On unconstrained MPC through multirate sampling ⋆

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Abstract: This paper formally highlights how the multirate sampled data equivalent model can be exploited for prediction in an MPC formulation in order to mitigate the possible instability arising from an MPC design while ensuring prefixed boundedness of the control amplitude. This last aspect is in particular addressed and solved with reference to the class of systems which admit, under feedback, a computable sampled model.

Keywords: Control of Sampled-Data Systems; Model-Predictive; Optimal Control.

1. INTRODUCTION

Model Predictive Control (MPC) has become a widely investigated research area in linear and nonlinear control and their applications (e.g., Boucher and Dumur (1996); Camacho and Alba (2013); Borrelli et al. (2017); Kwon et al. (1982)). Roughly speaking the control action is designed by solving a constrained optimization problem subject to the system dynamics and possibly additional requirements and bounds.

The simplest way to design and implement the resulting control law makes use of a sampled data model of the plant for prediction that is exploited for solving the optimization problem over a finite prediction horizon of length n_p. In this context, one implicitly assumes the dependence of the predicted future values of the output over n_c future controls actions, generally referred to as the control horizon.

Driving the output trajectory to a desired reference at the sampling instants via discrete time model predictive control while preserving stability in closed loop may induce difficulties especially when the plant (and the model used for prediction) is not minimum phase. In that case, the choice of the prediction and control horizons plays a key role. In fact, typically, one sets n_c much smaller than n_p to address this fact as well as minimizing the computations required (Clarke et al. (1987)).

As proven in Monaco and Normand-Cyrot (1988), single-rate (or standard) sampling generally induces unstable and extra zero-dynamics so that minimum-phaseness is lost independently of the continuous-time plant properties; to overcome such a pathology, a multirate sampling procedure has been properly introduced to preserve the continuous-time internal properties. With this in mind, it is proposed and shown in the sequel that the use of a multirate (MR) sampled equivalent model at the prediction and implementation level overcomes the aforementioned problems in an MPC control scheme.

Recalling moreover the peculiar properties of MR sampling when applied to nonholonomic systems (Monaco and Normand-Cyrot (1992)), it is shown that a MR based MPC control scheme can be fruitfully employed to limit the amplitude and increase the robustness of MR solution to the steering problem. This result relies on the fact that MPC restitutes the MR steering solution.

The work is organised as follows. In Section 2 recalls on single and multi-rate sampling are given and the problem under investigation is stated. Section 3 is devoted to the proposed MR-MPC control scheme and prove its effectiveness. Section 4 investigates the relation between MPC and MR controllers in the sampled data context with reference to the steering control of nonholonomic dynamics; the example of the unicycle dynamics is used to verify the effectiveness of the proposed control scheme. Concluding remarks end the paper.

2. RECALLS AND STATEMENT OF THE PROBLEM

2.1 Notation and definitions

All functions and vector fields defining the dynamics are assumed smooth and complete over the respective definition spaces. M_U (resp. M_U^I) denotes the space of measurable and locally bounded functions $u : \mathbb{R} \to U$ ($u : I \to U$, $I \subset \mathbb{R}$) with $U \subseteq \mathbb{R}$. $\mathcal{U}_\delta \subseteq M_U$ denotes the set of piecewise constant functions over time intervals of fixed length $\delta \in]0, T^*[$; i.e. $\mathcal{U}_\delta = \{u \in M_U \text{ s.t. } u(t) = u_k, \forall t \in [k\delta, (k+1)\delta[; k \geq 0\}$. When $u(t) \in \mathbb{R}^m$ then $u_k^j, (u_k)^j, u_k^{(j)}$ are the jth component of u for $t \in [k\delta, (k+1)\delta[; k \geq 0\}$, u_k raised to the power j and the jth derivative respectively. Given a vector field f, L_f denotes the Lie

⋆ Partially funded by *Université Franco-Italienne/Università Italo-Francese* (UFI/UIF) through the Vinci Program.

derivative operator, $\mathrm{L}_f = \sum_{i=1}^n f_i(\cdot)\nabla_{x_i}$ with $\nabla_{x_i} := \frac{\partial}{\partial x_i}$ while $\nabla = (\nabla_{x_1}, \ldots, \nabla_{x_n})$. The Lie exponential operator is denoted as e^{L_f} and defined as $e^{\mathrm{L}_f} := \mathrm{I} + \sum_{i\geq 1} \frac{\mathrm{L}_f^i}{i!}$. A function $R(x, \delta) = O(\delta^p)$ is said to be of order δ^p ($p \geq 1$) if whenever it is defined can be written as $R(x, \delta) = \delta^{p-1}\tilde{R}(x, \delta)$ and there exist function $\theta \in \mathcal{K}_\infty$ and $\delta^* > 0$ such that $\forall \delta \leq \delta^*$, $|\tilde{R}(x, \delta)| \leq \theta(\delta^*)$.

2.2 Sampled data systems and multirate sampling

The following recalls on sampled-data systems are given (see Monaco and Normand-Cyrot (2001) and the references therein). Given a SISO system

$$\dot{x} = f(x) + g(x)u, \quad y = h(x) \tag{1}$$

and considering $u(t) \in \mathcal{U}_\delta$ and $y(t) = y(k\delta)$ for $t \in [k\delta, (k+1)\delta[$ (δ the sampling period), the dynamics of (1) at the sampling instants is described by the single-rate sampled-data equivalent model

$$x_{k+1} = F^\delta(x_k, u_k), \quad y_k = h(x_k) \tag{2}$$

with $x_k := x(k\delta)$, $y_k := y(k\delta)$, $u_k := u(k\delta)$. The mapping $F^\delta(\cdot, u) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ gets the form of formal series expansion in powers of δ that is (dropping the time subscript for clarity)

$$F^\delta(x, u) = e^{\delta(\mathrm{L}_f + u\mathrm{L}_g)}x = x + \sum_{i>0} \frac{\delta^i}{i!}(\mathrm{L}_f + u\mathrm{L}_g)^i x. \tag{3}$$

It is a matter of computations to verify that if (1) has well defined relative degree $r \leq n$, the relative degree of the sampled-data equivalent (2) always falls to $r_d = 1$; namely, one has

$$y_{k+1} = \sum_{i=0}^{r} \frac{\delta^i}{i!}\mathrm{L}_f^i h(x)\big|_{x_k} + \frac{\delta^r}{r!}u_k \mathrm{L}_g \mathrm{L}_f^r h(x)\big|_{x_k} + O(\delta^{r+1})$$

so that $\nabla_{u_k} y_{k+1} = \frac{\delta^r}{r!}\mathrm{L}_g \mathrm{L}_f^r h(x)\big|_{x_k} + O(\delta^{r+1}) \neq 0$. As a consequence, whenever $r > 1$, the sampling process induces a further zero-dynamics of dimension $r-1$ (the so-called *sampling zero-dynamics*) that is in general unstable for small values of δ when $r > 1$. As a consequence, dynamics-inverting controllers via single-rate sampling do not guarantee internal stability.

Multirate sampling has been developed in a nonlinear context to overcome those issues. Namely, by setting $u(t) = u_k^i$ for $t \in [(k+i-1)\bar{\delta}, (k+i)\bar{\delta}[$ for $i = 1, \ldots, r$ and $y(t) = y_k$ for $t \in [k\delta, (k+1)\delta[$, the multirate equivalent model of order r of (1) gets the form

$$x_{k+1} = F_m^{\bar{\delta}}(x_k, \underline{u}_k), \quad Y_k = H(x_k) \tag{4}$$

with $\bar{\delta} = \frac{\delta}{r}$, $\underline{u} \in \mathbb{R}^r = \left(u^1 \ldots u^r\right)^\top$, a dummy output vector $H(x) = \left(h(x)\ \mathrm{L}_f h(x)\ \ldots\ \mathrm{L}_f^{r-1}h(x)\right)^\top$ and

$$F_m^{\bar{\delta}}(x_k, \underline{u}_k) = e^{\bar{\delta}(\mathrm{L}_f + u_k^1 \mathrm{L}_g)} \ldots e^{\bar{\delta}(\mathrm{L}_f + u_k^r \mathrm{L}_g)}x\big|_{x_k} = F_m^{\bar{\delta}}(\cdot, u_k^r) \circ \cdots \circ F^{\bar{\delta}}(x_k, u_k^1).$$

One gets so far a MIMO system possessing vector relative degree $r^\delta = (1, \ldots, 1)$ and zero dynamics which inherits the zero-dynamics stability properties of (1).

It must be recalled that exact computation of the sampled data equivalent model cannot in general be achieved, so that approximated models are usually computed by truncation in different ways of the expansion (3). Related to the properties of (3), the notion of exact and finite computability of the sampled data model, possibly under preliminary feedback, has also been introduced in Monaco and Normand-Cyrot (1992).

2.3 Multirate digital steering of chained forms

Multirate sampling has been shown to be of interest in the digital design of nonholonomic and under actuated mechanical systems (see Monaco and Normand-Cyrot (1992)). Under preliminary continuous-time feedback, a nonholonomic mechanical system admits the so-called single chained dynamics

$$\dot{\xi}_1 = u_1, \quad \dot{\xi}_2 = u_2, \quad \dot{\xi}_i = \xi_{i-1}u_1 \quad i = 3, \ldots, n \tag{5}$$

admitting a finitely computable sampled dynamics; moreover, the one-step ahead dynamics can be easily inverted with respect to the control input to deduce a multirate feedback ensuring exact deadbeat steering. For, consider (5) and $(\underline{\xi}_0, \underline{\xi}_f)$ then there exists a multirate control of order $(n-1)$ on u_2 and 1 on u_1 (i.e. $\bar{\delta} = \frac{\delta}{n-1}$) such that system (5) is *exactly steered* in one step δ from $\underline{\xi}_0$ to $\underline{\xi}_f$. The feedback ensuring exact steering for (5) can be easily deduced by inverting the multirate sampled data equivalent model of (5) provided by

$$\begin{aligned}
&\xi_{1,k+1} = \xi_{1,k} + \delta u_{1,k}, \quad \xi_{2,k+1} = \xi_{2,k} + \bar{\delta}\sum_i u_{2,k}^i \\
&\xi_{3,k+1} = \xi_{3,k} + \delta u_{1,k}\xi_{2,k} + \eta_1(\bar{\delta}^2, u_{1,k})\underline{u}_{2,k} \\
&\quad\vdots \\
&\xi_{n,k+1} = \xi_{n,k} + G(\delta, \xi_k, u_{1,k}) + \eta_{n-2}(\bar{\delta}^{n-1}, u_{1,k})\underline{u}_{2,k}
\end{aligned} \tag{6}$$

with $\eta_1(\cdot), G(\cdot)$ and $\eta_n(\cdot)$ being

$$\eta_1(u_{1,k}) = \frac{1}{2!}\left[c_1^1 u_{1,k}\ c_1^2 u_{1,k}\ \ldots\ c_1^{n-1}u_{1,k}\right]$$

$$G(\delta, \xi_{i,k}, u_{1,k}) = \xi_{n,k} + \delta u_{1,k}\xi_{n-1,k} + \frac{\delta^2}{2!}(u_{1,k})^2\xi_{n-2,k} + \ldots + \frac{\delta^{n-2}}{(n-2)!}(u_{1,k})^{n-2}\xi_{2,k}$$

$$\eta_{n-2}(u_{1,k}) = \frac{1}{(n-1)!}\left[c_{n-2}^1 u_{1,k}\ c_{n-2}^2 u_{1,k}\ \ldots\ c_{n-2}^{n-1}u_{1,k}\right]$$

with some suitable constants c_i^j.

For all fixed $\xi_0 = \xi[k\delta], \xi_f = \xi[k\delta + \delta]$, directly solving the above system in the unknowns $(u_{1,k}, \underline{u}_{2,k})$, one gets

$$\begin{aligned}
u_{1,k} &= \frac{1}{\delta}(\xi_{1,k+1}) - \xi_{1,k}) \\
\underline{u}_{2,k} &= \underline{\eta}^{-1}(\underline{\xi}_{k+1} - \underline{G}(\cdot))
\end{aligned} \tag{7}$$

with $\underline{\eta}(\cdot), \underline{G}(\cdot)$ and $\underline{\xi}(\cdot)$ being the compact forms of the corresponding elements in (6). Taking into account the preliminary feedback, steering is achieved under piecewise continuous control designed on the basis of the multirate sampled model. Roughly speaking, the multirate sampling is used as a *trajectory planner*.

A key thing to note on the control solution above is that, while it does steer the system to the desired final state, it is indeed an inverting controller and the control effort might grow unboundedly so making the feedback not implementable in practice. To overcome this issue, we shall improve such a feedback via MPC.

2.4 Problem statement

Consider the continuous-time system (1) under sampling, with relative degree $r \leq n$ being minimum phase . Hereinafter we shall address the problem of driving the output trajectory, to a desired reference $\nu(t)$ at the sampling instants $t = k\delta$, $k \geq 0$ via discrete time MPC (Camacho and Alba (2013)) while preserving stability in closed loop; that is $y_k = \nu_k$, $k \geq k^*$ with $\nu_k = \nu(k\delta)$ by minimizing the cost functional

$$J = \sum_{i=1}^{n_p} \left(\|e_{k+i}\|_Q^2 + \|\underline{u}_{k+i-1}\|_R^2\right) = \sum_{i=1}^{n_p} L(x_{k+i}, \nu_{k+i}, \underline{u}_{k+i-1}) \quad (8)$$

with $Q > 0, R \geq 0$ being appropriate penalizing weights on the tracking error and input magnitude and n_p being the prediction horizon; moreover, e is a suitably defined error map, such that $e_k = 0$ *iff* $y_k = \nu_k$.

MPC induces a constrained optimization problem subject to the dynamics (4) and possibly additional requirements and bounds. To solve this problem several methods are available, the simplest to implement of which is the so called *direct single shooting* (Hicks and Ray (1971)) by plugging (4) into (8) so getting

$$J = \sum_{i=1}^{n_p} L(\cdot, \underline{u}_{k+i-1}) \circ (F_m^{\bar{\delta}}(\cdot, \underline{u}_{k+i-1}) \circ \ldots \circ (F_m^{\bar{\delta}}(x_k, \underline{u}_k)).$$

Hence, an optimal solution $u_e = \left(\underline{u}_k \ \cdots \ \underline{u}_{k+n_c-1}\right)^\top$ is computed by solving $\nabla_{u_e} J = 0$ with n_c being the so-called *control horizon.*

For our purposes it is interesting to note that in its usual implementation MPC makes use of a single-rate sampled data model of the plant of the form (2) for prediction. This induces the loss of the minimum-phaseness, so forcing the designer to set $n_c < n_p$ to recover internal stability (or using terminal penalties and/or constraints sets) while also defining a dynamical controller (in the sense of using feedback on the states and also on the previous controls) to ensure off-set free tracking. In the approach we are proposing, the use of multirate sampled data model will provide a static feedback overcoming both issues.

3. PREDICTIVE MULTIRATE DIGITAL CONTROL OF NONLINEAR SYSTEMS

With reference to the problem statement in the previous section, and the augmented output vector, we set out to state our main result, however to do so one needs the following assumption;

Assumption 1. Measures of ν and its derivatives $\nu^{(i)}$ for $i = 1 \ldots r-1$ are available at all $t = k\delta$, $k \geq 0$.

For the sake of compactness, we shall define the extended output vector dynamics for $n_p = n_c$ future values as

$$Y_{e_{k+1}} = A_e Y_k + B_e(x_k)\underline{u}_{e_k} + \Theta(x_k, \underline{u}_{e_k})$$

with $Y_{e_{k+1}} = \left(Y_{k+1} \ Y_{k+2} \ \ldots \ Y_{k+n_p}\right)^\top$ and $\underline{u}_{e_k} = \left(\underline{u}_k \ \underline{u}_{k+1} \ \cdots \ \underline{u}_{k+n_c-1}\right)^\top$

$$B_e(\cdot) = \mathrm{L}_g \mathrm{L}_f^{r-1} h(\cdot) \begin{bmatrix} B & 0 & \ldots & 0 \\ AB & B & \ldots & 0 \\ \vdots & & & \\ A^{n_p-1}B & A^{n_p-2}B & \ldots & B \end{bmatrix},$$

$$B = D\Delta(r) \quad D = diag(\bar{\delta}^r/r!, \ldots, \bar{\delta})$$

$$\Delta(r+j) = \begin{bmatrix} (r+j)^{r+j} - (r+j-1)^{r+j} & \ldots & (j+1)^{r+j} - (j)^{r+j} \\ & \ddots & \\ (r+j)^j - (r+j-1)^j & \ldots & (j+1)^j - (j)^j \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & \delta & \ldots & \delta^{r-1}/(r-1)! \\ & \ddots & & \\ 0 & 0 & \ldots & 1 \end{bmatrix} \quad A_e = \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^{n_p} \end{bmatrix}$$

with $j \geq 0$ and $\Theta(\cdot)$ containing all higher order terms in $O(\bar{\delta}^{r+1})$ (**?**). The next result shows that the problem in Section 2.4 is always solvable with $n_p = n_c > 1$ under multirate feedback, provided the relative degree is well defined.

Theorem 2. Let (1) possess relative degree $r \leq n$, and (4) be its multirate equivalent model of order r. Consider the MPC problem with cost functional (8) and $e_k = Y_k - \underline{\nu}_k$ with $\underline{\nu}_k = \left(\nu_k \ \ldots \ \nu_k^{(r-1)}\right)^\top$. Then, there exists $\delta^* > 0$ such that for all $\delta \in [0, \delta^*[$, the MPC problem is solvable with internal stability for all $n_p = n_c \geq 1$. The feedback is defined as the unique solution to the equality

$$(K(x, \underline{u}_e)Q_e B_e(x) + R_e)\underline{u}_e = B_e(\cdot)(\underline{\nu}_e - A_e Y - \Theta(x, \underline{u}_e)) \quad (9)$$

with $Q_e = I \otimes Q$, $R_e = I \otimes R$

$$K(x, \underline{u}_e) = (B_e^\top(x) + \nabla_{u_e}\Theta(x, \underline{u}_e))$$
$$\underline{\nu}_{e_k} = \left(\underline{\nu}_{k+1} \ \cdots \ \underline{\nu}_{k+n_p}\right).$$

Sketch of proof. To prove that (9) is optimal, we first rewrite (8) as $J = \|Y_{e_{k+1}} - \nu_{e_k}\|_{Q_e} + \|u_{e_k}\|_{R_e}$ whose jacobian is clearly annihilated by the solution to (9). Existence of a feedback solution can be deduced by rewriting (9) a formal series expansion in powers of δ and applying the *implicit function theorem.* Indeed, the term $(D^{-1}K(x, \underline{u}_e)Q_e B_e(x) + R_e)$ is invertible as $\delta \to 0$ (Monaco and Normand-Cyrot (2001); Mattioni et al. (2017)). Internal stability is ensured by the minimum phaseness of the continuous-time plant which is consequently preserved under multirate sampling (Monaco and Normand-Cyrot (1988)).

Remark 3. The solution obtained in Theorem 2 is implicitly defined by the above equality and is a formal series in powers of $\bar{\delta}$. Such a solution cannot be generally exactly computed in practice although several procedures are available for deducing approximation up to any desired order so to guarantee the required performances (see Monaco and Normand-Cyrot (2001) for further details).

Remark 4. As $R \to 0$, the feedback defined by (9) coincides with the deadbeat inverting control that steers the output to the desired ν in one step of length δ. Such a feedback comes with an effort that is in general, inversely proportional to δ, thus by suitably setting R one can reduce the effort while still guaranteeing off-set free tracking in finite time.

Remark 5. It is rather straightforward to show that when (1) is linear (i.e. $f(x) = Fx, g(x) = G$ and $h(x) =$

Cx) one recovers the known output trajectory prediction of the discrete time model with $A_d = e^{F\delta}, B_d = [\bar{A}^{\bar{\delta}(r-1)}\bar{B}\ldots\bar{B}], \bar{A} = e^{F\bar{\delta}}, \bar{B} = \int_0^{\bar{\delta}} e^{Fs}\mathrm{d}sG$ and $C_d = C$; i.e,

$$\underline{Y}_{e_{k+1}} = \begin{bmatrix} C_dA_d \\ C_dA_d^2 \\ \vdots \\ C_dA_d^{n_p} \end{bmatrix} x_k +$$
$$\begin{bmatrix} C_dB_d & 0 & \ldots & 0 \\ C_dA_dB_d & C_dB_d & \ldots & 0 \\ \vdots & & & \\ C_dA_d^{n_p-1}B_d & C_dA_d^{n_p-2}B_d & \ldots & C_dA_d^{n_p-n_c}B_d \end{bmatrix} \underline{u}_{e_k}$$

which in compact form can be written as $Y_{e_{k+1}} = A_e x_k + B_e u_{e_k}$. Along the lines of Borrelli et al. (2017), the optimal control is $u_e^* = (B_e^\top Q_e B_e + R_e)^{-1} B_e^\top Q_e(\nu_e - A_e x)$. When implementing the above optimal control trajectory a receding horizon algorithm (i.e. selecting the first m components of $u_e[k]$ and discarding the rest and repeating at each sampling instant) one has

$$u_k^\star = (I_m \ 0 \ \ldots \ 0)\,(B_e^\top Q_e B_e + R_e)^{-1} B_e^\top Q_e(\nu_e - A_e x).$$

Thus, when $n_p = n_c, Q_e = I$ and $R_e = 0$ the MR-MPC feedback reduces to $u_k^\star = (C_dB_d)^{-1}(\nu - C_dA_dx)$, which is the classical dynamics inverting feedback.

Roughly speaking, with reference to a minimum-phase plant, Theorem 2 suggests the use of MR-MPC control law of order equal to the relative degree. In what follows, we explicitly define this solution for nonholonomic systems that are feedback-equivalent to chained forms (Brockett et al. (1983)). In doing so, we formally show that as $R \to 0$ one recovers the standard deadbeat control. As a byproduct, we also provide an extension of Theorem 2 to the case of MIMO systems for which the relative degree might not be defined.

4. PREDICTIVE MULTIRATE STEERING FOR CHAINED FORMS

For illustrative purposes, the following discussion will consider the chained form (5) with $n = 3$ albeit the arguments extend to the general case as highlighted. For, suppose one wants the state of the system to converge to a desired trajectory $\nu \in \mathbb{R}^n$. Considering $u_2(t) = u_{2,k}^i = u_2(k\delta + (i-1)\bar{\delta})$ for $i = 1, 2$, one can write the output prediction over $n_p = 1$ as

$$\begin{bmatrix} x_{1,k+1} \\ x_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & A_m^2(\cdot) \end{bmatrix} \begin{bmatrix} x_{1,k} \\ x_k \end{bmatrix} + \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & R_m(\cdot) \end{bmatrix} \begin{bmatrix} u_{1,k} \\ \underline{u}_{2,k} \end{bmatrix} \tag{10}$$

with $\underline{u}_2 = (u_2^1 \ u_2^2)^\top$ and

$$A_m(\cdot) = \begin{bmatrix} 1 & 0 \\ \bar{\delta}u_{1,k} & 1 \end{bmatrix}, b = \begin{bmatrix} \bar{\delta} \\ \frac{\bar{\delta}^2}{2!}u_{1,k} \end{bmatrix}, R_m(\cdot) = [A_m(\cdot)b \ \ b]$$

which compactly rewrites as $Y = F(\delta, u_1)X + G(\delta, u_1, \underline{u}_2)\underline{u}_2$. One can then proceed in a similar fashion to the previous section, using the cost index (8) with $e = \mathrm{col}(e_1, e_2, e_3) = \big(x_1 \ x^\top\big)^\top - \nu$ and setting $\nabla_u J = 0$, so getting (when $Q = I$) that the optimal control $u = u^\star$ is solution to (dropping time subscript for clarity)

$$\begin{aligned} u_1 &= \frac{-2\delta e_1 - e_3(3\bar{\delta}^2u_2^1 + \bar{\delta}^2u_2^2 + 4\bar{\delta}x_2)}{2\delta^2 + (3\bar{\delta}^2u_2^1 + \bar{\delta}^2u_2^2 + 4\bar{\delta}x_2)} \\ u_2^1 &= -\frac{2\bar{\delta}(e_2 + \bar{\delta}u_2^2) - 3\bar{\delta}^2u_1(e_3 + 2\bar{\delta}u_1x_2 - 0.5\bar{\delta}^2u_1u_2^2)}{2\bar{\delta}^2 + 4.5\bar{\delta}^4u_1^2} \\ u_2^2 &= -\frac{2\bar{\delta}(e_2 + \bar{\delta}u_2^1) - \bar{\delta}^2u_1(e_3 - 2\bar{\delta}u_1x_2 - 1.5\bar{\delta}^2u_1u_2^1)}{2\bar{\delta}^2 - 0.5\bar{\delta}^4u_1^2} \end{aligned} \tag{11}$$

with u_2^i $(i = 1, 2)$ being the two controls resulting from multirate of order 2 over u_2 and $\nu_i, i = 1, 2, 3$ being the reference values over the single step prediction horizon. To show that the solution to this system of equations coincides with that of the multirate inverting controller, it is sufficient to show that the multirate inverting solution, is indeed a solution of this system of equations.

Proposition 6. The multirate inverting controller (7) is a solution of the optimal control problem with the cost function (8) and $n_p = n_c = 1, Q = I, R = 0$.

Proof: Starting from (7), one has that

$$u_1 = \frac{\nu_1 - x_1}{\delta}, \quad u_2^1 = -\frac{\nu_2 - x_2}{2\,\bar{\delta}} - \frac{x_3 - \nu_3 + 2\,\bar{\delta}\,u_1\,x_2}{\bar{\delta}^2\,u_1}$$
$$u_2^2 = \frac{3\,(\nu_2 - x_2)}{2\,\bar{\delta}} + \frac{x_3 - \nu_3 + 2\,\bar{\delta}\,u_1\,x_2}{\bar{\delta}^2\,u_1}$$

solves (11). As a matter of fact, (11) admit two solutions for (u_1, u_2^1, u_2^2), one of which corresponds to the solution $u_1 = 0$ which is discarded[1] whereas the other one is $u_1 = \frac{e_1}{\delta}$ and

$$u_2^1 = \frac{-2\,\delta\,e_3 - \bar{\delta}\,\nu_1\,\nu_2 - 3\,\bar{\delta}\,\nu_1\,x_2 + \bar{\delta}\,\nu_2\,x_1 + 3\,\bar{\delta}\,x_1\,x_2}{2\,\bar{\delta}^2\,(\nu_1 - x_1)}$$
$$u_2^2 = -\frac{-2\,\delta\,e_3 - 3\,\bar{\delta}\,\nu_1\,\nu_2 - \bar{\delta}\,\nu_1\,x_2 + 3\,\bar{\delta}\,\nu_2\,x_1 + \bar{\delta}\,x_1\,x_2}{2\,\bar{\delta}^2\,(\nu_1 - x_1)}$$

clearly coinciding with the expression above from (7). ◁

4.1 The case of $n_p = n_c > 1$

We write the prediction model for the two components of the states vector as follows

$$\begin{aligned} x_{1,k+n_p} &= x_{1,k} + \delta \sum_{i=0}^{n_p-1} u_{1,k+i} \\ x_{k+n_p} &= \phi(k+n_p, k)x_k + \\ &\sum_{i=0}^{n_p-1} \phi(k+n_p-1, k+i+1)R(\cdot, u_{1,k+i})u_{2,k+i} \end{aligned} \tag{12}$$

where

$$\phi(k+n_p, k) = \prod_{i=0}^{n_p-1} A(\bar{\delta}, u_{1,k+n_p-i-1}) \tag{13}$$

and $A(\cdot) = A_m^2(\cdot)$.

We can then substitute this expression of prediction in our cost function and take the partial derivatives with respect to each u_i and, then prove that whenever $n_p = n_c$ the multirate inversion control is an optimum control with respect to our cost function. The following statement highlights this fact.

[1] Since, this solution doesn't bring the error on the state x_1 to zero.

Proposition 7. As $R \to 0$, the control minimizing (8) computed over the prediction model (12) reduces to the multirate plant inversion solution (7) if $n_p = n_c$ and $Q = I$.

Proof: The proof follows from induction starting with Proposition 6 which proves $n_p = n_c = 1$. By assuming that the statement holds for some $n_p = N$, we show it holds for $n_p = N + 1$. Let us split the cost functional (8) along the prediction model (12) as follows

$$J = \underbrace{J_1}_{\text{first } N \text{ steps terms}} + \underbrace{J_2}_{\text{last step terms}}$$

$$J_1 = \sum_{j=1}^{N} \Big((\nu_{1,k+j} - x_{1,k} - \delta \sum_{i=0}^{j-1} u_{1,k+i})^2 + (r(k+j) - \phi(k+j-1,k)x_k - \sum_{i=0}^{j-1} \phi(k+j-1,k+i)R(\cdot,u_{1,k+i})u_{2,k+i})^2 \Big)$$

$$J_2 = (\nu_{1,k+N+1} - x_{1,k} - \delta \sum_{i=0}^{N} u_{1,k+i})^2 + (r(k+N+1) - \phi(k+N+1,k)x_k - \sum_{i=0}^{N} \phi(k+N+1,k+i)R(\cdot,u_{1,k+i})u_{2,k+i})^2.$$

Denoting by u^*_{mr} the multirate inverse controller which satisfies by assumption $\nabla_u J_1(u^*_{mr}) = 0$, it remains to prove that that $\nabla_u J_2(u^*_{mr}) = 0$. For, notice that

$$\phi(k+N+1,k) = \begin{bmatrix} 1 & 0 \\ 2\bar{\delta} \sum_{s=0}^{N} u_{1,k+s} & 1 \end{bmatrix}$$

and recalling that u^*_{mr} is of the form (7), the proof proceeds as follows; one gets that as $\nabla_{u_2} J_2 = 0$ (for compactness we omit the time variable k and we write i for $k+i$)

$$-2n\phi(N,i+1)R(\cdot,u_{1,i})\Big(r(N+1) - \phi(N+1,.)x - \sum_{i=0}^{N-1} \phi(N,i+1)R(\cdot,u_1[i])u_{2,i}\Big)^\top = 0$$

Inspecting the second equation above, namely $\nabla_{u_2} J_2 = 0$ gives two possibilities, either the term $-2n\phi(N,i+1)R(\cdot,u_{1,i}) = 0$ or the term between the large brackets is 0, which upon inspection is 0 exactly when we set for $i = 1 \dots N-1$

$$\begin{aligned} u_{2,i} &= \Xi(\bar{\delta},u_1)^{-1}(r(i+1) - \phi(N+1,\cdot)x_k) \\ u_{2,N} &= R(\cdot,u_{1,N})^{-1}(r(N+1) - \phi(N+1,\cdot)x_k) \end{aligned} \tag{14}$$

with $\Xi(\bar{\delta},u_1)$ collecting the product terms of $\phi(N,i+1)R(\cdot,u_{1,i})$ which coincides with the solution obtained from (7). We then write only $\nabla_{u_{1,N}} J_2$, since by assumption and from the expression above for u_2, $\nabla_{u_{1,i}} J_2 = 0, \nabla_{u_{2,j}} J_2 = 0\ \forall i = 1 \dots N-1, j = 1 \dots N$ so getting

$$2\,\delta\,(x_{1,N} - \nu_{1,N+1} + \delta\,u_{1,N}) + \bar{\delta}\,\big(4\,x_{2,N} + 3\,\bar{\delta}\,u^1_{2,N} + \bar{\delta}\,u^2_{2,N}\big) (x_{3,N} - \nu_{3,N+1} + \frac{3\,\bar{\delta}^2\,u_{1,N}\,u^1_{2,N}}{2} + \frac{\bar{\delta}^2\,u_{1,N}\,u^2_{2,N}}{2} + 2\,\bar{\delta}\,u_{1,N}\,x_{2,N}) = 0.$$

By substituting $u^1_{2,N}, u^2_{2,N}$ as in (14) one recovers

$$u_{1,N} = \frac{\nu_{1,N+1} - x_{1,N}}{\delta}$$

which possesses the same form as in Proposition 6. ◁

It is rather intuitive to see that the discussion above holds for general chained forms, and the statements can be extended along the same lines, albeit the notations and algebraic manipulations will get rather long and cumbersome, the following statement summarizes this.

Theorem 8. Consider the dynamics of the form (5) admitting multirate equivalent model (6). Then, the multirate plant-inverting feedback (7) solves the MPC problem with (8) under prediction model (6) with perfect steering, whenever $n_p = n_c \geq 1, Q = I, R = 0$.

5. SIMULATION RESULTS AND COMMENTS

Simulations are performed to compare the proposed control scheme (MPC-MR) with respect to the standard MPC implementation (Figs 1,2) and the classical MR control (Figs 3,4). In all the cases MPC-MR is implemented with $n_p = n_c, \delta = 1$. Figure 1 clearly emphasises the pathology motivating this work; the MPC may fail when $n_p = n_c$ and no stability constraints are incorporated, even with no penalty on the control. To prevent this, as suggested in the literature, in Fig 2 MPC works with $n_p > n_c$; the comparison with the proposed MPC-MR with $R > 0$ in this case shows the better performance of our solution. A deeper comparison is proposed in Figs 3 and 4 where the proposed MPC-MR and the standard MR solutions are shown for steering and tracking maneuvers under the penalty $R > 0$. The proposed MPC-MR scheme appears to be the *natural* context to be adopted to account for the control amplitude in standard MR.

6. CONCLUSION

We establish an intuitive interpretation of MR inverting controllers, by highlighting the roles of the prediction and control horizons, and their relations to the relative degree. We then motivate the use of MPC with a MR prediction model through penalizing the controls, and obtaining comparable performance to the plant inversion controller, while maintaining low control effort. Future works concern the application of this improved MPC scheme to several case studies as in power systems or automotive control Giuseppi et al. (2018); Gionfra et al. (2016). Ongoing work is addressing the extension to other classes of systems, possibly non-minimum phase Mattioni et al. (2019).

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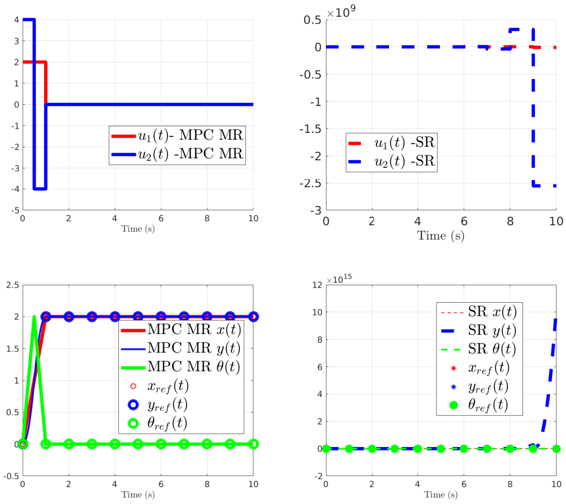

Fig. 1. MPC-MR vs MPC SR steering with $R = 0, Q = I$.

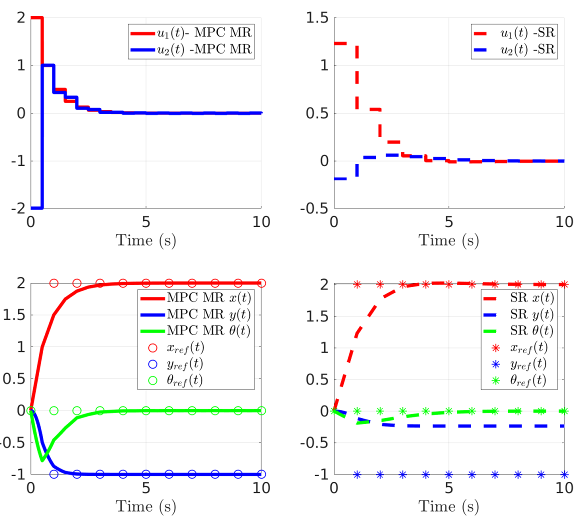

Fig. 2. MPC-MR vs MR steering with $R = 0.2I, Q = 2I$.

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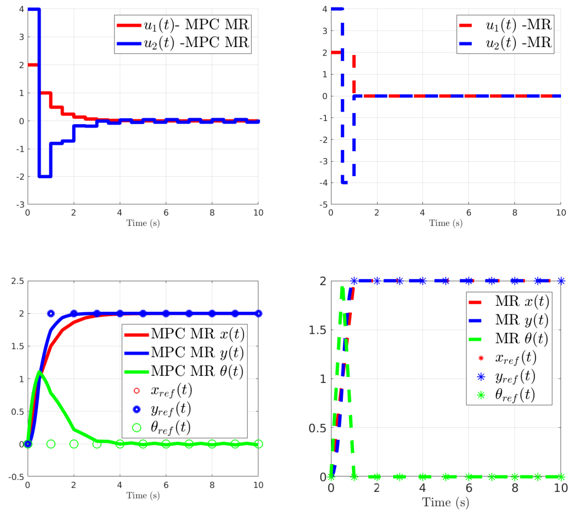

Fig. 3. MPC-MR vs MR steering with $R = 0.2I, Q = 10I$

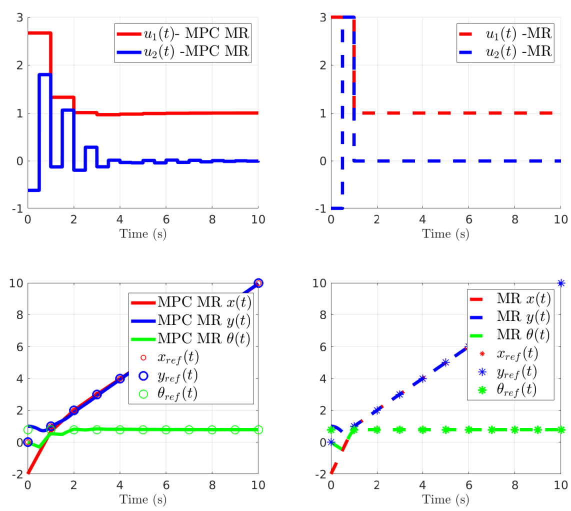

Fig. 4. MR-MPC Vs MR tracking with $R = 0.5I, Q = 2I, \theta_d = \frac{\pi}{4}, x_0 = (-2 \ \ -1 \ 0)^\top$.

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