MINIMAL VARIETIES OF PI-SUPERALGEBRAS WITH GRADED INVOLUTION

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Abstract. In the present paper it is proved that a variety of associative PI-superalgebras with graded involution of finite basic rank over a field of characteristic zero is minimal of fixed $*$-graded exponent if, and only if, it is generated by a subalgebra of an upper block triangular matrix algebra equipped with a suitable elementary $\mathbb{Z}_2$-grading and graded involution.

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1. Introduction

Let F be a field of characteristic zero. One of the most interesting problems in PI-Theory is that of finding numerical invariants allowing to classify varieties. Along this line, much research is focused on the sequence $\{c_n(A)\}_{n\geq 1}$ of the *codimensions* of an associative F-algebra A, introduced by Regev in the seminal paper [18], whose n-th term is the dimension of the space of multilinear polynomials in n variables in the corresponding relatively free algebra of countable rank. Indeed it gives in some sense a quantitative measure of all the polynomial identities satisfied by A, since that in characteristic zero they are completely determined by the multilinear ones. The celebrated result of [18] states that when A satisfies a non-zero polynomial identity (in the sequel we shall refer to these algebras as PI-algebras) $\{c_n(A)\}_{n\geq 1}$ is exponentially bounded. Later a key contribution of Giambruno and Zaicev ([11] and [12]) answered in a positive way a conjecture of Amitsur establishing that

$$\exp(A) := \lim_{m\to +\infty} \sqrt[m]{c_m(A)}$$

exists and is a non-negative integer, which is called the *exponent* of A.

In the recent decades the same approach has been applied to study the corresponding polynomial identities of different classes of algebras, such as group graded algebras and algebras with involution. Apart from their own interesting features, the main motivations for this type of work come from the significant information on quite general questions that the additional structure and related objects may provide. This is the case, for instance,

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for the solution of the Specht problem due to Kemer (see [17]) in which $\mathbb{Z}_2$-gradings play a fundamental role.

In the present paper we deal with $*$-*superalgebras*, namely superalgebras endowed with a graded involution, and the growth of their $*$-graded codimensions, as extensively made by several authors in the last few years (see, for instance, [7], [8], [10], [16] and [19]). Since a $*$-superalgebra can be viewed as an algebra with generalized FG-action, where $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ acts on it by automorphisms and antiautomorphisms, the existence of the $*$-graded exponent has been confirmed in [15] in the finite-dimensional case. Obviously this statement applies to commutative superalgebras (which can be seen as $*$-superalgebras with the trivial involution) and algebras with involutions (which are nothing but $*$-superalgebras endowed with the trivial grading), but it was independently proved for arbitrary finite group graded PI-algebras in [1] and for $*$-PI algebras with involution in [9] without further restrictions. In particular, the last mentioned paper is based on the Representability Theorem as presented in [2], where a crucial role is just played by graded involutions of a superalgebra.

These results are the most striking culminating points of quantitative investigations of the corresponding polynomial identities satisfied by algebras in these classes: in fact, the existence of the corresponding exponent allows to classify varieties on an integral scale whose steps are the *minimal varieties* of given exponent d, namely those varieties of exponent d such that every proper subvariety has exponent strictly less than d.

In this direction, a deep contribution of Giambruno and Zaicev ([13] and [14]) provided the classification in the ordinary case. Di Vincenzo and Spinelli completely described minimal varieties of finite-dimensional algebras with involution in [5] and, very recently, jointly with da Silva characterized minimal varieties of $\mathbb{Z}_p$-graded PI-algebras in the affine case ([4]).

Our goal here is to classify minimal varieties of PI $*$-superalgebras of finite basic rank (that is, generated by a finitely generated $*$-superalgebra satisfying an ordinary polynomial identity). In more detail, given an m-tuple of simple $*$-superalgebras $(A_1, \dots, A_m)$ we construct a subalgebra of an upper block triangular matrix algebra $UT^*_{\mathbb{Z}_2}(A_1, \dots, A_m)$ equipped with a suitable elementary $\mathbb{Z}_2$-grading and graded involution, in which each algebra A_i is embedded as a $*$-superalgebra. The main result we prove is that a variety of PI $*$-superalgebras of finite basic rank is minimal of $*$-graded exponent d if, and only if, it is generated by a $*$-superalgebra $UT^*_{\mathbb{Z}_2}(A_1, \dots, A_m)$ satisfying $\dim_F(A_1 \oplus \cdots \oplus A_m) = d$.

2. Preliminaries, $*$-graded exponent and announcement of the main result

Throughout the rest of the paper, unless otherwise stated, F is a field of characteristic zero and all the algebras are assumed to be associative and to have the same ground field F. For any pair of positive integers s and t the symbol $M_{s\times t}$ means the space of all matrices with s rows and t columns over F and set $M_s := M_{s\times s}$.

An algebra A is a $\mathbb{Z}_2$-*graded algebra* or a *superalgebra* if it has a vector space decomposition $A = A_{(0)} \oplus A_{(1)}$ such that $A_{(i)}A_{(j)} \subseteq A_{(i+j)}$. The elements of $A_{(0)}$ are called *homogeneous of degree* 0 and those of $A_{(1)}$ *homogeneous of degree* 1. An element w of A is *homogeneous* if it is homogeneous of degree 0 or 1 (and denote its degree by $|w|_A$), whereas a subalgebra or an ideal $V \subseteq A$ is *homogeneous* if $V = (V \cap A_{(0)}) \oplus (V \cap A_{(1)})$.

An *involution* $*$ on an algebra A is an antiautomorphism of order at most 2 of A. Write $A^+ := \{a \,|\, a \in A,\, a^* = a\}$ and $A^- := \{a \,|\, a \in A,\, a^* = -a\}$ for the subspaces of *symmetric* and *skew* elements of A, respectively. If A is a superalgebra, the involution $*$ is said to be *graded* if it preserves its homogeneous components. If this happens, we refer to A as a $*$-*superalgebra*. Clearly, A is a $*$-superalgebra if, and only if, the subspaces A^+ and A^- are homogeneous and in such an event, as $\operatorname{char} F = 0$, it can be written as

$$A = A^+_{(0)} \oplus A^-_{(0)} \oplus A^+_{(1)} \oplus A^-_{(1)}, \tag{1}$$

where $A^+_{(i)} := \{a \,|\, a \in A_{(i)},\, a^* = a\}$ and $A^-_{(i)} := \{a \,|\, a \in A_{(i)},\, a^* = -a\}$ for $i \in \{0, 1\}$.

Let $F\langle Y \cup Z\rangle$ be the free algebra on the disjoint countable sets of variables $Y := \{y_1, y_2, \ldots\}$ and $Z := \{z_1, z_2, \ldots\}$. It has a natural superalgebra structure if we require that the variables from Y have degree 0 and those from Z have degree 1 (and then extend this grading to the monomials on $Y \cup Z$). This algebra is said to be the *free superalgebra* over F. Let us consider the free algebra with involution $F\langle Y \cup Z, *\rangle$ which is a $*$-superalgebra if we assume also that for each variable $y \in Y$ ($z \in Z$, respectively) y^* is homogeneous of degree 0 (z^* is homogeneous of degree 1, respectively), called the *free* $*$-*superalgebra* over F. According to (1), sometimes it is useful to regard $F\langle Y \cup Z, *\rangle$ as generated by even and odd symmetric variables and by even and odd skew variables, namely

$$F\langle Y \cup Z, *\rangle = F\langle y_1^+, y_1^-, z_1^+, z_1^-, y_2^+, y_2^-, z_2^+, z_2^-, \ldots\rangle,$$

where $y_i^+ = y_i + y_i^*$, $y_i^- = y_i - y_i^*$, $z_i^+ = z_i + z_i^*$ and $z_i^- = z_i - z_i^*$ for every $i \geq 1$.

An element $f(y_1, y_1^*, \ldots, y_m, y_m^*, z_1, z_1^*, \ldots, z_n, z_n^*)$ of $F\langle Y \cup Z, *\rangle$ is a $*$-*graded polynomial identity* for a $*$-superalgebra $A = A_{(0)} \oplus A_{(1)}$ if $f(a_1, a_1^*, \ldots, a_m, a_m^*, b_1, b_1^* \ldots, b_n, b_n^*) = 0_A$ for every $a_1, \ldots, a_m \in A_{(0)}$ and $b_1, \ldots, b_n \in A_{(1)}$. We use the symbol $\mathrm{Id}^*_{\mathbb{Z}_2}(A)$ to indicate the set of all the $*$-graded polynomial identities satisfied by A, which is easily seen to be a $T^*_{\mathbb{Z}_2}$-*ideal* of $F\langle Y \cup Z, *\rangle$, namely a two-sided ideal of the free $*$-superalgebra stable under every graded endomorphism of $F\langle Y \cup Z\rangle$ commuting with $*$. Since F has characteristic zero, it is easily seen that $\mathrm{Id}^*_{\mathbb{Z}_2}(A)$ is completely determined by the multilinear polynomials it contains. Therefore, denoted by $P_n^{(\mathbb{Z}_2,*)}$ the space of multilinear elements of degree n of $F\langle Y \cup Z, *\rangle$ in the variables $y_1, y_1^*, \ldots, y_n, y_n^*, z_1, z_1^*, \ldots, z_n, z_n^*$, the study of $\mathrm{Id}^*_{\mathbb{Z}_2}(A)$ is equivalent to that of $P_n^{(\mathbb{Z}_2,*)} \cap \mathrm{Id}^*_{\mathbb{Z}_2}(A)$ for all $n \geq 1$. As made by Regev [18] in the ordinary case, one defines the n-th $*$-*graded codimension*, $c_n^{(\mathbb{Z}_2,*)}(A)$, of the $*$-superalgebra

A as

$$c_n^{(\mathbb{Z}_2,*)}(A) := \dim_F \frac{P_n^{(\mathbb{Z}_2,*)}}{P_n^{(\mathbb{Z}_2,*)} \cap \mathrm{Id}^*_{\mathbb{Z}_2}(A)}.$$

If A satisfies an ordinary polynomial identity, the sequence $\{c_n^{(\mathbb{Z}_2,*)}(A)\}_{n\geq 1}$ is exponentially bounded (see, for instance, Lemma 3.1 of [10]). Under the extra assumption that A is finite-dimensional, Gordienko in [15] captured its exponential growth proving the analogue of the result of Giambruno and Zaicev for PI-algebras announced in the Introduction, namely that

$$\exp^*_{\mathbb{Z}_2}(A) := \lim_{m\to+\infty} \sqrt[m]{c_m^{(\mathbb{Z}_2,*)}(A)}$$

exists and is a non-negative integer, which is called the *$*$-graded exponent* of the $*$-superalgebra A. He actually provides an explicit formula to compute the $*$-graded exponent which is a natural generalization of that for the ordinary PI-exponent. In more detail, since F has characteristic zero, $\mathrm{Id}^*_{\mathbb{Z}_2}(A) = \mathrm{Id}^*_{\mathbb{Z}_2}(A \otimes_F L)$ (in $F\langle Y \cup Z, *\rangle$) for any field extension L of F. Consequently also the $*$-graded codimensions of A do not change upon extension of the base field. Hence we can assume that F is algebraically closed. By the generalization of the Wedderburn-Malcev Theorem (see Theorem 7.3 of [10]) we can write

$$A = (A_1 \oplus \ldots \oplus A_m) + J(A), \tag{2}$$

where $A_1, \ldots, A_m$ are simple $*$-superalgebras and $J(A)$ is the Jacobson radical of A, which is a $*$-graded ideal of A (we recall that a *$*$-graded ideal* of A is just an ideal of A homogeneous in the $\mathbb{Z}_2$-grading and invariant under $*$, whereas a $*$-superalgebra is said to be *simple* if the multiplication is non-trivial and it has no non-trivial $*$-graded ideals). Consider all possible products of the type

$$B_1 J(A) B_2 J(A) \cdots J(A) B_r \neq 0_A \tag{3}$$

where $B_1, \ldots, B_r$ are distinct subalgebras taken from the set $\{A_1, \ldots, A_m\}$. Then

$$\exp^*_{\mathbb{Z}_2}(A) = \max \dim_F (B_1 \oplus \ldots \oplus B_r),$$

the maximal dimension of a subalgebra $B_1 \oplus \ldots \oplus B_r$ satisfying (3).

By virtue of Theorem 5.3 of [8], Gordienko's result on the existence of the $*$-graded exponent can be actually extended to any finitely generated PI $*$-superalgebra (since it satisfies the same $*$-graded polynomial identities of a finite-dimensional $*$-superalgebra).

As it is of interest to classify $*$-superalgebras up to $*$-graded PI-equivalence, it is more convenient to use the language of varieties. Given a $T^*_{\mathbb{Z}_2}$-ideal I of $F\langle Y \cup Z, *\rangle$, the *variety of $*$-superalgebras* $\mathcal{V}^*_{\mathbb{Z}_2}$ associated to I is the class of all the $*$-superalgebras A such that I is contained in $\mathrm{Id}^*_{\mathbb{Z}_2}(A)$. The $T^*_{\mathbb{Z}_2}$-ideal I is denoted by $\mathrm{Id}^*_{\mathbb{Z}_2}(\mathcal{V}^*_{\mathbb{Z}_2})$. The variety $\mathcal{V}^*_{\mathbb{Z}_2}$ is generated by the $*$-superalgebra A if $\mathrm{Id}^*_{\mathbb{Z}_2}(\mathcal{V}^*_{\mathbb{Z}_2}) = \mathrm{Id}^*_{\mathbb{Z}_2}(A)$, and in this case we write $\mathcal{V}^*_{\mathbb{Z}_2} = \mathcal{V}^*_{\mathbb{Z}_2}(A)$ and set $\exp^*_{\mathbb{Z}_2}(\mathcal{V}^*_{\mathbb{Z}_2}) := \exp^*_{\mathbb{Z}_2}(A)$, the *$*$-graded exponent* of the variety $\mathcal{V}^*_{\mathbb{Z}_2}$, if $\exp^*_{\mathbb{Z}_2}(A)$ does exist.

In the present paper we consider varieties of PI $*$-superalgebras of *finite basic rank*, i.e. generated by a finitely generated $*$-superalgebra satisfying an ordinary polynomial identity. In this case, according of Theorem 5.2 of [8], any subvariety of such a variety has finite basic rank as well. Our aim is to characterize those which are minimal with respect to their $*$-graded exponent. We recall the definition.

Definition 2.1. *A variety $\mathcal{V}^*_{\mathbb{Z}_2}$ of PI $*$-superalgebras of finite basic rank is said to be* minimal *of $*$-graded exponent d if $\exp^*_{\mathbb{Z}_2}(\mathcal{V}^*_{\mathbb{Z}_2}) = d$ and $\exp^*_{\mathbb{Z}_2}(\mathcal{U}^*_{\mathbb{Z}_2}) < d$ for every proper subvariety $\mathcal{U}^*_{\mathbb{Z}_2}$ of $\mathcal{V}^*_{\mathbb{Z}_2}$.*

To this end, as previously observed, since the $*$-graded exponent does not change upon extension of the base field, for the proofs of our main results we can assume that F is algebraically closed. As a first step, for any sequence $(A_1, \ldots, A_m)$ of finite-dimensional simple $*$-superalgebras, in Section 3 we construct a subalgebra, $UT^*_{\mathbb{Z}_2}(A_1, \ldots, A_m)$, of a suitable upper block triangular matrix algebra endowed with an elementary grading (depending on the nature of the gradings of the A_i's) preserved by the flip along the secondary diagonal, in which, roughly speaking, each $*$-superalgebra A_i appears at the i-th block of the main diagonal. We achieve our goal stating the following

Theorem 2.2. *A variety of PI $*$-superalgebras of finite basic rank is minimal of $*$-graded exponent d if, and only if, it is generated by a $*$-superalgebra $UT^*_{\mathbb{Z}_2}(A_1, \ldots, A_m)$ satisfying $\dim_F(A_1 \oplus \cdots \oplus A_m) = d$.*

Let us mention here the main steps of the proof. To establish the necessary conditions, we show that for any finite-dimensional $*$-superalgebra A there exists a so called *triangular* $*$-superalgebra B with the same $*$-graded exponent such that $\mathrm{Id}^*_{\mathbb{Z}_2}(A) \subseteq \mathrm{Id}^*_{\mathbb{Z}_2}(B)$. In particular, the $*$-superalgebra B, which is finite-dimensional as well, is defined by means of a sequence $(A_1, \ldots, A_m)$ of distinct subalgebras taken from the set of simple summands of the Wedderburn-Malcev decomposition of A satisfying $A_1J(A)A_2J(A)\cdots J(A)A_m \neq 0_A$ and $\exp^*_{\mathbb{Z}_2}(A) = \dim_F(A_1 \oplus \ldots \oplus A_m)$, and their interaction with some homogeneous radical elements, $w_{12}, \ldots, w_{m-1,m}$, whose product is non-zero. At this point, one has that that $UT^*_{\mathbb{Z}_2}(A_1, \ldots, A_m)$, endowed with a grading depending on the degree of the $w_{i,i+1}$'s, belongs to $\mathcal{V}^*_{\mathbb{Z}_2}(A)$, which is sufficient to conclude if we assume that $\mathcal{V}^*_{\mathbb{Z}_2}(A)$ is minimal, since $\exp^*_{\mathbb{Z}_2}(UT^*_{\mathbb{Z}_2}(A_1, \ldots, A_m)) = \exp^*_{\mathbb{Z}_2}(A)$. This is the content of Section 4.

The rest of the paper is devoted to prove their sufficiency: this is reduced to compare, in some sense, the structure of the $*$-superalgebras $A := UT^*_{\mathbb{Z}_2}(A_1, \ldots, A_m)$ and $B := UT^*_{\mathbb{Z}_2}(B_1, \ldots, B_n)$ satisfying $\exp^*_{\mathbb{Z}_2}(A) = \exp^*_{\mathbb{Z}_2}(B)$ and $\mathrm{Id}^*_{\mathbb{Z}_2}(A) \subseteq \mathrm{Id}^*_{\mathbb{Z}_2}(B)$. In more detail, it turns out that, under these assumptions, A must be isomorphic to B as a $*$-superalgebra. This is done in Section 6. In this way we also provide a contribution on the isomorphism question in PI-Theory, namely to decide whether having the same set of polynomial identities guarantees the isomorphism of algebras, recently investigated by several authors in different settings. As a main tool we use here a family of $*$-graded polynomials constructed in Section 5, the *Kemer* *polynomials* for $UT^*_{\mathbb{Z}_2}(A_1,\ldots,A_m)$ (heavily depending again on the interaction of the A_i's with the elements of the Jacobson radical), which are not in $\mathrm{Id}^*_{\mathbb{Z}_2}(UT^*_{\mathbb{Z}_2}(A_1,\ldots,A_m))$.

3. The $*$-superalgebra $UT^*_{\mathbb{Z}_2}(A_1,\ldots,A_m)$

Throughout this section assume that F is algebraically closed. We start the construction of the $*$-superalgebra $UT^*_{\mathbb{Z}_2}(A_1,\ldots,A_m)$. To this aim, some preliminary considerations are in order.

In the sequel, for every $q \le s$, set $[q,s] := \{q, q+1, \ldots, s\}$. Given positive integers $m_1,\ldots,m_n$, let $UT(m_1,\ldots,m_n)$ be the upper block triangular matrix algebra of size $m_1,\ldots,m_n$ over F. In particular,

$$UT(m_1,\ldots,m_n) := \{(a_{ij})_{i,j\in[1,n]} \mid a_{ij} \in M_{m_i\times m_j} \text{ if } 1 \le i \le j \le n \text{ and } a_{ij} = 0_{M_{m_i\times m_j}} \text{ otherwise}\}.$$

Set $U := UT(m_1,\ldots,m_n)$, for all $1 \le r \le t \le n$, let us define

$$U_{r,t} := \{(a_{ij}) \in U \mid a_{ij} = 0_{M_{m_i\times m_j}} \ \forall\,(i,j) \neq (r,t)\}.$$

A $\mathbb{Z}_2$-grading on the complete matrix algebra M_m is called *elementary* if there exists an m-tuple $(g_1,\ldots,g_m) \in \mathbb{Z}_2^m$ such that the matrix units E_{ij} of M_m are homogeneous and $E_{ij} \in (M_m)_{(g)}$ if, and only if, $g = g_j - g_i$. In an equivalent manner, we can define a map $\alpha : [1,m] \longrightarrow \mathbb{Z}_2$ inducing a grading on M_m setting the degree of E_{ij} equal to $\alpha(j)-\alpha(i)$. Obviously, the algebra of upper block triangular matrices admits an elementary grading: in fact, the embedding of such an algebra into a full matrix algebra with an elementary grading makes it a homogeneous subalgebra. To denote that M_m (as well as one of its homogeneous subalgebras) is equipped with an elementary grading induced by α, we write (M_m,α). Analogously, if $*$ is an involution on M_m we use $(M_m,*)$, and $(M_m,*,\alpha)$ if $*$ is a graded involution of (M_m,α).

In addition to the previously discussed things, to assume that F is algebraically closed is convenient because in this case it is easier to describe simple objects. In fact, up to graded isomorphisms, the only finite-dimensional simple superalgebras are of the following types:

(a) $M_{h,l} := M_{h+l}$ with $h \ge l \ge 0$, $h \neq 0$, endowed with the grading induced by the $(h+l)$-tuple $(\underbrace{0,\ldots,0}_{h \text{ times}},\underbrace{1,\ldots,1}_{l \text{ times}})$;

(b) $M_m + tM_m$, where $t^2 = 1_F$, with grading (M_m, tM_m).

The description of simple $*$-superalgebras is more involved.

Theorem 3.1 (7.6 of [10])**.** *Let A be a finite-dimensional simple $*$-superalgebra. Then A is isomorphic to one of the following $*$-superalgebras:*

(1) *$M_{h,l}$, with $h \ge l \ge 0$, $h \neq 0$, endowed with transpose or symplectic involution (the symplectic involution can occur only when $h = l$ if $l \neq 0$);*

(2) *$M_{h,l} \oplus M^{op}_{h,l}$ with $h \ge l \ge 0$, $h \neq 0$, endowed with induced grading and exchange involution;*

(3) *$M_n + tM_n$ with involution given by $(a+tb)^\star := a^\diamond - tb^\diamond$, where $\diamond$ denotes the transpose or symplectic involution;*
(4) *$M_n + tM_n$ with involution given by $(a+tb)^\star := a^\diamond + tb^\diamond$, where $\diamond$ denotes the transpose or symplectic involution;*
(5) *$(M_n+tM_n)\oplus(M_n+tM_n)^{op}$ with grading $(M_n \oplus M_n^{op}, t(M_n\oplus M_n^{op}))$ and exchange involution.*

It is possible to realize any finite-dimensional simple $*$-superalgebra as a subalgebra of an upper block triangular matrix algebra. To see this, let γ_m be the orthogonal involution on M_m, namely the map sending each $a \in M_m$ into the element $a^{\gamma_m} \in M_m$ obtained reflecting a along its secondary diagonal. In particular, for any matrix unit $E_{ij} \in M_m$, $E_{ij}^{\gamma_m} = E_{m-j+1,m-i+1}$. We use the same enumeration and notations as in Theorem 3.1.

(1) Let $\diamond$ be the involution on $M_{h,l}$ and consider $M_{2(h+l)}$ equipped with the grading induced by the map $\alpha : [1, 2(h+l)] \longrightarrow \mathbb{Z}_2$ such that

$$\alpha(j) := \begin{cases} 0 & \text{if } j \in [1,h] \cup [2l+h+1, 2(l+h)], \\ 1 & \text{otherwise.} \end{cases}$$

The map

$$M_{h+l} \longrightarrow M_{2(h+l)}, \qquad a \longmapsto \begin{pmatrix} a & 0 \\ 0 & (a^\diamond)^{\gamma_{h+l}} \end{pmatrix}$$

is an embedding of $*$-superalgebras from $M_{h,l}$ into $(M_{2(h+l)}, \gamma_{2(h+l)}, \alpha)$;

(2) if α is as in (1), the map

$$M_{h,l} \oplus M_{h,l}^{\mathrm{op}} \longrightarrow M_{2(h+l)}, \qquad a = (b,c) \longmapsto \begin{pmatrix} b & 0 \\ 0 & c^{\gamma_{h+l}} \end{pmatrix}$$

embeds the $*$-superalgebra $M_{h,l} \oplus M_{h,l}^{\mathrm{op}}$ into $(M_{2(h+l)}, \gamma_{2(h+l)}, \alpha)$;

(3) let

$$\beta' : [1,2n] \longrightarrow \mathbb{Z}_2, \qquad j \longmapsto \begin{cases} 0 & \text{if } j \in [1,n], \\ 1 & \text{otherwise.} \end{cases}$$

One preliminarily observes that there is an embedding of superalgebras τ from $M_n + tM_n$ into (M_{2n}, β') defining, for every $a, b \in M_n$,

$$\tau(a+tb) := \begin{pmatrix} a & b \\ b & a \end{pmatrix}.$$

One considers $\tau(M_n+tM_n)$ endowed with the involution $\perp$ defined by $\tau(a+tb)^\perp := \begin{pmatrix} a^\diamond & -b^\diamond \\ -b^\diamond & a^\diamond \end{pmatrix}$. The map

$$M_n + tM_n \longrightarrow M_{4n}, \qquad a+tb \longmapsto \begin{pmatrix} \begin{pmatrix} a & b \\ b & a \end{pmatrix} & 0 \\ 0 & \left(\begin{pmatrix} a & b \\ b & a \end{pmatrix}^\perp\right)^{\gamma_{2n}} \end{pmatrix}$$

is an embedding of $*$-superalgebras from $M_n + tM_n$ into $(M_{4n}, \gamma_{4n}, \beta)$ where

$$\beta : [1,4n] \longrightarrow \mathbb{Z}_2, \quad j \longmapsto \begin{cases} \beta'(j) & \text{if } j \in [1,2n], \\ \beta'(4n-j+1) & \text{otherwise;} \end{cases}$$

(4) one proceeds exactly as in (3) replacing the involution $\perp$ with $\top$ defined, for every $a, b \in M_n$, by

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}^{\top} := \begin{pmatrix} a^{\diamond} & b^{\diamond} \\ b^{\diamond} & a^{\diamond} \end{pmatrix};$$

(5) using the embedding τ constructed in (3), if β is as above the map from $(M_n + tM_n) \oplus (M_n + tM_n)^{\mathrm{op}}$ into M_{4n} such that

$$(a + tb, u + tv) \longmapsto \begin{pmatrix} \begin{pmatrix} a & b \\ b & a \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} u & v \\ v & u \end{pmatrix}^{\gamma_{2n}} \end{pmatrix}$$

is an embedding of $*$-superalgebras from $(M_n + tM_n) \oplus (M_n + tM_n)^{\mathrm{op}}$ into $(M_{4n}, \gamma_{4n}, \beta)$.

Given an m-tuple $(A_1, \ldots, A_m)$ of simple $*$-superalgebras, for every $k \in [1, m]$, let us denote the *size* of A_k by

$$s_k := \begin{cases} h_k + l_k & \text{if } A_k = M_{h_k,l_k} \text{ or } A_k = M_{h_k,l_k} \oplus M_{h_k,l_k}^{\mathrm{op}}, \\ 2n_k & \text{if } A_k = M_{n_k} + tM_{n_k} \text{ or } A_k = (M_{n_k} + tM_{n_k}) \oplus (M_{n_k} + tM_{n_k})^{\mathrm{op}} \end{cases}$$

and, set $\eta_0 := 0$, let $\eta_k := \sum_{1=1}^{k} s_i$ and $\mathrm{Bl}_k := [\eta_{k-1} + 1, \eta_k]$. We notice that

$$B_k := \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in M_{s_k} \right\}$$

is a subalgebra with involution of $(M_{2s_k}, \gamma_{2s_k})$ and for each algebra A_k we have actually constructed above a monomorphism of algebras with involution $\phi_k : A_k \longrightarrow B_k$. It is easily seen that there is a monomorphism of algebras with involution $\psi : \bigoplus_{k\in[1,m]} B_k \longrightarrow (M_{2\eta_m}, \gamma_{2\eta_m})$ defining

$$\psi\left(\sum_{k=1}^{m} \begin{pmatrix} a_k & 0 \\ 0 & b_k \end{pmatrix}\right) := \begin{pmatrix} a_1 & & & & & \\ & \ddots & & & & \\ & & a_m & & & \\ & & & b_m & & \\ & & & & \ddots & \\ & & & & & b_1 \end{pmatrix}.$$

The composition of the map

$$\phi : \bigoplus_{k\in[1,m]} A_k \longrightarrow \bigoplus_{k\in[1,m]} B_k, \quad (c_1, \ldots, c_m) \longmapsto (\phi_1(c_1), \ldots, \phi_m(c_m))$$

with ψ gives a monomorphism of algebras with involution from $\bigoplus_{k\in[1,m]} A_k$ into $(M_{2\eta_m}, \gamma_{2\eta_m})$, whose image let us call D.

For every $i, j \in [1, m]$ with $i \neq j$ let $V_{ij} := M_{s_i \times s_j}$ and set

$$V := \begin{pmatrix} 0 & V_{12} & \dots & V_{1m} & & & & \\ & \ddots & \ddots & \vdots & & & & \\ & & 0 & V_{m-1,m} & & & & \\ & & & 0 & & & & \\ & & & & 0 & V_{m,m-1} & \dots & V_{m1} \\ & & & & & \ddots & \ddots & \vdots \\ & & & & & & 0 & V_{21} \\ & & & & & & & 0 \end{pmatrix} \subseteq M_{2\eta_m}$$

and

$$UT^*(A_1, \dots, A_m) := D \oplus V \subseteq M_{2\eta_m}.$$

It is easily seen that $UT^*(A_1, \dots, A_m)$ is a subalgebra with involution of $(M_{2\eta_m}, \gamma_{2\eta_m})$ whose Jacobson radical coincides with V.

A $\mathbb{Z}_2$-grading on $UT^*(A_1, \dots, A_m)$. As discussed before, each simple $*$-superalgebra A_k is embedded in the superalgebra M_{2s_k} endowed with the elementary grading induced by the previously defined map from $[1, 2s_k]$ to $\mathbb{Z}_2$, which will be denoted from now on by α_k. For any m-tuple $\tilde{g} := (g_1, \dots, g_m) \in \mathbb{Z}_2^m$, let us define the maps

$$\alpha'_{\tilde{g}} : [1, \eta_m] \longrightarrow \mathbb{Z}_2, \quad i \longmapsto \alpha_k(i - \eta_{k-1}) + g_k,$$

where $k \in [1, m]$ is the (unique) integer such that $i \in \mathrm{Bl}_k$, and

$$\alpha_{\tilde{g}} : [1, 2\eta_m] \longrightarrow \mathbb{Z}_2, \quad i \longmapsto \begin{cases} \alpha'_{\tilde{g}}(i) & \text{if } i \in [1, \eta_m], \\ \alpha'_{\tilde{g}}(2\eta_m - i + 1) & \text{otherwise.} \end{cases}$$

In particular, $\alpha_{\tilde{g}}$ induces an elementary grading on $UT^*(A_1, \dots, A_m)$ with respect to which $\gamma_{2\eta_m}$ is a graded involution. Furthermore the k-th simple component of the maximal semisimple $*$-graded subalgebra of this $*$-superalgebra is isomorphic to A_k.

When convenient, any such $*$-superalgebra will be simply denoted by $UT^*_{\mathbb{Z}_2}(A_1, \dots, A_m)$, whereas we shall use the symbol $UT^*_{\mathbb{Z}_2, \tilde{g}}(A_1, \dots, A_m)$ to mean the $*$-superalgebra defined by the m-tuple $\tilde{g}$.

A canonical basis for $UT^*_{\mathbb{Z}_2}(A_1, \dots, A_m)$. We have seen that the algebra $\mathbf{R} := UT^*_{\mathbb{Z}_2}(A_1, \dots, A_m)$ is embedded in $\mathbf{U} := UT(s_1, \dots, s_m, s_m, \dots, s_1)$. For all $1 \leq r < t \leq 2m$, let

$$\mathbf{R}_{r,t} := \mathbf{R} \cap \mathbf{U}_{r,t},$$

whereas, when $r \in [1, m]$, let us define $\bar{r} := 2m - r + 1$ and

$$\mathbf{R}_{r,r} := \mathbf{R} \cap (\mathbf{U}_{r,r} \oplus \mathbf{U}_{\bar{r},\bar{r}}) =: \mathbf{R}_{\bar{r},\bar{r}}.$$

For every $1 \leq r \leq t \leq m$, $i \in [1, s_r]$ and $j \in [1, s_t]$, set

$$E^{(r,t)}_{ij} := E_{\eta_{r-1}+i, \eta_{t-1}+j}$$

and

$$E^{(\bar{t},\bar{r})}_{ji} := E_{2\eta_m - \eta_{t-1} - s_t + j, 2\eta_m - \eta_{r-1} - s_r + i},$$

where E_{pq} is the (p,q)-matrix unit of $M_{2\eta_m}$. When $r \neq t$ these elements form a basis of the Jacobson radical of $UT^*_{\mathbb{Z}_2}(A_1,\ldots,A_m)$.

Assume now that $k \in [1,m]$. A basis of $\mathbf{R}_{k,k}$ is given by the set

$$\{E^{(k,k)}_{ij} + ((E^{(\bar{k},\bar{k})}_{ij})^{\diamond})^{\gamma_{s_k}} \mid i,j \in [1,s_k]\}$$

if A_k is isomorphic to M_{h_k,l_k} with the transpose or symplectic involution on M_{s_k} (which we have denoted by $\diamond$),

$$\{E^{(r,r)}_{ij} \mid i,j \in [1,s_k],\, r \in \{k,\bar{k}\}\}$$

when A_k is isomorphic to $M_{h_k,l_k} \oplus M^{\mathrm{op}}_{h_k,l_k}$,

$$\{E^{(k,k)}_{ij} + E^{(k,k)}_{i+n_k,j+n_k} + ((E^{(\bar{k},\bar{k})}_{ij} + E^{(\bar{k},\bar{k})}_{i+n_k,j+n_k})^{\circ})^{\gamma_{s_k}} \mid i,j \in [1,n_k]\} \cup$$
$$\{E^{(k,k)}_{i,j+n_k} + E^{(k,k)}_{i+n_k,j} + ((E^{(\bar{k},\bar{k})}_{i,j+n_k} + E^{(\bar{k},\bar{k})}_{i+n_k,j})^{\circ})^{\gamma_{s_k}} \mid i,j \in [1,n_k]\}$$

if A_k is isomorphic to $M_{n_k} + tM_{n_k}$ and $\circ$ coincides with $\perp$ ($\top$, respectively) according to the fact that A_k is as in (3) ((4), respectively) of Theorem 3.1, and finally

$$\{E^{(r,r)}_{ij} + E^{(r,r)}_{i+n_k,j+n_k}, E^{(r,r)}_{i,j+n_k} + E^{(r,r)}_{i+n_k,j} \mid i,j \in [1,n_k],\, r \in \{k,\bar{k}\}\}$$

in the case in which A_k is isomorphic to $(M_{n_k} + tM_{n_k}) \oplus (M_{n_k} + tM_{n_k})^{\mathrm{op}}$.

Definition 3.2. *The (homogeneous) basis $\mathcal{B}$ of $\boldsymbol{R}$ consisting of the union of the basis of $\boldsymbol{R}_{r,t}$ constructed above is said to be the* canonical basis *of $\boldsymbol{R}$.*

For every $r \in [1,m]$ and $a \in \mathbf{R}_{r,r}$ there exist (and are unique) elements $b \in \mathbf{U}_{r,r}$ and $c \in \mathbf{U}_{\bar{r},\bar{r}}$ such that $a = b + c$. This determines two maps

$$p^{\uparrow}_r : \mathbf{R}_{r,r} \longrightarrow \mathbf{U}_{r,r}, \qquad a = b + c \longmapsto b$$

and

$$p^{\downarrow}_r : \mathbf{R}_{r,r} \longrightarrow \mathbf{U}_{\bar{r},\bar{r}}, \qquad a = b + c \longmapsto c.$$

Set

$$J(\mathbf{R})^{\uparrow} := \sum_{1 \le r < t \le m} \mathbf{R} \cap \mathbf{U}_{r,t} \text{ and } J(\mathbf{R})^{\downarrow} := \sum_{m+1 \le r < t \le 2m} \mathbf{R} \cap \mathbf{U}_{r,t},$$

and

$$\mathbf{R}^{\uparrow} := J(\mathbf{R})^{\uparrow} + \sum_{r \in [1,m]} p^{\uparrow}_r(\mathbf{R}_{r,r}) \text{ and } \mathbf{R}^{\downarrow} := J(\mathbf{R})^{\downarrow} + \sum_{r \in [1,m]} p^{\downarrow}_r(\mathbf{R}_{r,r}),$$

it is easily seen that all the above sums are direct ones and any element $a \in \mathbf{R}$ can be uniquely written as $a^{\uparrow} + a^{\downarrow}$, where $a^{\uparrow} \in \mathbf{R}^{\uparrow}$ and $a^{\downarrow} \in \mathbf{R}^{\downarrow}$.

Definition 3.3. *A non-zero element $a \in \boldsymbol{R}$ has* two parts *if $a^{\uparrow} \neq 0_{\boldsymbol{R}}$ and $a^{\downarrow} \neq 0_{\boldsymbol{R}}$. If this does not happen, the element a is said to have* one part.

Let $k \in [1,m]$. We notice that if the $*$-superalgebra A_k is either isomorphic to M_{h_k,l_k} or $M_{n_k} + tM_{n_k}$, then any element in $\mathcal{B} \cap \mathbf{R}_{k,k}$ has two parts. On the other hand, if A_k is either isomorphic to $M_{h_k,l_k} \oplus M^{\mathrm{op}}_{h_k,l_k}$ or $(M_{n_k} + tM_{n_k}) \oplus (M_{n_k} + tM_{n_k})^{\mathrm{op}}$, then any element in $\mathcal{B} \cap \mathbf{R}_{k,k}$ has one part. Based on this, we state the following

Definition 3.4. *For every $k \in [1, m]$ and $i \in \{1, 2\}$, the simple $*$-superalgebra A_k is of* Type i *if any basis element in $\mathcal{B} \cap \boldsymbol{R}_{k,k}$ has i parts.*

Hence, M_{h_k,l_k} and $M_{n_k} + tM_{n_k}$ are of Type 2, whereas $M_{h_k,l_k} \oplus M^{\mathrm{op}}_{h_k,l_k}$ and $(M_{n_k} + tM_{n_k}) \oplus (M_{n_k} + tM_{n_k})^{\mathrm{op}}$ are of Type 1.

Finally, for every $k \in [1, m]$, any element $b \in \mathcal{B} \cap \mathbf{R}_{k,k}$ such that $b^{\uparrow} \neq 0_{\mathbf{R}}$ is either a matrix unit or sum of matrix units. When it is convenient, set $b = \overline{E}^{(k,k)}_{ij}$ if either $b = E^{(k,k)}_{ij}$ or $E^{(k,k)}_{ij}$ is an addendum of b. In particular, one observes that, for every $i, j \in [1, s_k]$, there exists a unique element b of $\mathcal{B}$ such that $b = \overline{E}^{(k,k)}_{ij}$.

The following result is straightforward and we shall freely use.

Lemma 3.5. *Let $k \in [1, m]$ and b_1, b_2 elements of $\mathcal{B} \cap \boldsymbol{R}_{k,k}$ such that $b_1 b_2 \neq 0_{\boldsymbol{R}}$. Then, up to non-zero scalars, $b_1 b_2$ and b_1^* are in $\mathcal{B} \cap \boldsymbol{R}_{k,k}$. Moreover, set $S := \{b_1, b_2, b_1 b_2\}$, one has that*

(i) *if A_k is of Type 1 and at least one of the elements of S is in $\boldsymbol{R}^{\uparrow}$, then all the elements of S are in $\boldsymbol{R}^{\uparrow} \setminus \{0_{\boldsymbol{R}}\}$, whereas $b_1^* \in \boldsymbol{R}^{\downarrow} \setminus \{0_{\boldsymbol{R}}\}$;*

(ii) *if A_k is of Type 2, then b_1^* as well all the elements of S have two parts.*

In the sequel we are interested to evaluate special polynomials in the algebra $\mathbf{R}$ (and hence to compute products of elements of $\mathbf{R}$). Here we establish which are the possible values of the product of finitely many elements of $\mathcal{B}$. We start considering first only elements belonging to a unique finite-dimensional simple $*$-superalgebra. To this aim, if A is such a $*$-superalgebra, let us denote by $\mathcal{B}'_A$ the basis of A given by the following set:

- $\{E_{ij} \mid i, j \in [1, h + l]\}$, if $A = M_{h,l}$;
- $\{(E_{ij}, 0_{M_{h+l}}), (0_{M_{h+l}}, E_{ij}) \mid i, j \in [1, h + l]\}$, if $A = M_{h,l} \oplus M^{\mathrm{op}}_{h,l}$;
- $\{cE_{ij} \mid i, j \in [1, n], c \in \{1_F, t\}\}$, if $A = M_n + tM_n$;
- $\{(cE_{ij}, 0_{M_n}), (0_{M_n}, cE_{ij}) \mid i, j \in [1, n], c \in \{1_F, t\}\}$, if $A = (M_n + tM_n) \oplus (M_n + tM_n)^{\mathrm{op}}$.

Lemma 3.6. *Let A be a finite-dimensional simple $*$-superalgebra and $b_1, \ldots, b_s$ elements of $\mathcal{B}'_A$ such that $b := b_1 \cdots b_s \neq 0_A$.*

(a) *If $A = M_{h,l}$ with the transpose or symplectic involution $\diamond$, then there exist $i, j \in [1, h + l]$ such that $b = E_{ij}$. Furthermore, for every permutation π in the symmetric group S_s and $\boldsymbol{\lambda} := (\lambda_1, \ldots, \lambda_s) \in \{1, *\}^s$ such that $b^{\pi,\boldsymbol{\lambda}} := b^{\lambda_1}_{\pi(1)} \cdots b^{\lambda_s}_{\pi(s)} \neq 0_A$, one has that either $b^{\pi,\boldsymbol{\lambda}} = b$ or $b^{\pi,\boldsymbol{\lambda}} = b^{\diamond}$ when $i \neq j$, whereas $b^{\pi,\boldsymbol{\lambda}} = E_{\ell\ell}$ for some $\ell \in [1, h + l]$ otherwise;*

(b) *if $A = M_{h,l} \oplus M^{op}_{h,l}$, then there exist $i, j \in [1, h + l]$ such that either $b = (E_{ij}, 0_{M_{h+l}})$ or $b = (0_{M_{h+l}}, E_{ij})$. Moreover, if $b^{\pi,\boldsymbol{\lambda}} \neq 0_A$ for some $\pi \in S_s$ and $\boldsymbol{\lambda} := (\lambda_1, \ldots, \lambda_s) \in \{1, *\}^s$, then either $b^{\pi,\boldsymbol{\lambda}} =$*

$(E_{ij}, 0_{M_{h+l}})$ or $b^{\pi,\boldsymbol{\lambda}} = (0_{M_{h+l}}, E_{ij})$ when $i \neq j$, whereas either $b^{\pi,\boldsymbol{\lambda}} = (E_{\ell\ell}, 0_{M_{h+l}})$ or $b^{\pi,\boldsymbol{\lambda}} = (0_{M_{h+l}}, E_{\ell\ell})$ for some $\ell \in [1, h+l]$ otherwise;

(c) *if $A = M_n + tM_n$ with involution given by $(a+tb)^\star := a^\diamond - tb^\diamond$ ($(a+tb)^\star := a^\diamond + tb^\diamond$, respectively), where $\diamond$ denotes the transpose or symplectic involution on M_n, then there exist $i_1, \ldots, i_s, j_1, \ldots, j_s \in [1, n]$ and $c_1, \ldots, c_s \in \{1_F, t\}$ such that $b_1 = c_1 E_{i_1 j_1}, \ldots, b_s = c_s E_{i_s j_s}$. Consequently $b = cE_{i_1 j_s}$, where $c := c_1 \cdots c_s$ is equal to 1_F if $|\{c_l \mid c_l = t\}|$ is even, and $c = t$ otherwise. Furthermore, for every $\pi \in S_s$ and $\boldsymbol{\lambda} := (\lambda_1, \ldots, \lambda_s) \in \{1, *\}^s$ such that $b^{\pi,\boldsymbol{\lambda}} \neq 0_A$, set $m := |\{c_l \mid c_l = t \text{ and } \lambda_l = *\}|$ and $\alpha := (-1_F)^m$ ($\alpha := 1_F$ if $(a+tb)^\star := a^\diamond + tb^\diamond$, respectively), one has that either $b^{\pi,\boldsymbol{\lambda}} = \alpha c E_{i_1 j_s}$ or $b^{\pi,\boldsymbol{\lambda}} = \alpha c E^\diamond_{i_1 j_s}$ when $i_1 \neq j_s$, whereas $b^{\pi,\boldsymbol{\lambda}} = \alpha c E_{\ell\ell}$ for some $\ell \in [1, n]$ otherwise;*

(d) *if $A = (M_n + tM_n) \oplus (M_n + tM_n)^{op}$, then there exist $i, j \in [1, n]$ and $c \in \{1_F, t\}$ such that either $b = (cE_{ij}, 0_{M_n})$ or $b = (0_{M_n}, cE_{ij})$. Moreover, if $b^{\pi,\boldsymbol{\lambda}} \neq 0_A$ for some $\pi \in S_s$ and $\boldsymbol{\lambda} := (\lambda_1, \ldots, \lambda_s) \in \{1, *\}^s$, then either $b^{\pi,\boldsymbol{\lambda}} = (cE_{\ell\ell}, 0_{M_n})$ or $b^{\pi,\boldsymbol{\lambda}} = (0_{M_n}, cE_{\ell\ell})$ for some $\ell \in [1, n]$ when $i = j$, and either $b^{\pi,\boldsymbol{\lambda}} = (cE_{ij}, 0_{M_n})$ or $b^{\pi,\boldsymbol{\lambda}} = (0_{M_n}, cE_{ij})$ otherwise.*

Proof. The part (a) is a direct consequence of Lemma 1 of [6], whereas (b) easily follows from Lemma 3.3 of [4].

Assume now that $A = M_n + tM_n$. It is immediate to see that $b = cE_{i_1 j_s}$ with c as in the statement of the Lemma. When A is endowed with the graded involution $\star$ defined, for any $a, b \in M_n$, by $(a+tb)^\star := a^\diamond - tb^\diamond$, for every $\pi \in S_s$ and $\boldsymbol{\lambda} := (\lambda_1, \ldots, \lambda_s) \in \{1, *\}^s$ such that $b^{\pi,\boldsymbol{\lambda}} \neq 0_A$, invoking again Lemma 1 of [6] we get that either

$$b^{\pi,\boldsymbol{\lambda}} = (c_1^{\lambda_1} \cdots c_s^{\lambda_s}) E_{i_1 j_s} = (-1_F)^m c E_{i_1 j_s}$$

or

$$b^{\pi,\boldsymbol{\lambda}} = (c_1^{\lambda_1} \cdots c_s^{\lambda_s}) E^\diamond_{i_1 j_s} = (-1_F)^m c E^\diamond_{i_1 j_s}$$

if $i_1 \neq j_s$, whereas

$$b^{\pi,\boldsymbol{\lambda}} = (c_1^{\lambda_1} \cdots c_s^{\lambda_s}) E_{\ell\ell} = (-1_F)^m c E_{\ell\ell}$$

for some $\ell \in [1, n]$ otherwise.

When the graded involution $\star$ on $M_n + tM_n$ is defined by $(a+tb)^\star := a^* + tb^*$ one can follow the same line of reasoning taking into account that $c_1^{\lambda_1} \cdots c_s^{\lambda_s} = c$ for all $(\lambda_1, \ldots, \lambda_s) \in \{1, *\}^s$.

Finally, observe that similar arguments can be applied when A is as in (d) with an appeal to Lemma 3.3 of [4]. □

As a direct consequence of Lemma 3.6 one has the following

Lemma 3.7. *Let $b_1, \ldots, b_s \in \mathcal{B}$ and assume that $b := b_1 \cdots b_s \neq 0_{\boldsymbol{R}}$.*

(i) *If $b \in A_k$, then $b_i \in A_k$ for every $i \in [1, s]$. Moreover, for every $\pi \in S_s$ and $\boldsymbol{\lambda} := (\lambda_1, \ldots, \lambda_s) \in \{1, *\}^s$ such that $b^{\pi,\boldsymbol{\lambda}^\uparrow} \neq 0_{\boldsymbol{R}}$ there exist $c^{(k)}_{\pi,\boldsymbol{\lambda}} \in \{-1_F, 1_F\}$ and $i, j \in [1, s_k]$ such that $b^{\pi,\boldsymbol{\lambda}} = c^{(k)}_{\pi,\boldsymbol{\lambda}} \overline{E}^{(k,k)}_{ij}$.*

(ii) *If* $b \in J(\boldsymbol{R}) \cap \boldsymbol{R}^{\uparrow}$*, then there exist* $1 \leq r < t \leq m$*,* $i \in [1, s_r]$ *and* $j \in [1, s_t]$ *such that* $b = E_{ij}^{(r,t)}$*. Furthermore, for every* $\pi \in S_s$ *and* $\boldsymbol{\lambda} := (\lambda_1, \ldots, \lambda_s) \in \{1, *\}^s$ *such that* $b^{\pi, \boldsymbol{\lambda}^{\uparrow}} \neq 0_{\boldsymbol{R}}$ *one has that* $b^{\pi, \boldsymbol{\lambda}} = \pm b$*.*

Proof. The part (i) follows directly from Lemma 3.6 considering the restriction of the map $\psi \circ \phi$ introduced to define $UT^*(A_1, \ldots, A_m)$ to A_k, which establishes a $*$-superalgebras isomorphism from A_k into $\mathbf{R}_{k,k}$ mapping the elements of the basis $\mathcal{B}'_{A_k}$ into the elements of the canonical basis $\mathcal{B} \cap \mathbf{R}_{k,k}$.

The statement (ii) is consequence of (i) taking in account that, when one has M_n with the symplectic involution $\diamond$, if $E_{ij}^{\diamond} = E_{qj}$ ($E_{ij}^{\diamond} = E_{ir}$, respectively) then $i = q$ ($j = r$, respectively). $\square$

4. Triangular $*$-superalgebras and generators of minimal varieties

The goal here is to state that any minimal variety of PI $*$-superalgebras of finite basic rank is generated by a $*$-superalgebra of the form $UT^*_{\mathbb{Z}_2}(A_1, \ldots, A_m)$ (namely, the necessary conditions of Theorem 2.2). To this aim, we need certain arguments from [3].

Definition 4.1. *Let* F *be an algebraically closed field. A* $*$*-superalgebra* A *is called* triangular *if it is finite-dimensional and either* A *is simple or* $A = A_{ss} + J(A)$ *where*

(i) $A_{ss} = A_1 \oplus \cdots \oplus A_n$*, with* $A_1, \ldots, A_n$ *simple* $*$*-superalgebras and* $n \geq 2$*;*

(ii) *there exist homogeneous elements* $w_{12}, w_{12}^*, \ldots, w_{n-1,n}, w_{n-1,n}^* \in J(A)$ *and minimal homogeneous idempotents* $e_1 \in A_1, \ldots, e_n \in A_n$ *such that*

$$e_i w_{i,i+1} = w_{i,i+1} e_{i+1} = w_{i,i+1} \qquad i \in [1, n-1]$$

and

$$w_{12} w_{23} \cdots w_{n-1,n} \neq 0_A;$$

(iii) $w_{12}, w_{12}^*, \ldots, w_{n-1,n}, w_{n-1,n}^*$ *generate* $J(A)$ *as a two-sided* $*$*-graded ideal of* A*. In particular,* $J(A) = I(A) \oplus I(A)^*$ *where* $I(A)$ *is the two-sided (homogeneous) ideal generated by* $w_{12}, w_{23}, \ldots, w_{n-1,n}$*.*

We notice that, according to (3), for a triangular $*$-superalgebra A one has that $\exp^*_{\mathbb{Z}_2}(A) = \dim_F A_{ss}$.

The proof of the next result is essentially that of Theorem 4.5 of [3] and for this reason we confine ourselves to sketch it.

Lemma 4.2. *Let* F *be an algebraically closed field and* A *a finite-dimensional* $*$*-superalgebra. Then there exists a triangular* $*$*-superalgebra* B *such that* $Id^*_{\mathbb{Z}_2}(A) \subseteq Id^*_{\mathbb{Z}_2}(B)$ *and* $\exp^*_{\mathbb{Z}_2}(A) = \exp^*_{\mathbb{Z}_2}(B)$*. In particular, if* $\mathcal{V}^*_{\mathbb{Z}_2}(A)$ *is minimal, then* $Id^*_{\mathbb{Z}_2}(A) = Id^*_{\mathbb{Z}_2}(B)$*.*

Proof. Let $A := A_{ss} + J(A)$ be the Wedderburn decomposition of the $*$-superalgebra A. According to (3) there are distinct simple components $B_{i_1}, \ldots B_{i_k}$ of A_{ss} such that

$$B_{i_1} J(A) \cdots J(A) B_{i_k} \neq 0_A \text{ and } \exp^*_{\mathbb{Z}_2}(A) = \dim_F(B_{i_1} \oplus \ldots \oplus B_{i_k}).$$

In order to simplify the notation, assume that $i_j = j$ for all $j \in [1, k]$. Let us pick homogeneous elements $b_1 \in B_1, \ldots, b_k \in B_k$ and $x_1, \ldots, x_{k-1} \in J(A)$ satisfying

$$b_1 x_1 b_2 x_2 \cdots x_{k-1} b_k \neq 0_A.$$

Let $\sum_j \epsilon_j^{(i)}$ be the decompostion of the unit 1_{B_i} of B_i in minimal homogeneous idempotents. From the fact that $1_{B_1} b_1 x_1 1_{B_2} b_2 x_2 \cdots x_{k-1} 1_{B_k} b_k \neq 0_A$ it follows that there exist minimal idempotents $\epsilon_1 := \epsilon_{j_1}^{(1)}, \ldots, \epsilon_k := \epsilon_{j_k}^{(k)}$ such that

$$\epsilon_1 b_1 x_1 \epsilon_2 b_2 x_2 \cdots x_{k-1} \epsilon_k b_k \neq 0_A.$$

Set $\nu_j := b_j x_j \in J(A)$ for every $j \in [1, k]$ and $v_{l,l+1} := \epsilon_l \nu_l \epsilon_{l+1}$ for all $l \in [1, k-1]$, one has that $v_{12} v_{23} \cdots v_{k-1,k} \neq 0_A$. Let us define

$$A' : A \otimes F[t_1, \ldots, t_{k-1}]/(t_1^2, \ldots, t_{k-1}^2),$$

where $C := F[t_1, \ldots, t_{k-1}]/(t_1^2, \ldots, t_{k-1}^2)$ is considered as a $*$-superalgebra equipped with trivial grading and trivial involution. Then A' is a finite-dimensional $*$-superalgebra such that $\mathrm{Id}^*_{\mathbb{Z}_2}(A) = \mathrm{Id}^*_{\mathbb{Z}_2}(A')$. Let

$$e_i := \epsilon_i \otimes 1_C \in B_i \otimes 1_C =: B'_i, \qquad w_{i,i+1} := v_{i,i+1} \otimes t_i.$$

Using the same arguments and computations of the proof of Theorem 4.5 of [3] (for the details we refer to the original paper), we get that the subalgebra B of the $*$-superalgebra A' generated by $B'_1, \ldots, B'_k$ and $w_{12}, w^*_{12}, \ldots, w_{k-1,k}, w^*_{k-1,k}$ is a triangular $*$-superalgebra satisfying $\mathrm{Id}^*_{\mathbb{Z}_2}(A) \subseteq \mathrm{Id}^*_{\mathbb{Z}_2}(B)$ and $\exp^*_{\mathbb{Z}_2}(A) = \exp^*_{\mathbb{Z}_2}(B)$.

At this stage, the last statement of the Lemma trivially follows. □

We are in a position to prove the main result of this section.

Theorem 4.3. *Let A be a finitely generated PI $*$-superalgebra of $*$-graded exponent d. Then there exist finite-dimensional simple $*$-superalgebras $A_1, \ldots, A_m$ and $\tilde{g} \in \mathbb{Z}_2^m$ such that $\dim_F(A_1 \oplus \cdots \oplus A_m) = d$ and $UT^*_{\mathbb{Z}_2,\tilde{g}}(A_1, \ldots, A_m)$ belongs to $\mathcal{V}^*_{\mathbb{Z}_2}(A)$. In particular, if $\mathcal{V}^*_{\mathbb{Z}_2}(A)$ is minimal, then $\mathcal{V}^*_{\mathbb{Z}_2}(A) = \mathcal{V}^*_{\mathbb{Z}_2}(UT^*_{\mathbb{Z}_2,\tilde{g}}(A_1, \ldots, A_m))$.*

Proof. According to the deductions of Section 2, we can suppose that F is algebraically closed. Since A is a finitely generated PI $*$-superalgebra, by virtue of Theorem 5.3 of [8] there exists a finite-dimensional $*$-superalgebra B such that $\mathrm{Id}^*_{\mathbb{Z}_2}(A) = \mathrm{Id}^*_{\mathbb{Z}_2}(B)$.

Eventually replacing A with B, invoking Lemma 4.2, for our aims we can also assume that A is triangular and let $A_{ss} = A_1 \oplus \cdots \oplus A_m$. Using the same notation for the homogeneous radical elements defining A which appear in Definition 4.1, set

$$g_1 := 0, \qquad g_k := |w_{12} w_{23} \cdots w_{k-1,k}|_A \quad \forall k \in [2, m]$$

and $\tilde{g} := (g_1, \dots, g_m)$. Let us consider the $*$-superalgebra $UT^*_{\mathbb{Z}_2,\tilde{g}}(A_1, \dots, A_m)$ which is denoted, for convenience, by $\mathbf{R}$. We aim to prove that $\mathbf{R} \in \mathcal{V}^*_{\mathbb{Z}_2}(A)$. In order to do this, we proceed by induction on m.

If $m = 1$, then $\mathbf{R} = UT^*_{\mathbb{Z}_2,\tilde{g}}(A_1) \cong A_1 = A$ and we are done.

Thus, suppose that $m \geq 2$ and let $f = f(u_1, \dots, u_n, u_1^*, \dots, u_n^*) \in F\langle Y \cup Z, *\rangle \setminus \mathrm{Id}^*_{\mathbb{Z}_2}(\mathbf{R})$ (for simplicity, we are calling u_i the i-th variable from $Y \cup Z$ appearing in f). We want to show that f is not a $*$-graded polynomial identity for A. To this end, since the characteristic of the ground field is zero, we can assume that f is multilinear. Hence there exist $b_1, \dots, b_n$ in the canonical basis $\mathcal{B}$ of $\mathbf{R}$, with $|b_i|_{\mathbf{R}} = |u_i|_{F\langle Y\cup Z,*\rangle}$ for each $i \in [1, n]$, such that $f(b_1, \dots, b_n, b_1^*, \dots, b_n^*) \neq 0_{\mathbf{R}}$.

Let s be the number of the b_k's which are in $J(\mathbf{R})$. Since $J(\mathbf{R})$ is nilpotent of index $m-1$, one has that $s \leq m-1$. Assume first that $s < m-1$. Hence there exists $i \in [1, m-1]$ such that, for every $j \in [i+1, m]$, none of the elements $b_1, \dots, b_n$ is in $\mathbf{R}_{i,j} \cup \mathbf{R}_{\bar{j},\bar{i}}$. At this stage, we have two cases to distinguish. If there exists $\ell \in [1, n]$ such that $b_\ell \in \mathbf{R}_{r,i} \cup \mathbf{R}_{\bar{i},\bar{r}}$ for some $r \in [1, i]$, then all the b_k's are in

$$\sum_{1\leq r\leq i} \mathbf{R}_{r,r} \oplus \sum_{1\leq r<s\leq i} \mathbf{R}_{r,s} \oplus \sum_{1\leq r<s\leq i} \mathbf{R}_{\bar{s},\bar{r}}$$

which is a $*$-superalgebra isomorphic to $UT^*_{\mathbb{Z}_2,(g_1,\dots,g_i)}(A_1, \dots, A_i)$ and corresponding to the subalgebra of A generated by $A_1, \dots, A_i$ and w_{12}, $w_{12}^*, \dots,$ $w_{i-1,i}$, $w_{i-1,i}^*$, which is still a triangular $*$-superalgebra. Otherwise the basis elements $b_1, \dots, b_n$ are in

$$\sum_{\substack{1\leq r\leq m\\ r\neq i}} \mathbf{R}_{r,r} \oplus \sum_{\substack{1\leq r<s\leq m\\ r\neq i\neq s}} \mathbf{R}_{r,s} \oplus \sum_{\substack{1\leq r<s\leq m\\ r\neq i\neq s}} \mathbf{R}_{\bar{s},\bar{r}}$$

which is a $*$-superalgebra isomorphic to

$$UT^*_{\mathbb{Z}_2,(g_1,\dots,g_{i-1},g_{i+1},\dots,g_m)}(A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_m)$$

and corresponding to the triangular $*$-graded subalgebra of A generated by $A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_m$ and the radical elements $w_{12}, w_{12}^*, \dots, w_{i-2,i-1},$ $w_{i-2,i-1}^*, (w_{i-1,i}w_{i,i+1}), (w_{i-1,i}w_{i,i+1})^*, w_{i+1,i+2}, w_{i+1,i+2}^*, \dots, w_{m-1,m}, w_{m-1,m}^*$. By the induction assumption, we conclude that in any event f does not belong to $\mathrm{Id}^*_{\mathbb{Z}_2}(A)$.

Thus suppose that $s = m-1$ and $f(b_1, \dots, b_n, b_1^*, \dots, b_n^*)^\uparrow \neq 0_{\mathbf{R}}$ (indeed, when $f(b_1, \dots, b_n, b_1^*, \dots, b_n^*) = f(b_1, \dots, b_n, b_1^*, \dots, b_n^*)^\downarrow$ it is sufficient to replace f with f^*). Since the evaluation in $b_1, \dots, b_n, b_1^*, \dots, b_n^*$ of any monomial of f which is non-zero is either in $\mathbf{R}^\uparrow$ or in $\mathbf{R}^\downarrow$, we confine ourselves to consider only those whose evaluation belongs to $\mathbf{R}^\uparrow$. Let ξ be one of these monomials. Opportunely replacing the variables of f with their images under $*$, we can also assume that in ξ only $u_1, \dots, u_n$ appear. Hence there exist $t_1, \dots, t_{m-1} \in [1, n]$ such that

$$b_{t_1} = E^{(1,2)}_{i_1 j_2}, \ \dots, \ b_{t_{m-1}} = E^{(m-1,m)}_{i_{m-1} j_m},$$

where $i_k \in [1, s_k]$ and $j_{k+1} \in [1, s_{k+1}]$ for every $k \in [1, m-1]$. Moreover all the remaining elements of the set $\{b_1, \ldots, b_n\}$ are in the diagonal blocks of $\mathbf{R}$. According to Lemma 3.7,

$$f(b_1, \ldots, b_n, b_1^*, \ldots, b_n^*)^{\uparrow} = \beta E_{ij}^{(1,m)}$$

for some $\beta \in F \setminus \{0_F\}$, $i \in [1, s_1]$ and $j \in [1, s_m]$. Set $j_1 := i$ and $i_m := j$, let

$$b_0 := \overline{E}_{1j_1}^{(1,1)} \qquad \text{and} \qquad b_{n+1} := \overline{E}_{i_m 1}^{(m,m)}.$$

Then

$$b_{t_{k-1}+1} \cdots b_{t_k - 1} = \overline{E}_{j_k i_k}^{(k,k)} \qquad \forall\, k \in [1, m]$$

and

$$b_0 f(b_1, \ldots, b_n, b_1^*, \ldots, b_n^*)^{\uparrow} b_{n+1} = \overline{E}_{1i}^{(1,1)} (\beta E_{ij}^{(1,m)}) \overline{E}_{j1}^{(m,m)} = \beta E_{11}^{(1,m)}.$$

Now, for every $k \in [1, m]$, A_k is isomorphic as a $*$-superalgebra to $\mathbf{R}_{k,k}$. Furthermore, we can require that in this isomorphism the minimal homogeneous idempotent e_k of A_k corresponds to $\overline{E}_{11}^{(k,k)}$. In particular, if $k \in [1, m-1]$, let $v_k \in A_k$ and $\omega_{k+1} \in A_{k+1}$ be the elements corresponding to $\overline{E}_{i_k 1}^{(k,k)}$ and $\overline{E}_{1 j_{k+1}}^{(k+1,k+1)}$ in the above isomorphisms, respectively. Denoting by a_{t_k} the element $v_k w_{k,k+1} \omega_{k+1}$ of A (recall that $w_{k,k+1}$ is the k-th homogeneous radical element defining the triangular $*$-superalgebra A), from the equality

$$E_{i_k j_{k+1}}^{(k,k+1)} = \overline{E}_{i_k 1}^{(k,k)} E_{11}^{(k,k+1)} \overline{E}_{1 j_{k+1}}^{(k+1,k+1)}$$

it easily follows that

$$\begin{aligned} |a_{t_k}|_A &= |v_k|_A + |w_{k,k+1}|_A + |\omega_{k+1}|_A \\ &= |\overline{E}_{i_k 1}^{(k,k)}|_{\mathbf{R}} + |E_{11}^{(k,k+1)}|_{\mathbf{R}} + |\overline{E}_{1 j_{k+1}}^{(k+1,k+1)}|_{\mathbf{R}} = |E_{i_k j_{k+1}}^{(k,k+1)}|_{\mathbf{R}} = |b_{t_k}|_{\mathbf{R}}. \end{aligned}$$

Furthermore, setting $t_0 := 0$ and $t_m := n+1$, we notice that, for every $i \in [t_{k-1}+1, t_k - 1]$, $b_i \in \mathbf{R}_{k,k}$. Let $a_i \in A_k$ be the element corresponding to b_i, $\omega_1 := a_0$ that corresponding to b_0 in A_1 and $v_m := a_{n+1}$ that corresponding to b_{n+1} in A_m (again in the above isomorphisms). Finally, according to (iii) of Definition 4.1, let $J(A) = I(A) \oplus I(A)^*$.

We claim that, for every $\pi \in S_n$ and $\boldsymbol{\lambda} := (\lambda_1, \ldots, \lambda_n) \in \{1, *\}^n$, one has that $a_0 a_{\pi(1)}^{\lambda_1} \cdots a_{\pi(n)}^{\lambda_n} a_{n+1} \in I(A) \setminus \{0_A\}$ if, and only if, $(b_{\pi(1)}^{\lambda_1} \cdots b_{\pi(n)}^{\lambda_n})^{\uparrow} \neq 0_{\mathbf{R}}$. In fact, assume first that $a_0 a_{\pi(1)}^{\lambda_1} \cdots a_{\pi(n)}^{\lambda_n} a_{n+1}$ is a non-zero element of $I(A)$. Then $\pi(t_k) = t_k$ and $\lambda_{t_k} = 1$ for all $k \in [1, m-1]$. Moreover, for every $q \in [1, m]$, the inequality $t_{q-1} < l < t_q$ implies that $t_{q-1} < \pi(l) < t_q$ since $a_{\pi(l)}^{\lambda_l} \in A_q$. Thus

$$\begin{aligned} I(A) \setminus \{0_A\} \ni \omega_1 a_{\pi(1)}^{\lambda_1} \cdots a_{\pi(n)}^{\lambda_n} v_m = \omega_1 a_{\pi(1)}^{\lambda_1} \cdots a_{\pi(t_1-1)}^{\lambda_{t_1-1}} v_1 w_{12} \omega_2 a_{\pi(t_1+1)}^{\lambda_{t_1+1}} \cdots \\ \cdots a_{\pi(t_2-1)}^{\lambda_{t_2-1}} v_2 w_{23} \omega_3 \cdots v_{m-1} w_{m-1,m} \omega_m a_{\pi(t_{m-1}+1)}^{\lambda_{t_{m-1}+1}} \cdots a_{\pi(n)}^{\lambda_n} v_m, \end{aligned}$$

which yields

$$\omega_k a_{\pi(t_{k-1}+1)}^{\lambda_{t_{k-1}+1}} \cdots a_{\pi(t_k-1)}^{\lambda_{t_k-1}} v_k \neq 0_A \text{ for all } k \in [1, m],$$

and thus

$$\overline{E}^{(k,k)}_{1j_k} b^{\lambda_{t_{k-1}+1}}_{\pi(t_{k-1}+1)} \cdots b^{\lambda_{t_k-1}}_{\pi(t_k-1)} \overline{E}^{(k,k)}_{i_k 1} \neq 0_{\mathbf{R}} \text{ for all } k \in [1, m].$$

Noticing (when necessary) that $(\overline{E}^{(k,k)}_{1j_k})^{\uparrow} \neq 0_{\mathbf{R}}$ and invoking Lemma 3.5 one has that

$$(\overline{E}^{(k,k)}_{1j_k} b^{\lambda_{t_{k-1}+1}}_{\pi(t_{k-1}+1)} \cdots b^{\lambda_{t_k-1}}_{\pi(t_k-1)} \overline{E}^{(k,k)}_{i_k 1})^{\uparrow} \neq 0_{\mathbf{R}} \text{ for all } k \in [1, m].$$

At this stage, taking into account Lemma 3.7 and $|b^{\lambda_{t_{k-1}+1}}_{\pi(t_{k-1}+1)} \cdots b^{\lambda_{t_k-1}}_{\pi(t_k-1)}|_{\mathbf{R}} = |\overline{E}^{(k,k)}_{j_k i_k}|_{\mathbf{R}}$, we get that, for each $k \in [1, m]$, there exists $c^{(k)}_{\pi,\boldsymbol{\lambda}} \in \{-1_F, 1_F\}$ such that

$$b^{\lambda_{t_{k-1}+1}}_{\pi(t_{k-1}+1)} \cdots b^{\lambda_{t_k-1}}_{\pi(t_k-1)} = c^{(k)}_{\pi,\boldsymbol{\lambda}} \, \overline{E}^{(k,k)}_{j_k i_k} \text{ for all } k \in [1, m],$$

which yields

$$\overline{E}^{(1,1)}_{1j_1} b^{\lambda_1}_{\pi(1)} \cdots b^{\lambda_{t_1-1}}_{\pi(t_1-1)} E^{(1,2)}_{i_1 j_2} \cdots E^{(m-1,m)}_{i_{m-1} j_m} b^{\lambda_{t_{m-1}+1}}_{\pi(t_{m-1}+1)} \cdots b^{\lambda_n}_{\pi(n)} \overline{E}^{(m,m)}_{i_m 1} = (\Pi_{k=1}^m c^{(k)}_{\pi,\boldsymbol{\lambda}}) E^{(1,m)}_{11}$$

and therefore

$$b_0 b^{\lambda_1}_{\pi(1)} \cdots b^{\lambda_n}_{\pi(n)} b_{n+1} = (\Pi_{k=1}^m c^{(k)}_{\pi,\boldsymbol{\lambda}}) E^{(1,m)}_{11} \neq 0_{\mathbf{R}}.$$

Conversely, if $(b^{\lambda_1}_{\pi(1)} \cdots b^{\lambda_n}_{\pi(n)})^{\uparrow} \neq 0_{\mathbf{R}}$, then as above $\pi(t_k) = t_k$ and $\lambda_k = 1$ for all $k \in [1, m-1]$ and, for every $q \in [1, m]$, the inequality $t_{q-1} < l < t_q$ implies that $t_{q-1} < \pi(l) < t_q$. Still,

$$b^{\lambda_{t_{k-1}+1}}_{\pi(t_{k-1}+1)} \cdots b^{\lambda_{t_k-1}}_{\pi(t_k-1)} = c^{(k)}_{\pi,\boldsymbol{\lambda}} \, \overline{E}^{(k,k)}_{j_k i_k} \text{ for all } k \in [1, m],$$

from which it follows that

$$\overline{E}^{(k,k)}_{1j_k} b^{\lambda_{t_{k-1}+1}}_{\pi(t_{k-1}+1)} \cdots b^{\lambda_{t_k-1}}_{\pi(t_k-1)} \overline{E}^{(k,k)}_{i_k 1} = c^{(k)}_{\pi,\boldsymbol{\lambda}} \, \overline{E}^{(k,k)}_{11} \text{ for all } k \in [1, m].$$

As the minimal homogeneous idempotent e_k of A_k corresponds to $\overline{E}^{(k,k)}_{11}$, we have that

$$\omega_k a^{\lambda_{t_{k-1}+1}}_{\pi(t_{k-1}+1)} \cdots a^{\lambda_{t_k-1}}_{\pi(t_k-1)} v_k = c^{(k)}_{\pi,\boldsymbol{\lambda}} e_k \text{ for all } k \in [1, m]$$

and, consequently,

$$\begin{aligned} \omega_1 a_{\pi(1)} \cdots a_{\pi(n)} v_m &= (\Pi_{k=1}^m c^{(k)}_{\pi,\boldsymbol{\lambda}}) \, e_1 w_{12} e_2 w_{23} \cdots e_{m-1} w_{m-1,m} e_m \\ &= (\Pi_{k=1}^m c^{(k)}_{\pi,\boldsymbol{\lambda}}) \, w_{12} \cdots w_{m-1,m} \neq 0_A. \end{aligned}$$

The final outcome of these deductions is that

$$a_0 f(a_1, \ldots, a_n, a_1^*, \ldots, a_n^*) a_{n+1} = \beta w_{12} \cdots w_{m-1,m} \neq 0_A.$$

Hence f is not a $*$-graded polynomial identity for A, and this is enough to conclude that $UT^*_{\mathbb{Z}_2,\tilde{g}}(A_1, \ldots, A_m)$ is in $\mathcal{V}^*_{\mathbb{Z}_2}(A)$.

The final part of the statement is an immediate consequence of the fact that $\exp^*_{\mathbb{Z}_2}(A) = \exp^*_{\mathbb{Z}_2}(UT^*_{\mathbb{Z}_2,\tilde{g}}(A_1, \ldots, A_m))$. □

5. Kemer polynomials for $UT^*_{\mathbb{Z}_2}(A_1,\dots,A_m)$

Throughout this section assume that F is algebraically closed, $(A_1,\dots,A_m)$ is an m-tuple of simple $*$-superalgebras and $A := UT^*_{\mathbb{Z}_2}(A_1,\dots,A_m)$. The aim is to construct a family of $*$-graded polynomials which are not $*$-graded identities for the algebra A but, as we shall see in the sequel, they are in $\mathrm{Id}^*_{\mathbb{Z}_2}(UT^*_{\mathbb{Z}_2}(B_1,\dots,B_n))$ for any sequence of simple $*$-superalgebras $(B_1,\dots,B_n)$ such that $\exp^*_{\mathbb{Z}_2}(A) = \exp^*_{\mathbb{Z}_2}(UT^*_{\mathbb{Z}_2}(B_1,\dots,B_n))$ whenever either $n \neq m$ or $n = m$ but the sequence of the dimensions of the homogeneous skew and symmetric subspaces of the $*$-superalgebras $(A_1,\dots,A_m)$ are different from those of $(B_1,\dots,B_n)$ and $(B_n,\dots,B_1)$. This is a key point in the proof of the main result.

To this end, as a first step for each $*$-superalgebra A_k we find a suitable basis, we call *the $*$-standard basis* of A_k, in order to produce a product of all the elements of this basis, *the standard total product* (of the $*$-standard basis), whose result is a suitable non-zero element of A_k. To do this is more convenient to use for each $*$-superalgebra the decomposition (1). Once done, we build a monomial m_{A_k} of the free $*$-superalgebra $F\langle y_1^+, y_1^-, z_1^+, z_1^-, y_2^+, y_2^-, z_2^+, z_2^-, \dots\rangle$ (in pairwise different variables) replacing each element appearing in the standard total product of A_k with a variable of the same kind. We shall use the same notations introduced in Section 3.

CASE I. Let $A_k = M_{h_k,l_k}$ with the transpose or symplectic involution $\diamond$. Set, for all $i,j \in [1,s_k]$,

$$p_{ij}^{(k,k)} := E_{ij}^{(k,k)} + ((E_{ij}^{(\bar{k},\bar{k})})^{\diamond})^{\gamma_{s_k}},$$

one notices that

$$(p_{ij}^{(k,k)})^{\gamma_{2\eta_m}} = (p_{ij}^{(k,k)})^{\diamond}, \qquad p_{ij}^{(k,k)} p_{uv}^{(k,k)} = \delta_{ju} p_{iv}^{(k,k)} \ \ \forall\, u,v \in [1,s_k],$$

where δ_{ju} is the Kronecker delta, and, if $c = \pm 1_F$ and $i < j$, then

$$p_{uv}^{(k,k)}(p_{ij}^{(k,k)} + c p_{ji}^{(k,k)}) = \delta_{vi} p_{uj}^{(k,k)} + c\delta_{vj} p_{ui}^{(k,k)} \tag{4}$$

and at most one of the two add terms of the last equation is non-zero.

Assume now that $\diamond$ is the transpose involution, hence $(p_{ij}^{(k,k)})^{\gamma_{2\eta_m}} = p_{ji}^{(k,k)}$. It is easily seen that the set

$$\{p_{ij}^{(k,k)} + p_{ji}^{(k,k)} \mid 1 \le i < j \le h_k \text{ or } h_k + 1 \le i < j \le s_k\} \cup \{p_{ii}^{(k,k)} \mid i \in [1,s_k]\}$$

is a basis of $(A_k)^+_{(0)}$, the symmetric part of A_k which is homogeneous of degree 0. Analogously,

$$\{p_{ij}^{(k,k)} - p_{ji}^{(k,k)} \mid 1 \le i < j \le h_k \text{ or } h_k + 1 \le i < j \le s_k\}$$

is a basis of $(A_k)^-_{(0)}$,

$$\{p_{ij}^{(k,k)} + p_{ji}^{(k,k)} \mid 1 \le i \le h_k < j \le s_k\}$$

is a basis of $(A_k)^+_{(1)}$ and

$$\{p_{ij}^{(k,k)} - p_{ji}^{(k,k)} \mid 1 \le i \le h_k < j \le s_k\}$$

is a basis of $(A_k)^-_{(1)}$. The union of the above sets is said to be the $*$-standard basis of A_k. To construct the standard total product of the elements of this basis, we observe that for the matrix algebra M_{s_k} one obtains its matrix unit E_{11} as a product $E_{i_1j_1}\cdots E_{i_{s_k^2}j_{s_k^2}}$ of the s_k^2 matrix units of M_{s_k}. Let us fix such a product, which we denote by $\pi_{M_{s_k}}$, with the extra position in this case that $j_1 = 1$ as well. The standard total product of A_k, π_{A_k}, is obtained from $\pi_{M_{s_k}}$ replacing the matrix units E_{ij} appearing there with

- $p_{ii}^{(k,k)}$ if $1 \leq i = j \leq s_k$;
- $p_{ij}^{(k,k)} - p_{ji}^{(k,k)}$ if $1 \leq i < j \leq s_k$ and
- $p_{ji}^{(k,k)} + p_{ij}^{(k,k)}$ if $1 \leq j < i \leq s_k$

(according to (4), this choice is done to avoid a negative sign in π_{A_k}). Thus $\pi_{A_k} = p_{11}^{(k,k)}$.

Suppose now that $l_k \neq 0$ and $\diamond$ is the symplectic involution. Consequentely $h_k = l_k = \frac{s_k}{2}$. Set $s'_k := \frac{s_k}{2}$ and, for every $n \in [1, s_k]$,

$$\rho_k(n) := \begin{cases} n + s'_k & \text{if } n \in [1, s'_k], \\ n - s'_k & \text{otherwise,} \end{cases}$$

we notice that

$$(p_{i,\rho_k(j)}^{(k,k)})^{\gamma_{2\eta m}} = \begin{cases} -p_{j,\rho_k(i)}^{(k,k)} & \text{if } (i,j) \in [1, s'_k]^2 \cup [s'_k + 1, s_k]^2, \\ p_{j,\rho_k(i)}^{(k,k)} & \text{if } (i,j) \in [1, s'_k] \times [s'_k + 1, s_k] \cup [s'_k + 1, s_k] \times [1, s'_k]. \end{cases}$$

With these positions, it is easily checked that

$$\{p_{i,\rho_k(j)}^{(k,k)} + p_{j,\rho_k(i)}^{(k,k)} \mid 1 \leq i \leq s'_k < j \leq s_k\}$$

is a basis of $(A_k)^+_{(0)}$,

$$\{p_{i,\rho_k(j)}^{(k,k)} - p_{j,\rho_k(i)}^{(k,k)} \mid 1 \leq i \leq s'_k < j \leq s_k\}$$

is a basis of $(A_k)^-_{(0)}$,

$$\{p_{i,\rho_k(j)}^{(k,k)} - p_{j,\rho_k(i)}^{(k,k)} \mid 1 \leq i < j \leq s'_k \text{ or } s'_k + 1 \leq i < j \leq s_k\}$$

is a basis of $(A_k)^+_{(1)}$ and

$$\{p_{i,\rho_k(j)}^{(k,k)} + p_{j,\rho_k(i)}^{(k,k)} \mid 1 \leq i < j \leq s'_k \text{ or } s'_k + 1 \leq i < j \leq s_k\} \cup \{p_{i,\rho_k(i)}^{(k,k)} \mid i \in [1, s_k]\}$$

is a basis of $(A_k)^-_{(1)}$. The union of these sets is the $*$-standard basis of A_k. In order to produce the standard total product π_{A_k}, exactly as above consider the product of all matrix units of M_{s_k}, $\pi_{M_{s_k}} := E_{i_1,\rho_k(l_1)} \cdots E_{i_{s_k^2},\rho_k(l_{s_k^2})}$ with the extra condition that $i_1 = l_1 = \rho_k(l_{s_k^2}) = 1$, and replace each $E_{i,\rho_k(j)}$ with

- $p_{i,\rho_k(i)}^{(k,k)}$ if $1 \leq i = j \leq s_k$;
- $p_{i,\rho_k(j)}^{(k,k)} - p_{j,\rho_k(i)}^{(k,k)}$ if $1 \leq i < j \leq s_k$ and
- $p_{j,\rho_k(i)}^{(k,k)} + p_{i,\rho_k(j)}^{(k,k)}$ if $1 \leq j < i \leq s_k$.

Also in this case $\pi_{A_k} = p_{11}^{(k,k)}$.

Finally, assume that $\diamond$ is still the symplectic involution but $l_k = 0$. In such an event $A_k = (A_k)_{(0)}$ and the $*$-standard basis is union of the basis of $(A_k)^+_{(0)}$

$$\{p_{i,\rho_k(j)}^{(k,k)} + p_{j,\rho_k(i)}^{(k,k)} \,|\, 1 \le i \le s'_k < j \le s_k\} \cup$$
$$\{p_{i,\rho_k(j)}^{(k,k)} - p_{j,\rho_k(i)}^{(k,k)} \,|\, 1 \le i < j \le s'_k \text{ or } s'_k + 1 \le i < j \le s_k\}$$

and that of $(A_k)^-_{(0)}$

$$\{p_{i,\rho_k(j)}^{(k,k)} - p_{j,\rho_k(i)}^{(k,k)} \,|\, 1 \le i \le s'_k < j \le s_k\} \cup \{p_{i,\rho_k(i)}^{(k,k)} \,|\, i \in [1, s_k]\} \cup$$
$$\{p_{i,\rho_k(j)}^{(k,k)} + p_{j,\rho_k(i)}^{(k,k)} \,|\, 1 \le i < j \le s'_k \text{ or } s'_k + 1 \le i < j \le s_k\},$$

whereas the standard total product is constructed as when $l_k \neq 0$.

CASE II. When A_k is isomorphic to $M_{h_k,l_k} \oplus M_{h_k,l_k}^{\mathrm{op}}$ let us define, for all $i, j \in [1, s_k]$,

$$p_{ij}^{(k,k)} := E_{ij}^{(k,k)} + (E_{ij}^{(\bar{k},\bar{k})})^{\gamma_{s_k}} \text{ and } q_{ij}^{(k,k)} := E_{ij}^{(k,k)} - (E_{ij}^{(\bar{k},\bar{k})})^{\gamma_{s_k}}.$$

It is straightforward to check that

$$(p_{ij}^{(k,k)})^{\gamma_{2\eta_m}} = p_{ij}^{(k,k)} \text{ and } (q_{ij}^{(k,k)})^{\gamma_{2\eta_m}} = -q_{ij}^{(k,k)}.$$

Since the set $\{p_{ij}^{(k,k)}, q_{ij}^{(k,k)} \,|\, i, j \in [1, s_k]\}$ is a basis of A_k, the $*$-standard basis is given by the union of the basis of $(A_k)^+_{(0)}$

$$\{p_{ij}^{(k,k)} \,|\, (i, j) \in [1, h_k]^2 \cup [h_k + 1, s_k]^2\},$$

of that of $(A_k)^-_{(0)}$

$$\{q_{ij}^{(k,k)} \,|\, (i, j) \in [1, h_k]^2 \cup [h_k + 1, s_k]^2\},$$

of that of $(A_k)^+_{(1)}$

$$\{p_{ij}^{(k,k)} \,|\, (i, j) \in [1, h_k] \times [h_k + 1, s_k] \cup [h_k + 1, s_k] \times [1, h_k]\}$$

and of that of $(A_k)^-_{(1)}$

$$\{q_{ij}^{(k,k)} \,|\, (i, j) \in [1, h_k] \times [h_k + 1, s_k] \cup [h_k + 1, s_k] \times [1, h_k]\}.$$

To construct the standard total product, as made in Case I start from a product $E_{11}E_{1j_2}E_{i_3j_3} \cdots E_{i_{s_k^2}j_{s_k^2}}$ of all the matrix units of M_{s_k} whose result is E_{11}. Hence replace the matrix units E_{ij} appearing there with $p_{ij}^{(k,k)}$. After that, multiply this product with the product of all the elements of the $*$-standard basis involving the $q_{ij}^{(k,k)}$'s in the same order used for the $p_{ij}^{(k,k)}$'s. With these positions one has that

$$\pi_{A_k} = \begin{cases} q_{11}^{(k,k)} & \text{if } s_k = 1, \\ E_{11}^{(k,k)} & \text{otherwise.} \end{cases}$$

CASE III. When A_k is isomorphic to $M_{n_k}+tM_{n_k}$ with involution $\circ$ coinciding with $\perp$ or $\top$, let us consider the map

$$\iota_k:[1,s_k]\longrightarrow[1,s_k],\qquad l\longmapsto\begin{cases}l+n_k & \text{if } l\in[1,n_k],\\ l-n_k & \text{otherwise}\end{cases}$$

and, for every $i,j\in[1,s_k]$, the elements

$$p_{ij}^{(k,k)}:=E_{ij}^{(k,k)}+E_{\iota_k(i),\iota_k(j)}^{(k,k)}+((E_{ij}^{(\bar{k},\bar{k})}+E_{\iota_k(i),\iota_k(j)}^{(\bar{k},\bar{k})})^{\circ})^{\gamma_{s_k}}.$$

In particular,

$$p_{ij}^{(k,k)}=p_{\iota_k(i),\iota_k(j)}^{(k,k)},\qquad (p_{ij}^{(k,k)})^{\gamma_{2\eta m}}=(p_{ij}^{(k,k)})^{\circ},$$

for all $u,v\in[1,s_k]$

$$p_{ij}^{(k,k)}p_{uv}^{(k,k)}=\delta_{ju}p_{iv}^{(k,k)}+\delta_{j,\iota_k(u)}p_{i,\iota_k(v)}^{(k,k)}$$

and a basis of A_k is given by the set $\{p_{ij}^{(k,k)},p_{i,\iota_k(j)}^{(k,k)}\,|\,i,j\in[1,n_k]\}$.

Assume first that $\circ=\perp$. Then

$$(p_{ij}^{(k,k)})^{\gamma_{2\eta m}}=\begin{cases}(p_{ij}^{(k,k)})^{\diamond} & \text{if } (i,j)\in[1,n_k]^2\cup[n_k+1,s_k]^2,\\ (-p_{ij}^{(k,k)})^{\diamond} & \text{if } (i,j)\in[1,n_k]\times[n_k+1,s_k]\cup[n_k+1,s_k]\times[1,n_k],\end{cases}$$

where $\diamond$ is the transpose or symplectic involution. In the former case, the $*$-standard basis is obtained collecting the sets

$$\{p_{ij}^{(k,k)}+p_{ji}^{(k,k)}\,|\,1\le i<j\le n_k\}\cup\{p_{ii}^{(k,k)}\,|\,i\in[1,n_k]\},$$

basis of $(A_k)^+_{(0)}$,

$$\{p_{ij}^{(k,k)}-p_{ji}^{(k,k)}\,|\,1\le i<j\le n_k\},$$

basis of $(A_k)^-_{(0)}$,

$$\{p_{i,\iota_k(j)}^{(k,k)}-p_{j,\iota_k(i)}^{(k,k)}\,|\,1\le i<j\le n_k\},$$

basis of $(A_k)^+_{(1)}$ and

$$\{p_{i,\iota_k(j)}^{(k,k)}+p_{j,\iota_k(i)}^{(k,k)}\,|\,1\le i<j\le n_k\}\cup\{p_{i,\iota_k(i)}^{(k,k)}\,|\,i\in[1,n_k]\},$$

basis of $(A_k)^-_{(1)}$.

If $\diamond$ is the symplectic involution, n_k is even and, set $n'_k:=\frac{n_k}{2}$, using the same notations introduced in Case I (replacing s_k with n_k and s'_k with n'_k in the definition of ρ_k), the $*$-standard basis is given by the union of the sets

$$\{p_{i,\rho_k(j)}^{(k,k)}-p_{j,\rho_k(i)}^{(k,k)}\,|\,1\le i<j\le n'_k\text{ or }n'_k+1\le i<j\le n_k\}\cup$$
$$\{p_{i,\rho_k(j)}^{(k,k)}+p_{j,\rho_k(i)}^{(k,k)}\,|\,1\le i\le n'_k<j\le n_k\},$$
$$\{p_{i,\rho_k(j)}^{(k,k)}+p_{j,\rho_k(i)}^{(k,k)}\,|\,1\le i<j\le n'_k\text{ or }n'_k+1\le i<j\le n_k\}\cup$$
$$\{p_{i,\rho_k(j)}^{(k,k)}-p_{j,\rho_k(i)}^{(k,k)}\,|\,1\le i\le n'_k<j\le n_k\}\cup\{p_{i,\rho_k(i)}^{(k,k)}\,|\,i\in[1,n_k]\},$$
$$\{p_{i,\iota_k\rho_k(j)}^{(k,k)}+p_{j,\iota_k\rho_k(i)}^{(k,k)}\,|\,1\le i<j\le n'_k\text{ or }n'_k+1\le i<j\le n_k\}\cup$$
$$\{p_{i,\iota_k\rho_k(j)}^{(k,k)}-p_{j,\iota_k\rho_k(i)}^{(k,k)}\,|\,1\le i\le n'_k<j\le n_k\}\cup\{p_{i,\iota_k\rho_k(i)}^{(k,k)}\,|\,i\in[1,n_k]\}$$

and

$$\{p^{(k,k)}_{i,\iota_k\rho_k(j)}-p^{(k,k)}_{j,\iota_k\rho_k(i)}\,|\,1\le i<j\le n'_k \,\text{or}\, n'_k+1\le i<j\le n_k\}\cup$$

$$\{p^{(k,k)}_{i,\iota_k\rho_k(j)}+p^{(k,k)}_{j,\iota_k\rho_k(i)}\,|\,1\le i\le n'_k<j\le n_k\},$$

basis of $(A_k)^+_{(0)}, (A_k)^-_{(0)}, (A_k)^+_{(1)}, (A_k)^-_{(1)}$, respectively.

When $\circ=\top$, one has that

$$(p^{(k,k)}_{ij})^{\gamma_{2\eta_m}}=(p^{(k,k)}_{ij})^{\diamond}$$

and nothing changes for the $*$-standard basis of $(A_k)_{(0)}$, whereas the $*$-standard basis of $((A_k)^+_{(1)},\perp)$ and $((A_k)^-_{(1)},\perp)$ coincide with that of $((A_k)^-_{(1)},\top)$ and $((A_k)^+_{(1)},\top)$, respectively.

In all these cases the standard total product is constructed multiplying first all the elements of the $*$-standard basis of $(A_k)_{(0)}$ and then all the elements of the $*$-standard basis of $(A_k)_{(1)}$ exactly as before. The final result is that

$$\pi_{A_k}=\begin{cases} p^{(k,k)}_{11}p^{(k,k)}_{1,\iota_k(1)}=p^{(k,k)}_{1,\iota_k(1)} & \text{if } \diamond \text{ is the transpose involution and } n_k \text{ is odd,}\\ p^{(k,k)}_{11}p^{(k,k)}_{11}=p^{(k,k)}_{11} & \text{otherwise.}\end{cases}$$

CASE IV. Finally, let $A_k=(M_{n_k}+tM_{n_k})\oplus(M_{n_k}+tM_{n_k})^{\text{op}}$ and consider, for all $i,j\in[1,s_k]$,

$$p^{(k,k)}_{ij}:=E^{(k,k)}_{ij}+E^{(k,k)}_{\iota_k(i),\iota_k(j)}+(E^{(\bar{k},\bar{k})}_{ij}+E^{(\bar{k},\bar{k})}_{\iota_k(i),\iota_k(j)})^{\gamma_{s_k}}$$

and

$$q^{(k,k)}_{ij}:=E^{(k,k)}_{ij}+E^{(k,k)}_{\iota_k(i),\iota_k(j)}-(E^{(\bar{k},\bar{k})}_{ij}+E^{(\bar{k},\bar{k})}_{\iota_k(i),\iota_k(j)})^{\gamma_{s_k}}.$$

In this case, $\{p^{(k,k)}_{ij}, q^{(k,k)}_{ij}, p^{(k,k)}_{i,\iota_k(j)}, q^{(k,k)}_{i,\iota_k(j)}\,|\,i,j\in[1,n_k]\}$ is a basis of A_k. Standard computations yield

$$p^{(k,k)}_{ij}=p^{(k,k)}_{\iota_k(i),\iota_k(j)},\qquad q^{(k,k)}_{ij}=q^{(k,k)}_{\iota_k(i),\iota_k(j)},$$

$$(p^{(k,k)}_{ij})^{\gamma_{2\eta_m}}=p^{(k,k)}_{ij} \text{ and } (q^{(k,k)}_{ij})^{\gamma_{2\eta_m}}=-q^{(k,k)}_{ij}.$$

The $*$-standard basis of A_k is the union of the basis of $(A_k)^+_{(0)}$

$$\{p^{(k,k)}_{ij}\,|\,i,j\in[1,n_k]\},$$

of that of $(A_k)^-_{(0)}$

$$\{q^{(k,k)}_{ij}\,|\,i,j\in[1,n_k]\},$$

of that of $(A_k)^+_{(1)}$

$$\{p^{(k,k)}_{i,\iota_k(j)}\,|\,i,j\in[1,n_k]\}$$

and of that of $(A_k)^-_{(1)}$

$$\{q^{(k,k)}_{i,\iota_k(j)}\,|\,i,j\in[1,n_k]\}.$$

The standard total product is produced multiplying first all the elements of the $*$-standard basis of $(A_k)^+_{(0)}$, then those of $(A_k)^-_{(0)}$, hence those of $(A_k)^+_{(1)}$ and finally those of $(A_k)^-_{(1)}$ as made in Case II. The final outcome is that

$$\pi_{A_k} = \begin{cases} p_{11}^{(k,k)} & \text{if } n_k = 1, \\ E_{11}^{(k,k)} + E_{\iota_k(1),\iota_k(1)}^{(k,k)} & \text{otherwise.} \end{cases}$$

Definition 5.1. *The basis of the $*$-superalgebra $A := UT^*_{\mathbb{Z}_2}(A_1, \ldots, A_m)$ consisting of the union of the $*$-standard basis of the simple $*$-superalgebras $A_1, \ldots, A_m$ with the set $\{E_{ij}^{(r,t)} \pm (E_{ij}^{(r,t)})^{\gamma_{2\eta_m}} \mid 1 \leq r < t \leq m,\, i \in [1, s_r], j \in [1, s_t]\}$ is said to be the* $*$-standard basis *of A.*

As recalled at the beginning of the discussion, replacing each symmetric (skew, respectively) element of degree j in the standard total product π_{A_k} with a symmetric (skew, respectively) variable of the degree j in $F\langle y_1^+, y_1^-, z_1^+, z_1^-, y_2^+, y_2^-, z_2^+, z_2^-, \ldots\rangle$, we construct the monomial m_{A_k} (in pairwise different variables). In particular, an evaluation of m_{A_k} (by basis elements of A_k) is said to be *standard total* if each variable of m_{A_k} is evaluated by the corresponding element of the $*$-standard basis of A_k appearing in π_{A_k}.

Now, for each $k \in [1, m]$, $\epsilon \in \{0, 1\}$ and $\mu \in \{+, -\}$, set

$$d_k^A := \dim_F A_k, \quad d_{k,\epsilon,\mu}^A := \dim_F (A_k)^\mu_{(\epsilon)}$$

and

$$d_{ss,\epsilon,\mu}^A := \dim_F (A_{ss})^\mu_{(\epsilon)} = \sum_{k \in [1,m]} d_{k,\epsilon,\mu}^A.$$

For any integer $\nu \geq m$ and $k \in [1, m]$, consider ν copies of m_{A_k} in pairwise disjoint sets of variables. For each $i \in [1, \nu]$, let us denote by $m_{A_k}^{(i)}$ the i-th copy of m_{A_k} and by $S(k, i)$ the set of variables of $m_{A_k}^{(i)}$, respectively. Clearly we can write $S(k, i) = \cup_{\epsilon \in \{0,1\}, \mu \in \{+,-\}} S(k, i, \epsilon, \mu)$, where $S(k, i, \epsilon, \mu)$ is the set of symmetric (skew, respectively) variables of degree ϵ in $S(k, i)$ if $\mu = +$ (if $\mu = -$, respectively). Hence

$$|S(k, i)| = d_k^A \qquad \text{and} \qquad |S(k, i, \epsilon, \mu)| = d_{k,\epsilon,\mu}^A.$$

Furthermore, for every $i \in [1, \nu]$, $\epsilon \in \{0, 1\}$ and $\mu \in \{+, -\}$, let $T(i, \epsilon, \mu) := \cup_{k \in [1,m]} S(k, i, \epsilon, \mu)$. Clearly,

$$|T(i, \epsilon, \mu)| = \sum_{k \in [1,m]} d_{k,\epsilon,\mu}^A = d_{ss,\epsilon,\mu}^A. \tag{5}$$

We notice that the standard total evaluations of each monomial $m_{A_k}^{(i)}$ give a graded evaluation of $m_{A_k}^{(1)} \cdots m_{A_k}^{(\nu)}$ such that $E_{11}^{(k,k)}$ is an addendum of $(m_{A_k}^{(1)} \cdots m_{A_k}^{(\nu)})^\uparrow$ for every $k \in [1, m]$ except when ν is odd, A_k is as in Case III, $\diamond$ is the transpose involution and n_k is odd. If the latter case occurs, we shall say that (k, ν) *is an exception* (in particular, in the sequel we shall refer to A_k as an exception if it is of the form presented above, regardless of ν).

For each $j \in [1, m-1]$, consider the symmetric homogeneous radical element $E_{11}^{(j,j+1)} + (E_{11}^{(j,j+1)})^{\gamma_{2\eta_m}}$ of A if (j, ν) is not an exception, and $E_{\iota_j(1),1}^{(j,j+1)} + (E_{\iota_j(1),1}^{(j,j+1)})^{\gamma_{2\eta_m}}$ otherwise. Let us call θ_j its degree and choose a symmetric variable u_j of degree θ_j such that all the elements of $(\cup_{i\in[1,\nu],k\in[1,m]} S(k,i)) \cup (\cup_{j\in[1,m-1]}\{u_j\})$ are pairwise different. Setting $U_j := T(j, \theta_j, +) \cup \{u_j\}$, one has that

$$|U_j| = d_{ss,\theta_j,+}^A + 1. \tag{6}$$

Moreover the above considerations show that if $m > 1$ there exists an evaluation of the $*$-graded polynomial

$$m_{A,\nu} := m_{A_1}^{(1)} \cdots m_{A_1}^{(\nu)}\, u_1\, m_{A_2}^{(1)} \cdots m_{A_2}^{(\nu)}\, u_2 \cdots\, u_{m-1}\, m_{A_m}^{(1)} \cdots m_{A_m}^{(\nu)}$$

equal to $E_{11}^{(1,m)}$ if (m, ν) is not an exception, and to $E_{1,\iota_m(1)}^{(1,m)}$ otherwise.

We extend the monomial $m_{A,\nu}$ by inserting $\nu(\dim_F A_{ss}) + m$ pairwise different variables of degree zero not involved in $m_{A,\nu}$, bordering each variable appearing in $m_{A,\nu}$. We remark that the only thing we are requiring is that these variables have degree zero, and not that they are symmetric or skew as well (in other words, we are inserting elements of the form $y_r^+ + y_r^-$ of $F\langle y_1^+, y_1^-, z_1^+, z_1^-, y_2^+, y_2^-, z_2^+, z_2^-, \ldots\rangle$). Let us denote the polynomial obtained in this way by $\varpi_{A,\nu}$. For each $k \in [1, m]$ and $i \in [1, \nu]$, let $V(k, i)$ be the set of all the variables inserted on the left of the variables of the set $S(k, i)$. Still, for each $k \in [1, m]$, let $\widetilde{v}_k$ be the variable inserted on the right of the monomial $m_{A_k}^{(\nu)}$ (and therefore, for each $k \in [1, m-1]$, on the left of the variable u_k) and set $V_k := \cup_{i\in[1,\nu]} V(k, i) \cup \{\widetilde{v}_k\}$.

For each $j \in [1, m-1]$ we alternate in $\varpi_{A,\nu}$ the variables of the set U_j and, for each $i \in [1, \nu]$, $\epsilon \in \{0, 1\}$ and $\mu \in \{+, -\}$ such that $(\epsilon, \mu) \neq (\theta_i, +)$ if $i < m$, those of $T(i, \epsilon, \mu)$, respectively. We denote the $*$-graded polynomial obtained in this way (which clearly depends upon ν) by $f_{A,\nu}$ and we refer to it as a *Kemer polynomial* for A.

Proposition 5.2. *For every integer $\nu \geq m$ the polynomial $f_{A,\nu}$ is not a $*$-graded polynomial identity for the $*$-superalgebra $A := UT_{\mathbb{Z}_2}^*(A_1, \ldots, A_m)$.*

Proof. For every $k \in [1, m]$ and $i \in [1, \nu]$, consider the standard total evaluation of the monomial $m_{A_k}^{(i)}$ in A (here and in the sequel, given a $*$-graded evaluation of a $*$-graded polynomial h, if D is a set of variables appearing in h we use $\overline{D}$ to denote the evaluations of the variables in D and $\overline{q}$ for the evaluation of the variable $q \in D$). According to the above construction, for each variable $s_t^{(i)} \in S(k, i)$ one has that $\overline{s}_t^{(i)}$ is equal to one among $p_{hl}^{(k,k)} \pm p_{ab}^{(k,k)}$, $p_{hl}^{(k,k)}$ and $q_{hl}^{(k,k)}$ for suitable integers a, b, h, l. In particular, in any case there is only one add term of $\overline{s}_t^{(i)}$ which gives a non-zero contribution to π_{A_k} (and for our choice in the construction of the standard total product it appears with sign $+$), let us call it $p_{hl}^{(k,k)}$ or $q_{hl}^{(k,k)}$. When A_k is as in Cases I or II evaluate the variable $v_t^{(i)} \in V(k, i)$ appearing on the left of $s_t^{(i)}$ by the element $\overline{E}_{hh}^{(k,k)}$ of the canonical basis of A introduced in Section 3. If A_k is as in Cases III or IV, we observe that $(p_{hl}^{(k,k)})^{\uparrow}$ and $(q_{hl}^{(k,k)})^{\uparrow}$ are sum of two matrix units and

$$(\overline{s}_1^{(i)} \cdots \overline{s}_{d_k^A}^{(i)})^{\uparrow} = \begin{cases} E_{1,\iota_k(1)}^{(k,k)} + E_{\iota_k(1),1}^{(k,k)} & \text{if } A_k \text{ is an exception,} \\ E_{11}^{(k,k)} + E_{\iota_k(1),\iota_k(1)}^{(k,k)} & \text{otherwise.} \end{cases}$$

This means that for each element $(\overline{s}_t^{(i)})^{\uparrow}$ there is exactly one add term, $E_{h_t l_t}^{(k,k)}$, such that

$$E_{h_1 l_1}^{(k,k)} \cdots E_{h_{d_k^A} l_{d_k^A}}^{(k,k)} = \begin{cases} E_{1,\iota_k(1)}^{(k,k)} & \text{if } A_k \text{ is an exception,} \\ E_{11}^{(k,k)} & \text{otherwise.} \end{cases}$$

As before, evaluate the variable $v_t^{(i)} \in V(k,i)$ appearing on the left of $s_t^{(i)}$ by $\overline{E}_{h_t h_t}^{(k,k)}$.

In all the cases, evaluate the variable $\widetilde{v}_k$ by $\overline{E}_{11}^{(k,k)}$. Finally, for every $j \in [1, m-1]$, set

$$\overline{u}_j := \begin{cases} E_{11}^{(j,j+1)} + (E_{11}^{(j,j+1)})^{\gamma_{2\eta_m}} & \text{if } (j,\nu) \text{ is not an exception,} \\ E_{\iota_j(1),1}^{(j,j+1)} + (E_{\iota_j(1),1}^{(j,j+1)})^{\gamma_{2\eta_m}} & \text{otherwise.} \end{cases}$$

Let us denote by e_A this $*$-graded evaluation of the variables of $\varpi_{A,\nu}$ (and of all the monomials of $f_{A,\nu}$ as well). In this way if $m > 1$ we get that the evaluation e_A of $\varpi_{A,\nu}$ in A is equal to $E_{11}^{(1,m)}$ if (m,ν) is not an exception, and to $E_{1,\iota_m(1)}^{(1,m)}$ otherwise. When $m = 1$ this evaluation gives $\overline{E}_{11}^{(1,1)}$ if $(1,\nu)$ is not an exception, and $\overline{E}_{1,\iota_1(1)}^{(1,1)}$ otherwise.

Now, fixed $i \in [1,\nu]$, $\epsilon \in \{0,1\}$ and $\mu \in \{+,-\}$ such that $(\epsilon,\mu) \neq (\theta_i,+)$ if $i < m$, take a permutation σ of the variables of $\varpi_{A,\nu}$ which possibly moves only the elements of $T(i,\epsilon,\mu)$ and assume that the evaluation e_A of $\sigma(\varpi_{A,\nu})$ in A is non-zero. We claim that σ must be the identity permutation. In fact, if $m > 1$, since $u_1, \ldots, u_{m-1}$ are not moved by σ and $aa' = 0_A$ for any $a \in A_r$ and $a' \in A_s$ when $r \neq s$, then σ permutes the variables of $S(k,i,\epsilon,\mu)$ for each $k \in [1,m]$. Obviously this is true when $m = 1$ as well. Moreover in each summand of $f_{A,\nu}$ the variables in V_k appear in the same order. Therefore, by the choice of the evaluation of the elements in V_k, we conclude that, in order to not get zero, the evaluation of each variable $s_t^{(i)}$ is uniquely determined by those of the bordering variables and by the type of $s_t^{(i)}$. Since we do not alternate variables of different type, the claim is confirmed.

Finally, when $m > 1$, applying the same arguments presented above one has that, for each $j \in [1, m-1]$ and for each non-trivial permutation σ' of the variables of the set U_j in $\varpi_{A,\nu}$, the evaluation e_A of $\sigma'(\varpi_{A,\nu})$ is zero.

The outcome of all these deductions is that, in the evaluation e_A, $\varpi_{A,\nu}$ is the unique summand of $f_{A,\nu}$ which does not vanish, and this concludes the proof. □

6. Proof of the main result

According to Theorem 4.3, to provide a characterization of minimal varieties of PI $*$-superalgebras of finite basic rank of fixed $*$-graded exponent, we need to prove that any $*$-superalgebra of the form $UT^*_{\mathbb{Z}_2}(A_1, \dots, A_m)$ generates a minimal variety.

The first result in this direction concerns the algebraic structure of the $*$-superalgebras $A := UT^*_{\mathbb{Z}_2}(A_1, \dots, A_m)$ and $B := UT^*_{\mathbb{Z}_2}(B_1, \dots, B_n)$ under a condition a bit weaker than we need, namely $\exp^*_{\mathbb{Z}_2}(A) \geq \exp^*_{\mathbb{Z}_2}(B)$ (we shall freely use in the sequel that $\exp^*_{\mathbb{Z}_2}(UT^*_{\mathbb{Z}_2}(A_1, \dots, A_m)) = \dim_F(A_1 \oplus \dots \oplus A_m)$).

Lemma 6.1. *Let $A := UT^*_{\mathbb{Z}_2}(A_1, \dots, A_m)$ and $B := UT^*_{\mathbb{Z}_2}(B_1, \dots, B_n)$ be $*$-superalgebras satisfying $\exp^*_{\mathbb{Z}_2}(A) \geq \exp^*_{\mathbb{Z}_2}(B)$ and $\nu := m + n - 1$. If $f_{A,\nu}$ is not a $*$-graded polynomial identity for B, then the following conditions hold:*

(a) *$d^A_{ss,\epsilon,\mu} = d^B_{ss,\epsilon,\mu}$ for every $\epsilon \in \{0, 1\}$ and $\mu \in \{+, -\}$;*
(b) *$n = m$;*
(c) *either $(A_1, \dots, A_m) = (B_1, \dots, B_m)$ or $(A_1, \dots, A_m) = (B_m, \dots, B_1)$. In particular, if the non-zero $*$-graded evaluation in B of $f_{A,\nu} \in J(B)^{\uparrow}$, then $(A_1, \dots, A_m) = (B_1, \dots, B_m)$ otherwise $(A_1, \dots, A_m) = (B_m, \dots, B_1)$.*

Proof. Since $f_{A,\nu}$ is a multilinear polynomial which is not in $\mathrm{Id}^*_{\mathbb{Z}_2}(B)$, there exists a non-zero $*$-graded evaluation e_B of $f_{A,\nu}$ by elements of the $*$-standard basis of B. Moreover we may assume, without loss of generality, that e_B is a non-zero evaluation of the monomial $\varpi_{A,\nu}$ of $f_{A,\nu}$. As $J(B)$ is nilpotent of index n and $|[m, m+n-1]| = n$, one has that there exists $m \leq \ell \leq m+n-1$ such that all the variables of $\cup_{\epsilon\in\{0,1\},\mu\in\{+,-\}} T(\ell, \epsilon, \mu) = \cup_{k\in[1,m]} S(k, \ell)$ as well as all those of $\cup_{k\in[1,m]} V(k, \ell)$ must be necessarily evaluated only by semisimple elements.

The fact that, for every $\epsilon \in \{0, 1\}$ and $\mu \in \{+, -\}$, $f_{A,\nu}$ alternates in the set $T(\ell, \epsilon, \mu)$ forces $|T(\ell, \epsilon, \mu)| \leq d^B_{ss,\epsilon,\mu}$, and thus (5) yields

$$d^A_{ss,\epsilon,\mu} = |T(\ell, \epsilon, \mu)| \leq d^B_{ss,\epsilon,\mu} \qquad \text{for all } \epsilon \in \{0, 1\} \text{ and } \mu \in \{+, -\}.$$

Combining this inequality with the original assumption we obtain

$$\exp^*_{\mathbb{Z}_2}(A) = \sum_{\substack{\epsilon\in\{0,1\}\\ \mu\in\{+,-\}}} d^A_{ss,\epsilon,\mu} \leq \sum_{\substack{\epsilon\in\{0,1\}\\ \mu\in\{+,-\}}} d^B_{ss,\epsilon,\mu} = \exp^*_{\mathbb{Z}_2}(B) \leq \exp^*_{\mathbb{Z}_2}(A)$$

and, consequently, that $d^A_{ss,\epsilon,\mu} = d^B_{ss,\epsilon,\mu}$ for every $\epsilon \in \{0, 1\}$ and $\mu \in \{+, -\}$, which concludes the proof of item (a).

Now, $\cup_{\epsilon\in\{0,1\},\mu\in\{+,-\}} \overline{T(\ell, \epsilon, \mu)} = \cup_{k\in[1,m]} \overline{S(k, \ell)}$ is an evaluation of the product $m^{(\ell)}_{A_1} \cdots m^{(\ell)}_{A_m}$ involving all (and only) the elements of the $*$-standard basis of B_{ss} (and, hence, each one exactly once). Furthermore, as $bb' = 0_B$ for any $b \in B_i$ and $b' \in B_j$ whenever $i \neq j$, for each $k \in [1, m]$ the monomial $m^{(\ell)}_{A_k}$ must be evaluated in a unique block of B_{ss}. At this stage assume, if possible, that $m < n$. Then there exist $h \in \overline{[1, m]}$ and at least two elements from different B_t's appearing in the same $\overline{S(h, \ell)}$, which is a contradiction. Hence $n \leq m$. This means that $n = 1$ in the case in which $m = 1$.

Therefore, assume that $m > 1$. Since, for every $j \in [1, m-1]$, $f_{A,\nu}$ alternates in the set U_j, which, according to (6), has cardinality

$$|U_j| = d^A_{ss,\theta_j,+} + 1 = d^B_{ss,\theta_j,+} + 1,$$

we must have that in e_B at least $m-1$ elements must be evaluated in $J(B)$. But this implies that $m-1$ is smaller than the nilpotent index of $J(B)$. The combination with the previously stated inequality gives item (b).

Finally, we claim that, for every $k \in [1, m]$, $\epsilon \in \{0, 1\}$ and $\mu \in \{+, -\}$ either $d^A_{k,\epsilon,\mu} = d^B_{k,\epsilon,\mu}$ or $d^A_{k,\epsilon,\mu} = d^B_{m+1-k,\epsilon,\mu}$. In fact, as we have seen above, one has that for every $k \in [1, m]$ the monomial $m^{(\ell)}_{A_k}$ must be evaluated in a unique block of B_{ss}, let us say B_{i_k}. Moreover, we observe that $B_1 J(B) B_2 \cdots B_{m-1} J(B) B_m \neq 0_B$ and $B_m J(B) B_{m-1} \cdots B_2 J(B) B_1 \neq 0_B$, whereas you get 0_B for every rearrangement of the B_j's into the above sequences. This means that either $m^{(\ell)}_{A_k}$ must be evaluated in B_k or it must be evaluated in B_{m-k+1}. Since $\overline{S(k, \ell)}$ is an evaluation of $m^{(\ell)}_{A_k}$ involving all (and only) the elements (each one exactly once) of the $*$-standard basis of B_k or B_{m-k+1}, respectively, the claim follows looking at the number of symmetric and skew variables of $m^{(\ell)}_{A_k}$ of degree ϵ. At this stage, the first part of (c) is an easy consequence of the fact that two finite-dimensional simple $*$-superalgebras A_k and B_k are isomorphic if, and only if, $\dim_F (A_k)^\mu_{(\epsilon)} = \dim_F (B_k)^\mu_{(\epsilon)}$ for every $\epsilon \in \{0, 1\}$ and $\mu \in \{+, -\}$.

In particular, if the non-zero $*$-graded evaluation in B of $f_{A,\nu} \in J(B)^\uparrow$, then $m^{(\ell)}_{A_k}$ must be evaluated in B_k for every $k \in [1, m]$, and hence $(A_1, \dots, A_m) = (B_1, \dots, B_m)$; otherwise it must be evaluated in B_{m-k+1} and, consequently, $(A_1, \dots, A_m) = (B_m, \dots, B_1)$. □

After a series of reductions, the crucial step consists of showing that, for any $\widetilde{g}$ and $\widetilde{k}$ elements of $\mathbb{Z}_2^m$ the $*$-superalgebras $UT^*_{\mathbb{Z}_2,\widetilde{g}}(A_1, \dots, A_m)$ and $UT^*_{\mathbb{Z}_2,\widetilde{k}}(A_1, \dots, A_m)$ are isomorphic as soon as the $T^*_{\mathbb{Z}_2}$-ideal of $*$-graded polynomial identities of the first algebra is a subset of that of the second one. To this end, using the notations of Section 3, for any simple $*$-superalgebra A_j, set $s'_j := \lfloor \frac{s_j}{2} \rfloor$, let σ_j be the element of $S_{2\eta_m}$ defined by

$$\prod_{\substack{i \in [1, s'_j] \\ \alpha \in \{\eta_{j-1}, 2\eta_m - \eta_j\}}} (\alpha + i \qquad \alpha + s_j - i + 1)$$

and $\vartheta_j : M_{2\eta_m} \longrightarrow M_{2\eta_m}$ the endomorphism defined on the matrix units of $M_{2\eta_m}$ as

$$\vartheta_j(E_{uv}) := E_{\sigma_j(u), \sigma_j(v)}.$$

Proposition 6.2. *Let $\widetilde{g}, \widetilde{k}$ be elements of $\mathbb{Z}_2^m$, $A := UT^*_{\mathbb{Z}_2,\widetilde{g}}(A_1, \ldots, A_m)$ and $B := UT^*_{\mathbb{Z}_2,\widetilde{k}}(B_1, \ldots, B_m)$, where $(B_1, \ldots, B_m) = (A_1, \ldots, A_m)$. If $Id^*_{\mathbb{Z}_2}(A) \subseteq Id^*_{\mathbb{Z}_2}(B)$, then A is isomorphic to B (as a $*$-superalgebra).*

Proof. Let $\widetilde{g} := (g_1, \ldots, g_m)$ and $\widetilde{k} := (k_1, \ldots, k_m)$. For any simple $*$-superalgebra A_j which is not of the form M_{h_j,l_j} or $M_{h_j,l_j} \oplus M^{op}_{h_j,l_j}$ with $h_j > l_j$, it is easily seen that the restriction of ϑ_j to A is a $*$-superalgebra isomorphism from A into $UT^*_{\mathbb{Z}_2,(g_1,\ldots,g_{j-1},g_j+1,g_{j+1},\ldots,g_m)}(A_1, \ldots, A_m)$. This implies that, when no simple $*$-superalgebra A_j is of such a form, $\prod_{j\in[1,m],g_j\neq k_j} \vartheta_j$ determines an isomorphic image of A which coincides with B.

Assume now that there exists a unique integer $t \in [1, m]$ such that A_t is equal either to M_{h_t,l_t} or $M_{h_t,l_t} \oplus M^{op}_{h_t,l_t}$ with $h_t > l_t$. If $g_t = k_t$, as above $\prod_{j\in[1,m],g_j\neq k_j} \vartheta_j$ is a $*$-graded isomorphism from A to B, and we are done. Therefore suppose that $g_t \neq k_t$. Set $\widetilde{g}^+ := (g_1 + 1, \ldots, g_m + 1)$, clearly $A = UT^*_{\mathbb{Z}_2,\widetilde{g}^+}(A_1, \ldots, A_m)$ and, since $g_t + 1 = k_t$, exactly as before we get a $*$-graded isomorphism from $UT^*_{\mathbb{Z}_2,\widetilde{g}^+}(A_1, \ldots, A_m)$ to B.

Finally, we are left with the case in which there exist $t \geq 2$ and $j_1 < \ldots < j_t \in [1, m]$ such that A_{j_r} coincides with $M_{h_{j_r},l_{j_r}}$ or $M_{h_{j_r},l_{j_r}} \oplus M^{op}_{h_{j_r},l_{j_r}}$ with $h_{j_r} > l_{j_r}$ for all $r \in [1, t]$. If $g_{j_q} = k_{j_q}$ ($g_{j_q} \neq k_{j_q}$, respectively) for every $q \in [1, t]$ applying the same arguments used above one has that A is isomorphic to B ($UT^*_{\mathbb{Z}_2,\widetilde{g}^+}(A_1, \ldots, A_m) = A$ is isomorphic to B, respectively). Therefore we can assume that there exists $q, q' \in [1, t]$ such that $g_{j_q} \neq k_{j_q}$ and $g_{j_{q'}} = k_{j_{q'}}$. We aim to show that in this case it cannot be $\mathrm{Id}^*_{\mathbb{Z}_2}(A) \subseteq \mathrm{Id}^*_{\mathbb{Z}_2}(B)$. Let us call p the smallest index q such that $g_{j_q} \neq k_{j_q}$ and set $\ell_2 := j_p$. We notice that we may assume that $\ell_2 > j_1$: in fact, if $\ell_2 = j_1$, consider the grading on A induced by $\widetilde{g}^+$ and so $g_{j_1} + 1 = k_{j_1}$. Furthermore, applying the same previously used arguments, replacing A with the isomorphic $*$-superalgebra $UT^*_{\mathbb{Z}_2,(k_1,\ldots,k_{\ell_2-1},g_{\ell_2},g_{\ell_2+1},\ldots,g_m)}(A_1, \ldots, A_m)$, we may assume that $g_i = k_i$ for every $i \in [1, \ell_2 - 1]$. Put $\ell_1 := j_{p-1}$.

Let us consider the Kemer polynomial $f_{B,m}$ and the polynomial $\varpi_{B,m}$ from which $f_{B,m}$ was constructed and call v_1 the bordering variable (of the degree zero) inserted on the left of the first variable of the monomial $m^{(1)}_{B_{\ell_1}}$ and v_2 that inserted on the right of the last variable of the monomial $m^{(m)}_{B_{\ell_2}}$ in $\varpi_{B,m}$. Replace in $f_{B,m}$ the variable v_1 with the polynomial $St_{2h_{\ell_1}-1}(y_1, \ldots, y_{2h_{\ell_1}-1})v_1$, where $St_{2h_{\ell_1}-1}(y_1, \ldots, y_{2h_{\ell_1}-1})$ is the Standard polynomial of degree $2h_{\ell_1} - 1$ in the pairwise distinct variables of degree zero $y_1, \ldots, y_{2h_{\ell_1}-1}$ not involved in $f_{B,m}$. Analogously, substitute v_2 with the polynomial $v_2 St_{2h_{\ell_2}-1}(y'_1, \ldots, y'_{2h_{\ell_2}-1})$, where $y'_1, \ldots, y'_{2h_{\ell_2}-1}$ are still pairwise distinct variables of degree zero different from $y_1, \ldots, y_{2h_{\ell_1}-1}$ and not involved in $f_{B,m}$ (in the sequel, we simply write $St_{2h_{\ell_i}-1}$ for the Standard polynomials above introduced). Let us denote this new $*$-graded polynomial by g_B. Since that in the non-zero $*$-graded evaluation of $f_{B,m}$ constructed in the proof of Proposition 5.2 $\overline{v}_i$ coincides with the element $\overline{E}^{(\ell_i,\ell_i)}_{11}$ of the canonical basis of B and there exists a graded evaluation of $St_{2h_{\ell_i}-1}$, by canonical basis elements of B_{ℓ_i}, equal to $\overline{E}_{11}^{(\ell_i,\ell_i)}$, we conclude that g_B is not a $*$-graded polynomial identity for B as well.

Before proceeding, let us look in more detail at the polynomials $\varpi_{B,m}$, $f_{B,m}$ and g_B. Using the same notations introduced in Section 5 for $f_{B,m}$ (obviously replacing A with B), for every $i \in [1,m], \epsilon \in \{0,1\}$ and $\mu \in \{+,-\}$, let $\widetilde{T}(i,\epsilon,\mu) := \cup_{k\in[\ell_1,\ell_2]} S(k,i,\epsilon,\mu)$ and, for all $j \in [\ell_1, \ell_2 - 1]$,

$$\widetilde{U}_j := \widetilde{T}(j, \theta_j, +) \cup \{u_j\}.$$

For each $j \in [\ell_1, \ell_2 - 1]$ we alternate in $\varpi_{B,m}$ the variables of the set $\widetilde{U}_j$ and, for each $i \in [1,m]$, $\epsilon \in \{0,1\}$ and $\mu \in \{+,-\}$ such that $(\epsilon,\mu) \neq (\theta_i,+)$ if $i \in [\ell_1, \ell_2 - 1]$, those of $\widetilde{T}(i,\epsilon,\mu)$, respectively. If $f_{B,m}^{[\ell_1,\ell_2]}$ is the $*$-graded polynomial obtained in this way, it is easy to observe that

$$f_{B,m} = \sum_{\sigma\in S_H} (-1)^\sigma \sigma(f_{B,m}^{[\ell_1,\ell_2]}) = \sum_{\sigma\in S_H} (-1)^\sigma \sigma(v \hat{f}_{B,m}^{[\ell_1,\ell_2]} w) \tag{7}$$

where H is a suitable set, v and w are $*$-graded polynomials, and $\hat{f}_{B,m}^{[\ell_1,\ell_2]}$ is the Kemer polynomial for $B^{[\ell_1,\ell_2]} := UT^*_{\mathbb{Z}_2,\widetilde{k}^{[\ell_1,\ell_2]}}(B_{\ell_1},\dots,B_{\ell_2})$, where $\widetilde{k}^{[\ell_1,\ell_2]} := (k_{\ell_1},\dots,k_{\ell_2})$. Furthermore, set

$$\hat{g}_B^{[\ell_1,\ell_2]} := St_{2h_{\ell_1}-1} \hat{f}_{B,m}^{[\ell_1,\ell_2]} St_{2h_{\ell_2}-1},$$

from (7) it follows that

$$g_B = \sum_{\sigma\in S_H} (-1)^\sigma \sigma(v St_{2h_{\ell_1}-1} \hat{f}_{B,m}^{[\ell_1,\ell_2]} St_{2h_{\ell_2}-1} w) = \sum_{\sigma\in S_H} (-1)^\sigma \sigma(v \hat{g}_B^{[\ell_1,\ell_2]} w).$$

According to Proposition 5.2 and its proof, there exists a $*$-graded evaluation of $\hat{f}_{B,m}^{[\ell_1,\ell_2]}$ in $B^{[\ell_1,\ell_2]}$ equal to $E_{11}^{(\ell_1,\ell_2)}$ and, consequently, one of $\hat{g}_B^{[\ell_1,\ell_2]}$ with the same value (it is sufficient to take again the evaluations of $St_{2h_{\ell_i}-1}$ equal to $\overline{E}_{11}^{(\ell_i,\ell_i)}$). This means that $\hat{g}_B^{[\ell_1,\ell_2]}$ is a $*$-graded polynomial homogeneous of degree $|E_{11}^{(\ell_1,\ell_2)}|_{B^{[\ell_1,\ell_2]}} = k_{\ell_2} - k_{\ell_1}$ with respect the $\mathbb{Z}_2$-grading of the free $*$-superalgebra.

Now, we have proved that g_B is not in $\mathrm{Id}^*_{\mathbb{Z}_2}(B)$. As $\mathrm{Id}^*_{\mathbb{Z}_2}(A) \subseteq \mathrm{Id}^*_{\mathbb{Z}_2}(B)$, g_B is not in $\mathrm{Id}^*_{\mathbb{Z}_2}(A)$ as well.

At this stage, take a non-zero $*$-graded evaluation of g_B (here if h is a $*$-graded polynomial, for simplicity we use $\overline{h}$ for the value of a fixed evaluation of its) and assume first that $\overline{g_B}^{\uparrow} \neq 0_A$. Look at the $*$-graded evaluations of $\hat{g}_B^{[\ell_1,\ell_2]}$ in $A^{[\ell_1,\ell_2]} := UT^*_{\mathbb{Z}_2,\widetilde{g}^{[\ell_1,\ell_2]}}(A_{\ell_1},\dots,A_{\ell_2})$ such that $\overline{\hat{g}_B^{[\ell_1,\ell_2]}}^{\uparrow} \neq 0_{A^{[\ell_1,\ell_2]}}$. Since $\hat{f}_{B,m}^{[\ell_1,\ell_2]}$ is a Kemer polynomial for $B^{[\ell_1,\ell_2]}$, then any its non-zero $*$-graded evaluation with $\overline{\hat{f}_B^{[\ell_1,\ell_2]}}^{\uparrow} \neq 0_{A^{[\ell_1,\ell_2]}}$ is such that $\overline{\hat{f}_B^{[\ell_1,\ell_2]}}^{\uparrow}$ is a linear combination of the matrix units $E_{ij}^{(\ell_1,\ell_2)}$ of $A^{[\ell_1,\ell_2]}$ with $i \in [1, h_{\ell_1} + l_{\ell_1}]$ and $j \in [1, h_{\ell_2} + l_{\ell_2}]$. Thus in any possible $*$-graded evaluation of $\hat{g}_B^{[\ell_1,\ell_2]}$ in $A^{[\ell_1,\ell_2]}$ such that $\overline{\hat{g}_B^{[\ell_1,\ell_2]}}^{\uparrow} \neq 0_{A^{[\ell_1,\ell_2]}}$, $St_{2h_{\ell_i}-1}$ must be evaluated in A_{ℓ_i}. But from

Amitsur-Levitzki Theorem the non-zero evaluations of $St_{2h_{\ell_i}-1}$ are linear combination of the matrices $\overline{E}_{uv}^{(\ell_i,\ell_i)}$ with $u,v\in[1,h_{\ell_i}]$. As $\hat{g}_B^{[\ell_1,\ell_2]}$ is homogeneous of degree $k_{\ell_2}-k_{\ell_1}\neq g_{\ell_2}-g_{\ell_1}$, this forces to be $\overline{\hat{g}_B^{[\ell_1,\ell_2]}}^{\uparrow}=0_{A^{[\ell_1,\ell_2]}}$. Consequently, we must have $\overline{g_B}^{\uparrow}=0_A$, which contradicts the original assumption.

Finally, suppose that $\overline{g_B}^{\downarrow}\neq 0_A$. The same arguments used in the prooof of Lemma 6.1 (c) show that $(A_1,\dots,A_m)=(B_m,\dots,B_1)=(A_m,\dots,A_1)$. Set

$$K:=\left\{\begin{pmatrix}a&0\\0&b\end{pmatrix}\mid a,b\in M_{\eta_m}\right\}\subseteq M_{2\eta_m},$$

we notice that for any $\alpha:[1,2\eta_m]\longrightarrow\mathbb{Z}_2$ the map

$$\chi:K\longrightarrow K,\ \begin{pmatrix}a&0\\0&b\end{pmatrix}\longmapsto\begin{pmatrix}b&0\\0&a\end{pmatrix}$$

is a $\mathbb{Z}_2$-graded isomorphism of the subalgebras of $M_{2\eta_m}$ with elementary grading (K,α) and (K,β) where, for every $i\in[1,2\eta_m]$,

$$\beta(i):=\begin{cases}\alpha(i+\eta_m)&\text{if } i\in[1,\eta_m],\\ \alpha(i-\eta_m)&\text{otherwise.}\end{cases}$$

In particular, the product of the map χ with $\prod_{j\in[1,m]}\vartheta_j$ induces an isomorphism of $*$-superalgebras from A into $A':=UT^*_{\mathbb{Z}_2,(g_m,\dots,g_1)}(A_1,\dots,A_m)$. But now g_B has a non-zero $*$-graded evaluation in A' such that $\overline{g_B}^{\uparrow}\neq 0_{A'}$, and hence we can apply the same previously used considerations. □

We are now in a position to state the main result of the paper (we recall that, as remarked in Section 2, we can extend the basis field and this has no effect on the codimensions. Therefore assume throughout that F is algebraically closed).

Proof of Theorem 2.2. Let $A:=UT^*_{\mathbb{Z}_2,\widetilde{g}}(A_1,\dots,A_m)$ and consider a subvariety $\mathcal{U}^*_{\mathbb{Z}_2}\subseteq\mathcal{V}^*_{\mathbb{Z}_2}(A)$ such that $\exp^*_{\mathbb{Z}_2}(\mathcal{V}^*_{\mathbb{Z}_2}(A))=\exp^*_{\mathbb{Z}_2}(\mathcal{U}^*_{\mathbb{Z}_2})$. Since $\mathcal{V}^*_{\mathbb{Z}_2}(A)$ satisfies a Capelli identity of some rank, according to Theorem 5.2 of [8] we conclude that $\mathcal{U}^*_{\mathbb{Z}_2}$ has finite basic rank. Applying Theorem 4.3 we get that there exist finite-dimensional simple $*$-superalgebras $B_1,\dots,B_n$ and an element $\widetilde{k}:=(k_1,\dots,k_n)\in\mathbb{Z}_2^n$ such that, set $B:=UT^*_{\mathbb{Z}_2,\widetilde{k}}(B_1,\dots,B_n)$, $\mathrm{Id}^*_{\mathbb{Z}_2}(\mathcal{U}^*_{\mathbb{Z}_2})\subseteq\mathrm{Id}^*_{\mathbb{Z}_2}(B)$ and $\exp^*_{\mathbb{Z}_2}(\mathcal{U}^*_{\mathbb{Z}_2})=\exp^*_{\mathbb{Z}_2}(B)$. From Lemma 6.1 it follows that $m=n$ and either $(A_1,\dots,A_m)=(B_1,\dots,B_m)$ or $(A_1,\dots,A_m)=(B_m,\dots,B_1)$.

Now, as observed in the proof of Proposition 6.2, the product of χ with $\prod_{j\in[1,m]}\vartheta_j$ is a $*$-superalgebras isomorphism from $UT^*_{\mathbb{Z}_2,\widetilde{k}}(B_1,\dots,B_m)$ into $UT^*_{\mathbb{Z}_2,(k_m,\dots,k_1)}(B_m,\dots,B_1)$. This means that, eventually replacing $(k_1,\dots,k_m)$ with $(k_m,\dots,k_1)$, we can always assume that $A_i=B_i$ for all $i\in[1,m]$.

At this stage, the hypothesis of Proposition 6.2 are satisfied for A and B and hence we can conclude that they are isomorphic as $*$-superalgebras. Consequentely, $\mathrm{Id}^*_{\mathbb{Z}_2}(A)=\mathrm{Id}^*_{\mathbb{Z}_2}(B)$.

Conversely, if $\mathcal{V}^*_{\mathbb{Z}_2}$ is a minimal variety of PI $*$-superalgebras of finite basic rank of fixed $*$-graded exponent, the result has been proved in Theorem 4.3. $\square$

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