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Stochastic Representation Formulas for Viscosity Solutions to Nonlinear Partial Differential Equations

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Introduction

It is well known that viscosity solutions were conceived by Crandall and Lions (1982) in the framework of optimal control theory. The goal was to show well posedness of Hamilton–Jacobi–Bellman equations in the whole space, and to prove, via dynamic programming principle, the value function of a suitable optimal control problem being the unique solution.

When trying to extend viscosity methods to the analysis of second order *parabolic partial differential equations*, PDEs for short, and getting representation formulae, it appeared clear that some stochastic dynamics must be brought into play. Not surprisingly, this has been first done for stochastic control models. The hard work of a generation of mathematicians, [6, 13, 19, 21, 22, 25] among the others, allowed making effective dynamic programming approach to stochastic control problems.

Prompted by this body of investigations, a stream of research arose in the probabilistic community ultimately leading to the theory of *backward stochastic differential equations*, BSDEs for short, which was introduced by Pardoux and Peng in [26] (1990). Since then, it has attracted a great interest due to its connections with mathematical finance and PDEs, as well as with stochastic control. This theory has been in particular used to extend the classical Feynman–Kac formula, which establishes a link between linear parabolic PDEs and *stochastic differential equations*, SDEs for short, to semilinear and quasilinear equations, see for example [10, 11, 23]. See also [28] for a rather complete overview of the semilinear case.

For sake of clarity, let us consider the following semilinear parabolic PDE

problem coupled with final conditions,

$$
\begin{cases}
\dfrac{1}{2}\left\langle \sigma\sigma^{\dagger}, D_x^2 u \right\rangle (t,x) + (\nabla_x u b)(t,x) & \\
\qquad + \partial_t u(t,x) + f(t,x,u,\nabla_x u \sigma) = 0 & t \in [0,T], x \in \mathbb{R}^N, \\
u(T,x) = g(x), & x \in \mathbb{R}^N,
\end{cases}
\tag{1}
$$

then its *viscosity solution*, see [8, 18], can be written as $u(t,x) = \mathbb{E}\left(Y_t^{t,x}\right)$, where Y is given by the following system, called *forward backward stochastic differential equation* or FBSDE in short, which is made in turn of two equations, the first one is a SDE, and the second one a BSDE depending on the first one

$$
\begin{cases}
X_s^{t,x} = x + \displaystyle\int_t^s \sigma\left(r, X_r^{t,x}\right) dW_r + \int_t^s b\left(r, X_r^{t,x}\right) dr, \\
Y_s^{t,x} = g\left(X_T^{t,x}\right) + \displaystyle\int_s^T f\left(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\right) dr - \int_s^T Z_r^{t,x} dW_r,
\end{cases}
\quad s \in [t,T], x \in \mathbb{R}^N.
$$

As it can be intuitively seen, the SDE takes care of the linear operator defined by σ and b, also called the *infinitesimal generators* of the SDE, while the BSDE depends on f and g. In other words, this extension of the Feynman–Kac formula basically does not modify the treatment of the second order linear operator with respect to the completely linear case.

Subsequently, Peng introduced in [30] (2006) the notion of G–*expectation*, a nonlinear expectation generated by a fully nonlinear second order operator G via its viscosity solutions. This work has originated an active research field, with relevant applications to Mathematical Finance.

Peng has improved this theory in several papers and has given a comprehensive account of it in the book [31], where he highlights the role of the so–called *sublinear expectations*, namely G–expectations generated by sublinear operators. Finally in [12], Peng provides representation formulas for viscosity solutions using these expectations. More precisely, given a sublinear operators G and the G–heat equation

$$
\begin{cases}
\partial_t u(t,x) + G\left(D_x^2 u(t,x)\right) = 0, & t \in [0,T], x \in \mathbb{R}^N, \\
u(T,x) = g(x), & x \in \mathbb{R}^N.
\end{cases}
\tag{2}
$$

he represents the viscosity solution as

$$u(t,x) = \sup_{\sigma \in \mathcal{A}} \mathbb{E}\left(g\left(x + \int_t^T \sigma_s dW_s \right) \right),$$

where $\mathcal{A}$ is a family of stochastic process associated to G and W is a Brownian motion. The key to prove it is a *dynamic programming principle* that we will illustrate in section 2.2. We point out that here the novelty with respect to the Feynman–Kac formula is essentially given by the sublinearity of the operator G.

The main purpose of this thesis is to provide a method for representing viscosity solutions to second order nonlinear PDEs given as inf or sup–envelope function. This is the case of PDEs with sublinear operators or second order Hamilton–Jacobi equations. The method is based upon a generalized version of the dynamic programming principle of [12]. It will be tested on various problems, mostly using FBSDEs.

This is hopefully just a first step to further extend the Feynman–Kac formula to general fully nonlinear PDE using the BSDE theory. In this respect we also point out that Cheridito, Soner, Touzi and Victoir in [7] have introduced a second order BSDE to give stochastic representation of solutions to fully nonlinear parabolic PDEs.

The thesis is organized a follows: in chapter 1 we introduce the stochastic setting and give a characterization of sublinear operators preliminary to our analysis. Chapter 2 is devoted to present our method and to give us representation formulas of viscosity solutions to various problems. Starting with a problem which is slightly more general than (2), we then approach parabolic PDE problems of the type

$$\begin{cases} \partial_t u(t,x) + F\left(t, x, \nabla_x u, D_x^2 u\right) \\ \qquad\qquad + f(t, x, u, \nabla_x u) = 0, & t \in [0, T], x \in \mathbb{R}^N, \\ u(T, x) = g(x), & x \in \mathbb{R}^N, \end{cases} \tag{3}$$

where F is a sublinear operator, with respect the third and the fourth argument. This problem is clearly a blend between (1) and (2), where the

additional difficulty with respect to (1) is given by the sublinearity of the operator, while the generalization with respect to (2) is the dependence of F on (t, x) of and the presence of the term f. We subsequently deal with an elliptic version of (3), which are much harder to treat, due to the presence of stopping times. Then, using a sample problem, we show that it is possible by means of our method to retrieve already known representation formulas of viscosity solutions to some second order Hamilton–Jacobi problems. We finally conclude chapter 2 with the analysis of a problem with singular boundary conditions, inspired by the seminal paper of Lasry and Lions [20]. The appendices provide a compendium of the SDE and BSDE results we use.

Contents

Notation

Here we fix the notation and some conventions that we will use later. Assume that we have a arbitrary filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,\infty)}, \mathbb{P})$, then

- given two random variables X and Y we will write $X \overset{d}{=} Y$ if they are equal in distribution;

- we will say that a stochastic process $\{H_t\}_{t \in [0,\infty)}$ is adapted if H_t is $\mathcal{F}_t$–measurable for any $t \in [0, \infty)$;

- we will say that a stochastic process $\{H_t\}_{t \in [0,\infty)}$ is progressively measurable, or simply progressive, if, for any $T \in [0, \infty)$, the application that to any $(t, \omega) \in [0, T] \times \Omega$ associate $H_t(\omega)$ is $\mathcal{B}([0, T]) \times \mathcal{F}_T$–measurable, where $\mathcal{B}([0, T])$ is the Borel σ–algebra of $[0, T]$;

- a function on $\mathbb{R}$ is called *cadlag* if is right continuous and has left limit everywhere;

- a cadlag (in time) process is progressive if and only if is adapted;

- $\mathcal{F}_\infty := \sigma(\mathcal{F}_t | t \in [0, \infty))$;

- $\mathbb{L}_N^p(T) := \left\{ H \in L^p \left([0, T] \times \Omega; \mathbb{R}^N \right) : H \text{ is progressive} \right\}$;

- $\mathbb{L}_N^p := \bigcap_{T \in [0,\infty)} \mathbb{L}_N^p(T)$;

- given a stopping time τ we will also define $\mathbb{L}_N^p(\tau)$ as the set made up by the progressive processes H such that $\mathbb{E}\left(\int_0^\tau |H_t|^p dt \right) < \infty$;

- μ–a.e. denotes "almost everywhere for the measure μ", but we will usually omit the reference to μ when there is no ambiguity;

- $B_\delta(x)$ will denote an open ball centered in x with radius δ;

- given a function f defined on a set D and a subset E of D, we will denote with $f|_E$ the restriction of f to E;

- for any Lipschitz continuous function f we will denote its Lipschitz constant as $\mathrm{Lip}(f)$;

- for any $v, w \in \mathbb{R}^N$ we denote with $v \otimes w$ the matrix $(v_i w_j)$;

- if $A \in \mathbb{R}^{N \times M}$ then $A^\dagger$ will denote its transpose and eig_A its spectrum;

- (Frobenius product) if $A, B \in \mathbb{R}^{N \times M}$ then

$$\langle A, B \rangle := \mathrm{tr}\left(AB^\dagger\right) = \sum_{i=1}^{N} \sum_{j=1}^{M} A_{i,j} B_{i,j};$$

- if $A \in \mathbb{R}^{N \times M}$ then $|A|$ will denote the norm $\sqrt{\langle A, A \rangle} = \sqrt{\sum_{i=1}^{N} \sum_{j=1}^{M} A_{i,j}^2}$;

- $\mathbb{S}^N$ is the set made up by all the symmetric matrices of $\mathbb{R}^{N \times N}$ and $\mathbb{S}^N_+$ is its the subset made up by all the positive definite matrices.

Chapter 1

Preliminaries

1.1 The Stochastic Setting

We will work on the filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,\infty)}, \mathbb{P} \right)$, where $\mathcal{F}$ is a complete σ–algebra, $\{\mathcal{F}_t\}_{t\in[0,\infty)}$ is the filtration defined by W which satisfies the usual condition of completeness and right continuity and the stochastic process $\{W_t\}_{t\in[0,\infty)}$ denote the N dimensional Brownian motion on this space. Let $\mathcal{F}_\infty$ be the σ–algebra $\sigma(\mathcal{F}_t : t \in [0,\infty))$ and additionally define, for any t in $[0,\infty)$, the processes $\{W_s^t\}_{s\in[t,\infty)} := \{W_s - W_t\}_{s\in[t,\infty)}$, which are again Brownian motions, the filtration $\{\mathcal{F}_s^t\}_{s\in[t,\infty)}$ generated by these processes and, similarly as before, $\mathcal{F}_\infty^t := \sigma(\mathcal{F}_s^t : s \in [t,\infty))$. Notice that the filtrations $\{\mathcal{F}_s^t\}_{s\in[t,\infty)}$, which we assume satisfies the usual condition, are independent from $\mathcal{F}_t$ for any $t \in [0,\infty)$. Furthermore we define, for any non negative stopping time τ, the σ–algebra

$$\mathcal{F}_\tau := \{A \in \mathcal{F} : A \cap \{\tau \le t\} \in \mathcal{F}_t, t \in [0,\infty)\}.$$

It is well known that for the Brownian motion holds the *strong Markov property*, that is to say that, for any a.e. finite stopping time τ, the processes $\{W_t^\tau\}_{t\in[0,\infty)} := \{W_{\tau+t} - W_\tau\}_{t\in[0,\infty)}$ are Brownian motions independent from $\mathcal{F}_\tau$. Thus, as previously done, we can define the filtrations $\{\mathcal{F}_t^\tau\}_{t\in[0,\infty)}$ generated by these processes and the σ–algebras $\mathcal{F}_\infty^\tau := \sigma(\mathcal{F}_t^\tau : t \in [0,\infty))$ for

any a.e. finite stopping time τ. As always we assume that the filtration $\{\mathcal{F}_t^\tau\}_{t\in[0,\infty)}$ satisfies the usual condition. For these filtrations the following lemma holds.

Lemma 1.1.1. *If τ is an a.e. finite stopping time, then the σ–algebras $\mathcal{F}_\tau$ and $\mathcal{F}_t^\tau$ are independent for any $t \in [0, \infty)$. Furthermore $\mathcal{F}_\infty = \sigma(\mathcal{F}_\tau, \mathcal{F}_\infty^\tau)$.*

Proof. Consider the process $\{W_{\tau\wedge t}\}_{t\in[0,\infty)}$ and the filtration $\{\mathcal{F}_{\tau\wedge t}\}_{t\in[0,\infty)}$ generated by this. By construction we have that

$$
\begin{aligned}
\mathcal{F}_{\tau\wedge t} &= \{A \in \mathcal{F}_t : A \cap \{\tau = s\} \in \mathcal{F}_s, s \in [0, t]\} \\
&= \{A \in \mathcal{F}_t : A \cap \{\tau \le s\} \in \mathcal{F}_s, s \in [0, t]\},
\end{aligned}
\tag{1.1}
$$

which correspond to the definition of the σ–algebra defined by the stopping time $\tau \wedge t$. Note that, if $A \in \mathcal{F}_\tau$,

$$
A \cap \{\tau < \infty\} = \bigcup_{t=0}^{\infty}(A \cap \{\tau \le t\}) \in \bigcup_{t=0}^{\infty}\mathcal{F}_t,
$$

thus, since $A = A \cap \{\tau < \infty\} \cup B$ where B is a set of zero measure, $A \in \mathcal{F}_\infty$ and $\mathcal{F}_\tau \subseteq \mathcal{F}_\infty$. From this and (1.1) we get

$$
\sigma(\mathcal{F}_{\tau\wedge t} : t \in [0, \infty)) = \{A \in \mathcal{F}_\infty : A \cap \{\tau \le t\} \in \mathcal{F}_t, t \in [0, \infty)\} = \mathcal{F}_\tau,
$$

therefore, since for any $t, s \in [0, \infty)$ $\mathcal{F}_s^\tau$ and $\mathcal{F}_{\tau\wedge t}$ are independent, so are $\mathcal{F}_s^\tau$ and $\mathcal{F}_\tau$. Finally, to prove that $\mathcal{F}_\infty = \sigma(\mathcal{F}_\tau, \mathcal{F}_\infty^\tau)$, is enough to note that $\mathcal{F}_\infty \supseteq \sigma(\mathcal{F}_\tau, \mathcal{F}_\infty^\tau)$ and, since $W_t = W_t - W_{\tau\wedge t} + W_{\tau\wedge t}$ for any $t \in [0, \infty)$, $\mathcal{F}_\infty \subseteq \sigma(\mathcal{F}_\tau, \mathcal{F}_\infty^\tau)$. $\square$

The following is an important result on the filtration of Brownian motions:

Theorem 1.1.2 (Blumenthal's 0–1 law)**.** *For any a.e. finite stopping time τ and $A \in \mathcal{F}_0^\tau$, either $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = 1$.*

An important consequence of this theorem that we will exploit later is the fact that any $\mathcal{F}_0$–measurable function is an a.e. constant.

Let $\mathbb{L}_N^p(T)$ be the subset of $L^p\left([0, T] \times \Omega; \mathbb{R}^N\right)$ such that its elements are progressively measurable with respect to the filtration $\{\mathcal{F}_t\}_{t\in[0,\infty)}$ and define

$\mathbb{L}_N^p := \bigcup_{T \in [0,\infty)} \mathbb{L}_N^p(T)$. Since the limit of progressive processes is progressive, these are complete spaces. Moreover, for any $G \in \mathbb{L}_{M \times N}^2$, $\int_0^t G_s dW_s$ will denote the Itô's integral of G with respect to W. The properties of the Itô's integral are well known and can be found in many probability textbooks, the only result that we recall is the *Itô's formula*: given the Itô's process $X_t := X_0 + \int_0^t G_s dW_s + \int_0^t F_s ds$, where $G \in \mathbb{L}_{N \times N}^2$ and $F \in \mathbb{L}_N^1$, and a derivable function φ, we have that

$$\varphi(t, X_t) = \varphi(0, X_0) + \frac{1}{2} \int_0^t \langle G_s G_s^\dagger, D_x^2 \varphi(s, X_s) \rangle \, ds + \int_0^t \nabla_x \varphi(s, X_s) G_s dW_s$$
$$+ \int_0^t \nabla_x \varphi(s, X_s) F_s ds + \int_0^t \partial_t \varphi(s, X_s) ds.$$

1.2 Sublinear Operators

The purpose of this section is to characterize the operators of some problems with we will deal later.

We consider the space $\mathbb{R}^N \times \mathbb{S}^N$ with the inner product

$$((p, S), (p', S')) := \frac{1}{2} \langle S, S' \rangle + p^\dagger p'$$

and the norm $\|(p, S)\| := \sqrt{((p, S), (p, S))}$.

Assumptions 1.2.1. In this section we will focus on the study of continuous operators of the form

$$F : [0, \infty) \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{S}^N \to \mathbb{R}$$

that satisfy these properties hold true for any $(t, x) \in [0, \infty) \times \mathbb{R}^N$, (p, S) and (p', S') in $\mathbb{R}^N \times \mathbb{S}^N$:

(i) *(Convexity)* If $\delta \in [0, 1]$ then

$$F(t, x, \delta p + (1 - \delta)p', \delta S + (1 - \delta)S') \leq \delta F(t, x, p, S) + (1 - \delta)F(t, x, p', S');$$

(ii) *(Positive Homogeneity)* If $\delta \geq 0$ then $F(t, x, \delta p, \delta S) = \delta F(t, x, p, S)$;

(iii) *(Ellipticity)* If $S \leq S'$, $F(t, x, p, S) \leq F(t, x, p, S')$.

Note that items (i) and (ii) imply

(iv) *(Subadditivity)* $F(t, x, p + p', S + S') \leq F(t, x, p, S) + F(t, x, p', S')$.

The operators which satisfy conditions (ii) and (iv) are commonly known as *sublinear operators*. Notice that items (ii) and (iv) imply (i), i.e. every sublinear operator is also convex.

It is common to also ask that F is uniformly elliptic with ellipticity constants $\lambda \in (0, \infty)$ and $\Lambda \in (\lambda, \infty]$ (note that if $\lambda = 0$ and $\Lambda = \infty$ we have condition (iii)):

(v) *(Uniform Ellipticity)* If $S' \geq 0$,

$$\lambda |S'| \leq F(t, x, p, S + S') - F(t, x, p, S) \leq \Lambda |S'|,$$

for any $(t, x, p, S) \in [0, \infty) \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{S}^N$.

Finally we require a Lipschitz continuity condition: for any (t, x, p, S) in $[0, \infty) \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{S}^N$ we have that there exists a positive constant ℓ such that

(vi) *(Lipschitz Continuity)* for any y in $\mathbb{R}^N$

$$|F(t, x, p, S) - F(t, y, p, S)| \leq \ell |x - y| \|(p, S)\|.$$

The main result of this section is the following characterization theorem:

Theorem 1.2.2. *Let F be as in assumptions 1.2.1 and K_F be the set of the elements $(b, a) \in C^0 \left([0, \infty) \times \mathbb{R}^N; \mathbb{R}^N\right) \times C^0 \left([0, \infty) \times \mathbb{R}^N; \mathbb{S}_+^N\right)$ such that, for any $(t, x, p, S) \in [0, \infty) \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{S}^N$,*

$$\frac{1}{2} \langle a(t, x), S \rangle + p^\dagger b(t, x) \leq F(t, x, p, S),$$

$\mathrm{Lip}(b(t)) \leq 2\ell$, $\mathrm{Lip}(a(t)) \leq 2\sqrt{2}\ell$ *and the eigenvalues of $a(t, x)$ are in $[2\lambda, 2\Lambda]$. Then K_F is a non empty and convex set, and*

$$F(t, x, p, S) = \max_{(b, a) \in K_F} \frac{1}{2} \langle a(t, x), S \rangle + p^\dagger b(t, x)$$

for any $(t, x, p, S) \in [0, \infty) \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{S}^N$. Furthermore for each $(b, a) \in K_F$ the linear operator

$$(t, x, p, S) \mapsto \frac{1}{2} \langle a(t, x), S \rangle + p^\dagger b(t, x)$$

has the same ellipticity conditions of F.

That sublinear functions are support functions is a well known result in functional analysis and is the basis of theorem 1.2.2, however we will write down its proof for completeness in the next lemma. What theorem 1.2.2 adds to this is that if we a have sublinear function F which also depends on some parameters, $t \in [0, \infty)$ and $x \in \mathbb{R}^N$ in our case, then we have that it can be expressed as a supremum of functions which inherit some properties from F, like Lipschitz continuity in our case.

Lemma 1.2.3. *If F is like in assumptions 1.2.1 then the set*

$$\mathcal{L}_{F(t,x)} := \{L \in \left(\mathbb{R}^N \times \mathbb{S}^N\right)^* : L \leq F(t, x)\}$$

is non empty, compact and convex for any $(t, x) \in [0, \infty) \times \mathbb{R}^N$. Furthermore $F(t, x) = \max_{L \in \mathcal{L}_F(t,x)} L$.

Proof. Chosen $\left(\overline{p}, \overline{S}\right) \in \mathbb{R}^N \times \mathbb{S}^N$ define the space $V := \text{span}\left(\left(\overline{p}, \overline{S}\right)\right)$ and the application $I : V \to \mathbb{R}$ such that

$$I\left(t, x, c\overline{p}, c\overline{S}\right) := cF\left(t, x, \overline{p}, \overline{S}\right)$$

for all $c \in \mathbb{R}$. We want to show that $I(t, x) \leq F(t, x)$ on the space V for any $(t, x) \in [0, \infty) \times \mathbb{R}^N$, hence for the Hahn–Banach theorem there exists a linear application $L \in \mathcal{L}_{F(t,x)}$ that extends, for any $(t, x) \in [0, \infty) \times \mathbb{R}^N$, $I(t, x)$ on $\mathbb{R}^N \times \mathbb{S}^N$. This is trivial for the $(p, S) \in V$ such that $(p, S) := c\left(\overline{p}, \overline{S}\right)$ with $c \in [0, \infty)$, therefore we suppose that there exists a $c \in [0, \infty)$ which

$$-I\left(t, x, c\overline{p}, c\overline{S}\right) > F\left(t, x, -c\overline{p}, -c\overline{S}\right)$$

and we will find a contradiction, in fact if this is true we have that by item (iv) in assumptions 1.2.1

$$0 < -I\left(t, x, \overline{p}, \overline{S}\right) - F\left(t, x, -\overline{p}, -\overline{S}\right)$$

$$= - \left(F\left(t, x, \overline{p}, \overline{S} \right) + F\left(t, x, -\overline{p}, -\overline{S} \right) \right)$$
$$\leq - F(t, x, 0, 0) = 0.$$

We have shown that the set $\mathcal{L}_{F(t,x)}$ is not empty and that and for each (p, S) in $\mathbb{R}^N \times \mathbb{S}^N$ there exists an $L \in \mathcal{L}_{F(t,x)}$ such that $L(p, S) = F(t, x, p, S)$, otherwise said $F(t, x) = \max\limits_{L \in \mathcal{L}_{F(t,x)}} L$. The proof that $\mathcal{L}_{F(t,x)}$ is convex and compact is a simple verification, consequently the arbitrariness of (t, x) concludes the proof. $\qquad \square$

Another preliminary lemma needed to prove theorem 1.2.2 is an adaptation of [39, Lemma 1.8.14], which permits us to express the Hausdorff distance using support functions.

Lemma 1.2.4. *Given, A and B, two compact and convex subsets of $\mathbb{R}^N \times \mathbb{S}^N$ we define the application*

$$h_A, \; h_B : \mathbb{R}^N \times \mathbb{S}^N \to \mathbb{R}$$

as the support functions of A and B respectively, that is to say

$$h_A((p, S)) = \sup_{(p', S') \in A} ((p, S), (p', S')), \quad h_B((p, S)) = \sup_{(p', S') \in B} ((p, S), (p', S')),$$

for any $(p, S) \in \mathbb{R}^N \times \mathbb{S}^N$. Then, for the Hausdorff distance

$$d_H(A, B) :=$$
$$\max \left\{ \max_{(p,S) \in A} \min_{(p',S') \in B} \|(p, S) - (p', S')\|, \; \max_{(p',S') \in B} \min_{(p,S) \in A} \|(p, S) - (p', S')\| \right\},$$

we have that

$$d_H(A, B) = \max_{\substack{(p,S) \in \mathbb{R}^N \times \mathbb{S}^N \\ \|(p,S)\| = 1}} |h_A((p, S)) - h_B((p, S))|.$$

Proof. Assume that there exists a positive constant c such that $d_H(A, B) \leq c$. Then $A \subseteq B + B_c(0)$ and consequently, for any $(p, S) \in \mathbb{R}^N \times \mathbb{S}^N$ such that $\|(p, S)\| = 1$,

$$h_A((p, S)) \leq h_{B + B_c(0)}((p, S)) = h_B((p, S)) + c.$$

Similarly we also get that $h_B((p, S)) \le h_A((p, S)) + c$, therefore

$$\max_{\substack{(p,S)\in\mathbb{R}^N\times\mathbb{S}^N \\ \|(p,S)\|=1}} |h_A((p, S)) - h_B((p, S))| \le c.$$

Reversing the argument we can conclude the proof. $\qquad\qquad\square$

We can now prove theorem 1.2.2.

Proof of theorem 1.2.2. By lemma 1.2.3 and the Riesz representation theorem we then have that, for any $(t, x) \in [0, \infty) \times \mathbb{R}^N$, there exists the non empty convex and compact set

$$K_t^x := \left\{ \begin{matrix} (b, a) \in \mathbb{R}^N \times \mathbb{S}^N : \dfrac{1}{2}\langle a, S\rangle + p^\dagger b \le F(t, x, p, S) \\ \text{for any } (p, S) \in \mathbb{R}^N \times \mathbb{S}^N \end{matrix} \right\}.$$

Given a $(t, x) \in [0, \infty) \times \mathbb{R}^N$ and $(\overline{b}, \overline{a}) \in K_t^x$ we define the function

$$(b, a) : [0, \infty) \times \mathbb{R}^N \to \mathbb{R}^N \times \mathbb{S}^N$$

such that

$$(b, a)(s, y) := \begin{cases} (\overline{b}, \overline{a}), & \text{if } (s, y) = (t, x), \\ \arg\min_{(b,a)\in K_s^y} \left\|(\overline{b} - b, \overline{a} - a)\right\|, & \text{if } (s, y) \neq (t, x). \end{cases}$$

This function is well defined because is well known that the projection of a point onto a convex set, i.e. $\arg\min_{(b,a)\in K_s^y} \left\|(\overline{b} - b, \overline{a} - a)\right\|$, exists and is unique. We will show that $(b, a) \in K_F$, since this yields that K_F is a non empty convex (the convexity proof is trivial, hence we skip it) set such that, thanks to the arbitrariness of the construction,

$$F(t, x, p, S) = \max_{(b,a)\in K_F} \frac{1}{2}\langle a(t, x), S\rangle + p^\dagger b(t, x)$$

for any $(t, x, p, S) \in [0, \infty) \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{S}^N$.

As a consequence of the definition and lemma 1.2.4 we have, for any (s, y) in $[0, \infty) \times \mathbb{R}^N$,

$$\|(b, a)(t, x) - (b, a)(s, y)\| = \min_{(b,a)\in K_s^y} \|(b(t, x) - b, a(t, x) - a)\|$$

$$\leq \max_{(b_1,a_1)\in K_t^x} \min_{(b_2,a_2)\in K_s^y} \|(b_1 - b_2, a_1 - a_2)\|$$

$$\leq d_H(K_t^x, K_s^y)$$

$$= \max_{\substack{(p,S)\in\mathbb{R}^N\times\mathbb{S}^N: \\ \|(p,S)\|=1}} |F(t,x,p,S) - F(s,y,p,S)|.$$

Since $|a| \leq \sqrt{2}\|(b,a)\|$ for any $(b,a) \in \mathbb{R}^N \times \mathbb{S}^N$ and

$$|a(r,z) - a(s,y)| \leq |a(t,x) - a(s,y)| + |a(r,z) - a(t,x)|$$

the previous inequality yields that $\mathrm{Lip}(a(s)) \leq 2\sqrt{2}\ell$ for any $s \in [0,\infty)$, and similarly that $\mathrm{Lip}(b(s)) \leq 2\ell$ for any $s \in [0,\infty)$.

We now prove the ellipticity part of the statement and, as a consequence, that $\mathrm{eig}_{a(t,x)} \subset [2\lambda, 2\Lambda]^N$ for any $(t,x) \in [0,\infty) \times \mathbb{R}^N$, thus that $(b,a) \in K_F$. Let λ and Λ be the ellipticity constants of F and

$$L : [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{S}^N \to \mathbb{R}$$

$$(t,x,p,S) \mapsto \frac{1}{2}\langle a(t,x), S\rangle + p^\dagger b(t,x),$$

then, by its linearity, we only have to prove that for any $S \in \mathbb{S}_+^N$ and (t,x) in $[0,\infty) \times \mathbb{R}^N$

$$\lambda|S| \leq L(t,x,0,S) \leq \Lambda|S|. \tag{1.2}$$

Obviously we have, for any $S \in \mathbb{S}_+^N$ and $(t,x) \in [0,\infty) \times \mathbb{R}^N$,

$$\lambda|S| \leq F(t,x,0,0) - F(t,x,0,-S) \leq -L(t,x,0,-S)$$

$$= L(t,x,0,S) \leq F(t,x,0,S) - F(t,x,0,0) \leq \Lambda|S|,$$

hence (1.2). Finally, let q an element of $\mathbb{R}^N$ and define $Q := q \otimes q$, which is an element of $\mathbb{S}_+^N$ such that

$$|Q|^2 = \mathrm{tr}(q \otimes q\, q \otimes q) = \mathrm{tr}(|q|^2 q \otimes q) = |q|^4.$$

Therefore (1.2) yields, for any $(t,x) \in [0,\infty) \times \mathbb{R}^N$,

$$\lambda|q|^2 = \lambda|Q| \leq \frac{1}{2}\langle a(t,x), q \otimes q\rangle = \frac{1}{2}q^\dagger a(t,x)q \leq \Lambda|Q| = \Lambda|q|^2$$

and the Rayleigh quotient formula proves that $\mathrm{eig}_{a(t,x)} \subset [2\lambda, 2\Lambda]^N$, concluding the proof. $\qquad\square$

A converse of theorem 1.2.2 holds:

Proposition 1.2.5. *Given two non negative constants ℓ and λ, let K be the set of the elements $(b, a) \in C^0\left([0, \infty) \times \mathbb{R}^N; \mathbb{R}^N\right) \times C^0\left([0, \infty) \times \mathbb{R}^N; \mathbb{S}_+^N\right)$ such that, for any $(t, x, p, S) \in [0, \infty) \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{S}^N$, $\mathrm{Lip}(b(t)) \leq \dfrac{1}{\sqrt{2}}\ell$, $\mathrm{Lip}(a(t)) \leq \ell$ and the eigenvalues of $a(t, x)$ are in $\left[2\lambda, \dfrac{2\Lambda}{\sqrt{N}}\right]$. If we define the application*

$$F(t, x, p, S) := \sup_{(b,a) \in K} \frac{1}{2}\langle a(t, x), S\rangle + p^\dagger b(t, x),$$

then F is as in assumptions 1.2.1. Furthermore λ is the ellipticity constant of F.

Proof. The proof of items (i), (ii) and (vi) in assumptions 1.2.1 is a trivial verification. To prove item (v), and consequently (iii), note that if $A \in \mathbb{S}_+^N$ then there exists a unitary matrix U such that $D := UAU^\dagger$ is a diagonal matrix with diagonal elements $0 \leq d_1 \leq \cdots \leq d_N$ and, since the trace is invariant under similarities,

$$|A| = \sqrt{\langle A, A\rangle} = \sqrt{\mathrm{tr}(AA)} = \sqrt{\mathrm{tr}(DD)} = \sqrt{\sum_{i=1}^N d_i^2}$$

$$\leq \sum_{i=1}^N d_i = \mathrm{tr}(D) = \mathrm{tr}(A) \leq \sqrt{N \sum_{i=1}^N d_i^2} = \sqrt{N}|A|.$$

Moreover if we also have another matrix B in $\mathbb{S}_+^N$ then $\overline{B} := UBU^\dagger$ belong to $\mathbb{S}_+^N$ and, by the Rayleigh quotient formula, $\overline{B}_{i,i} \geq 0$ for any $i \in \{1, \cdots, N\}$. Therefore

$$d_1 \mathrm{tr}(B) = d_1 \mathrm{tr}\left(\overline{B}\right) \leq \sum_{i=1}^N d_i \overline{B}_{i,i} = \mathrm{tr}\left(D\overline{B}\right) = \mathrm{tr}(AB) = \langle A, B\rangle \leq d_N \mathrm{tr}(B).$$

Finally, as a consequence of the previous inequalities, we have that, for any $S' \in \mathbb{S}_+^N$, $(b, a) \in K$ and $(t, x, p, S) \in [0, \infty) \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{S}_+^N$,

$$\lambda|S'| \leq \lambda \mathrm{tr}(S') \leq \inf_{(b,a) \in K} \frac{1}{2}\langle a(t, x), S'\rangle$$

$$= \inf_{(b,a)\in K} \left(\frac{1}{2}\langle a(t,x), S+S'\rangle + p^\dagger b(t,x) - \frac{1}{2}\langle a(t,x), S\rangle - p^\dagger b(t,x) \right)$$

$$\leq F(t,x,p,S+S') - F(t,x,p,S)$$

$$\leq \sup_{(b,a)\in K} \left(\frac{1}{2}\langle a(t,x), S+S'\rangle + p^\dagger b(t,x) - \frac{1}{2}\langle a(t,x), S\rangle - p^\dagger b(t,x) \right)$$

$$\leq \sup_{(b,a)\in K} \frac{1}{2}\langle a(t,x), S'\rangle \leq \frac{\Lambda}{\sqrt{N}} \operatorname{tr}(S') \leq \Lambda |S'|.$$

$\square$

Remark 1.2.6. Note that we can also define a support function on sets, non necessarily convex, and it coincides with the support function of the convex hull, see theorem 1.2.2 and proposition 1.2.5.

We have characterized F as the support function of a set of linear operators. Usually, to obtain representation formulas for viscosity solutions to a second order PDE with linear operator, is useful to study a function σ such that $\sigma\sigma^\dagger$ is the diffusion part of that operator, hence we will do something similar: if we define the application from $\mathbb{S}_+^N$ to itself which associate via singular value decomposition the matrix A with its square root σ then it is well defined, as can be seen in [5, Section 6.5]. Moreover we know from [38, Lemma 2.1] that, on the space of matrices with eigenvalues equal or bigger than 2λ, this application is Lipschitz continuous with Lipschitz constant $\frac{1}{2\sqrt{\lambda}}$, therefore the application $(b,\sigma) \mapsto (b,\sigma\sigma)$ that maps the set $\mathcal{K}_F$, which is made up by the (b,σ) in $C^0\left([0,\infty)\times\mathbb{R}^N;\mathbb{R}^N\right) \times C^0\left([0,\infty)\times\mathbb{R}^N;\mathbb{S}_+^N\right)$ such that $(b,\sigma\sigma) \in K_F$ and $\operatorname{Lip}(\sigma(t)) \leq \frac{\sqrt{2}}{\sqrt{\lambda}}\ell$ for any $t \in [0,\infty)$, into K_F is surjective and consequently

$$F(t,x,p,S) = \max_{(b,\sigma)\in\mathcal{K}_F} \frac{1}{2}\langle \sigma(t,x)\sigma(t,x), S\rangle + p^\dagger b(t,x). \tag{1.3}$$

Our hope is that this characterization will help us find representation formulas for viscosity solutions to second order PDEs with sublinear operators as in the linear case.

We end this chapter with an useful property that sublinear operators inherit from the linear operators and which will play a key role further on, in

a proof of uniqueness. This lemma has already been proved in [17], here we will adapt the arguments of [8, Example 3.6], where the proof is just given for linear operators.

Lemma 1.2.7. *Let S and S' be two elements of $\mathbb{S}^N$ which admit a positive constant α such that*

$$\begin{pmatrix} S & 0 \\ 0 & -S' \end{pmatrix} \leq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix},$$

or equivalently

$$v^\dagger S v - w^\dagger S' w \leq 3\alpha |v - w|^2. \tag{1.4}$$

Then if F is as in assumptions 1.2.1, there exists a constant $c > 0$ depending only on ℓ and λ such that, for any $(t, x) \in [0, \infty) \times \mathbb{R}^N$,

$$F(t, y, \alpha(x - y), S') - F(t, x, \alpha(x - y), S) \leq c\alpha |x - y|^2.$$

Notice that (1.4) implies $S \leq S'$.

Proof. Thanks to theorem 1.2.2 we know that

$$F(t, y, \alpha(x - y), S') - F(t, x, \alpha(x - y), S)$$

$$\leq \sup_{(b,\sigma) \in K_F} \frac{1}{2} (\langle S', \sigma^2(t, y) \rangle - \langle S, a(t, x) \rangle) + \alpha(x - y)^\dagger (b(t, y) - b(t, x))$$

$$\leq \sup_{(b,\sigma) \in K_F} \frac{1}{2} \left(\langle S', \sigma^2(t, y) \rangle - \langle S, a(t, x) \rangle + 2\alpha\ell |x - y|^2 \right)$$

and this concludes the proof because

$$\begin{aligned}
\langle S', \sigma^2(t, y) \rangle - \langle S, \sigma^2(t, x) \rangle &= \operatorname{tr}(\sigma^2(t, y) S') - \operatorname{tr}(\sigma^2(t, x) S) \\
&= \operatorname{tr}(\sigma(t, y) S' \sigma(t, y) - \sigma(t, x) S \sigma(t, x)) \\
&= \sum_{i=1}^{N} e_i^\dagger \sigma(t, y) S' \sigma(t, y) e_i - e_i^\dagger \sigma_i(t, x) S \sigma(t, x) e_i \\
&\leq \sum_{i=1}^{N} 3\alpha |(\sigma_i(t, y) - \sigma_i(t, x)) e_i|^2 \\
&\leq \frac{6\alpha\ell^2 N}{\lambda} |x - y|^2.
\end{aligned}$$

$\square$

Chapter 2

Viscosity Solutions

Before we start this chapter we give two useful definition, bearing in mind the following second order PDE problem:

$$F\left(x, u(x), \nabla u(x), D^2 u(x)\right) = 0, \tag{2.1}$$

where F is a arbitrary continuous function. To ease notation we will usually write $F\left(x, u, \nabla u, D^2 u\right)$ instead of $F\left(x, u(x), \nabla u(x), D^2 u(x)\right)$.

Definition 2.0.1. Given an upper semicontinuous function u we say that a function φ is a *supertangent* to u at x if x is a local maximizer of $u - \varphi$. Similarly we say that a function ψ is a *subtangent* to a lower semicontinuous function v at x if x is a local minimizer of $v - \psi$.

Definition 2.0.2. An upper semicontinuous function u is called a *viscosity subsolution* to (2.1) if, for any suitable x and C^2 supertangent φ to u at x,

$$F\left(x, u(x), \nabla \varphi(x), D^2 \varphi(x)\right) \geq 0.$$

Similarly a lower semicontinuous function v is called a *viscosity supersolution* to (2.1) if, for any suitable x and C^2 subtangent ψ to v at x,

$$F\left(x, v(x), \nabla \psi(x), D^2 \psi(x)\right) \leq 0.$$

Finally a continuous function u is called a *viscosity solution* to (2.1) if it is both a super and a subsolution to (2.1).

Notice that these definitions can be naturally extended to parabolic problems.

Here are also given some known results in the viscosity solutions theory which will be used later, while for a detailed overview of this theory we refer to [8, 18].

Theorem 2.0.3. *Let* $\{u_\alpha\}_{\alpha \in \mathcal{A}}$ *be a family of locally equibounded viscosity subsolutions (resp. supersolutions) to* (2.1) *in a locally compact subset* K *of* $\mathbb{R}^N$. *Then if* $u := \sup_{\alpha \in \mathcal{A}} u_\alpha$ *is upper semicontinuous (resp.* $u := \inf_{\alpha \in \mathcal{A}} u_\alpha$ *is lower semicontinuous) it is a viscosity subsolution (resp. supersolution) to* (2.1) *in* K. *The same holds true for the parabolic version of* (2.1).

The proof can be found in [8, Lemma 4.2], see also [17], where the same statement is shown for discontinuous viscosity solutions.

Now consider the parabolic problem

$$\partial_t u(t,x) + F\left(t, x, u, \nabla u, D^2 u\right) = 0, \quad t \in (0,T), x \in \mathbb{R}^N, \qquad (2.2)$$

where F is a continuous elliptic operator which admits, for any $t \in [0,T]$, (x, r, p, S) and (y, r', p', S') in $\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N$, a $\mu \in \mathbb{R}$ and a positive constant ℓ such that

(i) $|F(t,x,r,p,S) - F(t,x,r,p,S')| \leq \ell \left(1 + |x|^2\right) |S - S'|$;

(ii) $|F(t,x,r,p,S) - F(t,x,r,p',S)| \leq \ell(1 + |x|)|p - p'|$;

(iii) $|F(t,x,r,p,S) - F(t,y,r,p,S)| \leq \ell(1 + |x| + |y|)|x - y| \|(p,S)\|$;

(iv) *(Monotonicity)* $(F(t,x,r,p,S) - F(t,x,r',p,S))(r - r') \leq \mu |r - r'|^2$;

(v) the continuity of the function $r \mapsto F(t,x,r,p,S)$ is independent from the fourth variable.

Notice that, given a compact set $K \subset \mathbb{R}^N \times \mathbb{R} \times \mathbb{S}^N$, item (v) and the Heine–Cantor theorem yield the existence of a modulus of continuity ω_K such that, if $(x, r, S), (x, r', S) \in K$,

$$|F(t,x,r,p,S) - F(t,x,r',p,S)| \leq \omega_K(|r - r'|) \qquad (2.3)$$

for any $t \in [0, T]$ and $p \in \mathbb{R}^N$.

In the next sections we will study some parabolic problems of this type, therefore we give here a comparison result, which is an adaptation of [31, Theorem C.2.3] for problem (2.2). To prove it we will need the following theorem, adaptation of [31, Theorem C.2.2] which can be proved similarly. Notice that condition **(G)** in [31] is replaced by (2.4).

Theorem 2.0.4. *Let $\{F_i\}_{i=1}^k$ be a collection of continuous functions from $[0, T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N$ to $\mathbb{R}^N$ and assume that, if $(x, r, S), (y, r, S)$ belong to a compact set $K \subset \mathbb{R}^N \times \mathbb{R} \times \mathbb{S}^N$, there exist a constant C_K and a modulus of continuity ω_K such that, for any $i \in \{1, \cdots, k\}$,*

$$|F_i(t, x, r, p, S) - F_i(t, y, r, p, S)| \leq C_K(1 + |p|)|x - y| + \omega_K(|x - y|). \quad (2.4)$$

Furthermore assume the following domination condition: there exists a collection of positive constants $\{\beta_i\}_{i=1}^k$ satisfying

$$\sum_{i=1}^k \beta_i F_i(t, x, r_i, p_i, S_i) \leq 0$$

for each $(t, x) \in [0, T] \times \mathbb{R}^N$ and (r_i, p_i, S_i) such that $\sum_{i=1}^k \beta_i r_i \geq 0$, $\sum_{i=1}^k \beta_i p_i = 0$ and $\sum_{i=1}^k \beta_i X_i \leq 0$.
For any $i \in \{1, \cdots, k\}$, let u_i be a viscosity subsolution of

$$\partial_t u(t, x) + F_i\left(t, x, u, \nabla u, D^2 u\right) = 0, \quad t \in (0, T), x \in \mathbb{R}^N,$$

and assume that $\sum_{i=1}^k \beta_i u_i(T, \cdot) \leq 0$ and $\left(\sum_{i=1}^k \beta_i u_i(\cdot, x)\right)^+ \to 0$ uniformly as $|x| \to \infty$. Then $\sum_{i=1}^k \beta_i u_i(t, \cdot) \leq 0$ for any $t \in (0, T)$.

Theorem 2.0.5. *Let u and v be, respectively, a viscosity subsolution and a viscosity supersolution to (2.2) satisfying polynomial growth condition. Then, if $u|_{t=T} \leq v|_{t=T}$, $u \leq v$ on $(0, T] \times \mathbb{R}^N$.*

Proof. We set $\phi(x) := (1 + |x|^2)^{\frac{c}{2}}$,

$$\gamma > \mu + \ell \sup_{x \in \mathbb{R}^N} \left((1 + |x|) \frac{|\nabla \phi(x)|}{\phi(x)} + (1 + |x|^2) \frac{|D^2 \phi(x)|}{\phi(x)} \right),$$

$u_1(t, x) := \dfrac{e^{-\gamma t} u(t, x)}{\phi(x)}$ and $u_2(t, x) := -\dfrac{e^{-\gamma t} v(t, x)}{\phi(x)}$, where c is such that both $|u_1|$ and $|u_2|$ converge uniformly to 0 as $|x| \to \infty$. Notice that

$$\nabla \phi(x) = c \frac{\phi(x) x^\dagger}{1 + |x|^2} \quad \text{and} \quad D^2 \phi(x) = \phi(x) \left(\frac{c}{1 + |x|^2} I + \frac{c(c-2)}{(1 + |x|^2)^2} x \otimes x \right),$$

therefore γ is well defined. We also set the operators $F_1(t, x, r, p, S)$, given by

$$\frac{e^{-\gamma t}}{\phi(x)} F\left(t, x, e^{\gamma t} r \phi, e^{\gamma t} \left(r \nabla \phi + \phi p^\dagger \right), e^{\gamma t} \left(r D^2 \phi + \nabla \phi \otimes p + p \otimes \nabla \phi + \phi S \right) \right),$$

and $F_2(t, x, r, p, S)$, given by

$$-\frac{e^{-\gamma t}}{\phi(x)} F\left(t, x, -e^{\gamma t} r \phi, -e^{\gamma t} \left(r \nabla \phi + \phi p^\dagger \right), -e^{\gamma t} \left(r D^2 \phi + \nabla \phi \otimes p + p \otimes \nabla \phi + \phi S \right) \right).$$

It is easy to check that, for $i \in \{1, 2\}$, F_i is still continuous, elliptic, Lipschitz continuous in p and S, that its monotonicity constant is

$$\mu + \ell \sup_{x \in \mathbb{R}^N} \left((1 + |x|) \frac{|\nabla \phi(x)|}{\phi(x)} + (1 + |x|^2) \frac{|D^2 \phi(x)|}{\phi(x)} \right),$$

i.e. is lower than γ, and u_i is a viscosity subsolution to

$$\partial_t u(t, x) - \gamma u(t, x) + F_i \left(t, x, u, \nabla u, D^2 u \right) = 0, \quad t \in (0, T), x \in \mathbb{R}^N.$$

It can also be checked that, if $(x, r, S), (y, r, S)$ belong to a compact set $K \subset \mathbb{R}^N \times \mathbb{R} \times \mathbb{S}^N$, there exist a constant C_K and a modulus of continuity $\widetilde{\omega}_K$ bigger than ω_K in (2.3) such that

$$|F_i(t, x, r, p, S) - F_i(t, y, r, p, S)| \leq C_K(1 + |p|)|x - y| + \widetilde{\omega}_K(|x - y|)$$

for any $i \in 1, 2$, $t \in [0, T]$ and $p \in \mathbb{R}^N$.

Furthermore $(u_1 + u_2)|_{t=T} \leq 0$ and $F_1(t, x, r, p, S) + F_2(t, x, -r, -p, -S) = 0$.

From these properties we have, for any $r_i \in \mathbb{R}$, $p_i \in \mathbb{R}^N$ and $S_i \in \mathbb{S}^N$ such that $r = r_1 + r_2 \geq 0$, $p_1 = -p_2$ and $S = S_1 + S_2 \leq 0$,

$$
-\gamma r + F_1(t, x, r_1, p_1, S_1) + F_2(t, x, r_2, p_2, S_2)
$$
$$
\begin{aligned}
&= -\gamma r + F_1(t, x, r_1, p_1, S_1) + F_2(t, x, -r_1, -p_1, -S_1) \\
&\quad + (F_2(t, x, r_2, p_2, S_2) - F_2(t, x, r_2 - r, p_2, S_2 - S))\frac{r}{r} \\
&\leq -\gamma r + (F_2(t, x, r_2, p_2, S_2) - F_2(t, x, r_2 - r, p_2, S_2))\frac{r}{r} \\
&\leq -\gamma r + \gamma r = 0.
\end{aligned}
$$

As a consequence we have that all the conditions of theorem 2.0.4 are satisfied, thus $u_1 + u_2 \leq 0$, or equivalently, $u \leq v$ in $(0, T] \times \mathbb{R}^N$. $\square$

2.1   The Cauchy Problem

In order to illustrate our methodology, we preliminarily study a problem which can be attacked only using the results previously given and some basic knowledge of the stochastic differential equation, see appendix A and the Feynman–Kac formula.

Problem 2.1.1. Let T be a terminal time F an elliptic operator as in assumptions 1.2.1 and

$$
g : \mathbb{R}^N \to \mathbb{R}
$$

a continuous functions for which there exists a constant $\ell \geq 0$ such that, for any $x, x' \in \mathbb{R}^N$,

$$
|g(x) - g(x')| \leq \ell|x - x'| \text{ and } |g(x)| \leq \ell(1 + |x|).
$$

Find the solution u to the parabolic PDE

$$
\begin{cases}
\partial_t u(t, x) + F\left(t, x, \nabla_x u, D_x^2 u\right) = 0 & t \in (0, T), x \in \mathbb{R}^N, \\
u(T, x) = g(x) & x \in \mathbb{R}^N.
\end{cases}
$$

Remark 2.1.2. To ease notation we can assume without loss of generality that the ℓ in problem 2.1.1 is the same of assumptions 1.2.1. Since F is continuous, we can also assume that $|b(t,0)| \leq \ell$ and $|\sigma(t,0)| \leq \ell$ for any $t \in [0,T]$.

Notice that theorem 2.0.5 yields a comparison result.

Theorem 2.1.3. *Let u and v be respectively a subsolution and a super-solution to problem 2.1.1 satisfying polynomial growth condition. Then, if $u|_{t=T} \leq v|_{t=T}$, $u \leq v$ on $(0,T] \times \mathbb{R}^N$.*

For any $(b,\sigma) \in \mathcal{K}_F$ define the linear operator $L_{(b,\sigma)}$ such that, for any $(t,x,p,S) \in [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{S}^N$,

$$L_{(b,\sigma)}(t,x,p,S) := \frac{1}{2}\langle \sigma(t,x)\sigma(t,x), S\rangle + p^\dagger b(t,x)$$

and the SDE $X_{(b,\sigma)}$ such that, for any $t \in [0,T]$ and $\zeta \in L^2(\Omega, \mathcal{F}_t; \mathbb{R}^N)$,

$$X_{(b,\sigma)}^{t,\zeta}(s) = \zeta + \int_t^s \sigma\left(r, X_{(b,\sigma)}^{t,\zeta}(r)\right)dW_r + \int_t^s b\left(r, X_{(b,\sigma)}^{t,\zeta}(r)\right)dr, \quad s \in [t,T].$$

We refer to appendix A for more details on SDEs. It is well known from the Feynman–Kac formula that the viscosity solution to the problem

$$\begin{cases} \partial_t v(t,x) + L_{(b,\sigma)}\left(t,x,\nabla_x v, D_x^2 v\right) = 0, & t \in (0,T), x \in \mathbb{R}^N \\ v(T,x) = g(x), & x \in \mathbb{R}^N. \end{cases}$$

is $v(t,x) := \mathbb{E}\left(g\left(X_{(b,\sigma)}^{t,x}(T)\right)\right)$. Moreover, since $F = \max_{(b,\sigma)\in\mathcal{K}_F} L_{(b,\sigma)}$, theorem 2.0.3 yields that

$$u(t,x) := \sup_{(b,\sigma)\in\mathcal{K}_F} \mathbb{E}\left(g\left(X_{(b,\sigma)}^{t,x}(T)\right)\right)$$

is a subsolution to problem 2.1.1. Thus the idea is that u could also be the viscosity solution to problem 2.1.1, but in reality we do not expect this supremum over $\mathcal{K}_F$ to produce a viscosity solution. In more detail: generally, we could have that, for some $x \in \mathbb{R}^N$, $t \in [0,T]$, $\delta > 0$ and for any $h \in (0,\delta)$,

$$u(t,x)+\varepsilon h = \sup_{(b,\sigma)\in\mathcal{K}_F} \mathbb{E}\left(g\left(X_{(b,\sigma)}^{t,x}(T)\right)\right) + \varepsilon h$$

$$< \sup_{(b,\sigma)\in\mathcal{K}_F} \mathbb{E}\left(\operatorname*{ess\,sup}_{(b',\sigma')\in\mathcal{K}_F} \mathbb{E}\left(g\left(X_{(b',\sigma')}^{t+h,y}(T)\right)\Big| y = X_{(b,\sigma)}^{t,x}(t+h)\right)\right) \quad (2.5)$$

$$\leq \sup_{(b,\sigma)\in\mathcal{K}_F} \mathbb{E}\left(u\left(t+h, X_{(b,\sigma)}^{t,x}(t+h)\right)\right).$$

Moreover we have that, under some additional assumptions (see [31, Section C.4], for example), problem 2.1.1 admits a $C^{1+\frac{\alpha}{2},2+\alpha}([0,T-k]\times\mathbb{R}^N)$ solution, where $\alpha\in(0,1)$ and $k\in(0,T)$. Thus, if we assume that u is this solution and h is small enough, from Itô's formula we get

$$\mathbb{E}\left(u\left(s, X_{(b,\sigma)}^{t,x}(t+h)\right)\right) = u(t,x) + \mathbb{E}\left(\int_t^{t+h} L_{(b,\sigma)}\left(s, X_{(b,\sigma)}^{t,x}(s), \nabla u, D_x^2 u\right)ds\right)$$

$$+ \mathbb{E}\left(\int_t^{t+h} \partial_t u\left(s, X_{(b,\sigma)}^{t,x}(s)\right)ds\right)$$

and by our assumptions, Jensen's inequality and corollary A.1.5 there exist three constants c_1, c_2 and c_3, independent from the choice of $(b,\sigma)\in\mathcal{K}_F$, such that

$$\frac{\mathbb{E}\left(\int_t^{t+h}\left(L_{(b,\sigma)}\left(s, X_{(b,\sigma)}^{t,x}(s), \nabla u, D_x^2 u\right) - L_{(b,\sigma)}\left(s, x, \nabla u, D_x^2 u\right)\right)ds\right)}{h}$$

$$+ \frac{\mathbb{E}\left(\int_t^{t+h}\left(\partial_t u\left(s, X_{(b,\sigma)}^{t,x}(s)\right) - \partial_t u(s,x)\right)ds\right)}{h}$$

$$\leq \frac{c_1\mathbb{E}\left(\int_t^{t+h}\left(\left|X_{(b,\sigma)}^{t,x}(s)-x\right| + |x|\vee 1\right)\left(\left|X_{(b,\sigma)}^{t,x}(s)-x\right| + \left|X_{(b,\sigma)}^{t,x}(s)-x\right|^\alpha\right)ds\right)}{h}$$

$$\leq c_2\left(\mathbb{E}\left(\sup_{s\in[t,t+h]}\left|X_{(b,\sigma)}^{t,x}(s)-x\right|^2\right)\right)^{\frac{1}{2}} + c_2\left(\mathbb{E}\left(\sup_{s\in[t,t+h]}\left|X_{(b,\sigma)}^{t,x}(s)-x\right|^2\right)\right)^{\frac{\alpha}{2}}$$

$$+ c_1\mathbb{E}\left(\sup_{s\in[t,t+h]}\left|X_{(b,\sigma)}^{t,x}(s)-x\right|^2\right) + c_1\left(\mathbb{E}\left(\sup_{s\in[t,t+h]}\left|X_{(b,\sigma)}^{t,x}(s)-x\right|^2\right)\right)^{\frac{\alpha+1}{2}}$$

$$\leq c_3\left(\int_t^{t+h} e^{\gamma(t+h-s)}ds\right)$$

which together with (2.5) imply that

$$
\begin{aligned}
0 < \lim_{h \downarrow 0} & \frac{\displaystyle\sup_{(b,\sigma) \in \mathcal{K}_F} \mathbb{E}\left(u\left(t+h, X^{t,x}_{(b,\sigma)}(t+h)\right)\right) - u(t,x)}{h} \\
= \lim_{h \downarrow 0} & \frac{\displaystyle\sup_{(b,\sigma) \in \mathcal{K}_F} \mathbb{E}\left(\int_t^{t+h}\left(L_{(b,\sigma)}\left(s, X^{t,x}_{(b,\sigma)}(s), \nabla_x u, D_x^2 u\right) + \partial_t u\left(s, X^{t,x}_{(b,\sigma)}(s)\right)\right)ds\right)}{h} \\
= & \partial_t u(t,x) + \sup_{(b,\sigma) \in \mathcal{K}_F} L_{(b,\sigma)}\left(t,x,\nabla_x u, D_x^2 u\right) = \partial_t u(t,x) + F\left(t,x,\nabla_x u, D_x^2 u\right).
\end{aligned}
$$

Therefore u is not a supersolution, in contradiction with our assumption that u is a solution.

In [12, Section 3.1] Denis, Hu and Peng find a representation formula for the viscosity solution to a version of problem 2.1.1, with a sublinear operator that depends only on the second order term, using a *dynamic programming principle* which assure that, thanks to an ad hoc construction, (2.5) never happens. Inspired by this paper, our method to obtain representation formulas relies on a dynamic programming principle, which is a generalization of the one presented in [12] and is based on a construction on a broader set than $\mathcal{K}_F$. This set, which we call $\mathcal{A}_F$, is made up of the progressive processes, sometimes referred as controls,

$$
(b,\sigma) : [0,\infty) \times \Omega \times \mathbb{R}^N \to \mathbb{R}^N \times \mathbb{S}_+^N
$$

which are cadlag, i.e. right continuous and left bounded, on $[0,\infty)$ and such that, for any $(t,x,p,S) \in [0,\infty) \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{S}^N$ and $\omega \in \Omega$,

$$
\frac{1}{2}\langle \sigma(t,\omega,x)\sigma(t,\omega,x), S\rangle + p^\dagger b(t,\omega,x) \leq F(t,x,p,S),
$$

$\mathrm{Lip}(b(t,\omega)) \leq 2\ell$, $\mathrm{Lip}(\sigma(t,\omega)) \leq \dfrac{\sqrt{2}\ell}{\sqrt{\lambda}}$, the eigenvalues of $(\sigma\sigma)(t,\omega,x)$ belong to $[2\lambda,\infty)^N$ and $(b(x),\sigma(x)) \in \mathbb{L}_N^2 \times \mathbb{L}_{N\times N}^2$. $\mathcal{A}_F$ is obviously non empty, since it contains $\mathcal{K}_F$. Notice that on $\mathcal{A}_F$ it is possible to define a topology induced by the convergence on compact sets, since, for any $k \in [0,\infty)$ and $(b,\sigma) \in \mathcal{A}_F$, $\mathbb{E}\left(\int_0^k \int_{\overline{B_k(0)}} \|(b,\sigma)\|^2\right) < \infty$, and that, by proposition A.1.6, the

solution to the SDE is continuous in $\mathcal{A}_F$ endowed with this topology.

For any $t \in [0, \infty)$, an useful subset of $\mathcal{A}_F$, which we will use later, is $\mathcal{A}_F^t$, which consists of the (b, σ) belonging to $\mathcal{A}_F$ such that $\{(b, \sigma)(t + s, x)\}_{s \in [0, \infty)}$ is progressive with respect to the filtration $\{\mathcal{F}_s^t\}_{s \in [0, \infty)}$. We point out that trivially $\mathcal{A}_F^0 = \mathcal{A}_F$ and $\mathcal{K}_F \subset \mathcal{A}_F^t$. Moreover, as a consequence of the definition, we have

$$F(t + s, x, p, S) = \max_{(b, \sigma) \in \mathcal{A}_F^\tau} \frac{1}{2} \langle \sigma(t + s, x) \sigma(t + s, x), S \rangle + p^\dagger b(t + s, x),$$

since this is true for each $\omega \in \Omega$, thanks to (1.3).

In what follows will be useful the next density result.

Lemma 2.1.4. *The set*

$$\mathcal{J} := \left\{ \begin{array}{l} (b, \sigma) \in \mathcal{A}_F : (b, \sigma)|_{[t, \infty)} = \sum_{i=0}^{n} \chi_{A_i}(b_i, \sigma_i)|_{[t, \infty)}, \\[2mm] \textit{where } \{(b_i, \sigma_i)\}_{i=0}^{n} \subset \mathcal{A}_F^t \textit{ and } \{A_i\}_{i=0}^{n} \textit{ is a } \mathcal{F}_t\textit{--partition of } \Omega \end{array} \right\}$$

is dense in $\mathcal{A}_F$ for any $t \in [0, T]$.

Proof. To prove this we will show that, fixed a $k \in \mathbb{N}$, we can approximate, in $L^2([0, k] \times \Omega \times B_k(0))$, any element of $\mathcal{A}_F$ with an element of $\mathcal{J}$.

Preliminarily notice that by our assumption each element of $\mathcal{A}_F$ can be approximated in $L^2([0, k] \times \Omega \times B_k(0))$ by a sequence of simple functions. We will denote with $\mathcal{B}([0, k] \times B_k(0))$ the Borel σ--algebra of $[0, k] \times B_k(0)$.

Furthermore, since the collection $\mathcal{I}$ of the rectangles $A \times B$ where $A \in \mathcal{F}_\infty$ and $B \in \mathcal{B}([0, k] \times B_k(0))$ is a π--system which contains the complementary of its sets and generate $\sigma(\mathcal{F}_\infty \times \mathcal{B}([0, k] \times B_k(0)))$, by [40, Dynkin's lemma A1.3] each set in $\sigma(\mathcal{F}_\infty \times \mathcal{B}([0, k] \times B_k(0)))$, which is the smallest d--system containing $\mathcal{I}$, can be approximate by a finite union of sets in $\mathcal{I}$. Similarly, each set in $\mathcal{F}_\infty$ can be approximated by finite intersection and union of sets in $\mathcal{F}_t$ and $\mathcal{F}_\infty^t$, since by lemma 1.1.1 $\mathcal{F}_\infty = \sigma(\mathcal{F}_t, \mathcal{F}_\infty^t)$.

Therefore, fixed $(b, \sigma) \in \mathcal{A}_F$, for any $\varepsilon > 0$ there exists a simple function s_ε such that $s_\varepsilon(t, \omega, x) = \sum_{i=1}^{n} \sum_{j=1}^{m} s_i^j(t, x) \chi_{A_i}(\omega) \chi_{A_j'}(\omega)$ where $\{A_i\}_{i=1}^{n}$ and $\{A_j'\}_{j=1}^{m}$

are respectively a $\mathcal{F}_t$–partition and a $\mathcal{F}_\infty^t$–partition of Ω and

$$\mathbb{E}\left(\int_0^k \int_{B_k(0)} |(b,\sigma)(t,x) - s_\varepsilon(t,x)|^2 dxdt\right) < \varepsilon. \tag{2.6}$$

Then, for each A_i and A'_j with $\mathbb{P}(A_i \cap A'_j) > 0$, there exists a $\omega_i^j \in A_i \cap A'_j$ such that

$$\int_0^k \int_{B_k(0)} \left|(b,\sigma)\left(t, \omega_i^j, x\right) - s_i^j(t,x)\right|^2 dxdt < \frac{\varepsilon}{\mathbb{P}(A_i \cap A'_j)},$$

otherwise we would have that

$$\mathbb{E}\left(\int_0^k \int_{B_k(0)} |(b,\sigma)(t,x) - s_\varepsilon(t,x)|^2 dxdt\right)$$

$$= \mathbb{E}\left(\sum_{i=1}^n \sum_{j=1}^m \int_0^k \int_{B_k(0)} \left|(b,\sigma)(t,x) - s_i^j(t,x)\right|^2 \chi_{A_i} \chi_{A'_j} dtdx\right)$$

$$\geq \mathbb{E}\left(\int_0^k \int_{B_k(0)} \left|(b,\sigma)(t,x) - s_i^j(t,x)\right|^2 \chi_{A_i} \chi_{A'_j} dtdx\right)$$

$$\geq \varepsilon,$$

in contradiction with (2.6). Finally, let ω_i^j be any elements of $A_i \cap A'_j$ if $\mathbb{P}(A_i \cap A'_j) = 0$ and $\left(b_i^k, \sigma_i^k\right) := \sum_{j=1}^m (b,\sigma)\left(\omega_i^j\right)\chi_{A'_j}$. Then $\left(b_i^k, \sigma_i^k\right) \in \mathcal{A}_F^t$ and $\left(b_\varepsilon^k, \sigma_\varepsilon^k\right) := \sum_{i=1}^n \left(b_i^k, \sigma_i^k\right)\chi_{A_i}$ is an element of $\mathcal{J}$ satisfying

$$\mathbb{E}\left(\int_0^k \int_{B_k(0)} \left|(b,\sigma)(t,x) - \left(b_\varepsilon^k, \sigma_\varepsilon^k\right)(t,x)\right|^2 dxdt\right) < 4\varepsilon.$$

This proves that $\mathcal{J}$ is dense in $\mathcal{A}_F$. $\qquad\qquad\qquad\square$

The dynamic programming principle is, in our case as in [12], an instrument that allows us to break a stochastic trajectory in two or more parts (this intuitively explain why we ask (b,σ) to be cadlag in time) avoiding (2.5), i.e., for any s and t such that $0 \leq s \leq t \leq T$ and $x \in \mathbb{R}^N$,

$$\sup_{(b,\sigma)\in\mathcal{A}_F} \mathbb{E}\left(g\left(X_{(b,\sigma)}^{s,x}(T)\right)\right)$$

$$
\begin{aligned}
&= \sup_{(b,\sigma)\in\mathcal{A}_F} \sup_{(b',\sigma')\in\mathcal{A}_F} \mathbb{E}\left(g\left(X^{t,X^{s,x}_{(b,\sigma)}(t)}_{(b',\sigma')}(T) \right) \right) \\
&= \sup_{(b,\sigma)\in\mathcal{A}_F} \mathbb{E}\left(\operatorname*{ess\,sup}_{(b',\sigma')\in\mathcal{A}_F} \mathbb{E}\left(g\left(X^{t,y}_{(b',\sigma')}(T) \right) \Big| y = X^{s,x}_{(b,\sigma)}(t) \right) \right),
\end{aligned}
\tag{2.7}
$$

therefore we proceed by steps analyzing, for any $\zeta \in L^2\left(\Omega, \mathcal{F}_t; \mathbb{R}^N\right)$, the function

$$
\Phi_t(\zeta) := \operatorname*{ess\,sup}_{(b,\sigma)\in\mathcal{A}_F} \mathbb{E}\left(g\left(X^{t,\zeta}_{(b,\sigma)}(T) \right) \Big| \mathcal{F}_t \right).
$$

We point out that the initial datum ζ here represent the first part of the trajectory defined by $X_{(b,\sigma)}$ broken off at t, i.e. is a generalization of the term $X^{s,x}_{(b,\sigma)}(t)$ in (2.7).

Remark 2.1.5. The reasons why we ask the elements of $\mathcal{A}_F$ to be cadlag in time can be seen in the first identity of (2.7):

$$
\sup_{(b,\sigma)\in\mathcal{A}_F} \mathbb{E}\left(g\left(X^{s,x}_{(b,\sigma)}(T) \right) \right) = \sup_{(b,\sigma)\in\mathcal{A}_F} \sup_{(b',\sigma')\in\mathcal{A}_F} \mathbb{E}\left(g\left(X^{t,X^{s,x}_{(b,\sigma)}(t)}_{(b',\sigma')}(T) \right) \right).
$$

While this is true in $\mathcal{A}_F$, since given two elements (b_1, σ_1), (b_2, σ_2) in $\mathcal{A}_F$

$$
(b_3(s), \sigma_3(s)) :=
\begin{cases}
(b_1(s), \sigma_1(s)) & \text{if } s \in [0, t], \\
(b_2(s), \sigma_2(s)) & \text{if } s \in [t, T]
\end{cases}
$$

belongs to $\mathcal{A}_F$, this could not be true in general for a space with time continuous elements.

Preliminarily we need the following lemma:

Lemma 2.1.6. *Under the assumptions of problem 2.1.1 we have that*

$$
u(t, x) := \sup_{(b,\sigma)\in\mathcal{A}_F} \mathbb{E}\left(g\left(X^{t,x}_{(b,\sigma)}(T) \right) \right)
$$

is $\frac{1}{2}$–Hölder continuous with respect to the first variable and Lipschitz continuous with respect to the second one. Furthermore there exists a constant c, which depends only on ℓ and T, such that

$$
\sup_{(b,\sigma)\in\mathcal{A}_F} \mathbb{E}\left(\left| g\left(X^{t,\zeta}_{(b,\sigma)}(T) \right) \right| \right) \leq c\left(1 + \left(\mathbb{E}\left(|\zeta|^2 \right) \right)^{\frac{1}{2}} \right)
$$

for any $t \in [0, T]$ and $\zeta \in L^2\left(\Omega, \mathcal{F}_t; \mathbb{R}^N\right)$.

Proof. Our assumption, Jensen's inequality and corollary A.1.5 yield, for any $t \in [0, T]$ and $\zeta \in L^2$,

$$
\begin{aligned}
\mathbb{E}\left(\left|g\left(X_{(b,\sigma)}^{t,\zeta}(T)\right)\right|\right) \leq & \ell\left(1 + \mathbb{E}\left(\left|X_{(b,\sigma)}^{t,\zeta}(T)\right|\right)\right) \\
\leq & \ell\left(1 + \left(\mathbb{E}\left(\left|X_{(b,\sigma)}^{t,\zeta}(T)\right|^2\right)\right)^{\frac{1}{2}}\right) \\
\leq & c_1\left(1 + \left(\mathbb{E}\left(|\zeta|^2\right)\right)^{\frac{1}{2}}\right),
\end{aligned}
$$

where c_1 is a constant which depends only on ℓ and T, and this prove the boundedness property. The continuity property can be proved in the same way, in fact, for any $t, s \in [0, T]$ and $x, y \in \mathbb{R}^N$, we obtain that there exists a constant c_2 which depends only on ℓ and T such that

$$
\begin{aligned}
|u(t,x) - u(t,y)| \leq & \sup_{(b,\sigma) \in \mathcal{A}_F} \mathbb{E}\left(\left|g\left(X_{(b,\sigma)}^{t,x}(T)\right) - g\left(X_{(b,\sigma)}^{t,y}(T)\right)\right|\right) \\
\leq & \sup_{(b,\sigma) \in \mathcal{A}_F} \ell\mathbb{E}\left(\left|X_{(b,\sigma)}^{t,x}(T) - X_{(b,\sigma)}^{t,y}(T)\right|\right) \\
\leq & \sup_{(b,\sigma) \in \mathcal{A}_F} \ell\left(\mathbb{E}\left(\left|X_{(b,\sigma)}^{t,x}(T) - X_{(b,\sigma)}^{t,y}(T)\right|^2\right)\right)^{\frac{1}{2}} \\
\leq & c_2\left(|x-y|^2\right)^{\frac{1}{2}} = c_2|x-y|,
\end{aligned}
$$

and, assuming $s \leq t$,

$$
\begin{aligned}
|u(t,x) - u(s,x)| \leq & \sup_{(b,\sigma) \in \mathcal{A}_F} \mathbb{E}\left(\left|g\left(X_{(b,\sigma)}^{t,x}(T)\right) - g\left(X_{(b,\sigma)}^{s,x}(T)\right)\right|\right) \\
\leq & \sup_{(b,\sigma) \in \mathcal{A}_F} \ell\mathbb{E}\left(\left|X_{(b,\sigma)}^{t,x}(T) - X_{(b,\sigma)}^{t,X_{(b,\sigma)}^{s,x}(t)}(T)\right|\right) \\
\leq & \sup_{(b,\sigma) \in \mathcal{A}_F} \ell\left(\mathbb{E}\left(\left|X_{(b,\sigma)}^{t,x}(T) - X_{(b,\sigma)}^{t,X_{(b,\sigma)}^{s,x}(t)}(T)\right|^2\right)\right)^{\frac{1}{2}} \\
\leq & c_3\left(\mathbb{E}\left(\left|x - X_{(b,\sigma)}^{s,x}(t)\right|^2\right)\right)^{\frac{1}{2}} \\
\leq & c_4(t-s)^{\frac{1}{2}},
\end{aligned}
$$

where c_3 and c_4 constants which depends only on ℓ and T. $\qquad\square$

We can now start to focus on the dynamic programming principle and our first step to prove it is a lattice property, which heavily relies on the randomness of the set $\mathcal{A}_F$. We point out that this method is a generalization of the one presented by Denis, Hu and Peng in [12, Section 3.1], and will be proved along the same lines. However in [12] the dynamic programming principle is applied to equation with sublinear operators only depending on the second order terms, consequently the controls take matrices as values, and not functions as in our case. Another difference is that we require controls to be cadlag because they are more suitable to our needs, as will appear clear in the next sections. While here we try to be as simple as possible, in the next section we will obtain a more general result.

Lemma 2.1.7. *For each (b_1, σ_1) and (b_2, σ_2) in $\mathcal{A}_F$ there exists a $(b, \sigma) \in \mathcal{A}_F$ such that*

$$\mathbb{E}\left(g\left(X^{t,\zeta}_{(b,\sigma)}(T)\right)\Big|\mathcal{F}_t\right) = \mathbb{E}\left(g\left(X^{t,\zeta}_{(b_1,\sigma_1)}(T)\right)\Big|\mathcal{F}_t\right) \vee \mathbb{E}\left(g\left(X^{t,\zeta}_{(b_2,\sigma_2)}(T)\right)\Big|\mathcal{F}_t\right). \quad (2.8)$$

Therefore exists a sequence $\{(b_i, \sigma_i)\}_{i\in\mathbb{N}}$ in $\mathcal{A}_F$ such that a.e.

$$\mathbb{E}\left(g\left(X^{t,\zeta}_{(b_i,\sigma_i)}(T)\right)\Big|\mathcal{F}_t\right) \uparrow \Phi_t(\zeta). \quad (2.9)$$

We also have

$$\mathbb{E}(|\Phi_t(\zeta)|) \leq \sup_{(b,\sigma)\in\mathcal{A}_F} \mathbb{E}\left(\left|g\left(X^{t,\zeta}_{(b,\sigma)}(T)\right)\right|\right) < \infty, \quad (2.10)$$

and, for any $s \in [0, t]$,

$$\mathbb{E}\left(\operatorname*{ess\,sup}_{(b,\sigma)\in\mathcal{A}_F} \mathbb{E}\left(g\left(X^{t,\zeta}_{(b,\sigma)}(T)\right)\Big|\mathcal{F}_t\right)\Big|\mathcal{F}_s\right) = \operatorname*{ess\,sup}_{(b,\sigma)\in\mathcal{A}_F} \mathbb{E}\left(g\left(X^{t,\zeta}_{(b,\sigma)}(T)\right)\Big|\mathcal{F}_s\right). \quad (2.11)$$

Proof. Given $(b_1, \sigma_1), (b_2, \sigma_2) \in \mathcal{A}_F$, we define

$$A := \left\{\omega \in \Omega : \mathbb{E}\left(g\left(X^{t,\zeta}_{(b_1,\sigma_1)}(T)\right)\Big|\mathcal{F}_t\right)(\omega) \geq \mathbb{E}\left(g\left(X^{t,\zeta}_{(b_2,\sigma_2)}(T)\right)\Big|\mathcal{F}_t\right)(\omega)\right\},$$

which belong to $\mathcal{F}_t$, and $(b, \sigma) := \chi_A(b_1, \sigma_1) + \chi_{A^c}(b_2, \sigma_2)$. We thus have $(b, \sigma) \in \mathcal{A}_F$ and (2.8). From this, by the properties of the essential supremum,

we obtain the existence of a sequence $\{(b_i, \sigma_i)\}_{i \in \mathbb{N}}$ in $\mathcal{A}_F$ such that (2.9) is true. Furthermore, from lemma 2.1.6, (2.8) and [41, Theorem 1], (2.10) and (2.11) follow. $\qquad\square$

Lemma 2.1.8. *For each* $x \in \mathbb{R}^N$, $\Phi_t(x)$ *is deterministic. Furthermore* $\Phi_t(x) = u(t, x)$.

At first sight stating that $\Phi_t(x)$ is deterministic may seem trivial, but we remember to the reader that in general the conditional expectation, hence $\mathbb{E}\left(g\left(X_{(b,\sigma)}^{t,x}(T)\right)\Big|\mathcal{F}_t\right)$, is not deterministic.

Proof. By lemma 2.1.4 $\mathcal{J}$ is dense in $\mathcal{A}_F$, hence, by lemma 2.1.7 and proposition A.1.6, there exists a sequence

$$\{(b_i, \sigma_i)\}_{i \in \mathbb{N}} = \left\{\sum_{j=0}^{n_i} \chi_{A_j}(b_{i,j}, \sigma_{i,j})\right\}_{i \in \mathbb{N}}$$

which belong to $\mathcal{J}$ such that $\mathbb{E}\left(g\left(X_{(b_i,\sigma_i)}^{t,x}(T)\right)\Big|\mathcal{F}_t\right) \uparrow \Phi_t(x)$ a.e.. But

$$\mathbb{E}\left(g\left(X_{(b_i,\sigma_i)}^{t,x}(T)\right)\Big|\mathcal{F}_t\right) = \sum_{j=0}^{n_i} \chi_{A_j}\mathbb{E}\left(g\left(X_{(b_{i,j},\sigma_{i,j})}^{t,x}(T)\right)\Big|\mathcal{F}_t\right)$$

$$= \sum_{j=0}^{n_i} \chi_{A_j}\mathbb{E}\left(g\left(X_{(b_{i,j},\sigma_{i,j})}^{t,x}(T)\right)\right)$$

$$\leq \max_{j=0}^{n_i}\mathbb{E}\left(g\left(X_{(b_{i,j},\sigma_{i,j})}^{t,x}(T)\right)\right)$$

$$= \mathbb{E}\left(g\left(X_{(b_{i,j_i},\sigma_{i,j_i})}^{t,x}(T)\right)\right),$$

where j_i is the index that realizes the maximum in the previous formula. This imply $\lim_{i \to \infty}\mathbb{E}\left(g\left(X_{(b_{i,j_i},\sigma_{i,j_i})}^{t,x}(T)\right)\right) = \Phi_t(x)$, therefore $\Phi_t(x)$ is deterministic and this concludes the proof. $\qquad\square$

Notice that, contrary to the appearances the equality $u(t, \zeta) = \Phi_t(\zeta)$, could in general be false, however it is true in our setting, as made clear by the next lemma.

Lemma 2.1.9. *For each* $\zeta \in L^2\left(\Omega, \mathcal{F}_t; \mathbb{R}^N\right)$, *we have that* $u(t, \zeta) = \Phi_t(\zeta)$ *a.e..*

Proof. Thanks to the continuity of u proved in lemma 2.1.6 and consequently, by lemma 2.1.8, of Φ, we only need to prove the statement when $\zeta = \sum_{j=0}^{n} \chi_{A_j} x_j$, where $x_j \in \mathbb{R}^N$ and $\{A_j\}_{j=0}^{n}$ is a $\mathcal{F}_t$–partition of Ω. As seen in the proof of the previous lemma, for each x_j, there exists a sequence $\{(b_{i,j}, \sigma_{i,j})\}$ in $\mathcal{A}_F^t$ such that

$$\lim_{i \to \infty} \mathbb{E}\left(g\left(X_{(b_{i,j}, \sigma_{i,j})}^{t, x_j}(T) \right) \right) = \Phi_t(x_j) = u(t, x_j).$$

Setting $(b_i, \sigma_i) := \sum_{j=0}^{n} \chi_{A_j}(b_{i,j}, \sigma_{i,j})$, we have

$$
\begin{aligned}
\Phi_t(\zeta) &\geq \mathbb{E}\left(g\left(X_{(b_i, \sigma_i)}^{t, \zeta}(T) \right) \Big| \mathcal{F}_t \right) \\
&= \sum_{j=0}^{n} \chi_{A_j} \mathbb{E}\left(g\left(X_{(b_{i,j}, \sigma_{i,j})}^{t, x_j}(T) \right) \Big| \mathcal{F}_t \right) \xrightarrow[i \to \infty]{} \sum_{j=0}^{n} \chi_{A_j} u(t, x_j) \\
&= u(t, \zeta).
\end{aligned}
$$

On the other hand, for any $(b, \sigma) \in \mathcal{A}_F$,

$$
\mathbb{E}\left(g\left(X_{(b, \sigma)}^{t, \zeta}(T) \right) \Big| \mathcal{F}_t \right) = \sum_{j=0}^{n} \chi_{A_j} \mathbb{E}\left(g\left(X_{(b, \sigma)}^{t, x_j}(T) \right) \Big| \mathcal{F}_t \right) \leq \sum_{j=0}^{n} \chi_{A_j} u(t, x_j)
$$
$$
= u(t, \zeta).
$$

Thus $\operatorname*{ess\,sup}_{(b, \sigma) \in \mathcal{A}_F} \mathbb{E}\left(g\left(X_{(b, \sigma)}^{t, \zeta}(T) \right) \Big| \mathcal{F}_t \right) \leq u(t, \zeta)$. This completes the proof. $\qquad\square$

Now we have all that we need to prove the dynamic programming principle:

Theorem 2.1.10 (Dynamic Programming Principle)**.** *Let t and s such that $0 \leq t \leq s \leq T$, then under our assumptions, for any $x \in \mathbb{R}^N$,*

$$
\sup_{(b, \sigma) \in \mathcal{A}_F} \mathbb{E}\left(g\left(X_{(b, \sigma)}^{t, x}(T) \right) \right)
$$
$$
= \sup_{(b, \sigma) \in \mathcal{A}_F} \mathbb{E}\left(\operatorname*{ess\,sup}_{(b', \sigma') \in \mathcal{A}_F} \mathbb{E}\left(g\left(X_{(b', \sigma')}^{s, X_{(b, \sigma)}^{t, x}(s)}(T) \right) \Big| \mathcal{F}_s \right) \right). \quad (2.12)
$$

Proof. By the uniqueness property of the SDEs and since the (b,σ) in $\mathcal{A}_F$ are cadlag, we have

$$\sup_{(b,\sigma)\in\mathcal{A}_F} \mathbb{E}\left(g\left(X^{t,x}_{(b,\sigma)}(T)\right)\right) = \sup_{(b,\sigma)\in\mathcal{A}_F}\sup_{(b',\sigma')\in\mathcal{A}_F} \mathbb{E}\left(g\left(X^{s,X^{t,x}_{(b,\sigma)}(s)}_{(b',\sigma')}(T)\right)\right).$$

Furthermore follows from (2.11) and lemma 2.1.9 that

$$\sup_{(b',\sigma')\in\mathcal{A}_F} \mathbb{E}\left(g\left(X^{s,X^{t,x}_{(b,\sigma)}(s)}_{(b',\sigma')}(T)\right)\right) = \mathbb{E}\left(\operatorname*{ess\,sup}_{(b',\sigma')\in\mathcal{A}_F} \mathbb{E}\left(g\left(X^{s,X^{t,x}_{(b,\sigma)}(s)}_{(b',\sigma')}(T)\right)\Big|\mathcal{F}_s\right)\right),$$

hence (2.12) is true. $\qquad\square$

We finish this section solving the problem 2.3.1 using the dynamic programming principle just proved.

Theorem 2.1.11. *The function* $u(t,x) := \sup_{(b,\sigma)\in\mathcal{A}_F} \mathbb{E}\left(g\left(X^{t,x}_{(b,\sigma)}(T)\right)\right)$ *is the unique viscosity solution to the problem 2.1.1 such that* $u(T,x) = g(x)$ *for any* x *in* $\mathbb{R}^N$.

Proof. Uniqueness follows from theorem 2.1.3 and we already proved in lemma 2.1.6 that u is well defined and continuous. Is also trivial that $u(T,x) = g(x)$ for any $x \in \mathbb{R}^N$, so we only need to prove that u is a viscosity solution. We start proving the subsolution property. Fixed (t,x) in $(0,T) \times \mathbb{R}^N$ and $(b,\sigma) \in \mathcal{A}_F$, we let $v(t,x) := \mathbb{E}\left(g\left(X^{t,x}_{(b,\sigma)}(T)\right)\right)$, φ be a supertangent to v in (t,x) and δ a positive constant such that

$$\varphi(s,y) - \varphi(t,x) \geq v(s,y) - v(t,x) \quad \text{for any } (s,y) \in [t,t+\delta) \times B_\delta(x).$$

By definition we know that

$$\partial_t\varphi(t,x) + F\left(t,x,\nabla_x\varphi, D^2_x\varphi\right) \geq \partial_t\varphi(t,x) + \mathbb{E}\left(L_{(b,\sigma)}\left(t,x,\nabla_x\varphi, D^2_x\varphi\right)\right)$$

and if we define

$$\tau_h := \inf\left\{s \in [t,T] : \left|X^{t,x}_{(b,\sigma)}(s) - x\right| \geq \delta\right\} \wedge (t+h),$$

Lebesgue's differentiation theorem and Itô's formula yield

$$\partial_t \varphi(t, x) + \mathbb{E}\left(L_{(b,\sigma)}\left(t, x, \nabla_x \varphi, D_x^2 \varphi\right)\right)$$

$$= \lim_{h \downarrow 0} \mathbb{E}\left(\int_t^h \left(\frac{\partial_t \varphi\left(s, X_{(b,\sigma)}^{t,x}(s)\right) + L_{(b,\sigma)}\left(s, X_{(b,\sigma)}^{t,x}(s), \nabla_x \varphi, D_x^2 \varphi\right)}{h}\chi_{\{\tau_h \geq s\}}\right) ds\right)$$

$$= \lim_{h \downarrow 0} \frac{\mathbb{E}\left(\varphi\left(\tau_h, X_{(b,\sigma)}^{t,x}(\tau_h)\right)\right) - \varphi(t, x)}{h}.$$

By the uniqueness property of SDEs we have that

$$\mathbb{E}\left(g\left(X_{(b,\sigma)}^{\tau_h, X_{(b,\sigma)}^{t,x}(\tau_h)}(T)\right)\right) := \mathbb{E}\left(g\left(X_{(b,\sigma)}^{t,x}(T)\right)\right),$$

therefore, thanks to the definition of v

$$\partial_t \varphi(t, x) + F\left(t, x, \nabla_x \varphi, D_x^2 \varphi\right) \geq \lim_{h \downarrow 0} \frac{\mathbb{E}\left(\varphi\left(\tau_h, X_{(b,\sigma)}^{t,x}(\tau_h)\right)\right) - \varphi(t, x)}{h}$$

$$\geq \lim_{h \downarrow 0} \frac{\mathbb{E}\left(v\left(\tau_h, X_{(b,\sigma)}^{t,x}(\tau_h)\right)\right) - v(t, x)}{h}$$

$$= 0.$$

This show that v is a viscosity subsolution to problem 2.1.1, thus theorem 2.0.3 yields that u is a viscosity subsolution to problem 2.1.1.

In a similar way we now show that u is a viscosity supersolution. Fixed (t, x) in $(0, T) \times \mathbb{R}^N$, let ψ be a subtangent to u in (t, x) and δ a positive constant such that

$$\psi(s, y) - \psi(t, x) \leq u(s, y) - u(t, x) \quad \text{for any } (s, y) \in [t, t+\delta) \times B_\delta(x).$$

By definition we know that

$$\partial_t \psi(t, x) + F\left(t, x, \nabla_x \psi, D_x^2 \psi\right) = \partial_t \psi(t, x)$$

$$+ \sup_{(b,\sigma) \in \mathcal{A}_F} \mathbb{E}\left(L_{(b,\sigma)}\left(t, x, \nabla_x \psi, D_x^2 \psi\right)\right)$$

and if we define

$$\tau_h := \inf\left\{s \in [t, T] : \left|X_{(b,\sigma)}^{t,x}(s) - x\right| \geq \delta\right\} \wedge (t + h)$$

Lebesgue's differentiation theorem and Itô's formula yield

$$
\partial_t \psi(t, x) + \mathbb{E}\left(L_{(b,\sigma)}\left(t, x, \nabla_x \psi, D_x^2 \psi\right)\right)
$$
$$
= \lim_{h \downarrow 0} \mathbb{E}\left(\int_t^h \left(\frac{\partial_t \psi\left(s, X_{(b,\sigma)}^{t,x}(s)\right) + L_{(b,\sigma)}\left(s, X_{(b,\sigma)}^{t,x}(s), \nabla_x \psi, D_x^2 \psi\right)}{h} \chi_{\{\tau_h \geq s\}}\right) ds\right)
$$
$$
= \lim_{h \downarrow 0} \frac{\mathbb{E}\left(\psi\left(\tau_h, X_{(b,\sigma)}^{t,x}(\tau_h)\right)\right) - \psi(t, x)}{h}.
$$

Therefore, using the subtangency property of ψ, we have that

$$
\partial_t \psi(t, x) + F\left(t, x, \nabla_x \psi, D_x^2 \psi\right) = \sup_{(b,\sigma) \in \mathcal{A}_F} \lim_{h \downarrow 0} \frac{\mathbb{E}\left(\psi\left(\tau_h, X_{(b,\sigma)}^{t,x}(\tau_h)\right)\right) - \psi(t, x)}{h}
$$
$$
\leq \sup_{(b,\sigma) \in \mathcal{A}_F} \lim_{h \downarrow 0} \frac{\mathbb{E}\left(u\left(\tau_h, X_{(b,\sigma)}^{t,x}(\tau_h)\right)\right) - u(t, x)}{h}
$$
$$
\leq \lim_{h \downarrow 0} \frac{\sup_{(b,\sigma) \in \mathcal{A}_F} \mathbb{E}\left(u\left(\tau_h, X_{(b,\sigma)}^{t,x}(\tau_h)\right)\right) - u(t, x)}{h}.
$$

Finally, by the dynamic programming principle 2.1.10,

$$
\lim_{h \downarrow 0} \frac{\sup_{(b,\sigma) \in \mathcal{A}_F} \mathbb{E}\left(u\left(\tau_h, X_{(b,\sigma)}^{t,x}(\tau_h)\right)\right) - u(t, x)}{h}
$$
$$
= \lim_{h \downarrow 0} \frac{\sup_{(b,\sigma) \in \mathcal{A}_F} \mathbb{E}\left(g\left(X_{(b,\sigma)}^{\tau_h, X_{(b,\sigma)}^{t,x}(\tau_h)}(T)\right)\right) - \sup_{(b,\sigma) \in \mathcal{A}_F} \mathbb{E}\left(g\left(X_{(b,\sigma)}^{t,x}(T)\right)\right)}{h} = 0
$$

and this concludes the proof. $\square$

2.2 Dynamic Programming Principle

The scope of this section is to provide a generalization of the dynamic programming principle 2.1.10 used on the Cauchy problem 2.1.1, so that we can employ it to derive representation formulas for the viscosity solutions to more general problems. We remember the reader that this dynamic programming

principle is a generalization of the one presented by Denis, Hu and Peng in [12, Section 3.1] and will be obtained in a similar way.

Let $\mathcal{A}$ be the set made up by the progressive stochastic controls

$$\alpha : [0, \infty) \times \Omega \times \mathbb{R}^N \to \mathbb{R}^M$$

such that $\alpha(x) \in \mathbb{L}^2$ for any $x \in \mathbb{R}^N$ and which are cadlag with respect to the first variable, uniformly Lipschitz continuous with respect to the third one with Lipschitz constant equal or lower than a positive constant ℓ and, for each $(t, x) \in [0, \infty) \times \mathbb{R}^N$, there exists a closed convex set K_t^x such that the $\alpha_t(\omega, x)$ belong to K_t^x for any $\omega \in \Omega$. Notice that under these assumptions $\mathcal{A}$ can be endowed with the topology of the convergence on compact sets as $\mathcal{A}_F$. Then, given an a.e. finite stopping time τ and a continuous application φ from $[0, \infty) \times \mathbb{R}^N \times \mathcal{A}$ into $\mathbb{R}$ such that

$$\varphi_\tau : \mathbb{R}^N \times \mathcal{A}|_{[\tau, \infty)} \to L^1\left(\Omega, \mathcal{F}_\infty; \mathbb{R}\right),$$

we define the set $\mathcal{A}^\tau$, which is made of the controls α belonging to $\mathcal{A}$ such that $\{\alpha_{\tau+t}\}_{t \in [0, \infty)}$ is progressive with respect to the filtration $\{\mathcal{F}_t^\tau\}_{t \in [0, \infty)}$, and, for any $\zeta \in L^2\left(\Omega, \mathcal{F}_\tau; \mathbb{R}^N\right)$, the function

$$\Phi_\tau(\zeta) := \operatorname*{ess\,sup}_{\alpha \in \mathcal{A}} \mathbb{E}(\varphi_\tau(\zeta, \alpha)|\mathcal{F}_\tau).$$

We assume that $\varphi_t(x, \alpha)$ is $\mathcal{F}_\infty^t$–measurable for any $t \in [0, \infty)$, $x \in \mathbb{R}^N$ and $\alpha \in \mathcal{A}^t$, and, for any a.e. finite stopping time τ and $\zeta \in L^2\left(\Omega, \mathcal{F}_\tau; \mathbb{R}^N\right)$,

$$\sup_{\alpha \in \mathcal{A}} \mathbb{E}(|\varphi_\tau(\zeta, \alpha)|) < \infty. \tag{2.13}$$

In this section the function Φ represents, roughly speaking, the viscosity solution, ζ is the first part of a stochastic trajectory broken off at τ (this is why we restrict φ_τ on $\mathcal{A}|_{[\tau, \infty)}$) and φ the function which we will use to build the viscosity solution.

Notice that under our assumptions the set $\mathcal{A}^\tau$ are non empty when τ is deterministic, but they could be empty for a generic a.e. finite stopping time. However it is easy to see that $\mathcal{A}^\tau$ is non empty for any a.e. finite stopping

time τ if the sets K_t^x do not depend on t, i.e. the following condition holds true

$$K_t^x = K_s^x =: K^x \quad \text{for any } t, s \in [0, \infty), \tag{2.14}$$

since deterministic function with value in K^x belongs to $\mathcal{A}^\tau$. In this case we will also assume that $\varphi_\tau(x, \alpha)$ is $\mathcal{F}_\infty^\tau$–measurable for any a.e. finite stopping time τ, $x \in \mathbb{R}^N$ and $\alpha \in \mathcal{A}^\tau$.

We proceed by steps, like the previous section, keeping in mind that the following results are a generalization of [12, Lemmas 41-44 and Theorem 45].

Lemma 2.2.1. *For each a.e. finite stopping time τ and α_1, α_2 in $\mathcal{A}$ there exists an $\alpha \in \mathcal{A}$ such that*

$$\mathbb{E}(\varphi_\tau(\zeta, \alpha)|\mathcal{F}_\tau) = \mathbb{E}(\varphi_\tau(\zeta, \alpha_1)|\mathcal{F}_\tau) \vee \mathbb{E}(\varphi_\tau(\zeta, \alpha_2)|\mathcal{F}_\tau). \tag{2.15}$$

Therefore there exists a sequence $\{\alpha_i\}_{i \in \mathbb{N}}$ in $\mathcal{A}$ such that a.e.

$$\mathbb{E}(\varphi_\tau(\zeta, \alpha_i)|\mathcal{F}_\tau) \uparrow \Phi_\tau(\zeta). \tag{2.16}$$

We also have

$$\mathbb{E}(|\Phi_\tau(\zeta)|) \leq \sup_{\alpha \in \mathcal{A}} \mathbb{E}(|\varphi_\tau(\zeta, \alpha)|) < \infty, \tag{2.17}$$

and, for any stopping time $\tau' \leq \tau$,

$$\mathbb{E}\left(\operatorname*{ess\,sup}_{\alpha \in \mathcal{A}} \mathbb{E}(\varphi_\tau(\zeta, \alpha)|\mathcal{F}_\tau) \middle| \mathcal{F}_{\tau'} \right) = \operatorname*{ess\,sup}_{\alpha \in \mathcal{A}} \mathbb{E}(\varphi_\tau(\zeta, \alpha)|\mathcal{F}_{\tau'}). \tag{2.18}$$

Proof. Given $\alpha_1, \alpha_2 \in \mathcal{A}$, we define

$$A := \{\omega \in \Omega : \mathbb{E}(\varphi_\tau(\zeta, \alpha_1)|\mathcal{F}_\tau)(\omega) \geq \mathbb{E}(\varphi_\tau(\zeta, \alpha_2)|\mathcal{F}_\tau)(\omega)\},$$

which belong to $\mathcal{F}_\tau$, and $\alpha := (\chi_A \alpha_1 + \chi_{A^c} \alpha_2)\chi_{\{t \geq \tau\}}$. We thus have $\alpha \in \mathcal{A}$ and (2.15). From this, by the properties of the essential supremum, we obtain the existence of a sequence $\{\alpha_i\}_{i \in \mathbb{N}}$ in $\mathcal{A}$ such that (2.16) is true. Furthermore, from (2.15), (2.13) and [41, Theorem 1], (2.17) and (2.18) follow. $\qquad \square$

The next is a density result similar to lemma 2.1.4.

Lemma 2.2.2. *The set*

$$
\mathcal{J}^{\tau} := \left\{
\begin{aligned}
&\alpha \in \mathcal{A} : \alpha|_{[\tau,\infty)} = \sum_{i=0}^{n} \chi_{A_i} \alpha_i|_{[\tau,\infty)}, \ \text{where} \ \{\alpha_i\}_{i=0}^{n} \subset \mathcal{A}^{\tau} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{and} \ \{A_i\}_{i=0}^{n} \ \text{is a} \ \mathcal{F}_{\tau}-\text{partition of} \ \Omega
\end{aligned}
\right\}
$$

is dense in $\mathcal{A}$ for any $\tau \in [0, \infty)$. If (2.14) holds, then this is true for any a.e. finite stopping time τ.

Proof. When τ is deterministic this lemma can be proved in the same way of lemma 2.1.4, hence we will just prove it when (2.14) holds true and τ is an a.e. finite stopping time.

Let

$$
\tau_n := \begin{cases}
\dfrac{j}{n}, & \text{if} \quad \dfrac{j-1}{n} \le \tau < \dfrac{j}{n}, j \in \{1, \cdots, n^2\}, \\
\infty, & \text{if} \quad \tau \ge n.
\end{cases}
$$

Clearly τ is the decreasing limit of $\{\tau_n\}$ and, fixed an $\alpha \in \mathcal{A}$, there exists a sequence $\left\{\alpha_i^{j,n}\right\}_{i \in \mathbb{N}} \subset \mathcal{J}^{\frac{j}{n}}$ converging to α for any $n \in \mathbb{N}$ and $j \in \{1, \cdots, n^2\}$. Now take an $\alpha' \in \mathcal{A}^{\tau}$ and define, for any $n \in \mathbb{N}$,

$$
\alpha_t^n := \alpha_t \chi_{\{t < \tau\}} + \left(\sum_{j=1}^{n^2} \alpha_{n,\tau_n - \tau + t}^{j,n} \chi_{\left\{\frac{j-1}{n} \le \tau < \frac{j}{n}\right\}} + \alpha_t' \chi_{\{n \le \tau\}} \right) \chi_{\{t \ge \tau\}}.
$$

It is easy to see that α^n belongs to $\mathcal{J}^{\tau}$ for any $n \in \mathbb{N}$. Fixed $k > 0$ we have that

$$
\mathbb{E}\left(\int_0^k \int_{B_k(0)} |\alpha_t(x) - \alpha_t^n(x)|^2 dx dt \right)
$$

$$
\le 2\mathbb{E}\left(\int_0^{(k-\tau)\vee 0} \int_{B_k(0)} |\alpha_{\tau+t}(x) - \alpha_{\tau_n + t}(x)|^2 \chi_{\{n > \tau\}} dx dt \right)
$$

$$
+ 2 \sum_{j=1}^{n^2} \mathbb{E}\left(\int_0^{(k-\tau)\vee 0} \int_{B_k(0)} \left|\alpha_{n,\tau_n + t}^{j,n}(x) - \alpha_{\tau_n + t}(x)\right|^2 \chi_{\left\{\frac{j-1}{n} \le \tau < \frac{j}{n}\right\}} dx dt \right)
$$

$$
+ 2\mathbb{E}\left(\int_0^{(k-\tau)\vee 0} \int_{B_k(0)} \left(|\alpha_{\tau+t}|^2 + |\alpha'_{\tau+t}|^2\right) \chi_{\{n \le \tau\}} dx dt \right),
$$

thus, thanks to the right continuity of α and the definition of τ, τ_n and $\alpha_n^{j,n}$, we have that for each positive ε and k there exists a $n_{k,\varepsilon}$ such that

$$\mathbb{E}\left(\int_0^k \int_{B_k(0)} |\alpha_t(x) - \alpha_t^n(x)|^2 dx dt\right) < \varepsilon$$

for any $n > n_{k,\varepsilon}$. Consequently $\mathcal{J}^\tau$ is dense in $\mathcal{A}$. $\square$

Lemma 2.2.3. *For each $\tau \in [0,\infty)$ and $x \in \mathbb{R}^N$, $\Phi_\tau(x)$ is deterministic. Furthermore*

$$\Phi_\tau(x) = \operatorname*{ess\,sup}_{\alpha \in \mathcal{A}} \mathbb{E}(\varphi_\tau(x,\alpha)|\mathcal{F}_\tau) = \operatorname*{ess\,sup}_{\alpha \in \mathcal{A}^\tau} \mathbb{E}(\varphi_\tau(x,\alpha)|\mathcal{F}_\tau). \qquad (2.19)$$

If (2.14) holds, then this is true for any a.e. finite stopping time τ.

We remember once again to the reader that in general the conditional expectation, hence $\mathbb{E}(\varphi_\tau(x,\alpha)|\mathcal{F}_\tau)$, is not deterministic.

Proof. By lemma 2.2.2 $\mathcal{J}^\tau$ is dense in $\mathcal{A}$, hence, by lemma 2.2.1, there exists a sequence

$$\{\alpha_i\}_{i\in\mathbb{N}} = \left\{\sum_{j=0}^{n_i} \chi_{A_j} \alpha_{i,j}\right\}_{i\in\mathbb{N}}$$

which belongs to $\mathcal{J}^\tau$ such that $\mathbb{E}(\varphi_\tau(x,\alpha_i)|\mathcal{F}_\tau) \uparrow \Phi_\tau(x)$ a.e.. But

$$\mathbb{E}(\varphi_\tau(x,\alpha_i)|\mathcal{F}_\tau) = \sum_{j=0}^{n_i} \chi_{A_j} \mathbb{E}(\varphi_\tau(x,\alpha_{i,j})|\mathcal{F}_\tau) = \sum_{j=0}^{n_i} \chi_{A_j} \mathbb{E}(\varphi_\tau(x,\alpha_{i,j}))$$

$$\leq \max_{j=0}^{n_i} \mathbb{E}(\varphi_\tau(x,\alpha_{i,j})) = \mathbb{E}(\varphi_\tau(x,\alpha_{i,j_i})),$$

where j_i is the index that realizes the maximum in the previous formula. This implies $\lim_{i\to\infty} \mathbb{E}(\varphi_\tau(x,\alpha_{i,j_i})) = \Phi_\tau(x)$, therefore $\Phi_\tau(x)$ is deterministic, and the inequality

$$\lim_{i\to\infty} \mathbb{E}(\varphi_\tau(x,\alpha_{i,j_i})) \leq \operatorname*{ess\,sup}_{\alpha \in \mathcal{A}^\tau} \mathbb{E}(\varphi_\tau(x,\alpha)|\mathcal{F}_\tau) \leq \Phi_\tau(x)$$

confirms the equation (2.19). $\square$

Remark 2.2.4. Note that by the previous lemma and (2.18) we have that

$$\Phi_\tau(x) = \mathbb{E}\left(\operatorname*{ess\,sup}_{\alpha \in \mathcal{A}} \mathbb{E}(\varphi_\tau(x, \alpha)|\mathcal{F}_\tau)\right) = \sup_{\alpha \in \mathcal{A}} \mathbb{E}(\varphi_\tau(x, \alpha))$$

$$= \sup_{\alpha \in \mathcal{A}^\tau} \mathbb{E}(\varphi_\tau(x, \alpha))$$

for any $\tau \in [0, \infty)$, or, if (2.14) holds true, for any a.e. finite stopping time τ.

Lemma 2.2.5. *We define the function*

$$u : [0, \infty) \times \mathbb{R}^N \longrightarrow \mathbb{R}$$

$$(t, x) \longmapsto \Phi_t(x)$$

and assume that it is continuous. Then, for each a.e. finite stopping time τ *and* $\zeta \in L_N^2\left(\Omega, \mathcal{F}_\tau; \mathbb{R}^N\right)$, *we have that* $u_\tau(\zeta) = \Phi_\tau(\zeta)$ *a.e..*

Remark 2.2.6. This lemma says, as a consequence of (2.19), that

$$\operatorname*{ess\,sup}_{\alpha \in \mathcal{A}} \mathbb{E}(\varphi_\tau(\zeta, \alpha)|\mathcal{F}_\tau) = \operatorname*{ess\,sup}_{\alpha \in \mathcal{A}^\tau} \mathbb{E}(\varphi_\tau(\zeta, \alpha)|\mathcal{F}_\tau)$$

for any $\tau \in [0, \infty)$, or, if (2.14) holds true, for any a.e. finite stopping time τ.

Proof. By the continuity of u, and consequently of Φ, we only need to prove the lemma when $\tau = \sum_{j=0}^n \chi_{A_j} t_j$ and $\zeta = \sum_{j=0}^n \chi_{A_j} x_j$, where $t_j \in [0, \infty)$, $x_j \in \mathbb{R}^N$ and $\{A_j\}_{j=0}^n$ is a $\mathcal{F}_\tau$–partition of Ω. As seen in the proof of the previous lemma, for each (t_j, x_j), there exists a sequence $\{\alpha_{i,j}\}_{i \in \mathbb{N}}$ in $\mathcal{A}^{t_j}$ such that

$$\lim_{i \to \infty} \mathbb{E}\left(\varphi_{t_j}(x_j, \alpha_{i,j})\right) = \Phi_{t_j}(x_j) = u_{t_j}(x_j).$$

Setting $\alpha_i := \sum_{j=0}^n \chi_{A_j} \alpha_{i,j}$, we have

$$\Phi_\tau(\zeta) \geq \mathbb{E}(\varphi_\tau(\zeta, \alpha_i)|\mathcal{F}_\tau)$$

$$= \sum_{j=0}^n \chi_{A_j} \mathbb{E}\left(\varphi_{t_j}(x_j, \alpha_{i,j})|\mathcal{F}_{t_j}\right) \xrightarrow[i \to \infty]{} \sum_{j=0}^n \chi_{A_j} u_{t_j}(x_j) = u_\tau(\zeta).$$

On the other hand, for any $\alpha \in \mathcal{A}$,

$$\mathbb{E}(\varphi_\tau(\zeta, \alpha)|\mathcal{F}_\tau) = \sum_{j=0}^n \chi_{A_j} \mathbb{E}\left(\varphi_{t_j}(x_j, \alpha)|\mathcal{F}_{t_j}\right) \leq \sum_{j=0}^n \chi_{A_j} u_{t_j}(x_j) = u_\tau(\zeta).$$

Thus $\operatorname*{ess\,sup}_{\alpha \in \mathcal{A}} \mathbb{E}(\varphi_\tau(\zeta, \alpha)|\mathcal{F}_\tau) \leq u_\tau(\zeta)$. This completes the proof. $\qquad\square$

Theorem 2.2.7 (Dynamic Programming Principle). *Let τ and τ' be two a.e. finite stopping times such that $\tau' \leq \tau$, ζ be a function from $\mathcal{A}|_{[\tau',\tau)}$ to $L^2(\Omega, \mathcal{F}_\tau; \mathbb{R}^N)$ and assume*

$$\sup_{\alpha \in \mathcal{A}} \mathbb{E}(\varphi_\tau(\zeta_\alpha, \alpha)|\mathcal{F}_{\tau'}) < \infty.$$

Then we have

$$\operatorname*{ess\,sup}_{\alpha \in \mathcal{A}} \mathbb{E}(\varphi_\tau(\zeta_\alpha, \alpha)|\mathcal{F}_{\tau'}) = \operatorname*{ess\,sup}_{\alpha \in \mathcal{A}} \operatorname*{ess\,sup}_{\beta \in \mathcal{A}} \mathbb{E}(\varphi_\tau(\zeta_\alpha, \beta)|\mathcal{F}_{\tau'})$$
$$= \operatorname*{ess\,sup}_{\alpha \in \mathcal{A}} \mathbb{E}(u_\tau(\zeta_\alpha)|\mathcal{F}_{\tau'}). \tag{2.20}$$

In particular

$$\sup_{\alpha \in \mathcal{A}} \mathbb{E}(u_\tau(\zeta_\alpha)) = \sup_{\alpha \in \mathcal{A}} \mathbb{E}(\varphi_\tau(\zeta_\alpha, \alpha)). \tag{2.21}$$

Proof. It follows from (2.18) and lemma 2.2.5 that

$$\operatorname*{ess\,sup}_{\beta \in \mathcal{A}} \mathbb{E}(\varphi_\tau(\zeta_\alpha, \beta)|\mathcal{F}_{\tau'}) = \mathbb{E}\left(\left.\operatorname*{ess\,sup}_{\beta \in \mathcal{A}} \mathbb{E}(\varphi_\tau(\zeta_\alpha, \beta)|\mathcal{F}_\tau)\right|\mathcal{F}_{\tau'}\right) = \mathbb{E}(u_\tau(\zeta_\alpha)|\mathcal{F}_{\tau'}),$$

therefore (2.20) is true (remember that the elements of $\mathcal{A}$ are cadlag, that is why we can break the essential supremum in two part in (2.20)). Finally from (2.20) and [41, Theorem 1(ii)] we get (2.21). $\qquad\square$

Remark 2.2.8. We point out that even if we required the elements of $\mathcal{A}$ to be cadlag, this condition can be weakened. Indeed if the elements of $\mathcal{A}$ are just progressive, each result of this section still holds true with the same proof.

2.2.1 On the continuity of φ

Here we briefly discuss on the continuity assumptions of φ.

It is straightforward to see that many results of this section are still true if φ is not assumed to be continuous. The continuity on the first two variables is actually never used, since we just need and assume the continuity of functions generated from φ by the essential supremum. However the continuity on the control set is used in the proofs of lemmas 2.2.3 and 2.2.5, which are crucial for the proof of dynamic programming principle 2.2.7. We point out that the continuity in probability is enough to prove those results, but we just require that $\varphi_\tau(x, \alpha)$ is in $L^1(\Omega, \mathcal{F}_\infty; \mathbb{R})$ for any $(x, \alpha) \in \mathbb{R}^N \times \mathcal{A}|_{[\tau, \infty)}$. Luckily we have, as a consequence of the density of the simple function in L^1 and that, by lemma 1.1.1, $\mathcal{F}_\infty = \sigma(\mathcal{F}_\tau, \mathcal{F}_\infty^\tau)$, that the set

$$\mathcal{J} := \left\{ \phi \in L^1 : \phi = \sum_{i=0}^n \chi_{A_i} \phi_i, \text{ where } \{\phi_i\}_{i=0}^n \subset L^1 \text{ is } \mathcal{F}_\infty^\tau\text{--measurable} \text{ and } \{A_i\}_{i=0}^n \text{ is a } \mathcal{F}_\tau\text{--partition of } \Omega \right\}$$

is dense in L^1, hence, since by (2.17) $\varphi_\tau(x, \alpha)$ is in L^1 for any $\alpha \in \mathcal{A}$, we have that there exists a sequence $\{\phi_i\}_{i \in \mathbb{N}} \subset \mathcal{J}$ such that $\lim_{i \to \infty} \mathbb{E}(\phi_i | \mathcal{F}_\tau) = \Phi_\tau(x)$ a.e.. So, define, for any $i \in \mathbb{N}$,

$$B_i := \{\omega \in \Omega : E(\phi_i | \mathcal{F}_\tau)(\omega) > \Phi_\tau(x)(\omega)\}, \qquad \overline{\phi}_1 := \phi_1 \chi_{\Omega \setminus B_1}$$

and recursively

$$C_i := \left\{ \omega \in \Omega : E(\phi_i | \mathcal{F}_\tau)(\omega) \geq E\left(\overline{\phi}_{i-1} \big| \mathcal{F}_\tau\right)(\omega) \right\},$$
$$\overline{\phi}_i := \phi_i \chi_{C_i \setminus B_i} + \overline{\phi}_{i-1} \chi_{\Omega \setminus C_i}.$$

By our assumptions $\lim_{i \to \infty} \mathbb{P}(B_i) = 0$ and the sets B_i, C_i belong to $\mathcal{F}_\tau$ for $i \in \mathbb{N}$, therefore $\left\{\overline{\phi}_i\right\}_{i \in \mathbb{N}}$ is a sequence in $\mathcal{J}$ such that $E\left(\overline{\phi}_i \big| \mathcal{F}_\tau\right) \uparrow \Phi_\tau(x)$ a.e.. We can then use this sequence to prove that $\Phi_\tau(x)$ is deterministic as in lemma 2.2.3 and similarly we can prove lemma 2.2.5 for measurable φ. Unfortunately this method can not be used to prove (2.19) and consequently we can not prove the last equality in remark 2.2.4 and remark 2.2.6, but any other result of this section still holds true.

2.3 Parabolic PDEs with Sublinear Operators

We analyze now the following problem:

Problem 2.3.1. Let T be a terminal time, F a uniformly elliptic operator as seen in assumptions 1.2.1 and

$$f : [0, T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R} \text{ and } g : \mathbb{R}^N \to \mathbb{R}$$

two continuous functions for which there exist two constants $\mu \in \mathbb{R}$ and $\ell \geq 0$ such that, for any $t \in [0, T]$, $x, x' \in \mathbb{R}^N$, $y, y' \in \mathbb{R}$ and $z, z' \in \mathbb{R}^N$,

(i) $|g(x) - g(x')| \leq \ell |x - x'|$;

(ii) $|g(x)| \leq \ell (1 + |x|)$;

(iii) $|f(t, x, y, z) - f(t, x', y, z')| \leq \ell (|x - x'| + |z - z'|)$;

(iv) $|f(t, x, y, z)| \leq \ell (1 + |x| + |y| + |z|)$;

(v) $(y - y')(f(t, x, y, z) - f(t, x, y', z)) \leq \mu |y - y'|^2$.

To obtain uniqueness of the solution we will also assume that

(vi) the continuity of the function $y \mapsto f(t, x, y, z)$ is independent from the fourth variable.

Find the solution u to the parabolic PDE

$$\begin{cases} \partial_t u(t, x) + F\left(t, x, \nabla_x u, D_x^2 u\right) + f(t, x, u, \nabla_x u) = 0, & t \in (0, T), x \in \mathbb{R}^N, \\ u(T, x) = g(x), & x \in \mathbb{R}^N. \end{cases}$$

Remark 2.3.2. To ease notation we can assume without loss of generality that the ℓ in problem 2.3.1 is the same of assumptions 1.2.1. Since F is continuous, we can also assume that, for any $(b, \sigma) \in \mathcal{A}_F$, $|b(t, 0)| \leq \ell$ and $|\sigma(t, 0)| \leq \ell$ for any $t \in [0, T]$. Furthermore for our analysis the operators $L_{(b, \sigma)}$ with $(b, \sigma) \in \mathcal{A}_F$ will be as important as they were for the Cauchy problem 2.1.1. Thus, to ease the notations, we define for any a.e. finite stopping time τ the sets $\mathcal{L}_F^\tau$ such that its elements are the operators $L_{(b, \sigma)}$ with (b, σ) in $\mathcal{A}_F^\tau$. As previously done with $\mathcal{A}$ we also define the set $\mathcal{L}_F := \mathcal{L}_F^0$.

As for problem 2.1.1, by theorem 2.0.5 we have the following comparison result.

Theorem 2.3.3. *Let u and v be respectively a subsolution and a super-solution to problem 2.3.1 satisfying polynomial growth condition. Then, if $u|_{t=T} \leq v|_{t=T}$, $u \leq v$ on $(0, T] \times \mathbb{R}^N$.*

When F is a linear operator it is known that the representation formula of its viscosity solution is build from a FBSDE, which we define below. Hence, as done for the Cauchy problem 2.1.1, we will use the dynamic programming principle to generalize this method for our case. We obviously start defining the FBSDE.

Definition 2.3.4. With the term *forward–backward differential equation*, or FBSDE for short, we will refer to a system composed of a SDE and a BSDE:

$$
\begin{cases}
X_s^{t,\zeta} = \zeta + \displaystyle\int_t^s \sigma\left(r, X_r^{t,\zeta}\right) dW_r + \int_t^s b\left(r, X_r^{t,\zeta}\right) dr, \\[2mm]
Y_s^{t,\zeta} = g\left(X_T^{t,\zeta}\right) + \displaystyle\int_s^T f_\sigma\left(r, X_r^{t,\zeta}, Y_r^{t,\zeta}, Z_r^{t,\zeta}\right) dr \qquad s \in [t, T], \\[2mm]
\qquad\quad - \displaystyle\int_s^T Z_r^{t,\zeta} dW_r,
\end{cases} \tag{2.22}
$$

where $\zeta \in L^2\left(\Omega, \mathcal{F}_t; \mathbb{R}^N\right)$, $(b, \sigma) \in \mathcal{A}_F$, the function f_σ is defined as

$$
f_\sigma(t, x, y, z) := f\left(t, x, y, z(\sigma(t,x))^{-1}\right),
$$

for any (t, x, y, z) in $[0, T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$ and the functions f and g are as in the assumptions of problem 2.3.1. Thanks to the uniformly ellipticity condition we know that f_σ is well defined and that the Lipschitz constant for the fourth argument of f_σ is $\ell\sqrt{\dfrac{N}{2\lambda}}$, but for simplicity we will just assume it is ℓ again, possibly increasing it.

Note that under these conditions the assumptions A.1.1 and B.1.1 hold for the SDE part and the BSDE part respectively, thus by theorems A.1.3 and B.1.3 there always exists a unique solution to (2.22). Thanks to remark A.1.8 and proposition B.1.9 this is true even if t is an a.e. finite stopping time.

We will call (X, Y, Z) a solution to the FBSDE if X is a solution to the SDE part of this system and $(Y^{t,\zeta}, Z^{t,\zeta})$ is a solution to the BSDE part for any $(t, \zeta) \in [0, T] \times L^2 \left(\Omega, \mathcal{F}_t; \mathbb{R}^N \right)$. Moreover we will simply write Y to denote the second term of the triplet (X, Y, Z) solution to the FBSDE (2.22), for (b, σ) that varies in $\mathcal{A}_F$. For notation's sake we will also omit the dependence of Y from X and σ or, equivalently, from (b, σ).

Remark 2.3.5. Notice that the uniqueness property of the FBSDE implies that, for any $0 \le t \le r \le s \le T$,

$$\left(X_s^{r, X_r^{t,\zeta}}, Y_s^{r, X_r^{t,\zeta}}, Z_s^{r, X_r^{t,\zeta}} \right) = \left(X_s^{t,\zeta}, Y_s^{t,\zeta}, Z_s^{t,\zeta} \right).$$

This holds true even if t, r and s are stopping time.

Remark 2.3.6. We point out that since each element of $\mathcal{L}_F^\tau$ can be uniquely determined by an element of $\mathcal{A}_F^\tau$, we can associate to each operator $L \in \mathcal{L}_F^\tau$ an FBSDE (2.22) and this connection is unique up to the initial conditions.

We will prove proceeding by steps that $u(t, x) := \sup\limits_{(b,\sigma) \in \mathcal{A}_F} \mathbb{E} \left(Y_t^{t,x} \right)$ is a viscosity solution to the problem 2.3.1.

Proposition 2.3.7. *The function* $u(t, x) := \sup\limits_{(b,\sigma) \in \mathcal{A}_F} \mathbb{E} \left(Y_t^{t,x} \right)$ *is* $\frac{1}{2}$–*Hölder continuous in the first variable and Lipschitz continuous in the second one. Furthermore we have that there exists a constant c, which depends only on ℓ, μ and T, such that*

$$\mathbb{E} \left(|u(\tau, \zeta)|^2 \right) \le \sup\limits_{(b,\sigma) \in \mathcal{A}_F} \mathbb{E} \left(\left| Y_\tau^{\tau, \zeta} \right|^2 \right) \le c \left(1 + \mathbb{E} \left(|\zeta|^2 \right) \right), \qquad (2.23)$$

for any stopping time τ bounded by T and $\zeta \in L^2 \left(\Omega, \mathcal{F}_\tau; \mathbb{R}^N \right)$.

We point out that this proposition permits us to use the results of section 2.2 on u. In particular Y_τ^τ, which is $\mathcal{F}_\tau^\tau$–measurable and therefore a.e. deterministic, has the same role of φ in section 2.2. We already know that Y_τ^τ is continuous, thanks to our assumptions, proposition A.1.6 and theorem B.1.4, furthermore we prove here that it satisfies (2.13) and the continuity of u, which is needed for lemma 2.2.5.

Proof. To prove our statement preliminarily note that by the definition and the Jensen's inequality

$$
\begin{aligned}
|u(t,x) - u(s,y)| &= \left| \sup_{(b,\sigma)\in\mathcal{A}_F} \mathbb{E}\left(Y_t^{t,x}\right) - \sup_{(b,\sigma)\in\mathcal{A}_F} \mathbb{E}\left(Y_s^{s,y}\right) \right| \\
&\leq \sup_{(b,\sigma)\in\mathcal{A}_F} \left(\mathbb{E}\left(\left| Y_t^{t,x} - Y_s^{s,y} \right|^2 \right) \right)^{\frac{1}{2}}
\end{aligned}
\tag{2.24}
$$

and

$$
\mathbb{E}\left(|u(\tau,\zeta)|^2 \right) = \mathbb{E}\left(\left| \operatorname*{ess\,sup}_{(b,\sigma)\in\mathcal{A}_F} \mathbb{E}\left(Y_\tau^{\tau,\zeta} | \mathcal{F}_\tau \right) \right|^2 \right) \leq \sup_{(b,\sigma)\in\mathcal{A}_F} \mathbb{E}\left(\left| Y_\tau^{\tau,\zeta} \right|^2 \right), \tag{2.25}
$$

for any $t,s \in [0,T]$, $x,y \in \mathbb{R}^N$ and $\zeta \in L^2\left(\Omega, \mathcal{F}_t; \mathbb{R}^N\right)$.

Our assumptions, theorems [A.1.4](#) and [B.1.4](#) yield

$$
\begin{aligned}
\mathbb{E}\left(\left| Y_t^{t,x} - Y_t^{t,y} \right|^2 \right) &\leq c_1 \mathbb{E}\left(\left| g\left(X_T^{t,x}\right) - g\left(X_T^{t,y}\right) \right|^2 \right. \\
&\quad \left. + \int_t^T \left| f_\sigma\left(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}\right) - f_\sigma\left(s, X_s^{t,y}, Y_s^{t,x}, Z_s^{t,x}\right) \right|^2 ds \right) \\
&\leq c_1 \ell^2 \mathbb{E}\left(\left| X_T^{t,x} - X_T^{t,y} \right|^2 + \int_t^T \left| X_s^{t,x} - X_s^{t,y} \right|^2 ds \right) \\
&\leq c_2 \left(|x-y|^2 + (T-t)|x-y|^2 \right) \leq c_3 |x-y|^2,
\end{aligned}
$$

where c_1, c_2 and c_3 are three constants which depends upon T, ℓ and μ. This shows that u is Lipschitz continuous in the second variable thanks to (2.24). Now, to prove that u is Hölder continuous in the first variable, we assume that $s \leq t$, hence, as previously done, we obtain, for a constant c_4 which depends on ℓ, μ and T,

$$
\begin{aligned}
\mathbb{E}\left(\left| Y_t^{t,x} - Y_s^{s,x} \right|^2 \right) &\leq 2\mathbb{E}\left(\left| Y_t^{s,x} - Y_s^{s,x} \right|^2 \right) + 2\mathbb{E}\left(\left| Y_t^{t,x} - Y_t^{s,x} \right|^2 \right) \\
&= 2\mathbb{E}\left(\left| Y_t^{s,x} - Y_s^{s,x} \right|^2 \right) + 2\mathbb{E}\left(\left| Y_t^{t,x} - Y_t^{t,X_t^{s,x}} \right|^2 \right) \\
&\leq c_4 \mathbb{E}\left(\int_s^t |f_\sigma(r, X_r^{s,x}, Y_r^{s,x}, Z_r^{s,x})|^2 dr + \left| g\left(X_T^{t,x}\right) - g\left(X_T^{t,X_t^{s,x}}\right) \right|^2 \right. \\
&\quad \left. + \int_t^T \left| f_\sigma\left(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\right) - f_\sigma\left(r, X_r^{t,X_t^{s,x}}, Y_r^{t,x}, Z_r^{t,x}\right) \right|^2 dr \right)
\end{aligned}
$$

$$\leq c_4 \ell^2 \mathbb{E}\Bigg(\int_s^t 3 \left(1 + |X_r^{s,x}|^2 + |Y_r^{s,x}|^2 + |Z_r^{s,x}|^2 \right) dr$$

$$+ \left| X_T^{t,x} - X_T^{t,X_t^{s,x}} \right|^2 + \int_t^T \left| X_r^{t,x} - X_r^{t,X_t^{s,x}} \right|^2 dr \Bigg).$$

Then, from corollaries A.1.5 and B.1.5, we have that there exist four constants c_5, c_6, c_7 and c_8 depending upon μ, ℓ and T such that

$$\mathbb{E}\left(\sup_{r \in [s,T]} |Y_r^{s,x}|^2 \right) \leq c_5 \mathbb{E}\left(|g\left(X_T^{s,x}\right)|^2 + \int_s^T |f_\sigma\left(r, X_r^{s,x}, 0, 0\right)|^2 dr \right)$$

$$\leq 2 c_5 \ell^2 \mathbb{E}\left(\left(1 + |X_T^{s,x}|^2\right) + \int_s^T \left(1 + |X_r^{s,x}|^2\right) dr \right) \quad (2.26)$$

$$\leq c_6 \left(1 + |x|^2 + T\left(1 + |x|^2\right) \right),$$

and

$$\mathbb{E}\left(\sup_{r \in [t,T]} \left| X_r^{t,x} - X_r^{t,X_t^{s,x}} \right|^2 \right) \leq c_7 \mathbb{E}\left(|x - X_t^{s,x}|^2 \right) \leq c_8(t-s),$$

therefore these three inequality together imply that

$$\mathbb{E}\left(\left| Y_t^{t,x} - Y_s^{s,x} \right|^2 \right) \leq c_9(t-s) \left(1 + |x|^2 \right),$$

where c_9 is a constants depending only on μ, ℓ and T, proving that u is $\frac{1}{2}$–Hölder continuous in the first variable thanks to (2.24).

Finally notice that, using remark A.1.8, as we got (2.26) we have that there exists a constants c_{10}, which depends only on μ, ℓ and T, such that

$$\mathbb{E}\left(\left| Y_\tau^{\tau,\zeta} \right|^2 \right) \leq c_{10} \left(1 + \mathbb{E}\left(|\zeta|^2 \right) \right).$$

This, thanks to (2.25), proves (2.23) and concludes the proof. $\qquad\square$

Now we proceed to show that u is a viscosity subsolution. In order to do so we need the following two lemmas:

Lemma 2.3.8. *Let $\{U_t\}_{t \in [0,\infty)}$ be a cadlag process, then for any $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$\mathbb{P}(\{|U_t - U_s| < \varepsilon, \text{ for any } s \in [t, t+\delta)\}) > 0.$$

Proof. Our argument is by contradiction. Assume that there exists an $\varepsilon > 0$ such that for any $\delta > 0$

$$\mathbb{P}(\{|U_t - U_s| < \varepsilon, \text{ for any } s \in [t, t+\delta)\}) = 0,$$

which is equivalent to

$$\mathbb{P}(\{|U_t - U_s| \geq \varepsilon, \text{ for any } s \in [t, t+\delta)\}) = 1.$$

Let, for any positive integer n,

$$A_n := \left\{ |U_t - U_s| \geq \varepsilon, \text{ for any } s \in \left[t, t + \frac{1}{n}\right) \right\},$$

then $A_n \subseteq A_k$ if $k \leq n$ and

$$A := \bigcap_{n=1}^{\infty} A_n = \left\{ \lim_{s \downarrow t} |U_t - U_s| \geq \varepsilon \right\}.$$

Since U is right continuous we know that $\mathbb{P}(A) = 0$ which contradicts our assumption, since $\mathbb{P}(A) = \lim_{n \to \infty} \mathbb{P}(A_n) = 1$. $\hspace{1cm}\square$

Lemma 2.3.9. *For any $t \in (0, T)$, let L be an element of $\mathcal{L}_F^t$ and (X, Y, Z) the solution to the FBSDE (2.22) associated to L as in remark 2.3.6. If we define, for any $x \in \mathbb{R}^N$ and $s \in [t, T]$, $u_L(s, x) := \mathbb{E}(Y_s^{s,x})$ we have that, for any supertangent φ to u_L at (t, x),*

$$L\left(t, x, \nabla_x \varphi, D_x^2 \varphi\right) \geq -\partial_t \varphi(t, x) - f\left(t, x, u_L, \nabla_x \varphi\right).$$

Proof. Preliminarily we denote (b, σ) as the element of $\mathcal{A}_F^t$ associated to L and point out that since (b, σ), restricted in $[t, T]$, is progressive with respect to the filtration $\{\mathcal{F}_s^t\}_{s \in [t,T]}$, so are L, X^t and Y^t, therefore they are constants a.e. in t. As a consequence $u_L(t, x) = Y_t^{t,x}$ a.e. for any $x \in \mathbb{R}^N$.

Given $x \in \mathbb{R}^N$ and a supertangent φ to u_L at (t, x) we can assume without loss of generality that $u_L(t, x) = \varphi(t, x)$, so we suppose that, a.e.,

$$\partial_t \varphi(t, x) + L\left(t, x, \nabla_x \varphi, D_x^2 \varphi\right) + f_\sigma(t, x, u_L, \nabla_x \varphi \sigma) < 0 \hspace{2cm} (2.27)$$

and we will find a contradiction. Note that, as a consequence of the Blumenthal's 0–1 law 1.1.2, this is a deterministic inequality a.e.. By the definition of supertangent, there exists a $\delta \in (0, T - t)$ such that, for any $s \in [t, t + \delta]$ and $y \in B_\delta(x)$,

$$u_L(s, y) \leq \varphi(s, y), \tag{2.28}$$

hence we define the stopping time

$$\tau := (t + \delta) \wedge \inf \left\{ s \in [t, \infty) : \left| X_s^{t,x} - x \right| \geq \delta \right\}$$

and assume, possibly taking a smaller τ, that

$$\begin{aligned}
\partial_t \varphi \left(s \wedge \tau, X_{s\wedge\tau}^{t,x} \right) &+ L \left(s \wedge \tau, X_{s\wedge\tau}^{t,x}, \nabla_x \varphi, D_x^2 \varphi \right) \\
&+ f_\sigma \left(s \wedge \tau, X_{s\wedge\tau}^{t,x}, \varphi, \nabla_x \varphi \sigma \right) < 0.
\end{aligned} \tag{2.29}$$

We point out that, by (2.27) and lemma 2.3.8, the previous inequality holds true on a set of positive measure for the $\chi_{[t,t+\delta]} dt \times d\mathbb{P}$ measure, thus $\tau > t$ on a set of positive measure.

Let $\left(\overline{Y}_s, \overline{Z}_s \right) := \left(Y_{s\wedge\tau}^{t,x}, Z_{s\wedge\tau}^{t,x} \right)$, which solve the BSDE

$$\overline{Y}_s = Y_\tau^{t,x} + \int_{s\wedge\tau}^\tau f_\sigma \left(r, X_r^{t,x}, \overline{Y}_r, \overline{Z}_r \right) dr - \int_{s\wedge\tau}^\tau \overline{Z}_r dW_r, \quad s \in [t, T],$$

and $\left(\hat{Y}_s, \hat{Z}_s \right) := \left(\varphi \left(s, X_{s\wedge\tau}^{t,x} \right), (\nabla_x \varphi \sigma) \left(s \wedge \tau, X_{s\wedge\tau}^{t,x} \right) \right)$ which, by Itô's formula, is solution to

$$\begin{aligned}
\hat{Y}_s = &\varphi \left(\tau, X_\tau^{t,x} \right) - \int_{s\wedge\tau}^\tau \hat{Z}_r dW_r \\
&- \int_{s\wedge\tau}^\tau \left(\partial_t \varphi \left(r, X_r^{t,x} \right) + L \left(r, X_r^{t,x}, \nabla_x \varphi, D_x^2 \varphi \right) \right) dr,
\end{aligned} \quad s \in [t, T].$$

By (2.28) we have that

$$u_L \left(\tau, X_\tau^{t,x} \right) - \varphi \left(\tau, X_\tau^{t,x} \right) = Y_\tau^{\tau, X_\tau^{t,x}} - \varphi \left(\tau, X_\tau^{t,x} \right) \leq 0$$

and (2.29) imply, thanks to corollary B.1.8, that $Y_t^{t,x} < \varphi(t, x)$ a.e., but this lead to a contradiction since we know that, by our assumptions, $\varphi(t, x) = Y_t^{t,x}$ a.e.. This concludes the proof. $\qquad\square$

Proposition 2.3.10. *The function $u(t,x)$ is a continuous viscosity subsolution to the problem 2.3.1.*

Proof. We know from proposition 2.3.7 that u is continuous, thus we just have to prove the subsolution property to conclude the proof.

Let L be an element of $\mathcal{L}_F^t$ and u_L as defined in lemma 2.3.9, then if φ is a supertangent to u_L in (t,x) we have that, by the definition of $\mathcal{L}_F^t$,

$$F\left(t,x,\nabla_x\varphi,D_x^2\varphi\right) \geq L\left(t,x,\nabla_x\varphi,D_x^2\varphi\right) \geq -\partial_t\varphi(t,x) - f(t,x,u_L,\nabla_x\varphi),$$

therefore u_L is a viscosity subsolution to the problem 2.3.1 at (t,x). Thanks to the arbitrariness of t, L and x we then have that u_L is a viscosity subsolution in (t,x) for any $L \in \mathcal{L}_F^t$, $x \in \mathbb{R}^N$ and $t \in (0,T)$. From remark 2.2.6 we have that

$$\sup_{L\in\mathcal{L}_F^t} u_L(t,x) = \sup_{(b,\sigma)\in\mathcal{A}_F^t} \mathbb{E}\left(Y_t^{t,x}\right) = \sup_{(b,\sigma)\in\mathcal{A}_F} \mathbb{E}\left(Y_t^{t,x}\right) = u(t,x),$$

therefore the family of functions $\{u_L\}$ is locally equibounded, thanks to proposition 2.3.7. Theorem 2.0.3 hence yields that

$$\sup_{L\in\mathcal{L}_F^t} u_L(t,x) = u(t,x)$$

is a viscosity subsolution for any $(t,x) \in (0,T) \times \mathbb{R}^N$. $\square$

Before concluding this section proving that u is also a viscosity supersolution, we need this preliminary lemma which expresses the dynamic programming principle for this problem.

Lemma 2.3.11. *For any $(b,\sigma) \in \mathcal{A}_F$ we let $\left(\overline{Y},\overline{Z}\right)$ be the solution of the BSDE*

$$\overline{Y}_s = u\left(\tau,X_\tau^{t,x}\right) + \int_{s\wedge\tau}^\tau f_\sigma\left(r,X_r^{t,x},\overline{Y}_r,\overline{Z}_r\right)dr - \int_{s\wedge\tau}^\tau \overline{Z}_r dW_r, \qquad (2.30)$$

where $s \in [t,T]$ and τ is a stopping time with value in $[t,T]$. Then we have that $\sup_{(b,\sigma)\in\mathcal{A}_F} \mathbb{E}\left(\overline{Y}_t\right) = u(t,x)$.

Proof. Fix $(\bar{b}, \bar{\sigma}) \in \mathcal{A}_F$ in (2.30) and define $\overline{X} := X_{(b,\sigma)}$ and the subset of $\mathcal{A}_F$

$$\overline{\mathcal{A}}_F := \left\{ (b, \sigma) \in \mathcal{A}_F : (b, \sigma)(s) = \left(\bar{b}, \bar{\sigma}\right)(s) \text{ for any } s \in [t, \tau] \right\}.$$

From lemma 2.2.5 we know that

$$\sup_{(b,\sigma) \in \overline{\mathcal{A}}_F} \mathbb{E}\left(Y_\tau^{t,x} \big| \mathcal{F}_\tau\right) = \sup_{(b,\sigma) \in \overline{\mathcal{A}}_F} \mathbb{E}\left(Y_\tau^{\tau, \overline{X}_\tau^{t,x}} \big| \mathcal{F}_\tau\right) = u\left(\tau, \overline{X}_\tau^{t,x}\right)$$

and lemma 2.2.1 yields the existence of a sequence $\{(b_n, \sigma_n)\}_{n \in \mathbb{N}}$ in $\overline{\mathcal{A}}_F$ and a corresponding sequence $\{Y_n\}_{n \in \mathbb{N}}$ such that

$$\lim_{n \to \infty} \mathbb{E}\left(Y_{n,\tau}^{t,x} \big| \mathcal{F}_\tau\right) = \sup_{(b,\sigma) \in \overline{\mathcal{A}}_F} \mathbb{E}\left(Y_\tau^{t,x} \big| \mathcal{F}_\tau\right) = u\left(\tau, \overline{X}_\tau^{t,x}\right).$$

Then, by theorem B.1.4, corollary B.1.5 and the dominated convergence theorem, there exists a constant c such that

$$\lim_{n \to \infty} \mathbb{E}\left(\left|\overline{Y}_t - Y_{n,t}^{t,x}\right|^2\right) \leq \lim_{n \to \infty} c\mathbb{E}\left(\left|u\left(\tau, \overline{X}_\tau^{t,x}\right) - Y_{n,\tau}^{t,x}\right|^2\right) = 0,$$

hence, up to subsequences,

$$\lim_{n \to \infty} \mathbb{E}\left(Y_{n,t}^{t,x}\right) = \mathbb{E}\left(\overline{Y}_t\right). \tag{2.31}$$

Furthermore, thanks to theorem B.1.7, $Y_t^{t,x} \leq \overline{Y}_t$ for any $(b, \sigma) \in \overline{\mathcal{A}}_F$, which together with (2.31) implies that $\sup_{(b,\sigma) \in \overline{\mathcal{A}}_F} \mathbb{E}\left(Y_t^{t,x}\right) = \mathbb{E}\left(\overline{Y}_t\right)$. Therefore we can use the arbitrariness of $\left(\bar{b}, \bar{\sigma}\right)$ and the dynamic programming principle 2.2.7 to obtain our conclusion:

$$\sup_{(b,\sigma) \in \mathcal{A}_F} \mathbb{E}\left(\overline{Y}_t\right) = \sup_{(b,\sigma) \in \mathcal{A}_F} \mathbb{E}\left(Y_t^{t,x}\right) = u(t,x).$$

$\square$

We can now prove the main statement of this section.

Theorem 2.3.12. *The function* $u(t, x) := \sup_{(b,\sigma) \in \mathcal{A}_F} \mathbb{E}\left(Y_t^{t,x}\right)$ *is the only viscosity solution to the problem 2.3.1 satisfying polynomial growth condition such that* $u(T, x) = g(x)$ *for any* x *in* $\mathbb{R}^N$.

Proof. The uniqueness is a consequence of theorem 2.3.3 and (2.23), hence we only have to show that u is a viscosity solution.

From proposition 2.3.10 we know that u is a continuous viscosity subsolution and it is easy to see that $u(T, x) = g(x)$ for any $x \in \mathbb{R}^N$, so we only need to prove the supersolution property of u. Fixed (t, x) in $(0, T) \times \mathbb{R}^N$, let ψ be a subtangent to u in (t, x) which we assume, without loss of generality, equal to u in (t, x) and δ a positive constant such that

$$\psi(s, y) \le u(s, y) \qquad \text{for any } (s, y) \in [t, t+\delta] \times B_\delta(x). \tag{2.32}$$

We know, thanks to theorem 1.2.2, that there exists a continuous and deterministic $L \in \mathcal{L}_F$ for which

$$F\left(t, x, \nabla_x \psi, D_x^2 \psi\right) = L\left(t, x, \nabla_x \psi, D_x^2 \psi\right)$$

and assume by contradiction

$$F\left(t, x, \nabla_x \psi, D_x^2 \psi\right) = L\left(t, x, \nabla_x \psi, D_x^2 \psi\right) > -\partial_t \psi(t, x) - f_\sigma(t, x, u, \nabla_x \psi \sigma).$$

Then, by continuity,

$$\partial_t \psi(s, y) + L\left(s, y, \nabla_x \psi, D_x^2 \psi\right) > -f_\sigma(s, y, \psi, \nabla_x \psi \sigma) \tag{2.33}$$

for any $(s, y) \in [t, t+\delta] \times B_\delta(x)$, possibly taking a smaller δ.

We denote with (b, σ) and (X, Y, Z), respectively, the element of $\mathcal{A}_F$, which, to repeat, is continuous and deterministic, and the solution to the FBSDE (2.22) associated to L. We define the stopping time

$$\tau := (t+\delta) \wedge \inf\left\{s \in [t, \infty) : \left|X_s^{t,x} - x\right| \ge \delta\right\},$$

let $\left(\overline{Y}_s, \overline{Z}_s\right)$ be the solution to the BSDE

$$\overline{Y}_s = u\left(\tau, X_\tau^{t,x}\right) + \int_{s \wedge \tau}^\tau f_\sigma\left(r, X_r^{t,x}, \overline{Y}_r, \overline{Z}_r\right) dr - \int_{s \wedge \tau}^\tau \overline{Z}_r dW_r, \quad s \in [t, T]$$

and $\left(\hat{Y}_s, \hat{Z}_s\right) := \left(\psi\left(s, X_{s \wedge \tau}^{t,x}\right), (\nabla_x \psi \sigma)\left(s, X_{s \wedge \tau}^{t,x}\right)\right)$ which, by Itô's formula, is solution to

$$\hat{Y}_s = \psi\left(\tau, X_\tau^{t,x}\right) - \int_{s \wedge \tau}^\tau \hat{Z}_r dW_r$$
$$- \int_{s \wedge \tau}^\tau \left(\partial_t \psi\left(r, X_r^{t,x}\right) + L\left(r, X_r^{t,x}, \nabla_x \psi, D_x^2 \psi\right)\right) dr, \qquad s \in [t, T].$$

We know from lemma 2.3.11 that

$$\sup_{(b,\sigma)\in\mathcal{A}_F} \mathbb{E}\left(\overline{Y}_t\right) = u(t,x) = \psi(t,x), \tag{2.34}$$

but by (2.32) we have $u\left(\tau, X_\tau^{t,x}\right) \geq \psi\left(\tau, X_\tau^{t,x}\right)$, which together with (2.33) imply, thanks to corollary B.1.8, that $\overline{Y}_t > \psi(t,x)$ a.e., in contradiction with (2.34). $\qquad\square$

Remark 2.3.13. Assume that F and f in problem 2.3.1 do not depend on t and let u be the viscosity solution to this problem. Then it is easy to see that $v(t,x) := u(T-t,x)$ is the viscosity solution of the following parabolic problem:

$$\begin{cases} F\left(x, \nabla_x v, D_x^2 v\right) - \partial_t v(t,x) + f(x, v, \nabla_x v) = 0, & t \in [0,\infty), x \in \mathbb{R}^N \\ v(0,x) = g(x), & x \in \mathbb{R}^N. \end{cases}$$

We thus have that, for any $T \in [0,\infty)$, $v(t,x) := \sup_{(b,\sigma)\in\mathcal{A}_F^{T-t}} \mathbb{E}\left(Y_{T-t}^{T-t,x}\right)$, where Y is the solution to the FBSDE

$$\begin{cases} X_s^{T-t,x} = x + \int_{T-t}^s \sigma\left(r, X_r^{T-t,x}\right) dW_r + \int_{T-t}^s b\left(r, X_r^{T-t,x}\right) dr, \\ Y_s^{T-t,x} = g\left(X_T^{T-t,x}\right) + \int_s^T f_\sigma\left(X_r^{T-t,x}, Y_r^{T-t,x}, Z_r^{T-t,x}\right) dr & \begin{array}{l} s \in [T-t, T], \\ x \in \mathbb{R}^N, \end{array} \\ \qquad\qquad - \int_s^T Z_r^{T-t,x} dW_r, \end{cases}$$

with $(b,\sigma) \in \mathcal{A}_F^{T-t}$. Note that even in the time independent case we still need to rely on time dependent σ and b in order to use the dynamic programming principle. To eliminate T from this representation formula we can, for example, use the fact that, in this case, for each $(b,\sigma) \in \mathcal{A}_F^{T-t}$ there exists an element $(b',\sigma') \in \mathcal{A}_F$ which has the same distribution of $(b,\sigma)(T-t+s,x)$ for any $(s,x) \in [0,t] \times \mathbb{R}^N$. From this, if we define the FBSDE, with notation changed to point out the terminal time instead of the initial time,

$$\begin{cases} X_s^{t,x} = x + \int_0^s \sigma\left(r, X_r^{t,x}\right) dW_r + \int_0^s b\left(r, X_r^{t,x}\right) dr, & s \in [t, T], \\ Y_s^{t,x} = g\left(X_t^{t,x}\right) + \int_s^t f_\sigma\left(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\right) dr - \int_s^t Z_r^{t,x} dW_r, & x \in \mathbb{R}^N, \end{cases}$$

with $(b, \sigma) \in \mathcal{A}_F$, we then have that $v(t, x) := \sup\limits_{(b,\sigma) \in \mathcal{A}_F} \mathbb{E}\left(Y_0^{t,x}\right)$.

2.4 The Dirichlet Problem with Sublinear Operators

The problem which will be examined in this section is static, i.e., time independent. It makes this problem more difficult to study, since we also have to analyze the behavior of the stopping times

$$\tau_{(b,\sigma)}^{t,\zeta} := \inf\left\{s \in [0, \infty) : X_{(b,\sigma),t+s}^{t,\zeta} \notin D\right\},$$

where D is the domain on which is defined the problem and X is a solution to the SDE in (2.22). We already know that this is an a.e. finite stopping times thanks to proposition A.2.1.

Before starting with this section's problem, we give the following definition, which will play a key role in the proof of the continuity of the exit times.

Definition 2.4.1. We will call a set $C \subset \mathbb{R}^N$ a *convex cone* if for every $x, y \in C$ then $x + y \in C$ and $\alpha x \in C$ for any non negative α.

We will say that a set D satisfies the *exterior cone condition* if, for any $x \in \partial D$, there exists a convex cone C with $\operatorname{int} C \neq \emptyset$ and a positive δ such that $(x + C) \cap \overline{D} \cap B_\delta(x) = \{x\}$.

Problem 2.4.2. Let $D \subset \mathbb{R}^N$ be an open bounded set which satisfies an exterior cone condition, F a uniformly elliptic operator as seen in assumptions 1.2.1, which we also assume time independent with

(i) $\max\limits_{(p,S) \in \mathbb{R}^N \times \mathbb{S}^N} \dfrac{F(x, p, S)}{|(p, S)|} \leq \ell$,

and

$$f : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R} \text{ and } g : \mathbb{R}^N \to \mathbb{R}$$

two continuous functions for which there exist three constants $\mu \in \mathbb{R}$, $l \geq 0$ and ℓ, which we assume to be the same one in assumptions 1.2.1 for F, such that, for any $x, x' \in \mathbb{R}^N$, $y, y' \in \mathbb{R}$ and $z, z' \in \mathbb{R}^N$,

(ii) $|g(x) - g(x')| \leq \ell|x - x'|$;

(iii) $|g(x)| \leq \ell$;

(iv) $|f(x, y, z) - f(x, y, z')| \leq l\sqrt{\dfrac{2\lambda}{N}}|z - z'|$;

(v) $|f(x, y, z) - f(x', y, z)| \leq \ell|x - x'|$;

(vi) $|f(x, 0, 0)| \leq \ell$;

(vii) $|f(x, y, z)| \leq \ell(1 + |y|)$;

(viii) $(y - y')(f(x, y, z) - f(x, y', z)) \leq \mu|y - y'|^2$.

Furthermore assume that there exists a constant $\vartheta > l^2 + \mu$ such that

(ix) $\displaystyle\sup_{(b,\sigma)\in\mathcal{A}_F} \sup_{x\in D} \mathbb{E}\left(e^{\vartheta\tau^{0,x}_{(b,\sigma)}}\right) < \infty.$

Find the solution u to the elliptic PDE

$$\begin{cases} F\left(x, \nabla u, D^2 u\right) + f(x, u, \nabla u) = 0, & x \in D, \\ u(x) = g(x), & x \in \partial D. \end{cases}$$

Remark 2.4.3. Here we comment on the assumptions for problem 2.4.2.

It is easy to see that, by item (i) in problem 2.4.2, $|b| \leq \ell$ and $|\sigma|^2 \leq 2\ell$ for any $(b, \sigma) \in \mathcal{A}_F$. It will also be convenient to assume f equal to 0 outside D. We point out that even if F do not depends on time, we still require that the elements of $\mathcal{A}_F$ are cadlag in time. This is because we need it to apply the dynamic programming principle 2.2.7.

A consequence of the fact that the operator F does not depend on time, is that for any $(b, \sigma) \in \mathcal{A}_F$ and a.e. finite stopping time ρ we can take a $\left(\bar{b}, \bar{\sigma}\right) \in \mathcal{A}_F^\rho$ such that $(b, \sigma)(t, x) \stackrel{d}{=} \left(\bar{b}, \bar{\sigma}\right)(\rho + t, x)$ for any $t \in [0, \infty)$ and $x \in \mathbb{R}^N$. Consequently, by remark A.1.9, we get that $X^{0,x}_{(b,\sigma),t} \stackrel{d}{=} X^{\rho,x}_{\left(\bar{b},\bar{\sigma}\right),\rho+t}$ for any $t \in [0, \infty)$ and $x \in D$, which also implies that $\tau^{0,x}_{(b,\sigma)} \stackrel{d}{=} \tau^{\rho,x}_{\left(\bar{b},\bar{\sigma}\right)}.$

Item (ix) is a standard assumption for stochastic representation formulas for viscosity solutions to elliptic problems in the linear case, as can be seen for

example in [3, 9, 27, 28]. The role of this condition is to assure us that our candidate viscosity solution does not explode at some point in $\overline{D}$, as we will see in lemma 2.4.7. We also note that while we know, thanks to proposition A.2.2, that there exists a constant ϑ which realize item (ix) in problem 2.4.2, it may not be greater than $l^2 + \mu$.

A comparison result holds for this problem as a consequence of [8, Theorem 3.3] and lemma 1.2.7.

Theorem 2.4.4. *Let u and v be respectively a subsolution and a supersolution to problem 2.4.2 such that $u \leq v$ on ∂D. Then, if $\mu < 0$, $u \leq v$ on $\overline{D}$.*

As previously done, we start our analysis defining the FBSDEs we will use to build our viscosity solution. The main difference with the parabolic problem is that this one has random terminal time.

Definition 2.4.5. Consider the FBSDE with random terminal time

$$\begin{cases} X_{t+s}^{t,\zeta} = \zeta + \int_t^{t+s} \sigma\left(r, X_r^{t,\zeta}\right) dW_r + \int_t^{t+s} b\left(r, X_r^{t,\zeta}\right) dr, \\ Y_{t+s}^{t,\zeta} = g\left(X_{t+\tau_{(b,\sigma)}^{t,\zeta}}^{t,\zeta}\right) + \int_{t+s\wedge\tau_{(b,\sigma)}^{t,\zeta}}^{t+\tau_{(b,\sigma)}^{t,\zeta}} f_\sigma\left(r, X_r^{t,\zeta}, Y_r^{t,\zeta}, Z_r^{t,\zeta}\right) dr \\ \qquad - \int_{t+s\wedge\tau_{(b,\sigma)}^{t,\zeta}}^{t+\tau_{(b,\sigma)}^{t,\zeta}} Z_r^{t,\zeta} dW_r, \\ Z_{t+s}^{t,\zeta} = 0, \qquad \text{on } \left\{\tau_{(b,\sigma)}^{t,\zeta} \leq s\right\}, \end{cases} \quad s \in [0, \infty), \quad (2.35)$$

where $\zeta \in L^2\left(\Omega, \mathcal{F}_t; \mathbb{R}^N\right)$, $(b, \sigma) \in \mathcal{A}_F$, the function f_σ is defined as

$$f_\sigma(t, x, y, z) := f\left(x, y, z(\sigma(t, x))^{-1}\right),$$

for any (t, x, y, z) in $[0, \infty) \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$ and the functions f and g are as in the assumptions of problem 2.4.2. Is easy to see that the Lipschitz constant for the fourth argument of f_σ is l.

Note that under these conditions the assumptions A.1.1, (A.4) and (A.5) hold for the SDE part, while assumptions B.2.1 hold for the BSDE part (item (viii)

is true by remark 2.4.3), thanks to lemma 2.4.9 which we will prove below. Therefore by remark B.2.3 and theorems A.1.3 and B.2.4 there always exists a unique solution to (2.35) and thanks to remark A.1.8 and proposition B.1.9 this is true even if t is an a.e. finite stopping time.

Once again we will call (X, Y, Z) a solution to this FBSDE with random terminal time if X is a solution to the SDE part of this system and $(Y^{t,\zeta}, Z^{t,\zeta})$ is a solution to the BSDE part for any $t \in [0, \infty)$ and $\zeta \in L^2(\Omega, \mathcal{F}_t; \mathbb{R}^N)$.

Remark 2.4.6. Notice that the uniqueness property of the FBSDE implies that, for any $0 \le t \le r \le s$,

$$\left(X_s^{r, X_r^{t,\zeta}}, Y_s^{r, X_r^{t,\zeta}}, Z_s^{r, X_r^{t,\zeta}} \right) = \left(X_s^{t,\zeta}, Y_s^{t,\zeta}, Z_s^{t,\zeta} \right).$$

Moreover $\tau^{r, X_r^{t,x}} = (\tau^{t,x} - (r - t)) \vee 0$. This holds true even if t, r and s are stopping time.

As before we will simply write Y to denote the second term of the triplet (X, Y, Z) solution to the FBSDE (2.35), for (b, σ) that varies in $\mathcal{A}_F$. For notation's sake we will omit the dependence from (b, σ) of X, Y, Z and even τ when obvious. Furthermore we will also write X^x, Y^x, Z^x and τ^x instead of $X^{0,x}$, $Y^{0,x}$, $Z^{0,x}$ and $\tau^{0,x}$ respectively.

We preliminarily analyze the exit times.

Lemma 2.4.7. *For any a.e. finite stopping time ρ, $x \in \mathbb{R}^N$ and $(b, \sigma) \in \mathcal{A}_F$, define the stopping time $\overline{\tau}^{\rho,x} := \inf \left\{ t \in [0, \infty) : X_{\rho+t}^{\rho,x} \notin \overline{D} \right\}$. Then*

$$\mathbb{P}(\tau^{\rho,x} = \overline{\tau}^{\rho,x}) = 1 \tag{2.36}$$

for any a.e. finite stopping time ρ, $(b, \sigma) \in \mathcal{A}_F$ and $x \in \mathbb{R}^N$.

The importance of this lemma is due to (2.36), which is used in theorems A.2.4 and A.2.5 to prove the continuity of the exit times and consequently will allow us to prove the continuity of our candidate viscosity solution. This lemma is the reason why we require D to satisfy an exterior cone condition and is a generalization of [3, Proposition III.3.1], where the author consider deterministic (b, σ).

Proof. Preliminarily note that if $x \notin \overline{D}$ this is obviously true, while, if $x \in \overline{D}$, by remark 2.4.6 our statement is equivalent to

$$\mathbb{P}\left(\overline{\tau}^{\tau^{\rho,x}, X_{\tau^{\rho,x}}^{\rho,x}} = 0\right) = 1, \tag{2.37}$$

for any a.e. finite stopping time ρ and $(b, \sigma) \in \mathcal{A}_F$. Instead of (2.37) we will prove the stronger result

$$\mathbb{P}\left(\overline{\tau}^{\rho,y} = 0\right) = 1 \tag{2.38}$$

for any a.e. finite stopping time ρ, $(b, \sigma) \in \mathcal{A}_F$ and $y \in \partial D$.

We will proceed by steps.

Step 1. Fixed $x \in \partial D$, in the first steps we will prove that $\mathbb{P}(\overline{\tau}^x = 0) = 1$ for any $(b, \sigma) \in \mathcal{A}_F$, and from this we will then prove (2.38) by approximation. We can assume without loss of generality that $x = 0$, since if X^x is solution to the SDE (b, σ), then $X^x - x$ is solution to the SDE $(b(\cdot + x), \sigma(\cdot + x))$.

Step 2. Thanks to our assumptions we have that there exists a convex cone C with $\operatorname{int} C \neq \emptyset$ and a positive δ such that $C \cap \overline{D} \cap B_\delta(0) = \{0\}$, hence there exists an $x \in C$ and a positive constant ε such that $B_\varepsilon(x) \subset C \cap B_\delta(0)$. Fixed $t \in [0, \infty)$ define the continuous function $\varphi : [0, t] \to \mathbb{R}^N$ such that $\varphi(0) = 0$ and $\varphi(t) = x$. Then, by proposition A.2.3, there exists a positive constant c, which depend on ℓ, λ, t and the modulus of continuity of φ, such that

$$\mathbb{P}\left(\sup_{s \in [0,t]} |X_s^x - \varphi(s)| < \varepsilon\right) \geq c.$$

Step 3. Now for any $\alpha > 1$ let $b_\alpha(t, x) := \alpha^{-1} b\left(\frac{t}{\alpha^2}, \frac{x}{\alpha}\right)$, $\sigma_\alpha(t, x) := \sigma\left(\frac{t}{\alpha^2}, \frac{x}{\alpha}\right)$ and $W_{\alpha,t} := \alpha W_{\frac{t}{\alpha^2}}$. Note that W_α is still a Brownian motion and, since $\alpha > 1$, for σ_α and b_α hold the same assumptions, with the same constants ℓ and λ of σ and b. Therefore

$$\begin{aligned} X_{\alpha,t}^0 :=& \alpha X_{\frac{t}{\alpha^2}}^0 = \int_0^{\frac{t}{\alpha^2}} \alpha \sigma\left(s, X_s^0\right) dW_s + \int_0^{\frac{t}{\alpha^2}} \alpha b\left(s, X_s^0\right) ds \\ =& \int_0^t \sigma_\alpha\left(s, X_{\alpha,s}^0\right) dW_s^\alpha + \int_0^t b_\alpha\left(s, X_{\alpha,s}^0\right) ds, \end{aligned}$$

that is to say that X_α^0 is solution to the SDE $(\sigma_\alpha, b_\alpha)$.

Step 4. By the previous steps and since cones are invariant under scaling, we

have for any $\alpha > 1$

$$\mathbb{P}\left(\overline{\tau}^0 \le \frac{t}{\alpha^2}\right) \ge \mathbb{P}\left(X^0_{\frac{t}{\alpha^2}} \in \text{int}\, C\right) = \mathbb{P}\left(X^0_{\alpha,t} \in \text{int}\, C\right) \ge c > 0.$$

Thus, sending α to ∞, we get $\mathbb{P}(\overline{\tau}^0 = 0) > 0$, so Blumenthal's 0–1 law 1.1.2 and the arbitrariness of x yield $\mathbb{P}(\overline{\tau}^x = 0) = 1$ for any $(b, \sigma) \in \mathcal{A}_F$ and $x \in \partial D$.

Step 5. Since for any a.e. finite stopping time ρ and $(b, \sigma) \in \mathcal{A}_F^\rho$ we can take an $(\overline{b}, \overline{\sigma}) \in \mathcal{A}_F$ such that, for any $y \in \partial D$, $X^{\rho,y}_{(b,\sigma),\rho+t}$ and $X^y_{(\overline{b},\overline{\sigma}),t}$ have the same distribution, we have that $\overline{\tau}^{\rho,y}_{(b,\sigma)}$ and $\overline{\tau}^y_{(\overline{b},\overline{\sigma})}$ have the same distribution, and consequently (2.38) is true for any a.e. finite stopping time ρ, $(b, \sigma) \in \mathcal{A}_F^\rho$ and $y \in \partial D$, but not for any $(b, \sigma) \in \mathcal{A}_F$.

Step 6. Consider the set

$$\mathcal{J} := \left\{ \begin{array}{l} (b, \sigma) \in \mathcal{A}_F : (b, \sigma)|_{[\rho,\infty)} = \sum_{i=0}^n \chi_{A_i}(b_i, \sigma_i)|_{[\rho,\infty)}, \\[2mm] \text{where } \{(b_i, \sigma_i)\}_{i=0}^n \subset \mathcal{A}_F^\rho \text{ and } \{A_i\}_{i=0}^n \text{ is a } \mathcal{F}_\rho\text{--partition of } \Omega \end{array} \right\}.$$

Moreover, for each $(b, \sigma) := \sum_{i=0}^n \chi_{A_i}(b_i, \sigma_i) \in \mathcal{J}$, (2.38) holds true for any a.e. finite stopping time ρ and $y \in \partial D$ since, by the previous step,

$$\mathbb{P}\left(\overline{\tau}^{\rho,y}_{(b,\sigma)} = 0\right) = \sum_{i=0}^n \mathbb{P}\left(A_i \cap \left\{\overline{\tau}^{\rho,y}_{(b_i,\sigma_i)} = 0\right\}\right) = \sum_{i=0}^n \mathbb{P}(A_i)\mathbb{P}\left(\overline{\tau}^{\rho,y}_{(b_i,\sigma_i)} = 0\right)$$

$$= \sum_{i=0}^n \mathbb{P}(A_i) = 1.$$

Step 7. We know by lemma 2.1.4 that $\mathcal{J}$ is dense in $\mathcal{A}_F$, therefore, fixed a $(\overline{b}, \overline{\sigma}) \in \mathcal{A}_F$ and two positive constants α and ε, we will prove that there exists a $(b, \sigma) \in \mathcal{J}$ such that

$$\mathbb{P}\left(\overline{\tau}^{\rho,y}_{(\overline{b},\overline{\sigma})} > \overline{\tau}^{\rho,y}_{(b,\sigma)} + \alpha\right) < \varepsilon. \tag{2.39}$$

Notice that by the previous step $\overline{\tau}^{\rho,y}_{(b,\sigma)} = 0$ a.e. for any $(b, \sigma) \in \mathcal{J}$, thus (2.39) is equivalent to

$$\mathbb{P}\left(\overline{\tau}^{\rho,y}_{(\overline{b},\overline{\sigma})} > \alpha\right) < \varepsilon,$$

therefore the arbitrariness of $\left(\overline{b},\overline{\sigma}\right)$, y, ρ, α and ε proves (2.38) for any a.e. finite stopping time ρ, $(b,\sigma) \in \mathcal{A}_F$ and $y \in \partial D$ concluding the proof.

Fix an a.e. finite stopping time ρ, $y \in \partial D$ and define for any $(b,\sigma) \in \mathcal{J}$ the stopping times

$$\tau^{\beta}_{(b,\sigma)} := \inf\left\{t \in [0,\infty) : \inf_{z \in D}\left|X^{\rho,y}_{\rho+t} - z\right| \geq \beta\right\}.$$

By proposition A.2.1 we can take a positive T depending only on D, ℓ, λ and ε such that $\mathbb{P}\left(\overline{\tau}^{\rho,y}_{(\overline{b},\overline{\sigma})} \geq T\right) < \dfrac{\varepsilon}{3}$. Similarly we can choose a β, depending on α and ε, such that $\mathbb{P}\left(\tau^{\beta}_{(b,\sigma)} > \overline{\tau}^{\rho,y}_{(b,\sigma)} + \alpha\right) < \dfrac{\varepsilon}{3}$ for any $(b,\sigma) \in \mathcal{J}$, in fact if that would not be true we should have, thanks to the reverse Fatou's lemma,

$$\mathbb{P}\left(\overline{\tau}^{\rho,y}_{(b,\sigma)} > \alpha\right) \geq \limsup_{\beta \to 0} \mathbb{P}\left(\tau^{\beta}_{(b,\sigma)} > \overline{\tau}^{\rho,y}_{(b,\sigma)} + \alpha\right) \geq \dfrac{\varepsilon}{3}$$

for some $y \in \overline{D}$, in contradiction with the previous step. Thus we have that

$$\begin{aligned}
\mathbb{P}\left(\overline{\tau}^{\rho,y}_{(\overline{b},\overline{\sigma})} > \overline{\tau}^{\rho,y}_{(b,\sigma)} + \alpha\right) \leq & \mathbb{P}\left(\tau^{\beta}_{(b,\sigma)} > \overline{\tau}^{\rho,y}_{(b,\sigma)} + \alpha\right) + \mathbb{P}\left(\overline{\tau}^{\rho,y}_{(\overline{b},\overline{\sigma})} \geq T\right) \\
& + \mathbb{P}\left(\left\{\overline{\tau}^{\rho,y}_{(\overline{b},\overline{\sigma})} > \tau^{\beta}_{(b,\sigma)}\right\} \cup \left\{\overline{\tau}^{\rho,y}_{(\overline{b},\overline{\sigma})} < T\right\}\right) \\
\leq & \mathbb{P}\left(\left\{\overline{\tau}^{\rho,y}_{(\overline{b},\overline{\sigma})} > \tau^{\beta}_{(b,\sigma)}\right\} \cup \left\{\overline{\tau}^{\rho,y}_{(\overline{b},\overline{\sigma})} < T\right\}\right) + \dfrac{2\varepsilon}{3}.
\end{aligned}$$

Finally Markov's inequality and theorem A.1.4 yield

$$\begin{aligned}
\mathbb{P}&\left(\left\{\overline{\tau}^{\rho,y}_{(\overline{b},\overline{\sigma})} > \tau^{\beta}_{(b,\sigma)}\right\} \cup \left\{\overline{\tau}^{\rho,y}_{(\overline{b},\overline{\sigma})} < T\right\}\right) \\
& \leq \mathbb{P}\left(\left\{\left|X^{\rho,y}_{(\overline{b},\overline{\sigma}),\rho+\tau^{\beta}_{(b,\sigma)}} - X^{\rho,y}_{(b,\sigma),\rho+\tau^{\beta}_{(b,\sigma)}}\right| \geq \beta\right\} \cap \left\{\overline{\tau}^{\rho,y}_{(\overline{b},\overline{\sigma})} < T\right\}\right) \\
& \leq \dfrac{1}{\beta^2}\mathbb{E}\left(\sup_{t \in [0,T]}\left|X^{\rho,y}_{(\overline{b},\overline{\sigma}),\rho+t} - X^{\rho,y}_{(b,\sigma),\rho+t}\right|^2\right) \\
& \leq \dfrac{c}{\beta^2}\mathbb{E}\left(\int_0^T e^{\gamma(T-t)}\left|\overline{b}\left(t, X^{\rho,y}_{(\overline{b},\overline{\sigma}),\rho+t}\right) - b\left(t, X^{\rho,y}_{(\overline{b},\overline{\sigma}),\rho+t}\right)\right|^2 dt\right) \\
& \quad + \dfrac{c}{\beta^2}\mathbb{E}\left(\int_0^T e^{\gamma(T-t)}\left|\overline{\sigma}\left(t, X^{\rho,y}_{(\overline{b},\overline{\sigma}),\rho+t}\right) - \sigma\left(t, X^{\rho,y}_{(\overline{b},\overline{\sigma}),\rho+t}\right)\right|^2 dt\right),
\end{aligned}$$

where c depends only on T, α and ℓ, hence, by proposition A.1.6, we can choose a $(b,\sigma) \in \mathcal{J}$ such that $\mathbb{P}\left(\left\{\overline{\tau}_{(\overline{b},\overline{\sigma})}^{\rho,y} > \tau_{(b,\sigma)}^{\beta}\right\} \cup \left\{\overline{\tau}_{(\overline{b},\overline{\sigma})}^{\rho,y} < T\right\}\right) < \dfrac{\varepsilon}{3}$ proving (2.39). $\qquad\square$

The previous lemma permits us to use theorems A.2.4 and A.2.5 to prove that:

Proposition 2.4.8. *The function $\tau^\rho : \mathbb{R}^N \times \mathcal{A}_F \to [0,\infty)$ is, under our assumptions, continuous in probability for any a.e. finite stopping time ρ.*

As previously said, the next lemma implies that the BSDE part of the FBSDE in definition 2.4.5 satisfies for any initial data assumptions B.2.1, or more precisely items (vi) and (vii) in assumptions B.2.1.

Lemma 2.4.9. *For any a.e. finite stopping time ρ and ζ in $L^2\left(\Omega, \mathcal{F}_\rho; \mathbb{R}^N\right)$ we have $\displaystyle\sup_{(b,\sigma)\in\mathcal{A}_F} \mathbb{E}\left(e^{\vartheta\tau^{\rho,\zeta}}\right) \leq \sup_{(b,\sigma)\in\mathcal{A}_F} \sup_{x\in D} \mathbb{E}\left(e^{\vartheta\tau^x}\right) < \infty.$*

Proof. By remark 2.4.3 we know that for any $(b,\sigma) \in \mathcal{A}_F$ we can take an $\left(\overline{b}, \overline{\sigma}\right) \in \mathcal{A}_F^\rho$ such that $\tau_{(b,\sigma)}^x \stackrel{d}{=} \tau_{(\overline{b},\overline{\sigma})}^{\rho,x}$ and vice versa, therefore item (ix) in problem 2.4.2 implies that $\displaystyle\sup_{(b,\sigma)\in\mathcal{A}_F^\rho} \sup_{x\in D} \mathbb{E}\left(e^{\vartheta\tau^{\rho,x}}\right) = \sup_{(b,\sigma)\in\mathcal{A}_F} \sup_{x\in D} \mathbb{E}\left(e^{\vartheta\tau^x}\right) < \infty.$
Thanks to the continuity of τ and the fact that we can approximate any $(b,\sigma) \in \mathcal{A}_F$ with a sequence in

$$\mathcal{J} := \left\{ \begin{array}{l} (b,\sigma) \in \mathcal{A}_F : (b,\sigma)|_{[\rho,\infty)} = \displaystyle\sum_{i=0}^{n} \chi_{A_i}(b_i,\sigma_i)|_{[\rho,\infty)}, \\[2mm] \text{where } \{(b_i,\sigma_i)\}_{i=0}^{n} \subset \mathcal{A}_F^\rho \text{ and } \{A_i\}_{i=0}^{n} \text{ is a } \mathcal{F}_\rho\text{--partition of } \Omega \end{array} \right\}.$$

we have that

$$\sup_{(b,\sigma)\in\mathcal{A}_F} \sup_{x\in D} \mathbb{E}\left(e^{\vartheta\tau^{\rho,x}}\right) = \sup_{(b,\sigma)\in\mathcal{J}} \sup_{x\in D} \mathbb{E}\left(e^{\vartheta\tau^{\rho,x}}\right) \leq \sup_{(b,\sigma)\in\mathcal{A}_F^\rho} \sup_{x\in D} \mathbb{E}\left(e^{\vartheta\tau^{\rho,x}}\right).$$

Finally, since each $\zeta \in L^2\left(\Omega, \mathcal{F}_\rho; \mathbb{R}^N\right)$ can be approximated by a sequence of simple functions, is easy to see that $\mathbb{E}\left(e^{\vartheta\tau^{\rho,\zeta}}\right) \leq \sup_{x\in D} \mathbb{E}\left(e^{\vartheta\tau^{\rho,x}}\right)$, and this concludes the proof. $\qquad\square$

We can now prove, proceeding by steps, that $u(x) := \sup\limits_{(b,\sigma)\in\mathcal{A}_F} \mathbb{E}\left(Y_0^x\right)$ is a viscosity solution to the problem 2.4.2. We start proving that we can apply the dynamic programming principle 2.2.7 on u, showing continuity and boundedness of Y, which is the analogue of φ in section 2.2.

Lemma 2.4.10. *Under our assumptions, for any a.e. finite stopping time ρ, the function $(x, (b, \sigma)) \in \mathbb{R}^N \times \mathcal{A}_F \mapsto Y_\rho^{\rho,x}$ is continuous in probability and uniformly equicontinuous in the L^1–norm on the first variable with respect to the second one. Furthermore there exists a constant c, which depends only on ℓ, μ, l and ϑ such that*

$$\mathbb{E}\left(\left|Y_\rho^{\rho,\zeta}\right|^2\right) \le c \sup_{(b,\sigma)\in\mathcal{A}_F} \sup_{x\in D} \mathbb{E}\left(e^{\vartheta\tau^x}\right) \tag{2.40}$$

for any a.e. finite stopping time ρ, $(b, \sigma) \in \mathcal{A}_F$ and $\zeta \in L^2\left(\Omega, \mathcal{F}_\rho; \mathbb{R}^N\right)$.

Proof. The continuity in probability is just a consequence of theorem B.2.5 and propositions A.1.6 and 2.4.8, while corollary B.2.6 and lemma 2.4.9 yield that there exist two constants c_1 and c_2 depending on ℓ, l, μ and ϑ such that, for any a.e. finite stopping time ρ, $(b, \sigma) \in \mathcal{A}_F$ and $\zeta \in L^2\left(\Omega, \mathcal{F}_\rho; \mathbb{R}^N\right)$,

$$\begin{aligned}
\mathbb{E}\left(\left|Y_\rho^{\rho,\zeta}\right|^2\right) \le & c_1 \mathbb{E}\left(e^{\vartheta\tau^{\rho,\zeta}}\left|g\left(X_{\rho+\tau^{\rho,\zeta}}^{\rho,\zeta}\right)\right|^2 + \int_0^{\tau^{\rho,\zeta}} e^{\vartheta t}\left|f\left(X_{\rho+t}^{\rho,\zeta}, 0, 0\right)\right|^2 dt\right) \\
\le & c_2 \mathbb{E}\left(e^{\vartheta\tau^{\rho,\zeta}}\right) \le c_2 \sup_{(b,\sigma)\in\mathcal{A}_F} \sup_{x\in D} \mathbb{E}\left(e^{\vartheta\tau^x}\right),
\end{aligned}$$

proving (2.40). Thus we only have to show the equicontinuity.
To prove the equicontinuity we fix a positive ε and find a δ for which

$$\mathbb{E}\left(\left|Y_\rho^{\rho,x} - Y_\rho^{\rho,y}\right|\right) < \varepsilon$$

for any $(b, \sigma) \in \mathcal{A}_F$, $x \in \mathbb{R}^N$ and $y \in B_\delta(x)$. Preliminarily fix an a.e. finite stopping time ρ and define $\tau_x := \tau^{\rho,x} \wedge T$, $\tau_y := (\tau_x - \alpha) \vee (\tau^{\rho,y} \wedge (\tau_x + \alpha))$, where T and α are positive number that will be chosen later. Let $\left(\overline{X}^x, \overline{Y}^x, \overline{Z}^x\right)$ and $\left(\overline{X}^y, \overline{Y}^y, \overline{Z}^y\right)$ be respectively the solutions in x and in y to the FBSDEs (2.35) but with stopping times τ_x and τ_y instead of $\tau^{\rho,x}$ and $\tau^{\rho,y}$, then

$$\mathbb{E}\left(\left|Y_\rho^{\rho,x} - Y_\rho^{\rho,y}\right|\right) \le \left(\mathbb{E}\left(\left|Y_\rho^{\rho,x} - \overline{Y}_\rho^x\right|^2\right)\right)^{\frac{1}{2}} + \left(\mathbb{E}\left(\left|\overline{Y}_\rho^y - Y_\rho^{\rho,y}\right|^2\right)\right)^{\frac{1}{2}}$$

$$+ \left(\mathbb{E} \left(\left| \overline{Y}^x_\rho - \overline{Y}^y_\rho \right|^2 \right) \right)^{\frac{1}{2}}$$

and by theorems A.2.4 and B.2.5 and proposition A.2.1 we can choose a T, an α and a δ such that

$$\mathbb{E} \left(\left| Y^{\rho,x}_\rho - \overline{Y}^x_\rho \right|^2 \right) \leq \frac{\varepsilon^2}{9} \quad \text{and} \quad \mathbb{E} \left(\left| \overline{Y}^y_\rho - Y^{\rho,y}_\rho \right|^2 \right) \leq \frac{\varepsilon^2}{9}. \tag{2.41}$$

Therefore we just have to prove that, for a suitable δ,

$$\mathbb{E} \left(\left| \overline{Y}^x_\rho - \overline{Y}^y_\rho \right|^2 \right) < \frac{\varepsilon^2}{9} \tag{2.42}$$

for any $(b, \sigma) \in \mathcal{A}_F$, $x \in \mathbb{R}^N$ and $y \in B_\delta(x)$.

Thanks to theorem B.2.5 we have for any fixed $(b, \sigma) \in \mathcal{A}_F$

$$\mathbb{E} \left(\left| \overline{Y}^x_\rho - \overline{Y}^y_\rho \right|^2 \right) \leq c_3 \mathbb{E} \left(\left| e^{\frac{\vartheta \tau_x}{2}} g \left(X^{\rho,x}_{\rho+\tau_x} \right) - e^{\frac{\vartheta \tau_y}{2}} g \left(X^{\rho,y}_{\rho+\tau_y} \right) \right|^2 \right.$$
$$\left. + \int_0^{\tau_x \vee \tau_y} e^{\vartheta t} \left| f_\sigma \left(t, X^{\rho,x}_{\rho+t}, \overline{Y}^x_{\rho+t}, \overline{Z}^x_{\rho+t} \right) - f_\sigma \left(t, X^{\rho,y}_{\rho+t}, \overline{Y}^x_{\rho+t}, \overline{Z}^x_{\rho+t} \right) \right|^2 dt \right),$$

where c_3 is a constant which depends on ϑ, l and μ, hence to prove (2.42) we will give an upper bound to both the elements on the right of the inequality. For the first one is easy to see that

$$\mathbb{E} \left(\left| e^{\frac{\vartheta \tau_x}{2}} g \left(X^{\rho,x}_{\rho+\tau_x} \right) - e^{\frac{\vartheta \tau_y}{2}} g \left(X^{\rho,y}_{\rho+\tau_y} \right) \right|^2 \right)$$
$$\leq 2\ell^2 \mathbb{E} \left(\left| e^{\frac{\vartheta \tau_x}{2}} - e^{\frac{\vartheta \tau_y}{2}} \right|^2 + e^{\vartheta \tau_x} \left| X^{\rho,x}_{\rho+\tau_x} - X^{\rho,y}_{\rho+\tau_y} \right|^2 \right)$$
$$\leq \frac{\alpha^2 \vartheta^2 \ell^2}{2} e^{\vartheta T + |\vartheta| \alpha} + 4\ell^2 e^{\vartheta T} \mathbb{E} \left(\left| X^{\rho,x}_{\rho+\tau_x} - X^{\rho,y}_{\rho+\tau_x} \right|^2 + \left| X^{\rho,y}_{\rho+\tau_x} - X^{\rho,y}_{\rho+\tau_y} \right|^2 \right).$$

Furthermore, thanks to theorem A.1.4 and corollary A.1.5, there exist a $\gamma > \ell^2 + 2\ell$ and two constants c_4 and c_5 depending on γ and ℓ with

$$\mathbb{E} \left(\left| X^{\rho,x}_{\rho+\tau_x} - X^{\rho,y}_{\rho+\tau_x} \right|^2 + \left| X^{\rho,y}_{\rho+\tau_x} - X^{\rho,y}_{\rho+\tau_y} \right|^2 \right)$$
$$\leq \mathbb{E} \left(\sup_{t \in [0,T]} \left| X^{\rho,x}_{\rho+t} - X^{\rho,y}_{\rho+t} \right|^2 + \sup_{t \in [0,\alpha]} \left| X^{\rho+\tau_x \wedge \tau_y, X^{\rho,y}_{\rho+\tau_x \wedge \tau_y}}_{\rho+\tau_x \wedge \tau_y + t} - X^{\rho,y}_{\rho+\tau_x \wedge \tau_y} \right|^2 \right)$$
$$\leq c_4 e^{\gamma T} |\delta|^2 + c_5 \alpha e^{\gamma \alpha},$$

so, taking α and δ satisfying (2.41) small enough, we have that

$$c_3 \mathbb{E}\left(\left| e^{\frac{\vartheta \tau_x}{2}} g\left(X^{\rho,x}_{\rho+\tau_x}\right) - e^{\frac{\vartheta \tau_y}{2}} g\left(X^{\rho,y}_{\rho+\tau_y}\right) \right|^2 \right) < \frac{\varepsilon^2}{18}. \tag{2.43}$$

For the second part note that by our assumptions and theorem A.1.4 there exists a constant c_6, depending on γ and ℓ, such that

$$\mathbb{E}\left(\int_0^{\tau_x \vee \tau_y} e^{\vartheta t} \left| f_\sigma\left(t, X^{\rho,x}_{\rho+t}, \overline{Y}^x_{\rho+t}, \overline{Z}^x_{\rho+t}\right) - f_\sigma\left(t, X^{\rho,y}_{\rho+t}, \overline{Y}^x_{\rho+t}, \overline{Z}^x_{\rho+t}\right) \right|^2 dt \right)$$

$$\leq \ell^2 e^{\vartheta T} \mathbb{E}\left(\int_0^T \left| X^{\rho,x}_{\rho+t} - X^{\rho,y}_{\rho+t} \right|^2 dt \right) + \ell^2 \frac{e^{\vartheta \alpha} - 1}{\vartheta}$$

$$\leq c_6 \ell^2 e^{(\vartheta+\gamma)T} |\delta|^2 + \ell^2 \frac{e^{\vartheta \alpha} - 1}{\vartheta}.$$

Thus, possibly taking a smaller α and δ, we get

$$c_3 \mathbb{E}\left(\int_0^{\tau_x \vee \tau_y} e^{\vartheta t} \left| f_\sigma\left(t, X^{\rho,x}_{\rho+t}, \overline{Y}^x_{\rho+t}, \overline{Z}^x_{\rho+t}\right) - f_\sigma\left(t, X^{\rho,y}_{\rho+t}, \overline{Y}^x_{\rho+t}, \overline{Z}^x_{\rho+t}\right) \right|^2 dt \right) < \frac{\varepsilon^2}{18},$$

which together with (2.43) yields (2.42). $\qquad \square$

As an immediate consequence of the previous lemma we have the following proposition.

Proposition 2.4.11. *The function* $u(x) := \sup\limits_{(b,\sigma) \in \mathcal{A}_F} \mathbb{E}\left(Y_0^x\right)$ *is bounded and continuous. In particular, for any* $x \in \mathbb{R}^N$ *and a.e. finite stopping time* ρ, $u(x) = \sup\limits_{(b,\sigma) \in \mathcal{A}_F} \mathbb{E}\left(Y_\rho^{\rho,x}\right).$

One of the consequences of this proposition is that it permits us to study $u(\zeta)$ as $\operatorname*{ess\,sup}\limits_{(b,\sigma) \in \mathcal{A}_F} \mathbb{E}\left(Y_\rho^{\rho,\zeta} \big| \mathcal{F}_\rho\right)$, if ζ is in $L^2\left(\Omega, \mathcal{F}_\rho; \mathbb{R}^N\right)$.

Proof. By definition and the Jensen's inequality, for any $x, y \in \mathbb{R}^N$,

$$|u(x) - u(y)| = \left| \sup_{(b,\sigma) \in \mathcal{A}_F} \mathbb{E}(Y_0^x) - \sup_{(b,\sigma) \in \mathcal{A}_F} \mathbb{E}(Y_0^y) \right| \leq \sup_{(b,\sigma) \in \mathcal{A}_F} \mathbb{E}(|Y_0^x - Y_0^y|)$$

and

$$|u(x)|^2 = \left| \sup_{(b,\sigma) \in \mathcal{A}_F} \mathbb{E}(Y_0^x) \right|^2 \leq \sup_{(b,\sigma) \in \mathcal{A}_F} \mathbb{E}\left(|Y_0^x|^2\right),$$

thus boundedness and continuity are a consequence of lemma 2.4.10.

To prove the second part of the statement note that, since F is time independent, for any $(b, \sigma) \in \mathcal{A}_F$ we can take an $(\bar{b}, \bar{\sigma}) \in \mathcal{A}_F^\rho$ such that $X_t^x = \overline{X}_{\rho+t}^{\rho,x}$ for any $t \in [0, \infty)$ and $x \in \mathbb{R}^N$. Hence for any solution Y to (2.4.5) we can take another solution $\overline{Y}$ such that $Y_t^x = \overline{Y}_{\rho+t}^{\rho,x}$ for any $t \in [0, \infty)$ and $x \in \mathbb{R}^N$. The statement is then a consequence of remark 2.2.6 and the reversibility of the argument. $\qquad\square$

Now we proceed to show that u is a viscosity subsolution. In order to do so we need the following lemma:

Lemma 2.4.12. *Let L be an element of $\mathcal{L}_F$ and (X, Y, Z) the solution to the FBSDE (2.35) associated to L as in remark 2.3.6. If we define, for any $x \in D$, $u_L(x) := \mathbb{E}(Y_0^x)$ we have that, for any supertangent φ to u_L at x,*

$$L\left(0, x, \nabla\varphi, D^2\varphi\right) \geq -f(x, u_L, \nabla\varphi).$$

Proof. Preliminarily we denote (b, σ) as the element of $\mathcal{A}_F$ associated to L and point out that since (b, σ) is progressive with respect to the filtration $\{\mathcal{F}_t\}_{t \in [0, \infty)}$, so are L, X and Y, therefore they are constants a.e. in 0. As a consequence $u_L(x) = Y_0^x$ a.e. for any $x \in D$.

Given $x \in D$ and a supertangent φ to u_L at x we can assume without loss of generality that $u_L(x) = \varphi(x)$, so suppose that, a.e.,

$$L\left(0, x, \nabla\varphi, D^2\varphi\right) + f_\sigma(0, x, u_L, \nabla\varphi\sigma) < 0 \tag{2.44}$$

and we will find a contradiction. Note that, as a consequence of the Blumenthal's 0–1 law 1.1.2, this is a deterministic inequality a.e.. By the definition of supertangent, there exists a $\delta > 0$ such that, for any $y \in B_\delta(x) \subseteq D$,

$$u_L(y) \leq \varphi(y), \tag{2.45}$$

hence we define the stopping time

$$\rho := \delta \wedge \inf\{t \in [0, \infty) : |X_t^x - x| \geq \delta\}$$

and assume, possibly taking a smaller ρ, that

$$L\left(t \wedge \rho, X_{t\wedge\rho}^x, \nabla\varphi, D^2\varphi\right) + f_\sigma\left(t \wedge \rho, X_{t\wedge\rho}^x, \varphi, \nabla\varphi\sigma\right) < 0. \qquad (2.46)$$

We point out that, by (2.44) and lemma 2.3.8, the previous inequality holds true on a set of positive measure for the $\chi_{[0,\delta]}dt \times d\mathbb{P}$ measure, thus $\rho > 0$ on a set of positive measure.

Let $\left(\overline{Y}_t, \overline{Z}_t\right) := \left(Y_{t\wedge\rho}^x, Z_{t\wedge\rho}^x\right)$, which solve the BSDE

$$\overline{Y}_t = Y_\rho^x + \int_{t\wedge\rho}^\rho f_\sigma\left(s, X_s^x, \overline{Y}_s, \overline{Z}_s\right) ds - \int_{t\wedge\rho}^\rho \overline{Z}_s dW_s, \quad t \in [0, \delta],$$

and $\left(\hat{Y}_t, \hat{Z}_t\right) := \left(\varphi\left(X_{t\wedge\rho}^x\right), (\nabla\varphi\sigma)\left(t, X_{t\wedge\rho}^x\right)\right)$ which, by Itô's formula, is solution to

$$\hat{Y}_t = \varphi\left(X_\rho^x\right) - \int_{t\wedge\rho}^\rho L\left(s, X_s^x, \nabla\varphi, D^2\varphi\right) ds - \int_{t\wedge\rho}^\rho \hat{Z}_s dW_s, \quad t \in [0, \delta].$$

By (2.45) and proposition 2.4.11 we have that

$$u_L\left(X_\rho^x\right) - \varphi\left(X_\rho^x\right) = Y_\rho^{\rho, X_\rho^x} - \varphi\left(X_\rho^x\right) \leq 0$$

and (2.46) imply, thanks to corollary B.1.8, that $Y_0^x < \varphi(x)$ a.e., but this lead to a contradiction since we know that, by our assumptions, $\varphi(x) = Y_0^x$ a.e.. This concludes the proof. $\qquad\qquad\square$

Proposition 2.4.13. *The function u is a continuous viscosity subsolution to the problem 2.4.2.*

Proof. We know from proposition 2.4.11 that u is continuous, thus we just have to prove the subsolution property to conclude the proof.

Let L be an element of $\mathcal{L}_F$ and u_L as defined in lemma 2.4.12, then if φ is a supertangent to u_L in $x \in D$ we have that, by the definition of $\mathcal{L}_F$,

$$F\left(x, \nabla\varphi, D^2\varphi\right) \geq L\left(0, x, \nabla\varphi, D^2\varphi\right) \geq -f(x, u_L, \nabla\varphi),$$

therefore u_L is a viscosity subsolution to the problem 2.4.2 at x. Thanks to the arbitrariness of L and x we then have that u_L is a viscosity subsolution

in x for any $L \in \mathcal{L}_F$ and $x \in D$. Lemma 2.4.10 yields that the family of functions $\{u_L\}$ is equibounded, therefore, thanks to theorem 2.0.3,

$$\sup_{L \in \mathcal{L}_F} u_L(x) = \sup_{(b,\sigma) \in \mathcal{A}_F} \mathbb{E}\left(Y_0^x\right) = u(x)$$

is a viscosity subsolution for any $x \in D$. $\square$

As for problem 2.3.1, to prove that u is viscosity solution we need to formulate the dynamic programming principle for this problem:

Lemma 2.4.14. *For any $(b, \sigma) \in \mathcal{A}_F$ we let $\left(\overline{Y}, \overline{Z}\right)$ be the solution of the BSDE*

$$\overline{Y}_s = u\left(X_\tau^x\right) + \int_{s \wedge \tau}^\tau f_\sigma\left(r, X_r^x, \overline{Y}_r, \overline{Z}_r\right) dr - \int_{s \wedge \tau}^\tau \overline{Z}_r dW_r, \qquad s \in [0, \infty),$$

where τ is a stopping time smaller than τ^x. Then $\displaystyle\sup_{(b,\sigma) \in \mathcal{A}_F} \mathbb{E}\left(\overline{Y}_0\right) = u(x)$.

The proof of this lemma is the same as lemma 2.3.11, we just point out that $X_{\tau \wedge \tau^x}^x \in L^2\left(\Omega, \mathcal{F}_\tau; \overline{D}\right)$, therefore the results of section 2.2 are still true in this case.

We can now prove the main statement of this section.

Theorem 2.4.15. *The function $u(x) := \displaystyle\sup_{(b,\sigma) \in \mathcal{A}_F} \mathbb{E}(Y_0^x)$ is a viscosity solution to the problem 2.4.2 such that $u(x) = g(x)$ for any $x \in \partial D$. If $\mu < 0$ it is also unique.*

Proof. The uniqueness is a consequence of theorem 2.4.4, hence we only have to show that u is a viscosity solution.

From proposition 2.4.13 we know that u is a continuous viscosity subsolution and it is easy to see that $u(x) = g(x)$ for any $x \in \partial D$, so we only need to prove the supersolution property of u. Fixed $x \in D$, let ψ be a subtangent to u in x which we assume, without loss of generality, equal to u in x and δ a positive constant such that

$$\psi(y) \leq u(y) \qquad \text{for any } y \in B_\delta(x) \subseteq D. \tag{2.47}$$

We know, thanks to theorem 1.2.2, that there exists a continuous and deterministic $L \in \mathcal{L}_F$ for which

$$F\left(x, \nabla\psi, D^2\psi\right) = L\left(0, x, \nabla\psi, D^2\psi\right)$$

and assume

$$F\left(x, \nabla\psi, D^2\psi\right) = L\left(0, x, \nabla\psi, D^2\psi\right) > -f_\sigma(0, x, u, \nabla\psi\sigma).$$

Then, by continuity,

$$L\left(t, y, \nabla\psi, D^2\psi\right) > -f_\sigma(t, y, \psi, \nabla\psi\sigma) \tag{2.48}$$

for any $(t, y) \in [0, \delta) \times B_\delta(x)$, possibly taking a smaller δ.

We proceed by contradiction, as in the proof of lemma 2.4.12 denoting with (b, σ) and (X, Y, Z), respectively, the element of $\mathcal{A}_F$, which we stress out that is continuous and deterministic, and the solution to the FBSDE (2.35) associated to L. We define the stopping time

$$\rho := \delta \wedge \inf\{t \in [0, \infty) : |X_t^x - x| \geq \delta\},$$

let $\left(\overline{Y}_t, \overline{Z}_t\right)$ be the solution to the BSDE

$$\overline{Y}_t = u\left(X_\rho^x\right) + \int_{t\wedge\rho}^\rho f_\sigma\left(s, X_s^x, \overline{Y}_s, \overline{Z}_s\right) ds - \int_{t\wedge\rho}^\rho \overline{Z}_s dW_s, \qquad t \in [0, \delta]$$

and $\left(\hat{Y}_t, \hat{Z}_t\right) := \left(\psi\left(X_{t\wedge\rho}^x\right), (\nabla\psi\sigma)\left(t, X_{t\wedge\rho}^x\right)\right)$ which, by Itô's formula, is solution to

$$\hat{Y}_t = \psi\left(X_\rho^x\right) - \int_{t\wedge\rho}^\rho L\left(s, X_s^x, \nabla\psi, D^2\psi\right) ds - \int_{t\wedge\rho}^\rho \hat{Z}_s dW_s, \qquad t \in [0, \delta].$$

We know from lemma 2.4.14 that

$$\sup_{(b,\sigma)\in\mathcal{A}_F} \mathbb{E}\left(\overline{Y}_0\right) = u(x) = \psi(x), \tag{2.49}$$

but by (2.47) we have $u\left(X_\rho^x\right) \geq \psi\left(X_\rho^x\right)$, which together with (2.48) imply, thanks to corollary B.1.8, that $\overline{Y}_0 > \psi(x)$ a.e., in contradiction with (2.49).

$$\square$$

2.5 Parabolic Second Order Hamilton–Jacobi PDEs

We apply the method developed in the previous sections for study PDEs with sublinear operators to the analysis of a generalized form of second order Hamilton–Jacobi PDEs. The classical Hamilton–Jacobi equations is

$$\partial_t u(t,x) + \sup_{a \in A} \left(\frac{1}{2} \left\langle \sigma\sigma^\dagger(t,x,a), S \right\rangle + p^\dagger b(t,x,a) + c(a)u(t,x) + f(t,x,a) \right)$$

and it has been deeply studied by many authors using stochastic formulas. In this setting we cite, among the others, [21, 22, 29, 32].

The problem we are going to study in this section will present a more general function f, also depending on the first and zero order term.

Problem 2.5.1. Let T be a terminal time, A a closed subset of $\mathbb{R}^N$ and

$$\sigma : [0,T] \times \mathbb{R}^N \times A \to \mathbb{R}^{N \times N}, \qquad b : [0,T] \times \mathbb{R}^N \times A \to \mathbb{R}^N,$$
$$f : [0,T] \times \mathbb{R}^N \times A \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R} \qquad g : \mathbb{R}^N \to \mathbb{R}$$

four measurable functions for which there exist two constants $\mu \in \mathbb{R}$ and $\ell \geq 0$ such that, for any $t \in [0,T]$, $x, x' \in \mathbb{R}^N$, $a \in A$, $y, y' \in \mathbb{R}$ and $z, z' \in \mathbb{R}^N$,

(i) $(s,v) \mapsto (b(s,x,a), \sigma(s,x,a), f(s,x,a,v,z))$ is continuous;

(ii) $|b(t,x,a) - b(t,x',a)| \leq \ell|x - x'|$;

(iii) $|\sigma(t,x,a) - \sigma(t,x',a)| \leq \ell|x - x'|$;

(iv) $|b(t,x,a)| \leq \ell(1 + |x|)$;

(v) $|\sigma(t,x,a)| \leq \ell(1 + |x|)$;

(vi) $|g(x) - g(x')| \leq \ell|x - x'|$;

(vii) $|g(x)| \leq \ell(1 + |x|)$;

(viii) $|f(t, x, a, y, z) - f(t, x', a, y, z')| \leq \ell(|x - x'| + |z - z'|)$;

(ix) $|f(t, x, a, y, z)| \leq \ell(1 + |x| + |y| + |z|)$;

(x) $(y - y')(f(t, x, a, y, z) - f(t, x, a, y', z)) \leq \mu|y - y'|^2$.

To obtain uniqueness of the solution we will also assume that

(xi) the continuity of the function $y \mapsto f(t, x, y, z)$ is independent from the fourth variable.

Define the elliptic functions

$$L : [0, T] \times \mathbb{R}^N \times A \times \mathbb{R}^N \times \mathbb{S}^N \to \mathbb{R}$$

and

$$H : [0, T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N \to \mathbb{R},$$

such that

$$L_a(t, x, p, S) := L(t, x, a, p, S) := \frac{1}{2} \left\langle \sigma \sigma^\dagger(t, x, a), S \right\rangle + p^\dagger b(t, x, a)$$

and

$$H(t, x, y, p, S) := \sup_{a \in A} \left(L_a(t, x, p, S) + f\left(t, x, a, y, p^\dagger \sigma\right) \right),$$

and further assume that, for any $(t, x, y, p, S) \in [0, T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N$,

(xii) $|H(t, x, y, p, S)| < \infty$;

(xiii) H is continuous.

Find the solution u to the parabolic PDE

$$\begin{cases} \partial_t u(t, x) + H\left(t, x, u, \nabla_x u, D_x^2 u\right) = 0, & t \in (0, T), x \in \mathbb{R}^N, \\ u(T, x) = g(x), & x \in \mathbb{R}^N. \end{cases}$$

It easy to see that items (iv), (v) and (ix) imply item (xii). Also notice that here we do not require uniform ellipticity, since this time we do not need a bound to the inverse matrix of σ.

Once again, we have that from theorem 2.0.5 follows a comparison result.

Theorem 2.5.2. *Let u and v be respectively a subsolution and a super-solution to problem 2.5.1 satisfying polynomial growth condition. Then, if $u|_{t=T} \leq v|_{t=T}$, $u \leq v$ on $(0, T] \times \mathbb{R}^N$.*

Similarly as what we have done in the previous sections we define the sets $\mathcal{A}_H^\tau$, which is made up by the progressive, with respect to $\{\mathcal{F}_t^\tau\}_{t \in [0,\infty)}$, processes with value in A. Notice that here we require the processes to be only progressive instead of also cadlag to be consistent with the previous literature, but everything we say in this section hold true also for cadlag processes. For any control a, $\zeta \in L^2\left(\Omega, \mathcal{F}_t; \mathbb{R}^N\right)$ and functions b, σ, f and g as in the assumptions of problem 2.5.1 we define the following controlled FBSDE:

$$
\begin{cases}
X_s^{t,\zeta,a} = \zeta + \int_t^s \sigma\left(r, X_r^{t,\zeta,a}, a_r\right) dW_r + \int_t^s b\left(r, X_r^{t,\zeta,a}, a_r\right) dr, \\
Y_s^{t,\zeta,a} = g\left(X_T^{t,\zeta,a}\right) + \int_s^T f\left(r, X_r^{t,\zeta,a}, a_r, Y_r^{t,\zeta,a}, Z_r^{t,\zeta,a}\right) dr \quad s \in [t, T]. \quad (2.50) \\
\qquad\quad - \int_s^T Z_r^{t,\zeta,a} dW_r,
\end{cases}
$$

Note that under these conditions the assumptions A.1.1 and B.1.1 hold for the SDE part and the BSDE part respectively, hence under our assumptions there always exists a unique solution to (2.50).

We then define the function $u_a(t, x) := \mathbb{E}\left(Y_t^{t,x,a}\right)$, which is obviously measurable in $\mathcal{A}_H$. We will prove that $u(t, x) := \sup_{a \in \mathcal{A}_H} u_a(t, x)$ is a viscosity solution to problem 2.5.1.

The natural question that arises, now that we have setted up the problem, is if the dynamic programming principle holds in this case, which presents some issues with respect to the previous problems. This question has a positive answer thanks to remark 2.2.8, section 2.2.1 and the next proposition, which shows the continuity and the boundedness of u. We only state the result, since the proof is identical to the one of proposition 2.3.7.

Proposition 2.5.3. *The function $u(t, x) := \sup_{a \in \mathcal{A}_H} \mathbb{E}\left(Y_t^{t,x,a}\right)$ is $\frac{1}{2}$–Hölder continuous in the first variable and Lipschitz continuous in the second one. Fur-*

thermore we have that there exists a constant c, which depends only on ℓ, μ and T, such that

$$\mathbb{E}\left(|u(\tau, \zeta)|^2\right) \leq \sup_{a \in \mathcal{A}_F} \mathbb{E}\left(\left|Y_\tau^{\tau, \zeta, a}\right|^2\right) \leq c\left(1 + \mathbb{E}\left(|\zeta|^2\right)\right), \tag{2.51}$$

for any stopping time τ bounded by T and $\zeta \in L^2\left(\Omega, \mathcal{F}_\tau; \mathbb{R}^N\right)$.

The next step is to prove that u is a viscosity subsolution. Since this time for a fixed a control we can not rely on right continuity in time, the proof will be slightly different from what we have seen in the previous sections.

Proposition 2.5.4. *The function $u(t, x)$ is a continuous viscosity subsolution to the problem 2.5.1.*

Proof. We know from proposition 2.5.3 that u is continuous, thus we just have to prove the subsolution property to conclude the proof. In order to do so we will show that, fixed $(t, x) \in (0, T) \times \mathbb{R}^N$, $u_a(t, x)$ is a subsolution in (t, x) to problem 2.5.1 for any $a \in \mathcal{A}_H$. Then u is the supremum of a family of locally equibounded subsolutions, hence is a subsolution by theorem 2.0.3. Given $(t, x) \in (0, T) \times \mathbb{R}^N$, $a \in \mathcal{A}_H$ and a supertangent φ to u_a at (t, x) we can assume without loss of generality that $u_a(t, x) = \varphi(t, x)$, so suppose that

$$\partial_t \varphi(t, x) + H\left(t, x, u_a, \nabla_x \varphi, D_x^2 \varphi\right) < 0$$

and we will find a contradiction. By the definition of supertangent, there exists a $\delta \in (0, T - t)$ such that, for any $s \in [t, t + \delta]$ and $y \in B_\delta(x)$,

$$u_a(s, y) \leq \varphi(s, y) \tag{2.52}$$

and, possibly taking a smaller δ,

$$\begin{aligned}
0 > &\partial_t \varphi(s, y) + H\left(s, y, \varphi, \nabla_x \varphi, D_x^2 \varphi\right) \\
\geq &\partial_t \varphi(s, y) + L_a\left(s, y, \nabla_x \varphi, D_x^2 \varphi\right) + f(s, y, a, \varphi, \nabla_x \varphi \sigma),
\end{aligned} \tag{2.53}$$

thanks to the continuity of H and φ. Now we define the stopping time

$$\tau := (t + \delta) \wedge \inf\left\{s \in [t, \infty) : \left|X_s^{t, x, a} - x\right| \geq \delta\right\},$$

let $\left(\overline{Y}_s, \overline{Z}_s\right) := \left(Y_{s\wedge\tau}^{t,x,a}, Z_{s\wedge\tau}^{t,x,a}\right)$, which solve the BSDE

$$\overline{Y}_s = Y_\tau^{t,x,a} + \int_{s\wedge\tau}^\tau f\left(r, X_r^{t,x,a}, a, \overline{Y}_r, \overline{Z}_r\right) dr - \int_{s\wedge\tau}^\tau \overline{Z}_r dW_r, \quad s \in [t,T],$$

and $\left(\hat{Y}_s, \hat{Z}_s\right) := \left(\varphi\left(s, X_{s\wedge\tau}^{t,x,a}\right), (\nabla_x\varphi\sigma(a))\left(s\wedge\tau, X_{s\wedge\tau}^{t,x,a}\right)\right)$ which, by Itô's formula, is solution to

$$\hat{Y}_s = \varphi\left(\tau, X_\tau^{t,x,a}\right) - \int_{s\wedge\tau}^\tau \hat{Z}_r dW_r$$
$$- \int_{s\wedge\tau}^\tau \left(\partial_t\varphi\left(r, X_r^{t,x,a}\right) + L_a\left(r, X_r^{t,x,a}, \nabla_x\varphi, D_x^2\varphi\right)\right) dr, \qquad s \in [t,T].$$

By (2.52) we have that, a.e.,

$$u_a\left(\tau, X_\tau^{t,x,a}\right) - \varphi\left(\tau, X_\tau^{t,x,a}\right) = Y_\tau^{\tau, X_\tau^{t,x,a}} - \varphi\left(\tau, X_\tau^{t,x,a}\right) \le 0$$

and (2.53) imply, thanks to corollary B.1.8, that $Y_t^{t,x,a} < \varphi(t,x)$ a.e., but this lead to a contradiction since, by our assumptions, $\varphi(t,x) = \mathbb{E}\left(Y_t^{t,x,a}\right)$. This concludes the proof. $\qquad\square$

Once again, to prove that u is the viscosity solution we need to express the dynamic programming principle for this problem:

Lemma 2.5.5. *For any $a \in \mathcal{A}_H$ we let $\left(\overline{Y}, \overline{Z}\right)$ be the solution of the BSDE*

$$\overline{Y}_s = u\left(\tau, X_\tau^{t,x,a}\right) + \int_{s\wedge\tau}^\tau f\left(r, X_r^{t,x,a}, \overline{Y}_r, \overline{Z}_r\right) dr - \int_{s\wedge\tau}^\tau \overline{Z}_r dW_r, \qquad s \in [t,T],$$

where τ is a stopping time with value in $[t,T]$. Then $\sup\limits_{a\in\mathcal{A}_H} \mathbb{E}\left(\overline{Y}_t\right) = u(t,x)$.

The proof of this lemma is similar to the proof of lemma 2.3.11, hence we skip it.

We can now prove the main statement of this section.

Theorem 2.5.6. *The function $u(t,x) := \sup\limits_{a\in\mathcal{A}_H} \mathbb{E}\left(Y_t^{t,x,a}\right)$ is the only viscosity solution to problem 2.5.1 satisfying polynomial growth condition such that $u(T,x) = g(x)$ for any x in $\mathbb{R}^N$.*

Proof. The uniqueness is a consequence of theorem 2.5.2 and (2.51), hence we only have to show that u is a viscosity solution.

From proposition 2.5.4 we know that u is a continuous viscosity subsolution and it is easy to see that $u(T, x) = g(x)$ for any $x \in \mathbb{R}^N$, so we only need to prove the supersolution property of u. Fixed (t, x) in $(0, T) \times \mathbb{R}^N$, let ψ be a subtangent to u in (t, x) which we assume, without loss of generality, equal to u in (t, x) and δ a positive constant such that

$$\psi(s, y) \leq u(s, y) \qquad \text{for any } (s, y) \in [t, t + \delta] \times B_\delta(x). \tag{2.54}$$

As in the proof of proposition 2.5.4 we proceed by contradiction, assuming

$$H\left(t, x, u, \nabla_x \psi, D_x^2 \psi\right) > 0.$$

Thanks to the definition of H we know that there exists an $a \in A$ for which

$$\frac{1}{2} \left\langle \sigma \sigma^\dagger(t, x, a), D_x^2 \psi(t, x) \right\rangle + \nabla_x \psi(t, x) b(t, x, a) + f\left(t, x, a, \psi, \nabla_x \psi \sigma\right) > 0,$$

thus if we define, with an abuse of notation, a as the control which is identically equal to a, then we have by continuity

$$L_a\left(s, y, \nabla_x \psi, D_x^2 \psi\right) + f\left(s, y, a, \psi, \nabla_x \psi \sigma\right) > 0, \tag{2.55}$$

for any $(s, y) \in [t, t + \delta] \times B_\delta(x)$, possibly taking a smaller δ.

Once again we define the stopping time

$$\tau := (t + \delta) \wedge \inf\left\{s \in [t, \infty) : \left|X_s^{t,x,a} - x\right| \geq \delta\right\},$$

let $\left(\overline{Y}_s, \overline{Z}_s\right)$ be the solution to the BSDE

$$\overline{Y}_s = u\left(\tau, X_\tau^{t,x,a}\right) + \int_{s \wedge \tau}^\tau f\left(r, X_r^{t,x,a}, \overline{Y}_r, \overline{Z}_r\right) dr - \int_{s \wedge \tau}^\tau \overline{Z}_r dW_r, \quad s \in [t, T]$$

and $\left(\hat{Y}_s, \hat{Z}_s\right) := \left(\psi\left(s, X_{s \wedge \tau}^{t,x,a}\right), (\nabla_x \psi \sigma)\left(s, X_{s \wedge \tau}^{t,x,a}\right)\right)$ which, by Itô's formula, is solution to

$$\hat{Y}_s = \psi\left(\tau, X_\tau^{t,x,a}\right) - \int_{s \wedge \tau}^\tau \hat{Z}_r dW_r$$
$$- \int_{s \wedge \tau}^\tau \left(\partial_t \psi\left(r, X_r^{t,x,a}\right) + L\left(r, X_r^{t,x,a}, \nabla_x \psi, D_x^2 \psi\right)\right) dr, \qquad s \in [t, T].$$

We know from lemma 2.5.5 that

$$\sup_{a \in \mathcal{A}_H} \mathbb{E}\left(\overline{Y}_t\right) = u(t,x) = \psi(t,x), \tag{2.56}$$

but by (2.54) we have $u\left(\tau, X_\tau^{t,x,a}\right) \geq \psi\left(\tau, X_\tau^{t,x,a}\right)$, which together with (2.55) imply, thanks to corollary B.1.8, that $\overline{Y}_t > \psi(t,x)$ a.e., in contradiction with (2.56). $\qquad\square$

2.6 A Nonlinear PDE Problem with Singular Boundary Conditions

This section is mostly inspired by the seminal paper [20] in which Lasry and Lions study a class of stochastic control problems using the associated semilinear PDEs. What they do is showing existence, uniqueness and comparison results for solutions to the PDE in a bounded domain D

$$\begin{cases} \dfrac{1}{2}\Delta u(x) - |\nabla u(x)|^\alpha - \mu u(x) + f(x) = 0, & x \in D, \\ u(x) \to \infty, & \text{as } \text{dist}(x, \partial D) \to 0^+ \end{cases} \tag{2.57}$$

under various conditions obtaining a complete solution to the stochastic control problem, which is also solution to the previous PDE,

$$\inf_{q \in \mathcal{A}} J(x,q) := \inf_{q \in \mathcal{A}} \mathbb{E}\left(\int_0^\infty e^{-\mu t}\left(f(X_t) + \frac{|q_t|^\beta}{\alpha^{(\beta-1)\beta}} \right) dt \right), \tag{2.58}$$

where $x \in D$, $\beta := \dfrac{\alpha}{\alpha - 1}$, $X_t := x + \int_0^t q_s ds + W_t$ and $\mathcal{A}$ is the set of controls which enforce the *state constraint conditions*, i.e. such that $X_t \in \overline{D}$ a.e. for any $t \in [0,\infty)$ and consequently $\tau^x := \inf\{t \in [0,\infty) : X_t \notin D\}$ is equal to infinity a.e..

What we will do here instead is, as we have done in the previous sections, to use the stochastic control problem to find the existence and a representation formula for the solution to a PDE problem with singular boundary conditions. This PDE problem is a generalization of a particular case studied in [20], where the first order term is an Hamiltonian subquadratic in the

first order term which presents a strong monotonicity with respect to the zero order term. The main difficulty of this problem is that it requires controls not uniformly bounded, which cause some issues when combined with the exit times necessary to study stationary problems. This force us to consider simpler operator than the ones used in the previous sections.

It is worth noticing that there exist some articles which study BSDEs with singular terminal condition and use them to find viscosity solutions to PDE problems ([34, 35]) or to solve optimal control problems ([2]), but all use generators which are independent from the first order term.

Before we can deal with the singular boundary conditions, it is necessary to study the associated Dirichlet problem.

2.6.1 The Dirichlet Problem for a PDE Concave and Coercive with respect to the First Order Term

Problem 2.6.1. Let $D \subset \mathbb{R}^N$ be an open bounded set which satisfies an exterior cone condition as in definition 2.4.1, σ an $N \times N$ real matrix, a constant G and $H : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ two continuous functions. Assume that there exist three positive number λ, μ and ℓ such that, for any $p \in \mathbb{R}^N$, $x, x' \in \mathbb{R}^N$ and $y, y' \in \mathbb{R}$,

(i) $|H(x, y, p) - H(x', y, p)| \leq \ell|x - x'|$;

(ii) $(y - y')(H(x, y, p) - H(x, y', p)) \leq -\mu|y - y'|^2$;

(iii) $|H(x, y, p) - H(x, 0, p)| \leq \nu(1 + |y|)$;

(iv) $p \mapsto H(x, y, p)$ is concave;

(v) $\lim\limits_{|p| \to \infty} \dfrac{H(x, y, p)}{|p|} = -\infty$;

(vi) $x^\dagger \sigma \sigma^\dagger x \geq \lambda|x|^2$.

Find the solution u to the elliptic PDE

$$\begin{cases} \dfrac{1}{2}\left\langle \sigma\sigma^{\dagger}, D^2 u(x)\right\rangle + H(x, u, \nabla u) = 0, & x \in D, \\ u(x) = G, & x \in \partial D. \end{cases}$$

As for problem 2.4.2, we have that holds a comparison result, which is a consequence of [8, Theorem 3.3].

Theorem 2.6.2. *Let u and v be respectively a subsolution and a supersolution to problem 2.6.1 such that $u \leq v$ on ∂D. Then $u \leq v$ on $\overline{D}$.*

Without the Lipschitz assumptions for the first order term is difficult to solve directly this problem using the method employed on the previous sections, therefore, in order to overcome this difficulty, we will use the convex conjugate of $-H$

$$f(x, y, q) = \sup_{p \in \mathbb{R}^N} \left(q^{\dagger} p + H(x, y, p)\right).$$

We point out that f is continuous, has superlinear growth at infinity and is convex in q, and satisfies items (i) to (iii) in problem 2.6.1. Furthermore

$$H(x, y, p) = \inf_{q \in \mathbb{R}^N} \left(f(x, y, q) - q^{\dagger} p\right)$$

and both the infimum and the supremum in the previous formulas is achieved at some point in $\mathbb{R}^N$, since both $-H$ and f have superlinear growth. We will conveniently assume that

$$\sup_{x \in D} |f(x, 0, 0)| \leq \ell. \tag{2.59}$$

The control sets that we will use this time to solve the problem are the $\mathcal{A}^{\rho}$, made up by the cadlag processes $q \in \mathbb{L}_N^2 \cap \mathbb{L}_N^{\beta}$ such that $\{q_{\rho+t}\}_{t \in [0,\infty)}$ is progressive with respect to the filtration $\{\mathcal{F}_t^{\rho}\}_{t \in [0,\infty)}$ and there exists a constant c such that $|q| < c$. As always $\mathcal{A} := \mathcal{A}^0$ and we have that, for any $\varphi \in C^2$ and $t \in [0, \infty)$,

$$\frac{1}{2}\left\langle \sigma\sigma^{\dagger}, D^2\varphi(x)\right\rangle + H(x, \varphi, \nabla\varphi)$$

$$= \inf_{q \in \mathcal{A}} \left(\frac{1}{2} \left\langle \sigma \sigma^\dagger, D^2 \varphi(x) \right\rangle - \nabla \varphi(x) q_t + f(x, u, q_t) \right).$$

The lack of a bound for our controls q constitute the main difference between this Dirichlet problem and the one presented in section 2.4.

The FBSDEs that we will use to build our viscosity solution this time assume the following form,

$$
\begin{cases}
X^{\rho,\zeta}_{\rho+s} = \zeta + \displaystyle\int_\rho^{\rho+s} \sigma d\widetilde{W}_r, \\[2mm]
Y^{\rho,\zeta}_{\rho+s} = G - \displaystyle\int_{\rho+s\wedge\tau^{\rho,\zeta}}^{\rho+\tau^{\rho,\zeta}} Z^{\rho,\zeta}_r d\widetilde{W}_r \\[2mm]
\qquad\quad + \displaystyle\int_{\rho+s\wedge\tau^{\rho,\zeta}}^{\rho+\tau^{\rho,\zeta}} \left(f\left(X^{\rho,\zeta}_r, Y^{\rho,\zeta}_r, q_r\right) - Z^{\rho,\zeta,q}_r \sigma_r^{-1} q_r \right) dr, \\[2mm]
Z^{\rho,\zeta,q}_{\rho+s} = 0, \qquad \text{on } \left\{ \tau^{\rho,\zeta} \leq s \right\},
\end{cases}
\qquad s \in [0,\infty). \quad (2.60)
$$

where ρ is an a.e. finite stopping time, $\zeta \in L^2\left(\Omega, \mathcal{F}_\rho; \mathbb{R}^N\right)$, $q \in \mathcal{A}$ and $\tau^{\rho,\zeta}$ is the exit time of $X^{\rho,\zeta}$ from D. As in the previous sections our candidate solution is $u(x) := \inf_{q \in \mathcal{A}} \mathbb{E}\left(Y^{\rho,x}_\rho\right)$.

Note that under these conditions the assumptions A.1.1, (A.4) and (A.5) hold for the SDE part, while assumptions B.2.1 holds for the BSDE part (we enforce item (viii)), thanks to the definition of $\mathcal{A}$, but this time some constants in these assumptions will vary depending on q, since the elements of $\mathcal{A}$ are not uniformly bounded. Furthermore under our assumptions, by remark B.2.3 and theorems A.1.3 and B.2.4, always exists a unique solution to (2.60). We will use the same notations and conventions used in section 2.4, explicitly stating the dependence from the controls in $\mathcal{A}$ only when needed.

We start proving that we can apply the dynamic programming principle 2.2.7 on u, showing first that it is bounded and later that it is continuous.

Proposition 2.6.3. *Let* $C := -G^- - \dfrac{\max\limits_{x \in \overline{D}} H^-(x, 0, 0)}{\mu}$. *Then we have that, for any* $q \in \mathcal{A}$ *and* $t \in [0, \infty)$, $Y^{x,q}_t \geq C$ *a.e.. In particular*

$$u(x) := \inf_{q \in \mathcal{A}} \mathbb{E}(Y^x_0) \geq C.$$

Proof. Is trivial that $G \geq C$ and $f(X_t^x, C, q_t) \geq H(X_t^x, C, 0) \geq 0$ for any $q \in \mathcal{A}$ and $t \leq \tau^x$. Moreover the couple $(C, 0)$ is solution of the BSDE $(C, 0, \tau^x)$, therefore theorem B.2.9 imply our statement. $\square$

Proposition 2.6.4. *The function* $u(x) := \inf\limits_{q \in \mathcal{B}} \mathbb{E}(Y_0^x)$ *is bounded from above by a constant that depends only on* D, G, λ *and* ℓ.

Proof. Fixed the control $q \equiv 0$ proposition A.2.2 yields that there exists a ϑ depending on ℓ, D and λ such that $\sup\limits_{x \in D} \mathbb{E}\left(e^{\vartheta \tau^x}\right) < \infty$. Then we can use corollary B.2.6 with $\gamma = \vartheta$ and $\mu = 0$ to obtain that there exists a constant c depending on ℓ and ϑ such that, thanks also to (2.59),

$$\sup_{x \in D} \mathbb{E}\left(|Y_0^x|^2\right) \leq \sup_{x \in D} c \mathbb{E}\left(e^{\vartheta \tau^x} G + \int_0^{\tau^x} e^{\vartheta t} \ell dt\right)$$

and this concludes the proof since $\sup\limits_{x \in D} \inf\limits_{q \in \mathcal{A}} \mathbb{E}(Y_0^{x,q}) \leq \sup\limits_{x \in D} \mathbb{E}\left(Y_0^{x,0}\right)$. $\square$

To study the continuity of u, we will use another formulation of our FBSDEs (2.60) given by the Girsanov's theorem, which is explained in detail in [37, Chapter VIII] and we summarize in the following.

Theorem 2.6.5 (Girsanov's Theorem). *Let* $H \in \mathbb{L}^2$, W *a Brownian motion under the probability measure* $\mathbb{P}$ *and assume that for any* $t \in [0, \infty)$ *the process*
$$\mathcal{E}(H)_t := e^{\int_0^t H_s dW_s - \frac{1}{2} \int_0^t |H_s|^2 ds}$$
is a martingale with $\mathbb{E}(\mathcal{E}(H)_t) = 1$. *Then if we define for any* $t \in [0, \infty)$ *the measure* $d\mathbb{Q}|_{\mathcal{F}_t} := \mathcal{E}(H)_t d\mathbb{P}|_{\mathcal{F}_t}$, $\mathbb{Q}$ *is a probability measure on* $\mathcal{F}_\infty$ *equivalent to* $\mathbb{P}$. *Furthermore* $\widetilde{W} := W - \int_0^\cdot H_s ds$ *is a Brownian motion for* $\mathbb{Q}$.

Since q and σ^{-1} are bounded and $\mathcal{E}(-\sigma^{-1}q)$ is solution to the SDE

$$\mathcal{E}\left(-\sigma^{-1}q\right)_t = 1 - \int_0^t \sigma_s^{-1} q_s \mathcal{E}\left(-\sigma^{-1}q\right)_s dW_s,$$

it is easy to see that $\mathcal{E}(-\sigma^{-1}q)$ satisfies the conditions of the Girsanov's theorem 2.6.5, therefore we have that under the measure $\mathbb{Q}^q$ defined by $\mathcal{E}(-\sigma^{-1}q)$

the FBSDE (2.60) take the form

$$
\begin{cases}
X^{\rho,\zeta}_{\rho+s} = \zeta + \displaystyle\int_{\rho}^{\rho+s} \sigma dW_r - \int_{\rho}^{\rho+s} q_r dr, \\[2mm]
Y^{\rho,\zeta}_{\rho+s} = G + \displaystyle\int_{\rho+s\wedge\tau^{\rho,\zeta}}^{\rho+\tau^{\rho,\zeta}} f\left(X^{\rho,\zeta}_r, Y^{\rho,\zeta}_r, q_r\right) dr \\[2mm]
\qquad\quad - \displaystyle\int_{\rho+s\wedge\tau^{\rho,\zeta}}^{\rho+\tau^{\rho,\zeta}} Z^{\rho,\zeta}_r dW_r, \\[2mm]
Z^{\rho,\zeta}_{\rho+s} = 0, \qquad \text{on } \left\{\tau^{\rho,\zeta} \leq s\right\},
\end{cases}
\qquad s \in [0,\infty). \qquad (2.61)
$$

In general the expectation under $\mathbb{P}$ of a stochastic variable is different from the expectation under an equivalent measure $\mathbb{Q}$ of the same variable, but, since $Y^{\rho,x}_\rho$ is a constant a.e. and $\mathbb{P}$ and $\mathbb{Q}$ are equivalent measures, $\mathbb{E}^{\mathbb{P}}\left(Y^{\rho,x}_\rho\right) = \mathbb{E}^{\mathbb{Q}}\left(Y^{\rho,x}_\rho\right) = Y^{\rho,x}_\rho$ a.e.. We point out that in this form the value of the SDE, and consequently of the exit time, depends on q.

Remark 2.6.6. As said before, we lack a uniform bound for the controls in $\mathcal{A}$ which cause a lack of a uniform bound for the exit times of the SDE in equation (2.61) from D. The bound for the exit times is an important tool to prove their equicontinuity and consequently the continuity of viscosity solutions, as we have done in section 2.4. To obtain the equicontinuity of the exit times in this setup we will use that the paths defined by the SDE part of (2.61) are easy to study, in fact we have that

$$
\sup_{q\in\mathcal{A}} \mathbb{E}\left(\sup_{s\in[0,\infty)} \left|X^{\rho,x}_{\rho+s} - X^{\rho,y}_{\rho+s}\right|^2\right) = |x-y|^2.
$$

Lemma 2.6.7. *Under our assumptions we have that fixed two positive constants ε and α, there exists a $\delta > 0$, depending on ε and α, such that*

$$
\mathbb{Q}^q\left(\left|\tau^{\rho,x} - \tau^{\rho,y}\right| > \alpha\right) < \varepsilon
$$

for any $x \in \mathbb{R}^N$ and $y \in B_\delta(x)$.

We recall that from lemma 2.4.7, which is true for each $q \in \mathcal{A}$, and theorem A.2.4 that the exit times are continuous for the measure $\mathbb{P}$ and hence for $\mathbb{Q}$. The purpose of this lemma is to give a bound, uniform with respect to q, that we will use to prove the continuity of u.

Proof. To ease notation assume that $\rho = 0$ and restrict x in $\overline{D}$, since for $x \notin \overline{D}$ this is obviously true. We start fixing $x \in \overline{D}$, defining for any $y \in \overline{D}$

$$\tau_\beta^y := \inf \left\{ t \in [0, \infty) : \inf_{z \in D} |X_t^y - z| \geq \beta \right\}$$

and noting that

$$
\begin{aligned}
\mathbb{Q}(|\tau^x - \tau^y| > \alpha) = & \mathbb{Q} \left(\{\tau^y > \tau^x + \alpha\} \cap \{\tau_\beta^x > \tau^x + \alpha\} \right) \\
& + \mathbb{Q} \left(\{\tau^y > \tau^x + \alpha\} \cap \{\tau_\beta^x \leq \tau^x + \alpha\} \right) \\
& + \mathbb{Q} \left(\{\tau^x > \tau^y + \alpha\} \cap \{\tau_\beta^y > \tau^y + \alpha\} \right) \\
& + \mathbb{Q} \left(\{\tau^x > \tau^y + \alpha\} \cap \{\tau_\beta^y \leq \tau^y + \alpha\} \right) \\
\leq & \mathbb{Q} \left(\tau_\beta^x > \tau^x + \alpha \right) + \mathbb{Q} \left(\{\tau^y > \tau_\beta^x\} \right) + \mathbb{Q} \left(\tau_\beta^y > \tau^y + \alpha \right) \\
& + \mathbb{Q} \left(\{\tau^x > \tau_\beta^y\} \right).
\end{aligned}
$$

We can choose a β, depending on α and ε, such that $\mathbb{Q} \left(\tau_\beta^y > \tau^y + \alpha \right) < \dfrac{\varepsilon}{4}$ for any $y \in \overline{D}$, in fact if that would not be true we should have, thanks to the reverse Fatou's lemma,

$$\mathbb{Q}(\overline{\tau}^y > \tau^y + \alpha) \geq \limsup_{\beta \to 0} \mathbb{Q} \left(\tau_\beta^y > \tau^y + \alpha \right) \geq \frac{\varepsilon}{4}$$

for some $y \in \overline{D}$, in contradiction with lemma 2.4.7, which holds true for each $q \in \mathcal{A}$.

Now, for the last terms, we can use Markov's inequality and theorem A.1.4 to get

$$\mathbb{Q} \left(\{\tau^y > \tau_\beta^x\} \right) \leq \mathbb{Q} \left(\left\{ \left| X_{\tau_\beta^x}^x - X_{\tau_\beta^x}^y \right| \geq \beta \right\} \right) \leq \frac{|x - y|^2}{\beta^2}$$

and similarly

$$\mathbb{Q} \left(\{\tau^x > \tau_\beta^y\} \right) \leq \frac{|x - y|^2}{\beta^2}.$$

Therefore there exists a $\delta > 0$ depending on ε and α such that

$$\mathbb{Q} \left(\{\tau^y > \tau_\beta^x\} \right) + \mathbb{Q} \left(\{\tau^x > \tau_\beta^y\} \right) \leq \frac{\varepsilon}{2}$$

for any $y \in B_\delta(x)$ and consequently $\mathbb{Q} \left(|\tau^{\rho,x} - \tau^{\rho,y}| > \alpha \right) < \varepsilon$. $\qquad \square$

Proposition 2.6.8. *Under our assumptions, for any a.e. finite stopping time ρ, the function $(x, q) \in \mathbb{R}^N \times \mathcal{A} \mapsto Y_\rho^{\rho,x}$ is continuous in probability and the function $u(x) := \inf\limits_{q \in \mathcal{A}} \mathbb{E}\left(Y_0^x\right)$ is bounded and continuous. In particular, for any $x \in \mathbb{R}^N$ and a.e. finite stopping time ρ, $u(x) = \inf\limits_{q \in \mathcal{A}} \mathbb{E}\left(Y_\rho^{\rho,x}\right)$.*

One of the consequences of this proposition is that it allows us to study $u(\zeta)$ as $\operatorname*{ess\,inf}\limits_{q \in \mathcal{A}} \mathbb{E}\left(Y_\rho^{\rho,\zeta}\big|\mathcal{F}_\rho\right)$, if ζ is in $L^2\left(\Omega, \mathcal{F}_\rho; \mathbb{R}^N\right)$.

Proof. To prove the last part of the statement note that for any $q \in \mathcal{A}$ we can take an $q' \in \mathcal{A}^\rho$ such that $Y_t^{x,q} \stackrel{d}{=} Y_{\rho+t}^{\rho,x,q'}$ for any $t \in [0, \infty)$ and $x \in \mathbb{R}^N$. Hence for any solution Y to (2.61) we can take another solution $\overline{Y}$ such that $Y_t^x \stackrel{d}{=} \overline{Y}_{\rho+t}^{\rho,x}$ for any $t \in [0, \infty)$ and $x \in \mathbb{R}^N$. The statement is then a consequence of remark 2.2.6 and the reversibility of the argument.

The continuity in probability of Y is just a consequence of remark A.1.7 and theorems A.2.4, A.2.5 and B.2.5. Moreover, thanks to propositions 2.6.3 and 2.6.4, we already know that u is bounded, thus we only have to show the continuity of u.

Since Y_0 is continuous in probability and $\mathcal{F}_0$–measurable, it is continuous a.e. and u is upper semicontinuous as the infimum of continuous functions: let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in D converging to a fixed $x \in D$, then we have that

$$\limsup_{n \to \infty} u(x_n) = \underline{u} \leq u(x).$$

We can extract a subsequence, which we denote again as $\{x_n\}_{n \in \mathbb{N}}$, such that $|u(x_n) - \underline{u}| < \dfrac{1}{n}$ and a sequence of controls $\{q^n\}_{n \in \mathbb{N}} \subset \mathcal{A}$ such that $\left|u(x_n) - Y_0^{x_n,q^n}\right| < \dfrac{1}{n}$ a.e., $q_t^n = q_{t \wedge \tau^{x_n,q^n}}^n$. We thus have that $\lim\limits_{n \to \infty} Y_0^{x_n,q^n} = \underline{u}$ a.e. and that the sequence $\left\{Y_0^{x_n,q^n}\right\}$ is equibounded, hence to prove the continuity of u we will show that $u(x) \leq \lim\limits_{n \to \infty} Y_0^{x_n,q^n} = \underline{u}$ a.e.. To do so we will build some BSDEs that will help us in this task.

Define, for any $t \in [0, \infty)$, the functions $\chi_t := \lim\limits_{s \downarrow t} \chi_{\{Y_s \leq G\}}$ and note that χ_t is cadlag with jumps of size 1, thus it has quadratic variation equal to 0 and

Itô's formula yields

$$
\begin{aligned}
Y_t^{x_n,q^n}\chi_t =& G\chi_{\tau^{x_n,q^n}} + \int_{t\wedge\tau^{x_n,q^n}}^{\tau^{x_n,q^n}} f\left(X_s^{x_n,q^n}, Y_s^{x_n,q^n}, q_s^n\right)\chi_s ds \\
&+ G\left(\left|J^0(t)\right| - \left|J^1(t)\right|\right) - \int_{t\wedge\tau^{x_n,q^n}}^{\tau^{x_n,q^n}} Z_s^{x_n,q^n}\chi_s dW_s,
\end{aligned}
\tag{2.62}
$$

where $J^0(t)$ and $J^1(t)$ are respectively the sets made up by the points of discontinuity s of χ between t and τ^{x_n,q^n} such that $\chi_s = 0$ and $\chi_s = 1$. We will now prove for almost any $\omega \in \Omega$ that

$$
Y_t^{x_n,q^n}\chi_t = G\chi_t + \int_{t\wedge\tau_t^{G,n}}^{\tau_t^{G,n}} f\left(X_s^{x_n,q^n}, Y_s^{x_n,q^n}, q_s^n\right)ds - \int_{t\wedge\tau_t^{G,n}}^{\tau_t^{G,n}} Z_s^{x_n,q^n}dW_s, \tag{2.63}
$$

where $\tau_t^{G,n} := \tau^{x_n,q^n}\wedge\inf\{s \in [t,\infty) : \chi_s = 0\}$. Notice that by definition B.2.2 $\chi_{\tau^{x_n,q^n}} = 1$, therefore $|J^0(t)| - |J^1(t)|$ is equal to 0 or 1 if we have respectively an even or an odd number of jumps. For a fixed $\omega \in \Omega$, assume that we have $2n$ jumps, then $\chi_s = 1$ for any $s \in \left[t, \tau_t^{G,n}\right]$, $|J^0(t)| - |J^1(t)| = 0$ and $Y_{\tau_t^{G,n}}^{x_n,q^n}\chi_{\tau_t^{G,n}} = G$, thus (2.63) follows from (2.62). Now assume instead that we have $2n+1$ jumps, then $\chi_t = 0$ and $\tau_t^{G,n} = t$, and (2.63) is obviously true. Finally define, for any $t \in [0,\infty)$ and for any $n \in \mathbb{N}$, the stopping time $\tau^n := \tau_{\tau^{x,q^n}}^{G,n}$, the BSDE

$$
\overline{Y}_t^n = G + \int_{t\wedge\tau^n}^{\tau^n} f\left(X_s^{x_n,q^n}, \overline{Y}_s^n, q_s^n\right)ds - \int_{t\wedge\tau^n}^{\tau^n} \overline{Z}_s^n dW_s,
$$

the sets

$$
A_t := \left\{f\left(X_t^{x_n,q^n}, \overline{Y}_t^n, q_t^n\right) > f\left(X_t^{x_n,q^n}, \overline{Y}_t^n, 0\right)\right\},
$$

the controls $\overline{q}_t^n := q_t^n\left(1 - \chi_{A_t}\chi_{\{\tau^{x,q^n}>t\}}\right)$ and the BSDE

$$
Y_t^n = G + \int_{t\wedge\tau^n}^{\tau^n} f\left(X_s^{x_n,q^n}, Y_s^n, \overline{q}_s^n\right)ds - \int_{t\wedge\tau^n}^{\tau^n} Z_s^n dW_s.
$$

We have by (2.63) that, for any $t \in [0,\infty)$ and $n \in \mathbb{N}$,

$$
\begin{aligned}
\overline{Y}_{t\vee\tau^{x,q^n}}^n =& Y_{(t\wedge\tau^n)\vee\tau^{x,q^n}}^{x_n,q^n}\chi_{(t\wedge\tau^n)\vee\tau^{x,q^n}} + G\left(1 - \chi_{(t\wedge\tau^n)\vee\tau^{x,q^n}}\right) \\
=& G \wedge Y_{(t\wedge\tau^n)\vee\tau^{x,q^n}}^{x_n,q^n}
\end{aligned}
\tag{2.64}
$$

and, by theorem B.2.9,

$$Y_t^n \leq \overline{Y}_t^n \quad \text{and} \quad Y_0^n \leq \overline{Y}_0^n \leq Y_0^{x_n, q^n}. \tag{2.65}$$

Using theorem B.2.5 we have that there exist a $\gamma \in (0, -\mu)$ and a constant c depending on μ and γ such that

$$\mathbb{E}\left(\left|Y_0^{x,q^n} - Y_0^n\right|^2\right)$$

$$\leq c\mathbb{E}\left(\left|e^{\gamma \tau^{x,q^n}} - e^{\gamma \tau^n}\right|^2 G^2 + \int_{\tau^{x,q^n}}^{\tau^n} e^{2\gamma t}\left|f\left(X_t^{x_n,q^n}, Y_t^{x,q^n}, \overline{q}_t^n\right)\right|^2 dt\right.$$

$$\left. + \int_0^{\tau^{x,q^n}} e^{2\gamma t}\left|f\left(X_t^{x,q^n}, Y_t^{x,q^n}, q_t^n\right) - f\left(X_t^{x_n,q^n}, Y_t^{x,q^n}, \overline{q}_t^n\right)\right|^2 dt\right).$$

By definition we have

$$f\left(X_t^{x_n,q^n}, Y_t^{x,q^n}, \overline{q}_t^n\right) \chi_{\{t > \tau^{x,q^n}\}} \geq H\left(X_t^{x_n,q^n}, G, 0\right)$$

and, thanks to (2.64), (2.65) and the monotonicity of f,

$$f\left(X_t^{x_n,q^n}, Y_t^{x,q^n}, \overline{q}_t^n\right) \chi_{\{t > \tau^{x,q^n}\}} = f\left(X_t^{x_n,q^n}, G, \overline{q}_t^n\right) \chi_{\{t > \tau^{x,q^n}\}}$$

$$\leq f\left(X_t^{x_n,q^n}, \overline{Y}_t^n, \overline{q}_t^n\right) \chi_{\{t > \tau^{x,q^n}\}}$$

$$\leq f\left(X_t^{x_n,q^n}, \overline{Y}_t^n, 0\right) \chi_{\{t > \tau^{x,q^n}\}}$$

$$\leq f\left(X_t^{x_n,q^n}, C, 0\right) \chi_{\{t > \tau^{x,q^n}\}},$$

where C is the lower bound defined in proposition 2.6.3. This means that

$$\int_{\tau^{x,q^n}}^{\tau^n} e^{2\gamma t}\left|f\left(X_t^{x_n,q^n}, Y_t^{x,q^n}, \overline{q}_t^n\right)\right|^2 dt \leq 2\int_{\tau^{x,q^n}}^{\tau^n} e^{2\gamma t}\left|H\left(X_t^{x_n,q^n}, G, 0\right)\right|^2 dt$$

$$+ 2\int_{\tau^{x,q^n}}^{\tau^n} e^{2\gamma t}\left|f\left(X_t^{x_n,q^n}, C, 0\right)\right|^2 dt,$$

thus the dominated convergence theorem and lemma 2.6.7 yield that

$$\lim_{n \to \infty} \mathbb{E}\left(\left|Y_0^{x,q^n} - Y_0^n\right|^2\right) = 0. \tag{2.66}$$

We can then conclude the proof since (2.65) and (2.66) imply, a.e.,

$$u(x) \leq \liminf_{n \to \infty} Y_0^{x,q^n} = \liminf_{n \to \infty} Y_0^n \leq \lim_{n \to \infty} Y_0^{x_n,q^n} = \underline{u}.$$

$$\square$$

Now that we have checked that u exists and is possible to apply the tools developed in section 2.2, we will focus on proving that u is a viscosity solution to problem 2.6.1.

Proposition 2.6.9. *The function u is a continuous viscosity supersolution to the problem 2.6.1.*

Proof. We know from proposition 2.6.8 that u is continuous, thus we just have to prove the supersolution property to conclude the proof. In order to do so we will show that, fixed $x \in D$, $Y_0^{0,\cdot,q}$ is a supersolution in x to problem 2.6.1 for any $q \in \mathcal{A}$. Then u is the infimum of a family of locally equibounded supersolutions, hence is a supersolution by theorem 2.0.3. In this proof the FBSDEs are intended in the (2.60) form.

Given $x \in D$, $q \in \mathcal{A}$ and a subtangent ψ to $Y_0^{x,q}$ a.e. at x we can assume without loss of generality that $Y_0^{x,q} = \psi(x)$ a.e., so suppose that

$$\frac{1}{2}\left\langle \sigma\sigma^\dagger, D^2\psi(x) \right\rangle + H(x, \psi, \nabla\psi) > 0$$

and we will find a contradiction. By the definition of subtangent, there exists a positive δ such that, for any $t \in [0, \delta]$ and $y \in B_\delta(x) \subseteq D$,

$$Y_t^{t,y,q} \geq \psi(y), \quad \text{a.e.,} \tag{2.67}$$

and, possibly taking a smaller δ,

$$\begin{aligned}
0 &< \frac{1}{2}\left\langle \sigma\sigma^\dagger, D^2\psi(y) \right\rangle + H(y, \psi, \nabla\psi) \\
&\leq \frac{1}{2}\left\langle \sigma\sigma^\dagger, D^2\psi(y) \right\rangle + f(y, \psi, q_t) - \nabla\psi(y)q_t,
\end{aligned} \tag{2.68}$$

thanks to the continuity of H and ψ. Now we define the stopping time

$$\tau := \delta \wedge \inf\{t \in [0, \infty) : |X_t^x - x| \geq \delta\},$$

let $\left(\overline{Y}_t, \overline{Z}_t\right) := (Y_{t\wedge\tau}^{x,q}, Z_{t\wedge\tau}^{x,q})$, which solve the BSDE

$$\overline{Y}_t = Y_\tau^{x,q} + \int_{t\wedge\tau}^\tau \left(f\left(X_s^x, \overline{Y}_s, q_s\right) - \overline{Z}_s\sigma^{-1}q_s \right) ds - \int_{t\wedge\tau}^\tau \overline{Z}_s dW_s,$$

for any $t \in [0, \delta]$, and $\left(\hat{Y}_t, \hat{Z}_t\right) := (\psi(X^x_{t\wedge\tau}), (\nabla\psi\sigma)(X^x_{t\wedge\tau}))$ which, by Itô's formula, is solution to

$$\hat{Y}_t = \psi\left(\tau, X^x_\tau\right) - \int_{t\wedge\tau}^\tau \hat{Z}_s dW_s - \int_{t\wedge\tau}^\tau \frac{1}{2}\left\langle\sigma\sigma^\dagger, D^2\psi(X^x_s)\right\rangle ds, \quad t \in [0, \delta].$$

By (2.67) we have that $Y^{\tau, X^x_\tau, q}_\tau \geq \psi\left(X^x_\tau\right)$ a.e. and (2.68) imply, thanks to corollary B.1.8, that $Y^{x,q}_0 > \psi(x)$ a.e., but this lead to a contradiction since, by our assumptions, $\psi(x) = Y^{x,q}_0$. This concludes the proof. □

As before, to finally prove that u is a viscosity solution we need a dynamic programming principle for this problem, which can be proved as in the proof of lemma 2.3.11.

Lemma 2.6.10. *For any $q \in \mathcal{A}$ we let $\left(\overline{Y}, \overline{Z}\right)$ be the solution of the BSDE*

$$\overline{Y}_t = u(X^x_\tau) + \int_{t\wedge\tau}^\tau \left(f\left(X^x_s, \overline{Y}_s, q_s\right) - \overline{Z}_s\sigma^{-1}q_s\right) ds - \int_{t\wedge\tau}^\tau \overline{Z}_s dW_s,$$

for any $t \in [0, \infty)$ and where τ is a stopping time smaller than τ^x. Then $\inf_{q\in\mathcal{A}} \mathbb{E}\left(\overline{Y}_0\right) = u(x)$.

Theorem 2.6.11. *The function $u(x) := \inf_{q\in\mathcal{A}} \mathbb{E}\left(Y^{x,q}_0\right)$ is the only viscosity solution to problem 2.6.1 such that $u(x) = G$ for any $x \in \partial D$.*

Proof. The uniqueness is a consequence of theorem 2.6.2, hence we only have to show that u is a viscosity solution.

From proposition 2.6.9 we know that u is a continuous viscosity supersolution and it is easy to see that $u(x) = G$ for any $x \in \partial D$, so we only need to prove the subsolution property of u. Fixed x in D, let φ be a supertangent to u in x which we assume, without loss of generality, equal to u in x and δ a positive constant such that

$$\varphi(y) \geq u(y) \qquad \text{for any } y \in B_\delta(x) \subseteq D. \tag{2.69}$$

As in the proof of proposition 2.6.9 we proceed by contradiction, assuming

$$\frac{1}{2}\left\langle\sigma\sigma^\dagger, D^2\varphi(x)\right\rangle + H(x, \varphi, \nabla\varphi) < 0$$

Since φ is a C^2 function we have that $\nabla\varphi$ is bounded, thus, if we define the stopping

$$\tau := \delta \wedge \inf\{t \in [0,\infty) : |X_t^x - x| \geq \delta\},$$

there exists a $q \in \mathcal{A}$ such that, for any $t \in [0,\delta]$,

$$f(X_{t\wedge\tau}^x, \varphi, q_{t\wedge\tau}) - \nabla\varphi(X_{t\wedge\tau}^x)q_{t\wedge\tau} = H(X_{t\wedge\tau}^x, \varphi, \nabla\varphi).$$

As a consequence of this we have that, by continuity,

$$\frac{1}{2}\left\langle \sigma\sigma^\dagger, D^2\varphi(X_{t\wedge\tau}^x)\right\rangle + f(X_{t\wedge\tau}^x, \varphi, q_{t\wedge\tau}) - \nabla\varphi(X_{t\wedge\tau}^x)q_{t\wedge\tau} < 0, \qquad (2.70)$$

for any $t \in [0,\delta]$, possibly taking a smaller δ.
Let $\left(\overline{Y}_t, \overline{Z}_t\right)$ be the solution to the BSDE

$$\overline{Y}_t = u(\tau, X_\tau^x) + \int_{t\wedge\tau}^\tau \left(f\left(X_s^x, \overline{Y}_s, q_s\right) - \overline{Z}_s\sigma^{-1}q_s\right)ds - \int_{t\wedge\tau}^\tau \overline{Z}_s dW_s,$$

for any $t \in [0,\delta]$ and $\left(\hat{Y}_t, \hat{Z}_t\right) := (\varphi(s, X_{t\wedge\tau}^x), (\nabla\varphi\sigma)(X_{t\wedge\tau}^x))$ which, by Itô's formula, is solution to

$$\hat{Y}_t = \varphi(X_\tau^x) - \int_{t\wedge\tau}^\tau \frac{1}{2}\left\langle \sigma\sigma^\dagger, D^2\varphi(X_{s\wedge\tau}^x)\right\rangle ds - \int_{t\wedge\tau}^\tau \hat{Z}_s dW_s, \quad t \in [0,\delta].$$

We know from lemma 2.6.10 that

$$\sup_{q\in\mathcal{A}} \mathbb{E}\left(\overline{Y}_0\right) = u(x) = \varphi(x), \qquad (2.71)$$

but by (2.69) we have $u(X_\tau^x) \leq \varphi(X_\tau^x)$, which together with (2.70) imply, thanks to corollary B.1.8, that $\overline{Y}_0 < \varphi(x)$ a.e., in contradiction with (2.71).

$\square$

2.6.2   A PDE with Blow Up Conditions

Here we will use the information gathered in the previous subsection to study an elliptic problem with blow up condition on the boundary. To ease notation we define the signed distance $d(x) := \inf_{y\in\mathbb{R}^N\setminus D} |x-y| - \inf_{y\in\overline{D}} |x-y|$ and the sets $\Gamma^\delta := \left\{x \in \mathbb{R}^N : |d(x)| < \delta\right\}$, $D^\delta := D \cap \Gamma^\delta$ and $D_\delta := D \setminus D^\delta$.

Problem 2.6.12. Under the same conditions of problem 2.6.1 assume that ∂D is C^2 and there exist an $\alpha \in (1,2]$, a $\Lambda \geq \lambda$ and two constants $K \geq k > 0$ such that, for any $x, p \in \mathbb{R}^N$,

(vii) $-K|p|^\alpha \leq \lim\limits_{c \to \pm\infty} \dfrac{H(x, 0, cp)}{c^\alpha} \leq -k|p|^\alpha$;

(viii) $x^\dagger \sigma \sigma^\dagger x \leq \Lambda |x|^2$.

Find the solution u to the elliptic PDE

$$\begin{cases} \dfrac{1}{2} \left\langle \sigma\sigma^\dagger, D^2 u(x) \right\rangle + H(x, u, \nabla u) = 0, & x \in D, \\ u(x) \to \infty, & \text{as } d(x) \to 0^+. \end{cases}$$

The idea to find a representation formula to solutions of this problem is simple: our candidate viscosity solution is $u := \lim\limits_{G \to \infty} u_G$, where u_G is the solution to problem 2.6.1 equal to G on ∂D (similarly we will denote with (Y_G, Z_G) as the (Y, Z) solution to the BSDE of (2.60) with final condition G). To do so we need a bound for this limit. The next theorem will provide this bound and is essentially a restatement adapted to our method of a result given in [20].

Remark 2.6.13. It is well known, see [15, Section 14.6], that if ∂D is C^k then there exists a δ_0 such that $d \in C^k \left(\Gamma^{\delta_0} \right)$ with bounded derivatives, in particular $|\nabla d| = 1$ in Γ^{δ_0}. Thus with an abuse of notation we will denote with d any function C^k equal to d in D^δ, for a $\delta \in (0, \delta_0)$, and bigger than δ in D_δ.

Theorem 2.6.14. *Under our assuming the limit function $u := \lim\limits_{G \to \infty} u_G$ exists and is locally bounded in D. Furthermore we have that*

$$\begin{cases} \dfrac{c_0}{(d(x))^\theta} - c^- \leq u(x) \leq \dfrac{C_0}{(d(x))^\theta} + c^+, & \text{if } \alpha \in (1, 2), \\ -c_0 \ln(d(x)) - c^- \leq u(x) \leq -C_0 \ln(d(x)) + c^+, & \text{if } \alpha = 2, \end{cases}$$

where $\theta := \frac{2-\alpha}{\alpha-1}$, c^+ and c^- are two positive constants depending on D, α, H

and respectively on k, Λ or K, λ, and

$$
(c_0, C_0) := \begin{cases} \left(\dfrac{1}{\theta} \left(\dfrac{\lambda(1+\theta)}{2K} \right)^{\frac{1}{\alpha-1}}, \dfrac{1}{\theta} \left(\dfrac{\Lambda(1+\theta)}{2k} \right)^{\frac{1}{\alpha-1}} \right), & \text{if } \alpha \in (1,2), \\[3mm] \left(\dfrac{\lambda}{2K}, \dfrac{\Lambda}{2k} \right), & \text{if } \alpha = 2. \end{cases}
$$

Proof. In this proof we will always assume that $\alpha \in (1,2)$, since the case $\alpha = 2$ can be handled similarly. Thanks to theorem B.2.9 we know that if $G \le G'$ then $Y_G^q \le Y_{G'}^q$ for any $q \in \mathcal{A}$ and consequently $u_G \le u_{G'}$. This implies that the limit function u exists, since for $G \to \infty$ the functions u_G converge monotonically to a function u.

Now, to prove that u is finite, we will define a function $\overline{w}$ finite in D and we will prove that $u \le w$. Let ε be any positive constant, c^+ a positive number that we will choose later and $C_\varepsilon := C_0 + \varepsilon$. Then we define the $C^2(D)$ function, see remark 2.6.13,

$$
\overline{w}(x) := \frac{C_\varepsilon}{(d(x))^\theta} + c^+.
$$

As a first step in proving this theorem, we will show that $\overline{w}$ is a supersolution to problem 2.6.12 starting with the following inequality:

$$
\begin{aligned}
\frac{1}{2} &\left\langle \sigma\sigma^\dagger, D^2\overline{w}(x) \right\rangle + H(x, \overline{w}, \nabla\overline{w}) \\
&\le \frac{1}{2} \left\langle \sigma\sigma^\dagger, D^2\overline{w}(x) \right\rangle + H(x, 0, \nabla\overline{w}) - \mu\overline{w}(x) \\
&= \frac{\theta(\theta+1)C_\varepsilon}{2(d(x))^{\theta+2}} \left\langle \sigma\sigma^\dagger, \nabla d(x) \otimes \nabla d(x) \right\rangle + H\left(x, 0, -\frac{\theta C_\varepsilon \nabla d(x)}{(d(x))^{\theta+1}} \right) \\
&\quad - \frac{\theta C_\varepsilon}{2(d(x))^{\theta+1}} \left\langle \sigma\sigma^\dagger, D^2 d(x) \right\rangle - \frac{\mu C_\varepsilon}{(d(x))^\theta} - \mu c^+.
\end{aligned}
$$

Now we define the function

$$
h(x) := \frac{\theta(\theta+1)C_\varepsilon}{2} \left\langle \sigma\sigma^\dagger, \nabla d(x) \otimes \nabla d(x) \right\rangle + (d(x))^{\theta+2} H\left(x, 0, -\frac{\theta C_\varepsilon \nabla d(x)}{(d(x))^{\theta+1}} \right),
$$

then, since $\theta + 2 = \alpha(\theta+1)$, for any $\overline{x} \in \partial D$ we obtain from item (vii) in problem 2.6.12 that

$$
\lim_{x \to \overline{x}} h(x) \le \frac{\theta(\theta+1)C_\varepsilon}{2} \left\langle \sigma\sigma^\dagger, \nabla d(\overline{x}) \otimes \nabla d(\overline{x}) \right\rangle - k\theta^\alpha (C_\varepsilon)^\alpha |\nabla d(\overline{x})|^\alpha.
$$

We point out that if $|\nabla d(x)| = 1$, which surely happens when $d(x)$ is little enough, then, if $\varepsilon > 0$, $\lim\limits_{x \to \overline{x}} h(x) < 0$ therefore, by continuity, there exists a $\delta > 0$ such that $h(x) < 0$ for any $x \in D^\delta$. We have from our assumptions that there exists a constant c_1 such that

$$\frac{1}{2}\left\langle \sigma\sigma^\dagger, D^2\overline{w}(x)\right\rangle + H(x, \overline{w}, \nabla\overline{w})$$
$$\leq \frac{h(x)}{(d(x))^{\theta+2}} + \frac{c_1}{(d(x))^{\theta+1}} - \frac{\mu C_\varepsilon}{(d(x))^\theta} - \mu c^+. \quad (2.72)$$

When $x \in D^\delta$ we have that h is negative, thus when $d(x)$ is little enough this guarantees us that (2.72) is negative, while in the other case we can choose c^+ big enough to achieve that. This prove that $\overline{w}$ is a supersolution and this yields, thanks to theorem 2.6.2, that $u_G \leq \overline{w}$ for any G, and consequently that $u \leq \overline{w}$. Then, by the arbitrariness of ε, $u(x) \leq \dfrac{C_0}{(d(x))^\theta} + c^+$.

To conclude we will proceed as before and prove that for an opportune constant c^- and $c_\varepsilon := c - \varepsilon$ that

$$\underline{w}_\epsilon(x) := \frac{c_\varepsilon}{(d(x) + \epsilon)^\theta} - c^-$$

is a subsolution to problem 2.6.1 for any positive ε and ϵ such that d is C^2 in Γ^ϵ. As previously done we start with the inequality

$$\frac{1}{2}\left\langle \sigma\sigma^\dagger, D^2\underline{w}_\epsilon(x)\right\rangle + H(x, \underline{w}_\epsilon, \nabla\underline{w}_\epsilon)$$
$$\geq \frac{1}{2}\left\langle \sigma\sigma^\dagger, D^2\underline{w}_\epsilon(x)\right\rangle + H(x, 0, \nabla\underline{w}_\epsilon) - \nu|\underline{w}_\epsilon(x)| - \nu$$
$$= \frac{\theta(\theta+1)c_\varepsilon}{2(d(x)+\epsilon)^{\theta+2}}\left\langle \sigma\sigma^\dagger, \nabla d(x) \otimes \nabla d(x)\right\rangle + H\left(x, 0, -\frac{\theta c_\varepsilon \nabla d(x)}{(d(x)+\epsilon)^{\theta+1}}\right)$$
$$- \frac{\theta c_\varepsilon}{2(d(x)+\epsilon)^{\theta+1}}\left\langle \sigma\sigma^\dagger, D^2 d(x)\right\rangle - \nu\left|\frac{c_\varepsilon}{(d(x)+\epsilon)^\theta} - c^-\right| - \nu$$

and define the function

$$h(x) := \frac{\theta(\theta+1)c_\varepsilon}{2}\left\langle \sigma\sigma^\dagger, \nabla d(x) \otimes \nabla d(x)\right\rangle$$
$$+ (d(x)+\epsilon)^{\theta+2}H\left(x, 0, -\frac{\theta c_\varepsilon \nabla d(x)}{(d(x)+\epsilon)^{\theta+1}}\right).$$

As before we obtain from item (vii) in problem 2.6.12 that, for any $\overline{x}$ such that $d(\overline{x}) = -\epsilon$,

$$\lim_{x \to \overline{x}} h(x) \geq \frac{\theta(\theta+1)c_\varepsilon}{2} \left\langle \sigma\sigma^\dagger, \nabla d(\overline{x}) \otimes \nabla d(\overline{x}) \right\rangle - K\theta^\alpha c_\varepsilon^\alpha |\nabla d(\overline{x})|^\alpha,$$

which is positive if $\varepsilon > 0$, therefore, by continuity, $h(x) > 0$ if $d(x) + \epsilon$ is little enough. Finally we have that there exists a constant c_2 such that

$$\frac{1}{2} \left\langle \sigma\sigma^\dagger, D^2 \underline{w}_\epsilon(x) \right\rangle + H(x, \underline{w}_\epsilon, \nabla \underline{w}_\epsilon)$$
$$\geq \frac{h(x)}{(d(x)+\delta)^{\theta+2}} + \frac{c_2}{(d(x)+\delta)^{\theta+1}} - \frac{\nu c_\varepsilon}{(d(x)+\delta)^\theta} - \nu c^-,$$

which can be made positive for any little enough ϵ choosing an opportune c^-, thanks to the positivity of $h(x)$ for $d(x) + \epsilon$ little enough. Then $\underline{w}_\epsilon$ is a bounded subsolution to problem 2.6.1, thus there exists a G_ϵ such that $\underline{w}_\epsilon \leq G_\epsilon$ on ∂D and by theorem 2.6.2 $\underline{w}_\epsilon \leq u_{G_\epsilon} \leq u$ on D. Hence the arbitrariness of ε and ϵ yields $u(x) \geq \frac{c_0}{(d(x))^\theta} - c^-$. $\square$

If ∂D is C^3 we can improve theorem 2.6.14.

Lemma 2.6.15. *If ∂D is C^3 we have that*

$$\begin{cases} \dfrac{c(x)}{(d(x))^\theta} - c^- \leq u(x) \leq \dfrac{C(x)}{(d(x))^\theta} + c^+, & \text{if } \alpha \in (1,2), \\ -c(x)\ln(d(x)) - c^- \leq u(x) \leq -C(x)\ln(d(x)) + c^+, & \text{if } \alpha = 2, \end{cases}$$

where $\theta := \frac{2-\alpha}{\alpha-1}$, c^+ and c^- are two positive constants depending on D, α, H, σ and respectively on k or K,

$$C(x) := \begin{cases} \dfrac{1}{\theta} \left(\dfrac{\left\langle \sigma\sigma^\dagger, \nabla d(x) \otimes \nabla d(x) \right\rangle (1+\theta)}{2k} \right)^{\frac{1}{\alpha-1}}, & \text{if } \alpha \in (1,2), \\ \dfrac{\left\langle \sigma\sigma^\dagger, \nabla d(x) \otimes \nabla d(x) \right\rangle}{2k}, & \text{if } \alpha = 2 \end{cases}$$

and

$$c(x) := \begin{cases} \dfrac{1}{\theta} \left(\dfrac{\left\langle \sigma\sigma^\dagger, \nabla d(x) \otimes \nabla d(x) \right\rangle (1+\theta)}{2K} \right)^{\frac{1}{\alpha-1}}, & \text{if } \alpha \in (1,2), \\ \dfrac{\left\langle \sigma\sigma^\dagger, \nabla d(x) \otimes \nabla d(x) \right\rangle}{2K}, & \text{if } \alpha = 2. \end{cases}$$

Proof. We will always assume that $\alpha \in (1, 2)$, since the case $\alpha = 2$ can be handled similarly. Let ε be any positive constant, c^+ a positive number that we will choose later and $C_\varepsilon(x) := C(x) + \varepsilon$. Then we define the $C^3(D)$ function, see remark 2.6.13,

$$\overline{w}(x) := \frac{C_\varepsilon(x)}{(d(x))^\theta} + c^+.$$

We will prove that $\overline{w}$ is a supersolution to problem 2.6.12, starting with the following inequality:

$$\frac{1}{2} \left\langle \sigma\sigma^\dagger, D^2\overline{w}(x) \right\rangle + H(x, \overline{w}, \nabla\overline{w})$$

$$\leq \frac{1}{2} \left\langle \sigma\sigma^\dagger, D^2\overline{w}(x) \right\rangle + H(x, 0, \nabla\overline{w}) - \mu\overline{w}(x)$$

$$= \frac{\theta(\theta+1)C_\varepsilon(x)}{2(d(x))^{\theta+2}} \left\langle \sigma\sigma^\dagger, \nabla d(x) \otimes \nabla d(x) \right\rangle + \frac{1}{2(d(x))^\theta} \left\langle \sigma\sigma^\dagger, D^2C_\varepsilon(x) \right\rangle$$

$$- \frac{\theta}{2(d(x))^{\theta+1}} \left\langle \sigma\sigma^\dagger, \nabla C_\varepsilon(x) \otimes \nabla d(x) + \nabla d(x) \otimes \nabla C_\varepsilon(x) \right\rangle$$

$$- \frac{\theta C_\varepsilon(x)}{2(d(x))^{\theta+1}} \left\langle \sigma\sigma^\dagger, D^2d(x) \right\rangle - \mu\frac{C_\varepsilon(x)}{(d(x))^\theta} - \mu c^+$$

$$+ H\left(x, 0, \frac{\nabla C_\varepsilon(x)}{(d(x))^\theta} - \frac{\theta C_\varepsilon(x)\nabla d(x)}{(d(x))^{\theta+1}}\right).$$

Now we define the function

$$h(x) := \frac{\theta(\theta+1)C_\varepsilon(x)}{2} \left\langle \sigma\sigma^\dagger, \nabla d(x) \otimes \nabla d(x) \right\rangle$$

$$+ (d(x))^{\theta+2}H\left(x, 0, \frac{\nabla C_\varepsilon(x)}{(d(x))^\theta} - \frac{\theta C_\varepsilon(x)\nabla d(x)}{(d(x))^{\theta+1}}\right),$$

then, since $\theta + 2 = \alpha(\theta + 1)$, for any $\overline{x} \in \partial D$ we obtain from item (vii) in problem 2.6.12 that

$$\lim_{x \to \overline{x}} h(x) \leq \frac{\theta(\theta+1)C_\varepsilon(\overline{x})}{2} \left\langle \sigma\sigma^\dagger, \nabla d(\overline{x}) \otimes \nabla d(\overline{x}) \right\rangle - k\theta^\alpha(C_\varepsilon(\overline{x}))^\alpha|\nabla d(\overline{x})|^\alpha.$$

We point out that if $|\nabla d(x)| = 1$, which surely happens when $d(x)$ is little enough, then, if $\varepsilon > 0$, $\lim_{x \to \overline{x}} h(x) < 0$ therefore, by continuity, there exists a $\delta > 0$ such that $h(x) < 0$ for any $x \in D^\delta$. We have from our assumptions that there exist two constants c_1 and c_2 such that

$$\frac{1}{2}\left\langle\sigma\sigma^\dagger, D^2\overline{w}(x)\right\rangle + H(x,\overline{w},\nabla\overline{w})$$

$$\leq \frac{h(x)}{(d(x))^{\theta+2}} + \frac{c_1}{(d(x))^{\theta+1}} + \frac{c_2}{(d(x))^\theta} - \mu c^+. \quad (2.73)$$

When $x \in D^\delta$ we have that h is negative, thus when $d(x)$ is little enough this guarantees us that (2.73) is negative, while in the other case we can choose c^+ big enough to achieve that. This prove that $\overline{w}$ is a supersolution and this yields, thanks to theorem 2.6.2, that $u_G \leq \overline{w}$ for any G, and consequently that $u \leq \overline{w}$. Then, by the arbitrariness of ε, $u(x) \leq \frac{C(x)}{(d(x))^\theta} + c^+$.

To conclude we will proceed as before and prove that for an opportune constant c^- and $c_\varepsilon(x) := c(x) - \varepsilon$ that

$$\underline{w}_\epsilon(x) := \frac{c_\varepsilon(x)}{(d(x) + \epsilon)^\theta} - c^-$$

is a subsolution to problem 2.6.1 for any positive ε and ϵ such that d is C^3 in Γ^ϵ. As previously done we start with the inequality

$$\frac{1}{2}\left\langle\sigma\sigma^\dagger, D^2\underline{w}_\epsilon(x)\right\rangle + H(x,\underline{w}_\epsilon, \nabla\underline{w}_\epsilon)$$

$$\geq \frac{1}{2}\left\langle\sigma\sigma^\dagger, D^2\underline{w}_\epsilon(x)\right\rangle + H(x,0,\nabla\underline{w}_\epsilon) - \nu|\underline{w}_\epsilon(x)| - \nu$$

$$= \frac{\theta(\theta+1)c_\varepsilon(x)}{2(d(x)+\epsilon)^{\theta+2}}\left\langle\sigma\sigma^\dagger, \nabla d(x)\otimes\nabla d(x)\right\rangle + \frac{1}{2(d(x)+\epsilon)^\theta}\left\langle\sigma\sigma^\dagger, D^2 c_\varepsilon(x)\right\rangle$$

$$- \frac{\theta}{2(d(x)+\epsilon)^{\theta+1}}\left\langle\sigma\sigma^\dagger, \nabla c_\varepsilon(x)\otimes\nabla d(x) + \nabla d(x)\otimes\nabla c_\varepsilon(x)\right\rangle$$

$$- \frac{\theta c_\varepsilon(x)}{2(d(x)+\epsilon)^{\theta+1}}\left\langle\sigma\sigma^\dagger, D^2 d(x)\right\rangle - \nu\left|\frac{c_\varepsilon(x)}{(d(x)+\epsilon)^\theta} - c^-\right| - \nu$$

$$+ H\left(x, 0, \frac{\nabla c_\varepsilon(x)}{(d(x)+\epsilon)^\theta} - \frac{\theta c_\varepsilon(x)\nabla d(x)}{(d(x)+\epsilon)^{\theta+1}}\right)$$

and define the function

$$h(x) := \frac{\theta(\theta+1)c_\varepsilon(x)}{2}\left\langle\sigma\sigma^\dagger, \nabla d(x)\otimes\nabla d(x)\right\rangle$$

$$+ (d(x)+\epsilon)^{\theta+2}H\left(x, 0, \frac{\nabla c_\varepsilon(x)}{(d(x)+\epsilon)^\theta} - \frac{\theta c_\varepsilon(x)\nabla d(x)}{(d(x)+\epsilon)^{\theta+1}}\right).$$

As before we obtain from item (vii) in problem 2.6.12 that, for any $\overline{x}$ such that $d(\overline{x}) = -\epsilon$,

$$\lim_{x \to \overline{x}} h(x) \geq \frac{\theta(\theta+1)c_\varepsilon(\overline{x})}{2} \left\langle \sigma\sigma^\dagger, \nabla d(\overline{x}) \otimes \nabla d(\overline{x}) \right\rangle - K\theta^\alpha (c_\varepsilon(\overline{x}))^\alpha |\nabla d(\overline{x})|^\alpha,$$

which is positive if $\varepsilon > 0$, therefore, by continuity, $h(x) > 0$ if $d(x) + \epsilon$ is little enough. Finally we have that there exist two constants c_3 and c_4 such that

$$\frac{1}{2} \left\langle \sigma\sigma^\dagger, D^2 \underline{w}_\epsilon(x) \right\rangle + H(x, \underline{w}_\epsilon, \nabla \underline{w}_\epsilon)$$
$$\geq \frac{h(x)}{(d(x)+\delta)^{\theta+2}} + \frac{c_3}{(d(x)+\delta)^{\theta+1}} + \frac{c_4}{(d(x)+\delta)^\theta} - \nu c^-,$$

which can be made positive for any little enough ϵ choosing an opportune c^-, thanks to the positivity of $h(x)$ for $d(x) + \epsilon$ little enough. Then $\underline{w}_\epsilon$ is a bounded subsolution to problem 2.6.1, thus there exists a G_ϵ such that $\underline{w}_\epsilon \leq G_\epsilon$ on ∂D and by theorem 2.6.2 $\underline{w}_\epsilon \leq u_{G_\epsilon} \leq u$ on D. Hence the arbitrariness of ε and ϵ yields $u(x) \geq \frac{c(x)}{(d(x))^\theta} - c^-$. $\qquad\square$

Remark 2.6.16. Using the same technique in the proof of lemma 2.6.15 we can prove that $v(x) \geq \frac{c(x)}{(d(x))^\theta} - c^-$, where v is a supersolution to problem 2.6.12 tending to ∞ as $d(x) \to 0$. Similarly, given a subsolution u to problem 2.6.12 finite in D and tending to ∞ as $d(x) \to 0$, we can use the functions

$$\overline{w}_\epsilon(x) := \frac{C_\varepsilon(x)}{(d(x)-\epsilon)^\theta} + c^+,$$

defined for $\epsilon > 0$ small enough, to prove that $u(x) \leq \frac{C(x)}{(d(x))^\theta} + c^+$, since $\overline{w}_\epsilon$ is a supersolution to problem 2.6.12 in D_ϵ which tends to ∞ as $d(x) \to \epsilon$.

From this we can obtain a comparison result.

Theorem 2.6.17. *Let u and v be respectively a subsolution and a supersolution to problem 2.6.12 tending to ∞ as $d(x) \to 0$. Then, if ∂D is C^3, $\mu = \nu$ and $k = K$, $u \leq v$ on D. Furthermore, if $\lambda = \Lambda$, this is true even if ∂D is just C^2.*

Proof. In this proof we will assume that $\alpha \in (1,2)$ and $\lambda < \Lambda$, since the cases $\alpha = 2$ and $\lambda = \Lambda$ can be handled similarly. Remark 2.6.16 yields that $v(x) \geq \frac{C(x)}{(d(x))^\theta} - c^-$ and $u(x) \leq \frac{C(x)}{(d(x))^\theta} + c^+$, since $k = K$. As a consequence of these inequalities we have in a neighborhood of ∂D that

$$v(x) \geq \gamma u(x) + (1 - \gamma)\frac{m}{\mu} =: u_\gamma(x), \tag{2.74}$$

where $\gamma \in (0,1)$ and $m := -\max_{x \in D} u^-(x) - \max_{x \in D} H^-(x, 0, 0) - \mu$. Notice that if φ is a supertangent to u in x, then $\gamma\varphi + (1 - \gamma)\frac{m}{\mu}$ is a supertangent to u_γ in x and vice versa. There exists a $q \in \mathbb{R}^N$ such that

$$\frac{\gamma}{2}\left\langle \sigma\sigma^\dagger, D^2\varphi(x) \right\rangle + H\left(x, \gamma u + (1 - \gamma)\frac{m}{\mu}, \gamma\nabla\varphi\right)$$

$$\geq \frac{\gamma}{2}\left\langle \sigma\sigma^\dagger, D^2\varphi(x) \right\rangle + f\left(x, \gamma u + (1 - \gamma)\frac{m}{\mu}, q\right) - \gamma\nabla\varphi(x)q$$

$$\geq \frac{\gamma}{2}\left\langle \sigma\sigma^\dagger, D^2\varphi(x) \right\rangle + f(x, u, q) - \gamma\nabla\varphi(x)q + (1 - \gamma)(\mu u(x) - m)$$

$$\geq \frac{\gamma}{2}\left\langle \sigma\sigma^\dagger, D^2\varphi(x) \right\rangle + \gamma H(x, u, \nabla\varphi) + (1 - \gamma)(f(x, u, q) + \mu u(x) - m)$$

$$\geq (1 - \gamma)(H(x, u, 0) + \mu u(x) - m),$$

where the last inequality is a consequence of u being a subsolution to problem 2.6.12. If $u(x) \leq 0$ then we can use the monotonicity of H to obtain

$$H(x, u, 0) + \mu u(x) - m \geq H(x, 0, 0) - m \geq 0$$

and if $u(x) > 0$ we have, since $\mu = \nu$,

$$H(x, u, 0) + \mu u(x) - m \geq H(x, 0, 0) - \mu - m \geq 0.$$

This implies that u_γ is a subsolution, hence (2.74) and theorem 2.6.2 yield that $v \geq u_\gamma$ in D_δ for any δ little enough and $\gamma \in (0,1)$. Finally letting $\delta \to 0$ and $\gamma \to 1$ we can conclude the proof. $\square$

Remark 2.6.18. Notice that the previous proof works also for $\nu < \mu$. But by our assumptions we have that, for any $y > 0$,

$$\mu y \leq H(0, 0, 0) - H(0, 0, y) \leq \nu(1 + y),$$

which implies that $\mu \leq \nu$.

Now we come back to our candidate viscosity solution u, discussing its representation formula. Preliminarily define, for any $x \in D$, the sets

$$\mathcal{S}_x := \left\{ \{q^n\}_{n \in \mathbb{N}} \subset \mathcal{A} : \lim_{n \to \infty} \tau^{x,q^n} = \infty \ a.e. \right\}$$

and the FBSDEs

$$\begin{cases} X_{\rho+s}^{\rho,\zeta} = \zeta + \int_{\rho}^{\rho+s} \sigma dW_r - \int_{\rho}^{\rho+s} q_r dr, \\ \widetilde{Y}_{\rho+s}^{\rho,\zeta} = \int_{\rho+s \wedge \tau^{\rho,\zeta}}^{\rho+\tau^{\rho,\zeta}} f\left(X_r^{\rho,\zeta}, \widetilde{Y}_r^{\rho,\zeta}, q_r\right) dr - \int_{\rho+s \wedge \tau^{\rho,\zeta}}^{\rho+\tau^{\rho,\zeta}} \widetilde{Z}_r^{\rho,\zeta} dW_r, \qquad s \in [0,\infty). \\ \widetilde{Z}_{\rho+s}^{\rho,\zeta} = 0, \qquad \text{on } \{\tau^{\rho,\zeta,q} \le s\}, \end{cases}$$

For any fixed $x \in D$ and $\{q^n\} \in \mathcal{S}_x$ theorem B.2.5 yields the existence of a constant c depending on γ and μ such that

$$\mathbb{E}\left(\left| Y_{G_n,0}^{x,q^n} - \widetilde{Y}_0^{x,q^n} \right|^2 \right) \le c\mathbb{E}\left(e^{-2\gamma \tau^{x,q^n}} G_n^2 \right),$$

hence, if we choose a $\gamma \in (0, \mu)$ and an increasing sequence $\{G_n\}_{n \in \mathbb{N}}$ for which $\lim\limits_{n \to \infty} G_n = \infty$ and $\lim\limits_{n \to \infty} \mathbb{E}\left(e^{-2\gamma \tau^{x,q^n}} \right) G_n^2 = 0$, we have that

$$\liminf_{n \to \infty} \mathbb{E}\left(Y_{G_n,0}^{x,q^n} \right) = \liminf_{n \to \infty} \mathbb{E}\left(\widetilde{Y}_0^{x,q^n} \right).$$

Since $\mathbb{E}\left(Y_{G_n,0}^{x,q^n} \right) \ge u_{G_n}(x)$, this means that $\inf\limits_{\{q^n\} \in \mathcal{S}_x} \liminf\limits_{n \to \infty} \mathbb{E}\left(\widetilde{Y}_0^{x,q^n} \right) \ge u(x)$. On the other hand, we can choose two sequences $\{G_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ and $\{q^n\}_{n \in \mathbb{N}}$ in $\mathcal{A}$ such that $\lim\limits_{n \to \infty} \left| Y_{G_n,0}^{x,q^n} - u(x) \right| = 0$ a.e.. From theorem B.2.9 we know that $Y_{G_n,0}^{x,q^n} \ge \widetilde{Y}_0^{x,q^n}$ a.e., which implies that $\liminf\limits_{n \to \infty} \mathbb{E}\left(\widetilde{Y}_0^{x,q^n} \right) \le u(x)$. Thus, if we can prove that $\{q^n\} \in \mathcal{S}_x$, we will have that

$$u(x) = \inf_{\{q^n\} \in \mathcal{S}_x} \liminf_{n \to \infty} \mathbb{E}\left(\widetilde{Y}_0^{x,q^n} \right). \qquad (2.75)$$

As in the proof of proposition 2.6.8 we can define $\chi_t^n := \lim\limits_{s \downarrow t} \chi_{\{Y_{G_n,s}^{x,q^n} \le G_n\}}$, the stopping times

$$\tau^n := \tau^{x,q^n} \wedge \inf\{t \in [0,\infty) : \chi_t^n = 0\}$$

and the BSDEs

$$\overline{Y}^n_t = G_n + \int_{t\wedge\tau^n}^{\tau^n} f\left(X^{x,q^n}_s, \overline{Y}^n_s, q^n_s\right) ds - \int_{t\wedge\tau^n}^{\tau^n} \overline{Z}^n_s dW_s,$$

which is equal to $G_n \wedge Y^{x,q^n}_{G_n,t\wedge\tau^n}$. Using Itô's formula it is easy to see that

$$e^{-\nu t}\overline{Y}^n_t = e^{-\nu\tau^n}G_n + \int_{t\wedge\tau^n}^{\tau^n} e^{-\nu s}\left(f\left(X^{x,q^n}_s, \overline{Y}^n_s, q^n_s\right) + \nu\overline{Y}^n_s\right) ds - \int_{t\wedge\tau^n}^{\tau^n} e^{-\nu s}\overline{Z}^n_s dW_s,$$

therefore, letting $\overline{\chi}^n_t := \chi_{\left\{\overline{Y}^n_t \leq 0\right\}}$ and C be the lower bound defined in proposition 2.6.3, we get

$$\begin{aligned}
\mathbb{E}\left(\overline{Y}^n_0\right) =& \mathbb{E}\left(e^{-\nu\tau^n}G_n + \int_0^{\tau^n} e^{-\nu s}\left(f\left(X^{x,q^n}_s, \overline{Y}^n_s, q^n_s\right) + \nu\overline{Y}^n_s\right) ds\right)\\
\geq& \mathbb{E}\left(e^{-\nu\tau^n}G_n + \int_0^{\tau^n} e^{-\nu s}\left(H\left(X^{x,q^n}_s, \overline{Y}^n_s, 0\right) + \nu\overline{Y}^n_s\right) ds\right)\\
\geq& \mathbb{E}\left(e^{-\nu\tau^n}G_n + \int_0^{\tau^n} e^{-\nu s}\left(H\left(X^{x,q^n}_s, 0, 0\right) + \nu\left(\overline{Y}^n_s - \left|\overline{Y}^n_s\right|\right) - \nu\right) ds\right)\\
\geq& \mathbb{E}\left(e^{-\nu\tau^n}G_n + \int_0^{\tau^n} e^{-\nu s}\left(H\left(X^{x,q^n}_s, 0, 0\right) + 2\nu C\overline{\chi}^n_s - \nu\right) ds\right).
\end{aligned}$$

By theorem 2.6.14 $Y^{x,q^n}_{G_n,0}$, and consequently $\overline{Y}^n_0$, are bounded by above, which, thanks to the last inequality, implies that $\mathbb{E}\left(e^{-\nu\tau^{x,q^n}}G_n\right)$ is bounded, hence $\lim_{n\to\infty} \tau^{x,q^n} \geq \lim_{n\to\infty} \tau^n = \infty$ a.e.. This prove (2.75).

Theorem 2.6.19. *Under our assumptions u is a continuous function in D.*

Proof. As a supremum of continuous function u is lower semicontinuous, thus fixed an $x \in D$ and a sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to it we have that

$$\liminf_{n\to\infty} u(x_n) = \overline{u} \geq u(x).$$

We can then choose a subsequence of $\{x_n\}$, which we will denote again with $\{x_n\}$, and an increasing sequence of positive numbers $\{G_n\}_{n\in\mathbb{N}}$, for which $\lim_{n\to\infty} G_n = \infty$, such that $u_{G_n}(x_n)$ converges to $\overline{u}$. Now let $\{q^n\}_{n\in\mathbb{N}}$ be a sequence of control in $\mathcal{S}_x$ such that $\lim_{n\to\infty} \widetilde{Y}^{x,q^n}_0 = u(x)$ a.e. and $q^n_t = q^n_{t\wedge\tau^{x,q^n}}$. Notice that for any $n \in \mathbb{N}$ we can find and $N > 0$ such that $\tau^{x,q^n} < \tau^{x,q^k}$

for any $k > N$, hence, by lemma 2.6.7, we can extract a subsequence, which we will denote again with $\{q^n\}$, such that not only $\lim_{n\to\infty} \tau^{x_n,q^n} = \infty$ a.e., but also

$$\lim_{n\to\infty} \mathbb{E}\left(e^{-2\gamma\tau^{x_n,q^n}}\right) G_n^2 = 0 \tag{2.76}$$

for a given $\gamma \in (0, \mu)$. It is obvious that $\liminf_{n\to\infty} \mathbb{E}\left(Y^{x,q^n}_{G_n,0}\right) \geq \overline{u}$, thus we will proceed as in the proof of proposition 2.6.8, defining a sequence of BSDE $\{Y^n\}_{n\in\mathbb{N}}$ such that $Y^n \leq \widetilde{Y}^{x,q^n}$ and $\lim_{n\to\infty} \mathbb{E}\left(\left|Y^{x_n,q^n}_{G_n,0} - Y^n_0\right|^2\right) = 0$. Define $\chi^n_t := \lim_{s\downarrow t} \chi_{\{\widetilde{Y}^{x,q^n}_s \leq G_n\}}$, the stopping times

$$\tau^n := \tau^{x,q^n} \wedge \inf\left\{t \geq \tau^{x_n,q^n} : \chi^n_t = 0\right\},$$

the BSDEs

$$\overline{Y}^n_t = \int_{t\wedge\tau^n}^{\tau^n} f\left(X^{x,q^n}_s, \overline{Y}^n_s, q^n_s\right) ds - \int_{t\wedge\tau^n}^{\tau^n} \overline{Z}^n_s dW_s,$$

the sets

$$A_t := \left\{f\left(X^{x,q^n}_t, \overline{Y}^n_t, q^n_t\right) > f\left(X^{x,q^n}_t, \overline{Y}^n_t, 0\right)\right\},$$

the controls $\overline{q}^n_t := q^n_t\left(1 - \chi_{A_t}\chi_{\{\tau^{x_n,q^n} > t\}}\right)$ and the BSDE

$$Y^n_t = \int_{t\wedge\tau^n}^{\tau^n} f\left(X^{x,q^n}_s, Y^n_s, \overline{q}^n_s\right) ds - \int_{t\wedge\tau^n}^{\tau^n} Z^n_s dW_s.$$

We have that, for any $t \in [0, \infty)$ and $n \in \mathbb{N}$,

$$\overline{Y}^n_{t\vee\tau^{x_n,q^n}} = G_n \wedge \widetilde{Y}^{x,q^n}_{(t\wedge\tau^n)\vee\tau^{x_n,q^n}} \tag{2.77}$$

and, by theorem B.2.9,

$$Y^n_t \leq \overline{Y}^n_t \quad \text{and} \quad Y^n_0 \leq \overline{Y}^n_0 \leq \widetilde{Y}^{x,q^n}_0. \tag{2.78}$$

Using theorem B.2.5 we have that there exists a constant c depending on μ and γ such that

$$\mathbb{E}\left(\left|Y^{x_n,q^n}_{G_n,0} - Y^n_0\right|^2\right)$$

$$\leq c\mathbb{E}\left(e^{-2\gamma\tau^{x_n,q^n}}G_n^2 + \int_{\tau^{x_n,q^n}}^{\tau^n} e^{-2\gamma t}\left|f\left(X_t^{x,q^n}, Y_{G_n,t}^{x_n,q^n}, \overline{q}_t^n\right)\right|^2 dt\right.$$

$$\left.+ \int_0^{\tau^{x_n,q^n}} e^{-2\gamma t}\left|f\left(X_t^{x_n,q^n}, Y_{G_n,t}^{x_n,q^n}, q_t^n\right) - f\left(X_t^{x,q^n}, Y_{G_n,t}^{x_n,q^n}, \overline{q}_t^n\right)\right|^2 dt\right).$$

By definition we have

$$f\left(X_t^{x,q^n}, Y_{G_n,t}^{x_n,q^n}, \overline{q}_t^n\right)\chi_{\{t>\tau^{x_n,q^n}\}} \geq H\left(X_t^{x,q^n}, G_n, 0\right)$$
$$\geq H\left(X_t^{x,q^n}, 0, 0\right) - \nu G_n - \nu$$

and, thanks to (2.77), (2.78) and the monotonicity of f,

$$f\left(X_t^{x,q^n}, Y_{G_n,t}^{x,q^n}, \overline{q}_t^n\right)\chi_{\{t>\tau^{x_n,q^n}\}} = f\left(X_t^{x,q^n}, G_n, \overline{q}_t^n\right)\chi_{\{t>\tau^{x_n,q^n}\}}$$
$$\leq f\left(X_t^{x,q^n}, \overline{Y}_t^n, \overline{q}_t^n\right)\chi_{\{t>\tau^{x_n,q^n}\}}$$
$$\leq f\left(X_t^{x,q^n}, \overline{Y}_t^n, 0\right)\chi_{\{t>\tau^{x_n,q^n}\}}$$
$$\leq f\left(X_t^{x,q^n}, C, 0\right)\chi_{\{t>\tau^{x_n,q^n}\}},$$

where C is the lower bound defined in proposition 2.6.3. This means that

$$\int_{\tau^{x_n,q^n}}^{\tau^n} e^{-2\gamma t}\left|f\left(X_t^{x,q^n}, Y_{G_n,t}^{x_n,q^n}, \overline{q}_t^n\right)\right|^2 dt \leq 4\int_{\tau^{x_n,q^n}}^{\tau^n} e^{-2\gamma t}\left|H\left(X_t^{x,q^n}, G_n, 0\right)\right|^2 dt$$
$$+ \frac{2\nu^2(G_n^2+1)}{\gamma}\left(e^{-2\gamma\tau^n} - e^{-2\gamma\tau^{x_n,q^n}}\right)$$
$$+ 4\int_{\tau^{x_n,q^n}}^{\tau^n} e^{-2\gamma t}\left|f\left(X_t^{x,q^n}, C, 0\right)\right|^2 dt,$$

thus lemma 2.6.7, the fact that $\tau^n \geq \tau^{x_n,q^n}$, (2.76) and the dominated convergence theorem yield that

$$\lim_{n\to\infty}\mathbb{E}\left(\left|Y_{G_n,0}^{x_n,q^n} - Y_0^n\right|^2\right) = 0.$$

From this follows that $u(x) = \overline{u}$, concluding the proof. $\qquad\square$

It is left to proving that u is a viscosity solution to problem 2.6.12. That u is a subsolution follows from theorem 2.0.3, since it is a supremum of solution, while the supersolution property can be showed with the exactly same

method used in theorem 2.6.11, we only need to prove a dynamic programming principle for u.

Lemma 2.6.20. *For any $q \in \mathcal{A}$ we let $\left(\overline{Y}, \overline{Z}\right)$ be the solution of the BSDE*

$$\overline{Y}_t = u(X_\tau^x) + \int_{t \wedge \tau}^\tau \left(f\left(X_s^x, \overline{Y}_s, q_s\right) - \overline{Z}_s \sigma^{-1} q_s \right) ds - \int_{t \wedge \tau}^\tau \overline{Z}_s dW_s,$$

for any $t \in [0, \infty)$ and where τ is a stopping time smaller than τ^x. Then $\operatorname*{ess\,inf}_{q \in \mathcal{A}} \overline{Y}_0 = u(x)$.

Proof. Define $\overline{Y}_G$ as the BSDE such that $\inf_{q \in \mathcal{A}} \mathbb{E}\left(\overline{Y}_{G,0}\right) = u_G(x)$ seen in lemma 2.6.10. Then corollary B.1.8 yields $\overline{Y}_G \leq \overline{Y}$ a.e., which implies

$$u(x) = \lim_{G \to \infty} u_G(x) = \lim_{G \to \infty} \operatorname*{ess\,inf}_{q \in \mathcal{A}} \overline{Y}_{G,0} \leq \operatorname*{ess\,inf}_{q \in \mathcal{A}} \overline{Y}_0.$$

On the other hand, if we let $\{G_n\}_{n \in \mathbb{N}}$ and $\{q^n\}_{n \in \mathbb{N}}$ be two sequences such that $\left| u(x) - Y_{G_n,0}^{x,q^n} \right| < \dfrac{1}{n}$ a.e., then we have that $\liminf_{n \to \infty} Y_{G_n,\tau}^{x,q^n} \geq u(X_\tau^x)$ a.e., since $Y_{G_n,\tau}^{x,q^n} \geq u_{G_n}(X_\tau^x)$ a.e.. Consequently

$$u(x) = \lim_{n \to \infty} Y_{G_n,0}^{x,q^n} \geq \operatorname*{ess\,inf}_{q \in \mathcal{A}} \overline{Y}_0,$$

concluding the proof. $\qquad \square$

Theorem 2.6.21. *The function $u := \lim_{G \to \infty} u_G$ is a continuous viscosity solution to problem 2.6.12. Furthermore, if the assumptions of theorem 2.6.17 are true, the solution is unique.*

Proof. As previously said we already know that u is a viscosity solution, while uniqueness is a consequence of theorem 2.6.17. $\qquad \square$

2.6.3   The Ergodic Problem

A natural use of the previous results is the study of the so called "ergodic problem", which is the stochastic control problem (2.58) with $\mu = 0$. This study is performed in [20] analyzing the behavior of the solutions to (2.57) as

μ tends to 0. A rather complete overview of this problem in the subquadratic case has been done by Porretta in [36] where, studying the solutions to

$$\begin{cases} \dfrac{1}{2}\Delta u_\mu(x) - |\nabla u_\mu(x)|^\alpha - \mu u_\mu(x) + f(x) = 0, & x \in D, \\ u_\mu(x) = 0, & x \in \partial D \end{cases} \tag{2.79}$$

as μ tends to 0, he obtain that either (2.79) with $\mu = 0$ admits a bounded subsolutions and u_μ tends to its solution, or $\lim_{\mu \to 0} u_\mu(x) = -\infty$ and the ergodic limit μu_μ tends to the unique ergodic constant $c_0 \geq 0$ such that

$$\begin{cases} \dfrac{1}{2}\Delta v(x) - |\nabla v(x)|^\alpha - c_0 + f(x) = 0, & x \in D, \\ v(x) \to \infty, & \text{as } d(x) \to 0^+ \end{cases} \tag{2.80}$$

admits solutions. Furthermore $u_\mu + \|u_\mu\|_\infty$ tends to the unique solution v to (2.80) such that $\min_{x \in D} v(x) = 0$.

The analysis of this problem in our framework is outside the scope of this thesis, however here we will briefly discuss about it. In our case the problem is, roughly speaking, to find representation formula of viscosity solutions to

$$\frac{1}{2}\left\langle \sigma\sigma^\dagger, D^2 u(x)\right\rangle + H(x, \nabla u) - c_0 = 0, \qquad x \in D, \tag{2.81}$$

where $H(x, p) := H(x, 0, p)$, which explodes as $d(x)$ go to 0. The idea to do that is to use the same method employed in [36] studying the solutions u_ν to problem 2.6.1 with $G = 0$ as ν, and by remark 2.6.18 also μ, tends to 0. First of all notice that if (2.81) admits a solution u equal to 0 in ∂D, then $u + c$ is a solution to (2.81) equal to c in ∂D, hence, since theorem 2.6.2 is still true for $\mu = 0$, it can not have a solution which blow up near the boundary and is also finite in D. Proposition 2.6.4 holds for $\mu = 0$, but the lower bound defined in proposition 2.6.3 explodes, thus, if u_ν has a lower bound as ν tends to 0, we expect that its limit converges to the unique viscosity solution to (2.81) with $c_0 = 0$ which is equal to 0 in ∂D. If instead $u_\nu \to -\infty$ we would like to get a result similar to the one obtained by Porretta in [36], where $u_\nu + \|u_\nu\|_\infty$ tends to a viscosity solution to (2.81) with singular boundary conditions. What is important is to understand what happen to

the term $f\left(X_r^{\rho,\zeta}, Y_r^{\rho,\zeta}, q_r\right)$ in (2.60) as ν tends to 0, in particular if it tends to $f\left(X_r^{\rho,\zeta}, q_r\right) - c_0$ i.e. if we get an ergodic constant as for (2.80). As for the other problems previously covered in this section, the main difficulty comes from the lack of a uniform bounds for our controls, with the added difficulty that this time we can not use μ as in the proof of proposition 2.6.8 and theorem 2.6.19 to deal with the exit times.

As a final note we point out the reader that the ergodic problem is not new in the BSDE literature and it has been studied in different framework still related to PDE and optimal control problems in, among the others, [1, 14, 16, 24].

Appendix A

Stochastic Differential Equations

In this appendix we expose some results on stochastic differential equations, SDEs for short, that are needed to our study.

A.1   Basic Results on SDEs

We will work under these assumptions:

Assumptions A.1.1. Let $t \in [0, \infty)$, $\zeta \in L^2\left(\Omega, \mathcal{F}_t; \mathbb{R}^N\right)$ and

$$b : [0, \infty) \times \Omega \times \mathbb{R}^N \to \mathbb{R}^N \quad \text{and} \quad \sigma : [0, \infty) \times \Omega \times \mathbb{R}^N \to \mathbb{R}^{N \times M}$$

be two functions which admit two positive constants ℓ, ν and a real number μ such that a.e., for any $s \in [0, \infty)$ and $x, y \in \mathbb{R}^N$,

(i) $(x - y)(b(s, x) - b(s, y)) \leq \mu |x - y|^2$;

(ii) $|b(s, x)| \leq \nu(1 + |x|)$;

(iii) $|\sigma(s, x) - \sigma(s, y)| \leq \ell |x - y|$;

(iv) $t \mapsto (b, \sigma)(t, 0) \in \mathbb{L}_N^2 \times \mathbb{L}_{N \times M}^2$.

Notice that under these assumptions (b, σ) belongs to the set of measurable functions such that, for any $k \in [0, \infty)$,

$$\mathbb{E}\left(\int_0^k \int_{B_k(0)} \left(|b|^2 + |\sigma|^2\right)(t, x) dx dt\right) < \infty.$$

Let $\mathscr{L}$ denote this set and endow it with the topology induced by the convergence on compact sets.

Definition A.1.2. We say that $X^{t,\zeta}$ is a solution to the SDE (b,σ) if, for the initial conditions $t \in [0,\infty)$ and $\zeta \in L^2\left(\Omega, \mathcal{F}_t; \mathbb{R}^N\right)$, the identity

$$X_s^{t,\zeta} = \zeta + \int_t^s \sigma\left(r, X_r^{t,\zeta}\right) dW_r + \int_t^s b\left(r, X_r^{t,\zeta}\right) dr, \qquad s \in [t,\infty), \quad \text{(A.1)}$$

is true and $X^{t,\zeta}$ is a continuous process in $\mathbb{L}_N^2$. We usually write X to denote the solution to the SDE (b,σ) for any initial conditions $t \in [0,\infty)$ and $\zeta \in L^2\left(\Omega, \mathcal{F}_t; \mathbb{R}^N\right)$.

We state here some classical result which can be found in many probability textbooks, such as [4, 19, 37]. We start with an existence and uniqueness theorem.

Theorem A.1.3 (Existence and uniqueness). *Under assumptions A.1.1 the SDE* (A.1) *has a unique solution for any* $t \in [0,\infty)$ *and* ζ *in* $L^2\left(\Omega, \mathcal{F}_t; \mathbb{R}^N\right)$.

The next one is a continuity result which will be used to study the continuity of our viscosity solutions.

Theorem A.1.4 (Continuity). *Let* X *and* $\hat{X}$ *be the solutions to the SDEs* (b,σ) *and* $\left(\hat{b}, \hat{\sigma}\right)$ *under assumptions A.1.1 with constants* ℓ, μ, ν *and* $\hat{\ell}$, $\hat{\mu}$, $\hat{\nu}$ *respectively. Then, for any* $\gamma > \hat{\ell}^2 + 2\hat{\mu}$, *there exists a positive c, depending upon* $\hat{\ell}$, $\hat{\mu}$ *and* γ, *such that, for any* $t \in [0,\infty)$, $\zeta, \hat{\zeta} \in L^2\left(\Omega, \mathcal{F}_t; \mathbb{R}^N\right)$ *and* $T \in [t,\infty)$,

$$\mathbb{E}\left(\sup_{s\in[t,T]} e^{-\gamma s}\left|X_s^{t,\zeta} - \hat{X}_s^{t,\hat{\zeta}}\right|^2 + \int_t^T e^{-\gamma s}\left|X_s^{t,\zeta} - \hat{X}_s^{t,\hat{\zeta}}\right|^2 ds\right)$$

$$\leq c\mathbb{E}\left(e^{-\gamma t}\left|\zeta - \hat{\zeta}\right|^2 + \int_t^T e^{-\gamma s}\left|b\left(s, X_s^{t,\zeta}\right) - \hat{b}\left(s, X_s^{t,\zeta}\right)\right|^2 ds\right)$$

$$+ c\mathbb{E}\left(\int_t^T e^{-\gamma s}\left|\sigma\left(s, X_s^{t,\zeta}\right) - \hat{\sigma}\left(s, X_s^{t,\zeta}\right)\right|^2 ds\right).$$

From the previous theorem follows this important corollary.

Corollary A.1.5. *Under assumptions [A.1.1](#), for any $\gamma > \ell^2 + 2\mu$, there exists a positive constant c, depending upon ℓ, μ and γ, such that, for any $t \in [0, \infty)$, $\zeta \in L^2\left(\Omega, \mathcal{F}_t; \mathbb{R}^N\right)$ and $T \in [t, \infty)$,*

$$\mathbb{E}\left(\sup_{s \in [t,T]} e^{-\gamma s}\left|X_s^{t,\zeta} - \zeta\right|^2 + \int_t^T e^{-\gamma s}\left|X_s^{t,\zeta} - \zeta\right|^2\right)$$

$$\leq c\mathbb{E}\left(\int_t^T e^{-\gamma s}\left(|b(s,0)|^2 + |\sigma(s,0)|^2\right)ds\right),$$

$$\mathbb{E}\left(\sup_{s \in [t,T]} e^{-\gamma s}\left|X_s^{t,\zeta}\right|^2 + \int_t^T e^{-\gamma s}\left|X_s^{t,\zeta}\right|^2\right)$$

$$\leq c\mathbb{E}\left(e^{-\gamma t}|\zeta|^2 + \int_t^T e^{-\gamma s}\left(|b(s,0)|^2 + |\sigma(s,0)|^2\right)ds\right).$$

As seen in assumptions [A.1.1](#), we have a topological structure on the set of the admissible generators (b, σ) inherited by $\mathscr{L}$. Thus, given the continuity result stated in theorem [A.1.4](#), is natural to ask if there is some sort of continuity for the solutions to the SDEs with respect to this topology. In the next proposition we give a positive answer to this question under some additional assumptions.

Proposition A.1.6. *Let $\{(b_n, \sigma_n)\}_{n \in \mathbb{N}}$ and (b, σ) be, respectively, a sequence and an element in $\mathscr{L}$ both satisfying assumptions [A.1.1](#) and denote with X_n and X the solutions to the SDEs (b_n, σ_n) and (b, σ) respectively. Assume that b, σ, $\{b_n\}$ and $\{\sigma_n\}$ are equi–Lipschitz continuous in the third variable with Lipschitz constant ℓ and that, as n tends to ∞, (b_n, σ_n) tends to (b, σ) in $\mathscr{L}$. Then, given a $\gamma > \ell^2 + 2\ell$ and an a.e. finite stopping time τ such that*

$$\sup_{n \in \mathbb{N}} \mathbb{E}\left(\int_0^\tau e^{\gamma(\tau - s)}\left(|b(s,0)|^2 + |\sigma(s,0)|^2 + |b_n(s,0)|^2 + |\sigma_n(s,0)|^2\right)ds\right) < \infty,$$

$$\lim_{n \to \infty} \mathbb{E}\left(\sup_{s \in [t,\infty)}\left|X_{n,s}^{t,\zeta} - X_s^{t,\zeta}\right|^2 \chi_{\{\tau \geq s\}}\right) = 0$$

for any $t \in [0, \infty)$ and $\zeta \in L^2\left(\Omega, \mathcal{F}_t; \mathbb{R}^N\right)$ such that

$$\mathbb{E}\left(e^{\gamma(\tau - t)}|\zeta|^2\right) < \infty. \tag{A.2}$$

Proof. To ease notation we preliminarily assume that b and each b_n are identically equal to 0. The general case can be proved similarly.

Fixed $t \in [0, \infty)$ and $\zeta \in L^2\left(\Omega, \mathcal{F}_t; \mathbb{R}^N\right)$, theorem A.1.4 and corollary A.1.5 yield that there exists a constant c satisfying, for each $n \in \mathbb{N}$,

$$
\mathbb{E}\left(\sup_{s \in [t, \infty)} \left|X_{n,s}^{t,\zeta} - X_s^{t,\zeta}\right|^2 \chi_{\{\tau \geq s\}}\right)
$$

$$
\leq c\mathbb{E}\left(\int_t^\tau e^{\gamma(\tau - s)} \left|\sigma\left(s, X_s^{t,\zeta}\right) - \sigma_n\left(s, X_s^{t,\zeta}\right)\right|^2 ds\right)
$$

$$
\leq 3c\mathbb{E}\left(\int_t^\tau e^{\gamma(\tau - s)} \left(|\sigma(s,0)|^2 + |\sigma_n(s,0)|^2 + 4\ell^2 \left|X_s^{t,\zeta}\right|^2\right) ds\right)
$$

$$
\leq 3c\mathbb{E}\left(4c\ell^2 e^{\gamma(\tau - t)}|\zeta|^2 + \int_t^\tau e^{\gamma(\tau - s)} \left(\left(1 + 4c\ell^2\right)|\sigma(s,0)|^2 + |\sigma_n(s,0)|^2\right) ds\right),
$$

which by our hypothesis is equibounded with respect to n. Thus there exist, for each $m \in \mathbb{N}$, a $T_m > 0$ and a compact set $K_m \subset \mathbb{R}^N$ such that, if $l \geq m$, $T_m \leq T_l$, $K_m \subseteq K_l$ and, for any $n \in \mathbb{N}$,

$$
\mathbb{E}\left(\sup_{s \in [t, \infty)} \left|X_{n,s}^{t,\zeta} - X_s^{t,\zeta}\right|^2 \chi_{\{\tau \geq \max\{T_m, s\}\}}\right) < \frac{1}{m},
$$
$$
\mathbb{E}\left(\sup_{s \in [t, \infty)} \left|X_{n,s}^{t,\zeta} - X_s^{t,\zeta}\right|^2 \chi_{\{T_m \geq \tau \geq s\}} \chi_{\left\{X_s^{t,\zeta} \notin K_m\right\}}\right) < \frac{1}{m}.
$$
(A.3)

Let $\chi_s := \chi_{\{T_m \geq \tau \geq s\}} \chi_{\left\{X_s^{t,\zeta} \in K_m\right\}}$ and c_δ be the volume of the ball with radius δ in $\mathbb{R}^N$, then by theorem A.1.4 we have

$$
\mathbb{E}\left(\sup_{s \in [t, \infty)} \left|X_{n,s}^{t,\zeta} - X_s^{t,\zeta}\right|^2 \chi_s\right)
$$

$$
\leq c\mathbb{E}\left(\int_t^\tau e^{\gamma(\tau - s)} \left|\sigma\left(s, X_s^{t,\zeta}\right) - \sigma_n\left(s, X_s^{t,\zeta}\right)\right|^2 \chi_s ds\right)
$$

$$
= \frac{c}{c_\delta} \mathbb{E}\left(\int_t^\tau \int_{B_\delta\left(X_s^{t,\zeta}\right)} e^{\gamma(\tau - s)} \left|\sigma\left(s, X_s^{t,\zeta}\right) - \sigma_n\left(s, X_s^{t,\zeta}\right)\right|^2 \chi_s dx ds\right)
$$

$$
\leq \frac{3c}{c_\delta} \mathbb{E}\left(\int_t^\tau \int_{B_\delta\left(X_s^{t,\zeta}\right)} e^{\gamma(\tau - s)} \left|\sigma\left(s, X_s^{t,\zeta}\right) - \sigma(s, x)\right|^2 \chi_s dx ds\right)
$$

$$
+ \frac{3c}{c_\delta} \mathbb{E}\left(\int_t^\tau \int_{B_\delta\left(X_s^{t,\zeta}\right)} e^{\gamma(\tau - s)} |\sigma(s, x) - \sigma_n(s, x)|^2 \chi_s dx ds\right)
$$

$$+ \frac{3c}{c_\delta} \mathbb{E} \left(\int_t^\tau \int_{B_\delta\left(X_s^{t,\zeta}\right)} e^{\gamma(\tau-s)} \left| \sigma_n(s,x) - \sigma_n\left(s, X_s^{t,\zeta}\right) \right|^2 \chi_s dx ds \right)$$

$$\leq \frac{3ce^{T_m}}{c_\delta} \mathbb{E} \left(\int_t^{T_m} \int_{B_\delta(K_m)} |\sigma(s,x) - \sigma_n(s,x)|^2 \chi_s dx ds \right)$$

$$+ 6c\ell^2 \delta^2 \int_t^{T_m} e^{\gamma(T_m-s)} ds.$$

Therefore, since σ_n converge to σ in $\mathscr{L}$, we can choose a δ_m and a N_m such that

$$\mathbb{E} \left(\sup_{s \in [t,\infty)} \left| X_{n,s}^{t,\zeta} - X_s^{t,\zeta} \right|^2 \chi_s \right) < \frac{1}{m}$$

for any $n > N_m$. This, together with (A.3), conclude the proof. □

Remark A.1.7. Notice that if the conditions (A.2) and

$$\sup_{n \in \mathbb{N}} \mathbb{E} \left(\int_0^\tau e^{\gamma(\tau-s)} \left(|b(s,0)|^2 + |\sigma(s,0)|^2 + |b_n(s,0)|^2 + |\sigma_n(s,0)|^2 \right) ds \right) < \infty$$

in proposition A.1.6 does not hold, we can use the last part of its proof to obtain that $\sup_{s \in [t,\infty)} \left| X_{n,s}^{t,\zeta} - X_s^{t,\zeta} \right|^2 \chi_{\{\tau \geq s\}}$ converge to 0 in probability.

Remark A.1.8. The results obtained in this section and in the next one hold even for SDEs with an a.e. finite stopping time τ as starting time. In fact if for any $\zeta \in L^2 \left(\Omega, \mathcal{F}_\tau; \mathbb{R}^N \right)$ we define

$$\bar{b}(t,x) := b(t, x+\zeta) \chi_{\{\tau \leq t\}} \text{ and } \bar{\sigma}(t,x) := \sigma(t, x+\zeta) \chi_{\{\tau \leq t\}},$$

then $X^{\tau,\zeta}$ is solution of the SDE (b, σ) if and only if $\overline{X}^{0,0} := X^{\tau,\zeta} - \zeta$ is solution of the SDE $(\bar{b}, \bar{\sigma})$.

Remark A.1.9. By the strong Markov property, for any a.e. finite stopping time τ, the processes $\{W_t^\tau\}_{t \in [0,\infty)} := \{W_{\tau+t} - W_\tau\}_{t \in [0,\infty)}$ are Brownian motions. Thus if b and σ are progressive with respect to the filtration $\{\mathcal{F}_t^\tau\}_{t \in [0,\infty)}$ then any solution to the SDE (b, σ) with initial data $\tau + t$ and $\zeta \in L^2 \left(\Omega, \mathcal{F}_t^\tau; \mathbb{R}^N \right)$ is also progressive with respect to that filtration. In fact, in this case, the stochastic integral with respect to W_t^τ is the same that the one with respect to $W_{\tau+t}$.

A.2   The Exit Times

In this section we will study a special kind of stopping times: the exit times of an SDE X from a set D, or more precisely, for t and x varying respectively in $[0, \infty)$ and $\mathbb{R}^N$, the stopping times

$$\tau^{t,x} := \inf \left\{ s \in [0, \infty) : X_{t+s}^{t,x} \notin D \right\}.$$

To ease notation we additionally define $\tau^x := \tau^{0,x}$.

We will usually require that, besides assumptions A.1.1, the SDEs will also satisfies, for any $(t, x) \in [0, \infty) \times \mathbb{R}^N$, the following condition:

$$\text{eig}_{(\sigma\sigma^\dagger)(t,x)} \subset [2\lambda, \infty)^N, \qquad |b(t,x)| \leq \ell, \tag{A.4}$$

and sometimes even

$$|\sigma(t,x)|^2 \leq 2\ell. \tag{A.5}$$

where λ is a positive constant and ℓ is, for simplicity, the same ℓ in assumptions A.1.1.

The following results are already known, see for example [3, 33]. What we add is a sort of uniformity with respect to the initial conditions, which is needed to our study of elliptic PDEs.

Proposition A.2.1. *Let D be a bounded set, X the solution to the SDE (b, σ) under assumptions A.1.1 and (A.4) and τ the exit time of X from D. Then there exists a constant c, which depends only on D, ℓ and λ, such that $\mathbb{E}\left(\tau^{t,x}\right) \leq c$ for any $t \in [0, \infty)$ and $x \in \mathbb{R}^N$.*

In particular $\tau^{t,x}$ is a.e. finite and for any positive ε there exists a $T \in [0, \infty)$, which depends only on D, ℓ, λ and ε, such that $\mathbb{P}\left(\tau^{t,x} \geq T\right) < \varepsilon$ for any $t \in [0, \infty)$ and $x \in \mathbb{R}^N$.

Proof. We will just study the case $x \in D$, since if $x \notin D$ then $\tau^{t,x} = 0$ for any $t \in [0, \infty)$. Thus fix $(t, x) \in [0, \infty) \times D$ and let X^1 be the first component of the vector $X^{t,x}$. Since D is bounded there exists a positive constant δ such that if $X_s^1 \notin [-\delta, \delta]$ then $X_s^{t,x} \notin D$. Hence, if we define

$$\tau_\delta := \inf \left\{ s \in [0, \infty) : X_{t+s}^1 \notin [-\delta, \delta] \right\},$$

by the arbitrariness of x and t will be enough to prove that $\mathbb{E}(\tau_\delta) \leq c$ for some constant c.

Thanks to Itô's formula we have, for any positive γ and s,

$$e^{\gamma x_1} = \mathbb{E}\left(e^{\gamma X_{t+s\wedge\tau_\delta}^1} - \int_t^{t+s\wedge\tau_\delta} \left(\frac{1}{2}\left|\sigma_1\left(r, X_r^1\right)\right|^2 \gamma^2 + b_1\left(r, X_r^1\right)\gamma\right) e^{\gamma X_r^1} dr\right),$$

where σ_1 denote the vector $(\sigma_{1,i})$, therefore, if we choose $\gamma > \dfrac{\ell}{\lambda}$, (A.4) yields

$$\begin{aligned}
e^{\gamma x_1} &\leq \mathbb{E}\left(e^{\gamma X_{t+s\wedge\tau_\delta}^1} - \left(\lambda\gamma^2 - \ell\gamma\right)\int_t^{t+s\wedge\tau_\delta} e^{\gamma X_r^1} dr\right)\\
&\leq \mathbb{E}\left(e^{\gamma X_{t+s\wedge\tau_\delta}^1} - \left(\lambda\gamma^2 - \ell\gamma\right) s \wedge \tau_\delta \inf_{x\in D} e^{\gamma x_1}\right).
\end{aligned}$$

Finally we can use the dominated and monotone convergence theorems to obtain

$$\left(\lambda\gamma^2 - \ell\gamma\right)\inf_{x\in D} e^{\gamma x_1}\mathbb{E}(\tau_\delta) \leq e^{\gamma\delta} - \inf_{x\in D} e^{\gamma x_1},$$

which implies $\mathbb{E}(\tau_\delta) \leq c$, where c is a constant depending on D, ℓ and λ. We can now conclude the proof using the Markov's inequality: fixed $\varepsilon > 0$ we have, for T big enough,

$$\mathbb{P}\left(\tau^{t,x} \geq T\right) \leq \frac{\mathbb{E}\left(\tau^{t,x}\right)}{T} \leq \frac{c}{T} < \varepsilon,$$

for any $t \in [0,\infty)$ and $x \in \mathbb{R}^N$. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proposition A.2.2. *Let D be a bounded set, X the solution to the SDE (b,σ) under assumptions A.1.1 and (A.4), and τ the exit time of X from D. Then there exists a positive ϑ, which depends only on D, ℓ and λ, such that*

$$\sup_{x\in D}\mathbb{E}\left(e^{\vartheta\tau^x}\right) < \infty.$$

Proof. By proposition A.2.1 we know that there is a positive T, which depends only on D, ℓ and λ, such that

$$\sup_{(t,x)\in[0,\infty)\times D}\mathbb{P}\left(\tau^{t,x} \geq T\right) < \frac{1}{2}.$$

Moreover we have that, for any $(t, x) \in [0, \infty) \times \mathbb{R}^N$ and $n \in \mathbb{N}$,

$$
\begin{aligned}
\mathbb{P}\left(\tau^{t,x} \geq nT\right) &= \mathbb{E}\left(\chi_{\{\tau^{t,x} \geq nT\}}\right) = \mathbb{E}\left(\chi_{\{\tau^{t,x} \geq T\}}\chi_{\{\tau^{t,x} \geq nT\}}\right) \\
&= \mathbb{E}\left(\chi_{\{\tau^{t,x} \geq T\}}\mathbb{E}\left(\chi_{\{\tau^{t,x} \geq nT\}}\big|X_{t+T}^{t,x}\right)\right) \\
&= \mathbb{E}\left(\chi_{\{\tau^{t,x} \geq T\}}\mathbb{E}\left(\chi_{\{\tau^{t+T,y} \geq (n-1)T\}}\big|X_{t+T}^{t,x} = y\right)\right) \\
&\leq \mathbb{E}\left(\chi_{\{\tau^{t,x} \geq T\}}\right)\sup_{y\in D}\mathbb{E}\left(\chi_{\{\tau^{t+T,y} \geq (n-1)T\}}\right) \\
&= \mathbb{P}\left(\tau^{t,x} \geq T\right)\sup_{y\in D}\mathbb{P}\left(\tau^{t+T,y} \geq (n-1)T\right) \\
&< \frac{1}{2}\sup_{y\in D}\mathbb{P}\left(\tau^{t+T,y} \geq (n-1)T\right),
\end{aligned}
$$

hence by induction

$$
\sup_{(t,x)\in[0,\infty)\times D}\mathbb{P}\left(\tau^{t,x} \geq nT\right) < \frac{1}{2^n}.
$$

Finally, if we choose a positive ϑ such that $\dfrac{e^{\vartheta T}}{2} < 1$, the previous inequality yields, for any $(t, x) \in [0, \infty) \times \mathbb{R}^N$,

$$
\begin{aligned}
\mathbb{E}\left(e^{\vartheta\tau^{t,x}}\right) &\leq \sum_{n=1}^{\infty}e^{\vartheta nT}\mathbb{P}\left(\tau^{t,x} \in [(n-1)T, nT)\right) \leq \sum_{n=1}^{\infty}e^{\vartheta nT}\mathbb{P}\left(\tau^{t,x} \geq (n-1)T\right) \\
&< e^{\vartheta T}\sum_{n=0}^{\infty}\left(\frac{e^{\vartheta T}}{2}\right)^n,
\end{aligned}
$$

which implies our statement. $\qquad\square$

Proposition A.2.3. *Let $\varphi : [0, T] \to \mathbb{R}^N$ be a continuous function such that $\varphi(0) = x \in \mathbb{R}^N$, $X^{0,x}$ the solution to the SDE (b, σ) under assumptions A.1.1, (A.4) and (A.5), and ε a positive constant. Then there is a positive constant c, depending only on ε, λ, ℓ, T and the modulus of continuity of φ, such that*

$$
\mathbb{P}\left(\sup_{t\in[0,T]}\left|X_t^{0,x} - \varphi(t)\right| < \varepsilon\right) \geq c.
$$

Proof. The proof of this is the same of [3, Theorem I.8.5]. $\qquad\square$

Theorem A.2.4. *Let D be a bounded set, X the solution to the SDE (b, σ) under assumptions A.1.1 and (A.4), τ the exit time of X from D and $\overline{\tau}$ the exit time of X from $\overline{D}$. Assume that, for some $t \in [0, \infty)$, $\mathbb{P}\left(\tau^{t,x} = \overline{\tau}^{t,x}\right) = 1$ for any $x \in \mathbb{R}^N$. Then, fixed two positive constants ε and α we have that there exists a $\delta > 0$, depending on D, μ, ℓ, λ, ε and α, such that*

$$\mathbb{P}\left(\left|\tau^{t,x} - \tau^{t,y}\right| > \alpha\right) < \varepsilon$$

for any $x \in \mathbb{R}^N$ and $y \in B_\delta(x)$. Therefore τ^t is continuous in probability with respect to x.

Proof. To ease notation assume that $t = 0$ and restrict x in $\overline{D}$, since for $x \notin \overline{D}$ this is obviously true. We start fixing $x \in \overline{D}$, defining for any $y \in \overline{D}$

$$\tau_\beta^y := \inf\left\{t \in [0, \infty) : \inf_{z \in D} |X_t^y - z| \geq \beta\right\}$$

and noting that

$$
\begin{aligned}
\mathbb{P}(|\tau^x - \tau^y| > \alpha) =& \mathbb{P}(\{|\tau^x - \tau^y| > \alpha\} \cap \{\tau^x \geq T\}) \\
& + \mathbb{P}\left(\{\tau^y > \tau^x + \alpha\} \cap \{\tau_\beta^x > \tau^x + \alpha\} \cap \{\tau^x < T\}\right) \\
& + \mathbb{P}\left(\{\tau^y > \tau^x + \alpha\} \cap \{\tau_\beta^x \leq \tau^x + \alpha\} \cap \{\tau^x < T\}\right) \\
& + \mathbb{P}\left(\{\tau^x > \tau^y + \alpha\} \cap \{\tau_\beta^y > \tau^y + \alpha\} \cap \{\tau^x < T\}\right) \\
& + \mathbb{P}\left(\{\tau^x > \tau^y + \alpha\} \cap \{\tau_\beta^y \leq \tau^y + \alpha\} \cap \{\tau^x < T\}\right) \\
\leq& \mathbb{P}(\tau^x \geq T) + \mathbb{P}\left(\tau_\beta^x > \tau^x + \alpha\right) \\
& + \mathbb{P}\left(\{\tau^y > \tau_\beta^x\} \cap \{\tau_\beta^x < T + \alpha\}\right) + \mathbb{P}\left(\tau_\beta^y > \tau^y + \alpha\right) \\
& + \mathbb{P}\left(\{\tau^x > \tau_\beta^y\} \cap \{\tau_\beta^y < T\}\right),
\end{aligned}
$$

where the last term is a consequence of the inequality

$$
\begin{aligned}
\mathbb{P}\left(\{\tau^x > \tau^y + \alpha\} \cap \{\tau_\beta^y \leq \tau^y + \alpha\} \cap \{\tau^x < T\}\right) \\
= \mathbb{P}\left(\{\tau^x > \tau^y + \alpha\} \cap \{\tau_\beta^y \leq \tau^y + \alpha\} \cap \{\tau^x < T\} \cap \{\tau^y < T - \alpha\}\right) \\
\leq \mathbb{P}\left(\{\tau^x > \tau_\beta^y\} \cap \{\tau_\beta^y < T\}\right).
\end{aligned}
$$

By proposition A.2.1 we can take a positive T depending only on D, ℓ, λ and ε such that $\mathbb{P}(\tau^x \geq T) < \dfrac{\varepsilon}{5}$. Similarly we can choose a β, depending on α and ε, such that $\mathbb{P}\left(\tau^y_\beta > \tau^y + \alpha\right) < \dfrac{\varepsilon}{5}$ for any $y \in \overline{D}$, in fact if that would not be true we should have, thanks to the reverse Fatou's lemma,

$$\mathbb{P}(\overline{\tau}^y > \tau^y + \alpha) \geq \limsup_{\beta \to 0} \mathbb{P}\left(\tau^y_\beta > \tau^y + \alpha\right) \geq \frac{\varepsilon}{5}$$

for some $y \in \overline{D}$, in contradiction with our hypothesis.

Now, for the last terms, we can use Markov's inequality and theorem A.1.4 to get

$$\mathbb{P}\left(\{\tau^y > \tau^x_\beta\} \cap \{\tau^x_\beta < T + \alpha\}\right) \leq \mathbb{P}\left(\left\{\left|X^x_{\tau^x_\beta} - X^y_{\tau^x_\beta}\right| \geq \beta\right\} \cap \{\tau^x_\beta < T + \alpha\}\right)$$

$$\leq \frac{1}{\beta^2} \mathbb{E}\left(\sup_{t \in [0, T+\alpha]} |X^x_t - X^y_t|^2\right)$$

$$\leq \frac{c_1}{\beta^2}|x - y|^2,$$

where c_1 depends only on T, α, μ and ℓ, and similarly

$$\mathbb{P}\left(\{\tau^x > \tau^y_\beta\} \cap \{\tau^y_\beta < T\}\right) \leq \frac{c_2}{\beta^2}|x - y|^2,$$

where c_2 depends only on T, μ and ℓ. Therefore there exists a $\delta > 0$ depending on D, μ, ℓ, λ, ε and α such that

$$\mathbb{P}\left(\{\tau^y > \tau^x_\beta\} \cap \{\tau^x_\beta < T + \alpha\}\right) + \mathbb{P}\left(\{\tau^x > \tau^y_\beta\} \cap \{\tau^y_\beta < T\}\right) \leq \frac{2\varepsilon}{5}$$

for any $y \in B_\delta(x)$ and consequently $\mathbb{P}\left(|\tau^{t,x} - \tau^{t,y}| > \alpha\right) < \varepsilon$. $\square$

We point out that under the same assumptions and with a similar proof, we can prove that the exit times are also continuous in probability with respect to b and σ. More precisely:

Theorem A.2.5. *Let D be a bounded set, $\mathcal{A}$ be a set made of equi–Lipschitz continuous (b, σ) as in assumptions A.1.1 and denote with $X_{(b,\sigma)}$ the solution to an SDE (b, σ), with $\tau_{(b,\sigma)}$ the exit time of $X_{(b,\sigma)}$ from D and with $\overline{\tau}_{(b,\sigma)}$ the exit time of $X_{(b,\sigma)}$ from $\overline{D}$. Assume that there exist a $\left(\overline{b}, \overline{\sigma}\right)$ satisfying*

(A.4) *and a $\delta_0 > 0$ such that $P\left(\tau_{(b,\sigma)}^{t,x} = \overline{\tau}_{(b,\sigma)}^{t,x}\right) = 1$ holds true for some fixed $(t,x) \in [0,\infty) \times \mathbb{R}^N$ and for any solution to the SDEs (b,σ) with (b,σ) in $\mathcal{A} \cap B_{\delta_0}\left(\overline{b},\overline{\sigma}\right)$. Then, fixed two positive constants ε and α we have that there exists a $\delta \in (0,\delta_0]$, depending on D, μ, ℓ, λ, ε and α, such that*

$$\mathbb{P}\left(\left|\tau_{\left(\overline{b},\overline{\sigma}\right)}^{t,x} - \tau_{(b,\sigma)}^{t,x}\right| > \alpha\right) < \varepsilon$$

for any $(b,\sigma) \in \mathcal{A} \cap B_\delta\left(\overline{b},\overline{\sigma}\right)$. Therefore $\tau^{t,x}$ is locally continuous in probability at $\left(\overline{b},\overline{\sigma}\right)$ in $\mathcal{A}$.

Appendix B

Backward Stochastic Differential Equations

In this appendix we expose some results on backward stochastic differential equations, BSDEs for short, that are needed to our study. Most of these results are well known in the BSDE field and hold even under more general assumptions. We refer to [9, 27, 28, 32], among the others, for the proof of most of these results.

B.1 BSDEs with Deterministic Terminal Time

We will work under the followings assumptions:

Assumptions B.1.1. Let $T \in [0, \infty)$, $\xi \in L^2\left(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^M\right)$ and

$$f : [0, T] \times \Omega \times \mathbb{R}^M \times \mathbb{R}^{M \times N} \to \mathbb{R}^M$$

a function which admits two positive constants ℓ, ν and a real number μ such that a.e., for any $t \in [0, T]$, $y, y' \in \mathbb{R}^M$ and $z, z' \in \mathbb{R}^{M \times N}$,

(i) $s \mapsto f(s, 0, 0) \in \mathbb{L}_M^2(T)$;

(ii) $|f(t, y, z)| \leq |f(t, 0, z)| + \nu(1 + |y|)$;

(iii) $|f(t, y, z) - f(t, y, z')| \leq \ell|z - z'|$;

(iv) $(y - y')(f(t, y, z) - f(t, y', z)) \le \mu |y - y'|^2$;

(v) $v \mapsto f(t, v, z)$ is continuous.

Definition B.1.2. A solution to the BSDE (ξ, f, T) is a pair (Y, Z) of processes belonging to $\mathbb{L}_M^2(T) \times \mathbb{L}_{M \times N}^2(T)$ such that

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \quad \text{for any } t \in [0, T]. \qquad \text{(B.1)}$$

ξ, f and T are commonly referred as the terminal or final condition, the generator and the terminal or final time respectively.

The followings are classical results of BSDE theory which we rewrite here for the reader's comfort. We start with an existence and uniqueness result.

Theorem B.1.3 (Existence and uniqueness). *Under assumptions B.1.1 the BSDE (B.1) has a unique solution (Y, Z).*

We are particularly interested in the study of the relationship between the solutions to BSDEs and their initial data, hence the next results will focus on that.

Theorem B.1.4 (Continuity). *Let (Y, Z) and (Y', Z') be the solutions to the BSDEs (ξ, f, T) and (ξ', f', T) under assumptions B.1.1 with constants ℓ, ν, μ and ℓ', ν', μ' respectively. Then, for any $\gamma > \ell'^2 + 2\mu'$, there exists a constant c, which depends upon ℓ', μ' and γ, such that*

$$\mathbb{E} \left(\sup_{t \in [0,T]} e^{\gamma t} |Y_t - Y_t'|^2 + \int_0^T e^{\gamma t} \left(|Y_t - Y_t'|^2 + |Z_t - Z_t'|^2 \right) dt \right)$$

$$\le c \mathbb{E} \left(e^{\gamma T} |\xi - \xi'|^2 + \int_0^T e^{\gamma t} |f(t, Y_t, Z_t) - f'(t, Y_t, Z_t)|^2 dt \right).$$

In particular there exists a constant C, which depends upon T, ℓ' and μ', such that

$$\mathbb{E} \left(\sup_{t \in [0,T]} |Y_t - Y_t'|^2 + \int_0^T |Z_t - Z_t'|^2 dt \right)$$

$$\le C \mathbb{E} \left(|\xi - \xi'|^2 + \int_0^T |f(t, Y_t, Z_t) - f'(t, Y_t, Z_t)|^2 dt \right).$$

Corollary B.1.5 (Boundedness)**.** *Under assumptions B.1.1, if (Y, Z) is a solution to the BSDE (B.1), then there exists a constant c, which depends upon T, μ and ℓ, such that*

$$\mathbb{E}\left(\sup_{t\in[0,T]} |Y_t|^2 + \int_0^T |Z_t|^2 dt\right) \leq c\mathbb{E}\left(|\xi|^2 + \int_0^T |f(t,0,0)|^2 dt\right).$$

The next theorem is a comparison result, which we will generalize in the successive corollary, that play an important role in our study, but to prove it we need this lemma:

Lemma B.1.6. *Assume $M = 1$. Given (Y, Z) solution of the linear BSDE*

$$Y_t = \xi + \int_t^T (A_s Y_s + Z_s B_s + C_s) ds - \int_t^T Z_s dW_s,$$

where $C \in \mathbb{L}^2(T)$ and A, B are bounded progressive processes valued in $\mathbb{R}$ and $\mathbb{R}^N$ respectively, we have that

$$Y_t = \mathbb{E}\left(\Gamma_{t,T}\xi + \int_t^T \Gamma_{t,s} C_s ds \,\Big|\, \mathcal{F}_t\right),$$

where $\Gamma_{s,t} := e^{\int_s^t \left(A_r - \frac{|B_r|^2}{2}\right) dr + \int_s^t B_r dW_r}$ belongs to $\mathbb{L}^2(T)$.

Proof. Preliminarily note that $\Gamma_{s,t}$ is solution to the SDE

$$\Gamma_{s,t} = 1 + \int_s^t \Gamma_{s,r} A_r dr + \int_s^t \Gamma_{s,r} B_r dW_r.$$

By Itô's formula we get

$$\Gamma_{0,t} Y_t = \Gamma_{0,T}\xi + \int_t^T \Gamma_{0,s} C_s ds - \int_t^T \Gamma_{0,s}(Z_s + B_s Y_s) dW_s, \qquad (B.2)$$

then if we prove that $\left\{\int_0^t \Gamma_{0,s}(Z_s + B_s Y_s) dW_s\right\}_{t\in[0,T]}$ is a martingale we can take the conditional expectation to obtain

$$\Gamma_{0,t} Y_t = \mathbb{E}\left(\Gamma_{0,T}\xi + \int_t^T \Gamma_{0,s} C_s ds \,\Big|\, \mathcal{F}_t\right)$$

and conclude the proof. Since by corollary A.1.5 $\mathbb{E}\left(\sup\limits_{t\in[0,T]}\Gamma_{0,t}^2\right) < \infty$, we have

$$\mathbb{E}\left(\left(\int_0^t \Gamma_{0,s}^2|Z_s + B_s Y_s|^2 ds\right)^{\frac{1}{2}}\right) \leq \frac{1}{2}\mathbb{E}\left(\sup_{t\in[0,T]}\Gamma_{0,t}^2 + 2\int_0^T \left(|B_t Y_t|^2 + |Z_t|^2\right)dt\right)$$
$$< \infty,$$

thus the local martingale in (B.2) is a martingale. $\qquad\square$

Theorem B.1.7 (Comparison 1). *Assume $M = 1$ and suppose that $\xi \leq \xi'$ a.e., $f(t,y,z) \leq f'(t,y,z)$ $dt \times d\mathbb{P}$ a.e. for any $(y,z) \in \mathbb{R} \times \mathbb{R}^N$ and let (Y,Z) and (Y',Z') be the solutions of the BSDEs, under assumptions B.1.1, (ξ, f, T) and (ξ', f', T) respectively. Then $Y_t \leq Y_t'$ a.e. for any $t \in [0,T]$. Furthermore if $Y_0 = Y_0'$ a.e., then $Y_t = Y_t'$ a.e. for any $t \in [0,T]$, or in other words, whenever either $\mathbb{P}(\{\xi < \xi'\}) > 0$ or $f(t,y,z) < f'(t,y,z)$, for any (y,z) in $\mathbb{R} \times \mathbb{R}^N$ on a set of positive $dt \times d\mathbb{P}$ measure, then $Y_0 < Y_0'$ a.e..*

Proof. For $t \in [0,T]$, define

$$A_t := \begin{cases} \dfrac{f(t, Y_t', Z_t) - f(t, Y_t, Z_t)}{Y_t' - Y_t}, & \text{if } Y_t \neq Y_t', \\ 0, & \text{if } Y_t = Y_t' \end{cases}$$

and the $\mathbb{R}^N$ valued process

$$B_t^i := \begin{cases} \dfrac{f\left(t, Y_t', Z_t^{(i)}\right) - f\left(t, Y_t', Z_t^{(i-1)}\right)}{Z_t'^i - Z_t^i}, & \text{if } Z_t^i \neq Z_t'^i, \\ 0, & \text{if } Z_t^i = Z_t'^i, \end{cases}$$

where $Z_t^{(i)} := \left(Z_t'^1, \ldots, Z_t'^i, Z_t^{i+1}, \ldots, Z_t^N\right)$. We note that A and B are progressively measurable and $|A_t| \leq \mu$, $|B_t| \leq \sqrt{N}\ell$. Now, for $s,t \in [0,T]$, define $\Gamma_{s,t} := e^{\int_s^t\left(A_r - \frac{|B_r|^2}{2}\right)dr + \int_s^t B_r dW_r}$, $\left(\overline{Y}, \overline{Z}\right) := (Y' - Y, Z' - Z)$, $\overline{\xi} := \xi' - \xi$ and $C_t := f'(t, Y_t', Z_t') - f(t, Y_t', Z_t')$. Is easy to see that $\left(\overline{Y}, \overline{Z}\right)$ solves the linear BSDE

$$\overline{Y}_t = \overline{\xi} + \int_t^T \left(A_s \overline{Y}_s + \overline{Z}_s B_s + C_s\right) ds - \int_t^T \overline{Z}_s dW_s,$$

therefore, by lemma B.1.6,

$$\overline{Y}_t = \mathbb{E}\left(\Gamma_{t,T}\overline{\xi} + \int_t^T \Gamma_{t,s}C_s ds \middle| \mathcal{F}_t\right).$$

Since $\overline{\xi}$, $\Gamma_{t,s}$ and C are a.e. non negative, then $Y_t \leq Y_t'$ a.e. for any $t \in [0,T]$. In particular, if $Y_0 = Y_0'$ a.e., $\overline{\xi}$ and C are equal to 0 a.e. for the measures $\Gamma_{0,T}d\mathbb{P}$ and $\Gamma_{0,t}dt \times d\mathbb{P}$, respectively, which are equivalent to $\mathbb{P}$ and $dt \times d\mathbb{P}$. Then $Y_t = Y_t'$ a.e. for all $t \in [0,T]$, ending the proof. $\qquad\square$

Corollary B.1.8 (Comparison 2). *In the $M = 1$ case, let (Y,Z) be the solution to the BSDE (ξ, f, T) under the assumptions B.1.1 and*

$$Y_t' = \xi' + \int_t^T V_s ds - \int_t^T Z_s' dW_s, \qquad t \in [0,T],$$

where $\xi' \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$, $Y', V \in \mathbb{L}^2(T)$ and $Z' \in \mathbb{L}_N^2(T)$. Suppose that $\xi \leq \xi'$ a.e. and $f(t, Y_t', Z_t') \leq V_t$ $dt \times d\mathbb{P}$ a.e.. Then, for any $t \in [0,T]$, $Y_t \leq Y_t'$ a.e..

Furthermore if $Y_0 = Y_0'$ a.e., then $Y_t = Y_t'$ a.e. for any $t \in [0,T]$, or in other words, whenever either $\mathbb{P}(\{\xi < \xi'\}) > 0$ or $f(s, Y_s', Z_s') < V_s$ on a set of positive $dt \times d\mathbb{P}$ measure, then $Y_0 < Y_0'$ a.e..

Proof. If we define $f'(t, y, z) := f(t, y, z) + V_t - f(t, Y_t', Z_t')$, then the corollary is a consequence of theorem B.1.7. $\qquad\square$

We conclude this section with a result concerning bounded stopping time.

Proposition B.1.9. *Let (Y,Z) be the solution to the BSDE (B.1) and assume that there exists a stopping time τ such that $\tau \leq T$, ξ is $\mathcal{F}_\tau$−measurable and $f(t, y, z) = 0$ on the set $\{\tau < t\}$. Then, for any $t \in [0,T]$, $Y_t = Y_{\tau \wedge t}$ and $Z_t = 0$ a.e. on the set $\{\tau < t\}$.*

Thanks to this proposition we can check that any result we proved on this section is still true for BSDE with random bounded terminal time.

Proof. Since Y is progressive

$$Y_\tau = \mathbb{E}(Y_\tau|\mathcal{F}_\tau) = \mathbb{E}\left(\xi - \int_\tau^T Z_s dW_s \middle| \mathcal{F}_\tau\right) = \xi.$$

On the other hand, by Itô's formula,

$$|Y_\tau|^2 + \int_\tau^T |Z_s|^2 ds = |\xi|^2 - 2\int_\tau^T Y_s^\dagger Z_s dW_s,$$

therefore, since $|Y_\tau|^2 = |\xi|^2$, we have that $\mathbb{E}\left(\int_\tau^T |Z_s|^2 ds\right) = 0$, and conse-
quently $\int_\tau^T |Z_s|^2 ds = 0$ a.e.. This concludes the proof. $\qquad\square$

B.2 BSDEs with Random Terminal Time

This section contains some basic results on BSDEs with random terminal
time, which are needed for the study of elliptic PDEs.

Assume the followings:

Assumptions B.2.1. Let τ be a stopping time, $\xi \in L^2\left(\Omega, \mathcal{F}_\tau, \mathbb{P}; \mathbb{R}^M\right)$ and

$$f : [0,\infty) \times \Omega \times \mathbb{R}^M \times \mathbb{R}^{M\times N} \to \mathbb{R}^M$$

a function which admits two positive constants ℓ, ν and a real number μ such
that a.e., for any $t \in [0,\infty)$, $y, y' \in \mathbb{R}^M$ and $z, z' \in \mathbb{R}^{M\times N}$,

(i) $s \mapsto f(s,0,0) \in \mathbb{L}_M^2$;

(ii) $|f(t,y,z)| \leq |f(t,0,z)| + \nu(1+|y|)$;

(iii) $|f(t,y,z) - f(t,y,z')| \leq \ell|z - z'|$;

(iv) $(y - y')(f(t,y,z) - f(t,y',z)) \leq \mu|y - y'|^2$;

(v) $v \mapsto f(t,v,z)$ is continuous.

Furthermore let $\vartheta > \ell^2 + 2\mu$ be a constant such that

(vi) $\mathbb{E}\left(\int_0^\tau e^{\vartheta t}\left(1 + |f(t,0,0)|^2\right) dt\right) < \infty$;

(vii) $\mathbb{E}(e^{\vartheta\tau}|\xi|^2) < \infty$.

To ease notation we will also assume, without loss of generality, since we are not interested in the behavior of $f(t)$ in the set $\{\tau < t\}$ and we can just study the function $f(t)\chi_{\{\tau \geq t\}}$ instead,

(viii) $f(t, y, z) = 0$ on the set $\{\tau \leq t\}$ for any $(y, z) \in \mathbb{R}^M \times \mathbb{R}^{M \times N}$.

Definition B.2.2. A solution to the BSDE (ξ, f, τ) is a pair (Y, Z) of processes belonging to $\mathbb{L}_M^2 \times \mathbb{L}_{M \times N}^2$ such that

(i) $\lim_{t \to \infty} \mathbb{E}\left(e^{\vartheta t}|Y_t - \xi|^2\right) = 0$;

(ii) $(Y_t, Z_t) = (\xi, 0)$ on the set $\{\tau \leq t\}$;

(iii) $Y_t = Y_T + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s$, for any $T \geq t \geq 0$.

Remark B.2.3. Intuitively we are solving the BSDE

$$Y_t = \xi + \int_t^\tau f(s, Y_s, Z_s)ds - \int_t^\tau Z_s dW_s, \qquad \text{on } \{t \leq \tau\}, \qquad \text{(B.3)}$$

but the integrals here may not make sense on the set $\{\tau = \infty\}$. In fact, if $\mathbb{P}(\tau < \infty) = 1$, asking that a pair of processes $(Y, Z) \in \mathbb{L}_M^2 \times \mathbb{L}_{M \times N}^2$ satisfies (B.3) and item (ii) in definition B.2.2 is equivalent to the definition B.2.2. Moreover, for consistency reason, we will require that a solution to a BSDE with deterministic terminal time will also satisfy item (ii) in definition B.2.2, so definition B.2.2 coincides with definition B.1.2 in that case.

As in the previous section we will show existence and continuity results.

Theorem B.2.4 (Existence and uniqueness)**.** *Under assumptions B.2.1 the BSDE (ξ, f, τ) admits a unique solution.*

Theorem B.2.5 (Continuity)**.** *Let (Y, Z) and (Y', Z') be the solutions to the BSDEs (ξ, f, τ) and (ξ', f', τ') under assumptions B.2.1 with constants ℓ, ν, μ and ℓ', ν', μ' respectively, but same ϑ. Then, for any $\gamma \in (\ell'^2 + 2\mu', \vartheta]$, there exists a constant c, which depends upon ℓ', μ' and γ, such that*

$$\mathbb{E}\left(\sup_{t\in[0,\infty)}\left|e^{\frac{\gamma t\wedge\tau}{2}}Y_t - e^{\frac{\gamma t\wedge\tau'}{2}}Y_t'\right|^2 + \int_0^{\tau\vee\tau'} e^{\gamma t}\left(|Y_t - Y_t'|^2 + |Z_t - Z_t'|^2\right)dt\right)$$

$$\leq c\mathbb{E}\left(\left|e^{\frac{\gamma\tau}{2}}\xi - e^{\frac{\gamma\tau'}{2}}\xi'\right|^2 + \int_0^{\tau\vee\tau'} e^{\gamma t}|f(t,Y_t,Z_t) - f'(t,Y_t,Z_t)|^2 dt\right).$$

We point out that condition (viii) in assumptions B.2.1 allows us to properly express the previous inequality.

Corollary B.2.6 (Boundedness). *Let (Y,Z) be the solution to the BSDE (ξ,f,τ) under assumptions B.2.1. Then, for any $\gamma\in(\ell^2 + 2\mu,\vartheta]$, there exists a constant c, which depends upon ℓ, μ and γ, such that*

$$\mathbb{E}\left(\sup_{t\in[0,\infty)} e^{\gamma t\wedge\tau}|Y_t|^2 + \int_0^\tau e^{\gamma t}\left(|Y_t|^2 + |Z_t|^2\right)dt\right)$$

$$\leq c\mathbb{E}\left(e^{\gamma\tau}|\xi|^2 + \int_0^\tau e^{\gamma t}|f(t,0,0)|^2 dt\right).$$

Remark B.2.7. By definition

$$\xi_t = \mathbb{E}(\xi) + \int_0^t \eta_s dW_s = \xi - \int_t^\infty \eta_s dW_s,$$

therefore is easy to check that $(\xi_t,\eta_t)_{t\in[0,\infty)}$ is solution to the BSDE $(\xi,0,\tau)$ and consequently

$$\mathbb{E}\left(\sup_{t\in[0,\infty)} e^{\gamma t\wedge\tau}|\xi_t|^2 + \int_0^\tau e^{\gamma t}\left(|\xi_t|^2 + |\eta_t|^2\right)dt\right) \leq c\mathbb{E}\left(e^{\gamma\tau}|\xi|^2\right) < \infty,$$

where c is an opportune constant.

Even in this case we have that holds a characterization for linear BSDEs and a comparison result.

Lemma B.2.8. *Assume $M = 1$. Given (Y,Z) solution of the linear BSDE $(\xi, A_t y + zB_t + C_t, \tau)$ under assumptions B.2.1, where $C\in\mathbb{L}^2$ and A, B are bounded progressive processes valued in $\mathbb{R}$ and $\mathbb{R}^N$ respectively, we have that*

$$Y_t = \mathbb{E}\left(\Gamma_{t,\tau}\xi + \int_t^\tau \Gamma_{t,s}C_s ds \,\Big|\, \mathcal{F}_t\right),$$

where $\Gamma_{s,t} := e^{\int_s^t\left(A_r - \frac{|B_r|^2}{2}\right)dr + \int_s^t B_r dW_r}$ belongs to $\mathbb{L}^2(\infty)$.

Proof. Preliminarily note that by our assumptions $A \leq \mu$, $|B| \leq \ell$ and both, together with C, are equal to 0 when t is bigger than τ. Moreover $e^{-\vartheta(t \wedge \tau - s)} \Gamma_{s,t}$ is solution to the SDE

$$e^{-\vartheta(t \wedge \tau - s)} \Gamma_{s,t} = 1 + \int_s^t (A_r - \vartheta) e^{-\vartheta(t \wedge \tau - s)} \Gamma_{s,t} dr + \int_s^t e^{-\vartheta(t \wedge \tau - s)} \Gamma_{s,t} B_r dW_r,$$

thus, by corollary A.1.5, there exists a constant c depending on ϑ, ℓ and μ such that

$$\mathbb{E} \left(\sup_{t \in [s,\infty)} e^{-2\vartheta(t \wedge \tau - s)} \Gamma_{s,t}^2 + \int_s^\infty e^{-2\vartheta(r \wedge \tau - s)} \Gamma_{s,r}^2 dr \right) \leq c. \tag{B.4}$$

By lemma B.1.6, for any $T \in [t, \infty)$,

$$Y_t = \mathbb{E} \left(\Gamma_{t,T} Y_T + \int_t^T \Gamma_{t,s} C_s ds \,\Big|\, \mathcal{F}_t \right),$$

and by the Young's inequality

$$\Gamma_{t,T} Y_T \leq \frac{1}{2} e^{-2\vartheta(T \wedge \tau - t)} \Gamma_{t,T}^2 + \frac{1}{2} e^{2\vartheta(T \wedge \tau - t)} Y_T^2$$

and

$$\Gamma_{t,s} C_s \leq \frac{1}{2} e^{-2\vartheta(s \wedge \tau - t)} \Gamma_{t,s}^2 + \frac{1}{2} e^{2\vartheta(s \wedge \tau - t)} C_s^2.$$

Then this lemma follows from corollary B.2.6, item (i) in definition B.2.2, item (vi) in assumptions B.2.1, (B.4) and the dominated convergence theorem. $\qquad \square$

A comparison result follows from lemma B.2.8 as corollary B.1.8 follows from lemma B.1.6.

Theorem B.2.9 (Comparison). *In the $M = 1$ case, let (Y, Z) be the solution to the BSDE (ξ, f, τ) under the assumptions B.2.1, $\xi' \in L^2(\Omega, \mathcal{F}_\tau, \mathbb{P}; \mathbb{R})$, $V \in \mathbb{L}^2(\tau)$ and assume that for some positive constant ϑ*

$$\mathbb{E} \left(\int_0^\tau e^{2\vartheta t} \left(1 + |V_t|^2 \right) dt \right) < \infty \quad and \quad \mathbb{E} \left(e^{2\vartheta \tau} |\xi'|^2 \right) < \infty.$$

Define (Y', Z') as the solution to the BSDE (ξ', V, τ) and suppose that $\xi \leq \xi'$ a.e. and $f(t, Y_t', Z_t') \leq V_t$ $dt \times d\mathbb{P}$ a.e.. Then, for any $t \in [0, \infty)$, $Y_t \leq Y_t'$

a.e..

Furthermore if $Y_0 = Y_0'$ a.e., then $Y_t = Y_t'$ a.e. for any $t \in [0, \infty)$, or in other words, whenever either $\mathbb{P}(\{\xi < \xi'\}) > 0$ or $f(s, Y_s', Z_s') < V_s$ on a set of positive $dt \times d\mathbb{P}$ measure, then $Y_0 < Y_0'$ a.e..

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