Bubbling nodal solutions for a large perturbation of the Moser-Trudinger equation on planar domains

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Abstract

In this work we study the existence of nodal solutions for the problem

$$-\Delta u = \lambda u e^{u^2+|u|^p} \text{ in } \Omega, u = 0 \text{ on } \partial\Omega,$$

where $\Omega \subseteq \mathbb{R}^2$ is a bounded smooth domain and $p \to 1^+$.

If Ω is ball, it is known that the case $p = 1$ defines a critical threshold between the existence and the non-existence of radially symmetric sign-changing solutions. In this work we construct a blowing-up family of nodal solutions to such problem as $p \to 1^+$, when Ω is an arbitrary domain and λ is small enough. As far as we know, this is the first construction of sign-changing solutions for a Moser-Trudinger critical equation on a non-symmetric domain.

1 Introduction

Let us consider the equation

$$\Delta u + \lambda u e^{u^2+a|u|^p} = 0 \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega, \tag{1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^2$, λ is a positive parameter and the nonlinear term $h(u) := ue^{a|u|^p}$, with $a \in \mathbb{R}$ and $p \in [0, 2)$, is a lower-order perturbation of e^{u^2} according to the definition given by Adimurthi in [2].

The nonlinearity $f(u) = h(u)e^{u^2}$ is critical from the view point of the Trudinger imbedding. Indeed, in view of the Moser-Trudinger inequality (see [25, 29, 24])

$$\sup\left\{\int_\Omega e^{u^2}\,dx \ :\ u \in H_0^1(\Omega),\ \|u\|^2_{H_0^1(\Omega)} \le 4\pi\right\} < +\infty, \tag{2}$$

the functional

$$J_\lambda(u) := \frac{1}{2}\int\limits_\Omega |\nabla u|^2 dx - \lambda \int\limits_\Omega F(u)dx,\ u \in H_0^1(\Omega), \tag{3}$$

where $F(t) = \int\limits_0^t f(s)ds$, is well defined and its critical points are solutions to problem (1). Adimurthi in [2] proved that J_λ satisfies the Palais-Smale condition in the infinite energy range $(-\infty, 2\pi)$ but, as observed by Adimurthi and Prashant in [5], the critical nature of $f(u)$ reflects in the failure of the Palais-Smale condition at the sequence of energy levels $2\pi k$ with $k \in \mathbb{N}$ (see also [7]).

In [2] Adimurthi proved the existence of a critical point of J_λ if the perturbation h is large, i.e. $a \geq 0$, and if $0 < \lambda < \lambda_1(\Omega)$, where $\lambda_1(\Omega)$ is the first eigenvalue of $-\Delta$ with Dirichlet boundary condition ((see also [1])). Such a critical point is a positive solution to problem (1). Successively, Adimurthi and Prashant in [6] showed that the condition $a \geq 0$ is necessary to get a positive solution to (1). Indeed, they proved that if the perturbation h is small, i.e. $a < 0$, then there are no positive solutions to problem (1) when the domain Ω is a ball provided λ is small. The case $a = 0$ in a general domain Ω has been studied by Del Pino, Musso and Ruf [14] using a perturbative approach. Indeed they find multiplicity of positive solutions which blow-up in one or more points of Ω (depending on the geometry) as $\lambda \to 0$. We point out that a general qualitative analysis of blowing-up families of positive solutions to problem (1) has been obtained by Druet in [15] (see also [3, 17, 16]).

As far as it concerns the existence of sign-changing solutions, Adimurthi and Yadava in [8] proved that problem (1) has a nodal solution when λ is small if there is the further restriction $p > 1$ on the growth of the large perturbation h (i.e. $a > 0$). Actually, this condition turns out to be optimal for the existence of nodal radial solutions in a ball. Indeed Adimurthi and Yadava in [9] proved that if $a > 0$ and Ω is a ball, problem (1) does not have any radial sign-changing solution when λ is small and $p \in [0, 1]$. If one drops the radial requirement, Adimurthi and Yadava in [8] proved the existence of infinitely many sign-changing solutions in a ball whatever $\lambda > 0$ is. We point out that, in the case $a = 0$, the approach of Del Pino, Musso and Ruf [14] allows to find sign-changing solutions which blow-up positively and negatively at least at two different points in any domain Ω as $\lambda \to 0$ (even if this is not explicitly said in their work).

According to the previous discussion, it turns out that when $a > 0$ the case $p = 1$ defines a critical threshold for the existence of radial sign-changing solutions in the ball. Indeed, when $\Omega = B(0, 1)$, (1) has radially symmetric sign-changing solutions which blow-up as $p \to 1^+$. The precise behavior of such solutions was studied by Grossi and Naimen in [19]. Therefore, when $a > 0$, it is natural to ask whether it is possible to find sign-changing solutions to problem (1) on an arbitrary planar domain Ω which blow-up at one point in Ω as $p \to 1^+$.

In this paper we give a positive answer. More precisely, let us consider the problem

$$\begin{cases} -\Delta u = \lambda u e^{u^2 + |u|^{1+\varepsilon}} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where ε is a positive small parameter. Set

$$f_\varepsilon(t) = te^{t^2+|t|^{1+\varepsilon}}. \tag{5}$$

For a given $0 < \lambda < \lambda_1(\Omega)$, let u_0 be a positive solution of the problem

$$\begin{cases} -\Delta u_0 = \lambda f_0(u_0) & \text{in } \Omega, \\ u_0 > 0 & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega, \end{cases} \tag{6}$$

whose existence has been established by Adimurthi in [2]. We make the following assumptions:

(A1) u_0 is non-degenerate, i.e. there is no non-trivial solution $\varphi \in H^1_0(\Omega)$ of the equation

$$-\Delta\varphi = \lambda f_0'(u_0)\varphi \text{ in } \Omega,\ \varphi = 0 \text{ on } \partial\Omega. \tag{7}$$

(A2) u_0 has a C^1-stable critical point $\xi_0 \in \Omega$ such that $u_0(\xi_0) > \frac{1}{2}$.

Then, we will show that (4) admits a family of sign-changing solutions which blow-up at ξ_0 with residual mass $-u_0$ as $\varepsilon \to 0$, namely:

Theorem 1.1 *For $0 < \lambda < \lambda_1(\Omega)$, let u_0 be a solution of (6) such that (A1) and (A2) are satisfied. Let also ξ_0 be as in (A2). Then there exist $\varepsilon_0 > 0$ and a family $(u_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ of sign-changing solutions to (4) such that:*

- $\max\limits_{B(\xi_0,r)} u_\varepsilon \to +\infty$ *as $\varepsilon \to 0$, for any $0 < r < d(\xi_0, \partial\Omega)$.*
- $u_\varepsilon \to -u_0$ *weakly in $H^1_0(\Omega)$ and in $C^1(\overline{\Omega} \setminus \{\xi_0\})$.*

Let us make some comments about assumtpions (A1) and (A2).

Remark 1.2
- *The solution u_0 to problem (6) turns out to be non-degenerate when Ω is the ball as proved by Adimurthi, Karthik and Giacomoni in [4]. In a work in progress, Grossi and Naimen are going to prove that the solution is also non-degenerate when Ω is convex and symmetric (see [20]). Actually, we believe that the non-degeneracy condition holds true for most domains Ω and positive parameters λ. Indeed, one could use similar arguments to those used by Micheletti and Pistoia in [23] for a class of singularly perturbed equations.*
- *We remind that ξ_0 is a C^1-stable critical point of u_0 if the Brouwer degree $\deg(\nabla u_0, B(\xi_0, r), 0) \neq 0$. In particular, any strict local maximum point of u_0 is C^1-stable. We point out that by Adimurthi and Druet [3] we can deduce that assumption (A2) holds true when the parameter λ is small enough.*

- *We strongly believe that the condition $u_0(\xi_0) > \frac{1}{2}$ is not purely technical, but it is necessary to build a solution which blows-up at ξ_0. Indeed, we conjecture that, if $u_0(\xi_0) \leq \frac{1}{2}$, there does not exist any sign-changing solution which blows-up at ξ_0 with non-trivial residual mass u_0 as $\varepsilon \to 0$. We point out that, in a different setting, a similar condition was proved by Mancini and Thizy [22] for problem* (1) *on a ball with $p = 1$ and $a < 0$: in fact, they show that the value at the origin of the residual mass of any non-compact sequence of radially symmetric positive solutions must be equal to $-\frac{a}{2}$ (and we get $\frac{1}{2}$, when $a = -1$).*

Actually, we can give a more precise description of the asymptotic behavior of the solution u_ε as $\varepsilon \to 0$, since it is build via a Lyapunov-Schmidt procedure. For $\delta, \mu > 0$, and $\xi \in \mathbb{R}^n$, let us consider the functions

$$U_{\delta,\mu,\xi}(x) = \log\left(\frac{8\mu^2\delta^2}{(\mu^2\delta^2 + |x-\xi|^2)^2}\right), \tag{8}$$

which describe the set of all the solutions to the Liouville equation

$$-\Delta U = e^U \quad \text{in } \mathbb{R}^2, \tag{9}$$

under the condition $e^U \in L^1(\mathbb{R}^2)$ (see [21, 12]). We further consider the projection $PU_{\delta,\mu,\xi} := (-\Delta)^{-1} e^{U_{\delta,\mu,\xi}}$, where $(-\Delta)^{-1} : L^2(\Omega) \to H^1_0(\Omega)$ is the inverse of $-\Delta$. Namely, $PU_{\delta,\mu,\xi}$ is defined as the unique solution to

$$\begin{cases} -\Delta PU_{\delta,\mu,\xi} = -\Delta U_{\delta,\mu,\xi} = e^{U_{\delta,\mu,\xi}} & \text{in } \Omega, \\ PU_{\delta,\mu,\xi} = 0 & \text{on } \partial\Omega. \end{cases} \tag{10}$$

Intuitively, we want to look for solutions of (4) that look like $\alpha PU_{\delta,\mu,\xi} - u_0$ for suitable choices of the parameters α, δ, μ, ξ. Unfortunately, in order to succesfully perform Lyapunov-Schmidt reduction, a more precise ansatz is necessary and we are forced to replace u_0 with a better approximation of the solutions. First, the non-degeneracy assumption (A1) allows to find a positive solution $v_\varepsilon \in C^1(\overline{\Omega})$ of (4) such that

$$v_\varepsilon \to u_0 \qquad \text{in } C^1(\overline{\Omega}),$$

as $\varepsilon \to 0$. Then, we consider the function

$$V_{\varepsilon,\alpha,\xi} := v_\varepsilon + \alpha w_{\varepsilon,\xi} + \alpha^2 z_{\varepsilon,\xi}, \tag{11}$$

where $\alpha \in (0,1)$ is a small positive parameter depending on ε, μ, ξ such that $\alpha \to 0$ as $\varepsilon \to 0$, and $w_{\varepsilon,\xi}$ and $z_{\varepsilon,\xi}$ are defined as the unique solutions to the couple of linear problems

$$\begin{cases} \Delta w_{\varepsilon,\xi} + \lambda f'_\varepsilon(v_\varepsilon) w_{\varepsilon,\xi} = 8\pi\lambda G_\xi f'_\varepsilon(v_\varepsilon) & \text{in } \Omega, \\ w_{\varepsilon,\xi} = 0 & \text{on } \partial\Omega, \end{cases} \tag{12}$$

and

$$\begin{cases} \Delta z_{\varepsilon,\xi} + \lambda f'_\varepsilon(v_\varepsilon) z_{\varepsilon,\xi} = \frac{\lambda}{2} f''_\varepsilon(-v_\varepsilon)(8\pi G_\xi - w_\varepsilon)^2 & \text{in } \Omega, \\ z_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{13}$$

with G_ξ denoting the Green function of Ω with singularity at ξ, namely the distributional solution to

$$\begin{cases} -\Delta G_\xi = \delta_\xi & \text{in } \Omega, \\ G_\xi = 0 & \text{on } \partial\Omega. \end{cases} \tag{14}$$

Problems (12) and (13) are nothing but the linearization of problem (4) around the solution v_ε and the R.H.S.'s are the terms of the second order Taylor's expansion with respect to α of $f_\varepsilon(\alpha PU_{\delta,\mu,\xi} - V_{\varepsilon,\alpha,\xi})$ far away from the concentration point ξ (indeed $PU_{\delta,\mu,\xi} \sim 8\pi G_\xi$ because of (23)).

Theorem 1.1 follows at once by the following result:

Theorem 1.3 *Let λ, u_0, ξ_0 be as in Theorem 1.1. There exists $\epsilon_0 > 0$ and functions $\alpha, \delta, \mu : (0, \varepsilon_0) \to (0, +\infty)$, $\xi : (0, \varepsilon_0) \to \Omega$ and $\varphi : (0, \varepsilon_0) \to H_0^1(\Omega)$ such that:*

- $u_\varepsilon := \alpha(\varepsilon) PU_{\delta(\varepsilon),\mu(\varepsilon),\xi(\varepsilon)} - V_{\varepsilon,\alpha(\varepsilon),\xi(\varepsilon)} + \varphi(\varepsilon)$ *is a solution* (4).
- $\alpha(\varepsilon) \to 0$, $\delta(\varepsilon) \to 0$, $\mu(\varepsilon) \to \sqrt{8}e^{-1}$, $\xi(\varepsilon) \to \xi_0$, *and* $u_\varepsilon(\xi(\varepsilon)) \to +\infty$ *as* $\varepsilon \to 0$.
- $\|\varphi(\varepsilon)\|_{H_0^1(\Omega)} + \|\varphi(\varepsilon)\|_{L^\infty(\Omega)} = O(e^{-\frac{\log(2u_0(\xi_0))}{\varepsilon}})$.

Let us briefly sketch the main steps of the proof of Theorem 1.3. First, in Section 2, we choose $\alpha = \alpha(\varepsilon, \mu, \xi)$ and $\delta = \delta(\varepsilon, \mu, \xi)$ such that the function

$$\omega_{\varepsilon,\mu,\xi} := \alpha PU_{\delta,\mu,\xi} - V_{\varepsilon,\alpha,\xi} \tag{15}$$

is an approximate solution of (4). Then, we look for solutions of (4) of the form $\omega_{\varepsilon,\mu,\xi} + \varphi$ with $\varphi \in H_0^1(\Omega)$. Clearly, (4) can be written in terms of φ as

$$-\Delta\varphi - \lambda f_\varepsilon'(\omega_{\varepsilon,\mu,\xi})\varphi = R + N(\varphi), \tag{16}$$

where the error term R is defined by

$$R = R_{\varepsilon,\mu,\xi} := \Delta\omega_{\varepsilon,\mu,\xi} + \lambda f_\varepsilon(\omega_{\varepsilon,\mu,\xi}), \tag{17}$$

and the higher order term N by

$$N(\varphi) = N_{\varepsilon,\mu,\xi}(\varphi) := \lambda \left(f_\varepsilon(\omega_{\varepsilon,\mu,\xi} + \varphi) - f_\varepsilon(\omega_{\varepsilon,\mu,\xi}) - f_\varepsilon'(\omega_{\varepsilon,\mu,\xi})\varphi \right). \tag{18}$$

Equivalently, introducing the linear operator

$$L\varphi = L_{\varepsilon,\mu,\xi}\varphi := \varphi - (-\Delta)^{-1}(\lambda f'(\omega_{\varepsilon,\mu,\xi})\varphi), \tag{19}$$

we need to solve

$$L\varphi = (-\Delta)^{-1}\left(R + N(\varphi)\right). \tag{20}$$

A careful and delicate estimate of the error R will be given in Section 3. The behaviour of the operator L will be studied in Section 4. On the one hand, for functions supported away from a suitable schrinking neighborhood of ξ, we will show that L is close to the operator $L_1\varphi := \varphi - (-\Delta)^{-1}(\lambda f_0'(u_0)\varphi)$, which is invertible on $H_0^1(\Omega)$ because of the non-degeneracy assumption (A1). On the other hand, near the point ξ, L is close to the operator $L_0\varphi := \varphi - (-\Delta)^{-1}(e^{U_{\delta,\mu,\xi}}\varphi)$. This operator appears in the analysis of several critical problems in dimension 2 (see for example [10, 13, 18]) and its behavior is well known: although L_0 is not invertible, it is possible to find an approximate three-dimensional kernel $K_{\delta,\mu,\xi}$ for L_0 by projecting on $H_0^1(\Omega)$ the three functions

$$Z_{0,\delta,\mu,\xi}(x) = \frac{\delta^2\mu^2 - |x-\xi|^2}{|x-\xi|^2+\delta^2\mu^2}, \qquad Z_{i,\delta,\mu,\xi}(x) = \frac{2\delta\mu(x_i-\xi_i)}{|x-\xi|^2+\delta^2\mu^2}, \qquad i = 1,2.$$

Such properties transfer to the operator L, which turns out to be invertible on the subspace $K^{\perp}_{\delta,\mu,\xi}$ orthogonal to $K_{\delta,\mu,\xi}$ in $H_0^1(\Omega)$. More precisely, denoting by π and $\pi^{\perp}$ the projections of $H_0^1(\Omega)$ respectively on $K_{\delta,\mu,\xi}$ and $K^{\perp}_{\delta,\mu,\xi}$, we will show that $\pi^{\perp}L$ is invertible on $K^{\perp}_{\delta,\mu,\xi}$. Then, it is natural to split equation (20) as

$$\begin{cases} \varphi = (\pi^{\perp}L)^{-1}\pi^{\perp}\,(-\Delta)^{-1}\,(R+N(\varphi))\,, \\ \pi L\varphi = \pi\,(-\Delta)^{-1}\,(R+N(\varphi))\,. \end{cases} \tag{21}$$

The first equation of (21) will be solved in Section 5, where for any $\mu > 0$, ξ close to ξ_0 and any small $\varepsilon > 0$, we will find a solution $\varphi_{\varepsilon,\mu,\xi}$ via a contraction mapping argument on a sufficiently small ball in $K^{\perp}_{\delta,\mu,\xi} \cap L^{\infty}(\Omega)$. Then, recalling that $\dim K_{\delta,\mu,\xi} = 3$ and using assumption (A2), we will show in Section 6 that it is possible to choose the three parameters $\mu = \mu(\varepsilon)$ and $\xi = \xi(\varepsilon) = (\xi_1(\varepsilon), \xi_2(\varepsilon))$ so that the second equation in (21) is also fullfilled. Clearly, for such choice of μ and ξ, the function $\varphi_{\varepsilon,\mu(\varepsilon),\xi(\varepsilon)}$ solves both the equations in (21) (or, equivalently (16) and (20)), and $u_\varepsilon := \omega_{\varepsilon,\mu(\varepsilon),\xi(\varepsilon)} + \varphi_{\varepsilon,\mu(\varepsilon),\xi(\varepsilon)}$ is a solution of (4).

It is important to point out that choice of the concentration point $\xi(\varepsilon)$ is extremely delicate since the scaling parameter δ turns out to be much smaller than the parameter α, whose powers control all the error terms. To overcome this difficulty, we introduce a new argument based on a precise Pohozaev-type identity. This allows us to bypass global a priori gradient estimates on the solution $\varphi_{\varepsilon,\mu,\xi}$, which are hard to obtain for Moser-Trudinger critical problems. Our argument requires a very precise ansatz of the approximate solution $\omega_{\varepsilon,\mu,\xi}$. In particular, the presence of the correction terms $w_{\varepsilon,\xi}$ and $z_{\varepsilon,\xi}$ in the expression of $V_{\varepsilon,\alpha,\xi}$ is not merely technical, but plays a crucial role both in the estimates of the error term R and in the choice of $\xi(\varepsilon)$.

2 Construction of the approximate solution

In this section we give the detailed construction of the approximate solution $\omega_{\varepsilon,\mu,\xi}$. Here and in the rest of the paper, we will assume that $(\mu,\xi) \in \mathcal{U} \times B(\xi_0,\sigma)$, where $\mathcal{U} \Subset \mathbb{R}^+$ is an open interval containing $\mu_0 := \sqrt{8}e^{-1}$, ξ_0 is as in the assumption (A2), and $0 < \sigma < \frac{1}{2}d(\xi_0,\partial\Omega)$. By (A2), we can also assume

$$\inf_{B(\xi_0,\sigma)} u_0(\xi) > \frac{1}{2}. \tag{22}$$

2.1 The main terms of the ansatz

Let us introduce the main property of the projection of the bubble $PU_{\delta,\mu,\xi}$ defined in (10), which gives the main term of the approximate solution close to the blow-up point ξ. Let $G_\xi(\cdot) = G(\cdot,\xi)$ be the Green's function of $-\Delta$ with Dirichlet boundary conditions introduced in (14) and let $H(\cdot,\xi)$ be its regular part, i.e.

$$H(x,\xi) := G_\xi(x) - \frac{1}{2\pi}\log\frac{1}{|x-\xi|}.$$

Lemma 2.1 *We have*

$$PU_{\delta,\mu,\xi}(x) = U_{\delta,\mu,\xi}(x) - \log(8\mu^2\delta^2) + 8\pi H(x,\xi) + \psi_{\delta,\mu,\xi}(x),$$

where

$$\|\psi_{\delta,\mu,\xi}\|_{C^1(\overline{\Omega})} = O(\delta^2),$$

uniformly with respect to $\mu \in \mathcal{U}$, $\xi \in B(\xi_0,\sigma)$.
In particular,

$$PU_{\delta,\mu,\xi} \to 8\pi G_\xi \text{ in } C^1_{loc}(\overline{\Omega}\setminus\{\xi\}). \tag{23}$$

Proof. See for example [11, Proosition 5.1]. □

Next, let us define the main term of the approximate solution in the whole domain as $\alpha PU_{\delta,\mu,\xi} - v_\varepsilon$ where α is a positive parameter approaching zero as $\varepsilon \to 0$ and v_ε is a non-degenerate solution to (4), whose existence is proved in the following lemma.

Lemma 2.2 *Let λ and u_0 be as in Theorems 1.1 and 1.3. There exists $\varepsilon_0 > 0$, and a family of functions $(v_\varepsilon)_{0<\varepsilon<\varepsilon_0} \subseteq C^1(\overline{\Omega})$ such that:*

i. v_ε is a non-degenerate weak solution of (4) for any $\varepsilon \in (0,\varepsilon_0)$.

ii. $v_\varepsilon \to u_0$ in $C^1(\overline{\Omega})$ as $\varepsilon \to 0$.

iii. There exists $c > 0$ such that $v_\varepsilon(x) \geq c d(x,\partial\Omega)$ for any $x \in \Omega$, $\varepsilon \in (0,\varepsilon_0)$.

Proof. Let $F : (-1,1) \times H^1_0(\Omega) \to H^1_0(\Omega)$ be defined by

$$F(\varepsilon,u) = F_\varepsilon(u) := u - (-\Delta)^{-1}(\lambda f_\varepsilon(u)), \tag{24}$$

where f_ε is defined as in (5). F is well defined because the Moser-Trudinger inequality (2) implies that $f_\varepsilon(u) \in L^p(\Omega)$ for any $1 \leq p < +\infty$ and $u \in H^1_0(\Omega)$. Moreover, it is a C^1-map and its partial derivative $DF_\varepsilon(u) : H^1_0(\Omega) \to H^1_0(\Omega)$ defined by

$$DF_\varepsilon(u)[\varphi] = \varphi - (-\Delta)^{-1}(\lambda f'_\varepsilon(u)\varphi)$$

is a Fredholm operator of index 0 (since the embedding $H^1_0(\Omega) \hookrightarrow L^p(\Omega)$ is compact).

Now, let u_0 be a non-degenerate weak solution of (6) such that (A1) holds true. In particular, $F_0(u_0) = 0$ and $DF_0(u_0)$ is invertible. Therefore, by the implicit function theorem, we can construct a C^1 curve $\varepsilon \mapsto v_\varepsilon \in H^1_0(\Omega)$, defined for $|\varepsilon| < \varepsilon_0$ such that $v_0 = u_0$, $F_\varepsilon(v_\varepsilon) = 0$, and $DF_\varepsilon(v_\varepsilon)$ is invertible for $|\varepsilon| < \varepsilon_0$. Then *i.* holds.

Applying the Moser-Trudinger inequality (2) and standard elliptic estimates, we obtain *ii.*.

Hopf's lemma and the compactness of $\partial\Omega$ give $\frac{\partial u_0}{\partial \nu} \leq -2c$ on $\partial\Omega$, for some $c > 0$. Then, for ε sufficiently small, we have $\frac{\partial v_\varepsilon}{\partial \nu} \leq -c$, which in turn gives $v_\varepsilon(x) \geq cd(x, \partial\Omega)$ for x in a neighborhood of $\partial\Omega$. Finally, since $v_\varepsilon \to u_0$ uniformly in $\overline{\Omega}$, and $u_0 > 0$ in Ω, we get *iii.*. □

2.2 The correction of the ansatz

We need to correct the ansatz in the whole domain by solving the following two linear problems (12) and (13):

$$\begin{cases} \Delta w_{\varepsilon,\xi} + \lambda f'_\varepsilon(v_\varepsilon) w_{\varepsilon,\xi} = 8\pi\lambda G_\xi f'_\varepsilon(v_\varepsilon) & \text{in } \Omega, \\ w_{\varepsilon,\xi} = 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} \Delta z_{\varepsilon,\xi} + \lambda f'_\varepsilon(v_\varepsilon) z_{\varepsilon,\xi} = \frac{\lambda}{2} f''_\varepsilon(-v_\varepsilon)(8\pi G_\xi - w_\varepsilon)^2 & \text{in } \Omega, \\ z_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

Lemma 2.3 *For any* $0 < \varepsilon < \varepsilon_0$ *and any* $\xi \in \Omega$*, there exist* $w_{\varepsilon,\xi}$*,* $z_{\varepsilon,\xi}$ *such that* (12) *and* (13) *hold. Moreover, there exists* $C > 0$ *such that*

$$\|w_{\varepsilon,\xi}\|_{C^1(\overline{\Omega})} + \|z_{\varepsilon,\xi}\|_{C^1(\overline{\Omega})} \leq C \tag{25}$$

for $\varepsilon \in (0, \varepsilon_0)$*,* $\xi \in \Omega$*.*

Proof. The existence of the solutions immediately follows from the non-degeneracy of the function v_ε proved in Lemma 2.2. Moreover, since for any $p \in [1, +\infty)$ one has

$$\sup_{\xi \in \Omega} \|G_\xi\|_{L^p(\Omega)} < +\infty \quad \text{and} \quad \sup_{0<\varepsilon<\varepsilon_0} \|v_\varepsilon\|_{C^1(\overline{\Omega})} < +\infty,$$

(25) follows by standard elliptic estimates. □

Finally, we introduce the corrected ansatz as

$$\omega_{\varepsilon,\mu,\xi} := \alpha P U_{\delta,\mu,\xi} - V_{\varepsilon,\alpha,\xi} \tag{26}$$

with

$$V_{\varepsilon,\alpha,\xi} := v_\varepsilon + \alpha w_{\varepsilon,\xi} + \alpha^2 z_{\varepsilon,\xi}, \tag{27}$$

where v_ε is defined in Lemma 2.2 and $w_{\varepsilon,\xi}$ and $z_{\varepsilon,\xi}$ as in Lemma 2.3.

2.3 The choice of parameters

It will be necessary to choose the parameters $\alpha = \alpha(\varepsilon, \mu, \xi)$ and $\delta = \delta(\varepsilon, \mu, \xi)$ such that $\lambda f_\varepsilon(\omega_{\varepsilon,\mu,\xi}) \sim \alpha e^{U_{\delta,\mu,\xi}}$ when $|x - \xi| \sim \delta$. We point out that one of the main difficulties in this problem is that this estimates holds true only at a very small scale.

Let us fix the values of α and δ according to the next lemma. The proof is based on the contraction mapping theorem and is postponed to the appendix.

Lemma 2.4 *There exist* $\varepsilon_0 > 0$ *and functions* $\alpha = \alpha(\varepsilon, \mu, \xi)$*,* $\beta = \beta(\varepsilon, \mu, \xi)$ *and* $\delta = \delta(\varepsilon, \mu, \xi)$*, defined in* $(0, \varepsilon_0) \times \mathcal{U} \times B(\xi_0, \sigma)$ *and continuous with respect to* μ *and* ξ*, such that*

$$\begin{cases} \lambda\beta e^{\beta^2+\beta^{1+\varepsilon}} = \frac{\alpha}{\delta^2}, \\ 2\alpha\beta + \alpha\beta^\varepsilon + \varepsilon\alpha\beta^\varepsilon = 1, \\ \beta = 4\alpha \log\frac{1}{\delta} - V_{\varepsilon,\alpha,\xi}(\xi) + \alpha c_{\mu,\xi}, \end{cases} \tag{28}$$

where $c_{\mu,\xi} := -\log(8\mu^2) + 8\pi H(\xi, \xi)$ *and* $V_{\varepsilon,\alpha,\xi}$ *is defined in* (11)*.*

Moreover, as $\varepsilon \to 0$*, we have that*

$$\alpha(\varepsilon, \mu, \xi) = \frac{1}{2} e^{-\frac{\log(2u_0(\xi))+o(1)}{\varepsilon}}, \tag{29}$$

$$\beta(\varepsilon, \mu, \xi) = \frac{1}{2\alpha} - u_0(\xi) + o(1), \tag{30}$$

$$\log \frac{1}{\delta(\varepsilon, \mu, \xi)} = \frac{1 + o(1)}{8\alpha^2}, \tag{31}$$

where $o(1) \to 0$ *as* $\varepsilon \to 0$*, uniformly for* $\mu \in \mathcal{U}$ *and* $\xi \in B(\xi_0, \sigma)$*.*

Remark 2.5 *Note that* (29)*-*(31) *and* (22) *give* $\alpha(\varepsilon, \mu, \xi), \delta(\varepsilon, \mu, \xi) \to 0$ *and* $\beta(\varepsilon, \mu, \xi) \to +\infty$ *as* $\varepsilon \to 0$*, uniformly for* $\mu \in \mathcal{U}$ *and* $\xi \in B(\xi_0, \sigma)$*.*

From now on we let $\alpha = \alpha(\varepsilon, \mu, \xi)$, $\beta = \beta(\varepsilon, \mu, \xi)$ and $\delta = \delta(\varepsilon, \mu, \xi)$ be as in Lemma 2.4.

It will be convenient to work on the scaled domain $\frac{\Omega-\xi}{\delta} := \left\{ \frac{x-\xi}{\delta}, \quad x \in \Omega \right\}$. Note that we have the scaling relation

$$U_{\delta,\mu,\xi}(x) = \bar{U}_\mu \left(\frac{x - \xi}{\delta} \right) - 2\log\delta, \tag{32}$$

where

$$\bar{U}_\mu(y) = U_{1,\mu,0}(y) = \log \left(\frac{8\mu^2}{(\mu^2 + |y|^2)^2} \right). \tag{33}$$

Lemma 2.6 *As* $\varepsilon \to 0$*, we have*

$$\omega_{\varepsilon,\mu,\xi}(\xi + \delta y) = \beta + \alpha\bar{U}_\mu(y) + O(\delta|y|) + O(\delta^2), \tag{34}$$

uniformly for $y \in B(0, \frac{\sigma}{\delta})$*,* $\mu \in \mathcal{U}$ *and* $\xi \in B(\xi_0, \sigma)$*.*
Moreover, for any $R > 0$ *it holds also true that*

$$\lambda f_\varepsilon(\omega_{\varepsilon,\mu,\xi})(\xi + \delta y) = \alpha e^{U_{\delta,\mu,\xi}(\xi+\delta y)}(1 + O(\alpha^2)), \tag{35}$$

as $\varepsilon \to 0$ *uniformly for* $y \in B(0, R)$*,* $\mu \in \mathcal{U}$ *and* $\xi \in B(\xi_0, \sigma)$*.*

Proof. Lemma 2.1 and the scaling relation (32) show that, as $\delta \to 0$, we have the following expansion uniformly for $\varepsilon \in (0, \varepsilon_0)$, $\mu \in \mathcal{U}$, $\xi \in B(\xi_0, \sigma)$ and $y \in B(0, \frac{\sigma}{\delta})$:

$$\omega_{\varepsilon,\mu,\xi}(\xi + \delta y) = \alpha \bar{U}_\mu + \underbrace{4\alpha \log \frac{1}{\delta} + \alpha c_{\mu,\xi} - V_{\varepsilon,\alpha,\mu}(\xi)}_{=\beta} + V_{\varepsilon,\alpha,\mu}(\xi) - V_{\varepsilon,\alpha,\xi}(\xi + \delta y)$$
$$+ 8\pi\alpha(H(\xi + \delta y, \xi) - H(\xi, \xi)) + O(\delta^2).$$

By Lemmas 2.2 and 2.3, we know that $V_{\varepsilon,\alpha,\mu}$ is uniformly bounded in $C^1(\overline{\Omega})$. Thus

$$V_{\varepsilon,\alpha,\mu}(\xi + \delta y) = V_{\varepsilon,\alpha,\mu}(\xi) + O(\delta|y|).$$

Similarly, since $H \in C^1(\overline{\Omega} \times B(\xi_0, \sigma))$, we have

$$H(\xi + \delta y, \xi) = H(\xi, \xi) + O(\delta|y|).$$

Then estimate (34) is proved.

Now, let us prove (35). Note that (29)-(31) yield $\beta = O(\frac{1}{\alpha})$, $\delta = O(e^{-\frac{1+o(1)}{8\alpha^2}})$, and $\beta^\varepsilon = 2u_0(\xi) + o(1) = O(1)$. For $|y| \le R$, (34) implies

$$\omega_{\varepsilon,\mu,\xi}(\xi + \delta y) = \beta + \alpha \bar{U}_\mu(y) + O(\delta).$$

In particular

$$\omega_{\varepsilon,\mu,\xi}(\xi + \delta y)^2 = \beta^2 + 2\alpha\beta \bar{U}_\mu(y) + O(\beta\delta), \tag{36}$$

and

$$\begin{aligned}
\omega_{\varepsilon,\mu,\xi}(\xi + \delta y)^{1+\varepsilon} &= (\beta + \alpha \bar{U}_\mu(y) + O(\delta))(\beta + \alpha \bar{U}_\mu(y) + O(\delta))^\varepsilon \\
&= (\beta + \alpha \bar{U}_\mu(y) + O(\delta))\beta^\varepsilon \left(1 + \frac{\alpha}{\beta}\bar{U}_\mu(y) + O(\alpha\delta)\right)^\varepsilon \\
&= \left(\beta^{1+\varepsilon} + \alpha\beta^\varepsilon \bar{U}_\mu(y) + O(\delta)\right)\left(1 + \frac{\varepsilon\alpha}{\beta}\bar{U}_\mu(y) + O(\varepsilon\alpha^4)\right) \\
&= \beta^{1+\varepsilon} + \alpha\beta^\varepsilon \bar{U}_\mu(y) + \varepsilon\alpha\beta^\varepsilon \bar{U}_\mu(y) + O(\varepsilon\alpha^3).
\end{aligned} \tag{37}$$

Then, using (28) we get

$$\begin{aligned}
\lambda f_\varepsilon(\omega_{\varepsilon,\mu,\xi})(\xi + \delta y) &= \lambda \omega_{\varepsilon,\mu,\xi}(\xi + \delta y) e^{\omega_{\varepsilon,\mu,\xi}(\xi+\delta y)^2 + \omega^{1+\varepsilon}_{\varepsilon,\mu,\xi}(\xi+\delta y)} \\
&= \lambda\beta(1 + O(\alpha^2)) e^{\beta^2 + \beta^{1+\varepsilon} + (2\alpha\beta + \alpha\beta^\varepsilon + \alpha\varepsilon\beta^\varepsilon)\bar{U}_\mu(y) + O(\alpha^2)} \\
&= \underbrace{\lambda\beta e^{\beta^2 + \beta^{1+\varepsilon}}}_{=\frac{\alpha}{\delta^2}} e^{\overbrace{(2\alpha\beta + \alpha\beta^\varepsilon + \alpha\varepsilon\beta^\varepsilon)}^{=1}\bar{U}_\mu(y)} (1 + O(\alpha^2)) e^{O(\alpha^2)} \\
&= \frac{\alpha}{\delta^2} e^{\bar{U}_\mu(y)} (1 + O(\alpha^2)) \\
&= \alpha e^{U_{\delta,\mu,\xi}(\xi+\delta y)} (1 + O(\alpha^2)),
\end{aligned}$$

which proves (35). □

It is also useful to point out the following result which will be used in the next sections.

Remark 2.7 *Lemma 2.1 and Lemma 2.4 give*

$$0 \leq \alpha PU_{\delta,\mu,\xi} \leq \beta + u_0(\xi) + o(1),$$

and

$$-V_{\alpha,\varepsilon,\xi} \leq \omega_{\varepsilon,\mu,\xi} \leq \beta + o(1),$$

uniformly for $x \in \Omega$, $\varepsilon \in (0, \varepsilon_0)$, $\mu \in \mathcal{U}$, $\xi \in B(\xi_0, \sigma)$.

Notation: In order to simplify the notation, we will write U_ε, $\bar{U}$, V_ε, ω_ε, w_ε and z_ε instead of $U_{\delta,\mu,\xi}$, $\bar{U}_\mu$, $V_{\varepsilon,\alpha,\xi}$, $\omega_{\varepsilon,\mu,\xi}$, $w_{\varepsilon,\xi}$ and $z_{\varepsilon,\xi}$, without specifying explicitly the dependence on the parameters. It is important to point out that all the estimates of the next sections will be uniform with respect to $\mu \in \mathcal{U}$ and $\xi \in B(\xi_0, \sigma)$. This will allow us to choose freely the values of μ and ξ in Section 6. Consistently, the notation $O(f(x, \varepsilon, \alpha, \beta, \delta))$ and $o(f(x, \varepsilon, \alpha, \beta, \delta))$ will be used for quantities depending on ε, ξ, μ (and the parameters α, β, δ of Lemma 2.4) and satisfying respectively

$$|O(f(x, \varepsilon, \mu, \xi\alpha, \beta, \delta))| \leq Cf(x, \varepsilon, \mu, \xi, \alpha, \beta, \delta)) \quad \text{and} \quad \frac{o(f(x, \varepsilon, \mu, \xi\alpha, \beta, \delta))}{f(x, \varepsilon, \mu, \xi, \alpha, \beta, \delta)} \to 0,$$

as $\varepsilon \to 0$, uniformly for $\mu \in \mathcal{U}$ and $\xi \in B(\xi_0, \sigma)$.

3 The estimate of the error term

In this section we give estimates for the error term R defined in (17)

$$R = R_{\varepsilon,\mu,\xi} := \Delta\omega_{\varepsilon,\mu,\xi} + \lambda f_\varepsilon(\omega_{\varepsilon,\mu,\xi}).$$

It will be convenient to split Ω into four different regions:

$$\Omega = B(\xi, \rho_0) \cup \Big(B(\xi, \rho_1) \setminus B(\xi, \rho_0)\Big) \cup \Big(B(\xi, \rho_2) \setminus B(\xi, \rho_1)\Big) \cup \Big(\Omega \setminus B(\xi, \rho_2)\Big), \tag{38}$$

where $\rho_0 = \rho_0(\varepsilon, \mu, \xi)$, $\rho_1 = \rho_1(\varepsilon, \mu, \xi)$, $\rho_2 = \rho_2(\varepsilon, \mu, \xi)$, are defined by

$$\rho_0 = \delta e^{\frac{\varepsilon}{\alpha}}, \quad \rho_1 = e^{-\frac{u_0(\xi)}{2\alpha}} \quad \text{and} \quad \rho_2 = e^{-\frac{\varepsilon}{\alpha}}. \tag{39}$$

Note that

$$\delta \ll \rho_0 \ll \rho_1 \ll \rho_2 \ll 1, \qquad \text{as } \varepsilon \to 0,$$

by (29) and (31). Roughly speaking, we have to split the error into four parts: in $B(\xi, \rho_0)$ we have $\lambda f_\varepsilon(\omega_\varepsilon) = \alpha e^{U_\varepsilon}(1 + o(1))$ (see (35)) and we can use a blow-up argument to get a uniform weighted estimate on R. This estimate does not hold anymore in the set $\Omega \setminus B(\xi, \rho_0)$, which we further split into three parts: the region $\Omega \setminus B(\xi, \rho_2)$, where $\alpha G_\xi = O(\varepsilon)$ and a uniform estimate on R can be obtained via a Taylor expansion of $f_\varepsilon(\omega_\varepsilon)$ (using that $\omega_\varepsilon = -V_\varepsilon + 8\pi\alpha G_\xi + o(\alpha^2)$), and the two annuli $B(\xi, \rho_1) \setminus B(\xi, \rho_0)$ and $B(\xi, \rho_2) \setminus B(\xi, \rho_1)$, where we give quite delicate integral estimates. The last two regions are treated separately since $\omega_\varepsilon \geq c_0 > 0$ in $B(\xi, \rho_1) \setminus B(\xi, \rho_0)$, while ω_ε changes sign in $B(\xi, \rho_2) \setminus B(\xi, \rho_1)$ (cfr. Lemma 3.2 and Lemma 3.11).

3.1 A uniform expansion in $B(\xi, \rho_1)$

In this section we give a more precise version of the expasions in (36)-(37).

Lemma 3.1 *For any $\varepsilon \in (0,1)$ and $x \geq -1$, we have*

$$|(1+x)^{1+\varepsilon} - 1 - (1+\varepsilon)x| \leq \varepsilon x^2.$$

Proof. According to Bernoulli's inequality we have

$$(1+x)^{\varepsilon} \leq 1 + \varepsilon x \tag{40}$$

and

$$(1+x)^{1+\varepsilon} \geq 1 + (1+\varepsilon)x. \tag{41}$$

Since $x \geq -1$, thanks to (40) we have that

$$(1+x)^{1+\varepsilon} \leq (1+x)(1+\varepsilon x) = 1 + (1+\varepsilon)x + \varepsilon x^2. \tag{42}$$

Then, the conclusion follows from (41) and (42). □

Lemma 3.2 *Set $c_0 := \frac{1}{2} \inf_{\xi \in B(\xi_0, \sigma)} u_0(\xi)$. For $x \in B(\xi, \rho_1)$, we have that*

$$\beta + \alpha \bar{U}\Big(\frac{x-\xi}{\delta}\Big) \geq c_0, \tag{43}$$

for sufficiently small ε. In particular, we have

$$c_0 \leq \omega_\varepsilon \leq \beta(1 + o(1)). \tag{44}$$

Proof. The definitons of $\overline{U}$ and ρ_1 (see (33) and (39)), and (30)-(31) give

$$\begin{aligned} \beta + \alpha \bar{U}\Big(\frac{x-\xi}{\delta}\Big) &\geq \beta + \alpha \bar{U}\Big(\frac{\rho_1}{\delta}\Big) \\ &= \beta - 4\alpha \log \frac{\rho_1}{\delta} + o(1) \\ &= u_0(\xi) + o(1), \end{aligned}$$

which implies (43) for sufficiently small ε. To get (44), it is sufficient to apply Lemma 2.6 and Remark 2.7. □

Lemma 3.3 *For $x \in B(\xi, \rho_1)$, we have*

$$\omega_\varepsilon^2(x) + \omega_\varepsilon^{1+\varepsilon}(x) = \beta^2 + \beta^{1+\varepsilon} + \bar{U}\left(\frac{x-\xi}{\delta}\right) + \alpha^2 \bar{U}^2\left(\frac{x-\xi}{\delta}\right) + O\left(\varepsilon\alpha^3\left(1 + \bar{U}^2\Big(\frac{x-\xi}{\delta}\Big)\right)\right).$$

Proof. Set $y = \frac{x-\xi}{\delta} \in B(0, \frac{\rho_1}{\delta})$. Noting that $\overline{U}(y) = O(\alpha^{-2})$ and using Lemma 2.6, we get

$$\omega_\varepsilon^2(x) = \omega_\varepsilon^2(\xi + \delta y) = \big(\beta + \alpha\bar{U}(y) + O(\rho_1)\big)^2$$
$$= \beta^2 + 2\alpha\beta\bar{U}(y) + \alpha^2\bar{U}(y)^2 + O(\beta\rho_1).$$

Similarly, since Lemma 3.2 gives $\frac{\alpha}{\beta}\bar{U}(y) \geq -1 + \frac{c_0}{\beta} \geq -1$, by Lemma 3.1 we infer

$$\begin{aligned}|\omega_\varepsilon|^{1+\varepsilon}(x) &= \beta^{1+\varepsilon}\left(1 + \frac{\alpha}{\beta}\bar{U}(y) + O(\alpha\rho_1)\right)^{1+\varepsilon} \\ &= \beta^{1+\varepsilon}\left(1 + (1+\varepsilon)\Big(\frac{\alpha}{\beta}\bar{U}(y) + O(\alpha\rho_1)\Big) + O\left(\varepsilon\left(\frac{\alpha}{\beta}\bar{U}(y) + O(\alpha\rho_1)\right)^2\right)\right) \\ &= \beta^{1+\varepsilon} + (1+\varepsilon)\alpha\beta^\varepsilon\bar{U}(y) + O(\varepsilon\alpha^3(1 + \bar{U}^2(y))).\end{aligned}$$

Then the conclusion follows from the second equation in (28). □

3.2 Expansions in $B(\xi, \rho_0)$

Let us now restrict our attention to the smaller ball $B(\xi, \rho_0)$. This allows to control the term $\alpha^2\bar{U}^2$ appearing in the expansion of Lemma 3.3. Indeed, since $|\bar{U}(y)| = -4\log|y| + O(1)$ as $|y| \to +\infty$, we have that

$$\bar{U}\left(\frac{x-\xi}{\delta}\right) = O\left(\frac{\varepsilon}{\alpha}\right) \quad \text{and} \quad \alpha^2\bar{U}^2\left(\frac{x-\xi}{\delta}\right) = O(\varepsilon^2) \quad \text{for } x \in B(\xi, \rho_0). \tag{45}$$

Lemma 3.4 *For $x \in B(\xi, \rho_0)$, we have*

$$R(x) = \alpha^3 e^{U_\varepsilon(x)}\left(2\bar{U}\Big(\frac{x-\xi}{\delta}\Big) + \bar{U}^2\Big(\frac{x-\xi}{\delta}\Big)\right) + \alpha^4 e^{U_\varepsilon(x)} O\left(1 + \bar{U}^4\Big(\frac{x-\xi}{\delta}\Big)\right).$$

Proof. Set $y = \frac{x-\xi}{\delta}$. First by Lemma 2.6, Lemma 3.3, and (28)-(32), we get that

$$\begin{aligned}\lambda f_\varepsilon(\omega_\varepsilon(x)) &= \lambda\beta\left(1 + \frac{\alpha}{\beta}\bar{U}(y) + O(\alpha\rho_1)\right)e^{\omega_\varepsilon^2(x) + \omega_\varepsilon^{1+\varepsilon}(x)} \\ &= \frac{\alpha}{\delta^2}\left(1 + 2\alpha^2\bar{U}(y) + O(\alpha^3(1 + |\bar{U}(y)|))\right)e^{\bar{U}(y) + \alpha^2\bar{U}^2(y) + O(\varepsilon\alpha^3(1+\bar{U}^2(y)))} \\ &= \alpha e^{U_\varepsilon(x)}\left(1 + 2\alpha^2\bar{U}(y) + O(\alpha^3(1 + |\bar{U}(y)|))\right)e^{\alpha^2\bar{U}^2(y) + O(\varepsilon\alpha^3(1+\bar{U}^2(y)))}.\end{aligned}$$

Now, by (45), we can expand the last exponential term, and find

$$\begin{aligned}e^{\alpha^2\bar{U}^2(y) + O(\varepsilon\alpha^3(1+\bar{U}^2(y)))} &= 1 + \alpha^2\bar{U}^2(y) + O(\varepsilon\alpha^3(1 + \bar{U}^2(y))) + O(\alpha^4(1 + \bar{U}^4(y))) \\ &= 1 + \alpha^2\bar{U}^2(y) + O(\varepsilon\alpha^3(1 + \bar{U}^4(y))).\end{aligned}$$

We can so conclude that

$$\lambda f_\varepsilon(\omega_\varepsilon(x)) = \alpha e^{U_\varepsilon(x)} + \alpha^3 e^{U_\varepsilon(x)}\left(2\bar{U}(y) + \bar{U}(y)^2\right) + \alpha^4 e^{U_\varepsilon(x)} O(1 + \bar{U}^4(y)). \tag{46}$$

Moreover, by (10)-(13), and Lemmas 2.2-2.3 we have

$$\Delta\omega_\varepsilon = -\alpha e^{U_\varepsilon} + O(1) = -\alpha e^{U_\varepsilon}\left(1 + O(\alpha)e^{-U_\varepsilon}\right) = -\alpha e^{U_\varepsilon}(1 + o(\alpha^3)), \tag{47}$$

where in the last equality we used that

$$e^{-U_\varepsilon(x)} = \frac{(\delta^2\mu^2 + |x-\xi|^2)^2}{8\delta^2\mu^2} = O(\delta^2 e^{\frac{4\varepsilon}{\alpha}}) = o(\alpha^3),$$

for $x \in B(\xi, \rho_0)$. Thanks to (46) and (47), we conclude that

$$R(x) = \alpha^3 e^{U_\varepsilon(x)}\left(2\bar U(y) + \bar U^2(y)\right) + \alpha^4 e^{U_\varepsilon(x)} O(1 + \bar U^4(y)).$$

□

As an immediate consequence of the previous lemma we obtain the estimate:

Corollary 3.5 *We have that*

$$R = O\left(\alpha^3 e^{U_\varepsilon}\left((1 + \bar U^4\Big(\frac{\cdot - \xi}{\delta}\Big)\right)\right)$$

in $B(\xi, \rho_0)$.

3.3 Estimates on $B(\xi, \rho_1) \setminus B(\xi, \rho_0)$

In this region, it is diffcult to provide pointwise estimates of R because the term $\alpha^2\bar U^2$ appearing in the expansion of Lemma 3.3 becomes very large. Then, we will look for integral estimates. Specifically we will show that R is (very) small in $L^p(B(\xi, \rho_1) \setminus B(\xi, \rho_0))$, for a suitable choice of $p = p(\alpha) > 1$, such that $p \to 1$ as $\varepsilon \to 0$, uniformly with respect to $\xi \in B(\xi_0, \sigma)$, $\mu \in \mathcal{U}$.

Lemma 3.6 *There exists* $c_1 > 0$ *such that*

$$0 \le \lambda f_\varepsilon(\omega_\varepsilon) \le \alpha e^{U_\varepsilon + \alpha^2(1 + c_1\varepsilon\alpha)\bar U^2(\frac{\cdot - \xi}{\delta})},$$

in $B(\xi, \rho_1) \setminus B(\xi, \rho_0)$.

Proof. Since $0 \le \omega_\varepsilon \le \beta$ in $B(\xi, \rho_1) \setminus B(\xi, \rho_0)$, from Lemma 3.3 and (28) we get

$$\begin{aligned}\lambda f_\varepsilon(\omega_\varepsilon) &\le \lambda\beta e^{\beta^2 + \beta^{1+\varepsilon} + \bar U(\frac{\cdot - \xi}{\delta}) + \alpha^2\bar U^2(\frac{\cdot - \xi}{\delta})(1 + O(\varepsilon\alpha))} \\ &= \frac{\alpha}{\delta^2} e^{\bar U(\frac{\cdot - \xi}{\delta}) + \alpha^2\bar U^2(\frac{\cdot - \xi}{\delta})(1 + O(\varepsilon\alpha))} \\ &= \alpha e^{U_\varepsilon + \alpha^2\bar U^2(\frac{\cdot - \xi}{\delta})(1 + O(\varepsilon\alpha))}.\end{aligned}$$

□

For c_1 as in Lemma 3.6, let us consider the function

$$\Gamma_\varepsilon(x) := e^{\bar U_\varepsilon(x) + \alpha^2\bar U(\frac{x - \xi}{\delta})^2(1 + c_1\varepsilon\alpha)}. \tag{48}$$

Lemma 3.7 *Set $p := 1+\alpha^2$. There exists $c_2>0$ such that*

$$\|\Gamma_\varepsilon\|_{L^p(B(\xi,\rho_1)\setminus B(\xi,\rho_0))} = O\left(\alpha^{-1}e^{-\frac{c_2}{\sqrt{\alpha}}}\right).$$

Proof. First of all, we observe that for $q\in(\frac{1}{2},+\infty)$, $R>0$, one has

$$\int_{\mathbb{R}^2\setminus B(0,R)} e^{q\bar{U}}dy \le \int_{\mathbb{R}^2\setminus B(0,R)} \frac{(8\mu^2)^q}{|y|^{4q}}dy = \frac{\pi(8\mu^2)^q}{(2q-1)R^{4q-2}}. \tag{49}$$

For $x\in B(\xi,\rho_1)\setminus B(\xi,\rho_0)$, set $y=\frac{x-\xi}{\delta}\in B(0,\frac{\rho_1}{\delta})\setminus B(0,\frac{\rho_0}{\delta})$. Clearly we have

$$\|\Gamma_\varepsilon\|_{L^p(B(\xi,\rho_1)\setminus B(\xi,\rho_0))} = \delta^{\frac{2-2p}{p}}\left(\int_{B(0,\frac{\rho_1}{\delta})\setminus B(0,\frac{\rho_0}{\delta})} e^{p\bar{U}(y)(1+\alpha^2\bar{U}(y)(1+c_1\varepsilon\alpha))}dy\right)^{\frac{1}{p}}. \tag{50}$$

Set $\bar\rho=\delta e^{\frac{1}{\alpha^{\frac{3}{2}}}}$, so that $\rho_0\ll\bar\rho\ll\rho_1$. For $\frac{\rho_0}{\delta}\le|y|\le\frac{\bar\rho}{\delta}$, we have

$$p\left(1+\alpha^2\bar{U}(y)(1+\varepsilon c_1\alpha)\right) = 1+O(\sqrt{\alpha}) \ge \frac{2}{3}.$$

Then, for ε small enough, (49) yields

$$\begin{aligned}\int_{B(0,\frac{\bar\rho}{\delta})\setminus B(0,\frac{\rho_0}{\delta})} e^{p\bar{U}(y)\left(1+\alpha^2\bar{U}(y)(1+c_1\varepsilon\alpha)\right)}dy &\le \int_{\mathbb{R}^2\setminus B(0,\frac{\rho_0}{\delta})} e^{\frac{2}{3}\bar{U}(y)}dy\\ &= O\left(\left(\frac{\rho_{0,\varepsilon}}{\delta}\right)^{-\frac{2}{3}}\right) = O(e^{-\frac{2\varepsilon}{3\alpha}}).\end{aligned} \tag{51}$$

For $\frac{\bar\rho}{\delta}\le|y|\le\frac{\rho_1}{\delta}$, by (30) and Lemma 3.2, we have

$$\begin{aligned}1+\alpha^2\bar{U}(y)\left(1+c_1\varepsilon\alpha\right) &= 1+\alpha(\beta+\alpha\bar{U}(y))\left(1+c_1\varepsilon\alpha\right)-\alpha\beta\left(1+c_1\varepsilon\alpha\right)\\ &\ge \frac{1}{2}+(c_0+u_0(\xi))\alpha+o(\alpha)\\ &\ge \frac{1}{2}+c_0\alpha.\end{aligned}$$

Hence, we get

$$\begin{aligned}\int_{B(0,\frac{\rho_1}{\delta})\setminus B(0,e^{\alpha^{-\frac{3}{2}}})} e^{p\bar{U}(y)\left(1+\alpha^2\bar{U}(y)(1+c_1\varepsilon\alpha)\right)}dy &\le \int_{\mathbb{R}^2\setminus B(0,e^{\alpha^{-\frac{3}{2}}})} e^{p(\frac{1}{2}+c_0\alpha)\bar{U}(y)}dy\\ &= O\Big(\alpha^{-1}e^{-\frac{4c_0}{\sqrt{\alpha}}}\Big).\end{aligned} \tag{52}$$

Thus, by (50),(51),(52), we obtain

$$\|\Gamma_\varepsilon\|_{L^p(B(\xi,\rho_1)\setminus B(\xi,\rho_0))} = O\left(\delta^{\frac{2-2p}{p}}\alpha^{-\frac{1}{p}}e^{-\frac{4c_0}{p\sqrt{\alpha}}}\right).$$

Since (29)-(31) give

$$\delta^{\frac{2-2p}{p}} = \delta^{-\frac{2\alpha^2}{1+\alpha^2}} = O(1), \qquad \alpha^{\frac{1}{p}} = \alpha\alpha^{\frac{1-p}{p}} = \alpha(1+o(1)), \qquad e^{-\frac{4c_0}{p\sqrt{\alpha}}} = O(e^{-\frac{4c_0}{\sqrt{\alpha}}}),$$

we get the conclusion. □

Lemma 3.8 *Let p and c_2 be as in Lemma 3.7, then*

$$\|R\|_{L^p(B(\xi,\rho_1)\setminus B(\xi,\rho_0))} = O(e^{-\frac{c_2}{\sqrt{\alpha}}}).$$

Proof. By Lemma 3.6 and Lemma 3.7 we get that

$$\|\lambda f_\varepsilon(\omega_\varepsilon)\|_{L^p(B(\xi,\rho_1)\setminus B(\xi,\rho_0))} = O(e^{-\frac{c_2}{\sqrt{\alpha}}}).$$

On the other hand, we have

$$\Delta\omega_\varepsilon(x) = -\alpha e^{U_\varepsilon(y)} + O(1),$$

so that

$$\begin{aligned}
\|\Delta\omega_\varepsilon\|_{L^p(B(\xi,\rho_1)\setminus B(\xi,\rho_0))} &\le \alpha\|e^{U_\varepsilon}\|_{L^p(B(\xi,\rho_1)\setminus B(\xi,\rho_0))} + O(\rho_1^{\frac{2}{p}}) \\
&\le \alpha\delta^{\frac{2-2p}{p}}\|e^{\bar{U}}\|_{L^p(\mathbb{R}^2\setminus B(0,\frac{\rho_0}{\delta}))} + O(\rho_1^{\frac{2}{p}}) \\
&= O\left(\frac{\alpha\delta^2}{\rho_0^2}\right) + O\left(\rho_1^2\right) \\
&= o(e^{-\frac{c_2}{\sqrt{\alpha}}}).
\end{aligned}$$

□

3.4 Estimates in $B(\xi,\rho_2)\setminus B(\xi,\rho_1)$

In $B(\xi,\rho_2)\setminus B(\xi,\rho_1)$ we can only say that ω_ε and R are uniformly bounded. Since ρ_2 is very small, we still get integral bounds for R.

Lemma 3.9 *We have $\omega_\varepsilon = O(1)$ and $R = O(1)$ in $\Omega\setminus B(\xi,\rho_1)$. In particular,*

$$\|R\|_{L^2(B(\xi,\rho_2)\setminus B(\xi,\rho_1))} = O(\rho_2) = O(e^{-\frac{\varepsilon}{\alpha}}).$$

Proof. Let us recall that $\omega_\varepsilon = \alpha PU_\varepsilon - V_\varepsilon$ with $V_\varepsilon = V_{\varepsilon,\alpha,\xi}$ defined as in (11). According to Lemma 2.2 and Lemma 2.3, we have $V_\varepsilon = O(1)$ in Ω. Besides Lemma 2.1 gives

$$\alpha PU_\varepsilon = \alpha\log\left(\frac{1}{(\mu^2\delta^2+|x-\xi|^2)^2}\right) + O(\alpha) = O(\alpha\log\frac{1}{\rho_1}) + O(\alpha) = O(1),$$

for $x\in\Omega\setminus B(\xi,\rho_1)$. Then, $\omega_\varepsilon = O(1)$ and $f_\varepsilon(\omega_\varepsilon) = O(1)$ in $\Omega\setminus B(\xi,\rho_1)$. Similarly

$$\begin{aligned}
\Delta\omega_\varepsilon &= -\alpha e^{U_\varepsilon} + O(1) \\
&= -\frac{\alpha\delta^2\mu^2}{(\delta^2\mu^2+|x-\xi|^2)^2} + O(1) \\
&= O(\delta^2\rho_1^{-4}) + O(1) = O(1).
\end{aligned}$$

Therefore $R = O(1)$. □

3.5 Estimates in $\Omega \setminus B(\xi, \rho_2)$

In $\Omega \setminus B(\xi, \rho_2)$ we will use that $\omega_\varepsilon \sim 8\pi\alpha G_\xi - V_\varepsilon$. Our choice of V_ε will make R uniformly small, namely of order α^3. Note further that the choice of ρ_2 gives $\alpha G_\xi = O(\varepsilon)$ on $\Omega \setminus B(\xi, \rho_2)$.

Lemma 3.10 *As $\varepsilon \to 0$ we have*

$$\|PU_\varepsilon - 8\pi G_\xi\|_{C^1(\overline{\Omega}\setminus B(\xi,\rho_2))} = O(\delta^2 \rho_2^{-3}).$$

Proof. By Lemma 2.1 we have

$$\begin{aligned} PU_\varepsilon &= \log\left(\frac{1}{(\delta^2\mu^2 + |x-\xi|^2)^2}\right) + 8\pi H(x,\xi) + \psi_{\delta,\mu,\xi} \\ &= -4\log|x-\xi| + 8\pi H(x,\xi) - 2\log\left(1 + \frac{\delta^2\mu^2}{|x-\xi|^2}\right) + \psi_{\delta,\mu,\xi} \\ &= 8\pi G_\xi(x) - 2\log\left(1 + \frac{\delta^2\mu^2}{|x-\xi|^2}\right) + \psi_{\delta,\mu,\xi} \end{aligned}$$

Since $\|\psi_{\delta,\mu,\xi}\|_{C^1(\overline{\Omega})} = O(\delta^2)$ as $\varepsilon \to 0$, it is sufficent to observe that

$$\| \log\left(1 + \frac{\delta^2\mu^2}{|\cdot - \xi|^2}\right) \|_{C^1(\overline{\Omega}\setminus B(\xi,\rho_2))} = O(\delta^2\rho_2^{-3}).$$

□

Lemma 3.11 *There exists a constant $c > 0$ such such that*

$$\omega_\varepsilon(x) \leq -c\, d(x, \partial\Omega) < 0,$$

for any $x \in \Omega \setminus B(\xi, \rho_2)$, provided ε is sufficiently small.

Proof. By Lemma 2.2, Lemma 2.3 and (11) we have

$$V_\varepsilon(x) \geq c(1 + O(\alpha))d(x, \partial\Omega) \qquad \forall x \in \Omega,$$

for some $c > 0$. Then, Lemma 3.10 implies that

$$\omega_\varepsilon(x) \leq -c(1 + O(\alpha))d(x, \partial\Omega) \tag{53}$$

in a neighborhood of $\partial\Omega$. By definiton of ρ_2, we have that $PU_\varepsilon = G_\xi + o(1) = O(\frac{\varepsilon}{\alpha})$ in $\Omega \setminus B(\xi, \rho_2)$. Then, using again Lemma 2.2 and Lemma 2.3, we get $\omega_\varepsilon = -u_0 + o(1)$ uniformly in $\Omega \setminus B(\xi, \rho_2)$. Since $u_0 > 0$ in Ω, this toghether with (53) yields the conclusion. □

Lemma 3.12 *In $\Omega \setminus B(\xi, \rho_2)$, we have $R = O(\alpha^3(1 + G_\xi^3))$. In particular,*

$$\|R\|_{L^2(\Omega\setminus B(\xi,\rho_2))} = O(\alpha^3).$$

Proof. Since $v_\varepsilon > 0$ in Ω, $\omega_\varepsilon < 0$ in $\Omega \setminus B(\xi, \rho_2)$, and $f_\varepsilon \in C^3((-\infty, 0))$, for any $x \in \Omega \setminus B(\xi, \rho_2)$ we can find $\theta(x) \in [0, 1]$ such that

$$\begin{aligned}
f_\varepsilon(\omega_\varepsilon) &= f_\varepsilon(-v_\varepsilon + \alpha PU_\varepsilon - \alpha w_\varepsilon - \alpha^2 z_\varepsilon) \\
&= f_\varepsilon(-v_\varepsilon) + f'_\varepsilon(-v_\varepsilon)(\alpha PU_\varepsilon - \alpha w_\varepsilon - \alpha^2 z_\varepsilon) + \frac{1}{2} f''_\varepsilon(-v_\varepsilon)(\alpha PU_\varepsilon - \alpha w_\varepsilon - \alpha^2 z_\varepsilon)^2 \\
&\quad + \frac{1}{6} f'''(-v_\varepsilon + \theta(\alpha PU_\varepsilon - \alpha w_\varepsilon - \alpha^2 z_\varepsilon))(\alpha PU_\varepsilon - \alpha w_\varepsilon - \alpha^2 z_\varepsilon)^3
\end{aligned}$$

According to Lemma 2.3 and Lemma 3.10, we have

$$|z_\varepsilon| + |w_\varepsilon| = O(G_\xi) \quad \text{ and } \quad \alpha PU_\varepsilon = 8\pi\alpha G_\xi(1 + o(\alpha^3)).$$

Thus we get

$$\begin{aligned}
f_\varepsilon(\omega_\varepsilon) = &-f_\varepsilon(v_\varepsilon) + \alpha f'_\varepsilon(v_\varepsilon)(8\pi G_\xi - w_\varepsilon) + \alpha^2 \left(\frac{1}{2} f''(-v_\varepsilon)(8\pi G_\xi - w_\varepsilon)^2 - f'(v_\varepsilon) z_\varepsilon \right) \\
&+ O(\alpha^3(1 + G_\xi^3)) + O(\alpha^3 |f'''(-v_\varepsilon + \theta(\alpha PU_{\delta,\mu} - \alpha w_\varepsilon - \alpha^2 z_\varepsilon))| G_\xi^3).
\end{aligned}$$

A direct computation shows the existence of a constant $C > 0$ such that

$$|f'''_\varepsilon(t)| \le C(|t|^{\varepsilon - 1} + t^4) e^{t^2 + |t|^{1+\varepsilon}} \quad \forall t \neq 0.$$

Since $-v_\varepsilon + \theta(\alpha PU_\varepsilon - \alpha w_\varepsilon - \alpha^2 z_\varepsilon) = O(1)$ uniformly in $\Omega \setminus B(\xi, \rho_2)$, and since Lemma 3.10 implies $-v_\varepsilon + \theta(\alpha PU_\varepsilon + \alpha w_\varepsilon + \alpha^2 z_\varepsilon) \le -cd(\cdot, \partial\Omega)$ in a neighborhood of $\partial\Omega$, we get

$$|f'''(-v_\varepsilon + \theta(\alpha PU_{\delta,\mu} - \alpha w_\varepsilon - \alpha^2 z_\varepsilon))| = O(1 + d(\cdot, \partial\Omega)^{\varepsilon - 1}).$$

Since $G_\xi = O(d(\cdot, \partial\Omega))$ near $\partial\Omega$, we deduce that

$$\begin{aligned}
f_\varepsilon(\omega_\varepsilon) = &-f_\varepsilon(v_\varepsilon) + \alpha f'_\varepsilon(v_\varepsilon)(8\pi G_\xi - w_\varepsilon) + \alpha^2 \left(\frac{1}{2} f''(-v_\varepsilon)(8\pi G_\xi - w_\varepsilon)^2 - f'_\varepsilon(v_\varepsilon) z_\varepsilon \right) \\
&+ O(\alpha^3(1 + G_\xi^3)).
\end{aligned}$$

Since by construction we have $\Delta\omega_\varepsilon = -\alpha e^{U_\varepsilon} - \Delta v_\varepsilon - \alpha \Delta w_\varepsilon - \alpha^2 \Delta z_\varepsilon$, with v_ε, w_ε, z_ε solving (4) and (12)-(13), we conclude that

$$\begin{aligned}
R &= -\alpha e^{U_\varepsilon} + O(\alpha^3(1 + G_\xi^3)) \\
&= O(\delta^2 \rho_2^{-4}) + O(\alpha^3(1 + G_\xi^3)) \\
&= O(\alpha^3(1 + G_\xi^3)).
\end{aligned}$$

□

3.6 The final estimate of the error in a mixed norm

We can summarize the estimates of the previous sections as follows:

In $B(\xi,\rho_0)$, Corollary 3.5 gives $|R| \le \alpha^3 j_\varepsilon$, where

$$j_\varepsilon(x) := e^{U_\varepsilon(x)} \left(1 + |\bar{U}\Big(\frac{x-\xi}{\delta}\Big)|^4\right). \tag{54}$$

In $B(\xi,\rho_1) \setminus B(\xi,\rho_0)$, Lemma 3.8 shows that the norm of R in $L^{1+\alpha^2}$ is exponentially small in α.

Finally, in $\Omega \setminus B(\xi,\rho_1)$, Lemma 3.9 and Lemma 3.12 give L^2 estimates on R. This suggests to introduce the norm

$$\|f\|_\varepsilon := \|j_\varepsilon^{-1} f\|_{L^\infty(B(\xi,\rho_0))} + \frac{1}{\alpha^2}\|f\|_{L^{1+\alpha^2}(B(\xi,\rho_1)\setminus B(\xi,\rho_0))} + \|f\|_{L^2(\Omega\setminus B(\xi,\rho_1))}. \tag{55}$$

The coefficient $\frac{1}{\alpha^2}$ is chosen in order to match the norm of $(-\Delta)^{-1}$ as a linear operator from $L^{1+\alpha^2}(B(\xi,\rho_1) \setminus B(\xi,\rho_0))$ into $L^\infty(B(\xi,\rho_1) \setminus B(\xi,\rho_0))$ (see Corollary B.4).

According to the estimates above we have:

Proposition 3.13 *There exists $D_1 > 0$, $\varepsilon_0 > 0$ such that*

$$\|R\|_\varepsilon \le D_1 \alpha^3,$$

for any $\varepsilon \in (0,\varepsilon_0)$, $\mu \in \mathcal{U}$, $\xi \in B(\xi_0,\sigma)$.

We conclude this section by stating some simple properties of the norm $\|\cdot\|_\varepsilon$ and the weight j_ε.

Lemma 3.14 *There exists a constant $C > 0$ such that*

$$\|\cdot\|_{L^1(\Omega)} \le C\|\cdot\|_\varepsilon$$

for any $\varepsilon > 0$, $\mu \in \mathcal{U}$, $\xi \in B(\xi_0,\sigma)$.

Proof. Let $f : \Omega \to \mathbb{R}$ be a Lebesgue measurable function. Then

$$\|f\|_{L^1(B(\xi,\rho_0))} \le \|f\|_\varepsilon \int_{B(\xi,\rho_0)} j_\varepsilon dx = \|f\|_\varepsilon \int_{B(0,\frac{\rho_0}{\delta})} e^{\bar{U}}(1+\bar{U}^4)dy \le C\|f\|_\varepsilon.$$

By Hölder's inequality

$$\|f\|_{L^1(B(\xi,\rho_1)\setminus B(\xi,\rho_0))} \le \|f\|_{L^{1+\alpha^2}(B(\xi,\rho_1)\setminus B(\xi,\rho_0))} \rho_1^{\frac{2\alpha^2}{1+\alpha^2}} \le C\|f\|_\varepsilon,$$

and

$$\|f\|_{L^1(\Omega\setminus B(\xi,\rho_1))} \le \|f\|_{L^2(\Omega\setminus B(\xi,\rho_1))} |\Omega \setminus B(\xi,\rho_1)|^{\frac{1}{2}} \le C\|f\|_\varepsilon.$$

Hence, the conclusion follows. □

Lemma 3.15 *For any $\varepsilon > 0$ let ρ_ε, σ_ε be such that $\rho_2 \le \sigma_\varepsilon \le \sigma$ and $\delta \ll \rho_\varepsilon \le \rho_0$ as $\varepsilon \to 0$. Let φ_ε of be the solution to*

$$\begin{cases} -\Delta\varphi_\varepsilon = j_\varepsilon & \text{in } B(\xi,\sigma_\varepsilon)\setminus B(\xi,\rho_\varepsilon), \\ \varphi_\varepsilon = 0 & \text{on } \partial B(\xi,\sigma_\varepsilon)\setminus B(\xi,\rho_\varepsilon). \end{cases}$$

As $\varepsilon \to 0$, we have

$$\|\varphi_\varepsilon\|_{L^\infty(B(\xi,\sigma_\varepsilon)\setminus B(\xi,\rho_\varepsilon))} = o(1).$$

Proof. Let us first note that there exists a constant $c > 0$, such that

$$\delta^2 j_\varepsilon(\xi+\delta\,\cdot) = e^{\bar U}(1+\bar U^4) = \frac{8\mu^2}{(\mu^2+|\cdot|^2)^2}\left(1+\log^4\left(\frac{8\mu^2}{(\mu^2+|\cdot|^2)^2}\right)\right) \le c\frac{\mu}{(\mu^2+|\cdot|^2)^{\frac{3}{2}}}$$

in $\mathbb{R}^2$. Then, by the maximum principle, we have

$$|\varphi_\varepsilon| \le c\psi\left(\frac{\cdot-\xi}{\delta}\right) \quad \text{in } B(\xi,\sigma_\varepsilon)\setminus B(\xi,\rho_\varepsilon), \tag{56}$$

where ψ satisfies

$$\begin{cases} -\Delta\psi = \frac{\mu}{(\mu^2+|\cdot|^2)^{\frac{3}{2}}} & \text{in } A_\varepsilon := B(0,\frac{\sigma_\varepsilon}{\delta})\setminus B(0,\frac{\rho_\varepsilon}{\delta}), \\ \psi = 0 & \text{on } \partial A_\varepsilon. \end{cases}$$

Since the function $W := -\log(\mu+\sqrt{|\cdot|^2+\mu^2})$ satisfies $-\Delta W = \frac{\mu}{(\mu^2+|\cdot|^2)^{\frac{3}{2}}}$, we have

$$\psi = a + b\log|\cdot| + W,$$

for suitable constants $a, b \in \mathbb{R}$. Denoting $R_1 = \frac{\rho_\varepsilon}{\delta}$ and $R_2 = \frac{\sigma_\varepsilon}{\delta}$ one can verify that

$$a = \frac{W(R_2)\log R_1 - W(R_1)\log R_2}{\log R_2 - \log R_1} \qquad \text{and} \qquad b = \frac{W(R_1)-W(R_2)}{\log R_2 - \log R_1}.$$

Since

$$|W + \log|\cdot|| \le \frac{C\mu}{|\cdot|} = O\left(\frac{1}{R_1}\right),$$

uniformly in $\overline{A_\varepsilon}$, one has $a = O\left(\frac{\log R_2}{R_1(\log R_2 - \log R_1)}\right)$ and $b = 1 + O\left(\frac{1}{R_1(\log R_2-\log R_1)}\right)$. Then

$$\begin{aligned} \psi &= a + (b-1)\log|\cdot| + O(\frac{1}{R_1}) \\ &= O\left(\frac{1}{R_1}\frac{\log R_2}{\log R_2 - \log R_1}\right) + O\left(\frac{1}{R_1}\right) \\ &= O\left(\frac{1}{R_1}\frac{1}{1-\frac{\log R_1}{\log R_2}}\right) + O\left(\frac{1}{R_1}\right). \end{aligned}$$

Since

$$\frac{\log R_1}{\log R_2} = \frac{\log\frac{\rho_\varepsilon}{\delta}}{\log\sigma_\varepsilon - \log\delta} \le \frac{\log\frac{\rho_0}{\delta}}{\log\rho_2 - \log\delta} = O(\alpha),$$

we conclude that $\psi_\mu = O(\frac{1}{R_1}) = o(1)$, uniformly in A_ε. Then, the conclusion follows by (56). □

4 The Linear Theory

Let us consider the linear operator

$$L\varphi = \varphi - (-\Delta)^{-1}(\lambda f_\varepsilon'(\omega_\varepsilon)\varphi)$$

introduced in (19). In this section we give a priori estimates for the operator L and we prove its invertibility on a suitable subspace of $H_0^1(\Omega)$.

Lemma 4.1 *The following expansions hold:*

1. $\lambda f_\varepsilon'(\omega_\varepsilon) = e^{U_\varepsilon}(1 + O(\varepsilon^2))$ *in* $B(\xi, \rho_0)$.
2. $\lambda f_\varepsilon'(\omega_\varepsilon) = O(\Gamma_\varepsilon)$ *in* $B(\xi, \rho_1)$, *with* Γ_ε *as in* (48).
3. $\lambda f_\varepsilon'(\omega_\varepsilon) = O(1)$ *in* $\Omega \setminus B(\xi, \rho_1)$.
4. $\|\lambda f_\varepsilon'(\omega_\varepsilon)\chi_{B(\xi,\rho_1)} - e^{U_\varepsilon}\|_\varepsilon = o(1)$ *as* $\varepsilon \to 0$.

Proof. For $x \in B(\xi, \rho_0)$, using (28)-(32), Lemma 3.3, (34), and (45), we have that

$$\begin{aligned}
\lambda f_\varepsilon'(\omega_\varepsilon) &= \lambda(1 + 2\omega_\varepsilon^2 + (1+\varepsilon)\omega_\varepsilon^{1+\varepsilon})e^{\omega_\varepsilon^2 + \omega_\varepsilon^{1+\varepsilon}} \\
&= \lambda\beta^2(2 + O(\alpha))e^{\beta^2 + \beta^{1+\varepsilon} + \bar{U}(\frac{\cdot - \xi}{\delta}) + O(\varepsilon^2)} \\
&= e^{U_\varepsilon}(1 + O(\varepsilon^2)).
\end{aligned}$$

For $x \in B(\xi, \rho_1)$, using Remark 2.7, Lemma 3.3 we have

$$\begin{aligned}
\lambda f_\varepsilon'(\omega_\varepsilon) &= \lambda(1 + 2\omega_\varepsilon^2 + (1+\varepsilon)\omega_\varepsilon^{1+\varepsilon})e^{\omega_\varepsilon^2 + \omega_\varepsilon^{1+\varepsilon}} \\
&= \lambda\beta^2(2 + O(\alpha))e^{\beta^2 + \beta^{1+\varepsilon} + \bar{U}(\frac{\cdot}{\delta}) + \bar{U}(\frac{\cdot - \xi}{\delta})^2(1 + O(\varepsilon\alpha))} \\
&= O(\Gamma_\varepsilon).
\end{aligned}$$

Claim *3* follows directly from Lemma 3.9. Finally, claim 4 follows by claims *1* and *2*, using also Lemma 3.7 and the estimates

$$\|e^{U_\varepsilon}\|_{L^{1+\alpha}(B(\xi,\rho_1)\setminus B(\xi,\rho_0))} = o(1), \quad \|e^{U_\varepsilon}\|_{L^2(\Omega\setminus B(\xi,\rho_1))} = o(1).$$

□

According to Lemma 4.1, for $|x - \xi| \leq \rho_0$, L approaches the operator $L_0\varphi := \varphi - (-\Delta)^{-1}(e^{U_\varepsilon}\varphi)$. Note that

$$\begin{aligned}
L_0\varphi = 0 \quad \text{in } \Omega \qquad &\Longleftrightarrow \qquad -\Delta\varphi = e^{U_\varepsilon}\varphi \quad \text{in } \Omega \\
&\Longleftrightarrow \qquad -\Delta\Phi = e^{\bar{U}}\Phi \quad \text{in } \frac{\Omega - \xi}{\delta}, \text{ where } \Phi = \varphi(\xi + \delta\,\cdot).
\end{aligned}$$

Let us recall the following known fact about L_0 (see for example [10]).

Proposition 4.2 *All bounded weak solutions of the problem*

$$-\Delta\Phi = e^{\bar U}\Phi \quad \text{in } \mathbb{R}^2 \tag{57}$$

have the form

$$\Phi = c_0 Z_0 + c_1 Z_1 + c_2 Z_2,$$

where $c_0, c_1, c_2 \in \mathbb{R}$ *and*

$$Z_0(y) := \frac{\mu^2 - |y|^2}{\mu^2 + |y|^2}, \qquad Z_1(y) := \frac{2\mu y_1}{\mu^2 + |y|^2}, \qquad Z_2(y) := \frac{2\mu y_2}{\mu^2 + |y|^2}.$$

Remark 4.3 *The functions* Z_0, Z_1, Z_2 *are orthogonal in* $D^{1,2}(\mathbb{R}^2)$, *that is*

$$\int_{\mathbb{R}^2} \nabla Z_i \cdot \nabla Z_j dy = \int_{\mathbb{R}^2} e^{\bar U} Z_i Z_j dy = \frac{8}{3}\pi\delta_{i,j}. \tag{58}$$

In the following we denote

$$Z_{i,\varepsilon}(x) := Z_i\left(\frac{x-\xi}{\delta}\right) \quad \text{and} \quad PZ_{i,\varepsilon} = (-\Delta)^{-1} Z_{i,\varepsilon}, \qquad i = 0,1,2.$$

Lemma 4.4 *It holds true that*

$$PZ_{0,\varepsilon} = Z_{0,\varepsilon} + 1 + O(\delta^2) \text{ and } PZ_{i,\varepsilon} = Z_{i,\varepsilon} + O(\delta),\ i = 1,2,$$

uniformly with respect to $\mu \in \mathcal{U}$, $\xi \in B(\xi_0, \sigma)$.

Proof. See for example Appendix A in [18]. □

Lemma 4.4 shows the smallness of $PZ_{i,\varepsilon} - Z_{i,\varepsilon}$ for $i = 1,2$, but not for $i = 0$. For this reason, in many cases it is convenient to replace $PZ_{0,\varepsilon}$ with the funtion

$$\widetilde{Z}_\varepsilon := \begin{cases} Z_{0,\varepsilon} & \text{if } |x-\xi| \le \rho_0, \\ Z_{0,\varepsilon}(\rho_0)\left(\frac{\log\rho_1 - \log|x-\xi|}{\log\rho_1 - \log\rho_0}\right) & \text{if } \rho_0 \le |x-\xi| \le \rho_1, \\ 0 & \text{if } |x-\xi| \ge \rho_1. \end{cases} \tag{59}$$

Lemma 4.5 *The function* $\widetilde{Z}_\varepsilon$ *satisfies the following properties:*

- $\widetilde{Z}_\varepsilon \in H_0^1(\Omega)$ *and* $|\widetilde{Z}_\varepsilon| \le 1$ *in* Ω.
- $\|\nabla(\widetilde{Z}_\varepsilon - Z_{0,\varepsilon})\|_{L^2(\Omega)} \to 0$, *uniformly for* $\mu \in \mathcal{U}$ *and* $\xi \in B(\xi_0, \sigma)$.

Proof. The first property follows trivially from the definition. Moreover we have

$$\begin{aligned} \|\nabla(\widetilde{Z}_\varepsilon - Z_{0,\varepsilon})\|^2_{L^2(\Omega)} &\le \frac{Z_{0,\varepsilon}(\rho_0)^2}{(\log\rho_1 - \log\rho_0)^2}\int_{B(\xi,\rho_1)\setminus B(\xi,\rho_0)} \frac{1}{|x-\xi|^2}dx + \|\nabla Z_{0,\varepsilon}\|^2_{L^2(\Omega\setminus B(\xi,\rho_0))} \\ &\le \frac{2\pi Z_{0,\varepsilon}(\rho_0)^2}{\log\rho_1 - \log\rho_0} + \|\nabla Z_0\|^2_{L^2(\mathbb{R}^2\setminus B(0,\frac{\rho_0}{\delta}))} \\ &= O(\alpha^2) + O(e^{-\frac{\varepsilon}{\alpha}}) \to 0, \end{aligned}$$

as $\varepsilon \to 0$. $\square$

We will denote by K_ε the subspace of $H_0^1(\Omega)$ spanned by $PZ_{i,\varepsilon}$, $i = 0,1,2$ and by $K_\varepsilon^\perp$ the subspaces of $H_0^1(\Omega)$ orthogonal to K_ε, i.e.

$$K_\varepsilon^\perp = \left\{ u \in H_0^1(\Omega) \;:\; \int_\Omega \nabla PZ_{i,\varepsilon} \cdot \nabla u \, dx = \int_\Omega e^{U_\varepsilon} Z_{i,\varepsilon} u \, dx = 0, \; i = 0,1,2 \right\}.$$

Let π and $\pi^\perp$ be the projections of $H_0^1(\Omega)$ respectively on K_ε and $K_\varepsilon^\perp$. Finally, we denote

$$Y_\varepsilon := \{ f \in L^1(\Omega) \;:\; \|f\|_\varepsilon < +\infty \}.$$

Proposition 4.6 *There exist $\varepsilon_0 > 0$ and a constant $D_0 > 0$ such that*

$$\|\varphi\|_{H_0^1(\Omega)} + \|\varphi\|_{L^\infty(\Omega)} \le D_0 \|h\|_\varepsilon, \tag{60}$$

for any $\varepsilon \in (0, \varepsilon_0)$, $\mu \in \mathcal{U}$, $\xi \in B(\xi_0, \sigma)$, $h \in Y_\varepsilon$ and $\varphi \in K_\varepsilon^\perp$ satisfying

$$\pi^\perp \left\{ L\varphi - (-\Delta)^{-1} h \right\} = 0. \tag{61}$$

Proof. We assume by contradiction that there exists $\varepsilon_n \to 0$, $\mu_n \in \mathcal{U}$, $\xi_n \in B(\xi_0, \sigma)$, $h_n \in Y_\varepsilon$ and a solution $\varphi_n \in K_{\varepsilon_n}^\perp$ of (61) such that

$$\frac{\|\varphi_n\|_{H_0^1(\Omega)} + \|\varphi_n\|_{L^\infty(\Omega)}}{\|h_n\|_{\varepsilon_n}} \to +\infty.$$

Let $\delta_n, \alpha_n, \beta_n$ be the parameters in Lemma 2.4 corresponding to ε_n, μ_n and ξ_n. Let also $\rho_{0,n}, \rho_{1,n}, \rho_{2,n}$ be defined as in (39). We denote $\omega_n := \omega_{\varepsilon_n}$, $U_n := U_{\varepsilon_n}$, $Z_{i,n} := Z_{i,\varepsilon_n}$ and $f_n := f_{\varepsilon_n}$. W.l.o.g we can assume that $\|\varphi_n\|_{H_0^1(\Omega)} + \|\varphi_n\|_{L^\infty(\Omega)} = 1$ and $\|h_n\|_{\varepsilon_n} \to 0$. Since φ_n satisfies (61), there exist $c_{i,n} \in \mathbb{R}$, $i = 0,1,2$, such that

$$-\Delta\varphi_n - \lambda f_n'(\omega_n)\varphi_n = h_n + \sum_{i=0}^{2} c_{i,n} e^{U_n} Z_{i,n}. \tag{62}$$

Step 1 *We have $c_{i,n} \to 0$ as $n \to +\infty$, $i = 0,1,2$.*

Let $\widetilde{Z}_n := \widetilde{Z}_{\varepsilon_n}$ be the function defined in (59). Testing equation (62) against $\widetilde{Z}_n$, we get

$$\sum_{j=0}^{2} c_{j,n} \int_\Omega e^{U_n} Z_{j,n} \widetilde{Z}_n dx = \int_\Omega \nabla \widetilde{Z}_n \cdot \nabla \varphi_n dx - \int_\Omega \lambda f_n'(\omega_n) \varphi_n \widetilde{Z}_n dx - \int_\Omega h_n \widetilde{Z}_n dx. \tag{63}$$

Since $\|\varphi_n\|_{H_0^1(\Omega)} \le 1$ and $\varphi_n \in K_{\varepsilon_n}^\perp$, using Lemma 4.5 we get

$$\int_\Omega \nabla \widetilde{Z}_n \cdot \nabla \varphi_n dx = \int_\Omega \nabla Z_{0,n} \cdot \nabla \varphi_n dx + o(1) = \underbrace{\int_\Omega e^{U_n} Z_{0,n} \varphi_n dx}_{=0} + o(1) = o(1),$$

as $n \to +\infty$. By Lemma 4.1 and Lemma 3.7, we find

$$\begin{aligned}\int_\Omega \lambda f_n'(\omega_n)\varphi_n \widetilde{Z}_n dx &= \int_{B(\xi_n,\rho_{0,n})} e^{U_n}\varphi_n Z_{0,n} dx + O(\varepsilon_n^2) + O\left(\|\Gamma_\varepsilon\|_{L^1(B(\xi_n,\rho_{1,n})\setminus B(\xi_n,\rho_{0,n}))}\right)\\ &= \underbrace{\int_\Omega e^{U_n}\varphi_n Z_{0,n} dx}_{=0} + o(1) = o(1).\end{aligned}$$

Finally, Lemma 4.5 and Lemma 3.14 give

$$|\int_\Omega h_n \widetilde{Z}_n dx| \le \|h_n\|_{L^1(\Omega)} \le C\|h_n\|_{\varepsilon_n} = o(1).$$

Then (63) rewrites as

$$\sum_{j=0}^{2} c_{j,n}\int_\Omega e^{U_n} Z_{j,n}\widetilde{Z}_n dx = o(1). \tag{64}$$

With similar arguments, testing equation (62) against $PZ_{i,n}$ for $i = 1, 2$, we get that

$$\begin{aligned}\sum_{j=0}^{2} c_{j,n}\int_\Omega e^{U_n} Z_{j,n} PZ_{i,n} dx &= -\int_\Omega \lambda f_n'(\omega_n)\varphi_n PZ_{i,n} dx - \int_\Omega h_n PZ_{i,n} dx\\ &= \underbrace{\int_\Omega e^{U_n}\varphi_n Z_{i,n} dx}_{=0} + o(1) = o(1).\end{aligned} \tag{65}$$

Note that, as in (58), we have

$$\begin{aligned}\int_\Omega e^{U_n} Z_{j,n}\widetilde{Z}_n dx &= \int_{B(\xi_n,\rho_{0,n})} e^{U_n} Z_{j,n} Z_{0,n} dx + O\left(\int_{\mathbb{R}^2\setminus B(\xi_n,\rho_{0,n})} e^{U_n}\right)\\ &= \int_{B(0,\frac{\rho_{0,n}}{\delta_n})} e^{\overline{U}} Z_j Z_0 dy + o(1)\\ &= \frac{8}{3}\pi\delta_{0j} + o(1),\end{aligned}$$

for $j = 0, 1, 2$. Similarly

$$\begin{aligned}\int_\Omega e^{U_n} Z_{j,n} PZ_{i,n} dx &= \int_\Omega e^{U_n} Z_{j,n} Z_{i,n} dx + o(1)\\ &= \frac{8}{3}\pi\delta_{ij} + o(1),\end{aligned}$$

for $i = 1, 2$, $j = 0, 1, 2$. Then, (63) and (64) rewrite as

$$\sum_{j=0}^{2} c_{j,n}(\delta_{ij} + o(1)) = o(1),$$

which implies the conclusion.

Step 2 *If* $\widetilde{h}_n := h_n + \left(\lambda f_n'(\omega_n)\chi_{B(\xi_n,\rho_{1,n})} - e^{U_n}\right)\varphi_n + \sum_{j=0}^{2} c_{j,n}e^{U_n}Z_{j,n}$, *then*

$$-\Delta\varphi_n = e^{U_n}\varphi_n + \lambda f_n'(\omega_n)\chi_{\Omega\setminus B(\xi_n,\rho_{1,n})}\varphi_n + \widetilde{h}_n \quad \text{in } \Omega, \qquad \text{and} \qquad \|\widetilde{h}_n\|_{\varepsilon_n} \to 0. \tag{66}$$

Since $\|h_n\|_{\varepsilon_n} \to 0$, $|Z_{i,n}| \leq 1$, and $\|\lambda f_n'(\omega_n)\chi_{B(\xi_n,\rho_{1,n})} - e^{U_n}\|_{\varepsilon_n} \to 0$ by Lemma 4.1, it is sufficient to observe that $\|e^{U_n}\|_{\varepsilon_n} = O(1)$ and apply Step 1.

Step 3 *There exists* $\delta_n \ll \rho_n \leq \rho_{0,n}$ *such that, up to a subsequence,* $\|\varphi_n\|_{L^\infty(B(\xi_n,\rho_n))} \to 0$ *as* $n \to +\infty$.

Let us consider the sequence $\Phi_n(y) := \varphi_n(\xi_n + \delta_n y)$, $y \in \frac{\Omega-\xi_n}{\delta_n}$. By (66) Φ_n satisfies

$$-\Delta\Phi_n = e^{\bar{U}}\Phi_n + \delta_n^2\widetilde{h}_n(\xi + \delta_n\cdot) \qquad \text{in } B\Big(0, \frac{\rho_{1,n}}{\delta_n}\Big).$$

We know that

$$\left|e^{\bar{U}(y)}\Phi_n(y)\right| \leq e^{\bar{U}(y)} \leq \frac{8}{\mu^2},$$

and, for $y \in B(0, \frac{\rho_{0,n}}{\delta_n})$, that

$$\delta_n^2|\widetilde{h}_n(\xi+\delta_n y)| \leq \delta_n^2 j_{\varepsilon_n}(\xi+\delta_n y)\|\widetilde{h}_n\|_{\varepsilon_n} = e^{\bar{U}(y)}(1+|\bar{U}(y)|^4)\|\widetilde{h}_n\|_{\varepsilon_n} \leq C\|\widetilde{h}_n\|_{\varepsilon_n} \to 0.$$

In particular Φ_n and $\Delta\Phi_n$ are uniformly bounded in $B(0, \frac{\rho_{0,n}}{\delta_n})$. By standard elliptic estimates, we can find $\Phi_0 \in C(\mathbb{R}^2) \cap H^1_{loc}(\mathbb{R}^2)$ and a sequence $R_n \to +\infty$, $R_n \leq \frac{\rho_{0,n}}{\delta_n}$, such that, up to a subsequence, $\|\Phi_n - \Phi_0\|_{L^\infty(B(0,R_n))} \to 0$. Moreover, $|\Phi_0| \leq 1$ and Φ_0 is a weak solution to

$$-\Delta\Phi_0 = e^{\bar{U}}\Phi_0 \quad \text{in } \mathbb{R}^2.$$

According to Proposition 4.2, we must have $\Phi_0 = \kappa_0 Z_0 + \kappa_1 Z_1 + \kappa_2 Z_2$, for some $\kappa_i \in \mathbb{R}$, $i = 0,1,2$. Keeping in mind (58) and using that $e^{\bar{U}} \in L^1(\mathbb{R}^2)$, we obtain

$$\begin{aligned} 0 = \int_\Omega e^{U_n} Z_{i,n}\phi_n\,dx &= \int_{\frac{\Omega-\xi_n}{\delta_n}} e^{\bar{U}} Z_i \Phi_n dy \\ &= \int_{B(0,R_n)} e^{\bar{U}} Z_i\Phi_n\,dy + O\left(\int_{\mathbb{R}^2\setminus B(0,R_n)} e^{\bar{U}} dy\right) \\ &\to \frac{8}{3}\pi\kappa_i, \end{aligned}$$

for $i = 0,1,2$. This implies $\kappa_i = 0$, $i = 0,1,2$. Then $\Phi_0 \equiv 0$ and we get the conclusion with $\rho_n = \delta_n R_n$.

Step 4 *Up to a subsequence,* $\xi_n \to \bar{\xi} \in \Omega$ *and* $\varphi_n \to 0$ *in* $L^\infty_{loc}(\Omega \setminus \{\bar{\xi}\})$, *as* $n \to \infty$.

We know that φ_n satisfies (66) in Ω. Since $|\varphi_n| \leq 1$, $\|e^{U_n}\|_{L^\infty(\Omega\setminus B(\xi_n,\rho_{1,n}))} \to 0$, $\|h_n\|_{L^2(\Omega\setminus B(\xi,\rho_{1,n}))} \to 0$, and $\|f_n'(\omega_n)\|_{L^\infty(\Omega\setminus B(\xi,\rho_{1,n}))} = O(1)$, by ellpitic estimates we find that φ_n is bounded in $C^{0,\gamma}_{loc}(\overline{\Omega} \setminus \{\bar{\xi}\})$, for some $\gamma \in (0,1)$. Therefore, there exists $\varphi_0 \in C(\overline{\Omega}) \cap H^1_0(\Omega)$, such that $\varphi_n \to \varphi_0$ locally uniformly on $\overline{\Omega} \setminus \{\bar{\xi}\}$ and weakly in

$H^1_0(\Omega)$. Noting that $\omega_n \to -u_0$ locally uniformly in $\overline{\Omega} \setminus \{\xi\}$ and that f_n' is even, we see that φ_0 satisfies $\Delta\varphi_0 + f_0'(u_0)\varphi_0$ in $\Omega \setminus \{\bar\xi\}$. Actually, since $\varphi_0, \Delta\varphi_0 \in L^\infty(\Omega)$, φ_0 is a weak solution of $\Delta\varphi_0 + f_0'(u_0)\varphi_0 = 0$ in Ω. Then, the non-degeneracy of u_0 implies $\varphi_0 \equiv 0$.

Step 5 $\|\varphi_n\|_{L^\infty(\Omega)} \to 0$.

By Step 4, we can find a sequence $\sigma_n \geq \rho_{2,n}$ such that $\|\varphi_n\|_{L^\infty(\Omega\setminus B(\xi_n,\sigma_n))} \to 0$ as $n \to +\infty$, up to a subsequence. Then, it is sufficient to show that $\|\varphi_n\|_{L^\infty(A_n)} \to 0$, where $A_n := B(\xi_n, \sigma_n) \setminus B(\xi_n, \rho_n)$ and ρ_n is as in Step 3. We can split $\varphi_n = \varphi_n^{(0)} + \varphi_n^{(1)} + \varphi_n^{(2)} + \varphi_n^{(3)}$, where

$$\begin{cases} \Delta\varphi_n^{(0)} = 0 & \text{in } A_n, \\ \varphi_n^{(0)} = \varphi_n & \text{on } \partial A_n, \end{cases} \qquad \text{and} \qquad \begin{cases} -\Delta\varphi_n^{(i)} = f_{i,n} & \text{in } A_n, \\ \varphi_n^{(i)} = 0 & \text{on } \partial A_n, \end{cases} \qquad \text{for } i = 1,2,3,$$

with

$$\begin{cases} f_{1,n} := e^{U_n}\varphi_n + \widetilde{h}_n \chi_{B(\xi_n,\rho_{0,n})}, \\ f_{2,n} := \widetilde{h}_n \chi_{B(\xi_n,\rho_{1,n})\setminus B(\xi_n,\rho_{0,n})}, \\ f_{3,n} := \widetilde{h}_n \chi_{B(\xi_n,\sigma_n)\setminus B(\xi_n,\rho_{1,n})} + \lambda f_n'(\omega_n)\chi_{B(\xi_n,\sigma_n)\setminus B(\xi_n,\rho_{1,n})}\varphi_n. \end{cases}$$

By the maximum principle

$$\|\varphi_n^{(0)}\|_{L^\infty(A_n)} \leq \|\varphi_n\|_{L^\infty(\partial A_n)} \to 0.$$

Since

$$|f_{1,n}| \leq e^{U_n} + \|\widetilde{h}_n\|_{\varepsilon_n} j_{\varepsilon_n} \leq j_{\varepsilon_n}(1 + o(1)) \leq 2 j_{\varepsilon_n},$$

we get that $|\varphi_n^{(1)}| \leq 2\psi_n$, where ψ_n satisfies

$$\begin{cases} -\Delta\psi_n = j_{\varepsilon_n} & \text{in } A_n \\ \psi_n = 0 & \text{on } \partial A_n. \end{cases}$$

Lemma 3.15 implies $\|\psi_n\|_{L^\infty(A_n)} \to 0$, hence $\|\varphi_n^{(1)}\|_{L^\infty(A_n)} \to 0$. Finally, since $|A_n|$ is uniformly bounded, elliptic estimates (see Corollaries B.3 and B.4) give

$$\|\varphi_n^{(2)}\|_{L^\infty(A_n)} \leq \frac{C}{\alpha^2}\|f_{2,n}\|_{L^{1+\alpha^2}(A_n)} = \frac{C}{\alpha^2}\|\widetilde{h}_n\|_{L^{1+\alpha^2}(B(\xi_n,\rho_{1,n})\setminus B(\xi_n,\rho_{0,n}))} \leq \|\widetilde{h}_n\|_{\varepsilon_n} \to 0,$$

and

$$\|\varphi_n^{(3)}\|_{L^\infty(A_n)} \leq C\|f_{3,n}\|_{L^2(A_n)} = O(\|h_n\|_{\varepsilon_n}) + O(\sqrt{\sigma_n}) \to 0.$$

Step 6 *Conclusion of the proof.*

By Step 5, we have that $\|\varphi_n\|_{H^1_0(\Omega)} = 1 - \|\varphi_n\|_{L^\infty(\Omega)} \to 1$. But (66) gives

$$\begin{aligned} \|\varphi_n\|^2_{H^1_0(\Omega)} &= \int_\Omega e^{U_n}\varphi_n^2\,dx + \int_{\Omega\setminus B(\xi,\rho_{1,n})} \lambda f_n'(\omega_n)\varphi_n^2\,dx + \int_\Omega \widetilde{h}_n\varphi_n\,dx \\ &= O(\|\varphi_n\|^2_{L^\infty(\Omega)}) + o(\|\varphi_n\|_{L^2(\Omega)}) \to 0. \end{aligned}$$

Then, we get a contadiction. □

As a consequence we have that $\pi^\perp L$ is invertible on $K_\varepsilon^\perp$.

Corollary 4.7 *$\pi^\perp L : K_\varepsilon^\perp \mapsto K_\varepsilon^\perp$ is invertible.*

Proof. This follows by standard Fredholm theory. Indeed, for any $\varepsilon > 0$ the map $F(\varphi) := \pi^\perp(-\Delta)^{-1}(f'(\omega_\varepsilon)\varphi)$ defines a compact operator on $K_\varepsilon^\perp$ (in fact on $H_0^1(\Omega)$). Then $\pi^\perp L = Id_{K_\varepsilon^\perp} - F$ is a Fredholm operator of index 0. Proposition 4.6 implies that $\pi^\perp L$ is injective, hence it is invertible on $K_\varepsilon^\perp$. □

5 The reduction to a finite dimensional problem

This section is devoted to reduce the problem to a finite dimensional one. More precisely, we prove:

Proposition 5.1 *There exist $\varepsilon_0 > 0$ and a map $(\varepsilon, \mu, \xi) \to \varphi_{\varepsilon,\mu,\xi} \in K_\varepsilon^\perp \cap L^\infty(\Omega)$ defined in $(0, \varepsilon_0) \times \mathcal{U} \times B(\xi_0, \sigma)$ and continuous with respect to μ and ξ, such that for some $D > 0$*

$$\|\varphi_{\varepsilon,\mu,\xi}\|_{H_0^1} + \|\varphi_{\varepsilon,\mu,\xi}\|_{L^\infty} \le D\alpha^3, \tag{67}$$

and

$$\pi^\perp\Big\{L\varphi_{\varepsilon,\mu,\xi} - (-\Delta)^{-1}(R + N(\varphi_{\varepsilon,\mu,\xi}))\Big\} = 0, \tag{68}$$

where the linear operator L is defined in (19), the error term R is defined in (17) and the quadratic term N is defined in (18).

5.1 Estimates on $N(\varphi)$

For a function $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, let $N(\varphi)$ be defined as in (18), i.e.

$$N(\varphi) = N_{\varepsilon,\mu,\xi}(\varphi) := \lambda\left(f_\varepsilon(\omega_{\varepsilon,\mu,\xi} + \varphi) - f_\varepsilon(\omega_{\varepsilon,\mu,\xi}) - f'_\varepsilon(\omega_{\varepsilon,\mu,\xi})\varphi\right).$$

Let us estimate $\|N(\varphi)\|_\varepsilon$, where $\|\cdot\|_\varepsilon$ is defined as in (55). Let us define

$$\mathcal{B}_\alpha := \{\varphi \in L^\infty(\Omega) \ : \ \|\varphi\|_{L^\infty(\Omega)} \le \alpha\}. \tag{69}$$

Lemma 5.2 *There exists $D_2 > 0$ such that*

$$\|N(\varphi_1) - N(\varphi_2)\|_\varepsilon \le D_2\alpha^{-1}\left(\|\varphi_1\|_{L^\infty(\Omega)} + \|\varphi_2\|_{L^\infty(\Omega)}\right)\|\varphi_1 - \varphi_2\|_{L^\infty(\Omega)},$$

for any $\varphi_1, \varphi_2 \in \mathcal{B}_\alpha$.

Proof. First, for any $x \in \Omega$ we can find $\theta_1 = \theta_1(x) \in [0,1]$ such that

$$\begin{aligned} N(\varphi_2) - N(\varphi_1) &= \lambda\left(f_\varepsilon(\omega_\varepsilon + \varphi_2) - f_\varepsilon(\omega_\varepsilon + \varphi_1) - f'_\varepsilon(\omega_\varepsilon)(\varphi_2 - \varphi_1)\right) \\ &= \lambda\left(f'_\varepsilon(\omega_\varepsilon + \theta_1\varphi_2 + (1-\theta_1)\varphi_1)(\varphi_2 - \varphi_1) - f'_\varepsilon(\omega_\varepsilon)(\varphi_2 - \varphi_1)\right) \\ &= \lambda\left(f'_\varepsilon(\omega_\varepsilon + \varphi_3) - f'_\varepsilon(\omega_\varepsilon)\right)(\varphi_2 - \varphi_1), \end{aligned}$$

where $\varphi_3 := \theta_1\varphi_2 + (1-\theta_1)\varphi_1$. Furthermore, there exists $\theta_2 = \theta_2(x)$ such that

$$f'_\varepsilon(\omega_\varepsilon + \varphi_3) = f'_\varepsilon(\omega_\varepsilon) + f''_\varepsilon(\omega_\varepsilon + \theta_2\varphi_3)\varphi_3.$$

Thus, we obtain

$$\begin{aligned} |N(\varphi_1) - N(\varphi_2)| &= \lambda |f''_\varepsilon(\omega_\varepsilon + \theta_2\varphi_3)||\varphi_3||\varphi_1 - \varphi_2| \\ &\leq \lambda |f''_\varepsilon(\omega_\varepsilon + \theta_2\varphi_3)| \left(\|\varphi_1\|_{L^\infty(\Omega)} + \|\varphi_2\|_{L^\infty(\Omega)}\right) \|\varphi_1 - \varphi_2\|_{L^\infty(\Omega)}. \end{aligned} \tag{70}$$

Then, in order to conclude the proof, we shall bound $\|f''_\varepsilon(\omega_\varepsilon + \theta_2\varphi_3)\|_\varepsilon$. Note that, there exists a universal constant $C_0 > 0$ such that

$$|f''_\varepsilon(t)| \leq C_0(1 + |t|^3)e^{t^2 + |t|^{1+\varepsilon}}, \quad \forall t \in \mathbb{R}.$$

By Remark 2.7 we have $\omega_\varepsilon = O(\beta) = O(\alpha^{-1})$. Since $|\varphi_3| \leq |\varphi_1| + |\varphi_2| \leq 2\alpha$, we get

$$(\omega_\varepsilon + \theta_2\varphi_3)^2 \leq \omega_\varepsilon^2 + 2|\omega_\varepsilon||\varphi_3| + \varphi_3^2 = \omega_\varepsilon^2 + O(1). \tag{71}$$

By convexity, we also have

$$|\omega_\varepsilon + \theta_2\varphi_3|^3 \leq (|\omega_\varepsilon| + |\varphi_3|)^3 \leq 4(|\omega_\varepsilon|^3 + |\varphi_3|^3) \leq 4(|\omega_\varepsilon|^3 + \alpha^3). \tag{72}$$

In $B(\xi, \rho_1)$ we have $\omega_\varepsilon \geq c_0$ by Lemma 3.2, so that

$$(\omega_\varepsilon + \theta_2\varphi_3)^{1+\varepsilon} \leq \omega_\varepsilon^{1+\varepsilon}\left(1 + \frac{\alpha}{c_0}\right)^{1+\varepsilon} = \omega_\varepsilon^{1+\varepsilon} + O(1). \tag{73}$$

Clearly (71)-(73) yield the existence of a constant $C_1 > 0$ such that

$$|f''_\varepsilon(\omega_\varepsilon + \theta_2\varphi_3)| \leq C_1\alpha^{-2}\omega_\varepsilon e^{\omega_\varepsilon^2 + |\omega_\varepsilon|^{1+\varepsilon}} = C_1\alpha^{-2} f_\varepsilon(\omega_\varepsilon),$$

in $B(\xi, \rho_1)$. Arguing as in Lemma 3.4 (see (46)) we get

$$\lambda |f''_\varepsilon(\omega_\varepsilon + \theta_2\varphi)| \leq C\alpha^{-1} j_\varepsilon \quad \text{in } B(\xi, \rho_0). \tag{74}$$

Lemma 3.6 and Lemma 3.7 yield

$$\lambda \|f''_\varepsilon(\omega_\varepsilon + \theta_2\varphi)\|_{L^{1+\alpha^2}(B(\xi,\rho_1)\setminus B(\xi,\rho_0))} = O(\alpha^{-2} e^{-\frac{c_2}{\sqrt{\alpha}}}). \tag{75}$$

Finally, thanks to Lemma 3.9, we know that

$$\lambda f''_\varepsilon(\omega_\varepsilon + \theta_2\varphi_3) = O(1) \quad \text{in } \Omega \setminus B(\xi, \rho_1). \tag{76}$$

Thanks to (74)-(76) we infer

$$\lambda \|f''_\varepsilon(\omega_\varepsilon + \theta_2\varphi_3)\|_\varepsilon = O(\alpha^{-1}),$$

and the conclusion follows from (70). □

Remark 5.3 *Applying Lemma 5.2 with $\varphi_2 = 0$, we obtain that*

$$\|N(\varphi)\|_\varepsilon \leq D_2\alpha^{-1}\|\varphi\|^2_{L^\infty(\Omega)},$$

for any $\varphi \in \mathcal{B}_\alpha$.

Remark 5.4 *The proof of Proposition 5.2 and Lemma 3.9 also shows that*

$$\|N(\varphi)\|_{L^\infty(\Omega\setminus B(\xi,\rho_1))} \leq D_3\|\varphi\|^2_{L^\infty(\Omega)},$$

for any $\varphi \in \mathcal{B}_\alpha$.

5.2 Proof of Proposition 5.1: a fixed point argument

Let us consider the operator

$$\mathcal{T} = \mathcal{T}_{\varepsilon,\mu,\xi} := (\pi^{\perp} L)^{-1}\pi^{\perp}\left[(-\Delta)^{-1}\Big(N(\varphi) + R\Big)\right] \tag{77}$$

on the space $X := K_{\varepsilon}^{\perp} \cap L^{\infty}(\Omega)$, which is a Banach space with respect to the norm

$$\|\cdot\|_X = \|\cdot\|_{H_0^1(\Omega)} + \|\cdot\|_{L^{\infty}(\Omega)}.$$

Let D_1 and D_0 be the constants defined in Proposition 3.13 and Proposition 4.6. Let us set

$$E_{\varepsilon} := \{\varphi \in X \ : \ \|\varphi\|_X \leq D_0(D_1+1)\alpha^3\}.$$

Proposition 5.1 is an immediate consequence of the following result.

Proposition 5.5 *There exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, $\mu \in \mathcal{U}$, $\xi \in B(\xi_0, \sigma)$, $\mathcal{T}$ has a fixed point $\varphi_{\varepsilon,\mu,\xi} \in E_{\varepsilon}$, which depends continuosly on μ and ξ.*

Proof. Since E_{ε} is a closed subspace of X and $\mathcal{T}$ depends continuously on μ and ξ, it is sufficient to verifry that

1. $\mathcal{T}$ maps E_{ε} into itself.
2. $\mathcal{T}$ is a contraction, i.e. $||\mathcal{T}(\varphi_1) - \mathcal{T}(\varphi_2)||_{H_0^1(\Omega)} \leq \theta||\varphi_1 - \varphi_2||_{H_0^1(\Omega)}$ for some positive constant $\theta < 1$ and for any $\varphi_1, \varphi_2 \in E_{\varepsilon}$.

Then the conclusion follows by the contraction mapping theorem.

Step 1 *$\mathcal{T}$ maps E_{ε} into itself.*

Let us denote $C_0 := D_0(D_1 + 1)$. Take $\varphi \in E_{\varepsilon}$ and set

$$h(\varphi) := R + N(\phi).$$

If ε is small enough, we have that $\alpha^2 C_0 \leq 1$, so that $E_{\varepsilon} \subseteq \mathcal{B}_{\alpha}$ (see (69)). By Proposition 3.13 and Remark 5.3 we get

$$\begin{aligned}\|h(\varphi)\|_{\varepsilon} &\leq \|R\|_{\varepsilon} + \|N(\varphi)\|_{\varepsilon} \\ &\leq D_1\alpha^3 + D_2\alpha^{-1}\|\varphi\|^2_{L^{\infty}(\Omega)} \\ &\leq D_1\alpha^3 + C_0^2 D_2 \alpha^5,\end{aligned}$$

for any $\varphi \in E_{\varepsilon}$. Then, if we take ε small enough so that $C_0^2 D_2\alpha^2 \leq 1$, we get that

$$\|h(\varphi)\|_{\varepsilon} \leq (D_1 + 1)\alpha^3.$$

Since by definition

$$\pi^{\perp} L(\mathcal{T}(\varphi)) = \pi^{\perp}(-\Delta)^{-1} h(\varphi),$$

we have by Proposition 4.6 that

$$\|\mathcal{T}(\varphi)\|_X \leq D_0\|h(\varphi)\|_{\varepsilon} \leq D_0(D_1+1)\alpha^3,$$

that is $\mathcal{T}(\varphi) \in E_{\varepsilon}$.

Step 2 $\mathcal{T}$ *is a contraction mapping in* E_ε.

Let us take ε small enough so that $D_0 D_2 C_0 \alpha^2 \le \frac{1}{4}$ and $E_\varepsilon \subseteq \mathcal{B}_\alpha$. By Propositions 4.6 and 5.2 we have

$$\begin{aligned}\|T(\varphi_1) - T(\varphi_2)\|_X &\le D_0 \|h(\varphi_1) - h(\varphi_2)\|_\varepsilon \\ &= D_0 \|N(\varphi_1) - N(\varphi_2)\|_\varepsilon \\ &\le D_0 D_2 \alpha^{-1} (\|\varphi_1\|_{L^\infty(\Omega)} + \|\varphi_2\|_{L^\infty(\Omega)}) \|\varphi_1 - \varphi_2\|_{L^\infty(\Omega)} \\ &\le 2 C_0 D_0 D_2 \alpha^2 \|\varphi_1 - \varphi_2\|_{L^\infty(\Omega)} \\ &\le \frac{1}{2} \|\varphi_1 - \varphi_2\|_{L^\infty(\Omega)},\end{aligned}$$

for any $\varphi_1, \varphi_2 \in E_\varepsilon$. Then, $\mathcal{T}$ is a contraction mapping on E_ε. □

6 The reduced problem: proof of Theorem 1.3 completed

Let $\varphi_\varepsilon := \varphi_{\varepsilon,\mu,\xi}$ be as in Proposition 5.1. By (68), we can find $\kappa_{\varepsilon,i} = \kappa_{\varepsilon,i}(\mu, \xi)$, $i = 0, 1, 2$ (which depend continuously on μ, and ξ), such that

$$-\Delta \varphi_\varepsilon = \lambda f'_\varepsilon(u_\varepsilon) \varphi_\varepsilon + R + N(\varphi_\varepsilon) + \sum_{j=0}^{2} \kappa_{\varepsilon,j} e^{U_\varepsilon} Z_{\varepsilon,j}. \tag{78}$$

Equivalently, setting $u_\varepsilon := \omega_\varepsilon + \varphi_\varepsilon$,

$$-\Delta u_\varepsilon = \lambda f_\varepsilon(u_\varepsilon) + \sum_{j=0}^{2} \kappa_{\varepsilon,j} e^{U_\varepsilon} Z_{\varepsilon,j}. \tag{79}$$

Our aim is to find the parameter $\mu = \mu(\varepsilon)$ and the point $\xi = \xi(\varepsilon)$ so that the $\kappa_{\varepsilon,i}$'s are zero provided ε is small enough.

Proposition 6.1 *It holds true that*

$$\kappa_{0,\varepsilon} = 6\pi\alpha^3 \left(2 - \log\left(\frac{8}{\mu^2}\right) + o(1) \right), \tag{80}$$

and

$$\kappa_{i,\varepsilon} = -\kappa_{0,\varepsilon} a_{i,\varepsilon} + \frac{3\mu}{2} \delta \frac{\partial v_\varepsilon}{\partial x_i}(\xi) + O(\alpha\delta), \;\; i = 1, 2 \tag{81}$$

as $\varepsilon \to 0$ *uniformly with respect to* $\mu \in \mathcal{U}$ *and* $\xi \in B(\xi_0, \sigma)$. *Here, the* $a_{i,\varepsilon}$*'s are continuous functions of* μ *and* ξ *and* $a_{i,\varepsilon} = O(\alpha^2)$ *uniformly for* $(\mu, \xi) \in \mathcal{U} \times B(\xi_0, \sigma)$.

Proof.

Step 1 *Let us prove that*

$$\kappa_{i,\varepsilon} = O(\alpha^3) \quad \textit{for } i = 0,1,2 \tag{82}$$

and

$$\|\varphi_\varepsilon\|_{C^1(\overline{\Omega}\setminus B(\xi_0,2\sigma))} = O(\alpha^3). \tag{83}$$

First, since (67) gives $\|\phi\|_{L^\infty(\Omega)} = O(\alpha^3)$, Proposition 3.13, Lemma 3.14, Remark 5.3 and Lemma 4.1 yield

$$\|R\|_{L^1(\Omega)} = O(\alpha^3), \quad \|N(\varphi_\varepsilon)\|_{L^1(\Omega)} = O(\alpha^5), \quad \|\lambda f'_\varepsilon(\omega_\varepsilon)\varphi_\varepsilon\|_{L^1(\Omega)} = O(\alpha^3).$$

Recalling that

$$\int_\Omega e^{U_n} Z_{j,n} PZ_{i,n} dx = \frac{8}{3}\pi\delta_{ij} + O(\delta), \text{ for } i,j = 0,1,2,$$

by Lemma 4.4 and (58), we get (82) by testing equation (78) with $PZ_{i,n}$, $i = 0,1,2$.

By Lemma 3.12, Remark 5.4, and Lemma 4.1, one has

$$\lambda f'_\varepsilon(\omega_\varepsilon) = O(1), \quad R = O(\alpha^3), \quad N(\varphi_\varepsilon) = O(\alpha^6),$$

uniformly in $\Omega \setminus B(\xi, \frac{\sigma}{2})$. Then

$$\|\Delta\varphi_\varepsilon\|_{L^\infty(\Omega\setminus B(\xi,\frac{\sigma}{2}))} + \|\varphi_\varepsilon\|_{L^\infty(\Omega)} = O(\alpha^3),$$

and we get (83) by standard elliptic estimates.

Step 2 *Proof of* (80).

Let $\widetilde{Z}_\varepsilon$ be the function defined in (59). We shall test equation (78) against $\widetilde{Z}_\varepsilon$. With the same arguments of the proof of Proposition 4.6 (Step 1), we obtain

$$\int_\Omega \nabla\varphi_\varepsilon \cdot \nabla\widetilde{Z}_\varepsilon \, dx = \int_\Omega \nabla\varphi_\varepsilon \cdot \nabla Z_{0,\varepsilon} \, dx + o(\|\varphi_\varepsilon\|_{H^1_0(\Omega)}) = o(\alpha^3).$$

Moreover

$$\begin{aligned}\int_\Omega \lambda f'_\varepsilon(\omega_\varepsilon)\varphi_\varepsilon \widetilde{Z}_\varepsilon dx &= \int_{B(\xi,\rho_0)} e^{U_\varepsilon} Z_{0,\varepsilon}\varphi_\varepsilon \, dx + O(\varepsilon^2\alpha^3) + O(\alpha^3\|\Gamma_\varepsilon\|_{L^1(B(\xi,\rho_1)\setminus B(\xi,\rho_0))}) \\ &= o(\alpha^3),\end{aligned}$$

and

$$\begin{aligned}\int_\Omega e^{U_n} Z_{j,\varepsilon}\widetilde{Z}_\varepsilon dx &= \int_{\mathbb{R}^2} e^{\bar{U}} Z_j Z_0 dy + O\left(\int_{\mathbb{R}^2\setminus B(0,\frac{\rho_0}{\delta})} e^{\bar{U}} dx\right) \\ &= \frac{8}{3}\pi\delta_{ij} + O(\delta^2\rho_0^{-2}).\end{aligned}$$

By Lemma 3.4 and Lemma 3.8, we get

$$\begin{aligned}\int_\Omega R\widetilde{Z}_n dx &= \int_{B(\xi,\rho_0)} RZ_{0,n} dx + O(\|R\|_{L^1(B(\xi,\rho_1)\setminus B(\xi,\rho_0))})\\ &= \alpha^3 \int_{B(0,\rho_0)} e^{\bar U}\left(2\bar U + \bar U^2\right) Z_0 dy + O\left(\alpha^4 \int_{\mathbb{R}^2} e^{\bar U}(1+\bar U^4)dy\right) + o(\alpha^4)\\ &= 16\pi\alpha^3\left(\log\left(\frac{8}{\mu^2}\right) - 2\right) + O(\alpha^4).\end{aligned}$$

Finally, we have that

$$\int_\Omega N(\varphi)\widetilde{Z}_\varepsilon dx = O(\|N(\varphi)\|_\varepsilon) = O(\alpha^5).$$

Then, testing (78) against $\widetilde{Z}_\varepsilon$ and using (82), one gets

$$0 = 16\pi\alpha^3\left(\log\left(\frac{8}{\mu^2}\right) - 2\right) + \frac{8}{3}\pi k_{0,\varepsilon} + o(\alpha^3),$$

from which we get (80).

Step 3 *Let us prove*

$$\sum_{j=0}^{2} \kappa_{j,\varepsilon}\int_\Omega e^{U_\varepsilon} Z_{j,\varepsilon}\frac{\partial u_\varepsilon}{\partial x_i}dx = -8\pi\alpha\frac{\partial v_\varepsilon}{\partial x_i}(\xi) + O(\alpha^2),\ i=1,2, \tag{84}$$

We multiply (79) and $\frac{\partial u_\varepsilon}{\partial x_i}$. Applying the Pohozaev identity (see e.g. [27, Proposition 2, Proof of Step 1]), we obtain

$$-\frac{1}{2}\int_{\partial\Omega}\frac{\partial u_\varepsilon}{\partial x_i}\frac{\partial u_\varepsilon}{\partial \nu}\nu_i\, d\sigma = \lambda\int_\Omega f_\varepsilon(u_\varepsilon)\frac{\partial u_\varepsilon}{\partial x_i}dx + \sum_{j=0}^{2}\kappa_{j,\varepsilon}\int_\Omega e^{U_n} Z_{j,\varepsilon}\frac{\partial u_\varepsilon}{\partial x_i}dx_i. \tag{85}$$

Since $u_\varepsilon = 0$ on $\partial\Omega$, the divergence theorem yields

$$\begin{aligned}\int_\Omega f_\varepsilon(u_\varepsilon)\frac{\partial u_\varepsilon}{\partial x_i}dx &= \int_\Omega \frac{d}{dx_i}\left(\int_0^{u_\varepsilon(x)} f_\varepsilon(t)dt\right)dx\\ &= \int_{\partial\Omega}\nu_i\left(\int_0^{u_\varepsilon(x)} f_\varepsilon(t)dt\right)d\sigma = 0.\end{aligned} \tag{86}$$

By (83), the definition of u_ε and ω_ε, Lemma 2.3, Lemma 3.10, we have

$$\frac{\partial u_\varepsilon}{\partial\nu} = -\frac{\partial v_\varepsilon}{\partial\nu} + \alpha\frac{\partial}{\partial\nu}(8\pi G_\xi - w_\varepsilon) + O(\alpha^2)$$

on $\partial\Omega$. Thus, keeping in mind that $|\nabla v_\varepsilon|$, $|\nabla w_\varepsilon|$ and $|\nabla G_\xi|$ are uniformly bounded on $\partial\Omega$ (see Lemma (2.2) and (2.3)) and that $\frac{\partial u_\varepsilon}{\partial x_i} = \frac{\partial u_\varepsilon}{\partial\nu}\nu_i$, we obtain

$$\int_{\partial\Omega}\frac{\partial u_\varepsilon}{\partial x_i}\frac{\partial u_\varepsilon}{\partial\nu}\,d\sigma = \int_{\partial\Omega}\frac{\partial v_\varepsilon}{\partial x_i}\frac{\partial v_\varepsilon}{\partial\nu}\,d\sigma + 2\alpha\int_{\partial\Omega}\frac{\partial v_\varepsilon}{\partial x_i}\frac{\partial}{\partial\nu}(w_\varepsilon - 8\pi G_\xi)\,d\sigma + O(\alpha^2). \tag{87}$$

Applying the Pohozaev identity to v_ε and arguing as in (86), we get that

$$\int_{\partial\Omega} \frac{\partial v_\varepsilon}{\partial x_i}\frac{\partial v_\varepsilon}{\partial \nu}\, d\sigma = -2\lambda \int_\Omega f_\varepsilon(v_\varepsilon)\frac{\partial v_\varepsilon}{\partial x_i} dx = 0. \tag{88}$$

Integrating by parts and noting that $-\Delta \frac{\partial v_\varepsilon}{\partial x_i} = \lambda f_\varepsilon'(v_\varepsilon)\frac{\partial v_\varepsilon}{\partial x_i}$ in Ω, we get

$$\begin{aligned}\int_{\partial\Omega} \frac{\partial v_\varepsilon}{\partial x_i}\frac{\partial}{\partial \nu}(w_\varepsilon - 8\pi G_\xi) d\sigma &= \int_\Omega \left(\frac{\partial v_\varepsilon}{\partial x_i}\Delta w_\varepsilon - w_\varepsilon \Delta \frac{\partial v_\varepsilon}{\partial x_i}\right) dx + 8\pi \frac{\partial v_\varepsilon}{\partial x_i}(\xi) \\ &\quad + 8\pi \int_\Omega G_\xi \Delta \frac{\partial v_\varepsilon}{\partial x_i} dx \\ &= \int_\Omega \frac{\partial v_\varepsilon}{\partial x_i} \underbrace{\left(\Delta w_\varepsilon + \lambda f_\varepsilon'(v_\varepsilon) w_\varepsilon - 8\pi\lambda f_\varepsilon'(v_\varepsilon) G_\xi\right)}_{=0 \text{ by } (12)} dx + 8\pi \frac{\partial v_\varepsilon}{\partial x_i}(\xi).\end{aligned}$$

This together with (87)-(88) gives

$$\frac{1}{2}\int_{\partial\Omega} \frac{\partial u_\varepsilon}{\partial x_i}\frac{\partial u_\varepsilon}{\partial \nu} d\sigma = 8\pi\alpha \frac{\partial v_\varepsilon}{\partial x_i}(\xi) + O(\alpha^2). \tag{89}$$

Finally, (84) follows by (85)-(86) and (89).

Step 4 *For $i = 1,2$, $j = 0,1,2$, we have*

$$\int_\Omega e^{U_\varepsilon} Z_{j,\varepsilon} \frac{\partial u_\varepsilon}{\partial x_i}\, dx = -\frac{\alpha}{\delta}\left(\frac{16}{3\mu}\pi\delta_{ij} + O(\alpha^2)\right). \tag{90}$$

For $i = 1,2$ and $j = 0,1,2$. Note that we have the identity

$$\frac{\partial}{\partial x_i} e^{U_\varepsilon} Z_{j,\varepsilon} = \frac{e^{U_\varepsilon}}{\delta\mu}\left(\delta_{ij}(Z_{0,\varepsilon}+1) - \delta_{j0} Z_{i,\varepsilon} - 3 Z_{i,\varepsilon} Z_{j,\varepsilon}\right).$$

Setting $\Psi_{ij} := \delta_{ij}(Z_0+1) - \delta_{j0}Z_i - 3Z_iZ_j$ and applying the divergence theorem, we find

$$\begin{aligned}\int_\Omega e^{U_\varepsilon} Z_{j,\varepsilon} \frac{\partial u_\varepsilon}{\partial x_i} dx &= -\int_\Omega u_\varepsilon \frac{d}{dx_i}\left(e^{U_\varepsilon} Z_{j,\varepsilon}\right) dx \\ &= -\frac{1}{\delta\mu}\int_\Omega u_\varepsilon e^{U_\varepsilon}\left(\delta_{ij}(Z_{0,\varepsilon}+1) - \delta_{j0}Z_{i,\varepsilon} - 3Z_{i,\varepsilon}Z_{j,\varepsilon}\right) dx \\ &= -\frac{1}{\delta\mu}\int_{\frac{\Omega-\xi}{\delta}} u_\varepsilon(\xi+\delta y) e^{\bar U}\Psi_{ij} dy \\ &= -\frac{1}{\delta\mu}\int_{B(0,\frac{\sigma}{\delta})} u_\varepsilon(\xi+\delta y) e^{\bar U}\Psi_{ij} dy + O(\beta\delta^2),\end{aligned}$$

where in the last equality we used that

$$u_\varepsilon = O(\beta) \qquad \text{and} \qquad e^{\bar U}\Psi_{ij} = O(|y|^{-5}), \tag{91}$$

for $|y| \geq \frac{\sigma}{\delta}$. By Lemma 2.6 we have

$$u_\varepsilon(\xi+\delta y) = \beta + \alpha \bar U(y) + O(\alpha^3) + O(\delta|y|), \tag{92}$$

for $y \in B(0, \frac{\sigma}{\delta})$. Using again (91), we get that

$$\int_{B(0,\frac{\sigma}{\delta})} e^{\bar{U}} \Psi_{ij} dy = \underbrace{\int_{\mathbb{R}^2} e^{\bar{U}} \Psi_{ij} dy}_{=0} + O(\delta^3).$$

Similarly, we have

$$\begin{aligned}\int_{B(0,\frac{\sigma}{\delta})} \overline{U} e^{\bar{U}} \Psi_{ij} dy &= \int_{\mathbb{R}^2} \overline{U} e^{\bar{U}} \Psi_{ij} dy + O(\beta^2 \delta^3) \\ &= \frac{16}{3} \pi \delta_{ij} + O(\beta^2 \delta^3),\end{aligned}$$

and (90) is proved.

Step 5 *Proof of* (81).

Let us set

$$a_{ij,\varepsilon} = a_{ij,\varepsilon}(\xi, \mu) := -\frac{3\mu}{16\pi} \frac{\delta}{\alpha} \int_{\Omega} e^{U_\varepsilon} Z_{j,\varepsilon} \frac{\partial u_\varepsilon}{\partial x_i} d\sigma.$$

According to Step 4, we have $a_{i0,\varepsilon} = O(\alpha^2)$ if $i = 1, 2$. Moreover the matrix $A = (a_{ij,\varepsilon})_{i,j \in \{1,2\}}$ is invertible and its inverse $A^{-1} = (a_\varepsilon^{ij})_{ij \in \{1,2\}}$ satisfies

$$a_\varepsilon^{ij} = \delta_{ij} + O(\alpha^2), \quad i, j = 1, 2.$$

Then (81) follows by (84), just setting

$$a_{i,\varepsilon} := \sum_{j=1}^{2} a_\varepsilon^{ij} a_{0j,\varepsilon}.$$

□

It is important to point out that (81) cannot be considered a precise uniform expansion of $\kappa_{i,\varepsilon}$. Indeed, (80) and the rough (but difficult to improve) estimate $a_{i,\varepsilon} = O(\alpha^2)$ yield only $\kappa_{0,\varepsilon} a_{i,\varepsilon} = O(\alpha^5)$. Since $\delta \ll \alpha^5$ it is not possible to identify the leading term in the RHS of (81). However, it is clear that the term involving $\frac{\partial v_\varepsilon}{\partial x_i}$ becomes dominant when $\kappa_{0,\varepsilon}$ vanishes. This is enough for our argument.

Proof of Theorem 1.3 completed

Proof. Let us consider the vector field

$$B_\varepsilon(\mu, \xi) = \left(\frac{1}{6\pi\alpha^3} \kappa_{0,\varepsilon}, \frac{2}{3\delta\mu} \left(\kappa_{1,\varepsilon} + \kappa_{0,\varepsilon} a_{1,\varepsilon} \right), \frac{2}{3\delta\mu} \left(\kappa_{2,\varepsilon} + \kappa_{0,\varepsilon} a_{2,\varepsilon} \right) \right).$$

By construction, for any $\varepsilon > 0$, B_ε depends continuously on μ and ξ. Moreover, thanks to (80), (81) and Lemma 2.2, we have

$$B_\varepsilon \to \bar{B}(\mu, \xi) := \left(2 - \log\left(\frac{8}{\mu^2} \right), \nabla u_0(\xi) \right)$$

as $\varepsilon \to 0$, uniformly for $\mu \in \mathcal{U}$ and $\xi \in B(\xi_0, \sigma)$. By assumption (A2), $\bar{B}$ has a C^0-stable zero at the point (μ_0, ξ_0), with $\mu_0 = \sqrt{8}e^{-1}$. Then, for ε small enough, there exist $\xi = \xi(\varepsilon) \to \xi_0$, $\mu = \mu(\varepsilon) \to \mu_0$ as $\varepsilon \to 0$ such that $B_\varepsilon(\mu(\varepsilon), \xi(\varepsilon)) = 0$. Clearly, this is equivalent to $\kappa_{i,\varepsilon,\mu(\varepsilon),\xi(\varepsilon)} = 0$, $i = 0, 1, 2$. That concludes the proof. $\square$

Appendix A. The proof of Lemma 2.4

Proof. The third equation in (28) allows to write δ as a function of $\alpha, \beta, \varepsilon, \mu, \xi$:

$$\log \frac{1}{\delta^2} = \frac{\beta}{2\alpha} + \frac{V_{\varepsilon,\alpha,\xi}(\xi)}{2\alpha} - \frac{c_{\mu,\xi}}{2},$$

and the second equation in (28) gives α as a function of $\beta, \varepsilon, \mu, \xi$:

$$\alpha = (2\beta + \beta^\varepsilon + \varepsilon\beta^\varepsilon)^{-1}.$$

Then, (after a simple computation) it is sufficient to prove that there exists $\beta = \beta(\varepsilon, \mu, \xi)$ such that

$$\begin{aligned} &\frac{1}{\beta}\left(\log\lambda + \frac{c_{\mu,\xi}}{2}\right) + 2\frac{\log\beta}{\beta} + \underbrace{\left(\frac{1}{2}\beta^\varepsilon - u_0(\xi)\right)}_{:=\theta_\varepsilon(\xi,\mu)} - (V_{\varepsilon,\alpha,\xi}(\xi) - u_0(\xi)) \\ &+ \frac{\log\left(2 + \beta^{\varepsilon-1} + \varepsilon\beta^{\varepsilon-1}\right)}{\beta} - \frac{1}{2}\varepsilon\beta^\varepsilon - \frac{1}{2}V_{\varepsilon,\alpha,\xi}(\xi)\left(\beta^{\varepsilon-1} + \varepsilon\beta^{\varepsilon-1}\right) = 0. \end{aligned} \tag{93}$$

Now, we choose $\beta^\varepsilon := 2u_0(\xi) + \theta_\varepsilon(\xi, \mu)$ with $\|\theta_\varepsilon\|_{C^0(\overline{B(\xi_0,\sigma)}\times\overline{\mathcal{U}})}$ so small that

$$2u_0(\xi) + \theta_\varepsilon(\xi, \mu) \geq \eta > 1 \text{ in } \overline{B(\xi_0, \sigma)} \times \overline{\mathcal{U}}.$$

This is possible because of (22). With this choice we have $\frac{1}{\beta} = O\left(\eta^{-\frac{1}{\varepsilon}}\right)$. It is easy to show that (93) has a solution θ_ε because of a simple fixed point argument. Indeed (93) rewrites as $\theta_\varepsilon = \mathcal{T}(\theta_\varepsilon)$ where $\mathcal{T}$ is a contraction mapping on the ball

$$\left\{\theta_\varepsilon \in C^0(\overline{B(\xi_0, \sigma)} \times \overline{\mathcal{U}}) \ : \|\theta_\varepsilon\|_{C^0(\overline{B(\xi_0,\sigma)}\times\overline{\mathcal{U}})} \leq \rho_\varepsilon\right\},$$

where $\rho_\varepsilon := \rho \min\left\{\frac{1}{\varepsilon}\eta^{-\frac{1}{\varepsilon}}, \|v_\varepsilon - u_0\|_{C^0(\overline{\Omega})}\right\}$ for some $\rho > 0$ and $\rho_\varepsilon \to 0$ as $\varepsilon \to 0$. Here we use the expression of $V_{\varepsilon,\alpha,\xi}(\xi)$ in (11) and (ii) of Lemma 2.2. $\square$

Appendix B. A Stampacchia type estimate

In this section we prove domain-independent estimates for solutions of the Poisson equation $-\Delta u = f$, under Dirichlet boundary conditions, with $f \in L^p(\Omega)$ and p approaching 1. Our strategy is based on the Stampacchia method.

Lemma B.1 ([28], Lemma 4.1) *Let $\psi : \mathbb{R}^+ \mapsto \mathbb{R}^+$ be a nonincreasing function. Assume that there exist $M > 0, \gamma > 0$, $\delta > 1$ such that*

$$\psi(h) \leq \frac{M\psi(k)^\delta}{(h-k)^\gamma} \qquad \forall\, h > k > 0.$$

Then $\psi(d) = 0$, where $d = M^{\frac{1}{\gamma}}\psi(0)^{\frac{\delta-1}{\gamma}} 2^{\frac{\delta}{\delta-1}}$.

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded smooth domain. For any $q > 1$, let $S_q(\Omega)$ be the Sobolev's constant for the embedding of $H_0^1(\Omega)$ in $L^q(\Omega)$, namely

$$S_q(\Omega) = \inf_{u \in H_0^1(\Omega)} \frac{\|u\|_{H_0^1(\Omega)}}{\|u\|_{L^q(\Omega)}}.$$

It is known that $0 < S_q(\Omega) < +\infty$ and that (see [26] Lemma 2.2)

$$\lim_{q \to +\infty} \sqrt{q} S_q(\Omega) = \sqrt{8\pi e}.$$

Theorem B.2 *Let Ω be a bounded smooth domain. For $p > 1$, $f \in L^p(\Omega)$, the unique solution $u \in H_0^1(\Omega)$ of the equation $-\Delta u = f$ satisfies*

$$\|u\|_{L^\infty(\Omega)} \leq 4 S_{\frac{3p+1}{p-1}}(\Omega)^{-2} \|f\|_{L^p} |\Omega|^{\frac{p^2-1}{3p^2+p}}.$$

Proof. We want to apply the previous lemma to the function

$$\psi(k) := |A_k|, \quad A_k := \{x \in \Omega \ : \ |u(x)| > k\}.$$

For any $k > 0$, let us consider the function

$$v_k(x) := \begin{cases} 0 & |u(x)| \leq k, \\ u(x) - k & u(x) > k, \\ -u(x) - k & u(x) < -k. \end{cases}$$

Note that $v_k \in H_0^1(\Omega)$ and $|\nabla v_k| = |\nabla u| \chi_{A_k}$. If we test the equation against v_k we get

$$\int_\Omega \nabla u \cdot \nabla v_k \, dx = \int_\Omega f v_k dx. \tag{94}$$

For any $q \in (1, p)$ Hölder's inequality gives

$$\int_\Omega f v_k \, dx = \int_{A_k} f v_k \, dx \leq \|f\|_{L^q(A_k)} \|v_k\|_{L^{\frac{q}{q-1}}(A_k)} \leq \|f\|_{L^p} |A_k|^{\frac{p-q}{pq}} \|v_k\|_{L^{\frac{q}{q-1}}(A_k)}. \tag{95}$$

By Sobolev's inequality, we have that

$$\int_\Omega \nabla u \cdot \nabla v_k \, dx = \int_{A_k} |\nabla v_k|^2 dx \geq S_{\frac{q}{q-1}}(\Omega)^2 \|v_k\|^2_{L^{\frac{q}{q-1}}}. \tag{96}$$

By (94)-(96), we have

$$\|v_k\|_{L^{\frac{q}{q-1}}} \leq S_{\frac{q}{q-1}}(\Omega)^{-2}\|f\|_{L^p}|A_k|^{\frac{p-q}{pq}}.$$

Now, for any $h > k$, we have that $A_h \subseteq A_k$ and $v_k \geq (h-k)$ in A_h, hence

$$\int_\Omega |v_k|^{\frac{q}{q-1}} dx = \int_{A_k} v_k^{\frac{q}{q-1}} dx \geq \int_{A_h} v_k^{\frac{q}{q-1}} dx \geq (h-k)^{\frac{q}{q-1}}|A_h|.$$

In conlcusion, we find

$$(h-k)|A_h|^{\frac{q-1}{q}} \leq S_{\frac{q}{q-1}}(\Omega)^{-2}\|f\|_{L^p}|A_k|^{\frac{p-q}{pq}},$$

or, equivalently,

$$\psi(h) \leq \frac{S_{\frac{q}{q-1}}(\Omega)^{-\frac{2q}{q-1}}\|f\|_{L^p}^{\frac{q}{q-1}}\psi(k)^{\frac{p-q}{p(q-1)}}}{(h-k)^{\frac{q}{q-1}}}.$$

Then, we are in position to apply Lemma B.1 to ψ with $M = S_{\frac{q}{q-1}}(\Omega)^{-\frac{2q}{q-1}}\|f\|_{L^p}^{\frac{q}{q-1}}$, $\gamma = \frac{q}{q-1}$, and $\delta = \frac{p-q}{p(q-1)}$. For this, we need to impose that $\delta = \frac{p-q}{p(q-1)}$, that is $q < \frac{2p}{p+1}$. Note that $1 < \frac{2p}{p+1} < p$. According to Stampacchia's Lemma, we have

$$\psi(d) = 0 \qquad \text{where} \qquad d = M^{\frac{1}{\gamma}}\psi(0)^{\frac{\delta-1}{\gamma}}2^{\frac{\delta}{\delta-1}} = S^2_{\frac{q}{q-1}}\|f\|_{L^p}|\Omega|^{\frac{2p-q(p+1)}{pq}}2^{\frac{p-q}{2p-q(p+1)}}.$$

This implies that

$$\|u\|_{L^\infty(\Omega)} \leq S_{\frac{q}{q-1}}(\Omega)^{-2}\|f\|_{L^p}|\Omega|^{\frac{2p-q(p+1)}{pq}}2^{\frac{p-q}{2p-q(p+1)}}.$$

This is true for any choice of $q \in (1, \frac{2p}{p+1})$. If we take for example p the midpoint of $(1, \frac{2p}{p+1})$, that is $q = \frac{1}{2} + \frac{p}{p+1} = \frac{3p+1}{2(p+1)}$, then we find that

$$\frac{q}{q-1} = \frac{3p+1}{p-1}, \quad \frac{2p-q(p+1)}{pq} = \frac{p^2-1}{3p^2+p}, \quad \frac{p-q}{2p-q(p+1)} = \frac{2p+1}{p+1} \leq 2,$$

and we get the conclusion. □

Corollary B.3 *Given $K > 0$ and $p > 1$, there exists a constant $C = C(K,p)$ such that, for any domain $\Omega \subseteq \mathbb{R}^2$ with $|\Omega| \leq K$ and any $f \in L^p(\Omega)$ the unique solution $u \in H_0^1(\Omega)$ of $-\Delta u = f$ satisfies*

$$\|u\|_{L^\infty(\Omega)} \leq C\|f\|_{L^p(\Omega)}.$$

Corollary B.4 *Given $K > 0$, there exist $p_0 = p_0(K)$ and $C = C(K)$ such that, for any $1 < p < p_0$, any domain $\Omega \subseteq \mathbb{R}^2$ with $|\Omega| \leq K$, and any $f \in L^p(\Omega)$, the unique solution $u \in H_0^1(\Omega)$ of $-\Delta u = f$ satisfies*

$$\|u\|_{L^\infty(\Omega)} \leq \frac{C}{p-1}\|f\|_{L^p(\Omega)}.$$

References

[1] Adimurthi, *Positive solutions of the semilinear Dirichlet problem with critical growth in the unit disc in* $\mathbb{R}^2$, Proc. Indian Acad. Sci., Mathematical Sciences 99 (1989), 49-–73, https://doi.org/10.1007/BF02874647.

[2] Adimurthi, *Existence of positive solutions of the semilinear Dirichlet problem with critical growth for the n-Laplacian*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 17 (1990), 393–413, http://www.numdam.org/item/ASNSP_1990_4_17_3_393_0.

[3] Adimurthi, O. Druet, *Blow-up analysis in dimension 2 and a sharp form of Trudinger-Moser inequality*, Comm. in PDE. 29 (2004), 295–322, https://doi.org/10.1081/PDE-120028854.

[4] Adimurthi, A. Karthik, J. Giacomoni, *Uniqueness of positive solutions of a n-Laplace equation in a ball in Rn with exponential nonlinearity*, J. Differential Equations 260 (2016), no. 11, 7739—7799,https://doi.org/10.1016/j.jde.2016.02.002.

[5] Adimurthi, S. Prashanth, *Failure of Palais-Smale condition and blow-up analysis for the critical exponent problem in* $\mathbb{R}^2$, Proc. Indian Acad. Sci. Math. Sci. 107 (1997), no.3, 283—317, https://doi.org/10.1007/BF02867260.

[6] Adimurthi, S. Prashanth, *Critical exponent problem in* $\mathbb{R}^2$*-border-line between existence and non-existence of positive solutions for Dirichlet problem*, Adv. Differential Equations 5 (2000), no. 1-3, 67–95, https://projecteuclid.org:443/euclid.ade/1356651379.

[7] Adimurthi, M. Struwe, *Global compactness properties of semilinear elliptic equations with critical exponential growth*, J. Funct. Anal. 175 (2000), 125-–167, https://doi.org/10.1006/jfan.2000.3602.

[8] Adimurthi, S.L. Yadava, *Multiplicity results for semilinear elliptic equations in a bounded domain of* $\mathbb{R}^2$ *involving critical exponents*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 17 (1990) 481—504, http://www.numdam.org/item/ASNSP_1990_4_17_4_481_0.

[9] Adimurthi, S.L. Yadava, *Nonexistence of Nodal Solutions of Elliptic Equations with Critical Growth in* $\mathbb{R}^2$, Trans. Amer. Math. Soc. 332 (1992), 449—458, https://doi.org/10.2307/2154041.

[10] S. Baraket, F. Pacard, *Construction of singular limits for a semilinear elliptic equation in dimension 2,* Calc. Var. Partial Differential Equations 6 (1998), no. 1, 1–38, https://doi.org/10.1007/s005260050080.

[11] D. Bartolucci, A. Pistoia, *Existence and qualitative properties of concentrating solutions for the sinh-Poisson equation*, IMA J. Appl. Math. 72 (2007), no 6, 706–729, https://doi.org/10.1093/imamat/hxm012.

[12] W. Chen, C. Li, *Qualitative properties of solutions to some nonlinear elliptic equations in* $\mathbb{R}^2$, Duke Math. J. 71 (1993), 427–439, https://doi.org/10.1215/S0012-7094-93-07117-7.

[13] M. del Pino, M. Kowalczyk, M. Musso, *Singular limits in Liouville-type equations*, Calc. Var. Partial Differential Equations 24 (2005), no. 1, 47–81, https://doi.org/10.1007/s00526-004-0314-5.

[14] M. Del Pino, M. Musso, B. Ruf, *New solutions for Trudinger–Moser critical equations in* $\mathbb{R}^2$, J. Funct. Anal. 258 (2010), no.2, 421—457, https://doi.org/10.1016/j.jfa.2009.06.018.

[15] O. Druet, *Multibumps analysis in dimension 2: quantification of blow-up levels*, Duke Math. J. 132 (2006), no. 2, 217-–269, https://doi.org/10.1215/S0012-7094-06-13222-2.

[16] O. Druet, A. Malchiodi, L. Martinazzi, P.D. Thizy, *Multi-bumps analysis for Trudinger–Moser nonlinearities II-existence of solutions of high energies*, in preparation.

[17] O. Druet, P.-D. Thizy, *Multi-bumps analysis for Trudinger–Moser nonlinearities I-quantification and location of concen-tration points*, J. Eur. Math. Soc. (2018), in press, arXiv :1710 .08811.

[18] P. Esposito, M. Grossi, A. Pistoia, *On the existence of blowing-up solutions for a mean field equation*, Ann. Inst. H. Poincaré Anal. Non Linéaire 22 (2005), no. 2, 227–257, https://doi.org/10.1016/j.anihpc.2004.12.001.

[19] M. Grossi, D. Naimen, *Blow-up analysis for nodal radial solutions in Moser-Trudinger critical equations in* $\mathbb{R}^2$, to appear in Annali della Scuola Normale Superiore di Pisa, https://doi.org/10.2422/2036-2145.201707_006.

[20] M. Grossi, D. Naimen, *Nondegeneracy of positive solutions to Moser-Trudinger problems in symmetric domains*, in preparation.

[21] J. Liouville, *Sur l'equation aux differences partielles* $\frac{\partial^2 \log \lambda}{\partial z \partial \bar{z}} \pm \frac{\lambda}{2a^2} = 0$, J. Math. Pures Appl. 36 (1853). 71–72,

[22] G. Mancini, P.-D. Thizy, *Glueing a peak to a non-zero limiting profile for a critical Moser–Trudinger equation*, J. Math. Anal. Appl. (2019), url: https://doi.org/10.1016/j.jmaa.2018.11.084.

[23] A. M. Micheletti, A. Pistoia, *Generic properties of singularly perturbed nonlinear elliptic problems on Riemannian manifold*, Adv. Nonlinear Stud. 9 (2009), no. 4, 803–813, https://doi.org/10.1515/ans-2016-6010.

[24] J.K. Moser, *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J. 20 (1970/71), 1077-1092, https://doi.org/10.1512/iumj.1971.20.20101.

[25] S. I. Pohozaev, *The Sobolev embedding in the case $pl = n$*, Proc. of the Technical Scientific Conference on Advances of Scientific Research 1964-1965, Mathematics Section, (Moskov. Energet. Inst., Moscow), (1965), 158-170.

[26] X. Ren, J. Wei, *Counting Peaks of Solutions to Some Quasilinear Elliptic Equations with Large Exponents*, J. Differential Equations 117 (1995), 28–55, https://doi.org/10.1006/jdeq.1995.1047.

[27] O. Rey, *The role of the Green's function in a nonlinear elliptic equation involving the critical Sobolev exponent*, J. Funct. Anal. 89 (1990), no 1, 1–52, https://doi.org/10.1016/0022-1236(90)90002-3.

[28] G. Stampacchia, *Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus,* Ann. Inst. Fourier (Grenoble) 15 (1965), 189–258, http://www.numdam.org/item?id=AIF_1965__15_1_189_0.

[29] N.S. Trudinger, *On imbeddings into Orlicz spaces and some applications*, J. Math. Mech. 17 (1967), 473–483, https://doi.org/10.1512/iumj.1968.17.17028.