorem 2.4.7. Hence by (8) we have

$$\mathcal{H}^j(\overline{B}(y_i, r)^j) \leq C(P_0) r^j,$$

where $C(P_0)$ is a constant depending only on P_0. Therefore, using the fact that the dimension of X is bounded above by $n_0 = \frac{P_0}{2}$ and $r \leq 1$, we get:

$$\mu_X(\overline{B}(y_i, r)) = \sum_{j=0}^{n_0} \mathcal{H}^j(\overline{B}(y_i, r)^j) \leq \frac{P_0}{2} \cdot C(P_0)$$

for any i. Finally

$$\mu_X(\overline{B}(x, R)) \leq P_0(1 + P_0)^{\frac{2R}{r} - 1} \cdot \frac{P_0}{2} \cdot C(P_0) = V(P_0, r_0, R, \kappa). \tag{28}$$

This shows that for any $x \in X$ and any R_0 we can find the desired uniform bound on the volume of the ball $B(x, R_0)$. This ends the proof of the implication (a) $\Rightarrow$ (b). Moreover this part of the proof, together with Theorem 3.1.1, shows that if (a) holds then the measure μ_X is a measure that satisfies condition (c) of the theorem. Assume now that X has dimension bounded above by n_0 and that the volume of the balls of radius R_0 are uniformly bounded above by V_0. We set $r_0 = \min\{\frac{R_0}{6}, 1, \frac{\rho_0}{6}\}$. The claim is that X satisfies $\text{Pack}(3r_0, \frac{r_0}{2}) \leq P_0$ for some P_0 depending only on

V_0, R_0, n_0 and ρ_0. We consider the ball of radius $\frac{R_0}{2}$ centered at a point $x \in X$. We take a r_0-separated subset of $\overline{B}\left(x, \frac{R_0}{2}\right)$ and we suppose its cardinality is bigger than some k. It means that there are k points $y_1, \ldots, y_k \in \overline{B}\left(x, \frac{R_0}{2}\right)$ such that $d(y_i, y_j) > r_0$ for any $i \neq j$. Hence the balls centered at y_i of radius $\frac{r_0}{2}$ are pairwise disjoint and satisfy $\overline{B}\left(y_i, \frac{r_0}{2}\right) \subseteq \overline{B}\left(x, \frac{R_0}{2} + \frac{r_0}{2}\right) \subset B(x, R_0)$, since $\frac{R_0}{2} + \frac{r_0}{2} \leq \frac{R_0}{2} + \frac{R_0}{6} < R_0$. We can apply Theorem 3.1.1 to get $\mu_X\left(\overline{B}(y_i, \frac{r_0}{2})\right) \geq c_{n_0}\left(\frac{r_0}{2}\right)^{n_0}$ for any i. Thus

$$V_0 \geq \mu_X(B(x, R_0)) \geq \sum_{i=1}^{k} \mu_X\left(\overline{B}\left(y_i, \frac{r_0}{2}\right)\right) \geq k \cdot c_{n_0}\left(\frac{r_0}{2}\right)^{n_0},$$

then

$$k \leq \frac{2^{n_0} V_0}{c_{n_0} r_0^{n_0}} = \frac{2^{n_0} V_0}{c_{n_0}} \cdot \max\left\{1, \left(\frac{6}{\rho_0}\right)^{n_0}, \left(\frac{6}{R_0}\right)^{n_0}\right\} = P_0.$$

It means that $\text{Pack}(\overline{B}(x, \frac{R_0}{2}), \frac{r_0}{2}) \leq P_0$. Since $R_0 \geq 6r_0$ we can conclude that $\text{Pack}(3r_0, \frac{r_0}{2}) \leq P_0$ that is what claimed.
Finally assume that there exists a measure μ such that for any $x \in X$ and for any $0 < r < \rho_0$ it holds

$$0 < c(r) \leq \mu(B(x, r)) \leq C(r) < +\infty.$$

We take any $r_0 < \frac{\rho_0}{3}$ and we fix any point $x \in X$. Let k be the maximal cardinality of a r_0-separated subset of $\overline{B}(x, 3r_0)$. Then, arguing as before, we can find k disjoint balls of radius $\frac{r_0}{2}$ contained in $B(x, 4r_0)$. Since $C(4r_0) \geq \mu(B(x, 4r_0)) \geq k \cdot c(\frac{r_0}{2})$ then $k \leq \frac{C(4r_0)}{c(\frac{r_0}{2})} = P_0$. This shows that X satisfies (a) with these choices of r_0 and P_0. □

3.3 Characterization of pure-dimensional spaces

In this section X will be a complete, geodesic GCBA-space. We say that X is *purely n-dimensional* if $\dim(x) = n$ for any $x \in X$. Moreover we say that a measure μ on X is:

- *D-doubling up to scale t at $x \in X$* if there exists a constant $D > 0$ such that for any $0 < t' \leq t$ it holds
$$\frac{\mu(B(x, 2t'))}{\mu(B(x, t'))} \leq D;$$

- *D-doubling up to scale t* if it is D-doubling up to scale t at any point $x \in X$ (for a uniform doubling constant D).

When uniformity of the constant and of the scale is not an issue we will simply say that μ is *locally doubling* on X: that is if for any $x \in X$ there exist $t_x > 0$ and $D_x > 0$ such that μ is D_x-doubling up to scale t_x at any point of $B(x, t_x)$.

Remark 3.3.1. Notice that any metric measure space (X, d, μ) satisfying a D_0-doubling condition up to scale t_0 is P_0-packed at scale $r_0 = \frac{t_0}{4}$ for $P_0 = D_0^4$ (provided that the measure gives positive mass to the balls of positive radius). Actually let $x \in X$ and take any r_0-separated subset $\{y_1, \ldots, y_k\}$ of $\overline{B}(x, 3r_0)$. So the balls $B(y_i, \frac{r_0}{2})$ are pairwise disjoint. From the doubling property we get:

$$\mu_X\left(B(x, 3r_0)\right) \geq \sum_{i=1}^{k} \mu\left(B(y_i, r_0/2)\right) \geq \sum_{i=1}^{k} \frac{1}{D_0^4} \mu(B(y_i, 8r_0))$$

and since $B(y_i, 8r_0) \supseteq B(x, 3r_0)$ we deduce that $k \leq D_0^4$.

The next result characterizes GCBA-spaces whose natural measure is locally doubling:

Theorem 3.3.2. *Let X be a proper, geodesic,* GCBA *metric space. Suppose μ_X is locally doubling: then X is purely n-dimensional for some n.*

We begin the proof of Theorem 3.3.2 with the following two preliminary results.

Lemma 3.3.3. *Let X be a proper, geodesic,* GCBA *metric space and $x \in X$. Let $v \in \Sigma_x X$ and assume that every point of $\overline{B}((v, 1), \varepsilon)$ is a k-regular point of $T_x X$, for some $\varepsilon > 0$. Then there exists $r > 0$ such that all points of the set*

$$A_{v,\varepsilon}(r) = \{y \in X \text{ s.t. } d_T(\log_x(y), (v, d(x, y))) \leq \varepsilon d(x, y)\} \cap B(x, r)$$

have dimension k.

We recall that, since $T_x X$ is a GCBA-space and since the set of k-regular points is open in $T_x X$, if $(v, 1)$ is k-regular point in $T_x X$ then it is always possible to find ε satisfying the assumptions of the lemma. The set $A_{v,\varepsilon}$, or better its projection on the tangent cone through the logarithm map at x, should be thought as a part of angular sector around v. So the statement of the lemma says that any possible geodesic segment starting at x with direction close to v stay in the k-dimensional part of X for a uniform time r.

Proof. Suppose the thesis is false. Then there exists a sequence of points y_n of dimension different from k at distance $r_n \to 0$ from x such that

$$d_T(\log_x(y_n), (v, r_n)) \leq \varepsilon r_n.$$

We consider rescaled tiny balls $Y_n = \frac{1}{r_n} B(x, r_0)$ as in Lemma 2.2.5, together with the approximating maps f_n; so for all n we have:

$$d_T(f_n(y_n), (v, 1)) \leq \varepsilon.$$

Moreover we are in the standard setting of convergence. Indeed the GCBA-space X is geodesic and complete, so the contraction maps φ_r^R are well-defined for any $R < \rho_{\text{cat}}(x)$ and they are surjective and $\frac{2r}{R}$-Lipschitz; therefore, by applying the same proof as in Lemma 2.4.9, we conclude that the rescaled balls are uniformly packed (the other properties follow from the discussion in Section 2.2). Moreover the sequence $y_n \in Y_n$ converges to some point $y_\infty \in \overline{B}((v, 1), \varepsilon)$. So y_∞ is k-regular by assumption. But, by Lemma 2.2.8, the points y_n must be k-dimensional for n large enough, which is a contradiction. □

Lemma 3.3.4. *Let $v \in \Sigma_x X$ and let γ be a geodesic starting at x defining v. For any $0 < \varepsilon < 1$ we have for all $r > 0$ small enough:*

$$B\left(\gamma\left(\frac{r}{2}\right), \frac{\varepsilon r}{8}\right) \subset A_{v,\varepsilon}(r)$$

Proof. As the logarithm map is 2-Lipschitz, for every $y \in B\left(\gamma\left(\frac{r}{2}\right), \frac{\varepsilon r}{8}\right)$ we have

$$\begin{aligned} d_T(\log_x(y), (v, d(x,y))) &\leq d_T\left(\log_x(y), \log_x\left(\gamma\left(\frac{r}{2}\right)\right)\right) + d_T\left(\left(v, \frac{r}{2}\right), (v, d(x,y))\right) \\ &\leq 2d\left(y, \gamma\left(\frac{r}{2}\right)\right) + \left|\frac{r}{2} - d(x,y)\right| \\ &\leq 3d\left(y, \gamma\left(\frac{r}{2}\right)\right) \leq \frac{3\varepsilon r}{8} \leq \varepsilon d(x,y) \end{aligned}$$

since $d(x,y) \geq \frac{r}{2} - \frac{\varepsilon r}{8}$. On the other hand if $y \in B(\gamma(\frac{r}{2}), \frac{\varepsilon r}{8})$ we have $d(x,y) \leq \frac{r}{2} + \frac{\varepsilon r}{8} < r$, so the ball $B(\gamma(\frac{r}{2}), \frac{\varepsilon r}{8})$ is included in $A_{v,\varepsilon}(r)$. □

Proof of Theorem 3.3.2. Let us suppose X is not pure dimensional. We take a point $x_0 \in X$ of minimal dimension d_0. Then we have by assumption

$$r_0 = \sup\{\rho > 0 \text{ s.t. } \mathrm{HD}(B(x_0, \rho)) = d_0\} < +\infty.$$

We can find a point $x \in X$ with dimension $d > d_0$ such that $d(x_0, x) = r_0$. Indeed for any n we can find a point x_n such that $d(x_0, x_n) < r_0 + \frac{1}{n}$ and $\dim(x_n) > d_0$. The sequence of points x_n converge, as the space is proper, to a point x at distance exactly r_0 from x_0. Assume that $\dim(x) = d_0$: then there would exist a small radius ρ such that the Hausdorff dimensions of $B(x, \rho)$ is exactly d_0. But x_n belongs to $B(x, \rho)$ for $n \gg 0$, and any open ball around x_n has Hausdorff dimension strictly greater than d_0; therefore $\mathrm{HD}(B(x, \rho)) > d_0$, a contradiction.
Now, the tangent cone $T_x X$ at x has dimension d. Hence there exists a point $v \in \Sigma_x X$ and $\varepsilon > 0$ such that any point of the ball $\overline{B}((v,1), \varepsilon)$ is regular and of dimension d. We take any geodesic γ starting at x and defining v and we set $y_r = \gamma(\frac{r}{2})$. Applying the two lemmas above we have that, for all r small enough, any point of the ball $B(y_r, \frac{\varepsilon r}{8})$ is d-dimensional. Since X satisfies a doubling condition around x we know by Remark 3.3.1 that a ball $B(x, r_0)$ is P_0-packed, for some r_0, P_0 depending on x. So, by Theorem 2.4.7 and by the properties of the natural measure recalled in Section 2.2.3, there exists a constant C, only depending on r_0 and P_0, such that for all sufficiently small r we have:

$$\mu_X\left(B\left(y_r, \frac{\varepsilon r}{8}\right)\right) \leq C \cdot \left(\frac{\varepsilon r}{8}\right)^d.$$

Consider now the ball $B(y_r, r)$: notice that there exists a ball of radius at least $\frac{r}{4}$ contained in $B(y_r, r) \cap B(x_0, r_0)$, so made only of d_0-dimensional points. In particular by Corollary (3.1.4) we have $\mu_X(B(y_r, r)) \geq c_{d_0}(\frac{r}{4})^{d_0}$, where c_{d_0} is a constant depending only on d_0. Thus

$$\frac{\mu_X(B(y_r, r))}{\mu_X(B(y_r, \frac{\varepsilon r}{4}))} \geq C' r^{d_0 - d},$$

where C' is a constant that does not depend on r. Since this is true for any r small enough and $d_0 < d$, this inequality contradicts the doubling assumption at y_r when r goes to 0. □

As a consequence we obtain the following:

Corollary 3.3.5. *Let X be a complete, geodesic,* GCBA$^\kappa$ *metric space with $\rho_{\text{ac}}(X) \geq \rho_0 > 0$. The following facts are equivalent:*

(a) there exist $D_0 > 0$ and $t_0 > 0$ such that the natural measure μ_X is D_0-doubling up to scale t_0;

(b) X is purely dimensional and there exist $P_0 > 0$ and $0 < r_0 < \rho_0/3$ such that $\text{Pack}(3r_0, \frac{r_0}{2}) \leq P_0$*;*

(c) there exist $n_0, V_0, R_0 > 0$ such that X is purely n_0-dimensional and $\mu_X(B(x, R_0)) \leq V_0$ for any $x \in X$.

Moreover each of the three sets of constants $(D_0, t_0, \rho_0, \kappa)$, $(P_0, r_0, \rho_0, \kappa)$, $(n_0, V_0, R_0, \rho_0, \kappa)$ can be expressed in terms of the others.
Finally if the conditions hold then X is proper and geodesically complete.

Proof of Corollary 3.3.5. The implication (a) ⇒ (b) follows from Theorem 3.3.2 and from Remark 3.3.1together with Theorem 2.4.7.
Assume now X purely n-dimensional and $\text{Pack}(3r_0, \frac{r_0}{2}) \leq P_0$. We recall that by Theorem 3.2.1 n can be bounded from above in terms of P_0. We fix $t_0 < \min\{1, R, \frac{1}{10}r_0, \frac{1}{100}D_\kappa\}$ as in the proof of Theorem 3.2.1. By Theorem 2.4.7 we know X is proper, so it is easy to check that $\rho_{\text{cat}}(X) \geq t_0$ by (6). Therefore by Theorem 3.1.1 we have

$$\mu_X(B(x,t)) \geq c_n t^n = c(P_0)t^n$$

for any $t \leq t_0$. Moreover, by the same estimate used in the proof of Theorem 3.2.1, and using the fact that μ_X is just the n-dimensional Hausdorff measure, we get

$$\mu_X(B(x, 2t)) \leq P_0(1 + P_0)^3 \cdot \frac{P_0}{2} \cdot C(P_0)t^n$$

for any $t \leq t_0$. Hence

$$\frac{\mu_X(B(x, 2t))}{\mu_X(B(x,t))} \leq \frac{P_0(1 + P_0)^3 \cdot \frac{P_0}{2} \cdot C(P_0)}{c(P_0)} = D_0$$

which shows the implication (b) ⇒ (a).
The equivalence between (b) and (c) is proved in Theorem 3.2.1. □

Finally the doubling condition also implies the uniform continuity of the natural measure of annuli:

Lemma 3.3.6. *Let X be a complete, geodesic,* GCBA$^\kappa$ *metric space which is D_0-doubling up to scale t_0 and satisfies $\rho_{\text{ac}}(X) \geq \rho_0$. There exists $\beta > 0$, only depending on D_0, such that for every $R > 0$ and for every positive $\varepsilon < \min\left\{\frac{t_0}{24R}, \frac{1}{9}\right\}$ it holds:*

$$\mu_X(A(x, R, (1-\varepsilon)R)) \leq \left(\max\left\{\frac{24R}{t_0}, 9\right\}\right)^\beta \cdot \varepsilon^\beta \cdot \mu_X(B(x, R)).$$

Proof. The proof is exactly the same as in Proposition 11.5.3 of [HKST15], with a minor modification due to the fact that we assume that μ_X is doubling only up to scale t_0. Actually, arguing as in the first part of the proof of Proposition 11.5.3 of [HKST15], one deduces that

$$\mu_X(A(x,R,R-t)) \leq D_0^4 \cdot \mu_X(A(x,R-t,R-3t)) \tag{29}$$

for all $x \in X$ and all positive $t \leq \min\left\{\frac{t_0}{8}, \frac{R}{3}\right\} =: t_R$. From (29) we deduce that for all $t \leq t_R$ it holds

$$\mu_X\left(A(x,R,R-t)\right) \leq D_0^4\Big(\mu_X(B(x,R)) - \mu_X\left(A(x,R,R-t)\right)\Big)$$

hence

$$\mu_X(A(x,R,R-t)) \leq \left(\frac{D_0^4}{1+D_0^4}\right) \cdot \mu_X(B(x,R))$$

Setting $t_m = \frac{1}{2\cdot 3^m}$ one then shows by induction as in [HKST15] that

$$\mu_X\Big(A(x,R,(1-t_m)R)\Big) \leq \left(\frac{D_0^4}{1+D_0^4}\right)^{m+1-m_0} \cdot \mu_X(B(x,R))$$

for all $m \geq m_0 = \left\lceil \log_3(\frac{R}{2t_R})\right\rceil$. Our claim then follows for $\varepsilon \leq \min\left\{\frac{t_0}{24R}, \frac{1}{9}\right\}$ choosing $\beta = \log_3\left(\frac{1+D_0^4}{D_0^4}\right)$. Indeed for every such ε we choose the unique integer $m \geq m_0$ such that $t_{m+1} \leq \varepsilon \leq t_m$. Therefore we have

$$\begin{aligned}\mu_X(A(x,R,(1-\varepsilon)R)) &\leq \mu_X(A(x,R,(1-t_m)R)) \\ &\leq \left(\frac{D_0^4}{1+D_0^4}\right)^{m+1-m_0} \cdot \mu_X(B(x,R)).\end{aligned}$$

Using the fact that $m+1 \geq -\log_3 2\varepsilon$ we get

$$\mu_X(A(x,R,(1-\varepsilon)R)) \leq (2\cdot 3^{m_0})^\beta \cdot \varepsilon^\beta \cdot \mu_X(B(x,R)).$$

Since $m_0 \leq \log_3\left(\frac{R}{2t_R}\right) + 1$ the thesis follows. □

As a consequence we deduce that for D-doubling GCBA-spaces the measure of balls is continuous under the Gromov-Hausdorff convergence, which sharpens Lemma 2.2.7:

Corollary 3.3.7. *Let X_n be a sequence of complete, geodesic,* GCBA$^\kappa$ *metric spaces which are D_0-doubling up to scale t_0 and satisfying $\rho_{\mathrm{ac}}(X_n) \geq \rho_0$. Assume that X_n converge in the pointed Gromov-Hausdorff sense to some* GCBA*-space X and let $x_n \in X_n$ be a sequence of points converging to $x \in X$. Then for any $R \geq 0$ it holds*

$$\mu_X(B(x,R)) = \lim_{n\to+\infty} \mu_{X_n}(B(x_n,R)).$$

Proof. By Remark 3.3.1 and Theorem 2.4.7 the spaces X_n are P_0-packed at some scale $r_0 \leq \rho_0/3$ for P_0, r_0 only depending on D_0, t_0, ρ_0 and κ. By Theorem 3.2.1, precisely by (28), the balls of radius R in X_n have uniformly bounded volume, that is

$$\mu_{X_n}(B(x_n, R)) \leq C(R)$$

for a universal function $C(R)$ only depending on D_0, t_0, ρ_0 and R. By the above Corollary for all $R > 0$ and $\varepsilon > 0$ there exists $\delta > 0$, depending only on D_0, t_0 and R such that for any $x_n \in X_n$ it holds $\mu_{X_n}(A(x_n, R+\delta, R)) \leq \varepsilon$. The proof then follows directly from (10). □

3.4 Examples

The first examples of GCBA$^\kappa$ spaces is given by complete Riemannian manifolds with sectional curvature bounded above by κ. Observe that if the sectional curvature is also bounded below then the manifold is P_0-packed at scale r_0, for some P_0, r_0 depending only on the lower bounds and on the dimension of the manifold. In the following we will introduce a class of non-manifold GCBA$^\kappa$ metric spaces satisfying a uniform packing condition.

3.4.1 M^κ-complexes

An important class of GCBA$^\kappa$ spaces is provided by M^κ-complexes with *bounded geometry*, in a sense we are going to explain. First of all we recall briefly the definitions and the properties of the class of simplicial complexes we are interested in. A *κ-simplex S* is the convex set generated by $n+1$ points $v_0, \ldots, v_n$ of M^κ_n in general position, where M^κ_n is the unique n-dimensional space-form with constant sectional curvature κ. If $\kappa > 0$ the points $v_0, \ldots, v_n$ are required to belong to an open emisphere. We say that S has dimension n. Each v_i is called a *vertex*. A *d-dimensional face T* of S is the convex hull of a subset $\{v_{i_0}, \ldots, v_{i_d}\}$ of $(d+1)$ vertices. The *interior* of S, denoted $\dot{S}$, is defined as S minus the union of its lower dimensional faces; the *boundary* ∂S is the union of its codimension 1 faces.
Let Λ be any set and $E = \bigsqcup_{\lambda \in \Lambda} S_\lambda$, where any S_λ is a κ-simplex. Let $\sim$ be an equivalence relation on E satisfying:

(i) for any $\lambda \in \Lambda$ the projection map $p\colon S_\lambda \to E/_\sim$ is injective;

(ii) for any $\lambda, \lambda' \in \Lambda$ such that $p(S_\lambda) \cap p(S_{\lambda'}) \neq \emptyset$ there exists an isometry $h_{\lambda,\lambda'}$ from a face $T \subseteq S_\lambda$ onto a face $T' \subseteq S_{\lambda'}$ such that $p(x) = p(x')$, for $x \in S_\lambda$ and $x' \in S_{\lambda'}$, if and only if $x' = h_{\lambda,\lambda'}(x)$.

The quotient space $K = E/_\sim$ is called a M^κ-simplicial complex or simply *M^κ-complex*; the set E is the *total space*. A subset $S \subseteq K$ is called a m-simplex of K if it is the image under p of a m-dimensional face of some S_λ; its *interior* and its *boundary* are, respectively, the image under p of the interior and the boundary of S_λ. The *support of a point* $x \in K$, denoted supp(x), is the unique simplex containing x in its interior (notice that supp$(v) = v$ when v is a vertex). The *open star* around a vertex v is the union of the interior of all simplices having v as a vertex.

Metrically K is equipped with the quotient pseudometric. By Lemma I.7.5 of [BH13] the pseudometric can be expressed using strings. A *m-string* in K from x to y is a sequence $\Sigma = (x_0, \ldots, x_m)$ of points of K such that $x = x_0$, $y = x_m$ and for each $i = 0, \ldots, m-1$ there exists a simplex S_i containing x_i and x_{i+1}. Moreover a m-string $\Sigma = (x_0, \ldots, x_m)$ from x to y is *taut* if

- there is no simplex containing $\{x_{i-1}, x_i, x_{i+1}\}$;
- if $x_{i-1}, x_i \in S_i$ and $x_i, x_{i+1} \in S_{i+1}$ then the concatenation of the segments $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$ is geodesic in the subcomplex $S_i \cup S_{i+1}$.

The *length* of Σ is defined as:

$$\ell(\Sigma) = \sum_{i=0}^{m-1} d_{S_i}(x_i, x_{i+1})$$

where d_S denotes the standard M^κ-metric on a geodesic simplex S of M^κ. Then any string can be identified to a path in K and the natural quotient pseudometric on K coincides with the following ([BH13], Lemma I.7.21):

$$d_K(x, y) = \inf\{\ell(\Sigma) \text{ s.t. } \Sigma \text{ is a taut string from } x \text{ to } y\}.$$

Moreover for any $x \in K$ one can define the number

$$\varepsilon(x) = \inf_{\substack{S \text{ simplex of } K \\ x \in S}} \left(\inf_{\substack{T \text{ face of } S \\ x \notin T}} d_S(x, T) \right) \tag{30}$$

which has the following fundamental property:

Lemma 3.4.1 (Lemma I.7.9 and Corollary I.7.10 of [BH13]).
If $\varepsilon(x) > 0$ for any x and K is connected then d_K is a metric and (K, d_K) is a length space. Moreover if $y \in K$ satisfies $d_K(x, y) < \varepsilon(x)$ then any simplex S containing y contains also x and $d_K(x, y) = d_S(x, y)$.

We remark that a M^κ-complex can be non-locally compact, even when the quotient pseudometric on it is a metric.
For any vertex $v \in K$ it is possible to define the *link* $\mathrm{Lk}(v, K)$ of K at v as follows. We fix any $\lambda \in \Lambda$ such that $v = p(v_\lambda)$, where v_λ is a vertex of S_λ. The set of unit vectors w of $T_{v_\lambda} M^\kappa_n$ such that the geodesic starting in direction w stays inside S_λ for a small time is a geodesic simplex of $M^1_{n-1} = \mathbb{S}^{n-1}$, denoted $\mathrm{Lk}(v_\lambda, S_\lambda)$. Consider the equivalence relation on the disjoint union $\bigsqcup_{p(S_\lambda)\ni v} S_\lambda$ given by $w_\lambda \sim w_{\lambda'}$ if and only if $p(S_\lambda) \cap p(S_{\lambda'}) \neq \emptyset$ and $(dh_{\lambda,\lambda'})_{v_\lambda}(w_\lambda) = w'_\lambda$: the link $\mathrm{Lk}(v, K)$ is the quotient space under this equivalence relation. It is clearly a M^1-complex.

We introduce now the class of simplicial complexes we are interested in. We say that K has *valency* at most N if for all $v \in K$ the number of simplices having v as a vertex is bounded above by N. Notice that if the valency is at most N then the maximal dimension of a simplex of K is at most N too. We say that a simplex S has *size* bounded by $R > 0$ if it contains a ball of radius $\frac{1}{R}$ and it is contained in a ball of radius R; accordingly we say the simplicial complex K has size bounded by R if all the simplices S_λ defining K have size bounded by R. The bound on the size avoids to have too thin simplices: it should be thought as a quantitative non-collapsing condition, as follows from the next results. Indeed the first one affirms that a bound on the size of a simplex gives uniform bounds on the size of any of its faces: for example its 1-dimensional faces are not too short (and not too long).

Lemma 3.4.2. *Let S be a M^κ-simplex of dimension n and size bounded by R. Then any face of S of dimension d has size bounded by $2^{n-d}R$.*

Proof. We prove the lemma by induction on the dimension n. If $n = 0, 1$ there is nothing to prove. Assume now that the bounds hold for all faces of M^κ-simplices of dimension $\leq n-1$ and consider a n-dimensional M^κ-simplex $S = \text{Conv}(v_0, \ldots, v_n)$ of size bounded by R. Let $S' = \text{Conv}(v_0, \ldots, v_{n-1})$ be the face of S opposite to v_n and identify M^κ_{n-1} with the κ-model space containing S'. It is clear that S' is contained in a ball $B_{M^\kappa_{n-1}}(x, 2R)$ of M^κ_{n-1}. On the other hand let $B_{M^\kappa_n}(x, \frac{1}{R})$ be the ball of M^κ_n which is contained in S. Call $\psi : S \to S'$ the map sending every point z of S to the intersection of the extension of the geodesic $[v_n, z]$ after z with S' and let $y = \psi(x)$; moreover let φ be the contraction map centered at v_n sending y to x. Notice that $\psi \circ \varphi(z) = z$ for all $z \in S'$. The map φ is 2-Lipschitz, so any point of $B_{M^\kappa_n}(y, \frac{1}{2R})$ is sent to $B_{M^\kappa_n}(x, \frac{1}{R})$ under φ. Therefore

$$B_{M^\kappa_{n-1}}\left(y, \tfrac{1}{2R}\right) = B\left(y, \tfrac{1}{2R}\right) \cap M^\kappa_{n-1} \subseteq \psi\left(B_{M^\kappa_n}(x, \tfrac{1}{R})\right) \subseteq S'$$

which proves the induction step. □

In the second result we prove the non-collapsing property: limit of n-dimensional simplices with uniform bound on the size is again n-dimensional (and satisfies the same bound on the size).

Proposition 3.4.3. *The class of n-dimensional M^κ-simplices of size bounded by R and having a fixed point o as a vertex is compact under the Hausdorff distance on M^κ_n. Moreover, under this convergence, any face of the limit space is limit of faces of the simplices in the sequence. Finally the same class is closed under ultralimits.*

Proof. We take a sequence of simplices S_l as in the assumption. We denote by $v_0^l = o, v_1^l \ldots, v_n^l$ the vertices of S_l. All the sequences (v_i^l) are contained in a compact subset of M^κ_n, so up to subsequence they converge to v_i for all $i = 0, \ldots, n$, in particular $v_0 = o$. Then the ε-neighbourhood $\text{Conv}(v_0, \ldots, v_n)_\varepsilon$ of $\text{Conv}(v_0, \ldots, v_n)$ is a convex subset of M^κ_n which definitely contains $v_0^l = o, v_1^l \ldots, v_n^l$, hence

$$\text{Conv}(v_0, \ldots, v_n)_\varepsilon \supseteq \text{Conv}(v_0^l = o, v_1^l \ldots, v_n^l).$$

Analogously $\text{Conv}(v_0, \ldots, v_n) \subseteq \text{Conv}(v_0^l = o, v_1^l \ldots, v_n^l)_\varepsilon$ definitely, hence $\text{Conv}(v_0, \ldots, v_n) \to \text{Conv}(v_0^l = o, v_1^l \ldots, v_n^l)$ for the Hausdorff distance. Similarly any face of S is limit of corresponding faces of S_l. We now claim that $v_0, \ldots, v_n$ are in general position. If not then there are three vertices, say v_0, v_1, v_2, belonging to the same 1-dimensional space. This means the faces $\text{Conv}(v_0^l, v_1^l, v_2^l)$ tend to a 1-dimensional face, therefore thay cannot have size bounded below uniformly, which contradicts Lemma 3.4.2. Therefore S is a n-dimensional simplex. Moreover it is clear it is contained in a ball of radius R and it contains a ball of radius $\frac{1}{R}$. Fix now any non-principal ultrafilter ω and a sequence S_l as above. Each S_l is proper and the sequence converges in the Gromov-Hausdorff sense to the proper space S. Then by Proposition 2.7.11 we get that the ultralimit S_ω is isometric to S. □

Clearly the same conclusion holds for the class of simplices of dimension at most n and size bounded by R since it is the finite union of compact classes.
Our aim is to use the compactness of this class of simplices to show uniform packing estimates for a M^κ-complex with bounded size and bounded valency.
In the following we will state the equivalent versions, in our setting, of a series of well-known results for M^κ-complexes with finite shapes proved in [BH13], where a M^κ-complex is of finite shape if the isometries classes of its simplices are finite. The original proofs are based on the finiteness of the class of simplices, while our proofs will be based on the compactness of the class of simplices we are considering: in this sense we can think our results as a generalization of the original ones.

Lemma 3.4.4. *Let K be a M^κ-complex of size bounded by R and $\dim(K) \leq n$. Then there exists a constant $\varepsilon_0(R,n) > 0$ depending only on R and n such that for all vertices v, w of K it holds $\varepsilon(v) > \varepsilon_0(R,n)$ and $d_K(v,w) \geq \varepsilon_0(R,n)$.*

Proof. The class of simplices with size bounded by $2^{n-d}R$ and dimension exactly d is compact with respect to the Hausdorff distance of M^κ_d by 3.4.3. Moreover the map $\text{Conv}(v_0,\ldots,v_d) \mapsto d_{M^\kappa_d}(v_0, \text{Conv}(v_1,\ldots,v_d))$ is continuous with respect to the Hausdorff distance and it is positive. Therefore it attains a global minimum $\varepsilon_d > 0$. Setting $\varepsilon_0(R,n) = \min_{d=0,\ldots,n} \varepsilon_d$, we have $\varepsilon(v) \geq \varepsilon_0(R,n) > 0$ for every vertex $v \in K$. Therefore every two vertices v, w of K satisfy $d_K(v,w) \geq \varepsilon_0(R,n)$ (or, by Lemma 3.4.1, there would exist a simplex S of K such that $d_K(v,w) = d_S(v,w) < \varepsilon_0(R,n)$, a contradiction). □

Lemma 3.4.5. *Let S be a M^κ-simplex of size bounded by R and $\dim(S) \leq n$. Let ∂T_τ denote the τ-neighbourhood of the boundary of any face T of S. For any positive τ there exists $\varepsilon(R,n,\tau) > 0$ such that for all faces T of S, for all $x \in T \setminus \partial T_\tau$ and all faces T' of S which do not contain x it holds:*

$$d(x, T') \geq \varepsilon(R,n,\tau)$$

Moreover for any integer $d \geq 0$ there exist $\eta_d = \eta_d(R,n)$, $\varepsilon_d = \varepsilon_d(R,n) > 0$, where $\varepsilon_0 = \varepsilon_0(R,n)$ is the function given by Lemma 3.4.4 and $\eta_0 = \frac{\varepsilon_0}{8(n+1)}$, satisfying the following conditions:

(a) for all d-dimensional faces T of S, for every $x \in T \setminus \partial T_{\eta_{d-1}}$ and every face T' of S not containing x it holds: $d(x,T') \geq \varepsilon_d$;

(b) $\eta_k + \eta_{k+1} + \cdots \eta_m \leq \frac{\varepsilon_k}{8}$, for all $0 \leq k \leq m \leq n$.

Proof. The proof follows same arguments of Lemma 3.4.4. Indeed it is sufficient to consider the positive, lower semicontinuous map

$$h(S) = \min_{T \text{ face of } S} \inf_{x \in T \setminus \partial T_\tau} \min_{\substack{T' \text{ face of } S \\ x \notin T'}} d(x, T')$$

on the compact set of M^κ-simplices of size bounded by R and dimension at most n, and take as $\varepsilon(R,n,\tau)$ its positive minimum.

To prove the second part of the Lemma we define $\varepsilon_1(R,n)$ as $\varepsilon(R,n,\eta_0)$, where this is the number given by the first statement with $\tau = \eta_0$. Then we choose $0 < \eta_1 = \min\{\frac{\varepsilon_0}{8(n+1)}, \frac{\varepsilon_1}{8(n+1)}\}$ and again we define $\varepsilon_2 > 0$ as $\varepsilon(R,n,\eta_1)$. We can continue choosing $0 < \eta_2 = \min\{\frac{\varepsilon_0}{8(n+1)}, \frac{\varepsilon_1}{8(n+1)}, \frac{\varepsilon_2}{8(n+1)}\}$ and so on. This process produces the announced ε_i, η_i, which clearly satisfy (b). □

As a consequence we get the following useful estimates:

Lemma 3.4.6. *Let K be a M^κ-complex of size bounded by R and* $\dim(K) \leq n$. *For all $\tau > 0$ there exists $\varepsilon(R,n,\tau) > 0$ with the following property: for all $x \in K$ whose support is S satisfying $d_S(x,\partial S) \geq \tau$ we have $\varepsilon(x) \geq \varepsilon(R,n,\tau)$. In particular if K is connected then (K,d_K) is a length metric space.*

Proof. Let $x \in K$. Any simplex containing x must contain $\mathrm{supp}(x)$ as a face. It is then enough to apply the first claim of Lemma 3.4.5 to get the estimate on $\varepsilon(x)$. The second part follows immediately from Lemma 3.4.1. □

Lemma 3.4.7. *Let K be a M^κ-complex of size bounded by R and* $\dim(K) \leq n$. *Then there exists $\delta = \delta(R,n) > 0$ depending only on R and n such that:*

(a) if two simplices S, S' of K are at distance $\leq \delta$, they share a face;

(b) moreover for every $x \in K$ the ball $\overline{B}(x,\delta)$ is contained in the open star of some vertex;

(c) finally for every $x \in K$ there exists $y \in K$ such that $\overline{B}(x,\delta) \subseteq \overline{B}(y, \frac{\varepsilon(y)}{4})$ (where $\varepsilon(y)$ is the function defined in (30)*).*

Proof. We start proving (c). Consider the numbers ε_d, η_d given by Lemma 3.4.5. The claim is that $\delta = \min_{d=0,\ldots,n} \eta_d$ satisfies the thesis of (c). Actually take any $x \in K$ and consider the d-dimensional simplex $S = \mathrm{supp}(x)$. There are two possibilities: either $x \in S \setminus \partial S_{\eta_{d-1}}$ or there exists a point $y_1 \in \partial S$ such that $d(x,y_1) \leq \eta_{d-1}$. In the first case we observe that any simplex S' containing x must have S has a face and by Lemma 3.4.5 we can conclude that $\varepsilon(x) \geq \varepsilon_d$. Hence in this case $\overline{B}(x,\delta) \subseteq \overline{B}(x, \frac{\varepsilon_d}{8}) \subseteq \overline{B}(x, \frac{\eta(x)}{4})$ as follows by Lemma 3.4.5.(b). Otherwise let $S_1 = \mathrm{supp}(y_1)$ and call $0 \leq d_1 \leq d-1$ its dimension. Arguing as before we find that either $\varepsilon(y_1) \geq \varepsilon_{d_1}$ or there exists again a point y_2 whose support S_2 has dimension $0 \leq d_2 < d_1$ such that $d(y_1,y_2) \leq \eta_{d_1-1}$. In the first case we have

$$\overline{B}(x,\delta) \subseteq \overline{B}(y_1, \eta_{d-1} + \eta_{d_1}) \subseteq \overline{B}\left(y_1, \frac{\varepsilon_{d_1}}{4}\right) \subseteq \overline{B}\left(y_1, \frac{\varepsilon(y_1)}{4}\right),$$

otherwise we continue the procedure inductively. Then either at some step we have the thesis or we find a vertex v of K such that

$$d(x,v) \leq \eta_{d-1} + \eta_{d-2} + \ldots + \eta_0 \leq \frac{\varepsilon_0}{8}.$$

Therefore $\overline{B}(x,\delta) \subseteq \overline{B}\left(v, \frac{\varepsilon_0(R,n)}{4}\right) \subseteq \overline{B}\left(v, \frac{\varepsilon(v)}{4}\right)$, which proves (c).

In order to prove (b) we fix $x \in K$ and we find the corresponding y given by (c). Then for all point $z \in \overline{B}(x, \delta)$ we can apply Lemma 3.4.1 and find that any simplex S containing z must contain also y. This means that any such S has the vertices of $\mathrm{supp}(y)$ as vertices. This concludes the proof of (b).
Finally the proof of (a) is an easy consequence: suppose to have two points x and x' belonging to two simplices S, S' respectively such that $d(x, x') \leq \delta$; then they belong to the open star of a same vertex by (b). In particular S and S' share a vertex. □

Another straightforward application of compactness and continuity yields the following, whose proof is omitted:

Lemma 3.4.8. *Let K be a M^κ-complex of size bounded by R and $\dim(K) \leq n$. Then there exists $R' = R'(R, n)$ depending only on R and n such that for every vertex v of K the M^1-complex $\mathrm{Lk}(v, K)$ has size bounded by R'.*

We start now considering M^κ-complexes with bounded size and valency:

Proposition 3.4.9. *Let K be a connected M^κ-complex of size bounded by R and valency at most N. Then K is locally finite (i.e. for all $x \in K$ there are a finite number of simplices containing x) and (K, d_K) is a proper, geodesic metric space.*

Proof. Any simplex S containing a point x must have $\mathrm{supp}(x)$ as a face; in particular if v is a vertex of $\mathrm{supp}(x)$ then it is also a vertex of S. So the number of simplices containing x is bounded by the number of simplices containing v, which is bounded by N by assumption. By Lemma 3.4.6 we know that (K, d_K) is a length metric space. Finally, by Lemma 3.4.7, for all $y \in K$ the ball $\overline{B}(y, \delta)$ belongs to the open star of a vertex, which is the union of a finite number of simplices, hence K is locally compact and complete. Then as K is a complete, locally compact, length metric space, it is proper and geodesic by Hopf-Rinow's Theorem. □

The following is the analogue of Theorem I.7.28 of [BH13]:

Proposition 3.4.10. *Let K be a connected M^κ-complex of size bounded by R and valency at most N. Then for any $\ell > 0$ there exists $m_0 = m_0(\ell, R, N)$ depending only on ℓ, R and N such that any m-taut string of length $\leq \ell$ satisfies $m \leq m_0$.*

Proof. We use the same proof of Theorem I.7.28 of [BH13] (which is for M^κ-complexes of *finite shape*), proceeding by induction on the dimension of K. The first step is to prove that if a m-string Σ is included in the open star of a vertex v then m is bounded by a function $m_0'(\ell, R, N)$. This is clear with $m_0' = 3$ if the geodesic associated to Σ passes through v, otherwise it follows by the inductive hypothesis by projecting radially Σ to $\mathrm{Lk}(v, K)$ (which has lower dimension) using Lemma 3.4.8. Now, if the bound stated in the proposition did not hold there would exist tout m-strings Σ_i in M^κ-complexes K_i with length $\leq \ell$ and arbitrary large m. Then there would exist also tout m'-substrings Σ_i' of the Σ_i, with $m' > m_0'(\ell, R, N)$, included in some ball $\bar{B}(x_i, \delta) \subset K_i$, for $\delta = \delta(R, N)$ defined in Lemma 3.4.7. By the same lemma Σ_i' would be included in the open star of some vertex which, by step one, implies that $m' \leq m_0'(\ell, R, N)$, a contradiction. □

Corollary 3.4.11. *Let K be a connected M^κ-complex of size bounded by R and valency at most N. Let $x, y \in K$ such that $d_K(x, y) \leq \ell$. Then there exists a geodesic joining x to y realized as the concatenation of at most $m_0(\ell, R, N)$ geodesic segments, each contained in a simplex of K.*

Proof. Immediate from the fact that K is a geodesic space (by 3.4.9), the characterization of d_K in terms of taut strings and Proposition 3.4.10. □

In order to establish if a M^κ-complex is a locally CAT(κ) space we use the following improvement of a well-known criteria. We recall that the *injectivity radius* of a complex K, denoted $\rho_{\text{inj}}(K)$, is defined as the supremum of the $r \geq 0$ such that any two points of K that are at distance at most r are joined by a unique geodesic.

Proposition 3.4.12. *Let K be a connected M^κ-complex of size bounded by R and valency at most N. The following facts are equivalent:*

(a) (K, d_K) is locally CAT(κ)*;*

(b) K satisfies the link condition, i.e. the link at any vertex is CAT(1)*;*

(c) (K, d_K) is locally uniquely geodesic;

(d) (K, d_K) has positive injectivity radius;

(e) $\rho_{\text{inj}}(K) \geq \delta(R, N)$, where $\delta(R, N)$ is the function defined in Lemma 3.4.7.

Moreover if K satisfies one of the equivalent conditions above then for any $x \in K$ the ball $B(x, \delta(R, N))$ is a CAT(κ) *space, i.e. the* CAT(κ)*-radius of K is at least $\delta(R, N)$.*

The equivalences between (a), (b) and (c) are quite standard. The equivalence of these conditions with (d) is known for simplicial complexes with finite shapes, see [BH13]. The main point of Proposition 3.4.12 is that the last equivalence continues to hold in our setting and moreover we can bound from below the injectivity radius of K in terms of R and N only.

Proof of Proposition 3.4.12. The equivalence between (a) and (b) follows from Theorem II.5.2 and Remark II.5.3 of [BH13], while (a) $\Rightarrow$ (c) is straightforward. The implication (c) $\Rightarrow$ (e) follows as in Proposition I.7.55 of [BH13]. Actually by Proposition 3.4.9 we have $\varepsilon(x) > 0$ for every $x \in K$, so the ball $B(x, \frac{\varepsilon(x)}{2})$ is isometric to the open ball $B(O, \frac{\varepsilon(x)}{2})$ of the κ-cone $C_\kappa(\text{Lk}(v, K))$ centered at the cone point O (cp. Theorem I.7.39 in [BH13]). Moreover by assumption a neighbourhood of O of the cone $C_\kappa(\text{Lk}(v, K))$ is uniquely geodesic, which implies that the whole $C_\kappa(\text{Lk}(v, K))$ is uniquely geodesic (cp. Corollary I.5.11, [BH13]) and this in turns implies that $B(x, \frac{\varepsilon(x)}{2})$ is. By Lemma 3.4.7(c) we conclude that the injectivity radius is bounded below by $\delta(R, N)$ (recall that the dimension of K is bounded above by N). The implication (e) $\Rightarrow$ (d) is obvious, while (c) $\Rightarrow$ (b) follows exactly as in Theorem II.5.4 of [BH13]. Finally the last remark follows from Theorem I.7.39 &Theorem II.3.14 of [BH13] together with Lemma 3.4.7(c). □

We recall that a locally compact, locally CAT(κ), M^κ-complex is locally geodesically complete if and only if it has no *free faces* (see II.5.9 and II.5.10 of [BH13] for the definition of having free faces and the proof of this fact). We can finally show that the class of metric spaces we are studying in this section is uniformly packed.

Proposition 3.4.13. *Let K be a connected M^κ-complex without free faces, of size bounded by R, valency at most N and positive injectivity radius. Then K is a proper, geodesic,* GCBA$^\kappa$ *metric space with $\rho_{\text{cat}}(K) \geq \rho_0$ and satisfying* $\text{Pack}(3r_0, \frac{r_0}{2}) \leq P_0$, *for constants ρ_0, P_0, r_0 depending only on R, N and κ, and $r_0 \leq \rho_0/3$.*

Proof. By the proof of Proposition 3.4.10 we know that K is proper and geodesic. Moreover since the injectivity radius is positive then K is locally CAT(κ) and by Proposition 3.4.12 the CAT(κ)-radius is at least $\rho_0 = \delta(N, R)$. Since K has no free faces then it is locally geodesically complete. This shows that K is also a GCBA$^\kappa$ metric space. We remark that clearly $\mathcal{H}^k(K) = 0$ if $k > N$ since the projection map from a simplex to K is 1-Lipschitz; this shows that there are no points of dimension greater than N, i.e. $\dim(K) \leq N$. We now use Lemma 3.4.7 to estimate the number of simplices intersecting a ball around any point $x \in K$. Any simplex S which intersect $\overline{B}(x, \delta(R, N))$ intersects the open star around some vertex v, by Lemma 3.4.7.(b). Therefore v must be a vertex of S. If follows that the number of simplices intersecting $\overline{B}(x, \delta(R, N))$ is bounded by N. Therefore, for any $x \in K$ we have

$$\mu_K(B(x, \delta(R, N))) \leq \sum_{d=0}^{N} N \cdot \mathcal{H}^d(B_{M_d^\kappa}(o, \delta(R, N))) \leq V_0,$$

where V_0 depends just on R, N and κ (here is o is any point of M_d^κ). The conclusion follows from Theorem 3.2.1. $\square$

3.4.2 Gromov-hyperbolic CAT(0)-cube complexes

Related to M^κ-complexes there is a special class of examples: geodesically complete, CAT(0)-cube complexes with bounded valency (i.e. such that the number of cubes having a common vertex is uniformly bounded by some constant N). These spaces can manifest a Gromov-hyperbolicity behaviour: for instance it is classical that any proper cocompact CAT(0)-space without 2-flats is Gromov-hyperbolic (see [Gro87], [BH13]).
For cube complexes with bounded valency we have the following hyperbolicity criterion. Recall that one says that a cube complex X *has L-thin rectangles* if all Euclidean rectangles $[0, a] \times [0, b]$ isometrically embedded in X satisfy $\min\{a, b\} \leq L$. Then:

Proposition 3.4.14. *Let X be a simply connected cube complex with no free faces, dimension $\leq n$ and valency $\leq N$, endowed with its canonical ℓ^2-metric (the length metric which makes any d-cube of X isometric to $[-1, 1]^d$):*

(a) if X has positive injectivity radius, then it is a complete, geodesically complete, CAT(0) *metric space which is P_0-packed at scale $r_0 = \frac{1}{3}$, for a packing constant $P_0 = P_0(N)$;*

(b) if moreover X has L-thin rectangles, then it is Gromov-hyperbolic with hyperbolicty constant $\delta = 4 \cdot \mathrm{Ram}\lceil L+1 \rceil$ (where $\mathrm{Ram}(m)$ *denotes the Ramsey number of m).*

Proof. The barycenter subdivision of the cube-complex gives a M^0-complex structure to X with valency bounded uniformly in terms of N and n and, clearly, with uniformly bounded size. Moreover since X has positive injectivity radius and it has no free fraces then the same is true for the metric induced by the complex structure (which is isometric), therefore we can apply Proposition 3.4.13 to conclude (a) (the fact that X is globally CAT(0) follows from the simply connectedness assumption).
The proof of (b) is presented in Theorem 3.3 of [Gen16]. □

Theorem J and its consequences will apply to this case. Notice that the quantitative Tits Alternative with specification is new for hyperbolic, CAT(0) cube complexes (compare with [SW05], [GJN20]).

Chapter 4

Entropies of convex, Gromov-hyperbolic metric spaces

From now on we will focus our attention on complete, convex, geodesically complete, packed metric spaces. We start introducing several notions of entropies with the idea to find invariants that give information on the complexity of the space. We do not need any group action at the moment.
We recall that given two functions $f, g\colon [0,+\infty) \to \mathbb{R}$ we say that *f and g have the same asymptotic behaviour*, and we write $f \asymp g$, if for all $\varepsilon > 0$ there exists $T_\varepsilon \geq 0$ such that if $T \geq T_\varepsilon$ then $|f(T) - g(T)| \leq \varepsilon$. The function T_ε is called the *threshold function*. The notation $f \underset{P_0,r_0,\delta,\ldots}{\asymp} g$ means that the threshold function can be expressed only in terms of ε and $P_0, r_0, \delta, \ldots$

4.1 Covering and volume entropy

In this section we will introduce the first two types of entropies: the covering entropy, defined in terms of the covering functions, and the volume entropy of a measure.

4.1.1 Properties of the covering entropy

Let X be a complete, convex, geodesically complete metric space that is P_0-packed at scale r_0. It is natural to define the *upper covering entropy* of X as the number

$$\overline{h_{\text{Cov}}}(X) = \limsup_{T\to+\infty} \frac{1}{T} \log \text{Cov}(\overline{B}(x,T), r_0),$$

where x is any point of X. The *lower covering entropy* is defined taking the limit inferior instead of the limit superior and it is denoted by $\underline{h_{\text{Cov}}}(X)$. By triangular inequality it is easy to show that the definitions of upper and lower covering entropy do not depend on the point $x \in X$. In the next proposition we can see that they do not depend on r_0 too and moreover we can replace the covering function with the packing function.

Proposition 4.1.1. *Let X be a complete, convex, geodesically complete metric space that is P_0-packed at scale r_0 and let $x \in X$. Then*

$$\frac{1}{T}\log \mathrm{Cov}(\overline{B}(x,T),r) \underset{P_0,r_0,r,r'}{\asymp} \frac{1}{T}\log \mathrm{Pack}(\overline{B}(x,T),r')$$

for all $r, r' > 0$. In particular any of these functions can be used in the definition of the upper and lower covering entropy.

Proof. For all $0 < r \leq r'$ and $x \in X$ clearly $\mathrm{Cov}(\overline{B}(x,T),r) \geq \mathrm{Cov}(\overline{B}(x,T),r')$ and $\mathrm{Cov}(\overline{B}(x,T),r) \leq \mathrm{Cov}(\overline{B}(x,T),r') \cdot \sup_{y\in X}\mathrm{Cov}(\overline{B}(y,r'),r)$. By Corollary 2.4.10 we have $\sup_{y\in X}\mathrm{Cov}(\overline{B}(y,r'),r) = \mathrm{Cov}(r',r)$ which is a finite number depending only on P_0, r_0, r, r'. Therefore we obtain

$$\frac{1}{T}\log \mathrm{Cov}(\overline{B}(x,T),r) \underset{P_0,r_0,r,r'}{\asymp} \frac{1}{T}\log \mathrm{Cov}(\overline{B}(x,T),r').$$

The thesis follows from (21). □

The upper and lower covering entropies can be computed also using the covering function of the metric spheres.

Proposition 4.1.2. *Let X be a complete, convex, geodesically complete metric space that is P_0-packed at scale r_0 and $x \in X$. Then for all $r > 0$*

$$\frac{1}{T}\log \mathrm{Cov}(\overline{B}(x,T),r) \underset{P_0,r_0,r}{\asymp} \frac{1}{T}\log \mathrm{Cov}(S(x,T),r)$$

Proof. Clearly it holds $\mathrm{Cov}(S(x,T),r) \leq \mathrm{Cov}(\overline{B}(x,T),r)$. The other estimate is more involved. We divide the ball $\overline{B}(x,T)$ in annulii $A(x,kr,(k+1)r)$ with $k = 0,\ldots,\frac{T}{r}-1$. We easily obtain

$$\mathrm{Cov}(\overline{B}(x,T),2r) \leq \sum_{k=0}^{\frac{T}{r}-1}\mathrm{Cov}(A(x,kr,(k+1)r),2r).$$

Now we claim that for any k it holds

$$\mathrm{Cov}(A(x,kr,(k+1)r),2r) \leq \mathrm{Cov}(S(x,T),r).$$

Indeed let $\{y_1,\ldots,y_N\}$ be a set of points realizing $\mathrm{Cov}(S(x,T),r)$. For all $i = 1,\ldots,N$ we consider the geodesic segment $\gamma_i = [x,y_i]$ and we call x_i the point along the geodesic γ_i at distance kr from x. Then $x_i \in A(x,kr,(k+1)r)$ for every $i = 1,\ldots,N$. We claim that $\{x_1,\ldots,x_N\}$ is a $2r$-dense subset of $A(x,kr,(k+1)r)$. We take any $y \in A(x,kr,(k+1)r)$ and we consider the geodesic segment $\gamma = [x,y]$. We extend this geodesic up to find a point $\gamma(T)$ at distance T from x. Then there exists i such that $d(\gamma(T),y_i) = d(\gamma(T),\gamma_i(T)) \leq r$. By convexity of the metric we have $d(\gamma(kr),\gamma_i(kr)) \leq r$, therefore we conclude that $d(y,x_i) \leq d(y,\gamma(kr)) + d(\gamma(kr),x_i) \leq 2r$. This ends the proof of the claim, so $\mathrm{Cov}(\overline{B}(x,T),2r) \leq \frac{R}{r}\mathrm{Cov}(S(x,T),r)$. The thesis follows from these estimates and Proposition 4.1.1. □

Combining Proposition 2.4.4 and Proposition 4.1.1 we can find an uniform upper bound to the covering entropy.

Lemma 4.1.3. *Let X be a complete, convex, geodesically complete metric space that is P_0-packed at scale r_0. Then*

$$\overline{h_{\text{Cov}}}(X) \leq \frac{\log(1+P_0)}{r_0}.$$

Proof. For every $x \in X$ it holds $\text{Pack}(\overline{B}(x,R), r_0) \leq P_0(1+P_0)^{\frac{R}{r_0}-1}$. The thesis follows immediately. □

4.1.2 Volume entropy of homogeneous measures

Let X be a complete, convex, geodesically complete metric space that is P_0-packed at scale r_0. The *upper volume entropy* of a measure μ on X is defined as

$$\overline{h_\mu}(X) = \limsup_{T\to+\infty} \frac{1}{T}\log \mu(\overline{B}(x,T)),$$

while the *lower volume entropy* $\underline{h_\mu}(X)$ is defined taking the limit inferior. These definitions do not depend on the choice of the point $x \in X$.
A measure μ on X is called *H-homogeneous at scale r* if

$$\frac{1}{H} \leq \mu(\overline{B}(x,r)) \leq H$$

for all $x \in X$. We remark that the condition must hold only at scale r.

Proposition 4.1.4. *Let X be a complete, convex, geodesically complete metric space that is P_0-packed at scale r_0 and let μ be a measure on X which is H-homogeneous at scale r. Then*

$$\frac{1}{T}\log \mu(\overline{B}(x,T)) \underset{P_0,r_0,H,r}{\asymp} \frac{1}{T}\log \text{Cov}(\overline{B}(x,T), r).$$

In particular the upper (resp. lower) volume entropy of μ coincides with the upper (resp. lower) covering entropy of X.

Proof. For all $x \in X$ it holds $\mu(\overline{B}(x,T)) \leq H \cdot \text{Cov}(\overline{B}(x,T), r)$ and $\mu(\overline{B}(x,T)) \geq \frac{1}{H}\cdot \text{Pack}(\overline{B}(x,T-r), r)$.
By Proposition 4.1.1 and since $\frac{T-r}{T} \underset{r}{\asymp} 1$ we have the thesis. □

Remark 4.1.5. *The proof of the proposition shows another fact: if a measure is H-homogeneous at scale r then it is $H(r')$-homogeneous at scale r' for all $r' \geq r$ and $H(r')$ depends just on H, P_0, r_0, r and r'.*

We provide here two examples of homogeneous measures. If X is a complete, geodesically complete, CAT(0) metric space that is P_0-packed at scale r_0 then the natural measure on X satisfies

$$c \leq \mu_X(\overline{B}(x,r_0)) \leq C$$

for all $x \in X$, where c and C are constants depending only on P_0 and r_0 (Theorem 3.2.1). It follows immediately the following result.

Corollary 4.1.6. *Let X be a complete, geodesically complete,* CAT(0) *metric space. If it is P_0-packed at scale r_0 for some P_0 and r_0 then $\overline{h_{\text{Cov}}}(X) = \overline{h_{\mu_X}}(X)$. The same holds for the lower entropies.*

The second example is the counting measure of a cocompact group of isometries on a δ-hyperbolic metric space.

Corollary 4.1.7. *Let X be a complete, convex, geodesically complete, Gromov-hyperbolic metric space X and Γ be a discrete, non-elementary, cocompact group of isometries of X. Then for all $x \in X$ it holds*

$$\overline{h_{\text{Cov}}}(X) = \overline{h_{\mu_x^\Gamma}}(X),$$

where μ_x^Γ is the counting measure of the orbit $\Gamma \cdot x$. The same holds for the lower entropies.

We will see in Section 7.2 that in this case the upper and lower entropies coincide.

Proof. By the cocompactness assumption we know that X is P_0-packed at scale r_0 for some P_0, r_0. Then by (42) we have $\text{sys}^\diamond(\Gamma, X) \geq s_0(P_0, r_0, \delta, D)$, where D is an upper bound on the codiameter of the action and δ is the Gromov-hyperbolicity constant. Then for all $y \in X$ it holds

$$1 \leq \mu_x^\Gamma(\overline{B}(y, D)) \leq \text{Pack}\left(D, \frac{s_0}{4}\right).$$

This shows, by Proposition 2.4.4, that μ_x^Γ is $H(P_0, r_0, \delta, D)$-homogeneous at scale D. The thesis follows from Proposition 4.1.4. □

4.2 Lipschitz-topological entropy

Let X be a complete, convex, geodesically complete metric space that is P_0-packed at scale r_0. The space Geod(X) is locally compact but not compact. The upper topological entropy of the geodesic flow is defined (see [Bow73], [HKR95]) as

$$\overline{h_{\text{top}}}(\text{Geod}(X)) = \inf_{\mathrm{d}'} \sup_K \lim_{r\to 0} \limsup_{T\to+\infty} \frac{1}{T} \log \text{Cov}_{\mathrm{d}'^T}(K, r),$$

where the infimum is taken among all metrics d' inducing the topology of Geod(X), the supremum is taken among all compact subsets of Geod(X) and $\text{Cov}_{\mathrm{d}'^T}(K, r)$ is the covering function of the compact subset K at scale r with respect to the metric d'^T defined by

$$\mathrm{d}'^T(\gamma, \gamma') = \max_{t\in[0,T]} \mathrm{d}'(\Phi_t\gamma, \Phi_t\gamma').$$

By the variational principle this quantity equals the measure-theoretic entropy defined as the supremum of the entropies of the flow-invariant probability measures on Geod(X) (cp. [HKR95], Lemma 1.5). An easy computation shows that the upper topological entropy is always zero.

Lemma 4.2.1. *There are no flow-invariant probability measures on* Geod(X)*. In particular the upper topological entropy of the geodesic flow is* 0*.*

Proof. Suppose there is a flow-invariant probability measure μ on $\text{Geod}(X)$. For $x \in X$ we define $A_R = \{\gamma \in \text{Geod}(X) \text{ s.t. } \gamma(0) \in \overline{B}(x, R)\}$, for every $R \geq 0$. Clearly there exists $R \geq 0$ such that $\mu(A_R) > \frac{1}{2}$. By flow-invariance of μ we have that the set

$$\Phi_{2R+1}^{-1}(A_R) = \{\gamma \in \text{Geod}(X) \text{ s.t. } \gamma(2R+1) \in \overline{B}(x, R)\}$$

has measure $> \frac{1}{2}$. This implies that $\mu(A_R \cap \Phi_{2R+1}^{-1}(A_R)) > 0$, but this intersection is empty. □

Looking at the proof of the variational principle given in [HKR95] we can observe that the sequence of metrics on $\text{Geod}(X)$ that approach the infimum in the definition of the upper topological entropy are the restriction to $\text{Geod}(X)$ of metrics defined on its one-point compactification. These metrics are not the natural ones on $\text{Geod}(X)$, since they are not geometric. We propose a more appropriate definition of topological entropy for proper, convex, geodesically complete metric spaces.
We define the *upper Lipschitz-topological entropy* of $\text{Geod}(X)$ as

$$\overline{h_{\text{Lip-top}}}(\text{Geod}(X)) = \inf_{\mathrm{d}} \sup_K \lim_{r \to 0} \limsup_{T \to +\infty} \frac{1}{T} \log \text{Cov}_{\mathrm{d}^T}(K, r),$$

where the infimum is now taken only among all geometric metrics on $\text{Geod}(X)$. The *lower Lipschitz-topological entropy* is defined by taking the limit inferior instead of the limit superior and it is denoted by $\underline{h_{\text{Lip-top}}}(\text{Geod}(X))$. The main result of this section is the following.

Theorem 4.2.2. *Let X be a complete, convex, geodesically complete metric space that is P_0-packed at scale r_0. Then*

$$\overline{h_{\text{Lip-top}}}(\text{Geod}(X)) = \overline{h_{\text{Cov}}}(X).$$

The same holds for the lower entropies.

One of the two inequalities is easy. In order to prove the other one we will show that for the distances induced by the functions $f \in \mathcal{F}$ the definition of topological entropy can be heavily simplified.

4.2.1 Topological entropy for the distances induced by $f \in \mathcal{F}$

For a metric $f \in \mathcal{F}$ we denote by $\overline{h_f}$ the upper metric entropy of the geodesic flow with respect to f, that is

$$\overline{h_f}(\text{Geod}(X)) = \sup_K \lim_{r \to 0} \limsup_{T \to +\infty} \frac{1}{T} \log \text{Cov}_{f^T}(K, r).$$

In the same way is defined the lower metric entropy with respect to f, $\underline{h_f}(\text{Geod}(X))$. For a subset Y of X we denote by $\text{Geod}(Y)$ the set of geodesic lines of X passing through Y at time 0.

Proposition 4.2.3. *Let X be a complete, convex, geodesically complete metric space that is P_0-packed at scale r_0 and let $f \in \mathcal{F}$. Then* *(a) for all $x, y \in X$ it holds $\overline{h_f}(\mathrm{Geod}(x)) = \overline{h_f}(\mathrm{Geod}(y))$;*

(b) for all $x \in X$ and $R \geq 0$ it holds $\overline{h_f}(\mathrm{Geod}(\overline{B}(x,R))) = \overline{h_f}(\mathrm{Geod}(x))$;

(c) for all $x \in X$ it holds $\overline{h_f}(\mathrm{Geod}(X)) = \overline{h_f}(\mathrm{Geod}(x)) \leq \overline{h_{\mathrm{Cov}}}(X)$;

(d) for all $x \in X$ the function $r \mapsto \limsup_{T\to+\infty} \frac{1}{T}\log \mathrm{Cov}_{f^T}(\mathrm{Geod}(x), r)$ is constant.

The same conclusions hold for the lower Lipschitz-topological entropy.

The proposition is a consequence of the following key lemma.

Lemma 4.2.4 (Key Lemma)**.** *Let $f \in \mathcal{F}$, $\gamma \in \mathrm{Geod}(X)$ and $0 < r \leq r'$. Then*

$$\frac{1}{T}\log \mathrm{Cov}_{f^T}(\overline{B}_{f^T}(\gamma, r'), r) \underset{P_0, r_0, r, r', f}{\asymp} 0,$$

where $\overline{B}_{f^T}(\gamma, r')$ is the closed ball of center γ and radius r' with respect to the metric f^T. As a consequence the convergence is uniform in γ.

Proof. Let $P > 0$ depending only on f and r such that

$$\int_{-\infty}^{-P} 2|u|f(u)du + \int_{P}^{+\infty} 2|u|f(u)du < \frac{r}{4}.$$

We fix $\varepsilon > 0$ and $T \geq \frac{P}{\varepsilon}$. Let $E_T = \{x_1, \ldots, x_N\}$ be a maximal $\frac{r}{16}$-separated subset of $B(\gamma(T), r' + \varepsilon T)$, so it is also $\frac{r}{16}$-dense, and $\{y_1, \ldots, y_M\}$ be a $\frac{r}{16}$-dense subset of $B(\gamma(-P), r' + 2P)$. For every $i = 1, \ldots, M$ and $j = 1, \ldots, N$ we take a geodesic line γ_{ij} extending the geodesic segment $[y_i, x_j]$. We parametrize γ_{ij} in such a way that $\gamma_{ij}(-P) = y_i$. The claim is that $\{\gamma_{ij}\}_{i,j}$ is a r-dense subset of $\overline{B}_{f^T}(\gamma, r')$ with respect to the metric f^T. We fix $\gamma' \in \overline{B}_{f^T}(\gamma, r')$. This means

$$\max_{t\in[0,T]} f^t(\gamma', \gamma) = \max_{t\in[0,T]} f(\Phi_t(\gamma'), \Phi_t(\gamma)) \leq r'.$$

In particular for all $t \in [0,T]$ we get $d(\gamma'(t), \gamma(t)) \leq r'$, since

$$d(\gamma'(t), \gamma(t)) = d(\Phi_t(\gamma')(0), \Phi_t(\gamma)(0)) \leq f(\Phi_t(\gamma'), \Phi_t(\gamma)) \leq r'.$$

Therefore $d(\gamma'(-P), \gamma(-P)) \leq r' + 2P$. Moreover

$$d(\gamma'(T + \varepsilon T), \gamma(T)) \leq d(\gamma'(T + \varepsilon T), \gamma'(T)) + d(\gamma'(T), \gamma(T)) \leq \varepsilon T + r'.$$

Thus there exists x_j such that $d(x_j, \gamma'(T+\varepsilon T)) \leq \frac{r}{16}$ and y_i such that $d(y_i, \gamma'(-P)) \leq \frac{r}{16}$. We have $d(\gamma_{ij}(-P), \gamma'(-P)) \leq \frac{r}{16}$, so if we denote with t_j the time such that $\gamma_{ij}(t_j) = x_j$ it holds $|t_j - (T + \varepsilon T)| \leq \frac{r}{8}$. Then

$$\begin{aligned} d(\gamma_{ij}(T + \varepsilon T), \gamma'(T + \varepsilon T)) &\leq d(\gamma_{ij}(T + \varepsilon T), \gamma_{ij}(t_j)) + d(\gamma_{ij}(t_j), \gamma'(T + \varepsilon T)) \\ &\leq \frac{r}{8} + \frac{r}{16} < \frac{r}{4}. \end{aligned}$$

From the convexity of the metric we have $d(\gamma'(u), \gamma_{ij}(u)) < \frac{r}{4}$ for all $u \in [-P, (1+\varepsilon)T]$. For $t \in [0, T]$ we have

$$\begin{aligned} f^t(\gamma', \gamma_{ij}) &= \int_{-\infty}^{+\infty} d(\gamma'(u), \gamma_{ij}(u)) f(u-t)du \\ &\leq \int_{-\infty}^{-P} \left(\frac{r}{4} + 2|u+P|\right) f(u-t)du + \\ &+ \int_{-P}^{(1+\varepsilon)T} \frac{r}{4} f(u-t)du + \\ &+ \int_{(1+\varepsilon)T}^{+\infty} \left(\frac{r}{4} + 2|u-(1+\varepsilon)T|\right) f(u-t)du. \end{aligned}$$

The first term can be estimated as follows

$$\begin{aligned} \int_{-\infty}^{-P} \left(\frac{r}{4} + 2|u+P|\right) f(u-t)du &\leq \frac{r}{4} + \int_{-\infty}^{-P-t} 2|v+t+P| f(v)dv \\ &\leq \frac{r}{4} + \int_{-\infty}^{-P} 2|v| f(v)dv. \end{aligned}$$

The second term is less than or equal to $\frac{r}{4}$. The third term can be controlled in this way:

$$\begin{aligned} \int_{(1+\varepsilon)T}^{+\infty} \left(\frac{r}{4} + 2|u-(1+\varepsilon)T|\right) f(u-t)du &\leq \frac{r}{4} + \int_{(1+\varepsilon)T-t}^{+\infty} 2|v-(1+\varepsilon)T+t| f(v)dv \\ &\leq \frac{r}{4} + \int_{(1+\varepsilon)T-t}^{+\infty} 2|v| f(v)dv \\ &\leq \frac{r}{4} + \int_{P}^{+\infty} 2|v| f(v)dv. \end{aligned}$$

The last inequality follows from $T \geq \frac{P}{\varepsilon}$. Therefore

$$f^t(\gamma', \gamma_{ij}) \leq \frac{r}{4} + \frac{r}{4} + \frac{r}{4} + \int_{-\infty}^{-P} 2|v| f(v)dv + \int_{P}^{+\infty} 2|v| f(v)dv \leq r.$$

We conclude that

$$\mathrm{Cov}_{f^T}(\overline{B}_{f^T}(\gamma, r'), r) \leq \mathrm{Cov}\left(r' + 2P, \frac{r}{16}\right) \cdot \#E_T.$$

From Proposition 2.4.4, if $\rho = \min\{r_0, \frac{r}{16}\}$, we get $\#E_T \leq P_0(1+P_0)^{\frac{r'+\varepsilon T}{\rho} - 1}$. Thus

$$\begin{aligned} \frac{1}{T} \log \mathrm{Cov}_{f^T}(\overline{B}_{f^T}(\gamma, r'), r) &\leq \frac{1}{T} K(P_0, r_0, r, r', f) \cdot \frac{\varepsilon T}{\rho} \log(1+P_0) \\ &= \varepsilon \cdot K'(P_0, r_0, r, r', f). \end{aligned}$$

Here K, K' are constants depending only on P_0, r_0, r, r', f and not on ε or γ. So from the arbitrariness of ε we achieve the proof. □

The computation of $\overline{h_f}$ requires to consider the supremum among all compact subsets of $\mathrm{Geod}(X)$. We notice that given a compact subset $K \subseteq \mathrm{Geod}(X)$ then the set $E(K)$ is compact since E is continuous. In particular it is bounded, hence contained in a ball $\overline{B}(x, R)$ centered at a reference point $x \in X$. We observe also that the set $\mathrm{Geod}(\overline{B}(x, R))$ is compact since the evaluation map E is proper. We conclude that any compact subset of $\mathrm{Geod}(X)$ is contained in a compact subset of the form $\mathrm{Geod}(\overline{B}(x, R))$ and therefore in order to compute $\overline{h_f}$ it is enough to take the supremum among these sets. The main consequence of Lemma 4.2.4 is the following result, which is the key ingredient in the proof of Proposition 4.2.3.

Corollary 4.2.5. *Let $f \in \mathcal{F}$, $x \in X$, $R \geq 0$ and $0 < r \leq r'$. Then*

$$\frac{1}{T}\log \mathrm{Cov}_{f^T}(\mathrm{Geod}(\overline{B}(x,R)), r) \underset{P_0,r_0,r,r',f}{\asymp} \frac{1}{T}\log \mathrm{Cov}_{f^T}(\mathrm{Geod}(\overline{B}(x,R)), r').$$

Proof. The quantity $\frac{1}{T}\log \mathrm{Cov}_{f^T}(\mathrm{Geod}(\overline{B}(x,R)), r)$ is

$$\leq \frac{1}{T}\log \mathrm{Cov}_{f^T}(\mathrm{Geod}(\overline{B}(x,R)), r') \cdot \sup_{\gamma \in \mathrm{Geod}(X)} \mathrm{Cov}_{f^T}(\overline{B}_{f^T}(\gamma, r'), r)$$
$$= \frac{1}{T}\big(\log \mathrm{Cov}_{f^T}(\mathrm{Geod}(\overline{B}(x,R)), r') + \log \sup_{\gamma \in X} \mathrm{Cov}_{f^T}(\overline{B}_{f^T}(\gamma, r'), r)\big)$$

The conclusion follows by Lemma 4.2.4. □

Proof of Proposition 4.2.3.(b). Let $\varepsilon > 0$ and $T > \frac{R}{\varepsilon}$. Let $\gamma_1, \ldots, \gamma_N$ be a r-dense subset of $\mathrm{Geod}(x)$ with respect to the metric $f^{(2+\varepsilon)T}$. The claim is that $\{\gamma_i\}$ is a K-dense subset of $\mathrm{Geod}(\overline{B}(x,R))$ with respect to f^T, where K depends only on r, R and f. We consider a geodesic line $\gamma \in \mathrm{Geod}(\overline{B}(x,R))$. Then there exists a geodesic line $\gamma' \in \mathrm{Geod}(x)$ extending $[x, \gamma((1+\varepsilon)T)]$. We call $t_{\gamma'}$ the time such that $\gamma'(t_{\gamma'}) = \gamma((1+\varepsilon)T)$. Then

$$\begin{aligned} t_{\gamma'} = d(x, \gamma((1+\varepsilon)T)) &\leq d(x, \gamma(0)) + d(\gamma(0), \gamma((1+\varepsilon)T)) \\ &\leq R + (1+\varepsilon)T \leq (1+2\varepsilon)T \end{aligned}$$

since $T \geq \frac{R}{\varepsilon}$. Moreover $|t_{\gamma'} - (1+\varepsilon)T| \leq R$. We know there exists γ_i such that $\max_{t\in[0,(1+2\varepsilon)T]} f(\Phi_t\gamma', \Phi_t\gamma_i) \leq r$. In particular $d(\gamma'(t_{\gamma'}), \gamma_i(t_{\gamma'})) \leq r$. Then $d(\gamma((1+\varepsilon)T), \gamma_i(t_{\gamma'})) \leq r$ and in conclusion:

$$d(\gamma((1+\varepsilon)T), \gamma_i((1+\varepsilon)T)) \leq d(\gamma((1+\varepsilon)T), \gamma_i(t_{\gamma'})) + d(\gamma_i(t_{\gamma'}), \gamma_i((1+\varepsilon)T)) \leq r + R.$$

From the convexity of the metric we have $d(\gamma(t), \gamma_i(t)) \leq R + r$ for all $t \in [0, (1+\varepsilon)T]$. We have to estimate $f^t(\gamma, \gamma_i) = \int_{-\infty}^{+\infty} d(\gamma(u), \gamma_i(u)) f(u-t)du$ for every $t \in [0, T]$. Since $d(\gamma(0), \gamma_i(0)) \leq R$ and $d(\gamma((1+\varepsilon)T), \gamma_i((1+\varepsilon)T)) \leq r + R$ then

$$\begin{aligned} \int_{-\infty}^{+\infty} d(\gamma(u), \gamma_i(u)) f(u-t)du \leq &\int_{-\infty}^{0} (R + 2|u|) f(u-t)du + \\ &+ \int_{0}^{(1+\varepsilon)T} (R + r) f(u-t)du + \\ &+ \int_{(1+\varepsilon)T}^{+\infty} (R + r + 2|u - (1+\varepsilon)T|) f(u-t)du \end{aligned}$$

$$\leq R + \int_{-\infty}^{-t} 2|v+t| f(v)dv + (R+r) + \int_{(1+\varepsilon)T-t}^{+\infty} (R+r+2|v-(1+\varepsilon)T+t|)f(v)dv.$$

We conclude that the above quantity is less than or equal to

$$3R + 2r + \int_{-\infty}^{0} 2|v|f(v)dv + \int_{0}^{+\infty} 2|v|f(v)dv \leq 3R + 2r + C(f) = K(R,r,f).$$

By the previous corollary $\overline{h_f}(\mathrm{Geod}(\overline{B}(x,R)))$can be computed as

$$\limsup_{T\to+\infty} \frac{1}{T} \log \mathrm{Cov}_{f^T}(\overline{B}(x,R), K)$$

which is

$$\begin{aligned}
&\leq \limsup_{T\to+\infty} \frac{1}{T} \log \mathrm{Cov}_{f^{(1+2\varepsilon)T}}(\mathrm{Geod}(x), r) \\
&= (1+2\varepsilon) \limsup_{T\to+\infty} \frac{1}{T} \log \mathrm{Cov}_{f^T}(\mathrm{Geod}(x), r).
\end{aligned}$$

Since this is true for all $\varepsilon > 0$ then we obtain the thesis.

□

Proof of Proposition 4.2.3.(a). We have $y \in \overline{B}(x,R)$, where $R = d(x,y)$, so $\mathrm{Geod}(y) \subseteq \mathrm{Geod}(\overline{B}(x,R))$. Therefore

$$\overline{h}_f(\mathrm{Geod}(y)) \leq \overline{h}_f(\mathrm{Geod}(\overline{B}(x,R))) = \overline{h}_f(\mathrm{Geod}(x)).$$

The other inequality can be proved in the same way. □

Finally we achieve the proof of the remaining parts of Proposition 4.2.3.

Proof of Proposition 4.2.3.(c) & (d). The equality in (c) follows directly from (b), so

$$\overline{h_f}(X) = \limsup_{T\to+\infty} \frac{1}{T} \log \mathrm{Cov}_{f^T}(\mathrm{Geod}(x), r_0),$$

where x is a point of X. We fix $T > 0$ and we consider a r_0-separated subset E_T of $S(x,T)$ of maximal cardinality, which is also r_0-dense. For all $y \in E_T$ we consider a geodesic line γ_y extending $[x,y]$ such that $\gamma_y(0) = x$ and $\gamma_y(T) = y$. We claim that $\{\gamma_y\}_{y\in E_T}$ is a $(r_0 + C(f))$-dense subset of $\mathrm{Geod}(x)$ with respect to f^T. We take a geodesic line $\gamma \in \mathrm{Geod}(x)$. Then there exists $y \in E_T$ such that $d(\gamma(T), y) = d(\gamma(T), \gamma_y(T)) \leq r_0$. From the convexity of the metric it holds $d(\gamma(u), \gamma_y(u)) \leq r_0$ for all $u \in [0,T]$. Moreover $d(\gamma(u), \gamma_y(u)) \leq r_0 + 2|u-T|$ for all $u \in [T, +\infty)$ and $d(\gamma(u), \gamma_y(u)) \leq 2|u|$ for all $u \in (-\infty, 0]$. Then for all $t \in [0,T]$ we get

$$\begin{aligned}
f^t(\gamma, \gamma_y) &= \int_{-\infty}^{+\infty} d(\gamma(u), \gamma_y(u)) f(u-t)du \\
&\leq \int_{-\infty}^{0} 2|u| f(u-t)du + \int_{0}^{T} r_0 f(u-t)du + \\
&+ \int_{T}^{+\infty} (r_0 + 2|u-T|) f(u-t)du \leq r_0 + C(f).
\end{aligned}$$

The last inequality follows from similar estimates given in the proofs of Lemma 4.2.4. Therefore applying Corollary 4.2.5 we have

$$\limsup_{T\to+\infty} \frac{1}{T} \log \mathrm{Cov}_{f^T}(\mathrm{Geod}(x), r_0) \leq \limsup_{T\to+\infty} \frac{1}{T} \log \mathrm{Cov}(S(x,T), r_0).$$

This, together with Proposition 4.1.2, proves (c). We observe that (d) is exactly Corollary 4.2.5 with $R = 0$. □

4.2.2 Proof of Theorem 4.2.2

We are ready to give the

Proof of Theorem 4.2.2. Proposition 4.2.3.(c) shows that $\overline{h_{\text{Lip-top}}}(\mathrm{Geod}(X))$ is less than or equal to $\overline{h_{\mathrm{Cov}}}(X)$.
In order to prove the other inequality we fix a geometric metric $\mathfrak{d}$ on $\mathrm{Geod}(X)$ and we denote by M the Lipschitz constant with respect to $\mathfrak{d}$ of the evaluation map E. Then we have

$$\sup_K \lim_{r\to 0} \limsup_{T\to+\infty} \frac{1}{T} \log \mathrm{Cov}_{\mathfrak{d}^T}(K, r) \geq \limsup_{T\to+\infty} \frac{1}{T} \log \mathrm{Cov}_{\mathfrak{d}^T}(\mathrm{Geod}(x), r_0),$$

where $x \in X$. Indeed the function $r \mapsto \frac{1}{T} \log \mathrm{Cov}_{\mathfrak{d}^T}(\mathrm{Geod}(x), r)$ is increasing. We fix $T \geq 0$ and we consider a set $\gamma_1, \ldots, \gamma_N$ realizing $\mathrm{Cov}_{\mathfrak{d}^T}(\mathrm{Geod}(x), r_0)$. The claim is that $\gamma_i(T)$ is a Mr_0-dense subset of $S(x,T)$. Indeed we take a point $y \in S(x,T)$ and we extend the geodesic $[x,y]$ to a geodesic line $\gamma \in \mathrm{Geod}(x)$. Then there exists γ_i such that $\mathfrak{d}^T(\gamma, \gamma_i) \leq r_0$. Since the evaluation map is M-Lipschitz we have

$$d(y, \gamma_i(T)) = d(\gamma(T), \gamma_i(T)) = d(\Phi_T\gamma(0), \Phi_T\gamma_i(0)) \leq L\mathfrak{d}(\Phi_T\gamma, \Phi_T\gamma_i) \leq Mr_0.$$

Therefore

$$\limsup_{T\to+\infty} \frac{1}{T} \log \mathrm{Cov}_{\mathfrak{d}^T}(\mathrm{Geod}(x), r_0) \geq \limsup_{T\to+\infty} \frac{1}{T} \log \mathrm{Cov}(S(x,T), Mr_0)$$

and the conclusion follows by Proposition 4.1.2. □

Remark 4.2.6. *By Proposition 4.2.3 and Theorem 4.2.2 the upper Lipschitz-topological entropy of X can be computed as*

$$\overline{h_{\text{Lip-top}}}(X) = \limsup_{T\to+\infty} \frac{1}{T} \log \mathrm{Cov}_{f^T}(\mathrm{Geod}(x), r)$$

independently of $f \in \mathcal{F}$, $x \in X$ and $r > 0$. Moreover

$$\frac{1}{T} \log \mathrm{Cov}_{f^T}(\mathrm{Geod}(x), r_0) \underset{P_0, r_0, f}{\asymp} \frac{1}{T} \log \mathrm{Cov}(\overline{B}(x,T), r_0)$$

by the proofs of Theorem 4.2.2 and Proposition 4.2.3 and by Proposition 4.1.2.

4.2.3 Lipschitz-topological entropy of the geodesic semi-flow

We consider now the space of geodesic rays $\mathrm{Ray}(X)$ and the corresponding geodesic semi-flow. The definition of upper and lower Lipschitz-topological entropy of the geodesic semi-flow can be given in an analogous way to the case of the geodesic flow and they are denoted by $\overline{h_{\text{Lip-top}}}(\mathrm{Ray}(X))$ and $\underline{h_{\text{Lip-top}}}(\mathrm{Ray}(X))$ respectively. We denote the space of geodesic rays with starting point belonging to $Y \subseteq X$ as $\mathrm{Ray}(Y)$.

Proposition 4.2.7. *Let X be a complete, convex, geodesically complete metric space that is P_0-packed at scale r_0. Then:*

(a) $\overline{h_{\text{Lip-top}}}(\mathrm{Ray}(X))$ *equals* $\limsup_{T\to\infty} \frac{1}{T}\log \mathrm{Cov}_{f^T}(\mathrm{Ray}(x), r)$ *independently of $f \in \mathcal{F}$, the point $x \in X$ and $r > 0$.*

(b) $\overline{h_{\text{Lip-top}}}(\mathrm{Ray}(X)) = \overline{h_{\text{Lip-top}}}(\mathrm{Geod}(X)) = \overline{h_{\mathrm{Cov}}}(X)$.

The same conclusions hold for the lower topological entropy.

Proof. The proof of $\overline{h_{\text{Lip-top}}}(\mathrm{Ray}(X)) \geq \overline{h_{\mathrm{Cov}}}(X)$ is the same given in the proof of Theorem 4.2.2. On the other hand it is clear that

$$\mathrm{Cov}_{f^T}(\mathrm{Ray}(\overline{B}(x,R)), r) \leq \mathrm{Cov}_{f^T}(\mathrm{Geod}(\overline{B}(x,R)), r),$$

therefore, using Theorem 4.2.2 and Proposition 4.2.3,

$$\begin{aligned}\overline{h_{\mathrm{Cov}}}(X) = \overline{h_{\text{Lip-top}}}(\mathrm{Geod}(X)) &= \limsup_{T\to+\infty} \frac{1}{T}\log \mathrm{Cov}_{f^T}(\mathrm{Geod}(\overline{B}(x,R)), r)\\ &\geq \limsup_{T\to+\infty} \frac{1}{T}\log \mathrm{Cov}_{f^T}(\mathrm{Ray}(\overline{B}(x,R)), r).\end{aligned}$$

Since this is true for all $R \geq 0$ and for all $r > 0$ we obtain $\overline{h_f}(\mathrm{Ray}(X)) = \overline{h_{\mathrm{Cov}}}(X)$, which shows (b).
Moreover the number $\limsup_{T\to+\infty} \frac{1}{T}\log \mathrm{Cov}_{f^T}(\mathrm{Ray}(\overline{B}(x,R)), r)$ does not depend on $f \in \mathcal{F}$, $x \in X$, $r > 0$ and $R \geq 0$, proving (a). □

We remark that we have:

$$\frac{1}{T}\log \mathrm{Cov}_{f^T}(\mathrm{Geod}(x), r) \underset{P_0, r_0, r, f}{\asymp} \frac{1}{T}\log \mathrm{Cov}_{f^T}(\mathrm{Ray}(x), r).$$

4.3 Dimension of the boundary

Let X be a complete, convex, geodesically complete metric space X that is P_0-packed at scale r_0. In this section we will show how the covering entropy equals the shadow dimension of the boundary, which is a sort of Minkowski dimension relative to appropriate open sets of the boundary: the shadows. In the second part we will see that if X is also Gromov-hyperbolic then there is a precise link between the shadow dimension and the Minkowski dimension of the visual metrics.

4.3.1 Shadow dimension

We fix a point $x \in X$. For all $y \in X$ and $r \geq 0$ we define the *shadow of radius r casted by y with center x* as

$$\text{Shad}_x(y, r) = \{z \in \partial X \text{ s.t. } [x, z] \cap B(y, r) \neq \emptyset\}.$$

We define the *r-shadow covering number* of ∂X at scale $\rho > 0$ as the minimum number of shadows of radius r casted by points at distance at least $\log \frac{1}{\rho}$ from x with center x needed to cover ∂X. It is denoted by $\text{Shad-Cov}_r(\partial X, \rho)$. The *upper shadow dimension* of ∂X is defined as

$$\overline{\text{Shad-D}}(\partial X) = \limsup_{\rho \to 0} \frac{\log \text{Shad-Cov}_r(\partial X, \rho)}{\log \frac{1}{\rho}}.$$

Taking the limit inferior instead of the limit superior we define the *lower shadow dimension*, denoted by $\underline{\text{Shad-D}}(\partial X)$. We can see that if we do the change of variable $\rho = e^{-T}$ we can write

$$\overline{\text{Shad-D}}(\partial X) = \limsup_{T \to +\infty} \frac{1}{T} \log \text{Shad-Cov}_r(\partial X, e^{-T}).$$

A priori the upper and the lower shadow dimension may depend on r, but we will see it is not the case.

Lemma 4.3.1. *Let X be a complete, convex, geodesically complete metric space that is P_0-packed at scale r_0. Then*

$$\text{Cov}(S(x, T), 4r) \leq \text{Shad-Cov}_{2r}(\partial X, e^{-T}) \leq \text{Cov}(S(x, T), r)$$

for all $x \in X$, $T \geq 0$ and $r > 0$.

Proof. Let $y_1, \ldots, y_N$ be a subset of $S(x, T)$ realizing $\text{Cov}(S(x, T), r)$. Then any geodesic ray starting at x passes through a closed ball of radius r and center some of the y_i and in particular it passes through the open ball of radius $2r$ and center y_i. In other words the boundary ∂X is covered by the shadows $\text{Shad}_x(y_i, 2r)$, showing the right inequality.
Now let $y_1, \ldots, y_N$ be a set realizing $\text{Shad-Cov}_{2r}(\partial X, e^{-T})$. This means that $d(x, y_i) \geq T$ for all i and that any geodesic ray $[x, z]$ passes through some open ball of radius $2r$ and center y_i. First of all it is possible to suppose $y_i \in S(x, T)$. Indeed we consider the geodesic $[x, y_i]$ and we take the point y_i' at distance T from x. By convexity of the metric it follows that also the shadows casted by y_i' of radius $2r$ and center x cover the boundary of X. So we suppose $y_i \in S(x, T)$ and we want to show that $\{y_i\}$ covers $S(x, T)$ at scale $4r$. We take $y \in S(x, T)$ and we extend the geodesic $[x, y]$ to a geodesic ray $[x, z]$. This ray passes through $B(y_i, 2r)$ for some i. Let $y' \in [x, y]$ be a point such that $d(y', y_i) < 2r$. Then $d(y, y') < 2r$ and so $d(y, y_i) < 4r$. This concludes the proof. □

As a consequence the covering entropy of X equals the shadow dimension of the boundary of X.

Proposition 4.3.2. *Let X be a complete, convex, geodesically complete metric space that is P_0-packed at scale r_0. Then*

$$\frac{1}{T}\log \text{Shad-Cov}_r(\partial X, e^{-T}) \underset{P_0,r_0,r}{\asymp} \frac{1}{T}\log \text{Cov}(S(x,T),r).$$

In particular the upper (resp. lower) shadow dimension of ∂X does not depend on r and equals the upper (resp. lower) covering entropy of X.

Proof. It follows directly from the previous lemma and Proposition 4.1.2. □

4.3.2 Minkowski dimension

In case X is also δ-hyperbolic the shadow dimension is equivalent to a modified version of the Minkowski dimension. We recall that *upper* and *lower Minkowski dimension* of ∂X with respect to a visual metric $D_{x,a}$ are respectively classically defined as

$$\overline{\text{MD}}_{D_{x,a}}(\partial X) = \limsup_{\rho\to 0} \frac{\log \text{Cov}_{D_{x,a}}(\partial X,\rho)}{\log\frac{1}{\rho}},$$

$$\underline{\text{MD}}_{D_{x,a}}(\partial X) = \liminf_{\rho\to 0} \frac{\log \text{Cov}_{D_{x,a}}(\partial X,\rho)}{\log\frac{1}{\rho}},$$

where the covering is considered with respect to the metric $D_{x,a}$. If we cover ∂X with generalized visual balls we define the *upper* and *lower visual Minkowski dimension* as

$$\overline{\text{MD}}(\partial X) = \limsup_{\rho\to 0} \frac{\log \text{Cov}(\partial X,\rho)}{\log\frac{1}{\rho}}, \qquad \underline{\text{MD}}(\partial X) = \liminf_{\rho\to 0} \frac{\log \text{Cov}(\partial X,\rho)}{\log\frac{1}{\rho}}$$

respectively, where $\text{Cov}(\partial X,\rho)$ denotes the minimal number of generalized visual balls of radius ρ needed to cover ∂X. Also in this case if we put $\rho = e^{-T}$ we have

$$\overline{\text{MD}}(\partial X) = \limsup_{T\to+\infty} \frac{1}{T}\log \text{Cov}(\partial X, e^{-T}),$$

and the analogous formula for the lower Minkowski dimension. The following result follows directly from Lemma 2.3.3.

Lemma 4.3.3. *Let $D_{x,a}$ be a visual metric of center x and parameter a. Then*

$$\overline{\text{MD}}(\partial X) = a\cdot \overline{\text{MD}}_{D_{x,a}}(\partial X), \qquad \underline{\text{MD}}(\partial X) = a\cdot \underline{\text{MD}}_{D_{x,a}}(\partial X).$$

The next well-known lemma highlights the relation between the Gromov product of two points of the boundary and the time until the corresponding geodesic rays stay close. We recall that given $z\in\partial_G X$ and $x\in X$ then ξ_z denotes any geodesic ray such that $\xi_z(0)=x$ and $\xi_z^+ = z$.

Lemma 4.3.4. *Let X be a proper, δ-hyperbolic metric space, $z,z'\in\partial_G X$ and $x\in X$. Then*

(a) if $(z,z')_x \geq T$ then $d(\xi_z(T-\delta),\xi_{z'}(T-\delta)) \leq 4\delta$;

(b) for all $b > 0$, if $d(\xi_z(T), \xi_{z'}(T)) < 2b$ then $(z, z')_x > T - b$.

Proof. Assume $(z, z')_x \geq T$ and suppose $d(\xi_z(T - \delta), \xi_{z'}(T - \delta)) > 4\delta$. We fix $S \geq T - \delta$ and we consider the triangle $\Delta(x, \xi_z(S), \xi_{z'}(S))$. We know there exist $a \in [x, \xi_z(S)], b \in [x, \xi_{z'}(S)], c \in [\xi_z(S), \xi_{z'}(S)]$ such that $d(a, b) < \delta$, $d(b, c) < \delta$, $d(a, c) < \delta$ and $T_\delta := d(x, a) = d(x, b)$, $d(\xi_z(S), a) = d(\xi_z(S), c)$, $d(\xi_{z'}(S), b) = d(\xi_{z'}(S), c)$. Since this triangle is 4δ-thin we conclude that $T - \delta > T_\delta$.
Moreover $d(\xi_z(S), \xi_{z'}(S)) = d(\xi_z(S), c) + d(c, \xi_{z'}(S)) = 2(S - T_\delta)$. Hence

$$(z, z')_x \leq \liminf_{S \to +\infty} \frac{1}{2}(2S - d(\xi_z(S), \xi_{z'}(S))) + \delta = T_\delta + \delta < T$$

where we have used (17). This contradiction concludes the first part.
Now we assume $d(\xi_z(T), \xi_{z'}(T)) < 2b$. Using $d(\xi_z(S), \xi_{z'}(S)) < 2(S - T) + 2b$ for all $S \geq T$ we obtain, again by (17),

$$(z, z')_x \geq \liminf_{S \to +\infty} \frac{1}{2}(2S - d(\xi_z(S), \xi_{z'}(S))) > T + b.$$

□

The shadows in a proper, δ-hyperbolic metric space are defined as

$$\text{Shad}_x(y, r) = \{z \in \partial_G X \text{ s.t.} [x, z] \cap B(y, r) \neq \emptyset \text{ for some } [x, z]\}.$$

If X is also convex clearly the two definitions of shadows coincide.

Lemma 4.3.5 (Shadow's Lemma, [Sul79])**.** *Let X be a proper, δ-hyperbolic metric space. Let $z \in \partial_G X$, $x \in X$ and $T \geq 0$. Then*

$$B(z, e^{-T}) \subseteq \text{Shad}_x(\xi_z(T), 7\delta) \subseteq B(z, e^{-T + 7\delta}).$$

Proof. Let $z' \in B(z, e^{-T})$, i.e. $(z, z')_{x_0} > T$. By the previous lemma we get $d(\xi_z(T - \delta), \xi_{z'}(T - \delta)) \leq 4\delta$. So $d(\xi_{z'}(T, \xi_z(T)) \leq 6\delta < 7\delta$. This implies $z' \in \text{Shad}_x(\xi_z(T), 7\delta)$, showing the first containment.
Now we fix $z' \in \text{Shad}_x(\xi_z(T), 7\delta)$, which means that there exists a geodesic ray $\xi_{z'} = [x, z']$ that passes through $B(\xi_z(T), 7\delta)$. Let $T' \geq 0$ such that $d(\xi_{z'}(T'), \xi_z(T)) < 7\delta$. Then it holds $|T' - T| < 7\delta$ and so $d(\xi_{z'}(T), \xi_z(T)) < 14\delta$. By the previous lemma we get $(z, z')_x > T - 7\delta$ implying the second containment. □

As a corollary we get another characterization of the covering entropy of X in case it is also δ-hyperbolic.

Proposition 4.3.6. *Let X be a complete, convex, geodesically complete, δ-hyperbolic metric space that is P_0-packed at scale r_0. Then*

$$\frac{1}{T} \log \text{Cov}(\partial X, e^{-T}) \underset{P_0, r_0, \delta}{\asymp} \frac{1}{T} \log \text{Cov}(S(x, T), r_0).$$

In particular the upper (resp. lower) visual Minkowski dimension of ∂X equals the upper (resp. lower) covering entropy of X.

Proof. It follows directly from the previous lemma and Proposition 4.3.2. □

Putting together Proposition 4.1.1, Proposition 4.1.4, Proposition 4.3.2, Proposition 4.3.6, Theorem 4.2.2 and Proposition 4.2.3 we get the proof of Theorem G and Theorem H.

Chapter 5

Quantitative Tits alternative

In this chapter we will prove Theorem J.
First remark that it is possible to assume $\ell(a) = \ell(b) =: \ell$. Indeed we can take $b' = bab^{-1}$ and $\langle a, b' \rangle$ is still a discrete and non-elementary group, which moreover is torsion-free if $\langle a, b \rangle$ was torsion-free. Futhermore $\ell(b') = \ell(a)$ and the length of b' as a word of a, b is 3.

The proof will then be divided into two cases: $\ell \leq \frac{\varepsilon_0}{3}$ and $\ell > \frac{\varepsilon_0}{3}$, where $\varepsilon_0 = \varepsilon_0(P_0, r_0)$ is the Margulis constant given by Corollary 2.6.2. The proof in the first case does not need the torsionless assumption and produces a true free subgroup; it heavily draws, in this case, from techniques introduced in [DKL18] and [BCGS17]. On the other hand in the case where $\ell > \frac{\varepsilon_0}{3}$ the proofs of the statements (a) and (b) diverge. In this last case producing a free sub-semigroup is quite standard, while producing a free subgroup is much more complicated and for this we need to properly modify the argument of [DKL18] to use it in our context.

5.1 Proof of Theorem J, case $\ell \leq \frac{\varepsilon_0}{3}$.

We assume here that a, b are non-elliptic isometries with $\ell(a)=\ell(b) = \ell \leq \frac{\varepsilon_0}{3}$ of a complete, convex, geodesically complete, δ-hyperbolic metric space X that is P_0-packed at scale r_0 and that the group $\langle a, b \rangle$ is non-elementary and discrete. In particular a and b are both parabolic or both hyperbolic.

In order to find a free subgroup in this case we will use a criterion which is the generalization of Proposition 4.21 of [BCGS17] to non-elliptic isometries. Recall that, given two isometries $a, b \in \mathrm{Isom}(X)$, the *Margulis constant of the couple* (a, b) is the number

$$L(a, b) = \inf_{x \in X} \inf_{(p,q) \in \mathbb{Z}^* \times \mathbb{Z}^*} \Big\{ \max\{d(x, a^p x), d(x, b^q x)\} \Big\}.$$

Proposition 5.1.1. *Let a, b be two non-elliptic isometries of X of the same type such that $\langle a, b \rangle$ is discrete and non-elementary. If a, b satisfy*

$$L(a, b) > \max\{\ell(a), \ell(b)\} + 56\delta$$

then $\langle a, b \rangle$ is a free group.

The proof closely follows (mutatis mutandis) the proof of Proposition 4.21 [BCGS17]; we report it here in the case of parabolic isometries for the sake of clarity and completeness; the case where one isometry is hyperbolic and the other one is parabolic can be proved in a similar way but we do not need it for our purposes.
As a first step notice that we can control the Margulis constant $L(a,b)$ by the distance of the corresponding generalized Margulis domains:

Lemma 5.1.2. *Let a,b two isometries of X and let $L > 0$:*

(a) if $d(\mathcal{M}_L(a), \mathcal{M}_L(b)) > 0$ then $L(a,b) \geq L$;

(b) conversely if $L(a,b) > L$ then $\overline{\mathcal{M}_L(a)} \cap \overline{\mathcal{M}_L(b)} = \emptyset$.

Proof. If $d(\mathcal{M}_L(a), \mathcal{M}_L(b)) > 0$ then $\mathcal{M}_L(a) \cap \mathcal{M}_L(b) = \emptyset$. In particular for every $x \in X$ and for all $p,q \in \mathbb{Z}^*$ we have $d(x, a^p x) > L$ or $d(x, b^q x) > L$. Taking the infimum over $x \in X$ we get $L(a,b) \geq L$, proving (a).
Suppose now $L(a,b) > L$ and $\overline{\mathcal{M}_L(a)} \cap \overline{\mathcal{M}_L(b)} \neq \emptyset$. Take x in the intersection. In particular $\forall \eta > 0$ there exist $x_\eta \in \mathcal{M}_L(a)$, $y_\eta \in \mathcal{M}_L(b)$ such that for some $(p_\eta, q_\eta) \in \mathbb{Z}^* \times \mathbb{Z}^*$

$$d(x, x_\eta) < \eta, \qquad d(x, y_\eta) < \eta, \qquad d(x_\eta, a^{p_\eta} x_\eta) \leq L, \qquad d(y_\eta, b^{q_\eta} y_\eta) \leq L.$$

By the triangle inequality we get $d(x, a^{p_\eta} x), d(x, b^{q_\eta} x) \leq L + 2\eta$. As this is true for every $\eta > 0$ we get $L(a,b) \leq L$. This contradiction proves (b). □

Proof of Proposition 5.1.1. We will assume that a,b are parabolic isometries, the hyperbolic case being covered in [BCGS17]. The aim is to show that there exists $x \in X$ such that

$$d(a^p x, b^q x) > \max\{d(x, a^p x), d(x, b^q x)\} + 2\delta \quad \forall (p,q) \in \mathbb{Z}^* \times \mathbb{Z}^*; \tag{31}$$

this will imply that a and b are in Schottky position by Proposition 4.6 of [BCGS17], so the group $\langle a, b \rangle$ is free.
With this in mind, choose L_0 and $0 < \varepsilon < \delta$ with $L(a,b) > L_0 > 56\delta + 2\varepsilon$, and set $\ell_0 = \delta + \varepsilon$. Since $L(a,b) > L_0$ then $\overline{\mathcal{M}_{L_0}(a)} \cap \overline{\mathcal{M}_{L_0}(b)} = \emptyset$ by Lemma 5.1.2. Moreover the two Margulis domains are non-empty since $L_0 > 0$.
Now fix points $x_0 \in \mathcal{M}_{l_0}(a)$ and $y_0 \in \mathcal{M}_{l_0}(b)$ which $\frac{\varepsilon}{2}$-almost realize the distance between the two generalized Margulis domains, that is:

$$d(x_0, y) \geq d(x_0, y_0) - \frac{\varepsilon}{2} \quad \forall y \in \mathcal{M}_{l_0}(b)$$
$$d(y_0, x) \geq d(y_0, x_0) - \frac{\varepsilon}{2} \quad \forall x \in \mathcal{M}_{l_0}(a).$$

Then we can find a point $x \in [x_0, y_0]$ such that:

$$d(x, a^p x) > L_0 \text{ and } d(x, b^q x) > L_0 \quad \forall (p,q) \in \mathbb{Z}^* \times \mathbb{Z}^*. \tag{32}$$

Indeed the sets $\overline{\mathcal{M}_{L_0}(a)}$ and $\overline{\mathcal{M}_{L_0}(b)}$ are non-empty, closed and disjoint; moreover the former contains x_0 and the latter contains y_0. Then their union cannot cover the whole geodesic segment $[x_0, y_0]$. Any x on this segment which does not belong to $\overline{\mathcal{M}_{L_0}(a)} \cup \overline{\mathcal{M}_{L_0}(b)}$ satisfies our requests. As x_0 and $a^p x_0$ belong to $\mathcal{M}_{l_0}(a)$ and $x \in X \setminus \mathcal{M}_{L_0}(a)$ we deduce by Lemma 2.5.5 that

$$d(x, x_0) \geq d(x, \mathcal{M}_{l_0}(a)) - \frac{\varepsilon}{2} \geq \frac{L_0 - \ell_0}{2} - \frac{\varepsilon}{2} > 27\delta \quad \forall p \in \mathbb{Z}^* \tag{33}$$

(notice that $x \in [x_0, y_0]$ and x_0 is a $\frac{\varepsilon}{2}$-almost projection of y_0 on $\mathcal{M}_{l_0}(a)$) and the same is true for $d(x, a^p x_0)$.
Now choose points $u \in [x, a^p x]$, $u' \in [x, a^p x_0]$ and $u'' \in [x, x_0]$ at distance 11δ from x (notice that this is possible as $d(x, a^p x) > L_0 > 56\delta$ and by (33)). Consider the approximating tripod $f_{\bar{\Delta}} : \Delta(x, x_0, a^p x_0) \to \bar{\Delta}$ and the preimage $c \in f_{\bar{\Delta}}^{-1}(\bar{c}) \cap [x_0, a^p x_0]$ of its center $\bar{c}$. By Lemma 2.5.4 we deduce that $d(c, \mathcal{M}_{l_0}(a)) \leq 12\delta$ and then, by (15) and Lemma 2.5.5, that

$$(a^p x_0, x_0)_x \geq d(x, c) - 4\delta \geq d(x, \mathcal{M}_{l_0}(a)) - 16\delta > 11\delta = d(x, u') = d(x, u'').$$

So $f_{\bar{\Delta}}(u') = f_{\bar{\Delta}}(u'')$ and, by the thinness of $\Delta(x, x_0, a^p x_0)$, we get $d(u', u'') \leq 4\delta$. Since $d(x, a^p x) > L_0$ and $d(x, a^p x_0) > d(a^p x, a^p x_0)$ we immediately deduce

$$(a^p x_0, a^p x)_x > \frac{1}{2} L_0 > d(u, x) = d(u', x).$$

So, again by thinness of the triangle $\Delta(x, a^p x_0, a^p x)$, we have $d(u, u') \leq 4\delta$. Therefore $d(u, u'') \leq 8\delta$. One analogously proves that choosing $v \in [x, b^q x]$, $v' \in [x, b^q y_0]$ and $v'' \in [x, y_0]$ at distance 11δ from x we have $d(v, v'') \leq 8\delta$.
Therefore (as x belongs to the geodesic segment $[x_0, y_0]$) we deduce that

$$d(u, v) \geq d(u'', x) + d(x, v'') - d(u, u'') - d(v, v'') \geq 6\delta.$$

Comparing with the tripod $\bar{\Delta}'$ which approximates the triangle $\Delta(a^p x, x, b^q x)$, we deduce by the 4δ-thinness that $f_{\bar{\Delta}'}(u) \neq f_{\bar{\Delta}'}(v)$. It follows that

$$(a^p x, b^q x)_x < d(x, u) = d(x, v) = 11\delta.$$

One then computes:

$$\begin{aligned} d(a^p x, b^q x) &= d(a^p x, x) + d(x, b^q x) - 2(a^p x, b^q x)_x \\ &\geq \max\{d(a^p x, x), d(x, b^q x)\} + \min\{d(a^p x, x), d(x, b^q x)\} - 22\delta \\ &\geq \max\{d(a^p x, x), d(x, b^q x)\} + L_0 - 22\delta \end{aligned}$$

which implies (31), by definition of L_0. □

We continue the proof of Theorem J in the case $\ell(a) = \ell(b) = \ell \leq \frac{\varepsilon_0}{3}$.
Set $b_i = b^i a b^{-i}$. Then $\ell(b_i) = \ell$ for all i and b_i is of the same type of a, in particular it is non-elliptic. Moreover for any $i \neq j$ the group $\langle b_i, b_j \rangle$ is discrete (as a subgroup of a discrete group) and non-elementary.
Indeed otherwise there would exist a subset $F \subseteq \partial X$ fixed by both b_i, b_j, so $b^i \mathrm{Fix}_\partial(a) = \mathrm{Fix}_\partial(b^i a b^{-i}) = F = \mathrm{Fix}_\partial(b^j a b^{-j}) = b^j \mathrm{Fix}_\partial(a)$. This implies that $b^{i-j}(F) = F$, hence $F \subseteq \mathrm{Fix}_\partial(b)$ and, as these sets have the same cardinality, they coincide. Therefore we deduce that $\mathrm{Fix}_\partial(a) = b^{-i}(F) = \mathrm{Fix}_\partial(b)$, which means that the group $\langle a, b \rangle$ is elementary, a contradiction.
Since for any $i \neq j$ the group $\langle b_i, b_j \rangle$ is discrete and non-elementary, then by definition of the Margulis constant ε_0 we have

$$\mathcal{M}_{\varepsilon_0}(b_i) \cap \mathcal{M}_{\varepsilon_0}(b_j) = \emptyset.$$

(otherwise there would exist a point $x \in X$ and powers k, h such that $d(x, b_i^k x) \leq \varepsilon_0$ and $d(x, b_j^h x) \leq \varepsilon_0$, and $\langle b_i^k, b_j^h \rangle$ would be virtually nilpotent, hence elementary; but we have just seen that this implies that $\langle a, b \rangle$ is elementary, a contradiction). Moreover each Margulis domain $\mathcal{M}_{\varepsilon_0}(b_i)$ is non-empty, since we assumed $\ell = \ell(b_i) \leq \frac{\varepsilon_0}{3}$.
We now need the following

Lemma 5.1.3. *Let $\mathcal{B}$ the set of all the conjugates $b_i = b^i a b^{-i}$, for $i \in \mathbb{Z}$. For any fixed $L > 0$ the cardinality of every subset S of $\mathcal{B}$ such that*

$$d(\mathcal{M}_{\varepsilon_0}(b_{i_h}), \mathcal{M}_{\varepsilon_0}(b_{i_k})) \leq L \qquad \forall\, b_{i_h}, b_{i_k} \in S$$

is bounded from above by a constant M_0 only depending on P_0, r_0, δ and L.

Proof. Let $S = \{b_{i_1}, \cdots, b_{i_M}\} \subseteq \mathcal{B}$ satisfying $d(\mathcal{M}_{\varepsilon_0}(b_{i_h}), \mathcal{M}_{\varepsilon_0}(b_{i_k})) \leq L$ for all $b_{i_h}, b_{i_k} \in S$. Fix $\eta > 0$ and consider the closed $(\frac{L}{2} + \eta)$-neighbourhoods of the generalized Margulis domains $B_k = \overline{B}(\mathcal{M}_{\varepsilon_0}(b_{i_k}), \frac{L}{2} + \eta)$, for $b_{i_k} \in S$. Since the domains are starlike then B_k is 20δ-quasiconvex for all k by Lemma 2.3.11. Moreover $B_h \cap B_k \neq \emptyset$ for every h, k. Indeed chosen points $x_{i_h} \in \mathcal{M}_{\varepsilon_0}(b_{i_h})$ and $x_{i_k} \in \mathcal{M}_{\varepsilon_0}(b_{i_k})$ which 2η-almost realize the distance between these domains then the midpoint of the geodesic segment $[x_{i_h}, x_{i_k}]$ is in $B_h \cap B_k$. Therefore we can apply Helly's Theorem (Proposition 2.3.12) to find a point x_0 at distance at most 419δ from each B_k. So

$$d(x_0, \mathcal{M}_{\varepsilon_0}(b_{i_k})) \leq R_0 = 419\delta + \frac{L}{2} + \eta, \qquad \text{for } k = 1, \ldots, M.$$

Notice that x_0 belongs at most to one of the domains $\mathcal{M}_{\varepsilon_0}(b_{i_k})$, since they are pairwise disjoint. So for each of the remaining $M - 1$ domains we can find points $x_k \in \mathcal{M}_{\varepsilon_0}(b_{i_k}) \cap \overline{B}(x_0, R_0 + \eta)$ and $p_k \in \mathbb{Z}^*$ such that $d(x_k, b_{i_k}^{p_k} x_k) = \varepsilon_0$. For this consider any $x'_k \in \mathcal{M}_{\varepsilon_0}(b_{i_k}) \cap \overline{B}(x_0, R_0 + \eta)$: by definition there exists $p_k \in \mathbb{Z}^*$ such that $d(x'_k, b_{i_k}^{p_k} x'_k) \leq \varepsilon_0$. On the other hand $d(x_0, b_{i_k}^{p_{i_k}} x_0) > \varepsilon_0$ since $x_0 \notin \mathcal{M}_{\varepsilon_0}(b_{i_k})$. Then, by continuity of the displacement function of the isometry $b_{i_k}^{p_k}$, we can find a point x_k along the geodesic segment $[x_0, x'_k]$ such that $d(x_k, b_{i_k}^{p_k} x_k) = \varepsilon_0$ precisely. Remark that $x_k \in \overline{B}(x_0, R_0 + \eta)$ as it belongs to the geodesic $[x_0, x'_k]$.
Now, since $\ell(b_{i_k}) \leq \frac{\varepsilon_0}{3}$ for all k, we can apply Proposition 2.5.6 and get

$$d\left(x_k, \mathcal{M}_{\frac{\varepsilon_0}{3}}(b_{i_k})\right) \leq K$$

for some K depending only on P_0, r_0, δ and ε_0, so (by Corollary 2.6.2) ultimately only on P_0, r_0 and δ.
So for each k we have some point $y_k \in X$ such that

$$d(y_k, b_{i_k}^{q_k} y_k) \leq \frac{\varepsilon_0}{3} \quad \text{and} \quad d(x_k, y_k) \leq K + \eta$$

for some $q_k \in \mathbb{Z}^*$. Set $R_1 = R_0 + \eta + K + \eta$, so that $y_k \in \overline{B}(x_0, R_1)$. We remark now that the ball $\overline{B}(y_k, \frac{\varepsilon_0}{3})$ is contained in $\mathcal{M}_{\varepsilon_0}(b_{i_k})$: indeed for every $z \in \overline{B}(y_k, \frac{\varepsilon_0}{3})$ it holds

$$d(z, b_{i_k}^{q_k} z) \leq d(z, y_k) + d(y_k, b_{i_k}^{q_k} y_k) + d(b_{i_k}^{q_k} y_k, b_{i_k}^{q_k} z) \leq \varepsilon_0.$$

Finally we set $R_2 = R_1 + \frac{\varepsilon_0}{3}$: then we have $\overline{B}(y_k, \frac{\varepsilon_0}{3}) \subset \overline{B}(x_0, R_2)$ for all k. All the balls $\overline{B}(y_k, \frac{\varepsilon_0}{3})$ are pairwise disjoint, so the points y_k are $\frac{\varepsilon_0}{3}$-separated. Hence the cardinality M of the set S satisfies $M \leq 1 + \mathrm{Pack}\left(R_2, \frac{\varepsilon_0}{6}\right) =: M_0$, which is a number depending only on P_0, r_0, δ, L and η by Proposition 2.4.4. Taking for instance the constant M_0 obtained for $\eta = 1$, we get the announced bound. □

To conclude the proof of the theorem in this case we will apply the previous lemma for an appropriate value of L. By Proposition 2.5.6 we get

$$\sup_{x \in \mathcal{M}_{\varepsilon_0+56\delta}(b_i)} d(x, \mathcal{M}_{\varepsilon_0}(b_i)) \leq K_0(P_0, r_0, \delta, \varepsilon_0) = K_0'(P_0, r_0, \delta),$$

where again Corollary 2.6.2 bounds the value of ε_0 in terms of P_0 and r_0.
We set $L = 2K_0'$ and apply Lemma 5.1.3: so there exist $i, j \leq M_0(P_0, r_0, \delta)$ such that

$$d(\mathcal{M}_{\varepsilon_0}(b_i), \mathcal{M}_{\varepsilon_0}(b_j)) > 2K_0'.$$

In particular

$$d(\mathcal{M}_{\varepsilon_0+56\delta}(b_i), \mathcal{M}_{\varepsilon_0+56\delta}(b_j)) > 0.$$

(otherwise we would find $x_i \in \mathcal{M}_{\varepsilon_0+56\delta}(b_i)$ and $x_j \in \mathcal{M}_{\varepsilon_0+56\delta}(b_j)$ at arbitrarily small distance; but there exists also points $y_i \in \mathcal{M}_{\varepsilon_0}(b_i)$ and $y_j \in \mathcal{M}_{\varepsilon_0}(b_j)$ with $d(x_i, y_i) \leq K_0'$ and $d(x_j, y_j) \leq K_0'$, which would yield $d(\mathcal{M}_{\varepsilon_0}(b_i), \mathcal{M}_{\varepsilon_0}(b_j)) \leq 2K_0'$, a contradiction).
Applying b^{-i} we deduce that $d(\mathcal{M}_{\varepsilon_0+56\delta}(a), \mathcal{M}_{\varepsilon_0+56\delta}(b^{j-i}ab^{i-j})) > 0$. This implies, by Lemma 5.1.2, that

$$L(a, b^{j-i}ab^{i-j}) \geq \varepsilon_0 + 56\delta > \max\{\ell(a), \ell(b^{j-i}ab^{i-j})\} + 56\delta.$$

By Proposition 5.1.1 we then deduce that the subgroup generated by a and $w = b^{j-i}ab^{i-j}$ is free. Remark that the length of w is bounded above by $3M_0$ that is a function depending only on P_0, r_0 and δ.

5.2 Proof of Theorem J, case $\ell > \frac{\varepsilon_0}{3}$.

We assume here that a, b are two isometries satisfying $\ell(a) = \ell(b) = \ell > \frac{\varepsilon_0}{3}$ of a complete, convex, geodesically complete, δ-hyperbolic metric space X that is P_0-packed at scale r_0 and the group $\langle a, b\rangle$ is non-elementary and discrete. In this case a and b are necessarily hyperbolic.

The proof of assertion (a) in Theorem J in this case stems directly from Proposition 4.9 of [BCGS17] (free sub-semigroup theorem for isometries with minimal displacement bounded below), since Corollary 2.6.2 bounds ε_0 in terms of P_0 and r_0. So *we will focus here on the proof of assertion (b), therefore assuming moreover $\langle a, b\rangle$ torsionless.*

Consider the closed, minimal displacement subsets $\mathrm{Min}(a), \mathrm{Min}(b)$ of a, b. Then the proof of assertion (b) of Theorem J will break down into three subcases according to the value of the distance $d_0 = d(\mathrm{Min}(a), \mathrm{Min}(b))$ between the minimal sets: the case where $d_0 \leq \frac{\varepsilon_0}{120}$, the case $\frac{\varepsilon_0}{120} < d_0 \leq 30\delta$ and the case $d_0 > 30\delta$. In all cases we will use a ping-pong argument which we will explain in the next subsection.

5.2.1 Ping-pong

The aim of this subsection is to prove the following:

Proposition 5.2.1. *Let X be a proper, convex, δ-hyperbolic space and let a, b be two hyperbolic isometries of X with minimal displacement $\ell(a) = \ell(b) = \ell$. Let α, β be two axis for a and b respectively satisfying $\partial\alpha \cap \partial\beta = \emptyset$. Finally let x_-, x_+ be respectively projections of β^-, β^+ on α and suppose that x_+ follows x_- along the (oriented) geodesic α. Assume $d(x_-, x_+) \leq M_0$: then the group $\langle a^N, b^N \rangle$ is free for any $N \geq (M_0 + 77\delta)/\ell$.*

For any $x \in \alpha$ and any $T \geq 0$ we will denote for short by $x \pm T$ the points along α at distance T from x according to the orientation of α; we will use the analogous notation $y \pm T$ for the points on β such that $d(y, y \pm T) = T$. By assumption we have $x_+ = x_- + T_0$ for some $T_0 \geq 0$.
Finally for $T > 0$ we define T-neighbourhoods of α^+ and α^- as:

$$\begin{aligned} A_+(T) &= \{z \in X \text{ s.t. } d(z, x_+ + T) \leq d(z, x_+)\} \\ A_-(T) &= \{z \in X \text{ s.t. } d(z, x_- - T) \leq d(z, x_-)\} \end{aligned} \tag{34}$$

and their analogues $B_\pm(T) = \{z \in X \text{ s.t. } d(z, y_\pm \pm T) \leq d(z, y_\pm)\}$ for β.

The proof will stem from a series of technical lemmas.

Lemma 5.2.2. *For any $T \geq 2t \geq 0$ one has:*

$$A_+(T) \subseteq \{z \in X \mid (\alpha^+, z)_{x_+} \geq t\}$$

$$A_-(T) \subseteq \{z \in X \mid (\alpha^-, z)_{x_-} \geq t\}.$$

Analogously, $B_\pm(T) \subseteq \{z \in X \mid (\beta^\pm, z)_{y_\pm} \geq t\}$ for $T \geq 2t \geq 0$.

Proof. Let $z \in A_+(T)$, i.e. $d(z, x_+ + T) \leq d(z, x_+)$. For any $S \geq T$ we have

$$d(z, x_+ + S) \leq d(z, x_+ + T) + (S - T) \leq d(z, x_+) + (S - T).$$

Hence

$$(\alpha^+, z)_{x_+} \geq \liminf_{S \to +\infty} \frac{1}{2}[d(z, x_+) + S - (d(z, x_+) + S - T)] = \frac{T}{2} \geq t.$$

The proof for $B_\pm(T)$ is analogous. □

Lemma 5.2.3. *For any $T \geq 2t + 8\delta$ one has:*

$$B_\pm(T) \subset \{z \in X \mid (\beta^\pm, z)_{x_\pm} \geq t\}.$$

Proof. Let $z \in B_+(T)$, i.e. $d(z, y_+) \geq d(z, y_+ + T)$. We have, again, $\forall S \geq T$,

$$d(z, y_+ + S) \leq d(z, y_+ + T) + (S - T) \leq d(z, y_+) + (S - T).$$

Since y_+ is also a projection of x_+ on the geodesic segment $[y_+, y_+ + S]$, by Lemma 2.3.5 it holds $(y_+, y_+ + S)_{x_+} \geq d(x_+, y_+) - 4\delta$. Expanding the Gromov product we get

$$d(x_+, y_+ + S) \geq d(x_+, y_+) + S - 8\delta.$$

Therefore

$$\begin{aligned}2(z,\beta^+)_{x_+} &\geq \liminf_{S\to+\infty}\left[d(x_+,z)+d(x_+,y_++S)-d(z,y_++S)\right]\\&\geq \liminf_{S\to+\infty}\left[d(x_+,z)+d(x_+,y_+)+S-8\delta-d(z,y_+)-(S-T)\right]\\&\geq T-8\delta\geq 2t.\end{aligned}$$

The proof for $B_-(T)$ is the same. □

Lemma 5.2.4. *For any $z \in X$ it holds:*

$$(\alpha^+,z)_{x_-}\geq(\alpha^\pm,z)_{x_+}-\delta,\qquad (\alpha^+,\beta^-)_{x_-}\geq(\alpha^+,\beta^-)_{x_+}-\delta,$$

$$(\alpha^-,z)_{x_+}\geq(\alpha^\pm,z)_{x_-}-\delta,\qquad (\alpha^-,\beta^+)_{x_+}\geq(\alpha^-,\beta^+)_{x_-}-\delta.$$

Proof. We have $(\alpha^+,z)_{x_-}\geq\liminf_{S\to+\infty}(x_++S,z)_{x_-}$. On the other hand for any $S\geq 0$ we get (as x_+ follows x_- along α):

$$\begin{aligned}2(x_++S,z)_{x_-}&=d(x_++S,x_-)+d(x_-,z)-d(x_++S,z)\\&=d(x_+,x_++S)+d(x_+,x_-)+d(x_-,z)-d(x_++S,z)\\&\geq d(x_+,x_++S)+d(z,x_+)-d(x_++S,z)\\&=2(x_++S,z)_{x_+}.\end{aligned}$$

So, by (17), we get $(\alpha^+,z)_{x_-}\geq\liminf_{S\to+\infty}(x_++S,z)_{x_+}\geq(\alpha^+,z)_{x_+}-\delta$. Taking any sequence z_n converging to β^- then proves the second formula. The proof for $(\alpha^-,z)_{x_+}$ and $(\alpha^-,\beta^+)_{x_+}$ is analogous. □

Lemma 5.2.5. *For any $u \in X$ it holds:*

$$(\beta^+,u)_{x_-}\geq(\beta^+,u)_{x_+}-13\delta,\quad (\beta^-,u)_{x_+}\geq(\beta^-,u)_{x_-}-13\delta.$$

Proof. Take a sequence (y_i) defining β^+ and let x_i be a projection of y_i on α. By Remark 2.3.7 we know that, up to a subsequence, the sequence x_i converges to x_∞ which is a projection on α of β^+. So $d(x_\infty,x_+)\leq 10\delta$ by Lemma 2.3.6. For any $\varepsilon>0$ and for every i large enough, by Lemma 2.3.5, we have

$$d(y_i,x_-)\geq d(y_i,x_i)+d(x_i,x_-)-8\delta\geq d(y_i,x_+)+d(x_+,x_-)-24\delta-\varepsilon.$$

Therefore we get

$$\begin{aligned}2(\beta^+,u)_{x_-}&\geq\liminf_{i\to+\infty}\left[d(x_-,u)+d(x_-,y_i)-d(u,y_i)\right]\\&\geq\liminf_{i\to+\infty}\left[d(x_-,u)+d(x_+,y_i)+d(x_-,x_+)-24\delta-d(u,y_i)\right]\\&\geq\liminf_{i\to+\infty}\left[d(x_+,y_i)+d(x_+,u)-d(u,y_i)-24\delta\right]\\&\geq 2(\beta^+,u)_{x_+}-26\delta.\end{aligned}$$

□

Lemma 5.2.6. *We have:*

$$(\alpha^+, \beta^+)_{x_+} \le 13\delta, \quad (\alpha^-, \beta^+)_{x_+} \le 13\delta,$$

$$(\alpha^-, \beta^-)_{x_-} \le 13\delta, \quad (\alpha^+, \beta^-)_{x_-} \le 13\delta$$

Proof. Take a sequence (y_i) defining β^+ and call x_i a projection of y_i on α. As before, x_i converges, up to a subsequence, to a projection x_∞ of β^+ on α and $d(x_\infty, x_+) \le 10\delta$. For any $\varepsilon > 0$, for any $S \ge 0$ and for every i large enough, by Lemma 2.3.5, we have

$$\begin{aligned} d(y_i, x_+ \pm S) &\ge d(y_i, x_i) + d(x_i, x_+ \pm S) - 8\delta \\ &\ge d(y_i, x_+) + d(x_+, x_+ \pm S) - 24\delta - \varepsilon. \end{aligned}$$

Therefore we get

$$2(\alpha^+, \beta^+)_{x_+} \le \liminf_{i,S \to +\infty} \left[d(x_+, x_+ + S) + d(x_+, y_i) - d(x_+ + S, y_i) \right] + 2\delta \le 26\delta.$$

The same computation with $-S$ instead of S proves the second inequality. The inequalities involving β^- are proved in the same way. □

Lemma 5.2.7. *The subsets $A_+(T)$, $A_-(T)$, $B_+(T)$ and $B_-(T)$ are pairwise disjoint for $T > 64\delta$.*

Proof. Fix some $t > 28\delta$. We claim that the subsets $A_+(T), A_-(T)$, $B_+(T)$ and $B_-(T)$ are pairwise disjoint provided that $T \ge 2t + 8\delta > 64\delta$. We first prove that $A_+(T) \cap A_-(T) = \emptyset$. If $z \in A_+(T) \cap A_-(T)$ then $(\alpha^+, z)_{x_+} \ge t > 28\delta$ and $(\alpha^-, z)_{x_-} \ge t > 28\delta$ by Lemma 5.2.2. Then by Lemma 5.2.4 we have $(\alpha^-, z)_{x_+} > 27\delta$. Thus we obtain a contradiction since

$$\delta \ge (\alpha^+, \alpha^-)_{x_+} \ge \min\{(\alpha^+, z)_{x_+}, (\alpha^-, z)_{x_+}\} - \delta > 27\delta.$$

Let us prove now that $A_+(T) \cap B_+(T) = \emptyset$. If $z \in A_+(T) \cap B_+(T)$ then we have $(\alpha^+, z)_{x_+} > 28\delta$ and $(\beta^+, z)_{x_+} > 28\delta$ by Lemma 5.2.2 and Lemma 5.2.3. We then obtain again a contradiction by Lemma 5.2.6, as

$$13\delta \ge (\alpha^+, \beta^+)_{x_+} \ge \min\{(\alpha^+, z)_{x_+}, (\beta^+, z)_{x_+}\} - \delta > 27\delta.$$

We now prove that $A_+(T) \cap B_-(T) = \emptyset$. Actually if $z \in A_+(T) \cap B_-(T)$ then $(\alpha^+, z)_{x_+} > 28\delta$ and $(\beta^-, z)_{x_-} > 28\delta$ by Lemma 5.2.2 and Lemma 5.2.3. Moreover by Lemma 5.2.5 we have $(\beta^-, z)_{x_+} > 15\delta$ and combining Lemma 5.2.6 with Lemma 5.2.4 we deduce that $(\beta^-, \alpha^+)_{x_+} \le (\beta^-, \alpha^+)_{x_-} + \delta \le 14\delta$. So we again get a contradiction since

$$14\delta \ge (\alpha^+, \beta^-)_{x_+} \ge \min\{(\alpha^+, z)_{x_+}, (\beta^-, z)_{x_+}\} - \delta > 14\delta.$$

The proof of $B_+(T) \cap B_-(T) = \emptyset$ can be done as for $A_+(T) \cap A_-(T) = \emptyset$, using Lemma 5.2.4. The remaining cases can be proved similarly. □

We are now in position to prove Proposition 5.2.1.

Proof of Proposition 5.2.1. We define the sets:

$$A_+ = \{z \in X \text{ s.t. } d(z, a^N x_-) \leq d(z, x_+)\},$$
$$A_- = \{z \in X \text{ s.t. } d(z, a^{-N} x_+) \leq d(z, x_-)\}$$

and their analogues $B_\pm = \{z \in X \text{ s.t. } d(z, b^N y_\mp) \leq d(z, y_\pm)\}$ for b.
By assumption we have $d(x_-, a^N x_-) = N\ell(a) \geq d(x_-, x_+) + 65\delta$.
In particular $A_+ \subseteq A_+(65\delta)$ as defined in (34), as follows directly from the convexity of the distance function from the geodesic line α using the fact that $x_+ + 45\delta$ is between x_+ and $a^N x_-$ (according to the chosen orientation of α), and $A_- \subseteq A_-(65\delta)$.
Moreover we have $d(y_-, y_+) \leq d(x_-, x_+) + 12\delta$ by Lemma 2.3.6, and we can prove in the same way that $B_+ \subseteq B_+(65\delta)$ and $B_- \subseteq B_-(65\delta)$.
Then by Proposition 5.2.7 the sets A_+, A_-, B_+, B_- are pairwise disjoint. We will prove now the following relations:

$$a^N(X \setminus A_-) \subseteq A_+, \quad a^{-N}(X \setminus A_+) \subseteq A_-,$$
$$b^N(X \setminus B_-) \subseteq B_+, \quad b^{-N}(X \setminus B_+) \subseteq B_-$$

Indeed if $z \in X \setminus A_-$ then $d(z, a^{-N}x_+) > d(z, x_-)$; applying a^N to both sides we get $d(a^N z, x_+) > d(a^N z, a^N x_-)$, i.e. $a^N u \in A_+$. The other relations are proved in the same way. As a consequence we have, for all $k \in \mathbb{N}^*$,

$$a^{kN}(X \setminus A_-) \subseteq A_+, \quad a^{-kN}(X \setminus A_+) \subseteq A_-,$$

$$b^{kN}(X \setminus B_-) \subseteq B_+, \quad b^{-kN}(X \setminus B_+) \subseteq B_-.$$

It is then standard to deduce by a ping-pong argument that the group generated by a^N and b^N is free. Actually no nontrivial reduced word w in $\{a^N, b^N\}$ can represent the identity, since it sends any point of $X \setminus (A_+ \cup A_- \cup B_+ \cup B_-)$ into the complement $A_+ \cup A_- \cup B_+ \cup B_-$ (notice that the former set is non-empty, as X is connected). □

5.2.2 Proof of Theorem J.(b) when $d(\mathrm{Min}(a), \mathrm{Min}(b)) \leq \frac{\varepsilon_0}{74}$.

Recall that we are assuming a, b isometries with $\ell(a) = \ell(b) = \ell > \frac{\varepsilon_0}{3}$.
Let $x_0 \in \mathrm{Min}(a), y_0 \in \mathrm{Min}(b)$ be points with $d(x_0, y_0) = d(\mathrm{Min}(a), \mathrm{Min}(b))$. Moreover we choose oriented geodesics $\alpha \subseteq \mathrm{Min}(a), \beta \subseteq \mathrm{Min}(b)$ with boundary points $\alpha^\pm = a^\pm$, $\beta^\pm = b^\pm$, and with $\alpha(0) = x_0$, $\beta(0) = y_0$ respectively (all these properties of the minimal set of an hyperbolic isometry in a convex metric space are well known and proved, for instance, in [Pap05]). In particular $d(\alpha, \beta) = d(x_0, y_0)$. Finally denote by π_α and π_β the projection maps on α and β respectively, and call $x_\pm$ the projections of $\beta^\pm$ on α. As in subsection 5.2.1, up to replacing b with b^{-1} we can assume that x_+ follows x_- along α.

Let now $[z_-, z_+]$ be the set of points of α at distance $d = \frac{\varepsilon_0}{37} < \frac{1}{3}\ell$ from β. It is a nonempty, finite geodesic segment since the metric space is convex and $\partial\alpha \cap \partial\beta = \emptyset$ (the group $\langle a, b \rangle$ being non-elementary). Clearly it holds:

$$d(z_-, \beta) = d(z_+, \beta) = d.$$

Call $2L$ the length of $[z_-, z_+]$. The following estimate of this length is due to Dey-Kapovich-Liu: since the proof is scattered in different papers (it appears in the discussion after Lemma 4.5 in [DKL18], using an argument of [Kap01] for trees), we consider worth to recall it for completeness:

Proposition 5.2.8. *With the notations above it holds:* $2L < 5\ell$.

For $0 \leq T \leq L$ let α^T denote the central segment of $[z_-, z_+]$ of length $2T$ (so $\alpha^L = [z_-, z_+]$). Then:

Lemma 5.2.9. *For any* $x \in \alpha^{L-\ell}$ *we have that either*

$$d(\pi_\beta(ax), \pi_\beta(bx)) \leq 2d \quad \textit{and} \quad d(\pi_\beta(a^{-1}x), \pi_\beta(b^{-1}x)) \leq 2d \tag{35}$$

or

$$d(\pi_\beta(ax), \pi_\beta(b^{-1}x)) \leq 2d \quad \textit{and} \quad d(\pi_\beta(a^{-1}x), \pi_\beta(bx)) \leq 2d. \tag{36}$$

Moreover the conditions (35) and (36) cannot hold together, and one of the two hold on the whole interval $\alpha^{L-\ell}$.

Proof. By assumption, as $\ell(a) = \ell$, the points $a^{\pm 1}x$ belong to α^L. Then, by definition of α^L, we have $d(\pi_\beta(x), x) \leq d$ and $d(\pi_\beta(a^{\pm 1}x), a^{\pm 1}x) \leq d$. Therefore:

$$|d(\pi_\beta(x), \pi_\beta(a^{\pm 1}x)) - \ell(a)| \leq 2d.$$

As $\pi_\beta(x), \pi_\beta(a^{\pm 1}x)$ belong to β and b translates $\pi_\beta(x)$ along β precisely by $\ell = \ell(b) = \ell(a)$, it follows that there exists $\tau, \tau' \in \{1, -1\}$ such that

$$d(\pi_\beta(ax), \pi_\beta(b^\tau x)) = d(\pi_\beta(ax), b^\tau \pi_\beta(x)) \leq 2d,$$

$$d(\pi_\beta(a^{-1}x), \pi_\beta(b^{\tau'} x)) \leq 2d.$$

Moreover, since $d(\pi_\beta(ax), \pi_\beta(a^{-1}x)) \geq 2\ell - 2d$, the above relations cannot hold together with $\tau = \tau'$ when $2\ell - 2d > 4d$. Therefore for our choice of $d < \frac{1}{3}\ell$ we have $\tau' = -\tau$ and the first part of the statement is proved.
Since $d(\pi_\beta(bx), \pi_\beta(b^{-1}x)) = 2\ell > 4d$, the relations (35) and (36) cannot hold at the same time. Finally the last assertion follows from the connectedness of the interval $\alpha^{L-\ell}$. □

Lemma 5.2.10. *Let* $\eta \geq 0$ *and* $x \in \overline{B}(\alpha^{L-\ell}, \eta)$. *Then either*

$$d(bx, \pi_\alpha(ax)) \leq 3\eta + 6d \quad \textit{and} \quad d(b^{-1}x, \pi_\alpha(a^{-1}x)) \leq 3\eta + 6d \tag{37}$$

or

$$d(b^{-1}x, \pi_\alpha(ax)) \leq 3\eta + 6d \quad \textit{and} \quad d(bx, \pi_\alpha(a^{-1}x)) \leq 3\eta + 6d \tag{38}$$

Moreover the first (resp. second) condition occurs if and only if the first (resp. second) condition in Lemma 5.2.9 holds.

Proof. We fix $x \in X$ such that $d(x, \alpha^{L-\ell}) \leq \eta$. Since every point of $\alpha^{L-\ell}$ is at distance at most d from β then $d(x, \pi_\beta(x)) \leq \eta + d$. We assume that (35) holds and we prove the first one in (37), the other cases being similar. We have:

$$d(bx, \pi_\alpha(ax)) \leq d(bx, \pi_\beta(bx)) + d(\pi_\beta(bx), \pi_\beta(\pi_\alpha(ax))) + d(\pi_\beta(\pi_\alpha(ax)), \pi_\alpha(ax)).$$

The first term equals $d(x, \pi_\beta(x)) \leq \eta + d$. We observe that from the choice of $L - \ell$ we have $\pi_\alpha(ax) \in \alpha^L$, so the third term is smaller than or equal to d. For the second term we have

$$d(\pi_\beta(bx), \pi_\beta(\pi_\alpha(ax))) \leq d(\pi_\beta(bx), \pi_\beta(b\pi_\alpha(x))) + d(\pi_\beta(b\pi_\alpha(x)), \pi_\beta(a\pi_\alpha(x))) \\ \leq d(\pi_\beta(x), x) + d(x, \pi_\beta(\pi_\alpha(x))) + 2d \leq 2\eta + 4d.$$

where we used (35) to estimate the term $d(\pi_\beta(b\pi_\alpha(x)), \pi_\beta(a\pi_\alpha(x)))$.
In conclusion $d(bx, \pi_\alpha(ax)) \leq 3\eta + 6d$. □

Lemma 5.2.11. *If $2L \geq 5\ell$ then for any commutator g of $\{a^{\pm 1}, b^{\pm 1}\}$ and any $x \in \alpha^{L/5}$ we have $d(x, gx) \leq 36d$.*

Proof. We give the proof for $[a, b] = aba^{-1}b^{-1}$, the other cases are similar. Assume that (37) holds. We have $d(b^{-1}x, a^{-1}x) \leq 6d$ by Lemma 5.2.10. Calling $x' = a^{-1}b^{-1}x$, we have

$$d([a, b]x, x) = d(bx', a^{-1}x) \leq d(bx', \pi_\alpha(b^{-1}x)) + d(\pi_\alpha(b^{-1}x), a^{-1}x). \tag{39}$$

By the second inequality in (37) we have $d(x', \alpha) = d(b^{-1}x, \alpha) \leq 6d$; hence applying the first one in (37) to x' yields $d(bx', \pi_\alpha(b^{-1}x)) \leq 24d$. The second term in (39) is less than or equal to

$$d(\pi_\alpha(b^{-1}x), b^{-1}x) + d(b^{-1}x, a^{-1}x) \leq 2d(b^{-1}x, a^{-1}x) \leq 12d.$$

So $d([a, b]x, x) \leq 36d$. The proof in case (38) holds is analogous. □

Proof of Proposition 5.2.8. If $2L \geq 5\ell$ then by Lemma 5.2.11 there exists a point $x \in \alpha^{L/5}$ which is displaced by all the commutators of $a^{\pm 1}, b^{\pm 1}$ by less than $36d < \varepsilon_0$. In particular $[a^{\pm 1}, b^{\pm 1}]$ and $[a^{\pm 1}, b^{\mp 1}]$ belong to the same elementary group. This in turns implies that $\langle a, b\rangle$ is elementary. Indeed if one commutator is the identity then $\langle a, b\rangle$ is abelian, hence elementary. If the commutators are all different from the identity then they do not have finite order (since $\langle a, b\rangle$ is assumed to be torsion-free); so, as they belong to the same elementary group, there exists a subset $F \subseteq \partial X$ made of one or two points that is fixed by all the commutators. But since $a^{-1}[a, b]a = [b, a^{-1}]$ and $b^{-1}[a, b]b = [b^{-1}, a]$ we deduce that $a^{-1}(F) = F$ and $b^{-1}(F) = F$. Therefore F is invariant for both a and b, hence $\langle a, b\rangle$ should be elementary. This contradiction concludes the proof. □

Let now $x \in [z_-, z_+]$ be any point whose distance from β is at most $\frac{\varepsilon_0}{74}$. It exists by assumption on the distance between the geodesics α and β. Now the distance function $d_\beta(\cdot) = d(\cdot, \beta)$ is convex with $d_\beta(x) \leq \frac{\varepsilon_0}{74}$ and $d_\beta(z_\pm) = d = \frac{\varepsilon_0}{37}$. Furthermore by Proposition 2.3.10 the value of d_β at x_+ and at x_- does not exceed $M = \max\{49\delta, \frac{\varepsilon_0}{74} + 19\delta\}$. Therefore by convexity we deduce

$$d(x, x_+) \leq \frac{d_\beta(x_+) - d_\beta(x)}{d_\beta(z_+) - d_\beta(x)} \cdot d(x, z_+) \leq \frac{74M}{\varepsilon_0} \cdot d(x, z_+)$$

and the same estimate holds for $d(x, x_-)$. Thus:

$$d(x_-, x_+) \leq 74 \max\left\{49\frac{\delta}{\varepsilon_0}, 19\frac{\delta}{\varepsilon_0} + 1\right\} \cdot 2L =: M_0$$

As $2L \leq 5\ell$ we conclude the proof in this case using Proposition 5.2.1. Recall that $\ell(a) = \ell > \frac{\varepsilon_0}{3}$, so it is enough to choose

$$N = \left\lceil 36491\frac{\delta}{\varepsilon_0} + 1 \right\rceil$$

which is bounded from above by a function depending only on P_0, r_0 and δ.

5.2.3 Proof of Theorem J.(b) when $\frac{\varepsilon_0}{74} < d(\mathrm{Min}(a), \mathrm{Min}(b)) \leq 30\delta$.

Recall that we are assuming a, b isometries with $\ell(a) = \ell(b) = \ell > \frac{\varepsilon_0}{3}$.
We use the same notations as in 5.2.2: $\alpha \subseteq \mathrm{Min}(a)$ and $\beta \subseteq \mathrm{Min}(b)$ are oriented geodesics invariant under the action of a, b respectively, with $\alpha^{\pm} = a^{\pm}$, $\beta^{\pm} = b^{\pm}$, $\alpha(0) = x_0$, $\beta(0) = y_0$ and $d(\alpha, \beta) = d(x_0, y_0) = d(\mathrm{Min}(a), \mathrm{Min}(b))$. Finally the points x_+, x_- are respectively projections of β^+, β^- on α, and x_+ follows x_- along α.

The strategy here is to use the packing assumption to reduce the proof to the previous subcase. For this we set

$$P := \mathrm{Pack}\left(60\delta, \frac{\varepsilon_0}{148}\right) + 1.$$

By assumption $d(x_0, y_0) = d(x_0, \beta) \leq 30\delta$. We define $[z_-, z_+]$ as the (non-empty) subsegment of α of points whose distance from β is at most 60δ. Moreover let w_- and w_+ be some projections of z_- and z_+ on β, respectively. There are two possibilities: $d(w_-, w_+) \geq 2P\ell$ or the opposite.

Assume that we are in the first case. Let w be the midpoint of the segment $[w_-, w_+]$. For every $i = 1, \ldots, P$ we consider the isometry b^i. Then:

$$d(b^i z_-, \beta) = d(z_-, \beta) = 60\delta, \quad d(b^i z_+, \beta) = d(z_+, \beta) = 60\delta.$$

Notice that the points $w_-^i = b^i w_-$ and $w_+^i = b^i w_+$ are respectively projections of $b^i z_-$ and $b^i z_+$ on β. So $d(w_-, w_-^i) = i \cdot \ell \leq P \cdot \ell \leq d(w_-, w)$. In particular w_-^i belongs to the segment $[w_-, w]$ for all $1 \leq i \leq P$. Hence w belongs to the segment $[w_-^i, w_+^i]$. Since the distance from $b^i\alpha$ of w_-^i and w_+^i is $\leq 60\delta$ then, by convexity, we get $d(w, b^i\alpha) \leq 60\delta$. Hence there exists a point $z_i \in b^i\alpha$ such that $d(z_i, w) \leq 60\delta$. If the distance between any two of these points z_i was greater than $\frac{\varepsilon_0}{74}$ then the subset $S = \{z_1, \cdots, z_P\}$ would be a $\frac{\varepsilon_0}{74}$-separated subset of $\overline{B}(w, 60\delta)$, but this is in contrast with the definition of P. So there must be two different indices $1 \leq i, j \leq P$ such that $d(z_i, z_j) \leq \frac{\varepsilon_0}{74}$. Therefore

$$d(\mathrm{Min}(a), \mathrm{Min}(b^{j-i}ab^{i-j})) = d(\mathrm{Min}(b^i ab^{-i}), \mathrm{Min}(b^j ab^{-j})) \leq \frac{\varepsilon_0}{74}.$$

Now the group $\langle a, b^{j-i}ab^{i-j}\rangle$ is clearly again discrete, non-elementary and torsion-free. Thus, from the proof of Theorem J in the case where the minimal sets have distance $\leq \frac{\varepsilon_0}{74}$ given in Subsection 5.2.2, we deduce that there exists an integer $N(P_0, r_0, \delta)$ such that the group $\langle a^N, (b^{j-i}ab^{i-j})^N\rangle$ is free. Remark that the length of $(b^{j-i}ab^{i-j})^N$, as a word in $\{a, b\}$, is at most $3PN$, and this number is bounded above in terms of P_0, r_0 and δ.

Assume now that we are in the case where $d(w_-, w_+) < 2P\ell$. Then:

$$d(z_-, z_+) \leq 2P\ell + 120\delta.$$

Starting from this inequality we want to bound the distance between the projections x_-, x_+ of β^- and β^+ on α, in order to apply Proposition 5.2.1. We look again at the distance d_β from β: we know that $d_\beta(x_0) \leq 30\delta$ and $d_\beta(z_-) = d_\beta(z_+) = 60\delta$. Moreover $d(x_0, z_+) \leq d(z_-, z_+) \leq 2P\ell + 120\delta$. By Proposition 2.3.10.(c) we know that $d_\beta(x_+) \leq \max\{49\delta, 19\delta + d(\alpha, \beta)\} = 49\delta$. Then we can conclude that $d(x_0, x_+) \leq d(x_0, z_+) \leq 2P\ell + 120\delta$, by convexity of the function d_β. The same estimate holds for x_-, so

$$d(x_-, x_+) \leq 4P\ell + 240\delta =: M_0.$$

We can therefore conclude, by Proposition 5.2.1, that the group $\langle a^N, b^N\rangle$ is free for any $N \geq 4P + 317\delta/\ell$. Again we remark that N can be bounded from above by a function depending only on P_0, r_0 and δ.

5.2.4 Proof of Theorem J.(b) when $d(\mathrm{Min}(a), \mathrm{Min}(b)) > 30\delta$.

Recall that we are always assuming a, b isometries with $\ell(a) = \ell(b) = \ell > \frac{\varepsilon_0}{3}$.
We use the same notations as in 5.2.2: $\alpha \subseteq \mathrm{Min}(a)$ and $\beta \subseteq \mathrm{Min}(b)$ are oriented geodesics invariant under the action of a, b respectively, with $\alpha^\pm = a^\pm$, $\beta^\pm = b^\pm$, $\alpha(0) = x_0$, $\beta(0) = y_0$ and $d(\alpha, \beta) = d(x_0, y_0) = d(\mathrm{Min}(a), \mathrm{Min}(b))$. Finally the points x_+, x_- are respectively projections of β^+, β^- on α, and x_+ follows x_- along α. By Remark 2.3.7 the points x_-, x_+ can be chosen in this way: for any time $t \geq 0$ we denote by x_t a projection on α of $\beta(t)$. The limit point of a convergent subsequence of (x_t), for $t \to +\infty$, defines a projection x_+ of β^+ on α. The point x_- can be similarly chosen as the limit point of a convergent subsequence of (x_t) for $t \to -\infty$. By Proposition 2.3.10.(b) we have $d(x_t, x_{t'}) \leq 9\delta$ for any $t, t' \in \mathbb{R}$ and this implies that $d(x_-, x_+) \leq 9\delta$. We can then apply again Proposition 5.2.1 to conclude that the group $\langle a^N, b^N\rangle$ is free for $N \geq 86\frac{\delta}{\ell}$, and the least N with this property can be bounded as before by a function depending only on P_0, r_0 and δ.

Chapter 6

Applications of the Tits Alternative

In this chapter we will always assume that X is a complete, convex, geodesically complete, δ-hyperbolic metric space that is P_0-packed at scale r_0. We will see several applications of the Quantitive Tits Alternative, as Theorem K, Theorem L, Theorem M and the description of the thin components. Other consequences will be studied in Chapter 8.

6.1 Lower bound for the entropy

We prove here the universal lower bounds for the entropy of any complete, convex, geodesically complete, δ-hyperbolic metric space X that is P_0-packed at scale r_0 and for the algebraic entropy of any finitely generated, non-elementary discrete group acting on X.
Recall that we defined in Section 2.6 the *nilpotence radius* of Γ at x as

$$\text{nilrad}(\Gamma, x) = \sup\{r \geq 0 \text{ s.t. } \Gamma_r(x) \text{ is virtually nilpotent}\}$$

and the *nilradius* of the action as $\text{nilrad}(\Gamma, X) = \inf_{x\in X} \text{nilrad}(\Gamma, x)$.

Theorem 6.1.1. *Let X be a complete, convex, geodesically complete, δ-hyperbolic metric space that is P_0-packed at scale r_0. Assume that X admits a non-elementary, discrete group of isometries Γ. Then:*

(a) $\text{EntAlg}(\Gamma) \geq C_0$*,*

(b) $\underline{h_{\text{Cov}}}(X) \geq C_0 \cdot \text{nilrad}(\Gamma, X)^{-1}$*,*

where $C_0 = C_0(P_0, r_0, \delta)$ is a constant depending only on P_0, r_0 and δ.

Proof. We fix any symmetric, finite generating set S of Γ. Clearly there exist $a, b \in S$ such that $\langle a, b\rangle$ is non-elementary, or Γ would be elementary. So, by Theorem J.(a), there exists a free semigroup $\langle v, w\rangle^+$ where $v, w \in S^N$, with N depending on P_0, r_0, δ. In particular $\text{card}(S^{kN}) \geq 2^k$ for all k, so $\text{Ent}(\Gamma, S) \geq \frac{\log 2}{N} = C_0$. Since this holds for any S, this proves (a). In order to show (b) notice that, if $\nu_0 = \text{nilrad}(\Gamma, X)$, there exist a point $x \in X$ and $g_1, g_2 \in \Gamma$ generating a non-virtually nilpotent subgroup such that $\max\{\ell(g_1), \ell(g_2)\} \leq \nu_0$. Since X is packed the subgroup $\langle g_1, g_2 \rangle$ is non-elementary by Corollary 2.6.5, so we can apply Theorem J.(a) and deduce the existence of a free semigroup $\langle v, w \rangle^+$ where $v, w \in S^N$, for $N = N(P_0, r_0, \delta)$. Let $s > 0$ such that $\mathrm{sys}^\diamond(\Gamma, x) > s$. Observe that the points of $\langle g_1, g_2 \rangle x$ are all distinct and s-separated. So we can apply Proposition 4.1.1 to conclude that

$$\underline{h_{\mathrm{Cov}}}(X) \geq \liminf_{T \to +\infty} \frac{1}{T} \log \#(\langle g_1, g_2 \rangle x \cap \overline{B}(x, T)).$$

As $\mathrm{card}(\langle g_1, g_2 \rangle x \cap B(x, kN\nu_0)) \geq 2^k$, we have $\underline{h_{\mathrm{Cov}}}(X) \geq \frac{\log 2}{N\nu_0} = C_0 \cdot \nu_0^{-1}$. □

6.2 Systolic estimates

Recall that we defined in Section 2.6 the *minimal free displacement* $\mathrm{sys}^\diamond(\Gamma, x)$ at x as the infimum of $d(x, gx)$ when g runs over the subset $\Gamma \setminus \Gamma^\diamond$ of the torsionless elements of Γ, and the *free systole* of the action as

$$\mathrm{sys}^\diamond(\Gamma, X) = \inf_{x \in X} \mathrm{sys}^\diamond(\Gamma, x).$$

Corollary 6.2.1. *Let X be a complete, convex, geodesically complete, δ-hyperbolic metric space that is P_0-packed at scale r_0. Then for any non-elementary discrete group of isometries Γ of X it holds:*

$$\mathrm{sys}^\diamond(\Gamma, x) \geq \min \left\{ \varepsilon_0, \frac{1}{H_0} e^{-H_0 \cdot \mathrm{nilrad}(\Gamma, x)} \right\} \tag{40}$$

where $H_0 = H_0(P_0, r_0, \delta)$ is a constant depending only on P_0, r_0 and δ.

The proof is a modification of the proof of Theorem 6.20 of [BCGS17]. We will use also Lemma 7.2.2 and in particular the fact that the Γ-entropy of X is at most $E_0 := \frac{\log(1+P_0)}{r_0}$.

Proof of Corollary 6.2.1. Suppose that $\mathrm{sys}^\diamond(\Gamma, x) < \varepsilon_0$: then we can choose a non-elliptic element $a \in \Gamma$ such that $d(x, ax) = \mathrm{sys}^\diamond(\Gamma, x) < \mathrm{nilrad}(\Gamma, x)$. Indeed the infimum in the definition of $\mathrm{sys}^\diamond(\Gamma, x)$ is attained since the action is discrete and $\mathrm{nilrad}(\Gamma, x) \geq \varepsilon_0$ by definition of ε_0. If $\mathrm{nilrad}(\Gamma, x) = +\infty$ there is nothing to prove. Otherwise we fix any arbitrary $\varepsilon > 0$ and set $R = (1 + \varepsilon) \cdot \mathrm{nilrad}(\Gamma, x)$. By definition $\Gamma_R(x)$ is not virtually nilpotent and contains a. This implies that there exists $b \in \Gamma_R(x)$ such that $d(x, bx) \leq R$ and $\langle a, b \rangle$ is not elementary. Indeed $\Gamma_R(x)$ is finitely generated by some $b_1, \ldots, b_k$ with $d(x, b_i x) \leq R$ for all $i = 1, \ldots, k$ (since Γ is discrete); if $\langle a, b_i \rangle$ was elementary for all i then each b_i would belong to the maximal, elementary subgroup containing g (Lemma 2.6.3), hence $\Gamma_R(x)$ would be elementary and virtually nilpotent (by Corollary 2.6.5), a contradiction. We can therefore apply Theorem J and infer that the semigroup $\langle a^{\tau N}, w \rangle^+$ is free, where w is a word on a and b of length at most $N = N(P_0, r_0, \delta)$ and $\tau \in \{\pm 1\}$. We now use Lemma 6.22.(ii) of [BCGS17] to deduce that

$$\overline{h_\Gamma}(X) \cdot \min \left\{ d(x, a^{\tau N} x), d(x, wx) \right\} \geq e^{-\overline{h_\Gamma}(X) \cdot \max\{d(x, a^{\tau N} x), d(x, wx)\}}.$$

As $d(x, wx) \le NR$ and $d(x, a^{\pm N}x) \le N \cdot \mathrm{sys}^\diamond(\Gamma, x) < N \cdot \mathrm{nilrad}(\Gamma, x)$, this implies, by the estimate on $\overline{h_\Gamma}(X)$ and by the arbitrariness of ε, that

$$E_0 N \cdot \mathrm{sys}^\diamond(\Gamma, x) \ge e^{-E_0 N \cdot \mathrm{nilrad}(\Gamma, x)}.$$

The conclusion follows by setting $H_0 = E_0 \cdot N$. □

We define the *upper nilradius* of Γ acting on X as the supremum over the ε_0-thin subset of X

$$\mathrm{nilrad}^+(\Gamma, X) = \sup_{x \in X_{\varepsilon_0}} \mathrm{nilrad}(\Gamma, x)$$

where $\varepsilon_0 = \varepsilon_0(P_0, r_0)$ always denotes the generalized Margulis constant. By convention we set $\mathrm{nilrad}^+(\Gamma, X) = -\infty$ if $X_{\varepsilon_0} = \emptyset$. The upper nilradius can be infinite. For instance we have:

Example 6.2.2. Let X be a complete, convex, geodesically complete, δ-hyperbolic metric space that is P_0-packed at scale r_0. Let Γ be a discrete group of isometries of X containing a parabolic element: then $\sup_{x\in X_r} \mathrm{nilrad}(\Gamma, x) = +\infty$ for all $r > 0$. Actually let g be a parabolic element of Γ, in particular $\ell(g) = 0$. We take a sequence of points $x_n \in X$ such that $d(x_n, gx_n) \le \frac{1}{n}$. Thus $\mathrm{sys}^\diamond(\Gamma, x_n) \le \frac{1}{n}$. So for any fixed $r > 0$ we have points x_n such that $\mathrm{sys}^\diamond(\Gamma, x_n) \le r$ and the nilpotence radius at x is larger and larger by Corollary 6.2.1.

By taking in (40) the supremum over all $x \in X_{\varepsilon_0}$ we deduce the formula

$$\mathrm{sys}^\diamond(X, \Gamma) \ge \min\left\{\varepsilon_0, \frac{1}{H_0} e^{-H_0 \cdot \mathrm{nilrad}^+(\Gamma, X)}\right\} \qquad (41)$$

which proves in particular Theorem L when Γ is torsionless.

Clearly if Γ is a cocompact group one has $\mathrm{nilrad}^+(\Gamma, X) \le 2 \cdot \mathrm{Diam}(\Gamma\backslash X)$. However there are many non-cocompact examples where the upper nilradius is finite and the estimate (41) is non-trivial:

Example 6.2.3 (Quasiconvex-cocompact groups)**.**
This example will be really important for the results of Chapter 7. We recall that a discrete group Γ of isometries of a δ-hyperbolic space X is called *quasiconvex-cocompact* if it acts cocompactly on the quasiconvex-hull of its limit set $\Lambda(\Gamma)$ and the *codiameter* of Γ is the infimum of the real numbers $D > 0$ such that for all $x, y \in$QC-Hull$(\Lambda(\Gamma))$ there exists $g \in \Gamma$ such that $d(y, gx) \le D$.
Consider now a non-elementary, quasiconvex-cocompact group Γ of a complete, convex, geodesically complete, δ-hyperbolic metric space that is P_0-packed at scale r_0. It is classical that, for all $x \in$QC-Hull$(\Lambda(\Gamma))$, the subset $\Sigma_{2D}(x)$ of elements g of Γ such that $d(x, gx) \le 2D$ generates Γ.
We affirm that there exists $s_0 = s_0(P_0, r_0, \delta, D)$ such that $\mathrm{sys}^\diamond(\Gamma, X) \ge s_0$. Indeed since the action is quasiconvex-cocompact then any element of Γ is either elliptic or hyperbolic. Therefore the infimum defining the free systole equals the infimum over points belonging to the axes of all hyperbolic elements of Γ. Any such axis is contained in QC-Hull$(\Lambda(\Gamma))$ by definition, so

$$\mathrm{sys}^\diamond(\Gamma, X) = \inf_{x \in \text{QC-Hull}(\Lambda(\Gamma))} \mathrm{sys}^\diamond(\Gamma, x).$$

By (40) we conclude that

$$\mathrm{sys}^\diamond(\Gamma, X) \geq \min\left\{\varepsilon_0, \frac{1}{H_0} e^{-2H_0 D}\right\} =: s_0. \tag{42}$$

Now, for any point $x \in X_{\varepsilon_0}$ by definition there exists $g \in \Gamma$ such that $d(x, gx) \leq \varepsilon_0$ and we denote one of its axis by $\mathrm{Ax}(g)$. By Proposition 2.5.6 we have

$$d(x, \mathrm{Ax}(g)) \leq K_0(P_0, r_0, \delta, \ell(g), \varepsilon_0) = K(P_0, r_0, \delta, D) =: K.$$

Moreover the axis of g belongs to $\text{QC-Hull}(\Lambda(\Gamma))$, therefore it is easy to conclude that $\mathrm{nilrad}(\Gamma, x) \leq 2K + 2D$. This shows that $\mathrm{nilrad}^+(\Gamma, X)$ is finite, bounded by $2(K + D)$.

Example 6.2.4 (Abelian covers)**.**
Let Γ be a torsionless group of isometries of a complete, convex, geodesically complete, δ-hyperbolic metric space that is P_0-packed at scale r_0. Let $\Gamma_0 = [\Gamma, \Gamma]$ be the commutator subgroup. We affirm that

$$\mathrm{nilrad}^+(\Gamma_0, X) \leq 6 \cdot \mathrm{nilrad}^+(\Gamma, X)$$

(in particular $\mathrm{nilrad}^+(\Gamma_0, X)$ is finite for any quasiconvex-cocompact Γ).
Actually assume that $\mathrm{nilrad}^+(\Gamma, X) < D$ and let $a, b \in \Gamma$ elements which generate a non-virtually nilpotent (hence non-elementary) subgroup and satisfy $d(x, ax) < D$, $d(x, bx) < D$. Then also the elements $a' = a^{-1}[a, b]a$ and $b' = b^{-1}[a, b]b$ generate a non-elementary (hence non-virtually nilpotent) subgroup of Γ, by the same argument used in the last lines of the proof of Proposition 5.2.8. However a' and b' belong to Γ_0 and both move x less than $6D$, which proves the claim.
Notice that X/Γ_0 is not compact provided that the abelianization Γ/Γ_0 of Γ is infinite.

6.3 Lower bound for the diastole

The estimate of the diastole given in Theorem M stems from the application of the classical Tits Alternative combined with Breuillard-Green-Tao's generalized Margulis Lemma. We state here the version allowing torsion, from which Theorem M easily follows. Recall that the *free diastole* of Γ acting on X is $\mathrm{dias}^\diamond(\Gamma, X) = \sup_{x \in X} \mathrm{sys}^\diamond(\Gamma, x)$, and that the *free r-thin subset* of X is defined as

$$X_r^\diamond = \{x \in X \mid \exists g \in \Gamma \setminus \Gamma^\diamond \text{ s.t. } d(x, gx) < r\}.$$

We then have:

Corollary 6.3.1. *Let X be a complete, convex, geodesically complete, δ-hyperbolic metric space that is P_0-packed at scale r_0. Then for any non-elementary, discrete group of isometries Γ of X we have:*

$$\mathrm{dias}^\diamond(\Gamma, X) \geq \varepsilon_0$$

where $\varepsilon_0 = \varepsilon_0(P_0, r_0)$ always denotes the generalized Margulis constant.

Proof. Consider the free r-thin subset $X_r^\diamond$. We first show that if $r \leq \varepsilon_0$ then $X_r^\diamond$ is not connected. By definition $\forall x \in X_r^\diamond$ the group $\Gamma_r(x)$ is virtually nilpotent and contains a torsionless element a. For any such x we denote by $N_r(x)$ the unique maximal elementary subgroup of Γ containing $\Gamma_r(x)$ given by Lemma 2.6.3. Moreover, since there are only finitely many $g \in \Gamma$ such that $d(x, gx) < r$, there exists $\eta > 0$ such that for all $g \in \Gamma_r(x)$ it holds $d(x, gx) \leq r - 2\eta$. In particular if $d(x, x') < \eta$ then $d(x', ax') < r$, so $x' \in X_r^\diamond$ too and $\Gamma_r(x) \subseteq \Gamma_r(x')$. This implies that the map $x \mapsto N_r(x)$ is locally constant. So if $X_r^\diamond$ was connected we would have that $N_r(x)$ is a fixed elementary subgroup N_r which does not depend on $x \in X_r^\diamond$. We show now that N_r is a normal subgroup of Γ: indeed $\forall g \in \Gamma$ and $\forall x \in X_r^\diamond$ we have that $gx \in X_r^\diamond$ and $\Gamma_r(gx) = g\Gamma_r(x)g^{-1}$, therefore $gN_rg^{-1} = N_r$. As Γ is non-elementary there exists a non-elliptic isometry $b \in \Gamma$ such that the group $\langle N_r, b\rangle$ is non-elementary. The group N_r is normal in Γ and amenable (since by Corollary 2.6.5 it is virtually nilpotent), therefore its cyclic extension $\langle N_r, b\rangle$ is amenable. However we know that N_r contains at least the non-elliptic element a, so by Theorem J the group $\langle N_r, b\rangle$ contains a free group and this is impossible for an amenable group. This shows that $X_r^\diamond$ is not connected and in particular $X_r^\diamond \neq X$. Therefore there exists a point $x \in X$ such that $d(x, gx) \geq \varepsilon_0$ for every $g \in \Gamma^\diamond$. □

Notice that, in the proof, we do exploit the existence of a true free *subgroup*, not just of a semi-group; actually there exist amenable groups with free semigroups, so the weak Tits Alternative would not suffice.

6.4 Geometry and topology of the thin subsets

In this part X is a complete, CAT(0), geodesically complete, δ-hyperbolic metric space that is P_0-packed at scale r_0. We will need the CAT(0) assumptions instead of the convexity since we will use Lemma 2.5.2. Moreover in this section we will assume that the group Γ acting on X is torsionless. The *ε-thin subset* of X is defined as the subset

$$X_\varepsilon = \{x \in X \mid \exists g \in \Gamma \text{ s.t. } d(x, gx) < \varepsilon\}.$$

We will denote in the following by $p : X \to \bar{X} = \Gamma\backslash X$ the natural covering projection, and we will call $\bar{X}_\varepsilon = p(X_\varepsilon)$ the *ε-thin subset of $\bar{X}$.*

The following theorems describe the geometric and topological structure of the thin subsets of X and of the quotient space $\bar{X} = \Gamma\backslash X$. They follow closely Theorems 6.25-6.26-6.29 of [BCGS17] for actions on CAT(0), Gromov-hyperbolic, packed metric spaces; however we precisely determine the group structure of the connected components of the thin subsets and extend those results to actions by groups which are not in the classes $\text{Hyp}_{\text{convex}}$ and $\text{Hyp}_{\text{thick}}$ considered there (for instance for groups with parabolics).

Proposition 6.4.1 (Group structure of components of the thin subset)**.**
Let X_ε^i be any non-empty, connected component of the ε-thin subset X_r and let $\Gamma_\varepsilon^i = \text{Stab}_\Gamma(X_\varepsilon^i)$. Then:

(a) the subsets X_ε^i are precisely invariant under the action of Γ, that is $gX_\varepsilon^i \cap X_\varepsilon^j = \emptyset$ unless $gX_\varepsilon^i = X_\varepsilon^j$ and $\Gamma_\varepsilon^j = g\Gamma_\varepsilon^i g^{-1}$;

(b) *if $\varepsilon < \varepsilon_0$ then Γ^i_ε is an elementary subgroup and coincides with the maximal elementary subgroup of Γ containing the ε-almost stabilizers $\Gamma_\varepsilon(x)$ for any $x \in X^i_\varepsilon$.*

Proof. The first assertion is classical: the X^i_ε are the connected components of $p^{-1}(\bar{X}_\varepsilon)$. As every $g \in \Gamma$ acts as an automorhism of the covering $p : X \to \bar{X}$, it permutes the connected components. The second assertion then follows from the fact that $\mathrm{Stab}_\Gamma(X^j_\varepsilon) = \mathrm{Stab}_\Gamma(gX^i_\varepsilon) = g\mathrm{Stab}_\Gamma(X^i_\varepsilon)g^{-1}$.
Let us now prove (b). Since ε is smaller than the Margulis constant ε_0 and X^i_ε is assumed to be non-empty and connected, we infer as in the proof of Corollary 6.3.1 the existence of a maximal elementary subgroup N^i_ε containing $\Gamma_\varepsilon(x)$ for any $x \in X^i_\varepsilon$. Moreover for every $g \in \Gamma^i_\varepsilon$ and every $x \in X^i_\varepsilon$ we have $g\Gamma_\varepsilon(x)g^{-1} = \Gamma_\varepsilon(gx)$, with $gx \in X^i_\varepsilon$ again, so we deduce analogously that N^i_ε is normal in Γ^i_ε. Following the same proof of 6.3.1 we deduce that, if Γ^i_ε was not elementary, there would exists $g \in \Gamma^i_\varepsilon \setminus N^i_\varepsilon$ such that the group $\langle N^i_\varepsilon, g\rangle$ is amenable but not elementary, a contradiction by the free subgroup theorem. □

The above proposition allows us to talk of *hyperbolic* and *parabolic* components of the ε-thin subset when ε is smaller than the Margulis constant ε_0, according to the type of the elementary subgroup Γ^i_ε. The following results give a geometric picture of these subsets:

Proposition 6.4.2 (Hyperbolic components)**.**
Let X^i_ε be a hyperbolic component of the ε-thin subset X_ε, for $\varepsilon < \min\{\varepsilon_0, r_0\}$, and let g_0 be a hyperbolic isometry generating the elementary, cyclic group Γ^i_ε. Let $\hat{p} : X \to \hat{X} = \Gamma^i_\varepsilon\backslash X$ and $p : X \to \bar{X} = \Gamma\backslash X$ denote the natural covering projections, let γ be an axis of g_0 and let $\hat{\gamma}$, $\bar{\gamma}$ be the closed geodesics obtained by projecting γ to $\hat{X}$ and $\bar{X}$ respectively. If $r = \ell(g_0) = \ell(\bar{\gamma})$, we have:

(a) *the neighbourhood $B(\gamma, L_\varepsilon(r))$ is entirely included in X^i_ε, where $L_\varepsilon(r)$ is the function defined by*
$$L_\varepsilon(r) = \frac{\log\left(\frac{2}{r} - 1\right)}{2\log(1 + P_0)} \cdot \varepsilon - \frac{1}{2} \tag{43}$$
(notice that the function $L_\varepsilon(r)$ tends to $+\infty$ when r tends to 0, the geometric parameters P_0, r_0, δ and ε being fixed);

(b) *the neighbourhood $B(\bar{\gamma}, R_\varepsilon(r))$ of $\bar{\gamma}$ in $\bar{X}$ is isometric to the neighbourhood $B(\hat{\gamma}, R_\varepsilon(r))$ of $\hat{\gamma}$ in $\hat{X}$, where R_ε is given by*
$$R_\varepsilon(r) = \frac{1}{4H_0} \cdot \log\left(\frac{1}{\varepsilon \cdot H_0}\right) - \frac{r}{4} \tag{44}$$
(notice that the function $R_\varepsilon(r)$ tends to $+\infty$ when ε tends to 0, and that $R_\varepsilon(r) \leq L_\varepsilon(r)$ when $r \to 0$ for fixed ε);

(c) *the geodesic $\bar{\gamma}$ is a deformation retract of $B(\bar{\gamma}, R_\varepsilon)$.*

For the proof we need a preliminary fact. We saw in Corollary 6.2.1 that, given $g \in \Gamma$ and $x \in X$, an upper bound of the displacement of this point by any other element of Γ which does not generate with g an elementary subgroup yields a corresponding lower bound of the displacement $d(x, gx)$. Reversing the inequality (40) we obtain:

Lemma 6.4.3. *Let ε be smaller than the generalized Margulis constant ε_0. For any given $x \in X_\varepsilon$ the group $\Gamma_{R_\varepsilon}(x)$ is elementary for*

$$R_\varepsilon = \frac{1}{H_0} \cdot \log\left(\frac{1}{\varepsilon \cdot H_0}\right)$$

where $H_0 = H_0(P_0, r_0, \delta)$ is the constant given in Corollary 6.2.1.

Proof of Proposition 6.4.2. As X_ε^i is a (non-empty) hyperbolic component, the elementary subgroup Γ_ε^i is cyclic (because Γ is assumed to be torsionless). To show (a) notice that γ is included in the generalized Margulis domain $\mathcal{M}_r(g_0)$. Then, by Proposition 2.5.5(b), for any $x \in \gamma$ we know that $\mathcal{M}_\varepsilon(g_0)$ contains the whole ball $B(x, L_\varepsilon(r))$, hence $B(x, L_\varepsilon(r)) \subseteq X_\varepsilon^i$.
To show (b) notice first that the map p is clearly Γ_ε^i-invariant and surjective, therefore it induces a well defined map $\pi : \Gamma_\varepsilon^i \backslash X \to \Gamma \backslash X$ satisfying $\pi \circ \hat{p} = p$. Let now $\bar{x}, \bar{y} \in B(\bar{\gamma}, R)$ and call $\bar{x}', \bar{y}'$ two projections of $\bar{x}, \bar{y}$ on $\bar{\gamma}$; let moreover $x, y, x', y' \in X$ projecting respectively to $\bar{x}, \bar{y}, \bar{x}', \bar{y}'$, with $x', y' \in \gamma$ and such that $d(x, x') = d(\bar{x}, \bar{x}')$, $d(y, y') = d(\bar{y}, \bar{y}')$ and $d(x', y') = d(\bar{x}', \bar{y}') \leq \frac{r}{2}$. Finally let $g \in \Gamma$ an element minimizing $d(x, g'y)$, that is

$$d(\bar{x}, \bar{y}) = \inf_{g' \in \Gamma} d(x, g'y) = d(x, gy) \leq 2R + \frac{r}{2}.$$

We therefore have $d(y, gy) \leq d(y, x) + d(x, gy) \leq 4R + r$ and, similarly, $d(y', gy') \leq d(y', x) + d(x, gy) + d(gy, gy') \leq 4R + r$. So we deduce that for $4R + r \leq R_\varepsilon$, that is R smaller than the function given in (44), the element g belongs to the group $\Gamma_{R_\varepsilon}(y')$, which is elementary by Lemma 6.4.3. As $g_0 \in \Gamma_{R_\varepsilon}(y')$ too, it follows that $\Gamma_{R_\varepsilon}(y') = \Gamma_\varepsilon^i$ by maximality. Then

$$d(\bar{x}, \bar{y}) = \inf_{g \in \Gamma} d(x, gy) = \inf_{g' \in \Gamma_\varepsilon^i} d(x, g'y) = d(\hat{p}(x), \hat{p}(y)).$$

As $\pi \circ \hat{p} = p$ we conclude that π is an isometry between $B(\hat{\gamma}, R_\varepsilon)$ and $B(\bar{\gamma}, R_\varepsilon)$. Notice that π restricts to an isometry between $\hat{\gamma}$ and $\bar{\gamma}$, since $\pi(\hat{p}(\gamma)) = p(\gamma) = \bar{\gamma}$. Consider now the map $Q\colon B(\gamma, R_\varepsilon) \times [0, 1] \to B(\gamma, R_\varepsilon)$ defined sending (x, t) to the point along $[x, q(x)]$ at distance t from x, where $q(x)$ is the projection of x on γ. The map Q yields a deformation retract of γ in $B(\gamma, R_\varepsilon)$. Observe that $B(\gamma, R_\varepsilon)$ is Γ_ε^i-invariant and that $q(g_0^k x) = g_0^k q(x)$; therefore Q is Γ_ε^i-equivariant and defines a quotient map $\hat{Q}\colon B(\hat{\gamma}, R_\varepsilon) \times [0, 1] \to B(\hat{\gamma}, R_\varepsilon)$ which is a deformation retract of $\hat{\gamma}$ in $B(\hat{\gamma}, R_\varepsilon)$. Composing $\hat{Q}$ with π we obtain the desired deformation of $\bar{\gamma}$ in $B(\bar{\gamma}, R_\varepsilon)$, which shows (c). □

The structure of parabolic components is similar, up to replace the tubular neighbourhood of the geodesic $\gamma \subseteq \mathrm{Min}(g_0)$ with a neighbourhood of any ray with endpoint the parabolic fixed point z of the subgroup Γ_ε^i.
In general there might be no horoball H_z entirely included in X_ε^i (for instance when Γ_ε^i is generated by a screw motion of a horosphere in $\mathbb{H}^3$). Nevertheless it remains true that there exists an open cone C_z (with uniform width, with respect to our geometric parameters P_0, r_0, δ), containing the end of any geodesic ray going to z, which is entirely included in one connected component X_ε^i. Moreover any $x \in C_z$ has a large neighbourhood (compared to the depth of x in X_ε^i) whose quotient by Γ is isometric to its quotient by the smaller group Γ_ε^i:

Proposition 6.4.4 (Parabolic components)**.**
Let X^i_ε be a parabolic component of the ε-thin subset X_ε, for $\varepsilon < \min\{\varepsilon_0, r_0\}$, with elementary, virtually nilpotent stabilizer Γ^i_ε and parabolic fixed point z. Let again $\hat{p} : X \to \hat{X} = \Gamma^i_\varepsilon \backslash X$, $p : X \to \bar{X} = \Gamma \backslash X$ be the projection maps and let $r < r(\varepsilon) = (1 + P_0)^{-\frac{56\delta+1}{\varepsilon}}$. Then:

(a) the set $X^i_r := X^i_\varepsilon \cap X_r$ is connected. In other words there exists a unique connected component of X_r inside X^i_ε, so the notation X^i_r is coherent.

Let H_z be any horosphere centered at z intersecting X^i_r and $H^i_{z,r} := H_z \cap X^i_r$. Let $C_r = C_r(H_z)$ be the set of points $x \in X$ satisfying

$$\mathcal{B}_z(H_z, x) > d(H^i_{z,r}, x) - L_\varepsilon(r) + 40\delta. \tag{45}$$

Then:

(b) the subset C_r is contained in X^i_ε;

(c) the subset C_r is a connected, Γ^i_ε-invariant geodesic cone of vertex z (i.e. for all $x \in C_r$ the whole geodesic ray $[x, z]$ belongs to C_r);

(d) any geodesic ray with endpoint z definitively belongs to C_r;

(e) calling $\hat{C}_r = \hat{p}(C_r)$ and $\bar{C}_r = p(C_r)$ the projections of C_r on $\hat{X}$ and $\bar{X}$, their $R_\varepsilon(\varepsilon)$-neighbourhoods $B(\hat{C}_r, R_\varepsilon(\varepsilon))$, $B(\bar{C}_r, R_\varepsilon(\varepsilon))$ are isometric;

(f) for any $x \in C_r$ the projections of $[x, z]$ in $\hat{X}$ and $\bar{X}$ are geodesic rays.

(*Here $L_\varepsilon(r)$ and $R_\varepsilon(\varepsilon)$ are the same functions as in* (43) *and* (44).)

Proof. The choice of $r < r(\varepsilon)$ gives $-L_\varepsilon(r) + 40\delta < -16\delta \leq 0$. So if $y \in H^i_{z,r}$ then any point y' of the geodesic ray $[y, z]$ satisfies

$$\mathcal{B}_z(H_z, y') > d(H^i_{z,r}, y') - L_\varepsilon(r) + 56\delta. \tag{46}$$

Indeed for any such point we have $\mathcal{B}_z(H_z, y') = d(H^i_{z,r}, y')$.

We are now going to show (b). Let $x \in X$ satisfying (45). Let $x_0 \in H_z$ be the intersection of a bi-infinite geodesic through x with endpoint z. Choose some y_0 and y in $H^i_{z,r}$ minimizing the distance to x_0 and x respectively, and call $d_0 = d(x_0, y_0) = d(x_0, H^i_{z,r})$ for short. Since $y_0 \in X^i_r$, by Proposition 6.4.1 there exists a parabolic isometry $g \in \Gamma^i_\varepsilon$ with fixed point z such that $y_0 \in M_r(g)$. Consider now the points x_1, y_1 on the geodesic rays $[x_0, z]$ and $[y_0, z]$ at distance $\frac{d_0}{2} + 4\delta$ from y_0, x_0 respectively: by Lemma 2.3.2 we know that $d(y_1, x_1) \leq 16\delta$. Indeed the two geodesic rays $\xi_1 = [x_0, z]$ and $\xi_2 = [x_0, z]$ satisfy $d(\xi_1(t + t_1), \xi_2(t + t_2)) \leq 8\delta$ for all $t \geq 0$, where $t_1 + t_2 = d_0$ and $|t_1 - t_2| \leq 8\delta$. This implies $\max\{t_1, t_2\} \leq \frac{d_0}{2} + 4\delta$, so $d(x_1, y_1) \leq 16\delta$ by triangular inequality. Clearly also $y_1 \in M_r(g)$ by convexity.
Assume now first that x does not belong to the horoball H^+_z. Since x_0 is the projection of x on the convex set H_z containing y, by the Projection Lemma 2.3.5, we have $d(x, y) \geq d(x, x_0) + d(x_0, y) - 8\delta$, therefore

$$d(y_1, x) \leq d(y_1, x_0) + d(x_0, x) \leq \left(\frac{d_0}{2} + 20\delta\right) + \left(d(y, x) - d(x_0, y) + 8\delta\right).$$

Moreover $d(x_0, y) \geq d_0$ by minimality, so we deduce from (45):

$$d(y_1, x) \leq \frac{d_0}{2} + d(H^i_{z,r}, x) - d_0 + 28\delta < \mathcal{B}_z(H_z, x) + L_\varepsilon(r).$$

Since $x \notin H_z^+$ the term $\mathcal{B}_z(H_z, x)$ is negative, hence x belongs to $B(y_1, L_\varepsilon(r))$ and is in $\mathcal{M}_\varepsilon(g)$ by Proposition 2.5.5.(b). So $x \in X^i_\varepsilon$.
Assume now that $x \in H_z^+$. If x is between x_1 and z along $[x_0, z]$ we can find a point y_2 along $[y_0, z]$ such that $d(x, y_2) \leq 8\delta$ and of course $y_2 \in M_r(g)$; so, by the choice of $L_\varepsilon(r)$, we conclude by Proposition 2.5.5.(b) that $x \in \mathcal{M}_\varepsilon(g)$ and so $x \in X^i_\varepsilon$. On the other hand, if $x \in [x_0, x_1]$, we want to show that $d(y_1, x) < L_\varepsilon(r)$. Suppose the opposite: $d(y_1, x) \geq L_\varepsilon(r)$. Then by (45) and the disposition of the points x_0, x, x_1 we get

$$\begin{aligned}
d_0 \leq d(x_0, x) + d(x, H^i_{z,r}) &= d(x_0, x_1) - d(x, x_1) + d(x, H^i_{z,r}) \\
&\leq d(x_0, y_1) + 16\delta - L_\varepsilon(r) + d(x, H^i_{z,r}) \\
&< d(x_0, y_1) + 16\delta - L_\varepsilon(r) + \mathcal{B}_z(H_z, x) + L_\varepsilon(r) - 40\delta \\
&\leq d(x_0, y_1) + 16\delta + d(x, x_1) - 40\delta \\
&\leq \left(\frac{d_0}{2} + 20\delta\right) + 16\delta + \left(\frac{d_0}{2} + 4\delta - 40\delta\right) \leq d_0
\end{aligned}$$

a contradiction. So $x \in B(y_1, L_\varepsilon(r))$, hence again we deduce that $x \in X^i_\varepsilon$ concluding the proof of (b).

Let us now show (c) and (d). First of all C_r is Γ^i_ε-invariant since X^i_r, H_z and $\mathcal{B}_z(H_z, \cdot)$ are, by Lemma 2.5.2 (observe that here we do not use (a)). Moreover if we consider any point x' belonging to a geodesic ray $[x, z]$ with $x \in C_r$ we notice that:

$$\mathcal{B}_z(H_z, x') = \mathcal{B}_z(H_z, x) + d(x, x'), \qquad d(H^i_{z,r}, x') \leq d(H^i_{z,r}, x) + d(x, x'),$$

so x' satisfies again (45), which proves that C_r is a geodesic cone of vertex z. Finally C_r is connected. Indeed choose any $y \in H^i_{z,r}$, pick a point $x \in C_r$ and consider the geodesic ray $[x, z]$, which is contained in C_r. The geodesic rays $[x, z]$ and $[y, z]$ are definitely 8δ-close, so by Lemma 2.3.2 we find two points x' along $[x, z]$ and $y' \in [y, z]$ satisfying $d(x', y') \leq 8\delta$. This implies $\mathcal{B}_z(H_z, y') \leq \mathcal{B}_z(H_z, x') + 8\delta$, therefore by (46):

$$\begin{aligned}
d(x', H^i_{z,r}) \leq d(x', y') + d(y', H^i_{z,r}) &< 8\delta + \mathcal{B}_z(H_z, y') + L_\varepsilon(r) - 56\delta \\
&\leq \mathcal{B}_z(H_z, x') + L_\varepsilon(r) - 40\delta.
\end{aligned}$$

A similar estimate is true for every point of the geodesic segment $[x', y']$. So every point of the path $[x, x'] \cup [x', y'] \cup [y', y]$ satisfies (45), which shows that every point of C_r can be connected to the chosen point y and thus that C_r is connected. In fact, the same proof shows that any geodesic ray with endpoint z definitely belongs to C_r, that is (d).

The proof of (a) uses the same ideas: let $r' > 0$ small enough to have $L_r(r') > 8\delta$, where $L_r(r')$ is given by Proposition 2.5.5.(b). We fix a point y that is displaced by some $g \in \Gamma^i_\varepsilon$ less than r'; by convexity the same is true for every point along $[y, z]$.

We take now a point $x \in X^i_\varepsilon \cap X_r$, so there exists $g' \in \Gamma^i_\varepsilon$ such that $d(x, g'x) < r$. Here the important fact is that the fixed point at infinity of g' is again z. As usual we can find two points $x' \in [x, z]$ and $y' \in [y, z]$ at distance $\leq 8\delta$ by Lemma 2.3.2. Since $L_r(r') > 8\delta$ we apply Proposition 2.5.5.(b) to conclude that every point of the geodesic segment $[x', y']$ is contained in $X^i_\varepsilon \cap X_r$. So the whole path $[x, x'] \cup [x', y'] \cup [y', y]$ is contained in this set. Notice that the isometry that displaces the points less than r along this path may change from point to point. However this shows that $X^i_\varepsilon \cap X_r$ is connected since y is fixed, proving (a).

Let us now show (e). The map p being Γ^i_ε-invariant, it induces a map $\pi\colon \hat{X} \to \bar{X}$ satisfying $\pi \circ \hat{p} = p$. This map is clearly surjective since p is. Let us now show that π is injective. Let $y, y' \in B(C_r, R)$, for $R < \frac{1}{4}(R_\varepsilon - \varepsilon)$ and R_ε as in (44), and assume that $p(y) = p(y')$. Then there exists $h \in \Gamma$ such that $y = hy'$. We take projections $x, x' \in C_r$ respectively of y and y'. From (b) we know that $d(x, gx) < \varepsilon$ and $d(x', g'x') < \varepsilon$ for some $g, g' \in \Gamma^i_\varepsilon$. This implies:

$$d(x', h^{-1}ghx') \leq 2R + d(hy', ghy') = 2R + d(y, gy) \leq 4R + \varepsilon.$$

Since $4R + \varepsilon < R_\varepsilon$ we conclude by Lemma 6.4.3 that g' and $h^{-1}gh$ belong to the same maximal, elementary group, namely Γ^i_ε. Now

$$h^{-1}\mathrm{Fix}_\partial(g) = \mathrm{Fix}_\partial(h^{-1}gh) = \mathrm{Fix}_\partial(g') = \mathrm{Fix}_\partial(g),$$

which implies $h \in \Gamma^i_\varepsilon$. This means that $\hat{p}(y) = \hat{p}(y')$, so π is injective. In conclusion, as π is the restriction of a local isometry and is bijective, it is an isometry.

Finally let us show (f). We first prove that the maps $\hat{p}$ and p are injective when restricted to $[x, z]$, with $x \in C_r$. Assume that there exists $x' \in [x, z]$ and $h \in \Gamma$ such that $hx' \in [x, z]$. Up to replacing h with h^{-1} we may suppose $hx' \in [x', z]$. Then let $g \in \Gamma^i_\varepsilon$ such that $d(x, gx) < \varepsilon$, which exists by (b). By convexity we have $d(x', gx') < \varepsilon$ and $d(x', h^{-1}ghx') = d(hx', ghx') < \varepsilon$. So $h^{-1}gh$ is an isometry that moves x' less than ε. Therefore $h^{-1}gh$ is in the same maximal elementary group containing g, namely Γ^i_ε. This implies that $h^{-1}gh$ is of parabolic type and, as $h^{-1}\mathrm{Fix}_\partial(g) = \mathrm{Fix}_\partial(h^{-1}gh) = \mathrm{Fix}_\partial(g)$, we deduce once again that $z \in \mathrm{Fix}_\partial(h)$, so h is parabolic with fixed point z. Moreover the isometry h sends the geodesic ray $[x', z]$ to the geodesic ray $[hx', z]$, but by Lemma 2.5.2 we necessarily have $hx' = x'$, which implies that $h = \mathrm{id}$, since Γ has no elliptic elements. This shows that the maps $\hat{p}$ and p, when restricted to $[x, z]$, are injective local isometries with images $\hat{p}([y, z])$ and $p([y, z])$ respectively. By Proposition I.3.28 of [BH13], the restriction of $\hat{p}$ and p to $[x, z]$ are locally isometric coverings, and since they are bijective then they are isometries, concluding the proof of (f). □

Chapter 7

Critical exponent of discrete groups of isometries

In this chapter we introduce the notion of critical exponent of a group of isometries Γ, showing how it can be seen as another notion of entropy. In order to highlight the relations between the critical exponent and the other definitions of entropies we gave in Chapter 4 we will first define the versions of those invariants relative to subsets of the boundary at infinity. In the second part of the chapter we will prove Theorem P which is one of the main results of this thesis.

7.1 Entropies of subsets of the boundary

Let X be a complete, convex, geodesically complete, δ-hyperbolic metric space that is P_0-packed at scale r_0. In this section we will consider a subset C of ∂X and we define the relative version, with respect to C, of all the different definitions of entropies introduced in Chapter 4. We observe that when $C = \partial X$ then we are in the case yet studied. We will be interested especially to the subsets C related to the limit set of a discrete group of isometries acting on X.
We start with a couple of basic, although fundamental, lemmas relating geodesic rays and lines with endpoints in C.

Lemma 7.1.1. *Let X be a proper, δ-hyperbolic metric space. Let γ be a geodesic line and $x \in X$ with $S := d(\gamma(0), x)$. Let x' be a projection of x on γ. Then*

(a) there exists an orientation of γ such that $[x,x'] \cup [x', \gamma^+]$ is a $(1,4\delta)$-quasigeodesic.

(b) with respect to the orientation of (a) there exists a geodesic ray ξ starting at x such that $d(\xi(S+t), \gamma(t)) \leq 76\delta$ for all $t \geq 0$. In particular $\xi^+ = \gamma^+$;

(c) for all orientations of γ there exists a geodesic ray ξ starting at x such that $d(\xi(S+t), \gamma(t)) \leq 2S + 76\delta$ for all $t \geq 0$. Also in this case $\gamma^+ = \xi^+$.

Proof. We choose the orientation of γ such that x' belongs to the negative ray $\gamma|_{(-\infty, 0]}$ and we take the geodesic ray $\xi = [x, \gamma^+]$. By Lemma 2.3.9 the path $\alpha = [x,x'] \cup [x', \gamma^+]$ is a $(1,4\delta)$-quasigeodesic and moreover it satisfies $d(\xi(S+t), \alpha(S+t)) \leq 72\delta$ for every $t \geq 0$. Furthermore the time t_0 such that $\alpha(t_0) = \gamma(0)$ is between S and

$S + 4\delta$ implying $d(\xi(S+t), \gamma(t)) \leq 76\delta$.
For the second part of the proof we suppose to be in the situation above and we consider the geodesic ray $\xi = [x, \gamma^-]$. By Lemma 2.3.9 the path $\alpha = [x, x'] \cup [x', \gamma^-]$ satisfies $d(\xi(S+t), \alpha(S+t)) \leq 72\delta$ for every $t \geq 0$. Furthermore for every $t \geq 0$ the point $\alpha(S+t)$ belongs to γ and $d(\alpha(S+t), \gamma(0)) \leq d(\alpha(S+t), x) + d(x, \gamma(0)) \leq 2S + t + 4\delta$. So $d(\xi(S+t), \gamma(t)) \leq 76\delta + 2S$. □

We remark that if $\gamma(0)$ is a projection of x on γ then the first part of the lemma holds for both the positive and negative rays of γ.

Lemma 7.1.2. *Let X be a proper, δ-hyperbolic metric space. Let $x \in X$ and $C \subseteq \partial X$ be a subset with at least two points. Then there exists $L > 0$ such that for every geodesic ray ξ with $\xi(0) = x$ and $\xi^+ \in C$ there exists a geodesic line γ with $\gamma^\pm \in C$ such that $d(\xi(t), \gamma(t)) \leq L$ for all $t \in [0, +\infty)$. Moreover if $x \in$ QC-Hull(C) then L depends only on δ.*

Proof. Let z, z' be two distinct points of C. Let $D_{x,a}$ be a standard visual metric of parameter a centered at x and let $m = \frac{D_{x,a}(z,z')}{2}$. We have that either $D_{x,a}(\xi^+, z) \geq m$ or $D_{x,a}(\xi^+, z') \geq m$. We suppose it holds the first case and we choose a geodesic line γ joining z and ξ^+. We parametrize γ in such a way that $\gamma(0)$ is a projection of x on γ. Then $m \leq D_{x,a}(z, \xi^+) \leq e^{-a(z,\xi^+)_x}$. If S denotes $d(x, \gamma(0)) = d(\xi(0), \gamma(0))$ and $\xi_z = [x, z]$ then by the previous lemma and the remark below we have $d(\xi(S+t), \xi_z(S+t)) \leq d(\gamma(t), \gamma(-t)) + 152\delta = 2t + 152\delta$. Therefore

$$(z, \xi^+)_x \geq \frac{1}{2} \liminf_{t \to +\infty} [2(S+t) - d(\xi^+(S+t), \xi_z(S+t))] \geq S - 76\delta$$

implying $e^{-a(S-76\delta)} \geq m$. This means $d(\gamma(0), \xi(0)) \leq \frac{1}{a} \log \frac{1}{m} + 76\delta =: L'$. The thesis follows with $L := L' + 76\delta$ by the previous lemma.
We observe that L depends on δ, a and m, and the choice of a depends only on δ. Moreover when $x \in$ QC-Hull(C) we can take $z, z' \in C$ such that $x \in [z, z']$. With this choice m equals $\frac{1}{2}$ showing that L depends only on δ. □

Remark 7.1.3. *If $C' \subseteq C \subseteq \partial X$ and $x \in X$ is fixed then the constant L given by the previous lemma relative to C works also for C', provided C' has at least two points, as follows by the proof.*

7.1.1 Covering and volume entropy

Let X be a complete, convex, geodesically complete, δ-hyperbolic metric space that is P_0-packed at scale r_0 and let C be a subset of ∂X. The *upper covering entropy* of C is defined as

$$\limsup_{T \to +\infty} \frac{1}{T} \log \mathrm{Cov}(\overline{B}(x, T) \cap \overline{B}(\text{QC-Hull}(C), \sigma), r),$$

where $r > 0$, $\sigma \geq 0$ and $x \in X$ and it is denoted by $\overline{h_{\mathrm{Cov}}}(C)$. The *lower covering entropy* of C, denoted by $\underline{h_{\mathrm{Cov}}}(C)$, is defined taking the limit inferior instead of the limit superior. These quantities do not depend on $x \in X$ as usual. The analogous of Proposition 4.1.1 holds.

Proposition 7.1.4. *Let X be a complete, convex, geodesically complete, δ-hyperbolic metric space that is P_0-packed at scale r_0, C be a subset of ∂X and $x \in X$. Then*

$$\frac{1}{T}\log \mathrm{Cov}(\overline{B}(x,T)\cap \overline{B}(\text{QC-Hull}(C),\sigma), r) \underset{P_0,r_0,r,r',\sigma,\sigma'}{\asymp} \frac{1}{T}\log \mathrm{Pack}(\overline{B}(x,T)\cap \overline{B}(\text{QC-Hull}(C),\sigma'), r')$$

for all $r, r' > 0$ and $\sigma, \sigma' \geq 0$. In particular any of these functions can be used in the definition of the upper and lower covering entropies of C.

Proof. Once σ is fixed the asymptotic estimate can be proved exactly as in Proposition 4.1.1. Moreover for all $\sigma \geq 0$ it is easy to prove that

$$\mathrm{Cov}(\overline{B}(x,T)\cap \overline{B}(\text{QC-Hull}(C),\sigma), r) \leq \mathrm{Cov}(\overline{B}(x,T)\cap \text{QC-Hull}(C), r_0)\cdot \mathrm{Cov}(r_0+\sigma, r_0).$$

and $\mathrm{Cov}(r_0+\sigma, r_0)$ is uniformly bounded in terms of P_0, r_0 and σ by Proposition 2.4.4. This concludes the proof. □

Clearly when $C = \partial X$ we have $\overline{h_{\mathrm{Cov}}}(\partial X) = \overline{h_{\mathrm{Cov}}}(X)$. Moreover if C is a closed subset of ∂X then $\overline{h_{\mathrm{Cov}}}(C) \leq \overline{h_{\mathrm{Cov}}}(\partial X)$, so $\overline{h_{\mathrm{Cov}}}(C) \leq \frac{\log(1+P_0)}{r_0}$ by Lemma 4.1.3. The analogous of Proposition 4.1.2 holds. We remark that in this case a dependence on δ appears.

Proposition 7.1.5. *Let X be a complete, convex, geodesically complete, δ-hyperbolic metric space that is P_0-packed at scale r_0, C be a subset of ∂X and $x \in X$. Then*

$$\frac{1}{T}\log \mathrm{Cov}(\overline{B}(x,T)\cap \text{QC-Hull}(C), r) \underset{P_0,r_0,r,\delta}{\asymp} \frac{1}{T}\log \mathrm{Cov}(S(x,T)\cap \text{QC-Hull}(C), r)$$

In particular any of these functions can be used in the definition of the upper and lower covering entropies of C.

Proof. As in the proof of Proposition 4.1.2 one inequality is obvious, so we are going to prove the other. We divide the ball $\overline{B}(x,T)$ in the annulii $A(x, kr, (k+1)r)$ with $k = 0, \ldots, \frac{T}{r}-1$. Therefore we can estimate the quantity $\mathrm{Cov}(\overline{B}(x,T)\cap \text{QC-Hull}(C), 72\delta + 2r)$ from above by

$$\sum_{k=0}^{\frac{T}{r}-1} \mathrm{Cov}(A(x,kr,(k+1)r)\cap \text{QC-Hull}(C), 72\delta+2r).$$

We claim that every element of the sum is $\leq \mathrm{Cov}(S(x,T)\cap \text{QC-Hull}(C), r)$. Indeed let $y_1, \ldots, y_N$ be a set of points realizing $\mathrm{Cov}(S(x,T)\cap \text{QC-Hull}(C), r)$. For all $i = 1, \ldots, N$ we consider the geodesic segment $\gamma_i = [x, y_i]$ and we call x_i the point along this geodesic at distance kr from x. We want to show that $x_1, \ldots, x_N$ is a $(72\delta + 2r)$-dense subset of $A(x, kr, (k+1)r)\cap \text{QC-Hull}(C)$. Given a point $y \in A(x, kr, (k+1)r)\cap \text{QC-Hull}(C)$ there exists a geodesic line γ with endpoints in C containing y. We parametrize γ so that $\gamma(0)$ is a projection of x on γ and $y \in \gamma|_{[0,+\infty)}$. We take a point $y_T \in \gamma|_{[0,+\infty)}$ at distance T from x, so that $y_T \in$

$S(x,T)\cap\text{QC-Hull}(C)$ and therefore there exists i such that $d(y_T, y_i) \leq r$. By Lemma 2.3.9 the path $\alpha = [x, \gamma(0)] \cup [\gamma(0), y_T]$ is a $(1, 4\delta)$-quasigeodesic and, if t_y denotes the real number such that $\alpha(t_y) = y$, it holds $t_y \in [kr, (k+1)r]$. By Lemma 2.3.9 we get $d(y, \gamma_i'(t_y)) \leq 72\delta$, where $\gamma_i' = [x, y_T]$. We conclude the proof of the claim since $d(y, x_i) \leq d(y, \gamma_i'(t_y)) + d(\gamma_i'(t_y), \gamma_i(t_y)) + d(\gamma_i(t_y), x_i) \leq 72\delta + 2r$, from the convexity of the metric. We remark that using the ideas of Lemma 4.3.4 it is possible to obtain a similar estimate without using the convexity. The thesis follows by Proposition 7.1.4. □

The *upper volume entropy* of C with respect to a measure μ is

$$\overline{h_\mu}(C) = \sup_{\sigma \geq 0} \limsup_{T \to +\infty} \frac{1}{T} \log \mu(\overline{B}(x,T) \cap \overline{B}(\text{QC-Hull}(C), \sigma)),$$

where $x \in X$. Taking the limit inferior instead of the limit superior is defined the *lower volume entropy* of C, $\underline{h_\mu}(C)$.

Proposition 7.1.6. *Let X be a complete, convex, geodesically complete, δ-hyperbolic metric space that is P_0-packed at scale r_0, let C be a subset of ∂X and let μ be a measure on X which is H-homogeneous at scale r. Then for all $\sigma \geq r$ it holds*

$$\frac{1}{T} \log \mu(\overline{B}(x,T) \cap \overline{B}(\text{QC-Hull}(C), \sigma)) \underset{H,P_0,r_0,r,\sigma}{\asymp} \frac{1}{T} \log \text{Cov}(\overline{B}(x,T) \cap \text{QC-Hull}(C), r_0).$$

In particular the upper (resp. lower) volume entropy of C with respect to μ coincides with the upper (resp. lower) covering entropy of C and it can be computed using $\sigma = r$ in place of the supremum.

Proof. By Remark 4.1.5 we know that μ is $H(\sigma)$-homogeneous at scale σ for all $\sigma \geq r$, where $H(\sigma)$ depends on P_0, r_0, σ, r, H. Therefore the proof of Proposition 4.1.4 works in this case. □

7.1.2 Lipschitz topological entropy

Let X be a complete, convex, geodesically complete, δ-hyperbolic metric space that is P_0-packed at scale r_0. For a subset C of ∂X and $Y \subseteq X$ we set

$$\text{Geod}(Y, C) = \{\gamma \in \text{Geod}(X) \text{ s.t. } \gamma^\pm \subseteq C \text{ and } \gamma(0) \in Y\}.$$

If $Y = X$ we simply write $\text{Geod}(C)$. Clearly $\text{Geod}(C)$ is a Φ-invariant subset of $\text{Geod}(X)$, so the geodesic flow is well defined on it. The *upper Lipschitz-topological entropy* of $\text{Geod}(C)$ is defined as

$$\overline{h_{\text{Lip-top}}}(\text{Geod}(C)) = \inf_{\mathrm{d}} \sup_{K} \lim_{r \to 0} \limsup_{T \to +\infty} \frac{1}{T} \log \text{Cov}_{\mathrm{d}^T}(K, r),$$

where the infimum is taken among all geometric metrics on $\text{Geod}(C)$. The *lower Lipschitz-topological entropy* is defined taking the limit inferior instead of the limit superior and it is denoted by $\underline{h_{\text{Lip-top}}}(\text{Geod}(C))$. In the following crucial result we observe the difference between closed and non-closed subsets of ∂X that is the basis of the difference between the Hausdorff and the Minkowski dimension of the limit set of a discrete group of isometries.

Theorem 7.1.7. *Let X be a complete, convex, geodesically complete, δ-hyperbolic metric space that is P_0-packed at scale r_0 and C be a subset of ∂X. Then*

$$\overline{h_{\text{Lip-top}}}(\text{Geod}(C)) = \sup_{C' \subseteq C} \overline{h_{\text{Cov}}}(C'),$$

where the supremum is among closed subsets C' of C. The same holds for the lower entropies.

We remark that the supremum of the covering entropies among the closed subsets of C can be strictly smaller than the covering entropy of C, marking the distance between the equivalences of the different notions of entropies in case of non-closed subsets of the boundary. We start with an easy lemma.

Lemma 7.1.8. *Let X and C be as in Theorem 7.1.7 and let $x \in X$. Then every compact subset of* $\text{Geod}(C)$ *is contained in* $\text{Geod}(\overline{B}(x, R), C')$ *for some $R \geq 0$ and some $C' \subseteq C$ closed. Moreover* $\text{Geod}(\overline{B}(x_0, R), C')$ *is compact for all $R \geq 0$ and all closed $C' \subseteq C$.*

Proof. We fix a compact subset K of $\text{Geod}(C)$. The continuity of the evaluation map gives that $E(K)$ is contained in some ball $\overline{B}(x, R)$. Moreover the maps $+, - \colon \text{Geod}(X) \to \partial X$, defined by $\gamma \mapsto \gamma^+, \gamma^-$ respectively, are continuous ([BL12], Lemma 1.6). This means that $C' = +(K) \cup -(K)$ is a closed subset of ∂X and clearly $K \subseteq \text{Geod}(\overline{B}(x, R), C')$. By a similar argument, and since the evaluation map is proper, it follows that the set $\text{Geod}(\overline{B}(x, R), C')$ is compact for all $R \geq 0$ and all $C' \subseteq C$ closed. □

For a metric $f \in \mathcal{F}$ and $C \subseteq \partial X$ we denote by $\overline{h_f}$ the upper metric entropy of $\text{Geod}(C)$ with respect to f, that is

$$\overline{h_f}(\text{Geod}(C)) = \sup_K \lim_{r \to 0} \limsup_{T \to +\infty} \frac{1}{T} \log \text{Cov}_{f^T}(K, r).$$

Taking the limit inferior instead of the limit superior we define the lower metric entropy of $\text{Geod}(C)$ with respect to f, denoted by $\underline{h_f}(\text{Geod}(C))$. The analogous of Proposition 4.2.3 is the following.

Proposition 7.1.9. *Let X be as in Theorem 7.1.7, C' be a closed subset of ∂X, $f \in \mathcal{F}$, $x \in X$ and L be the constant given by Lemma 7.1.2. Then*

(a) $\overline{h_f}(\text{Geod}(\overline{B}(x, R), C')) = \overline{h_f}(\text{Geod}(\overline{B}(x, L), C'))$ *for all $R \geq L$;*

(b) $\overline{h_f}(\text{Geod}(C')) = \overline{h_f}(\text{Geod}(\overline{B}(x, L), C')) \leq \overline{h_{\text{Cov}}}(C')$;

(c) The function $r \mapsto \limsup_{T \to +\infty} \frac{1}{T} \log \text{Cov}_{f^T}(\text{Geod}(\overline{B}(x, L), C'), r)$ *is constant.*

The same conclusions hold for the lower entropies.

We observe that applying the Key Lemma 4.2.4 we have directly the relative version of Corollary 4.2.5.

Corollary 7.1.10. *Let $f \in \mathcal{F}$, $x \in X$, $R \geq 0$ and $0 < r \leq r'$. Then*

$$\frac{1}{T}\log \mathrm{Cov}_{f^T}(\mathrm{Geod}(\overline{B}(x,R),C'),r') \underset{P_0,r_0,r,r',f}{\asymp} \frac{1}{T}\log \mathrm{Cov}_{f^T}(\mathrm{Geod}(\overline{B}(x,R),C'),r).$$

Proof of of Proposition 7.1.9. We fix $R \geq L$ and $T \geq 0$. We take a set $\gamma_1,\ldots,\gamma_N$ of geodesic lines realizing $\mathrm{Cov}_{f^T}(\mathrm{Geod}(\overline{B}(x,L),C'),r_0)$. Our aim is to show that $\gamma_1,\ldots,\gamma_N$ is a $(4R+2L+C(f)+76\delta+r_0)$-dense subset of $\mathrm{Geod}(\overline{B}(x,R),C')$. This, together with Corollary 7.1.10, will prove (a). We consider a geodesic line $\gamma \in \mathrm{Geod}(\overline{B}(x,R),C')$, so $d(\gamma(0),x) =: S \leq R$. By Lemma 7.1.1 there exists a geodesic ray ξ starting at x such that $d(\xi(S+t),\gamma(t)) \leq 2S+76\delta$ for all $t \geq 0$ and in particular ξ^+ belongs to C. Now we apply Lemma 7.1.2 to find a geodesic line $\gamma' \in \mathrm{Geod}(C')$ such that $d(\xi(t),\gamma'(t)) \leq L$ for all $t \geq 0$. Clearly we have $\gamma' \in \mathrm{Geod}(\overline{B}(x,L),C')$ and $d(\gamma'(S+t),\gamma(t)) \leq 2S+L+76\delta$ for all $t \geq 0$. Therefore $d(\gamma'(t),\gamma(t)) \leq 3S+L+76\delta$ for all $t \geq 0$. This implies that for all $t \in [0,T]$ we have

$$f^t(\gamma,\gamma') \leq \int_{-\infty}^{-t}(d(\gamma(0),\gamma'(0))+2|s|)f(s)ds + \int_{-t}^{+\infty}(3S+L+76\delta)f(s)ds.$$

Since $d(\gamma(0),\gamma'(0)) \leq L+S$ we get $f^t(\gamma,\gamma') \leq 4S+2L+C(f)+76\delta$ using the properties of f, and so $f^T(\gamma,\gamma') \leq 4R+2L+C(f)+76\delta$. Moreover, since $\gamma' \in \mathrm{Geod}(\overline{B}(x,L),C')$, there exists γ_i such that $f^T(\gamma',\gamma_i) \leq r_0$. This implies $f^T(\gamma,\gamma_i) \leq 4R+2L+C(f)+76\delta+r_0$.
We observe that (c) follows directly from the previous corollary.
The first equality in (b) follows by (a). In order to prove the inequality we fix $y_1,\ldots,y_N$ realizing $\mathrm{Cov}(S(x,T)\cap \text{QC-Hull}(C'))$. So there are $\gamma_i \in \mathrm{Geod}(C')$ such that $y_i \in \gamma_i$. By Lemma 7.1.1 there exists an orientation of γ_i such that, called $S_i = d(x,\gamma_i(0))$ and $T_i \geq 0$ such that $\gamma_i(T_i) = y_i$, we have $T \leq S_i + T_i \leq T + 4\delta$ and the geodesic ray $\xi_i = [x,\gamma_i^+]$ satisfies $d(\xi_i(S_i+t),\gamma_i(t)) \leq 76\delta$ for all $t \geq 0$. By Lemma 7.1.2 there exists $\gamma_i' \in \mathrm{Geod}(\overline{B}(x,L),C')$ such that $d(\gamma_i'(t),\xi_i(t)) \leq L$ for all $t \geq 0$. We claim that the set $\{\gamma_i'\}$ is $(6L+160\delta+2r_0+2C(f))$-dense in $\mathrm{Geod}(\overline{B}(x,L),C')$. By (a) and (c) this would imply the thesis. We fix $\gamma \in \mathrm{Geod}(\overline{B}(x,L),C')$, so there exists $y \in S(x,T)$ and $T_y \in [T-L,T+L]$ such that $\gamma(T_y) = y$ and therefore $d(y,y_i) \leq r_0$ for some i. We observe that we have $d(\gamma_i'(S_i+T_i),y_i) \leq L+76\delta$ and so $d(\gamma_i'(T),y_i) \leq L+80\delta$. Moreover $d(\gamma(T),y_i) \leq L+r_0$ implying $d(\gamma(T),\gamma_i'(T)) \leq 2L+80\delta+r_0$. Furthermore by definition $d(\gamma(0),\gamma_i'(0)) \leq 2L$, so by convexity $d(\gamma(t),\gamma_i'(t)) \leq 2L+80\delta+r_0$ for all $t \in [0,T]$. The thesis follows by the classical subdivision of the integral defining f into three parts, each estimated by the constants above. □

Proof of Theorem 7.1.7. We fix a geometric metric $\mathfrak{d}$ on $\mathrm{Geod}(C)$ and we denote by M the Lipschitz constant with respect to $\mathfrak{d}$ of the evaluation map E. By Remark 7.1.3 the constant L given by Lemma 7.1.2 can be chosen independently of $C' \subseteq C$, once x is fixed. Clearly we have

$$\sup_{R\geq 0, C'\subseteq C}\lim_{r\to 0}\limsup_{T\to+\infty}\frac{1}{T}\log \mathrm{Cov}_{\mathfrak{d}^T}(\mathrm{Geod}(\overline{B}(x,R),C'),r) \geq$$

$$\sup_{C'\subseteq C}\limsup_{T\to+\infty}\frac{1}{T}\log \mathrm{Cov}_{\mathrm{d}^T}(\mathrm{Geod}(\overline{B}(x,L),C'),r_0).$$

We fix geodesic lines $\gamma_1,\ldots,\gamma_N$ realizing $\mathrm{Cov}_{\mathrm{d}^T}(\mathrm{Geod}(\overline{B}(x,L),C'),r_0)$. Since $d(\gamma_i(0),x)\leq L$ for all $i=1,\ldots,N$ then there exists $t_i\in[T-L,T+L]$ such that $d(\gamma_i(t_i),x)=T$. We claim that the points $y_i=\gamma_i(t_i)\in S(x,T)\cap \text{QC-Hull}(C')$ are $(2L+80\delta+Mr_0)$-dense. By Proposition 7.1.5 this would imply

$$\overline{h_{\text{Lip-top}}}(\mathrm{Geod}(C))\geq \sup_{C'\subseteq C}\overline{h_{\text{Lip-top}}}(\mathrm{Geod}(C'))\geq \sup_{C'\subseteq C}\overline{h_{\mathrm{Cov}}}(C').$$

We fix $y\in S(x,T)\cap\text{QC-Hull}(C')$ and we select a geodesic line $\gamma\in\mathrm{Geod}(C')$ containing y. By Lemma 7.1.1, with an appropriate choice of the orientation of γ, the geodesic ray $\xi=[x,\gamma^+]$ satisfies $d(\xi(S+t),\gamma(t))\leq 76\delta$ for all $t\geq 0$, where $S=d(x,\gamma(0))$. By Lemma 7.1.2 there exists $\gamma'\in\mathrm{Geod}(\overline{B}(x,L),C')$ such that $d(\xi(t),\gamma'(t))\leq L$ for all $t\geq 0$, implying $d(\gamma'(S+t),\gamma(t))\leq L+76\delta$ for all $t\geq 0$. Denoting by T_y the real number such that $\gamma(T_y)=y$ we have by Lemma 7.1.1 that $T\leq S+T_y\leq T+4\delta$. Therefore we apply the previous estimate with $t=T_y$ obtaining $d(\gamma'(T),y)\leq d(\gamma'(T),\gamma'(S+T_y))+d(\gamma'(S+T_y),y)\leq L+80\delta$. Moreover there exists $i\in\{1,\ldots,N\}$ such that $\mathrm{d}^T(\gamma',\gamma_i)\leq r_0$ and in particular $d(\gamma'(T),\gamma_i(T))\leq Mr_0$. Therefore we get $d(y_i,y)\leq d(\gamma_i(t_i),\gamma_i(T))+d(\gamma_i(T),y)\leq 2L+80\delta+Mr_0$. The other inequality follows by Proposition 7.1.9. Indeed we have

$$\overline{h_{\text{Lip-top}}}(\mathrm{Geod}(C))\leq \sup_{C'\subseteq C}\overline{h_f}(\mathrm{Geod}(C'))\leq \sup_{C'\subseteq C}\overline{h_{\mathrm{Cov}}}(C').$$

□

Remark 7.1.11. *Let X be as in Theorem 7.1.7, $C\subseteq\partial X$ closed and $x\in$ QC-Hull(C). By the proof of Theorem 7.1.7, Lemma 7.1.2 and Remark 7.1.3 we obtain*

$$\frac{1}{T}\log\mathrm{Cov}(S(x,T)\cap\text{QC-Hull}(C),r_0)\underset{P_0,r_0,\delta,f}{\asymp}\frac{1}{T}\log\mathrm{Cov}_{f^T}(\mathrm{Geod}(\overline{B}(x,L),C),r_0)$$

for all $f\in\mathcal{F}$, where L depends only on δ.

For all $Y\subseteq X$ and $C\subseteq\partial X$ we denote by $\mathrm{Ray}(Y,C)$ the set of geodesic rays ξ with $\xi(0)\in Y$ and $\xi^+\in C$. When $Y=X$ we use the notation $\mathrm{Ray}(C)$ and in this case this set is invariant by the geodesic semi-flow. So it is defined in the usual way its upper and lower Lipschitz-topological entropy, denoted respectively by $\overline{h_{\text{Lip-top}}}(\mathrm{Ray}(C))$ and $\underline{h_{\text{Lip-top}}}(\mathrm{Ray}(C))$.

Proposition 7.1.12. *Let X and C be as in Theorem 7.1.7. Then*

(a) $\overline{h_{\text{Lip-top}}}(\mathrm{Ray}(C))$ equals $\sup_{C'\subseteq C}\limsup_{T\to\infty}\frac{1}{T}\log\mathrm{Cov}_{f^T}(\mathrm{Ray}(x,C'),r)$ independently of $f\in\mathcal{F}$, the point $x\in X$ and $r>0$, where the supremum is taken among the closed subsets of C.

(b) $\overline{h_{\text{Lip-top}}}(\mathrm{Ray}(C))=\overline{h_{\text{Lip-top}}}(\mathrm{Geod}(C))$;

(c) the equivalent asymptotic estimate of Remark 7.1.11 holds for the geodesic semi-flow.

The same conclusions hold for the lower entropies.

Proof. The inequality $\overline{h_{\text{Lip-top}}}(\text{Ray}(C')) \geq \overline{h_{\text{Cov}}}(C')$ for all closed $C' \subseteq C$ follows by the same proof of Theorem 7.1.7. The remaining part of the thesis can be proved in a similar way of Proposition 7.1.9 and we omit the details. □

7.1.3 Shadow and Minkowski dimension

The notions of shadow covering, shadow dimension and visual Minkowski dimension can be directly generalized to the case of subsets C of ∂X. The upper (resp. lower) shadow dimension of C will be denoted by $\overline{\text{Shad-D}}(C)$ (resp. $\underline{\text{Shad-D}}(C)$), while the upper (resp. lower) visual Minkowski dimension of C will be denoted by $\overline{\text{MD}}(C)$ (resp. $\underline{\text{MD}}(C)$).

Proposition 7.1.13. *Let X be a complete, convex, geodesically complete, δ-hyperbolic metric space that is P_0-packed at scale r_0. Let C be a subset of ∂X, $x \in X$ and L be the constant of Lemma 7.1.2. Then*

$$\frac{1}{T}\log \text{Cov}(S(x,T)\cap \text{QC-Hull}(C), r) \underset{P_0,r_0,\delta,r,L}{\asymp} \frac{1}{T}\log \text{Shad-Cov}_r(C, e^{-T}).$$

In particular the upper (resp. lower) shadow dimension of C equals the upper (resp. lower) covering entropy of C.

Proof. Let $y_1,\ldots,y_N$ be a set realizing $\text{Cov}(S(x,T)\cap \text{QC-Hull}(C), r)$. We fix $z \in C$ and we consider the geodesic ray $\xi = [x,z]$. By Lemma 7.1.2 there exists $\gamma \in \text{Geod}(\overline{B}(x,L), C)$ such that $d(\xi(t),\gamma(t)) \leq L$ for all $t \geq 0$. Let $t_y \in [T-L, t+L]$ such that $d(\gamma(t_y),x) = T$ and call $y = \gamma(t_y)$. Then there exists $i \in \{i,\ldots,N\}$ such that $d(y,y_i) \leq r$ and moreover $d(\xi(T),y) \leq 2L$, implying $[x,z]\cap B(y_i, 2L+2r) \neq \emptyset$. This shows that

$$\text{Shad-Cov}_{2L+2r}(C, e^{-T}) \leq \text{Cov}(S(x,T)\cap \text{QC-Hull}(C), r).$$

Now let $y_i,\ldots,y_N$ be points realizing $\text{Shad-Cov}_r(C,e^{-T})$. By the same argument used in the proof of Lemma 4.3.1 we can suppose $d(y_i,x) = T$. Let $y \in S(x,T)\cap \text{QC-Hull}(C)$ and let $\gamma \in \text{Geod}(C)$ such that $y \in \gamma$ oriented in such a way that, by Lemma 7.1.1, the geodesic ray $\xi = [x,\gamma^+]$ satisfies $d(\xi(S+t),\gamma(t)) \leq 76\delta$ for all $t \geq 0$, where $S = d(x,\gamma(0))$. By the same lemma we know, indicated by $t_y \geq 0$ the real number such that $\gamma(t_y) = y$, that $T \leq S + t_y \leq T + 4\delta$ implying $d(\xi(T),y) \leq 80\delta$. Moreover there exists $i \in \{1,\ldots,N\}$ such that $d(\xi(T),y_i) < 2r$, therefore $d(y,y_i) < 80\delta + 2r$. This shows that

$$\text{Cov}(S(x,T)\cap \text{QC-Hull}(C), 80\delta + 2r) \leq \text{Shad-Cov}_r(C, e^{-T}).$$

The thesis follows by Proposition 7.1.5 together with Proposition 7.1.4. □

By Lemma 4.3.1 we get immediately the following.

Proposition 7.1.14. *Let X be a complete, convex, geodesically complete, δ-hyperbolic metric space that is P_0-packed at scale r_0, let C be a subset of ∂X, $x \in X$ and L be the constant given by Lemma 7.1.2. Then*

$$\frac{1}{T}\log \text{Cov}(C, e^{-T}) \underset{P_0,r_0,\delta,L}{\asymp} \frac{1}{T}\log \text{Cov}(S(x,T)\cap \text{QC-Hull}(C), r_0).$$

In particular the upper (resp. lower) Minkowski dimension of C equals the upper (resp. lower) covering entropy of C.

We remark that the upper visual Minkowski dimension of C equals the upper visual Minkowski dimension of its closure $\overline{C}$ while it can happen that $\sup_{C'\subseteq C}\overline{\text{MD}}(C') < \overline{\text{MD}}(\overline{C})$, where the supremum is taken among the closed subsets of C.

The proofs of Theorems N and O follow by Proposition 7.1.4, Proposition 7.1.5, Proposition 7.1.6, Theorem 7.1.7, Proposition 7.1.9, Remark 7.1.11, Proposition 7.1.13 and Proposition 7.1.14.

7.2 Critical exponent of discrete groups of isometries

In this section we will specialize the study of the entropies to special subsets of the boundary at infinity of a complete, convex, geodesically complete, δ-hyperbolic metric space that is P_0-packed at scale r_0. In the first subsection we will introduce the entropy of X relative to a discrete group Γ of isometries of X and the critical exponent of Γ. In the second one we will study the special case of quasiconvex-cocompact groups.

7.2.1 General properties

Let X be a proper metric space and let Γ be a discrete group of isometries of X. The *critical exponent*[1] of Γ is

$$h_\Gamma := \inf\left\{ s \geq 0 \text{ s.t. } \sum_{g\in\Gamma} e^{-sd(x,gx)} < +\infty \right\}.$$

It does not depend on $x \in X$. We remark that for every $s \geq 0$ the series $\sum_{g\in\Gamma} e^{-sd(x,gx)}$, which is called the Poincaré series of Γ, is Γ-invariant. In other words $\sum_{g\in\Gamma} e^{-sd(x,gx)} = \sum_{g\in\Gamma} e^{-sd(x',gx')}$ for all $x' \in \Gamma x$.
The *upper* Γ*-entropy* of X is defined as

$$\overline{h_\Gamma}(X) = \limsup_{T\to+\infty} \frac{1}{T}\log \#\Gamma x \cap \overline{B}(x,T) = \limsup_{T\to+\infty}\frac{1}{T}\log\#\Sigma_T(x),$$

where the last equality follows from the finiteness of the stabilizers of a discrete group. The *lower* Γ*-entropy of* X is defined taking the limit inferior instead of the limit superior and it is denoted by $\underline{h_\Gamma}(X)$. They do not depend on $x \in X$. The following proposition is proved in the δ-hyperbolic case in [Coo93], but it remains true for proper metric spaces.

Lemma 7.2.1 (Proposition 5.3 of [Coo93]). *Let X be a proper metric space and let Γ be a discrete group of isometries of X. Then $\overline{h_\Gamma}(X) = h_\Gamma$.*

We remark that for CAT(-1) metric spaces X it holds $\overline{h_\Gamma}(X) = \underline{h_\Gamma}(X)$ for every discrete group of isometries of X, see [Rob02]. The Γ-entropy of X is also related to the covering entropy of the limit set $\Lambda(\Gamma)$.

[1] Usually it is denoted by $\delta(\Gamma)$ but we prefer the notation h_Γ in order to avoid the possibility of confusion with the hyperbolicity constant δ.

Lemma 7.2.2. *Let X be a complete, convex, geodesically complete, δ-hyperbolic metric space that is P_0-packed at scale r_0 and let Γ be a discrete group of isometries of X. Then:*

(a) $\overline{h_\Gamma}(X) \leq \overline{h_{\text{Cov}}}(\Lambda(\Gamma))$;

(b) if Γ is non-elementary and quasiconvex-cocompact with codiameter $\leq D$ and if $x \in \text{QC-Hull}(\Lambda(\Gamma))$ then

$$\frac{1}{T}\log \#\Gamma x \cap \overline{B}(x,T) \underset{P_0,r_0,\delta,D}{\asymp} \frac{1}{T}\log \text{Cov}(\overline{B}(x,T) \cap \text{QC-Hull}(\Lambda(\Gamma))).$$

In particular $\overline{h_\Gamma}(X) = \overline{h_{\text{Cov}}}(\Lambda(\Gamma))$.

The same conclusions hold for the lower entropies.

Proof. We fix $x \in \text{QC-Hull}(\Lambda(\Gamma))$ and $\varepsilon = \text{sys}^\diamond(\Gamma, x) > 0$. By the Γ-invariance of $\text{QC-Hull}(\Lambda(\Gamma))$ and the definition of the systole we get

$$\#\Gamma x \cap \overline{B}(x,T) \leq \text{Pack}\left(\overline{B}(x,T) \cap \text{QC-Hull}(\Lambda(\Gamma)), \frac{\varepsilon}{2}\right),$$

showing (a) by Proposition 7.1.4. In order to prove (b) we fix $x \in \text{QC-Hull}(\Lambda(\Gamma))$ and we call D the codiameter of the action. By (42) the free-systole of the action is bounded from below by a constant depending only on P_0, r_0, δ and D, so the proof of (a) shows the first half of the asymptotic estimate. Furthermore we claim that

$$\text{Pack}(\overline{B}(x,T) \cap \text{QC-Hull}(\Lambda(\Gamma)), D) \leq \#\Gamma x \cap \overline{B}(x, T+D).$$

Indeed let $y_1, \ldots, y_N$ be points realizing $\text{Pack}(\overline{B}(x,T) \cap \text{QC-Hull}(\Lambda(\Gamma)), D)$, so $d(y_i, y_j) > 2D$ for every $i \neq j$. For every i let x_i be a point of the orbit of x at distance at most D from y_i. It follows that the points x_i are all distinct, concluding the proof of the claim. The conclusion follows applying Proposition 7.1.4. □

Remark 7.2.3. *Under the assumptions of Lemma 7.2.2 then by Lemma 7.2.1 and the discussion after Proposition 7.1.4 we always have $h_\Gamma \leq \frac{\log(1+P_0)}{r_0} =: h^+$. Moreover if Γ is non-elementary and quasiconvex-cocompact with codiameter $\leq D$ then there exists $h^- > 0$ depending only on P_0, r_0, δ, D such that $h_\Gamma \geq h^-$. This follows from Lemma 7.2.1, Example 6.2.3 and the proof of Theorem 6.1.1.*

Let X be a proper, δ-hyperbolic metric space, let $x \in X$ and B be a Borelian subset of $\partial_G X$. Following [Pau96], for all $\alpha \geq 0$ and all $\eta > 0$ we set

$$\mathcal{H}^\alpha_\eta(B) = \inf\left\{\sum_{i\in\mathbb{N}} \rho_i^\alpha \text{ s.t. } B \subseteq \bigcup_{i\in\mathbb{N}} B(z_i,\rho_i) \text{ and } \rho_i \leq \eta\right\}.$$

As in the classical case the *visual α-dimensional Hausdorff measure* of B is defined as $\lim_{\eta\to 0}\mathcal{H}^\alpha_\eta(B) =: \mathcal{H}^\alpha(B)$, while the *visual Hausdorff dimension* of B is defined as the unique $\alpha \geq 0$ such that $\mathcal{H}^{\alpha'}(B) = 0$ for all $\alpha' > \alpha$ and $\mathcal{H}^{\alpha'}(B) = +\infty$ for all $\alpha' < \alpha$. The visual Hausdorff dimension of the borelian subset B is denoted by $\text{HD}(B)$. By Lemma 2.3.3, see also [Pau96], we have $\text{HD}(B) = a \cdot \text{HD}_{D_{x,a}}(B)$ for all visual metrics $D_{x,a}$ of center x and parameter a, where $\mathrm{HD}_{D_{x,a}}(B)$ denotes the classical Hausdorff dimension with respect to the metric $D_{x,a}$. Therefore we directly obtain the usual inequalities

$$\mathrm{HD}(B) \leq \underline{\mathrm{MD}}(B) \leq \overline{\mathrm{MD}}(B)$$

for all Borelian subsets B of $\partial_G X$. In Section 1.6.3 we have seen how these inequalities can be strict.

There is a canonical way to construct a measure on $\partial_G X$ starting from the Poincaré series. For every $s > h_\Gamma$ the measure

$$\mu_s = \frac{1}{\sum_{g\in\Gamma} e^{-sd(x,gx)}} \sum_{g\in\Gamma} e^{-sd(x,gx)} \Delta_{gx},$$

where Δ_{gx} is the Dirac measure at gx, is a probability measure on the compact space $X \cup \partial_G X$. Then there exists a sequence s_i converging to h_Γ such that μ_{s_i} converges $*$-weakly to a probability measure on $X \cup \partial_G X$. Any of these limits is called a Patterson-Sullivan measure and it is denoted by μ_{PS}.

Proposition 7.2.4 (Theorem 5.4 of [Coo93].). *Let X be a proper, δ-hyperbolic metric space and let Γ be a discrete group of isometries of X with $h_\Gamma < +\infty$. Then every Patterson-Sullivan measure is supported on $\Lambda(\Gamma)$. Moreover it is a Γ-quasi conformal density of dimension h_Γ, i.e. it satisfies*

$$\frac{1}{Q} e^{h_\Gamma(\mathcal{B}_z(x,x) - \mathcal{B}_z(x,g^{-1}x))} \leq \frac{d(g^*\mu_{\mathrm{PS}})}{d\mu_{\mathrm{PS}}}(z) \leq Q e^{h_\Gamma(\mathcal{B}_z(x,x) - \mathcal{B}_z(x,g^{-1}x))}$$

for every $g \in \Gamma$ and every $z \in \Lambda(\Gamma)$, where Q is a constant depending only on δ and an upper bound on h_Γ.

The quantification of Q is not explicitated in the original paper, but it follows from the proof therein.

7.2.2 The quasiconvex cocompact case

Let Γ be a discrete, quasiconvex-cocompact group of isometries of a proper, δ-hyperbolic metric space X. Then it is proved in [Coo93] that the Patterson-Sullivan measure on $\Lambda(\Gamma)$ is (A, h_Γ)-Ahlfors regular for some $A > 0$. We will precise this result quantifying the constant A in terms of universal constants in case X is also convex, geodesically complete and packed.

Theorem 7.2.5. *Let X be a complete, convex, geodesically complete, δ-hyperbolic metric space that is P_0-packed at scale r_0. Let Γ be a discrete, quasiconvex-cocompact group of isometries of X with codiameter $\leq D$ and x be a point of* QC-Hull$(\Lambda(\Gamma))$. *Then:*

(a) $\Lambda(\Gamma)$ *is visually (A, h_Γ)-Ahlfors regular with respect to any Patterson-Sullivan measure, where A depends only on P_0, r_0, δ and D.*

(b) it holds

$$\frac{1}{T} \log \mathrm{Cov}(\Lambda(\Gamma), e^{-T}) \underset{P_0,r_0,\delta,D}{\asymp} h_\Gamma;$$

(c) $\overline{\mathrm{MD}}(\Lambda(\Gamma)) = \underline{\mathrm{MD}}(\Lambda(\Gamma)) = h_\Gamma$.

We observe that (c) follows immediately from (b), while (b) is essentially straightforward once proved (a). Indeed we have

Lemma 7.2.6. *Suppose $C \subseteq \partial X$ is visually (A, s)-Ahlfors regular. Then*

$$\frac{1}{T} \log \mathrm{Cov}(C, e^{-T}) \underset{\delta, A, s}{\asymp} s.$$

We define the packing* number at scale ρ of a subset C of ∂X as the maximal number of disjoint generalized visual balls of radius ρ with center in C and we denote it by $\mathrm{Pack}^*(C, \rho)$.

Lemma 7.2.7. *For all $T \geq 0$ it holds* $\mathrm{Pack}^*(C, e^{-T+\delta}) \leq \mathrm{Cov}(C, e^{-T})$ *and* $\mathrm{Cov}(C, e^{-T+\delta}) \leq \mathrm{Pack}^*(C, e^{-T})$.

Proof. Let $z_1, \ldots, z_N$ be points of C realizing $\mathrm{Cov}(C, e^{-T})$. Suppose there exist points $w_1, \ldots, w_M$ of C such that $B(w_i, e^{-T+\delta})$ are disjoint, in particular $(w_i, w_j)_x \leq T - \delta$ for every $i \neq j$. If $M > N$ then two different points w_i, w_j belong to the same ball $B(z_k, \rho)$, i.e. $(z_k, w_i)_x > T$ and $(z_k, w_j)_x > T$. By (16) we have $(w_i, w_j)_x > T - \delta$ which is a contradiction. This shows the first inequality.
Now let $z_1, \ldots, z_N$ be a maximal collection of points of C such that $B(z_i, \rho)$ are disjoint. Then for every $z \in C$ there exists i such that $B(z, \rho) \cap B(z_i, \rho) \neq \emptyset$. Therefore there exists $w \in \partial X$ such that $(z_i, w)_x > T$ and $(z, w)_x > T$. As before we get $(z_i, z)_x > T - \delta$, proving the second inequality. □

Proof of Lemma 7.2.6. Since the measure μ in the definition of Ahlfors regularity is assumed to be of total measure one, we have

$$1 = \mu(C) \leq Ae^{-sT} \cdot \mathrm{Cov}(C, e^{-T}) \text{ and } 1 = \mu(C) \geq \frac{1}{A} e^{-sT} \cdot \mathrm{Pack}(C, e^{-T})$$

implying $\mathrm{Cov}(C, e^{-T}) \geq \frac{1}{A} e^{sT}$ and $\mathrm{Pack}(C, \rho) \leq Ae^{sT}$. Therefore

$$\frac{1}{T} \log \mathrm{Cov}(C, e^{-T}) \geq s + \frac{1}{T} \log \frac{1}{A}$$

and

$$\frac{1}{T} \log \mathrm{Cov}(C, e^{-T}) \leq \frac{1}{T} \log \mathrm{Pack}^*(C, e^{-T-\delta}) \leq s + \frac{1}{T} \log A + \frac{s\delta}{T}.$$

□

Proof of Theorem 7.2.5. As observed (c) follows from (b) and (b) follows from (a) applying Lemma 7.2.6 and the fact that $h_\Gamma \leq \frac{\log(1+P_0)}{r_0}$. In order to prove (a) we consider two cases: if Γ is elementary then $\#\Lambda(\Gamma) \in \{0, 2\}$ and $h_\Gamma = 0$. If this cardinality is 0 there is nothing to prove. If $\Lambda(\Gamma) = \{z^-, z^+\}$ then it is straightfoward to see that $\mu_{\mathrm{PS}}(z^-) = \mu_{\mathrm{PS}}(z^+) = \frac{1}{2}$.
If Γ is non-elementary we denote by $0 < h^- \leq h^+ < +\infty$ the numbers introduced in Remark 7.2.3. They depend only on P_0, r_0, δ and D. We will prove
(a') $\Lambda(\Gamma)$ is visually (A, h_Γ)-Ahlfors regular with respect to the Patterson-Sullivan measure, where A depends only on δ, h^-, h^+ and D.
We denote by L the constant given by Lemma 7.1.2 relative to x and $\Lambda(\Gamma)$, remarking that it depends only on δ.

Step 1: $\forall z \in \partial X$ *and* $\forall \rho > 0$ *it holds* $\mu_{\text{PS}}(B(z,\rho)) \leq e^{h_\Gamma(21\delta+3D+3L)}\rho^{h_\Gamma}$.

We suppose first $z \in \Lambda(\Gamma)$ and we take the set

$$\tilde{B}(z,\rho) = \left\{ y \in X \cup \partial X \text{ s.t. } (y,z)_x > \log\frac{1}{\rho} \right\}.$$

It is open (cp. Observation 4.5.2 of [DSU17]) and $\tilde{B}(z,\rho) \cap \partial X = B(z,\rho)$, so $\mu_{\text{PS}}(\tilde{B}(z,\rho)) = \mu_{\text{PS}}(B(z,\rho))$ since μ_{PS} is supported on $\Lambda(\Gamma) \subseteq \partial X$. Let $T = \log\frac{1}{\rho}$, $\xi_z = [x,z]$ and z_T be the point on ξ_z at distance T from x. For every $y \in \Gamma x \cap \tilde{B}(z,\rho)$ we have

$$d(x,y) > T - \delta \quad \text{and} \quad d(x,y) > d(x,z_T) + d(z_T,y) - 20\delta. \tag{47}$$

Indeed from $d(y, \xi_z(S)) \geq S - d(x,y)$ for all $S \geq 0$ we get $T < (y,z)_x \leq d(x,y) + \delta$. In order to prove the second inequality we extend $[x,y]$ to a geodesic ray ξ_w with $\xi_w^+ = w$. By the analogue of (16) we have $(w,z)_x \geq \min\{(y,z)_x, (y,w)_x\} - \delta > T - 2\delta$, where the last inequality follows from $d(x,y) > T - \delta$. By Lemma 4.3.4 we have $d(\xi_z(T-3\delta), \xi_w(T-3\delta)) \leq 4\delta$ and applying the triangular inequality we get $d(y, z_T) \leq 10\delta$ and the second estimate in (47).
Moreover by Lemma 7.1.2, since $x \in \text{QC-Hull}(\Lambda(\Gamma))$, there exists $\gamma \in \text{Geod}(\Lambda(\Gamma))$ such that $d(z_T, \gamma(T)) \leq L$. By the cocompactness of the action on QC-Hull$(\Lambda(\Gamma))$ we can find a point $x_1 \in \Gamma x$ such that $d(x_1, \gamma(T)) \leq D$, so $d(z_T, x_1) \leq L + D$. This actually implies $d(x,y) > d(x,x_1) + d(x_1,y) - 20\delta - 2D - 2L$ for all $y \in \Gamma x \cap \tilde{B}(z,\rho)$. Therefore

$$\begin{aligned}
\sum_{y \in \Gamma x \cap \tilde{B}(z,\rho)} e^{-sd(x,y)} &\leq \sum_{y \in \Gamma x \cap \tilde{B}(z,\rho)} e^{-s(d(x,x_1)+d(x_1,y)-20\delta-2D-2L)} \\
&= e^{s(20\delta+2D+2L)} e^{-sd(x,x_1)} \cdot \sum_{y \in \Gamma x \cap \tilde{B}(z,\rho)} e^{-sd(x_1,y)} \\
&\leq e^{s(20\delta+3D+3L)} e^{-sd(x,z_T)} \cdot \sum_{g \in \Gamma} e^{-sd(x_1,gx_1)} \\
&= e^{s(20\delta+3D+3L)} \cdot \rho^s \cdot \sum_{g \in \Gamma} e^{-sd(x,gx)}.
\end{aligned}$$

In other words we have $\mu_s(\tilde{B}(z,\rho)) \leq e^{s(20\delta+3D+3L)}\rho^s$, and by $*$-weak convergence we conclude that

$$\mu_{\text{PS}}(B(z,\rho)) = \mu_{\text{PS}}(\tilde{B}(z,\rho)) \leq \liminf_{i \to +\infty} \mu_{s_i}(\tilde{B}(z,\rho)) \leq e^{h_\Gamma(20\delta+3D+3L)}\rho^{h_\Gamma}.$$

In the general case of $z \in \partial X$ we observe that if $B(z,\rho) \cap \Lambda(\Gamma) = \emptyset$ then the thesis is obviously true since μ_{PS} is supported on $\Lambda(\Gamma)$. Otherwise there exists $w \in \Lambda(\Gamma)$ such that $(z,w)_x > \log\frac{1}{\rho}$. It is straightforward to check that $B(w,\rho) \subseteq B(z, \rho e^\delta)$ by (16). Then $\mu_{\text{PS}}(B(z,\rho)) \leq e^{h_\Gamma(21\delta+3D+3L)}\rho^{h_\Gamma}$.

Step 2: *for every* $R \geq R_0 := \frac{\log 2}{h_\Gamma} + 21\delta + 3D + 3L + 5\delta$ *and for every* $g \in \Gamma$ *it holds* $\mu_{\text{PS}}(\text{Shad}_x(gx,R)) \geq \frac{1}{2Q}e^{-h_\Gamma d(x,gx)}$*, where* Q *is the constant of Proposition 7.2.4 that depends only on* δ *and* h^+*.*

From the first step we know that for every $\rho \leq \rho_0 := 2^{-\frac{1}{h_\Gamma}} e^{-(21\delta+3D+3L)}$ and for every $z \in \partial X$ it holds $\mu_{\text{PS}}(B(z,\rho)) \leq \frac{1}{2}$. A direct computation shows that $R_0 = \log\frac{1}{\rho_0} + 5\delta$.

We claim that for every $R \geq R_0$ and every $g \in \Gamma$ the set $\partial X \setminus g(\text{Shad}_x(g^{-1}x, R))$ is contained in a generalized visual ball of radius at most ρ_0. Indeed if $z, w \in \partial X \setminus g(\text{Shad}_x(g^{-1}x, R))$ then the geodesic rays $[gx, z], [gx, w]$ do not intersect the ball $B(x, R)$. Therefore, setting $\xi = [gx, z]$, we get $(\xi(T), gx)_x \geq d(x, [gx, \xi(T)]) - 4\delta \geq R - 4\delta$ by Lemma 2.3.5. This implies $(z, gx)_x \geq \liminf_{T\to+\infty}(\xi(T), gx)_x \geq R - 4\delta$ and the same holds for w. Thus by (16) we get $(z, w)_x \geq R - 5\delta$, proving the claim. By Proposition 7.2.4 we get

$$\frac{\mu_{\text{PS}}(\text{Shad}_x(gx, R))}{\mu_{\text{PS}}(g^{-1}(\text{Shad}_x(gx, R)))} \geq \frac{1}{Q} e^{-h_\Gamma(\mathcal{B}_z(x,x) - \mathcal{B}_z(x,gx))}.$$

Since $R \geq R_0$ the measure of $g^{-1}(\text{Shad}_x(gx, R))$ is at least $\frac{1}{2}$ and the Busemann function is 1-Lipschitz (Lemma 2.3.1), so

$$\mu_{\text{PS}}(\text{Shad}_x(gx, R)) \geq \frac{1}{2Q} e^{-h_\Gamma d(x,gx)}.$$

Step 3. *For every* $z \in \text{QC-Hull}(\Lambda(\Gamma))$ *and every* $\rho > 0$ *the following is true:* $\mu_{\text{PS}}(B(z, \rho)) \geq \frac{1}{2Q} e^{-h_\Gamma(R_0+\delta+2D+2L)} \rho^{h_\Gamma}$.

For every $\rho > 0$ we set $T = \log \frac{1}{\rho}$. We first want to show that if $z \in \partial X$ and $R \geq 0$ then $\text{Shad}_x(\xi_z(T + R), R) \subseteq B(z, e^{-T})$. Indeed if $w \in \partial X$ is a point such that the geodesic ray $\xi_w = [x, w]$ passes through $B(\xi_z(T + R), R)$ then $d(\xi_z(T + R), \xi_w(T + R)) < 2R$ and by Lemma 4.3.4 we get $(z, w)_x > T$. We take $R = R_0 + L + D$, where R_0 is the constant of the second step and we conclude that $\text{Shad}_x(\xi_z(T + R), R)$ is contained in $B(z, \rho)$. By Lemma 7.1.2 it is possible to find a geodesic line $\gamma \in \text{Geod}(\Lambda(\Gamma))$ such that $d(\gamma(T + R), \xi_z(T + R)) \leq L$. Moreover there exists $g \in \Gamma$ such that $d(gx, \gamma(T + R)) \leq D$, implying $\text{Shad}_x(gx, R_0) \subseteq \text{Shad}_x(\xi_z(T + R), R) \subseteq B(z, \rho)$. From the second step we obtain $\mu_{\text{PS}}(B(z, \rho)) \geq \frac{1}{2Q} e^{-h_\Gamma d(x,gx)}$. Furthermore $d(x, gx) \leq T + R_0 + 2L + 2D$, so finally

$$\mu_{\text{PS}}(B(z, \rho)) \geq \frac{1}{2Q} e^{-h_\Gamma(R_0+\delta+2L+2D)} \rho^{h_\Gamma}.$$

The explicit description of the constants shows as they depend only on δ, h^-, h^+ and D, proving (a') and so the theorem. □

As a consequence we have a uniform asymptotic behaviour for the Γ-entropy of a discrete, non-elementary, quasiconvex-cocompact group. Indeed by Lemma 7.2.2 and Proposition 7.1.14 we get

$$\frac{1}{T} \log \#\Gamma x \cap \overline{B}(x, T) \underset{P_0, r_0, \delta, D}{\asymp} h_\Gamma,$$

where $x \in \text{QC-Hull}(\Lambda(\Gamma))$. We remark that similar results can be obtained for the covering entropy, the Lipschitz-topological entropy, and the shadow dimension. This uniform convergence to the limit will be the key of the continuity results proved in the last chapter.

Actually for the Γ-entropy we can improve the rate of convergence following again the ideas of [Coo93].

Theorem 7.2.8. *Let X be a complete, convex, geodesically complete, δ-hyperbolic metric space that is P_0-packed at scale r_0. Let Γ be a discrete, quasiconvex-cocompact group of isometries of X with codiameter $\leq D$ and x be a point of* QC-Hull($\Lambda(\Gamma)$)*. Then there exists $K > 0$ depending only on P_0, r_0, δ and D such that for all $T \geq 0$ it holds*

$$\frac{1}{K} \cdot e^{T \cdot h_\Gamma} \leq \Gamma x \cap \overline{B}(x, T) \leq K \cdot e^{T \cdot h_\Gamma}.$$

Proof. We denote by $s_0 = s_0(P_0, r_0, \delta, D)$ the number given by (42), by $R_0 = R_0(P_0, r_0, \delta, D)$ the number of Step 2 of Theorem 7.2.5, by Q the constant of Proposition 7.2.4 and by L the constant of Lemma 7.1.2 that depends only on δ. Moreover we set $N_0 = \text{Pack}\left(4R_0 + 1, \frac{s_0}{2}\right)$, which depends only on P_0, r_0, δ, D by Proposition 2.4.4. It is easy to check that if $[x, z] \cap B(y, R_0) \neq \emptyset$ and $[x, z] \cap B(y', R_0) \neq \emptyset$, where $z \in \partial X$ and y, y' are points of X with $|d(x, y) - d(x, y')| \leq 1$, then $d(y, y') \leq 4R_0 + 1$. Therefore for every $k \in \mathbb{N}$ we have

$$\#\{y \in \Gamma x \text{ s.t. } y \in A(x, k, k+1) \text{ and } z \in \text{Shad}_x(y, R_0)\} \leq N_0.$$

Step 1. *For all $k \in \mathbb{N}$ it holds $\#\Gamma x \cap \overline{B}(x, k) \leq 4QN_0 e^{h_\Gamma k}$.*

Let $A_j = \Gamma x \cap A(x, j, j+1)$. By the observation made before we conclude that among the set of shadows $\{\text{Shad}_x(y, R_0)\}_{y \in A_j}$ there are at least $\frac{\#A_j}{N_0}$ disjoint sets. Thus

$$1 \geq \mu_{\text{PS}}\left(\bigcup_{y \in A_j} \text{Shad}_x(y, R_0)\right) \geq \frac{\#A_j}{N_0} \cdot \frac{1}{2Q} e^{-h_\Gamma (j+1)},$$

where we used Step 2 of Theorem 7.2.5. This implies $\#A_j \leq 2QN_0 e^{h_\Gamma (j+1)}$ for every $j \in \mathbb{N}$. Finally we have

$$\#\Gamma x \cap \overline{B}(x, k) \leq \sum_{j=0}^{k-1} \#A_k \leq 4QN_0 e^{h_\Gamma k}.$$

Step 2. *For all $k \in \mathbb{N}$ it holds $\#\Gamma x \cap \overline{B}(x, k) \geq e^{-h_\Gamma (21\delta + 6D + 6L)} e^{h_\Gamma k}$.*

For every $z \in \Lambda(\Gamma)$ we consider the geodesic ray $\xi_z = [x, z]$. Then by Lemma 7.1.2 there exists $\gamma \in \text{Geod}(\Lambda(\Gamma))$ such that $d(\xi_z(t), \gamma(t)) \leq L$ for every $t \geq 0$. Moreover for every $t \geq 0$ we can find $y_t \in \Gamma x$ such that $d(y_t, \gamma(t)) \leq D$, so $d(\xi_z(t), y_t) \leq D + L$. This implies that $z \in \text{Shad}_x(y_t, D + L)$ and $|d(x, y_t) - t| \leq D + L$. Therefore for every $t \geq 0$ we can cover $\Lambda(\Gamma)$ with shadows casted by points of Γx at distance between $t - D - L$ and $t + D + L$ from x and with radius $D + L$. Choosing $t = k - D - L$ we get $\Lambda(\Gamma) \subseteq \bigcup_{y \in \Gamma x \cap A(x, k - 2D - 2L, k)} \text{Shad}_x(y, D + L)$. By the same argument of Lemma 4.3.5 we have $\text{Shad}_x(y, D + L) \subseteq B(z_y, e^{-d(x,y) + D + L}) \subseteq B(z_y, e^{-k + 3D + 3L})$ for every $y \in \Gamma x \cap A(x, k - 2D - 2L, k)$, where z_y is the point at infinity of an extension of $[x, y]$. So by Step 1 of Theorem 7.2.5 we conclude

$$1 = \mu_{\text{PS}}(\Lambda(\Gamma)) \leq \#\Gamma x \cap \overline{B}(x, k) \cdot e^{h_\Gamma (21\delta + 6D + 6L)} e^{-h_\Gamma k}.$$

The thesis follows by the bounded quantification of all the constants involved in terms of P_0, r_0, δ and D. $\square$

Chapter 8

Compactness and continuity

The aim of this chapter is to study properties that are stable under Gromov-Hausdorff convergence and ultralimits and to find a criteria for the continuity of the entropy under this kind of convergence.

8.1 Compactness of GCBA spaces

In the first part we focus on general GCBA spaces, while in the next sections we will specialize to the case of CAT(0) metric spaces.

8.1.1 Compactness of packed and doubling GCBA-spaces

Throughout the section we fix $P_0, r_0, \rho_0 > 0$ with $r_0 < \rho_0/3$ and $\kappa \in \mathbb{R}$. We denote by $\text{GCBA}^\kappa_{\text{pack}}(P_0, r_0; \rho_0)$ the class of complete, geodesic, GCBA^κ metric spaces X with $\rho_{\text{ac}}(X) \geq \rho_0$ and $\text{Pack}(3r_0, \frac{r_0}{2}) \leq P_0$. Then we have the following result which is strictly related to Gromov's Precompactness Theorem, see [Gro81]:

Theorem 8.1.1. *The class* $\text{GCBA}^\kappa_{\text{pack}}(P_0, r_0; \rho_0)$ *is closed under ultralimits and compact under pointed Gromov-Hausdorff convergence.*

Proof. Any space $X \in \text{GCBA}^\kappa_{\text{pack}}(P_0, r_0; \rho_0)$ is proper by Proposition 2.4.7, geodesic and geodesically complete. We consider any sequence (X_n, x_n) of elements of $\text{GCBA}^\kappa_{\text{pack}}(P_0, r_0; \rho_0)$ and any non-principal ultrafilter ω. For any n we have $\rho_{\text{cat}}(X_n) \geq \min\{\frac{D_\kappa}{2}, \rho_0\} = \rho_0' > 0$ from (6). Then by Corollary 2.7.10 we have that X_ω is a complete, locally geodesically complete, locally CAT(κ), geodesic metric space with again $\rho_{\text{cat}}(X_\omega) \geq \rho_0'$.
We want to prove now that $\text{Pack}(3r_0, \frac{r_0}{2}) \leq P_0$ holds on X_ω. We fix a point $y_\omega = \omega$-$\lim y_n \in X_\omega$: by Lemma 2.7.8 we have $\overline{B}(y_\omega, 3r_0) = \omega\text{-}\lim \overline{B}(y_n, 3r_0)$. Let $z^i_\omega = \omega$-$\lim z^i_n$, $i = 1, \ldots, N$ be a r_0-separated subset of $\overline{B}(y_\omega, 3r_0)$, that is $d(z^i_\omega, z^j_\omega) > r_0$ for all $i \neq j$. For any couple $i \neq j$ we have $d(z^i_n, z^j_n) > r_0$ for ω-a.e.(n). Since there are a finite number of couples, then for ω-a.e.(n) it holds $d(z^i_n, z^j_n) > r_0$ for any $i \neq j$. Moreover the points z^i_n belong to $\overline{B}(y_n, 3r_0)$ for any i. So, for ω-a.e.(n), there is a r_0-separated subset of $\overline{B}(y_n, 3r_0)$ of cardinality N. Therefore $N \leq P_0$ and in particular $\text{Pack}(3r_0, \frac{r_0}{2}) \leq P_0$ on X_ω. We can now apply again Proposition 2.4.7 to conclude that X_ω is proper, hence a GCBA^κ metric space.

To finish the first part of the proof we need to show that $\rho_{\mathrm{ac}}(X_\omega) \geq \rho_0$. This is the object of the following:

Proposition 8.1.2. *Let (X_n, x_n) be* GCBA^κ*-spaces converging to (X, x) with respect to the pointed Gromov-Hausdorff topology. Then:*

$$\rho_{\mathrm{ac}}(X) \geq \limsup_{n\to\infty} \rho_{\mathrm{ac}}(X_n)$$

We postpone the proof of this proposition to end the proof of Theorem 8.1.1. In order to prove the compactness under pointed Gromov-Hausdorff convergence we take a sequence of spaces $(X_n, x_n) \subseteq \mathrm{GCBA}^\kappa_{\mathrm{pack}}(P_0, r_0; \rho_0)$ and we fix any non-principal ultrafilter ω. Let $(X_\omega, x_\omega) \in \mathrm{GCBA}^\kappa_{\mathrm{pack}}(P_0, r_0; \rho_0)$ be the ultralimit. Since the limit is proper we can apply Proposition 2.7.11 to find a subsequence (X_{n_k}, x_{n_k}) that converges in the pointed Gromov-Hausdorff sense to (X_ω, x_ω), showing the compactness part of the statement. □

Proof of Proposition 8.1.2. Assume that $\rho_{\mathrm{ac}}(X_n) \geq \rho_0 > 0$ for infinitely many n. Take any non-principal ultrafilter ω: since by definition X is proper then by Proposition 2.7.11 we have $X = \omega\text{-}\lim X_n$. If $\rho_0 \leq \frac{D_\kappa}{2}$ we have $\rho_{\mathrm{cat}}(X_n) \geq \rho_0$ for all n, so by Corollary 2.7.10 we conclude immediately that $\rho_{\mathrm{ac}}(X_\omega) \geq \rho_{\mathrm{cat}}(X_\omega) \geq \rho_0$. Assume now that $\rho_0 > \frac{D_\kappa}{2}$; in particular as before we deduce $\rho_{\mathrm{cat}}(X_\omega) = \frac{D_\kappa}{2}$.
The strategy is the following: we claim that for any $y_\omega = \omega\text{-}\lim y_n \in X_\omega$ and for any point $z_\omega = \omega\text{-}\lim z_n$ at distance $< \rho_0$ from y_ω there exists a unique geodesic joining y_ω to z_ω. In particular this geodesic must coincide with the ultralimit of the geodesics $[y_n, z_n]$ of length $< \rho_0$. If this is true then for any two points $z_\omega = \omega\text{-}\lim z_n$, $w_\omega = \omega\text{-}\lim w_n$ of X_ω at distance $< \rho_0$ from y_ω and any $t \in [0, 1]$ we get

$$d((z_\omega)_t, (w_\omega)_t) = \omega\text{-}\lim d((z_n)_t, (w_n)_t) \leq \omega\text{-}\lim 2t \cdot d(z_n, w_n) = 2t \cdot d(z_\omega, w_\omega)$$

which implies that $\rho_{\mathrm{ac}}(y_\omega) \geq \rho_0$ for any $y_\omega \in X_\omega$.
So suppose our claim is not true: that is assume that there exists a point $y_\omega = \omega\text{-}\lim y_n \in X_\omega$, a radius $\rho_1 \in (\frac{D_\kappa}{2}, \rho_0)$ such that any point at distance $< \rho_1$ from y_ω is joined to y_ω by a unique geodesic, while for arbitrarily small values $\varepsilon > 0$ there exist two different geodesics $\gamma_\varepsilon, \gamma'_\varepsilon$ joining y_ω to the same point $z_{\varepsilon,\omega} = \omega\text{-}\lim z_{\varepsilon,n}$ with $d(y_\omega, z_{\varepsilon,\omega}) = \rho_1 + \varepsilon$.
We consider the points $w_\varepsilon = \gamma_\varepsilon(\rho_1 - \varepsilon)$, $w'_\varepsilon = \gamma'_\varepsilon(\rho_1 - \varepsilon)$ and set $\ell = d(w_\varepsilon, w'_\varepsilon)$. We observe we have $\ell \leq 4\varepsilon$ and $\ell > 0$ since the ball of radius $\frac{D_\kappa}{2}$ around $z_{\varepsilon,\omega}$ is $\mathrm{CAT}(\kappa)$ by assumption, so uniquely geodesic. Similarly we consider the points $u_\varepsilon = \gamma_\varepsilon(\rho_1 + \varepsilon - \frac{D_\kappa}{2})$, $u'_\varepsilon = \gamma'_\varepsilon(\rho_1 + \varepsilon - \frac{D_\kappa}{2})$ and we set $L = d(u_\varepsilon, u'_\varepsilon)$. Our first step is to prove that

$$L = d(u_\varepsilon, u'_\varepsilon) \geq \frac{D_\kappa}{8} \cdot \frac{\ell}{2\varepsilon} =: \tau. \tag{48}$$

So suppose by contradiction that (48) does not hold. First of all we remark that $\tau \leq \frac{D_\kappa}{2}$, since $\ell \leq 4\varepsilon$. Then, as the ball $B(z_{\varepsilon,\omega}, \frac{D_\kappa}{2})$ is $\mathrm{CAT}(\kappa)$, we can consider the κ-comparison triangle $\overline{\Delta}^\kappa(\overline{z_{\varepsilon,\omega}}, \overline{u_\varepsilon}, \overline{u'_\varepsilon})$. As usual we denote by $\overline{w_\varepsilon}, \overline{w'_\varepsilon}$ the comparison points of w_ε and w'_ε, respectively. By definition the edges of $\overline{\Delta}^\kappa(\overline{z_\varepsilon}, \overline{u_\varepsilon}, \overline{u'_\varepsilon})$ have length $\frac{D_\kappa}{2}, \frac{D_\kappa}{2}, L$. We consider another triangle $\Delta(Z, V, V')$ on M^κ_2 with edges $[Z, V], [Z, V'], [V, V']$ of length respectively $\frac{D_\kappa}{2}, \frac{D_\kappa}{2}, \tau$. We denote by W, W' the points along $[Z, V]$ and $[Z, V']$ at distance 2ε from Z. Since the contraction map φ_r^R towards Z is $\frac{2r}{R}$-Lipschitz and $d(W, Z) = d(W', Z) = 2\varepsilon$ we deduce

$$d(W, W') \leq 2 \cdot \frac{2\varepsilon}{(D_\kappa/2)} d(V, V') = \frac{8\varepsilon}{D_\kappa}\tau = \frac{\ell}{2}.$$

Since we are assuming by contradiction that $L < \tau$, we have by comparison that $d(\overline{w_\varepsilon}, \overline{w'_\varepsilon}) < d(W, W')$. So, applying the CAT($\kappa$) condition, we obtain

$$\ell = d(w_\varepsilon, w'_\varepsilon) \leq d(\overline{w_\varepsilon}, \overline{w'_\varepsilon}) < d(W, W') \leq \frac{\ell}{2}$$

a contradiction. Therefore (48) holds.
Now, by assumption there exists a unique geodesic from y to any point in $B(y, \rho_1)$. Since $d(y, w_\varepsilon) < \rho_1$ by construction, if $w_\varepsilon = \omega\text{-lim}\, w_{\varepsilon,n}$ then the ultralimit of the geodesics $\gamma_{\varepsilon,n} = [y_n, w_{\varepsilon,n}]$ is the unique geodesic joining y to w_ε, that is $\gamma_\varepsilon = \omega\text{-}\lim \gamma_{\varepsilon,n}$. Analogously, if $w'_\varepsilon = \omega\text{-lim}\, w'_{\varepsilon,n}$, we have $\gamma'_\varepsilon = \omega\text{-}\lim \gamma_{\varepsilon,n}$ where $\gamma'_{\varepsilon,n} = [y_n, w'_{\varepsilon,n}]$. Applying the contraction property on X_n from $R = \rho_1 - \varepsilon$ to $r = \rho_1 + \varepsilon - D_\kappa/2$ we get

$$\begin{aligned} L = d(u_\varepsilon, u'_\varepsilon) &= \omega\text{-}\lim\ d\left(\gamma_{\varepsilon,n}(\rho_1 + \varepsilon - D_\kappa/2),\, \gamma'_{\varepsilon,n}(\rho_1 + \varepsilon - D_\kappa/2)\right) \\ &\leq \omega\text{-}\lim\ \frac{2(\rho_1 + \varepsilon - D_\kappa/2)}{\rho_1 - \varepsilon} \cdot d(w_{\varepsilon,n}, w'_{\varepsilon,n}) \\ &= \frac{2(\rho_1 + \varepsilon - D_\kappa/2)}{\rho_1 - \varepsilon} \cdot \ell. \end{aligned} \tag{49}$$

As $\rho_1 > \frac{D_\kappa}{2}$, combining (48) and (49) gives a contradiction for $\varepsilon \to 0$. We have therefore proved that $\rho_{\mathrm{ac}}(X) \geq \rho_0$. This implies the upper semi-continuity of the almost-convexity radius since we can apply the same argument to any subsequence. □

Remark 8.1.3. In particular for any sequence of pointed metric spaces (X_n, x_n) in $\mathrm{GCBA}^\kappa_{\mathrm{pack}}(P_0, r_0; \rho_0)$ and for any non-principal ultrafilter ω the ultralimit X_ω is a proper space. Notice that, in general, the ultralimit of a sequence of proper spaces is not proper, even if the spaces are really mild. For instance let $(X_n, x_n) = (\mathbb{R}^n, 0)$ and ω be any non-principal ultrafilter. Then X_ω is isometric to $\ell^2(\mathbb{R})$, the spaces of sequences $\{a_n\}$ of real numbers such that $\sum a_n^2 < +\infty$. This is a non-proper space of infinite dimension.

The compactness of a class of proper metric spaces $\mathcal{C}$ is hard to achieve since properness and dimension are in general not stable under limits.
In the next theorem we precisely characterize the classes of proper, GCBA^κ, geodesic metric spaces with almost-convexity radius uniformly bounded from below that are precompact and compact under pointed Gromov-Hausdorff convergence. For this we need the following slight refinement of the packing condition at scale r_0. Given a function $P\colon [0, +\infty) \to \mathbb{N}$, we say that a pointed metric space (X, x) belongs to the class

$$\mathrm{GCBA}^\kappa_{\mathrm{pack}}(P(\cdot), r_0; \rho_0)$$

if X is a complete, geodesic, GCBA$^\kappa$ metric space with $\rho_{\text{ac}}(X) \geq \rho_0$ and it satisfies $\text{Pack}(\overline{B}(x,R), \frac{r_0}{2}) \leq P(R)$ for all $R > 0$. This is equivalent to asking that the packing costant P of Definition 2.4.1 possibly depends also on the distance of the center of the balls from x. The same argument used in Theorem 8.1.1 shows that the class $\text{GCBA}^\kappa_{\text{pack}}(P(\cdot), r_0; \rho_0)$ is closed under ultralimits and therefore compact under pointed Gromov-Hausdorff convergence. Moreover we have:

Theorem 8.1.4. *Let $\mathcal{C}$ be a class of proper,* GCBA$^\kappa$*, geodesic metric spaces X with $\rho_{\text{ac}}(X) \geq \rho_0 > 0$. Then $\mathcal{C}$ is precompact under the pointed Gromov-Hausdorff convergence if and only if there exist $P(\cdot)$ and $r_0 > 0$ such that*

$$\mathcal{C} \subseteq \text{GCBA}^\kappa_{\text{pack}}(P(\cdot), r_0; \rho_0).$$

Moreover $\mathcal{C}$ is compact if and only if it is precompact and closed under ultralimits.

We stress the "only if" part in the above statement: for GCBA$^\kappa$ spaces, a uniform packing assumption (depending only on the distance from the basepoint x) at some *fixed* scale is a *necessary* and sufficient condition in order to have precompactness (we recall that, in the general Gromov's Precompactness Theorem, one needs to have a uniform control of the packing function at every scale in order to achieve precompactness).

Proof of Theorem 8.1.4. Let $\mathcal{C}$ be a class of proper, GCBA$^\kappa$, geodesic spaces X with $\rho_{\text{ac}}(X) \geq \rho_0 > 0$. Let us prove the first equivalence stated in 8.1.4. So assume that it is precompact in the pointed Gromov-Hausdorff sense, i.e. the closure $\overline{\mathcal{C}}$ is compact under pointed Gromov-Hausdorff convergence. Suppose $\mathcal{C}$ is not contained in $\text{GCBA}^\kappa_{\text{pack}}(P(\cdot), r_0; \rho_0)$ for any choice of $P(\cdot)$ and r_0. Hence there exists $r_0 < \frac{\rho_0}{3}$ and $R > 0$ such that for any n there is a space $(X_n, x_n) \in \mathcal{C}$ with a set of r_0-separated points inside $\overline{B}(x_n, R)$ of cardinality at least n. By assumption there exists a subsequence, denoted again (X_n, x_n), converging in the pointed Gromov-Hausdorff sense to (X, x). The space X is proper, see the discussion at the beginning of Chapter 2. Fix now any non-principal ultrafilter ω. Then (X_ω, x_ω) is isometric to (X, x) by Proposition 2.7.11, and in particular it is proper. We are going to prove that inside $\overline{B}(x_\omega, R)$ there are infinitely many points that are at distance at least r_0 one from the other: therefore X_ω cannot be proper and this is a contradiction. For any n we denote the set of r_0-separated points of cardinality n inside $\overline{B}(x_n, R)$ by $\{z_n^1, \ldots, z_n^n\}$. Then, for any fixed $k \in \mathbb{N}$, we consider the admissible point $z_\omega^k = \omega\text{-}\lim z_n^k \in X_\omega$ (notice that z_n^k is defined only for $n \geq k$, but this suffices to define a point z_ω^k in the ultralimit). Clearly $z_\omega^k \in \overline{B}(x_\omega, R)$ for all k. Moreover if $k \neq l$ then $d(z_n^k, z_n^l) > r_0$ for all n, hence $d(z_\omega^k, z_\omega^l) \geq r_0$. This shows that $\mathcal{C}$ is a subclass of $\text{GCBA}^\kappa_{\text{pack}}(P(\cdot), r_0; \rho_0)$ for some $P(\cdot)$ and r_0. Viceversa if $\mathcal{C} \subseteq \text{GCBA}^\kappa_{\text{pack}}(P(\cdot), r_0; \rho_0)$ then its closure $\overline{\mathcal{C}}$ is contained in the compact space $\text{GCBA}^\kappa_{\text{pack}}(P(\cdot), r_0; \rho_0)$ by the analogue of Theorem 8.1.1, so $\overline{\mathcal{C}}$ is compact.
Let us show now the second equivalence. Suppose that $\mathcal{C}$ is precompact and closed under ultralimits. Applying the same proof of the second part of Theorem 8.1.1 we get that $\mathcal{C}$ is compact under pointed Gromov-Hausdorff convergence. Viceversa if $\mathcal{C}$ is compact under Gromov-Hausdorff convergence then it is contained in $\text{GCBA}^\kappa_{\text{pack}}(P(\cdot), r_0; \rho_0)$ for some $P(\cdot), r_0$. In particular for any non-principal ultrafilter ω and any sequence of spaces $(X_n, x_n) \in \mathcal{C}$ we have that X_ω is a proper metric

space. By Proposition 2.7.11 there exists a subsequence that converges in the pointed Gromov-Hausdorff sense to X_ω, hence $X_\omega \in \mathcal{C}$ since $\mathcal{C}$ is compact. □

As a consequence of Theorem 8.1.4 and of the estimates on volumes and packing proved in Sections 3.1 & 2.4, we deduce that the dimension is *almost* stable under pointed Gromov-Hausdorff convergence, in the following sense:

Proposition 8.1.5. *Let (X_n, x_n) be a sequence of GCBA$^\kappa$ metric spaces with almost convexity radius $\rho_{\text{ac}}(X_n) \geq \rho_0 > 0$ converging to (X, x) in the pointed Gromov-Hausdorff sense. Let $X_n^{\max}$ be the maximal dimensional subspace of X. Then*

$$\dim(X) \leq \liminf_{n \to +\infty} \dim(X_n)$$

and the equality $\dim(X) = \lim_{n\to+\infty} \dim(X_n)$ holds if and only if the distance $d(x_n, X_n^{\max})$ stays uniformly bounded when $n \to \infty$.

Proof. As the spaces (X_n, x_n) converge to (X, x), they form a precompact family and so they belong to $\text{GCBA}^\kappa_{\text{pack}}(P(\cdot), r_0, \rho_0)$, for some $P(\cdot)$ and r_0, by Theorem 8.1.4. Let us first show that we always have

$$\dim(X) \leq \liminf_{n \to +\infty} \dim(X_n) \tag{50}$$

Actually consider a subsequence, we we still denote (X_n), whose dimensions equal the limit inferior, denoted d_0. Now suppose that there exists a point $y \in X$ with $\dim(y) = d > d_0$. We may assume that y is d-regular, since $\text{Reg}^d(X)$ is dense in X^d. The point y is the limit of a sequence of points $y_n \in X_n$ and for any $r > 0$ the volume of the ball $\overline{B}(y, r)$ is bigger than or equal to the limit of the volumes of the balls $\overline{B}(y_n, \frac{r}{2})$, by (10). By Theorem 3.1.1 we have for all n:

$$\mu_X\left(\overline{B}\left(y_n, \frac{r}{2}\right)\right) \geq c_{d_0} \cdot \left(\frac{r}{2}\right)^{d_0}$$

where c_{d_0} is a constant depending only on d_0. Moreover, since y is d-regular, then for any r small enough the ball $\overline{B}(y, r)$ contains only d-dimensional points. We conclude by (8) & (9) that

$$\mu_X(\overline{B}(y, r)) \leq C \cdot r^d,$$

where C is a constant depending only on y and not on r. Therefore, as $d_0 < d$, we have a contradiction if r is small enough, and (50) is proved.
Assume now that $d(x_n, X_n^{\max}) < D$ for all n. Since the almost convexity radius is bounded below by ρ_0 both for X_n and for X, also the CAT(κ)-radius is bounded below by (6). So we can consider tiny balls $B(y_n, r_0)$ centered at regular points y_n of maximal dimension, all with the same radius r_0, such that the closed ball $\overline{B}(y_n, 10r_0)$ converge to some ball $\overline{B}(y, 10r_0)$ of X and satisfy the condition $\text{Pack}(P_0, \frac{r_0}{2})$ for some constant P_0 for all n, by Proposition 2.4.7. We are then in the standard setting of convergence, which implies by Lemma 2.2.8 that

$$\dim(X) \geq \dim(y) \geq \limsup_{n\to\infty} \dim(y_n) = \limsup_{n\to\infty} \dim(X_n).$$

Conversely, assuming $\dim(X) = \lim_{n\to+\infty} \dim(X_n)$, then in particular $\dim(X_n)$ is constant for $n \gg 0$ and equal to $d_0 = \dim(X)$. Consider a regular point $y = (y_n) \in X$

of dimension d_0: then the points y_n are admissible by definition (that is, $d(x_n, y_n)$ stays uniformly bounded); moreover we can choose as before uniformly packed tiny balls with $B(y_n, 10r_0)$ converging to $B(y, 10r_0)$, so the points y_n belong to $X_n^{\max}$, again by Lemma 2.2.8 (b). □

Example 8.1.6. Let $(X, x) \in \mathrm{GCBA}^\kappa_{\mathrm{pack}}(P_0, r_0, \rho_0)$ be any space. We consider the space Y obtained by gluing the half-line $[0, +\infty)$ to X at the point x. Clearly Y belongs to $\mathrm{GCBA}^\kappa_{\mathrm{pack}}(P_0', r_0', \rho_0')$. The pointed Gromov-Hausdorff limit of the sequence (Y, n), where $n \in [0, +\infty)$, is the real line. This is an example where the maximal dimension part escapes to infinity and the dimension is not preserved.

We are going now to explore some variations of Theorem 8.1.4.
We fix constants $\kappa \in \mathbb{R}$ and $P_0, r_0, V_0, R_0, D_0, t_0, \rho_0, n_0 > 0$, with $r_0 \leq \rho_0/3$, and consider the following classes of complete, geodesic GCBA^κ spaces X:

– the class $\mathrm{GCBA}^\kappa_{\mathrm{pack}}(P_0, r_0; \rho_0, n_0)$ of spaces which are P_0-packed at scale r_0, with almost-convexity radius $\rho_{\mathrm{ac}}(X) \geq \rho_0$ and dimension $\leq n_0$;

– the class $\mathrm{GCBA}^\kappa_{\mathrm{pack}}(P_0, r_0; \rho_0, \overset{\mathrm{pure}}{n_0})$ of spaces P_0-packed at scale r_0, with almost-convexity radius $\rho_{\mathrm{ac}}(X) \geq \rho_0$ and of *pure dimension* n_0;

– the classes $\mathrm{GCBA}^\kappa_{\mathrm{vol}}(V_0, R_0; \rho_0, n_0)$, $\mathrm{GCBA}^\kappa_{\mathrm{vol}}(V_0, R_0; \rho_0, \overset{\mathrm{pure}}{n_0})$ of those satisfying $\mu_X(B(x, R_0)) \leq V_0$, $\rho_{\mathrm{ac}}(X) \geq \rho_0$ and which have, respectively, dimension $\leq n_0$ and pure dimension n_0;

– the class $\mathrm{GCBA}^\kappa_{\mathrm{doub}}(D_0, t_0; \rho_0)$ of spaces D_0-doubled up to scale t_0, with $\rho_{\mathrm{ac}}(X) \geq \rho_0$.

Then:

Corollary 8.1.7. *All the above classes are compact with respect to the pointed Gromov-Hausdorff convergence.*

Proof. By Theorem 3.2.1 and Corollary 3.3.5, the above are all subclasses of $\mathrm{GCBA}^\kappa_{\mathrm{pack}}(P_0, r_0; \rho_0)$, for suitable P_0 and r_0. By the compactness Theorem 8.1.4, the proof then reduces to show that the additional conditions on the dimension, on the measure of balls of given radius or on the doubling constant are stable under Gromov-Hausdorff limits. By Lemma 2.2.7, if a sequence X_n in $\mathrm{GCBA}^\kappa_{\mathrm{vol}}(V_0, R_0; \rho_0, n_0)$ converges to X, then $\mu_X(B(y, R_0)) \leq V_0$ for any $y \in X$. On the other hand from Corollary 3.3.7 it follows that the doubling condition is preserved to the limit. The stability of the dimension is proved in Proposition 8.1.5. To conclude we need to show that pure-dimensionality is stable under Gromov-Hausdorff limits: this is the object of the proposition which follows. □

Proposition 8.1.8. *Let (X_n, x_n) be a sequence of* GCBA^κ *metric spaces with almost convexity radius $\rho_{\mathrm{ac}}(X_n) \geq \rho_0 > 0$ converging to (X, x) in the pointed Gromov-Hausdorff sense. Assume that X_n is pure-dimensional for all n: then X is pure-dimensional of dimension* $\dim(X) = \lim_{n \to +\infty} \dim(X_n)$.

Proof. The spaces (X_n, x_n) form a precompact family and so, by Theorem 8.1.4, they belong to $\mathrm{GCBA}^\kappa_{\mathrm{pack}}(P(\cdot), r_0, \rho_0)$, for suitable $P(\cdot)$ and r_0. Then, using the maps Ψ_{x_n} of Proposition 3.1.3 as in the first part of Theorem 3.2.1, the numbers $\dim(X_n)$ belong to the finite set $[0, n_0]$. Suppose to have two integers $d_1 \neq d_2$ and two infinite subsequences $X_{n_{i_1}}$, $X_{n_{i_2}}$ such that $\dim(X_{n_{i_1}}) = d_1$ for any i_1 and $\dim(X_{n_{i_2}}) = d_2$

for any i_2. We consider the sequences $x_{n_{i_1}}$ and $x_{n_{i_2}}$: for any $r > 0$ we have by (10) and Lemma 2.2.7

$$\limsup_{n\to+\infty} \mu_{X_n}\left(B\left(x_n, \frac{r}{2}\right)\right) \leq \mu_X(B(x,r)) \leq \limsup_{n\to+\infty} \mu_{X_n}(B(x_n,r)).$$

By (8) and Theorem 3.1.1 we have

$$\frac{1}{C}\left(\frac{r}{2}\right)^{d_1} \leq \mu_{X_{n_{i_1}}}\left(B\left(x_{n_{i_1}}, \frac{r}{2}\right)\right) \leq \mu_{X_{n_{i_1}}}(B(x_{n_{i_1}}, r)) \leq Cr^{d_1},$$

$$\frac{1}{C}\left(\frac{r}{2}\right)^{d_2} \leq \mu_{X_{n_{i_2}}}\left(B\left(x_{n_{i_2}}, \frac{r}{2}\right)\right) \leq \mu_{X_{n_{i_2}}}(B(x_{n_{i_2}}, r)) \leq Cr^{d_2},$$

where C is a constant depending only on $P(\cdot)$ and r_0. Since this is true for any arbitrarily small r we deduce that $d_1 = d_2$. Therefore $\lim_{n\to+\infty} \dim(X_n)$ exists and we denote it by d_0. We again apply the same estimate as before to conclude that for any $y \in X$ and for any small $r > 0$ we have

$$\frac{1}{C} r^{d_0} \leq \mu_X(B(y,r)) \leq Cr^{d_0},$$

where C is a constant depending on $P(\cdot)$ and r_0. Therefore the dimension of y is d_0, which concludes the proof. □

Finally we can specialize these theorems to subclasses of compact spaces. Clearly the subclasses of the above classes made of spaces with diameter less than or equal to some constant Δ will be compact with respect to the usual Gromov-Hausdorff distance. We state here just two particularly interesting cases, which are reminiscent of the classical finiteness theorems of Riemannian geometry. Consider the classes:

$$\text{GCBA}^\kappa_{\text{vol}}(V_0; \rho_0, n_0^{=}), \qquad \text{GCBA}^\kappa_{\text{vol}}(V_0; \rho_0, n_0^{\text{pure}})$$

of complete, geodesic GCBA$^\kappa$ with total measure $\mu(X) \leq V_0$, almost convexity radius $\rho_{\text{ac}}(X) \geq \rho_0$ and which are, respectively, precisely n_0-dimensional and purely n_0-dimensional.

Corollary 8.1.9. *The classes* $\text{GCBA}^\kappa_{\text{vol}}(V_0; \rho_0, n_0^{=})$ *and* $\text{GCBA}^\kappa_{\text{vol}}(V_0; \rho_0, n_0^{\text{pure}})$ *are compact under Gromov-Hausdorff convergence and contain only finitely many homotopy types.*

Proof. First we show that the diameter is uniformly bounded in both classes. Actually consider $X \in \text{GCBA}^\kappa_{\text{vol}}(V_0; \rho_0, n_0^{=})$ and take any two points $y, y' \in X$ such that $d(y, y') = \Delta > \rho := \min\{\rho_0, 2\}$. Let γ be a geodesic joining y to y'. Along γ we take points at distance ρ one from the other: they are at least $\frac{\Delta}{\rho} - 1$ and the balls of radius $\frac{\rho}{2}$ around these points are disjoint. Then by Theorem 3.1.1 we get

$$V_0 \geq \mu_X(X) \geq \frac{c_{n_0}}{2^{n_0}} \rho^{n_0} \left(\frac{\Delta}{\rho} - 1\right)$$

so the diameter of X is bounded from above in terms of n_0, ρ_0 and V_0 only.
Let R_0 such un upper bound. Then these classes are included in $\text{GCBA}^\kappa_{\text{vol}}(V_0, R_0; \rho_0, n_0)$,

whose compactness we have just proved. The conclusion follows from Propositions 8.1.5 and 8.1.8.
Finally notice that any element of both classes has local geometric contractibility function $\mathrm{LGC}(r) = r$ for $r \leq \rho_0$ (see [Pet90] for the definition). Moreover the covering dimension of any space in both classes coincides with the Hausdorff dimension, so it is uniformly bounded from above. We can then apply Corollary B of [Pet90] to conclude that there are only finitely many homotopy types inside any of the two classes. □

8.1.2 Compactness of M^κ-complexes

The aim of this section is to provide compactness and finiteness results for simplicial complexes. We denote by $M^\kappa(R, N)$ the class of M^κ-complexes without free faces, of size bounded by R, valency at most N and positive injectivity radius.

Theorem 8.1.10. *The class $M^\kappa(R, N)$ is compact under pointed Gromov-Hausdorff convergence.*

By Proposition 3.4.13 there exist P_0, r_0, ρ_0 such that any $K \in M^\kappa(R, N)$ belongs to $\mathrm{GCBA}^\kappa_{\mathrm{pack}}(P_0, r_0; \rho_0)$. So, by Theorem 8.1.4, the class $M^\kappa(R, N)$ is precompact and it is compact if and only if it is closed under ultralimits. We are going now to show that $M^\kappa(R, N)$ is closed under ultralimits.
We fix a non-principal ultrafilter ω and we take any sequence (K_n, o_n) in $M^\kappa(R, N)$. We denote by K_ω the ultralimit of this sequence. Our aim is to prove that K_ω is isometric to a M^κ-complex $\hat{K}_\omega$ satisfying the same conditions as the K_n's.

Step 1: construction of the simplicial complex $\hat{K}_\omega$.
Let us start definining who are the simplices of $\hat{K}_\omega$. Let (x_n) be any admissible sequence of points, with $x_n \in K_n$, and consider the unique simplex $\mathrm{supp}(x_n)$ of K_n containing x_n in its interior: we define $S_{(x_n)} = \omega\text{-}\lim \mathrm{supp}(x_n)$. The metric space $S_{(x_n)}$ is a M^κ-simplex with size bounded by R by 3.4.3. Notice that, a priori, if y_n is another sequence defining the same point as x_n in K_ω then $S_{(y_n)}$ might be different from $S_{(x_n)}$.
Now we define $\hat{K}_\omega$ as follows. Let $p_n : S \to K_n$ denote the projection of any simplex of the total space of K_n to K_n. The total space of $\hat{K}_\omega$ will be

$$\bigsqcup_{(x_n) \text{ admissible}} S_{(x_n)}$$

where (x_n) is *any* admissible sequence of points in K_n, and the equivalence relation is: if $z_\omega = \omega\text{-}\lim z_n \in S_{(x_n)}$ and $z'_\omega = \omega\text{-}\lim z'_n \in S_{(x'_n)}$ (i.e. (z_n), (z'_n) are admissible sequences of points *respectively in supp*(x_n) *and supp*(x'_n)), we say that $z_\omega \sim z'_\omega$ if and only if $\omega\text{-}\lim d_{K_n}(p_n(z_n), p_n(z'_n)) = 0$. That is, we compare the points z_n and z'_n in the common space K_n where they live. For simplicity we will abbreviate $d_{K_n}(p_n(z_n), p_n(z'_n))$ with $d_{K_n}(z_n, z'_n)$. First of all we need to check that the relation is well defined: given other admissible sequences $(w_n), (w'_n)$ with $w_n \in \mathrm{supp}(x_n)$ and $w'_n \in \mathrm{supp}(x'_n)$ such that $z_\omega = \omega\text{-}\lim w_n$ and $z'_\omega = \omega\text{-}\lim w'_n$, we have

$$d_{K_n}(w_n, w'_n) \leq d_{\mathrm{supp}(x_n)}(w_n, z_n) + d_{K_n}(z_n, z'_n) + d_{\mathrm{supp}(x'_n)}(z'_n, w'_n)$$

hence $\omega\text{-}\lim d_{K_n}(w_n, w_n') = 0$. Once proved it is well defined it is easy to show it is an equivalence relation. We call $\hat{K}_\omega$ the quotient space and denote $p_\omega : S_{(x_n)} \to \hat{K}_\omega$ the projections.

Step 2: $\hat{K}_\omega$ satisfies axiom (i) of M^κ-complexes.
We fix an admissible sequence (x_n) and the corresponding simplex $S_{(x_n)}$. We need to prove that the map $p_\omega\colon S_{(x_n)} \to \hat{K}_\omega$ is injective. For this consider points $z_\omega = \omega\text{-}\lim z_n$ and $z'_\omega = \omega\text{-}\lim z'_n$ in $S_{(x_n)}$, with $z_n, z'_n \in \operatorname{supp}(x_n)$ for all n; then there exists $\varepsilon_0 > 0$ such that $\omega\text{-}\lim d_{\operatorname{supp}(x_n)}(z_n, z'_n) > \varepsilon_0 > 0$. In particular $d_{\operatorname{supp}(x_n)}(z_n, z'_n) > \varepsilon_0$ ω-a.e.(n). Now, for any point z of a M^κ-complex define $\dim(z)$ as the dimension of $\operatorname{supp}(z)$. The strategy to prove the injectivity is by induction on

$$d = \max\{\omega\text{-}\lim \dim(z_n), \omega\text{-}\lim \dim(z'_n)\}.$$

Observe that if $\omega\text{-}\lim \dim(z_n) = k$ then we have $\dim(z_n) = k$ ω-a.e.(n) because the possible dimensions belong to a finite set. For $d = 0$ we have that z_n, z'_n are both vertices of $\operatorname{supp}(x_n)$, ω-a.e.(n). If p_ω is not injective then for every $\varepsilon > 0$ we have $d_{K_n}(z_n, z'_n) \leq \varepsilon$ for ω-a.e.(n). By Lemma 3.4.4 we know that if $d_{K_n}(z_n, z'_n) \leq \varepsilon_0(R, N)$ then $z_n = z'_n$ as points of $\operatorname{supp}(x_n)$.
We consider now the inductive step. We denote by T_n, T'_n the faces of S_n containing z_n and z'_n in their interior, respectively. We suppose there exists $\tau > 0$ such that for ω-a.e.(n) it holds $z_n \in T_n \setminus (\partial T_n)_\tau$. By Lemma 3.4.6 we have $\varepsilon(z_n) \geq \varepsilon(R, N, \tau)$ for ω-a.e.(n), and similarly for z'_n. Once again this fact implies the injectivity. Consider now the case where for all $\tau > 0$ the set

$$\{n \in \mathbb{N} \text{ s.t. } d(z_n, \partial T_n) \leq \tau \text{ and } d(z'_n, \partial T'_n) \leq \tau\}$$

belongs to ω. Therefore $\omega\text{-}\lim d(z_n, \partial T_n) = \omega\text{-}\lim d(z'_n, \partial T'_n) = 0$. This means that z_ω belongs to ∂T_ω and z'_ω belongs to $\partial T'_\omega$, by Proposition 3.4.3. Hence $z_\omega = \omega\text{-}\lim w_n$ and $z'_\omega = \omega\text{-}\lim w'_n$, where w_n and w'_n belong to a lower dimensional face of T_n and T'_n respectively. We then apply the inductive assumption to get the thesis.

Step 3: $\hat{K}_\omega$ satisfies axiom (ii) of M^κ-complexes.
Consider two simplices $S_{(x_n)}$, $S_{(x'_n)}$ and suppose $p_\omega(S_{(x_n)}) \cap p_\omega(S_{(x'_n)}) \neq \emptyset$. This means that for any $\varepsilon > 0$ there exist $y_\omega = \omega\text{-}\lim y_n$ and $y'_\omega = \omega\text{-}\lim y'_n$ with $y_n \in \operatorname{supp}(x_n)$ and $y'_n \in \operatorname{supp}(x'_n)$ such that $d_{K_n}(y_n, y'_n) < \varepsilon$ for ω-a.e.(n). If $\varepsilon < \delta(R, N)$ then by Lemma 3.4.7.(a) we know that $\operatorname{supp}(x_n)$ and $\operatorname{supp}(x'_n)$ share a face in K_n. Let then $T_n \subset \operatorname{supp}(x_n)$ and $T'_n \subset \operatorname{supp}(x'_n)$ such faces and $h_n\colon T_n \to T'_n$ an isometry such that $p_n(z) = p_n(z')$ for $z \in T_n$, $z' \in T'_n$ if and only if $z' = h_n(z)$. By assumption this holds for ω-a.e.(n). By Proposition 3.4.3 it is easy to see that the metric spaces $T_\omega = \omega\text{-}\lim T_n$ and $T'_\omega = \omega\text{-}\lim T'_n$ are, respectively, faces of $S_{(x_n)}$ and $S_{(x'_n)}$. Moreover the sequence of maps (h_n) defines a limit map $h_\omega\colon T_\omega \to T'_\omega$ which is an isometry, by Proposition 2.7.5. It remains to show that $p_\omega(z_\omega) = p_\omega(z'_\omega)$, for $z_\omega \in T_\omega$ and $z'_\omega \in T'_\omega$, if and only if $h_\omega(z_\omega) = z'_\omega$. But given $z_\omega = \omega\text{-}\lim z_n$ and $z'_\omega = \omega\text{-}\lim z'_n$ with $z_n \in \operatorname{supp}(x_n), z'_n \in \operatorname{supp}(x'_n)$ we have $p_\omega(z_\omega) = p_\omega(z'_\omega)$ by definition if and only if $\omega\text{-}\lim d_{K_n}(p_n(z_n), p_n(z'_n)) = 0$. This happens if and only if for any $\varepsilon > 0$ the inequality

$$d_{K_n}(p_n(z_n), p_n(z'_n)) < \varepsilon$$

holds for ω-a.e.(n). This means that $d_{K_n}(p_n(h_n(z_n)), p_n(z'_n)) < \varepsilon$ holds ω-a.e.(n), in particular $p_\omega(h_\omega(z_\omega)) = p_\omega(z'_\omega)$. By the injectivity of the projection map p_ω we then obtain $h_\omega(z_\omega) = z'_\omega$, which is the thesis.

Step 4: $\hat{K}_\omega$ belongs to $M^\kappa(R, N)$.
It is clear that $\hat{K}_\omega$ has size bounded by R by construction.
We want to show it has valency at most N. Fix a vertex v of $\hat{K}_\omega$ and parameterize by $\alpha \in A$ the set of simplices $S_{(x_n(\alpha))}$ of $\hat{K}_\omega$ having v as a vertex. For any fixed $\alpha \in A$ there is a vertex $v_n(\alpha)$ of $\mathrm{supp}(x_n(\alpha))$ such that the sequence $(v_n(\alpha))$ converges for ω-a.e.(n) to v, by Proposition 3.4.3. In particular for all $\alpha, \alpha' \in A$ we get $d_{K_n}(v_n(\alpha), v_n(\alpha')) < \varepsilon(R, N)$ for ω-a.e.(n), and then $v_n(\alpha) = v_n(\alpha')$ by Lemma 3.4.4. Let now $S_{(x_n(\alpha))} \neq S_{(x_n(\alpha'))}$ be distinct elements of $\hat{K}_\omega$, for $\alpha, \alpha' \in A$. Then there exists a vertex of the first simplex $u = \omega\text{-}\lim u_n$, with $u_n \in \mathrm{supp}(x_n(\alpha))$, which does not belong to the second one. So $d_{K_n}(u_n, \mathrm{supp}(x_n(\alpha'))) > 0$ for ω-a.e.(n), hence $\mathrm{supp}(x_n(\alpha)) \neq \mathrm{supp}(x_n(\alpha'))$ for ω-a.e.(n). Therefore if $\hat{K}_\omega$ has m different simplices $S_{(x_n(\alpha))}$ sharing the vertex v, there also exist m different simplices $\mathrm{supp}(x_n(\alpha))$ of K_n sharing the same vertex $v_n(\alpha)$ for ω-a.e.(n). This contradicts our assumptions if $m > N$.
Finally the fact that $\hat{K}_\omega$ has positive injectivity radius and has not free faces will follow from the last step, where we prove that $\hat{K}_\omega$ and K_ω are isometric. In fact K_ω is geodesically complete and locally CAT(κ), as ultralimit of complete, geodesically complete, locally CAT(κ) spaces with CAT(κ)-radius uniform bounded below; hence K_ω (and in turns $\hat{K}_\omega$) has positive injectivity radius and no free faces, by Proposition 3.4.12 and II.5.9&II.5.10 of [BH13].

Step 5: $\hat{K}_\omega$ is isometric to K_ω.
We define a map $\Phi : K_\omega \to \hat{K}_\omega$ as follows. Let $y_\omega = \omega\text{-}\lim y_n$ be the ω-limit of be an admissible sequence (y_n) of K_n. Any y_n belongs to $\mathrm{supp}(y_n)$: we will denote by $(y_n)_{\mathrm{supp}(y_n)}$ the point, in the ultralimit of the sequence of simplices $\mathrm{supp}(y_n)$, which is defined by the admissible sequence of points (y_n) [1].
We then define Φ as

$$\Phi(y_\omega) = p_\omega((y_n)_{\mathrm{supp}(y_n)}).$$

It is easy to see it is well defined and surjective.
It remains to prove it is an isometry. Let $y_n, z_n \in K_n$ define admissible sequences. So the distances $d_{K_n}(y_n, z_n)$ are uniformly bounded by some constant L. Therefore by Proposition 3.4.10 for any n there exists a geodesic between y_n and z_n which is the concatenation of at most $m_0(L, R, N)$ segments, each of them contained in a simplex. Since the number of segments is uniformly bounded we can define a path in $\hat{K}_\omega$ which is the concatenation of geodesic segments, each contained in a simplex of $\hat{K}_\omega$, and whose length is the limit of the lengths of the segments in K_n. This shows that

$$d_{\hat{K}_\omega}(p_\omega((y_n)_{\mathrm{supp}(y_n)}), p_\omega((z_n)_{\mathrm{supp}(z_n)}) \leq \omega\text{-}\lim d_{K_n}(y_n, z_n).$$

In order to prove the other inequality we fix two points $y = p_\omega((y_n)_{\mathrm{supp}(y_n)})$ and $z = p_\omega((z_n)_{\mathrm{supp}(z_n)})$ of $\hat{K}_\omega$. Notice that from the inequality above we deduce that

[1] The notation stresses the fact that we see $(y_n)_{\mathrm{supp}(y_n)}$ as limit of points in the abstract simplices $\mathrm{supp}(y_n)$ (not in K_n). Namely, $(y_n)_{\mathrm{supp}(y_n)}$ belongs to the total space of $\hat{K}_\omega$, while $y_\omega \in K_\omega$.

$\hat{K}_\omega$ is path-connected. Hence, by Proposition 3.4.10, we know that there exists a geodesic between y and z which is the concatenation of at most $m_0(\ell, R, N)$ geodesic segments, each of them contained in a simplex, where $\ell = d_{\hat{K}_\omega}(x, y)$. These segments cross finitely many simplices, each of which can be seen as the ω-limit of a sequence of simplices in K_n. Since the number is finite we can see the union of these simplices of $\hat{K}_\omega$ as the ultralimit of the union of the corresponding simplices in K_n. We can therefore approximate the geodesic in $\hat{K}_\omega$ with paths in K_n between y_n and z_n, whose total length tend to ℓ. So

$$d_{\hat{K}_\omega}(p_\omega((y_n)_{\mathrm{supp}(y_n)}), p_\omega((z_n)_{\mathrm{supp}(z_n)}) \geq \omega\text{-}\lim d_{K_n}(y_n, z_n).$$

which ends the proof of Theorem 8.1.10. □

We can specialize this compactness theorem to other families of M^κ-complexes, as done for $\mathrm{GCBA}^\kappa_{\mathrm{pack}}(P_0, r_0; \rho_0)$. Namely consider:

– the subclass $M^\kappa(R, N; \Delta) \subseteq M^\kappa(R, N)$ of complexes with diameter $\leq \Delta$;
– the class $M^\kappa(R; V, n)$ of M^κ-complexes without free faces, with size bounded by R, total volume $\leq V$, dimension bounded above by n and positive injectivity radius.

Remark 8.1.11. We should specify the measure on the complexes K of the class $M^\kappa(R; V, n)$ under consideration. Any such space is stratified in subspaces of different dimension, so it is naural to consider the measure which is the sum over $k = 0, \ldots, n$ of the k-dimensional Hausdorff measure on each k-dimensional part. This clearly coincides with the natural measure μ_K of K seen as GCBA-space.

Corollary 8.1.12. *For any choice of R, n, V, N and Δ, the above classes are compact under Gromov-Hausdorff convergence and contain only finitely many simplicial complexes up to simplicial homeomorphisms.*

Proof. The compactness of $M^\kappa(R, N; \Delta)$ is clear from the one of $M^\kappa(R, N)$. Moreover, by Proposition 3.4.13, we know that any $K \in M^\kappa(R, N; \Delta)$ satisfies the condition $\mathrm{Pack}(3r_0, \frac{r_0}{2}) \leq P_0$ for constants P_0, r_0 only depending on R and N. Furthermore, by Lemma 3.4.4, any two vertices of K are $\varepsilon(R, N)$-separated: in particular the number of vertices of K is bounded above by $\mathrm{Pack}(\frac{\Delta}{2}, \frac{\varepsilon(R,N)}{2})$ which is a number depending only on R, N, κ and Δ. Since the valency is bounded and the total number of vertices is bounded, we have only finitely many possible simplicial complexes up to simplicial homeomorphisms.
On the other hand it is straightforward to show that any $K \in M^\kappa(R; V, n)$ has valency bounded from above by a function depending only on R, V, n and κ, because any simplex of locally maximal dimension contributes to the total volume with a quantity greater than a universal function $v(R, n, \kappa) > 0$.
This also shows also that the total number of simplices of K is uniformly bounded in terms of R, V and n, hence the combinatorial finiteness of $M^\kappa(R; V, n)$. Moreover, since any simplex has uniformly bounded size, also the diameters of complexes in this class are uniformly bounded. Therefore $M^\kappa(R; V, n) \subseteq M^\kappa(R, N)$ for a suitable N and, as the class is made of compact metric spaces, it is actually precompact under (unpointed) Gromov-Hausdorff convergence. It remains to show that $M^\kappa(R; V, n)$ is closed. By the proof of Theorem 8.1.10 it is clear that the upper bound on the

dimension of the simplices is preserved under limits. The stability of the upper bound on the total volume is proved as for the class $\mathrm{GCBA}^{\kappa}_{\mathrm{vol}}(V_0, R_0; \rho_0, n_0)$ in Corollary 8.1.7. □

Finally we want to point out that the assumptions on size and diameter in the above compactness results are essential:

Examples 8.1.13. *Non-compact families of M^κ-complexes.*

(1) Let X_n be a wedge of n circles of radius 1. The family of M^0-complexes $\{X_n\}$ has uniformly bounded size and uniformly bounded diameter, but the valency is not bounded. Notice that this family is neither finite nor uniformly packed. In particular, it is not precompact.

(2) Let X_n be obtained from a circle of radius 1, then choosing n equidistant points on the circle and gluing n circles of radius 1 to them. The X_n's admit M^0-complex structures with uniformly bounded valency and uniformly bounded diameter, but the size of the simplices is not bounded. Again, this family is neither finite nor uniformly packed, hence not precompact.

8.2 Ultralimit of groups

In this section we investigate the convergence under ultralimits of group actions on CAT(0), δ-hyperbolic metric spaces that are P_0-packed at scale r_0.

Let us consider a sequence (X_n, x_n) of complete, geodesically complete, CAT(0), δ-hyperbolic metric spaces that are P_0-packed at scale r_0. For any non-principal ultrafilter ω the ultralimit (X_ω, x_ω) of the sequence (X_n, x_n) is again a complete, geodesically complete, CAT(0), δ-hyperbolic metric space that is P_0-packed at scale r_0: this follows from Theorem 8.1.1 and from the fact that the δ-hyperbolicity condition, as expressed in (12), is clearly stable under ultralimits.

Let moreover Γ_n be any group of isometries of the space X_n. We will now define a limit group of isometries Γ_ω of X_ω. We recall that a sequence of isometries (g_n), with each $g_n \in \Gamma_n$, is *admissible* if there exists $M < +\infty$ such that $d(x_n, g_n x_n) \leq M$ $\forall n \in \mathbb{N}$. Every admissible sequence (g_n) defines a limit isometry $g_\omega = \omega\text{-}\lim g_n$ of X_ω by the formula

$$g_\omega(y_\omega) = \omega\text{-}\lim(g_n(y_n))$$

where $y_\omega = \omega\text{-}\lim y_n$ is a generic point of X_ω, see Proposition 2.7.5. We then define:

$$\Gamma_\omega := \{\omega\text{-}\lim g_n \mid (g_n) \text{ admissible sequence, } g_n \in \Gamma_n \ \forall n\}.$$

The following lemma is straightforward:

Lemma 8.2.1. *The composition of admissible sequences of isometries is an admissible sequence of isometries and the limit of the composition is the composition of the limits.*

(Indeed if $g_\omega = \omega\text{-}\lim g_n$, $h_\omega = \omega\text{-}\lim h_n$ belong to Γ_ω then their composition belong to Γ_ω, as $d(g_n h_n \cdot x_n, x_n) \le d(g_n h_n \cdot x_n, g_n \cdot x_n) + d(g_n \cdot x_n, x_n) < +\infty$).
Analogously one proves that (id_n) belongs to Γ_ω and defines the identity map of X_ω, and that if $g_\omega = \omega\text{-}\lim g_n$ belongs to Γ_ω then also the sequence (g_n^{-1}) defines an element of Γ_ω, which is the inverse of g_ω.
So we have a well defined composition law on Γ_ω, that is for $g_\omega = \omega\text{-}\lim g_n$ and $h_\omega = \omega\text{-}\lim h_n$ we set

$$g_\omega \circ h_\omega = \omega\text{-}\lim(g_n \circ h_n)$$

With this operation Γ_ω is a group of isometries of X_ω and is called *the ultralimit group* of the sequence of groups Γ_n.
In the following proposition we describe the possible ultralimits of an admissible sequence of isometries:

Proposition 8.2.2. *Assume that (X_n, x_n) is a sequence of complete, geodesically complete,* CAT(0)*, δ-hyperbolic metric spaces that are P_0-packed at scale r_0. Let (g_n) be an admissible sequence of isometries.*

(a) If g_n is of hyperbolic type with axis γ_n for ω-a.e.(n) then

- *(a.1) if $\omega\text{-}\lim d(x_n, \gamma_n) < +\infty$ then g_ω is elliptic when $\omega\text{-}\lim \ell(g_n) = 0$, and hyperbolic with axis $\gamma_\omega = \omega\text{-}\lim \gamma_n$ and $\ell(g_\omega) = \omega\text{-}\lim \ell(g_n)$ otherwise;*
- *(a.2) if $\omega\text{-}\lim d(x_n, \gamma_n) = +\infty$ then g_ω is either elliptic or parabolic.*

(b) If g_n is parabolic for ω-a.e.(n) then g_ω is either elliptic or parabolic.

Notice that any two axes of a hyperbolic isometry are at uniformly bounded distance from each other, by the δ-hyperbolicity assumption, cp. Lemma 2.3.8, so (a) does not depend on the particular choice of γ. Moreover the ultralimit of a sequence of geodesics γ_n at uniformily bounded distance from the base points x_n is again a geodesic of X_ω (Proposition 2.7.5).

Example 8.2.3. Case (a.2) can actually occur for the limit g_ω of a sequence of hyperbolic isometries g_n. Let for instance Γ_n be a Schottky group of $X = \mathbb{H}^2$ generated by two hyperbolic isometries a_n, b_n with non-intersecting axes. The convex core of the quotient space $\bar{X}_n = \Gamma_n \backslash \mathbb{H}^2$ is a hyperbolic pair of pants, with boundary given by three periodic geodesics α_n, β_n and γ_n. These geodesics correspond, respectively, to the projections of the axes of the elements a_n, b_n and $c_n = a_n \cdot b_n$ (up to replacing b_n with its inverse). Letting the length $\ell(\gamma_n)$ tend to zero (which means pulling two of the isometry circles of a_n and b_n closer and closer), the sequence of hyperbolic isometries c_n tends to a parabolic isometry and $\bar{X}_n$ tends to a surface with one cusp.

Proof of Proposition 8.2.2. We start from g_n of hyperbolic type with axis γ_n for ω-a.e.(n). Assume first that $\omega\text{-}\lim d(x_n, \gamma_n) = C < +\infty$. If $\omega\text{-}\lim \ell(g_n) = 0$ then any point of the limit geodesic γ_ω is a fixed point of g_ω, so g_ω is elliptic. Otherwise $\omega\text{-}\lim \ell(g_n) = \ell > 0$ and it is immediate that g_ω translates γ_ω by ℓ, hence it is of hyperbolic type with axis γ_ω.
Suppose now that g_n is of hyperbolic type with axis γ_n for ω-a.e.(n) and that $\omega\text{-}\lim d(x_n, \gamma_n) = +\infty$. Let $M_0 \ge 0$ be an upper bound for $d(x_n, g_n x_n)$ for every

n. A direct application of Proposition 2.5.7 gives ω-lim $\ell(g_n) = 0$, otherwise the distance between x_n and the axis γ_n of g_n would be uniformly bounded. Suppose that g_ω is hyperbolic: in this case $\ell(g_\omega) = \ell_0$ would be strictly positive. Applying again Proposition 2.5.7 we would find, for ω-a.e.(n), a point $p_n \in X_n$ satisfying

$$d(p_n, x_n) \leq K_1(\ell_0, M_0, \delta), \qquad d(p_n, g_n p_n) \leq \frac{\ell_0}{2}.$$

The first condition implies that the sequence (p_n) defines a point p_ω of X_ω, while the second condition implies that $d(p_\omega, g_\omega p_\omega) \leq \frac{\ell_0}{2}$ which is impossible, so g_ω is no of hyperbolic type.
Finally suppose that g_n is of parabolic type for ω-a.e.(n). If g_ω was hyperbolic of translation length $\ell_0 > 0$ then arguing as before there would exist a point $p_n \in X_n$ satisfying

$$d(p_n, x_n) \leq K_1(\ell_0, M_0, \delta), \qquad d(p_n, g_n p_n) \leq \frac{\ell_0}{2}.$$

This is again a contradiction. □

The next theorem explains how the ultralimit of a sequence of torsion-free, discrete groups of isometries in our setting can degenerate, that is, when the limit is non-discrete or admits elliptic elements:

Theorem 8.2.4. *Let (X_n, x_n) be a sequence of complete, geodesically complete,* CAT(0)*, δ-hyperbolic metric spaces that are P_0-packed at scale r_0. Let Γ_n be a sequence of torsion-free, discrete groups of isometries of X_n. Let ω be a non-principal ultrafilter and Γ_ω be the limit group of isometries of X_ω. Then one of the following mutually exclusive possibilities holds:*

(a) $\forall L \geq 0\ \exists r > 0$ such that ω-$\lim d(x_n, (X_n)_r) > L$. In this case the group Γ_ω acts discretely on X_ω and it has no torsion;

(b) $\exists L \geq 0$ such that $\forall r > 0$ it holds ω-$\lim d(x_n, (X_n)_r) \leq L$. In this case the group Γ_ω is elementary (possibly non-discrete).

Proof. We start from case (a). We recall that X_ω is proper. Let $g_\omega = \omega$-lim g_n be an element of Γ_ω and $y_\omega = \omega$-lim y_n be a point of X_ω. By definition of y_ω there exists $L \geq 0$ such that $d(x_n, y_n) \leq L$ for all n. By assumption there exists r such that $d(y_n, g_n y_n) \geq r$ for ω-a.e.(n), if $g_n \neq \mathrm{id}$ for ω-a.e.(n). This implies $d(y_\omega, g_\omega y_\omega) \geq r$, so $\mathrm{sys}(\Gamma_\omega, y_\omega) \geq r$ for all $y_\omega \in \overline{B}(x_\omega, L)$. Since X_ω is proper we conclude that Γ_ω is discrete. Moreover it is torsion-free: indeed any elliptic element $g_\omega = (g_n)$ of Γ_ω must have a fixed point y_ω, hence, as just proved, g_n is the identity for ω-a.e.(n); so $g_\omega = \mathrm{id}$ necessarily, since $\mathrm{sys}(\Gamma_\omega, y_\omega)$ is strictly positive.
We now study case (b). In this case for all $r > 0$ there exists a point $y_n \in X_n$ with $d(x_n, y_n) \leq L$ and $\mathrm{sys}(\Gamma_n, y_n) \leq r$ for ω-a.e.(n). Observe that for all $r \leq \varepsilon_0$ the group $\Gamma_{R_r}(y_n)$ is elementary, with $R_r \to +\infty$ when $r \to 0$ by Lemma 6.4.3. Now, by definition, for every $g_\omega = \omega$-lim g_n of Γ_ω there exists M such that $d(x_n, g_n x_n) \leq M$ for ω-a.e.(n), so $d(y_n, g_n y_n) \leq 2L + M \leq R_r$ provided that r is small enough. This implies that g_n belongs, for ω-a.e.(n), to a fixed elementary subgroup $\Gamma'_n < \Gamma_n$ that does not depend on the element g_ω under consideration. Then there are two

possibilities: for ω-a.e.(n) either Γ'_n is of hyperbolic type or it is of parabolic type. Assume that the isometries in Γ'_n are all hyperbolic for ω-a.e.(n), so there exists a common axis γ_n for all of them. If ω-$\lim d(x_n, \gamma_n) < +\infty$, then we are in case (a.1) of Proposition 8.2.2: the limit g_ω is either hyperbolic with axis $\gamma_\omega = \omega$-$\lim \gamma_n$ or elliptic, for all $g_\omega \in \Gamma_\omega$, hence this group is elementary. In all the other cases Proposition 8.2.2 implies that Γ_ω does not contain any hyperbolic isometry, so the group is elementary by Gromov' classification of groups acting on hyperbolic spaces (cp. [Gro87],[DSU17]).

□

We examine now the case when the limit group is discrete:

Corollary 8.2.5. *Same assumptions as in Theorem 8.2.4.*
Let $\bar{X}_n = \Gamma_n \backslash X_n$ be the quotient metric spaces, let $p_n \colon X_n \to \bar{X}_n$ the projection maps and let $\bar{x}_n = p_n(x_n)$. Then:

(a) if the groups Γ_n are non-elementary, then one can always suitably choose the base points $x_n \in X_n$ so that case (a) of Theorem 8.2.4 occurs, hence the ultralimit group Γ_ω is discrete and torsion-free;

(b) if case (a) of Theorem 8.2.4 occurs, then the ultralimit space $\bar{X}_\omega$ of the sequence $(\bar{X}_n, \bar{x}_n)$ is isometric to $\Gamma_\omega \backslash X_\omega$.

Proof. By Corollary 6.3.1, if the groups Γ_n are non-elementary it is always possible to choose $x_n \in X_n$ in order that the pointwise systole of the Γ_n at x_n are uniformly bounded away from zero, i.e. $\text{sys}(\Gamma_n, x_n) \geq \varepsilon > 0$ for every n. The fact that case (a) occurs for this choice of the base points is then a direct consequence of Proposition 2.5.5.(b).
To show (b) notice that the projections $p_n \colon X_n \to \bar{X}_n$ form an admissible sequence of 1-Lipschitz maps and then, by Proposition 2.7.5, they yield a limit map $p_\omega \colon X_\omega \to \bar{X}_\omega$ defined as $p_\omega(y_\omega) = \omega$-$\lim p_n(y_n)$, for ω-$\lim y_n = y_\omega$. The map p_ω is clearly surjective. We want to show that it is Γ_ω-equivariant. We fix $g_\omega = \omega$-$\lim g_n \in \Gamma_\omega$ and $y_\omega = \omega$-$\lim y_n \in X_\omega$. Then:

$$p_\omega(\gamma_\omega y_\omega) = \omega\text{-}\lim p_n(\gamma_n y_n) = \omega\text{-}\lim p_n(y_n) = p_\omega(y_\omega).$$

Therefore we have a well defined, surjective quotient map $\bar{p}_\omega \colon \Gamma_\omega \backslash X_\omega \to \bar{X}_\omega$. We will now show that it is a local isometry. We fix an arbitrary point $y_\omega = \omega$-$\lim y_n \in X_\omega$, consider its class $[y_\omega] \in \Gamma_\omega \backslash X_\omega$ and set $L = d(x_\omega, y_\omega)$. By assumption there exists r, depending only on L, such that $\text{sys}(\Gamma_n, y_n) \geq r$ for ω-a.e.(n). In particular the systole of Γ_ω at y_ω is at least r, so the quotient map $X_\omega \to \Gamma_\omega \backslash X_\omega$ is an isometry between $\overline{B}(y_\omega, \frac{r}{2})$ and $\overline{B}([y_\omega], \frac{r}{2})$.
Moreover for ω-a.e.(n) we have that $\overline{B}(p_n(y_n), \frac{r}{2})$ is isometric to $\overline{B}(y_n, \frac{r}{2})$. By Lemma 2.7.8 we know that ω-$\lim \overline{B}(p_n(y_n), \frac{r}{2})$ is isometric to $\overline{B}(p_\omega(y_\omega), \frac{r}{2}) = \overline{B}(\bar{p}_\omega(y_\omega), \frac{r}{2})$ and that ω-$\lim \overline{B}(y_n, \frac{r}{2})$ is isometric to $\overline{B}(y_\omega, \frac{r}{2})$. Therefore $\overline{B}(\bar{p}_\omega(y_\omega), \frac{r}{2})$ is isometric to $\overline{B}([y_\omega], \frac{r}{2})$. By Proposition 3.28, Sec.I, of [BH13] we conclude that the map $\bar{p}_\omega$ is a locally isometric covering map. To conclude, it is enough to show that $\bar{p}_\omega$ is injective. Let $[z_\omega], [y_\omega] \in \Gamma_\omega \backslash X_\omega$. Then we have $\bar{p}_\omega([z_\omega]) = \bar{p}_\omega([y_\omega])$ if and only if $p_\omega(z_\omega) = p_\omega(y_\omega)$. This is equivalent to ω-$\lim d(p_n(z_n), p_n(y_n)) = 0$ and, as the

systole of y_n is uniformly bounded away from zero, this means $\omega\text{-lim}\, d(z_n, g_n y_n) = 0$ for some $g_n \in \Gamma_n$ and for ω-a.e.(n). We observe that the sequence (g_n) is admissible, therefore it defines an element $g_\omega = \omega\text{-lim}\, g_n \in \Gamma_\omega$ satisfying $d(z_\omega, g_\omega y_\omega) = 0$. This implies that $[z_\omega] = [y_\omega]$ and therefore $\bar{p}_\omega$ is an isometry. □

Non-elementary ultralimit groups are characterized in the next result:

Theorem 8.2.6. *Same assumptions as in Theorem 8.2.4.*

(a) If there exist two sequences of admissible isometries $(g_n), (h_n)$ *in* Γ_n *of the same type such that* $\langle g_n, h_n \rangle$ *is non-elementary for* ω*-a.e.*(n) *then the group* $\langle g_\omega, h_\omega \rangle$ *is non-elementary;*

(b) the group Γ_ω *is non-elementary if and only if there exist two sequences of admissible isometries* $(g_n), (h_n)$ *such that* $\langle g_n, h_n \rangle$ *is non-elementary for* ω*-a.e.*(n)*.*

Notice that Γ_ω non-elementary implies that it is also discrete and torsion-free, by Theorem 8.2.4.

Proof. Let $(g_n), (h_n)$ be as in (a) and let $M \geq 0$ such that for all n it holds $d(x_n, g_n x_n), d(x_n, h_n x_n) \leq M$. We first assume that $f_\omega = \omega\text{-}\lim f_n$, for some admissible sequence of isometries $f_n \in \langle g_\omega, h_\omega \rangle$, is elliptic. Then there would exist a point $y_\omega = \omega\text{-lim}\, y_n$ with $f_\omega y_\omega = y_\omega$. So for all $r > 0$ and for ω-a.e.(n) the following conditions would hold, for some $L \geq 0$:

$$d(x_n, y_n) \leq L, \qquad d(y_n, f_n y_n) \leq r.$$

The first condition implies that

$$d(y_n, g_n y_n) \leq d(y_n, x_n) + d(x_n, g_n x_n) + d(g_n x_n, g_n y_n) \leq 2L + M$$

and similarly for h_n. If r is small enough we then deduce that $\langle g_n, h_n \rangle$ is elementary by Lemma 6.4.3, a contradiction.
So we assume now that the elements g_ω, h_ω are both parabolic. If $\langle g_\omega, h_\omega \rangle$ was elementary then they would have the same fixed point at infinity z. We then choose $\varepsilon > 0$ small enough so that $R_\varepsilon \geq 16\delta + \varepsilon$, where R_ε is the quantity defined in Lemma 6.4.3. As $\ell(g_\omega) = \ell(h_\omega) = 0$ and X_ω is convex, there exist points $y_\omega = \omega\text{-lim}\, y_n$, $w_\omega = \omega\text{-lim}\, w_n$ of X_ω such that $d(y_\omega, g_\omega y_\omega) < \varepsilon$ and $d(w_\omega, h_\omega w_\omega) < \varepsilon$. By Lemma 2.3.2 we can also find points $y'_\omega \in [y_\omega, z]$ and $w'_\omega \in [w_\omega, z]$ such that $d(y_\omega, w_\omega) \leq 8\delta$. By convexity and by the triangular inequality we deduce:

$$d(y'_\omega, g_\omega y'_\omega) < \varepsilon, \qquad d(y'_\omega, h_\omega y'_\omega) < 16\delta + \varepsilon \leq R_\varepsilon.$$

Similar estimates hold for g_n and h_n for ω-a.e.(n), implying that $\langle g_n, h_n \rangle$ is elementary for ω-a.e.(n), a contradiction.
A similar argument works when one element is parabolic, say g_ω, and the other, h_ω, is hyperbolic. In this case, if $\langle g_\omega, h_\omega \rangle$ was elementary, the fixed point of g_ω would coincide with one point at infinity z of an axis γ of h_ω. In this case we choose $\varepsilon > 0$ so that $R_\varepsilon > \ell_0 + 16\delta$, where R_ε is again the number given by Lemma 6.4.3 and ℓ_0

is the minimal displacement of h_ω. We then take a point $z_\omega = \omega\text{-}\lim z_n$ of X_ω such that $d(z_\omega, g_\omega z_\omega) < \varepsilon$, a point $y_\omega = \omega\text{-}\lim y_n$ on γ, and a point $z'_\omega = \omega\text{-}\lim z'_n \in [z_\omega, z]$ such that $d(y_\omega, z'_\omega) \leq 8\delta$. By convexity and the triangular inequality we get

$$d(z'_\omega, g_\omega z'_\omega) < \varepsilon, \qquad d(z'_\omega, h_\omega z'_\omega) \leq 16\delta + \ell_0 < R_\varepsilon.$$

and again similar estimates hold for g_n, h_n for ω-a.e.(n), showing that $\langle g_n, h_n \rangle$ is elementary for ω-a.e.(n), a contradiction.
It remains to consider the case where both g_ω and h_ω are of hyperbolic type. Suppose they have the same axis. By Lemma 8.2.2 we know that the axis is the ultralimit of some axis γ_n of g_n, and of some axis η_n of h_n as well; therefore $\omega\text{-}\lim \gamma_n = \omega\text{-}\lim \gamma'_n$. This means that for all $C > 0$ and for ω-a.e.(n) the set of points of γ_n that are at distance at most $\frac{\varepsilon_0}{37}$ from γ'_n is a subsegment of length at least C, where ε_0 is the generalized Margulis constant. By Proposition 5.2.8 we conclude that $\ell(g_n) \geq \frac{1}{5}C$. Therefore the sequence (g_n) is not admissible, a contradiction. This implies that g_ω and h_ω do not have the same axis, therefore $\langle g_\omega, h_\omega \rangle$ is not elementary. This proves (a).
In order to prove (b) assume first that $(g_n), (h_n)$ are two admissible sequences such that $\langle g_n, h_n \rangle$ is not elementary for ω-a.e.(n). Up to replacing h_n with $h_n g_n h_n^{-1}$ we may suppose that g_n, h_n are of the same type, and still admissible. So Γ_ω is not elementary by (a).
Conversely assume that Γ_ω is not elementary. Then it contains at least a hyperbolic element g_ω and by Theorem 8.2.4 we know it is discrete. Therefore, by discreteness and non-elementarity, there exists another element $h_\omega \in \Gamma_\omega$ such that $\langle g_\omega, h_\omega \rangle$ is not elementary. Up to replacing h_ω with $h_\omega g_\omega h_\omega^{-1}$ we may again suppose that h_ω is of hyperbolic type (and the group genarated by g_ω and this element remains non-elementary). Hence the axes $\gamma_\omega, \gamma'_\omega$ respectively of $g_\omega = \omega\text{-}\lim g_n$ and $h_\omega = \omega\text{-}\lim h_n$ are not the same. By Lemma 8.2.2 the elements g_n, h_n are hyperbolic for ω-a.e.(n), and have axes γ_n, γ'_n such that $\gamma_\omega = \omega\text{-}\lim \gamma_n$, and $\gamma'_\omega = \omega\text{-}\lim \gamma'_n$. Now, if $\langle g_n, h_n \rangle$ was elementary for ω-a.e.(n) then we could choose $\gamma_n = \gamma'_n$ for ω-a.e.(n) and therefore $\gamma_\omega = \gamma'_\omega$, a contradiction. This shows that $\langle g_n, h_n \rangle$ is not elementary for ω-a.e.(n). □

For all $P_0, r_0, \delta, \Delta > 0$ we denote by

$$\text{CAT}^0_{\text{nil}}(P_0, r_0, \delta; \Delta)$$

the class of triples (X, x, Γ) where X is a complete, geodesically complete, CAT(0), δ-hyperbolic metric space that is P_0-packed at scale r_0, x is a point of X and Γ is a discrete and torsion-free group of isometries of X satisfying $\text{nilrad}^+(\Gamma, X) \leq \Delta$. Then we have:

Corollary 8.2.7. *The class* $\text{CAT}^0_{\text{nil}}(d, \delta, \sigma, R_0; \Delta)$ *is closed under ultralimits, hence compact under pointed Gromov-Hausdorff convergence.*

Proof. Consider any sequence $((X_n, x_n), \Gamma_n)$ in this class. As recalled at the beginning of this section the ultralimit (X_ω, x_ω) of the sequence (X_n, x_n) is again a complete, geodesically complete, CAT(0), δ-hyperbolic metric space that is P_0-packed at scale r_0. Then to prove that our class is closed under ultralimits we need only to

show that the bound of the upper nilradius is satisfied also by the limit group Γ_ω acting on X_ω, indeed by the estimate (41) we know that $\text{sys}(X_n, \Gamma_n) \geq s_0(P_0, r_0, \delta, \Delta)$ for all n. Therefore we are in case (a) of Theorem 8.2.4 and Γ_ω is a discrete and torsion-free group of isometries of X_ω.
Assume first that $\text{sys}(\Gamma_n, X_n)$ is greater than or equal to the the generalized Margulis constant ε_0 for ω-a.e.(n). Then $\text{sys}(\Gamma_\omega, X_\omega) \geq \varepsilon_0$, so $\text{nilrad}^+(\Gamma_\omega, X_\omega) = -\infty \leq \Delta$, and the conclusion holds.
Otherwise $\text{sys}(\Gamma_n, X_n) < \varepsilon_0$ for ω-a.e.(n). In this case we take any $y_\omega = \omega\text{-}\lim y_n$ such that $s = \text{sys}(\Gamma_\omega, y_\omega) < \varepsilon_0$. By the discreteness of Γ_ω there exists $g_\omega = \omega\text{-}\lim g_n \in \Gamma_\omega$ such that $d(y_\omega, g_\omega y_\omega) = s$. We fix $\varepsilon < \varepsilon_0 - s$ and we deduce that $d(y_n, g_n y_n) < s + \varepsilon < \varepsilon_0$ for ω-a.e.(n), so $\text{nilrad}^+(\Gamma_n, y_n) \leq \Delta$. This means that for all $\varepsilon > 0$ there is $h_n \in \Gamma_n$ such that $d(y_n, h_n y_n) \leq \Delta + \varepsilon$ and $\langle h_n, g_n \rangle$ is not elementary. To conclude we need to show that $\langle g_\omega, h_\omega \rangle$ is not elementary. Assume the contrary: then h_ω has the same type and the same fixed points at infinity as g_ω. If they were hyperbolic then by Lemma 8.2.2 also g_n, h_n would be hyperbolic for ω-a.e.(n), and by Theorem 8.2.6 we would obtain that $\langle g_n, h_n \rangle$ is elementary for ω-a.e.(n), a contradiction. On the other hand if both are parabolic then we have two possibilities: either g_n, h_n are of the same type for ω-a.e.(n), and arguing as before would give again a contradiction; or g_n is hyperbolic and h_n is parabolic for ω-a.e.(n). In this last case we consider the elementary group $\langle g_\omega, h_\omega g_\omega h_\omega^{-1} \rangle$ and apply Theorem 8.2.6 as before to deduce that the group $\langle g_n, h_n g_n h_n^{-1} \rangle$ is elementary for ω-a.e.(n). Therefore $h_n \text{Fix}_\partial(g_n) = \text{Fix}_\partial(h_n g h_n^{-1}) = \text{Fix}_\partial(g_n)$, so the fixed point of h_n coincides with one of the fixed points of g_n, which contradicts the fact that Γ_n is discrete. This shows that $\langle g_\omega, h_\omega \rangle$ is not elementary. By the arbitrariness of ε we then obtain $\text{nilrad}^+(\Gamma_\omega, y_\omega) \leq \Delta$. □

8.3 Convergence of the boundaries

We recall that $\text{CAT}^0(P_0, r_0)$ is the class of couples (X, x) where X is a complete, geodesically complete, CAT(0) metric space that is P_0-packed at scale r_0 and x is a point of X. Moreover $\text{CAT}^0(P_0, r_0, \delta)$ is the subclass of $\text{CAT}^0(P_0, r_0)$ made of δ-hyperbolic spaces. The boundary of the spaces in this class is stable under ultralimits.

Proposition 8.3.1. *Let $(X_n, x_n) \subseteq \text{CAT}^0(P_0, r_0, \delta)$ and let $D_{x_n,a}$ be a standard visual metric of parameter a and center x_n on ∂X_n. Let ω be a non-principal ultrafilter and let (X_ω, x_ω) be the ultralimit of the sequence (X_n, x_n). Then there exists a visual metric $D_{x_\omega,a}$ of parameter a and center x_ω on ∂X_ω such that ω-$\lim(\partial X_n, D_{x_n,a})$ is isometric to $(\partial X_\omega, D_{x_\omega,a})$.*

We observe that since the spaces ∂X_n are compact with diameter at most 1 then the ultralimit $\omega\text{-}\lim \partial X_n$ does not depend on the basepoints.

Proof. A point of $\omega\text{-}\lim \partial X_n$ is a class of a sequence of points $(z_n) \in \partial X_n$ and each point z_n is identified to the geodesic ray $\xi_n = [x_n, z_n]$. The sequence of geodesic rays (ξ_n) defines a geodesic ray ξ_ω of X_ω with $\xi_\omega(0) = x_\omega$ which provides a point of ∂X_ω. It is then defined the map $\Psi\colon \omega\text{-}\lim \partial X_n \to \partial X_\omega$ that sends the sequence (z_n) to the boundary point identified by the geodesic ray ξ_ω.

Good definition. We need to show that Ψ is well defined. Let (z'_n) be another sequence of points equivalent to (z_n), i.e. $\omega\text{-}\lim D_{x_n,a}(z_n, z'_n) = 0$. Since $D_{x_n,a}$ is a standard visual metric for every n this implies that for all $\varepsilon > 0$ and for ω-a.e.(n) it holds $(z_n, z'_n)_{x_n} > \log \frac{1}{\varepsilon} =: T_\varepsilon$. By Lemma 4.3.4 we have $d(\xi_n(T_\varepsilon - \delta), \xi'_n(T_\varepsilon - \delta)) \leq 4\delta$ and, by convexity of the metric, we have that $d(\xi_n(S_\eta), \xi'_n(S_\eta)) < \eta$, where $S_\eta = \eta \cdot \frac{T_\varepsilon}{4\delta}$ for all $\eta > 0$. This means that for every $T \geq 0$ and every $\eta > 0$ we have $d(\xi_n(T), \xi'_n(T)) < \eta$ for ω-a.e.(n). Since η is arbitrary we obtain that ξ_ω and ξ'_ω coincide up to time T for every $T \geq 0$ and therefore $\xi_\omega = \xi'_\omega$.

Bijectivity. The next step is to show that Ψ is bijective. It is clearly surjective since every geodesic ray of X_ω is ultralimit of geodesic rays of X_n by the CAT(0) condition. Let us show it is injective: if two sequence of points $(z_n), (z'_n)$ have the same image under Ψ then for all $T \geq 0$ and for every $\eta > 0$ we have that for ω-a.e.(n) the geodesic rays ξ_{z_n} and $\xi_{z'_n}$ stay at distance less than 2η up to time T. By Lemma 4.3.4 we conclude that $(z_n, z'_n)_{x_n} > T - \eta$ and therefore $D_{x_n,a}(z_n, z'_n) \leq e^{-a(T-\eta)}$. Since this is true for ω-a.e.(n) we get $\omega\text{-}\lim D_{x_n,a}(z_n, z'_n) \leq e^{-a(T-\eta)}$ implying ω-$\lim D_{x_n,a}(z_n, z'_n) = 0$, i.e. $(z_n) = (z'_n)$ as elements of $\omega\text{-}\lim \partial X_n$, by the arbitrariness of T and η.

Homeomorphism Let us show Ψ is continuous. Both $\omega\text{-}\lim \partial X_n$ and ∂X_ω are metrizable, then it is enough to check the continuity on sequences of points. We take a sequence $(z_n^k)_{k\in\mathbb{N}}$ converging to (z_n^∞) in $\omega\text{-}\lim \partial X_n$. This means that for every $\varepsilon > 0$ there exists $k_\varepsilon \geq 0$ such that if $k \geq k_\varepsilon$ then $\omega\text{-}\lim D_{x_n,a}(z_n^k, z_n^\infty) < \varepsilon$. Arguing as before we obtain that for every $\varepsilon > 0$ there exists $k_\varepsilon \geq 0$ such that for every fixed $k \geq k_\varepsilon$ it holds $(z_n^k, z_n^\infty)_{x_n} \geq \log \frac{1}{\varepsilon} =: T_\varepsilon$ for ω-a.e.(n). Therefore by the same argument used before we conclude that for every $T \geq 0$ there exists $k_T \geq 0$ such that for every fixed $k \geq k_T$ then $\xi_{z_n^k}$ and $\xi_{z_n^\infty}$ stay at distance at most 2 up to time T for ω-a.e.(n). So the same conclusion holds for $\xi_{z_\omega^k}$ and $\xi_{z_\omega^\infty}$ and by Lemma 4.3.4 we have $(\Psi(z_n^k), \Psi(z_n^\infty))_{x_\omega} \geq T - 1$. This implies exactly that the sequence $\Psi(z_n^k)$ converges to $\Psi(z_n^\infty)$. To prove the continuity of the inverse map we suppose $\Psi(z_n^k)$ converges to $\Psi(z_n^\infty)$. By similar arguments used before we get that the geodesic rays ξ_ω^k and ξ_ω^∞ stay at bounded distance up to time T, provided $k \geq k_T$. So the same happens for ξ_n^k and ξ_n^∞ for ω-a.e.(n) implying once again the convergence of (z_n^k) to (z_n^∞).

The metric on ∂X_ω. Since Ψ is an homeomorphism we can endow ∂X_ω with the metric induced by Ψ, i.e. $D(z_\omega, z'_\omega) = \omega\text{-}\lim D_{x_n,a}(z_n, z'_n)$, where z_n and z'_n are sequences such that $\Psi(z_n) = z_\omega$ and $\Psi(z'_n) = z'_\omega$. It remains to show it is a visual metric. We show one of the two conditions since the other is similar. We take $z_\omega = \Psi(z_n), z'_\omega = \Psi(z'_n)$ and we set $D_n := D_{x_n,a}(z_n, z'_n)$. By definition $D_\omega = \omega\text{-}\lim D_n = D(z_\omega, z'_\omega)$. Since each $D_{x_n,a}$ is a standard visual metric we get $(z_n, z'_n)_{x_n} \leq \frac{1}{a} \log \frac{1}{D_n} =: T_n$ for every n and by Lemma 4.3.4 we conclude that $d(\xi_{z_n}(T_n + 3\delta), \xi_{z_n}(T_n + 3\delta)) \geq 6\delta$ for every n. There are two possibilities: $T_\omega := \omega\text{-}\lim T_n$ is either $+\infty$ or a positive real number. In the first case we have $D_\omega = 0$ and so there is nothing to prove. In the second case we know that $d(\xi_{z_\omega}(T_\omega + 3\delta), \xi_{z_\omega}(T_\omega + 3\delta)) \geq 6\delta$ and so by Lemma 4.3.4 we conclude that $(z_\omega, z'_\omega)_{x_\omega} < T_\omega + \delta = \frac{1}{a} \log \frac{1}{D_\omega} + \delta$, implying $D_\omega < e^\delta e^{-a(z_\omega, z'_\omega)_{x_\omega}}$. □

We denote by $\mathrm{CAT}^0_{\mathrm{qc}}(P_0, r_0, \delta; D)$ the class of triples (X, x, Γ) such that $(X, x) \in \mathrm{CAT}^0(P_0, r_0, \delta)$, Γ is a discrete, non-elementary, quasiconvex-cocompact group of isometries of X with codiameter $\leq D$ and finally $x \in \text{QC-Hull}(\Lambda(\Gamma))$. This class is closed under ultralimits.

Theorem 8.3.2. *Let $(X_n, x_n, \Gamma_n) \subseteq \text{CAT}^0_{\text{qc}}(P_0, r_0, \delta; D)$, ω be a non-principal ultrafilter and let $(X_\omega, x_\omega, \Gamma_\omega)$ be the ultralimit space.*
Then $\Psi(\omega\text{-lim}\,\Lambda(\Gamma_n)) = \Lambda(\Gamma_\omega)$, where Ψ is the isometry of Proposition 8.3.1. Moreover Γ_ω is a discrete, non-elementary, quasiconvex-cocompact group of isometries of X_ω with codiameter $\leq D$ and $x_\omega \in \text{QC-Hull}(\Lambda(\Gamma_\omega))$.

Proof. Let L be the constant of Lemma 7.1.2, depending only on δ. We fix a sequence $z_n \in \Lambda(\Gamma_n)$ and we observe that by Lemma 7.1.2 and the cocompactness of the action of Γ_n on $\text{QC-Hull}(\Lambda(\Gamma_n))$ we can find a sequence $(g_n^k)_{k\in\mathbb{N}}$ such that

(a) $g_n^k x_n$ converges to z_n when k tends to $+\infty$;

(b) $g_n^0 = \text{id}$;

(c) $d(g_n^k x_n, g_n^{k+1} x_n) \leq 2L + 2D$;

(d) $d(g_n^k x_n, \xi_{z_n}(k)) \leq L + D$.

For every $k \in \mathbb{N}$ the sequence g_n^k is admissible by (b) and (c), so it defines a limit isometry $g_\omega^k \in \Gamma_\omega$. Moreover we have $d(g_\omega^k x_\omega, \xi_{\Psi(z_n)}(k)) \leq L + D$ for every $k \in \mathbb{N}$, as follows by the definition of Ψ. This clearly implies that the sequence $g_\omega^k x_\omega$ converges to $\Psi(z_n)$ and so $\Psi(z_n) \in \Lambda(\Gamma_\omega)$. In other words $\Psi(\omega\text{-}\lim \Lambda(\Gamma_n)) \subseteq \Lambda(\Gamma_\omega)$. It is easy to show that Γ_ω acts on $\omega\text{-}\lim \Lambda(\Gamma_n)$ by $(g_n)(z_n) = (g_n z_n)$ and that the action commutes with Ψ. Moreover the set $\omega\text{-}\lim \Lambda(\Gamma_n)$ is Γ_ω-invariant and closed, so it is $\Psi(\omega\text{-}\lim \Lambda(\Gamma_n))$. The Γ_ω-invariance is trivial, while if $(z_n^k)_{k\in\mathbb{N}} \in \omega\text{-}\lim \Lambda(\Gamma_n)$ is a sequence converging to (z_n^∞) and $z_n^\infty \notin \omega\text{-}\lim \Lambda(\Gamma_n)$ then there exists $\varepsilon_0 > 0$ such that for ω-a.e.(n) we have $D_{x_n,a}(z_n^\infty, \Lambda(\Gamma_n)) \geq \varepsilon_0$ and this is a contradiction. Therefore the set $\Psi(\omega\text{-}\lim \Lambda(\Gamma_n))$ is a closed Γ_ω-invariant subset of ∂X_ω, that implies it contains $\Lambda(\Gamma_\omega)$ and so the equality between these two sets. This also implies that $\omega\text{-lim}\,\text{QC-Hull}(\Lambda(\Gamma_n)) = \text{QC-Hull}(\Lambda(\Gamma_\omega))$ and so $x_\omega \in \text{QC-Hull}(\Lambda(\Gamma_\omega))$.
By Example 6.2.3 and Corollary 8.2.7 we know that Γ_ω is a non-elementary and discrete group. Moreover for every two points $y_\omega, y'_\omega \in \text{QC-Hull}(\Lambda(\Gamma_\omega))$ there exist sequences of points $y_n, y'_n \in \text{QC-Hull}(\Lambda(\Gamma_n))$ such that $y_\omega = \omega\text{-lim}\, y_n$ and $y'_\omega = \omega\text{-lim}\, y'_n$ and so there are $g_n \in \Gamma_n$ such that $d(g_n y_n, y'_n) \leq D$. The sequence g_n is clearly admissible so it defines an element $g_\omega = \omega\text{-lim}\, g_n$ of Γ_ω and $d(g_\omega y_\omega, y'_\omega) \leq D$, implying that the action of Γ_ω on $\text{QC-Hull}(\Lambda(\Gamma))$ is cocompact with codiameter $\leq D$. □

8.4 Continuity of the entropies

In this last section we will find sufficient conditions to ensure the continuity of the entropy under convergence of metric spaces. In general it is false that the upper (resp.lower) entropies of the ultralimit is the ultralimit of the upper (resp.lower) entropies of the spaces.

Example 8.4.1. Let X be any complete, geodesically complete, CAT(0) metric space X that is P_0-packed at scale r_0 and let X' be the metric space obtained by gluing a ray $[0, +\infty)$ to a point of X. The space X' is again complete, geodesically complete, CAT(0) and packed. We take the sequence $(X_n, x_n) = (X', n)$, where $n \in [0, +\infty)$. Clearly X_ω is isometric to $\mathbb{R}$ with respect to every non-principal ultrafilter ω, so $\overline{h_{\text{Cov}}}(X_\omega) = \underline{h_{\text{Cov}}}(X_\omega) = 0$. On the other hand $\overline{h_{\text{Cov}}}(X_n) = \overline{h_{\text{Cov}}}(X)$ and $\underline{h_{\text{Cov}}}(X_n) = \underline{h_{\text{Cov}}}(X)$ for every n. The same holds for all the other definition of entropies.

If we require an uniformity condition on the entropy function, as explained in the following theorem, then we have continuity. Later we will se a relative version of this result.

Theorem 8.4.2. *Let $(X_n, x_n) \subseteq \text{CAT}^0(P_0, r_0)$ and ω be a non-principal ultrafilter. Suppose that for every n it holds*

$$\frac{1}{T} \log \text{Pack}(\overline{B}(x_n, T), r_0) \asymp h_n$$

and that the threshold functions do not depend on n. Then the upper and lower covering entropies of X_ω coincide and equals $h_\omega = \omega\text{-}\lim h_n$.

Remark 8.4.3. *We remark that:*

(a) under the assumptions of the theorem then for every n the upper and lower covering entropies coincide and h_n is their common value. Moreover, and that is the important hypothesis, the rate of convergence to the limit is uniform in n.

(b) Furthermore by Proposition 4.1.1, Proposition 4.1.4, Theorem 4.2.2 and Remark 4.2.6, Proposition 4.3.2 and Theorem 4.3.6 the assumption of the theorem is equivalent to a control of the rate of convergence to the limit of the functions definining the volume entropies, the Lipschitz topological entropies, the shadow dimensions or the Minkowski dimensions. So if one has a uniform control on the rate of convergence of one of these functions then it has the continuity of all the entropies;

(c) by Proposition 2.7.12 under the assumptions of the theorem we have continuity under pointed Gromov-Hausdorff convergence.

Proof of Theorem 8.4.2. The first step is the following: we claim that for every $T \geq 0$ it holds

$$\omega\text{-}\lim \text{Pack}(\overline{B}(x_n, T), 2r_0) \leq \text{Pack}(\overline{B}(x_\omega, T), r_0) \leq \omega\text{-}\lim \text{Pack}(\overline{B}(x_n, T), r_0).$$

Let $y_\omega^1, \ldots, y_\omega^N$ be a maximal $2r_0$-separated subset of $\overline{B}(x_\omega, T)$. By Lemma 2.7.8 each y_ω^i can be written as $y_\omega^i = \omega\text{-}\lim y_n^i$ with $y_n^i \in \overline{B}(x_n, T)$. Since $d(y_\omega^i, y_\omega^j) > 2r_0$ for every $i \neq j$ and since they are a finite number then for ω-a.e. n it holds $d(y_n^i, y_n^j) > 2r_0$ for all $i \neq j$, so for ω-a.e.(n) there is a $2r_0$-separated subset of $\overline{B}(x_n, T)$ with at least N elements. This implies

$$\text{Pack}(\overline{B}(x_\omega, T), r_0) \leq \omega\text{-}\lim \text{Pack}(\overline{B}(x_n, T), r_0).$$

Now let $y_n^1, \ldots, y_n^{N_n}$ be a maximal $4r_0$-separated subset of $\overline{B}(x_n, T)$. We consider the set $A_\omega = \{\omega\text{-}\lim y_n^{i_n} \text{ s.t. } 1 \leq i_n \leq N_n\}$.
Clearly every element of A_ω belongs to $\overline{B}(x_\omega, T)$. Moreover for every two distinct points $y_\omega, z_\omega \in A_\omega$ it holds $d(y_\omega, z_\omega) \geq 4r_0$. Indeed ω-$\lim y_n^{i_n} = \omega$-$\lim y_n^{j_n}$ if and only if $\omega(\{n \in \mathbb{N} \text{ s.t. } i_n = j_n\}) = 1$, otherwise for ω-a.e. n it holds $d(y_n^{i_n}, y_n^{j_n}) > 4r_0$. This implies that if $y_\omega, z_\omega \in A_\omega$ are distinct points then $d(y_\omega, z_\omega) \geq 4r_0 > 2r_0$, so A_ω is a $2r_0$-separated subset of $\overline{B}(x_\omega, T)$. Since X_ω is proper the set A_ω is of finite cardinality N_ω. We claim that the set $I = \{n \in \mathbb{N} \text{ s.t. } N_n = N_\omega\}$ satisfies $\omega(I) = 1$. In order to prove it we rename the elements of A_ω as $y_\omega^1, \ldots, y_\omega^{N_\omega}$, where $y_\omega^k = \omega\text{-}\lim y_n^{i_n^k}$ for some $1 \leq i_n^k \leq N_n$. From what said before we know that for $k \neq l$ we have $\omega(\{n \in \mathbb{N} \text{ s.t. } i_n^k \neq i_n^l\}) = 1$. So

$$\begin{aligned}1 &= \omega\Bigg(\bigcap_{1 \leq k < l \leq N_\omega} \{n \in \mathbb{N} \text{ s.t. } i_n^k \neq i_n^l\}\Bigg) \\ &= \omega(\{n \in \mathbb{N} \text{ s.t. } i_n^k \neq i_n^l \text{ for all } 1 \leq k < l \leq N_\omega\}) \\ &\leq \omega(\{n \in \mathbb{N} \text{ s.t. } N_n \geq N_\omega\}) = \omega(I \cup J)\end{aligned}$$

where $J = \{n \in \mathbb{N} \text{ s.t. } N_n > N_\omega\}$. Then the claim is true if $\omega(J) = 0$. If $\omega(J) = 1$ then for all $1 \leq j \leq N_\omega + 1$ we can define $y_\omega^j = \omega$-$\lim y_n^j$ if $n \in J$. They are $N_\omega + 1$ distinct points of A_ω, which is impossible. In this way we conclude that for ω-a.e. n it holds $\operatorname{Pack}(\overline{B}(x_\omega, T), r_0) \geq N_\omega = N_n = \operatorname{Pack}(\overline{B}(x_n, T), 2r_0)$ and so $\omega\text{-}\lim \operatorname{Pack}(\overline{B}(x_n, T), 2r_0) \leq \operatorname{Pack}(\overline{B}(x_\omega, T), r_0)$.

Now let $\tilde{h} = \omega\text{-}\lim h_n \in \left[0, \frac{\log(1+P_0)}{r_0}\right]$ by Lemma 4.1.3. By assumption for every $\varepsilon > 0$ there exists T_ε depending on ε and not on n such that

$$\left|\frac{1}{T}\log \operatorname{Pack}(\overline{B}(x_n, T), r_0) - h_n\right| < \varepsilon \text{ for all } T \geq T_\varepsilon.$$

The sets

$$A_1 = \left\{n \in \mathbb{N} \text{ s.t. } \left|\frac{1}{T}\log \operatorname{Pack}(\overline{B}(x_n, T), r_0) - h_n\right| < \varepsilon \text{ for all } T \geq T_\varepsilon\right\}$$

and

$$A_2 = \{n \in \mathbb{N} \text{ s.t. } |h_n - \tilde{h}| < \varepsilon\}$$

belong to ω. Moreover by Proposition 4.1.1 and the first estimate we have that the set

$$A_3 = \left\{n \in \mathbb{N} \text{ s.t. } \left|\frac{1}{T}\log \operatorname{Pack}(\overline{B}(x_n, T), r_0) - \frac{1}{T}\log \operatorname{Pack}(\overline{B}(x_\omega, T), r_0)\right| \leq \varepsilon\right\}$$

belongs to ω for every $T \geq T'_\varepsilon$, where T'_ε depends only on ε, P_0 and r_0. Taking $T \geq \max\{T_\varepsilon, T'_\varepsilon\}$ and $n \in A_1 \cap A_2 \cap A_3 \in \omega$ we conclude that

$$\left|\frac{1}{T}\log \operatorname{Pack}(\overline{B}(x_\omega, T), r_0) - \tilde{h}\right| < 3\varepsilon,$$

finishing the proof, again using Proposition 4.1.1. □

We now state the relative version of theorem 8.4.2. For every sequence $(X_n, x_n) \in \text{CAT}^0(P_0, r_0, \delta)$, for every sequence of subsets $C_n \subseteq \partial X_n$ and every non-principal ultrafilter ω we denote by C_ω the set $\Psi(\omega\text{-}\lim C_n)$, where Ψ is the map of Proposition 8.3.1.

Theorem 8.4.4. *Let $(X_n, x_n) \subseteq \text{CAT}^0(P_0, r_0, \delta)$, $C_n \subseteq \partial X_n$ for every n and ω be a non-principal ultrafilter. Suppose that for every n it holds*

$$\frac{1}{T} \log \text{Pack}(\overline{B}(x_n, T) \cap \text{QC-Hull}(C_n), r_0) \asymp h_n$$

and that the threshold functions do not depend on n. Then the upper and lower covering entropies of C_ω coincide and equals $h_\omega = \omega\text{-}\lim h_n$.

Proof. The proof is the same of Theorem 8.4.2. The only delicate point is the first estimate on the packing number. But by definition of C_ω we observe that $\text{QC-Hull}(C_\omega) = \omega\text{-lim}\,\text{QC-Hull}(C_n)$, so that estimate can be proved in the same way. □

The analogue of Remark 8.4.3 holds for Theorem 8.4.4. As a consequence we get the continuity of the critical exponent of quasiconvex-cocompact groups.

Corollary 8.4.5. *Let $(X_n, x_n, \Gamma_n) \subseteq \text{CAT}^0_{\text{qc}}(P_0, r_0, \delta; D)$ and let ω be a non-principal ultrafilter. Then $h_{\Gamma_\omega} = \omega\text{-lim}\, h_{\Gamma_n}$.*

Proof. For every n we take $C_n = \Lambda(\Gamma_n)$. By Theorem 8.3.2 we have $C_\omega = \Lambda(\Gamma_\omega)$. By Theorem 7.2.5 and the analogue of Remark 8.4.3 the assumptions of Theorem 8.4.4 are satisfied. Since $(X_\omega, x_\omega, \Gamma_\omega) \in \text{CAT}^0_{\text{qc}}(P_0, r_0, \delta; D)$ then $h_n = h_{\Gamma_n}$ for every n and $h_\omega = h_{\Gamma_\omega}$ by Theorem 7.2.5. This concludes the proof. □

Bibliography

[AL06] Goulnara N Arzhantseva and IG Lysenok. A lower bound on the growth of word hyperbolic groups. *Journal of the London Mathematical Society*, 73(1):109–125, 2006.

[And08] PD Andreev. Geometry of ideal boundaries of geodesic spaces with nonpositive curvature in the sense of busemann. *Siberian Advances in Mathematics*, 18(2):95, 2008.

[Bab93] Ivan Konstantinovich Babenko. Asymptotic invariants of smooth manifolds. *Russian Academy of Sciences. Izvestiya Mathematics*, 41(1):1, 1993.

[BCG95] G. Besson, Gilles Courtois, and Sylvestre Gallot. Entropies et rigidités des espaces localement symétriques de courbure strictement négative. *Geometric and Functional Analysis*, 5:731–799, 09 1995.

[BCG03] Gérard Besson, Gilles Courtois, and Sylvestre Gallot. Un lemme de margulis sans courbure et ses applications. *Prép. Institut Fourier*, 595, 2003.

[BCGS17] Gérard Besson, Gilles Courtois, Sylvestre Gallot, and Andrea Sambusetti. Curvature-free margulis lemma for gromov-hyperbolic spaces. *arXiv preprint arXiv:1712.08386*, 2017.

[BF18] Emmanuel Breuillard and Koji Fujiwara. On the joint spectral radius for isometries of non-positively curved spaces and uniform growth. *arXiv preprint arXiv:1804.00748*, 2018.

[BG08] Emmanuel Breuillard and Tsachik Gelander. Uniform independence in linear groups. *Inventiones mathematicae*, 173(2):225, 2008.

[BGP92] Yu Burago, Mikhail Gromov, and Gregory Perel'man. Ad alexandrov spaces with curvature bounded below. *Russian mathematical surveys*, 47(2):1–58, 1992.

[BGS13] W. Ballmann, M. Gromov, and V. Schroeder. *Manifolds of Nonpositive Curvature*. Progress in Mathematics. Birkhäuser Boston, 2013.

[BGT11] Emmanuel Breuillard, Ben Green, and Terence Tao. The structure of approximate groups. *Publications mathématiques de l'IHÉS*, 116, 10 2011.

[BH13] Martin R Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319. Springer Science & Business Media, 2013.

[BJ97] Christopher J Bishop and Peter W Jones. Hausdorff dimension and kleinian groups. *Acta Mathematica*, 179(1):1–39, 1997.

[BL12] Arthur Bartels and Wolfgang Lück. Geodesic flow for CAT(0)–groups. *Geom. Topol.*, 16(3):1345–1391, 2012.

[Bow73] Rufus Bowen. Topological entropy for noncompact sets. *Transactions of the American Mathematical Society*, 184:125–136, 1973.

[Bow93] B. H. Bowditch. Discrete parabolic groups. *J. Differential Geom.*, 38(3):559–583, 1993.

[Bow95] B. H. Bowditch. Geometrical finiteness with variable negative curvature. *Duke Mathematical Journal*, 77(1):229–274, 1995.

[Bow06] Brian H Bowditch. A course on geometric group theory. 2006.

[Bre08] Emmanuel Breuillard. A strong tits alternative. *arXiv preprint arXiv:0804.1395*, 2008.

[Bru08] Michael Brunnbauer. Homological invariance for asymptotic invariants and systolic inequalities. *Geometric and Functional Analysis*, 18(4):1087–1117, 2008.

[BS20] Ivan Babenko and Stephane Sabourau. Minimal volume entropy of simplicial complexes. *arXiv preprint arXiv:2002.11069*, 2020.

[Cav21] Nicola Cavallucci. Variational principle for the geodesic flow of non-positively curved metric spaces. *In preparation*, 2021.

[CDP90] M. Coornaert, T. Delzant, and A. Papadopoulos. *Géométrie et théorie des groupes: les groupes hyperboliques de Gromov.* Lecture notes in mathematics. Springer-Verlag, 1990.

[CG00] C. Champetier and V. Guirardel. Monoides libres dans les groupes hyperboliques. *Séminaire de théorie spectrale et géométrie*, 18:157–170, 1999-2000.

[Coo93] Michel Coornaert. Mesures de patterson-sullivan sur le bord d'un espace hyperbolique au sens de gromov. *Pacific J. Math.*, 159(2):241–270, 1993.

[Cou16] Rémi B. Coulon. Partial periodic quotients of groups acting on a hyperbolic space. *Annales de l'Institut Fourier*, 66(5):1773–1857, 2016.

[Cro80] Christopher B Croke. Some isoperimetric inequalities and eigenvalue estimates. In *Annales scientifiques de l'École normale supérieure*, volume 13, pages 419–435, 1980.

[CS] Filippo Cerocchi and Andrea Sambusetti. Convergence of non-positively curved manifolds with acylindrical splittings. *In preparation.*

[CS19] Filippo Cerocchi and Andrea Sambusetti. Local topological rigidity of nongeometric 3–manifolds. *Geometry & Topology*, 23(6):2899–2927, 2019.

[Del96] Thomas Delzant. Sous-groupes distingués et quotients des groupes hyperboliques. *Duke Mathematical Journal*, 83(3):661–682, 1996.

[DK18] Cornelia Druţu and Michael Kapovich. *Geometric group theory*, volume 63. American Mathematical Soc., 2018.

[DKL18] Subhadip Dey, Michael Kapovich, and Beibei Liu. Ping-pong in hadamard manifolds. *arXiv preprint arXiv:1806.07020*, 2018.

[DPPS09] Françoise Dal'Bo, Marc Peigné, Jean-Claude Picaud, and Andrea Sambusetti. On the growth of nonuniform lattices in pinched negatively curved manifolds. *Journal für die reine und angewandte Mathematik*, 2009(627):31 – 52, 01 Feb. 2009.

[DPPS19] Françoise Dal'bo, Marc Peigné, Jean-Claude Picaud, and Andrea Sambusetti. Asymptotic geometry of negatively curved manifolds of finite volume. *Annales scientifiques de l'École normale supérieure*, 52:1459–1485, 01 2019.

[DPS12] F. Dal'Bo, M. Peigné, and A. Sambusetti. On the horoboundary and the geometry of rays of negatively curved manifolds. *Pacific J. Math.*, 259:55–100, 2012.

[DSU17] Tushar Das, David Simmons, and Mariusz Urbański. *Geometry and dynamics in Gromov hyperbolic metric spaces*, volume 218. American Mathematical Soc., 2017.

[Ebe72] Patrick Eberlein. Geodesic flows on negatively curved manifolds i. *Annals of Mathematics*, pages 492–510, 1972.

[EMO05] Alex Eskin, Shahar Mozes, and Hee Oh. On uniform exponential growth for linear groups. *Inventiones mathematicae*, 160(1):1–30, 2005.

[Gen16] Anthony Genevois. Coning-off cat (0) cube complexes. *arXiv preprint arXiv:1603.06513*, 2016.

[GJN20] Radhika Gupta, Kasia Jankiewicz, and Thomas Ng. Groups acting on cat (0) cube complexes with uniform exponential growth. *arXiv preprint arXiv:2006.03547*, 2020.

[Gro81] Mikhael Gromov. Groups of polynomial growth and expanding maps (with an appendix by jacques tits). *Publications Mathématiques de l'IHÉS*, 53:53–78, 1981.

[Gro82] Michael Gromov. Volume and bounded cohomology. *Publications Mathématiques de l'IHÉS*, 56:5–99, 1982.

[Gro87] Mikhael Gromov. Hyperbolic groups. In *Essays in group theory*, pages 75–263. Springer, 1987.

[Ham09] Ursula Hamenstädt. Rank-one isometries of proper cat (0)-spaces. *Contemporary Mathematics*, 501:43, 2009.

[Her16] David A Herron. Gromov–hausdorff distance for pointed metric spaces. *The Journal of Analysis*, 24(1):1–38, 2016.

[HKR95] Michael Handel, Bruce Kitchens, and Daniel J. Rudolph. Metrics and entropy for non-compact spaces. *Israel Journal of Mathematics*, 91(1):253–271, Oct 1995.

[HKST15] Juha Heinonen, Pekka Koskela, Nageswari Shanmugalingam, and Jeremy T. Tyson. *Sobolev spaces on metric measure spaces: An approach based on upper gradients*. Cambridge University Press, January 2015.

[Jan17] Dorothea Jansen. Notes on pointed gromov-hausdorff convergence. *arXiv preprint arXiv:1703.09595*, 2017.

[Kap01] Michael Kapovich. *Hyperbolic manifolds and discrete groups*, volume 183. Springer Science & Business Media, 2001.

[Kle99] Bruce Kleiner. The local structure of length spaces with curvature bounded above. *Math. Z.*, 231:409–456, 1999.

[Kle10] Bruce Kleiner. A new proof of gromov's theorem on groups of polynomial growth. *J. Amer. Math. Soc.*, 23(3):815–829, 2010.

[Kou98] Malik Koubi. Croissance uniforme dans les groupes hyperboliques. In *Annales de l'institut Fourier*, volume 48, pages 1441–1453, 1998.

[KS19] A. Kar and M. Sageev. Uniform exponential growth for cat(0) square complexes. *Algebr. Geom. Topol.*, 19(3):1229–1245, 2019.

[LN18] Alexander Lytchak and Koichi Nagano. Topological regularity of spaces with an upper curvature bound. *arXiv preprint arXiv:1809.06183*, 2018.

[LN19] Alexander Lytchak and Koichi Nagano. Geodesically complete spaces with an upper curvature bound. *Geometric and Functional Analysis*, 29(1):295–342, Feb 2019.

[LV09] John Lott and Cédric Villani. Ricci curvature for metric-measure spaces via optimal transport. *Annals of Mathematics*, pages 903–991, 2009.

[Man79] Anthony Manning. Topological entropy for geodesic flows. *Annals of Mathematics*, 110(3):567–573, 1979.

[Man10] Johanna Mangahas. Uniform uniform exponential growth of subgroups of the mapping class group. *Geometric and Functional Analysis*, 19(5):1468–1480, 2010.

[McC85] John McCarthy. A "tits-alternative" for subgroups of surface mapping class groups. *Transactions of the American Mathematical Society*, 291(2):583–612, 1985.

[Nag18] Koichi Nagano. Volume pinching theorems for cat (1) spaces. *arXiv preprint arXiv:1810.13056*, 2018.

[OP04] Jean-Pierre Otal and Marc Peigné. Principe variationnel et groupes kleiniens. *Duke Mathematical Journal*, 125(1):15–44, 2004.

[Osi16] D. Osin. Acylindrically hyperbolic groups. *Trans. Amer. Math. Soc.*, 368(2):851–888, 2016.

[Pap05] Athanase Papadopoulos. *Metric spaces, convexity and nonpositive curvature*, volume 6. European Mathematical Society, 2005.

[Pau96] Frédéric Paulin. Un groupe hyperbolique est déterminé par son bord. *Journal of the London Mathematical Society*, 54(1):50–74, 1996.

[Pau97] Frédéric Paulin. On the critical exponent of a discrete group of hyperbolic isometries. *Differential Geometry and its Applications*, 7(3):231–236, 1997.

[Pet90] Peter Petersen, V. A finiteness theorem for metric spaces. *J. Differential Geom.*, 31(2):387–395, 1990.

[Pie19] Erika Pieroni. Minimal entropy of 3-manifolds. *arXiv preprint arXiv:1902.09190*, 2019.

[PS19] Marc Peigné and Andrea Sambusetti. Entropy rigidity of negatively curved manifolds of finite volume. *Mathematische Zeitschrift*, 293(1-2):609–627, 2019.

[Rev05] Guillemette Reviron. *Espaces de longueur d'entropie majorée : rigidité topologique, adhérence des variétés, noyau de la chaleur.* Theses, Université Joseph-Fourier - Grenoble I, April 2005.

[Rob02] Thomas Roblin. Sur la fonction orbitale des groupes discrets en courbure négative. *Annales de l'Institut Fourier*, 52(1):145–151, 2002.

[Ros74] J.M. Rosenblatt. Invariant measures and growth conditions. *Trans. Amer. Math. Soc.*, 193:33–53, 1974.

[Sab17] Stéphane Sabourau. Small volume of balls, large volume entropy and the margulis constant. *Mathematische Annalen*, 369(3-4):1557–1571, 2017.

[Sam02] Andrea Sambusetti. Growth tightness of surface groups. *Expositiones Mathematicae*, 20(4):345–363, 2002.

[Sis18] A. Sisto. Contracting elments and random walks. *J. Reine Angew. Math.*, 742, 2018.

[ST10] Y. Shalom and T. Tao. A finitary version of gromov's polynomial growth theorem. *Geom. Funct. Anal.*, 20(6):1502–1547, 2010.

[SU96] Bernd Stratmann and Mariusz Urbański. The box-counting dimension for geometrically finite kleinian groups. *Fundamenta Mathematicae*, 149(1):83–93, 1996.

[Sul79] Dennis Sullivan. The density at infinity of a discrete group of hyperbolic motions. *Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, 50(1):171–202, 1979.

[Sul84] Dennis Sullivan. Entropy, hausdorff measures old and new, and limit sets of geometrically finite kleinian groups. *Acta Mathematica*, 153(1):259–277, 1984.

[SW05] Michah Sageev and Daniel T Wise. The tits alternative for cat (0) cubical complexes. *Bulletin of the London Mathematical Society*, 37(5):706–710, 2005.

[Thu97] William Thurston. *Three-dimensional geometry and topology*, volume 1. Princeton University Press, 1997.

[Tit72] Jacques Tits. Free subgroups in linear groups. *Journal of Algebra*, 20(2):250–270, 1972.

[Wu18] Yunhui Wu. Translation lengths of parabolic isometries of cat (0) spaces and their applications. *The Journal of Geometric Analysis*, 28(1):375–392, 2018.