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Candidate

Veronica Vignoli
ID number 1349834

Thesis Advisor

Prof. Alberto De Sole

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in front of a Board of Examiners composed by:

Nicoletta Cantarini
Matteo Casati
Alessandro D'Andrea (chairman)

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Author's email: vignoli@mat.uniroma1.it

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Contents

Introduction

A Poisson vertex algebra (we will abbreviate it with PVA) is a commutative associative algebra V, endowed with an even derivation ∂ and a bilinear λ-bracket $[\cdot_\lambda\cdot] : V \times V \to V[\lambda]$ that satisfies sesquilinearity $(a, b \in V)$:

$$[\partial a_\lambda b] = -\lambda[a_\lambda b]\,, \ [a_\lambda \partial b] = (\lambda + \partial)[a_\lambda b]\,, \tag{0.1}$$

skewsymmetry $(a, b \in V)$:

$$[a_\lambda b] = -[b_{-\lambda-\partial} a]\,, \tag{0.2}$$

the Jacobi identity $(a, b, c \in V)$:

$$[a_\lambda [b_\mu c]] - [b_\mu [a_\lambda, b]] = [[a_\lambda b]_{\lambda+\mu} c]\,, \tag{0.3}$$

and the left Leibniz rule

$$[a_\lambda bc] = [a_\lambda b]c + [a_\lambda c]b\,. \tag{0.4}$$

It is well known the relation between the theory of Poisson vertex algebras and the integrability of Hamiltonian partial differential equations [BDSK09]. The first appearence of a cohomological approach in order to study the integrability of an Hamiltonian system is due to Krasilshchik [Kra88] and Olver [Ol87] (see also [DSK13]).

More recently, in [BDSHK18], Bakalov, De Sole, Heluani and Kac, introduced a different point of view on cohomology complexes of algebraic structure. In particular, they translated De Sole and Kac's costruction of the variational Poisson cohomology ([DSK13]) in terms of superoperads. (We will call them operads for simplicity). Moreover, with these tools, they also define the classical operad $\mathcal{P}^{\mathrm{cl}}(V)$ associated to a vector superspace V.

Given a vector superspace V with an even endomorphism $\partial \in End(V)$, and $n > 0$, the classical operad $\mathcal{P}^{\mathrm{cl}}(V)(n)$ consists of maps

$$f : \mathcal{G}(n) \times V^{\otimes n} \longrightarrow V[\lambda_1, \dots, \lambda_n] \big/ \langle \partial + \lambda_1 + \dots + \lambda_n \rangle\,, \tag{0.5}$$

which are linear in the second factor. Here and below $\mathcal{G}(n)$ denotes the set of all oriented graphs (with no tadpoles) with set of vertices $V(\Gamma) = \{1, \dots, n\}$ and arbitrary set of edges $E(\Gamma)$. Given the graph $\Gamma \in \mathcal{G}(n)$, the map

$$f^\Gamma : V^{\otimes n} \longrightarrow V[\lambda_1, \dots, \lambda_n] \big/ \langle \partial + \lambda_1 + \dots + \lambda_n \rangle \tag{0.6}$$

in the classical operad $\mathcal{P}^{\mathrm{cl}}(V)(n)$ satisfies the cycle relations (2.26) and (2.27) (as regards the graphs), and the sesquilinearity conditions (2.29) and (2.30) (with respect to the derivation). The composition maps in the classical operad $\mathcal{P}^{\mathrm{cl}}(V)$ are described in section 2.8, see (2.33).

To an operad $\mathcal{P}$ (linear and symmetric) one associates a $\mathbb{Z}$- graded Lie superalgebra W. The $\mathbb{Z}$-graded Lie superalgebra associated to the classical operad $\mathcal{P}^{\mathrm{cl}}(V)$, denoted $W^{\mathrm{cl}}(V)$, is defined, as $\mathbb{Z}$-graded vector superspace, as

$$W^{\mathrm{cl}}(V) = \sum_{n\geq -1} W_n^{\mathrm{cl}}(V) = \sum_{n\geq -1} \mathcal{P}^{\mathrm{cl}}(V)(n+1)^{S_{n+1}}\,, \tag{0.7}$$

with Lie bracket given by

$$[f,g] = f\Box g - (-1)^{p(f)p(g)} g\Box f\,, \tag{0.8}$$

where, for $f\in W_n^{\mathrm{cl}}(V)$ and $g\in W_m^{\mathrm{cl}}(V)$, their $\Box$-product is:

$$f\Box g = \sum_{\sigma\in S_{m+1,n}} (f\circ_1 g)^{\sigma^{-1}} \in W_{m+n}^{\mathrm{cl}}(V)\,. \tag{0.9}$$

In [BDSHK18], the authors show that the structures of Poisson vertex algebras on V are in bijection with the odd elements $X\in W_1^{\mathrm{cl}}(\Pi V)$ such that $[X,X]=0$. Here ΠV stands for the same vector superspace V with reverse parity $\bar{p}$. This give us a cohomology complex (W^{cl}, adX), called the PVA cohomology complex.

The PVA cohomology and the variational Poisson cohomology studied in [DSK13], are related but defined differently. In particular, we have a canonical injective homomorphism of Lie superalgebras from the variational Poisson cohomology to the PVA cohomology, which is an isomorphism for the 0-th and 1-st cohomologies. The results of the present thesis are the first (and main) step needed for the proof that, under the assumption of smoothness for the PVA V, the PVA cohomology and the variational Poisson cohomology are isomorphic.

In this work we will focus on studying the PVA cohomology complex, associated to a Poisson vertex algebra V. Recall that on the space $W_{n-1}^{\mathrm{cl}}(\Pi V)$ we have the following grading: $\mathrm{gr}^r\, W_{n-1}^{\mathrm{cl}}(\Pi V)$ is the set of maps $Y\in W_{n-1}^{\mathrm{cl}}(\Pi V)$ such that $Y^\Gamma=0$ unless $\mid E(\Gamma)\mid = r$. By the cycle relations (2.26) and (2.27), the top degree in $grW_{n-1}^{\mathrm{cl}}(\Pi V)$ is $r=n-1$. It consists of collection of maps

$$Y^\Gamma : (\Pi V)^{\otimes n} \longrightarrow (\Pi V)\,,\ \text{for } \Gamma\in\mathcal{G}_0(n),\ \mid E(\Gamma)\mid = n-1,$$

satisfying cycle relations (2.26) and (2.27), the invariance under the action of the symmetric group , and $Y^\Gamma(\partial(v_1\otimes\ldots\otimes v_n)) = \partial Y^\Gamma(v_1\otimes\ldots\otimes v_n)$. The main result of the thesis is the following

Theorem 0.1. *Let V be a Poisson vertex algebra. There is a natural surjective morphism of cochain complexes*

$$(W_\bullet^{\mathrm{cl}}(\Pi V), adX) \to (C^\bullet_{\partial,Har}(V), d)\,, \tag{0.10}$$

mapping $Y \in W^{\mathrm{cl}}_{n-1}(\Pi V)$ to Y^{Λ_n}, where Λ_n is the standard n-line $\Lambda_n = 1 \to \ldots \to n$. The morphism (0.10) restricts to a bijection on the top degree:

$$gr^{n-1} W^{\mathrm{cl}}_{n-1}(\Pi V) \xrightarrow{\sim} C^n_{\partial,Har}(V)\,. \tag{0.11}$$

In Theorem 0.1, $(C^{\bullet}_{\partial,Har}(V), d)$ is the differential Harrison complex ([Har62]) associated to the commutative associative differential algebra V (and its action on V by left or right multiplication). It is defined as the subcomplex of the differential Hochschild complex, whose n-cochain are elements $F \in Hom_{\mathbb{F}[\partial]}(V^{\otimes n}, V)$ such that

$$F(v_1 \otimes \ldots \otimes v_n) = \sum_{\substack{\pi \in S_n \\ \pi \text{ monotone} \\ \pi(1)=k}} (-1)^{k-1} sgn(\pi) F(v_{\pi(1)} \otimes \ldots \otimes v_{\pi(n)})\,, \quad \forall k = 2, \ldots, n\,. \tag{0.12}$$

The differential of the Hochschild (hence the Harrison) cohomology complex is, $d : Hom_{\mathbb{F}[\partial]}(V^{\otimes n}, V) \to Hom_{\mathbb{F}[\partial]}(V^{\otimes n+1}, V)$, given by

$$\begin{aligned}(df)(v_1 \otimes \ldots \otimes v_{n+1}) &= v_1 f(v_2 \otimes \ldots \otimes v_{n+1}) \\ &+ \sum_{i=1}^{n} (-1)^i f(v_1 \otimes \ldots \otimes v_{i-1} \otimes v_i v_{i+1} \otimes v_{i+2} \otimes \ldots \otimes v_{n+1}) \\ &+ (-1)^{n+1} f(v_1 \otimes \ldots \otimes v_n) v_{n+1}\,.\end{aligned} \tag{0.13}$$

The key point of Theorem 0.1 is the relation between the Harrison's condition (0.12) and the property of invariance under the action of the symmetric group for an element $Y \in W^{\mathrm{cl}}_{n-1}(\Pi V)$. In fact, the key step for the proof of Theorem 0.1 is the following

Lemma 0.2. *If $Y \in W^{\mathrm{cl}}_{n-1}(\Pi V)$, then Y^{Λ_n} satisfies the Harrison's relations (1.12). Moreover, given $F \in C^n_{\partial,Har}(V)$, there exists a unique $Y \in gr^{n-1} W^{\mathrm{cl}}_{n-1}(\Pi V)$ such that $Y^{\Lambda_n} = F$. Hence, there is a bijective linear map*

$$gr^{n-1} W^{\mathrm{cl}}_{n-1}(\Pi V) \xrightarrow{\sim} C^n_{\partial,Har}(V)\,,$$

mapping $Y \mapsto Y^{\Lambda_n}$.

The outline of the thesis is as follows. In Chapter 1 we recall basic notions about Harrison cohomology. Since Harrison's complex is defined as a subcomplex of Hochschild's one, first we review the definition of Hochschild cohomology. Then, we follow the original paper of Harrison [Har62] to define his complex. The principal tools are monotone permutations, so, Section 1.2 is focused on them and their properties.

The second chapter is about the PVA cohomology. We follow the recent point of view of Bakalov, De Sole, Heluani and Kac's paper [BDSHK18], which involves the theory of linear unitary symmetric operads.

In Chapter 3 we present the main result of the thesis, Theorem 3.1. The proof of the theorem is divided in several lemmas. Sections 3.2 and 3.3 are focused on a particular class of graphs: lines. We study in a more deep way connected lines and we present the identity that relates the lines obtained by the action of all monotone permutations, starting at a fixed integer k, on the standard line Λ_n. This allows us to prove Lemma 0.2 (Section 3.4). Finally, in Section 3.5, we investigate the relation between the differentials of the two complexes.

In chapter 4, we give an overview of how Theorem 0.1 can be used towards linking the classical PVA cohomology fo [BDSHK18] and the variational Poisson cohomology of [DSK13]. We also recall the definition of variational Poisson complex ([DSK13]).

Throughout the thesis the base field $\mathbb{F}$ has characteristic 0, and, unless otherwise specified, all vector spaces, their tensor products and Hom's are over $\mathbb{F}$.

Chapter 1

Harrison cohomology

In this chapter we recall the definition of Harrison cohomology.

1.1 Hochschild complex

First, we need to define the Hoshschild complex, of which Harrison's is a subcomplex. Hochschild cohomology groups were introduced by Hochschild [Hoc45] in 1945. For more references, see also [W94].

Let A be a unital and associative algebra over the base field $\mathbb{F}$. Let M be an A-bimodule. We will write $A^{\otimes n}$ for the n-fold tensor product $A \otimes \ldots \otimes A$. The Hochschild cohomology is defined as follows. The space of n-cochain is

$$Hom(A^{\otimes n}, M) \,, \tag{1.1}$$

and the differential $d : Hom(A^{\otimes n}, M) \to Hom(A^{\otimes n+1}, M)$ is defined by

$$\begin{aligned}(df)(a_1 \otimes \ldots \otimes a_{n+1}) = a_1 f(a_2 \otimes \ldots \otimes a_{n+1}) \\ + \sum_{i=1}^{n} (-1)^i f(a_1 \otimes \ldots \otimes a_{i-1} \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \ldots \otimes a_{n+1}) \\ + (-1)^{n+1} f(a_1 \otimes \ldots \otimes a_n) a_{n+1} \,.\end{aligned} \tag{1.2}$$

It is not hard to check that $d^2 = 0$. We thus get a cohomology complex

$$0 \longrightarrow M \xrightarrow{d} Hom(A, M) \xrightarrow{d} Hom(A \otimes A, M) \xrightarrow{d} \cdots \tag{1.3}$$

Definition 1.1. The *Hochschild cohomology* of A with coefficients in M is

$$\begin{aligned}H^n(A, M) &:= H^n(Hom(A^{\otimes *}, M), d) \\ &= \frac{\mathrm{Ker}(d : Hom(A^{\otimes n}, M) \to Hom(A^{\otimes n+1}, M))}{Im(d : Hom(A^{\otimes n-1}, M) \to Hom(A^{\otimes n}, M))} \,.\end{aligned} \tag{1.4}$$

If A is a differential algebra, with derivation $\partial : A \to A$, and M is a differential module over A, we may consider the *differential Hochschild complex* by taking the subspace of n-cochains

$$Hom_{\mathbb{F}[\partial]}(A^{\otimes n}, M) \,. \tag{1.5}$$

It is clear by the definition (1.2) that the differential d maps $Hom_{\mathbb{F}[\partial]}(A^{\otimes n}, M)$ to $Hom_{\mathbb{F}[\partial]}(A^{\otimes n+1}, M)$. Hence, we have a cohomology subcomplex.

1.2 Monotone permutations

Let S_n be the symmetric group. Using Harrison's notation in [Har62] (see also [GS87]), we have the following definition:

Definition 1.2. Let $\pi \in S_n$ and $i \in \{1, \ldots, n\}$. π is called *monotone* if, for each $i \in \{1, \ldots, n\}$, one of the two following conditions holds:

(a) $\pi(j) < \pi(i)$ for all $j < i$;

(b) $\pi(j) > \pi(i)$ for all $j < i$.

(Not necessarely the same condition (a) or (b) holds for every i.) When (b) holds, we call i a *drop* with respect to π. Also, $\pi(1) = k$ is called the *start* of π.

We denote by $\mathcal{M}_n \subset S_n$ the set of monotone permutations and by $\mathcal{M}_n^k \subset \mathcal{M}_n$ the set of the ones starting at k.

Here is a simple description of all monotone permutations starting at k. Let us identify the permutation $\pi \in S_n$ with the n-tuple $[\pi(1), \ldots, \pi(n)]$. So, in the first position of $\pi \in \mathcal{M}_n^k$ we put k. Then, for every choice of $k-1$ positions in $\{2, \ldots, n\}$ we have a different monotone permutation. In the "selected" positions (that will be the drops with respect to π) we put numbers 1 to $k-1$ in decreasing order from left to right. In all the remaining positions we write numbers $k+1$ to n in increasing order from left to right.

According to the above description, we have a bijective correspondence

$$\mathcal{M}_n^k \xrightarrow{\sim} \{D \subset \{2, \ldots, n\} \text{ s. t. } \mid D \mid = k-1\} \;, \tag{1.6}$$

associating the monotone permutation $\pi \in \mathcal{M}_n^k$ to the set $D(\pi)$ of drops with respect to π, which are

$$\pi^{-1}(k-1) < \pi^{-1}(k-2) < \ldots < \pi^{-1}(1) \in \{2, \ldots, n\} \,.$$

Example 1.3. Obviously, the only monotone permutation starting at 1 is the identity, while the only monotone permutation starting at n is

$$\sigma_n = [n \;\; n-1 \;\; \ldots \;\; 2 \;\; 1] \,. \tag{1.7}$$

Example 1.4. Let $n = 5$ and $k = 3$. The monotone permutations starting at 3 are

$$[3 \;\; \underline{2} \;\; \underline{1} \;\; 4 \;\; 5] \,, [3 \;\; \underline{2} \;\; 4 \;\; \underline{1} \;\; 5] \,, [3 \;\; \underline{2} \;\; 4 \;\; 5 \;\; \underline{1}] \,, [3 \;\; 4 \;\; \underline{2} \;\; \underline{1} \;\; 5] \,, [3 \;\; 4 \;\; \underline{2} \;\; 5 \;\; \underline{1}] \,, [3 \;\; 4 \;\; 5 \;\; \underline{2} \;\; \underline{1}] \,,$$

where we underlined the positions of the drops.

Given a monotone permutation π, we denote by $dr(\pi)$ the sum of all the drops with respect to π. According to the previous description, we can easly see that

$$(-1)^{dr(\pi)} = (-1)^{k-1} sgn(\pi) \,, \tag{1.8}$$

if k is the start of π.

Note that the description (1.6) of $\mathcal{M}_n^k$ in terms of positions of drops allows us to count the elements in $\mathcal{M}_n^k$, for fixed n and k. We have:

$$| \mathcal{M}_n^k |= \binom{n-1}{k-1} . \tag{1.9}$$

Remark 1.5. Let us denote by $\mathcal{M}_n^{k,k-1} \subset \mathcal{M}_n^k$ the subset of all monotone permutations π starting at k with $\pi(2) = k-1$, and by $\mathcal{M}_n^{k,k+1} \subset \mathcal{M}_n^k$ the subset of all monotone permutations π starting at k with $\pi(2) = k+1$. We have

$$\mathcal{M}_n^k = \mathcal{M}_n^{k,k-1} \sqcup \mathcal{M}_n^{k,k+1} .$$

In accordance to (1.9), $| \mathcal{M}_n^{k,k-1} |= \binom{n-2}{k-2}$ and $| \mathcal{M}_n^{k,k+1} |= \binom{n-2}{k-1}$, so that $\binom{n-1}{k-1} = \binom{n-2}{k-2} + \binom{n-2}{k-1}$.

Lemma 1.6. *There are natural identifications $\mathcal{M}_n^{k,k-1} \simeq \mathcal{M}_{n-1}^{k-1}$ (respectively $\mathcal{M}_n^{k,k+1} \simeq \mathcal{M}_{n-1}^{k}$), mapping $\pi \in \mathcal{M}_n^{k,k-1}$ to $\bar{\pi} \in \mathcal{M}_{n-1}^{k-1}$ (resp. $\pi \in \mathcal{M}_n^{k,k+1}$ to $\bar{\pi} \in \mathcal{M}_{n-1}^{k}$), given by $\bar{\pi}(1) := k-1$ (resp. k), and, for $i = 2, \dots, n-1$,*

$$\bar{\pi}(i) := \begin{cases} \pi(i+1) & \text{if } \pi(i+1) < k \\ \pi(i+1) - 1 & \text{if } \pi(i+1) > k \end{cases} . \tag{1.10}$$

Moreover,

$$(-1)^{dr(\bar{\pi})} = (-1)^{dr(\pi)+k} \, (\textit{resp. } (-1)^{dr(\pi)+k-1}) .$$

Proof. The maps $\pi \mapsto \bar{\pi}$ are trivially injective maps between sets with same cardinality (cf. (1.9) and Remark 1.5). Hence, they are bijective. □

Remark 1.7. Observe that, given a monotone permutation π, either $\pi(n) = 1$ or $\pi(n) = n$. Let denote by ${}^1\mathcal{M}_n^k \subset \mathcal{M}_n^k$ the set of all the monotone permutations starting at k with $\pi(n) = 1$, and by ${}^n\mathcal{M}_n^k \subset \mathcal{M}_n^k$ the set of all the monotone permutations starting at k with $\pi(n) = n$. As in Remark 1.5, we have that

$$\mathcal{M}_n^k = {}^1\mathcal{M}_n^k \sqcup {}^n\mathcal{M}_n^k .$$

In accordance to (1.9), $| {}^1\mathcal{M}_n^k |= \binom{n-2}{k-2}$ and $| {}^n\mathcal{M}_n^k |= \binom{n-2}{k-1}$, so that $| \mathcal{M}_n^k |= \binom{n-1}{k-1} = \binom{n-2}{k-2} + \binom{n-2}{k-1}$.

Lemma 1.8. *There are natural identifications ${}^1\mathcal{M}_n^k \simeq \mathcal{M}_{n-1}^{k-1}$ (resp. ${}^n\mathcal{M}_n^k \simeq \mathcal{M}_{n-1}^{k}$), mapping $\pi \in {}^1\mathcal{M}_n^k$ to $\tilde{\pi} \in \mathcal{M}_{n-1}^{k-1}$ (resp. $\pi \in {}^n\mathcal{M}_n^k$ to $\tilde{\pi} \in \mathcal{M}_{n-1}^{k}$), given by, for $i = 1, \dots, n-1$, $\tilde{\pi}(i) := \pi(i) - 1$ (resp. $\tilde{\pi}(i) := \pi(i)$). Moreover,*

$$(-1)^{dr(\tilde{\pi})} = (-1)^{dr(\pi)+n} \, (\textit{resp. } (-1)^{dr(\pi)}) .$$

Proof. The maps $\pi \mapsto \tilde{\pi}$ are obviously bijective. □

1.3 Harrison cohomology

Let us now recall Harrison's original definition of his cohomology ([Har62]). Let A be a commutative, associative, unital algebra, and M be a symmetric A-bimodule, *i.e.* such that $am = ma$, for all $a \in A$ and $m \in M$. For every $k > 1$ and $F \in Hom(A^{\otimes n}, M)$ a cochain in the Hochschild complex, we use the following notation:

$$L_k F(a_1 \otimes \ldots \otimes a_n) \coloneqq \sum_{\pi \in \mathcal{M}_n^k} (-1)^{dr(\pi)} F(a_{\pi(1)} \otimes \ldots \otimes a_{\pi(n)}) \,, \tag{1.11}$$

The *Harrison's complex* is defined as the subcomplex of the Hochschild complex (1.3) consisting of those F such that

$$F = L_k F \,, \text{ for every } 2 \leq k \leq n \,. \tag{1.12}$$

We will denote by

$$C^n_{Har}(A, M) \subset Hom(A^{\otimes n}, M) \tag{1.13}$$

the space of n-cochain in Harrison cohomology, and by

$$H^\bullet_{Har}(A, M) \coloneqq H^\bullet(C^\bullet_{Har}(A, M), d) \tag{1.14}$$

the corresponding Harrison cohomology (where d is the Hochschild differential (1.2)).

Furthermore, if A is a differential algebra with derivation $\partial : A \to A$, and M is a differential module, we may consider the *differential Harrison subcomplex*

$$C^n_{\partial, Har}(A, M) \subset Hom_{\mathbb{F}[\partial]}(A^{\otimes n}, M) \,, \tag{1.15}$$

again defined by the Harrison's conditions (1.12).

Proposition 1.9 ([Har62]). *1. The Harrison complex $(C^\bullet_{Har}(A, M), d)$ is a subcomplex of the Hochschild complex.*

2. If A is a differential algebra, with a derivation $\partial : A \to A$, the differential Harrison complex $(C^\bullet_{\partial, Har}(A, M), d)$ is a subcomplex of the differential Hochschild complex.

Proof. We need to prove that the Harrison's conditions (1.12) are compatible with the Hochschild differential (1.2). Let $F \in Hom(A^{\otimes n-1}, M)$. We have to prove that

$$F = L_k F \,, \; \forall\, k = 2, \ldots, n-1 \quad \Rightarrow \quad dF = L_k(dF) \,, \; \forall\, k = 2, \ldots, n \,. \tag{1.16}$$

For $k = n$, recall that the only monotone permutation is σ_n as in (1.7). We have:

$$\begin{aligned} L_n(dF)(a_1 \otimes \ldots \otimes a_n) &= (-1)^{dr(\sigma_n)} (dF)(a_n \otimes \ldots \otimes a_1) \\ &= (-1)^{dr(\sigma_n)} a_n F(a_{n-1} \otimes \ldots \otimes a_1) \\ &+ \sum_{i=2}^{n} (-1)^{dr(\sigma_n)} (-1)^{n-i+1} F(a_n \otimes \ldots \otimes a_i a_{i-1} \otimes \ldots \otimes a_1) \end{aligned}$$

$$+ (-1)^{dr(\sigma_n)}(-1)^n F(a_n \otimes \ldots \otimes a_2)a_1 \,. \tag{1.17}$$

The first term in the right-hand side of (1.17) is

$$\begin{aligned}
&(-1)^{dr(\sigma_n)} a_n F(a_{n-1} \otimes \ldots \otimes a_1) \\
&= (-1)^n (-1)^{dr(\sigma_{n-1})} a_n F(a_{\sigma_{n-1}(1)} \otimes \ldots \otimes a_{\sigma_{n-1}(n-1)}) \\
&= (-1)^n (L_{n-1}F)(a_1 \otimes \ldots \otimes a_{n-1}) a_n \,.
\end{aligned} \tag{1.18}$$

For the second term appearing in the right-hand side of (1.17), let $b_j = a_j$ for $1 \leq j < i-1$, $b_{i-1} = a_i a_{i-1}$, and $b_j = a_{j+1}$ for $i \leq j \leq n-1$. We have:

$$\begin{aligned}
&\sum_{i=2}^{n} (-1)^{dr(\sigma_n)} (-1)^{n-i+1} F(a_n \otimes \ldots \otimes a_i a_{i-1} \otimes \ldots \otimes a_1) \\
&= \sum_{i=2}^{n} (-1)^{dr(\sigma_{n-1})} (-1)^{-i+1} F(b_{n-1} \otimes \ldots \otimes b_1) \\
&= \sum_{i=2}^{n} (-1)^{i-1} L_{n-1} F(b_1 \otimes \ldots \otimes b_{n-1}) \\
&= \sum_{i=2}^{n} (-1)^{i-1} L_{n-1} F(a_1 \otimes \ldots \otimes a_{i-1} a_i \otimes \ldots \otimes a_n)
\end{aligned} \tag{1.19}$$

Finally, the last term of the right-hand side of (1.17) is

$$\begin{aligned}
&(-1)^{dr(\sigma_n)} (-1)^n F(a_n \otimes \ldots \otimes a_2) a_1 \\
&= (-1)^{dr(\sigma_{n-1})} F(a_{\sigma_{n-1}(1)+1} \otimes \ldots \otimes a_{\sigma_{n-1}(n-1)+1}) a_1 \\
&= a_1 L_{n-1} F(a_2 \otimes \ldots \otimes a_n)
\end{aligned} \tag{1.20}$$

Combining (1.17), (1.18), (1.19), and (1.20), we get

$$\begin{aligned}
L_n(dF)(a_1 \otimes \ldots \otimes a_n) &= (-1)^n a_n L_{n-1} F(a_1 \otimes \ldots \otimes a_{n-1}) \\
&\quad + \sum_{i=1}^{n-1} (-1)^i L_{n-1} F(a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n) \\
&\quad + a_1 L_{n-1} F(a_2 \otimes \ldots \otimes a_n) \\
&= d(L_{n-1}F)(a_1 \otimes \ldots \otimes a_n) \,.
\end{aligned} \tag{1.21}$$

Using the assumption (1.12) on F, we get (1.16) for $k = n$.

Now, let $2 \leq k \leq n-1$. Using the definition (1.2) of the differential, and (1.11), we have:

$$\begin{aligned}
L_k(dF)(a_1 \otimes \ldots \otimes a_n) &= \sum_{\pi \in \mathcal{M}_n^k} (-1)^{dr(\pi)} dF(a_k \otimes a_{\pi(2)} \otimes \ldots \otimes a_{\pi(n)}) \\
&= \sum_{\pi \in \mathcal{M}_n^k} (-1)^{dr(\pi)} a_k F(a_{\pi(2)} \otimes \ldots \otimes a_{\pi(n)}) \\
&+ \sum_{\pi \in \mathcal{M}_n^k} \sum_{i=1}^{n-1} (-1)^{dr(\pi)+i} F(a_{\pi(1)} \otimes \ldots \otimes a_{\pi(i)} a_{\pi(i+1)} \otimes \ldots \otimes a_{\pi(n)})
\end{aligned}$$

$$+ \sum_{\pi \in \mathcal{M}_n^k} (-1)^{dr(\pi)+n} F(a_k \otimes a_{\pi(2)} \otimes \ldots \otimes a_{\pi(n-1)}) a_{\pi(n)} \,. \tag{1.22}$$

By Remark 1.5 and Lemma 1.6, the first term in the right-hand side of (1.22) is

$$\begin{aligned}
&\sum_{\pi \in \mathcal{M}_n^k} (-1)^{dr(\pi)} a_k F(a_{\pi(2)} \otimes \ldots \otimes a_{\pi(n)}) \\
&= a_k \sum_{\pi \in \mathcal{M}_n^{k,k-1}} (-1)^{dr(\pi)} F(a_{\pi(2)} \otimes \ldots \otimes a_{\pi(n)}) \\
&\quad + a_k \sum_{\pi \in \mathcal{M}_n^{k,k+1}} (-1)^{dr(\pi)} F(a_{\pi(2)} \otimes \ldots \otimes a_{\pi(n)}) \\
&= (-1)^k a_k \sum_{\bar{\pi} \in \mathcal{M}_{n-1}^{k-1}} (-1)^{dr(\bar{\pi})} F(a_{\bar{\pi}(1)} \otimes \ldots \otimes a_{\bar{\pi}(i)+\delta_{\bar{\pi}(i)\geq k}} \otimes \ldots) \\
&\quad + (-1)^{k-1} a_k \sum_{\bar{\pi} \in \mathcal{M}_{n-1}^{k}} (-1)^{dr(\bar{\pi})} F(a_{\bar{\pi}(1)+1} \otimes \ldots \otimes a_{\bar{\pi}(i)+\delta_{\bar{\pi}(i)\geq k}} \otimes \ldots) \\
&= (-1)^k a_k (L_{k-1}F)(a_1 \otimes \ldots \hat{a_k} \ldots \otimes a_n) \\
&\quad + (-1)^{k-1} a_k (L_k F)(a_1 \otimes \ldots \hat{a_k} \ldots \otimes a_n) \,,
\end{aligned} \tag{1.23}$$

where $\hat{a_k}$ denotes missing factor. We then use the assumption (1.12) to conclude that (1.23) vanishes.

The third term in the right-hand side of (1.22), is, by Remark 1.7 and Lemma 1.8,

$$\begin{aligned}
&\sum_{\pi \in \mathcal{M}_n^k} (-1)^{dr(\pi)} (-1)^n F(a_k \otimes a_{\pi(2)} \otimes \ldots \otimes a_{\pi(n-1)}) a_{\pi(n)} \\
&= \sum_{\pi \in {}^1\mathcal{M}_n^k} (-1)^{dr(\pi)} (-1)^n F(a_k \otimes a_{\pi(2)} \otimes \ldots \otimes a_{\pi(n-1)}) a_1 \\
&\quad + \sum_{\pi \in {}^n\mathcal{M}_n^k} (-1)^{dr(\pi)} (-1)^n F(a_k \otimes a_{\pi(2)} \otimes \ldots \otimes a_{\pi(n-1)}) a_n \\
&= \sum_{\tilde{\pi} \in \mathcal{M}_{n-1}^{k-1}} (-1)^{dr(\tilde{\pi})} F(a_{\tilde{\pi}(1)+1} \otimes \ldots \otimes a_{\tilde{\pi}(n-1)+1}) a_1 \\
&\quad + \sum_{\tilde{\pi} \in \mathcal{M}_{n-1}^{k}} (-1)^{dr(\tilde{\pi})+n} F(a_{\tilde{\pi}(1)} \otimes \ldots \otimes a_{\tilde{\pi}(n-1)}) a_n \\
&= (L_{k-1}F)(a_2 \otimes \ldots \otimes a_n) a_1 + (-1)^n (L_k F)(a_1 \otimes \ldots \otimes a_{n-1}) a_n \\
&= a_1 F(a_2 \otimes \ldots \otimes a_n) + (-1)^n F(a_1 \otimes \ldots \otimes a_{n-1}) a_n \,.
\end{aligned} \tag{1.24}$$

Finally, in the second summand in the right-hand side of (1.22), we compute separately the term with $i = 1$ and all other terms $i \neq 1$. By Remark 1.5 and by Lemma 1.6, the term with $i = 1$ in the second summand of the right-hand side of (1.22) is

$$- \sum_{\pi \in \mathcal{M}_n^k} (-1)^{dr(\pi)} F(a_{\pi(1)} a_{\pi(2)} \otimes a_{\pi(3)} \otimes \ldots \otimes a_{\pi(n)})$$

$$
\begin{aligned}
&= - \sum_{\pi \in \mathcal{M}_n^{k,k-1}} (-1)^{dr(\pi)} F(a_k a_{k-1} \otimes a_{\pi(3)} \otimes \ldots \otimes a_{\pi(n)}) \\
&\quad - \sum_{\pi \in \mathcal{M}_n^{k,k+1}} (-1)^{dr(\pi)} F(a_k a_{k+1} \otimes a_{\pi(3)} \otimes \ldots \otimes a_{\pi(n)}) \\
&= (-1)^{k-1} \sum_{\tilde{\pi} \in \mathcal{M}_{n-1}^{k-1}} (-1)^{dr(\tilde{\pi})} F(a_k a_{k-1} \otimes a_{\tilde{\pi}(2)+\delta_{\tilde{\pi}(2)\geq k}} \otimes \ldots \otimes a_{\tilde{\pi}(n-1)+\delta_{\tilde{\pi}(n-1)\geq k}}) \\
&\quad + (-1)^{k} \sum_{\tilde{\pi} \in \mathcal{M}_{n-1}^{k}} (-1)^{dr(\tilde{\pi})} F(a_k a_{k+1} \otimes a_{\tilde{\pi}(2)+\delta_{\tilde{\pi}(2)\geq k}} \otimes \ldots \otimes a_{\tilde{\pi}(n-1)+\delta_{\tilde{\pi}(n-1)\geq k}}) \\
&= (-1)^{k-1} (L_{k-1} F)(a_1 \otimes \ldots \otimes a_{k-2} \otimes a_{k-1} a_k \otimes a_{k+1} \otimes \ldots \otimes a_n) \\
&\quad + (-1)^{k} (L_k F)(a_1 \otimes \ldots \otimes a_{k-1} \otimes a_k a_{k+1} \otimes a_{k+2} \otimes \ldots \otimes a_n) \\
&= (-1)^{k-1} F(a_1 \otimes \ldots \otimes a_{k-1} a_k \otimes \ldots \otimes a_n) \\
&\quad + (-1)^{k} F(a_1 \otimes \ldots \otimes a_k a_{k+1} \otimes \ldots \otimes a_n) \,. \qquad (1.25)
\end{aligned}
$$

For the last equality we used the Harrison conditions assumption (1.12).

We are left to compute the sum over $i \in \{2, \ldots, n-1\}$ in the second summand of the right-hand side of (1.22). It is:

$$
\sum_{i=2}^{n-1} (-1)^i \sum_{\pi \in \mathcal{M}_n^k} (-1)^{dr(\pi)} F(a_{\pi(1)} \otimes \ldots \otimes a_{\pi(i)} a_{\pi(i+1)} \otimes \ldots \otimes a_{\pi(n)}) \,. \qquad (1.26)
$$

We have the following obvious set decomposition:

$$
\begin{aligned}
\mathcal{M}_n^k &= \mathcal{M}_n^k(i, i+1 = \text{drop}) \sqcup \mathcal{M}_n^k(i, i+1 = \text{not drop}) \sqcup \\
&\quad \sqcup \mathcal{M}_n^k(i = \text{drop}, i+1 = \text{not drop}) \sqcup \mathcal{M}_n^k(i = \text{not drop}, i+1 = \text{drop}) \,, \qquad (1.27)
\end{aligned}
$$

where

$$
\mathcal{M}_n^k(i, i+1 = \text{drop}) = \left\{ \pi \in \mathcal{M}_n^k \mid i \text{ and } i+1 \text{ are drops with respect to } \pi \right\} ,
$$

and similarly for all the other sets in (1.27). Hence, (1.26) splits as four sums, over the four sets in the right-hand side of (1.27). We have the obvious bijective correspondence

$$
\mathcal{M}_n^k(i = \text{drop}, i+1 = \text{not drop}) \xrightarrow{\sim} \mathcal{M}_n^k(i = \text{not drop}, i+1 = \text{drop}) \,,
$$

obtained by switching $\pi(i)$ and $\pi(i+1)$ in the permutation:

$$
\pi \mapsto \tilde{\pi} = \pi \circ (i, i+1) \,.
$$

Clearly,

$$
dr(\tilde{\pi}) = dr(\pi) + 1 \,.
$$

Moreover, since the algebra A is commutative,

$$
a_{\pi(i)} a_{\pi(i+1)} = a_{\tilde{\pi}(i)} a_{\tilde{\pi}(i+1)} \,.
$$

Hence, the two contributions in (1.26) obtained by running over π in $\mathcal{M}_n^k(i = \text{drop}, i+1 = \text{not drop})$ and in $\mathcal{M}_n^k(i = \text{not drop}, i+1 = \text{drop})$ are the same, with opposite sign, and they cancel each other. Hence, (1.26) is obtained by running over π only in the first two sets in the right-hand side of (1.27). We thus get the two contributions

$$\sum_{i=2}^{n-1}(-1)^i \sum_{\pi\in\mathcal{M}_n^k(i,i+1=\text{drop})} (-1)^{dr(\pi)} F(a_{\pi(1)}\otimes\ldots\otimes a_{\pi(i)}a_{\pi(i+1)}\otimes\ldots\otimes a_{\pi(n)}) \tag{1.28}$$

and

$$\sum_{i=2}^{n-1}(-1)^i \sum_{\pi\in\mathcal{M}_n^k(i,i+1=\text{not drop})} (-1)^{dr(\pi)} F(a_{\pi(1)}\otimes\ldots\otimes a_{\pi(i)}a_{\pi(i+1)}\otimes\ldots\otimes a_{\pi(n)}) \tag{1.29}$$

Let us compute (1.28). We can decompose

$$\mathcal{M}_n^k(i, i+1 = \text{drop}) = \bigsqcup_{j=2}^{k-1} \Big\{\pi \in \mathcal{M}_n^k \mid \pi(i) = j,\, \pi(i+1) = j-1\Big\}\,,$$

and we have a bijective correspondence

$$\Big\{\pi \in \mathcal{M}_n^k \mid \pi(i) = j,\, \pi(i+1) = j-1\Big\} \xrightarrow{\sim} \Big\{\bar\pi \in \mathcal{M}_{n-1}^{k-1} \mid \bar\pi(i) = j-1\Big\}$$

mapping π to $\bar\pi$, which is obtained by removing the drop in position $i+1$, and shifting accordingly all other indices. With a formula

$$\bar\pi(\alpha) = \begin{cases} \pi(\alpha)-1 & \text{if } \alpha \leq i \\ \pi(\alpha+1) & \text{if } \alpha \geq i+1 \text{ and } \pi(\alpha+1) < j-1 \\ \pi(\alpha+1)-1 & \text{if } \alpha \geq i+1 \text{ and } \pi(\alpha+1) > j-1 \end{cases}. \tag{1.30}$$

As for the drops, in passing from the permutation π to the permutation $\bar\pi$, we remove the drop $i+1$ and we shift by one position all the $(j-2)$ drops greater than $i+1$. Hence,

$$dr(\bar\pi) = dr(\pi) - i - j + 1\,. \tag{1.31}$$

Using (1.30) and (1.31), we can then rewrite (1.28) as

$$\begin{aligned}
&\sum_{i=2}^{n-1}(-1)^i \sum_{j=2}^{k-1} \sum_{\substack{\pi\in\mathcal{M}_n^k \\ \pi(i)=j,\,\pi(i+1)=j-1}} (-1)^{dr(\pi)} F(a_{\pi(1)}\otimes\ldots\otimes a_{\pi(i)}a_{\pi(i+1)}\otimes\ldots\otimes a_{\pi(n)}) \\
&= \sum_{i=2}^{n-1}(-1)^i \sum_{j=2}^{k-1} \sum_{\substack{\bar\pi\in\mathcal{M}_{n-1}^{k-1} \\ \bar\pi(i)=j-1}} (-1)^{dr(\bar\pi)+i+j-1} F(a_{\bar\pi(1)+\delta_{\bar\pi(1)\geq j}}\otimes\ldots\otimes a_{j-1}a_j\otimes\ldots\otimes a_{\bar\pi(n-1)+\delta_{\bar\pi(n-1)\geq j}}) \\
&= \sum_{j=2}^{k-1}(-1)^{j-1} \sum_{\bar\pi\in\mathcal{M}_{n-1}^{k-1}} (-1)^{dr(\bar\pi)} F(a_{\bar\pi(1)+\delta_{\bar\pi(1)\geq j}}\otimes\ldots\otimes a_{j-1}a_j\otimes\ldots\otimes a_{\bar\pi(n-1)+\delta_{\bar\pi(n-1)\geq j}})
\end{aligned}$$

$$= \sum_{j=2}^{k-1} (-1)^{j-1} (L_{k-1}F)(a_1 \otimes \ldots \otimes a_{j-1}a_j \otimes \ldots \otimes a_n)$$
$$= \sum_{j=2}^{k-1} (-1)^{j-1} F(a_1 \otimes \ldots \otimes a_{j-1}a_j \otimes \ldots \otimes a_n)\,, \tag{1.32}$$

by the Harrison conditions (1.12) on F.

Similarly, we compute (1.29). We decompose

$$\mathcal{M}_n^k(i, i+1 = \text{not drop}) = \bigsqcup_{j=k+1}^{n} \left\{ \pi \in \mathcal{M}_n^k \mid \pi(i) = j,\, \pi(i+1) = j+1 \right\}\,,$$

and we have a bijective correspondence

$$\left\{ \pi \in \mathcal{M}_n^k \mid \pi(i) = j,\, \pi(i+1) = j+1 \right\} \xrightarrow{\sim} \left\{ \bar\pi \in \mathcal{M}_{n-1}^k \mid \bar\pi(i) = j \right\}$$

mapping π to $\bar\pi$, given by:

$$\bar\pi(\alpha) = \begin{cases} \pi(\alpha) & \text{if } \alpha \leq i \\ \pi(\alpha+1) & \text{if } \alpha \geq i+1 \text{ and } \pi(\alpha+1) < j \\ \pi(\alpha+1) - 1 & \text{if } \alpha \geq i+1 \text{ and } \pi(\alpha+1) > j+1 \end{cases}\,. \tag{1.33}$$

As for the drops, in going from the permutation π to $\bar\pi$, we shift by one position all drops to the right of $i+1$. Since $\pi(i+1) = j+1$, of the $n-(i+1)$ positions to the right of $i+1$, precisely $n-(j+1)$ are not drops, and the remaining $j-i$ are drops. Hence,

$$dr(\bar\pi) = dr(\pi) - j + i\,. \tag{1.34}$$

Using (1.33) and (1.34), we can then rewrite (1.29) as

$$\sum_{i=2}^{n-1} (-1)^i \sum_{j=k+1}^{n} \sum_{\substack{\pi \in \mathcal{M}_n^k \\ \pi(i)=j,\, \pi(i+1)=j+1}} (-1)^{dr(\pi)} F(a_{\pi(1)} \otimes \ldots \otimes a_{\pi(i)} a_{\pi(i+1)} \otimes \ldots \otimes a_{\pi(n)})$$
$$= \sum_{i=2}^{n-1} (-1)^i \sum_{j=k+1}^{n} \sum_{\substack{\bar\pi \in \mathcal{M}_{n-1}^k \\ \bar\pi(i)=j}} (-1)^{dr(\bar\pi)+j-i} F(a_{\bar\pi(1)+\delta_{\bar\pi(1)\geq j+1}} \otimes \ldots \otimes a_j a_{j+1} \otimes \ldots \otimes a_{\bar\pi(n-1)+\delta_{\bar\pi(n-1)\geq j+1}})$$
$$= \sum_{j=k+1}^{n} (-1)^j \sum_{\bar\pi \in \mathcal{M}_{n-1}^k} (-1)^{dr(\bar\pi)} F(a_{\bar\pi(1)+\delta_{\bar\pi(1)\geq j+1}} \otimes \ldots \otimes a_j a_{j+1} \otimes \ldots \otimes a_{\bar\pi(n-1)+\delta_{\bar\pi(n-1)\geq j+1}})$$
$$= \sum_{j=k+1}^{n} (-1)^j (L_k F)(a_1 \otimes \ldots \otimes a_j a_{j+1} \otimes \ldots \otimes a_n)$$
$$= \sum_{j=k+1}^{n} (-1)^j F(a_1 \otimes \ldots \otimes a_j a_{j+1} \otimes \ldots \otimes a_n)\,. \tag{1.35}$$

Combining (1.24), (1.25), (1.32), and (1.35) we get

$$(dF)(a_1 \otimes \ldots \otimes a_n)\,,$$

as claimed. The last assertion of the proposition is obvious. $\square$

Chapter 2

PVA cohomology

In this chapter, we recall some basic notions that will be useful throughout the thesis, and we review the construction of the PVA cohomology complex as described in [BDSHK18].

2.1 Symmetric group actions

Let S_n be the symmetric group as in Section 1. There is a natural left action of S_n on an arbitrary n-tuple of objects $(\lambda_1, \dots, \lambda_n)$:

$$\sigma(\lambda_1, \dots, \lambda_n) = (\lambda_{\sigma^{-1}(1)}, \dots, \lambda_{\sigma^{-1}(n)}) , \quad \sigma \in S_n . \tag{2.1}$$

Also, given $V = V_{\bar{0}} \oplus V_{\bar{1}}$ a vector superspace with parity p, we have a linear left action of the symmetric group S_n on the tensor product $V^{\otimes n}$ ($\sigma \in S_n$, $v_1, \dots, v_n \in V$):

$$\sigma(v_1 \otimes \cdots \otimes v_n) := \epsilon_v(\sigma)\, v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)} , \tag{2.2}$$

where, following the Koszul-Quillen rule,

$$\epsilon_v(\sigma) = \prod_{i<j \,|\, \sigma(i)>\sigma(j)} (-1)^{p(v_i)p(v_j)} . \tag{2.3}$$

The corresponding right action of S_n on the the space $Hom(V^{\otimes n}, V)$ is given by ($f \in Hom(V^{\otimes n}, V)$, $\sigma \in S_n$)

$$f^{\sigma}(v_1 \otimes \ldots \otimes v_n) = f(\sigma(v_1 \otimes \ldots \otimes v_n)) . \tag{2.4}$$

The following lemma will be useful later:

Lemma 2.1. *Let $\sigma_n = [n\,, n-1\,, \dots\,, 2\,, 1] \in S_n$ (cf. (1.7)), R a commutative, unitary ring and A a commutative R-algebra. Let $f \in Hom_R(A^{\otimes n}, A)$ and d be the Hochschild differential (1.2). The following identity holds:*

$$d(f^{\sigma_n}) = (-1)^{n+1}(df)^{\sigma_{n+1}} , \tag{2.5}$$

where the symmetric action on f and df is given by (2.4).

Proof. Using the definition of the Hochschild differential (1.2) and the symmetric action (2.4), we get the following identities:

$$
\begin{aligned}
& d(f^{\sigma_n})(a_1 \otimes \ldots \otimes a_{n+1}) \\
& = a_1 f^{\sigma_n}(a_2 \otimes \ldots \otimes a_{n+1}) + \sum_{i=1}^{n} (-1)^i f^{\sigma_n}(a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_{n+1}) \\
& \qquad + (-1)^{n+1} f^{\sigma_n}(a_1 \otimes \ldots \otimes a_n) a_{n+1} \\
& = a_1 f(a_{n+1} \otimes \ldots \otimes a_2) + \sum_{i=1}^{n} (-1)^i f(a_{n+1} \otimes \ldots \otimes a_{i+1} a_i \otimes \ldots \otimes a_1) \\
& \qquad + (-1)^{n+1} f(a_n \otimes \ldots \otimes a_1) a_{n+1} \\
& = (-1)^{n+1} (df)(a_{n+1} \otimes \ldots \otimes a_1) \\
& = (-1)^{n+1} (df)^{\sigma_{n+1}}(a_1 \otimes \ldots \otimes a_{n+1}) \,.
\end{aligned}
$$

□

2.2 Composition of permutations and shuffles

Let $n \geq 1$ and $m_1, \ldots, m_n \geq 0$. We introduce the following notation: $M_0 = 0$ and

$$M_i = \sum_{j=1}^{i} m_j \,, \quad i = 1, \ldots, n \,. \tag{2.6}$$

Given $\sigma \in S_n$ and $\tau_1 \in S_{m_1}, \ldots, \tau_n \in S_{m_n}$, we describe the *composition*

$$\sigma(\tau_1, \ldots, \tau_n) \in S_{M_n}$$

saying how it acts on the tensor power V^{M_n} of a vector space V:

$$(\sigma(\tau_1, \ldots, \tau_n))(v_1 \otimes \ldots \otimes v_{M_n}) = \sigma(\tau_1(v_1 \otimes \ldots \otimes v_{M_1}) \otimes \ldots \otimes \tau_n(v_{M_{n-1}+1} \otimes \ldots \otimes v_{M_n})) \,. \tag{2.7}$$

Definition 2.2. A permutation $\sigma \in S_{m+n}$ is called an (m, n)-*shuffle* if

$$\sigma(1) < \ldots < \sigma(m) \,, \ \sigma(m+1) < \ldots < \sigma(m+n) \,.$$

The subset of (m, n)-shuffles is denoted by $S_{m,n} \subset S_{m+n}$.

Observe that, by definition, $S_{0,n} = S_{n,0} = 1$ for every $n \geq 0$. If either m or n is negative, we set $S_{m,n} = \emptyset$ by convention.

2.3 n-graphs

For an oriented graph Γ, we denoted by $V(\Gamma)$ the set of vertices of Γ, and by $E(\Gamma)$ the set of edges. We call Γ an *n-graph* if $V(\Gamma) = \{1, \ldots, n\}$. Denote by $\mathcal{G}(n)$ the collection of all n-graphs without tadpoles, and by $\mathcal{G}_0(n)$ the collection of all acyclic

n-graphs.

A graph L will be called an *n-line* if its set of edges is of the form $\{i_1 \to i_2,\, i_2 \to i_3, \dots, i_{n-1} \to i_n\}$ where $\{i_1, \dots, i_n\}$ is a permutation of $\{1, \dots, n\}$:

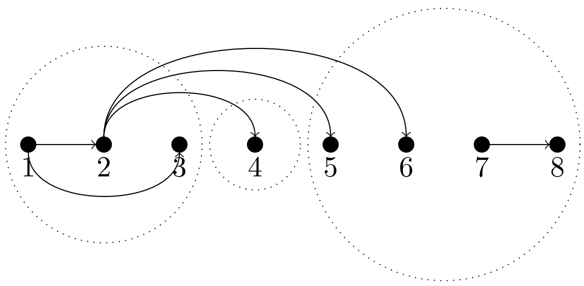
$$L = \underset{i_1}{\circ}\!\to\!\underset{i_2}{\circ}\!\to\cdots\to\!\underset{i_n}{\circ} \quad .$$

We have a natural left action of S_n on the set $\mathcal{G}(n)$: for the n-graph Γ and the permutation σ, the new graph $\sigma(\Gamma)$ is defined to be the same graph as Γ but with the vertex which was labelled as i relabelled as $\sigma(i)$, for every $i = 1, \dots, n$. So, if in Γ there is the edge $i \to j$, then in $\sigma(\Gamma)$ there is the oriented edge $\sigma(i) \to \sigma(j)$. Note that S_n permutes the set of n-lines.

Let us now recall the cocomposition of n-graphs, as described in [BDSHK18]. Given an n-tuple $(m_1, \dots, m_n)$ of positive integers, let M_i as in (2.6). If $\Gamma \in \mathcal{G}(M_n)$, define $\Delta_i^{m_1,\dots,m_n}(\Gamma) \in \mathcal{G}(m_i)$, $i = 1, \dots, n$, the subgraph of Γ associated to the set of vertices $\{M_{i-1}+1, \dots, M_i\}$, relabelled as $\{1, \dots, m_i\}$. Define also $\Delta_0^{m_1,\dots,m_n}(\Gamma)$ to be the graph obtained from Γ by collapsing the vertices and the edges of each $\Delta_i^{m_1,\dots,m_n}(\Gamma)$ into a single vertex, relabelled as i. Then the cocomposition map is the map

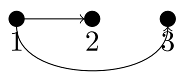
$$\begin{aligned} \Delta^{m_1,\dots,m_n} \colon \mathcal{G}(M_n) &\to \mathcal{G}(n) \times \mathcal{G}(m_1) \times \cdots \times \mathcal{G}(m_n) \\ \Gamma &\mapsto (\Delta_0^{m_1,\dots,m_n}(\Gamma), \Delta_1^{m_1,\dots,m_n}(\Gamma), \dots, \Delta_n^{m_1,\dots,m_n}(\Gamma)) \,. \end{aligned} \tag{2.8}$$

Example 2.3. Let $n = 3$, $(m_1, m_2, m_3) = (3, 1, 4)$, and $\Gamma \in \mathcal{G}(8)$ be the following graph

$\Gamma =$ 

(2.9)

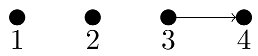
The cocomposition $\Delta^{3,1,4}(\Gamma) = \left(\Delta_0^{3,1,4}(\Gamma), \Delta_1^{3,1,4}(\Gamma), \Delta_2^{3,1,4}(\Gamma), \Delta_3^{3,1,4}(\Gamma)\right)$ is given by the following graphs. $\Delta_1^{3,1,4}(\Gamma)$ is the subgraph of Γ generated by the first three vertices:

$\Delta_1^{3,1,4}(\Gamma) =$  $\in \mathcal{G}(3)\,;$

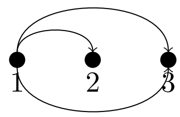
$\Delta_2^{3,1,4}(\Gamma)$ is the subgraph of Γ associated to the fourth vertex:

$\Delta_2^{3,1,4}(\Gamma) =$  $\in \mathcal{G}(1)\,;$

$\Delta_3^{3,1,4}(\Gamma)$ is the subgraph of Γ associated to the last four vertices:

$\Delta_3^{3,1,4}(\Gamma) =$  $\in \mathcal{G}(4)\,;$

and, finally, $\Delta_0^{3,1,4}(\Gamma)$ is:

$\Delta_0^{3,1,4}(\Gamma) =$  $\in \mathcal{G}(3)\,.$

From the construction of $\Delta_i^{m_1,\dots,m_n}(\Gamma)$, it is easy to see that there is a natural bijective correspondence

$$\Delta \colon E(\Gamma) \xrightarrow{\sim} E\big(\Delta_0^{m_1\dots m_n}(\Gamma)\big) \sqcup E\big(\Delta_1^{m_1\dots m_n}(\Gamma)\big) \sqcup \dots \sqcup E\big(\Delta_n^{m_1\dots m_n}(\Gamma)\big)\,. \tag{2.10}$$

Definition 2.4. Let $k \in \{1,\dots,M_n\}$ and $j \in \{1,\dots,n\}$. We say that j is *externally connected* to k (via the graph Γ and its cocomposition $\Delta^{m_1\dots m_n}(\Gamma)$) if there is an unoriented path (without repeating edges) of $\Delta_0^{m_1\dots m_n}(\Gamma)$ joining j to i, where $i \in \{1,\dots,n\}$ is such that $k \in \{M_{i-1}+1,\dots,M_i\}$, and the edge out of i is the image, via the map Δ in (2.10), of an edge which has its head or tail in k. Given a set of variables $x_1,\dots,x_n$, we denote

$$X(k) = \sum_{\substack{j \text{ externally} \\ \text{connected to } k}} x_j\,. \tag{2.11}$$

Example 2.5. For the graph (2.9), we have

$X(1)=0,\ X(2)=x_1+x_2+x_3,\ X(3)=0,\ X(4)=x_1+x_3,\ X(5)=x_1+x_2+x_3,$
$X(6)=x_1+x_2+x_3,\ X(7)=0,\ X(8)=0.$

2.4 Lie conformal algebras and PVA's

Definition 2.6. A Lie conformal algebra is a vector space V, endowed with an endomorphism $\partial \in \operatorname{End}(V)$ and a bilinear (over $\mathbb{F}$) λ-bracket $[\cdot\,_\lambda\,\cdot] \colon V \times V \to V[\lambda]$ satisfying sesquilinearity ($a,b \in V$):

$$[\partial a_\lambda b] = -\lambda [a_\lambda b]\,,\ [a_\lambda \partial b] = (\lambda+\partial)[a_\lambda b]\,, \tag{2.12}$$

skewsymmetry ($a,b \in V$):

$$[a_\lambda b] = -[b_{-\lambda-\partial} a]\,, \tag{2.13}$$

and the Jacobi identity ($a,b,c \in V$):

$$[a_\lambda[b_\mu c]] - [b_\mu[a_\lambda, b]] = [[a_\lambda b]_{\lambda+\mu} c]\,. \tag{2.14}$$

Definition 2.7. A Poisson vertex algebra (PVA) is a unital commutative associative algebra V endowed with a derivation ∂ and a Lie conformal algebra λ-bracket $[\cdot_\lambda\cdot]$ that satisfies the left Leibniz rule

$$[a_\lambda bc] = [a_\lambda b]c + [a_\lambda c]b\,. \tag{2.15}$$

Example 2.8 (Virasoro PVA)**.** An example of PVA is the Virasoro Poisson vertex algebra with central charge $c \in \mathbb{F}$. It is, as differential algebra,

$$\mathcal{V}ir^c = \mathbb{F}[L^{(n)} \mid n \in \mathbb{Z}] \,,$$

where L is the Virasoro element with λ-bracket

$$[L_\lambda L] = (\partial + 2\lambda)L + \frac{c}{12}\lambda^3 \,,$$

and the derivation is $L^{(n)} = \partial^n L$. The λ-bracket is extended uniquely to $\mathcal{V}ir^c$ by the left and right Leibniz rule and the sesquilinearity conditions.

Example 2.9 (Free boson)**.** Another example of a PVA is the Poisson vertex algebra of N free bosons. It is, as differential algebra, the algebra of differential polynomials in N generators

$$\mathcal{B} = \mathbb{F}[u_i^{(n)} \mid i = 1, \dots, N,\, n \in \mathbb{Z}_+] \,,$$

where the derivation ∂ is defined by $\partial u_i^{(n)} = u_i^{(n+1)}$. The Lie conformal algebra λ-bracket is given on the generators by

$$[u_{i\lambda} u_j] = \lambda \delta_{ij} \quad \text{for } i, j = 1, \dots, N \,,$$

and it is extended to $\mathcal{B} \otimes \mathcal{B}$ by the sesquilinearity (2.12) and the left (and right) Leibniz rule (2.15).

Example 2.10 (Free fermion)**.** The Poisson vertex algebra of N free fermions is, as differential algebra, the algebra of differential polynomials in N odd generators

$$\mathcal{F} = \bigwedge(u_i^{(n)} \mid i = 1, \dots, N,\, n \in \mathbb{Z}_+) \,.$$

As in Example 2.9, the derivation ∂ is defined by $\partial u_i^{(n)} = u_i^{(n+1)}$. The Lie conformal algebra λ-bracket is given on the generators by

$$[u_{i\lambda} u_j] = \delta_{ij} \quad \text{for } i, j = 1, \dots, N \,,$$

and it is extended by the sesquilinearity (2.12) and the left (and right) Leibniz rule (2.15).

2.5 Operad

Recall that a (linear, unital, symmetric) *superoperad* $\mathcal{P}$ is a collection of vector superspaces $\mathcal{P}(n)$, $n \geq 0$, with parity p, endowed, for every $f \in \mathcal{P}(n)$ and $m_1, \dots, m_n \geq 0$, with a *composition* parity preserving linear map,

$$\begin{aligned} \mathcal{P}(n) \otimes \mathcal{P}(m_1) \otimes \cdots \otimes \mathcal{P}(m_n) \;&\to\; \mathcal{P}(M_n) \,, \\ f \otimes g_1 \otimes \cdots \otimes g_n \;&\mapsto\; f(g_1 \otimes \cdots \otimes g_n) \,, \end{aligned} \tag{2.16}$$

where M_n is as in (2.6), satisfying the following associativity axiom:

$$f\big((g_1 \otimes \cdots \otimes g_n)(h_1 \otimes \cdots \otimes h_{M_n})\big) = \big(f(g_1 \otimes \cdots \otimes g_n)\big)(h_1 \otimes \cdots \otimes h_{M_n}) \in \mathcal{P}\Big(\sum_{j=1}^{M_n} \ell_j\Big) \,, \tag{2.17}$$

for every $f \in \mathcal{P}(n)$, $g_i \in \mathcal{P}(m_i)$ for $i = 1, \dots, n$, and $h_j \in \mathcal{P}(\ell_j)$ for $j = 1, \dots, M_n$. In the left-hand side of (2.17) the linear map

$$\bigotimes_{i=1}^{n} g_i \colon \bigotimes_{j=1}^{M_n} \mathcal{P}(\ell_j) \to \bigotimes_{i=1}^{n} \mathcal{P}\Big(\sum_{j=M_{i-1}+1}^{M_i} \ell_j \Big)$$

is the tensor product of composition maps applied to

$$h_1 \otimes \cdots \otimes h_{M_n} = (h_1 \otimes \cdots \otimes h_{M_1}) \otimes (h_{M_1+1} \otimes \cdots \otimes h_{M_2}) \otimes \cdots \otimes (h_{M_{n-1}+1} \otimes \cdots \otimes h_{M_n}).$$

We assume that $\mathcal{P}$ is endowed with a *unit* element $1 \in \mathcal{P}(1)$ satisfying the following unity axioms:

$$f(1 \otimes \cdots \otimes 1) = 1(f) = f\,, \quad \text{for every} \quad f \in \mathcal{P}(n)\,. \tag{2.18}$$

Furthermore, we assume that, for each $n \geq 1$, $\mathcal{P}(n)$ has a right action of the symmetric group S_n, denoted f^σ, for $f \in \mathcal{P}(n)$ and $\sigma \in S_n$, satisfying the following equivariance axiom ($f \in \mathcal{P}(n)$, $g_1 \in \mathcal{P}(m_1), \dots, g_n \in \mathcal{P}(m_n)$, $\sigma \in S_n$, $\tau_1 \in S_{m_1}, \dots, \tau_n \in S_{m_n}$):

$$f^\sigma(g_1^{\tau_1} \otimes \cdots \otimes g_n^{\tau_n}) = \big(f(\sigma(g_1 \otimes \cdots \otimes g_n))\big)^{\sigma(\tau_1,\dots,\tau_n)}\,, \tag{2.19}$$

where the left action of $\sigma \in S_n$ on the tensor product of vector superspaces was defined in (2.2), and the composition $\sigma(\tau_1, \dots, \tau_n)$ is described in (2.7).

For simplicity, from now on, we will use the term operad in place of superoperad. Given an operad $\mathcal{P}$, one defines, for each $i = 1, \dots, n$, the $\circ_i$-product $\circ_i \colon \mathcal{P}(n) \times \mathcal{P}(m) \to \mathcal{P}(n+m-1)$ by insertion in position i, i.e.

$$f \circ_i g = f(\overbrace{1 \otimes \cdots \otimes 1}^{i-1} \otimes \overset{i}{g} \otimes \overbrace{1 \otimes \cdots \otimes 1}^{n-i})\,. \tag{2.20}$$

Example 2.11. The simplest example of an operad is $\mathcal{P} = \mathcal{H}om$. Given a vector superspace V, $\mathcal{H}om = \mathcal{H}om(V)$ is defined as the collection of ($n \geq 0$)

$$\mathcal{H}om(n) := Hom(V^{\otimes n}, V)\,,$$

endowed with the composition maps ($f \in \mathcal{H}om(n)$, $g_i \in \mathcal{H}om(m_i)$ for $i = 1, \dots, n$, $v_j \in V$ for $j = 1, \dots, M_n$)

$$(f(g_1 \otimes \ldots \otimes g_n))(v_1 \otimes \ldots \otimes v_{M_n}) := f((g_1 \otimes \ldots \otimes g_n)(v_1 \otimes \ldots \otimes v_{M_n}))\,,$$

where M_n is as in (2.6). $\mathcal{H}om$ is a unital operad with unity $1 = \mathbb{1}_V \in End(V)$, and the right action of S_n on $\mathcal{H}om(n)$ is given by (2.4).

Example 2.12. Another example is the $\mathcal{L}ie$ operad. It is defined as follows: $\mathcal{L}ie(1) = \mathbb{F}1$, $\mathcal{L}ie(2) = \mathbb{F}\beta$ is the sign representation of S_2, and, for every $n > 2$, the elements in $\mathcal{L}ie(n)$ are obtained by composition of $\beta \in \mathcal{L}ie(2)$, subject to the Jacoby identity in $\mathcal{L}ie(3)$

$$\beta(\beta, 1) = \beta(1, \beta) - (\beta(1, \beta))^{(12)}\,.$$

One can show that Lie superalgebra structure on a vector superspace V is the same as a morphism of operads $\mathcal{L}ie \to \mathcal{H}om(V)$.

2.6 The associated $\mathbb{Z}$-graded Lie superalgebra

Recall that, given a superoperad $\mathcal{P}$, one can construct the associated $\mathbb{Z}$-graded Lie superalgebra $W(\mathcal{P})$. It is defined, as $\mathbb{Z}$-graded vector superspace, as

$$W(\mathcal{P}) = \sum_{n\geq -1} W_n(\mathcal{P}) = \sum_{n\geq -1} \mathcal{P}(n+1)^{S_{n+1}} . \tag{2.21}$$

For $f \in W_n(\mathcal{P})$ and $g \in W_m(\mathcal{P})$, their $\square$-product is:

$$f\square g = \sum_{\sigma\in S_{m+1,n}} (f \circ_1 g)^{\sigma^{-1}} \in W_{m+n}(\mathcal{P}) \tag{2.22}$$

and the Lie bracket on $W(\mathcal{P})$ is given by

$$[f, g] = f\square g - (-1)^{p(f)p(g)} g\square f . \tag{2.23}$$

2.7 The classical operad P^{cl}

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a vector superspace with parity p, endowed with an even endomorphism $\partial \in \operatorname{End} V$. For $n \geq 0$, define $P^{\mathrm{cl}}(n)$ as the vector superspace of all maps

$$f : \mathcal{G}(n) \times V^{\otimes n} \longrightarrow V[\lambda_1, \dots, \lambda_n] \big/ \langle \partial + \lambda_1 + \cdots + \lambda_n \rangle , \tag{2.24}$$

which are linear in the second factor, mapping the n-graph $\Gamma \in \mathcal{G}(n)$ and the monomial $v_1 \otimes \cdots \otimes v_n \in V^{\otimes n}$ to the polynomial

$$f^{\Gamma}_{\lambda_1,\dots,\lambda_n}(v_1 \otimes \cdots \otimes v_n) , \tag{2.25}$$

satisfying the *cycle relations* and the *sesquilinearity conditions* described as follows.

The *cycle relations* say that

$$\text{if } \Gamma \notin \mathcal{G}_0(n) \text{ then } f^{\Gamma} = 0 , \tag{2.26}$$

and if $C \subset E(\Gamma)$ is an oriented cycle of Γ, then

$$\sum_{e\in C} f^{\Gamma\backslash e} = 0 , \tag{2.27}$$

where $\Gamma\backslash e$ is the graph obtained from Γ by removing the edge e. Observe that for oriented cycles of length 2, the cycle relation (2.27) means that changing orientation of a single edge of the n-graph $\Gamma \in \mathcal{G}(n)$ amounts to a change of sign of f^{Γ}.

As for the *sesquilinearity conditions*, let $\Gamma = \Gamma_1 \sqcup \cdots \sqcup \Gamma_s$ be the decomposition of Γ as disjoint union of its connected components, and let $I_1, \dots, I_s \subset \{1, \dots, n\}$ be the sets of vertices associated to these connected components. Introducing the same notation as in [BDSHK18], for a graph $\tilde{\Gamma}$ and its set of vertices $\tilde{I} \subset \{1, \dots, n\}$, we write

$$\lambda_{\tilde{\Gamma}} = \sum_{i\in\tilde{I}} \lambda_i , \quad \partial_{\tilde{\Gamma}} = \sum_{i\in\tilde{I}} \partial_i , \tag{2.28}$$

with ∂_i the action of ∂ on the i-th factor in a tensor product $V^{\otimes n}$. Then, for every $\alpha = 1, \dots, s$, we have

$$\frac{\partial}{\partial \lambda_i} f^{\Gamma}_{\lambda_1,\dots,\lambda_n}(v_1 \otimes \cdots \otimes v_n) \text{ is the same for all } i \in I_\alpha \tag{2.29}$$

and

$$f^{\Gamma}_{\lambda_1,\dots,\lambda_n}(\partial_{\Gamma_\alpha}(v_1 \otimes \cdots \otimes v_n)) = -\lambda_{\Gamma_\alpha} f^{\Gamma}_{\lambda_1,\dots,\lambda_n}(v_1 \otimes \cdots \otimes v_n) \,. \tag{2.30}$$

Observe that the first sesquilinearity condition (2.29) is equivalent to state that the polynomial $f^{\Gamma}_{\lambda_1,\dots,\lambda_n}(v_1 \otimes \cdots \otimes v_n)$ is a function of the variables λ_{Γ_α}, $\alpha = 1,\dots,s$, defined in (2.28), and not of the variables $\lambda_1,\dots,\lambda_n$ separetely. Hence, in particular, when Γ is a connected graph $f^{\Gamma}_{\lambda_1,\dots,\lambda_n}(v_1 \otimes \cdots \otimes v_n)$ is indipendent of $\lambda_1,\dots,\lambda_n$. Whereas, the second sesquilinearity condition (2.30) implies

$$f^{\Gamma}_{\lambda_1,\dots,\lambda_n}(\partial v) = -\sum_{i=1}^{n} \lambda_i f^{\Gamma}_{\lambda_1,\dots,\lambda_n}(v) = \partial\big(f^{\Gamma}_{\lambda_1,\dots,\lambda_n}(v)\big), \qquad v \in V^{\otimes n} \,. \tag{2.31}$$

The classical operad $P^{\mathrm{cl}}(V)$ is defined as the collection of the vector superspaces $P^{\mathrm{cl}}(n)$, $n \geq 0$, endowed, for every $f \in P^{\mathrm{cl}}(n)$ and $m_1,\dots,m_n \geq 0$, with the composition parity preserving linear map

$$\begin{aligned} P^{\mathrm{cl}}(n) \otimes P^{\mathrm{cl}}(m_1) \otimes \cdots \otimes P^{\mathrm{cl}}(m_n) &\ \to\ P^{\mathrm{cl}}(M_n) \,, \\ f \otimes g_1 \otimes \cdots \otimes g_n &\ \mapsto\ f(g_1,\dots,g_n) \,, \end{aligned}$$

described as follows. Let M_i be as in (2.6), and

$$\Lambda_i = \sum_{j=M_{i-1}+1}^{M_i} \lambda_j \,, \quad i = 1,\dots,n \,. \tag{2.32}$$

If $\Gamma \in \mathcal{G}(M_n)$, let

$$(f(g_1,\dots,g_n))^{\Gamma} : V^{\otimes M_n} \to V[\lambda_1,\dots,\lambda_{M_n}] \big/ \langle \partial + \lambda_1 + \cdots + \lambda_{M_n} \rangle$$

be defined by the formula:

$$\begin{aligned} &(f(g_1,\dots,g_n))^{\Gamma}_{\lambda_1,\dots,\lambda_{M_n}}(v_1 \otimes \cdots \otimes v_{M_n}) \\ &= f^{\Delta_0^{m_1\dots m_n}(\Gamma)}_{\Lambda_1,\dots,\Lambda_n}\Big(\Big(\Big(\big|_{x_1=\Lambda_1+\partial} (g_1)^{\Delta_1^{m_1\dots m_n}(\Gamma)}_{\lambda_1+X(1),\dots,\lambda_{M_1}+X(M_1)}\Big) \otimes \cdots \\ &\qquad \cdots \otimes \Big(\big|_{x_n=\Lambda_n+\partial} (g_n)^{\Delta_n^{m_1\dots m_n}(\Gamma)}_{\lambda_{M_{n-1}+1}+X(M_{n-1}+1),\dots,\lambda_{M_n}+X(M_n)}\Big)\Big)(v_1 \otimes \cdots \otimes v_{M_n})\Big) \end{aligned} \tag{2.33}$$

where $\Delta^{m_1,\dots,m_n}(\Gamma)$ is the cocomposition of Γ described in Section 2.3, $X(1),\dots$ $\dots, X(M_n)$ are the variables as in (2.11), and the notation is as follows. For given graphs $\Gamma_1 \in \mathcal{G}(m_1)$, $\dots, \Gamma_n \in \mathcal{G}(m_n)$, we have:

$$\begin{aligned} &\Big((g_1)^{\Gamma_1}_{\lambda_1,\dots,\lambda_{M_1}} \otimes \cdots \otimes (g_n)^{\Gamma_n}_{\lambda_{M_{n-1}+1},\dots,\lambda_{M_n}}\Big)(v_1 \otimes \cdots \otimes v_{M_n}) \\ &:= (-1)^{\sum_{i<j} p(g_j)(p(v_{M_{i-1}+1})+\cdots+p(v_{M_i}))} (g_1)^{\Gamma_1}_{\lambda_1,\dots,\lambda_{M_1}}(v_1 \otimes \cdots \otimes v_{M_1}) \otimes \dots \\ &\qquad\qquad \cdots \otimes (g_n)^{\Gamma_n}_{\lambda_{M_{n-1}+1},\dots,\lambda_{M_n}}(v_{M_{n-1}+1} \otimes \cdots \otimes v_{M_n}), \end{aligned} \tag{2.34}$$

and for polynomials $P(\lambda)=\sum_m p_m\lambda^m$ and $Q(\mu)=\sum_n q_n\mu^n$ with coefficients in V, we write

$$\left(\big|_{x=\partial}P(\lambda+y)\right)\otimes\left(\big|_{y=\partial}Q(\mu+x)\right)=\sum_{m,n}((\mu+\partial)^n p_m)\otimes((\lambda+\partial)^m q_n)\,. \tag{2.35}$$

For each $n\geq 1$, $P^{\mathrm{cl}}(n)$ is assumed to have a natural right action of the symmetric group S_n. This is given by ($f\in P^{\mathrm{cl}}(n)$, $\Gamma\in\mathcal{G}(n)$, $v_1,\dots,v_n\in V$):

$$(f^\sigma)^\Gamma_{\lambda_1,\dots,\lambda_n}(v_1\otimes\ \dots\ \otimes v_n)=f^{\sigma(\Gamma)}_{\sigma(\lambda_1,\dots,\lambda_n)}(\sigma(v_1\otimes\dots\otimes v_n))\,, \tag{2.36}$$

where $\sigma(\lambda_1,\dots,\lambda_n)$ is defined by (2.1), $\sigma(v_1\otimes\dots\otimes v_n)$ is defined by (2.2), and $\sigma(\Gamma)$ is defined in Section 2.3.

On the space $P^{\mathrm{cl}}(n)$ we can also define a grading:

$$P^{\mathrm{cl}}(n)=\bigoplus_{r\geq 0} gr^r P^{\mathrm{cl}}(n)\,, \tag{2.37}$$

where $gr^r P^{\mathrm{cl}}(n)$ is the subspace of all maps in $P^{\mathrm{cl}}(n)$ vanishing on graphs with a number of edges not equal to r.

2.8 PVA cohomology

Given the vector superspace V with parity p, and the even endomorphism $\partial\in End(V)$, consider as usual ΠV the same vector space with reverse parity $\bar{p}$ and the corresponding classical operad $P^{\mathrm{cl}}(\Pi V)$ from Section 2.7. The associated $\mathbb{Z}$-graded Lie superalgebra is $W^{\mathrm{cl}}(\Pi V):=W(P^{\mathrm{cl}}(\Pi V))$, with Lie bracket defined by (2.23). We have the following

Theorem 2.13. *There is a bijective correspondence between the odd elements $X\in W^{\mathrm{cl}}_1(\Pi V)$ such that $X\Box X=0$ and the Poisson vertex superalgebra structures on V, defined as follows. The commutative associative product and the λ-bracket of the Poisson vertex superalgebra V corresponding to X are given by*

$$ab=(-1)^{p(a)}X^{\bullet\!\!\rightarrow\!\!\bullet}(a\otimes b)\ ,\quad [a_\lambda b]=(-1)^{p(a)}X^{\bullet\ \bullet}_{\lambda,-\lambda-\partial}(a\otimes b)\,. \tag{2.38}$$

Thanks to the Jacobi identity for the Lie superalgebra $W^{\mathrm{cl}}(\Pi V)$, if $X\in W^{\mathrm{cl}}_1(\Pi V)_{\bar 1}$ satisfies $X\Box X=0$, then $(adX)^2=0$. In view of Theorem 2.13, this means that we have a cohomology complex

$$(W^{\mathrm{cl}}(\Pi V),\mathrm{ad}\,X)\,,$$

called *PVA cohomology complex*, where $X\in W_1(\Pi V)_{\bar 1}$ is given by (2.38).

2.9 The finite analog P^{fn}

In [BDSHK18] is defined a finite analog P^{fn} of the operad P^{cl} as follows. For a vector superspace V, let $P^{\mathrm{fn}}(n)$ be the space of all maps

$$f\colon \mathcal{G}(n) \times V^{\otimes n} \longrightarrow V \tag{2.39}$$

$$\Gamma \times (v_1, \otimes \cdots \otimes v_n) \mapsto f^{\Gamma}(v_1 \otimes \cdots \otimes v_n)\,, \tag{2.40}$$

which are linear in the second factor and satisfy the cycle relations (2.26) and (2.27). Similar to (2.33), the composition maps are

$$(f(g_1,\dots,g_n))^{\Gamma} = f^{\Delta_0^{m_1\dots m_n}(\Gamma)}\left(g_1{}^{\Delta_1^{m_1\dots m_n}(\Gamma)} \otimes \cdots \otimes g_n{}^{\Delta_n^{m_1\dots m_n}(\Gamma)}\right)\,, \tag{2.41}$$

for $f \in P^{\mathrm{fn}}(n)$, $g_1 \in P^{\mathrm{fn}}(m_1)$, $\dots, g_n \in P^{\mathrm{fn}}(m_n)$, and $\Gamma \in \mathcal{G}(M_n)$. In this case, the action of the symmetric group S_n is given by

$$(f^{\sigma})^{\Gamma}(v_1 \otimes \cdots \otimes v_n) = f^{\sigma(\Gamma)}(\sigma(v_1 \otimes \cdots \otimes v_n))\,, \tag{2.42}$$

where again we are using the actions described in (2.2) and in Section 2.3.

The analog of Theorem 2.13 is the following:

Theorem 2.14. *There is a bijective correspondence between the odd elements $X \in W_1^{\mathrm{fn}}(\Pi V)$ such that $X \square X = 0$ and the Poisson superalgebra structures on V, given by*

$$ab = (-1)^{p(a)} X^{\bullet\!\!\longrightarrow\!\!\bullet}(a \otimes b)\ ,\quad \{a,b\} = (-1)^{p(a)} X^{\bullet\ \ \bullet}(a \otimes b)\,. \tag{2.43}$$

As we have for the classical operad, we can define the Lie superalgebra $W^{\mathrm{fn}}(\Pi V)$ associated to $P^{\mathrm{fn}}(\Pi V)$ as $W^{\mathrm{fn}}(\Pi V) \coloneqq \sum_{n\geq -1} P^{\mathrm{fn}}(\Pi V)(n+1)^{S_{n+1}}$ with Lie bracket described in (2.23). The corresponding cohomology complex is

$$\left(W^{\mathrm{fn}}(\Pi V), adX\right) \tag{2.44}$$

with X given by Theorem 2.14.

Chapter 3

Relation between PVA cohomology and Harrison cohomology

3.1 Main theorem

Let V be a Poisson vertex algebra. By Theorem 2.13, we have an odd element $X \in W^{\mathrm{cl}}(\Pi V)$ such that $[X, X] = 0$, which is associated to the PVA structure of V by (2.38). Thus, there is the PVA cohomology complex

$$\left(W^{\mathrm{cl}}_{\bullet}(\Pi V), adX\right) . \tag{3.1}$$

A classic n-cochain is an element $Y \in W^{\mathrm{cl}}_{n-1}(\Pi V)$, namely

$$Y : \mathcal{G}(n) \times (\Pi V)^{\otimes n} \longrightarrow (\Pi V)[\lambda_1, \dots, \lambda_n] \big/ \langle \partial + \lambda_1 + \dots + \lambda_n \rangle \tag{3.2}$$

satisfying relations (2.26), (2.27), (2.29), (2.30), and symmetry property (by definition (2.21))

$$Y^{\sigma} = Y \,, \quad \forall \sigma \in S_n \,. \tag{3.3}$$

Recall the grading of the superoperad $P^{\mathrm{cl}}(\Pi V)$ from (2.37): $\mathrm{gr}^r \ W^{\mathrm{cl}}_{n-1}(\Pi V)$ is the set of maps Y as in (3.2) such that

$$Y^{\Gamma} = 0 \ \text{ unless } \ | \ E(\Gamma) \ |= r \,.$$

Note that if $\Gamma \in \mathcal{G}(n)$ has $| \ E(\Gamma) \ | \geq n$, then necessarely Γ contains a cycle. Hence, by the cycle relation (2.26), $Y^{\Gamma} = 0$. Therefore the top degree in $grW^{\mathrm{cl}}_{n-1}(\Pi V)$ is $r = n-1$, *i.e.*

$$\mathrm{gr}^r \ W^{\mathrm{cl}}_{n-1}(\Pi V) = 0 \text{ if } r \geq n \,.$$

Let us consider the top degree subspace $gr^{n-1}W^{\mathrm{cl}}_{n-1}(\Pi V)$. It consists of collection of maps

$$Y^{\Gamma} : (\Pi V)^{\otimes n} \longrightarrow (\Pi V) \,, \ \text{ for } \Gamma \in \mathcal{G}_0(n), \ | \ E(\Gamma) \ |= n-1,$$

satisfying (2.26), (2.27), (3.3), and $Y^{\Gamma}(\partial(v_1 \otimes \dots \otimes v_n)) = \partial Y^{\Gamma}(v_1 \otimes \dots \otimes v_n)$. Note that, if $\Gamma \in \mathcal{G}_0(n)$, then $| \ E(\Gamma) \ |= n-1$ if and only if Γ is connected. If Γ is not connected we have $Y^\Gamma = 0$.

In addition, as explained in section 1.3, there is an another cohomology complex associated to V, as a commutative algebra, namely

$$(C^\bullet_{Har}(V), d)\,, \tag{3.4}$$

where $C^n_{Har}(V) \subset Hom(V^{\otimes n}, V)$ is defined by the Harrison's conditions (1.12), and d is the Hochschild differential (1.2).

The main result of this thesis is the following:

Theorem 3.1. *Let V be a Poisson vertex algebra. There is a natural surjective morphism of cochain complexes*

$$(W^{\mathrm{cl}}_\bullet(\Pi V), adX) \to (C^\bullet_{Har}(V), d)\,, \tag{3.5}$$

mapping $Y \in W^{\mathrm{cl}}_{n-1}(\Pi V)$ to Y^{Λ_n}, where Λ_n is the standard n-line

$$\Lambda_n = \underset{1}{\bullet}\!\longrightarrow\!\underset{2}{\bullet}\!\longrightarrow \cdots \longrightarrow\!\underset{n}{\bullet}\ . \tag{3.6}$$

The morphism (3.5) restricts to a bijection on the top degree:

$$gr^{n-1} W^{\mathrm{cl}}_{n-1}(\Pi V) \xrightarrow{\sim} C^n_{Har}(V)\,. \tag{3.7}$$

We will prove Theorem 3.1 in Section 3.6. For that, we will need some preliminary results.

3.2 Lines

We say that a graph $\Gamma \in \mathcal{G}(n)$ is a disjoint union of lines if it has the following form:

$$\Gamma = \underset{i^1_1}{\circ}\!\to\!\underset{i^1_2}{\circ}\!\to\cdots\to\!\underset{i^1_{k_1}}{\circ}\quad \underset{i^2_1}{\circ}\!\to\!\underset{i^2_2}{\circ}\!\to\cdots\to\!\underset{i^2_{k_2}}{\circ}\quad\cdots\quad \underset{i^s_1}{\circ}\!\to\!\underset{i^s_2}{\circ}\!\to\cdots\to\!\underset{i^s_{k_s}}{\circ} = L_1 \sqcup L_2 \sqcup \cdots \sqcup L_s\,, \tag{3.8}$$

where $1 \leq k_1 \leq \ldots \leq k_s$ are such that $k_1 + \cdots + k_s = n$, and the set of indices $\{i^a_b\}$ is a permutation of $\{1, \ldots, n\}$ such that

$$i^l_1 = min\{i^l_1, \ldots, i^l_{k_l}\} \quad \forall\, l = 1, \ldots, s\,. \tag{3.9}$$

If $k_l = k_{l+1}$, we also assume that $i^l_1 < i^{l+1}_1$. In particular, the connected lines are all of the form

$$\sigma(\Lambda_n)\,, \quad \sigma \in S_n \ \text{ s. t. } \sigma(1) = 1\,, \tag{3.10}$$

where Λ_n is the n-line (3.6). Let $\mathcal{L}(n) \subset \mathcal{G}(n)$ be the set of graphs that are disjoint union of lines. Let also $\mathbb{F}\mathcal{G}(n)$ be the vector space with basis the set of graphs $\mathcal{G}(n)$.

Definition 3.2. The *cycle relations* in $\mathbb{F}\mathcal{G}(n)$ are the following elements:

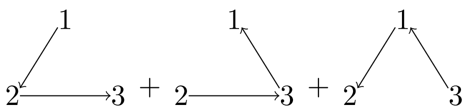
(i) all $\Gamma \in \mathcal{G}(n) \setminus \mathcal{G}_0(n)$ (*i.e.* graphs containing a cycle);

(ii) all linear combinations $\sum_{e\in C}\Gamma\backslash e$, where $\Gamma\in\mathcal{G}(n)$ and $C\subset E(\Gamma)$ is an oriented cycle.

Denote by $R(n)\subset\mathbb{F}\mathcal{G}(n)$ the subspace spanned by the cycle relations (i) and (ii).

Note that reversing an arrow in a graph $\Gamma\in\mathcal{G}(n)$ gives us, modulo cycle relations, the element $-\Gamma\in\mathbb{F}\mathcal{G}(n)$.

Example 3.3. For $n=3$, a cycle relation of type (ii) is:



(3.11)

Remark 3.4. The cycle relations (2.26) and (2.27) on $Y\in\mathcal{P}^{\mathrm{cl}}$, can be restated by saying that $Y^{\Gamma}=0$ for $\Gamma\in R(n)$.

From [BDSHK18b], we have:

Theorem 3.5 ([BDSHK18b, Theorem 4.7])**.** *The set $\mathcal{L}(n)$ is a basis for the quotient space $\mathbb{F}\mathcal{G}(n)/R(n)$.*

By Theorem 3.5 and (3.10), we can write every connected graph $\Gamma\in\mathcal{G}(n)$, uniquely, as follows:

$$\Gamma\equiv\sum_{\substack{\sigma\in S_n\\ \sigma(1)=1}} c_\sigma^\Gamma\;\sigma\Lambda_n\,, \tag{3.12}$$

modulo cycle relations, where the coefficients $c_\sigma^\Gamma\in\mathbb{F}$ and the action of the symmetric group on graphs is defined in Section 2.3.

Here and further, by $\Gamma\equiv\tilde{\Gamma}$ we mean equivalence modulo cycle relations, *i.e.* in the quotient space $\mathbb{F}\mathcal{G}(n)/R(n)$.

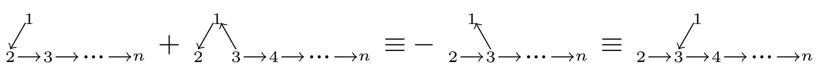
3.3 Connected lines

We have the following lemmas on connected lines.

Lemma 3.6. *For every n, the following identity on connected lines holds:*

$$\underset{1}{\bullet}\to\underset{2}{\bullet}\to\cdots\to\underset{n}{\bullet}+\underset{2}{\bullet}\to\underset{1}{\bullet}\to\underset{3}{\bullet}\to\cdots\to\underset{n}{\bullet}+\ldots+\underset{2}{\bullet}\to\underset{3}{\bullet}\to\cdots\to\underset{n-1}{\bullet}\to\underset{1}{\bullet}\to\underset{n}{\bullet}+\underset{2}{\bullet}\to\underset{3}{\bullet}\to\cdots\to\underset{n}{\bullet}\to\underset{1}{\bullet}\equiv 0\,. \tag{3.13}$$

Proof. Let us consider the first two terms in the left-hand side of (3.13). Reversing the edges $2\to1$ and $1\to3$ in the second graph, and by (3.11), we have:



(3.14)

Adding (3.14) to the third term appearing in the left-hand side of (3.13), and applying again (3.11), we obtain:

$$
\begin{tikzcd}[sep=small] & 1 \arrow[d] & & & \\ 2 \arrow[r] & 3 \arrow[r] & 4 \arrow[r] & \cdots \arrow[r] & n \end{tikzcd}
+
\begin{tikzcd}[sep=small] & & 1 \arrow[dl] & & & \\ 2 \arrow[r] & 3 & & 4 \arrow[ul] \arrow[r] & \cdots \arrow[r] & n \end{tikzcd}
\equiv -
\begin{tikzcd}[sep=small] & & 1 & & \\ 2 \arrow[r] & 3 \arrow[r] & 4 \arrow[u] \arrow[r] & \cdots \arrow[r] & n \end{tikzcd}
\equiv
\begin{tikzcd}[sep=small] & & 1 \arrow[d] & & \\ 2 \arrow[r] & 3 \arrow[r] & 4 \arrow[r] & \cdots \arrow[r] & n \end{tikzcd}
\tag{3.15}
$$

We proceed in the same way, up to

$$
\begin{tikzcd}[sep=small] & & & 1 \arrow[d] & \\ 2 \arrow[r] & 3 \arrow[r] & \cdots \arrow[r] & (n-1) \arrow[r] & n \end{tikzcd}
+
\begin{tikzcd}[sep=small] & & & & 1 \arrow[dl] & \\ 2 \arrow[r] & 3 \arrow[r] & \cdots \arrow[r] & (n-1) & & n \arrow[ul] \end{tikzcd}
+
\begin{tikzcd}[sep=small] & & & & 1 \\ 2 \arrow[r] & 3 \arrow[r] & \cdots \arrow[r] & (n-1) \arrow[r] & n \arrow[u] \end{tikzcd}
\equiv 0
\tag{3.16}
$$

$\square$

Remark 3.7. Observe that equation (3.13) can be viewed as a "local" identity: Lemma 3.6 holds even if we attach the same graph Γ at any vertex of every graph appearing in the identity.

Lemma 3.8. *Let Λ_n be as in (3.6). For every $k \in \{2, \dots, n\}$, the following identity holds:*

$$
\Lambda_n + (-1)^k \sum_{\pi \in \mathcal{M}_n^k} \pi \Lambda_n \equiv 0 \,, \tag{3.17}
$$

where the sum is over all the monotone permutations π starting at k, and the symmetric action on graphs is described in Section 2.3.

Proof. The proof is done by induction on k. Formula (3.13) is equivalent to (3.17) for $k = 2$, and this proves the base of the induction.

Fix $k > 2$. Recall the description 1.6 of the monotone permutations $\pi \in \mathcal{M}_n^k$ in terms of the set $D(\pi)$ of drops. Given the collection of drops $D = (2 \leq d_{k-1} < \ldots < d_1 \leq n)$, the corresponding monotone permutation $\pi^D \in \mathcal{M}_n^k$ is uniquely determined by $\pi^{-1}(i) = d_i$, $\forall i = 1 \dots, k-1$. Hence:

$$
\pi^D(\Lambda_n) = \quad k \to (k+1) \to \cdots \to \underset{[d_{k-1}]}{\underline{(k-1)}} \to \cdots \to \underset{[d_2]}{\underline{2}} \to \cdots \to \underset{[d_1]}{\underline{1}} \to \cdots \to n
$$

where the underlying positions correspond to drops, while all other positions have vertices in increasing order from $k+1$ to n.

We then have:

$$
\sum_{\pi \in \mathcal{M}_n^k} \pi \Lambda_n = \sum_{2 \leq d_{k-1} < \ldots < d_1 \leq n} \quad k \to (k+1) \to \cdots \to \underset{[d_{k-1}]}{\underline{(k-1)}} \to \cdots \to \underset{[d_2]}{\underline{2}} \to \cdots \to \underset{[d_1]}{\underline{1}} \to \cdots \to n \,. \tag{3.18}
$$

By Lemma 3.6, summing over $d_1 \in \{d_2 + 1, \dots, n\}$ in (3.18), we get:

$$
\sum_{d_1 = d_2 + 1}^{n} \quad k \to (k+1) \to \cdots \to \underset{[d_{k-1}]}{\underline{(k-1)}} \to \cdots \to \underset{[d_2]}{\underline{2}} \to \cdots \to \underset{[d_1]}{\underline{1}} \to \cdots \to n \;=\; - \quad k \to (k+1) \to \cdots \to \underset{[d_{k-1}]}{\underline{(k-1)}} \to \cdots \to \underset{[d_2]}{\overset{\overset{1}{\downarrow}}{\underline{2}}} \to \cdots \to n \,. \tag{3.19}
$$

Summing (3.19) over $d_2 \in \{d_3+1,\dots,n\}$ and using again Lemma 3.6 (cf. also Remark 3.7), we get:

$$-\sum_{d_2=d_3+1}^{n} k\to(k+1)\to\cdots\to\underset{[d_{k-1}]}{\underline{(k-1)}}\to\cdots\to\underset{[d_3]}{\underline{3}}\to\cdots\to\underset{[d_2]}{\overset{\begin{smallmatrix}1\\ \downarrow\end{smallmatrix}}{\underline{2}}}\to\cdots\to n \;=\; (-1)^2\; k\to(k+1)\to\cdots\to\underset{[d_{k-1}]}{\underline{(k-1)}}\to\cdots\to\underset{[d_3]}{\overset{\begin{smallmatrix}1\\ \downarrow\\ 2\\ \downarrow\end{smallmatrix}}{\underline{3}}}\to\cdots\to n\,. \tag{3.20}$$

Repeating the same argument k times, we conclude that

$$\begin{aligned}\sum_{\pi\in\mathcal{M}_n^k}\pi\Lambda_n &= (-1)^{k-2}\sum_{d_{k-1}=2}^{n} k\to(k+1)\to\cdots\to\underset{[d_{k-1}]}{\overset{\begin{smallmatrix}1\\ \downarrow\\ \vdots\\ \downarrow\\ (k-2)\\ \downarrow\end{smallmatrix}}{\underline{(k-1)}}}\to\cdots\to n\\ &= (-1)^{k-1}\;\overset{\begin{smallmatrix}1\\ \downarrow\\ \vdots\\ \downarrow\\ (k-1)\\ \downarrow\end{smallmatrix}}{k}\to(k+1)\to\cdots\to n\\ &= (-1)^{k-1}\Lambda_n\,.\end{aligned} \tag{3.21}$$

□

3.4 Relation between symmetry property and Harrison's conditions

Recall that $Y \in W_n^{\mathrm{cl}}(\Pi V)$ satisfies the symmetry property (3.3). Using equation (3.12) on connected lines, we get the following relation:

$$\forall\,\tau\in S_n \quad \tau\Lambda_n \equiv \sum_{\substack{\sigma\in S_n\\ \sigma(1)=1}} c_\sigma^{\tau\Lambda_n}\;\sigma\Lambda_n\,. \tag{3.22}$$

Hence, by Remark 3.4, we have:

$$\forall\,\tau\in S_n \quad Y^{\tau\Lambda_n} = \sum_{\substack{\sigma\in S_n\\ \sigma(1)=1}} c_\sigma^{\tau\Lambda_n}\;Y^{\sigma\Lambda_n}\,. \tag{3.23}$$

We denote for simplicity $f := Y^{\Lambda_n}$. By the symmetry condition (3.3) and the definition (2.36) of the action of S_n on $\mathcal{P}^{\mathrm{cl}}(n)$, equation (3.23) becomes:

$$sgn(\tau)f(v_{\tau(1)}\otimes\ldots\otimes v_{\tau(n)}) = \sum_{\substack{\sigma\in S_n\\ \sigma(1)=1}} c_\sigma^{\tau\Lambda_n}\; sgn(\sigma)f(v_{\sigma(1)}\otimes\ldots\otimes v_{\sigma(n)})\,. \tag{3.24}$$

Hence, the symmetry conditions for f coming from the lines are the following:

$$f(v_1\otimes\ldots\otimes v_n) = \sum_{\substack{\sigma\in S_n\\ \sigma(1)=1}} c_\sigma^{\tau\Lambda_n}\;\frac{sgn(\sigma)}{sgn(\tau)} f(v_{\tau^{-1}\sigma(1)}\otimes\ldots\otimes v_{\tau^{-1}\sigma(n)})\,. \tag{3.25}$$

Lemma 3.9. *If $Y \in W^{\mathrm{cl}}_{n-1}(\Pi V)$, then Y^{Λ_n} satisfies the Harrison's relations (1.12), hence it lies in the differential Harrison cohomology complex*

$$Y^{\Lambda_n} \in C^n_{\partial,Har}(V)\,.$$

Moreover, given $F \in C^n_{\partial,Har}(V)$, there exists a unique $Y \in gr^{n-1}W^{\mathrm{cl}}_{n-1}(\Pi V)$ such that

$$Y^{\Lambda_n} = F\,.$$

Hence, there is a bijective linear map

$$gr^{n-1}W^{\mathrm{cl}}_{n-1}(\Pi V) \xrightarrow{\sim} C^n_{\partial,Har}(V)\,, \ \ \textit{mapping } Y \mapsto Y^{\Lambda_n}\,.$$

Proof. Firstly, we prove that, since $Y \in W^{\mathrm{cl}}_{n-1}(\Pi V)$ satisfies the symmetry relations (3.3), $f = Y^{\Lambda_n}$ satisfies the Harrison's conditions (1.12). By Lemma 3.8 (cf. Remark 3.4), we get

$$Y^{\Lambda_n} = (-1)^{k-1} \sum_{\pi \in \mathcal{M}^k_n} Y^{\pi(\Lambda_n)}\,. \tag{3.26}$$

The left-hand side of this identity is simply $Y^{\Lambda_n}(v_1 \otimes \ldots \otimes v_n) = f(v_1 \otimes \ldots \otimes v_n)$. On the right-hand side we have

$$\begin{aligned}(-1)^{k-1} \sum_{\pi \in \mathcal{M}^k_n} Y^{\pi(\Lambda_n)}(v_1 \otimes \ldots \otimes v_n) &= (-1)^{k-1} \sum_{\pi \in \mathcal{M}^k_n} (Y^{\pi^{-1}})^{\pi(\Lambda_n)}(v_1 \otimes \ldots \otimes v_n) \\ &= (-1)^{k-1} \sum_{\pi \in \mathcal{M}^k_n} sgn(\pi) Y^{\Lambda_n}(v_{\pi(1)} \otimes \ldots \otimes v_{\pi(n)}) \\ &= L_k f(v_1 \otimes \ldots \otimes v_n)\,,\end{aligned}$$

by the defintion (1.11) of L_k. Hence, f satisfies the Harrison's conditions (1.12) as claimed.

We next turn to the second claim of the lemma. Let $F \in C^n_{\partial,Har}(V)$, *i.e.* $F : V^{\otimes n} \to V$ is an $\mathbb{F}[\partial]$-module homomorphism satisfying the Harrison's conditions (1.12). We want to construct the corresponding $Y \in gr^{n-1}W^{\mathrm{cl}}_{n-1}(\Pi V)$ such that $Y^{\Lambda_n} = F$. It is defined as follows. For $\Gamma \in R(n)$, or if $\Gamma \in \mathcal{L}(n)$ is not connected, we set

$$Y^{\Gamma} = 0\,. \tag{3.27}$$

For $\Gamma \in \mathcal{L}(n)$ connected, there exists a unique $\tau \in S_n$ such that $\tau(1) = 1$ and $\Gamma = \tau(\Lambda_n)$. We then set

$$Y^{\Gamma}(v_1 \otimes \ldots \otimes v_n) = sgn(\tau) F(v_{\tau(1)} \otimes \ldots \otimes v_{\tau(n)})\,. \tag{3.28}$$

We need to prove that equations (3.27) and (3.28) determine a unique element Y in $gr^{n-1}W^{\mathrm{cl}}_{n-1}(\Pi V)$, and that $Y^{\Lambda_n} = F$. The last assertion is obtained as the special case of (3.27) when $\Gamma = \Lambda_n$ (for which $\tau = 1$). By Theorem 3.5 and Remark 3.4, Y automatically satisfies the cycle relations, since by construction $Y^{\Gamma} = 0$ for $\Gamma \in R(n)$. We need to show that Y satisfies all symmetry conditions (3.3).

Obviously, the action of the symmetric group S_n preserves $R(n)$ and the set of non-connected lines. Hence, when we evaluate (3.3) on $\Gamma \in R(n)$ or on a non-connected n-line $\Gamma \in \mathcal{L}(n)$, we get $0 = 0$.

We are left to prove that (3.3) holds when evaluated on a connected line $\Gamma \in \mathcal{L}(n)$, which, as remarked above, can be obtained as $\Gamma = \tau(\Lambda_n)$, for a unique $\tau \in S_n$ such that $\tau(1) = 1$. The right-hand side of (3.3), when evaluated on such a Γ is given by (3.28). The left-hand side is, by (2.36),

$$\begin{aligned}(Y^\sigma)^\Gamma(v_1 \otimes \ldots \otimes v_n) &= Y^{\sigma\tau(\Lambda_n)}(\sigma(v_1 \otimes \ldots \otimes v_n)) \\ &= sgn(\sigma) Y^{\sigma\tau(\Lambda_n)}(v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma_n}) \,. \end{aligned} \tag{3.29}$$

By Lemma 3.8, we have, modulo $R(n)$,

$$\sigma\tau(\Lambda_n) \equiv (-1)^{\tau^{-1}\sigma^{-1}(1)-1} \sum_{\pi \in \mathcal{M}_n^{\tau^{-1}\sigma^{-1}(1)}} \sigma\tau\pi(\Lambda_n) \,.$$

Hence, by Remark 3.4, the right-hand side of (3.29) becomes

$$sgn(\sigma)(-1)^{\tau^{-1}\sigma^{-1}(1)-1} \sum_{\pi \in \mathcal{M}_n^{\tau^{-1}\sigma^{-1}(1)}} Y^{\sigma\tau\pi(\Lambda_n)}(v_{\sigma^{-1}(1)} \otimes \ldots \otimes v_{\sigma^{-1}(n)}) \,. \tag{3.30}$$

Note that, if $\pi \in \mathcal{M}_n^{\tau^{-1}\sigma^{-1}(1)}$, then $\sigma\tau\pi(1) = 1$. Hence, we can apply formula (3.28) to $\Gamma = \sigma\tau\pi(\Lambda_n)$ to get

$$Y^{\sigma\tau\pi(\Lambda_n)}(v_{\sigma^{-1}(1)} \otimes \ldots \otimes v_{\sigma^{-1}(n)}) = sgn(\sigma\tau\pi) F(v_{\tau\pi(1)} \otimes \ldots \otimes v_{\tau\pi(n)}) \,. \tag{3.31}$$

Combining (3.29), (3.30), and (3.31), we get, by the definition (1.11) of L_k, and by (1.8),

$$\begin{aligned}(Y^\sigma)^\Gamma(v_1 \otimes \ldots \otimes v_n) &= sgn(\tau)(-1)^{\tau^{-1}\sigma^{-1}(1)-1} \sum_{\pi \in \mathcal{M}_n^{\tau^{-1}\sigma^{-1}(1)}} sgn(\pi) F(v_{\tau\pi(1)} \otimes \ldots \otimes v_{\tau\pi(n)}) \\ &= sgn(\tau)(L_{\tau^{-1}\sigma^{-1}(1)}F)(v_{\tau(1)} \otimes \ldots \otimes v_\tau(n)) \,,\end{aligned}$$

which equals (3.28) by the Harrison's conditions (1.12).

Hence, Y is a well defined element of $gr^{n-1}W^{\mathrm{cl}}_{n-1}(\Pi V)$ such that $Y^{\Lambda_n} = F$, as required. The uniqueness of such a Y is obvious since, by Theorem 3.5, $Y \in gr^{n-1}W^{\mathrm{cl}}_{n-1}(\Pi V)$ is uniquely determined by its value on Λ_n. $\square$

3.5 Relation between adX and the Hochschild differential

Lemma 3.10. *For $Y \in W^{\mathrm{cl}}_{n-1}(\Pi V)$ and X defined in (2.38), we have*

$$[X, Y]^{\Lambda_{n+1}}(v_1 \otimes \ldots \otimes v_{n+1}) = (-1)^{n+1} d(Y^{\Lambda_n})(v_1 \otimes \ldots \otimes v_{n+1}) \,,$$

where Λ_n is as in (3.6) and d is the Hochschild differential (1.2).

Recall some notions that will be useful in the proof of the lemma. Let $X \in W_1^{\mathrm{cl}}(\Pi V)$ be the element given by (2.38) with parity $\bar{p}(X) = 1$. The adjoint action of X on $Y \in W_{n-1}^{\mathrm{cl}}(\Pi V)$ is, by definition (2.23), the following:

$$[X,Y] = X\Box Y - (-1)^{n-1} Y \Box X = \sum_{\sigma \in S_{n,1}} (X \circ_1 Y)^{\sigma^{-1}} + (-1)^n \sum_{\tau \in S_{2,n-1}} (Y \circ_1 X)^{\tau^{-1}} \tag{3.32}$$

since $\bar{p}(Y) = n-1$. The elements in $S_{n,1}$ are $(k = 1, \dots, n+1)$

$$\begin{aligned} \sigma_k &= \left(\begin{array}{cccccccc|c} 1 & 2 & \dots & k-1 & k & k+1 & \dots & n & n+1 \\ 1 & 2 & \dots & k-1 & k+1 & k+2 & \dots & n+1 & k \end{array} \right) \\ &= (k \;\; k+1 \;\; \dots \;\; n+1)\,, \end{aligned} \tag{3.33}$$

and the elements in $S_{2,n-1}$ are $(1 \leq i < j \leq n+1)$

$$\tau_{i,j} = \left(\begin{array}{cc|ccccccccc} 1 & 2 & 3 & \dots & i+1 & i+2 & \dots & j & j+1 & \dots & n+1 \\ i & j & 1 & \dots & i-1 & i+1 & \dots & j-1 & j+1 & \dots & n+1 \end{array} \right), \tag{3.34}$$

if $j \geq i+2$, and $(1 \leq i \leq n)$

$$\tau_{i,i+1} = \left(\begin{array}{cc|cccccc} 1 & 2 & 3 & \dots & i+1 & i+2 & \dots & n+1 \\ i & i+1 & 1 & \dots & i-1 & i+2 & \dots & n+1 \end{array} \right). \tag{3.35}$$

Proof of Lemma 3.10. First, using the definition (3.32) and (3.33), we evaluate

$$\begin{aligned} &\left((X \circ_1 Y)^{\sigma_k^{-1}} \right)^{\Lambda_{n+1}} (v_1 \otimes \dots \otimes v_{n+1}) \\ &= (-1)^{n+1-k} (X \circ_1 Y)^{\sigma_k^{-1}(\Lambda_{n+1})} (v_1 \otimes \dots \otimes v_{k-1} \otimes v_{k+1} \otimes \dots \otimes v_{n+1} \otimes v_k)\,. \end{aligned} \tag{3.36}$$

For $k = 1$, by the symmetric group's action described in Section 2.3, we have

$$\sigma_1^{-1}(\Lambda_{n+1}) = \underset{\sigma_1^{-1}(1)}{\bullet} \longrightarrow \underset{\sigma_1^{-1}(2)}{\bullet} \longrightarrow \cdots \longrightarrow \underset{\sigma_1^{-1}(n+1)}{\bullet}$$

$$= \begin{tikzcd} \underset{1}{\bullet} \arrow[r] & \cdots \arrow[r] & \underset{n}{\bullet} & \underset{n+1}{\bullet} \arrow[lll, bend right] \end{tikzcd}\,.$$

When we apply the cocomposition map $\Delta^{n,1}$ to this graph, we get

$$\Delta_0^{n,1}(\sigma_1^{-1}(\Lambda_{n+1})) \;=\; \underset{1}{\bullet} \longleftarrow \underset{2}{\bullet}$$

$$\Delta_1^{n,1}(\sigma_1^{-1}(\Lambda_{n+1})) \;=\; \underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longrightarrow \cdots \longrightarrow \underset{n}{\bullet} = \; \Lambda_n\,.$$

Hence, by the definition (2.33) of the composition map, (3.36) becomes

$$\left((X \circ_1 Y)^{\sigma_1^{-1}} \right)^{\Lambda_{n+1}} (v_1 \otimes \dots \otimes v_{n+1})$$

$$
\begin{aligned}
&= (-1)^n (X \circ_1 Y)^{\sigma_1^{-1}(\Lambda_{n+1})}(v_2 \otimes \ldots \otimes v_{n+1} \otimes v_1) \\
&= (-1)^n X^{\bullet\!\leftarrow\!\bullet}(Y^{\Lambda_n}(v_2 \otimes \ldots \otimes v_{n+1}) \otimes v_1) \\
&= (-1)^{n+1} Y^{\Lambda_n}(v_2 \otimes \ldots \otimes v_{n+1})\, v_1 \,. \qquad (3.37)
\end{aligned}
$$

Similarly, for $k = n+1$, we have $\sigma_{n+1} = 1$, and applying the cocomposition map $\Delta^{n,1}$ to $\sigma_{n+1}^{-1}(\Lambda_{n+1}) = \Lambda_{n+1}$, we get:

$$
\Delta_0^{n,1}(\sigma_{n+1}^{-1}(\Lambda_{n+1})) \;=\; \underset{1}{\bullet} \longrightarrow \underset{2}{\bullet}
$$

$$
\Delta_1^{n,1}(\sigma_{n+1}^{-1}(\Lambda_{n+1})) \;=\; \underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longrightarrow \cdots \longrightarrow \underset{n}{\bullet} = \;\Lambda_n \,.
$$

Hence, by the definition (2.33) of the composition maps, (3.36) becomes

$$
\begin{aligned}
&\left((X \circ_1 Y)^{\sigma_{n+1}^{-1}}\right)^{\Lambda_{n+1}} (v_1 \otimes \ldots \otimes v_{n+1}) \\
&= X^{\bullet\!\to\!\bullet}(Y^{\Lambda_n}(v_1 \otimes \ldots \otimes v_n) \otimes v_{n+1}) \\
&= Y^{\Lambda_n}(v_1 \otimes \ldots \otimes v_n)\, v_{n+1} \,. \qquad (3.38)
\end{aligned}
$$

Furthermore, for $2 \leq k \leq n$, we have

$$
\sigma_k^{-1}(\Lambda_{n+1}) = \underset{\sigma_k^{-1}(1)}{\bullet} \longrightarrow \underset{\sigma_k^{-1}(2)}{\bullet} \longrightarrow \cdots \longrightarrow \underset{\sigma_k^{-1}(n+1)}{\bullet}
$$

$$
= \begin{tikzcd}[column sep=small]
\underset{1}{\bullet} \arrow[r] & \cdots \arrow[r] & \underset{k-1}{\bullet} \arrow[rrrr, bend left] & \underset{k}{\bullet} \arrow[r] & \cdots \arrow[r] & \underset{n}{\bullet} & \underset{n+1}{\bullet} \arrow[lll, bend left]
\end{tikzcd} \,.
$$

Hence, applying the cocomposition map $\Delta^{n,1}$ we get

$$
\Delta_0^{n,1}(\sigma_k^{-1}(\Lambda_{n+1})) \;=\; \begin{tikzcd} \underset{1}{\bullet} \arrow[r, bend left] & \underset{2}{\bullet} \arrow[l, bend left] \end{tikzcd}
$$

which has a cycle. Therefore,

$$
\left((X \circ_1 Y)^{\sigma_k^{-1}}\right)^{\Lambda_{n+1}} = X^{\bullet \rightleftarrows \bullet}(\ldots) = 0 \,. \qquad (3.39)
$$

Next, we compute

$$
\left((Y \circ_1 X)^{\tau_{i,j}^{-1}}\right)^{\Lambda_{n+1}} (v_1 \otimes \ldots \otimes v_{n+1}) \,, \qquad (3.40)
$$

with $\tau_{i,j}$ defined in (3.34). By the symmetric group's action described in Section 2.3 we have

$$
\tau_{i,j}^{-1}(\Lambda_{n+1}) = \underset{\tau_{i,j}^{-1}(1)}{\bullet} \longrightarrow \underset{\tau_{i,j}^{-1}(2)}{\bullet} \longrightarrow \cdots \longrightarrow \underset{\tau_{i,j}^{-1}(n+1)}{\bullet}
$$

$$
= \begin{tikzcd}[column sep=small]
\underset{1}{\bullet} \arrow[rrrrr, bend left] & \underset{2}{\bullet} \arrow[rrrrrrr, bend right] & \underset{3}{\bullet} \arrow[r] & \cdots \arrow[r] & \underset{i+1}{\bullet} \arrow[llll, bend right] & \underset{i+2}{\bullet} \arrow[r] & \cdots \arrow[r] & \underset{j}{\bullet} \arrow[llllll, bend left] & \underset{j+1}{\bullet} \arrow[r] & \cdots \arrow[r] & \underset{n+1}{\bullet}
\end{tikzcd} .
$$

Hence, applying the cocomposition $\Delta^{2,1,\dots,1}$ we get

$$
\Delta_0^{2,1,\dots,1}(\tau_{i,j}^{-1}(\Lambda_{n+1})) = \begin{tikzcd}[column sep=small]
\underset{1}{\bullet} \arrow[rrrr, bend left] \arrow[rrrrrrr, bend right] & \underset{2}{\bullet} \arrow[r] & \cdots \arrow[r] & \underset{i}{\bullet} \arrow[lll, bend right] & \underset{i+1}{\bullet} \arrow[r] & \cdots \arrow[r] & \underset{j-1}{\bullet} \arrow[llllll, bend left] & \underset{j}{\bullet} \arrow[r] & \cdots \arrow[r] & \underset{n}{\bullet}
\end{tikzcd} ,
$$

which has a cycle. Therefore,

$$
\left((Y \circ_1 X)^{\tau_{i,j}^{-1}}\right)^{\Lambda_{n+1}} = Y^{\Delta_0^{2,1,\dots,1}(\tau_{i,j}^{-1}(\Lambda_{n+1}))}(\dots) = 0 . \tag{3.41}
$$

Finally, we evaluate

$$
\left((Y \circ_1 X)^{\tau_{i,i+1}^{-1}}\right)^{\Lambda_{n+1}} (v_1 \otimes \dots \otimes v_{n+1}) , \tag{3.42}
$$

with $\tau_{i,i+1}$ defined in (3.35). In this case, we have

$$
\tau_{i,i+1}^{-1}(\Lambda_{n+1}) = \underset{\tau_{i,i+1}^{-1}(1)}{\bullet} \longrightarrow \underset{\tau_{i,i+1}^{-1}(2)}{\bullet} \longrightarrow \cdots \longrightarrow \underset{\tau_{i,i+1}^{-1}(n+1)}{\bullet}
$$

$$
= \begin{tikzcd}[column sep=small]
\underset{1}{\bullet} \arrow[r] & \underset{2}{\bullet} \arrow[rrrr, bend right] & \underset{3}{\bullet} \arrow[r] & \cdots \arrow[r] & \underset{i+1}{\bullet} \arrow[llll, bend right] & \underset{i+2}{\bullet} \arrow[r] & \cdots \arrow[r] & \underset{n+1}{\bullet}
\end{tikzcd} .
$$

Hence, applying the cocomposition $\Delta^{2,1,\dots,1}$ we get

$$
\Delta_0^{2,1,\dots,1}(\tau_{i,i+1}^{-1}(\Lambda_{n+1})) = \begin{tikzcd}[column sep=small]
\underset{1}{\bullet} \arrow[rrrr, bend right] & \underset{2}{\bullet} \arrow[r] & \cdots \arrow[r] & \underset{i}{\bullet} \arrow[lll, bend right] & \underset{i+1}{\bullet} \arrow[r] & \cdots \arrow[r] & \underset{n}{\bullet}
\end{tikzcd} ,
$$

$$\Delta_1^{2,1,\dots,1}(\tau_{i,i+1}^{-1}(\Lambda_{n+1})) \;=\; \underset{1}{\bullet}\!\longrightarrow\!\underset{2}{\bullet} \qquad .$$

Therefore, by definition (2.33) of the composition map, (3.42) becomes:

$$\begin{aligned}
&\Big((Y \circ_1 X)^{\tau_{i,j}^{-1}}\Big)^{\Lambda_{n+1}}(v_1 \otimes \dots \otimes v_n) = \\
&= (Y \circ_1 X)^{\tau_{i,j}^{-1}(\Lambda_{n+1})}(v_i \otimes v_{i+1} \otimes v_1 \otimes \dots \otimes v_{i-1} \otimes v_{i+2} \otimes \dots \otimes v_{n+1}) \\
&= Y^{\Delta_0^{2,1,\dots,1}(\tau_{i,i+1}^{-1}(\Lambda_{n+1}))}(X^{\bullet\!\to\!\bullet}(v_i \otimes v_{i+1}) \otimes v_1 \otimes \dots \otimes v_{i-1} \otimes v_{i+2} \otimes \dots \otimes v_{n+1}) \\
&= Y^{\Delta_0^{2,1,\dots,1}(\tau_{i,i+1}^{-1}(\Lambda_{n+1}))}(v_i v_{i+1} \otimes v_1 \otimes \dots \otimes v_{i-1} \otimes v_{i+2} \otimes \dots \otimes v_{n+1}) \,. \qquad (3.43)
\end{aligned}$$

Note that

$$\Delta_0^{2,1,\dots,1}(\tau_{i,i+1}^{-1}(\Lambda_{n+1})) = \sigma(\Lambda_n)\,,$$

where $\sigma = (1\ 2\ \dots\ i) \in S_n$ is the i-cycle. Hence, by the symmetry property (3.3), we can replace Y by $Y^{\sigma^{-1}}$ in the right-hand side of (3.43) to get

$$(-1)^{i+1} Y^{\Lambda_n}(v_1 \otimes \dots \otimes v_{i-1} \otimes v_i v_{i+1} \otimes v_{i+2} \otimes \dots \otimes v_{n+1}) \,. \qquad (3.44)$$

By definition (3.32), and combining equations (3.37), (3.38), (3.39), (3.41), (3.43), and (3.44), we get:

$$\begin{aligned}
&[X,Y]^{\Lambda_{n+1}}(v_1 \otimes \dots \otimes v_n) \\
&= (-1)^{n+1} Y^{\Lambda_n}(v_2 \otimes \dots \otimes v_{n+1})\, v_1 \\
&\quad + Y^{\Lambda_n}(v_1 \otimes \dots \otimes v_n)\, v_{n+1} \\
&\quad + (-1)^n \sum_{i=1}^{n} (-1)^{i+1} Y^{\Lambda_n}(v_1 \otimes \dots \otimes v_{i-1} \otimes v_i v_{i+1} \otimes v_{i+2} \otimes \dots \otimes v_{n+1}) \\
&= (-1)^{n+1} d(Y^{\Lambda_n})(v_1 \otimes \dots \otimes v_n)\,,
\end{aligned}$$

completing the proof. □

Corollary 3.11. *Let X be defined in (2.38) and $Y \in W^{\mathrm{cl}}_{n-1}(\Pi V)$ such that $[X,Y] = 0$. Then Y^{Λ_n} is a cocyle in the Hochschild cohomology.*

Proof. Obvious, by Lemma 3.10. □

3.6 Proof of theorem 3.1

By Lemma 3.9, given $Y \in W^{\mathrm{cl}}_{n-1}(\Pi V)$, Y^{Λ_n} is a cochain in the Harrison complex, and also, for any $F \in C^n_{Har}(V)$, there is a unique $Y \in gr^{n-1} W^{\mathrm{cl}}_{n-1}(\Pi V)$ such that $F = Y^{\Lambda_n}$. So, the following diagram is well defined:

$$\begin{array}{ccccc}
Y \ni & W^{\mathrm{cl}}_{n-1}(\Pi V) & \xrightarrow{adX} & W^{\mathrm{cl}}_n(\Pi V) & \\
\downarrow & \downarrow & & \downarrow & , \\
Y^{\Lambda_n} \ni & C^n_{\partial,Har}(V) & \xrightarrow{d} & C^{n+1}_{\partial,Har}(V) &
\end{array} \qquad (3.45)$$

where the vertical maps are surjective and restrict to bijective maps on top degree: $gr^{n-1} W^{\mathrm{cl}}_{n-1}(\Pi V) \xrightarrow{\sim} C^n_{\partial,Har}(V)$. Lemma 3.10 says that, up to a sign, (3.45) is also commutative.

Chapter 4

Relation between variational Poisson cohomology and PVA cohomology

4.1 Variational Poisson cohomology

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a vector superspace with parity p, endowed with an even endomorphism $\partial \in \operatorname{End} V$. For $n \geq 0$, define $\mathcal{C}hom(n)$ the superspaces consisting of all linear maps

$$\begin{aligned} f\colon V^{\otimes n} &\longrightarrow V[\lambda_1,\dots,\lambda_n]\big/\langle \partial+\lambda_1+\cdots+\lambda_n\rangle \\ v_1\otimes\cdots\otimes v_n &\mapsto f_{\lambda_1,\dots,\lambda_n}(v_1\otimes\cdots\otimes v_n)\,, \end{aligned} \tag{4.1}$$

satisfying the sesquilinearity conditions:

$$f_{\lambda_1,\dots,\lambda_n}(v_1\otimes\cdots\partial v_i\cdots\otimes v_n) = -\lambda_i f_{\lambda_1,\dots,\lambda_n}(v_1\otimes\cdots\otimes v_n) \quad \text{for all } i=1,\dots,n\,. \tag{4.2}$$

Given an n-tuple $(m_1,\dots,m_n)$ of positive integers, let M_i and Λ_i be as in (2.6) and (2.32). The operad $\mathcal{C}hom(V)$ is defined as the collection of the vector superspaces $\mathcal{C}hom(n)$, $n \geq 0$, endowed, for every $f \in \mathcal{C}hom(n)$ and $m_1,\dots,m_n$, with composition parity preserving linear maps as follows. Let $g_1 \in \mathcal{C}hom(m_1),\dots,g_n \in \mathcal{C}hom(m_n)$, then we have $f(g_1\otimes\cdots\otimes g_n) \in \mathcal{C}hom(M_n)$,

$$\begin{aligned} &\big(f(g_1\otimes\cdots\otimes g_n)\big)_{\lambda_1,\dots,\lambda_{M_n}}(v_1\otimes\cdots\otimes v_{M_n}) \\ &:= f_{\Lambda_1,\dots,\Lambda_n}\big(((g_1)_{\lambda_1,\dots,\lambda_{M_1}}\otimes\cdots\otimes(g_n)_{\lambda_{M_{n-1}+1},\dots,\lambda_{M_n}})(v_1\otimes\cdots\otimes v_{M_n})\big)\,, \end{aligned} \tag{4.3}$$

where, recalling the definition of tensor product between linear maps of vector superspaces and notation (2.6),

$$\begin{aligned} &\Big((g_1)_{\lambda_1,\dots,\lambda_{M_1}}\otimes\cdots\otimes(g_n)_{\lambda_{M_{n-1}+1},\dots,\lambda_{M_n}}\Big)(v_1\otimes\cdots\otimes v_{M_n}) \\ &:= (-1)^{\sum_{i<j} p(g_j)(p(v_{M_{i-1}+1})+\cdots+p(v_{M_i}))}\,(g_1)_{\lambda_1,\dots,\lambda_{M_1}}(v_1\otimes\cdots\otimes v_{M_1})\otimes\dots \\ &\qquad\cdots\otimes(g_n)_{\lambda_{M_{n-1}+1},\dots,\lambda_{M_n}}(v_{M_{n-1}+1}\otimes\cdots\otimes v_{M_n})\,. \end{aligned} \tag{4.4}$$

The unity in the $\mathcal{C}hom$ operad is $1 = \mathbb{1}_V \in \mathcal{C}hom(1) = \mathrm{End}_{\mathbb{F}[\partial]} V$, and the right action of S_n on $\mathcal{C}hom(n)$ is given by (cf. (2.2) and (2.1)):

$$\begin{aligned}(f^\sigma)_{\lambda_1,\dots,\lambda_n}(v_1 \otimes \cdots \otimes v_n) &= f_{\sigma(\lambda_1,\dots,\lambda_n)}(\sigma(v_1 \otimes \cdots \otimes v_n)) \\ &= \epsilon_v(\sigma) f_{\lambda_{\sigma^{-1}(1)},\dots,\lambda_{\sigma^{-1}(n)}}(v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}) ,\end{aligned} \tag{4.5}$$

for every $\sigma \in S_n$, where $\epsilon_v(\sigma)$ is given by (2.3).

Denote by $W^\partial(\Pi V) := W(\mathcal{C}hom(\Pi V))$ the associated $\mathbb{Z}$-graded Lie superalgebra from (2.21) with Lie bracket (2.23). Let now V be a commutative, associative, unital, differential algebra, with an even derivation ∂. Following the notation in [DSK13], for $k \geq -1$, we let $W_k^{\partial,\mathrm{as}}(\Pi V)$ be the subspace of $W_k^\partial(\Pi V)$ consisting of all linear maps

$$\begin{aligned} f \colon V^{\otimes n} &\longrightarrow \mathbb{F}_-[\lambda_1, \dots, \lambda_n] \otimes_{\mathbb{F}[\partial]} V \\ v_1 \otimes \dots \otimes v_n &\mapsto f_{\lambda_1,\dots,\lambda_n}(v_1 \otimes \dots \otimes v_n) ,\end{aligned} \tag{4.6}$$

satisfying the sesquilinearity conditions (4.2) and the following Leibniz rules:

$$\begin{aligned} & f_{\lambda_1,\dots,\lambda_n}(v_1, \dots, u_i w_i, \dots, v_n) \\ &= (-1)^{p(w_i)(s_{i+1,k}+k-i)} f_{\lambda_1,\dots,\lambda_i+\partial,\dots,\lambda_n}(v_1, \dots, u_i, \dots, v_n)_{\to} w_i \\ &+ (-1)^{p(u_i)(p(w_i)+s_{i+1,k}+k-i)} f_{\lambda_1,\dots,\lambda_i+\partial,\dots,\lambda_n}(v_1, \dots, w_i, \dots, v_n)_{\to} u_i ,\end{aligned} \tag{4.7}$$

where the arrow means that ∂ is moved to the right and

$$\begin{cases} s_{ij} = p(v_i) + \dots + p(v_j) & \text{if } i \leq j \\ s_{ij} = 0 & \text{if } i > j . \end{cases} \tag{4.8}$$

Theorem 4.1 ([DSK13, Prop. 5.1-5.2])**.** *The space*

$$W^{\partial,\mathrm{as}}(\Pi V) = \bigoplus_{k \geq -1} W_k^{\partial,\mathrm{as}}(\Pi V)$$

is a subalgebra of the Lie superalgebra $W^\partial(\Pi V)$. Moreover, there is a bijective correspondence between the odd elements $\bar{X} \in W_1^{\partial,\mathrm{as}}(\Pi V)$ such that $[\bar{X}, \bar{X}] = 0$ and the Poisson vertex algebra λ-brackets on V, given by

$$[a_\lambda b] = (-1)^{p(a)} \bar{X}_{\lambda,-\lambda-\partial}(a \otimes b) . \tag{4.9}$$

As a consequence, given a Poisson vertex algebra λ-bracket on V, we have the corresponding cohomology complex $(W^{\partial,\mathrm{as}}(\Pi V), d_{\bar{X}})$ with differential $d_{\bar{X}} = \mathrm{ad}\,\bar{X}$.

4.2 Relation between PVA cohomology and variational Poisson cohomology

To a Poisson vertex algebra V we associate two cohomology complexes: the PVA cohomology complex $(W^{\mathrm{cl}}(\Pi V), adX)$ introduced in Section 2.8, and the variational

Poisson cohomology complex $(W^{\partial,as}(\Pi V), ad\bar{X})$ introduced above. It is natural to ask what is the relation between these two cohomology theories.

As explained in [BDSHK18], the complexes $(W^{\mathrm{cl}}(\Pi V), adX)$ and $(W^{\partial,\mathrm{as}}(\Pi V), ad\bar{X})$ are related by the following

Lemma 4.2 ([BDSHK18, Lemma 11.2]). *We have a natural Lie algebra isomorphism*

$$W^{\partial}(\Pi V) \xrightarrow{\sim} \mathrm{gr}^0 W^{\mathrm{cl}}(\Pi V)\,, \tag{4.10}$$

mapping $\bar{f} \in W^{\partial}(\Pi V)$ *to the element* $f \in \mathrm{gr}^0 W^{\mathrm{cl}}(\Pi V)$ *such that*

$$f^{\bullet\cdots\bullet} = \bar{f} \quad and \quad f^{\Gamma} = 0 \quad if \; |E(\Gamma)| \neq \emptyset\,.$$

Theorem 4.3 ([BDSHK18, Theorem 11.4]). *We have a canonical injective homomorphism of Lie superalgebras*

$$H(W^{\partial,\mathrm{as}}(\Pi V), d_{\bar{X}}) \hookrightarrow H(W^{\mathrm{cl}}(\Pi V), d_X) \tag{4.11}$$

induced by the map (4.10).

The map (4.11) is an isomorphism for the 0-th and 1-st cohomologies, and it is left as an open question in [BDSHK18] whether (4.11) is, in fact, an isomorphism.

The main application of Theorem 3.1 will be the proof that this is indeed the case, under some regularity assumption on V. Recall the following definitions (see [W94] for details):

Definition 4.4. Let R be a $\mathbb{F}$-algebra and M a R-module. A *squarezero extension* of R by M is a $\mathbb{F}$-algebra E, together with a surjective ring homomorphism $\epsilon : E \to R$ such that $\mathrm{Ker}(\epsilon)$ is an ideal of square zero, and an R-module isomorphism of M with $\mathrm{Ker}(\epsilon)$.

Definition 4.5. A commutative $\mathbb{F}$-algebra is *smooth* (over $\mathbb{F}$) if for every squarezero extension $0 \to M \to E \xrightarrow{\epsilon} T \to 0$ of commutative $\mathbb{F}$-algebras and every algebra map $\nu : R \to T$ there exists a $\mathbb{F}$-algebra homomorphism $u : R \to E$ lifting ν in the sense that $\epsilon u = \nu$.

For example, every polynomial algebra $R = \mathbb{F}[x_1, ..., x_n]$ is smooth over $\mathbb{F}$.

Theorem 4.6. *Assuming that the PVA V is finitely generated and smooth as a differential algebra, the Lie homomorphism* (4.11) *is an isomorphism.*

We shall not provide here a full proof of Theorem 4.6, which is deferred to a future research project. Here we outline the main ideas for the proof. Before, let us describe an enlightening example.

Example 4.7. Let V be an (even) algebra of polynomials in finitely many variables with a Poisson structure, *i.e.* a Lie bracket $[\cdot\,,\cdot]$ satisfying the Leibniz rule. We compute the second cohomology of the finite Poisson cohomology complex (2.44). Bearing the shift of the grading in mind, we have to calculate

$$H^2(W^{\mathrm{fn}}(\Pi V), adX) = \frac{\mathrm{Ker}(adX : W_1^{\mathrm{fn}}(\Pi V) \to W_2^{\mathrm{fn}}(\Pi V))}{Im(adX : W_0^{\mathrm{fn}}(\Pi V) \to W_1^{\mathrm{fn}}(\Pi V))}.$$

Recall that $Y \in W_1^{\mathrm{fn}}(\Pi V)$ is a map $Y : \mathcal{G}(2) \times V^{\otimes 2} \to V$, with parity $\bar{p}(Y) = 1$, and such that

$$Y^{\bullet\ \bullet}(v_1 \otimes v_2) = -Y^{\bullet\ \bullet}(v_2 \otimes v_1), \qquad Y^{\bullet\!\to\!\bullet}(v_1 \otimes v_2) = Y^{\bullet\!\to\!\bullet}(v_2 \otimes v_1) = -Y^{\bullet\!\leftarrow\!\bullet}(v_1 \otimes v_2)\,, \tag{4.12}$$

and $Y^\Gamma = 0$ if $\Gamma \in \mathcal{G}(2)$ containes a cycle. By (2.23), adX on such a Y is defined as:

$$\begin{aligned}
[X, Y] &= X \Box Y + Y \Box X \\
&= \sum_{\sigma \in S_{2,1}} (X \circ_1 Y)^{\sigma^{-1}} + \sum_{\tau \in S_{2,1}} (Y \circ_1 X)^{\tau^{-1}} \\
&= X \circ_1 Y + X \circ_2 Y + (X \circ_2 Y)^{(12)} + Y \circ_1 X + Y \circ_2 X + (Y \circ_2 X)^{(12)}.
\end{aligned} \tag{4.13}$$

$Y \in \mathrm{Ker}(adX : W_1^{\mathrm{fn}}(\Pi V) \to W_2^{\mathrm{fn}}(\Pi V))$ means that $[X, Y] = 0$ and, by the invariance of the $\Box$-product under the action of the symmetric group this is the same as to impose $[X, Y]^\Gamma = 0$ for each of the three graphs:

$$\underset{1}{\circ}\ \ \underset{2}{\circ}\ \ \underset{3}{\circ}\ , \qquad \underset{1}{\circ}\ \ \underset{2}{\circ}\!\!\longrightarrow\!\!\underset{3}{\circ}\ , \qquad \underset{1}{\circ}\!\!\longrightarrow\!\!\underset{2}{\circ}\!\!\longrightarrow\!\!\underset{3}{\circ}\ . \tag{4.14}$$

By Corollary 3.11, we know that evaluating $[X, Y]$ on the connected line, we find that $Y^{\bullet\!\to\!\bullet}$ is a cocyle in the Hochschild cohomology. Evaluating $[X, Y]$ on the second graph in (4.14), we get:

$$\begin{aligned}
(X \circ_1 Y)^{\bullet\ \ \bullet\!\to\!\bullet}(v_1 \otimes v_2 \otimes v_3) &= X^{\bullet\!\to\!\bullet}\,(Y^{\bullet\ \bullet}(v_1 \otimes v_2) \otimes v_3) \\
&= Y^{\bullet\ \bullet}(v_1 \otimes v_2)v_3\,, \\
(X \circ_2 Y)^{\bullet\ \ \bullet\!\to\!\bullet}(v_1 \otimes v_2 \otimes v_3) &= -X^{\bullet\ \bullet}\,(v_1 \otimes Y^{\bullet\!\to\!\bullet}(v_2 \otimes v_3)) \\
&= -[v_1, Y^{\bullet\!\to\!\bullet}(v_2 \otimes v_3)]\,, \\
((X \circ_2 Y)^{(12)})^{\bullet\ \ \bullet\!\to\!\bullet}(v_1 \otimes v_2 \otimes v_3) &= -(X \circ_2 Y)^{\overset{\curvearrowright}{\bullet\ \bullet\ \bullet}}(v_2 \otimes v_1 \otimes v_3) \\
&= v_2 Y^{\bullet\ \bullet}(v_1 \otimes v_3)\,, \\
(Y \circ_1 X)^{\bullet\ \ \bullet\!\to\!\bullet}(v_1 \otimes v_2 \otimes v_3) &= Y^{\bullet\!\to\!\bullet}\,([v_1, v_2] \otimes v_3)\,, \\
(Y \circ_2 X)^{\bullet\ \ \bullet\!\to\!\bullet}(v_1 \otimes v_2 \otimes v_3) &= -Y^{\bullet\ \bullet}\,(v_1 \otimes v_2 v_3)\,, \\
((Y \circ_2 X)^{(12)})^{\bullet\ \ \bullet\!\to\!\bullet}(v_1 \otimes v_2 \otimes v_3) &= -\,(Y \circ_2 X)^{\overset{\curvearrowright}{\bullet\ \bullet\ \bullet}}(v_2 \otimes v_1 \otimes v_3) \\
&= Y^{\bullet\!\to\!\bullet}(v_2 \otimes [v_1, v_3])\,.
\end{aligned}$$

Thus, we have:

$$\begin{aligned}
&[X, Y]^{\bullet\ \ \bullet\!\to\!\bullet}(v_1 \otimes v_2 \otimes v_3) \\
&= Y^{\bullet\ \bullet}(v_1 \otimes v_2)v_3 - [v_1, Y^{\bullet\!\to\!\bullet}(v_2 \otimes v_3)] + v_2 Y^{\bullet\ \bullet}(v_1 \otimes v_3) \\
&\qquad + Y^{\bullet\!\to\!\bullet}\,([v_1, v_2] \otimes v_3) - Y^{\bullet\ \bullet}\,(v_1 \otimes v_2 v_3) + Y^{\bullet\!\to\!\bullet}(v_2 \otimes [v_1, v_3]) = 0.
\end{aligned} \tag{4.15}$$

Next, evaluating all six summands of $[X, Y]$ on the disconnected graph in (4.14), we get:

$$(X \circ_1 Y)^{\bullet\ \bullet\ \bullet}(v_1 \otimes v_2 \otimes v_3) = X^{\bullet\ \bullet}\,(Y^{\bullet\ \bullet}(v_1 \otimes v_2) \otimes v_3)$$

$$
\begin{aligned}
&= [Y^{\bullet\ \bullet}(v_1 \otimes v_2), v_3]\,, \\
(X \circ_2 Y)^{\bullet\ \bullet\ \bullet}(v_1 \otimes v_2 \otimes v_3) &= -X^{\bullet\ \bullet}(v_1 \otimes Y^{\bullet\ \bullet}(v_2 \otimes v_3)) \\
&= -[v_1, Y^{\bullet\ \bullet}(v_2 \otimes v_3)]\,, \\
((X \circ_2 Y)^{(12)})^{\bullet\ \bullet\ \bullet}(v_1 \otimes v_2 \otimes v_3) &= -(X \circ_2 Y)^{\bullet\ \bullet\ \bullet}(v_2 \otimes v_1 \otimes v_3) \\
&= [v_2, Y^{\bullet\ \bullet}(v_1 \otimes v_3)]\,, \\
(Y \circ_1 X)^{\bullet\ \bullet\ \bullet}(v_1 \otimes v_2 \otimes v_3) &= Y^{\bullet\ \bullet}([v_1, v_2] \otimes v_3)\,, \\
(Y \circ_2 X)^{\bullet\ \bullet\ \bullet}(v_1 \otimes v_2 \otimes v_3) &= -Y^{\bullet\ \bullet}(v_1 \otimes [v_2, v_3])\,, \\
((Y \circ_2 X)^{(12)})^{\bullet\ \bullet\ \bullet}(v_1 \otimes v_2 \otimes v_3) &= Y^{\bullet\ \bullet}(v_2 \otimes [v_1, v_3])\,.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&[X, Y]^{\bullet\ \bullet\ \bullet}(v_1 \otimes v_2 \otimes v_3) \\
&= [Y^{\bullet\ \bullet}(v_1 \otimes v_2), v_3] - [v_1, Y^{\bullet\ \bullet}(v_2 \otimes v_3)] + [v_2, Y^{\bullet\ \bullet}(v_1 \otimes v_3)] \\
&\qquad + Y^{\bullet\ \bullet}([v_1, v_2] \otimes v_3) - Y^{\bullet\ \bullet}(v_1 \otimes [v_2, v_3]) + Y^{\bullet\ \bullet}(v_2 \otimes [v_1, v_3]) = 0.
\end{aligned}
\tag{4.16}
$$

By Theorem 3.1, $Y^{\bullet\!\to\!\bullet}$ is a cochain of the Harrison complex, and, since it also has zero Hochschild differential, $Y^{\bullet\!\to\!\bullet}$ is a cocycle in the Harrison cohomology. It is well known that the Harrison cohomology of V is trivial. So, up to a coboundary, we can set $Y^{\bullet\!\to\!\bullet}$ to be zero.

Assuming $Y^{\bullet\!\to\!\bullet} = 0$, equation (4.15) becomes:

$$
Y^{\bullet\ \bullet}(v_1 \otimes v_2)v_3 + v_2 Y^{\bullet\ \bullet}(v_1 \otimes v_3) - Y^{\bullet\ \bullet}(v_1 \otimes v_2 v_3) = 0\,, \tag{4.17}
$$

i.e. $Y^{\bullet\ \bullet}$ satisfies the finite analog of Leibniz rule in (4.7). So, $Y \in H^2(W^{\mathrm{fn}}(\Pi V))$ is as follows: $Y^{\bullet\!\to\!\bullet} = 0$, and $Y^{\bullet\ \bullet}$ satisfies the skewsymmetry property (4.12) and equations (4.16), (4.17). These relations are the same that describe the Lie algebra cohomology.

4.3 Sketch of the proof of Theorem 4.6

The proof of Theorem 4.6 will be divided in two claims.

4.3.1 Claim 1

The first step of the proof consists in showing that, under the smoothness assumption on V, the PVA cohomology is concentrated in degree 0, *i.e.* on the maps $Y \in W_n^{\mathrm{cl}}(\Pi V)$ vanishing on all graphs $\Gamma \in \mathcal{G}(n)$ with at least one edge. In other words, given Y a cocycle for the classical complex (3.1), and Γ a graph such that $\mid E(\Gamma) \mid \neq 0$, Y^Γ vanishes in the cohomology. More precisely,

Proposition 4.8. *For every $n \geq 0$, given $Y \in W_{n-1}^{\mathrm{cl}}(\Pi V)$ such that $[X, Y] = 0$, there exist $Z \in W_{n-2}^{\mathrm{cl}}(\Pi V)$ and $\tilde{Y} \in W_{n-1}^{\mathrm{cl}}(\Pi V)$ such that*

$$
Y = [X, Z] + \tilde{Y} \in W_{n-1}^{\mathrm{cl}}(\Pi V)\,, \quad [X, \tilde{Y}] = 0\,, \tag{4.18}
$$

and

$$
\tilde{Y}^\Gamma = 0, \quad \forall\, \Gamma \in \mathcal{G}(n) \text{ such that } |E(\Gamma)| \neq 0\,. \tag{4.19}
$$

By Lemma 3.5, we can write every graph $\Gamma \in \mathcal{G}(n)$, modulo cycle relations, uniquely, as linear combination of disjoint unions of lines

$$L_1 \sqcup \ldots \sqcup L_s$$

defined in Section 3.2. To prove Proposition 4.8, we proceed by induction on the multi-index $(s, k_1, \ldots, k_s)$ ordered lexicographically, *i.e.*

$$(s, |L_1|, \ldots, |L_s|) < (t, |l_1|, \ldots, |l_t|) \text{ if } \begin{cases} s < t \\ \text{ or} \\ s = t \text{ and } \exists\, i \,:\, |L_j| = |l_j| \,\forall\, j < i \text{ and } |L_i| < |l_i| \,. \end{cases} \tag{4.20}$$

At every step $h = (s, k_1, \ldots, k_s)$ of the induction, we produce an element $Z_h \in W^{\mathrm{cl}}_{n-2}(\Pi V)$ as follows. Let

$$\Gamma_h = \underset{1}{\bullet}\!\to\!\underset{2}{\bullet}\!\to\!\cdots\!\to\!\underset{k_1}{\bullet} \quad \underset{k_1+1}{\bullet}\!\to\!\bullet\!\to\!\cdots\!\to\!\underset{k_1+k_2}{\bullet} \quad \cdots \quad \underset{k_1+\ldots+k_{s-1}+1}{\bullet}\!\to\!\cdots\!\to\!\underset{n}{\bullet} \quad \in \mathcal{L}(n) \tag{4.21}$$

In particular, for $s = 1$ the corresponding graph is $\Gamma_h = \Lambda_n$, while for $s = n$ the corresponding graph is completely disconnected: $\Gamma_h = \bullet \quad \bullet \quad \cdots \quad \bullet$. Note that, for every disjoint union of s lines $\Gamma \in \mathcal{L}(n)$, with $|E(\Gamma)| = n - s$, there is a unique permutation $\sigma \in S_n$ such that

$$\Gamma = \sigma(\Gamma_h)\,. \tag{4.22}$$

We want to construct an element $Z_h \in gr^{n-s} W^{\mathrm{cl}}_{n-2}(\Pi V)$ such that

$$Y^{\Gamma} = [X, Z_h]^{\Gamma}\,. \tag{4.23}$$

for every other graph Γ such that $|E(\Gamma)| = n - s$. Note that, by Theorem (3.5), Remark 3.4, and the symmetry property (3.3), equation (4.23) holds as soon as

$$Y^{\Gamma_h} = [X, Y]^{\Gamma_h}\,. \tag{4.24}$$

The construction of these elements Z_h is obtained in Section 4.3.3 for the case $s = 1$ (the base of the induction), and in Section 4.3.4 for a generic h (the inductive step). Once we have all the elements Z_h, we set

$$Z = \sum_h Z_h\,.$$

Hence, by construction,

$$Y^{\Gamma} = [X, Z]^{\Gamma}$$

for all $\Gamma \in \mathcal{G}(n)$ with $|E(\Gamma)| \neq 0$. Therefore, (4.18) and (4.19) hold with $\tilde{Y} = Y - [X, Z]$.

4.3.2 Differential Harrison cohomology of smooth differential algebras

It is well known that the Harrison cohomology of a smooth algebra is trivial (see [L13] and [Hoc45]). An important key ingredient for the conctruction of the elements Z_h is a differential analog of this statement:

Proposition 4.9. *The differential Harrison cohomology $H_{\partial,Har}(V)$ of a smooth differential algebra is trivial in degree > 1.*

The proof of Proposition 4.9 will be part of a future research project.

4.3.3 Base of the induction

Let us go into detail explaining the base of the induction. It corresponds to $s = 1$. Given $Y \in \operatorname{Ker}(adX : W^{\mathrm{cl}}_{n-1}(\Pi V) \to W^{\mathrm{cl}}_n(\Pi V))$, we want to construct $Z_1 \in W^{\mathrm{cl}}_{n-2}(\Pi V)$ such that (cf. (4.24)):

$$Y^{\Lambda_n} = [X, Z_1]^{\Lambda_n} . \tag{4.25}$$

Let us denote $f = Y^{\Lambda_n}$. By the assumption $[X, Y] = 0$, and so, by Lemma 3.11, f is a cocycle in the Hochschild cohomology. Moreover, by Theorem 3.1, f is a cochain in the differential Harrison cohomology complex, namely $f \in C^n_{\partial,Har}(V)$. Hence, f is a cocycle in the differential Harrison cohomology. (Since we already know that Theorem 4.6 is true for the 0-th and the 1-st cohomologies, we can assume here that $n > 1$.)

By Proposition 4.9, there exists $g \in C^{n-1}_{\partial,Har}(V)$ such that $f = dg$. Again by Theorem 3.1, we then have an element $\bar{Z}_1 \in gr^{n-2}W^{\mathrm{cl}}_{n-2}(\Pi V)$ such that $\bar{Z}_1^{\Lambda_{n-1}} = g$. Hence, by Lemma 3.10

$$Y^{\Lambda_n} = d(\bar{Z}_1^{\Lambda_{n-1}}) = (-1)^n [X, \bar{Z}_1]^{\Lambda_n} . \tag{4.26}$$

Thus, $Z_1 \in W^{\mathrm{cl}}_{n-2}(\Pi V)$ is defined as an element in $gr^{n-2}W^{\mathrm{cl}}_{n-2}(\Pi V)$ (*i.e.* Z_1 vanishes when evaluated on a graph with number of edges not equal to $n-2$), $Z_1^{\Lambda_{n-1}} = (-1)^n \bar{Z}_1^{\Lambda_{n-1}}$, and, for $\Gamma' \in \mathcal{G}(n)$, Γ' connected, $Z_1^{\Gamma'}$ is defined combining identity (3.12), Remark 3.4, and symmetry property (3.3).

4.3.4 Inductive step

Here is a sketch of the inductive step. All the computations are deferred to a future research project. For the inductive step, let Γ_h be as in (4.21), and suppose that equations (4.18) and (4.19) are true for every $(t, h_1, \dots, h_t) < (s, k_1, \dots, k_s)$. Let also $\tilde{\Gamma}_h$ be the graph obtained by Γ_h by adding a vertex (labelled as $n+1$) and the edge $n \to n+1$, *i.e.*

$$\tilde{\Gamma}_h = \underbrace{\bullet_1 \to \bullet \to \cdots \to \bullet_{k_1}}_{\tilde{L}_1} \quad \underbrace{\bullet_{k_1+1} \to \bullet \to \cdots \to \bullet_{k_1+k_2}}_{\tilde{L}_2} \quad \cdots \quad \underbrace{\bullet_{k_1+\ldots+k_{s-1}+1} \to \bullet \to \cdots \to \bullet_n \to \bullet_{n+1}}_{\tilde{L}_s} , \tag{4.27}$$

By assumption, we have $[X,Y]=0$. Hence, in particular,

$$[X,Y]^{\tilde{\Gamma}_h}_{\lambda_1,\dots,\lambda_{n+1}}(v_1\otimes\dots\otimes v_{n+1})=0\,. \tag{4.28}$$

Computing explicitly the left-hand side of (4.28), using the definition (2.23) of the Lie bracket in $W^{\mathrm{cl}}(\Pi V)$ and (2.38) of the element $X\in W_1^{\mathrm{cl}}(\Pi V)$, we get the Hochschild differential of Y^Γ viewed as a function of only the last k_s+1 factors $v_{k_1+\dots+k_{s-1}+1},\dots,v_n,v_{n+1}$. Hence, we should be able to use Proposition 4.9, in a way similar to what we did in Section 4.3.3, to construct the corresponding element $Z_h\in gr^{n-s}W^{\mathrm{cl}}_{n-2}(\Pi V)$ such that (4.24) holds. We omit here the details of this computation, which are deferred to a future project.

4.3.5 Claim 2

Proposition 4.10. *Let V be a PVA. Suppose that, for every $n\geq 0$, for every $Y\in W^{\mathrm{cl}}_{n-1}(\Pi V)$ such that $[X,Y]=0$, $Y^\Gamma=0$ if Γ is a graph with at least one egde. Then, there exit an element $Z\in W^{\mathrm{cl}}_{n-2}(\Pi V)$ and an element $\tilde{Y}\in W^{\mathrm{cl}}_{n-1}(\Pi V)$ such that*

$$Y^{\bullet\,\cdots\,\bullet}=[X,Z]^{\bullet\,\cdots\,\bullet}+\tilde{Y}^{\bullet\,\cdots\,\bullet}\quad \textit{with } \tilde{Y}^{\bullet\,\cdots\,\bullet}=\bar{Y}\in W^{\partial,as}_{n-1}(\Pi V)\,. \tag{4.29}$$

Here is an outline of the proof. Evaluating $[X,Y]$ on the disconnected graph

$$\bullet\quad\bullet\quad\cdots\quad\bullet\quad,$$

and by the assumption $[X,Y]=0$ and $Y^\Gamma=0$ for every graph Γ with at least one edge, we get the Lie conformal algebra relation. Whereas, evaluating $[X,Y]$ on the graph

$$\bullet\quad\cdots\quad\bullet\!\to\!\bullet\quad,$$

we get the Leibniz rule (4.7).

4.4 Examples

In [BDSK19] the authors compute the variational Poisson cohomology of the most important examples of PVA's arising in conformal field theory. By Theorem 4.6, we can also obtain their Poisson vertex algebra cohomologies. Here, we just present two of them, and we refer to [BDSK19] for details and more examples.

Example 4.11. Let $\mathcal{V}ir^c$ the Virasoro PVA with central charge $c\in\mathbb{F}$, as in Example 2.8. We have ([BDSK19, Theorem 4.17]):

$$dimH^n_{\mathrm{cl}}(\mathcal{V}ir^c)=\begin{cases}1, & \text{for } n=0,2,3,\\ 0, & \text{otherwise.}\end{cases}$$

Example 4.12. Let $\mathcal{F}$ the PVA of free fermions described in Example 2.10. From [BDSK19], we have that the PVA cohomology of $\mathcal{F}$ is trivial ([BDSK19, Theorem 4.7]):

$$dimH^n_{\mathrm{cl}}(\mathcal{F})=\delta_{n,0}\qquad n\geq 0\,.$$

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