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Ph.D. Thesis in Mathematics

Quasi-classical Dynamics of Quantum Particles Interacting with Radiation

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*"A lesson without pain is meaningless.
For you cannot gain anything without
sacrificing something else in return."*

Hiromu Arakawa

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Introduction

In theoretical physics, Schrödinger operators with external electric and/or magnetic potentials are used to describe a huge variety of phenomena involving the interaction between quantum particles and radiation fields. Some examples are atoms confined in optical lattices created by superposition of laser beams, magneto-optical traps confining particles, or electrons interacting with intense electromagnetic fields (see [12, 24, 77]). In most of the physics models used to study the behaviour of these systems the fields are treated as classical objects acting on the subsystem of the particles only as external degrees of freedom.

The use of the effective models in theoretical physics, in place of more fundamental microscopic ones, is motivated by the fact that the latter are usually really complex and difficult to analyze, involving infinitely many degrees of freedom. However, the microscopic models of Quantum Field Theory take into account also the effects arising from the interaction between the atoms or the particles and the quantized radiation field, and thus there is the need to rigorously justify the approximation in terms of Schrödinger operators with classical fields in some appropriate regime. This is precisely the main question addressed in this thesis, and, mathematically, the solution is obtained by considering a *quasi-classical* regime of the full system, whose theory has been introduced in [26, 27].

More precisely, aim of the thesis is the derivation of an effective dynamics, in the quasi-classical limit, of three important models of interaction between quantum particles and bosonic fields, *i.e.*, the Nelson, polaron and Pauli-Fierz models. The goal is the identification of the effective dynamics for the particle subsystem in a mesoscopic regime, and we are going to prove that it is generated by an effective Schrödinger opeartor, averaged over all the classical configurations of the field.

In the quasi-classical regime quantum particles interact with a quantum field which is so intense that it can be considered macroscopic. This can be rephrased by assuming that the field is in a state $\Psi \in \Gamma_s(\mathfrak{H})$ with large average occupation number, *i.e.*,

$$\langle \Psi | \mathcal{N} | \Psi \rangle_{\Gamma_s(\mathfrak{H})} \gg 1, \tag{1}$$

where $\mathcal{N}$ is the number operator for the field excitations. Introducing a semiclassical parameter $\varepsilon > 0$, we can thus consider a state $\Psi_\varepsilon \in \Gamma_s(\mathfrak{H})$ such that

$$\langle \Psi_\varepsilon | \mathcal{N} | \Psi_\varepsilon \rangle_{\Gamma_s(\mathfrak{H})} \simeq \frac{1}{\varepsilon}, \tag{2}$$

which diverges as $\varepsilon \to 0$. Rescaling the variables, it is mathematically equivalent to study a semiclassical regime for field (see [6]): by (2), ε is proportional to the inverse of the expectation of the number of bosons and, if we set $\mathcal{N}_\varepsilon := \varepsilon\mathcal{N}$, then,

$$\langle \Psi_\varepsilon | \mathcal{N}_\varepsilon | \Psi_\varepsilon \rangle_{\Gamma_s(\mathfrak{H})} \simeq 1. \tag{3}$$

Coherently, we rescale the creation and annihilation operators setting $a_\varepsilon^\# = \sqrt{\varepsilon} a^\#$, which yields the $\varepsilon-$dependent canonical commutation relations (CCR)

$$[a_\varepsilon(\xi), a_\varepsilon^\dagger(\eta)] = \varepsilon \langle \xi \,|\, \eta \rangle_{\mathfrak{H}}, \qquad \text{for any } \xi, \eta \in \mathfrak{H}. \tag{4}$$

As $\varepsilon \to 0$, the canonical observables of the field commute, so that the field becomes classical. Note that only the quantum nature of the field is influenced, while the quantum particles remain unaffected, so that we obtain a quasi-classical system in the limit. At the level of states and observables the picture translates into the scheme

$$\begin{pmatrix} \text{quantum state} \\ \text{of the full system} \end{pmatrix} \xrightarrow[\varepsilon \to 0]{} \begin{pmatrix} \text{quantum state} \\ \text{of the particle} \end{pmatrix} \otimes \begin{pmatrix} \text{classical state} \\ \text{of the field} \end{pmatrix}$$

and

$$\begin{pmatrix} \text{quantum observables} \\ \text{of the full system} \end{pmatrix} \xrightarrow[\varepsilon \to 0]{} \begin{pmatrix} \text{quantum observables} \\ \text{of the particle} \end{pmatrix} \otimes \begin{pmatrix} \text{classical observables} \\ \text{of the field} \end{pmatrix}$$

where the classical states and observables are probability measures and functions over the phase space of the field, respectively (see Chapter 3). Therefore quasi-classical states are objects which show a quantum behaviour and, at the same time, properties of a measures. Their mathematical counterparts are indeed *state-valued measures*, whose definition was already provided in [28], so that the study of the quasi-classical limit reduces to the analysis of the convergence of quantum expectations to quasi-classical expectations of observables w.r.t. state-valued measures.

The rigorous derivation of effective dynamics has been studied in a wide class of problems: one of the first contributions in 1974 was the work of Hepp [54], where it was introduced a method based on coherent states. Since then, a lot of effort has been devoted to the derivation of effective models for many-body quantum systems, motivated also by the connection with Bose-Einstein condensation [23]. Two regimes have been mainly studied: the weak coupling, including the mean-field regime, leading to the derivation of the Hartree equation [1, 6, 14, 15, 16, 34, 70, 78] and the Gross-Pitaevskii regime to derive the NLS or GP equation [20, 35, 36, 68, 71]. The systems studied in this thesis, however, contain a variable number of bosons and call for the analysis of models with infinite degrees of freedom. We focus the attention on three important models: the Nelson model, describing the interaction between nucleons and a field of mesons; the polaron, *i.e.*, a pseudoparticle which models the behaviour of an electron interacting with phonons in a crystal lattice; the Pauli-Fierz model, which describes the interaction between extended charged particles and an electromagnetic field (see Chapter 1 for the details about the aforementioned models). The main results available about such models are mostly obtained in the following regimes:

- classical limit: approximating the behaviour of both the quantum particles and fields by their classical counterparts. Studied for the first time by Ginibre J. and Velo G. in [49] and later applied to derive the Schrödinger-Klein Gordon system from Nelson-Yukawa model [3, 4, 38, 48];

- mean-field limit: using the counting method proposed in [70], it was studied to get a rigorous derivation of the Schrödinger-Klein-Gordon [58, 61] and Schrödinger-Maxwell systems [60] from the Nelson and Pauli-Fierz models, respectively;

- strong coupling limit: mostly considered for the polaron, whose dynamics can be approximated by suitable effective Schrödinger equations [44, 45, 50, 62].

In the thesis we study the derivation of the effective dynamics, in quasi-classical regime, for the Nelson, polaron and Pauli-Fierz models. The Hilbert space for these models is

$$\mathscr{H} = L^2(\mathbb{R}^{dN}) \otimes \Gamma_s(\mathfrak{H})$$

where $L^2(\mathbb{R}^{dN})$ and $\Gamma_s(\mathfrak{H})$ are the space for N particles in a $d-$dimensional space and the symmetric Fock space for the field, respectively, with $\mathfrak{H}$ is the one-boson space. The Hamiltonians have the general form

$$H_\varepsilon = H_0 + H_I, \tag{5}$$

with H_0, free energy of the system, sum of two terms

$$H_0 = K \otimes \mathbb{1}_{\Gamma_s(\mathfrak{H})} + \mathbb{1}_{L^2(\mathbb{R}^{dN})} \otimes H_{\mathrm{f}}, \tag{6}$$

K and H_{f} being the free energies for the particle subsystem and the field, respectively, and H_I being the interaction term acting on the whole $\mathscr{H}$ (see Chapter 1).

Our main theorem (Theorem 1.3.1) states that, for quantum states ρ_ε of the full system, represented as trace-class operators on $\mathscr{H}$, whose field part satisfies (3), then

- the quantum state converges to a quasi-classical state, identified by a state-valued measure $\mathfrak{m}$,

$$\rho_\varepsilon \xrightarrow[\varepsilon\to 0]{} \mathfrak{m}, \tag{7}$$

- if $\rho_\varepsilon(t)$ is the quantum state evolved according to the dynamics of the Nelson, polaron or Pauli-Fierz models, then it converges as well to a time-dependent state-valued measure $\mathfrak{m}_t$

$$\rho_\varepsilon(t) \xrightarrow[\varepsilon\to 0]{} \mathfrak{m}_t, \tag{8}$$

  where $\mathfrak{m}_t$ is obtained by transporting the initial quasi-classical state $\mathfrak{m}$ along the effective dynamics.

The above convergences are up to extraction of a subsequence and the limit point might depend on such a choice. However, the convergence at time $t \neq 0$ is along the same subsequence chosen at time $t = 0$.

The quasi-classical dynamics can be characterized by means of the Radon-Nykodým decomposition of the state-valued measures (see Chapter 2): for any state-valued measure $\mathfrak{m}$,

$$\begin{aligned} \mathrm{d}\mathfrak{m}(z) &= \gamma_{\mathfrak{m}}(z)\, \mathrm{d}\mu(z), \\ \mathrm{d}\mathfrak{m}_t(z) &= \gamma_{\mathfrak{m}_t}(z)\, \mathrm{d}\mu_t(z), \end{aligned}$$

where γ, γ_t are functions on the space of classical fields with values in density matrices for the particles and μ, μ_t are classical probability measures on the phase space of classical fields. Then,

- $\mu_t = \Phi_t \sharp \mu$ is the measure obtained by transporting the initial measure along the free classical dynamics of the field;
- $\gamma_{\mathfrak{m}_t}(z) = U_{t,0}(z)\gamma_{\mathfrak{m}}(z)U^\dagger_{t,0}(z)$ is the density matrix, depending on the classical configuration of the field, evolved according to the unitary dynamics generated by a time-dependent Schrödinger operator. The action of the field is expressed by an effective external potential, playing the role of a classical environment.

The above results can be summed up in the commutativity of the following diagram

$$\begin{array}{ccc} \rho_\varepsilon & \xrightarrow{\text{micro dyn.}} & \rho_\varepsilon(t) \\ \Big\downarrow{\scriptstyle \varepsilon\to 0} & & \Big\downarrow{\scriptstyle \varepsilon\to 0} \\ \mathfrak{m} & \xrightarrow{\mathscr{W}_{\mathrm{eff}}(t)} & \mathfrak{m}_t \end{array}$$

where

$$\mathfrak{m}_t = \mathscr{W}_{\mathrm{eff}}(t)(\gamma_{\mathfrak{m}}, \mu) = (U_{t,0}(z)\gamma_{\mathfrak{m}}(z)U^\dagger_{t,0}(z), \Phi_t \sharp \mu). \tag{9}$$

Compared to the results available in the literature, which typically assume a factorization of the initial state, Theorem 1.3.1 allows us to treat also entangled initial states. It gives a qualitative method to identify the limit of quantum states, although it provides no information on the speed of convergence.

The proof of the main theorem makes use of semiclassical theory for systems with infinite degrees of freedom introduced in [6] by Ammari Z. and Nier F., adapted and developed for the quasi-classical setting (see Chapter 3.1.6).

The key assumption on the initial state is bound (3), which guarantees (Theorem 3.1.6 and Proposition 3.1.8) the quasi-classical convergence at time zero. To obtain convergence also at any later times, the bound on the expectation of the number operator has to be propagated along the dynamics (Section 5.1). The quantum dynamics can be characterized as the solution of the Heisenberg equation

$$i\frac{\mathrm{d}}{\mathrm{d}t}\rho_\varepsilon(t) = [H_\varepsilon, \rho_\varepsilon(t)], \tag{10}$$

where H_ε is the Nelson, polaron or Pauli-Fierz Hamiltonian. To study the limit of the previous equation, one needs to prove that quasi-classical convergence of observables like the interaction term H_I in the Hamiltonian: while for the Nelson model it is enough the bound on the number of excitations of the field, for the polaron and the Pauli-Fierz model a bound on the free part of the Hamiltonian and its propagation in time is also needed, due to the presence of gradient terms in the interaction (see Remark 1.3.2).

Taking the limit $\varepsilon \to 0$ of (10), we obtain a transport equation for the quasi-classical state $\mathfrak{m}_t$, then we prove (Corollary 5.3.8) that there exists a unique solution whose expression is (9), and this identifies the quasi-classical dynamics.

In the quasi-classical limit the quantum state can lose probability mass and, without additional conditions, it could converge to a measure with strictly smaller mass. However, assuming an a-priori bound on a suitable observable, *i.e.*,

$$\mathrm{Tr}(\rho_\varepsilon(-\Delta_x + W)) \leqslant C \tag{11}$$

uniformly in ε, with W a trapping potential, the no-loss of mass in the quasi-classical limit of the dynamics is guaranteed, so that $\mathfrak{m}_t$ is a probability measure over the field's phase space at any time. Furthermore, this also implies quasi-classical convergence of expectations of bounded particle observables.

The structure of the thesis is the following:

1. in Chapter 1, we define the Nelson, polaron and Pauli-Fierz models in detail, discussing self-adjointness of the Hamiltonian, and the associated quasi-classical effective models. We state there our main result about the quasi-classical convergence of the dynamics;

2. in Chapter 2, we introduce the mathematical framework of state-valued measures. We construct a theory of integration w.r.t. these measures, also considering operator-valued integrands, representing quasi-classical observables. Moreover, we study the Radon-Nykodým decomposition of the measures and generalize the notion of state-valued measures introducing the cylindrical state-valued measures;

3. in Chapter 3, we characterize cylindrical state-valued measures as cluster points of nets of states of the whole system in the quasi-classical limit. We also prove convergence of the expectations of observables, when gradient terms are involved too;

4. in Chapter 4, we study the quasi-classical limit in the stationary setting. We recall some results from [26, 27] about the convergence of the reduced Hamiltonian for the particle subsystem to the effective one on product states. We prove the convergence of the ground

state energy of the Nelson model to the ground state energy of the effective quasi-classical energy;

5. in Chapter 5, we provide the proof of our main result (Theorem 1.3.1).

In Appendix A, we recall the construction of Fock space and some useful estimates, while, in Appendix B, we recall the theory of Weyl and Wick quantizations for systems with infinitely many degrees of freedom.

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Notation

- We will denote by $C_0^\infty(\mathbb{R}^d)$ the space of smooth, compactly supported functions $f:\mathbb{R}^d\to\mathbb{R}$. If $f,g:\mathbb{R}^d\to\mathbb{R}$ are two functions, we denote their convolution by $f\star g$, *i.e.*,
$$(f\star g)(x) := \int_{\mathbb{R}^d} \mathrm{d}y\, f(y)\, g(x-y). \tag{12}$$
If (Ω,μ) is a measure space, for $p\in(0,+\infty)$ we denote by
$$L^p(\Omega;\mathrm{d}\mu) := \left\{ f:\Omega\to\mathbb{C} \;:\; \|f\|^p_{L^p(\Omega;\mathrm{d}\mu)} := \int_\Omega \mathrm{d}\mu(x)\,|f(x)|^p \;<+\infty \right\} \tag{13}$$
the $p-$th Lebesgue space, sometimes setting for short $\|\cdot\|_2 := \|\cdot\|_{L^2(\Omega;\mathrm{d}\mu)}$.
If $\mathfrak{H}$ is a Banach space, we set
$$L^\infty(\mathbb{R}^d;\mathfrak{H}) := \left\{ f:\mathbb{R}^d\to\mathfrak{H} \;:\; \operatorname*{ess\,sup}_{x\in\mathbb{R}^d}\|f\|_{\mathfrak{H}} < +\infty \right\}. \tag{14}$$
The norm $\|\cdot\|_\infty$ stands for the $L^\infty(\Omega;\mathrm{d}\mu)$-norm of the usual a.e. bounded functions, or, more in general,
$$\|\cdot\|_\infty := \operatorname*{ess\,sup}_{x\in\mathbb{R}^d}\|\cdot\|_{\mathfrak{H}}. \tag{15}$$
- If V is a topological vector space, we denote by V^* its algebraic dual and by $V'\subseteq V^*$ its continuous dual; if W is another topological vector space which is dual of V, then $\sigma(V,W)$ is the topology on V generated by the seminorms $\{p_F\}_{F\subseteq W \text{finite}}$
$$p_F(v) = \sum_{w\in F} |\langle v\,|\,w\rangle|.$$
  - The topology $\sigma(V,V')$ is the weak topology on V.
  - The topology $\sigma(V',V)$ is the weak-* topology on V'.
- If $\mathscr{H}$ is a Hilbert space with scalar product $\langle\cdot\,|\,\cdot\rangle_{\mathfrak{H}}$, that we always assume to be anti-linear in the first variable and linear in the second one, we set
$$\langle z\rangle := (1+\|z\|^2_{\mathfrak{H}})^{1/2}. \tag{16}$$
We introduce the following classes of bounded operators on a Hilbert space:
  - $\mathscr{L}(\mathscr{H})$ the space of bounded operators on $\mathscr{H}$;
  - $\mathscr{L}_p(\mathscr{H})$ the $p-$th Schatten ideal of $\mathscr{L}(\mathscr{H})$; in particular, $\mathscr{L}_1(\mathscr{H})$ and $\mathscr{L}_\infty(\mathscr{H})$ are the spaces of trace-class and compact operators, respectively, and
$$\mathscr{L}_1(\mathscr{H})\subseteq\mathscr{L}_p(\mathscr{H})\subseteq\mathscr{L}_\infty(\mathscr{H})\subseteq\mathscr{L}(\mathscr{H}), \qquad \text{for any } p\in\mathbb{N}, p\geqslant 2.$$
  We denote by $\mathscr{L}_{p,+}(\mathscr{H})$ the positive elements of the $p-$th Schatten ideal.

- C stands for a positive, finite constant whose value may change from line to line but is always independent on the key parameters into play.
- If H is a self-adjoint operator acting on a Hilbert space $\mathscr{H}$, we denote by
  - $\mathscr{D}(H)$ its domain of self-adjointness;
  - $\mathscr{Q}(H)$ its form domain;
  - $\varrho(H), \sigma(H)$ its resolvent set and spectrum, respectively.
- We denote by $\mathcal{K}_{\ll}$ the space of operators acting on a $L^2(\mathbb{R}^d)$ that are infinitesimally bounded w.r.t. the Laplacian $-\Delta$

$$\mathcal{K}_{\ll} := \{V : \mathscr{D}(V) \to \mathscr{H} \, : \, \forall \alpha \in (0,1), \exists \beta \in [0,+\infty) \text{ s.t. } \|V\psi\|_2 \leqslant \alpha\| - \Delta\psi\|_2 + \beta\|\psi\|_2^2\}.$$

- If (X, Σ) is a measurable space, we denote by

$$\mathscr{M}(X; E) := \{\mu : \Sigma \to E \,|\, \mu(\varnothing) = 0, \mu \text{ is } \sigma - \text{additive}\}$$

  the space of measures on X with values in the normed space E which are $\sigma-$additive in the sense given in Chapter 2.
- In the Appendix we define the usual canonical observables in the Fock space.

1 Setting and Main Results

In this Chapter we describe in detail the three microscopic models we plan to study, in the quasi-classical regime, to derive the effective dynamics. Afterwards, we introduce the associated effective Hamiltonians, and, finally, we present the main results concerning the convergence of the dynamics.

First, we give an overview of their general structure: the systems under analysis are composed by two subsystems, one being a fixed number of quantum, non-relativistic particles, whose degrees of freedom are described by separable Hilbert space $\mathcal{K}$, which is invariant under the action of the effective limit we will take. The other subsystem is a quantum field, which can be interpreted or as an external field or a force-carrier field and that, at a microscopic description level, is composed by a variable number of bosons, whose relative dispersion relation depends on the type of the field. The field subsystem is studied in a regime justifying the approximation of its behaviour by its classical counterpart, as it will be explained in the theory developed in the following chapters. Its canonical observables are thus assumed to be dependent on a semiclassical parameter $\varepsilon > 0$. The chosen representation of its observables is the usual Fock representation, and so the associated Hilbert space is the $\varepsilon-$dependent Fock space $\Gamma_s(\mathfrak{H})$, with $\mathfrak{H}$ a separable Hilbert space encoding information on both the classical phase and one-boson spaces, and depending on the kind of field considered (mesons, phonons, photons, etc.).

1.1 Microscopic Hamiltonians

In this Section we define the Nelson, polaron and Pauli-Fierz model, at first presenting the general structure common to these three models, and then describing them in detail separately.

Let us start with the general framework: the Hilbert space of the full system is the tensor product of the aforementioned Hilbert spaces, *i.e.*,

$$\mathscr{H} := \mathcal{K} \otimes \Gamma_s(\mathfrak{H}). \tag{1.1}$$

Note that the results that we will obtain applies also to particles with spin, or following a specific statistics. Nevertheless, for simplicity of exposition, and to concentrate the attention on concrete models, we consider only quantum, non-relativistic, spin-less particles, without specifying their statistics. From now on, we denote their number by $N \in \mathbb{N}_*$. Considering their motion in a $d-$dimensional space, the natural choice for the space $\mathcal{K}$ is then

$$\mathcal{K} = L^2(\mathbb{R}^{dN}). \tag{1.2}$$

The Hamiltonians H_ε of the models of interest, acting on the full Hilbert space $\mathscr{H}$, are formal sum of two terms:

$$H_\varepsilon := H_0 + H_I, \tag{1.3}$$

where H_0 is the Hamiltonian describing the free dynamics of both the particles and the field:

$$H_0 := K \otimes \mathbb{1}_{\Gamma_s(\mathfrak{H})} + \nu_\varepsilon \mathbb{1}_{\mathcal{K}} \otimes \mathrm{Op}_\varepsilon^{\mathrm{Wick}}(\omega), \tag{1.4}$$

where K is a self-adjoint operator on its domain $\mathscr{D}(K) \subseteq \mathcal{K}$ and describes the free motion of the particles, while $\omega : \mathfrak{H} \to \mathbb{R}^+$ is a non-negative symbol defined on the one-boson space $\mathfrak{H}$, corresponding to the dispersion relation of the field. Its second quantization in Wick[1] formalism $\mathrm{Op}_\varepsilon^{\mathrm{Wick}}(\omega) = \mathrm{d}\Gamma_\varepsilon(\omega)$ is assumed to be a self-adjoint, densely defined operator on the Fock space that expresses the free energy of the field.

For $\varepsilon \in (0,1)$, the scaling factor ν_ε in front of the field's free energy varies, in general, in the interval $[1, \varepsilon^{-1}]$, but it has to be such that

$$\lim_{\varepsilon\to 0} \varepsilon\nu_\varepsilon =: \nu \in \{0,1\} \tag{1.5}$$

and determines whether, in the quasi-classical limit, the field is freezed or evolves freely, corresponding to the cases $\nu = 0$ and $\nu = 1$, respectively. For the sake of simplicity, we will thus consider only the cases for which $\nu_\varepsilon \in \{1, \varepsilon^{-1}\}$.

The interaction term between the particles and the field is of the form

$$H_I = \mathrm{Op}_\varepsilon^{\mathrm{Wick}}(\mathcal{V}). \tag{1.6}$$

This operator acts on the whole space $\mathscr{H}$ and is the second quantization of an operator-valued, polynomial symbol

$$\mathcal{V}(z) : \mathscr{D}(\mathcal{V}(z)) \subseteq \mathcal{K} \to \mathcal{K}, \qquad \text{for any } z \in \mathfrak{H}. \tag{1.7}$$

More precisely, we are going to focus on the sets of symbols

$$\mathscr{O}^{(p,q)} := \left\{ \mathfrak{H} \ni z \longmapsto \prod_{j=1}^{p} \left\langle z \middle| \lambda_x^{(j)} \right\rangle \prod_{k=1}^{q} \left\langle \eta_x^{(k)} \middle| z \right\rangle \;\middle|\; \left\{\lambda_\cdot^{(j)}\right\}_{j=1}^{p}, \left\{\eta_\cdot^{(k)}\right\}_{k=1}^{q} \subseteq L^\infty(\mathbb{R}^{dN}; \mathfrak{H}) \right\}. \tag{1.8}$$

Any symbol in $\mathscr{O}^{(p,q)}$, acting as bounded multiplication operator on $\mathcal{K}$, can be Wick-quantized yielding an operator in

$$\mathscr{O}_\varepsilon^{(p,q)} := \left\{ \prod_{j=1}^{p} a_\varepsilon^\dagger(\lambda_x^{(j)}) \prod_{k=1}^{q} a_\varepsilon(\lambda_x^{(k)}) \;\middle|\; \left\{\lambda_\cdot^{(j)}\right\}_{j=1}^{p}, \left\{\eta_\cdot^{(k)}\right\}_{k=1}^{q} \subseteq L^\infty(\mathbb{R}^{dN}; \mathfrak{H}) \right\}. \tag{1.9}$$

For any $z \in \mathfrak{H}$, we thus assume that

$$\mathcal{V}(z) \in \bigoplus_{(p,q)\in\mathbb{N}^2} T_{p,q}\, \mathscr{O}^{(p,q)} S_{p,q}, \tag{1.10}$$

so that,

$$H_I \in \bigoplus_{(p,q)\in\mathbb{N}^2} T_{p,q}\, \mathscr{O}_\varepsilon^{(p,q)} S_{p,q}, \tag{1.11}$$

where $T_{p,q}, S_{p,q}$ are (possibly unbounded) operators acting on $\mathcal{K}$ for every $(p,q) \in \mathbb{N}^2$.

In the following we present some precise models that fit the general form described above: the Nelson, polaron and Pauli-Fierz models.

1.1.1 Nelson model

The Nelson model was introduced by E. Nelson in [67] and was meant to describe the interaction between a finite number of non-relativistic nucleons and a meson field. The model has proved to be useful also in condensed matter in the description of phenomena arising from the interaction of a system of atoms with an optical lattice, created by superposition of laser beams.

In the same paper, Nelson also applied the first procedure of renormalization to a model of quantum field theory, obtaining the so-called renormalized Nelson model: it is proved how the

[1]For Wick and other kind of quantizations, see Appendix B.

Nelson Hamiltonian with a ultra-violet cut-off in the momenta converges, when suitably dressed by a unitary transformation, to what is defined as the renormalized Nelson Hamiltonian.

The same model was also studied then also in [21], where it was proved to satisfy some of the Wightman axioms of the quantum field theory. In [46] its ground state properties and its Haag-Ruelle scattering theory were studied. Since there, several works in the mathematical physics literature have been devoted to the Nelson model, like [3], [4], [29], [38], [48], [59], [61], [74].

In this thesis, we analyse the Nelson model with ultraviolet cut-off, postponing to future works the study of the renormalized Nelson model.

Among the models considered, the Nelson model is the most “regular" one and the simplest to study, due to the fact that the interaction term of the Hamiltonian is linear in the creation and annihilation operators and the arguments of the latters are the quantization of bounded symbols.

The Hilbert space associated to the system is

$$\mathscr{H} = L^2(\mathbb{R}^{dN}; \mathrm{d}x) \otimes \Gamma_s(\mathfrak{H}). \tag{1.12}$$

We denote by $x = (x_1, \ldots, x_N) \in \mathbb{R}^{dN}$ the vector of particle positions. The Fock space describes the radiation, with one-boson space given by

$$\mathfrak{H} = L^2(\mathbb{R}^d; \mathrm{d}k). \tag{1.13}$$

The dynamics of the system is generated by the Nelson Hamiltonian (with suitable hypotheses guaranteeing self-adjointness, see below), whose explicit form is

$$H_\varepsilon = H_0 + H_I. \tag{1.14}$$

The free part of the Hamiltonian is

$$H_0 = (-\Delta_x + W(x_1, \ldots, x_N)) \otimes \mathbb{1}_{\Gamma_s(\mathfrak{H})} + \mathbb{1}_{L^2(\mathbb{R}^{dN})} \otimes \nu_\varepsilon \mathrm{d}\Gamma_\varepsilon(\omega), \tag{1.15}$$

where $-\Delta_x := \sum_{j=1}^N -\Delta_{x_j}$, and W is an interaction potential between the particles satisfying the following assumption:

$$W = W_+ + W_\ll, \qquad W_+ \in L^1_{\mathrm{loc}}(\mathbb{R}^{dN}; \mathbb{R}^+), \quad W_\ll \in \mathcal{K}_\ll. \tag{1.16}$$

The free energy of the field $\mathrm{d}\Gamma_\varepsilon(\omega)$ is the Wick quantization of the symbol $z \mapsto \langle z|\, \omega\, |z\rangle$ with ω that acts as a multiplication operator on the space $\mathfrak{H}$ by the function

$$\omega(k) \geqslant m > 0, \qquad \text{for a.e. } k \in \mathbb{R}^d. \tag{1.17}$$

The previous hypothesis restricts our analysis to the case of the Nelson model with massive bosons in the radiation, excluding the more delicate case of Nelson massless model, for $m = 0$.

The interaction term between the particles and the field is in the form of a Segal field, *i.e.*,

$$H_I = \phi_\varepsilon(\lambda_x) = a_\varepsilon(\lambda_x) + a^\dagger_\varepsilon(\lambda_x), \tag{1.18}$$

and the argument of the creation and annihilation operators belongs uniformly w.r.t. x to the one-boson space:

$$\lambda_\cdot \in L^\infty(\mathbb{R}^{dN}; \mathfrak{H}). \tag{1.19}$$

By Wick calculus it is easy to see that H_I is the Wick quantization of the effective potential

$$\mathcal{V}(z) = \sum_{j=1}^N 2\Re e \left\langle \lambda_{x_j} \,|z \right\rangle_{\mathfrak{H}} \tag{1.20}$$

which is a bounded multiplication operator on $L^2(\mathbb{R}^{dN})$. The Nelson Hamiltonian has, therefore, the following expression

$$\boxed{H_\varepsilon = -\Delta_x + W(x_1, \ldots, x_N) + \nu_\varepsilon \mathrm{d}\Gamma_\varepsilon(\omega) + a_\varepsilon(\lambda_x) + a^\dagger_\varepsilon(\lambda_x)} \tag{1.21}$$

Remark 1.1.1. • *An explicit choice of the dispersion relation may be*

$$\omega(k) = \sqrt{k^2 + m^2}, \qquad m > 0, \tag{1.22}$$

representing a relativistic massive field. A physically meaningful choice for the argument of the Segal field is

$$\lambda_x(k) = \chi(k) \sum_{j=1}^{N} e^{ikx_j}, \tag{1.23}$$

where $\chi \in \mathfrak{H}$ is a cut-off at high momenta, i.e., for a fixed $R > 0$,

$$\chi(k) = \chi_{\{|k|<R\}}(k) = \begin{cases} 1, & \text{if } |k| < R; \\ 0, & \text{if } |k| \geqslant R. \end{cases}$$

Hence, only the interaction of the particles with bosons under a certain fixed speed is taken into account.

- *Another interesting possibility to consider is $\mathfrak{H} = \ell^2(\mathbb{Z}^d; \mathbb{C})$, i.e., a field with only a countable number of modes. Such a model can be used to describe a system of particles in an optical lattice. In this case the full Hamiltonian has the form*

$$H_\varepsilon = -\Delta_x + W(x_1, \ldots, x_N) + \nu_\varepsilon \sum_{n \in \mathbb{Z}^d} \omega_n a_n^\dagger a_n + \sum_{n \in \mathbb{Z}^d} (a_n^\dagger + a_n)\lambda_{x,n} \tag{1.24}$$

with $\{\omega_n\}_{n\in\mathbb{Z}^d} \subseteq \mathbb{R}^+$, $\{\lambda_{\cdot,n}\}_{n\in\mathbb{Z}^d} \in \ell^\infty(\mathbb{R}^{dN}; \ell^2(\mathbb{Z}^d))$ and $a_\cdot^\#$ the operator-valued distributions associated to the creation and annihilation operators satisfying

$$a_\varepsilon^\#(\lambda) = \sum_{n\in\mathbb{Z}^d} a_n^\# \lambda_n, \qquad \textit{for any } \lambda \in \ell^2(\mathbb{Z}^d; \mathbb{C}).$$

In the next Theorem we prove that the assumptions above are sufficient for the essential self-adjointness of the Hamiltonian on a suitable core, and, adding a further control on the λ, also for self-adjointness. We denote by $\Gamma_{\text{fin}}(\mathfrak{H}) \subset \Gamma_s(\mathfrak{H})$ the space of vectors with finitely many non-zero components.

Theorem 1.1.2. *Under assumptions* (1.16) *and* (1.19)*, the Nelson Hamiltonian H_ε is essentially self-adjoint on $\Gamma_{\text{fin}}(\mathfrak{H}) \cap \mathscr{D}(H_0)$, with $\mathscr{D}(H_0) = \mathscr{D}(-\Delta_x + W_+) \cap \mathscr{D}(\mathrm{d}\Gamma_\varepsilon(\omega))$. Furthermore, if we assume that $\omega^{-1/2}\lambda_\cdot \in L^\infty(\mathbb{R}^{dN}; \mathfrak{H})$, then, H_ε is self-adjoint on the domain $\mathscr{D}(H_\varepsilon) = \mathscr{D}(H_0)$ and bounded from below.*

Proof. H_ε satisfies the assumptions of [41, Theorem 3.1], which directly implies the essential self-adjointness.

By the canonical commutation relations, we have, for any $\Psi \in \mathscr{D}(\mathrm{d}\Gamma_\varepsilon(\mathbb{1})^{1/2})$,

$$\|a_\varepsilon^\dagger(\lambda_x)\Psi\|_{\Gamma_s}^2 = \|a_\varepsilon(\lambda_x)\Psi\|_{\Gamma_s}^2 + \|\lambda_\cdot\|_\infty^2 \|\Psi\|_{\Gamma_s}^2 \tag{1.25}$$

where we denoted by $\|\cdot\|_\infty$ the norm in the space $L^\infty(\mathbb{R}^{dN}; \mathfrak{H})$. Using the previous inequality and Proposition A.1.2, we have

$$\|\phi_\varepsilon(\lambda_x)\Psi\|_{\mathscr{H}} \leqslant 2\|\omega^{-1/2}\lambda_\cdot\|_\infty \, \|\mathrm{d}\Gamma_\varepsilon(\omega)^{1/2}\Psi\|_{\mathscr{H}} + \|\lambda_\cdot\|_\infty \|\Psi\|_{\mathscr{H}}. \tag{1.26}$$

Now, using the inequality $2ab \leqslant \alpha a^2 + \frac{1}{\alpha}b^2$ for any $a, b, \alpha > 0$, and Cauchy-Schwarz inequality yield (remember that $\nu_\varepsilon \geqslant 1$)

$$2\|\omega^{-1/2}\lambda_.\|_\infty \, \|\mathrm{d}\Gamma_\varepsilon(\omega)^{1/2}\Psi\|_{\mathscr{H}} \leqslant 2\|\omega^{-1/2}\lambda_.\|_\infty \langle \Psi | \left(-\Delta_x + W_+ + \nu_\varepsilon \mathrm{d}\Gamma_\varepsilon(\omega)\right) |\Psi\rangle^{1/2} \leqslant$$
$$\leqslant \alpha \|(-\Delta_x + W_+ + \nu_\varepsilon \mathrm{d}\Gamma_\varepsilon(\omega))\Psi\|_{\mathscr{H}} + \frac{1}{\alpha}\|\omega^{-1/2}\lambda_.\|_\infty^2 \|\Psi\|_{\mathscr{H}}$$

so that,

$$\|\phi_\varepsilon(\lambda_x)\Psi\|_{\mathscr{H}} \leqslant \alpha \|(-\Delta_x + W_+ + \nu_\varepsilon \mathrm{d}\Gamma_\varepsilon(\omega))\Psi\|_{\mathscr{H}} + b(\alpha)\|\Psi\|_{\mathscr{H}} \tag{1.27}$$

with $b(\alpha) = \alpha^{-1}\|\omega^{-1/2}\lambda_.\|_\infty + \|\lambda_.\|_\infty$. Furthermore, since $W_\ll \in \mathcal{K}_\ll$, there exist $\alpha' \in (0,1), b'(\alpha') \in [0, +\infty)$, such that

$$\|W_\ll \Psi\|_{\mathscr{H}} \leqslant \alpha' \| - \Delta_x \Psi\|_{\mathscr{H}} + b'(\alpha')\|\Psi\|_{\mathscr{H}} \leqslant$$
$$\leqslant \alpha' \|(-\Delta_x + W_+ + \nu_\varepsilon \mathrm{d}\Gamma_\varepsilon(\omega))\Psi\|_{\mathscr{H}} + b'(\alpha')\|\Psi\|_{\mathscr{H}}.$$

The proof is concluded by choosing $\alpha + \alpha' < 1$ and using the Kato-Rellich Theorem, which also implies that the operator H_ε is bounded from below by

$$-M := \sup_{1/4 < \alpha, \alpha' < 1/2} -(b(\alpha) + b'(\alpha')) > -\infty. \tag{1.28}$$

□

Remark 1.1.3. *We remark that the Theorem holds true also in the case of the Nelson massless model. Defining*

$$\mathfrak{H}_\omega := L^2(\mathbb{R}^d; \omega(k)\mathrm{d}k) \tag{1.29}$$

the assumptions (1.19) *and* $\omega^{-1/2}\lambda_. \in L^\infty(\mathbb{R}^{dN}; \mathfrak{H})$ *can be reformulated by requiring that*

$$\lambda_. \in L^\infty(\mathbb{R}^{dN}; \mathfrak{H} \cap \mathfrak{H}_{\omega^{-1}}). \tag{1.30}$$

From now on we assume the hypotheses above guaranteeing the self-adjointness of the Nelson Hamiltonian, together with (1.17) *about coerciveness of the dispersion relation.*

Furthermore, our results will hold true with the following assumption, requiring an explicit formula for the field argument of the Nelson model:

$$\lambda_x(k) := \chi(k) \sum_{j=1}^{N} e^{ik\cdot x_j}, \qquad x \in \mathbb{R}^{dN}, \chi \in L^2(\mathbb{R}^d; \mathrm{d}k) \quad \textit{with } \chi \geqslant m > 0. \tag{A$_{\mathrm{Nel}}$}$$

1.1.2 Polaron model

An interesting example that fits in the class of models of interaction between quantum fields and particles described in Section 1.1 is the one describing an electron moving in a crystal of ions. Indeed, the electron induces a polarization of the crystal by electro-magnetic attraction-repulsion forces: the ions are perturbed around their equilibrium positions and vibrate. The modes of vibration of the lattice can be interpreted as a field of excitations whose force-carriers are bosons called phonons. The electron is so considered as dressed by a cloud of phonons, and can be modeled as a new quasi-particle with a larger effective mass and a lower speed, which moves in the crystal and deforms the lattice creating or destroying excitations, *i.e.*, phonons. Such a quasi-particle is called polaron, a notion used for the first time by L. D. Landau in [57]. The theoretical model for the dynamics of the polaron that we are going to present was introduced by H. Fröhlich in [46] and it is known as the (Fröhlich) polaron model.

The full Hilbert space for the system is

$$\mathscr{H} = L^2(\mathbb{R}^{dN}; \mathrm{d}x) \otimes \Gamma_s(\mathfrak{H}), \tag{1.31}$$

where the first factor is associated to N polarons moving in $\mathbb{R}^d$ and interacting with a field of phonons with one-boson space

$$\mathfrak{H} = L^2(\mathbb{R}^d; \mathrm{d}k). \tag{1.32}$$

The Hamiltonian is given by the formal expression

$$H_\varepsilon = H_0 + H_I \tag{1.33}$$

with

$$H_0 = (-\Delta_x + W(x_1, \ldots, x_N)) \otimes \mathbb{1}_{\Gamma_s(\mathfrak{H})} + \nu_\varepsilon \mathbb{1}_{L^2(\mathbb{R}^{dN})} \otimes \mathrm{d}\Gamma_\varepsilon(\mathbb{1}), \tag{1.34}$$

W being a potential satisfying the same hypotheses as for the Nelson model:

$$W = W_+ + W_\ll, \qquad W_+ \in L^1_{\mathrm{loc}}(\mathbb{R}^{dN}; \mathbb{R}^+), \quad W_\ll \in \mathcal{K}_\ll, \tag{1.35}$$

and $\mathrm{d}\Gamma_\varepsilon(\mathbb{1})$ the number operator corresponding to the free energy of the phonons with dispersion relation

$$\omega(k) = 1, \qquad \text{for any } k \in \mathbb{R}^d. \tag{1.36}$$

Note that $\mathrm{d}\Gamma_\varepsilon(\mathbb{1})$ is in fact the Wick quantization of the symbol $z \mapsto \|z\|^2_{\mathfrak{H}}$. Setting

$$g_{x_j}(k) = \frac{e^{ikx_j}}{|k|^{\frac{d-1}{2}}} \tag{A$_{\mathrm{Pol}}$}$$

the interaction term is

$$H_I = \sum_{j=1}^{N} (a_\varepsilon(g_{x_j}) + a^\dagger_\varepsilon(g_{x_j})). \tag{1.37}$$

As we see, $g_{x_j} \notin L^2(\mathbb{R}^d)$, and therefore, unlike in the Nelson model, the term H_I is not a well-posed operator on $\mathscr{H}$. However, it is possible to interpret H_ε as a quadratic form Q_H on the full Hilbert space whose action is $Q_H[\Psi] = \langle \Psi | \, H_\varepsilon \, |\Psi\rangle_{\mathscr{H}}$ with

$$\boxed{H_\varepsilon = -\Delta_x + W(x_1, \ldots, x_N) + \nu_\varepsilon \mathrm{d}\Gamma_\varepsilon(\mathbb{1}) + \sum_{j=1}^{N} (a_\varepsilon(g_{x_j}) + a^\dagger_\varepsilon(g_{x_j})).} \tag{1.38}$$

Next, we associate a self-adjoint operator to the quadratic form by KLMN Theorem, showing that the interaction is infinitesimally form-bounded w.r.t. the free part. To this purpose, we need the following Lemma which is going to be used throughout the thesis.

Lemma 1.1.4. *One can decompose* $g_y = g_{<,y} + g_{>,y}$ *for any* $y \in \mathbb{R}^d$, *with*

$$g_{<,\cdot} \in L^\infty(\mathbb{R}^d; \mathfrak{H}),$$
$$g_{>,y}(k) = [-i\nabla_y \,,\, \tilde{g}_y(k)], \qquad \tilde{g}_\cdot \in L^\infty(\mathbb{R}^d; \mathfrak{H}^d)$$

by taking, e.g.,

$$g_{<,y}(k) = \frac{e^{iky}}{|k|^{\frac{d-1}{2}}} \chi_{\{|k|<r\}}(k), \qquad \tilde{g}_y(k) = \frac{k}{|k|^{\frac{d+3}{2}}} e^{iky} \chi_{\{|k|\geqslant r\}}(k) \tag{1.39}$$

with $r \in (0, +\infty)$.

Proof. Let us proceed by direct calculation:

$$\|g_{<,y}\|^2_{\mathfrak{H}} = \int_{|k|<r} \mathrm{d}k\, \frac{1}{|k|^{d-1}} < +\infty; \qquad \|\tilde{g}_y\|^2_{\mathfrak{H}\otimes\mathbb{C}^d} = \int_{|k|\geqslant r} \mathrm{d}k\, \frac{|k|^2}{|k|^{d+3}} < +\infty \tag{1.40}$$

and so the functions $g_{<,\cdot}$ and $\tilde{g}_\cdot$ belong to $\mathfrak{H}$ and $\mathfrak{H}\otimes\mathbb{C}^d$, respectively, uniformly in y. Furthermore, since

$$\begin{aligned} -i\nabla_y \cdot \tilde{g}_y(k) &= \sum_{l=1}^{d} \left(\frac{|k_l|^2}{|k|^{\frac{d+3}{2}}} e^{iky} + \frac{k_l}{|k|^{\frac{d+3}{2}}} e^{iky}(-i\partial_y) \right) \chi_{\{|k|\geqslant r\}}(k) = \\ &= \frac{e^{iky}}{|k|^{\frac{d-1}{2}}} \chi_{\{|k|\geqslant r\}}(k) + \tilde{g}_y(k)\cdot(-i\nabla_y) \end{aligned}$$

setting $g_{>,y}(k) := \frac{e^{iky}}{|k|^{\frac{d-1}{2}}}\chi_{\{|k|\geqslant r\}}(k)$, the decomposition is proven. □

Thanks to the previous Lemma we are now ready to prove the existence of a self-adjoint operator associated with the polaron quadratic form.

Theorem 1.1.5. *Under assumptions* (1.35), (1.36), $(\mathrm{A_{Pol}})$, *the quadratic form* Q_H *is closed and bounded from below and identifies a unique self-adjoint operator, still denoted by* H_ε, *on*

$$\mathscr{D}(H_\varepsilon) \subset \mathscr{Q}(H_\varepsilon) = \mathscr{Q}(H_0) = \mathscr{D}((-\Delta_x + W_+)^{1/2}) \cap \mathscr{D}(\mathrm{d}\Gamma_\varepsilon(\mathbb{1})^{1/2}). \tag{1.41}$$

Proof. Let us prove the infinitesimal form boundedness of the w.r.t. the positive part. Since $W_\ll$ is infinitesimally $-\Delta_x-$bounded, there exist $a \in (0,1), b \in [0,+\infty)$, such that

$$|\langle \Psi \,|\, W_\ll \Psi \rangle| \leqslant a \langle \Psi | -\Delta_x + W_+ + \nu_\varepsilon \mathrm{d}\Gamma_\varepsilon(\mathbb{1}) \,|\Psi\rangle + b\|\Psi\|^2. \tag{1.42}$$

By Lemma 1.1.4, for any $j = 1,\dots,N$,

$$g_{x_j} = g_{<,x_j} + [-i\nabla_{x_j}\,,\, \tilde{g}_{x_j}] \tag{1.43}$$

so that, by the trivial inequality,

$$\left| \sum_{j=1}^{N} \left\langle \Psi \,\middle|(a_\varepsilon(g_{x_j}) + a_\varepsilon^\dagger(g_{x_j}))\Psi \right\rangle \right| \leqslant 2 \sum_{j=1}^{N} \left|\left\langle \Psi \,\middle| a_\varepsilon(g_{x_j})\Psi \right\rangle\right|$$

we obtain that, for $\Psi \in \mathscr{D}((-\Delta_x + W_+)^{1/2}) \cap \mathscr{D}(\mathrm{d}\Gamma_\varepsilon(\mathbb{1})^{1/2})$,

- for every $\alpha \in (0,+\infty)$ we have

$$\begin{aligned} 2\sum_{j=1}^{N} \left|\left\langle \Psi \,\middle| a_\varepsilon(g_{<,x_j})\Psi \right\rangle\right| &\leqslant 2N\|\Psi\|\|g_{<,x_j}\|_{L^\infty(\mathbb{R}^d;\mathfrak{H})}\|\mathrm{d}\Gamma_\varepsilon(\mathbb{1})^{1/2}\Psi\| \leqslant \\ &\leqslant \alpha \langle \Psi | -\Delta_x + W_+ + \nu_\varepsilon \mathrm{d}\Gamma_\varepsilon(\mathbb{1}) \,|\Psi\rangle + \frac{N^2}{\alpha}\|g_{<,\cdot}\|^2_\infty \|\Psi\|^2, \end{aligned}$$

- while for the second term we have

$$\begin{aligned} 2\sum_{j=1}^{N} \left|\left\langle \Psi \,\middle|[i\nabla_{x_j}, a_\varepsilon(\tilde{g}_{x_j})]\Psi \right\rangle\right| &\leqslant 4\sum_{j=1}^{N} \|\nabla_{x_j}\Psi\|\|\tilde{g}_\cdot\|_{L^\infty(\mathbb{R}^d;\mathfrak{H}\otimes\mathbb{C}^d)}\|\mathrm{d}\Gamma_\varepsilon(\mathbb{1})^{1/2}\Psi\| \leqslant \\ &\leqslant 2N\|\tilde{g}_\cdot\|_{L^\infty(\mathbb{R}^d;\mathfrak{H}\otimes\mathbb{C}^d)} \langle \Psi | -\Delta_x + W_+ + \nu_\varepsilon \mathrm{d}\Gamma_\varepsilon(\mathbb{1}) \,|\Psi\rangle. \end{aligned}$$

Since a is fixed, choosing α and $r \in (0, +\infty)$ such that $\alpha + a + 2N\|\tilde{g}_\cdot\|_\infty =: \alpha' < 1$ and setting $b' := b + N^2\alpha^{-1}\|g_\cdot\|_\infty^2$, we get

$$\left|\left\langle \Psi \middle| \left(W_\ll + \sum_{j=1}^{N} (a_\varepsilon(g_{x_j}) + a_\varepsilon^\dagger(g_{x_j})) \right) \Psi \right\rangle\right| \leqslant \alpha' \langle \Psi | -\Delta_x + W_+ + \nu_\varepsilon \mathrm{d}\Gamma_\varepsilon(\mathbb{1}) |\Psi\rangle + b'\|\Psi\|^2, \quad (1.44)$$

and the result follows by KLMN Theorem. □

As a byproduct, Lemma 1.1.4 shows that the interaction in the polaron model can be written as the Wick quantization of the symbol

$$\mathcal{V}(z) = \sum_{j=1}^{N} \left\{ 2\Re e \left\langle g_{<,x_j} | z \right\rangle + [-i\nabla_{x_j}, 2\Im m \left\langle \tilde{g}_{x_j} | z \right\rangle] \right\}. \quad (1.45)$$

From now on so, we will denote by H_ε the self-adjoint Hamiltonian identified in Theorem 1.1.5, whose associated quadratic form has the explicit expression (1.38), and we will always assume the following form for the argument of the field operators:

$$g_y(k) = \frac{e^{iky}}{|k|^{\frac{d-1}{2}}} = g_{<,y}(k) + [-i\nabla_y, \tilde{g}_y(k)], \qquad y, k \in \mathbb{R}^d,$$

$$g_{<,y}(k) = \frac{e^{iky}}{|k|^{\frac{d-1}{2}}} \chi_{\{|k|<r\}}(k), \qquad \tilde{g}_y(k) = \frac{k}{|k|^{\frac{d+3}{2}}} e^{iky} \chi_{\{|k|\geqslant r\}}(k). \quad (\mathrm{A_{Pol}})$$

1.1.3 Pauli-Fierz model

In quantum electrodynamics, the Pauli-Fierz model is often used to describe systems of non-relativistic charged quantum particles interacting with a quantized electromagnetic field. It was introduced by W. Pauli and M. Fierz in [69] to investigate the problem of infrared divergences appearing in the radiative corrections to the scattering of an electron by the Coulomb field generated by a nucleus. Indeed, F. Bloch and A. Nordsieck in [17] showed how this infrared catastrophe could be avoided using a Born approximation method. Pauli and Fierz, in their work, followed this strategy and attacked other difficulties such as the divergence of the electrostatic energy for point charges: they considered instead the quantization of the classical Abraham model (see, for example, [77] for its derivation) with extended charges, in order to obtain a bounded electrostatic energy.

The Hilbert space of the model is

$$\mathscr{H} = L^2(\mathbb{R}^{dN}; \mathrm{d}x) \otimes \Gamma_s(\mathfrak{H}) \quad (1.46)$$

where, as before, $L^2(\mathbb{R}^{dN})$ is the Hilbert space of N non-relativistic, spinless, extended charged quantum particles in $\mathbb{R}^d$. Unlike the previous models, however, the one-photon space is now

$$\mathfrak{H} = \mathbb{C}^{d-1} \otimes L^2(\mathbb{R}^d; \mathrm{d}k) = L^2(\mathbb{R}^d; \mathbb{C}^{d-1}; \mathrm{d}k) \quad (1.47)$$

i.e., vectors with $d-1$ components, given by square integrable functions in $\mathbb{R}^d$. Every component corresponds to a polarization direction of the photons. We use the following notation: for a vector $f = (f_\gamma)_{\gamma=1}^{d-1} \in \mathfrak{H}$, we set

$$a_\varepsilon^\#(f) := (a_1^\#(f_1), \ldots, a_{d-1}^\#(f_{d-1})), \quad (1.48)$$

so that one has the canonical commutation relations

$$[a_\varepsilon(f), a_\varepsilon^\dagger(g)] = \varepsilon \sum_{\gamma=1}^{d-1} \langle f_\gamma | g_\gamma \rangle_2 = \varepsilon \langle f | g \rangle_{\mathfrak{H}}. \quad (1.49)$$

The Pauli-Fierz Hamiltonian is

$$H_\varepsilon = H_0 + H_I, \tag{1.50}$$

where for the free part we have

$$H_0 = (-\Delta_x + W(x_1, \ldots, x_N)) \otimes \mathbb{1}_{\Gamma_s(\mathfrak{H})} + \nu_\varepsilon \mathbb{1}_{L^2(\mathbb{R}^{dN})} \otimes \mathrm{d}\Gamma_\varepsilon(\omega) \tag{1.51}$$

with $\omega = (\omega_\gamma)_{\gamma=1}^{d-1} : \mathfrak{H} \to \mathfrak{H}$ acts as, componentwise, a multiplication by a vector of positive functions (*e.g.*, $\omega_\gamma(k) = |k|$, for any $\gamma = 1, \ldots d-1$), satisfying

$$\exists\, \omega^{-1} > 0, \text{ densely defined, self-adjoint operator on } \mathscr{D}(\omega^{-1}) \subseteq \mathfrak{H}. \tag{1.52}$$

The free field energy has explicit action, sectorwise,

$$\mathrm{d}\Gamma_\varepsilon(\omega)\Psi_1 \otimes_s \ldots \otimes_s \Psi_n = \sum_{j=1}^{n} \Psi_1 \otimes_s \ldots \otimes_s \omega\Psi_j \otimes_s \ldots \otimes_s \Psi_n \tag{1.53}$$

where

$$\omega\Psi_j = \sum_{\gamma=1}^{d-1} \omega_\gamma \Psi_{j,\gamma} \tag{1.54}$$

with $\Psi_{j,\gamma}$ the $\gamma-$th component of the vector $\Psi_j \in L^2(\mathbb{R}^d; \mathbb{C}^{d-1})$. The potential between particles satisfies

$$W = W_+ + W_\ll, \qquad W_+ \in L^1_{\mathrm{loc}}(\mathbb{R}^{dN}; \mathbb{R}^+), \quad W_\ll \in \mathcal{K}_\ll. \tag{1.55}$$

The definition of the interaction term, on the other hand, is more involved due to the polarizations we set and the fact that it contains quadratic terms in creation and annihilation operators:

$$H_I = \sum_{j=1}^{N} \left(2i\nabla_{x_j} \cdot q_j \phi_\varepsilon(\lambda_{x_j}) + q_j^2 \phi_\varepsilon(\lambda_{x_j})^2\right) \tag{1.56}$$

where $\{q_j\}_{j=1}^N \subseteq \mathbb{R}$ are the charges of the particles,

$$\lambda_{x_j}(k) = \Big(e_\gamma(k) f_{x_j}(k)\Big)_{\gamma=1,\ldots,d-1} \tag{1.57}$$

is the form factor with $e_\gamma(k) \in \mathbb{R}^d$ for any $k \in \mathbb{R}^d$, so that

$$a_\varepsilon^\#(\lambda_{x_j}) = \sum_{\gamma=1}^{d-1} a_\gamma^\#(e_\gamma f_{x_j}). \tag{1.58}$$

We assume that

$$f_\cdot \in L^\infty(\mathbb{R}^d; L^2(\mathbb{R}^d)), \qquad \lambda_\cdot \in L^\infty(\mathbb{R}^d; \mathbb{R}^d \otimes \mathscr{D}(\omega^{1/2}) \cap \mathscr{D}(\omega^{-1/2})). \tag{1.59}$$

The set of vectors $\{\frac{k}{|k|}, (e_\gamma(k))_{\gamma=1}^{d-1}\}$ identifies an orthonormal basis of $\mathbb{R}^d$ for a.e. $k \in \mathbb{R}^d$, *i.e.*,

$$e_\sigma(k) \cdot e_\tau(k) = \delta_{\sigma,\tau}, \qquad k \cdot e_\gamma(k) = 0. \tag{1.60}$$

The vector k yields the direction of propagation of the field's wave, and e_γ are the vectors of polarization of the transverse waves associated with the photons[2]. Condition (1.60) is the choice of the *Coulomb gauge* for the magnetic potential $\phi_\varepsilon(\lambda_x)$, *i.e.*, encodes

$$\nabla_y \cdot \lambda_y = 0, \qquad \forall y \in \mathbb{R}^d. \tag{1.61}$$

[2]In dimension 3 they provide the two directions of oscillation of the electric and magnetic fields, respectively.

From the explicit expression of the interaction term of the Hamiltonian one can easily see that it is the Wick quantization of the effective potential

$$\mathcal{V}(z) = \sum_{j=1}^{N} \left(2i\nabla_{x_j} \cdot q_j \, 2\Re e \left\langle z \,\middle|\, \lambda_{x_j} \right\rangle_{\mathfrak{H}} + \left(q_j \, 2\Re e \left\langle z \,\middle|\, \lambda_{x_j} \right\rangle_{\mathfrak{H}} \right)^2 \right), \tag{1.62}$$

up to an additional term

$$\varepsilon B_\varepsilon = \varepsilon \sum_{j=1}^{N} \|\lambda_{x_j}\|_{\mathfrak{H}}^2 \tag{1.63}$$

which disappear in the limit $\varepsilon \to 0$, needed to take into account the normal ordering. In conclusion, the Pauli-Fierz Hamiltonian can be rewritten as

$$\boxed{H_\varepsilon = \sum_{j=1}^{N} (-i\nabla_{x_j} - q_j \, \phi_\varepsilon(\lambda_{x_j}))^2 + W + \nu_\varepsilon \mathrm{d}\Gamma_\varepsilon(\omega).} \tag{1.64}$$

We prove now that under the previous assumptions the Pauli-Fierz Hamiltonian is self-adjoint on its domain. We state two results: the first one follows from an application of the KLMN theorem, which requires a condition on the charge. The second requires less stringent assumptions (results on self-adjointness with more restrictive assumptions are however available in literature: [52], [55], [77]).

Theorem 1.1.6. *Let assumptions* (1.52)*,* (1.55)*,* (1.59) *and* (1.60) *hold. Suppose, furthermore that* $|q_j| \leqslant q_0$ *for any* $j = 1, \ldots, N$*, for a certain* $q_0 > 0$ *(specified in the proof). Then, the Pauli-Fierz Hamiltonian* H_ε *is self-adjoint and bounded from below on its domain* $\mathscr{D}(H_\varepsilon) := \mathscr{D}(H_0) = \mathscr{D}(-\Delta_x + W_+) \cap \mathscr{D}(\mathrm{d}\Gamma_\varepsilon(\omega))$.

Proof. Consider the quadratic form $Q_H(\Psi) := \langle \Psi \,|\, H_\varepsilon \Psi \rangle$ for $\Psi \in \mathscr{D}(H_0)$. We want to prove the form boundedness of the interaction w.r.t. the free part. We already know it for $W_\ll$: there exist $\alpha' \in (0,1), \beta \in [0, +\infty)$ such that, for any $\Phi_\varepsilon \in \mathscr{D}(H_0)$,

$$\langle \Phi_\varepsilon | \, W_\ll \, | \Phi_\varepsilon \rangle \leqslant \alpha' \, \langle \Phi_\varepsilon | - \Delta_x + W_+ + \nu_\varepsilon \mathrm{d}\Gamma_\varepsilon(\omega) \, | \Phi_\varepsilon \rangle + \beta' \|\Phi_\varepsilon\|^2. \tag{1.65}$$

On the other hand, for any $\Phi_\varepsilon \in \mathscr{D}(H_0)$, we can bound the second term on the r.h.s. of (1.56) as

$$\sum_{j=1}^{N} \langle \Phi_\varepsilon | \, q_j^2 (a_\varepsilon(\lambda_{x_j}) + a_\varepsilon^\dagger(\lambda_{x_j}))^2 \, | \Phi_\varepsilon \rangle \leqslant$$

$$\leqslant \sum_{j=1}^{N} \left(2q_j^2 \, \langle \Phi_\varepsilon | \, a_\varepsilon^\dagger(\lambda_{x_j}) a_\varepsilon(\lambda_{x_j}) \, | \Phi_\varepsilon \rangle + 2q_j^2 \Re e \left\langle \Phi_\varepsilon \,\middle|\, a_\varepsilon^\dagger(\lambda_{x_j}) a_\varepsilon^\dagger(\lambda_{x_j}) \Phi_\varepsilon \right\rangle + \varepsilon q_j^2 \|\lambda_{x_j}\|^2 \|\Phi_\varepsilon\|^2 \right).$$

Now, using Proposition A.1.2 and Cauchy-Schwarz inequality, we obtain the bound for the previous expression:

$$\sum_{j=1}^{N} \langle \Phi_\varepsilon | \, q_j^2 (a_\varepsilon(\lambda_{x_j}) + a_\varepsilon^\dagger(\lambda_{x_j}))^2 \, |\Phi_\varepsilon\rangle \leqslant \tag{1.66}$$

$$\leqslant \sum_{j=1}^{N} q_j^2 \Big(2\|a_\varepsilon(\lambda_{x_j})\Phi_\varepsilon\|^2 + 2\|a_\varepsilon(\lambda_{x_j})\Phi_\varepsilon\|\|a_\varepsilon^\dagger(\lambda_{x_j})\Phi_\varepsilon\| + \varepsilon\|\lambda_{x_j}\|^2\|\Phi_\varepsilon\|^2 \Big) \leqslant$$

$$\leqslant \sum_{j=1}^{N} q_j^2 \Big(4\|a_\varepsilon(\lambda_{x_j})\Phi_\varepsilon\|^2 + 2\varepsilon\|\lambda_{x_j}\|^2\|\Phi_\varepsilon\|^2 \Big) \leqslant$$

$$\leqslant \sum_{j=1}^{N} \Big\{ 4q_j^2 \|\omega^{-1/2}\lambda_{x_j}\|_{\mathfrak{H}}^2 \langle \Phi_\varepsilon | \, \nu_\varepsilon \mathrm{d}\Gamma_\varepsilon(\omega) \, |\Phi_\varepsilon\rangle + 2\varepsilon q_j^2 \|\lambda_{x_j}\|^2 \|\Phi_\varepsilon\|^2 \Big\} \leqslant$$

$$\leqslant 4Nq_0^2 \|\omega^{-1/2}\lambda_\cdot\|_\infty^2 \langle \Phi_\varepsilon | \, (-\Delta_x + W_+ + \nu_\varepsilon \mathrm{d}\Gamma_\varepsilon(\omega)) \, |\Phi_\varepsilon\rangle + 2\varepsilon N q_0^2 \|\lambda_\cdot\|_{L^\infty}^2 \|\Phi_\varepsilon\|^2 \tag{1.67}$$

Moreover, following the same strategy for the first term, one has

$$\sum_{j=1}^{N} 2|q_j| \, \big| \big\langle \nabla_{x_j}\Phi_\varepsilon \, \big| (a_\varepsilon(\lambda_{x_j}) + a_\varepsilon^\dagger(\lambda_{x_j}))\Phi_\varepsilon \big\rangle \big| \leqslant$$

$$\leqslant \sum_{j=1}^{N} 2|q_j| \|\nabla_{x_j}\Phi_\varepsilon\| \|(a_\varepsilon(\lambda_{x_j}) + a_\varepsilon^\dagger(\lambda_{x_j}))\Phi_\varepsilon\| \leqslant$$

$$\leqslant \sum_{j=1}^{N} \Big(\langle \Phi_\varepsilon | - \Delta_{x_j} \, |\Phi_\varepsilon\rangle + q_j^2 \big\langle \Phi_\varepsilon \, \big| (a_\varepsilon(\lambda_{x_j}) + a_\varepsilon^\dagger(\lambda_{x_j}))^2 \Phi_\varepsilon \big\rangle \Big) \leqslant$$

$$\leqslant \Big(1 + 4Nq_0^2 \|\omega^{-1/2}\lambda_\cdot\|_{L^\infty}^2 \Big) \langle \Phi_\varepsilon | - \Delta_x + W_+ + \nu_\varepsilon \mathrm{d}\Gamma_\varepsilon(\omega) \, |\Phi_\varepsilon\rangle + 2\varepsilon N q_0^2 \|\lambda_\cdot\|_{L^\infty}^2 \|\Phi_\varepsilon\|^2. \tag{1.68}$$

Putting together the three inequalities (1.65), (1.67), (1.68) we obtain finally a bound of the interaction part of the form w.r.t. the free part: for any $\Phi_\varepsilon \in \mathscr{D}(H_0)$

$$\big\langle \Phi_\varepsilon \, \big| (W_\ll + H_I)\Phi_\varepsilon \big\rangle \leqslant \alpha \langle \Phi_\varepsilon | - \Delta_x + W_+ + \nu_\varepsilon \mathrm{d}\Gamma_\varepsilon(\omega) \, |\Phi_\varepsilon\rangle + \beta\|\Phi_\varepsilon\|^2, \tag{1.69}$$

where $\alpha := \alpha' + \frac{1}{2} + 6Nq_0^2\|\omega^{-1/2}\lambda_\cdot\|_{L^\infty}$ and $\beta = \beta' + 3\varepsilon N q_0^2 \|\lambda_\cdot\|_{L^\infty}^2$. Choosing $q_0 > 0$ such that $\alpha \in (0,1)$, we obtain the result thanks to KLMN theorem. □

As already mentioned, the following theorem has more general assumptions.

Theorem 1.1.7. *[65, Theorem 5.7] Let S be an open subset of $\mathbb{R}^d$ and let $\lambda_\cdot \in L^\infty(S;\mathfrak{H})$, with $\nabla_\cdot \cdot \lambda_\cdot \in L^\infty(\Lambda;\mathfrak{H})$. Assume also that* (1.52) *is satisfied. Then, denoting by $-\Delta_D$ the Dirichlet Laplacian, one has*

- *H_ε is essentially self-adjoint on $\mathscr{D}(-\Delta_D + W_+) \otimes \mathscr{D}(\mathrm{d}\Gamma_\varepsilon(\omega)) \cap \Gamma_{\mathrm{fin}}(\mathfrak{H})$;*
- *if, in addition, $\lambda_\cdot \in L^\infty(\Lambda; \mathbb{R}^d \otimes \mathscr{D}(\omega^{1/2}) \cap \mathscr{D}(\omega^{-1/2}))$ and $\nabla_\cdot \cdot \lambda_\cdot \in L^\infty(\Lambda; \mathscr{D}(\omega^{1/2}))$, then H_ε is self-adjoint on $\mathscr{D}(-\Delta_D + W_+) \otimes \mathscr{D}(\mathrm{d}\Gamma_\varepsilon(\omega))$ and bounded from below, with bound independent on ε.*

Note that, in the Coulomb gauge $\nabla_y \cdot \lambda_y = 0$ for any $y \in \mathbb{R}^d$, and therefore one of the conditions involving this term is automatically satisfied. Hence, assuming (1.52), (1.55), (1.59), (1.60), the Pauli-Fierz Hamiltonian H_ε is self-adjoint in $\mathscr{D}(H_\varepsilon) := \mathscr{D}(-\Delta_x + W_+) \cap \mathscr{D}(\mathrm{d}\Gamma_\varepsilon(\omega)) = \mathscr{D}(H_0)$.

Remark 1.1.8. *The Hamiltonian constructed above belongs to the class of Pauli-Fierz-type Hamiltonians but, the operator obtained from the quantization of the Abraham model is slightly different. Following [77], one indeed would get the operator acting on $\mathscr{H} = L^2(\mathbb{R}^{3N}) \otimes \Gamma_s(\mathfrak{H})$, with $\mathfrak{H} = \mathbb{C}^2 \otimes L^2(\mathbb{R}^3)$,*

$$H_\varepsilon = \sum_{j=1}^{N} \frac{1}{2m_j} \Big(-i\nabla_{x_j} - \frac{q_j}{c}\phi_\varepsilon(\lambda_{x_j}) \Big)^2 + W + \mathrm{d}\Gamma_\varepsilon(|k|), \tag{1.70}$$

with m_j, q_j, being, respectively, the mass and the charge of the $j-$th electron moving in $\mathbb{R}^3$, c the speed of light, W the Coulomb potential between particles

$$W(x_1, \ldots, x_N) = \sum_{j\neq k}^{N} \frac{q_j q_k}{|x_j - x_k|}, \tag{1.71}$$

and where the interaction can be written as

$$\phi_\varepsilon(\lambda_{x_j}) = \sum_{\gamma=1}^{2} \int \mathrm{d}k \, \frac{c}{|k|^{1/2}} \, e_\gamma(k) \, (\hat{\rho}_j(k) e^{ikx_j} a_\gamma(k) + \hat{\rho}_j^*(k) e^{-ikx_j} a_\gamma^\dagger(k)). \tag{1.72}$$

Here, $\hat{\rho}_j-$ stands for the Fourier transform of the signed charge distribution of the $j-$th particle. This is reminiscent of the Abraham model: the point distributions are indeed replaced by smooth charge distributions $e_j\rho_j$, with ρ_j radial, smooth, normalized and vanishing outside a ball of fixed radius. In this way, one can avoid the divergences due to the coupling between Maxwell and Newton equations for point charged particles.

From now on we set the charges of the particles to be equal to 1 and always assume the following explicit formula for the argument of the fields for the Pauli-Fierz model, and our results will be stated provided this assumption is satisfied:

$$\lambda_{x_j}(k) := \Big(e_\gamma(k)\chi(k)e^{ikx_j} \Big)_{\gamma=1}^{d-1}, \qquad \chi \in L^2(\mathbb{R}^d; \mathrm{d}k). \tag{$\mathrm{A_{PF}}$}$$

1.2 Effective Hamiltonians

In this section we study the effective operators obtained in the quasi-classical limit. The notion of convergence will be discussed in details in Chapter 4, but we heuristically expect a common behaviour for all the models of interest. Recalling the general form of the microscopic Hamiltonians

$$H_\varepsilon = K \otimes \mathbb{1}_{\Gamma_s(\mathfrak{H})} + \nu_\varepsilon \mathbb{1}_{\mathcal{K}} \otimes \mathrm{d}\Gamma_\varepsilon(\omega) + H_I, \tag{1.73}$$

we observe that by taking the limit $\varepsilon \to 0$ on the field degrees of freedom, it is reasonable to obtain

$$K \xrightarrow[\varepsilon\to 0]{} K,$$
$$H_I = \mathrm{Op}_\varepsilon^{\mathsf{Wick}}(\mathcal{V}) \xrightarrow[\varepsilon\to 0]{} \mathcal{V}(z).$$

For the moment, we neglect the field energy, since we are interested in the effective Hamiltonian for the particle system. We thus expect to find an effective Schrödinger operator of the form

$$\mathcal{H}(z) = K + \mathcal{V}(z), \tag{1.74}$$

depending on the classical state $z \in \mathfrak{H}$ of the field. Although the structure of the microscopic models is similar, the properties of the potential $\mathcal{V}(z)$ obtained in the limit strongly depends on the model chosen. Therefore, we analyse each model separately.

Nelson model: in this case the effective Hamiltonian takes the form

$$\boxed{\mathcal{H}(z) = -\Delta_x + W(x_1, \ldots, x_N) + 2\Re e \langle z \,|\, \lambda_x \rangle} \tag{1.75}$$

for any $z \in \mathfrak{H}$, and the effective potential $\mathcal{V}(z) = 2\Re e \langle z \,|\, \lambda_x \rangle$ is uniformly bounded w.r.t. x:

$$|\mathcal{V}(z)| \leqslant \|z\|_{\mathfrak{H}} \|\lambda_\cdot\|_{L^\infty(\mathbb{R}^{dN};\mathfrak{H})}. \tag{1.76}$$

Hence, the sum of the terms $W_\ll + \mathcal{V}(z)$ is infinitesimally bounded w.r.t. $-\Delta_x + W_+$, therefore, by Kato-Rellich theorem the operator $\mathcal{H}(z)$ is self-adjoint on the domain $\mathscr{D}(\mathcal{H}(z)) = \mathscr{D}(-\Delta_x + W_+)$, for any $z \in \mathfrak{H}$.

Polaron model: the following formal expression

$$\boxed{\mathcal{H}(z) = -\Delta_x + W(x_1, \ldots, x_N) + \sum_{j=1}^N 2\Re e \left\langle z \,|\, g_{x_j} \right\rangle} \tag{1.77}$$

can be interpreted as a quadratic form on the domain $\mathscr{D}((-\Delta_x + W_+)^{1/2})$. Given a function $\psi \in \mathscr{D}((-\Delta_x + W_+)^{1/2})$, by (1.45) following Lemma 1.1.4 and Cauchy-Schwarz, we have

$$\begin{aligned} |\langle \psi \,|\, \mathcal{V}(z)\psi \rangle| &\leqslant 2N \|g_{<,\cdot}\|_\infty \|z\|_{\mathfrak{H}} \|\psi\|^2 + 4 \sum_{j=1}^N \|\nabla_{x_j} \psi\|_{L^2(\mathbb{R}^d;\mathbb{C}^d)} \|\tilde{g}_\cdot\|_\infty \|z\|_{\mathfrak{H}} \|\psi\| \leqslant \\ &\leqslant \alpha \langle \psi | -\Delta_x + W_+ |\psi\rangle + \left(2N \|g_{<,\cdot}\|_\infty \|z\|_{\mathfrak{H}} + \frac{16N^2}{\alpha} \|\tilde{g}_\cdot\|_\infty^2 \|z\|_{\mathfrak{H}}^2 \right) \|\psi\|^2 \end{aligned}$$

for any $\alpha > 0$. Since $W_\ll \in \mathcal{K}_\ll$, it is infinitesimally bounded w.r.t. $-\Delta_x + W_+$, for a certain constant $\alpha' \in (0,1)$. Choosing α such that $\alpha + \alpha' \in (0,1)$, we can use KLMN theorem and deduce that the form $\langle \psi | \, \mathcal{H}(z) \, |\psi\rangle$ is closed and bounded from below. Hence, there exists a self-adjoint operator, still denoted $\mathcal{H}(z)$, such that $\mathscr{D}(\mathcal{H}(z)) \subseteq \mathscr{D}((-\Delta_x + W_+)^{1/2})$.

Pauli-Fierz model: the effective Hamiltonian is

$$\mathcal{H}(z) = -\Delta_x + W(x_1, \ldots, x_N) + \sum_{j=1}^N \left(2i\nabla_{x_j} \cdot 2\Re e \left\langle z \,\middle|\, \lambda_{x_j} \right\rangle + \left(2\Re e \left\langle z \,\middle|\, \lambda_{x_j} \right\rangle \right)^2 \right), \tag{1.78}$$

which can be rewritten as

$$\boxed{\mathcal{H}(z) = \sum_{j=1}^N \left(-i\nabla_{x_j} - 2\Re e \left\langle z \,\middle|\, \lambda_{x_j} \right\rangle \right)^2 + W(x_1, \ldots, x_N).} \tag{1.79}$$

We thus recognize a magnetic Schrödinger operator with magnetic potential

$$A(x_j) := 2\Re e \left\langle z \,\middle|\, \lambda_{x_j} \right\rangle. \tag{1.80}$$

Since $2\Re e \langle z \,|\lambda_\cdot \rangle \in L^\infty(\mathbb{R}^d)$, by perturbation theory in combination with Kato-Rellich for the $W_\ll$, $\mathcal{H}(z)$ is self-adjoint on $\mathscr{D}(\mathcal{H}(z)) = \mathscr{D}(-\Delta_x + W_+)$ for any $z \in \mathfrak{H}$.

Hence, we can summarize the results obtained in this section in the following theorem:

Theorem 1.2.1. *The families of Schrödinger operators $\{\mathcal{H}(z)\}_{z \in \mathfrak{H}}$, acting on $L^2(\mathbb{R}^{dN})$,*

- $\mathcal{H}(z) = -\Delta_x + W(x_1, \ldots, x_N) + 2\Re e \langle z \,|\, \lambda_x \rangle$,
- $\mathcal{H}(z) = -\Delta_x + W(x_1, \ldots, x_N) + \sum_{j=1}^N 2\Re e \left\langle z \,|\, g_{x_j} \right\rangle$,
- $\mathcal{H}(z) = \sum_{j=1}^N \left(-i\nabla_{x_j} - 2\Re e \left\langle z \,\middle|\, \lambda_{x_j} \right\rangle \right)^2 + W(x_1, \ldots, x_N)$,

are self-adjoint on the domain $\mathscr{D}(-\Delta_x + W_+)$ for the Nelson and Pauli-Fierz models, and on a domain $\mathscr{D}(\mathcal{H}(z)) \subseteq \mathscr{D}((-\Delta_x + W_+)^{1/2})$, for the polaron, respectively.

1.3 Main results

We are now ready to state the main results of the thesis about the convergence of the microscopic dynamics to the effective dynamics in the quasi-classical regime, that will be proved in Chapter 5.

We consider as initial states for the microscopic system a family of $\varepsilon-$dependent mixed states described by a normalized, positive density matrix over the full Hilbert space $\{\rho_\varepsilon\}_{\varepsilon\in(0,1)} \subseteq \mathscr{L}_{1,+}(\mathscr{H})$. In the quasi-classical limit $\varepsilon \to 0$, only the field's degrees of freedom are affected: the limits of sequences of field states are positive measure $\mu \in \mathscr{M}(\mathfrak{H};\mathbb{R}^+)$ over the one-boson space, while the particle subsystem is described by a density matrix $\gamma(z) \in \mathscr{L}_{1,+}(L^2(\mathbb{R}^{dN}))$, depending on the field configuration $z \in \mathfrak{H}$. Therefore, one obtains the convergence to a state-valued measure $\mathfrak{m} \in \mathscr{M}(\mathfrak{H};\mathscr{L}_{1,+}(L^2(\mathbb{R}^{dN})))$ over a subsequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$, in a sense explained in Chapter 3:

$$\rho_{\varepsilon_n} \xrightarrow[n\to+\infty]{} \mathrm{d}\mathfrak{m}(z) = \gamma(z)\mathrm{d}\mu(z). \tag{1.81}$$

Given one of the self-adjoint Hamiltonians H_ε introduced above with a scaling ν_ε such that $\varepsilon\nu_\varepsilon \to \nu \in \{0,1\}$, we will show that

- if $\nu = 0$, the field is stationary;
- if $\nu = 1$, the field evolves freely in time, by a classical dynamics generated by the field's dispersion relation ω.

Let e^{-itH_ε} be the one-parameter group generated by H_ε, then we define the Heisenberg evolution of the initial state ρ_ε as

$$\rho_\varepsilon(t) := e^{-itH_\varepsilon}\,\rho_\varepsilon\,e^{itH_\varepsilon} \in \mathscr{L}_{1,+}(\mathscr{H}). \tag{1.82}$$

The main result of the thesis is about the convergence of the previous family of states in the quasi-classical limit: if $\rho_\varepsilon \to \mathfrak{m}$, then we prove that there exists a time-dependent family of state-valued measures $\{\mathfrak{m}_t\}_{t\in\mathbb{R}} \subseteq \mathscr{M}(\mathfrak{H};\mathscr{L}_{1,+}(L^2(\mathbb{R}^{dN})))$, such that

$$\boxed{\rho_\varepsilon(t) \to \mathrm{d}\mathfrak{m}_t(z) = \gamma_{\mathfrak{m}_t}(z)\;\mathrm{d}\mu_t(z).} \tag{1.83}$$

Furthermore, we characterize the effective dynamics that yields the evolution of $\mathfrak{m}_t$. If $\mathrm{d}\mathfrak{m}(z) = \gamma_{\mathfrak{m}}(z)\,\mathrm{d}\mu(z)$ is the state-valued measure obtained by the quasi-classical limit of the initial states ρ_ε, then its time evolution $\mathfrak{m}_t$ is given by the decomposition (1.83), where

- μ_t is the measure transported along the flow $\Phi_t(z) = e^{-it\nu\omega}z$, which can be either a free dynamics or the identity, depending on the scaling of the field energy:

$$\boxed{\mu_t = \Phi_t \sharp \mu;} \tag{1.84}$$

- the density matrix $\gamma_{\mathfrak{m}_t}(z)$ evolves by the action of the two-parameter group $U_{t,s}(z)$ weakly generated by the effective Hamiltonian $\mathcal{H}(\Phi_t z)$:

$$\boxed{\gamma_{\mathfrak{m}_t}(z) = U_{t,0}(z)\,\gamma_{\mathfrak{m}}(z)\,U_{t,0}^\dagger(z).} \tag{1.85}$$

In order to formulate the assumption on the initial state needed to prove the next Theorem, we make use of the operator

$$H_+ := -\Delta_x + W_+(x_1,\ldots,x_N) + \mathrm{d}\Gamma_\varepsilon(\omega). \tag{1.86}$$

We split the needed assumptions in three groups:

- the form factors satisfy the assumptions in Section 1.1, *i.e.*, (A_{Nel}), (A_{Pol}) and (A_{PF}), that we recall for reader's convenience:

$$\boxed{\begin{cases} \lambda_x(k) = \chi(k)\sum_{j=1}^N e^{ikx_j}, \quad \chi \in L^2(\mathbb{R}^d; \mathrm{d}k), & \text{for the Nelson model;} \\ g_{x_j}(k) = \frac{e^{iky}}{|k|^{\frac{d-1}{2}}}\chi_{\{|k|<r\}}(k) + \left[-i\nabla_y, \frac{k}{|k|^{\frac{d+3}{2}}}e^{iky}\chi_{\{|k|\geqslant r\}}(k)\right], & \text{for the polaron;} \\ \lambda_{x_j}(k) = \left(e_\gamma(k)\chi(k)e^{ikx_j}\right)_{\gamma=1}^{d-1}, \qquad \chi \in L^2(\mathbb{R}^d; \mathrm{d}k), & \text{for the Pauli-Fierz model.} \end{cases}} \tag{A0}$$

- for any $\delta > 0$ there exists a $C_\delta \in (0, +\infty)$ such that

$$\boxed{\begin{cases} \sup_{\varepsilon\in(0,1)} \mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon(-\Delta_x + W_+ + (\mathrm{d}\Gamma_\varepsilon(\mathbb{1}) + 1)^\delta)) \leqslant C_\delta, & \text{for the Nelson model;} \\ \sup_{\varepsilon\in(0,1)} \mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon(-\Delta_x + W_+ + (\mathrm{d}\Gamma_\varepsilon(\mathbb{1}) + 1)^\delta) \leqslant C_\delta, & \text{for the polaron;} \\ \sup_{\varepsilon\in(0,1)} \mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon(H_+ + (\mathrm{d}\Gamma_\varepsilon(\mathbb{1}) + 1)^\delta)) \leqslant C_\delta, & \text{for the Pauli-Fierz model.} \end{cases}} \tag{A1}$$

- the resolvent of $(-\Delta_x + W_+)$ is compact: for any $z \in \varrho(-\Delta_x + W_+)$

$$\boxed{(-\Delta_x + W_+ - z)^{-1} \in \mathscr{L}_\infty(L^2(\mathbb{R}^{dN})),} \tag{A2}$$

  and therefore the spectrum of $-\Delta_x + W_+$ is discrete. This is in particular the case if W_+ is trapping.

Theorem 1.3.1. *Let $\{\rho_\varepsilon\}_{\varepsilon\in(0,1)} \subseteq \mathscr{L}_{1,+}(\mathscr{H})$ be a family of normalized states satisfying assumptions* (A1) *and* (A2)*. Let also*

$$\rho_\varepsilon(t) = e^{-itH_\varepsilon}\rho_\varepsilon e^{itH_\varepsilon}, \tag{1.87}$$

be their time evolution generated either by the Nelson, polaron or Pauli-Fierz Hamiltonian, satisfying assumption (A0) *with $\lim_{\varepsilon\to0}\varepsilon\nu_\varepsilon = \nu \in \{0,1\}$. Then, there exists a subsequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ and a state-valued measure $\mathfrak{m}$, such that*

$$\rho_{\varepsilon_n} \xrightarrow[n\to+\infty]{} \mathfrak{m} \in \mathscr{M}(\mathfrak{H}; \mathscr{L}_{1,+}(L^2(\mathbb{R}^{dN}))), \quad \textit{with} \quad \mathrm{d}\mathfrak{m}(z) = \gamma_{\mathfrak{m}}(z)\mathrm{d}\mu(z), \tag{1.88}$$

and

$$\rho_{\varepsilon_n}(t) \xrightarrow[n\to+\infty]{} \mathrm{d}\mathfrak{m}_t(z) \in \mathscr{M}(\mathfrak{H}; \mathscr{L}_{1,+}(L^2(\mathbb{R}^{dN}))), \quad \textit{with} \quad \mathrm{d}\mathfrak{m}(z) = \gamma_{\mathfrak{m}_t}(z)\mathrm{d}\mu_t(z), \tag{1.89}$$

for any $t \in \mathbb{R}$, where the family of state-valued measures $\{\mathfrak{m}_t\}_{t\in\mathbb{R}}$ is given by

$$\mu_t = \Phi_t \sharp \mu, \qquad \gamma_{\mathfrak{m}_t}(z) = U_{t,0}(z)\,\gamma_{\mathfrak{m}}(z)\,U_{t,0}^\dagger(z), \tag{1.90}$$

with $\Phi_t z = e^{-it\nu\omega}z$ and $U_{t,s}(z)$ the two-parameter group of unitary operators weakly generated by $\mathcal{H}(\Phi_t z) = -\Delta_x + W + \mathcal{V}(\Phi_\tau z)$. Furthermore, the previous convergence holds true also in weak sense, i.e.,

$$\lim_{n\to+\infty} \mathrm{Tr}_{\mathscr{H}}(\rho_{\varepsilon_n}(t)\, B \otimes W_{\varepsilon_n}(\xi)) = \mathrm{Tr}_{L^2}\left(\int_{\mathfrak{H}} \mathrm{d}\mathfrak{m}_t(z)\, B\, e^{2i\Re e\langle \xi \,|\, z\rangle}\right), \tag{1.91}$$

for any $\xi \in \mathfrak{H}$ and $B \in \mathscr{L}(L^2(\mathbb{R}^{dN}))$.

The above results are proven in Chapter 5. In order to make the result more explicit (see also Chapter 3), we observe that the convergence stated above is equivalent to the following: for any $B \in \mathscr{L}(L^2(\mathbb{R}^{dN}))$ and any $\xi \in \mathfrak{H}$,

$$\begin{aligned}\lim_{n\to+\infty} \mathrm{Tr}_{\mathscr{H}}(\rho_{\varepsilon_n}(t)\, B \otimes W_{\varepsilon_n}(\xi)) = \mathrm{Tr}_{L^2}\left(\int_{\mathfrak{H}} \mathrm{d}\mathfrak{m}_t(z)\, B\, e^{2i\Re e\langle \xi \,|\, z\rangle}\right) = \\ = \int_{\mathfrak{H}} \mathrm{d}\mu_t(z)\, \mathrm{Tr}_{L^2}(\gamma_{\mathfrak{m}_t}(z)\, B)\, e^{2i\Re e\langle \xi \,|\, z\rangle} = \\ = \int_{\mathfrak{H}} \mathrm{d}\mu(z)\, \mathrm{Tr}_{L^2}(\gamma_{\mathfrak{m}}(z)\, U^{\dagger}_{t,0}(z)\, B\, U_{t,0}(z))\, e^{2i\Re e\langle \xi \,|\, \Phi_t z\rangle}.\end{aligned}$$

Remark 1.3.2. *We point out here the relevance of the assumptions and their role in the proof of the convergence.*

- *The uniform bound on the expectation of the number operator: for some $\delta > 0$,*

$$\mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon(1 + \mathrm{d}\Gamma_\varepsilon(\mathbb{1}))^\delta) \leqslant C_\delta. \tag{1.92}$$

  Such an assumption is important for the convergence of the dynamics of the Nelson model, since the control on the number operator propagates in time (Proposition 5.1.2) and allows us to obtain the convergence of the evolved state, at any time, to a state-valued measure in a weak- sense. For the Pauli-Fierz and polaron models, this assumption on the initial datum is not enough, since the propagation of this condition requires also a control on the positive part of the Hamiltonian (Propositions 5.1.4 and 5.1.7), i.e.,*

$$\mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon(-\Delta_x + W_+ + \mathrm{d}\Gamma_\varepsilon(\omega))) \leqslant C, \tag{1.93}$$

  (for the polaron this is actually needed only for $-\Delta + W_+$).

- *The weak-* convergence is however not totally satisfactory, because the test against the identity operator may not converge. As a consequence, even at initial time, there may be a loss of mass along the quasi-classical limit. Indeed, under the previous assumptions, even if $\|\rho_\varepsilon\|_{\mathscr{L}_1} = 1$, we only know that*

$$\|\mathfrak{m}(\mathfrak{H})\|_{\mathscr{L}_1(L^2)} \leqslant 1. \tag{1.94}$$

  To guarantee the conservation of the probability mass it is necessary to have a uniform bound on the expectation of a particle observable K with compact resolvent, i.e., such that

$$z \in \varrho(K), \qquad (K - z)^{-1} \in \mathscr{L}_\infty(L^2). \tag{1.95}$$

  The most natural choice for such an observable is the Schrödinger operator $-\Delta_x + W_+$ for a trapping W_+. We thus require that W_+ is a trapping potential and assume that

$$\mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon(-\Delta_x + W_+)) \leqslant C, \tag{1.96}$$

  which guarantees that the measure obtained in quasi-classical limit is a probability state-valued measure (Corollary 3.2.3). Furthermore, by conservation of the probability along the dynamics, this holds true at any time:

$$\|\mathfrak{m}_t(\mathfrak{H})\|_{\mathscr{L}_1(L^2)} = \|\mathfrak{m}(\mathfrak{H})\|_{\mathscr{L}_1(L^2)} = 1, \tag{1.97}$$

  Moreover, the convergence can be lifted to weak topology.

The convergence can be extended, imposing stronger assumptions on the states, also to the expectations of polynomial symbols in the creation and annihilation operators. We choose $\{\rho_\varepsilon\}_{\varepsilon\in(0,1)} \subseteq \mathscr{L}_{1,+}(\mathscr{H})$ such that

$$\boxed{\begin{cases} \exists\delta > 1/4, \exists C_\delta \geqslant 1 \text{ s.t. } \mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon(\mathrm{d}\Gamma_\varepsilon(\mathbb{1})+1)^{2\delta}) \leqslant C_\delta, & \text{for the Nelson model;} \\ \exists\delta \in \mathbb{N}_*, \exists C_\delta > 0 \text{ s.t. } \mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon(H_\varepsilon^{2\delta})) \leqslant C_\delta \nu(\varepsilon)^{2\delta}, & \text{for the polaron;} \\ \exists C > 0 \text{ s.t. } \mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon(H_+^2 + \mathrm{d}\Gamma_\varepsilon(\mathbb{1})^2)) \leqslant C, & \text{for the Pauli-Fierz model.} \end{cases}} \tag{A3}$$

Theorem 1.3.3. *If ρ_ε satisfies assumption* (A3)*, and $\rho_{\varepsilon_n} \to \mathfrak{m} \in \mathscr{M}(\mathfrak{H}; \mathscr{L}_{1,+}(L^2))$, then for every $f \in \mathscr{O}^{(p,q)}$, such that*

$$\begin{cases} \frac{p+q}{2} < 2\delta, & \textit{for Nelson and polaron models,} \\ \frac{p+q}{2} < 2, & \textit{for the Pauli-Fierz model,} \end{cases} \tag{1.98}$$

and any $\kappa \in \mathscr{L}_\infty(L^2)$, $\xi \in \mathfrak{H}$,

$$\boxed{\begin{aligned} &\lim_{n\to+\infty} \mathrm{Tr}_{\mathscr{H}}(\rho_{\varepsilon_n}(t)\,\mathrm{Op}^{\mathrm{Wick}}_{\varepsilon_n}(f)\,\kappa \otimes W_{\varepsilon_n}(\xi)) = \\ &= \int_{\mathfrak{H}} \mathrm{d}\mu(z)\,\mathrm{Tr}_{L^2}(\gamma_{\mathfrak{m}}(z) U^\dagger_{t,0}(z)\,\kappa\, f(e^{-it\nu\omega}z) U_{t,0}(z)) e^{2i\Re e\langle \xi \,|\, \Phi_t z\rangle}. \end{aligned}}$$

Furthermore, if assumptions (A1) *and* (A2) *are satisfied, then the previous convergence holds true in weak convergence.*

Proof. As mentioned above, the assumptions on the initial state are stronger than the ones of Theorem 1.3.1. Hence, we can use the latter to deduce the convergence to a time-dependent family of state-valued measures $\mathfrak{m}_t \in \mathscr{M}(\mathfrak{H}; \mathscr{L}_{1,+}(L^2))$ on the subsequence $\varepsilon_n \to 0$, *i.e.*,

$$\rho_{\varepsilon_n}(t) \xrightarrow[n\to+\infty]{} \mathfrak{m}_t. \tag{1.99}$$

Furthermore, for any time $t \in \mathbb{R}$,

$$\mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon(t)(1+\mathrm{d}\Gamma_\varepsilon(\mathbb{1}))^{2\delta}) \leqslant C(t), \text{ with } \begin{cases} \delta > 1/4, & \text{for the Nelson model;} \\ \delta \in \mathbb{N}_*, & \text{for the polaron;} \\ \delta = 1, & \text{for the Pauli-Fierz model.} \end{cases} \tag{1.100}$$

Indeed, for the Nelson model, Proposition 5.1.2 implies that

$$\mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon(t)(1+\mathrm{d}\Gamma_\varepsilon(\mathbb{1}))^{2\delta}) \leqslant (N^2+1)^{2\delta} e^{m_\delta(1)|t||\delta|\|\lambda.\|_\infty} C_\delta = C(t).$$

For the polaron model, we make use of the pull-through formula stated in Lemma 5.1.3: we deduce that there exist constants $b, c > 0$ such that

$$\|(\mathrm{d}\Gamma_\varepsilon(\mathbb{1})+\varepsilon)^\delta (H_\varepsilon + b)^{-\delta}\|_{\mathscr{L}} \leqslant c\,\nu_\varepsilon^{-\delta}, \tag{1.101}$$

so that

$$\begin{aligned} \mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon(t)(1+\mathrm{d}\Gamma_\varepsilon(\mathbb{1}))^{2\delta}) &\leqslant C[\mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon(t)(\varepsilon+\mathrm{d}\Gamma_\varepsilon(\mathbb{1}))^{2\delta}) + 1] \leqslant \\ &\leqslant C\mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon(H_\varepsilon+b)^{2\delta})\|(\mathrm{d}\Gamma_\varepsilon(\mathbb{1})+\varepsilon)^\delta(H_\varepsilon+b)^{-\delta}\|^2 + C \leqslant C_\delta \nu_\varepsilon^{2\delta}\nu_\varepsilon^{-2\delta} + C = C. \end{aligned}$$

Analogously, for the Pauli-Fierz model, we apply Proposition 5.1.7, which gives that, for any $t \in \mathbb{R}$, there exists $K(t) > 0$ such that, for any $\Psi_\varepsilon \in \mathscr{D}(H_+) \cap \mathscr{D}(\mathrm{d}\Gamma_\varepsilon(\mathbb{1}))$,

$$\|(\mathrm{d}\Gamma_\varepsilon(\mathbb{1}) + 1)e^{-itH_\varepsilon}\Psi_\varepsilon\|_{\mathscr{L}} \leqslant K(t)(\|\mathrm{d}\Gamma_\varepsilon(\mathbb{1})\Psi_\varepsilon\| + \|(H_+ + 1)\Psi_\varepsilon\|) \tag{1.102}$$

i.e.,

$$\|(\mathrm{d}\Gamma_\varepsilon(\mathbb{1}) + 1)e^{-itH_\varepsilon}(\mathrm{d}\Gamma_\varepsilon(\mathbb{1}) + H_+)^{-1}\| \leqslant K(t). \tag{1.103}$$

This allows to bound the expectation of the square of the number operator at time t, as

$$\mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon(t)(1 + \mathrm{d}\Gamma_\varepsilon(\mathbb{1}))^2) \leqslant \mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon(H_+^2 + \mathrm{d}\Gamma_\varepsilon(\mathbb{1})^2))\|\mathrm{d}\Gamma_\varepsilon(\mathbb{1})e^{-itH_\varepsilon}(\mathrm{d}\Gamma_\varepsilon(\mathbb{1}) + H_+)^{-1}\|^2 \leqslant C(t).$$

Thanks to (1.100) we can apply Corollary 3.3.2 and obtain the desired weak-* convergence for the polynomial, operator valued symbols f in the class $\mathscr{O}^{(p,q)}$, such that $\frac{p+q}{2} < 2\delta$ for the Nelson and polaron models and $\frac{p+q}{2} < 2$ for Pauli-Fierz model.

To strengthen the convergence, we apply again Corollary 3.3.2 to the time evolved density matrices. Therefore, we have to verify that the condition on the expectation value of the Laplacian is satisfied at any time. However, this holds true thanks to Propositions 5.1.2, 5.1.4 and 5.1.7, which implies that, for any $t \in \mathbb{R}$,

$$\mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon(t)\,(-\Delta_x + W_+)) \leqslant C(t). \tag{1.104}$$

□

Remark 1.3.4. *We observe that, while we were able to prove the above convergence for the Nelson and polaron models for polynomial-valued symbols of every degree, for the Pauli-Fierz the result applies only to polynomials of degree at most 3. The reason is technical and is due to the fact that the propagation estimate for the norm of the number operator is available only for power* 1 *(Proposition 5.1.7). An adaptation of the pull-through formula to higher powers is proven in [2], but it works under the restriction to massive fields.*

2 State-valued measures and integration of operator-valued symbols

Bohr's correspondence principle is one of the cornerstones of quantum theory, providing a recipe of how classical mechanics has to be recovered from quantum mechanics in a limit where certain key physical quantities, as the energy of the system, are "large". Analogously, the classical behaviour can also be obtained when the Planck constant $\hbar$ is small compared to characteristic action, which is typically of order 1 in natural units, *i.e.*, $\hbar \ll 1$. In this case one speaks of a semiclassical regime, and two famous applications of the correspondence principle are the Ehrenfest Theorem and the WKB approximation.

Among the many results of semiclassical analysis, one of the most relevant is the convergence of the expectations of quantum observables to their classical counterpart in the limit of $\hbar \to 0$. It is known that, for systems with a finite dimensional phase space (see [64], [79]), the classical counterparts of quantum states and self-adjoint operators are probability measures and real-valued functions over the phase space, a.k.a."symbols", respectively. Quantum observables may in turn be recovered from classical symbols via quantization procedures. The Bohr correspondence can be summarized in the following scheme:

quantum system	**classical system**
quantum states	probability measures
self-adjoint operators	symbols on phase space

The semiclassical framework was extended and generalized to the case of systems with infinite dimensional phase space, *e.g.*, quantum fields, in [6]: semiclassical states still converge to scalar measures over the phase space, called Wigner measures, as in the finite dimensional case, and expectations of quantum observables tend to the classical ones.

Under suitable assumptions (see, *e.g.*, Chapter 3) any family of vectors $\{\Psi_\varepsilon\}_{\varepsilon\in(0,1)}$ in the Fock space $\Gamma_s(\mathfrak{H})$ of the field depending on the semiclassical parameter ε (which plays the role of $\hbar \ll 1$), converges to a measure over $\mathfrak{H}$. More precisely,

$$\Psi_\varepsilon \xrightarrow[\varepsilon\to 0]{} \mu \tag{2.1}$$

where $\mu \in \mathscr{M}(\mathfrak{H}; \mathbb{R}_+)$ is a positive, scalar measure on the one-boson space and the convergence is meant at the level of expectations

$$\langle \Psi_\varepsilon | \operatorname{Op}_\varepsilon(b) \Psi_\varepsilon \rangle \xrightarrow[\varepsilon\to 0]{} \int_{\mathfrak{H}} \mathrm{d}\mu(z)\, b(z) \tag{2.2}$$

with $b : \mathfrak{H} \to \mathbb{C}$ a measurable function and $\mathrm{Op}_\varepsilon(b) : \Gamma_s(\mathfrak{H}) \to \Gamma_s(\mathfrak{H})$ the associated quantum operator obtained via a chosen quantization.

Adding further degrees of freedom, *e.g.*, making the field interacting with particles, whose pure states are vectors in a suitable L^2 space, the above scheme trivially extends to product states. Indeed, denoting by $\mathscr{H} = L^2 \otimes \Gamma_s(\mathfrak{H})$ the Hilbert space of the full system, if we consider a product state $\psi \otimes \Psi_\varepsilon \in \mathscr{H}$, then

$$\psi \otimes \Psi_\varepsilon \xrightarrow[\varepsilon \to 0]{} \langle \psi | \, \mu \, | \psi \rangle , \tag{2.3}$$

i.e.,

$$\langle \psi \otimes \Psi_\varepsilon \, | \, \mathrm{Op}_\varepsilon(b) \psi \otimes \Psi_\varepsilon \rangle \xrightarrow[\varepsilon \to 0]{} \left\langle \psi \, \middle| \int_{\mathfrak{H}} \mathrm{d}\mu(z) b(z) \psi \right\rangle . \tag{2.4}$$

If, however, the states considered are mixed, and thus described by a density matrix $\rho_\varepsilon \in \mathscr{L}_1(\mathscr{H})$, in this case one cannot split the semiclassical part from the one that retains its quantum nature. Heuristically speaking consider a particular density matrix of the form

$$\rho_\varepsilon = \sum_{j,k \in \mathbb{N}} \lambda_{j,k}(\varepsilon) \left| \psi_j \otimes \Psi_\varepsilon^{(k)} \right\rangle \left\langle \psi_j \otimes \Psi_\varepsilon^{(k)} \right| , \tag{2.5}$$

and assume that the vectors $\{\psi_j\}_{j\in\mathbb{N}}$ are $\varepsilon-$independent and that $\Psi_\varepsilon^{(k)} \to \mu_k$ for any $k \in \mathbb{N}$, with $\{\mu_k\}_{k\in\mathbb{N}}$ a sequence of positive measures on $\mathfrak{H}$, then

$$\rho_\varepsilon \xrightarrow[\varepsilon \to 0]{} \sum_{j,k \in \mathbb{N}} \lambda_{j,k} \, \mu_k \, |\psi_j\rangle\langle\psi_j| =: \mathfrak{m} \tag{2.6}$$

The object so obtained behaves as a density matrix on the particle space L^2, and, symmetrically, it gives a measure on the Hilbert space $\mathfrak{H}$ when the particles are discarded. This heuristic analysis suggests the need of introducing new mathematical objects called *state-valued measures*. Besides their precise definition, in the next chapters we are going to provide a characterization of state-valued measures as quasi-classical cluster points of quantum states of systems composed by two interacting subsystems, *i.e.*, a semiclassical field and a particle system. Thus, Bohr correspondence can be summarized in the following way:

quantum system	**quasi-classical system**
quantum states of the full system	probability state-valued measures
self-adjoint operators	operator-valued symbols on phase space

The chapter has the following structure: in Section 2.1 we adapt the definitions of cone-valued measures [66] and of Bartle's Banach-valued measures [13] to our case, to give a definition of state-valued measures, and show how these definitions are equivalent in our setting. After the discussion of some important properties of state-valued measures, in Section 2.2 we define the integration of scalar, and then, of operator-valued functions, which will prove to be useful to deal with interaction term of the Hamiltonians. Finally, we state a version of dominated convergence theorem for integrals of operator-valued functions w.r.t. state-valued measures and then prove a convenient decomposition of measures when the space of states has the Radon-Nykodým property. We conclude the Chapter generalizing in Section 2.3 the definition to cylindrical state-valued measures.

We consider the abstract setting of measures with values in positive, continuous functionals over a C*-algebra $\mathfrak{A}$, and observables with values in a W*-algebra $\mathfrak{B}$, of which $\mathfrak{A} \subseteq \mathfrak{B}$ is a bilateral ideal. We do not assume that the C*-algebra has an identity element, which require to use approximate identities. We recall thus the definition and some properties of approximate identities (see [18]).

Definition. *[18, Definition 2.2.17] If $\mathfrak{A}$ is a C*-algebra, then a net $\{e_\alpha\}_{\alpha\in I} \subseteq \mathfrak{A}_+$ is an **approximate identity**, if and only if*

- $\|e_\alpha\|_{\mathfrak{A}} \leqslant 1$*, for any* $\alpha \in I$*;*
- $e_\alpha \leqslant e_\beta$*, if* $\alpha \leqslant \beta$*;*
- $\lim_{\alpha\in I} \|e_\alpha x - x\|_{\mathfrak{A}} = 0$*, for any* $x \in \mathfrak{A}$*.*

Existence of approximate identities is guaranteed by [18, Proposition 2.2.18].

It is well known that, for a unital C*-algebra, the norm of a positive continuous functional over the C*-algebra is obtained evaluating the functional on the identity element, *i.e.*,

$$\|\omega\|_{\mathfrak{A}'} = \omega(\mathbb{1}). \tag{2.7}$$

If the C*-algebra has no identity element, the previous property can be recovered by using approximate identities.

Proposition. *[18, Proposition 2.3.11] Let us consider $\omega \in \mathfrak{A}'_+$, then*

$$\|\omega\|_{\mathfrak{A}'} = \lim_{\alpha\in I} \omega(e_\alpha). \tag{2.8}$$

2.1 State-Valued Measures

Let $\mathfrak{A}$ be a C*-algebra and denote by $\mathfrak{A}'_+$ the cone of positive elements in the dual of $\mathfrak{A}$. In addition, let (X, Σ) be a measurable space. We want to define a measure

$$\mathfrak{m} : \Sigma \longrightarrow \mathfrak{A}'_+ \tag{2.9}$$

There are, in fact, two *equivalent* ways of defining an $\mathfrak{A}'_+$-valued measure on (X, Σ). Let us first adapt the definition of cone-valued measures [66].

2.1.1 Cone-valued measures

If V is a real vector space, a subset $\mathcal{C} \subseteq V$ is a *cone*, if its elements multiplied by any positive scalar still belong to the cone. We consider pointed and generating cones, *i.e.*, cones for which it holds[1]

$$\mathcal{C} \cap -\mathcal{C} = \{0\}, \qquad \mathcal{C} - \mathcal{C} = V, \tag{2.10}$$

respectively. Consider a subspace $V' \subseteq V^*$ in the algebraic dual of V, and denote by $\mathcal{C}'$ the dual cone defined by

$$\mathcal{C}' := \{\kappa \in V' \,|\, \kappa(\mathcal{C}) \geqslant 0\}.$$

We will need the following property

$$\mathcal{C} = \mathcal{C}'' := \{v \in \mathcal{C} \,|\, v(\mathcal{C}') \geqslant 0\}. \tag{2.11}$$

If V is locally convex and V' the continuous dual of V, then the Hahn-Banach separation theorem automatically implies the previous condition.

Now we denote by $\mathbb{R}^+_\infty = [0, \infty]$ the extended real half-line with semigroup structure

$$\infty + v = v + \infty = \infty, \qquad \text{for any } v \in \mathbb{R}^+_\infty, \tag{2.12}$$

[1] By $-\mathcal{C}$ we denote the opposite cone composed by elements of the form $-v$, for any $v \in \mathcal{C}$.

and by $C_\infty := \mathrm{Hom}_{\mathrm{mon}}(\mathcal{C}'; \mathbb{R}^+_\infty)$ the monoid homomorphisms w.r.t. pointwise addition. Furthermore, define the map

$$\begin{aligned} i_\mathcal{C} : \mathcal{C} &\longrightarrow \mathcal{C}_\infty \\ c &\longmapsto i_\mathcal{C}(c), \qquad i_\mathcal{C}(c)(\kappa) = \kappa(c) \end{aligned}$$

For any $c_1, c_2 \in \mathcal{C}$, such that $i_\mathcal{C}(c_1) = i_\mathcal{C}(c_2)$, we have $i_\mathcal{C}(c_1 - c_2) = i_\mathcal{C}(c_2 - c_1) = 0$, and then, if (2.10) and (2.11) hold, $c_1 - c_2 \in \mathcal{C} \cap -\mathcal{C} = \{0\}$, *i.e.*, $c_1 = c_2$. So $i_\mathcal{C}$ is injective, and $i_\mathcal{C}(\mathcal{C}) \cong \mathcal{C}$. Another important condition that we will require is

$$i_\mathcal{C}(\mathcal{C}) = \mathcal{C}_\infty. \tag{2.13}$$

Now we are ready to state the definition for cone-valued measures [66, Definition I.8].

Definition 2.1.1 (Cone-valued measures)**.** *Suppose the triple* $(V, V', \mathcal{C})$ *satisfies conditions* (2.10)*,* (2.11)*,* (2.13)*, in which case we say that it is compatible, and let* (X, Σ) *be a measurable space. Then we say that*

$$\mathfrak{m} : \Sigma \longrightarrow \mathcal{C}_\infty \tag{2.14}$$

*is a **cone-valued measure** if* $\mathfrak{m}(\varnothing) = 0$ *and it is* $\sigma-$*additive pointwise*[2]*, i.e., for any collection of mutually disjoint measurable sets* $\{S_j\}_{j\in\mathbb{N}} \subseteq \Sigma$*,*

$$\mathfrak{m}\left(\bigcup_{j\in\mathbb{N}} S_j\right)(\kappa) = \sum_{j\in\mathbb{N}} \mathfrak{m}(S_j)(\kappa) \qquad \text{for any } \kappa \in \mathcal{C}'. \tag{2.15}$$

We now show a useful characterization of state-valued measures, which we are going to use frequently in the rest of the thesis: a cone-valued measure can also be viewed as a collection of positive scalar measures obtained by testing against all the elements of the dual cone $\mathcal{C}'$. We will refer to this result as “Neeb's Lemma".

Lemma 2.1.2. *[66, Theorem I.10] The map* $\mathfrak{m} : \Sigma \to \mathcal{C}_\infty$ *defines a cone-valued measure if and only if there exists a family of positive, scalar measures* $\{\mathfrak{m}^\kappa\}_{\kappa\in\mathcal{C}'} \subseteq \mathscr{M}(X; \mathbb{R}^+_\infty)$ *such that*

$$\mathfrak{m}^\kappa(S) = \mathfrak{m}(S)(\kappa), \qquad \text{for any } S \in \Sigma,\ \kappa \in \mathcal{C}', \tag{2.16}$$

and the map $\kappa \mapsto \mathfrak{m}^\kappa$ *is a monoid homomorphism.*

A compatible triple fitting the above conditions is $(V, V', \mathcal{C}) = (\mathfrak{A}'_*, \mathfrak{A}_*, \mathfrak{A}'_+)$, where

- $\mathfrak{A}$ is a C*-algebra, $\mathfrak{A}_*$ the set of its self-adjoint elements, and $\mathfrak{A}_+$ the set of its positive elements;
- $\mathfrak{A}'$ is the continuous dual of the C*-algebra, $\mathfrak{A}'_*$ the dual of the set of self-adjoint elements, and $\mathfrak{A}'_+$ the set of functionals which are positive when acting on $\mathfrak{A}_+$.

The fact that such a triple satisfies the conditions (2.10), (2.11), (2.13) is proven in [40, Appendix A.4].

Definition 2.1.3 (State-valued measures-I)**.** *A **state-valued measure*** $\mathfrak{m} : \Sigma \to \mathfrak{A}'_+$ *is a cone-valued measure with the compatible triple given by* $(\mathfrak{A}'_*, \mathfrak{A}_*, \mathfrak{A}'_+)$*.*

Hence, by Neeb's Lemma, to define a state-valued measure it is sufficient to provide a collection of positive, scalar measures indexed by the positive elements of the C*-algebra (representing the dual cone w.r.t. $\mathfrak{A}'_+$), *i.e.*,

$$\mathfrak{m} \quad \longleftrightarrow \quad \{\mathfrak{m}^\kappa\}_{\kappa\in\mathfrak{A}_+} \tag{2.17}$$

where $\mathfrak{m}(S)(\kappa) = \mathfrak{m}^\kappa(S)$, for any $S \in \Sigma$, and the map $\kappa \mapsto \mathfrak{m}^\kappa$ is a homomorphism of monoids.

[2] We will see that this, in the case of positive functionals over the C^*-algebras, this implies $\sigma-$additivity in weak-* topology

2.1.2 Banach-valued measures

As anticipated, we provide an alternative definition of state-valued measures following [13].

Definition 2.1.4 (State-valued measures-II)**.** *A map* $\mathfrak{m} : \Sigma \to \mathfrak{A}'_+$ *is a* ***state-valued measure*** *iff* $\mathfrak{m}(\varnothing) = 0$ *and for any family* $\{S_n\}_{n\in\mathbb{N}} \subset \Sigma$ *of mutually disjoint measurable sets,*

$$\mathfrak{m}\left(\bigcup_{n\in\mathbb{N}} S_n\right) = \sum_{n\in\mathbb{N}} \mathfrak{m}(S_n) \ ,$$

where the right hand side converges unconditionally in the norm of $\mathfrak{A}'$*, i.e., for any permutation* $\tau : \mathbb{N} \to \mathbb{N}$*,*

$$\lim_{N\to+\infty} \left\| \mathfrak{m}\left(\bigcup_{n\in\mathbb{N}} S_n\right) - \sum_{n=1}^{N} \mathfrak{m}(S_{\tau(n)}) \right\|_{\mathfrak{A}'} = 0. \tag{2.18}$$

Now we show that the two definitions above are equivalent. To this purpose, we need the following result due to Dunford [32, Theorem 32], that we recall below for the reader's convenience:

Theorem 2.1.5. *Let T be a set, Y a Banach space, and $\Gamma \subseteq Y'$ a closed linear submanifold. Let also*[3] $\{y_n(t)\}_{n\in\mathbb{N}} \subseteq Y^T$ *be a sequence of functions from T to Y, such that*

1. *for every set of integers $I \subseteq \mathbb{N}$, there is an element $y_I \in Y^T$ such that*
$$\sum_{n\in I} \gamma(y_n(t)) = \gamma(y_I(t)), \quad t \in T, \gamma \in \Gamma; \tag{2.19}$$
2. *for every $\gamma \in \Gamma$, $\sup_{t\in T} \sum_{n=1}^{\infty} |\gamma(y_n(t))| < +\infty$;*
3. *for every n and I, the sets $y_n(T), y_I(T)$ are relatively compact;*
4. *Γ is a linear closed submanifold of Y' such that*
$$\|y\|_Y = \sup_{\gamma\in\Gamma,\, \|\gamma\|_{Y'}=1} |\gamma(y)|. \tag{2.20}$$

Then for every set I of integers, the following series

$$\sum_{n\in I} y_n(t) = y_I(t) \tag{2.21}$$

converges in norm on T.

Proposition 2.1.6. *Definition 2.1.3 and Definition 2.1.4 are equivalent.*

Proof. If $\mathfrak{m}$ is a state-valued measure according to Definition 2.1.4, then it is so also in the sense of Definition 2.1.3, because the latter is weaker, being unconditional $\sigma-$additivity in norm stronger than weak-* $\sigma-$additivity.

To prove the converse, we use Theorem 2.1.5 and exploit uniform boundedness in Banach spaces which assures that convergence of the series in weak-*-topology implies unconditional convergence of the series in strong topology in $\mathfrak{A}'$. Let us prove that all the conditions are met. Choosing $T = \{t\}$, $Y = \mathfrak{A}'$, $\Gamma = \mathfrak{A} \subseteq Y' = \mathfrak{A}''$, $y_n = \mathfrak{m}(S_n)$, for every n we have:

[3] We denote here by Y^T the space of functions $f : T \to Y$.

1. the definition $\mathfrak{m}^\kappa(\,\cdot\,) = \mathfrak{m}(\,\cdot\,)(\kappa)$ yields a scalar positive measure for every $\kappa \in \mathfrak{A}_+$. Thanks to the decomposition of elements of C*-algebra, for any $\kappa \in \mathfrak{A}$, there exist four positive elements $\kappa^{+,R}, \kappa^{-,R}, \kappa^{+,I}, \kappa^{-,I}$, such that

$$\kappa = (\kappa^{+,R} - \kappa^{-,R}) + i(\kappa^{+,I} - \kappa^{-,I}).$$

Then, there exists a complex measure $\mathfrak{m}^\kappa$, such that

$$\mathfrak{m}^\kappa = (\mathfrak{m}^{\kappa^{+,R}} - \mathfrak{m}^{\kappa^{-,R}}) + i(\mathfrak{m}^{\kappa^{+,I}} - \mathfrak{m}^{\kappa^{-,I}}).$$

By definition of the state-valued measures, we know that the positive scalar measures in the previous decomposition are $\sigma-$additive and the same holds for the measure $\mathfrak{m}^\kappa$ for every $\kappa \in \mathfrak{A}$. Then, there exists an element $y_I := \mathfrak{m}\left(\bigcup_{n\in\mathbb{N}} S_n\right) \in \mathfrak{A}'$ such that

$$\sum_{n\in I} \mathfrak{m}(S_n)(\kappa) = \sum_{n\in I} \mathfrak{m}^\kappa(S_n) = \mathfrak{m}^\kappa\left(\bigcup_{n\in\mathbb{N}} S_n\right) = \mathfrak{m}\left(\bigcup_{n\in\mathbb{N}} S_n\right)(\kappa);$$

2. for every $\kappa \in \mathfrak{A}$, $\sum_{n=1}^\infty |\mathfrak{m}(S_n)(\kappa)| < \infty$, by convergence of the four series with positive terms obtained in the decomposition of κ;

3. since $T = \{t\}$, the sets $\{\mathfrak{m}(S_n)(t)\}_{t\in T}, \{\mathfrak{m}\left(\bigcup_{n\in\mathbb{N}} S_n\right)(t)\}_{t\in T}$ are composed by only one element, and therefore they are compact;

4. $\mathfrak{A}$ is a linear closed submanifold of $\mathfrak{A}''$, such that

$$\|y\|_{\mathfrak{A}'} = \sup_{\|\kappa\|_{\mathfrak{A}''}=1} |y(\kappa)| = \sup_{\|\kappa\|_{\mathfrak{A}}=1} |y(\kappa)|.$$

we can conclude that Theorem 2.1.5 applies and we obtain that

$$\mathfrak{m}\left(\bigcup_{n=1}^\infty S_n\right) = \sum_{n=1}^\infty \mathfrak{m}(S_n) \tag{2.22}$$

unconditionally converges in the $\mathfrak{A}'-$norm, since the argument holds for every permutation of the set of indices I. □

The proof above also suggests how to define the action of the state-valued measures on complex compact operators $\kappa \in \mathscr{L}_\infty(L^2)$, simply exploiting the decomposition in positive operators of the latter and then using the linearity w.r.t. the test operators, *i.e.*,

$$\mathfrak{m}(S)(\kappa) := \mathfrak{m}(S)(\kappa^{+,R}) - \mathfrak{m}(S)(\kappa^{-,R}) + i(\mathfrak{m}(S)(\kappa^{+,I}) - \mathfrak{m}(S)(\kappa^{-,I})). \tag{2.23}$$

From now on, we will denote by $\mathscr{M}(X;\mathfrak{A}'_+)$ the space of state-valued measures. We give below some examples of these measures.

Example. 1. Every scalar valued measure is trivially a state-valued measure: choosing as $\mathfrak{A} = \mathbb{C}$, we have that $\mathfrak{m} : \Sigma \to \mathfrak{A}'_+ = \mathbb{R}^+$ and the action of the measure over $\mathfrak{A}_+ = \mathbb{R}^+$ simply by multiplication by positive numbers:

$$\mathfrak{m}(S)(\kappa) = \mathfrak{m}(S) \cdot \kappa, \qquad S \in \Sigma, \kappa \in \mathbb{R}^+.$$

Neeb condition for $\sigma-$additivity reads

$$\left|\mathfrak{m}\Big(\bigcup_{n\in\mathbb{N}} S_n\Big) \cdot \kappa - \sum_{n\in\mathbb{N}} \mathfrak{m}(S_n) \cdot \kappa\right| = 0 \quad \Rightarrow \quad \mathfrak{m}\Big(\bigcup_{n\in\mathbb{N}} S_n\Big) = \sum_{n\in\mathbb{N}} \mathfrak{m}(S_n)$$

which is the usual condition of $\sigma-$additivity for scalar measures.

2. Choosing $(X, \Sigma) = (\mathfrak{H}, \text{Borel}(\mathfrak{H}))$, $\mathfrak{A} = \mathscr{L}_\infty(L^2(\mathbb{R}^{dN}))$ and $\mathfrak{A}'_+ = \mathscr{L}_{1,+}(L^2(\mathbb{R}^{dN}))$, a state-valued measure is a norm (or weak-*)$\sigma-$additive map

$$\mathfrak{m} : \text{Borel}(\mathfrak{H}) \longrightarrow \mathscr{L}_{1,+}(L^2) \tag{2.24}$$

which may also be defined by a family of positive scalar measures $\{\mathfrak{m}^\kappa\}_{\kappa \in \mathscr{L}_{\infty,+}(L^2(\mathbb{R}^{dN}))}$, parametrized by positive, compact operators, such that

$$\mathfrak{m}^\kappa(S) = \mathfrak{m}(S)(\kappa) = \text{Tr}_{L^2}(\mathfrak{m}(S)\,\kappa). \tag{2.25}$$

We give an example of state-valued measures with values in positive density matrices: let $\{z_1, \dots, z_n\} \subseteq \mathfrak{H}$ and $\{\rho_1, \dots, \rho_n\} \subseteq \mathscr{L}_{1,+}(L^2(\mathbb{R}^{dN}))$,

$$\mathfrak{m}(S) = \sum_{j=1}^{n} \rho_j\, \chi_S(z_j), \qquad S \in \text{Borel}(\mathfrak{H}),$$

where χ_S is the indicator function of the set S.

3. Another important example of state-valued measures is given by measures with associated triple $(\mathscr{L}(\mathscr{Z})_*, \mathscr{L}_1(\mathscr{Z})_*, \mathscr{L}(\mathscr{Z})_+)$, composed by self-adjoint bounded operators, self-adjoint trace-class operators and positive bounded operators on a Hilbert space $\mathscr{Z}$, respectively. The fact that the aforementioned triple is compatible with the definition of state-valued measures is proven in [66]. The so-obtained measures are of the form

$$\mathfrak{m} : \Sigma \longrightarrow \mathscr{L}(\mathscr{Z})_+$$

with action on trace-class operators: $\mathfrak{m}^\kappa(S) = \mathfrak{m}(S)(\kappa) = \text{Tr}_{\mathscr{Z}}(\mathfrak{m}(S)\,\kappa)$, for any $S \in \Sigma$, $\kappa \in \mathscr{L}_1(\mathscr{Z})_+$. Since κ is a trace-class operator, we can write

$$\kappa = \sum_{n \in \mathbb{N}} \lambda_n\, |\psi_n\rangle\langle\psi_n|$$

with $\{\psi_j\}_{j \in \mathbb{N}}$ an orthonormal basis in $\mathscr{Z}$ and $\sum_{n \in \mathbb{N}} |\lambda_n| < +\infty$. Then, we get

$$\mathfrak{m}^\kappa(S) = \sum_{n \in \mathbb{N}} \lambda_n\, \langle\psi_n|\, \mathfrak{m}(S)\, |\psi_n\rangle$$

so that the measure is a linear combination of scalar measures.

Let us remark that with the above definitions, we are implicitly restricting the discussion to finite measures, since $\mathfrak{m}(X) \in \mathfrak{A}'_+$. In fact, for most of the thesis, we are going to consider probability measures, *i.e.*, such that $\|\mathfrak{m}(X)\|_{\mathfrak{A}'} = 1$.

A state-valued measure is also *monotone*.

Lemma 2.1.7. *For any* $S_1 \subseteq S_2 \in \Sigma$,

$$\mathfrak{m}(S_1) \leqslant \mathfrak{m}(S_2)\,,$$

i.e., $\mathfrak{m}(S_2) - \mathfrak{m}(S_1) \in \mathfrak{A}'_+$.

Proof. The scalar measures $\{\mathfrak{m}^\kappa\}_{\kappa \in \mathfrak{A}_+}$, are monotone. Therefore, for all $\kappa \in \mathfrak{A}_+$,

$$\mathfrak{m}(S_1)(\kappa) = \mathfrak{m}^\kappa(S_1) \leqslant \mathfrak{m}^\kappa(S_2) = \mathfrak{m}(S_2)(\kappa)\,.$$

Hence, for all $\kappa \in \mathfrak{A}_+$, the functional $\mathfrak{m}(S_2) - \mathfrak{m}(S_1)$ is positive. □

We now introduce the norm measure $\|\mathfrak{m}\|$, a scalar measure associated to every $\mathfrak{m}$, which we will show is absolutely continuous w.r.t. the measure and which will be useful to control trace-class integrals with scalar integrals.

Definition 2.1.8. *Let* $\mathfrak{m}$ *be a state-valued measure. Then its* ***norm-measure*** $\|\mathfrak{m}\| : \Sigma \to \mathbb{R}_+$ *is*

$$\|\mathfrak{m}\|(S) := \|\mathfrak{m}(S)\|_{\mathfrak{A}'} \qquad \text{for any } S \in \Sigma.$$

In the following we prove some properties about the norm measure.

Proposition 2.1.9. *Let* $\mathfrak{m} : \Sigma \to \mathfrak{A}'_+$ *be a state-valued measure and* $\|\mathfrak{m}\| : \Sigma \to \mathbb{R}^+$ *the associated norm measure. Then,*

(i) *for any* $S \in \Sigma$ *and* $\kappa \in \mathfrak{A}_+$,

$$\mathfrak{m}^\kappa(S) \leqslant \|\mathfrak{m}\|(S)\|\kappa\|_{\mathfrak{A}}; \tag{2.26}$$

(ii) $\|\mathfrak{m}\|$ *is a finite,* $\sigma-$*additive positive scalar measure;*

(iii) $\mathfrak{m}$ *is absolutely continuous w.r.t.* $\|\mathfrak{m}\|$, *i.e.,* $\mathfrak{m} \ll \|\mathfrak{m}\|$.

Proof. (i) Since $\mathfrak{m}(S) \in \mathfrak{A}'$, then, for any $S \in \Sigma, \kappa \in \mathfrak{A}_+$,

$$\mathfrak{m}^\kappa(S) = \mathfrak{m}(S)(\kappa) \leqslant \|\mathfrak{m}(S)\|_{\mathfrak{A}'} \|\kappa\|_{\mathfrak{A}} = \|\mathfrak{m}\|(S) \|\kappa\|_{\mathfrak{A}}.$$

(ii) The proof that $\|\mathfrak{m}\|(\varnothing) = 0$ and $\|\mathfrak{m}\|(X) < +\infty$ follows immediately from the definition. To prove σ-additivity, let us consider a family of mutually disjoint measurable sets $\{S_n\}_{n\in\mathbb{N}} \subseteq \Sigma$. For first we prove additivity: if $\{e_\alpha\}_{\alpha\in I} \subseteq \mathfrak{A}_+$ are approximate identities, then, by (2.8),

$$\|\omega\|_{\mathfrak{A}'} = \lim_{\alpha\in I} \omega(e_\alpha) \tag{2.27}$$

for any $\omega \in \mathfrak{A}'_+$, and therefore, for any $N \in \mathbb{N}$,

$$\|\mathfrak{m}\|\Big(\bigcup_{n=1}^N S_n\Big) = \lim_{\alpha\in I} \mathfrak{m}\Big(\bigcup_{n=1}^N S_n\Big)(e_\alpha) = \lim_{\alpha\in I} \mathfrak{m}^{e_\alpha}\Big(\bigcup_{n=1}^N S_n\Big) = \lim_{\alpha\in I} \sum_{n=1}^N \mathfrak{m}^{e_\alpha}(S_n) = \sum_{n=1}^N \|\mathfrak{m}\|(S_n).$$

By $\sigma-$additivity of the measures $\mathfrak{m}^{e_\alpha}$, for any $\alpha \in I$, we have

$$\lim_{\alpha\in I} \lim_{N\to+\infty} \sum_{n=1}^N \mathfrak{m}^{e_\alpha}(S_n) = \lim_{\alpha\in I} \mathfrak{m}^{e_\alpha}\Big(\bigcup_{n=1}^\infty S_n\Big) = \|\mathfrak{m}\|\Big(\bigcup_{n=1}^\infty S_n\Big). \tag{2.28}$$

Hence it remains to show that the limits in N and α can be exchanged. In order to do that, it is sufficient to show that the limit in α exists uniformly w.r.t. N:

$$\lim_{\alpha\in I} \sup_{N\in\mathbb{N}} \left| \|\mathfrak{m}\|\Big(\bigcup_{n=1}^N S_n\Big) - \sum_{n=1}^N \mathfrak{m}^{e_\alpha}(S_n)\right| = \lim_{\alpha\in I} \sup_{N\in\mathbb{N}} \left| \|\mathfrak{m}\|\Big(\bigcup_{n=1}^N S_n\Big) - \mathfrak{m}^{e_\alpha}\Big(\bigcup_{n=1}^N S_n\Big)\right| =$$

$$= \lim_{\alpha\in I} \sup_{N\in\mathbb{N}} (\|\mathfrak{m}\| - \mathfrak{m}^{e_\alpha})\Big(\bigcup_{n=1}^N S_n\Big) \leqslant \lim_{\alpha\in I} (\|\mathfrak{m}\| - \mathfrak{m}^{e_\alpha})(X) = 0\ ,$$

where we have used finite additivity of $\|\mathfrak{m}\|$ and $\|\mathfrak{m}\| - \mathfrak{m}^{e_\alpha}$. Notice that the latter is a positive measure because, for any $S \in \Sigma$, $\mathfrak{m}^{e_\alpha}(S) \leqslant \|\mathfrak{m}\|(S)$ by point (i).

(iii) Since both $\mathfrak{m}$ and $\|\mathfrak{m}\|$ are $\sigma-$additive, using the definition of absolute continuity in [31], we get

$$\lim_{\|\mathfrak{m}\|(S)\to 0} \|\mathfrak{m}(S)\|_{\mathfrak{A}'} = \lim_{\|\mathfrak{m}\|(S)\to 0} \|\mathfrak{m}\|(S) = 0, \tag{2.29}$$

and thus $\mathfrak{m} \ll \|\mathfrak{m}\|$.

□

2.2 Integration with respect to state-valued measures

In this section we introduce the notion of integration with respect to state-valued measures. The theory of integration can be constructed for both the scalar and vector-valued functions, *i.e.*, functions with values in a W*-algebra $\mathfrak{B}$ for which $\mathfrak{A} \subseteq \mathfrak{B}$ is a bilateral ideal. Since the measures take values in positive functionals over the C*-algebra, which is an ideal of the W*-algebra, the integral is naturally defined as a positive functional in $\mathfrak{A}'_+$ as well.

We discuss first the integration of scalar functions. Given the two equivalent definitions of weak-* or norm $\sigma-$additive state-valued measures, we provide two different definitions of integration.

2.2.1 Integration of scalar functions

Let us recall that a function $g : X \to \mathbb{R}^+$ is simple if there exist a number $N \in \mathbb{N}$, mutually disjoint measurable sets $S_1, \ldots, S_N \in \Sigma$ and non-negative numbers $c_1, \ldots, c_N \in \mathbb{R}_+$, such that, for all $z \in X$,

$$g(z) = \sum_{j=1}^{N} c_j \chi_{S_j}(z) \ ,$$

where χ_{S_j} is the characteristic function of the set S_j. Integration of simple functions w.r.t. a state-valued measure $\mathfrak{m}$ is straightforwardly defined as the positive functional

$$\int_X \mathrm{d}\mathfrak{m}(z)\, g(z) = \sum_{j=1}^{N} c_j\, \mathfrak{m}(S_j) \in \mathfrak{A}'_+ \ .$$

Using simple functions and Bartle's definition of state-valued measures, we can construct the integral via approximation in terms of simple functions (see [13, Theorem 9]).

Definition 2.2.1 (Integration of scalar functions-I)**.** *A measurable function $f : X \to \mathbb{R}^+$ is $\mathfrak{m}-$**integrable** if and only if, for any sequence of simple scalar functions $\{f_n\}_{n\in\mathbb{N}}$ approximating f pointwise, the sequence of simple integrals*

$$\left\{ \int_X \mathrm{d}\mathfrak{m}(z)\, f_n(z) \right\}_{n\in\mathbb{N}} \subseteq \mathfrak{A}'_+, \tag{2.30}$$

is Cauchy w.r.t. the norm in $\mathfrak{A}'$. The integral is then defined as

$$\int_S \mathrm{d}\mathfrak{m}(z)\, f(z) := \|\cdot\|_{\mathfrak{A}'} - \lim_{n\to+\infty} \int_X \mathrm{d}\mathfrak{m}(z)\, f_n(z)\, \chi_S(z), \tag{2.31}$$

and it is independent on the choice of the approximating functions $\{f_n\}_{n\in\mathbb{N}}$.

While the first definition above is constructive and rather standard in measure theory, the second one below is useful to prove important properties of integration.

Definition 2.2.2 (Integral of scalar functions-II)**.** *A measurable scalar function $f : X \to \mathbb{R}^+$ is $\mathfrak{m}-$**integrable** if and only if f is $\mathfrak{m}^\kappa-$integrable for every $\kappa \in \mathfrak{A}_+$. For a fixed $S \in \Sigma$, the integral of f w.r.t. $\mathfrak{m}$ is then defined as the functional $I_\mathfrak{m}(f) \in \mathfrak{A}'_+$, acting as*

$$I_\mathfrak{m}(f)(\kappa) = \int_S \mathrm{d}\mathfrak{m}^\kappa(z)\, f(z), \qquad \textit{for any } \kappa \in \mathfrak{A}_+. \tag{2.32}$$

The previous definition can be extended to general elements of the C*-algebra: any $\kappa \in \mathfrak{A}$ can be decomposed into a sum of positive operators $\kappa = \kappa_1 - \kappa_2 + i\kappa_3 - i\kappa_4$, with $\kappa_j \in \mathfrak{A}_+$, $j = 1, \ldots, 4$, and, therefore, the action of the integral on the C*-algebra is

$$\left(\int_S \mathrm{d}\mathfrak{m}(z)\, f(z)\right)(\kappa) = \int_S \mathrm{d}\mathfrak{m}^{\kappa_1}(z)\, f(z) - \int_S \mathrm{d}\mathfrak{m}^{\kappa_2}(z)\, f(z) + i\int_S \mathrm{d}\mathfrak{m}^{\kappa_3}(z)\, f(z) - i\int_S \mathrm{d}\mathfrak{m}^{\kappa_4}(z)\, f(z).$$

Furthermore, a complex-valued measurable function $f : X \to \mathbb{C}$ is $\mathfrak{m}-$integrable if and only if $|f|$ is $\mathfrak{m}-$integrable, as a positive-valued function. The integral of the complex-valued function is then defined as the integral of its decomposition into real/imaginary, positive/negative parts:

$$I_{\mathfrak{m}}(f) := I_{\mathfrak{m}}(\Re e(f)_+) - I_{\mathfrak{m}}(\Re e(f)_-) + iI_{\mathfrak{m}}(\Im m(f)_+) - iI_{\mathfrak{m}}(\Im m(f)_-).$$

An important result proven in [66] is the following inequality which is inherited from the continuity of the integral as a functional over the C*-algebra: if f is positive-valued, there exists a constant C, depending only on $S, \mathfrak{m}$ and f, such that, for any $\kappa \in \mathfrak{A}_+$,

$$\int_S \mathrm{d}\mathfrak{m}^{\kappa}(z)\, f(z) \leqslant C\, \|\kappa\|_{\mathfrak{A}}. \tag{2.33}$$

If a function f is integrable in both senses of Definitions 2.2.1 and 2.2.2, then the integrals must necessarily coincide. We show now that the definitions of integrability are equivalent. To this purpose, we first need to bound the integral using scalar integrals w.r.t. the norm measure $\|\mathfrak{m}\|$.

Lemma 2.2.3. *If a non-negative function f is $\mathfrak{m}$-integrable in the sense of Definition 2.2.2, then it is $\|\mathfrak{m}\|$-integrable as well.*

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be a simple, pointwise and non-decreasing approximation of f from below. By assumptions f is $\mathfrak{m}^{e_\alpha}-$integrable for any $\alpha \in I$, with $\{e_\alpha\}_{\alpha\in I}$ approximate identities. By the properties of norm-measure

$$\int_X \mathrm{d}\|\mathfrak{m}\|(z)\, f_n(z)\, \chi_S(z) = \lim_{\alpha\in I} \int_X \mathrm{d}\mathfrak{m}^{e_\alpha}(z)\, f_n(z)\, \chi_S(z) \leqslant \lim_{\alpha\in I} \int_S \mathrm{d}\mathfrak{m}^{e_\alpha}(z)\, f(z) \leqslant C \lim_{\alpha\in I} \|e_\alpha\|_{\mathfrak{A}} \leqslant C\ .$$

Taking the limit $n \to +\infty$ of the previous expressions, we get the $\|\mathfrak{m}\|-$integrability of f by monotone convergence theorem:

$$\int_S \mathrm{d}\|\mathfrak{m}\|(x) f(x) = \lim_{n\to\infty} \int_X \mathrm{d}\|\mathfrak{m}\|(z)\, f_n(z)\chi_S(z) \leqslant C\ .$$

□

Proposition 2.2.4. *Let $f : X \to \mathbb{R}^+$ be a measurable function. Then f is $\mathfrak{m}$-integrable in the sense of Definition 2.2.1 if and only if it is $\mathfrak{m}-$integrable in the sense if Definition 2.2.2. In addition, for any $S \in \Sigma$,*

$$\left\|\int_S \mathrm{d}\mathfrak{m}(z) f(z)\right\|_{\mathfrak{A}'} \leqslant \int_S \mathrm{d}\|\mathfrak{m}\|(z) f(z)\ .$$

Proof. If f is $\mathfrak{m}-$integrable according to Definition 2.2.1, then $\{I_{\mathfrak{m}}(f_n)\}_{n\in\mathbb{N}}$ is Cauchy in the norm topology of $\mathfrak{A}'$ and thus in weak-* topology. Then, it is possible to define a positive functional $I_{\mathfrak{m}}(f) \in \mathfrak{A}'_+$ such that $I_{\mathfrak{m}}(f)(\kappa) := \lim_{n\to+\infty} I_{\mathfrak{m}}(f_n)(\kappa) < +\infty$ for any $\kappa \in \mathfrak{A}$.

Let us prove the converse: consider a sequence of non-decreasing, positive simple functions $\{f_n\}_{n\in\mathbb{N}}$ approximating f for $\mathfrak{m}-$a.e. $z \in X$. Then $f_n - f_m$, for $n \geqslant m$, is a simple, positive function of the form

$$(f_n - f_m)(z) = \sum_{j=1}^{N(n,m)} c_j^{(n,m)} \chi_{S_j^{(n,m)}}(z)$$

where $\{S_j^{(n,m)}\}_{j\in\mathbb{N}} \subseteq \Sigma$ are mutually disjoint measurable sets obtained by the intersection of the sets $\{S_k^{(l)}\}_{k\in\mathbb{N}}$ for $l = n, m$. Hence, since $f_n - f_m \leqslant 2f$, for $\mathfrak{m}-$a.e. $z \in X$, and f is $\|\mathfrak{m}\|-$integrable thanks to Lemma 2.2.3, we can use the dominated convergence theorem to obtain

$$\left\| \int_S \mathrm{d}\mathfrak{m}(z) \big(f_n(z) - f_m(z)\big) \right\|_{\mathfrak{A}'} \leqslant \sum_{j=1}^{N(n,m)} c_j^{(n,m)} \|\mathfrak{m}\|(S_j^{(n,m)} \cap S) = \int_S \mathrm{d}\|\mathfrak{m}\|(z)(f_n - f_m)(z) \underset{\substack{n\to\infty \\ m\to\infty}}{\longrightarrow} 0\ .$$

This proves the $\mathfrak{m}-$integrability in the sense of Definition 2.2.1. Furthermore, we have $I_{\mathfrak{m}}(f) = \|\cdot\|_{\mathfrak{A}'} - \lim_{n\to+\infty} I_{\mathfrak{m}}(f_n)$ and

$$\left\| \int_S \mathrm{d}\mathfrak{m}(z)\, f(z) \right\|_{\mathfrak{A}'} \leqslant \lim_{n\to+\infty} \left\| \int_S \mathrm{d}\mathfrak{m}(z)\, f_n(z) \right\|_{\mathfrak{A}'} \leqslant \lim_{n\to+\infty} \sum_{j=1}^{N(n)} c_j^{(n)} \|\mathfrak{m}\|(S_j^{(n)} \cap S) \leqslant \int_S \mathrm{d}\|\mathfrak{m}\|(z)\, f(z).$$

□

As usual in measure theory, the integrability of a complex-valued function is equivalent to the integrability of its positive parts as positive functions. We will denote the space of $\mathfrak{m}-$integrable, scalar functions by $L^1(X; \mathbb{C}; \mathrm{d}\mathfrak{m})$. The integral defined is linear and monotone:

Lemma 2.2.5. *Let $f, g : X \to \mathbb{R}$ be two $\mathfrak{m}$-integrable functions. If, for $\mathfrak{m}$-a.e. $z \in X$,*

$$g(z) \leqslant f(z)\ ,$$

then

$$\int_X \mathrm{d}\mathfrak{m}(z)\, g(z) \leqslant \int_X \mathrm{d}\mathfrak{m}(z)\, f(z)\ ,$$

i.e., $\int_X \mathrm{d}\mathfrak{m}(z)\big(f(z) - g(z)\big) \in \mathfrak{A}'_+$.

Proof. The function $f - g$ is $\mathfrak{m}-$almost everywhere positive and, by Definition 2.2.2, we have

$$\left(\int_X \mathrm{d}\mathfrak{m}(z)\, \big(f(z) - g(z)\big) \right)(\kappa) = \int_X \mathrm{d}\mathfrak{m}^{\kappa}(z)\, \big(f(z) - g(z)\big) \geqslant 0\ ,$$

for any $\kappa \in \mathfrak{A}_+$, because $\{\mathfrak{m}^{\kappa}\}_{\kappa\in\mathfrak{A}_+}$ are positive measures. □

Theorem 2.2.6 (Dominated Convergence - I)**.** *Let $f_n : X \to \mathbb{R}_+$, $n \in \mathbb{N}$, be a sequence of measurable functions that converges $\mathfrak{m}$-a.e. to $f : X \to \mathbb{R}^+$. If there exists a $\mathfrak{m}$-integrable function $g : X \to \mathbb{R}_+$, such that $f_n \leqslant g$ $\mathfrak{m}$-a.e., then, f_n, $n \in \mathbb{N}$, and f are $\mathfrak{m}$-integrable, and, for all $S \in \Sigma$,*

$$\int_S \mathrm{d}\mathfrak{m}(z)\, f(z) = \|\cdot\|_{\mathfrak{A}'} - \lim_{n\to\infty} \int_S \mathrm{d}\mathfrak{m}(z)\, f_n(z)\ .$$

Proof. Since $f_n \leqslant g$ and g is $\mathfrak{m}$-integrable, then, for all $n \in \mathbb{N}$, for all $\kappa \in \mathfrak{A}_+$, and for all $S \in \Sigma$,

$$\int_S \mathrm{d}\mathfrak{m}^\kappa(z)\, f_n(z) \leqslant \int_S \mathrm{d}\mathfrak{m}^\kappa(z)\, g(z) < +\infty\ .$$

Therefore, f_n is $\mathfrak{m}$-integrable for all $n \in \mathbb{N}$ and, by triangular inequality, f is integrable as well. [13, Theorem 6] allows to exchange the limit $n \to +\infty$ with the integral, provided that the following condition is fulfilled: for all $S \in \Sigma$ and for all $n \in \mathbb{N}$,

$$\Big\|\int_S \mathrm{d}\mathfrak{m}(z)\, f_n(z)\Big\|_{\mathfrak{A}'} \leqslant \Big\|\int_S \mathrm{d}\mathfrak{m}(z)\, g(z)\Big\|_{\mathfrak{A}'}\ .$$

In order to prove this inequality, we use the approximate identities $\{e_\alpha\}_{\alpha\in I} \subseteq \mathfrak{A}_+$:

$$\Big(\int_S \mathrm{d}\mathfrak{m}(x) f_n(x)\Big)(e_\alpha) = \int_S \mathrm{d}\mathfrak{m}^{e_\alpha}(z)\, f_n(z) \leqslant \int_S \mathrm{d}\mathfrak{m}^{e_\alpha}(z)\, g(z) = \Big(\int_S \mathrm{d}\mathfrak{m}(z)\, g(z)\Big)(e_\alpha)\ .$$

Hence, it follows that

$$\Big\|\int_S \mathrm{d}\mathfrak{m}(z)\, f_n(z)\Big\|_{\mathfrak{A}'} = \lim_{\alpha\in I}\Big(\int_S \mathrm{d}\mathfrak{m}(x) f_n(x)\Big)(e_\alpha) \leqslant \lim_{\alpha\in I}\Big(\int_S \mathrm{d}\mathfrak{m}(z)\, g(z)\Big)(e_\alpha) = \Big\|\int_S \mathrm{d}\mathfrak{m}(z)\, g(z)\Big\|_{\mathfrak{A}'}\ .$$

□

2.2.2 Integration of operator-valued functions

In order to develop a theory of integration for operator-valued functions, we follow the same steps as for scalar functions, defining simple functions.

An operator valued function $g : X \to \mathfrak{B}$ is simple if there exist mutually disjoint measurable sets $\{S_n\}_{n\in\mathbb{N}} \subseteq \Sigma$, and $\{c_n\}_{n\in\mathbb{N}} \subseteq \mathfrak{B}$ such that for all $z \in X$,

$$g(z) = \sum_{j=1}^{N} c_j\, \chi_{S_j}(z)\ .$$

The integral w.r.t. $\mathfrak{m}$ of the simple function g is therefore

$$\int_X \mathrm{d}\mathfrak{m}(z)\, g(z) := \sum_{j=1}^{N} \mathfrak{m}(S_j)(\, c_j\ \cdot\,) \in \mathfrak{A}'.$$

Let us remark that, being $\mathfrak{A}$ a bilateral ideal of $\mathfrak{B}$, we could define as well the simple integral as

$$\int_X \mathrm{d}\mathfrak{m}(z)\, g(z) := \sum_{j=1}^{N} \mathfrak{m}(S_j)(\,\cdot\ c_j) \in \mathfrak{A}'.$$

From now on, we focus on the first choice, the other being perfectly analogous.

In general, it is not guaranteed that a measurable function $f : X \to \mathfrak{B}$ can be approximated by a sequence of simple functions of the type introduced above. Luckily, for a function with separable range, that is the case.

Proposition 2.2.7. *[25, Proposition E.2] Let $f : X \to \mathfrak{B}$ be a measurable function. If $f(X)$ is separable, then there exists a sequence $\{f_n\}_{n\in\mathbb{N}}$ of simple functions such that, for all $z \in X$,*

$$\lim_{n\to\infty} \|f(z) - f_n(z)\|_{\mathfrak{B}} = 0\ ,$$

and such that, for any $n \in \mathbb{N}$,

$$\|f_n(z)\|_{\mathfrak{B}} \leqslant \|f(z)\|_{\mathfrak{B}}\ .$$

Hence, for measurable functions with separable range we can give a definition of integrability by using simple functions, following the idea of Definition 2.2.1.

Definition 2.2.8. *A measurable function $f : X \to \mathfrak{B}$ with separable range is $\mathfrak{m}-$integrable if and only there exists a sequence of simple functions $\{f_n\}_{n\in\mathbb{N}}$ approximating f according to Proposition 2.2.7, such that*

$$\left\{\int_X \mathrm{d}\mathfrak{m}(z)\, f_n(z)\right\}_{n\in\mathbb{N}} \subseteq \mathfrak{A}' \tag{2.34}$$

is Cauchy in norm. The integral is then defined as

$$\int_S \mathrm{d}\mathfrak{m}(z)\, f(z) := \|\cdot\|_{\mathfrak{A}'} - \lim_{n\to+\infty}\int_X \mathrm{d}\mathfrak{m}(z)\, f_n(z)\,\chi_S(z), \tag{2.35}$$

and it can be shown to be independent from the choice of the approximating simple functions.

For the rest of the section, if not specified otherwise, we will consider observables with separable range.

Continuing to follow the analogy with the scalar case, we now control the vector integrals by scalar integrals, by means of norm measure. To do so, we first introduce the notion of absolute integrability.

Definition 2.2.9 (Absolute integrability)**.** *A measurable function $f : X \to \mathfrak{B}$ is $\mathfrak{m}$-**absolutely integrable** if and only if $\|f(\cdot)\|_{\mathfrak{B}} : X \to \mathbb{R}^+$ is $\|\mathfrak{m}\|$-integrable. We will denote by $L^1(X;\mathfrak{B};\mathrm{d}\mathfrak{m})$ the space of $\mathfrak{m}-$absolutely integrable functions.*

Proposition 2.2.10. *Let $f : X \to \mathfrak{B}$ be a $\mathfrak{m}$-absolutely integrable function, then f is also $\mathfrak{m}$-integrable. In addition, for all $S \in \Sigma$,*

$$\left\|\int_S \mathrm{d}\mathfrak{m}(z)\, f(z)\right\|_{\mathfrak{A}'} \leqslant \int_S \mathrm{d}\|\mathfrak{m}\|(z)\,\|f(z)\|_{\mathfrak{B}}\ .$$

Proof. The proof is analogous to the proof of Proposition 2.2.4. By Proposition 2.2.7 there exists a sequence of simple functions $\{f_n\}_{n\in\mathbb{N}}$ with non-decreasing norm approximating f pointwise in the norm of $\mathfrak{B}$. For any $n \geqslant m \in \mathbb{N}$, $f_n - f_m$ is a simple function, that we write as

$$(f_n - f_m)(z) = \sum_{j=1}^{N^{(n,m)}} c_j^{(n,m)}\,\chi_{S_j^{(n,m)}}(z)\ .$$

In addition, observe that for any simple function $g = \sum_{j=1}^{N} c_j\chi_{S_j}$,

$$\|g(z)\|_{\mathfrak{B}} = \sum_{j=1}^{N}\|c_j\|_{\mathfrak{B}}\,\chi_{S_j}(z)\ .$$

Using this fact we can write

$$\left\|\int_S \mathrm{d}\mathfrak{m}(x)\big(f_n(z) - f_m(z)\big)\right\|_{\mathfrak{A}'} = \left\|\sum_{j=1}^{N^{(n,m)}} \mathfrak{m}(S_j^{(n,m)}\cap S)(c_j^{(n,m)})\right\|_{\mathfrak{B}} \leqslant$$

$$\leqslant \sum_{j=1}^{N^{(n,m)}} \|\mathfrak{m}\|(S_j^{(n,m)}\cap S)\|c_j^{(n,m)}\|_{\mathfrak{B}} = \int_S \mathrm{d}\|\mathfrak{m}\|(z)\,\|(f_n - f_m)(z)\|_{\mathfrak{B}} \underset{\substack{n\to\infty\\ m\to\infty}}{\longrightarrow} 0\ ,$$

where in the last limit we used the dominated convergence theorem, since $\|f_n(z) - f_m(z)\|_{\mathfrak{B}} \leqslant 2\|f(z)\|_{\mathfrak{B}}$ for any $z \in X$, and the $\|\mathfrak{m}\|-$integrability of f. ☐

Corollary 2.2.11. *Any function with separable range $f : X \to \mathfrak{B}$ such that $\|f(\,\cdot\,)\|_{\mathfrak{B}}$ is $\|\mathfrak{m}\|$-a.e. uniformly bounded is $\mathfrak{m}$-integrable.*

Finally, we are ready to state a dominated convergence theorem for operator-valued functions with separable range.

Theorem 2.2.12 (Dominated Convergence - II)**.** *Let $\{f_n\}_{n\in\mathbb{N}}$, $f_n : X \to \mathfrak{B}$ for all $n \in \mathbb{N}$, be a sequence of operator-valued functions that strongly converges $\mathfrak{m}$-a.e. to $f : X \to \mathfrak{B}$. If there exists a $\|\mathfrak{m}\|$-integrable function $G : X \to \mathbb{R}^+$ such that $\mathfrak{m}$-a.e.*

$$\|f_n(z)\|_{\mathfrak{B}} \leqslant G(z) \ ,$$

then, for any $n \in \mathbb{N}$, f_n, f are $\mathfrak{m}$-absolutely integrable, and

$$\int_S \mathrm{d}\mathfrak{m}(z)\, f(z) = \|\cdot\|_{\mathfrak{A}'} - \lim_{n\to\infty} \int_S \mathrm{d}\mathfrak{m}(z)\, f_n(z) \ .$$

Proof. By dominated convergence theorem for scalar measures and functions it follows that the $\|f_n(\,\cdot\,)\|_{\mathfrak{B}}, \|f(\,\cdot\,)\|_{\mathfrak{B}}$ are all $\|\mathfrak{m}\|$-integrable. By definition, it means that f_n, for any $n \in \mathbb{N}$, and f are absolutely $\mathfrak{m}-$integrable, and so, by Proposition 2.2.10, also $\mathfrak{m}$-integrable. Now, for any $S \in \Sigma$, again by Proposition 2.2.10

$$\Big\|\int_S \mathrm{d}\mathfrak{m}(z)(f - f_n)(z)\Big\|_{\mathfrak{A}'} \leqslant \int_S \mathrm{d}\|\mathfrak{m}\|(z)\, \|(f - f_n)(z)\|_{\mathfrak{B}} \ .$$

Hence the proof is reduced to the application of dominated convergence for $\|\mathfrak{m}\|$, applied to the sequence of scalar functions $\{\|(f - f_n)(\,\cdot\,)\|\}_{n\in\mathbb{N}}$. Since $\|(f - f_n)(\,\cdot\,)\|_{\mathfrak{B}} \leqslant 2\|f(\,\cdot\,)\|_{\mathfrak{B}}$ and we have proved that the last one is $\|\mathfrak{m}\|-$integrable, it follows that in the strong topology of $\mathfrak{A}'$,

$$\int_S \mathrm{d}\mathfrak{m}(z)\, f(z) = \lim_{n\to\infty} \int_S \mathrm{d}\mathfrak{m}(z)\, f_n(z) \ .$$

□

2.2.3 Radon-Nikodým property and integration of functions with non-separable range

State-valued measures are rather difficult to manage in their general form. However, if $\mathfrak{A}'$ is separable, it is possible to write them in a form that is very convenient for concrete applications. Thanks to a result of Dunford and Pettis (see [33, Theorem 2.1.1]) the so-called *Radon-Nykodým property* holds and therefore it is possible to separate the measure in a scalar measure and an integrable, state-valued function. Such decomposition allows to construct a good theory of integration for operator-valued observables with non-separable range, enlarging thus the class of integrable functions.

The aforementioned result assure that, given a vectorial measure $\mathfrak{m}$ absolutely continuous w.r.t. a scalar measure m, then there exists the Radon-Nykodým derivative $\frac{\mathrm{d}\mathfrak{m}}{\mathrm{d}m}$. In our case we consider as scalar measure $m = \|\mathfrak{m}\|$ the norm measure, and obtain the correspondence

$$\mathscr{M}(X; \mathfrak{A}'_+) \ni \mathfrak{m} \quad \longleftrightarrow \quad \left(\|\mathfrak{m}\| \,,\, \frac{\mathrm{d}\mathfrak{m}}{\mathrm{d}\|\mathfrak{m}\|}(\,\cdot\,)\right) \in \mathscr{M}(X; \mathbb{R}^+) \times L^1(X; \mathfrak{A}'_+; \mathrm{d}\|\mathfrak{m}\|). \tag{2.36}$$

It is however convenient that the $\mathfrak{A}'$−valued function are uniformly normalized w.r.t. the variable in X. Therefore, we include the norm of the Radon-Nykodým derivative in the scalar measure and we denote by μ the new measure:

$$\mathrm{d}\mu(\,\cdot\,) = \mathrm{d}\|\mathfrak{m}\|(\,\cdot\,)\left\|\frac{\mathrm{d}\mathfrak{m}}{\mathrm{d}\|\mathfrak{m}\|}(\,\cdot\,)\right\|_{\mathfrak{A}'} \tag{2.37}$$

Theorem 2.2.13. *If $\mathfrak{A}'$ is separable, then, for every state-valued measure $\mathfrak{m}$, there exists a scalar measure $\mu \in \mathscr{M}(X;\mathbb{R}^+)$ and a function $\gamma_{\mathfrak{m}} : X \to \mathfrak{A}'_+$ that is μ-Bochner-integrable, such that, uniformly in $z \in X$*

$$\|\gamma_{\mathfrak{m}}(z)\|_{\mathfrak{A}'} = 1, \tag{2.38}$$

and such that, for all $S \in \Sigma$,

$$\mathfrak{m}(S) = \int_S \mathrm{d}\mu(z)\,\gamma_{\mathfrak{m}}(z)\;.$$

Therefore, there is a natural way to define integrability in the case of spaces with the Radon-Nykodým property: the integration is of Bochner-type and with respect to a scalar measure. We recall below the action on elements of $\mathfrak{A}$:

$$\left(\int_S \mathrm{d}\mathfrak{m}(z)\,f(z)\right)(\kappa) = \int_S \mathrm{d}\mu(z)\,\gamma_{\mathfrak{m}}(z)(f(z)\kappa), \qquad \kappa \in \mathfrak{A}, \tag{2.39}$$

and with the Definition below we relate the integrability of functions with separable range, w.r.t. state-valued measures, with the Bochner theory of integration.

Definition 2.2.14. *Let us consider a state-valued measure $\mathfrak{m} \in \mathscr{M}(X;\mathfrak{A}')$ with $\mathfrak{A}'$ separable and decomposition $\mathfrak{m} = (\gamma_{\mathfrak{m}}, \mu)$, and let $f : X \to \mathfrak{B}$ be a measurable function. Then f is $\mathfrak{m}$−**integrable** if and only if $\gamma_{\mathfrak{m}}(\,\cdot\,)f(\,\cdot\,) \in \mathfrak{A}'$ is μ−Bochner integrable, and for any $S \in \Sigma$,*

$$\boxed{\int_S \mathrm{d}\mathfrak{m}(z)\,f(z) := \int_S \mathrm{d}\mu(z)\,\gamma_{\mathfrak{m}}(z)f(z) \in \mathfrak{A}'.} \tag{2.40}$$

It is easy to see that, in the case of function f with separable range, the Definition 2.2.14 is equivalent to the Definition 2.2.8 à la Bartle simply applying the definitions and the decomposition. Furthermore, since $\|\mathfrak{m}\|$-Bochner-integrability is equivalent to $\mathfrak{m}$-absolute integrability, it follows that if $\mathfrak{A}'$ is separable then $\mathfrak{m}$-integrability is equivalent to $\mathfrak{m}$-absolute-integrability. Hence all the results of Section 2.2 extend, if $\mathfrak{A}'$ is separable, to functions with non-separable range; let us give a useful explicit example.

Proposition 2.2.15. *Let $\mathfrak{m} \in \mathscr{M}(X;\mathfrak{A}'_+)$ such that $\mathfrak{A}'$ is separable. Then if $U : X \to \mathfrak{B}$ is such that there exists $C > 0$ such that for $\mathfrak{m}$−a.e. $z \in X$, $\|U(z)\|_{\mathfrak{B}} \leqslant C$, then both U and $U \cdot U^\dagger$ are $\mathfrak{m}$-integrable, and, for any $S \in \Sigma$,*

$$\int_S U(z)\,\mathrm{d}\mathfrak{m}(z)\,U^\dagger(z) = \int_S \mathrm{d}\mu(z)\,U(z)\,\gamma_{\mathfrak{m}}(z)\,U^\dagger(z) \in \mathfrak{A}'\,. \tag{2.41}$$

2.2.4 Push-forward of measures

Let us now define the push-forward of state-valued measures.

Definition 2.2.16. *Let $\Phi : X \to Y$, be a measurable function with $(X, \Sigma), (Y, \Sigma')$ two measurable spaces, and $\mathfrak{m} \in \mathscr{M}(X; \mathfrak{A}'_+)$ a state-valued measure. Then the **push-forward** of $\mathfrak{m}$ by the function Φ is defined, for every measurable set $S \in \Sigma'$, as*

$$(\Phi \sharp \mathfrak{m})(S) := \mathfrak{m}(\Phi^{-1} S) \tag{2.42}$$

The explicit action of a push-forward measure is the following: since for every characteristic function $\chi_S(y)$

$$\int_Y \mathrm{d}(\Phi \sharp \mathfrak{m})(z)\, \chi_S(z) = (\Phi \sharp \mathfrak{m})(S) = \mathfrak{m}(\Phi^{-1} S) = \int_X \mathrm{d}\mathfrak{m}(z)\, \chi_{\Phi^{-1}S}(z) = \int_X \mathrm{d}\mathfrak{m}(z)\, \chi_S(\Phi z)$$

then, by density of simple functions, if $f \circ \Phi \in L^1(X; \mathbb{C}; \mathrm{d}\mathfrak{m})$,

$$\int_Y \mathrm{d}(\Phi \sharp \mathfrak{m})(z)\, f(z) = \int_X \mathrm{d}\mathfrak{m}(z)\, f(\Phi z)$$

If $\mathfrak{A}'$ has the Radon-Nykodým property, we have the following action for the decomposition associated to $\mathfrak{m}$:

$$\int_Y \mathrm{d}(\Phi \sharp \mathfrak{m})(z)\, f(z) = \int_X \mathrm{d}\mu(z)\, \gamma_{\mathfrak{m}}(z)\, f(\Phi z) \tag{2.43}$$

that is, the push-forward *does not act* on the Radon-Nykodým derivative of the measure.

2.3 Cylindrical state-valued measures

Let us extend the concept of state-valued measures to the so-called cylindrical state-valued measures. Cylindrical state-valued measures are the quasi-classical counterpart of a very general class of microscopic states.

There are equivalent definitions for cylindrical state-valued measures, and here we adopt the most suitable for our purposes (see [40, Appendix A]). We denote by $\mathscr{F}(X)$ the set of finite codimensional, weakly closed, subspaces of X:

$$\mathscr{F}(X) := \{E \text{ subspace of } X \,|\, \dim(X/E) < +\infty,\ E \text{ is } \sigma(X, X') - \text{closed}\}.$$

It is possible to make the following decomposition

$$X = \prod_{E \in \mathscr{F}(X)} X/E. \tag{2.44}$$

Definition 2.3.1 (Cylindrical state-valued measures). *We call $\mathfrak{m} = \{\mathfrak{m}_E\}_{E \in \mathscr{F}(X)}$ a **cylindrical state-valued measure**[4] iff it is a projective system of state-valued measures on finite dimensional quotients of X, that is:*

- *for any $E \in \mathscr{F}(X)$, $\mathfrak{m}_E \in \mathscr{M}(X/E;\, \mathfrak{A}'_+)$ is a state-valued measure;*
- *defining the family of maps $\{p_{E,F}\}_{F \subseteq E \in \mathscr{F}(X)}$*

$$\begin{aligned} p_{E,F} :& X/F \longrightarrow X/E \\ & [x]_F \longmapsto [x]_E \end{aligned}$$

then $\mathfrak{m}_E = p_{E,F} \sharp \mathfrak{m}_F$.

[4]whose set we denote by $\mathscr{M}_{\mathrm{cyl}}(X; \mathfrak{A}'_+)$.

In the case of $\mathfrak{A}'_+ = \mathbb{R}_+$ we find the usual definition for the scalar cylindrical measures.

Given a cylindrical state-valued measure, it is possible to define the **Fourier transform** of that measure. Indeed, the function $z \mapsto e^{2i\Re e\langle z\,|\,\xi\rangle}$ is cylindrical (see Definition 5.3.3), and bounded, and therefore the following map is well defined:

$$\mathscr{M}_{\mathrm{cyl}}(\mathfrak{H};\mathscr{L}_{1,+}(L^2)) \ni \mathfrak{m} \longmapsto \hat{\mathfrak{m}}(\xi) := \int_{\mathfrak{H}} \mathrm{d}\mathfrak{m}(z)\; e^{2i\Re e\langle \xi\,|\,z\rangle} \in \mathscr{L}_1(L^2). \tag{2.45}$$

In the following Chapters it will be explained how this map uniquely characterize the associated state-valued measure and how topologies of convergence for the Fourier transforms induce topology on the state-valued measures.

An interesting question is when it is possible to obtain a concentration of cylindrical measures as pure state-valued measures. In general, cylindrical measures on X are state-valued measures in a bigger space $\bar{X}$. We can construct this last space in the following way: consider $E \in \mathscr{F}(X)$ and the space X/E. There is a canonical injection $j_E : X/E \to \overline{X/E}$, where the latter is the Čech compactification[5] of X/E, and it induces another injection $j = \prod_{E\in\mathscr{F}(X)} j_E$ of X into

$$\bar{X} := \prod_{E\in\mathscr{F}(X)} \overline{X/E}. \tag{2.46}$$

which is a compact space thanks to Tychonoff theorem. If $\mathfrak{m} \in \mathscr{M}_{\mathrm{cyl}}(X;\mathfrak{A}'_+)$ is a cylindrical measure, we can define the push-forward measure

$$\bar{\mathfrak{m}} := j\,\sharp\,\mathfrak{m} \in \mathscr{M}_{\mathrm{cyl}}(\bar{X};\mathfrak{A}'_+),$$

as the projective family $\bar{\mathfrak{m}} := \{j_E \,\sharp\, \mathfrak{m}_E\}_{E\in\mathscr{F}(X)}$, $j_E\,\sharp\,\mathfrak{m}_E \in \mathscr{M}(\overline{X/E};\mathfrak{A}'_+)$. A key result for the concentration of cylindrical measures is the following Prokhorov's criterion for tightness of measures:

Lemma 2.3.2 (Prokhorov criterion). *A cylindrical measure $\mu \in \mathscr{M}_{\mathrm{cyl}}(\mathscr{Z};\mathbb{R}^+)$ concentrates as a probability measure on $\mathscr{Z}$ if and only if, for any $\eta > 0$, there exists $R_\eta > 0$ such that, for any finite dimensional projector $\mathbb{P}$ on $\mathfrak{H}$,*

$$\mu(\{z \in \mathscr{Z} : \|\mathbb{P}z\|_{\mathfrak{H}} \leqslant R_\eta\}) \geqslant 1 - \eta. \tag{2.47}$$

By [75], $\bar{\mathfrak{m}}$ concentrates as a state-valued measure on $\bar{X}$: $\bar{\mathfrak{m}} \in \mathscr{M}(\bar{X};\mathfrak{A}'_+)$, and then every cylindrical measure on a space can be viewed also as a state-valued measure on $\bar{X}$. The problem with this last space is that sometimes it is so "big" that it is not useful to express properties of the measures. For example, cylindrical measures on $X = L^2(\mathbb{R}^d)$ concentrates as state-valued measures on $\bar{X} = \overline{L^2(\mathbb{R}^2)}_{L^2}$, that is, the completion in weak topology of the L^2 space. It would be so preferable to have a finer criterion for the concentration in a smaller space (that may or may not coincide with X) that we will present in the next chapter for cylindrical measures that are cluster points of net of semiclassical states.

Below we give a standard example of cylindrical state-valued measures: the *Gaussian measure*. Heuristically speaking, one wants to give a good definition of the expression

$$\mathrm{d}\mathfrak{g}(u) = \text{“}\, Z^{-1} e^{-\frac{\langle u\,|\,u\rangle}{2}} \mathrm{d}u \,\text{”} \tag{2.48}$$

for some scalar product $\langle\cdot\,|\,\cdot\rangle$. In order to do that, we follow the construction outlined in [47, Chap.3] and [63, Sec.3]: suppose that $X = \mathfrak{H}$ is a complex Hilbert space with scalar product $\langle\cdot\,|\,\cdot\rangle_{\mathfrak{H}}$. Let $h > 0$ be a positive operator on $\mathfrak{H}$ with compact resolvent. Then by spectral theorem

[5] If V is a topological space, its Čech compactification, denoted by $\bar{V}$, is a compact space containing V such that V is dense in $\bar{V}$ and for every continuous function $f : X \to K$, with K a Hausdorff compact space, there exists a continuous extension $\bar{f} : \bar{X} \to K$.

it admits a basis of eigenvectors $\{u_j\}_{j\in\mathbb{N}} \subseteq \mathfrak{H}$ with relative eigenvalues $\{\lambda_j\}_{j\in\mathbb{N}}$ such that every element in the Hilbert space can be decomposed

$$u = \sum_{j\in\mathbb{N}} \langle u_j \,|\, u\rangle_{\mathfrak{H}}\, u_j =: \sum_{j\in\mathbb{N}} \alpha_j u_j. \tag{2.49}$$

If, for any $s \in \mathbb{R}$, we denote by

$$\mathfrak{H}^{(s)} := \mathscr{D}(h^{s/2}) = \Big\{ u \in \mathfrak{H} \;:\; \|u\|^2_{\mathfrak{H}_s} := \sum_{j\in\mathbb{N}} \lambda_j^s |\alpha_j|^2 < +\infty \Big\} \tag{2.50}$$

there is a unitary equivalence with the weighted ℓ^2 space

$$\ell^2_s(\mathbb{N};\mathbb{C}) := \Big\{ \{\alpha_j\}_{j\in\mathbb{N}} \;:\; \sum_{j\in\mathbb{N}} \lambda_j^s |\alpha_j|^2 < +\infty \Big\}. \tag{2.51}$$

Trivially, $\mathfrak{H} = \mathfrak{H}^{(0)}$. Now, for any $E \in \mathscr{F}(\mathfrak{H}/E)$, $\mathfrak{H}/E$ is finite dimensional, say $\dim_{\mathbb{C}}(\mathfrak{H}/E) = n$, and so we can define the isometry

$$\begin{aligned} i_E : &\mathbb{C}^n \longrightarrow \mathfrak{H}/E \\ (\alpha_1, \dots, \alpha_n) &\mapsto u = \sum_{j=1}^{n} \alpha_j u_j \end{aligned}$$

provided that $\mathfrak{H}/E$, as a subspace[6] of $\mathfrak{H}$, is generated by the eigenvectors $\{u_j\}_{j=1}^n$. So we are ready to define the weighted Gaussian (probability) measure as the push-forward measure, for any $S \in \mathrm{Borel}(\mathfrak{H}/E)$,

$$\mathfrak{g}_E^{(h)}(S) := \frac{1}{Z_E} \int_S \frac{\mathrm{d}u}{(2\pi)^n}\, e^{-\frac{1}{2}\langle u|h|u\rangle} = \frac{1}{Z_E} \int_{i_E^{-1}S} \frac{\mathrm{d}\alpha_1 \dots \mathrm{d}\alpha_n}{(2\pi)^n} e^{-\frac{1}{2}\sum_{j=1}^n \lambda_j |\alpha_j|^2} \tag{2.52}$$

with

$$Z_E := \int_{\mathfrak{H}/E} \frac{\mathrm{d}u}{(2\pi)^n}\, e^{-\frac{1}{2}\langle u|h|u\rangle} = \int_{\mathbb{C}^n} \frac{\mathrm{d}\alpha_1 \dots \mathrm{d}\alpha_n}{(2\pi)^n} e^{-\frac{1}{2}\sum_{j=1}^n \lambda_j |\alpha_j|^2} = \prod_{j=1}^{n} \lambda_j^{-1}, \tag{2.53}$$

so that we can express the measure in the following way

$$\mathfrak{g}_E^{(h)}(S) = \int_{i_E^{-1}S} \prod_{j=1}^{n} \left(\frac{\mathrm{d}\alpha_j}{2\pi}\, \lambda_j\, e^{-\frac{\lambda_j |\alpha_j|^2}{2}} \right). \tag{2.54}$$

This defines a collection of positive scalar measures on the finite-dimensional quotient spaces. To conclude that $\mathfrak{g} = \{\mathfrak{g}_E\}_{E\in\mathscr{F}(\mathfrak{H})}$ is a cylindrical measure we have to prove the compatibility condition: let $F \subseteq E$, with $E, F \in \mathscr{F}(\mathfrak{H})$ and consider the map $p_{E,F} : \mathfrak{H}/F \to \mathfrak{H}/E$. The quotient spaces are finite-dimensional, and without loss of generality we can consider (at most, applying a permutation of the basis),

$$\begin{aligned} \mathfrak{H}/E &= \mathrm{Span}\{u_1, \dots, u_m\} \\ \mathfrak{H}/F &= \mathrm{Span}\{u_1, \dots, u_m, \dots, u_n\} \end{aligned}$$

[6] Here we mean, recalling that

$$\mathfrak{H} = \prod_{E\in\mathscr{F}(\mathfrak{H})} \mathfrak{H}/E,$$

its image under the injection $\mathfrak{H}/E \hookrightarrow \mathfrak{H}$.

with $m \leqslant n$, and we have the following isometries

$$i_F : \mathbb{C}^n \to \mathfrak{H}/F, \qquad i_E : \mathbb{C}^m \to \mathfrak{H}/E$$
$$(\alpha_1, \dots, \alpha_n) \mapsto [x]_F, \qquad (\alpha_1, \dots, \alpha_m) \mapsto [x]_E$$

so that we can interpret $i_E = i_F \upharpoonright_{\mathbb{C}^m}$, and so $i_F^{-1}([x]_E) = (\alpha_1, \dots, \alpha_m, 0, \dots, 0) \in \mathbb{C}^n$. Then, for any $S \in \text{Borel}(\mathfrak{H}/E)$,

$$\begin{aligned} p_{E,F} \sharp \mathfrak{g}_F^{(h)}(S) &= \frac{1}{Z_F} \int_{\mathfrak{H}/F} \mathrm{d}\mathfrak{g}_F^{(h)}([x]_F)\, \chi_S(p_{E,F}[x]_F) = \\ &= \prod_{j=1}^n \lambda_j \int_{\mathbb{C}^n} \frac{\mathrm{d}\alpha_1 \dots \mathrm{d}\alpha_n}{(2\pi)^n} e^{-\frac{1}{2}\sum_{j=1}^n \lambda_j |\alpha_j|^2} \chi_{i_E^{-1}S}(\alpha_1, \dots, \alpha_m, 0, \dots, 0) \end{aligned}$$

where we used the fact that $p_{E,F}[x]_F = [x]_E = i_F(\alpha_1, \dots, \alpha_m, 0, \dots, 0)$. Now, integrating in the free $n-m$ variables and having in mind that

$$\int_{\mathbb{C}^{n-m}} \mathrm{d}\alpha_{m+1} \dots \mathrm{d}\alpha_n e^{-\frac{1}{2}\sum_{j=m+1}^n \lambda_j |\alpha_j|^2} = (2\pi)^{n-m} \prod_{j=m+1}^n \lambda_j^{-1},$$

we have that

$$p_{E,F} \sharp \mathfrak{g}_F^{(h)}(S) = \prod_{j=1}^m \lambda_j \int_{\mathbb{C}^m} \frac{\mathrm{d}\alpha_1 \dots \mathrm{d}\alpha_m}{(2\pi)^m}\, e^{-\frac{1}{2}\sum_{j=1}^m \lambda_j |\alpha_j|^2} \chi_{i_E^{-1}S}(\alpha_1, \dots, \alpha_m) = \mathfrak{g}_E^{(h)}(S).$$

Thus, $\mathfrak{g}^{(h)} = \{\mathfrak{g}_E^{(h)}\}_{E \in \mathscr{F}(\mathfrak{H})}$ is a cylindrical measure.

Gaussian measures are a paradigmatic example of cylindrical measures that do not concentrate as Radon measures on the phase space $\mathfrak{H}$. The proof of this fact, that we recall below, is presented in [63] (Sec. 3.1), using the Prokhorov criterion. Indeed, for any $E \in \mathscr{F}(\mathfrak{H})$ and $R > 0$ we have, by Markov inequality and for any $p \in \mathbb{R}$,

$$\begin{aligned} \mathfrak{g}_E^{(h)}(\{z \in \mathfrak{H}/E \,:\, \|z\|_{\mathfrak{H}^{(1-p)}} \geqslant R\}) &\leqslant \frac{1}{R^2} \int_{\mathfrak{H}/E} \mathrm{d}\mathfrak{g}_E^{(h)}(z)\, \|z\|^2_{\mathfrak{H}^{(1-p)}} = \\ &= \frac{1}{R^2} \int_{\mathbb{C}^n} \prod_{j=1}^n \left(\frac{\mathrm{d}\alpha_j}{2\pi} \lambda_j \right) e^{-\frac{1}{2}\sum_{j=1}^n \lambda_j |\alpha_j|^2} \sum_{k=1}^n \lambda_k^{1-p} |\alpha_k|^2 = \\ &= \frac{2}{R^2} \sum_{k=1}^n \frac{(2\pi)^n}{(2\pi)^n} \frac{\lambda_k^{1-p}}{\lambda_k} \frac{\prod_{j=1}^n \lambda_j^{-1}}{\prod_{j=1}^n \lambda_j^{-1}} = \frac{2}{R^2} \sum_{k=1}^n \lambda_k^{-p} \leqslant \frac{2}{R^2} \operatorname{Tr}_{\mathfrak{H}}(h^{-p}), \end{aligned}$$

and the bound is uniform in the dimension of the quotient space. Therefore, if we assume that $\operatorname{Tr}_{\mathfrak{H}}(h^{-p}) < +\infty$, then Prokhorov criterion guarantees that $\mathfrak{g}^{(h)}$ concentrates on $\mathfrak{H}^{(1-p)}$ as a Radon measure.

A relevant case is if we consider $\mathfrak{H} = L^2(\Lambda)$, with Λ a bounded open subset of $\mathbb{R}^d$, and $h = -\Delta + C$, with $C \in \mathbb{R}$ such that $h > 0$ and fixed boundary conditions. Then in this case $\operatorname{Tr}_{L^2}((-\Delta + C)^{-p}) < +\infty$ for $p > d/2$, and so the weighted Gaussian measure concentrates as a Radon measure on $\mathfrak{H}^{(1-p)} = H^{1-p}(\Lambda)$ a Sobolev space, with $p > d/2$. If the dimension is greater than one, we have an example of a cylindrical measure that concentrates as a pure measure outside of the phase space $\mathfrak{H}$.

3 Semiclassical analysis of regular states

In this chapter we develop the mathematical theory used to study interacting systems.

The aim is to characterize state-valued measures as cluster points, w.r.t. suitable topologies, of a class of states for the microscopic systems satisfying hypotheses of uniform boundedness w.r.t. a semiclassical paramter ε.

We make use of semiclassical techniques applied to Quantum Field Theories (QFT) developed in [6], [40] and references thereof contained. It is first convenient to consider abstract algebraic models of bosonic QFTs, and then turn to concrete situations choosing the suitable representations.

Given a symplectic phase space (X', σ), it is possible to define a C*-algebra $\mathfrak{W}_\varepsilon(X', \sigma)$ and the regular states on the C*-algebra $\mathfrak{S}_\varepsilon(X', \sigma)$, representing, respectively, the observables and the states of the composite system. We will show that quasi-classical states and observables are represented by measures and functions on the phase space.

The rest of the Chapter is organized as follows: in Section 3.1, we recall the construction of the semiclassical theory given in [40], and give a meaning to the limit

$$\omega_\varepsilon \to \mathfrak{m}. \tag{3.1}$$

In Sections 3.2 and 3.3, we extend the semiclassical analysis developed in [6] to state-valued measures and operator-valued symbols.

3.1 Topologies of convergence for semiclassical states

Bosonic QFTs that encode the canonical commutation relations are defined starting from a *real symplectic space*, that is, a vector space V endowed with a bilinear, non-degenerate, alternate form σ on V, called the symplectic form. The couple (V, σ) is the phase space of the system and the space of classical fields and the phase space have to be in duality: a natural choice is to consider the space of classical fields as the algebraic dual of V endowed with the weak-* topology, as $V_V^* = X$, where X is a topological vector space, and so $V = (V_V^*)' = X'$ the continuous dual of X. An example of such space X is given by the cotangent bundle of a reflexive, locally convex space Σ (so-called space of configurations):

$$X' = T^*\Sigma \cong \Sigma \oplus \Sigma',$$

The previous identifications also highlight the nature of the phase space X': as we can see, X' carries the structure of the space of Hamiltonian coordinates, while X that of Lagrangian coordinates, and, in infinite dimension, they are not in general equivalent. The aforementioned choice is suitable due to the cylindrical measures setting, that will require spaces in compatible duality: thanks to the reflexivity of the space Σ, it is assured that $X' \hookrightarrow X'' \cong X$ (see [40] for further details).

Consider now the Heisenberg group $\mathbb{H}(X', \sigma)$ associated to (X', σ), that is, the space $X' \times \mathbb{R}$ endowed with the group operation

$$(\xi, t) \cdot (\eta, s) = (\, \xi + \eta,\, t + s - \sigma(\xi, \eta)\,).$$

It is possible to define a family of $C*-$algebraic representations indexed by a parameter $\varepsilon > 0$:

$$\begin{aligned} \mathbb{H}(X', \sigma) &\longrightarrow \mathscr{W}_\varepsilon(X', \sigma) \\ (\xi, t) &\longmapsto e^{it}\, W_\varepsilon(\xi) \end{aligned}$$

where $W_\varepsilon(\xi)$ are the *Weyl operators* satisfying, for every $\xi, \eta \in X'$,

$$W_\varepsilon(\xi) \neq 0, \tag{3.2}$$

$$W_\varepsilon^\dagger(\xi) = W_\varepsilon(-\xi), \tag{3.3}$$

$$W_\varepsilon(\xi) W_\varepsilon(\eta) = e^{-i\varepsilon\sigma(\xi,\eta)} W_\varepsilon(\xi + \eta), \tag{3.4}$$

and the last formula expresses the *bosonic canonical commutation relations*[1] . $\{\mathscr{W}_\varepsilon(X', \sigma)\}_{\varepsilon \geqslant 0}$ is called a *deformation of Weyl C*-algebras*, where every $\mathscr{W}_\varepsilon(X', \sigma)$ is the smallest C*-algebra, w.r.t. inclusion, containing the set $\{W_\varepsilon(\xi), \xi \in X'\}$. $\mathscr{W}_\varepsilon(X', \sigma)$ is unique up to $*-$isomorphism, once (X', σ) is fixed. In addition, for every ε, $W_\varepsilon(\xi) = W_1(\varepsilon^{1/2}\xi)$.

The semiclassical parameter $\varepsilon > 0$ that has been introduced gives the "degree" of non commutativity of the quantum observables.

We are interested in systems composed by a quantum field, that is approximated by its classical counterpart in the limit of $\varepsilon \to 0$, and a subsystem that does not behave semiclassically. To describe this further ε-independent degree of freedom we consider the deformation given by the tensor product[2] of the algebra generated by Weyl operators with another non-semiclassically behaved C^*-algebra $\mathfrak{A}$:

$$\boxed{\{\mathfrak{W}_\varepsilon(X', \sigma)\}_{\varepsilon \geqslant 0} := \{\mathfrak{A} \otimes \mathscr{W}_\varepsilon(X', \sigma)\}_{\varepsilon \geqslant 0},}$$

that from now on we will refer to as the *algebra of observables* of our system.

States of the system are linear, continuous, positive functionals over the algebra of observables: $\omega_\varepsilon \in (\mathfrak{W}_\varepsilon(X', \sigma))'_+$. To every state we can associate its *generating map*, and this object allows to define the notion of regularity for states.

Definition 3.1.1 (Generating map). *Given a state $\omega_\varepsilon \in (\mathfrak{W}_\varepsilon(X', \sigma))'_+$, then the associated* ***generating map*** *is the function $G_{\omega_\varepsilon} : X' \longrightarrow \mathfrak{A}'$, whose action on $a \in \mathfrak{A}$ is*

$$G_{\omega_\varepsilon}(\xi)(a) = \omega_\varepsilon(\, a \otimes W_\varepsilon(\xi)\,).$$

Definition 3.1.2 (Regular states). *A state $\omega_\varepsilon \in (\mathfrak{W}_\varepsilon(X', \sigma))'_+$ is* ***regular*** *iff, for any $\xi \in X'$, the action*

$$\begin{aligned} \mathbb{R} &\longrightarrow \mathfrak{A}' \\ t &\longmapsto G_{\omega_\varepsilon}(t\,\xi) \end{aligned}$$

is continuous when $\mathfrak{A}'$ is endowed with the $\sigma(\mathfrak{A}', \mathfrak{A})$-topology. The set of regular states on the algebra of X' will be denoted by $\mathfrak{S}_\varepsilon(X', \sigma)$.

[1]We can see that, if $\varepsilon = 0$, the definition identifies a commutative Weyl C^*-algebra $\mathscr{W}_0(X', \sigma)$ that is the algebra of almost-periodic functions.

[2]We could choose different norms for the tensor product of C*-algebra. However, it is convenient to consider the so-called maximal norm: for $A \in \mathfrak{A}_1 \otimes \mathfrak{A}_2$

$$\|A\|_{\max} = \sup\{\|A\|_\gamma \,:\, \|\cdot\|_\gamma \text{ C*-seminorm on } \mathfrak{A}_1 \otimes \mathfrak{A}_2\}.$$

Regular states are uniquely defined by their generating functional. Therefore, regular states behave as non-commutative measures, as the next result shows.

Theorem 3.1.3 (Non-commutative Bochner Theorem)**.** *There exists a bijection between the set of regular quantum states $\mathfrak{S}_\varepsilon(X',\sigma)$ with partial trace $\alpha_\varepsilon = \omega_\varepsilon(\,\cdot \otimes \mathbb{1}) \in \mathfrak{A}'_+$ and the set of functions G_ε between X' and $\mathfrak{A}'$ (dual in weak*-topology) such that*

- $G_\varepsilon(0) = \alpha_\varepsilon$;
- $G_\varepsilon \restriction_R$ *is continuous in $\sigma(\mathfrak{A}',\mathfrak{A})$ topology for every finite dimensional subspace $R \subseteq X'$;*
- G_ε *is of almost positive type:*

$$\sum_{j,k=1}^{n} e^{i\varepsilon\sigma(\xi_j,\xi_k)}\, G_\varepsilon(\xi_j - \xi_k)(a_k^\dagger a_j) \geqslant 0,$$

  for every $\{a_j\}_{j=1}^n \subseteq \mathfrak{A}$, $\{\xi_j\}_{j=1}^n \subseteq X'$,

and the bijection is given by the generating map: $\omega_\varepsilon \mapsto G_\varepsilon := G_{\omega_\varepsilon}$.

That is, a function in $(\mathfrak{A}')^{X'}$ is the generating map of a regular state if and only if it is ultraweakly continuous when restricted to finite dimensional subspaces and it is of almost positive type.

We are now ready to give a characterization of the states, that we will call semiclassical. First of all, let us consider families of states whose index varies in the bounded interval $(0,1)$.

Definition 3.1.4. *A net $\{\omega_\varepsilon\}_{\varepsilon\in(0,1)}$ is a family of* ***semiclassical states*** *if*

- $\omega_\varepsilon \in \mathfrak{S}(X',\sigma)$*, i.e., each ω_ε is a regular state;*
- $\sup_{\varepsilon\in(0,1)} \|\omega_\varepsilon\|_{\mathfrak{W}'_\varepsilon} < +\infty$*, where $\|\cdot\|_{\mathfrak{W}'_\varepsilon}$ is the norm of C^*-algebraic states.*

The next step is to define suitable topologies for which we have convergence of semiclassical states to cylindrical state-valued measures. First of all, let us denote by $\mathscr{F}(X)$ the set of finite codimensional, weakly closed, subspaces of X:

$$\mathscr{F}(X) := \{E \text{ subspace of } X \,|\, \dim(X/E) < +\infty,\; E \text{ is } \sigma(X,X') - \text{closed}\}.$$

If we denote by $\mathscr{F}(X)^\circ$ the set of polars of the spaces belonging to $\mathscr{F}(X)$, it is in bijection with the set of the finite dimensional subspaces of X':

$$\mathscr{F}(X)^\circ \xleftrightarrow{1:1} \{\Phi \subseteq X' \,|\, \Phi \text{ finite dimensional subspace of } X'\}.$$

Therefore, we have the following decompositions

$$X = \bigoplus_{E\in\mathscr{F}(X)} X/E, \qquad X' = \bigoplus_{E\in\mathscr{F}(X)} E^\circ,$$

that allows to use the familiar tools of finite dimensional semiclassical analysis.

Consider now the disjoint union of sets

$$S(X,X',\mathfrak{A}) := \mathscr{M}_{\text{cyl}}(X;\mathfrak{A}'_+) \sqcup \bigsqcup_{\varepsilon\in(0,1)} \mathfrak{S}_\varepsilon(X',\sigma).$$

We can endow it with two topologies.

- *Topology* $\mathfrak{P}$:
$$\mathfrak{S}_\varepsilon(X', \sigma) \ni \omega_\varepsilon \xrightarrow[\varepsilon\to 0]{\mathfrak{P}} \mathfrak{m} \in \mathscr{M}_{\mathrm{cyl}}(X; \mathfrak{A}'_+)$$
iff, for every $E \in \mathscr{F}(X)$, $a \in \mathfrak{A}_+$, and $f \in C_0^\infty(X/E)$,
$$\omega_\varepsilon(a \otimes \mathrm{Op}^\varepsilon_{1/2}(f)) \xrightarrow[\varepsilon\to 0]{} \int_{X/E} \mathrm{d}\mathfrak{m}^a_E(x)\, f(x)$$
with $\mathrm{d}\mathfrak{m}^a_E(x) = \mathrm{d}\mathfrak{m}_E(x)(a)$, $\mathfrak{m}_E \in \mathscr{M}_{\mathrm{cyl}}(X/E; \mathfrak{A}'_+)$, and the standard Weyl quantization
$$\mathrm{Op}^\varepsilon_{1/2}(f) = \int_{E^\circ} \mathrm{d}\xi\, \hat{f}(\xi)\, W_\varepsilon(2\pi\xi).$$

- *Topology* $\mathfrak{T}$*:*
$$\mathfrak{S}_\varepsilon(X', \sigma) \ni \omega_\varepsilon \xrightarrow[\varepsilon\to 0]{\mathfrak{T}} \mathfrak{m} \in \mathscr{M}_{\mathrm{cyl}}(X; \mathfrak{A}'_+)$$
iff, for every $\xi \in X'$,
$$G_{\omega_\varepsilon}(\xi) \xrightarrow[\varepsilon\to 0]{\mathrm{w}^*} \hat{\mathfrak{m}}(\xi)$$
in weak*-topology, that is,
$$\omega_\varepsilon(\, a \otimes W_\varepsilon(\xi)) \xrightarrow[\varepsilon\to 0]{} \hat{\mathfrak{m}}^a(\xi), \qquad \text{for every } a \in \mathfrak{A}_+.$$

Topology $\mathfrak{P}$ is physically significant, since it describes convergence of expectations for cylindrical smooth symbols, the second is more useful to identify the measures that are cluster points of the semiclassical net, thanks to the classic Bochner's Theorem.

Theorem 3.1.5 (Commutative Bochner Theorem)**.** *There exists a bijection between the set of cylindrical state-valued measures* $\mathscr{M}_{\mathrm{cyl}}(X; \mathfrak{A}'_+)$ *and the set of functions* G *between* X' *and* $\mathfrak{A}'$ *such that*

- $G\restriction_R$ *is continuous in* $\sigma(\mathfrak{A}', \mathfrak{A})$ *topology for every finite dimensional vector subspace* $R \subseteq X'$*;*
- G *is of positive type:*
$$\sum_{j,k=1}^{n} G(\xi_j - \xi_k)(a_k^\dagger a_j) \geqslant 0,$$
for every $\{a_j\}_{j=1}^n \subseteq \mathfrak{A}$, $\{\xi_j\}_{j=1}^n \subseteq X'$,

and the bijection is given by the generating map: $\mathfrak{m} \mapsto G := \hat{\mathfrak{m}}$.

That is, a function in $(\mathfrak{A}')^{X'}$ is the Fourier transform of a cylindrical state-valued measure if and only if it is ultraweakly continuous when restricted to finite dimensional subspaces and it is of positive type. To summarize,

$$\begin{array}{ccc}
\left\{\begin{array}{c}\text{Functions of}\\ \text{almost positive type}\end{array}\right\} & \xleftrightarrow{1:1} & \left\{\begin{array}{c}\text{Generating maps of}\\ \text{regular states}\end{array}\right\} \\
{\scriptstyle \varepsilon\to 0}\big\downarrow & & \big\downarrow{\scriptstyle \varepsilon\to 0} \\
\left\{\begin{array}{c}\text{Functions of}\\ \text{positive type}\end{array}\right\} & \xleftrightarrow{1:1} & \left\{\begin{array}{c}\text{Fourier transform of}\\ \text{cyl. state-val. measures}\end{array}\right\}
\end{array}$$

Unfortunately, the topologies $\mathfrak{P}$ and $\mathfrak{T}$ are not comparable. Nonetheless (see [40, Theorem 5.1]), semiclassical states always admit a cylindrical measure as $\mathfrak{P}-$cluster point, and the convergence in the joint topology[3] $\mathfrak{P} \vee \mathfrak{T}$, provided that the following *no-loss of mass condition* holds: for every family of mollifiers $\{\varphi_m\}_{m\in\mathbb{N}} \subseteq C_0^\infty(X/E)$, with $E \in \mathscr{F}(X)$, and every $\xi \in X', a \in \mathfrak{A}_+$, the following statements are equivalent

$$(i)\ \omega_\varepsilon \xrightarrow[\varepsilon\to 0]{\mathfrak{P}\vee\mathfrak{T}} \mathfrak{m};$$

$$(ii)\ G_{\omega_\varepsilon}(0) \xrightarrow[\varepsilon\to 0]{\mathrm{w}^*} \hat{\mathfrak{m}}(0);$$

$$(iii)\ \lim_{m\to+\infty} \lim_{k\in B} \omega_{\varepsilon_k}\left(a\otimes\left(\mathrm{Op}_{1/2}^{\varepsilon_k}(\varphi_m \star 1) - 1\right)\right) = 0;$$

$$(iv)\ \lim_{m\to+\infty} \lim_{k\in B} \omega_{\varepsilon_k}\left(a\otimes\left(\mathrm{Op}_{1/2}^{\varepsilon_k}(\varphi_m \star e^{i\xi(\,\cdot\,)}) - W_{\varepsilon_k}(\xi)\right)\right) = 0.$$

Theorem 3.1.6. *[40, Theorem 5.1] Let us consider a net of semiclassical states $\{\omega_\varepsilon\}_{\varepsilon\in(0,1)} \subseteq \mathfrak{S}_\varepsilon(X',\sigma)$. Then there exist a subnet $\{\omega_{\varepsilon_n}\}_{n\in B\subseteq(0,1)}$ and a cylindrical state-valued measure $\mathfrak{m} \in \mathscr{M}_{\mathrm{cyl}}(X;\mathfrak{A}'_+)$ such that*

$$\omega_{\varepsilon_n} \xrightarrow[n\in B]{\mathfrak{P}} \mathfrak{m}. \tag{3.5}$$

Furthermore, if the net satisfies the no-loss of mass condition*, then*

$$\omega_{\varepsilon_n} \xrightarrow[n\in B]{\mathfrak{P}\vee\mathfrak{T}} \mathfrak{m}. \tag{3.6}$$

Since the convergence in joint topology (3.6) is the most convenient to study semiclassical states, from now on it will be denoted simply by

$$\boxed{\omega_{\varepsilon_n} \to \mathfrak{m}.}$$

We remark that the uniqueness of the limit measure is guaranteed only once the subsequence has been fixed: changing the chosen subsequence other measures can be reached as cluster points.

It is possible to extend the previous convergence also to complex states. We denote the space of complex states with their decomposition in positive elements that are regular states by $\mathfrak{S}(X',\sigma)_\mathbb{C}$:

$$\mathfrak{S}_\varepsilon(X',\sigma)_\mathbb{C} := \left\{ \varpi_\varepsilon \in (\mathfrak{W}_\varepsilon(X',\sigma))' \;\middle|\; \begin{array}{l} \varpi_\varepsilon = \varpi^+_{R,\varepsilon} - \varpi^-_{R,\varepsilon} + i\,(\varpi^+_{I,\varepsilon} - \varpi^-_{I,\varepsilon}), \\ \varpi^+_{R,\varepsilon}, \varpi^-_{R,\varepsilon}, \varpi^+_{I,\varepsilon}, \varpi^-_{I,\varepsilon} \in \mathfrak{S}_\varepsilon(X',\sigma) \end{array} \right\} \tag{3.7}$$

and call them *complex regular states*.

Corollary 3.1.7. *Let $\{\varpi_\varepsilon\}_{\varepsilon\in(0,1)} \subseteq \mathfrak{S}(X',\sigma)_\mathbb{C}$ be a net of complex regular states uniformly bounded w.r.t. ε and such that every element of their decomposition in positive elements satisfy the no-loss of mass condition. Then, there exist a subnet $\{\varpi_{\varepsilon_n}\}_{n\in B}$ and a cylindrical (complex) state-valued measure $\mathfrak{w} \in \mathscr{M}(X;\mathfrak{A}')$ such that*

$$\varpi_{\varepsilon_n} \to \mathfrak{w}, \qquad |\varpi_{\varepsilon_n}| \to |\mathfrak{w}| \tag{3.8}$$

with the first convergence holding in $\mathfrak{P}\vee\mathfrak{T}$ topology for every positive term of the decomposition.

[3] It is the coarsest topology which is finer than both $\mathfrak{P}$ and $\mathfrak{T}$.

Proof. By definition, for every complex regular state $\varpi_\varepsilon \in \mathfrak{S}_\varepsilon(X', \sigma)_\mathbb{C}$ there exists a unique decomposition in positive operators

$$\varpi_\varepsilon = \varpi^+_{R,\varepsilon} - \varpi^-_{R,\varepsilon} + i\,(\varpi^+_{I,\varepsilon} - \varpi^-_{I,\varepsilon})$$
$$\varpi^+_{R,\varepsilon},\ \varpi^-_{R,\varepsilon},\ \varpi^+_{I,\varepsilon},\ \varpi^-_{I,\varepsilon} \in \mathfrak{S}_\varepsilon(X', \sigma)$$

and from this it is possible to define the modulus of the complex state as

$$|\varpi_\varepsilon| = \varpi^+_{R,\varepsilon} + \varpi^-_{R,\varepsilon} + \varpi^+_{I,\varepsilon} + \varpi^-_{I,\varepsilon}. \tag{3.9}$$

Since $\varpi^+_{R,\varepsilon} \leqslant |\varpi_\varepsilon|$ we have

$$\|\varpi^+_{R,\varepsilon}\|_{\mathscr{L}_1} \leqslant \|\varpi_\varepsilon\|_{\mathscr{L}_1} \leqslant C \tag{3.10}$$

uniformly in ε, and the same for any term of the decomposition. Therefore, by Theorem 3.1.6,

$$\varpi^+_{R,\varepsilon_n} \to \mathfrak{w}^+_R, \qquad \varpi^-_{R,\varepsilon_n} \to \mathfrak{w}^-_R,$$
$$\varpi^+_{I,\varepsilon_n} \to \mathfrak{w}^+_I, \qquad \varpi^-_{I,\varepsilon_n} \to \mathfrak{w}^-_I.$$

Then, by linearity, also we have

$$|\varpi_{\varepsilon_n}| \to |\mathfrak{w}|. \tag{3.11}$$

□

3.1.1 Quasi-classical analysis in the Fock representation

In this section we restrict to the important Fock representation. We start by recasting the standard results of semiclassical analysis in a form suitable for QFT, and then discuss the infinite dimensional Fock representation.

Finite dimensional quasi-classical analysis

Let us consider X' to be finite dimensional. Since X' has a symplectc structure, $\dim(X') = 2d$, for some $d \in \mathbb{N}$, and thus $X' \cong X \cong \mathbb{R}^{2d}$. For convenience, we choose the standard symplectic form on $\mathbb{R}^{2d}$:

$$\sigma((q,p),(q',p')) = q \cdot p' - p \cdot q', \qquad q, q', p, p' \in \mathbb{R}^d.$$

By Stone-Von Neumann Theorem, every irreducible unitary representation of $\mathscr{W}_\varepsilon(\mathbb{R}^{2d}, \sigma)$ is equivalent to the Schrödinger representation

$$\pi_S : \mathscr{W}_\varepsilon(\mathbb{R}^{2d}, \sigma) \longrightarrow \mathcal{U}(L^2(\mathbb{R}^d; \mathrm{d}x))$$
$$W_\varepsilon(q,p) \longmapsto \pi_S(W_\varepsilon(q,p)) := e^{-ix\cdot q} e^{-i\varepsilon p \cdot \nabla_x} e^{i\varepsilon q \cdot p},$$

Also, let us consider, for the sake of clarity, the case $\mathfrak{A} = \mathbb{C}$. Therefore,

$$\mathfrak{W}_\varepsilon(\mathbb{R}^{2d}, \sigma) = \mathscr{W}_\varepsilon(\mathbb{R}^{2d}, \sigma), \qquad \{\omega_\varepsilon\}_{\varepsilon\in(0,1)} \subseteq (\mathscr{W}_\varepsilon(\mathbb{R}^{2d}, \sigma))'_+ \tag{3.12}$$

Every family of regular states is normal[4] when restricted to a finite dimensional subspace of X' (see [19, Section 5.2.3]), but since X is finite dimensional, $\{\omega_\varepsilon\}_{\varepsilon\in(0,1)}$ are automatically normal. Then, there exists a family of density matrices $\{\rho_\varepsilon\}_{\varepsilon\in(0,1)} \subseteq \mathscr{L}_1(L^2)_+$ such that

$$\omega_\varepsilon(A_\varepsilon) = \mathrm{Tr}_{L^2}(\rho_\varepsilon\, \pi_S(A_\varepsilon)), \qquad \text{for every } A_\varepsilon \in \mathscr{W}_\varepsilon(\mathbb{R}^{2d}, \sigma).$$

[4] By normal we mean that the states can be represented as trace-class operators w.r.t. a fixed representation, in this case Schrödinger representation.

By spectral theorem, in addition, there exist $\{\psi_j^{(\varepsilon)}\}_{j\in\mathbb{N}} \subseteq L^2(\mathbb{R}^d)$ and $\{\lambda_j^{(\varepsilon)}\}_{j\in\mathbb{N}} \subseteq (0,1]$, $\sum_j \lambda_j^{(\varepsilon)} = 1$ such that

$$\rho_\varepsilon = \sum_{j\in\mathbb{N}} \lambda_j^{(\varepsilon)} \left|\psi_j^{(\varepsilon)}\right\rangle\left\langle\psi_j^{(\varepsilon)}\right|.$$

Then the analysis of semiclassical convergence can be restricted, thanks to the linearity of the trace, to families of vectors $\{\psi_\varepsilon\}_{\varepsilon\in(0,1)} \subseteq L^2(\mathbb{R}^d)$.

If we consider normalized vectors, then they are trivially a semiclassical net. By Theorem 3.1.6, there exists a subsequence $\{\psi_{\varepsilon_n}\}_{n\in\mathbb{N}}$ and a cylindrical measure $\mathfrak{m} \in \mathscr{M}_{\mathrm{cyl}}(\mathbb{R}^{2d};\mathbb{R}_+) = \mathscr{M}(\mathbb{R}^{2d};\mathbb{R}_+)$ such that $\psi_{\varepsilon_n} \xrightarrow[n\to+\infty]{\mathfrak{P}} \mathfrak{m}$, that is,

$$\omega_{\varepsilon_n}(\mathrm{Op}_{1/2}^{\varepsilon_n}(f)) = \left\langle \psi_{\varepsilon_n} \middle| \pi_S(\mathrm{Op}_{1/2}^{\varepsilon_k}(f))\,\psi_{\varepsilon_n} \right\rangle \xrightarrow[n\to+\infty]{} \int_{\mathbb{R}^{2d}} \mathrm{d}\mathfrak{m}(q,p)\, f(q,p), \tag{3.13}$$

for every $f \in C_0^\infty(\mathbb{R}^{2d})$, where

$$\pi_S(\mathrm{Op}_{1/2}^{\varepsilon}(f)) = \int_{\mathbb{R}^{2d}} \mathrm{d}\xi\mathrm{d}\eta\, \hat{f}(\xi,\eta)\,\pi_S(W_\varepsilon(2\pi(\xi,\eta))).$$

This is a standard result of semiclassical theory in finite dimension, that can be found, for example, in [79, Theorem 5.2].

Infinite dimensional quasi-classical analysis: Fock representation

Let us consider $X = \mathfrak{H} \cong X'$ a separable real Hilbert space, for example[5] $\mathfrak{H} = (L^2_{\mathbb{R}}(\mathbb{R}^d)\,,\,\Re e\langle\,\cdot\mid\cdot\,\rangle_2)$ with symplectic form $\sigma(\,\cdot\,,\cdot\,) = \Im m\langle\,\cdot\mid\cdot\,\rangle_2$. We choose as further non-semiclassical degrees of freedom the C^*-algebra

$$\mathfrak{A} = \mathscr{L}_\infty(L^2(\mathbb{R}^{dN})),$$

for some $N \in \mathbb{N}_*$, so that $\mathfrak{A}' = \mathscr{L}_1(L^2(\mathbb{R}^{dN}))$. In this case

$$\mathfrak{W}_\varepsilon(\mathfrak{H},\sigma) = \mathscr{L}_\infty(L^2(\mathbb{R}^{dN})) \otimes \mathscr{W}_\varepsilon(\mathfrak{H},\sigma),$$

and among all the infinitely inequivalent possible choices, we represent the Weyl algebra by the symmetric Fock representation:

$$\begin{aligned} \pi_F : \mathscr{W}_\varepsilon(\mathfrak{H},\sigma) &\longrightarrow \mathcal{U}(\Gamma_s(\mathfrak{H})) \\ W_\varepsilon(\xi) &\longmapsto \pi_F(W_\varepsilon(\xi)) := e^{i(a_\varepsilon(\xi)+a_\varepsilon^\dagger(\xi))}. \end{aligned}$$

Therefore, $\mathscr{W}_\varepsilon$ is represented by a set of bounded operators on $\mathscr{H} = L^2(\mathbb{R}^{dN})\otimes\Gamma_s(\mathfrak{H})$:

$$\begin{aligned} \pi : \mathfrak{W}_\varepsilon(\mathfrak{H},\sigma) &\to \mathscr{L}(\mathscr{H}) \\ \kappa\otimes W_\varepsilon(\xi) &\longmapsto \kappa\otimes e^{i(a_\varepsilon(\xi)+a_\varepsilon^\dagger(\xi))}. \end{aligned}$$

Consider a family of normal states $\{\omega_\varepsilon\}_{\varepsilon\in(0,1)} \subseteq (\mathfrak{W}_\varepsilon(\mathfrak{H},\sigma))'_+$ for the whole system, so that they can be represented as density matrices:

$$\omega_\varepsilon(\kappa\otimes W_\varepsilon(\xi)) = \mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon\,\kappa\otimes\pi_F(W_\varepsilon(\xi))) \tag{3.14}$$

where $\{\rho_\varepsilon\}_{\varepsilon\in(0,1)} \subseteq \mathscr{L}_{1,+}(\mathscr{H})$ are normalized in trace norm: $\|\rho_\varepsilon\|_{\mathscr{L}_1} = 1$ for every $\varepsilon \in (0,1)$. Then, it is a semiclassical family and Theorem 3.1.6 holds: there exists a subsequence $\{\rho_{\varepsilon_n}\}_{n\in\mathbb{N}}$, that converges in $\mathfrak{P}$ topology, to a cylindrical state-valued measure $\mathfrak{m} \in \mathscr{M}_{\mathrm{cyl}}(\mathfrak{H};\mathscr{L}_{1,+}(L^2(\mathbb{R}^{dN})))$:

$$\mathrm{Tr}_{\mathscr{H}}(\rho_{\varepsilon_n}\kappa\otimes\pi_F(\mathrm{Op}_{1/2}^{\varepsilon_k}(f))) \xrightarrow[n\to+\infty]{} \int_{\mathfrak{H}/E} \mathrm{d}\mathfrak{m}_E^a(x)\, f_E(x) \tag{3.15}$$

[5] Where by $L^2_{\mathbb{R}}$ we intend the space L^2 as a real Hilbert space, with scalar product $\Re e\langle\cdot\mid\cdot\rangle_2$.

for every $f \in C^\infty_{0,\mathrm{cyl}}(\mathfrak{H})$, that is, such that, for every $E \in \mathscr{F}(\mathfrak{H})$, there exists $f_E \in C^\infty_0(\mathfrak{H}/E)$, $f(x) = f_E([x]_E)$, with $\mathfrak{m}^a_E(\,\cdot\,) = \mathrm{Tr}_{L^2(\mathbb{R}^{dN})}(\mathfrak{m}_E(\,\cdot\,)\,a) \in \mathscr{M}(\mathfrak{H}/E;\mathbb{R}^+)$, and

$$\pi_F(\mathrm{Op}^\varepsilon_{1/2}(f)) = \int_{E^\circ} \mathrm{d}\xi\; \hat{f}(\xi)\,\pi_F(W_\varepsilon(2\pi\xi)).$$

We would like to have the convergence in the joint topology $\mathfrak{P} \vee \mathfrak{T}$. In order to do that, a uniform control of the number operator is sufficient. In addition, the corresponding cluster points aer Radon measures.

Proposition 3.1.8. *Let $\{\rho_\varepsilon\}_{\varepsilon\in(0,1)} \subseteq \mathscr{L}_{1,+}(\mathscr{H})$ be a net of normalized states such that there exist a $\delta > 0$ and a constant $C_\delta \in (0,+\infty)$ such that*

$$\sup_{\varepsilon\in(0,1)} \mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon\,(\mathrm{d}\Gamma_\varepsilon(\mathbb{1}) + 1)^\delta) \leqslant C_\delta. \tag{3.16}$$

Then, if $\rho_{\varepsilon_n} \xrightarrow{\mathfrak{P}} \mathfrak{m} \in \mathscr{M}_{\mathrm{cyl}}(X;\mathfrak{A}'_+)$, for the net of states the no-loss of mass condition holds, that is,

$$\rho_{\varepsilon_n} \xrightarrow{\mathfrak{P}\vee\mathfrak{T}} \mathfrak{m}. \tag{3.17}$$

Proof. Denoting by $G_{\rho_{\varepsilon_n}}(\xi) := \rho_{\varepsilon_n} W_{\varepsilon_n}(\xi)$ the generating map of ρ_{ε_n} in Fock representation, with action on any $\kappa \in \mathscr{L}_\infty(L^2)$:

$$G_{\rho_{\varepsilon_n}}(\xi)(\kappa) = \mathrm{Tr}_{\mathscr{H}}(\rho_{\varepsilon_n}\kappa \otimes \pi_F(W_{\varepsilon_n}(\xi))). \tag{3.18}$$

Since $\|G_{\rho_{\varepsilon_n}}(\xi)\|_{\mathscr{L}_1} \leqslant 1$ for any $\xi \in \mathfrak{H}$, we can use Banach-Alaoglu theorem and obtain that there exists a subsequence $\{\varepsilon_{n_j}\}_{j\in\mathbb{N}}$ and a trace-class operator $G \in \mathscr{L}_1(L^2)$ such that $\|G(\xi)\|_{\mathscr{L}_1} \leqslant 1$ and

$$G_{\rho_{\varepsilon_{n_j}}}(\xi) \xrightarrow{\mathrm{w}^*} G(\xi). \tag{3.19}$$

Since G_{ρ_ε} is of almost positive type, applying the limit $\varepsilon_{n_j} \to 0$, we obtain that G is of positive type. For $\kappa \in \mathscr{L}_\infty(L^2)$ and $\xi,\eta \in I$, the last being a bounded set in $\mathfrak{H}$, we have, by hypothesis and Corollary A.1.4,

$$\begin{aligned}
|G_{\rho_\varepsilon}(\xi)(\kappa) - G_{\rho_\varepsilon}(\eta)(\kappa)| &= |\mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon\,\kappa \otimes (W_\varepsilon(\xi) - W_\varepsilon(\eta))| \leqslant \\
&\leqslant C_\kappa \mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon\,(1 + \mathrm{d}\Gamma_\varepsilon(\mathbb{1}))^\delta)\times \\
&\times \|(1+\mathrm{d}\Gamma_\varepsilon(\mathbb{1}))^{-\delta/2}(W_\varepsilon(\xi) - W_\varepsilon(\eta))(1+\mathrm{d}\Gamma_\varepsilon(\mathbb{1}))^{-\delta/2}\|_{\mathscr{L}} \leqslant \\
&\leqslant C_{\kappa,\delta}\,(1 + \|\xi\|_{\mathfrak{H}})\,\|\xi - \eta\|_{\mathfrak{H}} \leqslant C_{\kappa,\delta,I}\|\xi - \eta\|_{\mathfrak{H}}
\end{aligned}$$

that is, $G_{\rho_{\varepsilon_{n_j}}}$ is weak-* continuous, uniformly in ε, on every bounded set of $\mathfrak{H}$. Thus, we can apply the limit to the previous inequality and obtain that G is a weak-* continuous map on every bounded set of $\mathfrak{H}$, that can be extended uniquely to a weak-* continuous map on the whole $\mathfrak{H}$, still denoted by G. By commutative Bochner theorem (Theorem 3.1.5) we have that G is the Fourier transform of a cylindrical state-valued measure $\mathfrak{n} \in \mathscr{M}_{\mathrm{cyl}}(\mathfrak{H};\mathscr{L}_{1,+}(L^2))$:

$$G(\xi) = \hat{\mathfrak{n}}(\xi). \tag{3.20}$$

By uniqueness of the limit in $\mathfrak{P} \vee \mathfrak{T}$ topology, we obtain that $\mathfrak{n} = \mathfrak{m}$. The same reasoning holds for any convergent subsequence, and therefore $\rho_{\varepsilon_n} \xrightarrow{\mathfrak{P}\vee\mathfrak{T}} \mathfrak{m}$.

□

The same can be proved for complex states, this time requiring a symmetric control on the power of the expectation of the number operator.

Corollary 3.1.9. *Let $\{\sigma_\varepsilon\}_{\varepsilon\in(0,1)} \subseteq \mathscr{L}_1(\mathscr{H})$ be a net of normalized complex states such that there exists a $\delta > 0$ and a constant $C_\delta \in (0, +\infty)$ such that*

$$\sup_{\varepsilon\in(0,1)} \|(1 + \mathrm{d}\Gamma_\varepsilon(\mathbb{1}))^{\delta/2}\sigma_\varepsilon(1 + \mathrm{d}\Gamma_\varepsilon(\mathbb{1}))^{\delta/2}\|_{\mathscr{L}_1} \leqslant C_\delta. \tag{3.21}$$

Then, if $\sigma_{\varepsilon_n} \xrightarrow{\mathfrak{P}} \mathfrak{w} \in \mathscr{M}_{\mathrm{cyl}}(\mathfrak{H}; \mathfrak{A}'_+)$, the net of states converge in the joint topology:

$$\sigma_{\varepsilon_n} \xrightarrow{\mathfrak{P}\vee\mathfrak{T}} \mathfrak{w}. \tag{3.22}$$

Proof. By Corollary 3.1.7 we have only to prove the no-loss of mass condition for every term of the decomposition. Consider the decomposition of σ_ε

$$\sigma_\varepsilon = \sigma^+_{R,\varepsilon} - \sigma^-_{R,\varepsilon} + i(\sigma^+_{I,\varepsilon} - \sigma^-_{I,\varepsilon}). \tag{3.23}$$

Let us prove it, without loss of generality, for $\sigma^+_{R,\varepsilon}$:

$$\begin{aligned}\mathrm{Tr}_{\mathscr{H}}(\sigma^+_{R,\varepsilon}(1 + \mathrm{d}\Gamma_\varepsilon(\mathbb{1}))^\delta) &= \|(1 + \mathrm{d}\Gamma_\varepsilon(\mathbb{1}))^{\delta/2}\sigma^+_{R,\varepsilon}(1 + \mathrm{d}\Gamma_\varepsilon(\mathbb{1}))^{\delta/2}\|_{\mathscr{L}_1} \leqslant \\ &\leqslant \|(1 + \mathrm{d}\Gamma_\varepsilon(\mathbb{1}))^{\delta/2}\sigma_\varepsilon(1 + \mathrm{d}\Gamma_\varepsilon(\mathbb{1}))^{\delta/2}\|_{\mathscr{L}_1} \leqslant C_\delta.\end{aligned}$$

The proof is concluded by Proposition 3.1.8, a proper subsequence extraction for every part of the decomposition and the uniqueness of the limit in the joint topology. □

So we have proved that, for semiclassical states satisfying assumption (3.16), we also have the convergence in topology $\mathfrak{T}$: for every $\xi \in \mathfrak{H}$ and $\kappa \in \mathscr{L}_\infty(L^2(\mathbb{R}^{dN}))$

$$G_{\rho_{\varepsilon_n}}(\xi)(\kappa) = \mathrm{Tr}_{\mathscr{H}}(\rho_{\varepsilon_n}\kappa \otimes \pi_F(W_{\varepsilon_n}(\xi))) \xrightarrow[n\in B]{} \mathrm{Tr}_{L^2(\mathbb{R}^{dN})}(\kappa \otimes \hat{\mathfrak{m}}(\xi)) = \hat{\mathfrak{m}}^\kappa(\xi). \tag{3.24}$$

Let us now show that the uniform control also implies that the cluster points are Radon measures. We follow closely [6, 39].

Theorem 3.1.10. *[39, Theorem 3.3] Let $\{\rho_\varepsilon\}_{\varepsilon\in(0,1)} \subseteq \mathscr{L}_{1,+}(\mathscr{H})$ be normalized, such that $\rho_{\varepsilon_n} \to \mathfrak{m} \in \mathscr{M}_{\mathrm{cyl}}(\mathfrak{H}; \mathscr{L}_{1,+}(L^2))$. If there exist a $\delta > 0$ and $C_\delta \in (0, +\infty)$ such that*

$$\sup_{\varepsilon\in(0,1)} \mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon \mathrm{d}\Gamma_\varepsilon(\omega)^\delta) \leqslant C_\delta, \tag{3.25}$$

then the cylindrical measure $\mathfrak{m}$ is concentrated as a state-valued measure $\mathfrak{m} \in \mathscr{M}(\mathfrak{H}_\omega; \mathscr{L}_{1,+}(L^2))$. There ω is a multiplication operator on $\mathfrak{H}$ by a positive function, and $\mathfrak{H}_\omega$ is the following weighted Lebesgue space:

$$\begin{aligned}&\mathfrak{H}_\omega = L^2(\mathbb{R}^d; \omega(k)\mathrm{d}k), && \text{if } \mathfrak{H} = L^2(\mathbb{R}^d; \mathrm{d}k);\\ &\mathfrak{H}_\omega = L^2(\mathbb{R}^d; \omega(k)\mathrm{d}k) \otimes \mathbb{C}^{d-1}, && \text{if } \mathfrak{H} = L^2(\mathbb{R}^d; \mathrm{d}k) \otimes \mathbb{C}^{d-1}.\end{aligned}$$

Furthermore, the following bound holds:

$$\int_{\mathfrak{H}_\omega} \mathrm{d}\mu(z)\, \|z\|^{2\delta}_{\mathfrak{H}_\omega} \leqslant C_\delta. \tag{3.26}$$

In particular, let us consider the following explicit examples:

1. If $\omega = 1$, we have the already mentioned bound by means of the number operator

$$\sup_{\varepsilon \in (0,1)} \mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon \mathrm{d}\Gamma_\varepsilon(\mathbb{1})^\delta) \leqslant C_\delta, \tag{3.27}$$

and the measure concentrates on itself. In addition, the moments up to the power 2δ are bounded:

$$\int_{\mathfrak{H}} \mathrm{d}\mu(z) \, \|z\|_{\mathfrak{H}}^{2\delta} \leqslant C_\delta. \tag{3.28}$$

2. In the case $\omega \geqslant m > 0$, we have the inclusion

$$\mathfrak{H}_\omega \subseteq \mathfrak{H}, \qquad m \, \mathrm{d}\Gamma_\varepsilon(\mathbb{1}) \leqslant \mathrm{d}\Gamma_\varepsilon(\omega) \tag{3.29}$$

and the measure thus concentrates on a subspace of $\mathfrak{H}$.

3. If $\omega(k) = |k|$, there is no inclusion of $\mathfrak{H}$ in $\mathfrak{H}_\omega$ or vice-versa. In fact, the measure $\mathfrak{m}$ concentrates on $\mathfrak{H}_{|k|} = L^2(\mathbb{R}^d; |k|\mathrm{d}k)$ and

$$\mathfrak{H} \smallsetminus \mathfrak{H}_{|k|} \neq \varnothing, \qquad \mathfrak{H}_{|k|} \smallsetminus \mathfrak{H} \neq \varnothing. \tag{3.30}$$

3.1.2 Concrete examples of quasi-classical states

In this section we want to apply the theory of quasi-classical convergence to some examples of states, obtaining interesting state-valued measures in the limit.

We restrict our attention to density matrices on the Hilbert space $\mathscr{H} = L^2(\mathbb{R}^{dN}) \otimes \Gamma_s(\mathfrak{H})$:

$$\rho_\varepsilon \in \mathscr{L}_{1,+}(\mathscr{H}). \tag{3.31}$$

Product states. These states describe physical composite systems that are unentangled, and thus they reduce to the semiclassical theory for the field's degrees of freedom (see [6, 7, 8, 9, 10, 26, 27]). More precisely,

$$\rho_\varepsilon = |\psi \otimes \Psi_\varepsilon\rangle\langle\psi \otimes \Psi_\varepsilon| \tag{3.32}$$

where $\psi \in L^2(\mathbb{R}^{dN})$, $\Psi_\varepsilon \in \Gamma_s(\mathfrak{H})$, both normalized. Suppose furthermore that Ψ_ε satisfies

$$\langle\Psi_\varepsilon| \, \mathrm{d}\Gamma_\varepsilon(\mathbb{1})^\delta \, |\Psi_\varepsilon\rangle \leqslant C_\delta \tag{3.33}$$

for $\delta > 0$ and $C_\delta \in (0+, \infty)$. By Proposition 3.1.8, there exists a subsequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ such that $\rho_{\varepsilon_n} \to \mathfrak{m} \in \mathscr{M}(\mathfrak{H}; \mathscr{L}_{1,+}(L^2))$. We want to characterize $\mathfrak{m}$ explicitly. There exists a positive, scalar, probability measure $\mu \in \mathscr{M}(\mathfrak{H}; \mathbb{R}^+)$ such that $\Psi_{\varepsilon_{n_k}} \to \mu$, along $\{\varepsilon_{n_k}\}_{k\in\mathbb{N}}$ by the scalar theory (see [6, Theorem 6.13]). Therefore we obtain

$$\mathrm{Tr}_{\mathscr{H}}(\rho_{\varepsilon_{n_k}} \kappa \otimes W_{\varepsilon_{n_k}}(\xi)) = \left\langle \Psi_{\varepsilon_{n_k}} \middle| W_{\varepsilon_{n_k}}(\xi) \Psi_{\varepsilon_{n_k}} \right\rangle \langle \psi \,|\, \kappa\psi \rangle \xrightarrow[k \to +\infty]{} \int_{\mathfrak{H}} \mathrm{d}\mu(z) \, e^{2i\Re e\langle \xi \,|\, z\rangle} \, \langle \psi \,|\, \kappa\psi \rangle,$$

for any $\kappa \in \mathscr{L}_\infty(L^2)$ and $\xi \in \mathfrak{H}$, that is

$$\left|\psi \otimes \Psi_{\varepsilon_{n_k}}\right\rangle \left\langle \psi \otimes \Psi_{\varepsilon_{n_k}}\right| \longrightarrow \mu \; |\psi\rangle\langle\psi| \, . \tag{3.34}$$

By uniqueness of weak-* limit and since compact operators separate points of the space of trace-class operators, necessarily $\mathfrak{m} = \mu \, |\psi\rangle\langle\psi|$. So the convergence holds true over the whole subsequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$.

Coherent states. These product states play a very important role, and thus we mention them explicitly. Coherent states have minimal uncertainty, and this property is reflected in the limit by their convergence to deterministic classical states, that is, measures concentrated in a single point. Indeed, consider

$$\rho_\varepsilon = |\psi \otimes \Xi_\varepsilon(\eta)\rangle\langle\psi \otimes \Xi_\varepsilon(\eta)| \tag{3.35}$$

with $\psi \in L^2(\mathbb{R}^{dN})$ normalized and

$$\Xi_\varepsilon(\eta) = W_\varepsilon\left(\frac{\eta}{i\varepsilon}\right)\Omega \qquad \eta \in \mathfrak{H}, \tag{3.36}$$

where $\Omega = (1, 0, 0, \ldots) \in \Gamma_s(\mathfrak{H})$ is the Fock vacuum. By construction, Ξ_ε are normalized and satisfy the no-loss of mass condition:

$$\langle \Xi_\varepsilon(\eta)|\, \mathrm{d}\Gamma_\varepsilon(\mathbb{1})\, |\Xi_\varepsilon(\eta)\rangle = \|\eta\|^2_{\mathfrak{H}}. \tag{3.37}$$

Since they are product states, the limit measure is $\mathfrak{m} = \mu\, |\psi\rangle\langle\psi|$. Let us prove that μ is supported in one point. An explicit computation yields

$$\begin{aligned}
&\langle \Xi_\varepsilon \,|\, W_\varepsilon(\xi)\Xi_\varepsilon(\eta)\rangle = \left\langle \Omega \middle| W_\varepsilon\left(\frac{\eta}{i\varepsilon}\right)^\dagger W_\varepsilon(\xi)\, W_\varepsilon\left(\frac{\eta}{i\varepsilon}\right)\Omega \right\rangle = \\
&= e^{-i\sigma(\xi,-i\eta)} \left\langle \Omega \middle| W_\varepsilon\left(\frac{\eta}{i\varepsilon}\right)^\dagger W_\varepsilon\left(\xi + \frac{\eta}{i\varepsilon}\right)\Omega \right\rangle = \\
&= e^{-i\sigma(\xi,-i\eta)} e^{-i\sigma\left(i\eta,\, \xi + \frac{\eta}{i\varepsilon}\right)} \langle \Omega \,|W_\varepsilon(\xi)\Omega\rangle = \\
&= e^{2i\Re e\langle \xi \,|\, \eta\rangle} e^{-i\frac{\varepsilon}{2}\|\xi\|^2_{\mathfrak{H}}} \xrightarrow[\varepsilon\to 0]{} e^{2i\Re e\langle \xi \,|\, \eta\rangle} = \int_{\mathfrak{H}} \mathrm{d}\delta_\eta(z)\, e^{2i\Re e\langle \xi \,|\, z\rangle},
\end{aligned}$$

where we used the Weyl relations, $\sigma(\xi, \eta) = \Im m\langle \xi \,|\, \eta\rangle$, and $\langle \Omega \,|W_\varepsilon(\xi)\Omega\rangle = e^{-i\frac{\varepsilon}{2}\|\xi\|^2_{\mathfrak{H}}}$, which implies

$$\rho_\varepsilon \to \delta_\eta\, |\psi\rangle\langle\psi|\,. \tag{3.38}$$

Statistical mixture of product states. Let us consider statistical mixtures of the form

$$\rho_\varepsilon = \sum_{j\in\mathbb{N}} \lambda_j \left|\psi^{(j)} \otimes \Psi^{(j)}_\varepsilon\right\rangle\left\langle\psi^{(j)} \otimes \Psi^{(j)}_\varepsilon\right|, \tag{3.39}$$

where

- $\lambda_j \in [0, 1]$ for any $j \in \mathbb{N}$ and $\sum_{j\in\mathbb{N}} \lambda_j = 1$;
- $\psi^{(j)} \in L^2(\mathbb{R}^{dN})$ and $\Psi^{(j)}_\varepsilon \in \Gamma_s(\mathfrak{H})$ are both normalized for any $j \in \mathbb{N}$, and $\Psi^{(j)}_\varepsilon$ satisfy

$$\left\langle \Psi^{(j)}_\varepsilon \middle| \mathrm{d}\Gamma_\varepsilon(\mathbb{1})^\delta \middle| \Psi^{(j)}_\varepsilon \right\rangle \leqslant C_\delta \tag{3.40}$$

  uniformly in ε and j for some $\delta > 0$ and $C_\delta \in (0, +\infty)$.

There exists a subsequence $\{\varepsilon_n\}_{n\in\mathbb{N}}, \varepsilon_n \to 0$, such that, for any $j \in \mathbb{N}$, we have $\psi^{(j)}_{\varepsilon_n} \to \mu^{(j)}$. Therefore,

$$\sum_{j\in\mathbb{N}} \lambda_j \left|\psi^{(j)} \otimes \Psi^{(j)}_{\varepsilon_n}\right\rangle\left\langle\psi^{(j)} \otimes \Psi^{(j)}_{\varepsilon_n}\right| \longrightarrow \sum_{j\in\mathbb{N}} \lambda_j\, \mu^{(j)} \left|\psi^{(j)}\right\rangle\left\langle\psi^{(j)}\right|, \tag{3.41}$$

where the latter is a state-valued measure convex combination of projectors on particles space coupled with the limit scalar measures.

3.2 Convergence for scalar symbols

Using quasi-classical analysis for states developed in Section 3.1, it is possible to study the quasi-classical behaviour of observables as well. From now on we focus on the space

$$\mathscr{H} = L^2(\mathbb{R}^{dN}) \otimes \Gamma_s(\mathfrak{H})$$

where the first factor is the Hilbert space of N particles moving in $\mathbb{R}^d$, that do not behave semi-classically, and the second factor is the Fock space associated to the boson field, with $\mathfrak{H}$ a separable Hilbert space.

We study a net of semiclassical, positive states of the system with norm uniformly equal to 1: $\{\rho_\varepsilon\}_{\varepsilon\in(0,1)} \subseteq \mathscr{L}_{1,+}(\mathscr{H})$, with $\|\rho_\varepsilon\|_{\mathscr{L}_1} = 1$ for every $\varepsilon \in (0,1)$. By Theorem 3.1.6, all the corresponding Wigner meausures are cylindrical state-valued measure on the one-boson space

$$\mathfrak{m} \in \mathscr{M}_{\mathrm{cyl}}(\mathfrak{H}; \mathscr{L}_{1,+}(L^2(\mathbb{R}^{dN}))).$$

If $\mathfrak{m}$ is a true measure, since $\mathscr{L}_1(L^2(\mathbb{R}^{dN}))$ has the Radon-Nikodým property (see Chapter 2), by Theorem 2.2.13 it can be decomposed in:

- A scalar probability measure $\mu \in \mathscr{M}(\mathfrak{H}; \mathbb{R}^+)$, with
$$\mathrm{d}\mu(\,\cdot\,) = \left\| \frac{\mathrm{d}\mathfrak{m}}{\mathrm{d}\|\mathfrak{m}\|}(\,\cdot\,) \right\|_{\mathscr{L}_1} \mathrm{d}\|\mathfrak{m}\|(\,\cdot\,); \tag{3.42}$$
- A function of density matrices on the particle space $\gamma_{\mathfrak{m}} \in L^1(\mathfrak{H}; \mathscr{L}_{1,+}(\mathscr{H}); \mathrm{d}\mu)$, with $\|\gamma_{\mathfrak{m}}(z)\|_{\mathscr{L}_1} = 1$ for $\mu-$a.e. $z \in \mathfrak{H}$.

Thus we can write, for any $S \in \mathrm{Borel}(\mathfrak{H})$ and $f : \mathfrak{H} \to \mathscr{L}(L^2(\mathbb{R}^{dN}))$

$$\mathrm{Tr}_{L^2(\mathbb{R}^{dN})}\left(\int_{\mathfrak{H}} \mathrm{d}\mathfrak{m}(z) f(z)\right) = \int_{\mathfrak{H}} \mathrm{d}\mu(z)\, \mathrm{Tr}_{L^2(\mathbb{R}^{dN})}(\gamma_{\mathfrak{m}}(z) f(z)).$$

Theorem 3.2.1. *Let $\{\rho_\varepsilon\}_{\varepsilon\in(0,1)} \subseteq \mathscr{L}_{1,+}(\mathscr{H})$ be normalized, and suppose that, for some $\delta > 0$, there exists $C_\delta \in [1, +\infty)$ such that*

$$\sup_{\varepsilon\in(0,1)} \mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon\, (\mathrm{d}\Gamma_\varepsilon(\mathbb{1}) + 1)^\delta) \leqslant C_\delta. \tag{3.43}$$

Then, there exist a subsequence $\varepsilon_n \to 0$ and a probability state-valued measure $\mathfrak{m} := (\gamma_{\mathfrak{m}}, \mu) \in \mathscr{M}(\mathfrak{H}; \mathscr{L}_{1,+}(L^2))$ such that $\rho_{\varepsilon_n} \to \mathfrak{m}$, and, for every $\kappa \in \mathscr{L}_\infty(L^2)$,

1. *The limit*
$$\lim_{n\to+\infty} \mathrm{Tr}_{\mathscr{H}}(\rho_{\varepsilon_n}\, \mathrm{Op}_{1/2}^{\varepsilon_n}(b)\, \kappa \otimes W_{\varepsilon_n}(\xi)) = \int_{\mathfrak{H}} \mathrm{d}\mu(z)\, \mathrm{Tr}_{L^2}\,(\gamma_{\mathfrak{m}}(z)\, \kappa)\; b(z)\, e^{2i\Re e\langle \xi \,|\, z\rangle},$$
holds true for all $b \in \bigcup_{\mathbb{P}:\dim \mathbb{P}\mathfrak{H}<+\infty} S_{\mathbb{P}\mathfrak{H}}(\langle z\rangle^{2\delta})$ (defined in Section B.1);

2. *The limit*
$$\lim_{n\to+\infty} \mathrm{Tr}_{\mathscr{H}}(\rho_{\varepsilon_n}\, (b)_{\varepsilon_n}^{\mathit{Wick}}\, \kappa \otimes W_{\varepsilon_n}(\xi)) = \int_{\mathfrak{H}} \mathrm{d}\mu(z)\, \mathrm{Tr}_{L^2}\,(\gamma_{\mathfrak{m}}(z)\, \kappa)\; b(z)\; e^{2i\Re e\langle \xi \,|\, z\rangle},$$
holds true for all $b \in \mathscr{P}_{p,q}^{\infty}(\mathfrak{H})$ such that $p + q < 2\delta$ (defined in Section B.2).

Furthermore, the map $\mathfrak{H} \ni z \mapsto \|z\|_{\mathfrak{H}}^{2\bar{\delta}} \mathbb{1} \in \mathscr{L}(L^2)$, with $\bar{\delta} \leqslant \delta$, is $\mathfrak{m}$-integrable and the following control holds:

$$\left\| \int_{\mathfrak{H}} \mathrm{d}\mu(z)\, \gamma_{\mathfrak{m}}(z)\, (1 + \|z\|_{\mathfrak{H}}^2)^{\bar{\delta}} \right\|_{\mathscr{L}_1} = \int_{\mathfrak{H}} \mathrm{d}\mu(z)\, (1 + \|z\|_{\mathfrak{H}}^2)^{\bar{\delta}} \leqslant C_\delta. \tag{3.44}$$

Proof. By Theorem 3.1.10 and its remarks, $\mathfrak{m}$ concentrates as a state-valued measure over the space $\mathfrak{H}$: $\mathfrak{m} \in \mathscr{M}(\mathfrak{H}; \mathscr{L}_{1,+}(L^2(\mathbb{R}^{dN})))$. We will prove in order: (i) the convergence of quantum expectations to classical ones testing with compact operators; (ii) the control over the integral of the norm of z.

(i) Let us restrict to $\kappa \in \mathscr{L}_{\infty,+}(L^2)$ and to Wick quantized symbols. The extension to general compact operators and other quantization procedures is straightforward. By Theorem 3.1.6, there exists a subsequence $\varepsilon_n \to 0$ and a cylindrical state-valued measure $\mathfrak{m} \in \mathscr{M}_{\mathrm{cyl}}(\mathfrak{H}; \mathscr{L}_{1,+}(L^2))$ such that $\rho_{\varepsilon_n} \to \mathfrak{m}$. Recall that, in particular, this implies the weak-* convergence of the generating map to the Fourier transform of the measure. Define now, for every $\kappa \in \mathscr{L}_{\infty,+}(L^2)$,

$$\rho_{\varepsilon_n}^{\kappa} := \mathrm{Tr}_{L^2}(\rho_{\varepsilon_n} \kappa) \in \mathscr{L}_{1,+}(\Gamma_s(\mathfrak{H})),$$
$$\mathfrak{m}^{\kappa}(\,\cdot\,) := \mathrm{Tr}_{L^2}(\mathfrak{m}(\,\cdot\,)\, \kappa) \in \mathscr{M}_{\mathrm{cyl}}(\mathfrak{H}; \mathbb{R}_+),$$

then, by construction, we have that

$$\sup_{\varepsilon \in (0,1)} \mathrm{Tr}_{\Gamma_s(\mathfrak{H})}(\rho_{\varepsilon}^{\kappa}(1 + \mathrm{d}\Gamma_{\varepsilon}(\mathbb{1}))^{\delta})) \leqslant C_\delta \|\kappa\|_{\mathscr{L}} \tag{3.45}$$

and, for every $\rho_{\varepsilon_n}^{\kappa}$, $\mathfrak{m}^{\kappa}$ is the unique associated Wigner measure (since the subsequence is fixed): $\rho_{\varepsilon_n}^{\kappa} \to \mathfrak{m}^{\kappa}$, for any $\kappa \in \mathscr{L}_{\infty,+}(L^2)$. By [6, Corollary 6.14], and thanks to the assumptions, we have that the convergence at the level of generating maps extends to the convergence of expectations of operators to the relative classical expectations of symbols w.r.t. scalar measures:

$$\lim_{n \to +\infty} \mathrm{Tr}_{\Gamma_s}(\rho_{\varepsilon_n}^{\kappa}(b)_{\varepsilon_n}^{\mathrm{Wick}} W_{\varepsilon_n}(\xi)) = \int_{\mathfrak{H}} \mathrm{d}\mathfrak{m}^{\kappa}(z)\, b(z)\, e^{2i\Re e\langle \xi \,|\, z\rangle}. \tag{3.46}$$

As a byproduct, by [6, Theorem 6.2] we have also the following bound, for every $\kappa \in \mathscr{L}_{\infty,+}(L^2)$:

$$\int_{\mathfrak{H}} \mathrm{d}\mathfrak{m}^{\kappa}(z)\, (1 + \|z\|_{\mathfrak{H}}^2)^{\delta} \leqslant C_\delta \|\kappa\|_{\mathscr{L}}. \tag{3.47}$$

(ii) Let us consider now a non-decreasing sequence of approximate identities $\{\mathbb{1}_n\}_{n\in\mathbb{N}} \subseteq \mathscr{L}_{\infty,+}(L^2)$, $\mathbb{1}_n \nearrow \mathbb{1}$ in strong topology in $L^2(\mathbb{R}^{dN})$. Then, by the previous point we have

$$\mathrm{Tr}_{L^2}\left(\int_{\mathfrak{H}} \mathrm{d}\mathfrak{m}(z)\, (1 + \|z\|_{\mathfrak{H}}^2)^{\delta}\, \mathbb{1}_n \right) = \int_{\mathfrak{H}} \mathrm{d}\mathfrak{m}^{\mathbb{1}_n}(z)\, (1 + \|z\|_{\mathfrak{H}}^2)^{\delta} \leqslant C_\delta \|\mathbb{1}_n\|_{\mathscr{L}} \leqslant C_\delta.$$

In particular the following estimate holds, recalling that $\mathrm{d}\mathfrak{m}^{\mathbb{1}_n}(z) = \mathrm{d}\mu(z) \mathrm{Tr}_{L^2}(\gamma_{\mathfrak{m}}(z)\mathbb{1}_n)$ and considering $\{\|z\|_m\}_{m\in\mathbb{N}}$ as simple non-decreasing functions approximating the norm of z:

$$\int_{\mathfrak{H}} \mathrm{d}\mu(z)\, (1 + \|z\|_m^2)^{\delta} = \lim_{n \to +\infty} \int_{\mathfrak{H}} \mathrm{d}\mathfrak{m}^{\mathbb{1}_n}(z)\, (1 + \|z\|_m^2)^{\delta} \leqslant C_\delta.$$

Applying the Fatou Lemma for the $\liminf$ in $m \to +\infty$ we obtain the desired control

$$\int_{\mathfrak{H}} \mathrm{d}\mu(z)\, (1 + \|z\|_{\mathfrak{H}}^2)^{\delta} \leqslant C_\delta,$$

and the μ-integrability implies $\mathfrak{m}$ integrability.

□

If we make an additional assumption on the particle part of the state, it is possible to extend the convergence to the test with bounded operators. This ensures that no mass is lost in the limit due to the particle-field interaction. In order to extend the convergence, we need the following preparatory Lemma introducing a new class of test operators composed by regularized compact and bounded operators endowed with the spectral cut-off of a suitable self-adjoint operator.

Lemma 3.2.2. *Let T be a densely defined, self-adjoint operator on a Hilbert space $\mathscr{Z}$, and denote by $\mathbb{1}_m(T)$ its spectral projector in the interval $[-m, m], m \in \mathbb{N}$. Then the sets of operators*

$$\mathcal{K} := \mathcal{K}_T = \{\kappa_m := \mathbb{1}_m(T)\, \kappa\, \mathbb{1}_m(T), \kappa \in \mathscr{L}_{\infty,+}(\mathscr{Z}), m \in \mathbb{N}\};$$
$$\mathcal{B} := \mathcal{B}_T = \{B_m := \mathbb{1}_m(T)\, B\, \mathbb{1}_m(T), B \in \mathscr{L}(\mathscr{Z})_+, m \in \mathbb{N}\}$$

separate points of $\mathscr{L}_1(\mathscr{Z})$ in weak- and weak topology respectively.*

Proof. Let us prove it for $\mathcal{K}$, the proof for $\mathcal{B}$ being perfectly analogous. Let $\rho \in \mathscr{L}_{1,+}(\mathscr{H})$ such that for all $\kappa_m \in \mathcal{K}$,

$$\mathrm{Tr}_{\mathscr{H}}(\rho\, \kappa_m) = 0.$$

By spectral theorem, we can decompose $\rho = \sum_{j\in\mathbb{N}} \lambda_j |\psi_j\rangle\langle\psi_j|$, with $\lambda_j \in [0,1], \sum_{j\in\mathbb{N}} \lambda_j = 1$ and $\{\psi_j\}_{j\in\mathbb{N}}$ an orthonormal basis for $\mathscr{Z}$. Then it follows, since it is a sum of positive terms, that for all $j \in \mathbb{N}$,

$$\lambda_j \langle\psi_j|\, \kappa_m\, |\psi_j\rangle_{\mathscr{Z}} = 0\ .$$

Suppose that there is at least one j for which $\lambda_j \neq 0$, then we have that the second factor has to be zero. Taking the limit $m \to \infty$ for that factor we obtain that, for all $\kappa \in \mathscr{L}_{\infty,+}(\mathscr{Z})$,

$$\langle\psi_j|\, \kappa\, |\psi_j\rangle_{\mathscr{Z}} = 0.$$

Hence $\psi_j = 0$ for that j-s such that $\lambda_j \neq 0$, and therefore $\rho = 0$. □

Corollary 3.2.3. *Under the same assumptions of Theorem 3.2.1, and if, furthermore, there exists a densely defined, positive operator $T > 0$ on $L^2(\mathbb{R}^{dN})$ such that*

$$\begin{aligned} &T^{-1} \text{ is compact,} \qquad \sup_{\varepsilon\in(0,1)} \mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon T) \leqslant C, \\ &\text{and } \sup_{\varepsilon\in(0,1)} \mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon\, (\mathrm{d}\Gamma_\varepsilon(\mathbb{1}) + 1)^{2\delta}) \leqslant C_{2\delta}, \end{aligned} \tag{A$'$}$$

and if $\rho_{\varepsilon_n} \to \mathfrak{m}$, then, for any $B \in \mathscr{L}(L^2(\mathbb{R}^{dN}))$,

1. *The limit*

$$\lim_{n\to+\infty} \mathrm{Tr}_{\mathscr{H}}(\rho_{\varepsilon_n}\, \mathrm{Op}^{\varepsilon_n}_{1/2}(b)\, B \otimes W_{\varepsilon_n}(\xi)) = \int_{\mathfrak{H}} \mathrm{d}\mu(z)\, \mathrm{Tr}_{L^2}\left(\gamma_{\mathfrak{m}}(z)\, B\right)\, b(z)\, e^{2i\Re e\langle\xi\,|\,z\rangle},$$

holds true for all $b \in \bigcup_{\mathbb{P}:\dim\mathbb{P}\mathfrak{H}<+\infty} S_{\mathbb{P}\mathfrak{H}}(\langle z\rangle^{2\delta})$;

2. *The limit*

$$\lim_{n\to+\infty} \operatorname{Tr}_{\mathscr{H}}(\rho_{\varepsilon_n}\,(b)^{Wick}_{\varepsilon_n}\, B\otimes W_{\varepsilon_n}(\xi)) = \int_{\mathfrak{H}} \mathrm{d}\mu(z)\, \operatorname{Tr}_{L^2}\left(\gamma_{\mathfrak{m}}(z)\, B\right)\, b(z)\; e^{2i\Re e\langle \xi\,|\,z\rangle},$$

holds true for all $b \in \mathscr{P}^{\infty}_{p,q}(\mathfrak{H})$ *such that* $p+q \leqslant 2\delta$.

Furthermore, $\mathfrak{m}$ *concentrates as a probability measure:*

$$\|\mathfrak{m}(\mathfrak{H})\|_{\mathscr{L}_1(L^2)} = 1. \tag{3.48}$$

Remark 3.2.4. *A natural choice for the operator* T *is*

$$T = -\Delta_x + W_+ \tag{3.49}$$

with a trapping W_+*, which gives automatically compactness of its inverse.*

Proof. Let us fix the sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ such that $\rho_{\varepsilon_n} \to \mathfrak{m}$. Let us denote by

$$\rho^T_{\varepsilon} := T^{1/4}\rho_\varepsilon T^{1/4} \tag{3.50}$$

which is a family of density matrices satisfying, thanks to the hypotheses, for any $\delta > 0$,

$$\operatorname{Tr}_{\mathscr{H}}(\rho^T_\varepsilon(\mathrm{d}\Gamma_\varepsilon(\mathbb{1})+1)^{\delta}) \leqslant \operatorname{Tr}_{\mathscr{H}}(\rho_\varepsilon(T + (\mathrm{d}\Gamma_\varepsilon(\mathbb{1})+1)^{2\delta})) < C + C_{2\delta} \tag{3.51}$$

uniformly in ε. Then, thanks to Theorems 3.1.6 and 3.1.10, there exists a subsequence $\{\varepsilon_{n_j}\}_{j\in\mathbb{N}}$ and a state-valued measure $\mathfrak{m}^T \in \mathscr{M}(\mathfrak{H};\mathscr{L}_{1,+}(L^2))$ such that

$$\rho^T_{\varepsilon_{n_j}} \to \mathfrak{m}^T. \tag{3.52}$$

We want to identify $\mathfrak{m}^T$ by a standard argument: let us consider compact operators of the form $\kappa^T := T^{-1/4}\kappa T^{-1/4}$, then

$$\begin{aligned}\lim_{j\to+\infty} \operatorname{Tr}_{\mathscr{H}}(\rho^T_{\varepsilon_{n_j}}\kappa^T \otimes W_{\varepsilon_{n_j}}(\xi)) &= \lim_{j\to+\infty} \operatorname{Tr}_{\mathscr{H}}(\rho_{\varepsilon_{n_j}}\kappa \otimes W_{\varepsilon_{n_j}}(\xi)) = \\ = \operatorname{Tr}_{L^2}\left(\int_{\mathfrak{H}} \mathrm{d}\mathfrak{m}(z)\,\kappa\, e^{2i\Re e\langle\xi\,|\,z\rangle}\right) &= \operatorname{Tr}_{L^2}\left(\int_{\mathfrak{H}} T^{1/4}\mathrm{d}\mathfrak{m}^T(z)T^{1/4}\kappa e^{2i\Re e\langle\xi\,|\,z\rangle}\right)\end{aligned}$$

and so, since compact operators separate points of $\mathscr{L}_1$, necessarily $\mathfrak{m}^T = T^{1/4}\mathfrak{m}T^{1/4}$. For any other convergent subsequence $\{\rho^T_{\varepsilon_{n_j}}\}_{j\in\mathbb{N}}$ to another measure $\mathfrak{m}^T_{(j)}$, the same identification can be portrayed, and so, as a consequence, we obtain the convergence over the whole sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$:

$$T^{1/4}\rho_{\varepsilon_n}T^{1/4} \to T^{1/4}\mathfrak{m}T^{1/4}, \tag{3.53}$$

and by (3.51) we have

$$\operatorname{Tr}_{L^2}\left(\int_{\mathfrak{H}} \mathrm{d}\mathfrak{m}(z)\,(T + \|z\|^{4\delta}_{\mathfrak{H}}\mathbb{1}_{L^2})\right) < C + C_{2\delta}. \tag{3.54}$$

Now, let us consider $B \in \mathscr{L}(L^2)$, then $T^{-1/4}BT^{-1/4} \in \mathscr{L}_\infty(L^2)$, and so it holds true the following convergence:

$$\begin{aligned}\lim_{n\to+\infty} \operatorname{Tr}_{\mathscr{H}}(\rho_{\varepsilon_n}(b)^{\mathrm{Wick}}_{\varepsilon_n}\, B\otimes W_{\varepsilon_n}(\xi)) &= \lim_{n\to+\infty} \operatorname{Tr}_{\mathscr{H}}(\rho^T_{\varepsilon_n}\,(b)^{\mathrm{Wick}}_{\varepsilon_n}\, T^{-1/4}BT^{-1/4}\otimes W_{\varepsilon_n}(\xi)) = \\ = \operatorname{Tr}_{L^2}\left(\int_{\mathfrak{H}} \mathrm{d}\mathfrak{m}^T(z)\, T^{-1/4}BT^{-1/4}\, b(z)\, e^{2i\Re e\langle\xi\,|\,z\rangle}\right) &= \operatorname{Tr}_{L^2}\left(\int_{\mathfrak{H}} \mathrm{d}\mathfrak{m}(z)\, B\, b(z)\, e^{2i\Re e\langle\xi\,|\,z\rangle}\right)\end{aligned}$$

that proves the convergence of symbols belonging to the class $\mathscr{P}^{\infty}_{p,q}$, $p+q<2\delta$, testing with bounded operators (the case of Weyl quantization being totally the same). To deal with the case $p+q=2\delta$, we use the regularized class of test operators. Consider a sequence $\{B_m\}_{m\in\mathbb{N}}\subseteq\mathcal{B}$, $B_m=\mathbb{1}_m(T)B\mathbb{1}_m(T)$, such that $\mathrm{s}-\lim_{m\to+\infty}B_m=B$ and, by functional calculus,

$$\|T^{-1/4}(B-B_m)\,T^{-1/4}\|\xrightarrow[m\to+\infty]{}0. \tag{3.55}$$

We use the latter to estimate the following:

$$\begin{aligned}|\mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon(b)^{\mathrm{Wick}}_\varepsilon(B-B_m))\otimes W_\varepsilon(\xi))| &\leqslant \|T^{1/4}(1+\mathrm{d}\Gamma_\varepsilon(\mathbb{1}))^{\frac{p+q}{4}}\rho_\varepsilon(1+\mathrm{d}\Gamma_\varepsilon(\mathbb{1}))^{\frac{p+q}{4}}T^{1/4}\|_{\mathscr{L}_1}\times\\ &\times\|T^{-1/4}(B-B_m)T^{-1/4}\|\|(1+\mathrm{d}\Gamma_\varepsilon(\mathbb{1}))^{-\frac{p+q}{4}}(b)^{\mathrm{Wick}}_\varepsilon(1+\mathrm{d}\Gamma_\varepsilon(\mathbb{1}))^{-\frac{p+q}{4}}\|\leqslant\\ &\leqslant C\|T^{-1/4}(B-B_m)T^{-1/4}\|\xrightarrow[m\to+\infty]{}0,\end{aligned}$$

uniformly in ε. Therefore we can exchange the limits in ε and m, obtaining

$$\lim_{n\to+\infty}\mathrm{Tr}_{\mathscr{H}}(\rho_{\varepsilon_n}(b)^{\mathrm{Wick}}_{\varepsilon_n}B_m\otimes W_{\varepsilon_n}(\xi))=\mathrm{Tr}_{L^2}\left(\int_{\mathfrak{H}}\mathrm{d}\mathfrak{m}(z)\,B_m\,b(z)\,e^{2i\Re e\langle z\,|\,\xi\rangle}\right) \tag{3.56}$$

and conclude by the bound (3.54) and the Dominated Convergence Theorem.

Now, testing with the identity element we have the following limit:

$$1=\lim_{n\to+\infty}\mathrm{Tr}_{\mathscr{H}}(\rho_{\varepsilon_n}\mathbb{1}_{\mathscr{H}})=\int_{\mathfrak{H}}\mathrm{d}\mu(z)\,\mathrm{Tr}_{L^2}(\gamma_{\mathfrak{m}}(z)\,\mathbb{1}_{L^2})=\|\mathfrak{m}(\mathfrak{H})\|_{\mathscr{L}_1(L^2)}. \tag{3.57}$$

□

3.3 Convergence for operator-valued symbols

In order to study the interacting theories, it is necessary to study operator-valued symbols and their quantization. If we consider a product state $|\psi\otimes\Psi_\varepsilon\rangle\langle\psi\otimes\Psi_\varepsilon|$, with $\psi\otimes\Psi_\varepsilon\in\mathscr{H}$, the limit of the expectation of creation and annihilation operator with $x-$dependent argument (x being the variable in $\mathbb{R}^{dN}$ for the particles) is known (see [26, 27]). In this case, if $\Psi_\varepsilon\to\mu\in\mathscr{M}(\mathfrak{H};\mathbb{R}_+)$ then

$$\begin{aligned}\lim_{\varepsilon\to0}\mathrm{Tr}_{\mathscr{H}}(|\psi\otimes\Psi_\varepsilon\rangle\langle\psi\otimes\Psi_\varepsilon|\;a^{\#}_\varepsilon(\lambda_x))&=\left\langle\psi\,\middle|\lim_{\varepsilon\to0}\left\langle\Psi_\varepsilon\,\middle|a^{\#}_\varepsilon(\lambda_x)\Psi_\varepsilon\right\rangle\psi\right\rangle=\\ &=\left\langle\psi\,\middle|\int_{\mathfrak{H}}\mathrm{d}\mu(z)\,\langle\lambda_x\,|\,z\rangle^{\#}\,\psi\right\rangle.\end{aligned} \tag{3.58}$$

The variable x in the operators argument is treated as a parameter and is not involved in the limit process. In the case of non-product state, however, the limit does not act separately on the particle and field part. In this case, we expect a result of the following type:

$$\lim_{\varepsilon\to0}\mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon\,a^{\#}_\varepsilon(\lambda_x))=\int_{\mathfrak{H}}\mathrm{d}\mu(z)\,\mathrm{Tr}_{L^2}(\gamma_{\mathfrak{m}}(z)\,\langle\lambda_x\,|\,z\rangle^{\#}). \tag{3.59}$$

Clearly, here $\langle\lambda_x\,|\,z\rangle^{\#}$ shall be considered an operator-valued symbol, *i.e.*, it has the form

$$\begin{aligned}\mathcal{V}:\mathfrak{H}&\longrightarrow\mathscr{L}(L^2(\mathbb{R}^{dN}))\\ z&\longmapsto\mathcal{V}(z):\psi\longmapsto(\mathcal{V}(z)\psi)(x).\end{aligned}$$

Quasi-classical analysis for operator-valued symbols may be very complicated in general. For our purposes, it is sufficient to consider polynomial symbols of the form

$$\mathscr{O}^{(p,q)}:=\left\{z\longmapsto\prod_{j=1}^{p}\left\langle z\,\middle|\lambda^{(j)}_x\right\rangle\prod_{k=1}^{q}\left\langle\eta^{(k)}_x\middle|\,z\right\rangle,\;\{\lambda^{(j)}_x\}_{j=1}^p,\{\eta^{(k)}_x\}_{k=1}^q\subseteq\mathfrak{H}\right\}, \tag{3.60}$$

and Wick quantizations are operators of the following type

$$\mathscr{O}_\varepsilon^{(p,q)} := \left\{ \prod_{j=1}^{p} a_\varepsilon^\dagger(\lambda_x^{(j)}) \prod_{k=1}^{q} a_\varepsilon(\lambda_x^{(k)}),\ \{\lambda_x^{(j)}\}_{j=1}^p, \{\eta_x^{(k)}\}_{k=1}^q \subseteq \mathfrak{H} \right\}. \tag{3.61}$$

The key idea is to approximate λ_x by simple functions λ_N with separated variables. In this way it is possible, by linearity, to quantize only the scalar product part and obtain "truncated" creation/annihilation operators for which the limit of the expectation is already known by Theorem 3.2.1:

$$\begin{array}{ccc} \mathrm{Tr}(\rho_\varepsilon\, a_\varepsilon^\#(\lambda_M)) & \xrightarrow[\text{Thm 3.2.1}]{\varepsilon\to 0} & \mathrm{Tr}_2 \int \mathrm{d}\mathfrak{m}(z)\ \langle \lambda_M \,|\, z\rangle^\# \\ \big\downarrow{\scriptstyle M\to+\infty} & & \big\downarrow{\scriptstyle M\to+\infty} \\ \mathrm{Tr}(\rho_\varepsilon\, a_\varepsilon^\#(\lambda_x)) & \overset{\varepsilon\to 0}{\dashrightarrow} & \mathrm{Tr}_2 \int \mathrm{d}\mathfrak{m}(z)\ \langle \lambda_x \,|\, z\rangle^\# \end{array}$$

and closing the previous diagram proving the limit expressed by the dashed arrow.

Theorem 3.3.1. *Let $\{\rho_\varepsilon\}_{\varepsilon\in(0,1)} \in \mathscr{L}_{\infty,+}(L^2)\mathscr{L}_{1,+}(\mathscr{H})$ be normalized such that there exists $\delta > 1/2$ and $C_\delta \in [1,+\infty)$ such that*

$$\sup_{\varepsilon\in(0,1)} \mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon\,(\mathrm{d}\Gamma_\varepsilon(\mathbb{1})+1)^\delta) \leqslant C_\delta.$$

Then, if $\rho_{\varepsilon_n} \to \mathfrak{m} \in \mathscr{M}(\mathfrak{H};\mathscr{L}_{1,+}(L^2))$, we have that, for $\lambda_x(k) = \chi(k)\sum_{j=1} e^{ikx_j}$, $\chi \in L^2(\mathbb{R}^d;\mathrm{d}k)$ and every $\kappa \in \mathscr{L}_\infty(L^2)$, $\xi \in \mathfrak{H}$,

$$\lim_{n\to+\infty} \mathrm{Tr}_{\mathscr{H}}\left(\rho_{\varepsilon_n}\, a_{\varepsilon_n}^\#(\lambda_x)\,\kappa \otimes W_{\varepsilon_n}(\xi)\right) = \int_{\mathfrak{H}} \mathrm{d}\mu(z)\,\mathrm{Tr}_{L^2}\left(\gamma_{\mathfrak{m}}(z)\ \langle \lambda_x \,|\, z\rangle_{\mathfrak{H}}^\#\ \kappa\right) e^{2i\Re e\langle \xi \,|\, z\rangle}. \tag{3.62}$$

If assumption (A′) *holds with $\delta \geqslant 1/2$, then the convergence is true when testing on bounded observables as well.*

It is possible to generalize the previous Theorem to operator-valued polynomial symbols of any degree, with suitable assumptions on the states.

Corollary 3.3.2. *Let $\{\rho_\varepsilon\}_{\varepsilon\in(0,1)} \in \mathscr{L}_{1,+}(\mathscr{H})$ be normalized such that there exists $\delta > \frac{p+q}{2}$ for some $p,q \in \mathbb{N}$, and $C_\delta \in [1,+\infty)$ such that*

$$\sup_{\varepsilon\in(0,1)} \mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon\,(\mathrm{d}\Gamma_\varepsilon(\mathbb{1})+1)^\delta) \leqslant C_\delta.$$

Then, if $\rho_{\varepsilon_n} \to \mathfrak{m} \in \mathscr{M}(\mathfrak{H};\mathscr{L}_1(L^2)_+)$, we have that, for any $\mathcal{V} \in \mathscr{O}^{(p,q)}$, with the $\lambda_\cdot$−s and $\eta_\cdot$−s of the same form as the in Theorem 3.3.1, and every $\kappa \in \mathscr{L}_\infty(L^2), \xi \in \mathfrak{H}$,

$$\lim_{n\to+\infty} \mathrm{Tr}_{\mathscr{H}}\left(\rho_{\varepsilon_n}\,(\mathcal{V})_{\varepsilon_n}^{\mathrm{Wick}}\,\kappa \otimes W_{\varepsilon_n}(\xi)\right) = \int_{\mathfrak{H}} \mathrm{d}\mu(z)\,\mathrm{Tr}_{L^2}\left(\gamma_{\mathfrak{m}}(z)\,\mathcal{V}(z)\,\kappa\right) e^{2i\Re e\langle \xi \,|\, z\rangle}. \tag{3.63}$$

If assumption (A′) *holds with $\delta \geqslant \frac{p+q}{2}$, the previous convergence is extended to the test with bounded operators.*

Proof of Corollary 3.3.2. We prove the result by induction. The cases $(p,q) = (1,0)$ and $(p,q) = (0,1)$ are covered by Theorem 3.3.1. Suppose the result holds true for generic $(p-1,q) \in \mathbb{N}^2$, and denote by

$$\sigma_\varepsilon = \rho_\varepsilon \, a_\varepsilon^\dagger(\lambda_x^{(p)}). \tag{3.64}$$

The state σ_ε is complex, and satisfies the bound

$$\|(1+\mathrm{d}\Gamma_\varepsilon(\mathbb{1}))^{\frac{\delta-1}{2}}\sigma_\varepsilon(1+\mathrm{d}\Gamma_\varepsilon(\mathbb{1}))^{\frac{\delta-1}{2}}\|_{\mathscr{L}_1} \leqslant C_\delta\|(1+\mathrm{d}\Gamma_\varepsilon(\mathbb{1}))^{-\frac{\delta}{2}}a_\varepsilon^\dagger(\lambda_x^{(p)})(1+\mathrm{d}\Gamma_\varepsilon(\mathbb{1}))^{\frac{\delta-1}{2}}\|_{\mathscr{L}} \leqslant C,$$

thanks to Lemma B.2.1, therefore it converges, up to a subsequence extraction, to a complex state-valued measure $\mathfrak{w} \in \mathscr{M}(\mathfrak{H};\mathscr{L}_1(L^2))$. In addition, by Theorem 3.3.1 it follows that $\mathrm{d}\mathfrak{w}(z) = \mathrm{d}\mathfrak{m}(z)\left\langle z \middle| \lambda_x^{(p)} \right\rangle$ and the subsequence is exactly $\{\varepsilon_n\}_{n\in\mathbb{N}}$. Furthermore, by Corollary 3.1.9 we know that this implies also convergence for the parts in the decomposition in positive operators. Let us consider the real positive part $\sigma_{R,\varepsilon_n}^+ \to \mathfrak{w}_R^+$. By the induction step at $p-1$,

$$\begin{aligned}\lim_{n\to+\infty}\mathrm{Tr}_{\mathscr{H}}\left(\sigma_{R,\varepsilon_n}^+\prod_{j=1}^{p-1}a_{\varepsilon_n}^\dagger(\lambda_x^{(j)})\prod_{k=1}^{q}a_{\varepsilon_n}(\eta_x^{(k)})\kappa\otimes W_{\varepsilon_n}(\xi)\right) = \\ = \mathrm{Tr}_{L^2}\left(\int_{\mathfrak{H}}\mathrm{d}\mathfrak{w}_R^+(z)\prod_{j=1}^{p-1}\left\langle z \middle| \lambda_x^{(j)}\right\rangle\prod_{k=1}^{q}\left\langle \eta_x^{(k)} \middle| z\right\rangle \kappa\, e^{2i\Re e\langle z \,|\, \xi\rangle}\right).\end{aligned} \tag{3.65}$$

Analogously, one deals with the other positive parts and obtains the result by linearity. To deal with the case from $(p,q-1)$ to (p,q), let us observe that, thanks to canonical commutation relations

$$\prod_{j=1}^{p}a_\varepsilon^\dagger(\lambda_x^{(j)})\prod_{k=1}^{q}a_\varepsilon(\eta_x^{(k)}) = a_{\varepsilon_n}(\eta_x^{(q)})\prod_{j=1}^{p}a_\varepsilon^\dagger(\lambda_x^{(j)})\prod_{k=1}^{q-1}a_\varepsilon(\eta_x^{(k)}) + \varepsilon R_\varepsilon^{(p-1,q-1)} \tag{3.66}$$

where $R_\varepsilon^{(p-1,q-1)}$ is a sum of terms of degree $(p-1,q-1)$ uniformly bounded w.r.t. ε. Therefore, we can consider the complex state $\tau_\varepsilon = \rho_\varepsilon\, a_\varepsilon(\xi_x^{(q)})$, and proceed as above. □

The proof of Theorem 3.3.1 requires some preliminary steps.

Lemma 3.3.3. *Consider $\lambda_x(k) = \chi(k)\sum_{j=1}^N e^{ikx_j}$, with $\chi \in \mathfrak{H} := L^2(\mathbb{R}^d;\mathrm{d}k), x \in \mathbb{R}^{dN}$. Then, there exists a sequence of Hilbert-valued simple functions $\{\lambda_{M,\cdot}\}_{M\in\mathbb{N}} \subseteq L^\infty(\mathbb{R}^{dN};\mathfrak{H})$, whose explicit formula is*

$$\lambda_{M,x}(k) := \chi(k)\sum_{\ell=1}^{M}c_\ell(k)\chi_{B_\ell}(x) \tag{3.67}$$

such that

$$\lim_{M\to+\infty}\|\lambda_\cdot - \lambda_{M,\cdot}\|_{L^\infty(\mathbb{R}^{dN};\mathfrak{H})} = 0. \tag{3.68}$$

Proof. The function $\mathbb{R}^{dN} \ni x \mapsto \sum_{j=1}^N e^{ikx_j}$ is measurable and bounded, so it can be approximated, uniformly in x, by a sequence of simple functions, with non-decreasing norm, of the form

$$f_M(k,x) = \sum_{\ell=1}^{M}c_\ell(k)\chi_{B_\ell}(x). \tag{3.69}$$

that is

$$\lim_{M\to+\infty} \operatorname*{ess\,sup}_{x\in\mathbb{R}^{dN}} \left| \sum_{j=1}^{N} e^{ikx_j} - f_M(k,x) \right| = 0;$$
$$\operatorname*{ess\,sup}_{k\in\mathbb{R}^{d}} \operatorname*{ess\,sup}_{x\in\mathbb{R}^{dN}} \left| \sum_{j=1}^{N} e^{ikx_j} - f_M(k,x) \right| \leqslant 2C.$$

Defining $\lambda_{M,x}(k) := \chi(k) f_M(k,x)$, we have that

$$\int_{\mathbb{R}^d} \mathrm{d}k \operatorname*{ess\,sup}_{x\in\mathbb{R}^{dN}} |\lambda_x(k) - \lambda_{M,x}(k)|^2 \leqslant 2C \|\chi\|^2_{L^2(\mathbb{R}^d)}. \tag{3.70}$$

Therefore, by dominated convergence theorem

$$\lim_{M\to+\infty} \|\lambda_x - \lambda_{M,x}\|^2_{L^\infty(\mathbb{R}^{dN};L^2(\mathbb{R}^d))} \leqslant \lim_{M\to+\infty} \int_{\mathbb{R}^d} \mathrm{d}k \operatorname*{ess\,sup}_{x\in\mathbb{R}^{dN}} |\lambda_x(k) - \lambda_{M,x}(k)|^2 = 0.$$

□

Remark 3.3.4. *The result of Lemma 3.3.3 extends easily to the vector-valued arguments in polaron and Pauli-Fierz interactions.*

Corollary 3.3.5. *Define the operator-valued symbols $\mathcal{V} \in \mathscr{O}^{(0,1)}, \mathcal{V}^\dagger \in \mathscr{O}^{(1,0)}$ in the following way:*

$$\begin{aligned} \mathcal{V}^\# : \mathfrak{H} &\longrightarrow \mathscr{L}(L^2(\mathbb{R}^{dN})) \\ z &\longmapsto \langle \lambda_\cdot \,|\, z\rangle^\#_{\mathfrak{H}}, \qquad (\mathcal{V}^\#(z)\psi)(x) = \langle \lambda_x \,|\, z\rangle^\# \psi(x) \end{aligned}$$

Then, there exist sequences of simple operator-valued functions $\{\mathcal{V}^\#_M\}_{M\in\mathbb{N}} \subseteq \mathscr{L}(L^2)^{\mathfrak{H}}$ such that $\lim_{N\to+\infty} \|\mathcal{V}^\#(z) - \mathcal{V}^\#_N(z)\|_{\mathscr{L}(L^2)} = 0$, for all $z \in \mathfrak{H}$. Furthermore, the approximating sequences can be chosen of the form

$$\mathcal{V}^\#_M(z) = \sum_{j=1}^{M} \langle c_j \,|\, z\rangle^\# \chi_{B_j}(x), \quad \{c_j\}_{j=1}^M \subseteq \mathfrak{H},\ \{B_j\}_{j=1}^M \subseteq \mathit{Borel}(\mathfrak{H}). \tag{3.71}$$

Proof. Let us prove it for $\mathcal{V}$, the other case being analogous. Since $\mathcal{V}(z)$ is a multiplication operator, its norm can be rewritten as

$$\|\mathcal{V}(z) - \mathcal{V}_M(z)\|_{\mathscr{L}(L^2)} = \operatorname*{ess\,sup}_{x\in\mathbb{R}^{dN}} |\langle \lambda_x(\,\cdot\,) - \lambda_{M,x}(\,\cdot\,) \,|\, z\rangle_{\mathfrak{H}}|, \tag{3.72}$$

where λ_M is the approximation given by Lemma 3.3.3. Maximizing over all the vectors with norm coinciding with the one of z, we get

$$\|\mathcal{V}(z) - \mathcal{V}_M(z)\|_{\mathscr{L}(L^2)} \leqslant \operatorname*{ess\,sup}_{x\in\mathbb{R}^{dN}} \sup_{\|\eta\|_{\mathfrak{H}}=\|z\|_{\mathfrak{H}}} |\langle \lambda_x(\,\cdot\,) - \lambda_{M,x}(\,\cdot\,) \,|\, \eta\rangle_{\mathfrak{H}}| \leqslant \|z\|_{\mathfrak{H}} \|\lambda_\cdot - \lambda_{M,\cdot}\|_{L^\infty(\mathbb{R}^{dN};\mathfrak{H})}.$$

The r.h.s. converges to zero by Lemma 3.3.3. □

Proof of Theorem 3.3.1. Let us prove the result for the annihilation operator, the other case being analogous. We know that $a_\varepsilon(\lambda_x) = (\mathcal{V})^{\mathrm{Wick}}_\varepsilon$, where $\mathcal{V}(z) = \langle \lambda_x \,|\, z \rangle$. Then by Corollary 3.3.5, there exists a sequence of operators of multiplication by simple functions $\{\mathcal{V}_M\}_{M\in\mathbb{N}} \subseteq L^\infty(\mathbb{R}^{dN};\mathfrak{H})$, defined in (3.71), approximating $\mathcal{V}(z)$:

$$\lim_{M\to+\infty} \|\mathcal{V}(z) - \mathcal{V}_M(z)\|_{\mathscr{L}(L^2)} = 0. \tag{3.73}$$

In addition

$$(\mathcal{V}_M)^{\mathrm{Wick}}_\varepsilon = \sum_{j=1}^{M} a_\varepsilon(c_j)\chi_{B_j}(x), \tag{3.74}$$

Therefore

$$\lim_{n\to+\infty} \mathrm{Tr}_{\mathscr{H}}\left(\rho_{\varepsilon_n}\, a_{\varepsilon_n}(\lambda_x)\, \kappa \otimes W_{\varepsilon_n}(\xi)\right) = \lim_{n\to+\infty}\lim_{M\to+\infty} (I_1(\varepsilon_n, M) + I_2(\varepsilon_n, M)) \tag{3.75}$$

where

$$\begin{aligned} I_1(\varepsilon, M) &= \mathrm{Tr}_{\mathscr{H}}\big(\rho_\varepsilon\, ((\mathcal{V})^{\mathrm{Wick}}_\varepsilon - (\mathcal{V}_M)^{\mathrm{Wick}}_\varepsilon)\, \kappa \otimes W_\varepsilon(\xi)\big), \\ I_2(\varepsilon, M) &= \mathrm{Tr}_{\mathscr{H}}\big(\rho_\varepsilon\, (\mathcal{V}_M)^{\mathrm{Wick}}_\varepsilon\, \kappa \otimes W_\varepsilon(\xi)\,\big). \end{aligned}$$

Using the regularity assumptions on the state, and Lemma B.2.1:

$$\begin{aligned} |I_1(\varepsilon, M)| &= |\mathrm{Tr}_{\mathscr{H}}\left(\rho_\varepsilon\, a_\varepsilon(\lambda_x - \lambda_{M,x})\, \kappa \otimes W_\varepsilon(\xi)\,\right)| \leqslant \\ &\leqslant \sup_{\varepsilon\in(0,1)} \mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon\, (\mathrm{d}\Gamma_\varepsilon(\mathbb{1}) + 1)^{1/2}) \times \\ &\quad \times \|(\mathrm{d}\Gamma_\varepsilon(\mathbb{1}) + 1)^{-1/4} a_\varepsilon(\lambda_x - \lambda_{M,x})(\mathrm{d}\Gamma_\varepsilon(\mathbb{1}) + 1)^{-1/4}\|_{\mathscr{L}(\mathscr{H})}\, \|\kappa\| \leqslant \\ &\leqslant C_\delta\, \|\lambda_\cdot - \lambda_{M,\cdot}\|_{L^\infty(\mathbb{R}^{dN};\mathfrak{H})}\, \|\kappa\|, \end{aligned}$$

and thus, by Lemma 3.3.3,

$$\lim_{M\to+\infty} |I_1(\varepsilon, M)| = 0, \tag{3.76}$$

uniformly in ε. Hence, it is sufficient to prove the existence of the limit in ε of

$$\begin{aligned} I_2(\varepsilon, M) &= \mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon\, (\mathcal{V}_M)^{\mathrm{Wick}}_\varepsilon\, \kappa \otimes W_\varepsilon(\xi)) = \\ &= \sum_{j=1}^{M} \mathrm{Tr}_{\mathscr{H}}\left(\rho_\varepsilon\, (\langle c_j \,|\, \cdot \rangle)^{\mathrm{Wick}}_\varepsilon \chi_{B_j}(x)\, \kappa \otimes W_\varepsilon(\xi)\right). \end{aligned}$$

Since $\chi_{B_j}\, \kappa \in \mathscr{L}_\infty(L^2)$ and $\langle c_j \mid \cdot \rangle \in \mathscr{P}^\infty_{0,1}$ for every $j \in \mathbb{N}$, we can apply Theorem 3.2.1:

$$\begin{aligned} \lim_{n\to+\infty} \mathrm{Tr}_{\mathscr{H}}\left(\rho_{\varepsilon_n}\, (\mathcal{V}_M)^{\mathrm{Wick}}_{\varepsilon_n}\, \kappa \otimes W_{\varepsilon_n}(\xi)\right) &= \lim_{n\to+\infty} I_2(\varepsilon_n, M) = \\ &= \lim_{n\to+\infty} \sum_{j=1}^{M} \mathrm{Tr}_{\mathscr{H}}\left(\rho_{\varepsilon_n}\, (\langle c_j \,|\, \cdot \rangle)^{\mathrm{Wick}}_{\varepsilon_n}\, \chi_{B_j}(x)\, \kappa \otimes W_{\varepsilon_n}(\xi)\right) = \\ &= \sum_{j=1}^{M} \int_{\mathfrak{H}} \mathrm{d}\mu(z)\, \mathrm{Tr}_{L^2}\left(\gamma_{\mathfrak{m}}(z)\, \chi_{B_j}(x)\, \kappa\right)\, \langle c_j \,|\, z \rangle\, e^{2i\Re e\langle \xi \,|\, z \rangle} = \\ &= \int_{\mathfrak{H}} \mathrm{d}\mu(z)\, \mathrm{Tr}_{L^2}\left(\gamma_{\mathfrak{m}}(z)\, \mathcal{V}_M(z)\, \kappa\right)\, e^{2i\Re e\langle \xi \,|\, z \rangle}. \end{aligned}$$

Now, since

$$\|\mathcal{V}_M(z)\|_{\mathscr{L}} \leqslant \|z\|_{\mathfrak{H}}\, \|\lambda_\cdot\|_{L^\infty(\mathbb{R}^{dN};\mathfrak{H})},$$

uniformly in M, and $z \mapsto \|z\|_{\mathfrak{H}}$ is a $\mu-$integrable function by Theorem 3.2.1, we can use Dominated Convergence Theorem and conclude the proof. If (A′) holds, by Corollary 3.2.3 we extend the convergence to the test with bounded operators. □

In order to study the polaron and Pauli-Fierz model, we need also to study the convergence of gradient terms. This is done in the remaining part of this section.

Proposition 3.3.6. *Let $\{\rho_\varepsilon\}_{\varepsilon\in(0,1)} \subseteq \mathscr{L}_{1,+}(\mathscr{H})$ be normalized, let $T > 0$ be a positive operator on $L^2(\mathbb{R}^{dN})$ such that $|\nabla_y| T^{-1/2} \in \mathscr{L}(L^2(\mathbb{R}^d))$ and suppose that, for some $\delta > 1$, there exists $C'_\delta \in [1, +\infty)$ such that*

$$\sup_{\varepsilon\in(0,1)} \mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon\, (T + (\mathrm{d}\Gamma_\varepsilon(\mathbb{1}) + 1)^\delta)) \leqslant C'_\delta. \tag{3.77}$$

Then, if $\rho_{\varepsilon_n} \to \mathfrak{m}$, we have, for every $\kappa \in \mathcal{K}$, $\xi \in \mathfrak{H}$ and $\vec{\lambda}_\cdot = (\lambda_\cdot^{(\ell)})_{\ell=1}^d$, with $\lambda_y^{(\ell)}(k) = \chi^{(\ell)}(k)e^{iky}$ and $\chi^{(\ell)} \in \mathfrak{H}$, for any $\ell \in \{1, \ldots, d\}$,

$$\lim_{n\to+\infty} \mathrm{Tr}_{\mathscr{H}}(\rho_{\varepsilon_n}\, a^{\#}_{\varepsilon_n}(\vec{\lambda}_y) \cdot \nabla_y\, \kappa \otimes W_\varepsilon(\xi)) = \int_{\mathfrak{H}} \mathrm{d}\mu(z)\, \mathrm{Tr}_{L^2}\left(\gamma_{\mathfrak{m}}(z) \left\langle \vec{\lambda}_y \middle| z \right\rangle^{\#} \cdot \nabla_y \kappa\right) e^{2i\Re e\langle z \,|\, \xi\rangle}, \tag{3.78}$$

$$\lim_{n\to+\infty} \mathrm{Tr}_{\mathscr{H}}(\rho_{\varepsilon_n}\, \nabla_y \cdot a^{\#}_{\varepsilon_n}(\vec{\lambda}_y)\, \kappa \otimes W_\varepsilon(\xi)) = \int_{\mathfrak{H}} \mathrm{d}\mu(z)\, \mathrm{Tr}_{L^2}\left(\gamma_{\mathfrak{m}}(z) \nabla_y \cdot \left\langle \vec{\lambda}_y \middle| z \right\rangle^{\#} \kappa\right) e^{2i\Re e\langle z \,|\, \xi\rangle}. \tag{3.79}$$

Proof. Let us denote by $\rho^{(T)}_{\varepsilon_n} := T^{1/2}\rho_{\varepsilon_n}T^{1/2}$. By assumption, it is a family of density matrices with uniformly bounded trace-class norm and so, by Theorem 3.1.6, there exists a cylindrical state-valued measure $\mathfrak{m}^{(T)} \in \mathscr{M}_{\mathrm{cyl}}(\mathfrak{H}; \mathscr{L}_1(L^2)_+)$ and a generalized subsequence $\{\varepsilon_{n_k}\}_{k\in B} \subseteq \{\varepsilon_n\}_{n\in\mathbb{N}}$ such that $\rho^{(T)}_{\varepsilon_{n_k}} \xrightarrow{\mathfrak{P}} \mathfrak{m}^{(T)}$. Let us show that also its generating map, tested on suitable compact operators, converges. To do so, let us test convergence with elements $\kappa \in \mathcal{K}$. Since $\rho_{\varepsilon_{n_k}} \to \mathfrak{m}$ we have that, for $f \in C^\infty_{0,\mathrm{cyl}}(\mathfrak{H})$ and $\xi \in \mathfrak{H}$,

$$\begin{aligned}\lim_{k\to+\infty} \mathrm{Tr}_{\mathscr{H}}(\rho^{(T)}_{\varepsilon_{n_k}}\kappa \otimes \mathrm{Op}^{\varepsilon_{n_k}}_{1/2}(f)) &= \lim_{k\to+\infty} \mathrm{Tr}_{\mathscr{H}}(\rho_{\varepsilon_{n_k}} T^{1/2}\kappa T^{1/2} \otimes \mathrm{Op}^{\varepsilon_{n_k}}_{1/2}(f)) = \\ &= \mathrm{Tr}_{L^2}\left(\int_{\mathfrak{H}} \mathrm{d}(T^{1/2}\mathfrak{m}T^{1/2})(z)\, \kappa\, f(z)\right);\end{aligned}$$

$$\begin{aligned}\lim_{k\to+\infty} G_{\rho^{(T)}_{\varepsilon_{n_k}}}(\xi)(\kappa) &= \lim_{k\to+\infty} \mathrm{Tr}_{\mathscr{H}}(\rho_{\varepsilon_{n_k}} T^{1/2}\kappa T^{1/2} \otimes W_{\varepsilon_{n_k}}(\xi)) = \\ &= \int_{\mathfrak{H}} \mathrm{d}\mu(z)\, \mathrm{Tr}_{L^2}(\rho(z)T^{1/2}\kappa T^{1/2})\, e^{2i\Re e\langle \xi \,|\, z\rangle} = (\widehat{T^{1/2}\mathfrak{m}T^{1/2}})(\xi)(\kappa),\end{aligned}$$

where we used that $T^{1/2}\kappa T^{1/2} \in \mathscr{L}_\infty(L^2)$. Hence $\rho^{(T)}_{\varepsilon_{n_k}} \to T^{1/2}\mathfrak{m}T^{1/2}$ when testing with elements of $\mathcal{K}$. Since this last set separates points of $\mathscr{L}_1(L^2)$ by Lemma 3.2.2 it follows that $\mathfrak{m}^{(T)} = T^{1/2}\mathfrak{m}T^{1/2}$. The same proof applies to any $\mathfrak{P} \vee \mathfrak{T}$ convergent generalized subsequence $\{\rho^{(T)}_{\varepsilon_{n_j}}\}_{j\in B}$. Therefore, the cluster point is unique, and thus

$$T^{1/2}\rho_{\varepsilon_n}T^{1/2} \to T^{1/2}\mathfrak{m}T^{1/2}. \tag{3.80}$$

As a byproduct of the previous convergence we also obtain the following bound:

$$\mathrm{Tr}_{L^2}\left(\int_{\mathfrak{H}} \mathrm{d}\mathfrak{m}(z)T\right) \leqslant \liminf_{\varepsilon\to 0} \mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon T) \leqslant C. \tag{3.81}$$

Let us now prove (3.78): since $\kappa \in \mathcal{K}$ then $\nabla_y\kappa \in \mathscr{L}_\infty(L^2) \otimes \mathbb{C}^d$, and by the fact that $\rho_{\varepsilon_n} \to \mathfrak{m}$ we can apply Theorem 3.3.1 and obtain the desired convergence. Let us verify that the limiting expression is well defined:

$$\left|\int_{\mathfrak{H}} \mathrm{d}\mu(z)\, \mathrm{Tr}_{L^2}(\gamma_{\mathfrak{m}}(z)\, \langle \vec{\lambda}_y \,|\, z\rangle \cdot \nabla_y\kappa)\, e^{2i\Re e\langle z \,|\, \xi\rangle}\right| \leqslant \|\nabla_y\kappa\|_{\mathscr{L}} \|\vec{\lambda}_\cdot\|_\infty \int_{\mathfrak{H}} \mathrm{d}\mu(z)\, \|z\|_{\mathfrak{H}} \leqslant C_\delta.$$

Let us now consider (3.79). By Corollary 3.3.5 we know that there exists a sequence of simple approximating functions $\{\vec{\mathcal{V}}_M(z)\}_{M\in\mathbb{N}}$, $\vec{\mathcal{V}}_M(z) = \left\langle \vec{\lambda}_{M,y} \middle| z \right\rangle$. By Lemma B.2.1 and Proposition A.1.5, and considering the expansion of ρ_{ε_n} in singular values $\{d_\ell^{(\varepsilon)}\}_{\ell\in\mathbb{N}}$, such that $d_\ell^{(\varepsilon)} \in (0,1]$, $\sum_{\ell\in\mathbb{N}} d_\ell^{(\varepsilon)} = 1$, and $\{\Psi_\ell^{(\varepsilon)}\}_{\ell\in\mathbb{N}}$ being the set of corresponding eigenfunctions:

$$\begin{aligned}
&\left|\mathrm{Tr}_{\mathscr{H}}(\rho_{\varepsilon_n} \nabla_y \cdot \left(a_{\varepsilon_n}(\vec{\lambda}_y) - (\vec{\mathcal{V}}_M)_{\varepsilon_n}^{\mathsf{Wick}}\right) \kappa \otimes W_\varepsilon(\xi))\right| \leqslant \\
&\leqslant \sum_{\ell\in\mathbb{N}} d_\ell^{(\varepsilon_n)} \left|\left\langle \nabla_y \Psi_\ell^{(\varepsilon)} \middle| \left(a_{\varepsilon_n}(\vec{\lambda}_y) - (\vec{\mathcal{V}}_M)_{\varepsilon_n}^{\mathsf{Wick}}\right) \kappa \otimes W_{\varepsilon_n}(\xi))\Psi_\ell^{(\varepsilon_n)} \right\rangle\right| \leqslant \\
&\leqslant C\|\vec{\lambda}_\cdot - \vec{\lambda}_{M,\cdot}\|_\infty \sum_{\ell\in\mathbb{N}} d_\ell^{(\varepsilon_n)} \|\nabla_y T^{-1/2}\|_{\mathscr{L}} \|T^{1/2}\Psi_\ell^{(\varepsilon)}\| \|(1+\mathrm{d}\Gamma_\varepsilon(\mathbb{1}))^{1/2}\Psi_\ell^{(\varepsilon_n)}\| \leqslant \\
&\leqslant C\|\vec{\lambda}_\cdot - \vec{\lambda}_{M,\cdot}\|_\infty \sum_{\ell\in\mathbb{N}} d_\ell^{(\varepsilon_n)} \left\langle \Psi_\ell^{(\varepsilon_n)} \middle| (T+1+\mathrm{d}\Gamma_\varepsilon(\mathbb{1})) \middle| \Psi_\ell^{(\varepsilon_n)} \right\rangle = \\
&= C\|\vec{\lambda}_\cdot - \vec{\lambda}_{M,\cdot}\|_\infty \mathrm{Tr}_{\mathscr{H}}(\rho_{\varepsilon_n}(T+1+\mathrm{d}\Gamma_\varepsilon(\mathbb{1}))) \leqslant C\|\vec{\lambda}_\cdot - \vec{\lambda}_{M,\cdot}\|_\infty, \qquad (3.82)
\end{aligned}$$

where this time we denoted by $\|\cdot\|_\infty := \|\cdot\|_{L^\infty(\mathbb{R}^{dN};\mathfrak{H}\otimes\mathbb{C}^d)}$. Therefore, we have convergence to zero as $M \to +\infty$, uniformly w.r.t. ε by Lemma 3.3.3. Let us denote by $(y_j)_{j=1}^d, (\lambda_{y,j})_{j=1}^d$ the components of, respectively, $y \in \mathbb{R}^d$ and $\vec{\lambda}_\cdot \in L^\infty(\mathbb{R}^d;\mathfrak{H}^d)$, and by $\mathcal{V}_M^{(j)}(z)$ the components of the simple functions approximating $\left\langle \vec{\lambda}_y \middle| z \right\rangle$. Now it remains to study the convergence for

$$\mathrm{Tr}_{\mathscr{H}}(\rho_{\varepsilon_n} \partial_{y_j} (\mathcal{V}_M^{(j)})_{\varepsilon_n}^{\mathsf{Wick}} \kappa \otimes W_{\varepsilon_n}(\xi)) = \sum_{l=1}^M \mathrm{Tr}_{\mathscr{H}}(\rho_{\varepsilon_n} \partial_{y_j} \chi_{B_l}(y) \kappa \otimes a_{\varepsilon_n}(c_l^{(j)}) W_{\varepsilon_n}(\xi)). \qquad (3.83)$$

We want to use convergence in topology $\mathfrak{P}$ for expectations of cylindrical, compactly supported symbols. However, the operator $a_{\varepsilon_n}(c_l^{(j)})W_{\varepsilon_n}(\xi)$ is the quantization of a product of cylindrical, but not compactly supported symbols. So we need an additional approximation: by finite-dimensional pseudo-differential calculus, for every $M \in \mathbb{N}$, there exists a sequence of smooth, compactly supported cylindrical symbols $\{\beta_m^{(l,j,\xi)}\}_{m\in\mathbb{N}} \subseteq C_{0,\mathsf{cyl}}^\infty(\mathfrak{H})$ whose elements, we simply denote by β_m, such that, for every $\delta > 1/2$:

$$\|(1+\mathrm{d}\Gamma_\varepsilon(\mathbb{1}))^{-\delta/2}(a_\varepsilon(c_l^{(j)})W_\varepsilon(\xi) - (\beta_m)_\varepsilon^{\mathsf{Weyl}})(1+\mathrm{d}\Gamma_\varepsilon(\mathbb{1}))^{-\delta/2}\|_{\mathscr{L}(\Gamma_s(\mathfrak{H}))} \leqslant C(o_m(1) + o_\varepsilon(1)) \qquad (3.84)$$

$$|(\langle c_l^{(j)} \,|\, z\rangle e^{2i\Re e\langle z\,|\,\xi\rangle} - \beta_m(z))(\|z\|_{\mathfrak{H}}^2 + 1)^\delta| = o_m(1). \qquad (3.85)$$

Therefore,

$$\begin{aligned}
&\left|\mathrm{Tr}_{\mathscr{H}}(\rho_{\varepsilon_n} \partial_{y_j} \chi_{B_l}(y) B \otimes (a_{\varepsilon_n}(c_l^{(j)}) W_{\varepsilon_n}(\xi) - (\beta_m)_{\varepsilon_n}^{\mathsf{Weyl}}))\right| \leqslant \\
&\qquad \leqslant C'_\delta \|B\|_{\mathscr{L}} \|(1+\mathrm{d}\Gamma_\varepsilon(\mathbb{1}))^{-\delta/2}(a_{\varepsilon_n}(c_l^{(j)})W_{\varepsilon_n}(\xi) - (\beta_m)_{\varepsilon_n}^{\mathsf{Weyl}})(1+\mathrm{d}\Gamma_\varepsilon(\mathbb{1}))^{-\delta/2}\|_{\mathscr{L}(\Gamma_s(\mathfrak{H}))} \leqslant \\
&\qquad \leqslant C'_\delta \|B\|_{\mathscr{L}}(o_m(1) + o_\varepsilon(1)), \qquad (3.86)
\end{aligned}$$

and, thanks to the fact that $T^{1/2}\rho_\varepsilon T^{1/2} \to T^{1/2}\mathfrak{m}T^{1/2}$ in $\mathfrak{P}$ topology and that $\beta_m \in C_{0,\mathsf{cyl}}^\infty(\mathfrak{H})$ for every $m \in \mathbb{N}$, we obtain, denoting $\kappa_T = T^{-1/2}\partial_{y_j}\chi_{B_l}(x_j)\kappa T^{-1/2} \in \mathscr{L}_\infty(L^2)$,

$$\begin{aligned}
\lim_{n\to+\infty} \sum_{l=1}^M \mathrm{Tr}_{\mathscr{H}}(\rho_{\varepsilon_n} \partial_{y_j} \chi_{B_l}(y) \kappa \otimes (\beta_m)_{\varepsilon_n}^{\mathsf{Wick}}) &= \lim_{n\to+\infty} \sum_{l=1}^M \mathrm{Tr}_{\mathscr{H}}(\rho_{\varepsilon_n}^{(T)} \kappa_T \otimes (\beta_m)_{\varepsilon_n}^{\mathsf{Wick}}) = \\
&= \sum_{l=1}^M \int_{\mathfrak{H}} \mathrm{d}\mu(z) \mathrm{Tr}_{L^2}(\gamma_{\mathfrak{m}}(z) \partial_{y_j} \chi_{B_l}(y) \kappa)\, \beta_m(z).
\end{aligned}$$

Now we use, in order, (3.86) for the approximation with the trace of the quantized compactly supported symbol, the previous limit in ε and a combination of (3.85) and a bound similar to (3.82), to obtain, by dominated convergence:

$$\begin{aligned}
&\lim_{n\to+\infty}\sum_{j=1}^{d}\sum_{l=1}^{M}\mathrm{Tr}_{\mathscr{H}}(\rho_{\varepsilon_n}\partial_{y_j}a_{\varepsilon_n}(c_l^{(j)})\chi_{B_l}(y)\kappa\otimes W_{\varepsilon_n}(\xi)) = \\
&= \lim_{m\to+\infty}\lim_{n\to+\infty}\sum_{l=1}^{M}\mathrm{Tr}_{\mathscr{H}}(\rho_{\varepsilon_n}\,\partial_{y_j}\chi_{B_l}(y)\kappa\otimes(\beta_m)^{\mathsf{Wick}}_{\varepsilon_n}) = \\
&= \sum_{l=1}^{M}\int_{\mathfrak{H}}\mathrm{d}\mu(z)\mathrm{Tr}_{L^2}(\gamma_{\mathfrak{m}}(z)\partial_{y_j}\chi_{B_l}(y)\kappa)\,e^{2i\Re e\langle z\,|\,\xi\rangle}.
\end{aligned}$$

Using the aforementioned bound, we apply again dominated convergence for $M\to+\infty$ and conclude the proof. □

Concretely we use

$$T = -\Delta_x + W_+ \tag{3.87}$$

in the applications, and thus, as space of test operators,

$$\mathcal{K} = \mathcal{K}_T = \{\kappa_m := \mathbb{1}_m(-\Delta_x + W_+)\kappa\mathbb{1}_m(-\Delta_x + W_+), \kappa\in\mathscr{L}_{\infty,+}(L^2), m\in\mathbb{N}\}. \tag{3.88}$$

Obviously, $|\nabla_x|T^{-1/2}\in\mathscr{L}(\mathscr{H})$, and the regularity requirement becomes

$$\mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon(-\Delta_x + W_+ + (\mathrm{d}\Gamma_\varepsilon(\mathbb{1})+1)^\delta)) \leqslant C_\delta, \qquad \text{for } \delta > 1. \tag{3.89}$$

Hence, thanks to (3.81), we have the bound

$$\mathrm{Tr}_{L^2}\left(\int_{\mathfrak{H}}\mathrm{d}\mathfrak{m}(z)(-\Delta_x + W_+)\right) \leqslant C. \tag{3.90}$$

4 Quasi-classical Limit: Stationary Setting

We discuss now the analysis of the quasi-classical limit of the models introduced above, focusing on their stationary properties. More precisely, in Section 4.1 and 4.2, we prove the convergence of the expectations of the microscopic Hamiltonian to the classical expectation of the effective model introduced in Chapter 1. Furthermore, we prove that the microscopic ground state energy of the Nelson model converges to the ground state energy of the associated quasi-classical model, minimized over all the possible classical configurations of the field.

For reader convenience, we recall the explicit expressions of both the microscopic and effective Hamiltonians. We stress that, in this Chapter only, we choose the factor $\nu_\varepsilon = 1$ in order to put the field and particle energies on the same ground.

Nelson model

$$\begin{cases} H_\varepsilon = -\Delta_x + W + \mathrm{d}\Gamma_\varepsilon(\omega) + \phi_\varepsilon(\lambda_x), \\ \mathcal{H}(z) = -\Delta_x + W + 2\Re e \langle z \,|\lambda_x \rangle . \end{cases} \tag{4.1}$$

Polaron[1]

$$\begin{cases} H_\varepsilon = -\Delta_x + W + \mathrm{d}\Gamma_\varepsilon(\mathbb{1}) + \displaystyle\sum_{j=1}^{N} \phi_\varepsilon(g_{x_j}), \\ \mathcal{H}(z) = -\Delta_x + W + \displaystyle\sum_{j=1}^{N} 2\Re e \big\langle z \,\big|g_{<,x_j} \big\rangle + \big[-i\nabla_{x_j}, 2\Im m \big\langle z \,\big|\tilde{g}_{x_j} \big\rangle\big] . \end{cases} \tag{4.2}$$

Pauli-Fierz model

$$\begin{cases} H_\varepsilon = \displaystyle\sum_{j=1}^{N} \big(- i\nabla_{x_j} - \phi_\varepsilon(\lambda_{x_j})\big)^2 + W + \mathrm{d}\Gamma_\varepsilon(\omega), \\ \mathcal{H}(z) = \displaystyle\sum_{j=1}^{N} \big(- i\nabla_{x_j} - 2\Re e \big\langle z \,\big|\lambda_{x_j} \big\rangle\big)^2 + W. \end{cases} \tag{4.3}$$

4.1 Quasi-classical limit of the Hamiltonians

The first stationary feature we consider is the behaviour of the energy of the composite systems in the quasi-classical limit. We prove that, once the degrees of freedom of the field are traced out, the energy of the particle subsystem can be approximated in norm resolvent sense by the energy of the particles under the action of an external classical potential.

[1] Here we denote improperly by H_ε the formal expression of the quadratic form.

Consider a vector $\Psi_\varepsilon \in \Gamma_s(\mathfrak{H})$ describing a product state of the total system. Setting

$$\mathcal{H}_\varepsilon = \langle \Psi_\varepsilon | \, H_\varepsilon \, | \Psi_\varepsilon \rangle - \langle \Psi_\varepsilon | \, \mathrm{d}\Gamma_\varepsilon(\omega) \, | \Psi_\varepsilon \rangle , \tag{4.4}$$

the convergence results proved in Chapter 3 imply that, if $\Psi_{\varepsilon_n} \to \mu \in \mathscr{M}(\mathfrak{H}; \mathbb{R}^+)$, then,

$$\mathcal{H}_{\varepsilon_n} \xrightarrow[n \to +\infty]{\mathrm{w}} \int_{\mathfrak{H}} \mathrm{d}\mu(z) \, \mathcal{H}(z) \tag{4.5}$$

in weak operator topology. In other words, the quadratic form of the energy of the particles coupled to the quantum field converges to the form of the effective Hamiltonian averaged w.r.t. the measure identifying the classical configurations of the field.

Due to technical issues concerning the convergence of expectations of bounded operators (in this case the identity operator), the proof is restricted to product states. In this case, the expectation of the Hamiltonian reads

$$\langle \psi \otimes \Psi_\varepsilon | \, H_\varepsilon \, | \psi \otimes \Psi_\varepsilon \rangle = \langle \psi | - \Delta_x + W \, | \psi \rangle + \langle \Psi_\varepsilon | \, \mathrm{d}\Gamma_\varepsilon(\omega) \, | \Psi_\varepsilon \rangle + \langle \psi \otimes \Psi_\varepsilon | \, H_I \, | \psi \otimes \Psi_\varepsilon \rangle \tag{4.6}$$

and the free part H_0 is unaffected by the limit.

As shown in [26, Theorems 2.2-3] and [27, Theorem 1.1], however, it is possible to prove that the convergence of the Hamiltonian to the effective one is not only at the level of expectations, but also in norm-resolvent sense. We report the results below: the set of assumptions chosen in Chapter 1 would be enough, but we prefer to stick to the result proven in [26], [27] and then we make different assumptions. If we had a mixed state, we should have used Corollary 3.2.3 for the convergence testing with the identity of the state $(-\Delta_x + W_+)^{1/2} \rho_\varepsilon (-\Delta_x + W_+)^{1/2}$, but the requirement for the convergence would have been a control over the square of the Laplacian, which is a request physically non-well motivated, and that cannot be used in the following for the convergence of ground state energies.

Theorem 4.1.1. *Let $\{\Psi_\varepsilon\}_{\varepsilon \in (0,1)} \subseteq \Gamma_s(\mathfrak{H})$ be a family of normalized states satisfying*

$$\exists \delta > 0, \quad \exists C, C_\delta < +\infty \qquad \langle \Psi_\varepsilon | \, \mathrm{d}\Gamma_\varepsilon(\mathbb{1})^\delta \, | \Psi_\varepsilon \rangle \leqslant C_\delta, \qquad \langle \Psi_\varepsilon | \, \mathrm{d}\Gamma_\varepsilon(\omega) \, | \Psi_\varepsilon \rangle \leqslant C, \tag{4.7}$$

uniformly in ε, and let $\Psi_\varepsilon \to \mu \in \mathscr{M}(\mathfrak{H} \cap \mathfrak{H}_\omega; \mathbb{R}^+)$. Then, if additionally

$$W \in L^1_{\mathrm{loc}}(\mathbb{R}^{dN}; \mathbb{R}^+) + \mathcal{K}_{\ll},$$
$$\omega(k) \geqslant 0, \qquad \textit{for a.e. } k \in \mathbb{R}^d,$$

and

$$\begin{cases} \lambda_\cdot \in L^\infty(\mathbb{R}^d; \mathfrak{H}), & \textit{for the Nelson model;} \\ g_\cdot = |k|^{-\frac{d-1}{2}} \, e^{ik(\cdot)}, & \textit{for the polaron;} \\ \lambda_\cdot \in L^\infty(\mathbb{R}^d; \mathbb{R}^d \otimes \mathscr{D}(\omega^{1/2}) \cap \mathscr{D}(\omega^{-1/2})), & \textit{for the Pauli-Fierz model;} \end{cases}$$

then, for any $\varepsilon \in (0,1)$, the particle Hamiltonian $\mathcal{H}_\varepsilon$, defined as (4.4) *is self-afdjoint on $\mathscr{D}(\mathcal{H}_\varepsilon)$, where*

- *$\mathscr{D}(\mathcal{H}_\varepsilon) = \mathscr{D}(-\Delta_x + W_+)$ for the Nelson and Pauli-Fierz models;*
- *$\mathscr{D}(\mathcal{H}_\varepsilon) \subseteq \mathscr{Q}(-\Delta_x + W_+)$ for the polaron.*

Furthermore,

$$\mathcal{H}_\varepsilon \xrightarrow[\varepsilon \to 0]{\| \cdot \| - \mathrm{res}} \mathcal{H}_{\mathrm{eff}}(\mu) := \int_{\mathfrak{H} \cap \mathfrak{H}_\omega} \mathrm{d}\mu(z) \, \mathcal{H}(z), \tag{4.8}$$

where $\mathcal{H}_{\mathrm{eff}(\mu)}$ is self-afdjoint on $\mathscr{D}(\mathcal{H}_{\mathrm{eff}}) = \mathscr{D}(\mathcal{H}_\varepsilon)$ for the Nelson and the Pauli-Fierz models, and on $\mathscr{D}(\mathcal{H}_{\mathrm{eff}}) \subseteq \mathscr{Q}(-\Delta_x + W_+)$ for the polaron. Finally, both the microscopic and the effective Hamiltonian are bounded from below.

The following technical result plays an important role in the proof of the Theorem above

Lemma 4.1.2. *There exists* $\delta > 0$ *such that*

$$\|(-\Delta_x + W + (\|z\|_{\mathfrak{H}}^2 + 1)^{\delta})^{-1/2}\mathcal{V}(z)(-\Delta_x + W + (\|z\|_{\mathfrak{H}}^2 + 1)^{\delta})^{-1/2}\| \leqslant C; \tag{4.9}$$

Proof. Let us consider each model separately:

Nelson model: $\mathcal{V}(z)$ is a bounded operator on $L^2(\mathbb{R}^{dN})$, for any $z \in \mathfrak{H}$, and satisfies the bound

$$\|\mathcal{V}(z)\| \leqslant C\|\lambda.\|_{\infty}\|z\|_{\mathfrak{H}}, \tag{4.10}$$

so that (4.9) holds true with $\delta \geqslant 1/2$.

Polaron: Recalling the explicit expression for the effective potential

$$\mathcal{V}(z) = \sum_{j=1}^{N} 2\Re e \left\langle g_{x_j,<} \,|z \right\rangle + [-i\nabla_{x_j}, 2\Im m \left\langle \tilde{g}_{x_j} \,|\, z \right\rangle], \tag{4.11}$$

and setting $S_\delta = (-\Delta_x + W + (\|z\|_{\mathfrak{H}}^2 + 1)^{\delta})$, we have

$$\begin{aligned}
&\left\| S_\delta^{-1/2}\mathcal{V}(z)S_\delta^{-1/2} \right\| \leqslant \\
&\leqslant 2\sum_{j=1}^{N} \left\| S_\delta^{-1/2}\nabla_{x_j} \cdot 2\Im m \left\langle z \,|\, \tilde{g}_{x_j} \right\rangle S_\delta^{-1/2} \right\| + 2\sum_{j=1}^{N} \left\| S_\delta^{-1/2} 2\Re e \left\langle z \,|g_{<,x_j} \right\rangle S_\delta^{-1/2} \right\| \leqslant \\
&\leqslant 2\sum_{j=1}^{N} \left\| S_\delta^{-1/2}(-\Delta_{x_j} + 4\|z\|_{\mathfrak{H}}^2\|\tilde{g}_{x_j}\|^2)S_\delta^{-1/2} \right\|_{\mathscr{L}} + 2N\|g_{<,\cdot}\|_{\infty}(\|z\|_{\mathfrak{H}}^2 + 1)^{-\delta}\|z\|_{\mathfrak{H}} \leqslant C,
\end{aligned}$$

for $\delta = 1$.

Pauli-Fierz model: the expression for the effective potential

$$\mathcal{V}(z) = \sum_{j=1}^{N} \left(2i\nabla_{x_j} \cdot 2\Re e \left\langle z \,|\lambda_{x_j} \right\rangle_{\mathfrak{H}} + \left(2\Re e \left\langle z \,|\lambda_{x_j} \right\rangle_{\mathfrak{H}}\right)^2 \right), \tag{4.12}$$

and we get

$$\begin{aligned}
&\left\| S_\delta^{-1/2}\mathcal{V}(z)S_\delta^{-1/2} \right\| \leqslant \\
&\leqslant 2\sum_{j=1}^{N} \left\| S_\delta^{-1/2}\nabla_{x_j} \cdot 2\Re e \left\langle z \,|\lambda_{x_j} \right\rangle S_\delta^{-1/2} \right\| + \sum_{j=1}^{N} \left\| S_\delta^{-1/2}\left(2\Re e \left\langle z \,|\lambda_{x_j} \right\rangle\right)^2 S_\delta^{-1/2} \right\| \leqslant \\
&\leqslant 2\sum_{j=1}^{N} \left\| S_\delta^{-1/2}(-\Delta_{x_j} + \|z\|_{\mathfrak{H}}^2\|\lambda_{x_j}\|_{\mathfrak{H}}^2)S_\delta^{-1/2} \right\| + 4N\|\lambda.\|_{\infty}^2\|z\|_{\mathfrak{H}}^2(1 + \|z\|_{\mathfrak{H}}^2)^{-\delta} \leqslant C,
\end{aligned}$$

for $\delta = 1$.

□

For product states, the limit measure is a scalar measure and the effective operators, which do not actually depend on the field configurations, can be rewritten as

$$\mathcal{H}_{\text{eff}}(\mu) = -\Delta_x + W(x_1, \ldots, x_N) + \mathcal{V}_\mu(x) \tag{4.13}$$

with

$$\mathcal{V}_\mu(x) = \int_{\mathfrak{H}\cap\mathfrak{H}_\omega} \mathrm{d}\mu(z)\, \mathcal{V}(z). \tag{4.14}$$

Therefore, Theorem 4.1.1 provides a first rigorous justification of the physical approximation of quantized fields by the classical counterparts, *i.e.*, suitable external potentials. Imposing stronger conditions on the interaction parts of the microscopic Hamiltonians, we can deduce more information on the effective potentials obtained in the limit.

Nelson model: if we set the form factor to be

$$\lambda_x(k) = \lambda(k) \sum_{j=1}^{N} e^{-ik\cdot x_j}, \qquad \lambda \in L^2(\mathbb{R}^d; \mathrm{d}k), \tag{4.15}$$

then, the external potential reads

$$\mathcal{V}_\mu(x) = (2\pi)^{d/2} \sum_{j=1}^{N} \int_{\mathfrak{H}\cap\mathfrak{H}_\omega} \mathrm{d}\mu(z)\, 2\Re e \left(\widecheck{z\overline{\lambda}} \right)(x_j) \tag{4.16}$$

where we denoted by $\check{\ }$ the inverse Fourier transform. Since by assumptions and Cauchy-Schwarz inequality, $z\overline{\lambda}$ is in $L^1(\mathbb{R}^d; \mathrm{d}k)$, its inverse Fourier transform belongs to $L^\infty(\mathbb{R}^d; \mathrm{d}x_j)$ for any j. Then, by Riemann-Lebesgue lemma, $\mathcal{V}(x)$ vanishes for $|x| \to +\infty$ and is continuous;

Polaron: the explicit expression yields

$$\mathcal{V}_\mu(x) = (2\pi)^{d/2} \sum_{j=1}^{N} \int_{\mathfrak{H}\cap\mathfrak{H}_\omega} \mathrm{d}\mu(z)\, 2\Re e \left(\widecheck{|k|^{\frac{d-1}{2}} z} \right)(x_j) \tag{4.17}$$

which is infinitesimally form-bounded w.r.t. $-\Delta$, although not necessarily vanishing at infinity;

Pauli-Fierz model: the effective potential is

$$\mathcal{V}_\mu(x) = \sum_{j=1}^{N} \int_{\mathfrak{H}\cap\mathfrak{H}_\omega} \mathrm{d}\mu(z) \left(-i\nabla_{x_j} \cdot 2\Re e \left\langle z \,\middle|\, \lambda_{x_j} \right\rangle_{\mathfrak{H}} + \left(2\Re e \left\langle z \,\middle|\, \lambda_{x_j} \right\rangle_{\mathfrak{H}}\right)^2 \right) \tag{4.18}$$

and one can thus reconstruct a magnetic Schrödinger operator, by completing the square:

$$\begin{aligned}
\mathcal{H}_{\text{eff}} &= -\Delta_x + W + \sum_{j=1}^{N} \int_{\mathfrak{H}\cap\mathfrak{H}_\omega} \mathrm{d}\mu(z) \left(-i\nabla_{x_j} \cdot 2\Re e \left\langle z \,\middle|\, \lambda_{x_j} \right\rangle_{\mathfrak{H}} + \left(2\Re e \left\langle z \,\middle|\, \lambda_{x_j} \right\rangle_{\mathfrak{H}}\right)^2 \right) = \\
&= \sum_{j=1}^{N} \left(-\Delta_{x_j} - i\nabla_{x_j} \cdot \mu(2\Re e \langle z \mid \lambda_{x_j} \rangle) + \mu(2\Re e \langle z \mid \lambda_{x_j} \rangle)^2 \right) + W + \mathbb{V}_\mu(x) = \\
&= \sum_{j=1}^{N} \left(-i\nabla_{x_j} + \mathbf{A}_\mu(x_j) \right)^2 + W + \mathbb{V}_\mu(x),
\end{aligned}$$

where we denoted by

$$\mathbb{V}_\mu(x) = \sum_{j=1}^{N} \Big(\mu\big((2\Re e \big\langle z \,|\, \lambda_{x_j} \big\rangle)^2\big) - \mu(2\Re e \big\langle z \,|\, \lambda_{x_j} \big\rangle)^2 \Big) \tag{4.19}$$

the variance term w.r.t. the measure μ that appears as a correction term and can be interpreted as an additional electric potential, while

$$\mathbf{A}_\mu(y) := \int_{\mathfrak{H} \cap \mathfrak{H}_\omega} \mathrm{d}\mu(z)\, 2\Re e \langle z \,|\, \lambda_y \rangle \tag{4.20}$$

is the magnetic vector potential, which is continuous and vanishing at infinity. As discussed before, the derivation of the effective Hamitlonian above justifies once more the approximation of the action fo the electromagnetic field by means of classical magnetic potentials. A specific example in which the correction term is missing, is when μ concentrates on a point, *i.e.*, it is a Dirac delta: the variance indeed vanishes and $\mathcal{H}_{\mathrm{eff}}$ becomes the usual magnetic Schrödinger operator (see [26, 27] for further details).

The effective potentials $\mathcal{V}_\mu$ and $\mathbf{A}_\mu$ generated in the quasi-classical limit are continuous and vanishing at infinity (Nelson and Pauli-Fierz models) or infinitesimally small perturbations of the kinetic energy (polaron). Such a regularity is due to the assumptions on the states, and, in particular, the uniform boundedness (4.7) of the expectation of the number operator. If we drop such an assumption, more singular potentials (like trapping potentials) can be obtained. For further details, on how to derive harmonic potentials in the Nelson case or a uniform magnetic field from the Pauli-Fierz model see again [26, 27].

4.2 Quasi-classical limit of the ground state energies

We study now the variational properties of the systems under consideration. More precisely, our aim is to show that the ground state energy of the microscopic Nelson model is approximated by the ground state energy of the corresponding effective model, minimized over all possible classical configuration. To this purpose, it will be key the use of the machinery of state-valued measures.

If $\mathcal{H}$ is a self-adjoint Hamiltonian on $\mathscr{D}(\mathcal{H})$, let us denote by

$$\underline{\sigma}(\mathcal{H}) := \inf \sigma(\mathcal{H}) = \inf_{\substack{\psi \in \mathscr{D}(\mathcal{H}) \\ \|\psi\|=1}} \langle \psi |\, \mathcal{H} \,| \psi \rangle \tag{4.21}$$

the bottom of its spectrum. We also denote by $\mathcal{K}(z)$ the operator

$$\mathcal{K}(z) := \mathcal{H}(z) + \langle z |\, \omega \,| z \rangle_{\mathfrak{H}} \tag{4.22}$$

whose Wick quantization is the microscopic Hamiltonian H_ε, *i.e.*,

$$H_\varepsilon = \mathrm{Op}_\varepsilon^{\mathrm{Wick}}(\mathcal{K}). \tag{4.23}$$

The previous results suggest to introduce the effective operators

$$\mathcal{K}_{\mathrm{eff}}(\mathfrak{m}) := \mathrm{Tr}_{L^2} \left(\int_{\mathfrak{H}} \mathrm{d}\mathfrak{m}(z)\, \mathcal{K}(z) \right) = \mathcal{H}_{\mathrm{eff}}(\mathfrak{m}) + c(\mathfrak{m}), \tag{4.24}$$

$$\mathcal{H}_{\mathrm{eff}}(\mathfrak{m}) := \mathrm{Tr}_{L^2} \left(\int_{\mathfrak{H}} \mathrm{d}\mathfrak{m}(z)\, \mathcal{H}(z) \right), \qquad c(\mathfrak{m}) := \int_{\mathfrak{H}} \mathrm{d}\mu(z)\, \|\omega^{1/2} z\|_{\mathfrak{H}}^2. \tag{4.25}$$

Theorem 4.2.1. *Let H_ε be the Nelson Hamiltonian and let assumptions* (A0) *and* (A2) *be satisfied. Then we have*

$$\lim_{\varepsilon\to 0}\underline{\sigma}(H_\varepsilon) = \inf_{\mathfrak{m}\in\mathscr{M}_2(\mathfrak{H}_\omega;\mathscr{L}_1(L^2)_+)} \underline{\sigma}(\mathcal{K}_{\mathrm{eff}}(\mathfrak{m})) \tag{4.26}$$

where

$$\mathscr{M}_2(\mathfrak{H}_\omega;\mathscr{E}'_+) := \{\mathfrak{m}\in\mathscr{M}(\mathfrak{H}_\omega;\mathscr{E}'_+)\ :\ c(\mathfrak{m}) < +\infty\} \tag{4.27}$$

is the space of measures concentrated on $\mathfrak{H}_\omega$ taking values on the positive functionals of the C^*-algebra $\mathscr{E}$, with finite second moment.

The result follows from the following Proposition, which exploits a chain of variational identities. From now on, we denote by

$$\mathcal{E}_z[\,\cdot\,] := \langle\,\cdot\,|\,\mathcal{K}(z)\,|\,\cdot\,\rangle_{\mathscr{H}} \tag{4.28}$$

the quadratic form of the energy with fixed classical configuration of the field.

Proposition 4.2.2. *For any $\mathfrak{m}\in\mathscr{M}(\mathfrak{H};\mathscr{L}_{1,+}(L^2))$,*

$$\inf_{\mathfrak{m}\in\mathscr{M}_2(\mathfrak{H}_\omega;\mathscr{L}_{1,+}(L^2))}\underline{\sigma}(\mathcal{K}_{\mathrm{eff}}(\mathfrak{m})) = \inf_{\mathfrak{m}\in\mathscr{M}_2(\mathfrak{H}_\omega;\mathscr{L}_1(L^2)_+)}\mathrm{Tr}_{L^2}\left(\int_{\mathfrak{H}_\omega}\mathrm{d}\mathfrak{m}(z)\,\mathcal{K}(z)\right) = \tag{4.29}$$

$$= \inf_{\mu\in\mathscr{M}_2(\mathfrak{H}_\omega;\mathbb{R}^+)}\inf_{\|\psi\|=1}\left\langle\psi\,\middle|\,\int_{\mathfrak{H}_\omega}\mathrm{d}\mu(z)\,\mathcal{K}(z)\,\middle|\,\psi\right\rangle = \tag{4.30}$$

$$= \inf_{z\in\mathfrak{H}_\omega}\inf_{\|\psi\|=1}\mathcal{E}_z[\psi] = \inf_{\|\psi\|=1}\inf_{z\in\mathfrak{H}_\omega}\mathcal{E}_z[\psi]. \tag{4.31}$$

Proof. By [26, Lemma 3.20], we have the following equivalences (setting $\mathscr{M}_2 = \mathscr{M}_2(\mathfrak{H}_\omega;\mathbb{R}^+)$ for short):

$$\inf_{\mathfrak{m}\in\mathscr{M}_2(\mathfrak{H}_\omega;\mathscr{L}_{1,+}(L^2))}\mathrm{Tr}_{L^2}\left(\int_{\mathfrak{H}_\omega}\mathrm{d}\mathfrak{m}(z)\,\mathcal{K}(z)\right) = \inf_{\mathfrak{m}\in\mathscr{M}_{\mathrm{fin}}}\mathrm{Tr}_{L^2}\left(\int_{\mathfrak{H}_\omega}\mathrm{d}\mathfrak{m}(z)\,\mathcal{K}(z)\right); \tag{4.32}$$

$$\inf_{\mu\in\mathscr{M}_2}\inf_{\|\psi\|=1}\left\langle\psi\,\middle|\,\int_{\mathfrak{H}_\omega}\mathrm{d}\mathfrak{m}(z)\,\mathcal{K}(z)\,\middle|\,\psi\right\rangle = \inf_{\mu\in\mathscr{M}_{\mathrm{fin}}(\mathfrak{H}_\omega;\mathbb{R}^+)}\inf_{\|\psi\|=1}\left\langle\psi\,\middle|\,\int_{\mathfrak{H}_\omega}\mathrm{d}\mathfrak{m}(z)\,\mathcal{K}(z)\,\middle|\,\psi\right\rangle; \tag{4.33}$$

where $\mathscr{M}_{\mathrm{fin}}$ is the space of atomic measures, *i.e.*, measures concentrated on a finite set of points:

$$\mathfrak{m}\in\mathscr{M}_{\mathrm{fin}}\subseteq\mathscr{M}(\mathfrak{H}_\omega;\mathscr{L}_{1,+}(L^2)) \;\overset{\cdot}{\Longleftrightarrow}\; \mathfrak{m}(z) = \sum_{j\in I}\alpha_j\gamma_j(z)\delta(z-z_j), \tag{4.34}$$

with I a finite subset of $\mathbb{N}$, $\{z_j\}_{j\in\mathbb{N}}\subseteq\mathfrak{H}_\omega$, $\{\gamma_j(z)\}_{j\in I}\subseteq\mathscr{L}_1(L^2)_+$, $\sum_{j\in I}\alpha_j = 1$, $\alpha_j\geqslant 0$ and $\|\gamma_j(z)\|_{\mathscr{L}_1} = 1$, for any $j\in I$. Similarly, $\mathscr{M}_{\mathrm{fin}}(\mathfrak{H}_\omega;\mathbb{R}^+)$ stands for the space of the atomic scalar measures, *i.e.*,

$$\mu\in\mathscr{M}_{\mathrm{fin}}(\mathfrak{H}_\omega;\mathbb{R}^+)\subseteq\mathscr{M}(\mathfrak{H}_\omega;\mathbb{R}^+) \;\overset{\cdot}{\Longleftrightarrow}\; \mathfrak{m}(z) = \sum_{j\in I}\alpha_j\delta(z-z_j), \tag{4.35}$$

with analogous conditions on $\{z_j\}_{j\in I}$ and $\{\alpha_j\}_{j\in I}$.

Now, for any $\delta>0$, there exists $\mathfrak{m}_\delta\in\mathscr{M}_{\mathrm{fin}}$, such that

$$\inf_{\mathfrak{m}\in\mathscr{M}_{\mathrm{fin}}}\mathrm{Tr}_{L^2}\left(\int_{\mathfrak{H}_\omega}\mathrm{d}\mathfrak{m}(z)\,\mathcal{K}(z)\right)+\delta > \mathrm{Tr}_{L^2}\left(\int_{\mathfrak{H}_\omega}\mathrm{d}\mathfrak{m}_\delta(z)\,\mathcal{K}(z)\right) = \sum_{j\in I}\alpha_j^{(\delta)}\mathrm{Tr}_{L^2}(\gamma_j^{(\delta)}(z_j)\mathcal{K}(z_j)); \tag{4.36}$$

and, since $\gamma_j^{(\delta)}$ are normalized density matrices,

$$\sum_{j\in I}\alpha_j \mathrm{Tr}_{L^2}(\gamma_j^{(\delta)}(z_j)\mathcal{K}(z_j)) \geqslant \inf_{\|\psi\|=1}\sum_{j\in I}\alpha_j^{(\delta)}\langle\psi\,|\,\mathcal{K}(z_j)\,|\,\psi\rangle. \tag{4.37}$$

Minimizing over the scalar atomic measures, we obtain

$$\inf_{\mu\in\mathscr{M}_{\mathrm{fin}}(\mathfrak{H}_\omega;\mathbb{R}^+)}\inf_{\|\psi\|=1}\left\langle\psi\,\middle|\int_{\mathfrak{H}_\omega}\mathrm{d}\mu(z)\,\mathcal{K}(z)\middle|\,\psi\right\rangle = \inf_{\mu\in\mathscr{M}_{\mathrm{fin}}(\mathfrak{H}_\omega;\mathbb{R}^+)}\inf_{\|\psi\|=1}\sum_{j\in I}\alpha_j\langle\psi\,|\,\mathcal{K}(z_j)\,|\,\psi\rangle <$$

$$< \inf_{\mathfrak{m}\in\mathscr{M}_{\mathrm{fin}}}\mathrm{Tr}_{L^2}\left(\int_{\mathfrak{H}_\omega}\mathrm{d}\mathfrak{m}(z)\,\mathcal{K}(z)\right)+\delta.$$

Since this is true for every $\delta>0$, equivalences (4.32) imply that

$$\inf_{\mathfrak{m}\in\mathscr{M}_2(\mathfrak{H}_\omega;\mathscr{L}_1(L^2)_+)}\mathrm{Tr}_{L^2}\left(\int_{\mathfrak{H}_\omega}\mathrm{d}\mathfrak{m}(z)\,\mathcal{K}(z)\right)\geqslant\inf_{\mu\in\mathscr{M}_2(\mathfrak{H}_\omega;\mathbb{R}^+)}\inf_{\|\psi\|=1}\left\langle\psi\,\middle|\int_{\mathfrak{H}_\omega}\mathrm{d}\mu(z)\,\mathcal{K}(z)\middle|\,\psi\right\rangle. \tag{4.38}$$

To prove the opposite inequality we proceed as follows: for any $\delta>0$ there exists $\psi_\delta\in L^2(\mathbb{R}^{dN})$, $\mu_\delta=\sum_{j\in I}\alpha_j^{(\delta)}\delta_{z_j^{(\delta)}}$ such that

$$\inf_{\mu\in\mathscr{M}_{\mathrm{fin}}(\mathfrak{H}_\omega;\mathbb{R}^+)}\inf_{\|\psi\|=1}\left\langle\psi\,\middle|\int_{\mathfrak{H}_\omega}\mathrm{d}\mu(z)\,\mathcal{K}(z)\middle|\,\psi\right\rangle+\delta>\sum_{j\in I}\alpha_j^{(\delta)}\left\langle\psi_\delta(z_j^{(\delta)})\,\middle|\,\mathcal{K}(z_j)\,\middle|\,\psi_\delta(z_j^{(\delta)})\right\rangle\geqslant$$

$$\geqslant\inf_{\mathfrak{m}\in\mathscr{M}_{\mathrm{fin}}}\mathrm{Tr}_{L^2}\left(\int_{\mathfrak{H}_\omega}\mathrm{d}\mathfrak{m}(z)\,\mathcal{K}(z)\right)$$

where we used the fact that $\mathfrak{m}_\delta:=\sum_{j\in I}\alpha_j^{(\delta)}\,|\psi_\delta(z)\rangle\langle\psi_\delta(z)|\,\delta_{z_j^{(\delta)}}(z)$ is an atomic measure. Then, by arbitrariness of δ, and by the inequality proved above, we conclude the proof of equivalence (4.29).

To prove (4.30), let us consider again the atomic state-valued measure

$$\mathfrak{m}_\delta=\sum_{j\in I}\alpha_j^{(\delta)}\,|\psi_\delta(z)\rangle\langle\psi_\delta(z)|\,\delta_{z_j^{(\delta)}}$$

defined above, and we have that

$$\inf_{\mu\in\mathscr{M}_{\mathrm{fin}}}\inf_{\|\psi\|=1}\left\langle\psi\,\middle|\int_{\mathfrak{H}_\omega}\mathrm{d}\mu(z)\,\mathcal{K}(z)\middle|\,\psi\right\rangle+\delta>\sum_{j\in I}\alpha_j^{(\delta)}\left\langle\psi_\delta(z_j^{(\delta)})\,\middle|\,\mathcal{K}(z_j^{(\delta)})\,\middle|\,\psi_\delta(z_j^{(\delta)})\right\rangle\geqslant$$

$$\geqslant\sum_{j\in I}\alpha_j^{(\delta)}\inf_{\|\psi\|=1}\langle\psi\,|\,\mathcal{K}(z_j^{(\delta)})\,|\,\psi\rangle>\inf_{z\in\mathfrak{H}_\omega}\inf_{\|\psi\|=1}\langle\psi\,|\,\mathcal{K}(z)\,|\,\psi\rangle$$

where, in the last inequality, we used the fact that the term on the left is a convex combination and then strictly greater than the infimum.

In order to get the opposite inequality we write

$$\inf_{z\in\mathfrak{H}_\omega}\inf_{\|\psi\|=1}\langle\psi\,|\,\mathcal{K}(z)\,|\,\psi\rangle=\inf_{z\in\mathfrak{H}_\omega}\inf_{\|\psi\|=1}\mathrm{Tr}_{L^2}\left(\int_{\mathfrak{H}_\omega}\mathrm{d}\delta_z(y)\ |\psi\rangle\langle\psi|\,\mathcal{K}(y)\right)\geqslant$$

$$\geqslant\inf_{\mu\in\mathscr{M}_{\mathrm{fin}}(\mathfrak{H}_\omega;\mathbb{R}^+)}\inf_{\|\psi\|=1}\left\langle\psi\,\middle|\int_{\mathfrak{H}_\omega}\mathrm{d}\mu(y)\,\mathcal{K}(y)\middle|\,\psi\right\rangle$$

and, recalling that $\mathcal{E}_z[\,\cdot\,]=\langle\cdot\,|\,\mathcal{K}(z)\,|\,\cdot\rangle$, we finally obtain

$$\inf_{z\in\mathfrak{H}_\omega}\inf_{\|\psi\|=1}\mathcal{E}_z[\psi]\geqslant\inf_{\mu\in\mathscr{M}_{\mathrm{fin}}(\mathfrak{H}_\omega;\mathbb{R}^+)}\inf_{\|\psi\|=1}\left\langle\psi\,\middle|\int_{\mathfrak{H}_\omega}\mathrm{d}\mu(y)\,\mathcal{K}(y)\middle|\,\psi\right\rangle \tag{4.39}$$

which completes the proof of (4.30).

Finally, (4.31) is a trivial consequence of the fact that the infima can be always exchanged, whenever the two minimizing domains are independent. □

Before proving the Theorem, we have to show that the field energy is uniformly bounded over a minimizing sequence. This will allow us to exploit the results about quasi-classical convergence proved in the previous chapter.

Lemma 4.2.3. *Let us consider a normalized, minimizing sequence $\Psi_{\varepsilon,n} \subseteq \mathscr{D}(H_\varepsilon)$ for $\underline{\sigma}(H_\varepsilon)$. Under the assumptions of Lemma 4.1.2, there exists δ', such that $\delta' > \delta > 0$ and*

$$\langle \Psi_{\varepsilon,n} | \, H_0 + (\mathrm{d}\Gamma_\varepsilon(\mathbb{1}) + 1)^{\delta'} \, | \Psi_{\varepsilon,n} \rangle \leqslant C. \tag{4.40}$$

Proof. By Theorem 1.1.2, H_ε is self-adjoint on $\mathscr{D}(H_\varepsilon) = \mathscr{D}(-\Delta_x + W_+) \cap \mathscr{D}(\mathrm{d}\Gamma_\varepsilon(\omega))$, and, for any minimizing sequence, it holds $\|H_\varepsilon \Psi_{\varepsilon,n}\| \leqslant C \in (0, +\infty)$, uniformly in ε, n. By Kato-Rellich theorem there exist $\alpha \in (0,1), \beta \in [0, +\infty)$, such that

$$\begin{aligned} \langle \Psi_{\varepsilon,n} | \, H_0 \, | \Psi_{\varepsilon,n} \rangle \leqslant \|H_0 \Psi_{\varepsilon,n}\| &\leqslant \|H_\varepsilon \Psi_{\varepsilon,n}\| + \|H_I \Psi_{\varepsilon,n}\| \leqslant \\ &\leqslant C + \alpha \|H_0 \Psi_{\varepsilon,n}\| + \beta \end{aligned}$$

which gives a uniform bound on the free part of the Hamiltonian. In the massive Nelson model, however, the number operator is controlled by the free energy of the field, *i.e.*, $m\mathrm{d}\Gamma_\varepsilon(\mathbb{1}) \leqslant \mathrm{d}\Gamma_\varepsilon(\omega)$, and therefore (4.40) is satisfied for $\delta' = 1 > 1/2 = \delta$.

□

We are now ready to prove the convergence of the ground state energies.

Proof of Theorem 4.2.1 . We split the proof into two parts.

($\leqslant$) The proof of the upper bound is done by using coherent states, *i.e.*,

$$\Xi_\varepsilon(\xi) = W_\varepsilon \left(\frac{\xi}{i\varepsilon} \right) \Omega, \qquad \xi \in \mathfrak{H} \tag{4.41}$$

with $\Omega = (1, 0, 0, \ldots) \in \Gamma_s(\mathfrak{H})$ the vacuum in the Fock space. Then, taking $\psi \in \mathscr{Q}(-\Delta_x + W)$ and $z \in \mathfrak{H}_\omega$, by direct calculation and using the properties of the Weyl operators,

$$\langle \psi \otimes \Xi_\varepsilon(z) | \, H_\varepsilon \, | \psi \otimes \Xi_\varepsilon(z) \rangle = \langle \psi | \, \mathcal{K}(\xi) \, | \psi \rangle . \tag{4.42}$$

Hence,

$$\begin{aligned} \limsup_{\varepsilon \to 0} \underline{\sigma}(H_\varepsilon) &\leqslant \limsup_{\varepsilon \to 0} \inf_{\|\psi\|=1} \inf_{z \in \mathfrak{H}_\omega} \langle \psi \otimes \Xi_\varepsilon(z) | \, H_\varepsilon \, | \psi \otimes \Xi_\varepsilon(z) \rangle = \\ &= \inf_{\|\psi\|=1} \inf_{z \in \mathfrak{H}_\omega} \langle \psi | \, \mathcal{K}(z) \, | \psi \rangle = \inf_{\|\psi\|=1} \inf_{z \in \mathfrak{H}_\omega} \mathcal{E}_z[\psi] = \underline{\sigma}(\mathcal{K}_{\mathrm{eff}}(\mathfrak{m})), \end{aligned}$$

where the last identities follow from Proposition 4.2.2.

($\geqslant$) For the lower bound, let $\{\Psi_{\varepsilon,n}\}_{n \in \mathbb{N}} \subseteq \mathscr{D}(H_\varepsilon)$ be a normalized minimizing sequence for the microscopic Hamiltonian, *i.e.*, for any $n \in \mathbb{N}$,

$$\langle \Psi_{\varepsilon,n} | \, H_\varepsilon \, | \Psi_{\varepsilon,n} \rangle \leqslant \underline{\sigma}(H_\varepsilon) + \frac{1}{n}. \tag{4.43}$$

Now, we want to bound from below the quantity on the left-hand side by an operator whose expectation admits an explicit limit: due to a possible lack of compactness, indeed, we only know that the expectation of the field energy is uniformly bounded, but this does not imply that it converges to the average of the classical field energy. To overcome such a difficulty, we exploit a result of pseudodifferential calculus implying the existence of a smooth, cylindrical, compactly supported symbol $b_r \in C^{\infty}_{0,\mathrm{cyl}}(\mathfrak{H}_\omega)$, such that

$$\left|\mathrm{Op}^{\mathrm{Weyl}}_{\varepsilon}(b_r) - \mathrm{d}\Gamma_{\varepsilon}(\omega)\right| \leqslant C\varepsilon \mathrm{d}\Gamma_{\varepsilon}(\omega) \tag{4.44}$$

and, pointwise in $\mathfrak{H}_\omega$, $\lim_{r\to+\infty} b_r(z) = \langle z|\,\omega\,|z\rangle$. Using the previous inequality we can write

$$\langle \Psi_{\varepsilon,n}| -\Delta_x + W + \mathrm{Op}^{\mathrm{Weyl}}_{\varepsilon}(b_r) + H_I\,|\Psi_{\varepsilon,n}\rangle - \varepsilon C \langle \Psi_{\varepsilon,n}|\, \mathrm{d}\Gamma_{\varepsilon}(\omega)\,|\Psi_{\varepsilon,n}\rangle \leqslant \langle \Psi_{\varepsilon,n}|\, H_\varepsilon\,|\Psi_{\varepsilon,n}\rangle . \tag{4.45}$$

Now, we take the $\liminf$ over a subsequence ε_k, which we omit for simplicity, of the previous expression. By Lemma 4.2.3 and bound (4.40), we are in condition to use Theorem 3.1.6 and Theorem 3.1.10 to deduce the existence of a sequence of state-valued measures $\{\mathfrak{m}_n\}_{n\in\mathbb{N}} \subseteq \mathscr{M}(\mathfrak{H}\cap\mathfrak{H}_\omega; \mathscr{L}_1(L^2(\mathbb{R}^{dN}))$, such that, for any $n \in \mathbb{N}$, there is a subsequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$, depending on n, so that

$$|\Psi_{\varepsilon_k,n}\rangle\langle\Psi_{\varepsilon_k,n}| \to \mathfrak{m}_n. \tag{4.46}$$

Let now $\{\mathbb{1}_m\}_{m\in\mathbb{N}}$ be a sequence of operators with $\mathbb{1}_m := \mathbb{1}_m(-\Delta_x + W_+) \in \mathcal{K}$ converging strongly to the identity. Note that the sequence is non-decreasing and the operators are compact because $-\Delta + W_+$ has discrete spectrum, due to the compactness of the resolvent. Then, we have that, for any $m \in \mathbb{N}$,

$$\langle \Psi_{\varepsilon,n}|\, \mathbb{1}_m(-\Delta_x + W)\mathbb{1}_m + H_I + \mathrm{Op}^{\mathrm{Weyl}}_{\varepsilon}(b_r)\,|\Psi_{\varepsilon,n}\rangle \leqslant \langle \Psi_{\varepsilon,n}| -\Delta_x + W + \mathrm{Op}^{\mathrm{Weyl}}_{\varepsilon}(b_r) + H_I\,|\Psi_{\varepsilon,n}\rangle .$$

We now plug this inequality into (4.45) and take the $\liminf$ of the obtained expression.

Thanks to Lemma 4.2.3, Theorem 3.2.1 and Theorem 3.3.1 and, since $\mathbb{1}_m \in \mathcal{K}$,

$$\lim_{k\to+\infty} \langle \Psi_{\varepsilon_k,n}|\, \mathbb{1}_m(-\Delta_x + W)\mathbb{1}_m + H_I\,|\Psi_{\varepsilon_k,n}\rangle = \mathrm{Tr}_{L^2}\left(\int_{\mathfrak{H}_\omega} \mathrm{d}\mathfrak{m}_n(z)\mathbb{1}_m(-\Delta_x + W)\mathbb{1}_m + \mathcal{V}(z)\right). \tag{4.47}$$

By Theorem 3.1.6 for quantized cylindrical symbols we also have that

$$\lim_{k\to+\infty} \langle \Psi_{\varepsilon_k,n}|\, \mathrm{Op}^{\mathrm{Weyl}}_{\varepsilon_k}(b_r)\,|\Psi_{\varepsilon_k,n}\rangle = \mathrm{Tr}_{L^2}\left(\int_{\mathfrak{H}_\omega} \mathrm{d}\mathfrak{m}(z)\, b_r(z)\right) \tag{4.48}$$

so that, putting together the previous inequalities, we get

$$\begin{aligned}
&\liminf_{k\to+\infty} \langle \Psi_{\varepsilon_k,n}|\, \mathbb{1}_m(-\Delta_x + W)\mathbb{1}_m + H_I + \mathrm{Op}^{\mathrm{Weyl}}_{\varepsilon_k}(b_r)\,|\Psi_{\varepsilon_k,n}\rangle = \\
&= \mathrm{Tr}_{\mathscr{H}}\left(\int_{\mathfrak{H}_\omega} \mathrm{d}\mathfrak{m}_n(z)\big(\mathbb{1}_m(-\Delta_x + W)\mathbb{1}_m + \mathcal{V}(z) + b_r(z)\big)\right) \leqslant \liminf_{k\to+\infty} \langle \Psi_{\varepsilon_k,n}|\, H_{\varepsilon_k}\,|\Psi_{\varepsilon_k,n}\rangle .
\end{aligned} \tag{4.49}$$

which gives a uniform bound w.r.t. both r and m. By dominated convergence, we can take the limits $r, m \to +\infty$ and obtain

$$\mathrm{Tr}_{\mathscr{H}}\left(\int_{\mathfrak{H}_\omega} \mathrm{d}\mathfrak{m}_n(z)\mathcal{K}(z)\right) \leqslant \liminf_{k\to+\infty} \langle \Psi_{\varepsilon_k,n}|\, H_{\varepsilon_k}\,|\Psi_{\varepsilon_k,n}\rangle . \tag{4.50}$$

Finally, the infimum of both sides yields the inequality

$$\begin{aligned}
\inf_{\mathfrak{m}\in\mathscr{M}_2(\mathfrak{H}_\omega;\mathscr{L}_1(L^2)_+)} \underline{\sigma}(\mathcal{K}_{\mathrm{eff}}(\mathfrak{m})) &\leqslant \inf_{\mathfrak{m}_n:\Psi_{\varepsilon,n}\to\mathfrak{m}_n} \mathrm{Tr}_{\mathscr{H}}\left(\int_{\mathfrak{H}_\omega} \mathrm{d}\mathfrak{m}_n(z)\mathcal{K}(z)\right) \leqslant \\
&\leqslant \liminf_{k\to+\infty} \langle \Psi_{\varepsilon_k,n}|\, H_{\varepsilon_k}\,|\Psi_{\varepsilon_k,n}\rangle \leqslant \liminf_{k\to+\infty} \underline{\sigma}(H_{\varepsilon_k}) + \frac{1}{n},
\end{aligned}$$

which, since it holds for every $n \in \mathbb{N}$, concludes the proof.

□

5 Quasi-classical Limit: Dynamics

In this chapter, we derive the effective dynamics, of the state-valued measure, obtained in the quasi-classical limit of the microscopic system.

The strategy consists in writing an integral equation for the density matrix and, then, applying the results of Chapter 3 on the convergence of quantum expectations to classical ones. In this respect, the estimates in time proved in Section 5.1 play a key role, allowing to derive an equation for the Fourier transform of the measure. In Section 5.2, it is proven the existence of a common subsequence along which there is convergence at all times and that such subsequence coincides with the one singled out at initial time. In Section 5.3, existence and uniqueness of the solution of the effective equation for the measure are proven, and this solution is characterized as the time evolution via the dynamics generated by an effective Hamiltonian, averaged over all the classical configurations of the field.

The main result is stated in Theorem 1.3.1 that we reproduce below for convenience of the reader. The proof is a consequence of various Lemmas and Propositions, that will be proved in the rest of the Chapter and that we will recall below.

We recall also the assumptions: setting

$$H_+ := -\Delta_x + W_+ + \mathrm{d}\Gamma_\varepsilon(\omega), \tag{5.1}$$

- the form factors are

$$\begin{cases} \lambda_x(k) = \chi(k)\sum_{j=1}^N e^{ikx_j}, \quad \text{with } \chi \in L^2(\mathbb{R}^d;\mathrm{d}k), & \text{for the Nelson model;} \\ g_{x_j}(k) = \frac{e^{iky}}{|k|^{\frac{d-1}{2}}}\chi_{\{|k|<r\}}(k) + \left[-i\nabla_y, \frac{k}{|k|^{\frac{d+3}{2}}}e^{iky}\chi_{\{|k|\geqslant r\}}(k)\right], & \text{for the polaron;} \\ \lambda_{x_j}(k) = \left(e_\gamma(k)\chi(k)e^{ikx_j}\right)_{\gamma=1}^{d-1}, \quad \text{with } \chi \in L^2(\mathbb{R}^d;\mathrm{d}k), & \text{for the Pauli-Fierz model;} \end{cases} \tag{A0}$$

- for any $\delta > 0$,

$$\begin{cases} \sup_{\varepsilon\in(0,1)} \mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon(-\Delta_x + W_+ + (\mathrm{d}\Gamma_\varepsilon(\mathbb{1})+1)^\delta) \leqslant C, & \text{for the Nelson model and polaron;} \\ \sup_{\varepsilon\in(0,1)} \mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon(H_+ + (\mathrm{d}\Gamma_\varepsilon(\mathbb{1})+1)^\delta)) \leqslant C, & \text{for the Pauli-Fierz model.} \end{cases} \tag{A1}$$

- the resolvent of $-\Delta_x + W_+$ is compact: for any $z \in \varrho(-\Delta_x + W_+)$,

$$(-\Delta_x + W_+ - z)^{-1} \in \mathscr{L}_\infty(L^2(\mathbb{R}^{dN})). \tag{A2}$$

Theorem. *Let* $\{\rho_\varepsilon\}_{\varepsilon\in(0,1)} \subseteq \mathscr{L}_{1,+}(\mathscr{H})$ *be a family of normalized states satisfying assumptions* (A1) *and* (A2)*. Let also*

$$\rho_\varepsilon(t) = e^{-itH_\varepsilon}\rho_\varepsilon e^{itH_\varepsilon}, \tag{5.2}$$

be their time evolution generated either by the Nelson, polaron or Pauli-Fierz Hamiltonian, satisfying assumption (A0) *with* $\lim_{\varepsilon\to 0}\varepsilon\nu_\varepsilon = \nu \in \{0,1\}$*. Then, there exists a subsequence* $\{\varepsilon_n\}_{n\in\mathbb{N}}$ *and a state-valued measure* $\mathfrak{m}$*, such that*

$$\rho_{\varepsilon_n} \xrightarrow[n\to+\infty]{} \mathfrak{m} \in \mathscr{M}(\mathfrak{H};\mathscr{L}_{1,+}(L^2(\mathbb{R}^{dN}))), \quad \text{with} \quad \mathrm{d}\mathfrak{m}(z) = \gamma_{\mathfrak{m}}(z)\mathrm{d}\mu(z), \tag{5.3}$$

and

$$\rho_{\varepsilon_n}(t) \xrightarrow[n\to+\infty]{} \mathrm{d}\mathfrak{m}_t(z) \in \mathscr{M}(\mathfrak{H};\mathscr{L}_{1,+}(L^2(\mathbb{R}^{dN}))), \quad \text{with} \quad \mathrm{d}\mathfrak{m}(z) = \gamma_{\mathfrak{m}_t}(z)\mathrm{d}\mu_t(z), \tag{5.4}$$

for any $t\in\mathbb{R}$*, where the family of state-valued measures* $\{\mathfrak{m}_t\}_{t\in\mathbb{R}}$ *is given by*

$$\mu_t = \Phi_t \sharp \mu, \qquad \gamma_{\mathfrak{m}_t}(z) = U_{t,0}(z)\,\gamma_{\mathfrak{m}}(z)\,U_{t,0}^\dagger(z), \tag{5.5}$$

with $\Phi_t z = e^{-it\nu\omega}z$ *and* $U_{t,s}(z)$ *the two-parameter group of unitary operators weakly generated by* $\mathcal{H}(\Phi_t z) = -\Delta_x + W + \mathcal{V}(\Phi_\tau z)$*. Furthermore, the previous convergence holds true also in weak sense, i.e.,*

$$\lim_{n\to+\infty} \mathrm{Tr}_{\mathscr{H}}(\rho_{\varepsilon_n}(t)\, B\otimes W_{\varepsilon_n}(\xi)) = \mathrm{Tr}_{L^2}\left(\int_{\mathfrak{H}} \mathrm{d}\mathfrak{m}_t(z)\, B\, e^{2i\Re e\langle \xi\,|\,z\rangle}\right), \tag{5.6}$$

for any $\xi\in\mathfrak{H}$ *and* $B\in\mathscr{L}(L^2(\mathbb{R}^{dN}))$*.*

Proof. We prove the statement of the Theorem initially for a smaller class of states, with additional assumptions, and then complete the proof by a density argument.

Let then $\{\rho_\varepsilon\}_{\varepsilon\in(0,1)}$ be a family of positive and normalized states, such that, for any $\delta\geqslant 1$,

$$\begin{cases} \|(\rho_\varepsilon\,(-\Delta_x + W_+ + (\mathrm{d}\Gamma_\varepsilon(\mathbb{1})+1)^\delta)\|_{\mathscr{L}_1} \leqslant C, & \text{for the Nelson model,}\\ \mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon\,(H_+ + \nu_\varepsilon^{-2}H_\varepsilon^2)) \leqslant C, & \text{for the polaron,}\\ \mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon\,((H_+ +1)^2 + \mathrm{d}\Gamma_\varepsilon(\mathbb{1})^2)) \leqslant C, & \text{for the Pauli-Fierz model,} \end{cases} \tag{A_R}$$

and let $\{\varepsilon_n\}_{n\in\mathbb{N}}$, with $\varepsilon_n\to 0$, be a suitable subsequence. Then,

- by Theorem 3.2.1 and Corollary 3.2.3, $\rho_{\varepsilon_n}\to\mathfrak{m}$, and $\mathfrak{m}$ concentrates as a probability state-valued measure over $\mathfrak{H}$, *i.e.*,

$$\|\mathfrak{m}(\mathfrak{H})\|_{\mathscr{L}_1(L^2)} = 1, \tag{5.7}$$

  and there is no loss of mass in the limit;

- by Propositions 5.1.2, 5.1.4 and 5.1.7, and assumptions (A_R), we have uniform bounds w.r.t. ε of expectations of the particle Hamiltonian and of a power strictly larger than one of the number operator at any time $t\in\mathbb{R}$, *i.e.*, for some $\delta>0$

$$\begin{cases} \|\rho_\varepsilon(t)\,(-\Delta_x + W_+ + (\mathrm{d}\Gamma_\varepsilon(\mathbb{1})+1)^\delta)\|_{\mathscr{L}_1} \leqslant C\,(e^{c_\delta|t|}+1), & \text{for the Nelson model;}\\ \mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon(t)\,(-\Delta_x + W_+ + \mathrm{d}\Gamma_\varepsilon(\mathbb{1})^2)) \leqslant C((|t|+1)e^{|t|}+1), & \text{for the polaron;}\\ \mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon(t)\,(-\Delta_x + W_+ + \mathrm{d}\Gamma_\varepsilon(\mathbb{1})^2)) \leqslant C(e^{|t|}+|t|+1), & \text{for the Pauli-Fierz model;} \end{cases} \tag{5.8}$$

- by Proposition 5.1.8, we can extract subsequences $\{\varepsilon_{n_j(t)}\}_{n\in\mathbb{N}}$ at any time, such that $\rho_{\varepsilon_{n_j(t)}}(t) \to \mathfrak{m}_t$, with $\mathfrak{m}_t$ a probability state-valued measure over $\mathfrak{H}$, *i.e.*,

$$\|\mathfrak{m}_t(\mathfrak{H})\|_{\mathscr{L}_1(L^2)} = 1, \tag{5.9}$$

  because the mass is conserved by the dynamics;

- by Corollary 5.2.4, we can extract at least one subsequence $\{\varepsilon_{n_k}\}_{k\in\mathbb{N}}$, common to all the times $t \in \mathbb{R}$, such that $\rho_{\varepsilon_{n_k}}(t) \to \mathfrak{m}_t$;

- by Theorem 5.3.4, every measure $\mathfrak{m}_t$, which is a cluster point of ρ_ε, has to satisfy the integral equation in interaction representation

$$\mathrm{Tr}_{L^2}\left(\int_{\mathfrak{H}} \mathrm{d}\tilde{\mathfrak{m}}_t(z)\, f(z)\,\kappa\right) = \mathrm{Tr}_{L^2}\left(\int_{\mathfrak{H}} \mathrm{d}\tilde{\mathfrak{m}}_s(z)\, f(z)\,\kappa\right) - i\int_s^t \mathrm{d}\tau\, \mathrm{Tr}_{L^2}\left(\int_{\mathfrak{H}} [\tilde{\mathcal{V}}_\tau(\Phi_\tau z),\, \mathrm{d}\tilde{\mathfrak{m}}_\tau(z)]\, f(z)\,\kappa\right)$$

  for any $f \in C^\infty_{0,\mathrm{cyl}}(\mathfrak{H})$ and $\kappa \in \mathscr{L}_\infty(L^2)$ or $\mathcal{K}$, depending on the model;

- Proposition 5.3.7 shows that there exists a unique solution of the equation above, which gives, thanks to Corollary 5.3.8, the following expression for $\mathfrak{m}_t$

$$\mu_t = \Phi_t \sharp \mu, \qquad \gamma_{\mathfrak{m}_t}(z) = U_{t,0}(z)\, \gamma_{\mathfrak{m}}(z)\, U^\dagger_{t,0}(z).$$

  This implies that every convergent subsequence of $\rho_{\varepsilon_n}(t)$ converges to $\mathfrak{m}_t$, *i.e.*,

$$\rho_{\varepsilon_n}(t) \to \mathfrak{m}_t; \tag{5.10}$$

- in Section 5.4, we extend the results by density to states satisfying assumption (A1);

- finally, by the bounds (5.8), we can apply Theorem 3.2.1 to density matrices at time t and deduce the convergence (5.10) against bounded test operators.

□

5.1 Propagation of technical estimates

In this section we discuss some useful technical estimates that allow to propagate uniform bounds on quantum states at initial time. Such bounds will be crucial to ensure the existence at any time of at least one Wigner measure associated with the quantum state. The estimates depend on the model and the scaling of the field energy, so we discuss them separately.

Let

$$\rho_\varepsilon(t) := e^{-itH_\varepsilon}\rho_\varepsilon\, e^{itH_\varepsilon} \tag{5.11}$$

be the time evolution for a state $\rho_\varepsilon \in \mathscr{L}_{1,+}(\mathscr{H})$. We observe that $\mathscr{D}(H_+) = \mathscr{D}(H_0)$, for fixed ε and for any scaling ν_ε, where H_+ is defined in (5.1) and we recall that $H_0 = -\Delta_x + W + \nu_\varepsilon \mathrm{d}\Gamma_\varepsilon(\omega)$ is the unperturbed Hamiltonian.

5.1.1 Nelson model

Let us recall the expression of the Nelson Hamiltonian:

$$H_\varepsilon = -\Delta_x + W(x_1, \ldots, x_N) + \nu_\varepsilon \mathrm{d}\Gamma_\varepsilon(\omega) + a_\varepsilon(\lambda_x) + a_\varepsilon^\dagger(\lambda_x).$$

Lemma 5.1.1. *[38, Proposition IV.2] For any $N \in \mathbb{N}_*$, $\varepsilon > 0$, $t \in \mathbb{R}$ and $\delta \in \mathbb{R}$,*

$$\mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon(t)\,(\mathrm{d}\Gamma_\varepsilon(\mathbb{1}) + N^2 + \varepsilon)^\delta) \leqslant e^{m_{\delta/2}(\varepsilon)\sqrt{\varepsilon}|\delta||t|\|\lambda_\cdot\|_{L^\infty(\mathbb{R}^{dN};\mathfrak{H})}} \mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon\,(\mathrm{d}\Gamma_\varepsilon(\mathbb{1}) + N^2 + \varepsilon)^\delta), \quad (5.12)$$

$$\|\rho_\varepsilon(t)\,(\mathrm{d}\Gamma_\varepsilon(\mathbb{1}) + N^2 + \varepsilon)^\delta\|_{\mathscr{L}_1} \leqslant e^{m_\delta(\varepsilon)\sqrt{\varepsilon}|\delta||t|\|\lambda_\cdot\|_{L^\infty(\mathbb{R}^{dN};\mathfrak{H})}} \|\rho_\varepsilon\,(\mathrm{d}\Gamma_\varepsilon(\mathbb{1}) + N^2 + \varepsilon)^\delta\|_{\mathscr{L}_1}, \quad (5.13)$$

where $m_\delta(\varepsilon) := \max\{2+\varepsilon, 1 + (1+\varepsilon)^\delta\}$.

Thanks to the previous result, we can easily derive an estimate of the expectation of powers of the number operator at any time.

Proposition 5.1.2. *For any $\delta > 0$, there exists $c \in (0, +\infty)$, such that, for any $t \in \mathbb{R}$,*

$$\|\rho_\varepsilon(t)\,(-\Delta_x + W_+ + (\mathrm{d}\Gamma_\varepsilon(\mathbb{1}) + 1)^\delta)\|_{\mathscr{L}_1} \leqslant C(e^{c|t|} + 1)\|\rho_\varepsilon\,(-\Delta_x + W_+ + (\mathrm{d}\Gamma_\varepsilon(\mathbb{1}) + 1)^\delta)\|_{\mathscr{L}_1}. \quad (5.14)$$

Proof. By Lemma 5.1.1, for any $N \in \mathbb{N}_*, \varepsilon > 0$ and $\delta > 0$,

$$\begin{aligned}\|\rho_\varepsilon(t)\,(\mathrm{d}\Gamma_\varepsilon(\mathbb{1}) + 1)^\delta\|_{\mathscr{L}_1(\mathscr{H})} &\leqslant \|\rho_\varepsilon(t)\,(\mathrm{d}\Gamma_\varepsilon(\mathbb{1}) + N^2 + \varepsilon)^\delta\|_{\mathscr{L}_1(\mathscr{H})} \leqslant \\ &\leqslant e^{m_\delta(\varepsilon)\sqrt{\varepsilon}|\delta||t|\|\lambda_\cdot\|_\infty} \|\rho_\varepsilon\,(\mathrm{d}\Gamma_\varepsilon(\mathbb{1}) + N^2 + \varepsilon)^\delta\|_{\mathscr{L}_1(\mathscr{H})} \leqslant \\ &\leqslant e^{c|t|}\|\rho_\varepsilon\,(\mathrm{d}\Gamma_\varepsilon(\mathbb{1}) + N^2 + 1)^\delta\|_{\mathscr{L}_1(\mathscr{H})} \leqslant e^{c|t|}(N^2+1)^\delta\|\rho_\varepsilon\,(\mathrm{d}\Gamma_\varepsilon(\mathbb{1}) + 1)^\delta\|_{\mathscr{L}_1(\mathscr{H})}.\end{aligned}$$

Furthermore, by [4, Lemma 3.4], there exist constants $C_1, C_2 \in (0, +\infty)$, such that

$$\|(H_+ + \mathrm{d}\Gamma_\varepsilon(\mathbb{1}) + 1)e^{-itH_\varepsilon}(H_+ + \mathrm{d}\Gamma_\varepsilon(\mathbb{1}) + 1)^{-1}\| \leqslant C_1(1 + e^{C_2 t}) \quad (5.15)$$

uniformly in ε. Then we conclude that

$$\begin{aligned}\|\rho_\varepsilon(t)H_+\|_{\mathscr{L}_1} \leqslant \|\rho_\varepsilon(H_+ + \mathrm{d}\Gamma_\varepsilon(\mathbb{1}) + 1)\|_{\mathscr{L}_1}\,\|(H_+ + \mathrm{d}\Gamma_\varepsilon(\mathbb{1}) + 1)^{-1}e^{itH_\varepsilon}(H_+ + \mathrm{d}\Gamma_\varepsilon(\mathbb{1}) + 1)\|\times \\ \times \|(H_+ + \mathrm{d}\Gamma_\varepsilon(\mathbb{1}) + 1)^{-1}H_+\| \leqslant \\ \leqslant C_1(1 + e^{C_2 t})\|\rho_\varepsilon(H_+ + \mathrm{d}\Gamma_\varepsilon(\mathbb{1}) + 1)\|_{\mathscr{L}_1}.\end{aligned}$$

□

Therefore any state ρ_ε satisfying assumptions (A_R) is such that, for any $t \in \mathbb{R}$,

$$\|\rho_\varepsilon(t)\,(-\Delta_x + W_+ + (\mathrm{d}\Gamma_\varepsilon(\mathbb{1}) + 1)^\delta)\|_{\mathscr{L}_1} \leqslant C(e^{c|t|} + 1). \quad (5.16)$$

5.1.2 Polaron model

We consider the polaron Hamiltonian H_ε associated with the quadratic form induced by the formal expression

$$-\Delta_x + W(x_1, \dots, x_N) + \nu_\varepsilon \mathrm{d}\Gamma_\varepsilon(\mathbb{1}) + \sum_{j=1}^{N} (a_\varepsilon(g_{x_j}) + a_\varepsilon^\dagger(g_{x_j})). \tag{5.17}$$

The idea is to exploit the pull-through formula [2, Theorem 4.7] which applies to the Nelson model (see Appendix A), and extend it to the polaron model, which leads to the following inequality.

Lemma 5.1.3. *For any $\delta \in \mathbb{N}_*$, there exists $b > 0$ such that, for any $\varepsilon \in (0,1)$ and $\Psi_\varepsilon \in \mathscr{D}(H_\varepsilon^\delta)$,*

$$\|(\mathrm{d}\Gamma_\varepsilon(\mathbb{1}) + \nu_\varepsilon^{-1})^\delta \Psi_\varepsilon\|_{\mathscr{H}} \leqslant C\nu_\varepsilon^{-\delta} \|(H_\varepsilon + b)^\delta \Psi_\varepsilon\|_{\mathscr{H}}.$$

Proof. Let us denote by $H_\varepsilon(\Lambda) := -\Delta_x + W + \mathrm{d}\Gamma_\varepsilon(\mathbb{1}) + H_I(\Lambda),\ \Lambda > 0$ the polaron Hamiltonian with scaling $\nu_\varepsilon = 1$ and cut-off for the high momenta:

$$H_I(\Lambda) := a_\varepsilon(g_\Lambda) + a_\varepsilon^\dagger(g_\Lambda),$$
$$g_\Lambda(x,k) := \sum_{j=1}^{N} \frac{e^{ik\cdot x_j}}{|k|^{\frac{d-1}{2}}} \chi_{\{|k|\leqslant \Lambda\}}(k).$$

Since $g_\Lambda \in L^\infty(\mathbb{R}^{dN}; \mathfrak{H})$, H_Λ is a Nelson-type Hamiltonian. Then, we can use the estimate (A.47) derived by pull-through formula from Appendix A to get that, for any $\delta \in \mathbb{N}_*$, there exists $b \in (0, +\infty)$ such that, for any $\Psi_\varepsilon \in \mathscr{D}(H_\varepsilon(\Lambda)^\delta)$,

$$\|(\mathrm{d}\Gamma_\varepsilon(\mathbb{1}) + 1)^\delta \Psi_\varepsilon\|_{\mathscr{H}} \leqslant C\, \|(H_\varepsilon(\Lambda) + b)^\delta \Psi_\varepsilon\|_{\mathscr{H}}. \tag{5.18}$$

By [51, Theorem 2.2],

$$H_\varepsilon(\Lambda) \xrightarrow[\Lambda\to+\infty]{\|\cdot\|-\mathrm{res}} H_\varepsilon, \tag{5.19}$$

where now the r.h.s. stands for the polaron Hamiltonian with scaling $\nu_\varepsilon = 1$. By induction on δ, one can prove that norm-resolvent convergence holds true also for powers of H_ε: let us assume that the result is true for $\delta - 1 \in \mathbb{N}_*$, then, by the second resolvent identity,

$$\begin{aligned} &\|((H_\varepsilon(\Lambda) + b)^\delta - z)^{-1} - ((H_\varepsilon + b)^\delta - z)^{-1}\| = \\ &= \|((H_\varepsilon(\Lambda) + b)^\delta - z)^{-1}((H_\varepsilon(\Lambda) + b)^\delta - (H_\varepsilon + b)^\delta)((H_\varepsilon + b)^\delta - z)^{-1})\| \leqslant \\ &\leqslant C_z \|H_\varepsilon(\Lambda) - H_\varepsilon\| + C'_z \|(H_\varepsilon(\Lambda) + b)^{\delta-1} - (H_\varepsilon + b)^{\delta-1}\|, \end{aligned}$$

for any $z \in \sigma(H_\varepsilon(\Lambda)) \cap \sigma(H_\varepsilon)$. By induction, we deduce the result, *i.e.*, for any $\delta \in \mathbb{N}_*$,

$$(H_\varepsilon(\Lambda) + b)^\delta \xrightarrow[\Lambda\to+\infty]{\|\cdot\|-\mathrm{res}} (H_\varepsilon + b)^\delta. \tag{5.20}$$

The cut-off independence stated in [2, Lemma 2.1] however yields that

$$\|(\mathrm{d}\Gamma_\varepsilon(\mathbb{1}) + 1)^\delta \Psi_\varepsilon\|_{\mathscr{H}} \leqslant C\, \|(H_\varepsilon + b)^\delta \Psi_\varepsilon\|_{\mathscr{H}} \tag{5.21}$$

for any $\Psi_\varepsilon \in \mathscr{D}((H_\varepsilon)^\delta)$, which proves the Lemma for $\nu_\varepsilon = 1$.

To prove it for $\nu_\varepsilon = \varepsilon^{-1}$, we consider the Hamiltonian $\tilde{H}_\varepsilon$ associated to the quadratic form induced by the formal expression

$$-\Delta_x + W + \mathrm{d}\Gamma_\varepsilon(\mathbb{1}) + \sqrt{\varepsilon} H_I. \tag{5.22}$$

The pull-through formula applies to this operator as well, yielding

$$\|(\mathrm{d}\Gamma_\varepsilon(\mathbb{1})+1)^\delta\Psi_\varepsilon\|_{\mathscr{H}} \leqslant C\,\|(\tilde{H}_\varepsilon+b)^\delta\Psi_\varepsilon\|_{\mathscr{H}}, \tag{5.23}$$

for any $\Psi_\varepsilon \in \mathscr{D}(\tilde{H}_\varepsilon^\delta)$. Rescaling now $a_\varepsilon^\# \mapsto \varepsilon^{-1/2}a_\varepsilon^\#$, we get

$$\|(\mathrm{d}\Gamma_\varepsilon(\mathbb{1})+\varepsilon)^\delta\Psi_\varepsilon\|_{\mathscr{H}} \leqslant c\varepsilon^\delta\,\|(H_\varepsilon+b)^\delta\Psi_\varepsilon\|_{\mathscr{H}} \tag{5.24}$$

for any $\Psi_\varepsilon \in \mathscr{D}(\tilde{H}_\varepsilon^\delta) = \mathscr{D}(H_\varepsilon^\delta)$. This completes the proof. □

The previous estimate allows to derive a bound on the evolved density matrix.

Proposition 5.1.4. *For any time $t \in \mathbb{R}$,*

$$\begin{aligned}&\mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon(t)(-\Delta_x+W_++\mathrm{d}\Gamma_\varepsilon(\mathbb{1})^2)) \leqslant \\ &\qquad\leqslant C\left[(e^{|t|}+1)\,\mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon(\nu_\varepsilon^{-2}H_\varepsilon^2+H_+))+(|t|+1)e^{|t|}+1\right].\end{aligned}$$

Proof. Let us deal with the term with the particle energy and the number operator, separately. In the scaling $\nu_\varepsilon = \varepsilon^{-1}$, we use [22, Lemma 3.4] to obtain that, for any $t \in \mathbb{R}$,

$$\mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon(t)(-\Delta_x+W_+)) \leqslant C\left[e^{|t|}\mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon H_+)+(|t|+1)e^{|t|}+1\right]. \tag{5.25}$$

For the scaling $\nu_\varepsilon = 1$, the proof is even simpler, and obtained by reconstructing H_ε to get rid of time-evolution and using the form-boundedness of H_I w.r.t. H_+.

Consider now the number operator term: by Lemma 5.1.3, setting $m = \inf\sigma(H_\varepsilon)$,

$$\|(\mathrm{d}\Gamma_\varepsilon(\mathbb{1})+\varepsilon)\,e^{-itH_\varepsilon}(H_\varepsilon+|m|+1)^{-1}\| \leqslant C\varepsilon. \tag{5.26}$$

Then, we get

$$\begin{aligned}\mathrm{Tr}_{\mathscr{H}}\left(\rho_\varepsilon(t)\mathrm{d}\Gamma_\varepsilon(\mathbb{1})^2\right) &\leqslant \mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon(H_\varepsilon+|m|+1)^2)\|\mathrm{d}\Gamma_\varepsilon(\mathbb{1})(H_\varepsilon+|m|+1)^{-1}\|_{\mathscr{H}}^2 \leqslant \\ &\leqslant C\left[\mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon\varepsilon^2H_\varepsilon^2)+\mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon H_+)+1\right] \leqslant \\ &\leqslant C\left[\mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon(\varepsilon^2H_\varepsilon^2+H_+))+1\right].\end{aligned}$$

where we used the form-boundedness of H_ε w.r.t. H_+. For the scaling $\nu_\varepsilon = 1$ the proof is completely analogous. □

Therefore, for states satisfying assumption (A_R), we have that, for any $t \in \mathbb{R}$,

$$\mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon(t)\,(-\Delta_x+W_++\mathrm{d}\Gamma_\varepsilon(\mathbb{1})^2)) \leqslant C((|t|+1)e^{|t|}+1). \tag{5.27}$$

5.1.3 Pauli-Fierz model

Consider the Pauli-Fierz Hamiltonian

$$H_\varepsilon = \sum_{j=1}^{N} \left(-i\nabla_{x_j} + \phi_\varepsilon(\lambda_{x_j})\right)^2 + W(x_1, \dots, x_N) + \nu_\varepsilon \mathrm{d}\Gamma_\varepsilon(\omega) \tag{5.28}$$

which can be rewritten in the Coulomb gauge as

$$H_\varepsilon = -\Delta_x + W(x_1, \dots, x_N) + \nu_\varepsilon \mathrm{d}\Gamma_\varepsilon(\omega) + H_I \tag{5.29}$$

with

$$H_I = \sum_{j=1}^{N} \left(-2i\nabla_{x_j} \cdot \sum_{\gamma=1}^{d-1} \phi_\varepsilon(e_\gamma \lambda_{x_j}) + \left(\sum_{\gamma=1}^{d-1} \phi_\varepsilon(e_\gamma \lambda_{x_j}) \right)^2 \right).$$

We recall a result from [5] for the control of the norm of the number operator in time and also a preliminary Lemma to bound the expectation value of the particle Hamiltonian for the scaling $\nu_\varepsilon = \varepsilon^{-1}$.

Lemma 5.1.5. *[5] There exists* $K \in (0, +\infty)$ *such that, for any* $\varepsilon \in (0,1), \nu_\varepsilon \in \{\varepsilon^{-1}, 1\}$ *and* $\Psi_\varepsilon \in \mathscr{D}(H_+) \cap \mathscr{D}(\mathrm{d}\Gamma_\varepsilon(\mathbb{1}))$,

$$\|\mathrm{d}\Gamma_\varepsilon(\mathbb{1})\, e^{-itH_\varepsilon} \Psi_\varepsilon\|_{\mathscr{H}} \leqslant C e^{K|t|} \left(\|\mathrm{d}\Gamma_\varepsilon(\mathbb{1})\Psi_\varepsilon\|_{\mathscr{H}} + \|(H_+ + 1)\Psi_\varepsilon\|_{\mathscr{H}} \right). \tag{5.30}$$

The same holds true in the scaling $\nu_\varepsilon = \varepsilon^{-1}$ *with* H_0 *in place of* H_+. *Furthermore, there exists* $c \in (0, +\infty)$ *such that, for any* $\Psi_\varepsilon \in \mathscr{D}(H_\varepsilon)$,

$$c\|(H_\varepsilon + 1)\Psi_\varepsilon\|_{\mathscr{H}} \leqslant \|(H_0 + 1)\Psi_\varepsilon\|_{\mathscr{H}} \leqslant C\|(H_\varepsilon + 1)\Psi_\varepsilon\|_{\mathscr{H}}. \tag{5.31}$$

Lemma 5.1.6. *There exist two constants* $c, c' \in (0, +\infty)$ *such that, for any* $t \in \mathbb{R}$ *and any normalized* $\Phi_\varepsilon \in \mathscr{Q}(H_\varepsilon)$,

$$\langle \Phi_\varepsilon |\, e^{itH_\varepsilon} (-\Delta_x + W_+)\, e^{-itH_\varepsilon}\, |\Phi_\varepsilon\rangle \leqslant C \Big[(e^{c|t|} + 1) \langle \Phi_\varepsilon |\, (H_+ + 1)^2 + \mathrm{d}\Gamma_\varepsilon(\mathbb{1})^2 \,|\Phi_\varepsilon\rangle + \\ + (e^{c'|t|} + |t| + 1) \Big]. \tag{5.32}$$

Proof. The proof follows the analogous one for the polaron [22, Lemma 3.4], but it is more involved due to the complicated form of the interaction term in the Pauli-Fierz Hamiltonian. The idea is to use Gronwall lemma, so we restrict the set of Φ_ε to ensure differentiability and then extend the result by density: let $\Phi_\varepsilon \in \mathscr{D} := \mathscr{D}((H_\varepsilon + |m| + 1)^{3/2}) \cap \mathscr{D}(\mathrm{d}\Gamma_\varepsilon(\mathbb{1}))$, where $m < 0$ is the bottom of the spectrum of H_ε, which we know is is finite by Theorem 1.1.6. The one-parameter group e^{-itH_ε} is weakly differentiable on $\mathscr{Q}(H_0) \supseteq \mathscr{D}$, and leaves $\mathscr{D}$ invariant by its action, thanks to Lemma 5.1.5. Let us set $M(t) := |\langle \Phi_\varepsilon(t)| - \Delta_x + W_+ |\Phi_\varepsilon(t)\rangle|$, with $\Phi_\varepsilon(t) := e^{-itH_\varepsilon}\Phi_\varepsilon$. Then, we can write

$$M(t) \leqslant \langle \Phi_\varepsilon(t) \,|\, H_+ \,|\, \Phi_\varepsilon(t)\rangle \leqslant |\langle \Phi_\varepsilon(t) \,|\, H_+ + H_I \,|\, \Phi_\varepsilon(t)\rangle| + |\langle \Phi_\varepsilon(t) \,|\, H_I \,|\, \Phi_\varepsilon(t)\rangle|. \tag{5.33}$$

Let us bound the last expression term by term, starting from the interaction:

$$|\langle \Phi_\varepsilon(t) \,|\, H_I \,|\, \Phi_\varepsilon(t)\rangle| \leqslant \sum_{j=1}^{N} \left(2\left|\left\langle \nabla_{x_j}\Phi_\varepsilon(t) \,\middle|\, \phi_\varepsilon(\lambda_{x_j})\Phi_\varepsilon \right\rangle\right| + \left|\left\langle \Phi_\varepsilon(t) \,\middle|\, (\phi_\varepsilon(\lambda_{x_j}))^2 \Phi_\varepsilon(t) \right\rangle\right| \right),$$

so that

- for any $\alpha \in (0, +\infty)$ we have

$$\sum_{j=1}^{N} 2\left|\left\langle \nabla_{x_j}\Phi_\varepsilon(t) \,\middle|\, \phi_\varepsilon(\lambda_{x_j})\Phi_\varepsilon \right\rangle\right| \leqslant \alpha \left\langle \Phi_\varepsilon(t)\right| -\Delta_x \left|\Phi_\varepsilon(t)\right\rangle + \frac{16}{\alpha}\sum_{j=1}^{N}\|a_\varepsilon(\lambda_{x_j})\Phi_\varepsilon(t)\|^2 \leqslant$$
$$\leqslant \alpha \left\langle \Phi_\varepsilon(t)\right| -\Delta_x \left|\Phi_\varepsilon(t)\right\rangle + \frac{16}{\alpha}N\|\lambda_\cdot\|_\infty^2\|\Phi_\varepsilon\|\|\mathrm{d}\Gamma_\varepsilon(\mathbb{1})\Phi_\varepsilon(t)\| \leqslant$$
$$\leqslant \alpha \left\langle \Phi_\varepsilon(t)\right| -\Delta_x + W_+ \left|\Phi_\varepsilon(t)\right\rangle + \frac{C}{\alpha}e^{c|t|}(\|\mathrm{d}\Gamma_\varepsilon(\mathbb{1})\Phi_\varepsilon\| + \|(H_+ + 1)\Phi_\varepsilon\|)$$

  where we have used Lemma 5.1.5 in the last step;

- Cauchy-Schwarz inequality yields that

$$\sum_{j=1}^{N}\left|\left\langle \Phi_\varepsilon(t) \,\middle|\, (\phi_\varepsilon(\lambda_{x_j}))^2\Phi_\varepsilon(t) \right\rangle\right| \leqslant \sum_{j=1}^{N}\left(4\|a_\varepsilon(\lambda_{x_j})\Phi_\varepsilon(t)\|^2 + 2\|\lambda_{x_j}\|^2\|\Phi_\varepsilon\|^2\right) \leqslant$$
$$\leqslant 2N\|\lambda_\cdot\|_\infty^2\left(\|\mathrm{d}\Gamma_\varepsilon(\mathbb{1})\Phi_\varepsilon(t)\|\|\Phi_\varepsilon\| + \|\Phi_\varepsilon\|^2\right) \leqslant$$
$$\leqslant C\left(e^{c|t|}\left(\|\mathrm{d}\Gamma_\varepsilon(\mathbb{1})\Phi_\varepsilon\| + \|(H_+ + 1)\Phi_\varepsilon\|\right) + 1\right),$$

  again by Lemma 5.1.5.

Putting together the above inequalities, we obtain

$$|\left\langle \Phi_\varepsilon(t)\right| H_I \left|\Phi_\varepsilon(t)\right\rangle| \leqslant \alpha \left\langle \Phi_\varepsilon(t)\right| -\Delta_x + W_+ \left|\Phi_\varepsilon(t)\right\rangle + C\left[\frac{e^{c|t|}}{\alpha}\left(\|\mathrm{d}\Gamma_\varepsilon(\mathbb{1})\Phi_\varepsilon\| + \|(H_+ + 1)\Phi_\varepsilon\|\right) + 1\right].$$

Choosing $\alpha \in \left[\frac{1}{2}, 1\right)$, and plugging the above inequality into (5.33), we obtain

$$M(t) \leqslant \frac{1}{1-\alpha}|\left\langle \Phi_\varepsilon(t)\right| H_+ + H_I \left|\Phi_\varepsilon(t)\right\rangle| + C\left[e^{c|t|}\left(\|\mathrm{d}\Gamma_\varepsilon(\mathbb{1})\Phi_\varepsilon\| + \|(H_+ + 1)\Phi_\varepsilon\|\right) + 1\right]. \tag{5.34}$$

We turn our attention now to the sum of the free and interaction parts and, by differentiability,

$$|\left\langle \Phi_\varepsilon(t)\right| H_+ + H_I \left|\Phi_\varepsilon(t)\right\rangle| \leqslant |\left\langle \Phi_\varepsilon\right| H_+ + H_I \left|\Phi_\varepsilon\right\rangle| + \left|\int_0^t \mathrm{d}s\,\frac{\mathrm{d}}{\mathrm{d}s}\left\langle \Phi_\varepsilon(s)\right| H_+ + H_I \left|\Phi_\varepsilon(s)\right\rangle\right| \leqslant$$
$$\leqslant |\left\langle \Phi_\varepsilon\right| H_+ + H_I \left|\Phi_\varepsilon\right\rangle| + \frac{1}{\varepsilon}\int_0^t \mathrm{d}s\,|\left\langle \Phi_\varepsilon(s)\,\middle|\,[\mathrm{d}\Gamma_\varepsilon(\omega), H_I]\Phi_\varepsilon(s)\right\rangle|.$$

Let us bound the last expression. For the first term, by form boundedness of the interaction w.r.t. the free part, there exist $a, b \in (0, +\infty)$ such that

$$|\left\langle \Phi_\varepsilon\right| H_+ + H_I \left|\Phi_\varepsilon\right\rangle| \leqslant (1+a)|\left\langle \Phi_\varepsilon\right| H_+ \left|\Phi_\varepsilon\right\rangle| + b\|\Phi_\varepsilon\|^2.$$

Next, we estimate each term in the integrand:

- computing the commutator between the field energy and the canonical observables, we obtain

$$\left|\left\langle \Phi_\varepsilon(s) \,\middle|\, \left[\mathrm{d}\Gamma_\varepsilon(\omega), \sum_{j=1}^{N} 2i\nabla_{x_j}(a_\varepsilon(\lambda_{x_j}) + a_\varepsilon^\dagger(\lambda_{x_j}))\right]\Phi_\varepsilon(s) \right\rangle\right| \leqslant$$
$$\leqslant 2\varepsilon\sum_{j=1}^{N}\left|\left\langle \nabla_{x_j}\Phi_\varepsilon(s) \,\middle|\, (a_\varepsilon^\dagger(\omega\lambda_{x_j}) - a_\varepsilon(\omega\lambda_{x_j}))\Phi_\varepsilon(s) \right\rangle\right| \leqslant$$
$$\leqslant \varepsilon\left\langle \Phi_\varepsilon(s)\,\middle|\,(-\Delta_x + W_+)\Phi_\varepsilon(s)\right\rangle + C\varepsilon e^{c|s|}\left(\|\mathrm{d}\Gamma_\varepsilon(\mathbb{1})\Phi_\varepsilon\| + \|(H_+ + 1)\Phi_\varepsilon\|\right),$$

  where we have used Lemma 5.1.5 and the assumptions on $\lambda_\cdot$.

- For the quadratic terms:

$$\Big|\Big\langle \Phi_\varepsilon(s)\Big|\Big[\mathrm{d}\Gamma_\varepsilon(\omega),\sum_{j=1}^N (a_\varepsilon(\lambda_{x_j})+a_\varepsilon^\dagger(\lambda_{x_j}))^2\Big]\Phi_\varepsilon(s)\Big\rangle\Big| \leqslant$$
$$\leqslant \varepsilon\sum_{j=1}^N \big|\big\langle \Phi_\varepsilon(s)\big|\big(a_\varepsilon(\omega\lambda_{x_j})a_\varepsilon(\lambda_{x_j})+a_\varepsilon(\lambda_{x_j})a_\varepsilon(\omega\lambda_{x_j})+a_\varepsilon^\dagger(\omega\lambda_{x_j})a_\varepsilon^\dagger(\lambda_{x_j})+$$
$$+a_\varepsilon^\dagger(\lambda_{x_j})a_\varepsilon^\dagger(\omega\lambda_{x_j})+2a_\varepsilon^\dagger(\omega\lambda_{x_j})a_\varepsilon(\lambda_{x_j})-2a_\varepsilon^\dagger(\lambda_{x_j})a_\varepsilon(\omega\lambda_{x_j})\big)\Phi_\varepsilon(s)\big\rangle\big| \leqslant$$
$$\leqslant C\varepsilon\sum_{j=1}^N \left(\|a_\varepsilon(\lambda_{x_j})\Phi_\varepsilon(s)\|^2+\|a_\varepsilon(\omega\lambda_{x_j})\Phi_\varepsilon(s)\|^2+\|\omega\lambda_{x_j}\|^2+\|\lambda_{x_j}\|^2\right) \leqslant$$
$$\leqslant C\varepsilon\left[\|\mathrm{d}\Gamma_\varepsilon(\mathbb{1})\Phi_\varepsilon(s)\|+1\right] \leqslant C\varepsilon\Big[e^{c|s|}(\|\mathrm{d}\Gamma_\varepsilon(\mathbb{1})\Phi_\varepsilon\|+\|(H_++1)\Phi_\varepsilon\|)+1\Big].$$

  using again Lemma 5.1.5 and assumptions on $\lambda_\cdot$.

Putting together the previous inequalities we have

$$|\big\langle \Phi_\varepsilon(t)\,\big|\,H_++H_I\,\big|\,\Phi_\varepsilon(t)\big\rangle| \leqslant (1+a)|\langle\Phi_\varepsilon|\,H_+\,|\Phi_\varepsilon\rangle|+b+$$
$$+\int_0^t \mathrm{d}s\Big[M(s)+C\Big(e^{c|s|}(\|\mathrm{d}\Gamma_\varepsilon(\mathbb{1})\Phi_\varepsilon\|+\|(H_++1)\Phi_\varepsilon\|)+1\Big)\Big] \leqslant$$
$$\leqslant (1+a)|\langle\Phi_\varepsilon|\,H_+\,|\Phi_\varepsilon\rangle|+C\Big[(e^{c|t|}-1)(\|\mathrm{d}\Gamma_\varepsilon(\mathbb{1})\Phi_\varepsilon\|+\|(H_++1)\Phi_\varepsilon\|)+|t|\Big]+\int_0^t \mathrm{d}s\,M(s).$$

Plugging it into (5.34) we obtain the following:

$$M(t) \leqslant B(t)+\frac{1}{1-\alpha}\int_0^t \mathrm{d}s\,M(s), \tag{5.35}$$

where

$$B(t) := C\Big[(e^{c|t|}+1)(\|\mathrm{d}\Gamma_\varepsilon(\mathbb{1})\Phi_\varepsilon\|+\|(H_++1)\Phi_\varepsilon\|)+|t|+1\Big]. \tag{5.36}$$

We are ready to use Gronwall Lemma to obtain

$$M(t) \leqslant B(t)+\frac{1}{1-\alpha}\int_0^t \mathrm{d}s\,B(s)\,e^{\frac{t-s}{1-\alpha}}. \tag{5.37}$$

Doing all the calculation we finally derive the inequality

$$|\langle\Phi_\varepsilon(t)|-\Delta_x\,|\Phi_\varepsilon(t)\rangle| \leqslant C\Big[(e^{c|t|}+1)(\|(H_++1)\Phi_\varepsilon\|+\|\mathrm{d}\Gamma_\varepsilon(\mathbb{1})\Phi_\varepsilon\|)+(e^{c'|t|}+|t|+1)\Big] \tag{5.38}$$

and from this easily find the desired result for $\Phi_\varepsilon \in \mathscr{D}$. A density argument can be used to conclude the proof for $\Phi_\varepsilon \in \mathscr{Q}(H_\varepsilon)$. □

Proposition 5.1.7. *There exists $c \in (0,+\infty)$ such that, for any $t \in \mathbb{R}$,*

$$\mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon(t)\,(-\Delta_x+W_++(\mathrm{d}\Gamma_\varepsilon(\mathbb{1})+1)^2)) \leqslant C\Big[(e^{c|t|}+1)\,\mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon\,(H_++1)^2+\mathrm{d}\Gamma_\varepsilon(\mathbb{1})^2)+$$
$$+(e^{|t|}+|t|+1)\Big]. \tag{5.39}$$

Proof. Let us bound the free energy and the number operator separately. Consider the scaling $\nu_\varepsilon = \varepsilon^{-1}$. For the first term, we use Lemma 5.1.6, which gives

$$\mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon(t)(-\Delta_x + W_+)) \leqslant C_1(e^{C_3|t|}+1)\mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon\,((H_+ +1)^2 + \mathrm{d}\Gamma_\varepsilon(\mathbb{1})^2)) + C_2(e^{C_4|t|}+|t|+1).$$

By first part of Lemma 5.1.5 we have, instead, the following inequality:

$$\begin{aligned}&\mathrm{Tr}_{\mathscr{H}}\,(\rho_\varepsilon(t)\,\mathrm{d}\Gamma_\varepsilon(\mathbb{1})^2) = \mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon\, e^{itH_\varepsilon}\mathrm{d}\Gamma_\varepsilon(\mathbb{1})^2 e^{-itH_\varepsilon}) \leqslant\\ &\leqslant \mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon(H_+ + \mathrm{d}\Gamma_\varepsilon(\mathbb{1}))^2)\,\|\mathrm{d}\Gamma_\varepsilon(\mathbb{1})e^{-itH_\varepsilon}(H_+ + \mathrm{d}\Gamma_\varepsilon(\mathbb{1}))^{-1}\|^2 \leqslant\\ &\leqslant C\Big[\mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon(H_+ + \mathrm{d}\Gamma_\varepsilon(\mathbb{1}))^2)\,e^{c|t|}(\|\mathrm{d}\Gamma_\varepsilon(\mathbb{1})(H_+ + \mathrm{d}\Gamma_\varepsilon(\mathbb{1}))^{-1}\|+\\ &\quad +\|(H_+ +1)(H_+ + \mathrm{d}\Gamma_\varepsilon(\mathbb{1}))^{-1}\|)\Big] \leqslant\\ &\leqslant Ce^{c|t|}\,\mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon(H_+^2 + \mathrm{d}\Gamma_\varepsilon(\mathbb{1})^2)),\end{aligned}$$

which completes the proof.

For the scaling $\nu_\varepsilon = 1$, the bound for the square of the number operator is the same, while we bound the expectation of H_+ as

$$\mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon(t)H_+) \leqslant C\|H_+\rho_\varepsilon H_+\|_{\mathscr{L}_1}\,\|H_+^{-1}(H_\varepsilon + |m|+1)\|\,\|(H_\varepsilon+|m|+1)^{-1}H_+\| \leqslant C\,\mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon H_+^2),$$

where we used the fact that e^{-itH_ε} commutes with its own generator and that H_ε and H_+ have the same domain. □

Therefore, for states satisfying assumption (A_R), we have, for any $t \in \mathbb{R}$,

$$\mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon(t)\,(-\Delta_x + W_+ + \mathrm{d}\Gamma_\varepsilon(\mathbb{1})^2)) \leqslant C(e^{|t|}+|t|+1). \tag{5.40}$$

The previous estimates can be used to prove that, at any fixed time, we can extract a subsequence from the family $\{\rho_\varepsilon(t)\}_{\varepsilon\in(0,1)}$ that converges to a state-valued measure $\mathfrak{m}_t$.

Proposition 5.1.8. *Let* $\{\rho_\varepsilon(t)\}_{\varepsilon\in(0,1)} \subseteq \mathscr{L}_{1,+}(\mathscr{H})$ *be the family of density matrices defined in* (5.11). *If, for any* $\delta \geqslant 1$,

$$\begin{cases}\|\rho_\varepsilon\,((-\Delta_x + W_+ + (\mathrm{d}\Gamma_\varepsilon(\mathbb{1})+1)^\delta)\|_{\mathscr{L}_1} \leqslant C, & \text{for the Nelson model},\\ \mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon\,(H_+ + \nu_\varepsilon^{-2}H_\varepsilon^2)) \leqslant C, & \text{for the polaron},\\ \mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon\,((H_+ +1)^2 + \mathrm{d}\Gamma_\varepsilon(\mathbb{1})^2)) \leqslant C, & \text{for the Pauli-Fierz model},\end{cases} \tag{5.41}$$

then, for any $t \in \mathbb{R}$, *there exist a subsequence* $\{\varepsilon_{n_j(t)}\}_{j\in\mathbb{N}}$, $\varepsilon_{n_j(t)} \xrightarrow[j\to+\infty]{} 0$, *and a probability state-valued measure* $\mathfrak{m}_t \in \mathscr{M}(\mathfrak{H};\mathscr{L}_{1,+}(L^2))$, *such that*

$$\rho_{\varepsilon_{n_j(t)}}(t) \xrightarrow[j\to+\infty]{} \mathfrak{m}_t. \tag{5.42}$$

Proof. By Theorem 3.1.6, since the state is normalized, there exists a cylindrical state-valued measure $\mathfrak{m}_t$ such that $\rho_\varepsilon(t)$ converges to it in topology $\mathfrak{P}$ over a fixed subsequence $\varepsilon_{n_j(t)}$. To obtain also the convergence in topology $\mathfrak{T}$, we need to prove that the no-loss of mass condition is satisfied, but the latter follows from a uniform bound on the expectation of the number operator by Proposition 3.1.8. By the combination of the assumptions with Propositions 5.1.2, 5.1.4 and 5.1.7 we obtain that

$$\mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon(t)\,(\mathrm{d}\Gamma_\varepsilon(\mathbb{1})+1)) \leqslant C_t, \tag{5.43}$$

for a certain constant $C_t \in (0,+\infty)$, for any $t \in \mathbb{R}$. By Theorem 3.1.10, this condition guarantees that every $\mathfrak{m}_t$ concentrates as a state-valued measure on $\mathfrak{H}$. Finally, Theorem 3.2.1 implies that $\mathfrak{m}_t$ is a probability state-valued measure. □

5.2 Extraction of a common subsequence

Let us recall the formula for the time evolution of states, according to the dynamics generated by a self-adjoint Hamiltonian H_ε:

$$\rho_\varepsilon(t) := e^{-itH_\varepsilon}\, \rho_\varepsilon\, e^{itH_\varepsilon}, \tag{5.44}$$

which, at least formally, solves the equation

$$i\frac{\mathrm{d}}{\mathrm{d}t}\rho_\varepsilon(t) = [H_\varepsilon, \rho_\varepsilon(t)]. \tag{5.45}$$

The main goal of this Section is to identify the cluster points of the above family of states as the solution of a certain transport equation by taking the limit $\varepsilon \to 0$ of (5.45). It is more convenient to work with the modified version of the dynamics given by the *interaction representation*: we set

$$\tilde{\rho}_\varepsilon(t) := e^{itH_0}\, \rho_\varepsilon(t)\, e^{-itH_0}, \tag{5.46}$$

which, at least formally, solves

$$i\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\rho}_\varepsilon(t) = \left[\tilde{H}_I(t)\,,\, \tilde{\rho}_\varepsilon(t)\right] \tag{5.47}$$

where

$$\tilde{H}_I(t) := e^{itH_0}\, H_I\, e^{-itH_0}.$$

Before discussing the formula for $\tilde{\rho}_\varepsilon(t)$, let us recall the spaces of test operators : we set $T = (-\Delta_x + W_+)$, and (see also Lemma 3.2.2)

$$\mathcal{K} := \{\kappa_m := \mathbb{1}_m(T)\,\kappa\,\mathbb{1}_m(T), \kappa \in \mathscr{L}_{\infty,+}(\mathscr{Z}), m \in \mathbb{N}\}; \tag{5.48}$$

$$\mathcal{B} := \{B_m := \mathbb{1}_m(T)\,B\,\mathbb{1}_m(T), B \in \mathscr{L}(\mathscr{Z})_+, m \in \mathbb{N}\}. \tag{5.49}$$

Proposition 5.2.1. *If ρ_ε satisfies* (A_R)*, then, for any $B \in \mathscr{L}(L^2)$ ($B \in \mathcal{B}$ for the polaron), $\xi \in \mathfrak{H}$ and $t \in \mathbb{R}$:*

$$\mathrm{Tr}_{\mathscr{H}}(\tilde{\rho}_\varepsilon(t)\, B \otimes W_\varepsilon(\xi)) = \mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon\, B \otimes W_\varepsilon(\xi)) - i\int_0^t \mathrm{d}\tau\, \mathrm{Tr}_{\mathscr{H}}(\,[\tilde{H}_I(\tau), \tilde{\rho}_\varepsilon(\tau)]\, B \otimes W_\varepsilon(\xi)). \tag{5.50}$$

Proof. We only have to prove the differentiability in time of the function

$$t \longmapsto \mathrm{Tr}_{\mathscr{H}}(\tilde{\rho}_\varepsilon(t)\, B \otimes W_\varepsilon(\xi)),$$

since, then, the result follows from the fundamental theorem of calculus. Let us study each model separately.

Nelson model: We prove the result for the set of states ρ_ε such that

$$\|\rho_\varepsilon\,(H_+ + 1)\|_{\mathscr{L}_1} \leqslant C. \tag{5.51}$$

This set is dense in $\mathscr{L}_1$ norm in the space of quantum states satisfying the assumptions for any $\delta \geqslant 1$, so it is enough to prove the result for states satisfying(5.51).

Now, $(H_+ + 1)^{-1}$ maps $\mathscr{H}$ into $\mathscr{D}(H_0)$, so that the functions $t \mapsto e^{\pm itH_0}(H_+ + 1)^{-1}$, $e^{\pm itH_\varepsilon}(H_+ + 1)^{-1}$ are strongly differentiable thanks to Stone theorem. Furthermore, by assumptions, $(H_+ + 1)\rho_\varepsilon$ and $\rho_\varepsilon(H_+ + 1)$ are trace-class operators in $\mathscr{H}$, while $e^{\pm itH_0}(H_+ + 1)^{-1}$ and

$(H_+ + 1)\, e^{\pm itH_\varepsilon}(H_+ + 1)^{-1}$ are bounded operators on $\mathscr{H}$, the last thanks to [4, Lemma 3.4]. Since

$$\begin{aligned}\mathrm{Tr}_{\mathscr{H}}(\tilde\rho_\varepsilon(t)\, B \otimes W_\varepsilon(\xi)) = \mathrm{Tr}_{\mathscr{H}}(e^{itH_0}(H_+ + 1)^{-1}(H_+ + 1)e^{-itH_\varepsilon}(H_+ + 1)^{-1} \times \\ \times (H_+ + 1)\, \rho_\varepsilon\, e^{itH_\varepsilon} e^{-itH_0}\, B \otimes W_\varepsilon(\xi)),\end{aligned}$$

the previous remarks yield the differentiability of the product above.

Pauli-Fierz model: Since, by assumptions,

$$\|(H_+ + \mathrm{d}\Gamma_\varepsilon(\mathbb{1}))\rho_\varepsilon(H_+ + \mathrm{d}\Gamma_\varepsilon(\mathbb{1}))\|_{\mathscr{L}_1} \leqslant C, \tag{5.52}$$

the proof is analogous to the one for Nelson model with no need of using a density argument.

Polaron: Let $m := \inf \sigma(H_\varepsilon)$ and consider $(H_\varepsilon + |m| + 1)^{-1}$: it maps $\mathscr{H}$ into $\mathscr{D}(H_\varepsilon)$, and then $t \mapsto e^{\pm itH_\varepsilon}(H_\varepsilon + |m| + 1)^{-1}$ is strongly differentiable. Furthermore, since $\mathscr{D}(H_\varepsilon) \subseteq \mathscr{D}(H_0^{1/2})$, $H_+^{1/2} e^{-itH_\varepsilon}(H_\varepsilon + |m| + 1)^{-1}$ is bounded, while $\|(H_\varepsilon + |m| + 1)\, \rho_\varepsilon\, (H_\varepsilon + |m| + 1)\|_{\mathscr{L}_1} \leqslant C(\varepsilon^{-2} + 1)$ and therefore $(H_\varepsilon + |m| + 1)\rho_\varepsilon(H_\varepsilon + |m| + 1)$ is trace-class. Setting $\tilde B(t) := e^{it(-\Delta+W)} B\, e^{it(-\Delta+W)}$, and using the properties of Weyl operators, we get

$$\begin{aligned}&\mathrm{Tr}_{\mathscr{H}}(\tilde\rho_\varepsilon(t) B \otimes W_\varepsilon(\xi)) = \\ &\quad = \mathrm{Tr}_{\mathscr{H}}(H_+^{1/2} e^{-itH_\varepsilon}(H_\varepsilon + |m| + 1)^{-1}(H_\varepsilon + |m| + 1)\, \rho_\varepsilon\, (H_\varepsilon + |m| + 1) \times \\ &\qquad \times (H_\varepsilon + |m| + 1)^{-1} e^{itH_\varepsilon} H_+^{1/2}\, H_+^{-1/2}\, \tilde B(t) \otimes W_\varepsilon(e^{-it}\xi) H_+^{-1/2}).\end{aligned}$$

Since $B \in \mathcal{B}$, the function $t \mapsto \tilde B(t) \otimes W_\varepsilon(e^{-it}\xi)$ is weakly differentiable in $\mathscr{D}(H_+^{1/2})$, which allows to deduce the differentiability of the product.

□

Thanks to the previous representation formula, we can prove equicontinuity in weak and weak-* topology for the generating map of the Nelson model on bounded sets in phase space. A similar result holds true also for the Pauli-Fierz model and the polaron by restricting the test operators to the spaces $\mathcal{K}, \mathcal{B}$.

Proposition 5.2.2 (Equicontinuity of generating map). *Let $\{\rho_\varepsilon\}_{\varepsilon\in(0,1)} \subseteq \mathscr{L}_{1,+}(\mathscr{H})$ satisfy (A_R) and let G_ε be the generating map for the state $\tilde\rho_\varepsilon(t)$, i.e.,*

$$\begin{aligned}G_\varepsilon :\mathbb{R} \times \mathfrak{H} &\longrightarrow \mathscr{L}_1(L^2) \\ (t,\xi) &\longmapsto \tilde\rho_\varepsilon(t)\, W_\varepsilon(\xi) =: G_\varepsilon(t,\xi)\end{aligned}$$

acting on bounded and on compact operators as

$$\begin{aligned}G_\varepsilon(t,\xi)(\kappa) &= \mathrm{Tr}_{\mathscr{H}}(\, \tilde\rho_\varepsilon(t)\, \kappa \otimes W_\varepsilon(\xi)\,), \quad \kappa \in \mathscr{L}_\infty(L^2), \\ G_\varepsilon(t,\xi)(B) &:= \mathrm{Tr}_{\mathscr{H}}(\, \tilde\rho_\varepsilon(t)\, B \otimes W_\varepsilon(\xi)\,), \quad B \in \mathscr{L}(L^2).\end{aligned}$$

Then,

- *the function $(t,\xi) \mapsto G_\varepsilon(t,\xi)$ is uniformly weak/weak* equicontinuous w.r.t. ε on bounded sets of $\mathbb{R} \times \mathfrak{H}$, for the Nelson model;*
- *the functions $(t,\xi) \mapsto G_\varepsilon(t,\xi)(\kappa), G_\varepsilon(t,\xi)(B)$, for any $\kappa \in \mathcal{K}, B \in \mathcal{B}$ are uniformly equicontinuous w.r.t. ε on bounded sets in $\mathbb{R} \times \mathfrak{H}$, for the Pauli-Fierz and polaron models.*

Proof. Let us prove the result for compact operators, since the case of bounded operators is analogous. Let $\kappa \in \mathscr{L}_\infty(L^2)$ and $(t,\xi),(s,\eta) \in I \subseteq \mathbb{R}\times\mathfrak{H}$, where I is a bounded set. Then,

$$|G_\varepsilon(t,\eta)(\kappa) - G_\varepsilon(s,\xi)(\kappa)| \leqslant X_1 + X_2,$$

where

$$X_1 := |G_\varepsilon(t,\xi)(\kappa) - G_\varepsilon(s,\xi)(\kappa)|, \quad X_2 := |G_\varepsilon(s,\xi)(\kappa) - G_\varepsilon(s,\eta)(\kappa)|.$$

For the first term we use the integral representation formula given by Proposition 5.2.1: setting $\tilde{\kappa}(t) := e^{it(-\Delta+W)}\kappa\, e^{-it(-\Delta+W)}$ and using (B.26), we have that

$$\begin{aligned} X_1 &= |\mathrm{Tr}_{\mathscr{H}}((\tilde{\rho}_\varepsilon(t) - \tilde{\rho}_\varepsilon(s))\,\kappa\otimes W_\varepsilon(\xi)\,)| \leqslant \int_s^t \mathrm{d}\tau\, |\mathrm{Tr}_{\mathscr{H}}([\tilde{H}_I(\tau),\, \tilde{\rho}_\varepsilon(\tau)]\,\kappa\otimes W_\varepsilon(\xi))| = \\ &= \int_s^t \mathrm{d}\tau\, |\mathrm{Tr}_{\mathscr{H}}(\,[H_I,\, \rho_\varepsilon(\tau)]\,\tilde{\kappa}(\tau)\otimes W_\varepsilon(e^{-i\tau\omega}\xi))|. \end{aligned}$$

We now bound the integrand for each model separately.

Nelson model: by Proposition 5.1.2, Lemma B.2.1 and by assumption (A_R) on the state,

$$\begin{aligned} \|[\rho_\varepsilon(\tau),\, H_I]\|_{\mathscr{L}_1} &\leqslant 2e^{m_\delta\sqrt{\varepsilon}|\delta||\tau|\|\lambda\|_\infty}\|\rho_\varepsilon\,(\mathrm{d}\Gamma_\varepsilon(\mathbb{1})+1)^\delta\|_{\mathscr{L}_1}\|(\mathrm{d}\Gamma_\varepsilon(\mathbb{1})+1)^{-\delta}H_I\|_{\mathscr{L}} \leqslant \\ &\leqslant 2C\,e^{c|\tau|}\|\lambda_\cdot\|_{L^\infty(\mathbb{R}^{dN};\mathfrak{H})}, \end{aligned}$$

for a certain constant $c>0$ independent on ε. W.l.o.g. we can assume that $0\leqslant s<t$, which implies

$$\begin{aligned} X_1 &\leqslant C\|\lambda_\cdot\|_{L^\infty(\mathbb{R}^{dN};\mathfrak{H})}\|\kappa\|_{\mathscr{L}}\int_s^t \mathrm{d}\tau\, e^{c_\delta\tau} = \\ &= C\|\lambda_\cdot\|_{L^\infty(\mathbb{R}^{dN};\mathfrak{H})}\|\kappa\|_{\mathscr{L}}\,c^{-1}(e^{ct}-e^{cs}) \leqslant C\,|t-s|. \end{aligned}$$

Polaron: by Lemma 1.1.4, there exist functions $\tilde{g}_\cdot \in L^\infty(\mathbb{R}^d;\mathfrak{H}^d)$, $g_{<,\cdot}\in L^\infty(\mathbb{R}^d;\mathfrak{H})$, such that

$$g_{x_j} = g_{<,x_j} + [-i\nabla_{x_j}\,,\,\tilde{g}_{x_j}], \qquad \text{for any } j=1,\dots,N. \tag{5.53}$$

Therefore, by Proposition 5.1.4, since we are testing against $\kappa\in\mathcal{K}$,

$$\begin{aligned} &\left|\sum_{j=1}^N \mathrm{Tr}_{\mathscr{H}}([\phi_\varepsilon(g_{<,x_j}) + [-i\nabla_{x_j},(a_\varepsilon^\dagger(\tilde{g}_{x_j}) - a_\varepsilon(\tilde{g}_{x_j}))],\rho_\varepsilon(\tau)]\,\tilde{\kappa}(\tau)\otimes W_\varepsilon(e^{-i\tau}\xi))\right| \leqslant \\ &\leqslant CN\Big((\|g_{<,\cdot}\|_\infty\|\kappa\| + \|\tilde{g}_\cdot\|_\infty\|\kappa(-i\nabla_\cdot)\|)\,\mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon(\tau)(\mathrm{d}\Gamma_\varepsilon(\mathbb{1})+1)^{1/2})+ \\ &\qquad + \|\tilde{g}_\cdot\|_\infty\mathrm{Tr}_{\mathscr{H}}((-\Delta_x + \mathrm{d}\Gamma_\varepsilon(\mathbb{1})^2)\rho_\varepsilon(\tau))\,\|\kappa\|\Big) \leqslant \\ &\leqslant C((|\tau|+1)e^{|\tau|}+1). \end{aligned}$$

Integrating w.r.t. τ, we obtain that $X_1 \leqslant C\,|t-s|$.

Pauli-Fierz model: Recalling the explicit expression for the Pauli-Fierz interaction term: $H_I = \sum_{j=1}^N(2i\nabla_{x_j}\cdot\phi_\varepsilon(\lambda_{x_j}) + \phi_\varepsilon(\lambda_{x_j})^2)$, we bound both terms of the sum separately. By Proposition 5.1.7, Lemma B.2.1, the fact that $\kappa\in\mathcal{K}$ and assumption (A_R), we have

$$\begin{aligned} &\left|\sum_{j=1}^N \mathrm{Tr}_{\mathscr{H}}([\phi_\varepsilon(\lambda_{x_j})\cdot\nabla_{x_j}\,,\,\rho_\varepsilon(\tau)\,]\,\kappa\otimes W_\varepsilon(\xi))\right| \leqslant \\ &\leqslant 2N\|\lambda_\cdot\|_\infty\mathrm{Tr}_{\mathscr{H}}((-\Delta_x + (\mathrm{d}\Gamma_\varepsilon(\mathbb{1})+1)^2)\rho_\varepsilon(\tau))\|\kappa\| \leqslant \\ &\leqslant C\left(e^{c|\tau|} + |\tau| + 1\right), \end{aligned}$$

and

$$\left|\sum_{j=1}^{N} \mathrm{Tr}_{\mathscr{H}}([\phi_\varepsilon^2(\lambda_{x_j}),\, \rho_\varepsilon(\tau)]\,\kappa \otimes W_\varepsilon(\xi))\right| \leqslant C\left(e^{c'|\tau|} + |\tau| + 1\right)$$

for some $c, c' > 0$. Integrating between s and t, we finally get

$$X_1 \leqslant C|t-s|.$$

Let us now turn our attention to the second term: using Corollary A.1.4 and the fact that for every model we have a bound on the number operator (5.43) at any time, we get

$$\begin{aligned} X_2 &= |\mathrm{Tr}_{\mathscr{H}}(\tilde{\rho}_\varepsilon(t)\,\kappa \otimes (W_\varepsilon(\xi) - W_\varepsilon(\eta))| \leqslant \\ &\leqslant C\mathrm{Tr}_{\mathscr{H}}((1 + \mathrm{d}\Gamma_\varepsilon(\mathbb{1}))\,\tilde{\rho}_\varepsilon(t))\|(1+\mathrm{d}\Gamma_\varepsilon(\mathbb{1}))^{-1/2}(W_\varepsilon(\xi) - W_\varepsilon(\eta))(1+\mathrm{d}\Gamma_\varepsilon(\mathbb{1}))^{-1/2}\|_{\mathscr{L}} \leqslant \\ &\leqslant C\mathrm{Tr}_{\mathscr{H}}((1 + \mathrm{d}\Gamma_\varepsilon(\mathbb{1}))\,\rho_\varepsilon(t))\,(1 + \|\xi\|_{\mathfrak{H}})\,\|\xi - \eta\|_{\mathfrak{H}} \leqslant C\|\xi-\eta\|_{\mathfrak{H}}, \end{aligned}$$

where we used that $(t,\xi) \in I$. Combining the above results, we complete the proof, obtaining

$$|G_\varepsilon(t,\xi)(\kappa) - G_\varepsilon(s,\eta)(\kappa)| \leqslant C(\|\xi-\eta\|_{\mathfrak{H}} + |t-s|), \quad (t,\xi),(s,\eta) \in I. \tag{5.54}$$

□

Proposition 5.2.2 gives equicontinuity of the generating map, that allows an extraction of a common subsequence in time from the family of states, evolved according to the microscopic dynamics of the models, converging so to the family of probability state-valued measures parametrized by the time $t \in \mathbb{R}$.

Proposition 5.2.3 (Subsequence extraction). *Let $\{\rho_\varepsilon\}_{\varepsilon\in(0,1)} \subseteq \mathscr{L}_{1,+}(\mathscr{H})$ be a family of normalized states satisfying* (A_R). *Then, for any sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}, \varepsilon_n \to 0$, there exist a subsequence $\{\varepsilon_{n_k}\}_{k\in\mathbb{N}}$ and a family of probability state-valued measures $\{\tilde{\mathfrak{m}}_t\}_{t\in\mathbb{R}} \subseteq \mathscr{M}(\mathfrak{H}; \mathscr{L}_1(L^2)_+)$, $\mathrm{d}\tilde{\mathfrak{m}}_t(z) = \gamma_{\tilde{\mathfrak{m}}_t}(z)\mathrm{d}\tilde{\mu}_t(z)$ such that, for any $t \in \mathbb{R}$,*

$$\tilde{\rho}_{\varepsilon_{n_k}}(t) \xrightarrow[k\to+\infty]{} \tilde{\mathfrak{m}}_t. \tag{5.55}$$

Furthermore, for any finite $T > 0$, there exists $C_T \in (0,+\infty)$ such that

$$\int_{\mathfrak{H}} \mathrm{d}\tilde{\mu}_t(z)\,(1 + \|z\|_{\mathfrak{H}}^2)^{\bar{\delta}} \leqslant C_T, \tag{5.56}$$

for any $t \in [-T,T]$ and any

$$\begin{cases} \bar{\delta} \leqslant \delta, & \textit{for the Nelson model,} \\ \bar{\delta} \leqslant 2, & \textit{for the polaron and Pauli-Fierz model.} \end{cases}$$

Corollary 5.2.4. *Under the same assumptions as Proposition 5.2.3,*

$$\rho_{\varepsilon_{n_k}}(t) \xrightarrow[k\to+\infty]{} \mathfrak{m}_t \tag{5.57}$$

for any $t \in \mathbb{R}$, where $\{\mathfrak{m}_t\}_{t\in\mathbb{R}} \subseteq \mathscr{M}(\mathfrak{H}; \mathscr{L}_{1,+}(L^2))$ is the family of probability state-valued measures

$$\mathfrak{m}_t = e^{-it(-\Delta+W)}\,(\Phi_t \,\sharp\, \tilde{\mathfrak{m}}_t)\,e^{it(-\Delta+W)} \tag{5.58}$$

with $\Phi_t(z) = e^{-it\nu\omega}z$. Furthermore, (5.56) *holds for $\mathfrak{m}_t$ as well.*

Proof. By Proposition 5.2.3, $\tilde{\rho}_{\varepsilon_{n_k}}(t) \to \tilde{\mathfrak{m}}_t$, for any $t \in \mathbb{R}$. Since $\rho_\varepsilon(t) = e^{-itH_0}\,\tilde{\rho}_\varepsilon(t)\,e^{itH_0}$, setting $\tilde{\kappa}(t) = e^{it(-\Delta+W)}\kappa\, e^{-it(-\Delta+W)}$ and using (B.26) for Weyl operators, we get

$$\lim_{k\to+\infty} \mathrm{Tr}_{\mathscr{H}}(\rho_{\varepsilon_{n_k}}(t)\,\kappa \otimes W_{\varepsilon_{n_k}}(\xi)) = \lim_{k\to+\infty} \mathrm{Tr}_{\mathscr{H}}(\tilde{\rho}_{\varepsilon_{n_k}}(t)\,\tilde{\kappa}(t) \otimes W_{\varepsilon_{n_k}}(\Phi_{-t}\,\xi)) =$$
$$= \int_{\mathfrak{H}} \mathrm{d}\tilde{\mu}_t(z)\,\mathrm{Tr}_{L^2}(\gamma_{\tilde{\mathfrak{m}}_t}(z)\,\tilde{\kappa}(t))\,e^{2i\Re\mathrm{e}\langle \xi \,|\, \Phi_t z\rangle} = \int_{\mathfrak{H}} \mathrm{d}\mu_t(z)\,\mathrm{Tr}_{L^2}(\gamma_{\mathfrak{m}_t}(z)\,\kappa)\,e^{2i\Re\mathrm{e}\langle \xi \,|\, z\rangle}$$

where $\mu_t = \Phi_t \sharp \tilde{\mu}_t$ and $\gamma_{\mathfrak{m}_t} = e^{-it(-\Delta+W)}\,\gamma_{\tilde{\mathfrak{m}}_t}\,e^{it(-\Delta+W)}$, and the convergence is in topology $\mathfrak{T}$. Convergence in topology $\mathfrak{P}$ is proved in the same way, exploiting the properties of Weyl quantization. We thus conclude that $\rho_{\varepsilon_{n_k}} \to \mathfrak{m}_t$ and that the latter is a probability state-valued measure, since its norm is not affected by the action of unitary operators and the push-forward. The analogue of (5.56) can be obtained along the same lines of the proof of Proposition 5.2.3. □

Proof of Proposition 5.2.3. Let $E := \{t_\ell\}_{\ell\in\mathbb{N}} \subseteq \mathbb{R}$ be a countable dense subset of $\mathbb{R}$. By the fact that H_0 commutes with the number operator for every model, we have that (5.43) holds with the same constants for $\tilde{\rho}_\varepsilon(t)$ in place of $\rho_\varepsilon(t)$. Hence, we can apply Proposition 5.1.8: there exist a subsequence $\{\varepsilon_{n_j(t_1)}\}_{n\in\mathbb{N}}$ and a probability state-valued measure $\tilde{\mathfrak{m}}_{t_1}$, such that

$$\tilde{\rho}_{\varepsilon_{n_j(t_1)}}(t_1) \xrightarrow[j\to+\infty]{} \tilde{\mathfrak{m}}_{t_1}. \tag{5.59}$$

Now, $\tilde{\rho}_{\varepsilon_{n_j(t_1)}}(t_2)$, still satisfies the assumptions of Proposition 5.1.8, so that there exist a subsequence $\{\varepsilon_{n_j(t_2)}\}_{j\in\mathbb{N}} \subseteq \{\varepsilon_{n_j(t_1)}\}_{j\in\mathbb{N}}$ and a probability state-valued measure $\tilde{\mathfrak{m}}_{t_2}$, such that

$$\tilde{\rho}_{\varepsilon_{n_j(t_2)}}(t_2) \xrightarrow[j\to+\infty]{} \tilde{\mathfrak{m}}_{t_2}. \tag{5.60}$$

We can thus apply a diagonal extraction to obtain a subsequence $\{\varepsilon_{n_k}\}_{k\in\mathbb{N}}$, common to every subsequence $\{\varepsilon_{n_j(t_\ell)}\}_{j\in\mathbb{N}}$, such that

$$\tilde{\rho}_{\varepsilon_{n_k}}(t_\ell) \xrightarrow[k\to+\infty]{} \tilde{\mathfrak{m}}_{t_\ell} \tag{5.61}$$

for any $t_\ell \in E$. Furthermore, since

$$0 \leqslant \|G_\varepsilon(t_\ell,\xi)\|_{\mathscr{L}_1(L^2)} \leqslant 1, \tag{5.62}$$

the Banach-Alaoglu theorem guarantees that the sequence $G_{\varepsilon_{n_k}}(t_\ell,\xi)$ weak-* converges to the Fourier transform $\hat{\tilde{\mathfrak{m}}}_{t_\ell}(\xi)$ for any $\ell \in \mathbb{N}$ and $\xi \in \mathfrak{H}$. Furthermore, the inequality is inherited at the level of state-valued measures, *i.e.*,

$$0 \leqslant \|\hat{\tilde{\mathfrak{m}}}_{t_\ell}(\xi)\|_{\mathscr{L}_1(L^2)} \leqslant 1. \tag{5.63}$$

By Proposition 5.2.2, we have that, for $t_i, t_j \in E, \xi \in \mathfrak{H}$, and $\kappa \in \mathscr{L}_\infty(L^2)$ or $\mathcal{K}$,

$$|G_{\varepsilon_{n_k}}(t_i,\xi)(\kappa) - G_{\varepsilon_{n_k}}(t_j,\xi)(\kappa)| \leqslant C|t_i - t_j|. \tag{5.64}$$

Since the constants are independent on ε, we can take the limit $k \to +\infty$, obtaining, thanks to (5.61),

$$\left|\hat{\tilde{\mathfrak{m}}}^{\kappa}_{t_i}(\xi) - \hat{\tilde{\mathfrak{m}}}^{\kappa}_{t_j}(\xi)\right| \leqslant C_\kappa |t_i - t_j|. \tag{5.65}$$

The previous estimate shows that the sequence $\{\hat{\tilde{\mathfrak{m}}}^{\kappa}_{t_j}(\xi)\}_{j\in\mathbb{N}}$ is Cauchy, and therefore convergent in $\mathbb{C}$: since for every $t \in \mathbb{R}$ there exists a sequence $\{t_j\}_{j\in\mathbb{N}} \subseteq E$ such that $t_j \to t$, then the following limit exists

$$G_0(t,\xi)(\kappa) := \lim_{j\to+\infty} \hat{\tilde{\mathfrak{m}}}^{\kappa}_{t_j}(\xi) \tag{5.66}$$

Furthermore, by (5.61) and uniform continuity, we have also that $\tilde{\rho}_{\varepsilon_k}(t) \xrightarrow{\mathfrak{P}} \tilde{\mathfrak{n}}_t$, for a time dependent family of state-valued measures $\{\mathfrak{n}_t\}_{t\in\mathbb{R}} \subseteq \mathscr{M}(\mathfrak{H}; \mathscr{L}_{1,+}(L^2))$. Let us now consider for concreteness the Nelson model, with $\kappa \in \mathscr{L}_\infty(L^2)$. The functional G_0 has three important properties:

1. G_0 is weak-* continuous. Fix $R > 0$ and consider $I_R := \{(t,\xi) \in E \times \mathfrak{H} \mid |t| + \|\xi\|_{\mathfrak{H}} \leqslant R\}$: thanks to (5.54) applied to the pair $G_{\varepsilon_k}(t_j,\xi)$, $G_{\varepsilon_k}(s_l,\eta)$, where $t_j \to t, s_l \to s$, taking the limit $k \to +\infty$ and then the limits $j, l \to +\infty$ we get

$$|G_0(t,\xi)(\kappa) - G_0(s,\eta)(\kappa)| \leqslant C(|t-s| + \|\xi-\eta\|_{\mathfrak{H}}), \qquad (t,\xi),(s,\eta) \in I_R. \tag{5.67}$$

Hence, $G_0 \restriction_{I_R}$ is weak* continuous in I_R, for any $R > 0$, and thus it admits a unique weak* continuous extension to $\mathbb{R} \times \mathfrak{H}$, that coincides with G_0 in (5.66) by uniqueness of the weak-* limit;

2. $\|G_0(t,0)\|_{\mathscr{L}_1} = 1$;

3. G_0 is a function of complete positive type: indeed, for $\{\kappa_j\}_{j=1}^n \subseteq \mathscr{L}_{\infty,+}(L^2)$ and $\{\xi_j\}_{j=1}^n \subseteq \mathfrak{H}$,

$$\sum_{i,k=1}^n G_0(t,\xi_i-\xi_k)(\kappa_k^*\kappa_i) = \sum_{i,k=1}^n \lim_{j\to+\infty}\lim_{\varepsilon\to 0} \mathrm{Tr}_{\mathscr{H}}(\tilde{\rho}_\varepsilon(t_j)(e^{2i\Re e\langle \xi_i-\xi_k\,|\,z\rangle}\kappa_k^*\kappa_i)_\varepsilon^{\mathrm{Weyl}}) =$$
$$= \lim_{j\to+\infty}\lim_{\varepsilon\to 0} \mathrm{Tr}_{\mathscr{H}}\left(\tilde{\rho}_\varepsilon(t_j)\left|\sum_{k=1}^n \left(e^{2i\Re e\langle \xi_k\,|\,z\rangle}\kappa_k\right)_\varepsilon^{\mathrm{Weyl}}\right|^2\right) \geqslant 0.$$

Then, Bochner theorem (Theorem 3.1.5) for cylindrical state-valued measures yields that G_0 is the Fourier transform of a unique cylindrical, probability state-valued measure $\tilde{\mathfrak{m}}_t$:

$$G_0(t,\xi)(\kappa) = \mathrm{Tr}_{L^2}(\hat{\tilde{\mathfrak{m}}}_t(\xi)\,\kappa). \tag{5.68}$$

On the other hand, we know that

$$G_\varepsilon(t,\xi) = \mathrm{w}^* - \lim_{j\to+\infty} G_\varepsilon(t_j,\xi) \tag{5.69}$$

and that G_ε is uniformly equicontinuous w.r.t. ε. Then, it is possible to exchange the limits in ε_{n_k} and t_j and obtain

$$\hat{\tilde{\mathfrak{m}}}_t(\xi) = \mathrm{w}^* - \lim_{k\to+\infty} G_{\varepsilon_{n_k}}(t,\xi), \tag{5.70}$$

which gives convergence in topology $\mathfrak{T}$. Considering the joint topology $\mathfrak{P} \vee \mathfrak{T}$, by uniqueness of limit, we obtain that $\tilde{\mathfrak{n}}_t = \tilde{\mathfrak{m}}_t$ and therefore

$$\tilde{\rho}_{\varepsilon_{n_k}}(t) \to \tilde{\mathfrak{m}}_t. \tag{5.71}$$

The only difference for the polaron and Pauli-Fierz model is that G_0 is tested against an element $\kappa \in \mathcal{K}$. In this case, considering the generating functional

$$G_0^\kappa(t,\xi) := G_0(t,\xi)(\kappa) \tag{5.72}$$

for every $\kappa \in \mathcal{K}$, it is possible to prove that its value is 1 when evaluated in $\xi = 0$, it is a function of complete positive type and continuous in $\mathbb{R} \times \mathfrak{H}$. Therefore, we can use Bochner theorem for the scalar measures and state that there exists only one probability scalar measure $\tilde{\mathfrak{m}}_t^\kappa$, such that

$$G_0^\kappa(t,\xi) = \hat{\tilde{\mathfrak{m}}}_t^\kappa(\xi) \tag{5.73}$$

for any $\kappa \in \mathcal{K}$. By using uniform equicontinuity w.r.t. ε of $G_\varepsilon(\,\cdot\,)(\kappa)$, we have that

$$\hat{\tilde{\mathfrak{m}}}_t^\kappa(\xi) = \lim_{k\to+\infty} G_{\varepsilon_{n_k}}(t,\xi)(\kappa). \tag{5.74}$$

However, by Proposition 5.1.8, there exist subsequences $\{\varepsilon_{n_{\ell(t)}}\}_{\ell\in\mathbb{N}} \subseteq \{\varepsilon_{n_k}\}_{k\in\mathbb{N}}$, such that

$$\lim_{\ell\to+\infty} G_{\varepsilon_{n_{\ell(t)}}}(t,\xi)(\kappa) = \mathrm{Tr}_{L^2}(\hat{\tilde{\mathfrak{m}}}_t(\xi)\,\kappa) \tag{5.75}$$

for any $\kappa \in \mathscr{L}_\infty(L^2)$, and then $\mathrm{Tr}_{L^2}(\hat{\tilde{\mathfrak{m}}}_t(\xi)\,\kappa) = \hat{\tilde{\mathfrak{m}}}_t^\kappa(\xi)$ for every $\kappa \in \mathcal{K}$. In conclusion, we have convergence

$$\lim_{k\to+\infty} G_{\varepsilon_{n_k}}(t,\xi)(\kappa) = \mathrm{Tr}_{L^2}(\hat{\tilde{\mathfrak{m}}}_t(\xi)\,\kappa) \tag{5.76}$$

along the subsequence $\{\varepsilon_{n_k}\}_{k\in\mathbb{N}}$ independent of time, for every $\kappa \in \mathcal{K}$.

Now, by uniform boundedness of $\tilde{\rho}_\varepsilon(t)$ and Theorem 3.1.6, we have also convergence to $\tilde{\mathfrak{m}}_t$ in topology $\mathfrak{P}$, up to a subsequence extraction. Therefore, $\tilde{\rho}_{\varepsilon_{n_k}}(t) \to \tilde{\mathfrak{m}}_t$ and $\tilde{\rho}_\varepsilon(t)$ satisfies

$$\mathrm{Tr}_{\mathscr{H}}(\tilde{\rho}_\varepsilon(t)(\mathrm{d}\Gamma_\varepsilon(\mathbb{1})+1)^{\bar{\delta}}) \leqslant C, \tag{5.77}$$

for

$$\begin{cases} \bar{\delta} \leqslant \delta, & \text{for the Nelson model,} \\ \bar{\delta} \leqslant 2, & \text{for the polaron and Pauli-Fierz model,} \end{cases}$$

for any $t \in [-T,T]$, $T > 0$, thanks to the assumptions (A_R) and Propositions 5.1.2, 5.1.7, 5.1.4. Finally, from Theorem 3.2.1, we obtain that $\tilde{\mathfrak{m}}_t$ concentrates as a probability state-valued measure on $\mathfrak{H}$ and the bound (5.56) easily follows. □

5.3 Equation for the state-valued measure

Once the existence of a common subsequence along which there is convergence in the quasi-classical limit has been established, we can take the limit and derive the effective equation for the Fourier transform of the state-valued measure. As before, we work in the interaction representation and subsequently go back to the original state-valued measure.

Proposition 5.3.1. *Let $\{\rho_\varepsilon\}_{\varepsilon\in(0,1)} \subseteq \mathscr{L}_{1,+}(\mathscr{H})$ be a family of normalized states satisfying* (A_R) *and such that*

$$\rho_{\varepsilon_n} \xrightarrow[n\to+\infty]{} \mathfrak{m}.$$

Then, for all the subsequences $\{\varepsilon_{n_k}\}_{k\in\mathbb{N}}$ as in Proposition 5.2.3, such that

$$\tilde{\rho}_{\varepsilon_{n_k}}(t) \xrightarrow[k\to+\infty]{} \tilde{\mathfrak{m}}_t$$

for any $t \in \mathbb{R}$, the map $t \mapsto \tilde{\mathfrak{m}}_t$ satisfies the following equation

$$\hat{\tilde{\mathfrak{m}}}_t(\xi) = \hat{\tilde{\mathfrak{m}}}_s(\xi) - i\int_s^t \mathrm{d}\tau \int_{\mathfrak{H}} [\,\tilde{\mathcal{V}}_\tau(\Phi_\tau z)\,,\mathrm{d}\tilde{\mathfrak{m}}_\tau(z)]\, e^{2i\Re\mathrm{e}\langle z\,|\,\xi\rangle} \tag{5.78}$$

for any $\xi \in \mathfrak{H}$ and any $s \in \mathbb{R}$, where $\Phi_t z = e^{-it\nu\omega} z$ and

$$\tilde{\mathcal{V}}_\tau(z) = e^{i\tau(-\Delta+W)}\,\mathcal{V}(z)\,e^{-i\tau(-\Delta+W)},$$

and the identity is meant in weak- sense for the trace-class operators when tested against $\kappa \in \mathscr{L}(L^2)$ for the Nelson model and $\kappa \in \mathcal{K}$ for polaron and Pauli-Fierz model, respectively.*

Proof. The representation formula from Proposition 5.2.1 holds true, *i.e.*, for any $\kappa \in \mathscr{L}_\infty(L^2)$, or $\mathcal{K}$, depending on the model, $\xi \in \mathfrak{H}$ and $t \in \mathbb{R}$,

$$\mathrm{Tr}_{\mathscr{H}}(\tilde{\rho}_\varepsilon(t)\,\kappa\otimes W_\varepsilon(\xi)) = \mathrm{Tr}_{\mathscr{H}}(\tilde{\rho}_\varepsilon(s)\,\kappa\otimes W_\varepsilon(\xi)) - i\int_s^t \mathrm{d}\tau\,\mathrm{Tr}_{\mathscr{H}}(\,[\tilde{H}_I(\tau),\tilde{\rho}_\varepsilon(\tau)]\,\kappa\otimes W_\varepsilon(\xi)). \tag{5.79}$$

Let $\{\varepsilon_n\}_{n\in\mathbb{N}}$ be a sequence such that $\rho_{\varepsilon_n} \to \mathfrak{m}$, then, by Proposition 5.1.8, we know that there exists a subsequence $\{\varepsilon_{n_k}\}_{k\in\mathbb{N}}$, such that $\tilde{\rho}_{\varepsilon_{n_k}}(t) \to \tilde{\mathfrak{m}}_t$, for any time $t \in \mathbb{R}$. Therefore, we can take the limit $\varepsilon_{n_k} \to 0$ term by term in (5.79): the term on the left hand side and the first term on the right hand side can be dealt with in the same way, by applying Theorem 3.2.1, which yields, *e.g.*,

$$\lim_{k\to+\infty} \mathrm{Tr}_{\mathscr{H}}(\tilde{\rho}_{\varepsilon_{n_k}}(t)\, \kappa \otimes W_{\varepsilon_{n_k}}(\xi)) = \mathrm{Tr}_{L^2}(\,\hat{\tilde{\mathfrak{m}}}_t(\xi)\kappa\,). \tag{5.80}$$

The second term on the right hand side, on the other hand, has already been studied in the proof of Proposition 5.2.2, when estimating the term X_1, where it is shown that it is uniformly bounded w.r.t. $\tau \in [s,t]$. Hence, by dominated convergence theorem, we can exchange the limit with the integral. Setting $\tilde{\kappa}(\tau) = e^{i\tau(-\Delta+W)}\,\kappa\, e^{-i\tau(-\Delta+W)}$, by ciclicity of the trace, we have that

$$\mathrm{Tr}_{\mathscr{H}}(\,[\tilde{H}_I(\tau),\, \tilde{\rho}_\varepsilon(\tau)]\, \kappa \otimes W_\varepsilon(\xi)) = \mathrm{Tr}_{\mathscr{H}}(\,[H_I,\, \rho_\varepsilon(\tau)]\, \tilde{\kappa}(\tau) \otimes W_\varepsilon(e^{-i\tau\nu\omega}\xi)). \tag{5.81}$$

We study now each model separately:

Nelson model: thanks to Proposition 5.1.2 and assumptions on the state, we are in condition to use Theorem 3.3.1 to obtain convergence of quantum expectations of creation and annihilation operators at any time to the classical expectations on the state-valued measure $\mathfrak{m}_\tau$, identified by Corollary 5.2.4, with $\tilde{\kappa}(\tau)$ as compact test operator:

$$\begin{aligned}\lim_{k\to+\infty} \mathrm{Tr}_{\mathscr{H}}(\,[H_I, \rho_{\varepsilon_{n_k}}(\tau)]\, \tilde{\kappa}(\tau) \otimes W_{\varepsilon_{n_k}}(e^{-i\tau\nu\omega}\xi)) &= \\ &= \int_{\mathfrak{H}} \mathrm{d}\mu_\tau(z)\, \mathrm{Tr}_{L^2}([\mathcal{V}(z), \gamma_{\mathfrak{m}_\tau}(z)]\, \tilde{\kappa}(\tau))\, e^{2i\Re e\langle \xi | e^{it\nu\omega}z\rangle} = \\ &= \int_{\mathfrak{H}} \mathrm{d}\tilde{\mu}_\tau(z)\, \mathrm{Tr}_{L^2}([\tilde{\mathcal{V}}_\tau(\Phi_\tau z, \tau), \gamma_{\tilde{\mathfrak{m}}_\tau}(z)]\, \kappa)\, e^{2i\Re e\langle \xi \,|\, z\rangle}\end{aligned}$$

with $\mathcal{V}(z) = 2\Re e\,\langle \lambda_x \,|\, z\rangle$ and we have used the fact that $\tilde{\mathfrak{m}}_t = \Phi_{-t}\,\sharp\,(e^{it(-\Delta+W)}\,\mathfrak{m}_t\, e^{-it(-\Delta+W)})$.

Polaron: by the decomposition $g_{x_j} = g_{<,x_j} + [-i\nabla_{x_j},\, \tilde{g}_{x_j}]$,

$$\begin{aligned}&\mathrm{Tr}_{\mathscr{H}}(\,[H_I,\, \rho_{\varepsilon_{n_k}}(\tau)]\, \tilde{\kappa}(\tau) \otimes W_{\varepsilon_{n_k}}(e^{-i\tau\nu\omega}\xi)) = \\ &= \sum_{j=1}^N \mathrm{Tr}_{\mathscr{H}}\Big(\big([\phi_{\varepsilon_{n_k}}(g_{<,x_j}), \rho_{\varepsilon_{n_k}}(\tau)] + [[-i\nabla_{x_j}, a^\dagger_{\varepsilon_{n_k}}(\tilde{g}_{x_j}) - a_{\varepsilon_{n_k}}(\tilde{g}_{x_j})], \rho_\varepsilon(\tau)]\big)\tilde{\kappa}(\tau) \otimes W_{\varepsilon_{n_k}}(e^{-i\tau\nu}\xi)\Big).\end{aligned}$$

By Proposition 5.1.4, we can use Theorem 3.3.1 for the first summand and Proposition 3.3.6 for the second one. Thus, we get in the limit $\varepsilon_{n_k} \to 0$

$$\begin{aligned}&\sum_{j=1}^N \mathrm{Tr}_{L^2}\Big(\int_{\mathfrak{H}} [2\Re e\,\langle z\,|g_{<,x_j}\rangle, \mathrm{d}\mathfrak{m}_\tau(z)]\, \tilde{\kappa}(\tau)\, e^{2i\Re e\langle \xi | e^{i\tau\nu}z\rangle}\Big) + \\ &+ \sum_{j=1}^N \mathrm{Tr}_{L^2}\Big(\int_{\mathfrak{H}} [[-i\nabla_{x_j},\, 2\Im m\,\langle z\,|\,\tilde{g}_{x_j}\rangle], \mathrm{d}\mathfrak{m}_\tau(z)]\, \tilde{\kappa}(\tau)\, e^{2i\Re e\langle \xi | e^{i\tau\nu}z\rangle}\Big) = \\ &= \sum_{j=1}^N \mathrm{Tr}_{L^2}\Big(\int_{\mathfrak{H}} [2\Re e\,\langle z\,|g_{x_j}\rangle, \mathrm{d}\mathfrak{m}_\tau(z)\,]\tilde{\kappa}(\tau)\, e^{2i\Re e\langle \xi | e^{i\tau\nu}z\rangle}\Big) = \\ &= \mathrm{Tr}_{L^2}\Big(\int_{\mathfrak{H}} [\,\mathcal{V}(z), \mathrm{d}\mathfrak{m}_\tau(z)\,]\tilde{\kappa}(\tau)\ e^{2i\Re e\langle \xi | e^{i\tau\nu}z\rangle}\Big)\end{aligned}$$

with $\mathcal{V}(z) = \sum_{j=1}^N 2\Re e\,\langle g_{x_j} \,|\, z\rangle$, where the latter is short for

$$\mathcal{V}(z) = \sum_{j=1}^N \big(2\Re e\,\langle z\,|g_{<,x_j}\rangle + [-i\nabla_{x_j}, 2\Im m\,\langle z\,|\tilde{g}_{x_j}\rangle]\big).$$

Using the same argument as for the Nelson model, we can complete the proof.

Pauli-Fierz model: since $H_I = -2i\nabla_x \cdot \phi_\varepsilon(\lambda_x) + \phi_\varepsilon^2(\lambda_x)$,

$$\begin{aligned}\mathrm{Tr}_{\mathscr{H}}\big(\,[H_I,\, \rho_{\varepsilon_{n_k}}(\tau)]\, \tilde{\kappa}(\tau) \otimes W_{\varepsilon_{n_k}}(e^{-i\tau\nu\omega}\xi)\big) = \\ = \sum_{j=1}^N \mathrm{Tr}_{\mathscr{H}}\big(\,[-2i\nabla_{x_j} \cdot \phi_{\varepsilon_{n_k}}(\lambda_{x_j}),\, \rho_{\varepsilon_{n_k}}(\tau)]\, \tilde{\kappa}(\tau) \otimes W_{\varepsilon_{n_k}}(e^{-i\tau\nu\omega}\xi)\big) + \\ + \sum_{j=1}^N \mathrm{Tr}_{\mathscr{H}}\big(\,[\phi^2_{\varepsilon_{n_k}}(\lambda_{x_j}),\, \rho_{\varepsilon_{n_k}}(\tau)]\, \tilde{\kappa}(\tau) \otimes W_{\varepsilon_{n_k}}(e^{-i\tau\nu\omega}\xi)\big).\end{aligned}$$

By Proposition 5.1.7, we can apply Proposition 3.3.6 to the first term of the sum and Corollary 3.3.2 to the second one, so that we obtain, in the limit $\varepsilon_{n_k} \to 0$,

$$\begin{aligned}\sum_{j=1}^N \mathrm{Tr}_{L^2}\left(\int_{\mathfrak{H}} \left[-2i\nabla_{x_j} \cdot 2\Re e\left\langle z \,\middle|\lambda_{x_j}\right\rangle,\, \mathrm{d}\mathfrak{m}_\tau(z)\right]\, \tilde{\kappa}(\tau)\, e^{2i\Re e\left\langle \xi \middle| e^{i\tau\nu\omega} z\right\rangle}\right) + \\ + \sum_{j=1}^N \mathrm{Tr}_{L^2}\left(\int_{\mathfrak{H}} \left[\left(2\Re e\left\langle z \,\middle|\lambda_{x_j}\right\rangle\right)^2,\, \mathrm{d}\mathfrak{m}_\tau(z)\right]\, \tilde{\kappa}(\tau)\, e^{2i\Re e\left\langle \xi \middle| e^{i\tau\nu\omega} z\right\rangle}\right).\end{aligned}$$

By the fact that $\mathcal{V}(z) = \sum_{j=1}^N \big(-2i\nabla_{x_j} \cdot 2\Re e\left\langle z \,\middle|\lambda_{x_j}\right\rangle + \left(2\Re e\left\langle z \,\middle|\lambda_{x_j}\right\rangle\right)^2\big)$ and by the aforementioned relation between $\tilde{\mathfrak{m}}_\tau$ and $\mathfrak{m}_\tau$, we can conclude the proof.

□

Remark 5.3.2. *Thanks to the a-priori information on the states, we can prove that the previous equation is well-defined for any model:*

Nelson model: *by properties of integration w.r.t. state-valued measures,*

$$\begin{aligned}\left\|\int_{\mathfrak{H}} [\,\tilde{\mathcal{V}}_\tau(\Phi_\tau z)\,, \mathrm{d}\tilde{\mathfrak{m}}_\tau(z)]\, e^{2i\Re e\langle z\,|\,\xi\rangle}\right\|_{\mathscr{L}_1} \leqslant \int_{\mathfrak{H}} \mathrm{d}\tilde{\mu}_\tau(z)\, \|[\,\tilde{\mathcal{V}}_\tau(\Phi_\tau z)\,, \gamma_{\tilde{\mathfrak{m}}_\tau}(z)]\|_{\mathscr{L}_1} \leqslant \\ \leqslant 2\int_{\mathfrak{H}} \mathrm{d}\tilde{\mu}_\tau(z)\, \|\gamma_{\tilde{\mathfrak{m}}_\tau}(z)\|_{\mathscr{L}_1}\, \|\mathcal{V}(\Phi_\tau z)\|_{\mathscr{L}} \leqslant 4N\|\lambda_{\cdot}\|_{L^\infty(\mathbb{R}^{dN};\mathfrak{H})} \int_{\mathfrak{H}} \mathrm{d}\tilde{\mu}_\tau(z)\, \|z\|_{\mathfrak{H}}\end{aligned}$$

and, by (5.56)*, the previous integral is uniformly bounded w.r.t.* $\tau \in [s,t]$.

Polaron: *in this case it is relevant that we are testing against an operator in* $\mathcal{K}$*:*

$$\begin{aligned}\left|\int_{\mathfrak{H}} \mathrm{d}\tilde{\mu}_\tau(z)\, \mathrm{Tr}_{L^2}\big([\,\tilde{\mathcal{V}}_\tau(\Phi_\tau z)\,, \gamma_{\tilde{\mathfrak{m}}_\tau}(z)]\, \kappa \otimes W_\varepsilon(\xi)\big)\right| \leqslant \\ \leqslant 2\sum_{j=1}^N \int_{\mathfrak{H}} \mathrm{d}\tilde{\mu}_\tau(z)\, \|\nabla_{x_j}\, \gamma_{\tilde{\mathfrak{m}}_\tau}(z)\|_{\mathbb{C}^d \otimes \mathscr{L}_1}\, \|2\Im m\left\langle \Phi_\tau z \,|\, \tilde{g}_{x_j}\right\rangle\|_{\mathbb{C}^d \otimes \mathscr{L}} \|\kappa\| + \\ + 2\sum_{j=1}^N \int_{\mathfrak{H}} \mathrm{d}\tilde{\mu}_\tau(z)\, \|\gamma_{\tilde{\mathfrak{m}}_\tau}(z)\|_{\mathscr{L}_1} \|2\Im m\left\langle \Phi_\tau z \,|\, \tilde{g}_{x_j}\right\rangle\|_{\mathscr{L}} \|\nabla_{x_j}\kappa\|_{\mathbb{C}^d \otimes \mathscr{L}} + \\ + 2\sum_{j=1}^N \int_{\mathfrak{H}} \mathrm{d}\tilde{\mu}_\tau(z) \|\gamma_{\tilde{\mathfrak{m}}_\tau}(z)\|_{\mathscr{L}_1}\, \|2\Re e\left\langle \Phi_\tau z \,\middle|g_{<,x_j}\right\rangle\|_{\mathscr{L}} \|\kappa\| \leqslant \\ \leqslant C\left[\int_{\mathfrak{H}} \mathrm{d}\tilde{\mu}_\tau(z)\, \mathrm{Tr}_{L^2}\big(\,\gamma_{\tilde{\mathfrak{m}}_\tau}(z)(-\Delta_x)\big) + \int_{\mathfrak{H}} \mathrm{d}\tilde{\mu}_\tau(z)\, \|z\|^2_{\mathfrak{H}}\right].\end{aligned}$$

By estimates (3.90) *and* (5.56)*, the integral is uniformly bounded w.r.t.* $\tau \in [s,t]$.

Pauli-Fierz model: *in this case we have*

$$\begin{aligned}
&\left|\int_{\mathfrak{H}} \mathrm{d}\tilde{\mu}_\tau(z)\, \mathrm{Tr}_{L^2}\big([\,\tilde{\mathcal{V}}_\tau(\Phi_\tau z)\,,\gamma_{\tilde{\mathfrak{m}}_\tau}(z)]\,\kappa\otimes W_\varepsilon(\xi)\big)\right| \leqslant \\
&\leqslant \sum_{j=1}^N \int_{\mathfrak{H}} \mathrm{d}\tilde{\mu}_\tau(z)\,\|\nabla_{x_j}\gamma_{\tilde{\mathfrak{m}}_\tau}(z)\|_{\mathbb{C}^d\otimes\mathscr{L}_1}\,\big\|2\Re\mathrm{e}\big\langle \Phi_\tau z\,\big|\lambda_{x_j}\big\rangle\big\|_{\mathbb{C}^d\otimes\mathscr{L}}\,\|\kappa\|_{\mathscr{L}}+\\
&\quad+\sum_{j=1}^N \int_{\mathfrak{H}} \mathrm{d}\tilde{\mu}_\tau(z)\,\|\gamma_{\tilde{\mathfrak{m}}_\tau}(z)\|_{\mathscr{L}_1}\,\big\|2\Re\mathrm{e}\big\langle \Phi_\tau z\,\big|\lambda_{x_j}\big\rangle\big\|_{\mathscr{L}}\,\|\nabla_{x_j}\kappa\|_{\mathbb{C}^d\otimes\mathscr{L}}+\\
&\quad+2\sum_{j=1}^N \int_{\mathfrak{H}} \mathrm{d}\tilde{\mu}_\tau(z)\,\|\gamma_{\tilde{\mathfrak{m}}_\tau}(z)\|_{\mathscr{L}_1}\,\big\|2\Re\mathrm{e}\big\langle \Phi_\tau z\,\big|\lambda_{x_j}\big\rangle\big\|_{\mathscr{L}}^2\,\|\kappa\|_{\mathscr{L}} \leqslant\\
&\leqslant C_1\int_{\mathfrak{H}} \mathrm{d}\tilde{\mu}_\tau(z)\,\mathrm{Tr}_{L^2}(\gamma_{\tilde{\mathfrak{m}}_\tau}(z)(-\Delta_x))\,+C_2\int_{\mathfrak{H}} \mathrm{d}\tilde{\mu}_\tau(z)\,\|z\|_{\mathfrak{H}}^2.
\end{aligned}$$

Combining (3.90) *and 5.2.3 we deduce that the integral is uniformly bounded w.r.t.* $\tau\in[s,t]$.

We are ready to derive the effective equation for the state-valued measure by taking the inverse Fourier transform of the previous equation, testing against cylindrical smooth functions.

Definition 5.3.3. *A function $f:\mathfrak{H}\to\mathbb{C}$ is a smooth, compactly supported,* ***cylindrical function****, if there exist a finite dimensional orthogonal projector $\mathbb{P}$ on $\mathfrak{H}$ and a function $g\in C_0^\infty(\mathbb{P}\mathfrak{H})$, such that, for all $z\in\mathfrak{H}$,*

$$f(z)=g(\mathbb{P}z). \tag{5.82}$$

We denote this set by $C_{0,\mathrm{cyl}}^\infty(\mathfrak{H})$.

For such functions, the Fourier transform is still a cylindrical function on $\mathbb{P}\mathfrak{H}$. If $\{e_j\}_{j\in\mathbb{N}}$ is an orthonormal basis of $\mathfrak{H}$ and $\mathbb{P}\mathfrak{H}=\mathrm{span}\{e_j\}_{j=1}^n$,

$$\hat{f}(\xi)=\int_{\mathfrak{H}}\mathrm{d}z\,e^{-2\pi i\Re\mathrm{e}\langle\xi\,|\,z\rangle}f(z)=\int_{\mathbb{P}\mathfrak{H}}\mathrm{d}z_1\dots\mathrm{d}z_n\,e^{-2\pi i\Re\mathrm{e}\sum_{j=1}^n\bar{\xi}_j z_j}g(z_1,\dots,z_n)=\hat{g}(\xi_1,\dots,\xi_n),$$

where $z_i=\langle e_i\,|\,z\rangle$ for $z\in\mathfrak{H}$.

We are finally ready to state one of the main theorem of this Chapter.

Theorem 5.3.4. *Let $\{\rho_\varepsilon\}_{\varepsilon\in(0,1)}\subseteq\mathscr{L}_{1,+}(\mathscr{H})$ be such that there exists $\delta>1$ such that*

$$\begin{cases}\|\rho_\varepsilon\,(\mathrm{d}\Gamma_\varepsilon(\mathbb{1})+1)^\delta\|_{\mathscr{L}_1}\leqslant C, & \text{for the Nelson model},\\ \mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon\,(H_++\nu_\varepsilon^{-2}H_\varepsilon^2))\leqslant C, & \text{for the polaron},\\ \mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon\,((H_++1)^2+\mathrm{d}\Gamma_\varepsilon(\mathbb{1})^2))\leqslant C, & \text{for the Pauli-Fierz model.}\end{cases} \tag{5.83}$$

Let also $\{\varepsilon_{n_k}\}_{k\in\mathbb{N}}$ be a sequence such that $\tilde{\rho}_{\varepsilon_{n_k}}(t)\to\tilde{\mathfrak{m}}_t$ as in Proposition 5.2.3. Then, for every $f\in C_{0,\mathrm{cyl}}^\infty(\mathfrak{H})$,

$$\int_{\mathfrak{H}}\mathrm{d}\tilde{\mathfrak{m}}_t(z)\,f(z)=\int_{\mathfrak{H}}\mathrm{d}\tilde{\mathfrak{m}}_s(z)\,f(z)-i\int_s^t\mathrm{d}\tau\int_{\mathfrak{H}}[\tilde{\mathcal{V}}_\tau(\Phi_\tau z),\,\mathrm{d}\tilde{\mathfrak{m}}_\tau(z)]\,f(z) \tag{5.84}$$

where the identity is meant in weak- sense, i.e., testing against compact operators in $\mathscr{L}_\infty(L^2)$ or $\mathcal{K}$, depending on the model.*

Proof. By Proposition 5.3.1, for any $f \in C^\infty_{0,\text{cyl}}(\mathfrak{H})$ with associated function $g \in C^\infty_0(\mathbb{P}\mathfrak{H})$,

$$\int_{\mathfrak{H}} \mathrm{d}\xi \; \widehat{\tilde{\mathfrak{m}}}_t(\pi\xi)\, \hat{f}(\xi) = \int_{\mathfrak{H}} \mathrm{d}\xi \; \widehat{\tilde{\mathfrak{m}}}_s(\pi\xi)\, \hat{f}(\xi) - i \int_{\mathfrak{H}} \mathrm{d}\xi \; \hat{f}(\xi) \int_s^t \mathrm{d}\tau \int_{\mathfrak{H}} [\, \tilde{\mathcal{V}}_\tau(\Phi_\tau z)\, , \mathrm{d}\tilde{\mathfrak{m}}_\tau(z)]\, e^{2i\Re e\langle z \,|\, \pi\xi\rangle}.$$

Fubini theorem and the fact that the integrand functions are cylindrical imply that

$$\begin{aligned}
&\int_{\mathfrak{H}} \mathrm{d}\xi \; \widehat{\tilde{\mathfrak{m}}}_t(\pi\xi)\, \hat{f}(\xi) = \int_{\mathfrak{H}} \mathrm{d}\tilde{\mathfrak{m}}_t(z) \int_{\mathbb{P}\mathfrak{H}\oplus\mathbb{P}\mathfrak{H}} \mathrm{d}L(\mathbb{P}y)\, \mathrm{d}L(\mathbb{P}\xi)\; e^{2\pi i \Re e\langle z-y \,|\, \xi\rangle}\, f(y) = \\
&= \int_{\mathfrak{H}} \mathrm{d}\tilde{\mathfrak{m}}_t(z) \int_{\mathbb{P}\mathfrak{H}} \mathrm{d}L(\mathbb{P}y)\, \delta(\mathbb{P}(z-y))\, g(\mathbb{P}y) = \int_{\mathfrak{H}} \mathrm{d}\tilde{\mathfrak{m}}_t(z)\, f(z),
\end{aligned}$$

where we denoted by $\mathrm{d}L(\mathbb{P}y)$ the Lebesgue measure on $\mathbb{P}\mathfrak{H}$. The second term on the right hand side can be treated in the same way:

$$\begin{aligned}
&\int_{\mathfrak{H}} \mathrm{d}\xi \; \hat{f}(\xi) \int_s^t \mathrm{d}\tau \int_{\mathfrak{H}} [\, \tilde{\mathcal{V}}_\tau(\Phi_\tau z)\, , \mathrm{d}\tilde{\mathfrak{m}}_\tau(z)]\, e^{2i\Re e\langle z \,|\, \pi\xi\rangle} = \\
&\quad = \int_s^t \mathrm{d}\tau \int_{\mathbb{P}\mathfrak{H}\oplus\mathbb{P}\mathfrak{H}} \mathrm{d}L(\mathbb{P}y)\, \mathrm{d}L(\mathbb{P}\xi) \int_{\mathfrak{H}} [\tilde{\mathcal{V}}_\tau(\Phi_\tau z),\, \mathrm{d}\tilde{\mathfrak{m}}_\tau(z)]\, e^{2\pi i \Re e\langle z-y \,|\, \xi\rangle}\, f(y) = \\
&\quad = \int_s^t \mathrm{d}\tau \int_{\mathfrak{H}} [\tilde{\mathcal{V}}_\tau(\Phi_\tau z),\, \mathrm{d}\tilde{\mathfrak{m}}_\tau(z)]\, f(z).
\end{aligned}$$

Putting together what we have proven, we get the result. □

5.3.1 Existence and uniqueness of the solution

Given (5.58), we can address the question of existence and uniqueness equivalently for $\mathfrak{m}_t$ or $\tilde{\mathfrak{m}}_t$. Before entering in the discussion, however, let us make the notion of solution more precise.

Definition 5.3.5. *Let us consider the integral equation*

$$\begin{cases} \mathrm{d}\mathfrak{n}_t(z) = \mathrm{d}\mathfrak{n}_s(z) - i \displaystyle\int_s^t \mathrm{d}\tau\, [\, \tilde{\mathcal{V}}_\tau(\Phi_\tau z),\, \mathrm{d}\mathfrak{n}_\tau(z)\,] \\ \mathrm{d}\mathfrak{n}_s(z) = \mathrm{d}\mathfrak{n}(z) \end{cases} \tag{5.85}$$

with $\mathfrak{n}_\cdot : \mathbb{R} \to \mathscr{M}(\mathfrak{H}; \mathscr{L}_{1,+}(L^2))$, $\mathcal{V}(z) : \mathscr{D}(\mathcal{V}(z)) \subseteq \mathscr{H} \to \mathscr{H}$, *and* $s, t \in \mathbb{R}$. *Then, setting*

$$\mathrm{d}\mathscr{E}(z) := \mathrm{d}\mathfrak{n}_t(z) - \mathrm{d}\mathfrak{n}_s(z) + i \int_s^t \mathrm{d}\tau\, [\, \tilde{\mathcal{V}}_\tau(\Phi_\tau z),\, \mathrm{d}\mathfrak{n}_t(z)\,], \tag{5.86}$$

a map $t \mapsto \mathfrak{n}_t$ *is called:*

- *a* weak-solution *of* (5.85), *if and only if, for any* $B \in \mathscr{L}(L^2)$ *and* $f \in C^\infty_{0,\text{cyl}}(\mathfrak{H})$,

$$\mathrm{Tr}_{L^2}\left(\int_{\mathfrak{H}} \mathrm{d}\mathscr{E}(z)\, f(z)\, B \right) = 0; \tag{5.87}$$

- *a* weak-* solution *of* (5.85), *if and only if, for any* $\kappa \in \mathscr{L}_\infty(L^2)$ *and* $f \in C^\infty_{0,\text{cyl}}(\mathfrak{H})$,

$$\mathrm{Tr}_{L^2}\left(\int_{\mathfrak{H}} \mathrm{d}\mathscr{E}(z)\, f(z)\, \kappa \right) = 0; \tag{5.88}$$

Every weak-solution is obviously also a weak- solution.*

In order to give the explicit expression of the solution we have to construct the two-parameters group of unitary operators generated by the Schrödinger operators with the effective potentials obtained in the limit. We thus denote by $U_{t,s}(z)$ such a two-parameter group weakly generated by $\mathcal{H}(\Phi_\tau z) = -\Delta_x + W + \mathcal{V}(\Phi_\tau z)$: for any $\varphi_t, \psi_s \in \mathscr{Q}(\mathcal{H}(\Phi_\tau z))$,

$$\begin{cases} \left\langle \varphi_t \,\middle|\, i\frac{\mathrm{d}}{\mathrm{d}t} U_{t,s}(z)\psi_s \right\rangle = \left\langle \varphi_t \,\middle|\, \mathcal{H}(\Phi_t z)\, U_{t,s}(z)\psi_s \right\rangle \\ \left\langle \varphi_t \,\middle|\, i\frac{\mathrm{d}}{\mathrm{d}s} U_{t,s}(z)\psi_s \right\rangle = -\left\langle \varphi_t \,\middle|\, U_{t,s}(z)\mathcal{H}(\Phi_s z)\, \psi_s \right\rangle, \\ U_{s,s}(z) = \mathbb{1}. \end{cases} \tag{5.89}$$

We also denote by $\tilde{U}_{t,s}(z)$ the group weakly generated by $\tilde{\mathcal{V}}_\tau(\Phi_\tau z)$: for any $\varphi_t, \psi_s \in \mathscr{Q}(\tilde{\mathcal{V}}_\tau(\Phi_\tau z))$

$$\begin{cases} \left\langle \varphi_t \,\middle|\, i\frac{\mathrm{d}}{\mathrm{d}t} \tilde{U}_{t,s}(z)\psi_s \right\rangle = \left\langle \varphi_t \,\middle|\, \tilde{\mathcal{V}}_t(\Phi_t z)\, \tilde{U}_{t,s}(z)\psi_s \right\rangle, \\ \left\langle \varphi_t \,\middle|\, i\frac{\mathrm{d}}{\mathrm{d}s} \tilde{U}_{t,s}(z)\psi_s \right\rangle = -\left\langle \varphi_t \,\middle|\, \tilde{U}_{t,s}(z)\tilde{\mathcal{V}}_s(\Phi_s z)\, \psi_s \right\rangle, \\ \tilde{U}_{s,s}(z) = \mathbb{1}. \end{cases} \tag{5.90}$$

Being existence of the previous groups perfectly equivalent, we study $\tilde{U}_{t,s}$ for the scaling $\nu_\varepsilon = \varepsilon^{-1}$ in the Nelson model and the existence of $U_{t,s}$ afterwards. For the polaron and Pauli-Fierz model we follow the opposite strategy, proving the existence of $U_{t,s}$ for $\nu_\varepsilon = \varepsilon^{-1}$ by means of results from [76] and [73], and then for $\tilde{U}_{t,s}$ in the scaling $\nu_\varepsilon = 1$.

Proposition 5.3.6. *There exists a two-parameter group of unitary operators $\{U_{t,s}(z)\}_{t,s\in\mathbb{R}}$ (resp. $\{\tilde{U}_{t,s}(z)\}_{t,s\in\mathbb{R}}$) satisfying* (5.89) *(resp.* (5.90)*) for every model and $\nu \in \{0,1\}$. Furthermore, we have*

$$U_{t,s}(z) = e^{-it(-\Delta+W)}\, \tilde{U}_{t,s}(z)\, e^{is(-\Delta+W)}. \tag{5.91}$$

Proof. We first prove (5.91):

$$i\frac{\mathrm{d}}{\mathrm{d}t} U_{t,s}(z) = (-\Delta + W)\, U_{t,s}(z) + e^{-it(-\Delta+W)}\, i\frac{\mathrm{d}}{\mathrm{d}t} \tilde{U}_{t,s}(z)\, e^{is(-\Delta+W)}$$

which implies that $U_{t,s}(z)$ is a solution of (5.89) if and only if $\tilde{U}_{t,s}(z)$ solves (5.90).

If $\nu = 0$, the radiation does not evolve in time and $\mathcal{V}(\Phi_\tau z) = \mathcal{V}(z)$ is time-independent. In this case $U_{t,s}(z) = U_{t-s}(z)$ is the one-parameter group generated by the self-adjoint operator $\mathcal{H}(z) = -\Delta_x + W + \mathcal{V}(z)$, and existence follows from Stone Theorem.

If $\nu = 1$, we consider each model separately:

Nelson model:

$\tilde{\mathcal{V}}_\tau(\Phi_\tau z) = e^{i\tau(-\Delta+W)}\, 2\Re e \left\langle \lambda_x \,\middle|\, e^{-it\omega} z \right\rangle\, e^{-i\tau(-\Delta+W)}$ is a bounded operator in $L^2(\mathbb{R}^{dN})$ and the existence of the two-parameter group $\tilde{U}_{t,s}(z)$ weakly generated by it is trivial.

Polaron:

We want to use [76, Theorem II.27 + Corollary II.28], so let us verify that the conditions are satisfied: consider the operator $\mathcal{H}_+ = -\Delta_x + W_+(x_1, \ldots, x_N)$ on $L^2(\mathbb{R}^{dN})$ with domain $\mathscr{D}(\mathcal{H}_+) = \mathscr{D}(-\Delta_x + W_+)$ and also $H_I(\tau) = W_{\ll} + \sum_{j=1}^N \left(2\Re e \left\langle \Phi_\tau z \,\middle|\, g_{<,x_j} \right\rangle + [-i\nabla_{x_j}, 2\Im m \left\langle \Phi_\tau z \,\middle|\, \tilde{g}_{x_j} \right\rangle]\right)$, then

1. $W_{\ll}$ is infinitesimally bounded w.r.t. $\mathcal{H}_+$: for any $\psi \in \mathscr{D}(\mathcal{H}_+^{1/2})$,

$$\begin{aligned}
|\langle\psi\,|\,H_I(t)\psi\rangle| &\leqslant \sum_{j=1}^{N}\Big(\big|\big\langle\psi\,\big|2\Re e\big\langle\Phi_t z\,|\,g_{<,x_j}\big\rangle\phi\big\rangle\big| + 2\big|\big\langle\nabla_{x_j}\psi\,|\,2\Im m\big\langle\Phi_t z\,|\,\tilde{g}_{x_j}\big\rangle\psi\big\rangle\big|\Big) \leqslant \\
&\leqslant 2N\|z\|_{\mathfrak{H}}\|g_{<,\cdot}\|_\infty\|\psi\|^2 + 4\sum_{j=1}^{N}\|\nabla_{x_j}\psi\|\|\psi\|\|z\|_{\mathfrak{H}}\|\tilde{g}_\cdot\|_\infty \leqslant \\
&\leqslant a\langle\psi\,|-\Delta_x\psi\rangle + 2N(8a^{-1}\|z\|^2_{\mathfrak{H}}\|\tilde{g}_\cdot\|^2_\infty + \|z\|_{\mathfrak{H}}\|g_{<,\cdot}\|_\infty)\|\psi\|^2 \leqslant \\
&\leqslant a\langle\psi\,|\,\mathcal{H}_+\psi\rangle + C(N,a,z)\|\psi\|^2.
\end{aligned}$$

   Choosing $a > 0$ small enough, we obtain that $H_I(t)$ is infinitesimally form-bounded w.r.t. $\mathcal{H}_+$;
2. for any $\psi \in \mathscr{Q}(\mathcal{H}_+) = (\mathcal{H}_+ + 1)^{-1/2}L^2(\mathbb{R}^{dN})$, since $i\frac{\mathrm{d}}{\mathrm{d}\tau}\Phi_\tau z = \Phi_\tau z$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\psi\,|\,H_I(t)\psi\rangle = \sum_{j=1}^{N}\Big(\big\langle\psi\,|\,2\Im m\big\langle g_{<,x_j}\,|\Phi_t z\big\rangle\psi\big\rangle + \big\langle\psi\,\big|[-i\nabla_{x_j}, 2\Re e\big\langle\Phi_t z\,|\tilde{g}_{x_j}\big\rangle]\psi\big\rangle\Big)$$

   which can bounded as we did above, obtaining that $\dot{H}_I(t)$ is infinitesimal form-bounded w.r.t. $\mathcal{H}_+$.

Since both the conditions are satisfied, we can apply the aforementioned result and obtain that there exists a two-parameter group of unitary operators $\{U_{t,s}(z)\}_{t,s\in\mathbb{R}}$ weakly generated by $\mathcal{H}(t)$ in $\mathscr{Q}(H_0)$.

Pauli-Fierz model:

In this case, set $\mathcal{H}(\tau) := -\Delta_x + W + \mathcal{V}(\Phi_\tau)$. Then, the map $\tau \mapsto \mathcal{H}(\tau)$ is strongly continuously differentiable in $L^2(\mathbb{R}^{dN})$. Let us show explicitly only the continuity of the derivative, the existence of the latter and the continuity of the function being proven in the very same way: in strong sense

$$\dot{\mathcal{H}}(t) = \sum_{j=1}^{N}\left(2i\nabla_{x_j}\cdot 2\Im m\left\langle\omega\lambda_{x_j}\middle|\,e^{-it\omega}z\right\rangle + 8\,\Re e\left\langle\lambda_{x_j}\middle|\,e^{-it\omega}z\right\rangle\cdot\Im m\left\langle\omega\lambda_{x_j}\middle|\,e^{-it\omega}z\right\rangle\right). \quad (5.92)$$

Hence, if $\psi \in \mathscr{D}(\mathcal{H}(\tau))$,

$$\begin{aligned}
\|(\dot{\mathcal{H}}(t+h) - \dot{\mathcal{H}}(t))\psi\|_{L^2} \leqslant 4\sum_{j=1}^{N}\Bigg(&\|\partial_{x_j}\psi_s\|_{L^2}\|\omega\lambda_\cdot\|_{L^\infty}\|(e^{-i(t+h)\omega} - e^{-it\omega})z\|_{\mathfrak{H}} + \\
&+ 16\,\|\psi\|_{L^2}\,\|\lambda_\cdot\|_{L^\infty}\,\|\omega\lambda_\cdot\|_{L^\infty}\,\|z\|_{\mathfrak{H}}\,\|(e^{-i(t+h)\omega} - e^{-it\omega})z\|_{\mathfrak{H}}\Bigg) \xrightarrow[h\to 0]{} 0
\end{aligned}$$

Hence, since all the operators $\{\mathcal{H}(\tau)\}_{\tau\in\mathbb{R}}$ share a common domain of self-adjointness $\mathscr{D}(\mathcal{H}(t)) = \mathscr{D}(-\Delta + W)$, we can use [73, Theorem 2.1] that ensures the existence of a two-parameter group of unitary operators $U_{t,s}(z)$ solving (5.89).

□

We are now ready to give the solution to the integral equation for the state-valued measure.

Proposition 5.3.7. *Consider the integral equation, for any $s, t \in \mathbb{R}$,*

$$\begin{cases} \mathrm{d}\tilde{\mathfrak{m}}_t(z) = \mathrm{d}\tilde{\mathfrak{m}}_s(z) \; - i \displaystyle\int_s^t \mathrm{d}\tau \, [\tilde{\mathcal{V}}_\tau(\Phi_\tau z), \, \mathrm{d}\tilde{\mathfrak{m}}_\tau(z)], \\ \mathrm{d}\tilde{\mathfrak{m}}_s(z) = \mathrm{d}\mathfrak{m}(z), \end{cases} \tag{5.93}$$

where $\mathfrak{m} \in \mathscr{M}(\mathfrak{H}; \mathscr{L}_{1,+}(L^2))$, and for any $t \in \mathbb{R}$,

$$\begin{cases} \displaystyle\int_{\mathfrak{H}} \mathrm{d}\tilde{\mu}(z) \, \|z\|_{\mathfrak{H}} \leqslant C, & \textit{for the Nelson model;} \\ \displaystyle\int_{\mathfrak{H}} \mathrm{d}\tilde{\mu}(z) \, \mathrm{Tr}_{L^2}(\gamma_{\tilde{\mathfrak{m}}}(z)(-\Delta_x + \|z\|^2_{\mathfrak{H}})) \leqslant C, & \textit{for the polaron;} \\ \displaystyle\int_{\mathfrak{H}} \mathrm{d}\tilde{\mu}(z) \, \mathrm{Tr}_{L^2}(\gamma_{\tilde{\mathfrak{m}}}(z)(-\Delta_x + \|z\|^2_{\mathfrak{H}})) \leqslant C, & \textit{for the Pauli-Fierz model.} \end{cases} \tag{5.94}$$

Then, there exists a unique weak- solution to the equation* (5.93)*, which has the explicit expression*

$$\mathrm{d}\tilde{\mathfrak{m}}_t(z) = \tilde{U}_{t,s}(z) \, \mathrm{d}\mathfrak{m}(z) \, \tilde{U}^\dagger_{t,s}(z) \tag{5.95}$$

i.e., its Radon-Nykodym decomposition is

$$\tilde{\mu}_t = \mu, \qquad \gamma_{\tilde{\mathfrak{m}}_t}(z) = \tilde{U}_{t,s}(z) \, \gamma_{\mathfrak{m}}(z) \, \tilde{U}^\dagger_{t,s}(z). \tag{5.96}$$

Furthermore, this solution is continuously differentiable on every Borel set of $\mathfrak{H}$ w.r.t. the strong topology in $\mathscr{L}_1(L^2)$.

Proof. Recalling Remark 5.3.2 about well-posedness of equation (5.78), we see that, if $\tilde{\mathfrak{m}}_t$ satisfies conditions (5.94), then the term inside the integral is integrable and so the map

$$t \mapsto \int_s^t \mathrm{d}\tau \int_{\mathfrak{H}} [\, \tilde{\mathcal{V}}_\tau(\Phi_\tau z), \, \mathrm{d}\tilde{\mathfrak{m}}_\tau(z) \,] \tag{5.97}$$

is $C^1(\mathbb{R}; \mathscr{L}_1(L^2))$ in trace-class norm topology, which implies continuous differentiability of the solution.

Therefore, we can differentiate and obtain that $\tilde{\mathfrak{m}}_t$ is a solution of the integral equation, if and only if it is a solution of the differential equation

$$i \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathfrak{H}} \mathrm{d}\tilde{\mathfrak{m}}_t(z) \; = \int_{\mathfrak{H}} [\, \tilde{\mathcal{V}}_t(\Phi_t z), \, \mathrm{d}\tilde{\mathfrak{m}}_t(z) \,]. \tag{5.98}$$

It is however easy to see that, by direct inspection, (5.95) solves the previous equation, and therefore it is also a weak and weak* solution of (5.93).

To prove uniqueness, let us define the measure

$$\mathrm{d}\mathfrak{n}_t(z) = \tilde{U}^\dagger_{t,s}(z) \, \mathrm{d}\tilde{\mathfrak{m}}_t(z) \, \tilde{U}_{t,s}(z) \tag{5.99}$$

where $\tilde{\mathfrak{m}}_t$ stands here for a generic weak solution of (5.93) and (5.98). Let $A \in \mathrm{Borel}(\mathfrak{H})$ be a Borel set, then

$$\begin{aligned} i \frac{\mathrm{d}}{\mathrm{d}t} \mathfrak{n}_t(A) = &- \int_A \tilde{U}^\dagger_{t,s}(z) \, [\, \tilde{\mathcal{V}}_t(\Phi_t z) \, , \, \mathrm{d}\tilde{\mathfrak{m}}_t(z)] \, \tilde{U}_{t,s}(z) + \\ &+ \int_A \tilde{U}^\dagger_{t,s}(z) \, i \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{d}\tilde{\mathfrak{m}}_t(z) \, \tilde{U}_{t,s}(z) = 0, \end{aligned}$$

i.e., $\mathfrak{n}_t(A) = \mathfrak{n}_0(A) = \mathfrak{m}(A)$. This means that $\mathfrak{n}_t$ is independent of time and therefore $\tilde{\mathfrak{m}}_t$ is unique. □

Proposition 5.3.7 permits to identify $\mathfrak{m}_t$, as shown in the next

Corollary 5.3.8. *Let $\{\rho_\varepsilon\}_{\varepsilon\in(0,1)} \subseteq \mathscr{L}_{1,+}(\mathscr{H})$ be a family of normalized states satisfying* (A_R)*, then there exists a subsequence $\varepsilon_n \to 0$ such that*

$$\rho_{\varepsilon_n}(t) \xrightarrow[n\to+\infty]{} \mathfrak{m}_t \tag{5.100}$$

for any $t \in \mathbb{R}$, with

$$\mathrm{d}\mathfrak{m}_t(z) = U_{t,0}(z)\,\mathrm{d}(\Phi_t \,\sharp\, \mathfrak{m})(z)\,U^\dagger_{t,0}(z). \tag{5.101}$$

Proof. Convergence (5.100) holds at time zero over a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ thanks to Theorem 3.2.1. By Corollary 5.2.4 we have (5.100) at any times, over a subsequence $\{\varepsilon_{n_k}\}_{k\in\mathbb{N}}$, with

$$\mathrm{d}\mathfrak{m}_t(z) = e^{-it(-\Delta+W)}\,\mathrm{d}(\Phi_t \,\sharp\, \tilde{\mathfrak{m}}_t)(z)\,e^{it(-\Delta+W)}, \tag{5.102}$$

where, by Theorem 5.3.4, $\tilde{\mathfrak{m}}_t$ is solution of equation (5.84). Thanks to Proposition 5.3.7, the solution of the latter is unique and given by expression (5.96) and this is true for any convergent subsequence $\{\rho_{\varepsilon_{n_k}}(t)\}_{k\in\mathbb{N}}$, which implies that the convergence holds true over the original $\{\varepsilon_n\}_{n\in\mathbb{N}}$. Combining these information, we have

$$\begin{aligned}
&\mathrm{Tr}_{L^2}\left(\int_{\mathfrak{H}} \mathrm{d}\mathfrak{m}_t(z)\,f(z)\,\kappa\right) = \mathrm{Tr}_{L^2}\left(\int_{\mathfrak{H}} \mathrm{d}\tilde{\mathfrak{m}}_t(z)\,f(\Phi_t z)\,\tilde{\kappa}(t)\right) = \\
&= \mathrm{Tr}_{L^2}\left(\int_{\mathfrak{H}} \mathrm{d}\mathfrak{m}(z)\,f(\Phi_t z)\,\tilde{U}^\dagger_{t,0}(z)\,e^{it(-\Delta+W)}\,\kappa\,e^{-it(-\Delta+W)}\,\tilde{U}_{t,0}(z)\right) = \\
&= \mathrm{Tr}_{L^2}\left(\int_{\mathfrak{H}} \mathrm{d}\mathfrak{m}(z)\,f(\Phi_t z)\,U^\dagger_{t,0}(z)\,\kappa\,U_{t,0}(z)\right),
\end{aligned}$$

where we used (5.91). □

We have thus completed the derivation of the effective equation for initial states satisfying assumptions (A_R). In the next section we relax such conditions and show that the result applies to a larger class of states.

5.4 Weaker assumptions on the initial states

This is the last step of the proof of Theorem 1.3.1: we proved that, assuming (A_R), if $\rho_{\varepsilon_n} \to \mathrm{d}\mathfrak{m}(z) = \mathrm{d}\mu(z)\gamma_{\mathfrak{m}}(z)$, then

$$\rho_{\varepsilon_n}(t) \to \mathrm{d}\mathfrak{m}_t(z) = \mathrm{d}(\Phi_t\sharp\mu)(z)U_{t,0}(z)\gamma_{\mathfrak{m}}(z)U_{t,0}(z) \tag{5.103}$$

Let now $\rho_\varepsilon \in \mathscr{L}_{1,+}(\mathscr{H})$ be a family of normalized states such that, for any $\delta > 0$, the original assumption (A1) is satisfied, *i.e.*,

$$\begin{cases}
\sup_{\varepsilon\in(0,1)} \mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon(-\Delta_x + W_+ + (\mathrm{d}\Gamma_\varepsilon(\mathbb{1})+1)^\delta)) \leqslant C, & \text{for the Nelson model;} \\
\sup_{\varepsilon\in(0,1)} \mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon(-\Delta_x + W_+ + (\mathrm{d}\Gamma_\varepsilon(\mathbb{1})+1)^\delta) \leqslant C, & \text{for the polaron;} \\
\sup_{\varepsilon\in(0,1)} \mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon(H_+ + (\mathrm{d}\Gamma_\varepsilon(\mathbb{1})+1)^\delta)) \leqslant C, & \text{for the Pauli-Fierz model.}
\end{cases} \tag{A1}$$

Define

$$\rho^{(r)}_\varepsilon := \frac{1}{Z_r}\chi_r\,\rho_\varepsilon\,\chi_r, \qquad Z_r = \mathrm{Tr}_{\mathscr{H}}(\chi_r\,\rho_\varepsilon\,\chi_r), \tag{5.104}$$

where $\chi_r := \chi_r(S)\chi_r(T)$, with $\chi_r(x) := \chi(x/r)$, $\chi \in C_0^\infty(\mathbb{R})$ a suitable cut-off function equal to 1 around the origin and vanishing far from it, and

$$S := (\mathrm{d}\Gamma_\varepsilon(\mathbb{1}) + 1)^{\delta/2}, \qquad T := \begin{cases} (-\Delta_x + W_+)^{1/2}, & \text{for the Nelson model,} \\ (-\Delta_x + W_+)^{1/2}, & \text{for the polaron,} \\ H_+^{1/2}, & \text{for the Pauli-Fierz model.} \end{cases} \tag{5.105}$$

The family of semiclassical states $\{\rho_\varepsilon^{(r)}\}_{\varepsilon\in(0,1)}$ satisfies assumption (A_R), and furthermore

$$\begin{aligned}
&\|\rho_\varepsilon(t) - \rho_\varepsilon^{(r)}(t)\|_{\mathscr{L}_1(\mathscr{H})} \leqslant \|\rho_\varepsilon - \rho_\varepsilon^{(r)}\|_{\mathscr{L}_1(\mathscr{H})} \leqslant \\
&\leqslant \left\| \rho_\varepsilon - \frac{\chi_r}{Z_r}\rho_\varepsilon + \frac{\chi_r}{Z_r}\rho_\varepsilon - \frac{\chi_r}{Z_r}\rho_\varepsilon\chi_r \right\|_{\mathscr{L}_1} \leqslant \\
&\leqslant \left\| \left(1 - \frac{\chi_r}{Z_r}\right) T^{-1/2}S^{-1/2} \right\| \|T^{1/2}S^{1/2}\rho_\varepsilon T^{1/2}S^{1/2}\|_{\mathscr{L}_1} \|T^{-1/2}S^{-1/2}\| + \\
&\qquad + \left\| \frac{\chi_r}{Z_r} T^{-1/2}S^{-1/2} \right\| \|T^{1/2}S^{1/2}\rho_\varepsilon T^{1/2}S^{1/2}\|_{\mathscr{L}_1} \left\| T^{-1/2}S^{-1/2}(1-\chi_r) \right\| \leqslant \\
&\leqslant C \sup_{\varepsilon\in(0,1)} \mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon(S^2+T^2)) \left[\left\| \left(1 - \frac{\chi_r}{Z_r}\right) T^{-1/2}S^{-1/2} \right\| + \left\| T^{-1/2}S^{-1/2}(1-\chi_r) \right\| \right] \leqslant \\
&\leqslant C \left[\left\| \left(1 - \frac{\chi_r}{Z_r}\right) T^{-1/2}S^{-1/2} \right\| + \left\| T^{-1/2}S^{-1/2}(1-\chi_r) \right\| \right] = o_r(1)
\end{aligned} \tag{5.106}$$

uniformly in ε, where we used the estimate

$$\left\| T^{-1/2}S^{-1/2}(1-\chi_r) \right\| \leqslant C_1 \sup_{\lambda\in\sigma(S)} \left| \frac{1-\chi_r(\lambda)}{\lambda^{1/2}} \right| + C_2 \sup_{\lambda\in\sigma(T)} \left| \frac{1-\chi_r(\lambda)}{\lambda^{1/2}} \right| = o_r(1). \tag{5.107}$$

By Theorem 3.2.1, for every r, there exists a sequence $\{\varepsilon_{n(r)}\}_{n\in\mathbb{N}}$ and a probability state-valued measure $\mathfrak{m}^{(r)} \in \mathscr{M}(\mathfrak{H}; \mathscr{L}_{1,+}(L^2))$, such that $\rho^{(r)}_{\varepsilon_{n(r)}} \to \mathfrak{m}_t^{(r)}$. By Corollary 5.2.4, there exists a subsequence $\{\varepsilon_{n_k(r)}\}_{k\in\mathbb{N}}$ such that

$$\rho^{(r)}_{\varepsilon_{n_k(r)}}(t) \xrightarrow[k\to+\infty]{} \mathrm{d}\mathfrak{m}_t^{(r)}(z) = U_{t,0}(z)\, \mathrm{d}(\Phi_t \sharp \mathfrak{m}^{(r)})(z)\, U_{t,0}^\dagger(z). \tag{5.108}$$

Now, again by Theorem 3.2.1, we can extract a time-dependent subsequence $\{\varepsilon_{n_j(r,t)}\}$ from $\{\varepsilon_{n(r)}\}$ such that

$$\rho_{\varepsilon_{n_j(r,t)}}(t) \xrightarrow[j\to+\infty]{} \mathfrak{m}'_t \in \mathscr{M}(\mathfrak{H}; \mathscr{L}_1(L^2)_+). \tag{5.109}$$

We want now to prove that $\mathfrak{m}'_t = \mathfrak{m}_t$. To this purpose, we adapt an argument from [8, Proposition 2.10]: consider $b \in \mathscr{S}_{\mathrm{cyl}}(\mathfrak{H})$ with base $\mathbb{P}\mathfrak{H}$ for a finite dimensional orthogonal projector $\mathbb{P}$, then by pseudodifferential calculus in finite dimension (see Appendix B.1),

$$\|\mathrm{Op}^\varepsilon_{1/2}(b)\|_{\mathscr{L}(\mathscr{H})} \leqslant \|b\|_{L^\infty(\mathbb{P}\mathfrak{H})} + O(\varepsilon). \tag{5.110}$$

Hence, by Corollary 3.1.7 and Theorem 3.2.1, for any $\kappa \in \mathscr{L}_{\infty,+}(L^2)$,

$$\left| \int_{\mathfrak{H}} \mathrm{d}\left|\mu'_t - \mu_t^{(r)}\right|(z)\, \mathrm{Tr}_{L^2}\left((\gamma_{\mathfrak{m}'_t}(z) - \gamma_{\mathfrak{m}_t^{(r)}}(z))\,\kappa \right)\, b(z) \right| = \tag{5.111}$$

$$\begin{aligned}
&= \lim_{\varepsilon_{n_j(r,t)}\to 0} \left| \mathrm{Tr}_{\mathscr{H}}\left(\left| \rho_{\varepsilon_{n_j(r,t)}}(t) - \rho^{(r)}_{\varepsilon_{n_j(r,t)}}(t) \right| \mathrm{Op}_{1/2}^{\varepsilon_{n_j(r,t)}}(b)\,\kappa \right) \right| \leqslant \\
&\leqslant \lim_{\varepsilon_{n_j(r,t)}\to 0} \left\| \rho_{\varepsilon_{n_j(r,t)}}(t) - \rho^{(r)}_{\varepsilon_{n_j(r,t)}}(t) \right\|_{\mathscr{L}_1} \|b\|_{L^\infty(\mathbb{P}\mathfrak{H})} \|\kappa\| = o_r(1)
\end{aligned} \tag{5.112}$$

uniformly in ε by (5.106). Setting $\mathfrak{m}_1 = \mathfrak{m}'_t, \mathfrak{m}_2 = \mathfrak{m}_t^{(r)}$, and $\mathfrak{n} = \mathfrak{m}'_t - \mathfrak{m}_t^{(r)}$ for short, for any $\kappa \in \mathscr{L}_{\infty,+}(L^2)$, $|\mathfrak{n}|^\kappa$ is absolutely continuous w.r.t. $\left|\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right|^\kappa$, the latter being a Radon positive measure on $\mathfrak{H}$. Then, there exists a measurable function $\lambda^\kappa : \mathfrak{H} \to \mathbb{R}_+$, such that

$$\mathrm{d}|\mathfrak{n}|^\kappa(z) = |\lambda^\kappa|(z)\,\mathrm{d}\left|\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right|^\kappa(z),$$

and the previous relation holds true for any $\kappa \in \mathscr{L}_\infty(L^2)$, using the decomposition in positive operators. Furthermore,

$$\sup_{\kappa\in\mathscr{L}_\infty(L^2)} \int_{\mathfrak{H}} \mathrm{d}\left|\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right|^\kappa(z)\,|\lambda^\kappa(z)| = \sup_{\kappa\in\mathscr{L}_\infty(L^2)} |\mathfrak{n}|^\kappa(\mathfrak{H}) = \||\mathfrak{m}_1-\mathfrak{m}_2|(\mathfrak{H})\|_{\mathscr{L}_1} \leqslant 2.$$

Then, $|\lambda^\kappa(z)| \leqslant 2$ for $\left|\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right|^\kappa$ a.e. z, uniformly in κ. By [8], $\mathscr{S}_{\mathrm{cyl}}(\mathfrak{H})$ is dense in $L^p(\mathbb{P}\mathfrak{H}; \mathrm{d}\left|\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right|^\kappa)$, with $p \in [1,\infty)$, so that there exists a sequence $\{\beta_n^\kappa\}_{n\in\mathbb{N}} \subseteq \mathscr{S}_{\mathrm{cyl}}(\mathfrak{H})$, such that

$$\lim_{n\to+\infty}\left\|\beta_n^\kappa - \frac{|\lambda^\kappa|}{\lambda^\kappa}\mathbb{1}_{\{\lambda^\kappa\neq 0\}}\right\|_{L^1(\mathfrak{H};\,\mathrm{d}(\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2})^\kappa)} = 0.$$

Moreover, there exists a subsequence $\{\beta_{n_j}^\kappa\}_{j\in\mathbb{N}}$, such that $\beta_{n_j}^\kappa(z) \to \frac{|\lambda^\kappa(z)|}{\lambda^\kappa(z)}\mathbb{1}_{\{\lambda^\kappa\neq 0\}}(z)$, for $\left|\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right|^\kappa$ a.e. z in $\mathfrak{H}$. Setting $b_j^\kappa(z) = 2\frac{\beta_{n_j}^\kappa}{1+|\beta_{n_j}^\kappa|^2}$, we have obtained a sequence in $\mathscr{S}_{\mathrm{cyl}}(\mathfrak{H})$ such that $\|b_j^\kappa\|_{L^\infty(\mathbb{P}\mathfrak{H})} \leqslant 1$ uniformly in κ and

$$b_j(z) \xrightarrow[j\to+\infty]{} \frac{\lambda^\kappa(z)}{|\lambda^\kappa(z)|}\mathbb{1}_{\{\lambda^\kappa\neq 0\}}(z), \quad \text{for } \left|\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right|^\kappa \text{ a.e. } z \in \mathfrak{H}.$$

Using now a net of approximate identities $\{e_\alpha\}_{\alpha\in I} \subseteq \mathscr{L}_{\infty,+}(L^2)$ and, denoting by ν and $\varrho_{\mathfrak{n}} = \left\|\frac{\mathrm{d}|\mathfrak{n}|}{\mathrm{d}\|\mathfrak{n}\|}\right\|$ the scalar measure and the Radon-Nykodým derivative of $|\mathfrak{n}|$, respectively, we get

$$\begin{aligned}\int_{\mathfrak{H}} \mathrm{d}\nu(z) &= \lim_{\alpha\in I}\int_{\mathfrak{H}} \mathrm{d}|\mathfrak{n}|^{e_\alpha}(z)\,\varrho_{\mathfrak{n}}(z) = \lim_{\alpha\in I}\int_{\mathfrak{H}} \mathrm{d}\left|\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right|^{e_\alpha}(z)\,|\lambda^{e_\alpha}(z)|\,\varrho_{\mathfrak{n}}(z) = \\ &= \lim_{\alpha\in I}\lim_{j\to+\infty}\int_{\mathfrak{H}} \mathrm{d}\left|\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right|^{e_\alpha}(z)\,\lambda^{e_\alpha}(z)\,b_j^\kappa(z)\,\varrho_{\mathfrak{n}}(z) = \\ &\leqslant \lim_{\alpha\in I}\lim_{j\to+\infty}\int_{\mathfrak{H}} \mathrm{d}|\mathfrak{n}|^{e_\alpha}(z)\,b_j^\kappa(z)\,\varrho_{\mathfrak{n}}(z) \leqslant o_r(1)\lim_{\alpha\in I}\lim_{j\to+\infty}\|b_j^\kappa\|_\infty\,\|e_\alpha\|_{\mathscr{L}} = o_r(1)\end{aligned}$$

by (5.112). Therefore, denoting with a little abuse of notation by $|\mu_\cdot - \mu_\star|$ the scalar measure in the Radon-Nykodým decomposition of $|\mathfrak{m}_\cdot - \mathfrak{m}_\star|$, we have proven that

$$\int_{\mathfrak{H}} \mathrm{d}\left|\mu'_t - \mu_t^{(r)}\right|(z) = o_r(1). \tag{5.113}$$

The density matrix ϱ in the Radon-Nykodým decomposition of $|\mathfrak{m}^{(r)} - \mathfrak{m}|$ is such that $\|\varrho\|_{\mathscr{L}_1} = 1$ and therefore

$$\int_{\mathfrak{H}} \mathrm{d}\left|\mu_t^{(r)} - \mu_t\right|(z) \leqslant \liminf_{\varepsilon\to 0}\mathrm{Tr}_{\mathscr{H}}|\rho_\varepsilon^{(r)}(t) - \rho_\varepsilon(t)| = o_r(1). \tag{5.114}$$

Putting together (5.113) and (5.114), we finally have

$$\int_{\mathfrak{H}} \mathrm{d}|\mu'_t - \mu_t|(z) \leqslant \int_{\mathfrak{H}} \mathrm{d}|\mu'_t - \mu_t^{(r)}|(z) + \int_{\mathfrak{H}} \mathrm{d}|\mu_t^{(r)} - \mu_t|(z) = o_r(1), \tag{5.115}$$

which implies that $\mathfrak{m}_t = \mathfrak{m}'_t$. Since the previous argument applies to any subsequence $\varepsilon_{n(r)}$, then the convergence holds along ε_n, *i.e.*, $\rho_{\varepsilon_n} \to \mathfrak{m}_t$.

5.5 Properties of the quasi-classical dynamics

We now discuss some important properties of the effective dynamics. Let us sum up the result proven: if $\{\rho_\varepsilon\}_{\varepsilon\in(0,1)}$ is a family of normalized density matrices satisfying

$$\mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon\,((1+\mathrm{d}\Gamma_\varepsilon(\mathbb{1}))^\delta + H_+)) \leqslant C, \qquad \text{for some} \quad \delta > 0,$$

so that

$$\rho_{\varepsilon_n} \xrightarrow[n\to+\infty]{} \mathrm{d}\mathfrak{m}(z) = \gamma_{\mathfrak{m}}(z)\mathrm{d}\mu(z),$$

then,

$$e^{-itH_\varepsilon}\rho_{\varepsilon_n}e^{itH_\varepsilon} = \rho_{\varepsilon_n}(t) \xrightarrow[n\to+\infty]{} \mathrm{d}\mathfrak{m}_t(z) = \gamma_{\mathfrak{m}_t}(z)\mathrm{d}\mu_t(z), \tag{5.116}$$

with

$$\mu_t = \Phi_t \sharp \mu, \qquad \gamma_{\mathfrak{m}_t}(z) = U_{t,0}(z)\,\gamma_{\mathfrak{m}}(z)\,U^\dagger_{t,0}(z). \tag{5.117}$$

The corresponding Heisenberg picture reads as follows: if $f : \mathfrak{H} \to \mathscr{L}(L^2(\mathbb{R}^{dN}))$,

$$\begin{aligned}\mathrm{Tr}_{L^2}\left(\int_{\mathfrak{H}} \mathrm{d}\mathfrak{m}_t(z)\, f(z)\right) &= \int_{\mathfrak{H}} \mathrm{d}\mu(z)\, \mathrm{Tr}_{L^2}\left(\gamma_{\mathfrak{m}_t}(z)\, f(\Phi_t z)\right) = \\ &= \int_{\mathfrak{H}} \mathrm{d}\mu(z)\, \mathrm{Tr}_{L^2}\left(\gamma_{\mathfrak{m}}(z)\, U^\dagger_{t,0}(z)\, f(\Phi_t z)\, U_{t,0}(z)\right).\end{aligned}$$

As we have seen, the normalized Radon-Nykodým derivative of the measure evolves, pointwise for every classical configuration of the field $z \in \mathfrak{H}$, according to a two-parameter unitary group, but this structure does not apply to the total evolution. Indeed, the particle evolution has to be averaged over all the possible configurations of the field, taking also into account, if $\nu = 1$, the free evolution of the field.

Let us analyse in more detail some specific example. Consider, for instance, a product state $\rho_\varepsilon = |\psi\otimes\Psi_\varepsilon\rangle\langle\psi\otimes\Psi_\varepsilon|$, with $\psi \in L^2(\mathbb{R}^{dN})$, $\Psi_\varepsilon \in \Gamma_s(\mathfrak{H})$, such that $\|\rho_\varepsilon\|_{\mathscr{L}_1} = 1$ for every $\varepsilon \in (0,1)$, and satisfying the bound on the expectation of the number operator. The quasi-classical limit of such states, up to a subsequence extraction, is trivial to take because of the factorization and, using results for Wigner scalar measures [6] or Theorem 3.2.1 for $|\Psi_\varepsilon\rangle\langle\Psi_\varepsilon| \in \mathscr{L}_1(\Gamma_s(\mathfrak{H}))$, we get that $\Psi_\varepsilon \to \mu \in \mathscr{M}(\mathfrak{H};\mathbb{R}_+)$,

$$\begin{aligned}&\lim_{\varepsilon\to0}\mathrm{Tr}_{\mathscr{H}}(\rho_\varepsilon\, B\otimes W_\varepsilon(\xi)) = \lim_{\varepsilon\to0}\langle\Psi_\varepsilon|\, B\,|\psi\rangle\langle\psi|\otimes W_\varepsilon(\xi)\Psi_\varepsilon\rangle = \\ &= \int_{\mathfrak{H}} \mathrm{d}\mu(z)\,\mathrm{Tr}_{L^2}(|\psi\rangle\langle\psi|\; B)\, e^{2i\Re e\langle\xi\,|\,z\rangle}\end{aligned}$$

and

$$e^{-itH_\varepsilon}\,|\psi\otimes\Psi_\varepsilon\rangle\langle\psi\otimes\Psi_\varepsilon|\,e^{itH_\varepsilon} \to \mathfrak{m}_t = \Phi_t\sharp\mu\;|\psi(t)\rangle\langle\psi(t)|$$

with $\psi(t) = U_{t,0}(z)\psi$. Since the Radon-Nykodým derivative is constant, *i.e.*, $\gamma(z) = |\psi\rangle\langle\psi|$ for any $z \in \mathfrak{H}$, we can define reduced effective dynamics

$$\mathscr{W}_{t,s} : \mathscr{L}_1(L^2(\mathbb{R}^{dN})) \longrightarrow \mathscr{L}_1(L^2(\mathbb{R}^{dN}))$$

via

$$\mathscr{W}_{t,s}\gamma = \int_{\mathfrak{H}} \mathrm{d}(\Phi_{t-s}\sharp\mu_s)(z)\, U_{t,s}(z)\,\gamma\, U^\dagger_{t,s}(z). \tag{5.118}$$

In this case, one recover the group structure for $\mathscr{W}_{t,s}$, but some additional condition is needed, *e.g.*, the scalar measure has to concentrate at one point (Dirac delta). Indeed, for $r < s < t$,

$$\mathscr{W}_{t,s}\mathscr{W}_{s,r}\gamma = \int_{\mathfrak{H}\times\mathfrak{H}} \mathrm{d}(\Phi_{t-r} \sharp \mu_s \times \mu_s)(z,y)\, U_{t,s}(z)U_{s,r}(y)\,\gamma\, U^{\dagger}_{t,s}(z)U^{\dagger}_{s,r}(y)$$

differs from

$$\mathscr{W}_{t,r}\gamma = \int_{\mathfrak{H}} \mathrm{d}(\Phi_{t-r} \sharp \mu_r)(z,y)\, U_{t,r}(z)\,\gamma\, U^{\dagger}_{t,r}(z), \tag{5.119}$$

except when $\mu_s \times \mu_s$ is concentrated on the diagonal of $\mathfrak{H}\times\mathfrak{H}$. This is, however, possible only when the measure is supported at one point, *i.e.*, when $\mu = \delta_{z_0}$, $z_0 \in \mathfrak{H}$, which entails

$$\begin{aligned}\mathscr{W}_{t,s}\mathscr{W}_{s,r}\gamma &= U_{t,s}(z_0)U_{s,r}(z_0)\,\gamma\, U^{\dagger}_{t,s}(z_0)U^{\dagger}_{s,r}(z_0) = \\ &= U_{t,r}(z_0)\,\gamma\, U^{\dagger}_{t,r}(z_0) = \mathscr{W}_{t,r}\gamma.\end{aligned}$$

Dirac deltas are known to be the semiclassical counterpart of coherent states:

$$\Xi_\varepsilon(\xi) = W_\varepsilon\left(\frac{\xi}{i\varepsilon}\right)\Omega \quad \underset{\varepsilon\to 0}{\longrightarrow} \quad \delta_\xi, \tag{5.120}$$

with Ω the vacuum in the Fock space and $\xi \in \mathfrak{H}$. Note that such states satisfy out assumptions and therefore their quasi-classical limit fits in the work presented so far.

A Fock space estimates

A.1 Standard Fock space estimates

Let us start by introducing the ($\varepsilon-$dependent) symmetric Fock space, and its most important properties (see [42] and [11] for a detailed analysis).

The symmetric Fock space is commonly used to describe a system of particles following the Bose statistics and subjected to creation and annihilation. From an algebraic quantum field theory point of view, the symmetry of the wave functions can be encoded in the properties of commutation of the Weyl C*-algebra $\mathscr{W}_\varepsilon(X', \sigma)$, with (X', σ) a real, symplectic vector space. The Fock space is the natural space to represent free bosonic theories:

$$\begin{aligned}\pi_F : \mathscr{W}_\varepsilon(X', \sigma) &\to \mathscr{L}(\Gamma_s(\mathfrak{H}))\\ W_\varepsilon(\xi) &\mapsto \pi_F(W_\varepsilon(\xi)),\end{aligned}$$

where

$$\Gamma_s(\mathfrak{H}) := \bigoplus_{n=0}^{\infty} \mathfrak{H}^{\otimes_s n}, \tag{A.1}$$

and $\pi_F(W_\varepsilon(\xi))$ to be defined precisely later. For convenience, we drop the π_F and denote the represented Weyl operators by $W_\varepsilon(\xi)$, if no confusion arises.

The space $\mathfrak{H}^{\otimes_s n}$, whose vectors describes n bosons, is defined as follows. Denote by P_n the group of permutations of order n, and define the projector S_n, whose action on pure tensors is:

$$\Psi_1 \otimes_s \ldots \otimes_s \Psi_n := S_n(\Psi_1 \otimes \ldots \otimes \Psi_n) = \frac{1}{n!} \sum_{\sigma \in P_n} \Psi_{\sigma(1)} \otimes \ldots \otimes \Psi_{\sigma(n)}. \tag{A.2}$$

Hence, $\mathfrak{H}^{\otimes_s n} := S_n(\mathfrak{H}^{\otimes n})$ is the space of all the finite linear combinations of symmetric pure tensors, closed w.r.t. the norm of $\mathfrak{H}$. The symmetric product of Hilbert spaces $\mathfrak{H}^{\otimes_s n}$ is endowed with the following scalar product:

$$\begin{aligned}\text{if} \quad \Psi^{(n)} &= \sum_{j_1,\ldots,j_n} a_{j_1,\ldots,j_n} \Psi_{j_1} \otimes_s \ldots \otimes_s \Psi_{j_n} \in \mathfrak{H}^{\otimes_s n},\\ \Phi^{(n)} &= \sum_{k_1,\ldots,k_n} b_{k_1,\ldots,k_n} \Phi_{k_1} \otimes_s \ldots \otimes_s \Phi_{k_n} \in \mathfrak{H}^{\otimes_s n},\end{aligned}$$

then

$$\left\langle \Phi^{(n)} \middle| \Psi^{(n)} \right\rangle_{\mathfrak{H}^{\otimes_s n}} = \sum_{j_1,\ldots,j_n,k_1,\ldots,k_n} \bar{b}_{k_1,\ldots,k_n} a_{j_1,\ldots,j_n} \langle \Phi_{k_1} | \Psi_{j_1} \rangle_{\mathfrak{H}} \ldots \langle \Phi_{k_n} | \Psi_{j_n} \rangle_{\mathfrak{H}}. \tag{A.3}$$

The space $\mathfrak{H}^{\otimes_s 0}$ is simply $\mathbb{C}$.

The Fock space is thus defined as the closure of the algebraic sum of the aforementioned symmetric spaces, as n varies, w.r.t. the norm

$$\|\Psi\|^2_{\Gamma_s(\mathfrak{H})} := \sum_{n=0}^{\infty} \|\Psi^{(n)}\|^2_{\mathfrak{H}^{\otimes_s n}}. \tag{A.4}$$

The vacuum vector $\Omega \in \Gamma_s(\mathfrak{H})$ is of particular importance and it describes the state with no bosons:

$$\Omega = (1, 0, 0, \dots). \tag{A.5}$$

The $\varepsilon-$dependence comes into play when one defines the canonical quantum observables, *i.e.*, the creation and annihilation operators. We define them on the $n-$fold tensor product to be, respectively

$$\begin{aligned} a_\varepsilon^\dagger(\xi) : \mathfrak{H}^{\otimes_s n} &\longrightarrow \mathfrak{H}^{\otimes_s (n+1)} \\ \Psi_1 \otimes_s \dots \otimes_s \Psi_n &\mapsto \sqrt{\varepsilon(n+1)}\, \xi \otimes_s \Psi_1 \otimes_s \dots \otimes_s \Psi_n, \end{aligned}$$

$$\begin{aligned} a_\varepsilon(\xi) : \mathfrak{H}^{\otimes_s n} &\longrightarrow \mathfrak{H}^{\otimes_s (n-1)} \\ \Psi_1 \otimes_s \dots \otimes_s \Psi_n &\mapsto \sqrt{\varepsilon n}\, \langle \xi \,|\, \Psi_1 \rangle_{\mathfrak{H}}\, \Psi_2 \otimes_s \dots \otimes_s \Psi_n. \end{aligned}$$

They extend to densely defined, closed operators on the Fock space that will be denoted in the same way. It can be proved that they are one the adjoint of the other, and furthermore they satisfy the canonical commutation relations:

$$[a_\varepsilon(\xi), a_\varepsilon^\dagger(\eta)] = \varepsilon \langle \xi \,|\, \eta \rangle_{\mathfrak{H}}, \qquad \xi, \eta \in \mathfrak{H}. \tag{A.6}$$

Therefore, both a_ε and $a_\varepsilon^\dagger$ scale as $\sqrt{\varepsilon}$.

If $\mathfrak{H}$ is a function space, *e.g.*, $\mathfrak{H} = L^2(\mathbb{R}^d; \mathrm{d}k)$, it is convenient to define formal objects operator-valued distributions $a_\varepsilon(k)$ and $a_\varepsilon^\dagger(k)$ such that:

$$a_\varepsilon(\xi) = \int_{\mathbb{R}^d} \mathrm{d}k\, a_\varepsilon(k)\, \bar{\xi}(k), \qquad a_\varepsilon^\dagger(\xi) = \int_{\mathbb{R}^d} \mathrm{d}k\, a_\varepsilon^\dagger(k)\, \xi(k), \qquad \xi \in \mathfrak{H}. \tag{A.7}$$

These distributions satisfy the canonical commutation relations:

$$[a_\varepsilon(h), a_\varepsilon^\dagger(k)] = \varepsilon \delta(h-k), \qquad h, k \in \mathbb{R}^d. \tag{A.8}$$

An important role is also played by the so-called Segal fields, defined, for any $\xi \in \mathfrak{H}$, as

$$\begin{aligned} \phi_\varepsilon(\xi) &:= a_\varepsilon(\xi) + a_\varepsilon^\dagger(\xi), \\ \pi_\varepsilon(\xi) &:= i(a_\varepsilon^\dagger(\xi) - a_\varepsilon(\xi)) = \phi_\varepsilon(i\xi). \end{aligned}$$

For any linear contraction $U \in \mathscr{L}(\mathfrak{H})$ that is, such that $\|U\|_{\mathscr{L}} \leqslant 1$, one can define the operator

$$\Gamma(U) \restriction_{\mathfrak{H}^{\otimes_s n}} = U \otimes \dots \otimes U \tag{A.9}$$

which is a linear contraction on $\Gamma_s(\mathfrak{H})$. Thus, for any self-adjoint operator $A : \mathscr{D}(A) \subseteq \mathfrak{H} \to \mathfrak{H}$, since it is possible to define the second quantization of A as the operator $\mathrm{d}\Gamma_\varepsilon(A)$ such that

$$e^{i\frac{t}{\varepsilon}\mathrm{d}\Gamma_\varepsilon(A)} = \Gamma(e^{itA}). \tag{A.10}$$

Explicitly,

$$\mathrm{d}\Gamma_\varepsilon(A) \restriction_{\mathscr{D}(A)^{\otimes n}} \Psi_1 \otimes_s \dots \otimes_s \Psi_n = \varepsilon \sum_{j=1}^{n} \Psi_1 \otimes_s \dots \otimes_s A\Psi_j \otimes_s \dots \otimes_s \Psi_n. \tag{A.11}$$

By means of the operator-valued distributions for creation and annihilation operators, the previous operator can be written as

$$\mathrm{d}\Gamma_\varepsilon(A) = \int_{\mathbb{R}^d} \mathrm{d}k\; a_\varepsilon^\dagger(k)(A\, a_\varepsilon)(k). \tag{A.12}$$

Therefore,

$$\mathrm{d}\Gamma_\varepsilon(A) \simeq \varepsilon. \tag{A.13}$$

The second quantization of the identity operator is called number operator due to its action on the $n-$fold tensor product:

$$\mathrm{d}\Gamma_\varepsilon(\mathbb{1})\Psi_1 \otimes_s \ldots \otimes_s \Psi_n = \varepsilon n\, \Psi_1 \otimes_s \ldots \otimes_s \Psi_n. \tag{A.14}$$

In terms of operator-valued distributions it is written as

$$\mathrm{d}\Gamma_\varepsilon(\mathbb{1}) = \int_{\mathbb{R}^d} \mathrm{d}k\, a_\varepsilon^\dagger(k)\, a_\varepsilon(k). \tag{A.15}$$

Furthermore, we will denote by

$$\Gamma_{\text{fin}}(\mathfrak{H}) := \bigcup_{N\in\mathbb{N}} \mathbb{1}_{[0,N)}(\mathrm{d}\Gamma_\varepsilon(\mathbb{1}))\,\Gamma_s(\mathfrak{H}), \tag{A.16}$$

the space of vectors with a finite number particles.

Proposition A.1.1. *The symmetric Fock space can be defined in the following equivalent ways:*

- $\Gamma_s(\mathfrak{H}) = \bigoplus_{n=0}^{\infty} \mathfrak{H}^{\otimes_s n}$;
- $\Gamma_s(\mathfrak{H})$ *is the Hilbert space generated by the total family* $\{\Psi_1 \otimes_s \ldots \otimes_s \Psi_n; \Psi_j \in \mathfrak{H}\}_{n\in\mathbb{N}}$;
- $\Gamma_s(\mathfrak{H})$ *is the Hilbert space generated by the total family* $\{z^{\otimes_s n}; z \in \mathfrak{H}\}_{n\in\mathbb{N}}$;
- $\Gamma_s(\mathfrak{H})$ *is the Hilbert space generated by the total family* $\{a_\varepsilon^\dagger(\xi_1)\ldots a_\varepsilon^\dagger(\xi_n)\Omega;\ \xi_j \in \mathfrak{H}\}_{n\in\mathbb{N}}$,

recalling that $\{\Phi_n\}_{n\in\mathbb{N}}$ *is a total family for the Hilbert space* $\mathscr{H}$ *if the latter is obtained by the closure w.r.t. its norm of finite linear combinations of elements of the total family:*

$$\mathscr{H} = \overline{\mathrm{Span}\{\Phi_n\}}^{\|\cdot\|_{\mathscr{H}}}. \tag{A.17}$$

Let us now discuss some basic Fock space inequalities, that we used throughout the thesis.

First of all, let us define, for a Hilbert space $\mathfrak{H}$, the associated "weighted" $\mathfrak{H}_\omega$ with $\omega : \mathscr{D}(\omega) \subseteq \mathfrak{H} \to \mathfrak{H}$ self-adjoint and positive with

$$\mathfrak{H}_\omega := \{\xi\ :\ \langle \xi |\, \omega\, |\xi\rangle_{\mathfrak{H}} < +\infty\}, \tag{A.18}$$

and inner product $\langle \cdot\,|\,\cdot\rangle_{\mathfrak{H}_\omega} := \langle \cdot\,|\,\omega\,|\,\cdot\rangle_{\mathfrak{H}}$.

Proposition A.1.2. *For any* $\Psi_\varepsilon \in \mathscr{D}(\mathrm{d}\Gamma_\varepsilon(\omega)^{1/2})$ *and* $\xi \in \mathfrak{H} \cap \mathfrak{H}_{\omega^{-1}}$ *we have that*

$$\|a_\varepsilon(\xi)\Psi_\varepsilon\|_{\Gamma_s(\mathfrak{H})} \leqslant \|\omega^{-1/2}\xi\|_{\mathfrak{H}}\, \|\mathrm{d}\Gamma_\varepsilon(\omega)^{1/2}\Psi_\varepsilon\|_{\Gamma_s(\mathfrak{H})}$$
$$\|a_\varepsilon^\dagger(\xi)\Psi_\varepsilon\|_{\Gamma_s(\mathfrak{H})} \leqslant \|\omega^{-1/2}\xi\|_{\mathfrak{H}}\, \|\mathrm{d}\Gamma_\varepsilon(\omega)^{1/2}\Psi_\varepsilon\|_{\Gamma_s(\mathfrak{H})} + \varepsilon\|\xi\|_{\mathfrak{H}}\|\Psi_\varepsilon\|_{\mathfrak{H}}$$

Proof. Let us prove for first for the annihilation operator: by Cauchy-Schwarz inequality and symmetry of the functions

$$\begin{aligned}
\|a_\varepsilon(\xi)\Psi_\varepsilon\|^2_{\Gamma_s(\mathfrak{H})} &= \sum_{n\in\mathbb{N}} \varepsilon(n+1)|\langle \xi\,|\,\Psi_1\rangle|^2\, \|\Psi_2 \otimes_s \ldots \otimes_s \Psi_{n+1}\|^2_{\mathfrak{H}^{\otimes_s n}} \leqslant \\
&\leqslant \sum_{n\in\mathbb{N}} \varepsilon(n+1)\|\omega^{-1/2}\xi\|_{\mathfrak{H}}\|\omega^{1/2}\Psi_1\|_{\mathfrak{H}}\|\Psi_2 \otimes_s \ldots \otimes_s \Psi_{n+1}\|^2_{\mathfrak{H}^{\otimes_s n}} = \\
&= \sum_{n\in\mathbb{N}} \|\omega^{-1/2}\xi\|^2_{\mathfrak{H}}\|\mathrm{d}\Gamma_\varepsilon(\omega)^{1/2}\Psi_1 \otimes_s \ldots \otimes_s \Psi_{n+1}\|^2_{\mathfrak{H}^{\otimes_s (n+1)}} = \\
&= \|\omega^{-1/2}\xi\|^2_{\mathfrak{H}}\, \|\mathrm{d}\Gamma_\varepsilon(\omega)^{1/2}\Psi_\varepsilon\|^2_{\Gamma_s(\mathfrak{H})}.
\end{aligned}$$

For the creation operator, it follows from the canonical commutation relations:

$$\|a_\varepsilon^\dagger(\xi)\Psi_\varepsilon\|^2_{\Gamma_s(\mathfrak{H})} = \|a_\varepsilon(\xi)\Psi_\varepsilon\|^2_{\Gamma_s(\mathfrak{H})} + \varepsilon^2\|\xi\|^2\|\Psi_\varepsilon\|^2_{\Gamma_s(\mathfrak{H})}. \tag{A.19}$$

□

In the case of $\omega = 1$ we can obtain the following useful inequality involving the number operator, for all $\varepsilon \leqslant 1$,

$$\|a_\varepsilon^\#(\xi)\Psi_\varepsilon\|_{\Gamma_s(\mathfrak{H})} \leqslant \|\xi\|_{\mathfrak{H}} \;\|(\mathrm{d}\Gamma_\varepsilon(\mathbb{1}) + 1)^{1/2}\Psi_\varepsilon\|_{\Gamma_s(\mathfrak{H})}. \tag{A.20}$$

It is easy to see that if we consider a function $\lambda_\cdot \in L^\infty(\mathbb{R}^d;\mathfrak{H})$ in place of $\xi \in \mathfrak{H}$, the inequality has the same form using the uniform norm $\|\cdot\|_\infty := \|\cdot\|_{L^\infty(\mathbb{R}^d;\mathfrak{H})}$:

$$\|a_\varepsilon^\#(\xi)\Psi_\varepsilon\|_{\Gamma_s(\mathfrak{H})} \leqslant \|\lambda_\cdot\|_\infty \;\|(\mathrm{d}\Gamma_\varepsilon(\mathbb{1}) + 1)^{1/2}\Psi_\varepsilon\|_{\Gamma_s(\mathfrak{H})}, \tag{A.21}$$

that is,

$$\|a_\varepsilon^\#(\lambda_x)(\mathrm{d}\Gamma_\varepsilon(\mathbb{1}) + 1)^{-1/2}\|_{\mathscr{L}(\Gamma_s(\mathfrak{H}))} \leqslant C, \tag{A.22}$$

uniformly in ε. In Appendix B we generalize these inequalities to Wick quantized symbols.

Weyl operators are represented on Fock space as the unitary operators

$$W_\varepsilon(\xi) = e^{i\phi_\varepsilon(\xi)} = e^{i(a_\varepsilon(\xi)+a_\varepsilon^\dagger(\xi))}. \tag{A.23}$$

Let us recall that the Weyl operators satisfy the Weyl relations: for every $\xi, \eta \in \mathfrak{H}$,

$$\begin{aligned}
&W_\varepsilon(\xi) \neq 0,\\
&W_\varepsilon^\dagger(\xi) = W_\varepsilon(-\xi),\\
&W_\varepsilon(\xi)W_\varepsilon(\eta) = e^{-i\varepsilon\sigma(\xi,\eta)}W_\varepsilon(\xi+\eta).
\end{aligned}$$

Proposition A.1.3. *For any $\lambda_\cdot \in L^\infty(\mathbb{R}^d;\mathfrak{H})$ and any $\xi \in \mathfrak{H}, \eta \in \mathfrak{H}_\omega \cap \mathfrak{H}_{\omega^{1/2}}$, with $\omega : \mathscr{D}(\omega) \subseteq \mathfrak{H} \to \mathfrak{H}$ self-adjoint,*

$$W_\varepsilon^\dagger(\xi)\, a_\varepsilon^\dagger(\lambda_x)\, W_\varepsilon(\xi) = a_\varepsilon^\dagger(\lambda_x) - i\varepsilon\,\langle \xi \,|\, \lambda_x\rangle; \tag{A.24}$$

$$W_\varepsilon^\dagger(\xi)\, a_\varepsilon(\lambda_x)\, W_\varepsilon(\xi) = a_\varepsilon(\lambda_x) + i\varepsilon\,\langle \lambda_x \,|\, \xi\rangle; \tag{A.25}$$

$$W_\varepsilon^\dagger(\eta)\, \mathrm{d}\Gamma_\varepsilon(\omega)\, W_\varepsilon(\eta) = \mathrm{d}\Gamma_\varepsilon(\omega) + i\varepsilon(a_\varepsilon^\dagger(\omega\eta) - a_\varepsilon(\omega\eta)) + \varepsilon^2\|\omega^{1/2}\eta\|^2_{\mathfrak{H}}. \tag{A.26}$$

Proof. The first two equations are simply obtained as follows:

$$\begin{aligned}
W_\varepsilon^\dagger(\xi)\, a_\varepsilon(\lambda_x)\, W_\varepsilon(\xi) - a_\varepsilon(\lambda_x) &= \int_0^1 \mathrm{d}s\, \frac{\mathrm{d}}{\mathrm{d}s} e^{-is(a_\varepsilon(\xi)+a_\varepsilon^\dagger(\xi))} a_\varepsilon(\lambda_x) e^{is(a_\varepsilon(\xi)+a_\varepsilon^\dagger(\xi))} =\\
&= i\int_0^1 \mathrm{d}s\; e^{-is(a_\varepsilon(\xi)+a_\varepsilon^\dagger(\xi))}[a_\varepsilon(\lambda_x), a_\varepsilon^\dagger(\xi)] e^{is(a_\varepsilon(\xi)+a_\varepsilon^\dagger(\xi))} = i\varepsilon\,\langle \lambda_x \,|\, \xi\rangle,
\end{aligned}$$

the calculation for the creation operator being analgous (for details see [37, Lemma 5.3]). Using operator-valued distributions, we get

$$W_\varepsilon^\dagger(\xi)\, a_\varepsilon^\dagger(k)\, W_\varepsilon(\xi) = a_\varepsilon^\dagger(\lambda_x) - i\varepsilon\overline{\xi}(k); \tag{A.27}$$

$$W_\varepsilon^\dagger(\xi)\, a_\varepsilon(k)\, W_\varepsilon(\xi) = a_\varepsilon(\lambda_x) + i\varepsilon\xi(k). \tag{A.28}$$

Hence,

$$W_\varepsilon^\dagger(\xi)\,\mathrm{d}\Gamma_\varepsilon(\omega)\,W_\varepsilon(\xi) = \int_{\mathbb{R}^d} \mathrm{d}k\ \omega(k) W_\varepsilon^\dagger(\xi) a_\varepsilon^\dagger(k) W_\varepsilon(\xi) W_\varepsilon^\dagger(\xi) a_\varepsilon(k) W_\varepsilon(\xi) =$$
$$= \int_{\mathbb{R}^d} \mathrm{d}k\ \omega(k)\,(a_\varepsilon^\dagger(\lambda_x) - i\varepsilon\overline{\xi}(k))(a_\varepsilon(\lambda_x) + i\varepsilon\xi(k)),$$

from which (A.26) follows by commutation. □

The Weyl operators are continuous, in Fock representation, w.r.t. their argument.

Corollary A.1.4. *For any $\delta > 0$, there exists a constant $C_\delta \in (0, +\infty)$ such that, for every $\xi, \eta \in \mathfrak{H}$*

$$\|(W_\varepsilon(\xi) - W_\varepsilon(\eta))(\mathrm{d}\Gamma_\varepsilon(\mathbb{1}) + 1)^{-\delta}\|_{\mathscr{L}} \leqslant C_\delta(1 + \|\xi\|_{\mathfrak{H}})\|\xi - \eta\|_{\mathfrak{H}} \tag{A.29}$$

Proof. Following the proof in [6, Lemma 3.1]

$$\|(W_\varepsilon(\xi) - W_\varepsilon(\eta))(\mathrm{d}\Gamma_\varepsilon(\mathbb{1}) + 1)^{-\delta}\| \leqslant$$
$$\leqslant \|(1 - e^{i\varepsilon\sigma(\xi,\eta)} W_\varepsilon(\eta - \xi))(\mathrm{d}\Gamma_\varepsilon(\mathbb{1}) + 1)^{-\delta}\| \leqslant$$
$$\leqslant |e^{i\varepsilon\sigma(\xi,\eta)} - 1| + \|(1 - W_\varepsilon(\eta - \xi))(\mathrm{d}\Gamma_\varepsilon(\mathbb{1}) + 1)^{-\delta}\|.$$

Now we use the following estimate: for every $\alpha \in [0, 1]$ there exists $C_\alpha \in (0, +\infty)$ sucht that

$$|e^{is} - 1| \leqslant C_\alpha |s|^\alpha, \quad s \in \mathbb{R} \tag{A.30}$$

yielding

$$|e^{i\varepsilon\sigma(\xi,\eta)} - 1| = |e^{i\varepsilon\sigma(\xi,\eta-\xi)} - 1| \leqslant C\varepsilon\|\xi - \eta\|_{\mathfrak{H}}\|\xi\|_{\mathfrak{H}}. \tag{A.31}$$

For the second term, we use complex interpolation: for $\delta = 0$ we have $\|(1 - W_\varepsilon(\eta - \xi))\| \leqslant 2$, while for $\delta = 1$ we use (A.30), functional calculus and Proposition A.1.2 to obtain, for any $\psi \in \Gamma_s(\mathfrak{H})$,

$$\|(W_\varepsilon(\eta - \xi) - 1)(1 + \mathrm{d}\Gamma_\varepsilon(\mathbb{1}))^{-1/2}\Psi\|_{\Gamma_s} \leqslant \left\|\phi_\varepsilon(\eta - \xi)(1 + \mathrm{d}\Gamma_\varepsilon(\mathbb{1}))^{-1/2}\Psi\right\| \leqslant$$
$$\leqslant C\|\eta - \xi\|_{\mathfrak{H}}\|\Psi\|_{\Gamma_s}.$$

Therefore,

$$\|(W_\varepsilon(\eta - \xi) - 1)(1 + \mathrm{d}\Gamma_\varepsilon(\mathbb{1}))^{-\delta}\|_{\mathscr{L}} \leqslant C\|\xi - \eta\|_{\mathfrak{H}} \tag{A.32}$$

is true for any $\delta \in [0, 1]$. Putting together inequalities (A.31) and (A.32), we conclude the argument. □

Proposition A.1.5. *For any $\delta \in \mathbb{R}$ and $\xi \in \mathfrak{H}$,*

$$(1 + \mathrm{d}\Gamma_\varepsilon(\mathbb{1}))^\delta W_\varepsilon(\xi)(1 + \mathrm{d}\Gamma_\varepsilon(\mathbb{1}))^{-\delta} \in \mathscr{L}(\Gamma_s(\mathfrak{H})). \tag{A.33}$$

Proof. See [4, Corollary A.2 (ii)]. □

A.2 Trace-class operators

Let us consider in this section some properties of the density matrices in Fock space.

Definition A.2.1. *An operator ρ on a separable Hilbert space $\mathscr{H}$ is trace-class if and only if, for any set of orthonormal basis $\{\phi_j\}_{j\in\mathbb{N}} \subseteq \mathscr{H}$,*

$$\|\rho\|_{\mathscr{L}_1(\mathscr{H})} = \mathrm{Tr}_{\mathscr{H}}(|\rho|) := \sum_{j\in\mathbb{N}} \langle\psi_j| \, |\rho| \, |\psi_j\rangle_{\mathscr{H}} < +\infty. \tag{A.34}$$

The space of trace class operators is denoted by $\mathscr{L}_1(\mathscr{H})$.

If $\rho \in \mathscr{L}_1(\mathscr{H})$, then its trace is independent on the chosen orthonormal basis, and reads

$$\mathrm{Tr}_{\mathscr{H}}(\rho) := \sum_{j\in\mathbb{N}} \langle\psi_j| \, \rho \, |\psi_j\rangle_{\mathscr{H}} \, . \tag{A.35}$$

Furthermore, the function $\|\cdot\|_{\mathscr{L}_1(\mathscr{H})} : \mathscr{L}_1(\mathscr{H}) \to \mathbb{R}^+$ is a norm and the set of trace-class operators with such norm $(\mathscr{L}_1(\mathscr{H}), \|\cdot\|_{\mathscr{L}_1(\mathscr{H})})$ is a Banach space.

For any $\rho \in \mathscr{L}_1(\mathscr{H}) \subseteq \mathscr{L}(\mathscr{H})$ and $B \in \mathscr{L}(\mathscr{H})$, the following inequality holds

$$|\mathrm{Tr}_{\mathscr{H}}(\rho B)| \leqslant \|\rho B\|_{\mathscr{L}_1} \leqslant \|\rho\|_{\mathscr{L}_1} \|B\|_{\mathscr{L}}. \tag{A.36}$$

In addition,

$$\mathrm{Tr}_{\mathscr{H}}(\rho B) = \mathrm{Tr}_{\mathscr{H}}(B\rho). \tag{A.37}$$

Thus trace class operators are a bilateral ideal of the bounded operators, that is, $B\rho, \rho B \in \mathscr{L}_1(\mathscr{H})$.

A useful remark is that, in general, for any $\delta > 0$ and $A \geqslant 0$,

$$\mathrm{Tr}_{\mathscr{H}}(\rho A^{\delta}) = \mathrm{Tr}_{\mathscr{H}}(A^{\delta/2}\rho A^{\delta/2}) = \|A^{\delta/2}\rho_\varepsilon A^{\delta/2}\|_{\mathscr{L}_1} \leqslant \|\rho_\varepsilon A^{\delta}\|_{\mathscr{L}_1}. \tag{A.38}$$

Trace-class operators are compact operators: $\mathscr{L}_1(\mathscr{H}) \subseteq \mathscr{L}_\infty(\mathscr{H})$, and therefore for positive operators ρ_+ there exists a basis of orthonormal eigenvectors $\{\psi_j\}_{j\in\mathbb{N}} \subseteq \mathscr{H}$, and a sequence of eigenvalues $\{\lambda_j\}_{j\in\mathbb{N}} \subseteq \mathbb{R}_+$ such that

$$\rho_+ = \sum_{j\in\mathbb{N}} \lambda_j \, |\psi_j\rangle\langle\psi_j| \, . \tag{A.39}$$

In particular,

$$\|\rho_+\|_{\mathscr{L}_1} = \sum_{j\in\mathbb{N}} \lambda_j < +\infty. \tag{A.40}$$

In general, for a trace-class operator ρ the following equality holds

$$\mathrm{Tr}_{\mathscr{H}}(\rho) = \sum_{j\in\mathbb{N}} \lambda_j \tag{A.41}$$

with each eigenvalue counted with its multiplicity, a result known also as the Lidskii's Theorem.

Quantum states are net of families of density matrices dependent on a semiclassical parameter ε, and are trace-class operators on the tensor product of the Fock space with a space of square integrable functions associated to the particles:

$$(0,1) \ni \varepsilon \mapsto \rho_\varepsilon \in \mathscr{L}_1(L^2(\mathbb{R}^{dN}) \otimes \Gamma_s(\mathfrak{H})). \tag{A.42}$$

We mostly consider positive states that are normalized in the trace-class norm:

$$\rho_\varepsilon > 0, \qquad \|\rho_\varepsilon\|_{\mathscr{L}_1(L^2\otimes\Gamma_s)} = 1, \qquad \text{for any } \varepsilon. \tag{A.43}$$

Therefore, there exist $\{\psi_j\}_{j\in\mathbb{N}}$ and $\{\Psi_k\}_{k\in\mathbb{N}}$ being orthonormal bases, respectively, of $L^2(\mathbb{R}^{dN})$ and $\Gamma_s(\mathfrak{H})$ such that,

$$\rho_\varepsilon = \sum_{j,k\in\mathbb{N}} \lambda_{j,k}(\varepsilon)\,|\psi_j\otimes\Psi_k\rangle\langle\psi_j\otimes\Psi_k|\,, \tag{A.44}$$

with

$$\sum_{j,k\in\mathbb{N}} \lambda_{j,k}(\varepsilon) = 1. \tag{A.45}$$

A.3 Proof of the pull-through formula

In this section we discuss how to control the number operator with the Hamitlonian for the Nelson model. This control is based on the pull-through formula that has been proved in full generality for the Renormalized Nelson model with positive mass in [2] and [3].

We present here a proof given in [29, Lemma 3.2] that, thanks to the fact that we are dealing with the Nelson Hamitlonian with cut-off, is easier. The Hamiltonian is

$$H_\varepsilon = -\Delta_x + W(x_1,\dots,x_N) + \mathrm{d}\Gamma_\varepsilon(\omega) + \phi_\varepsilon(\lambda_x), \tag{A.46}$$

with $\omega(k) \geqslant m > 0$ for a.e. $k\in\mathbb{R}^d$.

Proposition A.3.1. *For any $\delta\in\mathbb{N}_*$, there exist $b,c>0$ such that, for any $\Psi_\varepsilon\in\mathscr{D}(H_\varepsilon^\delta)$,*

$$\|(\mathrm{d}\Gamma_\varepsilon(\mathbb{1})+1)^\delta\Psi_\varepsilon\|_{\mathscr{H}} \leqslant c\,\|(H_\varepsilon+b)^\delta\Psi_\varepsilon\|_{\mathscr{H}}. \tag{A.47}$$

Proof. For simplicity, let us denote $N := \mathrm{d}\Gamma_\varepsilon(\mathbb{1})$. Let us define the adjoint action of the number operator as, recursively for $j\in\mathbb{N}_*$,

$$\mathrm{ad}_N(\,\cdot\,) = [N, i(\,\cdot\,)], \qquad \mathrm{ad}_N^{(j)}(\,\cdot\,) = [N,\, i\,\mathrm{ad}_N^{(j-1)}(\,\cdot\,)\,]. \tag{A.48}$$

Using canonical commutation relations we obtain:

$$\mathrm{ad}_N^{(j)}(H_\varepsilon) = \varepsilon^j\phi_\varepsilon(i^j\lambda_x). \tag{A.49}$$

H_ε is bounded from below, and so there exists $b>0$ such that $H_\varepsilon+b>0$ is invertible. Hence, for $k\in\mathbb{N}$,

$$\begin{aligned}(H_\varepsilon+b)^{-1}N^k &= (H_\varepsilon+b)^{-1}N(H_\varepsilon+b)(H_\varepsilon+b)^{-1}N^{k-1} = \\ &= N(H_\varepsilon+b)^{-1}N^{k-1} + (H_\varepsilon+b)^{-1}\phi_\varepsilon(i\lambda_x)(H_\varepsilon+b)^{-1}N^{k-1}.\end{aligned}$$

Repeating the commutation k times,

$$(H_\varepsilon+b)^{-1}N^k = N^k(H_\varepsilon+b)^{-1} + \sum_{j=1}^k N^{k-j}(H_\varepsilon+b)^{-1}B_j(b) \tag{A.50}$$

where $B_j(b)$ is a polynomial in the variable $\mathrm{ad}_N^{(\ell)}(H_\varepsilon)(H_\varepsilon+b)^{-1} = \varepsilon^\ell\phi_\varepsilon(i^\ell\lambda_x)(H_\varepsilon+b)^{-1}\in\mathscr{L}(\mathscr{H})$, for $\ell\leqslant j$. Applying N^{-k+1} to the previous inequality we obtain:

$$N^{-k+1}(H_\varepsilon+b)^{-1}N^k = N(H_\varepsilon+b)^{-1} + \sum_{j=1}^k N^{1-j}(H_\varepsilon+b)^{-1}B_j(b) \tag{A.51}$$

which, since $N(H_\varepsilon + b)^{-1} \in \mathscr{L}(\mathscr{H})$, yields

$$N^{-k+1}(H_\varepsilon + b)^{-1} N^k \in \mathscr{L}(\mathscr{H}). \tag{A.52}$$

Observing that, for $\gamma \in \mathbb{N}$

$$N^{\gamma+\delta}(H_\varepsilon + b)^{-\delta} N^{-\gamma} = \prod_{j=1}^{2\gamma+\delta} N^{\gamma+\delta-j+1}(H_\varepsilon + b)^{-1} N^{\gamma+\delta-j}, \tag{A.53}$$

we have that, for any $\gamma \in \mathbb{N}$,

$$N^{\gamma+\delta}(H_\varepsilon + b)^{-\delta} N^{-\gamma} \in \mathscr{L}(\mathscr{H}),$$

and so the proof is concluded choosing $\gamma = 0$. □

B Quantizations in infinite dimension

In this Appendix we introduce quantization of observables on an infinite dimensional phase space. There is not a unique quantization procedure, due to ordering ambiguities in the multiplication of non-commutative observables.

In Section B.1 we review some properties of the Weyl quantization, useful to quantize cylindrical symbols, that is, with support in a finite dimensional space.

In Section B.2, we introduce Wick quantization suitable for polynomial non-cylindrical symbols. The creation and annihilation operators, in particular, can be seen as Wick quantizations.

Let us recall first that a symbol is a function

$$b : \mathfrak{H} \to \mathbb{C}, \tag{B.1}$$

where $\mathfrak{H}$ is the phase space of the classical fields.

B.1 Weyl quantization

In our discussion of Weyl quantization we follow mostly [6], [43] and [79].

Weyl calculus is the most commonly used to quantize symbols in finite dimensional case, and it can be suitably extended to the infinite dimensional case. Weyl quantization corresponds to a symmetric ordering of products. If (x, ξ) are classical variables, quantized to $(\hat{x}, \hat{\xi})$, then the Weyl quantization of $x \cdot \xi$ is

$$\frac{1}{2}(\hat{x}\,\hat{\xi} + \hat{\xi}\,\hat{x}).$$

Let us now consider a finite dimensional projector on the Hilbert space $\mathbb{P} : \mathfrak{H} \to \mathfrak{H}$, $\dim(\mathbb{P}\mathfrak{H}) < +\infty$. Then it is possible to decompose the Fock space in the following way:

$$\Gamma_s(\mathfrak{H}) = \Gamma_s(\mathbb{P}\mathfrak{H}) \otimes \Gamma_s(\mathbb{P}^{\perp}\mathfrak{H}), \tag{B.2}$$

with $\mathbb{P}^{\perp} := \mathbb{1} - \mathbb{P}$. Keeping this in mind, suppose to have a cylindrical Schwartz symbol $b \in \mathscr{S}_{\mathrm{cyl}}(\mathfrak{H})$, that is, such that there exists a Schwartz function $\tilde{b} \in \mathscr{S}(\mathbb{P}\mathfrak{H})$ that coincides with the original one restricted to the finite dimensional space and that, by abuse of notation, we will denote in the same way. We can define the Weyl quantization of b as the operator acting on the Fock space $\Gamma_s(\mathfrak{H})$:

$$\mathrm{Op}^{\varepsilon}_{1/2}(b) := \int_{\mathbb{P}\mathfrak{H}} \mathrm{d}\xi\, \hat{b}(\xi)\, W_{\varepsilon}(2\pi\xi) \otimes \mathbb{1}_{\Gamma_s(\mathbb{P}\mathfrak{H})} \tag{B.3}$$

where we denoted by $\mathrm{d}\xi$ the Lebesgue measure on the finite dimensional space $\mathbb{P}\mathfrak{H}$, and by $\hat{b}$ the Fourier transform of the base function of b. The index $\frac{1}{2}$ reflects the symmetric balancing of products in Weyl quantization.

Identifying complex and real variables in the right way (for details, see [6], Section 3.2) the Weyl quantization inherits all the properties of the finite dimensional Weyl-Hörmander pseudo-differential calculus, for any fixed projector $\mathbb{P}$. Thus, it is possible so to quantize cylindrical symbols of the Hörmander classes: we define, for any $\delta \in \mathbb{R}$ and any order function[1] m the space (and recall the correspondence $z = x + i\xi, \bar{z} = z - i\xi$):

$$S_{\mathbb{P}\mathfrak{H}}(m^\delta) := \{b \in C^\infty(\mathbb{P}\mathfrak{H}) : \forall \alpha, \beta \in \mathbb{N}, \exists C_{\alpha,\beta} > 0, |\partial_z^\alpha \partial_{\bar{z}}^\beta b(z)| \leqslant C_{\alpha,\beta} m(z)^\delta\} \tag{B.4}$$

The Hörmander classes are Fréchet spaces with family of seminorms

$$\|b\|_{m^\delta,\gamma} := \sup_{\alpha+\beta\leqslant\gamma} \sup_{z\in\mathbb{P}\mathfrak{H}} m(z)^{-\delta} |\partial_z^\alpha \partial_{\bar{z}}^\beta b(z)|. \tag{B.5}$$

Symbols in this classes have important properties.

- *Calderon-Vaillancourt theorem:* If $b \in S_{\mathbb{P}\mathfrak{H}}(1)$, then

$$\mathrm{Op}_{1/2}^\varepsilon(b) \in \mathscr{L}(\Gamma_s(\mathfrak{H})); \tag{B.6}$$

- *Garding inequality:* If $b > 0$, in the sense that it is defined as a map $b : \mathbb{P}\mathfrak{H} \to \mathbb{R}^+$, and $b \in S_{\mathbb{P}\mathfrak{H}}(1)$,

$$\left\langle \psi \otimes \psi^\perp \middle| \mathrm{Op}_{1/2}^\varepsilon(b)\, \psi \otimes \psi^\perp \right\rangle_{\Gamma_s(\mathfrak{H})} \geqslant -C\varepsilon \|\psi\|^2_{\Gamma_s(\mathbb{P}\mathfrak{H})} \|\psi^\perp\|^2_{\Gamma_s(\mathbb{P}^\perp\mathfrak{H})} \tag{B.7}$$

  for any $\psi \in C_0^\infty(\mathbb{R}^d) \subseteq L^2(\mathbb{R}^d) \cong \Gamma_s(\mathbb{P}\mathfrak{H}), d = \dim\mathbb{P}\mathfrak{H}$, and any $\psi^\perp \in \Gamma_s(\mathbb{P}^\perp\mathfrak{H})$.

- If $b \in S_{\mathbb{P}\mathfrak{H}}(m_1^{\delta_1}), c \in S_{\mathbb{P}\mathfrak{H}}(m_2^{\delta_2})$, then the composition of their quantizations $\mathrm{Op}_{1/2}^\varepsilon(b) \circ \mathrm{Op}_{1/2}^\varepsilon(c)$ can be written

$$\mathrm{Op}_{1/2}^\varepsilon(b) \circ \mathrm{Op}_{1/2}^\varepsilon(c) = \mathrm{Op}_{1/2}^\varepsilon(b \#_\varepsilon c) \tag{B.8}$$

  where

$$b(z) \#_\varepsilon c(z) = b(z) \cdot c(z) + \varepsilon R_\varepsilon(z), \tag{B.9}$$

  with R_ε uniformly bounded on the space $S_{\mathbb{P}\mathfrak{H}}(m_1^{\delta_1} m_2^{\delta_2})$.

Finally we show a control of the Weyl operators by powers of the number operator. Defining $N_\mathbb{P} := \mathrm{d}\Gamma_\varepsilon(\mathbb{1}_{\mathbb{P}\mathfrak{H}})$ as the finite dimensional number operator, by [53] we have that, for any $s \in \mathbb{R}$,

$$\left(1 + \frac{\varepsilon \dim\mathbb{P}\mathfrak{H}}{2} + N_\mathbb{P}\right)^{s/2} = \mathrm{Op}_{1/2}^\varepsilon(b(z,\varepsilon)), \tag{B.10}$$

with

$$\frac{1}{\varepsilon} \|b(z,\varepsilon) - \langle z \rangle^s\|_{*,\gamma} \leqslant C \in (0,+\infty), \qquad \text{for any } \gamma \in \mathbb{N} \tag{B.11}$$

and the previous seminorms defined as

$$\|b\|_{*,\gamma} := \sup_{\alpha+\beta\leqslant\gamma} \sup_{z\in\mathbb{P}\mathfrak{H}} \left(\langle z \rangle^{-(s-2)} \left| \partial_z^\alpha \partial_{\bar{z}}^\beta \frac{b(z)}{\langle z \rangle^2} \right| \right). \tag{B.12}$$

Thus, for any $\delta > 0$ and any symbol $b \in S_{\mathbb{P}\mathfrak{H}}(\langle z \rangle^{2\delta})$,

$$\left(1 + \frac{\varepsilon}{2}\dim(\mathbb{P}\mathfrak{H}) + N_\mathbb{P}\right)^{-\delta/2} \mathrm{Op}_{1/2}^\varepsilon(b) \left(1 + \frac{\varepsilon}{2}\dim(\mathbb{P}\mathfrak{H}) + N_\mathbb{P}\right)^{-\delta/2} \in \mathscr{L}(\Gamma_s(\mathfrak{H})) \tag{B.13}$$

uniformly in ε. This leads directly, since $\left(1 + \frac{\varepsilon}{2}\dim(\mathbb{P}\mathfrak{H}) + N_\mathbb{P}\right)^{\delta/2} (\mathrm{d}\Gamma_\varepsilon(\mathbb{1}) + 1)^{-\delta/2} \in \mathscr{L}(\Gamma_s(\mathfrak{H}))$, to the fact that

$$(\mathrm{d}\Gamma_\varepsilon(\mathbb{1}) + 1)^{-\delta/2} \mathrm{Op}_{1/2}^\varepsilon(b) (\mathrm{d}\Gamma_\varepsilon(\mathbb{1}) + 1)^{-\delta/2} \tag{B.14}$$

is a bounded operator on the Fock space, uniformly in ε.

[1] $m : \mathbb{P}\mathfrak{H} \to \mathbb{R}^+$ is an order function if there exist $C, D > 0$ such that $m(z) \leqslant C\langle z - w \rangle^D m(w)$. Examples are $m(z) = 1$ and $m(z) = \langle z \rangle$.

B.2 Wick quantization

In this section we introduce the Wick quantization and Wick calculus for polynomial symbols, following [6] and [30].

We say that, for any $p, q \in \mathbb{N}$, the symbol $b : \mathfrak{H} \to \mathbb{C}$ is a $(p,q)-$homogeneous polynomial if and only if there exists a $\tilde{b} \in \mathscr{L}(\mathfrak{H}^{\otimes_s p}; \mathfrak{H}^{\otimes_s q})$ such that

$$b(z) = \left\langle z^{\otimes q} \,\middle|\, \tilde{b}\, z^{\otimes p} \right\rangle_{\mathfrak{H}^{\otimes_s q}}. \tag{B.15}$$

Hence

$$\tilde{b}(z) = \frac{1}{q!}\frac{1}{p!}\partial_z^p \partial_{\bar{z}}^q b(z), \tag{B.16}$$

where the derivatives above have to be intended in Gateaux sense.

The space of $(p,q)-$homogeneous polynomials will be denoted by $\mathscr{P}_{p,q}(\mathfrak{H})$. We say that a polynomial of this kind is compact if and only if $\tilde{b} \in \mathscr{L}_\infty(\mathfrak{H}^{\otimes_s p}; \mathfrak{H}^{\otimes_s q})$, and we denote the space of compact polynomials by $\mathscr{P}^\infty_{p,q}(\mathfrak{H})$.

Now we introduce the Wick quantization for polynomials symbols: for any $b \in \mathscr{P}_{p,q}(\mathfrak{H})$ we can define

$$\begin{aligned}
&(b)^{\mathrm{Wick}}_\varepsilon : \Gamma_{\mathrm{fin}}(\mathfrak{H}) \to \Gamma_{\mathrm{fin}}(\mathfrak{H}) \\
&(b)^{\mathrm{Wick}}_\varepsilon \restriction_{\mathfrak{H}^{\otimes_s n}} := \mathbb{1}_{[p,+\infty)}(n) \frac{\sqrt{n!(n+q-p)!}}{(n-p)!} \varepsilon^{\frac{p+q}{2}}\, \tilde{b} \otimes_s \mathbb{1}_{\mathfrak{H}^{\otimes_s (n-p)}},
\end{aligned}$$

where $\mathbb{1}_{[p,+\infty)}(n)$ is zero for $n < p$, 1 otherwise.

Paradigmatic examples of operators obtained by Wick quantization are the creation and annihilation operators that we introduced in Appendix A: for any $\xi \in \mathfrak{H}$

$$(\langle \xi \,|\, z \rangle)^{\mathrm{Wick}}_\varepsilon = a_\varepsilon(\xi), \qquad (\langle z \,|\, \xi \rangle)^{\mathrm{Wick}}_\varepsilon = a^\dagger_\varepsilon(\xi), \tag{B.17}$$

since $\langle \xi \,|\, \cdot \rangle \in \mathscr{P}_{1,0}(\mathfrak{H})$ and $\langle \cdot \,|\, \xi \rangle \in \mathscr{P}_{0,1}(\mathfrak{H})$. In general, normal ordered products of creation and annihilation operators are Wick quantizations:

$$b(z) = \prod_{j=1}^{p} \langle z \,|\, \xi_j \rangle \prod_{k=1}^{q} \langle \eta_k \,|\, z \rangle \;\mapsto\; (b)^{\mathrm{Wick}}_\varepsilon = \prod_{j=1}^{p} a^\dagger_\varepsilon(\xi_j) \prod_{k=1}^{q} a_\varepsilon(\eta_k). \tag{B.18}$$

In addition,

$$\begin{aligned}
\mathrm{d}\Gamma_\varepsilon(\omega) &= \int_{\mathbb{R}^d} \mathrm{d}k\; \omega(k)\, a^\dagger_\varepsilon(k)\, a_\varepsilon(k) = (\langle z |\, \omega \,| z \rangle_{\mathfrak{H}})^{\mathrm{Wick}}_\varepsilon, \\
\langle z |\, \omega \,| z \rangle_{\mathfrak{H}} &= \int_{\mathbb{R}^d} \mathrm{d}k\; \omega(k)\, \bar{z}(k)\, z(k).
\end{aligned}$$

We wrote the previous formulas in that way to put in evidence how, intuitively, Wick quantizations can be obtained by the correspondence $z(k)^\# \mapsto a^\#_\varepsilon(k)$, and putting all the creation operators to the left of annihilations. Hence, if $\tilde{b}$ has an integral kernel, one can write, formally,

$$\begin{aligned}
b(z) &= \int_{\mathbb{R}^{(p+q)d}} \mathrm{d}k_1 \ldots \mathrm{d}k_q \mathrm{d}h_1 \ldots \mathrm{d}h_p\; \bar{z}(k_1) \ldots \bar{z}(k_q)\, \tilde{b}(k_1, \ldots, \ldots, h_p)\, z(h_1) \ldots z(h_p); \\
(b)^{\mathrm{Wick}}_\varepsilon &= \int_{\mathbb{R}^{(p+q)d}} \mathrm{d}k_1 \ldots \mathrm{d}k_q \mathrm{d}h_1 \ldots \mathrm{d}h_p\; a^\dagger_\varepsilon(k_1) \ldots a^\dagger_\varepsilon(k_q)\, \tilde{b}(k_1, \ldots, \ldots, h_p)\, a_\varepsilon(h_1) \ldots a_\varepsilon(h_p).
\end{aligned}$$

From the definition, for $b \in \mathscr{P}_{p,q}(\mathfrak{H})$,

$$\left\langle z^{\otimes j} \,\middle|\, (b)^{\mathrm{Wick}}_\varepsilon z^{\otimes k} \right\rangle = \delta_{k-p,j-q} \mathbb{1}_{[0,+\infty)}(k-p) \sqrt{\frac{k!j!}{(k-p)!(j-q)!}}\, \varepsilon^{\frac{p+q}{2}} \|z\|^{k-p+j-q} b(z), \tag{B.19}$$

with $\delta_{\ell,m}$ being the Kronecker delta. Therefore $(b)^{\mathrm{Wick}}_{\varepsilon}$ maps $\mathfrak{H}^{\otimes_s k} \to \mathfrak{H}^{\otimes_s j}$, with $j = k - p + q$, and this is used in the following Lemma.

Lemma B.2.1. *Let $b \in \mathscr{P}_{p,q}(\mathfrak{H})$, then, for any $\delta \geqslant p + q$,*

$$\|(\mathrm{d}\Gamma_{\varepsilon}(\mathbb{1}) + 1)^{-\delta/4}(b)^{\mathrm{Wick}}_{\varepsilon}(\mathrm{d}\Gamma_{\varepsilon}(\mathbb{1}) + 1)^{-\delta/4}\|_{\mathscr{L}(\Gamma_s(\mathfrak{H}))} \leqslant \|\tilde{b}\|_{\mathscr{L}(\mathfrak{H}^{\otimes_s p};\mathfrak{H}^{\otimes_s q})}. \tag{B.20}$$

Equation (B.20) *holds for every $(m,n) \in \mathbb{Z}$ such that $m + n \leqslant -\frac{p+q}{2}$, one replacing the $\delta/4$ on the left and the other the one on the right.*

Proof. Let us remark that $(b)^{\mathrm{Wick}}_{\varepsilon}(\mathfrak{H}^{\otimes_s k}) \subseteq \mathfrak{H}^{\otimes_s j}$, with $j = k - p + q$. Let $\psi \in \mathfrak{H}^{\otimes_s k}$, for $k \in \mathbb{N}$. Using the functional calculus for $\mathrm{d}\Gamma_{\varepsilon}(\mathbb{1})$, we obtain

$$\begin{aligned}
&\|(\mathrm{d}\Gamma_{\varepsilon}(\mathbb{1}) + 1)^{-\delta/4}(b)^{\mathrm{Wick}}_{\varepsilon}(\mathrm{d}\Gamma_{\varepsilon}(\mathbb{1}) + 1)^{-\delta/4}\psi\|_{\mathfrak{H}^{\otimes_s j}} = \\
&= \frac{\sqrt{j!k!}}{(k-p)!}\frac{\varepsilon^{\frac{p+q}{2}}}{(\varepsilon k + 1)^{\delta/4}(\varepsilon j + 1)^{\delta/4}}\|\tilde{b} \otimes_s \mathbb{1}_{\mathfrak{H}^{\otimes_s (k-p)}}\|_{\mathfrak{H}^{\otimes_s j}} \leqslant \\
&\leqslant \frac{(\varepsilon k)^{\frac{p}{2}}(\varepsilon j)^{\frac{q}{2}}}{(\varepsilon k + 1)^{\frac{p+q}{4}}(\varepsilon j + 1)^{\frac{p+q}{4}}}\sqrt{\frac{j!}{(j-q)!j^q}}\sqrt{\frac{k!}{(k-p)!k^p}}\|\tilde{b}\|_{\mathscr{L}(\mathfrak{H}^{\otimes_s p};\mathfrak{H}^{\otimes_s q})}\|\psi\|_{\mathfrak{H}^{\otimes_s k}} \leqslant \\
&\leqslant C_{p,q}\,\|\tilde{b}\|_{\mathscr{L}(\mathfrak{H}^{\otimes_s p};\mathfrak{H}^{\otimes_s q})}\|\psi\|_{\mathfrak{H}^{\otimes_s k}}
\end{aligned}$$

with $C_{p,q}$ independent of ε. The proof (m,n) is completely analogous. □

Finally, let us show the explicit action of one-parameter groups generated by second quantized operator on Wick quantized symbols. Let us recall the definition of rescaled coherent states: if $\Omega \in \Gamma_s(\mathfrak{H})$ is the vacuum vector,

$$\Xi_{\varepsilon}(\xi) := W_{\varepsilon}\left(\frac{\xi}{i\varepsilon}\right)\Omega, \qquad \text{for any } \xi \in \mathfrak{H}. \tag{B.21}$$

Proposition B.2.2. (i) *For every $b \in \bigoplus_{p,q\in\mathbb{N}} \mathscr{P}_{p,q}(\mathfrak{H})$*

$$e^{-i\frac{t}{\varepsilon}\mathrm{d}\Gamma_{\varepsilon}(\omega)}(b(z))^{\mathit{Wick}}_{\varepsilon}e^{i\frac{t}{\varepsilon}\mathrm{d}\Gamma_{\varepsilon}(\omega)} = (b(e^{it\omega}z))^{\mathit{Wick}}_{\varepsilon}. \tag{B.22}$$

In particular, for every $\xi \in \mathfrak{H}$

$$e^{-i\frac{t}{\varepsilon}\mathrm{d}\Gamma_{\varepsilon}(\omega)}\,a^{\#}_{\varepsilon}(\xi)\,e^{i\frac{t}{\varepsilon}\mathrm{d}\Gamma_{\varepsilon}(\omega)} = a^{\#}_{\varepsilon}(e^{-it\omega}\xi). \tag{B.23}$$

(ii) *For every $z \in \mathfrak{H}$,*

$$\left\langle \Xi_{\varepsilon}(z) \middle| (b)^{\mathrm{Wick}}_{\varepsilon}\Xi_{\varepsilon}(z) \right\rangle = b(z). \tag{B.24}$$

In addition,

$$\langle \Xi_{\varepsilon}(z) \,|\, W_{\varepsilon}(\xi)\Xi_{\varepsilon}(z)\rangle = e^{2i\Re e\langle \xi \,|\, z\rangle}e^{-\frac{\varepsilon\|\xi\|^2_{\mathfrak{H}}}{4}}, \tag{B.25}$$

and

$$e^{-i\frac{t}{\varepsilon}\mathrm{d}\Gamma_{\varepsilon}(\omega)}W_{\varepsilon}(\xi)\,e^{i\frac{t}{\varepsilon}\mathrm{d}\Gamma_{\varepsilon}(\omega)} = W_{\varepsilon}(e^{-it\omega}\xi). \tag{B.26}$$

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