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Variational methods in the Landau-de Gennes theory of liquid crystals

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Contents

Chapter 1

Introduction

1.1 Introduction

Nematic liquid crystals are mesophases between the liquid and the solid phases. Nematic molecules typically have elongated shape, approximately rod-like, and can translate freely, like in a liquid, but, in doing so, their long axes tend to align along some common direction. This feature is the key for the extreme responsivity of nematics to external stimuli, which in turn is the reason why they are so useful in technological applications (some of which are perhaps very surprising, cfr. [118] and references therein). Macroscopic configurations of nematics are better described by continuum theories rather than by molecular models [52, 113, 6, 7]. Among continuum models, the most successful is the phenomenological Landau-de Gennes (LdG) theory [52, 113].

Call $\Omega \subset \mathbb{R}^3$ the region of space containing the nematic liquid crystal under study; Ω will be assumed to be bounded and simply-connected, with smooth boundary. In the LdG theory the state of the system is described by a map $Q : \Omega \to \mathcal{S}_0$, where

$$\mathcal{S}_0 = \{M \in \mathcal{M}_{3\times 3}(\mathbb{R}) : M = M^t \text{ and } \operatorname{Tr} M = 0\}. \tag{1.1.1}$$

Note that $\mathcal{S}_0 \simeq \mathbb{R}^5$ as linear spaces; we endow $\mathcal{S}_0$ with the norm $|M| = \sqrt{M_{ij}M_{ij}}$, where summation convention is understood. The map Q is called *Q-tensor order parameter*. The state of the system at $x \in \Omega$ is said to be

- *isotropic*, if $Q(x) = 0$;
- *uniaxial*, if $Q(x) \neq 0$ and $Q(x)$ has two equal eigenvalues;
- *biaxial*, if $Q(x)$ has three distinct eigenvalues.

Following a common convention, we shall often include the isotropic case into the uniaxial one; moreover, we shall always label the eigenvalues $\lambda_1(x), \lambda_2(x), \lambda_3(x)$ of $Q(x)$ in the increasing order.

A convenient measure of biaxiality is provided by the *biaxiality parameter* β^2 [81], defined as

$$\beta^2(Q(x)) = 1 - 6\frac{\operatorname{Tr}(Q^3(x))^2}{\operatorname{Tr}(Q^2(x))^3}. \tag{1.1.2}$$

It holds $\beta^2(Q(x)) = 0$ if and only if $Q(x)$ is uniaxial and $\beta^2(Q(x)) = 1$ if and only if $Q(x)$ is *maximally biaxial* [106].

Observable configurations of nematics are described in the LdG theory as minimizers of an appropriate energy functional $E_{\mathrm{LdG}}(Q;\Omega)$, whose precise form depends on the physical effects one wish to include, defined on an appropriate space of functions subject to the imposed boundary conditions. Thus, amazingly complicated choices are possible but we will work only with the simplest form for $E_{\mathrm{LdG}}(Q;\Omega)$, assuming Dirichlet boundary conditions. We take

$$E_{\mathrm{LdG}}(Q;\Omega) := \int_{\Omega} e(\nabla Q, Q)\,\mathrm{d}x, \tag{1.1.3}$$

where

$$e(\nabla Q, Q) = \frac{L}{2}\,|\nabla Q|^2 - \frac{A(T^*-T)}{2}\operatorname{Tr}Q^2 - \frac{b}{3}\operatorname{Tr}Q^3 + \frac{c}{4}(\operatorname{Tr}Q^2)^2. \tag{1.1.4}$$

Here T is the absolute temperature, T^* is a critical temperature, depending on the material. $A, b, c > 0$ are material constants. $L > 0$ is a constant approximately depending only on the material [109]. We will work at fixed temperature, so that $a := A(T^*-T), b, c$ can be considered constant. When $T > T^*$, the system behave like an isotropic fluid, while $T < T^*$ is called the *nematic regime*. In this case, $a > 0$.

Since we aim to do a variational theory, a natural class of functions on which defining the functional E_{LdG} is $W^{1,2}(\Omega, \mathcal{S}_0)$; as we assume Dirichlet boundary condition, we further suppose that $Q = Q_b$ on $\partial\Omega$ in the trace sense, with $Q_b \in C^\infty(\partial\Omega, \mathcal{S}_0)$. We set

$$F(Q) := -\frac{a}{2}\operatorname{Tr}Q^2 - \frac{b}{3}\operatorname{Tr}Q^3 + \frac{c}{4}(\operatorname{Tr}Q^2)^2 \tag{1.1.5}$$

Actually, F is well-defined on matrices and one can show that it is bounded below [102] on $\mathcal{S}_0$. Since F is bounded below, the functional

$$\tilde{F}(Q) := F(Q) - \inf_{Q} F(Q) \tag{1.1.6}$$

is nonnegative. The infimum of F is achieved on the subclass of matrices in $\mathcal{S}_0$ of the special form

$$M = s_+\left(n \otimes n - \frac{1}{3}I\right), \quad n \in S^2, \tag{1.1.7}$$

where I is the identity 3×3-matrix and s_+ a positive constant given by [106]

$$s_+ = \frac{b + \sqrt{b^2 + 24ac}}{4c} \tag{1.1.8}$$

Notice that on simply connected domains, thanks to a lifting theorem of Ball&Zarnescu [9], the set of minimizers of F can be written as the following manifold:

$$\mathcal{Q}_{\min} := \left\{Q \in \mathcal{S}_0 : Q = s_+\left(n \otimes n - \frac{1}{3}I\right),\, n \in W^{1,2}(\Omega, S^2)\right\} \simeq \mathbb{R}P^2. \tag{1.1.9}$$

The infimum of $F(Q)$ will be achievied on the classes of maps that will be of our interest [102, 106]; this will allow us to subtract the minimum of $F(Q)$ in (1.1.6).

Notation. With a slight abuse of notation, from now on we will write $F(Q)$ meaning $\tilde{F}(Q)$. Furthermore, we will drop the subscript "LdG" and write simply $E(Q;\Omega)$ for the energy functional.

For a long time, it was widely believed that equilibrium configurations of nematics should be uniaxial. In 2004, experiments revealed that they could also be biaxial [101]. The possibility of having biaxial minimizers was already seriously considered since long time before, both in theoretical [100, 120] and in numerical studies [133, 84, 85, 47]. In particular, the numerical simulations [84, 85, 47] shows that in the regime called *deep nematic phase* the energy minimizers subject to certain physically-significant boundary conditions are particular configurations, called *biaxial torus solutions*. The deep nematic phase may be characterized in terms of the material costants as the situation in which the *reduced temperature*[1]

$$t := \frac{27ac}{b^2} = \frac{27A(T^* - T)c}{b^2} \tag{1.1.10}$$

is much greater than 1. More naively, it is often said that "b is much smaller than both a and c" or that the temperature is "sufficiently lower" than T^*.

In this work, we follow two main threads. The first is producing some rigorous arguments in the direction of minimality biaxial torus solutions both in similar situations to those studied in numerical simulations and with different boundary data. Recent interest in this kind of problems is witnessed by [5, 142, 143]. The second thread is studying the behavior of minimizers when the boundary data are suitably rearranged.

In order to do this, as a preliminary step we have to clarify what we mean by "biaxial torus solution". Since apparently there are no codified definitions of this concept in the mathematical literature, we extracted the features we judged essential from the phenomenological picture, see Section 2.5.7 for an account and Section 2.5.8 for some discussion. Before giving our definition of biaxial torus solution, we find it convenient to redefine the biaxiality parameter in order it can distinguish between *positive uniaxiality* (the two lowest eigenvalues are identical) and *negative uniaxiality* (the two highest eigenvalues are identical). This is done by setting

$$\tilde{\beta}(Q(x)) = \sqrt{6}\frac{\mathrm{Tr}(Q^3)}{(\mathrm{Tr}(Q^2))^{\frac{3}{2}}}. \tag{1.1.11}$$

Since $\mathrm{Tr}(Q^3) = 3\lambda_1\lambda_2\lambda_3$ because of the tracelessness constraint, we have $-1 \leq \tilde{\beta} \leq +1$ and $\tilde{\beta}(Q(x)) = -1$ if and only if $\lambda_2(x) = \lambda_3(x)$, $\tilde{\beta}(Q(x)) = +1$ if and only if $\lambda_1(x) = \lambda_2(x)$ and $\tilde{\beta}(Q(x)) = 0$ if and only if Q is maximally biaxial at $x \in \Omega$.

Next, we establish the following definition of *linking compact sets.*

Definition 1.1 (Linking compact sets)**.** Let $\Omega \subset \mathbb{R}^n$ be a set and let $K_1, K_2 \subset \Omega$ be compact sets. We say that K_1, K_2 are *linking* if we have both that K_1 is not contractible in $\Omega \setminus K_2$ and K_2 is not contractible in $\Omega \setminus K_1$.

Now we can state

Definition 1.2 (Biaxial torus solution)**.** Let Q be a smooth critical point of the LdG energy functional $E(\cdot;\Omega)$, defined as in (1.1.3), on some admissible class of functions. We call Q a *biaxial torus solution in* Ω if Q has the following properties:

(a) $Q \neq 0$ everywhere;

(b) there exist linking compact sets $\mathcal{U}_+, \mathcal{U}_- \subset \overline{\Omega}$ so that $\tilde{\beta}(Q) \equiv 1$ in $\mathcal{U}_+$, $\tilde{\beta}(Q) \equiv -1$ in $\mathcal{U}_-$.

[1]The are various ways of defining a quantity playing the same rôle; we take the one of [105].

To comply our program, in this work we study three kind of minimum problems for the functional $E(Q; B_1)$. **We will always work deep in the nematic phase**. In this situation, both physically-grounded arguments (due to Lyuksyutov [100]) and numerical studies [84, 85] (and, to some extent, rigorous arguments [105]) suggest that the norm of the Q-tensor order parameter may be considered constant inside Ω. Thus, in particular, Q cannot melt inside Ω and reacts to strong deformations through exchanges in the eigenvalues and entering biaxial states [84, 85]. **We shall always assume the validity of this norm constraint**, customary called the *Lyuksyutov constraint*, and further we rescale Q-tensors so that

$$|Q|^2 = Q_{ij}Q_{ij} = 1. \tag{1.1.12}$$

Thus, Q takes values in the unit sphere S^4 into $\mathcal{S}_0$. In each case under studying, we set

$$\Omega = B_1,$$

as in most numerical simulations [85, 47, 32, 75], where B_1 denotes the open unit ball in $\mathbb{R}^3$ centered at the origin. Taking the map

$$Q_b(x) = \sqrt{\frac{3}{2}}\left(\frac{x}{|x|} \otimes \frac{x}{|x|} - \frac{1}{3}I\right) \in C^\infty(S^2; S^4) \tag{1.1.13}$$

as (physically-significant) boundary condition and defining the admissible class

$$\mathcal{A}_{Q_b} = \left\{Q \in W^{1,2}(B_1; S^4) : Q = Q_b \text{ on } S^2 \text{ in the trace sense}\right\}, \tag{1.1.14}$$

we first study, in Chapter 4, the regularity of minimizers of the LdG energy functional $E(\cdot; B_1)$ in the class $\mathcal{A}_{Q_b}$. Although the analysis here does not present particular elements of novelty, details appear to be missing in the literature; moreover, the ε-regularity theorem (Theorem 4.6) and higher regularity theorems (Section 4.5) proven there are actually valid for any minimizer appearing in this work. The main outcome here is the following complete regularity theorem.

Theorem 1. *Let $Q \in \mathcal{A}_{Q_b}$ be a minimizer of the LdG energy* (1.1.3) *in the class* (1.1.14)*, with Q_b as in* (1.1.13)*. Then Q is real-analytic in $\overline{B_1}$.*

The second problem we shall study is the regularity of minimizers of the LdG energy (1.1.3) in the class

$$\mathcal{A}^{\mathrm{ax}}_{Q_b} = \mathcal{A}_{Q_b} \cap \{Q \in \mathcal{A}_{Q_b} : Q \text{ is } S^1\text{-equivariant}\}, \tag{1.1.15}$$

with Q_b as in (1.1.13). By S^1-equivariance, we mean that the Q-tensor satisfies

$$\boxed{Q(\mathcal{R}x) = \mathcal{R}Q(x)\mathcal{R}^t, \quad \text{for a.e. } x \in B_1,} \tag{1.1.16}$$

for any rotation $\mathcal{R}$ about some fixed axis, which we always identify with the z-axis of a cartesian coordinate system centered at the origin.

Our motivation for approaching this problem is that we have a simple topological argument (Theorem 5.1) proving that smooth S^1-equivariant minimizers of $E(\cdot; B_1)$ in the class (1.1.15) with Q_b as in (1.1.13) are biaxial torus solutions in B_1, in the sense of Definition 1.2. Our main result here is the following dichotomy.

Theorem 2. *Let $Q \in \mathcal{A}^{ax}_{Q_b}$ be a minimizer of the LdG energy* (1.1.3) *in the class* (1.1.15)*, with Q_b as in* (1.1.13)*. Then there exists a $\delta > 0$ so that Q is smooth in a δ-neighborhood of S^2 and, in the interior, either Q has a finite number of singularities of dipole kind or Q is a biaxial torus solution in B_1, in the sense of Definition 1.2.*

Here the term *dipole* refers to a specific structure of the singularities, which in particular have to come out in pairs; we use it in analogy to [21], as it will become clear at the end of the analysis in Chapter 7. Even if the theorem cannot exclude singularities, contrary to our plan, it allows only for a peculiar kind of singular solution, bearing a remarkable resemblance with the metastable *split core solutions* found in [47] (see also Section 2.5.7 for a comparison to biaxial torus solutions). In analogy to [47], we call the singular S^1-equivariant minimizers allowed by the theorem *split solutions* and we extend this name also to *any* S^1-equivariant minimizer of the LdG energy having a finite number of dipoles.

The last problem we tackle, in Chapter 8, is the examination of the behavior of minimizers when the boundary data are arranged in suitable ways. We show that there exist smooth boundary data whose minimizers are biaxial torus solutions (Theorem 8.2), smooth boundary data whose minimizers necessarly have singularities (Theorem 8.3) and also smooth boundary data whose minimizers are biaxial torus solutions in a subregion of B_1 but have singularities outside (Theorem 8.4). We summarize these results in the following

Theorem 3. *Let $E(\cdot; B_1)$ the LdG energy functional defined in* (1.1.3) *over the class $\mathcal{A}^{ax}_{Q_b}$ defined in* (1.1.15)*. Then*

(i) There exist boundary data Q_b such that the corresponding minimizers are biaxial torus solutions in B_1.

(ii) There exist boundary data Q_b such that the corresponding minimizers have singularities in B_1.

(iii) There exist boundary data Q_b such that the corresponding minimizers have singularities in B_1 but they are biaxial torus solutions in $\Omega \subset\subset B_1$.

We also complement the analysis by extending some well-known theorems in harmonic maps [2, 65], such as generic uniqueness of minimizers, uniform distance between singular points, convergence of singularities to singularities and we discuss the possibility of having smooth boundary data with at least two minimizers of different character: one a biaxial torus solution in B_1 and the other a split solution.

Our approach to regularity will be in the spirit of geometric measure theory, more specifically, of very classic works by Schoen & Uhlenbeck [130, 131, 132] and of more recent works with similar structure of the Euler-Lagrange equations associated to the energy functional, such as [119]. Thus, the main issue in the quest for regularity will be ruling out all nonconstant minimizing tangent maps. In this direction, the symmetry constraint is a major cause of troubles, since it prevent us from applying well-known results in the literature and, especially, the Liouville theorem of Schoen&Uhlenbeck [132, Theorem 2.7], which is by far the most delicate point. We partly remedy this lack by classifying, in Chapter 6, all possible S^1-equivariant tangent maps. Next, we exploit the form of S^1-action to identify, for most tangent maps, directions along which push them to lower their energy. Anyway, in the S^1-equivariant case there are nonconstant minimizing tangent maps, Theorem 7.6. This circumstance yields both the obstruction in proving full regularity under the boundary condition (1.1.13) and the interesting phenomenon of multiple minimizers mentioned above.

This dissertation is organized as follows. Chapter 2 and Chapter 3 contain review material on nematic liquid crystals and on harmonic maps respectively. In this work, harmonic maps appear as asymptotic objects in the blow-up analysis (i.e., as tangent maps), thus the discussion of this huge topic is geared exclusively towards relevant aspects to our needs. Chapter 4 deals with the minimization problem in the class (1.1.14). Chapters 5, 7 tackle the minimization problem in the class (1.1.15). The classification of the S^1-equivariant harmonic maps in Chapter 6 is a necessary step in the analysis in Chapter 7 but it is also of strong independent interest. In Chapter 8 we study the behavior of minimizers for different boundary data.

The results in this Ph.D. thesis will constitute the core of publications in preparation with V. Millot (Paris Diderot) and A. Pisante (Sapienza – Univerità di Roma). Some problems are still open, such as (for instance) proving that split solutions are not minimizing w.r.t. the hedgehog boundary condition and the question whether gap happens or not in this setting (in analogy to [63]). These are part of a line of research, involving also the uniqueness issue, worth to investigate, possibly in the near future.

Notations. We will use quite standard notations. $B^n_R(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < R\}$ denotes the n-ball of radius R centered at x_0. For balls centered at the origin, we drop the specification of the center. The unit ball in $\mathbb{R}^3$ centered at the origin will be denoted B_1. $S^d_R(x_0)$ denotes the d-sphere of radius R with center at x_0; we drop both x_0 and the subscript 1 for unit spheres and, as for balls, the specification of the center when $x_0 = 0$.

Vectors in $\mathbb{R}^n$ will usually be written in small Latin letters. Sometimes, vectors will be emphasized writing them in bold letters; anyway, this will rarely happen and mostly when it is useful to distinguish them from other kind of quantities or they stands for quantities universally written in bold capital letters (as for the magnetic field $\mathbf{H}$). Scalar product of vectors are usually denoted with a dot and the euclidean norm with $|\cdot|$.

Matrices will usually be written in capital Latin letters. The symbol $\operatorname{Tr} M$ denotes the trace of the matrix M. M^t is the transpose of M. Scalar products of matrices are usually denoted with brackets: $\langle A, B\rangle = \operatorname{Tr}(A^t B) = \sum_{i,j} A_{ji}B_{ij}$. $|A| = \sqrt{\operatorname{Tr}(A^t A)}$ indicates the Hilbert-Schmidt norm of A (no confusion will arise with the euclidean norm).

As a general rule, summation convention on repeated indexes will be understood, but sometimes we will explicitly indicate the sum.

The symbol $W^{1,2}_\varphi(M, N)$ denotes the space of $W^{1,2}$-functions $u : M \to N$ that agree φ on ∂M, in the trace sense. We will also write

$$u|_{\partial M} = \varphi \quad \text{or } \operatorname{tr}(u) = \varphi$$

to mean the same thing. Here the only trace spaces occurring are that of traces of maps in $W^{1,2}(M, N)$; for brevity, they are denoted $H^{\frac{1}{2}}(M, N)$ rather than $W^{\frac{1}{2},2}(M, N)$.

The gradient on open domains in $\mathbb{R}^n$ is denoted ∇. ∇_T denotes the tangential gradient on a surface, almost always a sphere. The subscript T will be dropped when no confusion can arise.

Often, following a common convention, area and volume elements will be dropped in integrals appearing in fairly long formulae.

Most often, the material constant L will not play any significant rôle and thus will be tacitly set to 1 for convenience.

Chapter 2

Liquid crystals: main theories for nematics

Synopsis. In this chapter we give a quick introduction, by no means intended to be exhaustive, to the main continuum theories for nematic liquid crystals (or, shortly, *nematics*). These are the Oseen-Frank theory (Section 2.3), the Leslie-Ericksen theory (Section 2.4) and the Landau-de Gennes theory (Section 2.5). The main difference between them is the definition of the order parameter, the mathematical tool encoding the physical concept of orientational ordering. This difference is highlighted in Section 2.2 while the relation between the above-mentioned theories is a key field of research, briefly exposed in Section 2.5. The focus is however on Landau-de Gennes theory and, more specifically, on those topic directly connected to our problem. In particular, we discuss two asymptotic limits: the *large body limit* (§ 2.5.4) and the *Lyuksyutov limit* (also termed *low-temperature limit*) (§ 2.5.4); we give an account of the phenomenology of biaxial torus solutions as evinced from numerical simulations (see Section 2.5.7) and, in light of this account, we comment on our definition of biaxial torus solution (Section 2.5.8).

2.1 Nematic liquid crystals

Some organic materials do not show a single transition from solid to liquid but rather they exhibit a certain number of intermediate phases (technically, *mesomorphic phases*), often called *liquid crystals* phases. Among them, the main ones are the *nematic phase*, the *smectic phase* and the *columnar phase*. Liquid crystals are said to be *thermotropic* when the main physical parameter governing phase transitions is temperature and *lyotropic* if such parameter is instead the concentration of the liquid crystals molecules into a solution. Here we shall concerned exclusively with thermotropic nematics, which are simplest to study and yet sources of very challenging mathematical problems; for information about all other types we refer to the classic textbook [52].

Nematic liquid crystals are anisotropic fluids whose constituent molecules typically have elongated shape; the molecules are free to translate (i.e., they flow like in liquids) but their distinghuished axes tend to align, in average, along a common preferred direction, labelled by a unit vector $\mathbf{n}$, usually called the *director*. The direction $\mathbf{n}$ is arbitrary in space [52, § 1.3.1] and $\mathbf{n}$ and $-\mathbf{n}$ are indistinguishable; as a consequence, nematic liquid crystals are not ferroelectric even if they carry a permanent electric dipole and diamagnetic [52]. Nematic phases occurs only with achiral materials.

Nematic liquid crystals are said *uniaxial* if $\mathbf{n}$ is an axis of complete rotational symmetry of the system and *biaxial* otherwise [113]. Biaxiality in thermotropic nematics has been observed only relatively recently [101]. Rigid rods are the simplest type of objects giving rise to nematic behavior and this is the way we will think of nematic molecules in the sequel.

Nematics are characterized by an extreme responsivity to external inputs. This makes them suitable for many technological applications, among which the most known is certainly in the industry of digital displays; a perhaps surprising list of various other employments, with references to the literature, can be found in [118]. Another consequence is that nematics may be observed making recourse to a number of different techniques, in particular microscopy and NMR (Nuclear Magnetic Resonance).

Liquid crystals may be modelled at various levels of detail; in principle, one would aim at deriving their macroscopic properties starting from molecular processes but such a detailed description appear to be unviable up to now [6, 7]. Continuum model are instead much more suitable to describe macroscopic configurations of nematics. In continuum models, the main quantity of interest is the *order parameter*, which provides a measurament of orientational order. There are various continuum theories available for nematics, each with a different definition of the order parameter (and, consequently, of the function space where the theory takes place). Continuum theories may be classified into *mean-field* approaches and *phenomenological* approaches [8]. The most successful continuum theory is the phenomenological mesoscopic *Landau-de Gennes theory*, the development of which was a major reason[1] to award P.J. de Gennes with the Nobel prize in physics in 1991. The order parameter in the Landau-de Gennes theory is a second-order tensor, called the *Q-tensor order parameter*. In the next sections, we will briefly introduce also other notable continuum theories, such as the *Oseen-Frank* theory, the *Leslie-Ericksen* theory (which can be viewed as particular cases of Landau-de Gennes theory) and we will spend some words about the relation between mean field approach and the phenomenological approach to the Q-tensor order parameter.

A striking feature of liquid crystals is that they show a vast variety of *defects*. Accordingly to [6], by a defect we mean a point, curve or surface in the neighborhood of which the order parameter varies very rapidly (depending on the theory on use, this abrupt change may or may not result in a mathematical singularity of the order parameter). Defects are observed optically and often forms impressively suggestive patterns. An example of such patterns (a so-called *Schlieren texture*) is shown in Figure 2.1.

The Landau-de Gennes theory can account for all types of defects observed in real nematics [52]. In general, the analysis of defects in ordered media is one of most active fields of physics [84] and liquid crystals stand out from other ordered media for the multiplicity of possible defects and the chance of controlling their appearance and morphology by choosing suitably the confining geometry and the boundary conditions [84]. Landau-de Gennes theory looks particularly appropriate to investigate the fine structure of defects [84]. A detailed account on defects can be found in [52, Chapter 4] while in Section 2.5.2 some notions directly relevant to this work can be found.

[1] The Nobel Prize in Physics 1991. NobelPrize.org. Nobel Media AB 2018. Sun. 14 Oct 2018. <https://www.nobelprize.org/prizes/physics/1991/summary/>

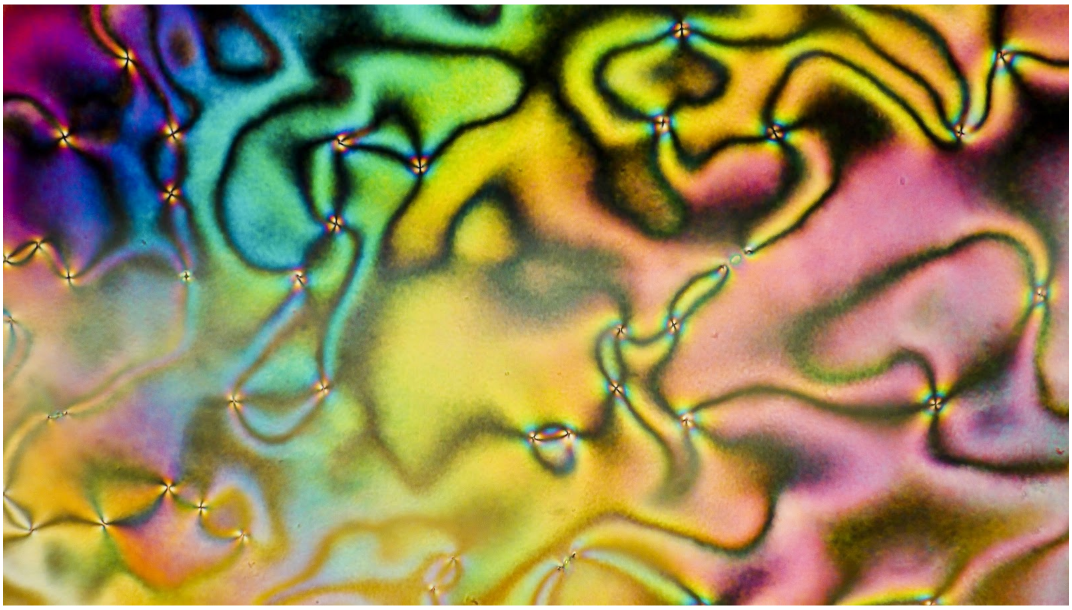


Figure 2.1: Example of Schlieren texture in nematic 5CB. Image downloaded from `http://blitiri.blogspot.com/search/label/schlieren`.

2.2 Order parameters and continuum theories for nematics

Broadly speaking, in condensed matter physics the orientational order of the constituent molecules arises from first principles of entropy and energy[2] and can be usefully quantified, in continuum theories, by the means of well chosen quantities, depending on the problem, generally called *order parameters*.

Remark 2.2.1. We shall not insist more on the relation between molecular models and continuum theories, which is in fact not precisely known at the moment being [6]. As a consequence of this uncertainty, the genesis of order parameters from molecular models is not completely clear.

Remark 2.2.2. In this work, we are not interested in dynamics but only in equilibrium configurations, so that there will be no dependence on the time anywhere. Of course, dynamics is a topic of great interest. Some comments will be given at the end of Section 2.5.2.

The choice of the order parameter is essential, as it forces one to choose an appropriate function space, thus conditioning the whole subsequent theory. We shall deal with three important continuum theories for nematics, the following:

- *Oseen-Frank theory*, whose order parameter is a vector field $n : \Omega \subset \mathbb{R}^3 \to S^2/\{\pm 1\} \simeq \mathbb{R}P^2$, termed the *director*, which assigns, to each point of the domain, the preferred direction of alignment. Note that we take the quotient by $\{\pm 1\}$ because of the statistical head-to-tail symmetry of the molecules.

- *Leslie-Ericksen theory*, whose order parameter is the director n as before coupled with a scalar field $s : \Omega \to \mathbb{R}$, describing the local average degree of orientation.

[2] Cfr. [118].

- *Landau-de Gennes theory*, whose order parameter is a field of 3×3-traceless matrices, also called *Q-tensor order parameter*. Roughly speaking, the eigenvectors of the matrices indicates the preferred directions of alignment and the two independent eigenvalues the degree of alignment.

Note that, the higher the dimensions of the order parameter, the richer the information contained in it. The director n has three degree of freedom and we can think of it as a special case of the couple (s, n), so that Oseen-Frank theory is conceivably a special case of Leslie-Ericksen theory. In the same fashion, the five-dimensional Q-tensors include the four-dimensional Leslie-Ericksen couples as a special case. Indeed, the above inclusions can be made more precise, meaning that all the results known for the Oseen-Frank theory can be obtained by means of the Leslie-Ericksen theory and all those of the Leslie-Ericksen theory can be recovered by the Landau-de Gennes theory, at least from a physical point of view. In fact, there is a general consensus on that the Landau-de Gennes theory is the most effective continuum theory for nematics.

2.3 Oseen-Frank theory

The *Oseen-Frank theory* is the oldest and the simplest continuum theory for nematics still of use today. As we already said, the order parameter of the Oseen-Frank theory is a line field $n : \Omega \subset \mathbb{R}^3 \to \mathbb{R}P^2$, where Ω represent the region of space in which the sample is enclosed, which we will suppose to be simply-connected with Lipschitz boundary. Defects are identified in this theory with the (mathematical) singularities of n. The Oseen-Frank free energy is customary assumed to be[3] (see, for instance, [60])

$$\begin{aligned} W(\nabla n, n) :=& \frac{1}{2}\Big\{K_1(\operatorname{div} n)^2 + K_2(n \cdot \operatorname{curl} n)^2 K_3 \left|n \times (\operatorname{curl} n)\right|^2 \\ &+(K_2 + K_4)\Big[\operatorname{Tr}(\nabla n)^2 - (\operatorname{div} n)^2\Big]\Big\}, \end{aligned} \tag{2.3.1}$$

where K_1, K_2, K_3, K_4 are generally assumed to satisfy [60] the *Ericksen inequalities*

$$K_1 > 0, \quad K_2 > 0, \quad K_3 > 0, \quad K_2 > |K_4|, \quad 2K_1 > K_2 + K_4. \tag{2.3.2}$$

Fixed $n_0 : \partial\Omega \to S^2$ a Lipschitz function, the equilibrium configurations are the solutions to the problem

$$\inf_{u \in \mathcal{A}^{\mathrm{OF}}_{n_0}} \int_\Omega W(\nabla n, n), \tag{2.3.3}$$

where

$$\mathcal{A}^{\mathrm{OF}}_{n_0} := \Big\{u \in W^{1,2}(\Omega, S^2) : u = n_0 \text{ on } \partial\Omega\Big\}. \tag{2.3.4}$$

The reason for asking n_0 Lipschitz is that, in this case, $\mathcal{A}^{\mathrm{OF}}_{n_0}$ is always nonempty [60, Lemma 1.1].

[3]This form for the energy density was firstly proposed by Frank [43], who get it under the request that W must be a quadratic function of its arguments and the assumptions of *frame indifference*: $W(\nabla n, n) = W(\mathcal{R}\nabla n \mathcal{R}^t, \mathcal{R}n)$ for any rotation $\mathcal{R}$ in $\mathbb{R}^3$ and statistical head-to-tail symmetry: $W(\nabla n, n) = W(-\nabla n, -n)$.

Remark 2.3.1. Although from the mathematical point of view it is always legitimate to consider Dirichlet boundary conditions, one may wonder whether this imposition is really physical significant. The answer is in the affirmative, because there are various physical and chemical treatments of the walls of the container that permit to essentially prescribe values of n on $\partial\Omega$ (see [52, Section 3.1.4] for a justification).

Note that the last term in $W(\nabla n, n)$ is formally a divergence:

$$\mathrm{Tr}(\nabla n)^2 - (\operatorname{div} n)^2 = \operatorname{div}\Big((\nabla n)^2 n - (\operatorname{div} n)n\Big). \tag{2.3.5}$$

This is the reason why it is very common the assumption $K_2 = -K_4$ in the physical literature. From a rigorous point of view, it can be proven [60, Lemma 1.2] that such a term is a number depending only on n_0, thus neglecting it in $W(\nabla n, n)$ or changing the value of K_4 will not affect equilibria. Indeed, we shall assume $K_4 = 0$ below.

Remark 2.3.2. Although we shall not consider other liquid crystals than nematics, let us remark that a similar treatment can be done for cholesterics [60] (and references therein).

In [60, Theorem 1.5] it is demonstrated that the minimum problem (2.3.3) has always a solution in the class (2.3.4).

The main concern is now the regularity of equilibrium configurations. To start with, let us note that in the *equal elastic constant case* $K := K_1 = K_2 = K_3$, $K_4 = 0$ the energy of n reduces to

$$E(n;\Omega) = K\int_\Omega |\nabla n|^2 \,\mathrm{d}x, \tag{2.3.6}$$

so that its critical points are harmonic maps from Ω into S^2. For such maps, the regularity theory has been developed by Schoen & Uhlenbeck in [130, 131, 132]. Recalling fundamental results in [21], we can state the following

Theorem 2.1. *Suppose that $K_1 = K_2 = K_3$, $K_4 = 0$, in the energy density $W(\nabla n, n)$ in* (2.3.1)*. Let $n_0 : \partial\Omega \to S^2$ be a Lipschitz function and let $n \in W^{1,2}(\Omega, S^2)$ be a minimizer of the energy in the class $\mathcal{A}^{OF}_{n_0}$. Then the singular set* $\operatorname{sing} n$ *of n is discrete and n is real-analytic in $\Omega \setminus \operatorname{sing} n$. Moreover, in the vicinity of each singularity $a \in \operatorname{sing} n$, n behaves like $\mathcal{R}_a\left(\frac{x-a}{|x-a|}\right)$, where $\mathcal{R}_a$ is a rotation in $\mathbb{R}^3$.*

Remark 2.3.3. The regularity at the boundary depends on n_0 ad on the regularity of the boundary. For sufficiently regular n_0 and boundaries (i.e., $C^{2,\alpha}$ for some $\alpha \in (0,1)$), the minimizers inherit the same regularity in the vicinity of the boundary. For less regular boundary and boundary data, the situation is slightly more complicated and the interested reader can consult [60, Section 6] for more information.

A similar statement [60, Theorem 2.6] holds also in the general case of elastic constant satisfying (2.3.2). In the subsequent sections of the quoted paper, various generalizations are taken into account (including also the presence of electric and magnetic fields).

Although we are reporting here only those results that are directly relevant to our purposes, we have to remark that there is a plently of rigorous statements for the Oseen-Frank theory, and the reason for this is twofold. The first, as we already mentioned above, is the reduction (essentially) to the harmonic maps problem. The other is the resemblance with the well-studied Ginzburg-Landau theory of superconductivity (see, for instance, [13]), which was a rich source of analogies and inspiration for rigorous developments in the Oseen-Frank theory.

Remark 2.3.4. In particular, one may wonder whether some of the nonuniqueness results for the Dirichlet problem for harmonic maps carry over to the Dirichlet problem for the Euler-Lagrange equations associated to the Oseen-Frank energy. Hong [71] has shown that this is indeed the case under suitable hypotheses on the elastic constants K_i (such restrictions are consistent in a significant range of elastic constants with Ericksen inequalities).

To conclude this short review of the Oseen-Frank theory, we mention some results on the axially symmetric case, which are essentially due to Hardt, Kinderlehrer & Lin [61].

Let $(r, \phi, z) \in \mathbb{R}^+ \times [0, 2\pi] \times \mathbb{R}$ be the standard cylindrical coordinates on $\mathbb{R}^3$. Call $u : \Omega \subset \mathbb{R}^3 \to S^2$ *k-axially symmetric* if there exists a real-valued function $\vartheta = \vartheta(r, z)$, usually called an *angle function* [61, 63], such that

$$u = u_\vartheta = (\cos k\phi \cos\vartheta, \sin k\phi \cos\vartheta, \sin\phi).$$

Note that we have

$$u \circ R_\phi = R_{k\phi} \circ u$$

for any $\phi \in [0, 2\pi]$. This implies that u cannot have concentration points[4] off the z-axis. Indeed, recall that the concentration set of a $W^{1,2}(\Omega, S^2)$ function has always Hausdorff measure strictly smaller than 1 [53, Proposition 9.21]. Thus, if a was a concentration point located off the z-axis, the whole orbit of a under the S^1-action defined before would be of concentration points and this is impossible. In the sequel of this short exposition, let us set $k = 1$.

Now consider the Oseen-Frank energy in the one-elastic constant approximation and set $K = 1$ for convenience. Then the Oseen-Frank energy of u reduces to the Dirichlet energy of u and u is a critical point of this energy iff the angle function ϑ satisfies (in the distribution sense) the Euler-Lagrange equations

$$\frac{\partial}{\partial r}\left(r\frac{\partial\vartheta}{\partial r}\right) + \frac{\partial}{\partial z}\left(r\frac{\partial\vartheta}{\partial z}\right) + \frac{1}{2}r^{-1}\sin 2\vartheta = 0 \tag{2.3.7}$$

in the half-disk

$$D = \left\{(r, z) : 0 \le r^2 \le 1 - z^2\right\}.$$

Also, let B_1 the unit ball in $\mathbb{R}^3$ and observe that B_1 is generated by rotating D of an angle 2π about the z-axis.

Note that (2.3.7) is a system of semilinear elliptic equations, the solutions of which are real-analytic off the z-axis (where the differential operator becomes degenerate) by standard elliptic regularity arguments. In particular, this leads quite easily to a small-energy regularity theorem (see [61, Lemma 4.2]) which in turn yields that a concentration point is a singular point for u (i.e., a point at which u is not continuous) and viceversa. Moreover, in [61] the following partial regularity theorem for minimizers among axially symmetric maps is proven.

Theorem 2.2 ([61, Theorem 4.2])**.** *Suppose $u : B_1 \to S^2$ is energy minimizing among axially symmetric maps. Then u is real-analytic in B_1 away from a set of isolated points on the z-axis. If $u|\partial_{B_1}$ is Lipschitz, then u is Hölder continuous in a neighborhood of ∂B.*

[4] See Eq. (3.3.1).

The technique used by the quoted authors suits the spirit of the Schoen & Uhlenbeck theory, with the modifications needed to account for the symmetry. The authors first note that axially symmetric minimality gives again a monotonicity formula as in [130], then they blow-up the given minimizer around points on the z-axis and, by constructing suitable comparison maps exploiting a clever slicing of the ball, they show that the blown-up maps converge strongly (on a subsequence) in $W^{1,2}_{\text{loc}}$ to a minimizing tangent map. The strong convergence also allows to use the Federer's reduction principle [53, Theorem 10.18] to conclude that the singular set of u must consist of isolated point. They were also able to classify all possible tangent maps which, up to a sign, are of the form [61, Lemma 4.3]

$$\Lambda^{+}(x) = \frac{(x_1, x_2, x_3)}{|x|} \quad \text{or} \quad \Lambda^{-}(x) = \frac{(x_1, x_2, -x_3)}{|x|}.$$

Reference to [61] will also be done in the sequel, especially in the proof of the strong compactness theorem, Theorem 5.13.

Observe that all the regularity theorems above exclude, in particular, a line of singularities for the director. We already pointed out that a defect may or may not correspond to a singularity of the order parameter depending on the order parameter itself. In the case of the Oseen-Frank theory, a defect correspond to a singularity (i.e., a discontinuity) of the director. The previous theorems imply that the Oseen-Frank theory can account only for point defects. This feature is for sure the main drawback of the Oseen-Frank theory but such a problem is solved by the Leslie-Ericksen and Landau-de Gennes theories.

2.4 Leslie-Ericksen theory

The Leslie-Ericksen theory, elaborated by Ericksen [37] in 1991, reminescent of older works by Leslie [89], dating back to 1968, and Fan [41], 1971, overcomes the main trouble of the Oseen-Frank theory and can accomodate all types of defects actually observed in experiments until that time [139].

The main idea of Ericksen consists in coupling the director[5] n with a variable scalar order parameter $s = s(x)$, $x \in \Omega$. Defects are then defined within this theory as the sets

$$\mathcal{D}_{\text{LE}} = \{x \in \Omega : s(x) = 0\}.$$

In other words, defects are interpreted in Leslie-Ericksen theory as points, lines or surfaces at which the liquid crystal melts, performing a transition from a uniaxial state to the isotropic state. The singularities of the director n are cured by letting $s \to 0$ at the defect site, in such a way that melting happens.

In the simplest setting, the free energy in the Leslie-Eriscksen theory can be written[6]

$$E_{\text{LE}}(s, n; \Omega) = K_E \int_{\Omega} (K |\nabla s|^2 + s^2 |\nabla n|^2) + W_0(s), \tag{2.4.1}$$

where $W_0(s)$ is a potential satisfying [37, § 5], [92]

(i) $\lim_{s \to 1} W_0(s) = \lim_{s \to -1/2} W_0(s) = +\infty$;

[5] The original idea of Leslie was allowing the director having arbitrary norm [139].

[6] The wonderful derivation of Ericksen of the complete free energy can be found in [37] and [139]. The form we are reporting for the energy has been established by Lin [92].

(ii) $W_0(0) > W_0(s^*) := \min_{s\in[-1/2,1]} W_0(s)$, for some $s^* \in (0,1)$;

(iii) ${W_0}'(0) = 0$.

Rigorous results have been derived mostly by Lin [92] and Hardt [58]. The important novelty, with respect to the Oseen-Frank theory, is that now minimizers are locally Hölder continuous in the interior and smooth off $\mathcal{D}_{\mathrm{LE}}$, as one may expect on the basis of the ansatz on how interpreting defects. The regularity at the boundary depends on the regularity of the boundary and on that of the datum. Details can be found in [92, Theorem 6.1]. Some improvements to these results were given by Hardt & Lin [62]. Generalizations to less restrictive forms of the energy can be found in [92] and in [94].

2.5 Landau-de Gennes theory

2.5.1 The Q-tensor order parameter

Before entering into some details about Landau-de Gennes theory, let us say some words about the Q-tensor order parameter.

There are two main approaches to define the Q-tensor: the first is phenomenological and the second is a mean-field approach.

Let us start with the phenomenological method. Within this, there are several different ways of defining the Q-tensor in terms of quantities measurable by a macroscopic observer [52]. Some are based on *static response functions* and others on *dynamical response functions*. It is a common convention [52] to take the *magnetic susceptibility* χ as static response function and then define Q as its anisotropic part ([52, Eq. 2.32]):

$$Q = G\left(\chi - \frac{1}{3}\operatorname{Tr}\chi\right), \tag{2.5.1}$$

where G is a normalization constant. Thus, Q is real, symmetric and traceless and can be represented by a real, symmetric and traceless 3×3 matrix. Recall that the tensor χ relates the magnetic moment per unit volume $\mathbf{M}$ (due to the diamagnetism of nematic molecules) and the magnetic field $\mathbf{H}$ through

$$M_i = \chi_{ij} H_j.$$

The magnetic field $\mathbf{H}$ is under the control of the experimenter; we then will suppose $\mathbf{H}$ is static, so that χ is symmetric.

As a dynamical response function, it is usually considered the dynamical dielectric tensor $\varepsilon(\omega)$ at some standard frequency ω. This has the advantage of being directly related to refractive indices, which can be accurately obtained [52]. The advantage of the approach through static response functions is instead that the relation between the macroscopic quantity χ and relevant microscopic quantities (known as *ordering matrices*, see [52, §2.1.1.3]) is much better understood, at least when the molecules can be considered as rigid rods [52]; nonetheless, the relation between Q and microscopic quantities is more involved, cfr. [52, §2.1.3] for a detailed discussion.

In the mean-field framework, the state of alignment of the nematic molecules is described by a probability distribution function ϱ on the unit sphere and Q is defined in terms of the second moment of ϱ. The passage from microscopic to macroscopic quantities is made by means of a *coarse graining* procedure which actually is not perfectly rigorous [6]. The argument goes as follows [6].

Let us consider rod-like molecules stored in a regular (say, C^2) bounded domain $\Omega \subset \mathbb{R}^3$. The main point for the sequel of the construction is that, picked $x \in \Omega$, we can find[7] $\delta > 0$ such that the ball $B(x, \delta)$ contains a statistically significant number of molecules and, nonetheless, $B(x, \delta)$ can be identified, from the macroscopic point of view, with the material point x. Let $N = N(x)$ the number of molecules entirely contained in $B(x, \delta)$. Picking at random molecules from those N, we obtain a probability measure μ_x on the unit sphere S^2 given by

$$\mu_x = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{2} (\delta_{p_i} + \delta_{-p_i}), \tag{2.5.2}$$

where $\pm p_i$ denotes the orientation of the ith molecule. We require that

$$\mu_x(E) = \mu_x(-E) \text{ for all } \mu_x\text{-measurable } E \subset S^2. \tag{2.5.3}$$

Exploiting again the smallness of statistically significant regions on the macroscopic scale, we can consider μ to be a continuously distributed measure $\mathrm{d}\mu(p) = \varrho(p)\mathrm{d}p$, where $\mathrm{d}p$ denotes the surface area element on S^2 and $\varrho \in L^1(S^2)$ satisfies

$$\varrho \geq 0, \quad \int_{S^2} \varrho(p)\,\mathrm{d}p = 1, \quad \varrho(p) = \varrho(-p) \text{ for a.e. } p \in S^2.$$

In particular, if the orientation of molecules is equally distributed, we say that μ is *isotropic* and we clearly have $\mu = \mu_0$, with

$$\mathrm{d}\mu_0(p) = \frac{1}{4\pi}\mathrm{d}p,$$

i.e., with $\varrho_0 = \frac{1}{4\pi}$.

Now, in principle μ contains all the information about the orientation of the molecules, and it would be natural to take μ as the order parameter. However, μ represent an infinite-dimensional state at each point $x \in \Omega$, so that it is convenient to employ, as order parameter, a finite-dimensional approximation consisting of a finite number of moments of μ. This is what we are going to do. We note that the first moment

$$m_1 = \int_{S^2} p\,\mathrm{d}\mu(p)$$

vanishes identically, because of the head-to-tail symmetry. Next, we note, according to Ball [6, p. 6], that, due to the head-to-tail symmetry, n is better understood as a line field, i.e., as a map $\Omega \to \mathbb{R}P^2$. As precisely proven in [6, pp. 6-7], elements of $\mathbb{R}P^2$ can be identified with matrices $p \otimes p$, with $p \in S^2$. Thus, the second moment of μ can be written

$$M = \int_{S^2} p \otimes p\,\mathrm{d}\mu(p).$$

Observe that M is a symmetric non-negative tensor satisfying $\operatorname{Tr} M = 1$ and that, for $\mu = \mu_0$ we have

[7] See, for instance, [6, p.6]; however, such typical numbers can be found spread all over the literature on nematics.

$$M_0 = \frac{1}{3}I.$$

Since we want to measure the deviation from isotropy, we take as the principal quantity of interest in our study

$$\tilde{Q} := M - M_0 = \int_{S^2} \left(p \otimes p - \frac{1}{3}I \right) \mathrm{d}\mu(p). \tag{2.5.4}$$

In this framework, $\tilde{Q}$ defined in 2.5.4 is termed the *Q-tensor order parameter*. It is immediate from the definition that $\tilde{Q}$ is a symmetric traceless 3×3-matrix on $\mathbb{R}$. Noting that

$$\left(\tilde{Q} + \frac{1}{3}I \right) e \cdot e \geq 0 \text{ for all } e \in S^2,$$

it is justified the common notation $\tilde{Q} \geq -\frac{1}{3}I$. We remark that, if $\mu = \mu_0$, then $\tilde{Q} = 0$ but the converse does not hold: one can have $\tilde{Q} = 0$ even if $\mu \neq \mu_0$ (see [6, p. 9]). This is possible because higher order moments are neglected in this approach. Following the common practice of labeling the eigenvalues λ_i, $i = 1, 2, 3$, of $\tilde{Q}$ in the increasing order, we have

$$\lambda_1 = \lambda_{\min}, \quad \lambda_2 = \lambda_{\text{mid}}, \quad \lambda_3 = \lambda_{\max}.$$

Note that, since $\tilde{Q} \geq -\frac{1}{3}I$, each λ_i satisfies $\lambda_i \geq -\frac{1}{3}$ and, due to the tracelessness constraint, we also have

$$-\frac{1}{3} \leq \lambda_i \leq \frac{2}{3}, \quad i = 1, 2, 3. \tag{2.5.5}$$

Note that there is no hint of the eigenvalue constraints (2.5.5) in (2.5.1). This circumstance marks a significant difference between Q and $\tilde{Q}$ and may have consequences in that (2.5.5) are often view as physicality constraints on nematic states [6, §4.3], [8]. Accepting $\tilde{Q}$ as Q-tensor order parameter may therefore imply ruling out some states, for instance biaxial torus solutions as obtained in many simulations[8] (see §2.5.7). Thus, the question whether the two approaches are in contrast arises naturally.

To investigate this issue, we first need to understand in terms of which measured quantities ϱ is inferred. According to de Gennes & Prost [52, § 2.1.1], *average* molecular orientations are deduced *via* analysis of NMR spectra; microscopic *ordering matrices* $S_{ij}^{\alpha\beta}$ are built this way:

$$S_{ij}^{\alpha\beta} = \frac{1}{2} \langle 3 i_\alpha j_\beta - \delta_{\alpha\beta}\delta_{ij} \rangle .$$

Here, $\alpha, \beta = x, y, z$ are indexes referring to the laboratory frame, $i, j = a, b, c$ are related to an eigenframe of the molecule and $\delta_{\alpha\beta}$, δ_{ij} Kroenecker symbols. The brackets $\langle \cdot \rangle$ represent thermal averages.

The ordering matrices may be connected to the magnetic susceptibility χ *making the hypothesis that the macroscopic response function is simply the sum of individual molecule responses* [52, § 2.1.3.1]. This leads to set

[8] Anyway, this does not mean that biaxial torus solutions are incompatible with (2.5.5): in [32] the eigenvalue constraints are imposed and biaxial torus solutions are found.

$$\chi_{\alpha\beta} - \frac{1}{3}\delta_{\alpha\beta}\chi_{\gamma\gamma} = cA_{ij}S_{ij}^{\alpha\beta},$$

where c denotes the number of molecules per unit volume and A_{ij} the magnetic polarizability tensor of one molecule.

Relating microscopic ordering matrices to dynamical response functions is more difficult, because dynamical response functions depend not only on the angular distribution function but also on a correlation function g for (at least) two molecules, as a function of their relative distance and orientation, on which arbitrary assumptions are done in the literature [52, § 2.1.3.2].

Now, the distribution ϱ should be expressed, in the mean-field approach, in terms of the gran canonical partition function Z, see [8] for details, which in turn should be related to the ordering matrices through appropriate *self-consistency* conditions (see [52, 136]). Because of simplifying assumptions anyway made to link microscopic and macroscopic quantities, it is not really clear (to the writer, at least) whether Q and $\tilde{Q}$ may really be identified. It appears to us that (2.5.1) has simpler connections with experimentally measured quantities and that accepting it as the definition of the Q-tensor may avoid to deal with many subtleties requiring a deep meditation rooted on the physical ground. **We will then stick by the phenomenological definition** (2.5.1). The whole following discussion relies only on the fact that the Q-tensor order parameter can be represented by a symmetric traceless 3×3-matrix, and this is true for both Q and $\tilde{Q}$. For very interesting developments in the mean-field approach, we address the reader to [8], [6] and references therein.

Symmetric traceless 3×3-matrix on $\mathbb{R}$ form a vector space which we indicate $\mathcal{S}_0$ (already defined in (1.1.1)):

$$\mathcal{S}_0 = \{M \in \mathcal{M}_{3\times 3}(\mathbb{R}) : M = M^t \text{ and } \operatorname{Tr} M = 0\}$$

We endow $\mathcal{S}_0$ with the norm $|\cdot|$ induced by the scalar product

$$\langle A, B\rangle := \operatorname{Tr}(A^t B) = \operatorname{Tr}(AB), \quad A, B \in \mathcal{S}_0.$$

Note that $\mathcal{S}_0 \simeq \mathbb{R}^5$ as linear spaces. Thus, Q-tensor order parameter may be seen as a map

$$\mathbb{R}^3 \supset \Omega \ni x \mapsto Q(x) \in \mathcal{S}_0.$$

In order the terms in the energy functional make sense (see Section 2.5.2), Q will be assumed belonging to $W^{1,2}(\Omega; \mathcal{S}_0)$.

Recall that Q describes the *local* state of the liquid crystal at any point $x \in \Omega \subset \mathbb{R}^3$. Such a state is said to be

- *isotropic*, if $Q(x) = 0$;
- *uniaxial*, if $Q(x)$ has exactly two equal eigenvalues;
- *biaxial*, if $Q(x)$ has three distinct eigenvalues.

Often, according to a common convention, we shall include the isotropic case in the uniaxial case for convenience (Q would then be more precisely characterized by saying that Q has two equal eigenvalues). As before, the eigenvalues of Q will be always labeled in the increasing order. A nematic liquid crystal is then said to be (a) *isotropic*, if $Q(x) = 0$ for a.e. $x \in \Omega$; (b) *uniaxial*, if $Q(x)$ is uniaxial for a.e. $x \in \Omega$; (c) *biaxial*, otherwise.

A convenient measure of the biaxiality of Q ($\neq 0$) at a point $x \in \Omega$ is provided by the so-called *biaxiality parameter*, introduced in [81] (see also [47] and [106]):

$$\beta^2(Q(x)) = 1 - 6\frac{\mathrm{Tr}(Q^3(x))^2}{\mathrm{Tr}(Q^2(x))^3}. \tag{2.5.6}$$

Indeed, it can be proven [106], [6]

Proposition 2.3 ([106, Lemma 1, (i)])**.** *Let $Q \in S_0 \setminus \{0\}$. Then $\beta^2(Q) \in [0,1]$ and $\beta(Q) = 0$ iff Q is uniaxial.*

A point $x \in \Omega$ is said to be of *maximal biaxiality* if $\beta^2(Q(x)) = 1$. Note that the traceless condition then implies $\lambda_1(x) = -\lambda_3(x)$, with $\lambda_3 \in (0, 1/3]$, so that $\lambda_2(x) = 0$ if x is a point of maximal biaxiality.

The following proposition, although elementary, is very useful [104]

Proposition 2.4. *Let $Q \in \mathcal{S}_0$. Then, for any $n \in \mathbb{N}$,*

$$\mathrm{Tr}\, Q^n = \sum_{i=1}^{3} \lambda_i^n. \tag{2.5.7}$$

Moreover, an easy calculation yields

$$\mathrm{Tr}\, Q^3 = 3\lambda_1\lambda_2\lambda_3, \tag{2.5.8}$$

thus, if $\beta^2(Q) = 1$, then $\mathrm{Tr}(Q^3) = 0$ (and viceversa, if $Q \neq 0$).

Note that $\beta^2(Q)$ gives no information on the sign of the eigenvalues of Q. Defining

$$\tilde{\beta}(Q(x)) := \sqrt{6}\frac{\mathrm{Tr}\, Q^3(x)}{(\mathrm{Tr}\, Q^2(x))^{3/2}}, \tag{2.5.9}$$

we have $-1 \leq \tilde{\beta(Q)} \leq +1$ and

- $\tilde{\beta}(Q(x)) = -1$ if and only if $Q(x)$ is uniaxial at x with $\lambda_2(x) = \lambda_3(x)$;
- $\tilde{\beta}(Q(x)) = 0$ if and only if $Q(x)$ is maximally biaxial at x;
- $\tilde{\beta}(Q(x)) = +1$ if and only if $Q(x)$ is uniaxial at x with $\lambda_1(x) = \lambda_2(x)$.

The representation formulae for Q-tensors below are proven with the aid of the spectral theorem and the tracelessness condition.

Proposition 2.5 ([106, Proposition 1])**.** *A matrix $Q \in \mathcal{S}_0$ can be represented in the form*

$$Q = s\left(n \otimes n - \frac{1}{3}I\right) + r\left(m \otimes m - \frac{1}{3}I\right), \tag{2.5.10}$$

with n and m unit-length eigenvectors of Q, $n \cdot m = 0$ and

$$0 \leq r \leq \frac{s}{2} \text{ or } \frac{s}{2} \leq r \leq 0. \tag{2.5.11}$$

In particular, a uniaxial $Q \in \mathcal{S}_0$ can be written

$$Q = s\left(n \otimes n - \frac{1}{3}I\right), \tag{2.5.12}$$

with $s \in \mathbb{R} \setminus \{0\}$ and $n \in S^2$ the distinguished unit-length eigenvector of Q.

The scalar parameter s is said *scalar order parameter*. An equivalent characterization of uniaxiality is given by:

Proposition 2.6 ([6, Proposition 1])**.** *A matrix $Q \in \mathcal{S}_0$ is uniaxial with scalar order parameter s iff*

$$\operatorname{Tr}(Q^2) = \frac{2s^2}{3}, \quad \det Q = \frac{2s^3}{27}. \tag{2.5.13}$$

The following Proposition is proved in [106].

Proposition 2.7 ([106, Proposition 14])**.** *Let Q be a real analytic function $Q : \Omega \subset \mathbb{R}^3 \to \mathcal{S}_0$. Then the set where Q is uniaxial or isotropic is either Ω itself or has zero Lebesgue measure.*

Remark 2.5.1. We aware the reader that, in the physical and chemical literature, the terms *uniaxial* and *biaxial* are used both in the sense above, i.e., for arrangements of molecules, and for the molecules themselves. A molecule is said to be *uniaxial* when it has an axis of rotational symmetry and it is said to be *biaxial* when there is no axis of rotational symmetry but there are two axes of reflective symmetry. The picture to have in mind is a rod for uniaxial molecules and a plank for biaxial molecules. We are here interested only in uniaxial molecules and in their uniaxial or biaxial arrangements.

2.5.2 Generalities

The Landau-de Gennes (LdG) theory can handle both uniaxial and biaxial nematics, differently from Oseen-Frank and Leslie-Ericksen theories, which can account only for uniaxiality. The order parameter of the LdG theory is a map $Q \in W^{1,2}(\Omega, \mathcal{S}_0)$, where Ω, as usual, is a smooth bounded domain in $\mathbb{R}^3$, representing the region occupied by the material. The energy functional may be amazingly complicated, because of the rich variety of interactions in which nematics can be involved. In any technological application, at least the following terms should be considered:

- A thermotropic energy, related to the bulk, dictating the preferred state of the liquid crystal when there are no external influences;
- an elastic energy penalizing distortions from the preferred state;
- an electromagnetic term, describing the interaction with external fields and/or the self-interaction due to dielectric and spontaneous polarization effects;
- a surface energy, accounting for the interaction of the nematic molecules with the walls of the container.

A readable account of all these terms can be found in [113] and a standard very deep reference for the LdG theory is, of course, [52]. See also [25] and [140].

We shall deliberately ignore the electromagnetic term, in the sense that in this work we will be concerned only with free liquid crystals (i.e., not subject to external fields). Next, there will be no surface energy in our treatment, since we will suppose the walls coated so that the contribution of the surface energy becomes equivalent to prescribing a Dirichlet boundary condition (also called a *strong/infinite anchoring* in this context in the physical literature [113]).

Thus, we are left with an elastic term and a thermotropic term. The energy functional will then be of the form

$$E_T(Q;\Omega) = \int_\Omega \psi(Q(x), \nabla Q(x), T)\,\mathrm{d}x, \tag{2.5.14}$$

where T denotes temperature and $\psi : \mathcal{S}_0 \times (\mathcal{S}_0)^3 \times \mathbb{R} \to \mathbb{R}$ is the total *free energy density*, whose form we are going to determine. The physical principles helping us to do this are frame-indifference and material symmetry, already exploited for determining the elastic energy in the Oseen-Frank and Leslie-Ericksen theories, that is, the invariance of the material under rotations and translations of the sample. Adopting the passive point of view, we can say that two observers, differing for a rigid transformation, must measure the same free energy density. More precisely, let $x = (x_1, x_2, x_3)$ be the Cartesian coordinates used by the first observer and $z = \bar{x} + \mathcal{R}(x - \bar{x})$, where $\bar{x}$ is a fixed point in Ω and $\mathcal{R}$ a fixed rotation in SO(3); we then require that

$$\psi(Q^*(z), \nabla Q^*(z), T) = \psi(Q(x), \nabla Q(x), T). \tag{2.5.15}$$

A function ψ satisfying (2.5.15) is said *hemitropic*. Both nematics and cholesterics have hemitropic free energies but for nematics (2.5.15) actually holds for any $\mathcal{R} \in \mathrm{O}(3)$. A function ψ satisfying (2.5.15) for any $\mathcal{R} \in \mathrm{O}(3)$ is called *isotropic*.

Thus, we are looking for an isotropic function $\psi(Q, \nabla Q, T)$. We decompose

$$\begin{aligned}\psi(Q, \nabla Q, T) &= \psi(Q, 0, T) + (\psi(Q, \nabla Q, T) - \psi(Q, 0, T)) \\ &= \psi_B(Q, T) + \psi_{\mathrm{El}}(Q, \nabla Q, T).\end{aligned} \tag{2.5.16}$$

The term $\psi_B(Q, T) := \psi(Q, 0, T)$ is the *bulk energy density* while $\psi_{\mathrm{EL}}(Q, \nabla Q, T)$ is the *elastic energy density*. Usually it is assumed that $\psi_{\mathrm{EL}}(Q, \nabla Q, T)$ is quadratic in ∇Q. There are three linearly independent invariant isotropic functions:

$$I_1 = Q_{ij,k}Q_{ij,k}, \quad I_2 = Q_{ij,j}Q_{ik,k}, \quad I_3 = Q_{ik,j}Q_{ij,k}.$$

Further, there are 6 possible linearly independent cubic terms quadratic in ∇Q, and the one usually included in the nematic elastic energy is

$$I_4 = Q_{lk}Q_{ij,l}Q_{ij,k}$$

which is invariant.

The Landau-de Gennes elastic energy is thus a linear combinantions of the terms I_i with coefficients $L_i > 0$. These elastic coefficients depend on the material but are approximately independent of temperature [109]. Anyway, we shall consider, as before and besides this introductory section, a one-constant approximation, so that for us the elastic energy density will reduce to

$$\psi_{\mathrm{El}}(Q, \nabla Q) = \frac{L}{2} Q_{ij,k}Q_{ij,k} \equiv \frac{L}{2} |\nabla Q|^2 . \tag{2.5.17}$$

Now we have to determine the form of the bulk term. We appeal to the following results, whose proofs can be found in [6].

Proposition 2.8 ([6, Proposition 3])**.** *A function $f(Q)$ of a real, symmetric,* $3\times$ *matrix Q is isotropic if and only if $f(Q) = g(\operatorname{Tr} Q, \operatorname{Tr} Q^2, \operatorname{Tr} Q^3)$ for some function g, and if f is a polynomial so is g.*

The above proposition readily yields

Proposition 2.9 ([6, Proposition 4]). *The bulk energy $\psi_B(Q,T)$ satisfies the frame-indifference condition* (2.5.15) *if and only if*

$$\psi_B(Q,T) = g(\operatorname{Tr} Q^2, \operatorname{Tr} Q^3, T) \tag{2.5.18}$$

for some function g. If, for a given temperature T, $\psi_B(Q,T)$ is a polynomial in Q, then $g(\operatorname{Tr} Q^2, \operatorname{Tr} Q^3, T)$ is a polynomial in $\operatorname{Tr} Q^2$, $\operatorname{Tr} Q^3$.

Recall that we work at fixed temperature; for this reason, we will drop the dependence on T henceforth (in particular, the coefficients L_i will be constant for us).

To step further in the determination of ψ_B, we make a couple of remarks. The first observation is that, at high enough temperature, ψ_B should have a unique minimum for $Q = 0$, i.e., in the isotropic state. At lower temperatures, the minima of the thermotropic bulk energy must be uniaxial states, as shown in [100] on the physical ground. The possible biaxial character is then due to the competition between the bulk term and the elastic term. Elastic distortion is typically induced by the constraint of satisfying the assigned boundary condition(s)[9].

Experiments show that bulk minima move in a continuous way with temperature, a typical picture is as in Fig. 2.2. This means, of course, that there are three characteristic temperatures for nematics. In the increasing order, the first, usually denoted T^*, is the temperature at which the isotropic phase loses its stability; the second, T_{NI}, is that at which the energy of the isotropic and nematic states are exactly equal; beyond the third, T^+, the nematic phase disappears.

The second observation is that, for the above behavior to be possible, $\psi_B(Q)$ should behave like a quartic (at least) polynomial. Next, note that, being Q symmetric and traceless, we have $\operatorname{Tr} Q^4 = \frac{1}{2}\left(\operatorname{Tr} Q^2\right)^2$. Taking into account also Proposition 2.9, the easiest thing to do, in order to obtain a bulk energy density function suitable for our purposes, is a Taylor expansion of the putative complete bulk function ψ_B near $Q = 0$, truncated to the fourth order. We then have

$$\psi_B(Q) = \frac{A}{2} \operatorname{Tr} Q^2 + \frac{B}{3} \operatorname{Tr} Q^3 + \frac{C}{4} \left(\operatorname{Tr} Q^2\right)^2. \tag{2.5.19}$$

In the above formula, we neglected the zero-order term since it does not affect the minima. The coefficients A, B, C are determined experimentally. Experiments show that $B < 0$ and $C > 0$ are both (approximately) independent of temperature, while A has a linear dependence on temperature:

$$A = \alpha(T - T^*), \tag{2.5.20}$$

where $\alpha > 0$ is a material constant and $T < T^*$ (thus, $A < 0$). Relabeling

$$a = -\alpha(T - T^*), \quad b = -B, \quad c = C,$$

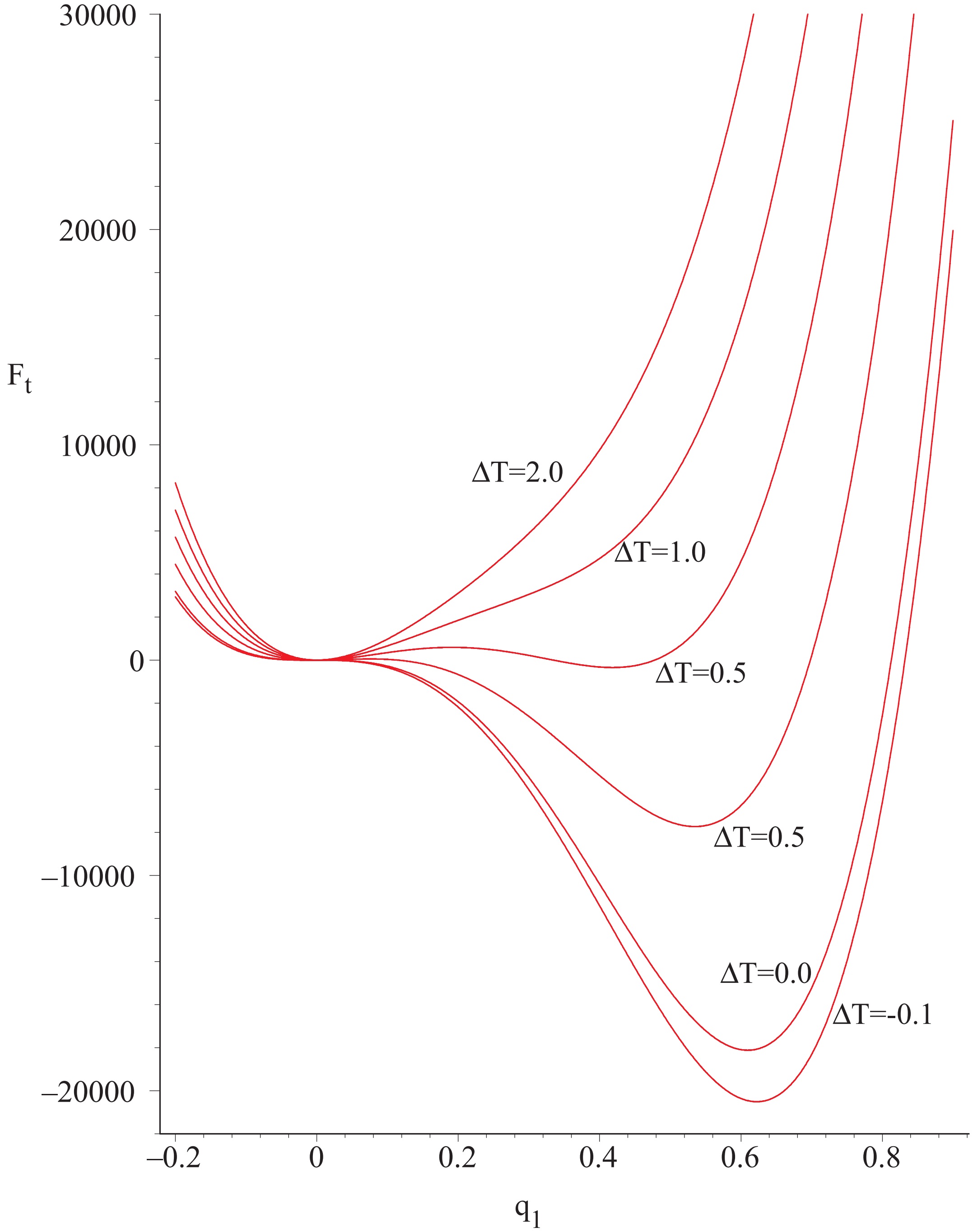
we then have $a, b, c > 0$ and

$$F(Q) := \psi_B(Q) = -\frac{a}{2} \operatorname{Tr} Q^2 - \frac{b}{3} \operatorname{Tr} Q^3 + \frac{c}{4} \left(\operatorname{Tr} Q^2\right)^2,$$

which is exactly the potential energy already introduced in (1.1.5).

[9] Obviously, it may also arise under the influence of external fields, when present.

Figure 2.2: Typical plot of $F(Q)$ vs temperature. Picture taken from the review [112].



Remark 2.5.2. It is worth stressing that, due to our construction of the bulk potential function, the Landau-de Gennes theory can be *a priori* valid only for $Q \approx 0$, that is, only for temperatures close to T_{NI}. Anyway, this turns out to be not an inconvenient: experimental data on heat of transitions (see references quoted in [100]) strongly suggest that the transition nematic-isotropic is of the first kind, nearly of the second, and, furthermore, the temperature range of nematic phase is much smaller than the transition temperature [100]. Therefore, it is believed that Landau-de Gennes theory is a reliable description of a nematic over its whole range of existence [100].

Note that $F(Q)$ has good properties with respect to the physical requests above. Indeed, it can be proved

Proposition 2.10 ([104, Proposition 1])**.** *The stationary points of the bulk energy density $F(Q)$ are given by either uniaxial or isotropic Q-tensors of the form*

$$Q = s\left(n \otimes n - \frac{1}{3}I\right),$$

where $s \in \mathbb{R}$ is a scalar order parameter and n one of eigenvectors of Q.

Moreover, the minimum of $F(Q)$ is attained on the class of uniaxial Q-tensors with constant scalar order parameter [104, 106]. Thus, we can subtract this minimum to $F(Q)$ and then redefine the bulk energy density to be nonnegative. With a slight abuse of notation, we continue to denote $F(Q)$ the nonnegative bulk energy density. Further, for $T > T^*$, the first term in $F(Q)$ is positive so that the unique minimum of $F(Q)$ becomes $Q = 0$.

We now look at the complete free energy

$$E_{\mathrm{LdG}}(Q;\Omega) := \int_{\Omega} \left\{ \sum_{i=1}^{4} L_i I_i + F(Q) \right\} \mathrm{d}x \tag{2.5.21}$$

and we ask about the existence of minimizers. A quite general result has been proven by Davies & Gartland [31] (we quote below the statement in [6]).

Proposition 2.11 ([6, Proposition 11])**.** *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary $\partial\Omega$. For fixed $T > 0$, let $F(Q)$ in* (2.5.21) *be continuous and bounded below on $\mathcal{S}_0$ and assume that the constants L_i satisfy*

$$L_1 > 0, \quad -L_1 < L_3 < 2L_1, \quad L_1 + \frac{5}{3}L_3 + \frac{1}{6}L_3 > 0, \quad L_4 = 0. \tag{2.5.22}$$

Let $Q_b : \partial\Omega \to \mathcal{S}_0$ belong to $H^{\frac{1}{2}}(\Omega, \mathcal{S}_0)$. Then the energy functional $E_{LdG}(Q;\Omega)$ in (2.5.21) *attains a minimum on*

$$\mathcal{A} = \left\{ Q \in W^{1,2}(\Omega, \mathcal{S}_0) : Q|_{\partial\Omega} = Q_b \right\}.$$

The condition $L_4 = 0$ is delicate. Indeed, an extremely desirable feature of the LdG model is the equivalence with Oseen-Frank theory, when both are applicable. To understand the problem caused by $L_4 = 0$, let us consider Q a uniaxial Q-tensor with constant scalar order parameter, i.e., of the form

$$Q = s\left(n \otimes n - \frac{1}{3}I\right), \quad n \in S^2,\ s \in \mathbb{R} \setminus \{0\},$$

and let us calculate formally the elastic energy for Q in terms on n, ∇n. This gives [6, § 6] the Oseen-Frank energy functional (2.3.6) up to an additive constant, enforcing the following relations between the coefficients L_i and the Frank's elastic constants K_i:

$$\begin{aligned} K_1 &= 2L_1 s^2 + L_2 s^2 + L_3 s^2 - \frac{2}{3} L_4 s^3, \\ K_2 &= 2L_1 s^2 - \frac{2}{3} L_4 s^2, \\ K_3 &= 2L_1 s^2 + L_2 s^2 + L_3 s^2 + \frac{4}{3} L_4 s^3, \\ K_4 &= L_3 s^2. \end{aligned}$$

Setting $L_4 = 0$ implies $K_1 = K_3$, which is generally untrue [6, p. 28]. On the other hand, it is a result of Ball & Majumdar (see [6, Theorem 11]) that allowing $L_4 \neq 0$ under the same hypothesis of the Davies & Gartland theorem yields $E_{\mathrm{LdG}}(\cdot;\Omega)$ unbounded below for any boundary condition [8].

There are expedients to overcome this difficulty [6] but we do not enter in greater detail. For us, the important thing is that in the one-constant approximation this problem does not arise and indeed we can prove existence for any boundary condition we will deal with, as we shall see in Chapters 4, 5, 8.

Relation between LdG theory and OF theory. Going back to the relation between the Landau-de Gennes theory and the Oseen-Frank theory, it is readily realized that a major difference between the two theories is the fact that the Q-tensor order parameter is invariant under the transformation $n \mapsto -n$, and so is the Landau-de Gennes theory, while the invariance of the Oseen-Frank theory is *a priori* unclear. Said another way, the issue is whether a line field can be *oriented*, i.e., turned into a vector field by assigning an orientation at each point. It is clear that this can always be done but the problem is that if we can do this in a smooth way.

To settle the question a little more precisely, let us define, for $n \in S^2$ and $s \neq 0$ constant, the set

$$\mathcal{Q} = \left\{ Q \in \mathcal{S}_0 : Q = s\left(n \otimes n - \frac{1}{3} I\right) \right\}.$$

When the target of Q-tensor parameters is restricted to $\mathcal{Q}$, we speak of *constrained Landau-de Gennes theory*. More specifically, our concern will be understand under what conditions the constrained Landau-de Gennes theory and the Oseen-Frank theory are equivalent.

Given $Q \in W^{1,1}(\Omega, \mathcal{Q})$, we say that Q is *orientable* if we can write

$$Q(x) = s\left(n(x) \otimes n(x) - \frac{1}{3} I\right),$$

where $n \in W^{1,1}(\Omega, S^2)$. When Q is orientable, we also say that it has a *lifting*, the lifting being n, to $W^{1,1}(\Omega, S^2)$. In particular, since $n \in L^\infty(\Omega)$, if $Q \in W^{1,p}(\Omega, S^2)$ is orientable for some $1 \leq p \leq \infty$, then $n \in W^{1,p}(\Omega, S^2)$. An orientable Q has exactly two liftings (see [6, Theorem 8]).

Ball & Zarnescu [9] exhibited examples of nonorientable smooth line fields in domains that are not simply-connected. Thus, simple connectivity plays an important rôle in the lifting problem for Q-tensors. Indeed, Ball & Zarnescu also proved

Proposition 2.12 ([9]). *If $\Omega \subset \mathbb{R}^3$ is a bounded simply-connected domain of class C^0 and $Q \in W^{1,2}(\Omega, \mathcal{Q})$, then Q is orientable.*

As a straighforward corollary, we have that

> *In a simply-connected domain the constrained Landau-de Gennes theory and the Oseen-Frank theory are equivalent.*

The above results explain why people often work in simply-connected domains in the context of Landau-de Gennes theory.

Remark 2.5.3. The above observation notwithstanding, there are however interesting and physically meaningful situations in which the domain is not simply-connected, as the case of a nematic film spread on an annular region.

Relation between LdG theory and LE theory. The way of interpreting defects in the Ericksen theory has the side effect that s and n are not independent. One can pair s and n to form a uniaxial Q-tensor order parameter, i.e., a Q-tensor order parameter of the special form

$$Q(x) = s(x)\left(n \otimes n - \frac{1}{3}I\right).$$

However, strictly speaking, it is incorrect to say that two paradigma are equivalent. They are so when consistency conditions are imposed, so that the energy does not blow-up around defects. These conditions, due to Ericksen (see [37, § 7,8] or [139, § 6.2.3]) are always assumed in LE theory but they do not explicitly appear in the energy functional (2.4.1) because we assumed the simplest form of the free energy, i.e., the one coming from a *one-constant approximation*. Another reason for the importance of the constistency conditions lies in the fact that there are physical reasons, correlated to the way of experimentally observing the optic properties of liquid crystals [52, Chapter 4], for according to Q the status of preferred variable to describe the state of nematic liquid crystals, cfr. [37]. Thus, Leslie-Ericksen theory can be viewed as a particular case of Landau-de Gennes theory.

Defects. There appear to be no general agreement on how interpreting defects in Landau-de Gennes theory, because the rôle played by the two additional degrees of freedom offered by biaxiality w.r.t. uniaxiality is not always clear. However, following the point of view of P. de Gennes [51], the isotropic/uniaxial, isotropic/biaxial and uniaxial/biaxial interfaces are usually viewed as defects; their common feature is, of course, the change in the eigenvalues structure of the Q-tensor. This ansantz includes that of Ericksen, thus it is somewhat corroborated by experiments, though the discussion is still open (arguibly, mainly because biaxiality has been experimentally investigated only recently). In a mathematical way, one can then take the point of view [143, 142] that defects are discontinuities in the eigenframe. To be precise [144], a point x_0 has to be regarded as a discontinuity in the eigenframe if it is not possible to find in a neighborhood of x_0 some continuous $e_i(x)$, $i = 1, 2, 3$, with $e_i \cdot e_j = \delta_{ij}$, so that $Q(x)e_i(x) = \lambda_i(x)e_i(x)$ for some eigenvalue λ_i. An explicit example can be found in [144].

Defects in liquid crystal may be points, lines or surfaces. Actually, surface defects do not occur in free nematic liquid crystals [52, § 4.2.1], thus we will ignore them hereafter. Point and line defects are classified by a number M, called *disclination index* or also *topological charge* [52, Chapter 4]. M is defined w.r.t. the surrounding nematic director field n, which is singular (i.e., discontinuous) exactly at the defect site. Around the defect the director rotates of an angle $2\pi M$. In dimension 3, M is a relative integer for point defects while for line defects it can also take half-integer values. Lowest values of $|M|$ are preferred. Defects with $|M| > 1$ appear rarely. Line defects are often called *disclinations* or *disclination lines*. This way of classifying defects reflects the point of view, adopted by workers, of adopting Oseen-Frank theory "far enough" from the defects and the LdG theory to study the cores of defects, when needed.

Remarks about dynamics. Although in this work only static configurations are considered, dynamics is actually a topic of great interest, deserving a particularly careful treatment. Here we write down some short remarks for the convenience of the interest reader: indeed, several different approaches can be found in the literature and the situation may look rather intricate at first sight. Here we follow [7], where a short but very focused discussion can be found.

Liquid crystals must be viewed as complex non-Newtonian fluids. The first step for a dynamical theory is establishing if the fluid may be considered incompressible or not. Indeed, the corresponding variational problems are very different and thus also the related theory of existence and regularity of solutions. Often, the fluid is considered incompressible.

Next, the set of dynamical equations is very different when the Leslie-Ericksen model is used to describe the system and when a Q-tensor model is employed. In the first case, the dynamical equations are similar to the Navier-Stokes system but with added difficulties, such as the norm constraint on the director. Within the framework of the Q-tensor theory, a model currently studied is that of Beris&Edwards, which is structurally related to a forced Navier-Stokes system, coupled with a parabolic system. It has desirable features also in the direction of its relation with the Leslie-Ericksen model. In both cases, are known results about the existence of global weak solutions and about the existence of strong solutions, in some cases. Apparently, higher regularity results are not known. We refer to [7] for an up-dated account of the relevant literature.

2.5.3 One-constant approximation

Let us consider the LdG energy functional

$$E_{\mathrm{LdG}}(Q;\Omega) = \int_{\Omega} \left\{ \frac{L}{2} |\nabla Q|^2 + F(Q) \right\} \mathrm{d}x,$$

where $L > 0$, $|\nabla Q|^2$ and $F(Q)$ are as introduced in the previous section, on the class

$$\mathcal{A} = \left\{ Q \in W^{1,2}(\Omega, \mathcal{S}_0) : Q = Q_b \text{ on } \partial\Omega \text{ in the trace sense } \right\},$$

where $Q_b \in C^\infty(\partial\Omega, \mathcal{S}_0)$ is an assigned boundary datum.

The first question is that of existence of minimizers of $E_{\mathrm{LdG}}(\cdot;\Omega)$ in the class $\mathcal{A}$. The answer is clearly in the affirmative, whenever $\mathcal{A}$ is nonempty: indeed, the energy density is convex in ∇Q, coercive in Q and bounded below (recall that $F(Q) \geq 0$, see the previous section); moreover, the admissible class $\mathcal{A}$ is weakly closed in $W^{1,2}(\Omega, \mathcal{S}_0)$ (which is a Hilbert space) because Q_b is a Dirichlet boundary condition. The direct method (see, e.g., [40, Theorem 8.2.1]) applies, giving the existence of minimizers of $E_{\text{LdG}}(\cdot;\Omega)$ in the class $\mathcal{A}$.

The second, natural, question is that of regularity of minimizers. To answer this question, we note that $Q \in \mathcal{A}$ is a critical point of the functional $E_{\text{LdG}}(\cdot;\Omega)$ if and only if is a solution of the Euler-Lagrange equations [104, 106]

$$L\Delta Q_{ij} = aQ_{ij} + b\left(\frac{1}{3}\delta_{ij}\operatorname{Tr}Q^2 - Q_{ik}Q_{kj}\right) + cQ_{ij}\operatorname{Tr}Q^2, \tag{2.5.23}$$

subject to the boundary condition $Q = Q_b$ on $\partial\Omega$. The derivation of Eqs. (2.5.23) is straightforward, the only point to which pay attention being the tracelessness constraint (which is the reason for the first term in round brackets at r.h.s.).

Eqs. (2.5.23) form a system of *semilinear* elliptic equations of second order. Using the embedding $W^{1,2} \hookrightarrow L^6$ (in $\mathbb{R}^3$) and Hölder inequality to obtain the r.h.s. of (2.5.23) is in L^2, elliptic regularity gives $Q \in W^{2,2} \hookrightarrow W^{1,6} \hookrightarrow L^\infty$, hence the r.h.s. of (2.5.23) is actually in $W^{1,2}$. Elliptic regularity gives back $Q \in W^{1,3}$ and bootstrapping we obtain $Q \in C^\infty$. Now, due to results of Friedman [44], any smooth solution is real-analytic. Hence, any critical point (not only minimizers) of the functional $E_{\text{LdG}}(\cdot;\Omega)$ on the class $\mathcal{A}$ is completely smooth (more precisely, real-analytic) in the interior and smooth as the boundary datum and boundary allow at the boundary.

Remark 2.5.4. In contrast, we will add a constraint on the norm of Q-tensors and turn Eqs. (2.5.23) into a *quasilinear* system of elliptic equations (see (4.2.1)), to which the elliptic regularity scheme cannot be applied as before, since the nonlinearity will be no more polynomial but will involve also the gradient of Q.

To say something more about minimizers, we now restrict to a class of physically-significant boundary conditions. Let $\mathcal{Q}_{\min} \simeq \mathbb{R}P^2$ the subset of $\mathcal{S}_0$, already defined in (1.1.9), on which $F(Q)$ attains its minimum. As in [106], we shall consider boundary conditions Q_b such that $Q_b(x) \in \mathcal{Q}_{\min}$ is smooth and given by

$$Q_b = s_+\left(n_b \otimes n_b - \frac{1}{3}I\right), \quad n_b \in C^\infty(\partial\Omega, S^2). \tag{2.5.24}$$

Then it can be proven

Proposition 2.13 ([106, Proposition 3])**.** *Let $\Omega \subset \mathbb{R}^3$ be a bounded and simply-connected domain with smooth boundary. Let Q be a global minimizer of $E_{LdG}(\cdot;\Omega)$, in the space $\mathcal{A}$, w.r.t. a boundary condition Q_b as in* (2.5.24)*. Then*

$$\|Q\|_{L^\infty(\Omega)} \le \sqrt{\frac{2}{3}}s_+, \tag{2.5.25}$$

where s_+ is defined in (1.1.8)*.*

Further, the following monotonicity inequality holds.

Proposition 2.14 ([106, Lemma 2])**.** *Let Q be a global minimizer of $E_{LdG}(\cdot;\Omega)$, in the space $\mathcal{A}$, w.r.t a boundary condition Q_b as in* (2.5.24)*. Then*

$$E_{LdG}(Q,x,r) \le E_{LdG}(Q,x,R), \quad \forall x \in \Omega, r \le R \text{ so that } B(x,R) \subset \Omega, \tag{2.5.26}$$

where

$$E_{LdG}(Q,x,r) := \frac{1}{r}\int_{B_r(x)} \left\{\frac{1}{2}|\nabla Q|^2 + \frac{F(Q)}{L}\right\} dx.$$

2.5.4 Asymptotic limits

Probing the extremal regions of the parameters space is often an interesting source of information and helps to describe, at least qualitatively, the behavior of true minimizers in an easy way. However, this kind of analysis usually involves performing some limiting process, which must be taken with care.

The material parameters entering the Landau-de Gennes energy density are, in first place, established once and for all by the physical and chemical properties of the material itself, at a level of atomic and molecular interactions, thus we are not allowed to change their value (the material and the external conditions being fixed), performing a limit. Further, the parameters **are not pure numbers**, they have appropriate physical dimensions and thus it does not make any sense to speak about smallness or comparing the absolute values of parameters having different units of measure in order to decide whether some are negligible with respect to others.

Note that Q-tensor order parameters are dimensionless by definition but, since $E_{\mathrm{LdG}}(\cdot;\Omega)$ must have the dimensions of energy, we have that $[L] = \mathrm{J\,m^{-1}}$ and $[\alpha] = [b] = [c] = \mathrm{J\,m^{-3}}$, while $a = -\alpha(T - T^*)$ has $[a] = \mathrm{J\,m^{-3}\,K^{-1}}$. Rough typical numbers for such constants are

$$L \approx 10 \times 10^{-11}\,\mathrm{J\,m^{-1}}, \quad \alpha, b, c \approx 10 \times 10^{-5}\,\mathrm{J\,m^{-3}}.$$

As a preliminary step to any asymptotic analysis, we have to *non-dimensionalize* the problem, that is, we have to suitably rescale the parameters in such a way that the new rescaled parameters become pure numbers and a sensible notion of smallness may be afforded. The most effective way to do this depends on the specific problem at hand. In the case of Landau-de Gennes theory, the above observations have been explicitly highlighted by Gartland [45], although practitioners were well-aware of these issues since long time before. The main contribution of the Gartland's paper [45] is giving a clear interpretation of two important asymptotic limits in the context of the Landau-de Gennes theory: the commonly called *vanishing elastic constant* (which should be named, more properly, *large-body limit*), studied in [106] (see also [143]), and the so-called *Lyuksyutov limit*, analyzed in [105] (whose authors already explicitly non-dimensionalized the problem) and in [29].

Remark 2.5.5. The paper [106] is not the only one studying the large-body limit (see [45] and references therein) but it is the most directly relevant to our purposes.

Remark 2.5.6. We must observe that physicists are usually very skeptics about asymptotic limits involving the material parameters. In particular, also rescalings are to be taken with special care, in order to not fall into unphysical regimes. Then, the following discussion must not be taken too literary and should be considered only as a first step towards a deep understanding of the issues they refer to.

Vanishing elastic constant limit

As explained before, the name *vanishing elastic constant limit* is misleading, since associated to an incorrect limiting process from the physical point of view, and we should not use it; we do so only because this name has acquired some popularity in force of the similarity of the limit with the well-known London limit in the context of Ginzburg-Landau theory. Below we follow Gartland [45] (but see also [143]).

Consider the energy functional (1.1.3) and let R denote a characteristic geometric length scale of the problem; since the domain Ω is assumed to be bounded, we can take $R = \operatorname{diam}\Omega$. Rescale lengths by R:

$$\bar{x}_i = \frac{x_i}{R}, \quad i = 1, 2, 3,$$

so that

$$\nabla = \frac{1}{R}\bar{\nabla}, \quad \mathrm{d}x = R^3\,\mathrm{d}\bar{x}, \quad \operatorname{diam}\bar{\Omega} = 1.$$

Let a_{NI} the value of a at $T = T_{\mathrm{NI}}$ and let us accord to a_{NI} the status of characteristic bulk parameter[10]. Although Q is dimensionless, it is convenient to rescale it as

$$Q = \gamma\bar{Q}, \quad \gamma := \frac{1}{\sqrt{27}}\frac{b}{c}.$$

We then obtain

$$\bar{E}(\bar{Q};\bar{\Omega}) = \int_{\bar{\Omega}} \left[\frac{1}{2}\bar{\xi}_{\mathrm{NI}}^2 \left|\bar{\nabla}\bar{Q}\right|^2 + \frac{t}{2}\operatorname{Tr}(\bar{Q}^2) - \sqrt{3}\operatorname{Tr}(\bar{Q}^3) + \frac{1}{4}\operatorname{Tr}(\bar{Q}^2)^2\right] \mathrm{d}\bar{x}, \tag{2.5.27}$$

where

$$\bar{E} = \frac{E}{\gamma^2 a_{\mathrm{NI}} R^3}, \quad \bar{\xi}_{\mathrm{NI}} = \frac{\xi_{\mathrm{NI}}}{R}, \quad \xi_{\mathrm{NI}} := \sqrt{\frac{L}{a_{\mathrm{NI}}}}, \quad t := \frac{a}{a_{\mathrm{NI}}} = \frac{T - T^*}{T_{\mathrm{NI}} - T^*}. \tag{2.5.28}$$

The parameter ξ_{NI} is usually called *nematic correlation length* while t is known as *reduced temperature*. Note that we have the following correspondences:

$$T = T^*, T_{\mathrm{NI}}, T^+ \leftrightarrow t = 0, 1, 9/8.$$

Typical numbers for ξ_{NI} and R are [45, 84, 85]

$$\xi_{\mathrm{NI}} \approx 10\,\mathrm{nm}, \quad R \approx 10\,\mu\mathrm{m},$$

thus $\bar{\xi}_{\mathrm{NI}}^2$ is typically of order 10^{-6}.

Remark 2.5.7. The nematic correlation has a statistical-physics interpretation: it is the characteristic distance at which thermal fluctuations (that try to get the order parameter out of its equilibrium value) are exponentially damped, see [140, Chapter 10] for a satisfactory discussion of this topic.

The rescaled EL equations are

$$-\bar{\xi}_{\mathrm{NI}}^2 \Delta\bar{Q}_{ij} = t\bar{Q}_{ij} - 3\sqrt{3}\bar{Q}_{ik}\bar{Q}_{kj} + \bar{Q}_{ij}\operatorname{Tr}(\bar{Q})^2. \tag{2.5.29}$$

The (non-physical) limit $L \to 0$, considered in [106], then corresponds to the physically meaningful[11] limiting situation in which

$$0 < \bar{\xi}_{\mathrm{NI}} \ll 1 \iff 0 < \xi_{\mathrm{NI}} \ll R,$$

that is, when the nematic correlation length is much smaller than the typical geometric length of the system, explaining why we called this the *large-body limit*.

Remark 2.5.8. We considered only the one-elastic constant case; a similar treatment carry over also to more general situations [45].

[10] This choice is arbitrary; we could take, for instance, b, c or even ratios of them.

[11] Or at least conceivable.

Below we report the main results of [106]. For ease of comparison, we do not change the statements of Majumdar & Zarnescu to incorporate the above considerations. Anyway, their conclusions continue holding true, although they are better interpreted in light of the non-dimensional analysis above; the interested reader may write down details easily. We shall give a more physically grounded interpretation of their results at the end of the short presentation below.

Call a *limiting uniaxial harmonic map* a map $Q^{(0)} : \Omega \to \mathcal{Q}_{\min}$, where $\mathcal{Q}_{\min}$ is defined in (1.1.9), which is a minimizer of the energy functional (1.1.3) in the restricted class

$$\mathcal{A}^{(0)}_{Q_b} := \left\{ Q \in W^{1,2}_{Q_b}(\Omega, S^4) : Q(x) \in \mathcal{Q}_{\min} \text{ almost everywhere } x \in \Omega \right\}, \tag{2.5.30}$$

where $\mathcal{Q}_b$ is as in (2.5.24). Then $Q^{(0)}$ is of the form

$$Q^{(0)} = s_+ \left(n^{(0)} \otimes n^{(0)} - \frac{1}{3} I \right), \tag{2.5.31}$$

where $n^{(0)}$ is a global minimizer of the corresponding one-elastic constant Oseen-Frank functional, i.e.,

$$\int_\Omega \left| \nabla n^{(0)}(x) \right|^2 \, \mathrm{d}x = \min_{n \in \mathcal{A}_n} \int_\Omega |\nabla n(x)|^2 \, \mathrm{d}x,$$

in the admissible class

$$\mathcal{A}_n = \left\{ n \in W^{1,2}(\Omega; S^2) : n = n_b \text{ on } \partial\Omega \right\},$$

with n_b and Q_b related as in (2.5.24). It follows from standard results in harmonic maps that $Q^{(0)}$ has at most a finite number of isolated point singularities (some of the theory of harmonic map is reviewed in the next Chapter).

The first important result of [106] — see Lemma 3 therein — is the $W^{1,2}$-convergence to the limiting harmonic map as $L \to 0$, up to subsequences. Using the $W^{1,2}$-convergence $Q^{(L)} \to Q^{(0)}$, the authors are able to prove the uniform convergence of the bulk energy density to its minimum (i.e, to zero) on those compact sets $K \subset \Omega$ not containing any singularities of $Q^{(0)}$ (Proposition 4). Then they obtain a Bochner-type inequality for the energy density $e_L(Q^{(L)})$ (Lemma 6):

$$-\Delta e_L(Q^{(L)})(x) \le C e_L^2(Q^{(L)}(x)) \tag{2.5.32}$$

on those balls $B_{\rho(x)}(x)$ such that $\left| Q^{(L)}(y) - s_+ \left(m(y) \otimes m(y) - \frac{1}{3} I \right) \right| < \varepsilon_0$, with $m(y) \in S^2$, for all $y \in B_{\rho(x)}(x)$, with suitable $\varepsilon_0 > 0$ and the constant $C > 0$ both independent of L.

The relevance of the Bochner-inequality (2.5.32) stays in its crucial rôle in proving uniform energy density estimates (Lemma 7), which in turn are used to yield uniform convergence (up to subsequences) $Q^{(L)} \to Q^{(0)}$ on the compact sets $K \subset \Omega$ free of the singularities of $Q^{(0)}$ (Proposition 5). The uniform convergence results may be sharpened to give smooth convergence in the compact sets $K \subset \Omega$ not containing any singuarities of the limiting harmonic map, see [114].

Established the uniform convergence into the interior, Majumdar & Zarnescu then study the situation near the boundary. In doing this, they are reminiscent of techniques used in Ginzburg-Landau theory (see references in [106]). As a first step, they obtain a boundary monotonicity formula (Lemma 9), which is the main ingredient to show the uniform convergence up to the boundary of $F(Q^{(L)})$ to its minimum (i.e, to zero) as $L \to 0$ (up to subsequences), see Proposition 6.

Various consequences of the convergence results are examined in Section 5 of [106] while in Section 6 the authors obtain estimates on the size of the regions where minimizers deviate from uniaxiality and on that of the regions where they are *strongly biaxial*, that is, where their biaxiality parameter is strictly positive, and also of the regions where it is greater than an assigned threshold.

We now elucidate a little the interpretation of the above results. Recalling that the unphysical limit $L \to 0$ corresponds to the more physically reasonable limit $\frac{\xi_{\mathrm{NI}}}{R} \to 0$, $R = \operatorname{diam}\Omega$, and observing that L is fixed a phyisical quantity (approximately) depending only on the material, we see that we must regard the limiting harmonic map as the object to which minimizers tend when the sample is sufficiently enlarged. Larger the sample, better the approximation, at least well-away from the singular set of the limiting harmonic map. This is consistent with the fact that Landau-de Gennes theory is *mesoscopic* while Oseen-Frank theory is *macroscopic* [45]. Although equivalent in a mathematical sense in simply-connected domains, they have different ranges of reliability in accounting for experimental results, the Oseen-Frank theory being appropriate when the characteristic geometric lengths of the system are large enough compared to intrinsic length scales. Thus it makes sense that the (simpler) minimizers of the OF-energy may approximate the (more complex) minimizers of the LdG-energy only when the typical geometric length-scale of the physical system is large. However, accordingly to Gartland [45, § 3], we note that, when the sample is **too** large, thermal fluctuations may destroy any orientational order, thus the limit $\bar{\xi}_{\mathrm{NI}} \to 0$ must be thought of as an idealization which, anyway, may yield interesting information.

Lyuksyutov limit

Besides the extension of the sample, there is another physical parameter under the control of the experimentalist and it is clearly temperature. In [105] and in [29] the regime of low-temperature is studied.

The notion of low-temperature regime requires some words of explaination: indeed, the nematic phase takes usually place in a relatively narrow range of temperatures (few Kelvin degrees, often one or two dozens [52, 139]); for lower temperatures, the liquid crystal will change mesophase or become a crystal. However, an appropriate rescaled variable connected to temperature can be found so that an asymptotic analysis involving limiting values of this variable is actually a physically meaningful idealization of real situations. It turns out that the *reduced temperature* defined in (2.5.28) is (almost) suitable for playing this rôle. This notwithstanding, the non-dimensionalization below will be different, even if related, from the previous one, and there are good physical reasons for this. On the other hand, we already remarked that the method of non-dimensionalization is problem-dependent.

Lowering the temperature sufficiently below T_{NI}, they are observed: an increasing of the orientational order; a deepening of the potential wells in $F(Q)$ (while the barriers between the wells become smaller); a reduction of correlation lengths and defect core sizes. The net result of the combination of these features are a weaker penalization of biaxiality and encouragment of localized biaxiality as a way to avoid isotropic melting. A good scaling should take all these features into account. Below we use the rescaling of [105]; that in [29] differs from this by constant factors, thus there is no a significant difference between them.

Let $\Omega \subset \mathbb{R}^3$ be a simply-connected smooth bounded domain, let R be a characteristic length scale of the Ω and define

$$\tilde{Q} = \frac{1}{s_+}\sqrt{\frac{3}{2}}Q.$$

Let

$$t := \frac{27ac}{b^2}, \quad h_+ = \frac{3+\sqrt{9+8t}}{4}, \quad \tilde{x}_i = \frac{x_i}{R}, \quad \tilde{L} := \frac{27c}{2R^2b^2}L. \tag{2.5.33}$$

and set

$$\tilde{E}(\tilde{Q};\Omega) := \frac{3\tilde{L}}{2Ls_+^2R}E(\tilde{Q};\Omega),$$

where

$$\frac{3\tilde{L}}{2Ls_+^2R}E(\tilde{Q};\Omega) = \int_\Omega \frac{\tilde{L}}{2}\left|\tilde{\nabla}\tilde{Q}\right|^2 + \frac{t}{8}\left(1-\operatorname{Tr}\tilde{Q}^2\right)^2 + \frac{h_+}{8}\left(1+3\left(\operatorname{Tr}\tilde{Q}^2\right)^2 - 4\sqrt{6}\operatorname{Tr}\tilde{Q}^3\right)\mathrm{d}\tilde{x}. \tag{2.5.34}$$

The corresponding rescaled EL equations are

$$\tilde{L}\Delta\tilde{Q}_{ij} = \frac{t}{2}\tilde{Q}_{ij}\left(\operatorname{Tr}\tilde{Q}^2-1\right) + \frac{3h_+}{2}\left[\tilde{Q}_{ij}\operatorname{Tr}\tilde{Q}^2 - \sqrt{6}\tilde{Q}_{ik}\tilde{Q}_{kj} + \sqrt{\frac{2}{3}}\delta_{ij}\operatorname{Tr}\tilde{Q}^2\right].$$

Define the *biaxial correlation lenght*

$$\xi_b = \sqrt{\frac{L}{bs_+}}. \tag{2.5.35}$$

The regime studied in [105] and [29] is $\xi_{\mathrm{NI}} \ll \xi_b \ll R$, corresponding to taking

$$t \to \infty \text{ and } \frac{\tilde{L}}{h_+} \to 0 \textbf{ simultaneously}.$$

Note that this limit and the large-body limit are not unrelated, being $\bar{Q}, \tilde{Q}$ and $\bar{E}, \tilde{E}$ proportional by direct comparison of the definitions. The scaling above is however more adapt to our purposes, see comments in [45, § 2.2].

For each $t > 0$, let $\tilde{Q}_t$ be a global minimizer of the Landau-de Gennes energy (2.5.34) subject to boundary conditions that are rescaled versions of those encompassed by (2.5.24). Relying also on techniques inspired by [106], the authors of [105] can prove strong $W^{1,2}$-convergence, up to subsequences, $\tilde{Q}_{t_j} \to \tilde{Q}^{(0)}$. The map $\tilde{Q}^{(0)}$ is the same limiting harmonic map in (2.5.31), up to rescaling. The strong $W^{1,2}$-convergence is then improved, by the same arguments as in [106], to locally uniform convergence, away from the singular set of the limiting harmonic map. Moreover, appealing to arguments in [29] they can prove **uniform** convergence $\left|Q_{t_j}\right| \to 1$ on Ω as $j \to \infty$ and they also have an estimate on the size of strongly biaxial regions showing that their diameters behaves like $t_j^{-1/4}$ for j large enough. In addition, both uniaxiality and maximal biaxiality are achieved in the closure of strongly biaxial regions, the uniaxial subregions having vanishing Lebesgue measure. In particular, a minimizer cannot be purely uniaxial at sufficiently low temperatures (note that "sufficiently low" should not be understood in absolute terms but in terms of the reduced temperature; it may be that a large range in terms of reduced temperature turns out to be a narrow range in term of absolute temperature).

Remark 2.5.9. In an earlier version of [105] published as preprint before the publication of [29], the statement on the attainment of maximal biaxiality was missing. Indeed, its proof constitutes a major novelty of [29]. In more recent versions of [105], the maximal biaxiality statement is proven *via* a slighty different reasoning than in [29] (whose authors deduce it as a consequence of a lemma of Canevari [24], stated in [24] for planar domains but with a proof actually valid also in dimension 3, as observed in [29]).

We want to stress the uniform convergence $\left|Q_{t_j}\right| \to 1$ in Ω: this result represents the first rigorous justification of the *Lyuksyutov constraint* (see next section), whose introduction was legitimated by Lyuksyutov [100] by energy comparisons, i.e., on the physical ground.

2.5.5 Lyuksyutov constraint

In his as short as remarkable paper [100], dating back to 1978, I. F. Lyuksyutov firstly argued that it may be not restrictive setting the norm of Q-tensor order parameters equal to a constant. The argument exposed by Lyuksyutov goes as follows. Suppose the material constant $b = 0$. Then, the potential energy reduces to

$$F_{b=0}(Q) = -\frac{a(T-T^*)}{2}\operatorname{Tr}(Q^2) + \frac{c}{4}\left(\operatorname{Tr}(Q^2)\right)^2.$$

Minimizing $F_{b=0}(Q)$ gives

$$\operatorname{Tr} Q^2 = \frac{a(T^*-T)}{2c} = \text{const.}, \tag{2.5.36}$$

so that Q takes values in the unitary sphere $S^4 \subset \mathbb{R}^5 \simeq \mathcal{S}_0$.

Now suppose that $b \neq 0$ and that its value is small w.r.t. to $a(T-T^*)$ and c. Here "small" means that the nematic-isotropic correlation lenght ξ_{NI}, defined in (2.5.28), is small w.r.t. the biaxial correlation lenght ξ_b, defined in (2.5.35). This makes sense from the physical point of view, because of the fact that measurements of the latent heat during the isotropic-nematic transition indicate that the transition is only weakly first order, meaning that the contribution from the cubic term to the energy is modest. According to [84], in order to make quantitative this hint, let us set

$$\xi(\tau) := \begin{cases} \frac{\xi_{\text{NI}}}{\sqrt{\tau}}, & \text{for } \tau > 1, \\ \frac{\sqrt{2}\xi_{\text{NI}}}{\sqrt{-4\tau + \frac{9}{2} + \frac{3}{2}\sqrt{9-8\tau}}}, & \text{for } \tau \leq 1, \end{cases}$$

and let us define

$$\mu(\tau) := \frac{\xi_b(\tau)}{\xi(\tau)}.$$

The numerics in [84] strongly suggest that the Lyuksyutov constraint (2.5.36) may be taken as valid when $\mu(\tau) \gg 1$. Assuming the constraint (2.5.36) amounts to assume that the liquid crystals responds to distortions, even strong, in a way that leaves $\operatorname{Tr} Q^2$ unchanged, typically by exchange of the eigenvalues. This is plausible deep in the nematic phase, because $a(T^* - T)\operatorname{Tr} Q^2$, $c(\operatorname{Tr} Q^2)^2$ are large enough to remain comparable to the elastic term $L\,|\nabla Q|^2$, even for large values of $|\nabla Q|^2$, due to the typical smallness of the elastic constant L.

Remark 2.5.10. Recalling (2.5.5), we see that assuming the Lyuksyutov constraint may drive minimizers out of the range $[-1/3, 2/3]$ for the eigenvalues. Indeed, along the 180°-disclination ring of the biaxial torus solutions of [84, 85] (see also §2.5.7), we have $\lambda_1 = -2/3$ and $\lambda_2 = \lambda_3 = 1/3$. Apparently, this problematic has never been reported in the literature. We content ourselves to highlight it and to suggest cautiously that this incompatibility may be the signal that in some regimes more multiples of the distribution function than only the second are needed to fully specify the nematic state at some points.

The possibility to escape to biaxiality in order to avoid melting motivated the conjecture that defect cores may posses a biaxial fine structure. Taking all the above considerations into account, Lyuksyutov argued that, deep in the nematic phase, melting at the defect sites may be exceedingly costing. In this direction, he noted that, when the characteristic geometric lenght R of the system is greater than ξ_b, then the Oseen-Frank theory may be applied and, due to the topology of the target space (i.e., $\mathbb{R}P^2$) of the order parameters, point singularities and line singularities alike are allowed. On the contrary, when $\xi_b > R$, a description involving the full Q-tensor order parameter is needed. In this regime, the constraint (2.5.36) may be assumed. He then remarks that the topology of S^4 does not allow point nor line singularities and then he also discussed an example of removal of a singularity on a disclination line in the range $\xi_b > R$, suggesting a way to experimentally verify such a prediction.

Since Lyuksyutov's arguments sound reasonable from a physical point of view, his conclusions were often assumed as a starting point in many works in the physical literature, especially the numerical simulations [47, 84, 85, 137]. However, a rigorous – although partial – justification of the Lyuksyutov constraint, deep in the nematic phase, lacked until the papers [105, 29], as we saw above.

2.5.6 The hedgehog solution

Let us consider the LdG energy (1.1.3) on the admissible class

$$\mathcal{A}_H := \left\{ Q \in W^{1,2}(B_R; \mathcal{S}_0) : Q = \sqrt{\frac{3}{2}} \left(\frac{x}{|x|} \otimes \frac{x}{|x|} - \frac{1}{3} I \right) \text{ on } \partial B_R \right\},$$

where $R > 0$. The name *radial hedgehog solution* is customarily given to solutions of the EL equations associated to the LdG energy functional (1.1.3) on $\mathcal{A}_H$ of the form

$$H_h = \sqrt{\frac{3}{2}} h(|x|) \left(\frac{x}{|x|} \otimes \frac{x}{|x|} - \frac{1}{3} I \right), \tag{2.5.37}$$

where h is a radial function minimizing the energy functional

$$I(h; [0, R]) = \int_0^R r^2 \left(\frac{1}{2} \left(\frac{dh}{dr} \right)^2 + \frac{3h^2}{r^2} + f(h) \right) \mathrm{d}r,$$

where, using the notation introduced in 2.5.33,

$$f(h) = -\frac{h^2}{2} - \frac{h_+}{t} h^3 + \frac{h_+^2}{2t} h^4,$$

in the admissible class

$$\mathcal{A}_h = \left\{ h \in W^{1,2}([0,R],\mathbb{R}); h(0) = 0 \text{ and } h(R) = 1 \right\}.$$

Note that H_h is SO(3)-equivariant. It can be proven [103] that

$$E(H_h; B_R) \leq 12\pi R.$$

A natural question is whether the radial hedgehog solution is actually a global minimizer of the LdG energy in the class $\mathcal{A}_h$. To investigate this question, one is naturally led to examine the simpler issue of the stability of the radial hedgehog. Various analyses have been carried out over the years [46, 125, 103, 77] in various ranges of the parameter R and of reduced temperature t. Without entering too specific details, it turns out that radial hedgehog solutions are stable for sufficiently high values of t and for sufficiently small R [103, 77], or in narrow classes of perturbations [125], while thery are *unstable* w.r.t. more general biaxial perturbations for sufficiently low temperatures and sufficiently large balls [47, 103, 77]. In particular, under the Lyuksyutov constraint we have $h \equiv 1$ and H_1 is unstable also w.r.t. S^1-equivariant perturbations in the unit ball (we prove this explicitly in Chapter 7 but it is not a new result as it was already contained in the quoted works above).

2.5.7 Biaxial torus solutions

Though early recognized, the possibility of allowing biaxiality in Q-tensor theory was not really exploited, or even explicitly ignored, for a long time, because of the lacking of any experimental evidence of biaxial phases in nematics. According to [14], the earliest experimental hint of biaxiality is maybe contained in [33]. Anyway, the first commonly recognized evidences of biaxiality in thermotropic nematics were produced only in 2004 [99, 101]. Nevertheless, Lyuksyutov [100] early suggested in 1978 that escaping to biaxiality may save energy in equilibrium configurations, deep in the nematic phase, and avoid the isotropic melting. The mechanism proposed by him, already mentioned at the end of §2.5.5, exploits the additional room provided by biaxiality to remove the isotropic core on a disclination line through, in few words, a broadening of the isotropic core to a 180°-disclination ring[12] linking the axis containing the singularity line (which we always identify with the z-axis). Importantly enough, this should happen even when *homeotropic* boundary conditions[13] are imposed, against the accepted fact [87, 120] that such boundary conditions enforce the arising of point singularities, and despite the fact that the thermotropic nematic bulk can have only uniaxial minima. Since both these facts induced to believe that the equilibrium configuration should be the hedgehog in all nematic regimes, Lyuksyutov's argument raised the doubt that this may not actually be the case.

Lyuksyutov's predictions were then later (1987) confirmed numerically by Schopohl & Sluckin[14] [133], who were able to describe the picture more accurately. They worked in an infinite domain and, choosing uniaxial asymptotic values for the Q-tensor order parameter, they observed a sort of symmetry breaking in approaching the core, in the sense that far from the core three parameters were sufficient to specify the Q-tensor

[12] The ring is thus a line defect of strenght $|M| = \frac{1}{2}$ corresponding to a uniaxial/biaxial interface.

[13] In the language commonly adopted in the physical literature, homeotropic boundary conditions are Dirichlet boundary conditions such that the boundary Q-tensor is uniaxial with leading eigenvector everywhere normal to the boundary.

[14] Actually, Schopohl and Sluckin do not quote the work of Lyuksyutov. Since the two papers are separated by a decade, it may be plausible that Lyuksyutov's ideas were already become part of the common lore of theoretical physics.

(so that it was uniaxial there) while near the core five parameters, that is a full biaxial Q-tensor, were needed. In particular, they found a 180°-disclination ring around the core on which the Q-tensor parameter becomes again uniaxial but whose scalar order parameter changed sign w.r.t. to that it had on the z-axis and far (i.e., outside $10\xi_b$ from the defect line, where ξ_b is the biaxial correlation lenght introduced in (2.5.35)) from the core. Around the core they found another ring of diameter $\approx 2\xi_b$ where the maximal biaxiality was attained. Moreover, the analysis of the eigenvalues showed that melting never happens. Perhaps, the first rigorous revision of Lyuksyutov arguments is contained in [15], whose authors also detail much more thoroughly biaxial escaping and explain that, when escaping happens, then a negative uniaxial disclination line is naturally surrounded by a toroidal surface of maximal biaxiality. Furthermore, [15] puts Lyuksyutov argument in a more precise perspective. However, the authors of [15] are not able to say whether escaping actually happens.

In 1986 Lavrentovich & Terent'ev [87] observed in real nematic droplets a related phenomenon but of different origin (it was due to changes in elastic constants caused by tuning temperature) and a much larger radius of the ring described above, strengthening however the idea of a somewhat universal attitude of singularities towards broadening.

Later, in 1989, Penzenstadler & Trebin [120] elaborated on the above results and other experimental works and assuming the Lyuksyutov constraint showed theoretically that there are classes of Q-tensors exhibiting the 180°-disclination ring as above that are also more stable than the radial hedgehog solution. The natural question then became whether real minimizers have this structure.

Various numerical analyses have been produced studying this issue, mostly in the setting in which the domain is axially symmetric [137, 84] or, more specifically, a ball (i.e., a droplet) [137, 85, 47], and the Dirichlet boundary condition is the hedgehog. In [84, 85] the Lyuksyutov constraint is assumed while in [137, 47, 32] it is not. In any case, it turns out that, deep in the nematic phase, the minimizers have the structure already encountered by Schophol & Sluckin and Penzenstadler & Trebin. At higher temperatures, very near to the isotropic-nematic transition, the hedgehog appear to be preferred. Accordingly to [47], there is also another configuration, called by the authors of [47] the *split core solution*, which is strongly biaxial near the z-axis and uniaxial along the z-axis with a disclination segment ending in two isotropic points, see Fig. 2.5c. This solution is only metastable and only within a narrow intermediate range of temperatures. The estimates on the size of the 180°-disclination ring and on critical lenght scales agree with that in [133]. In [137] no mention is made of the ring of maximal biaxiality while in [84] it is shown that the disclination ring is actually enrolled by a solid torus on the surface of which the maximal biaxiality is attained. The torus, like the ring, is linking the z-axis, as can be appreciated from Fig. 2.4a. This kind of solution is called in [84] a *biaxial torus solution.* We tried to encode their features in Definition 1.2, stressing the linking property. Fig. 2.4a and Fig. 2.4b will clarify the structure of biaxial torus solutions as derived in the above-quoted works. The three possible competing equilibria are sketched in Fig. 2.3. Their tensor fields can be compared looking at Fig. 2.5. Protoypical plots of the biaxiality parameter β^2 for biaxial torus solutions and split core solutions are reported to comparison in Fig. 2.6 and the corresponding eigenvalues are plotted in Fig. 2.7.

In [85] (and later in [32]) the further theme of the universality of such structures was tackled, specifically of their independence of the confining geometry, and it was there seen that they are independent of the confining geometry when the typical geometric scale of the system is large enough.

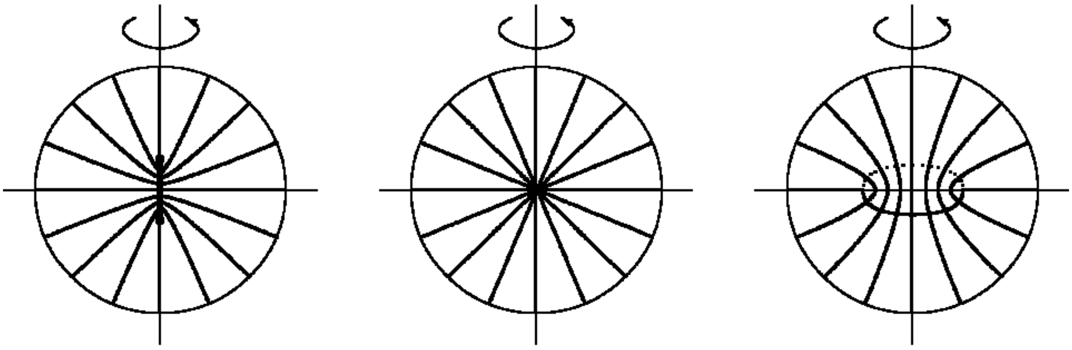


Figure 2.3: Figure from the preprint of [47] available at `http://icm.mcs.kent.edu/reports/2000/ICM-200002-0001.pdf` — Reproduced with permission of E.C. Gartland.

Remark 2.5.11. Recalling (2.5.5), we see that, if one accepts *both* the mean-field derivation of Q *and* the Lyuksyutov constraint, then biaxial torus solutions have to be regarded as unphysical states, since their eigenvalues on the negative uniaxiality ring lie outside the physical range. Anyway, we use the phenomenological definition of Q given by (2.5.1), which does not entail constraints on the eigenvalues. Remarkably, in [32] the eigenvalue constraints are assumed and yet biaxial torus solutions are retrieved. It would be a very interesting (and, presumably, very difficult) problem in the Calculus of Variation finding biaxial torus solutions assuming the mean-field definition of Q.

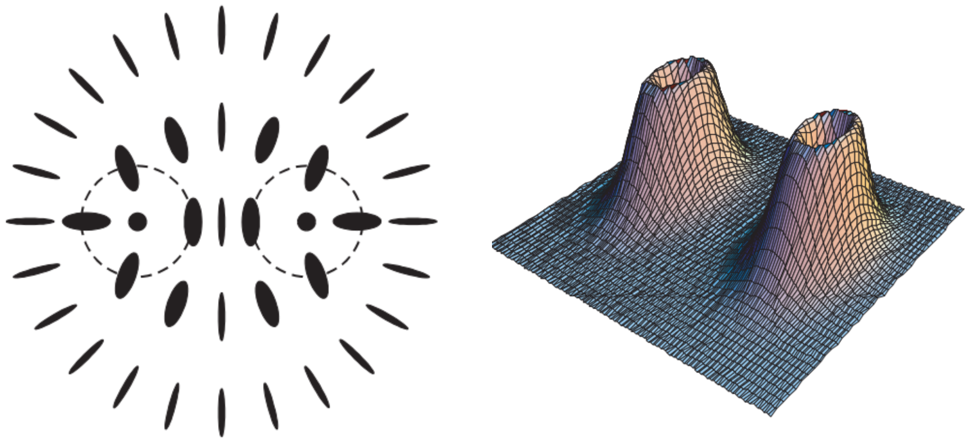
Remark 2.5.12. In order to make tractable a computationally expensive problem, the authors of [84, 85, 47] restricted to S^1-equivariant Q-tensors. Such a restriction has been removed in [32, 75], so that nowadays biaxial torus solutions appear as a fairly general feature of nematics deep in the nematic phase.

2.5.8 Comments on the definition of biaxial torus solution

Here we make some remarks on the definition we chose of biaxial torus solution. First of all, we note that biaxial torus solutions as obtained in numerical simulations [47, 84, 85, 137] have a high degree of symmetry. It is unclear whether this feature is universal or due to the symmetry of the domain and/or of the boundary condition (although [85] strongly suggest that the rôle of the confining geometry is negligible, at least for large enough domain), so we preferred not to encode it directly into the definition.

Next, biaxial torus solutions appear as smooth solutions in the quoted simulations and we required smoothness in Definition 1.2 for a twofold reason: on the one hand, the EL equations associated to the LdG energy functional (1.1.3) should be semilinear in physically reasonable regimes (recall that the imposition of the Lyuksyutov constraint, to which the quasilinear character of our EL equations[15] is due, is a mathematical idealization), thus their solutions have to be smooth (by arguments in §2.5.3). On the other hand, we need smoothness in the semidisk argument, Theorem 5.1. Further, the analysis of the eigenvalues of biaxial torus solutions show that isotropic melting is avoided; again, this is also needed for the semidisk argument, hence we asked $Q \neq 0$

[15]See Chapters 4 and 5.



(a) *Schematic representation of biaxial torus solution. The Q-tensor is represented as a field of ellipses in this section. The points where the ellipses degenerate into discs, are those traversed by the ring of negative uniaxiality.*

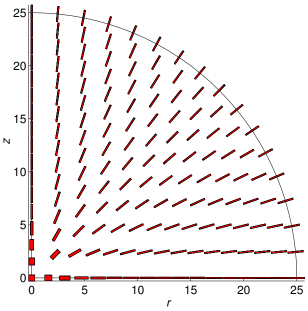
(b) *Logarithmic plot of the biaxiality parameter β^2 on a plane through the symmetry axis of the core.*

Figure 2.4: Figures from the version of [85] freely available at `https://www.researchgate.net/publication/230987935_Universal_fine_structure_of_nematic_hedgehogs` — Reproduced with permission of S. Kralj.

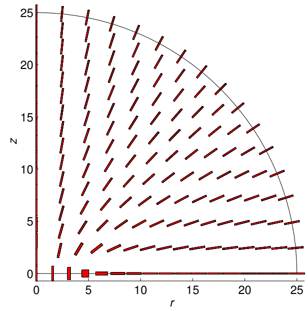
in Definition 1.2. Let us stress that the semidisk argument is a mere technical tool; nonetheless, we need it, in the sense that we at the moment we have no other ways to convert minimizers into biaxial torus solutions.

The peculiarities of biaxial torus solution are, however, the fact that there are regions of positive and negative uniaxiality that are linking and the fact that these are separated by a surface on which the maximal biaxiality is attained; moreover, this surface is topologically the surface of a torus of revolution. There are various subtle points in translating the phenomenological picture into mathematical language.

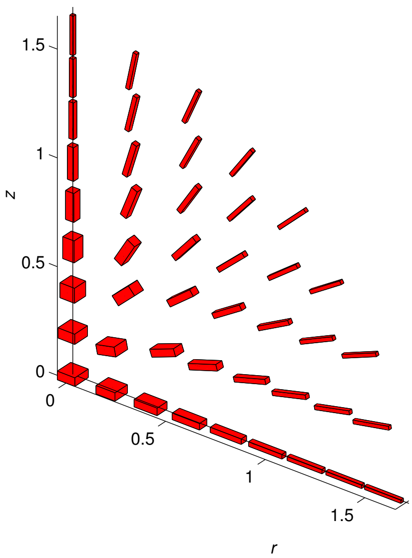
In the first place, numerical simulations take only axially symmetric domains (capillary tubes or spheres) with the hedgehog as the boundary condition into account. The picture coming out of them is very neat. The positive uniaxial region is made up by the boundary and the symmetry axis, while the negative uniaxial region is a ring in a plane orthogonal to the symmetry axis and linking the symmetry axis. The maximal biaxiality is attained on the surface of a solid torus of circular section and the negative uniaxial ring passes exactly through its center; biaxiality increases from the center towards the surface. On the other hand, [85] suggest that biaxial torus solutions should be minimizers also in more general domains, at least when they are large enough. Because of this, in Definition 1.2 we did not specify any particular domain. Lacking the symmetry of the domain, it is not clear whether the final picture may be neat as before. Said another way, the *geometrical* properties of the biaxial torus solutions resulting from simulations may be a byproduct of the various symmetries employed to make the problem tractable but their *topological* features appear to be universal. This point of view seems to be confirmed by [32, 75]. This is the main reason why we required the existence of two linking compact sets, one of positive uniaxiality and the other of negative uniaxiality, without ask anything about their shape or their regularity. In particular, $\mathcal{U}_-$ in Definition 1.2 need not be



(a) *Radial hedgehog.*

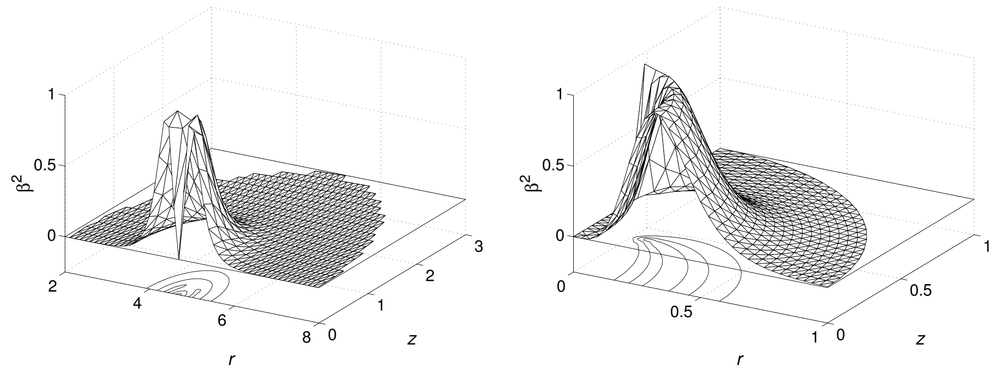


(b) *Biaxial torus solution.*



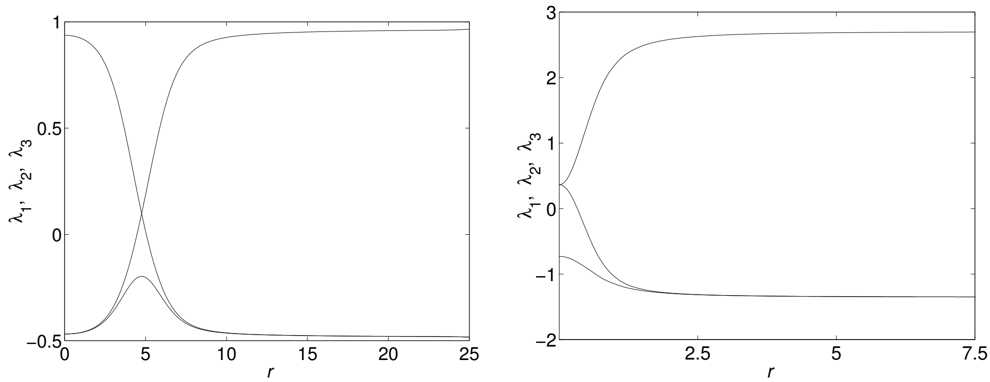
(c) *Split core.*

Figure 2.5: Tensor fields of competing equilibria. Tensor fields are visualized as fields of rectangular boxes aligned with the eigenframe of the Q-tensors with axes scaled proportionally to eigenvalues; a fixed constant is added to the eigenvalues, so that they are all nonnegative. Lenghts are expressed in units of $\xi_0 := \sqrt{\frac{27cL}{b^2}}$. For more information about scaling and the physical parameters, see [47]. Figures from the preprint of [47] available at `http://icm.mcs.kent.edu/reports/2000/ICM-200002-0001.pdf` — Reproduced with permission of E.C. Gartland.



(a) *Biaxial torus solution.* (b) *Split core solution.*

Figure 2.6: β^2 for biaxial torus solutions and split core solutions. Lenghts are expressed in units of $\xi_0 := \sqrt{\frac{27cL}{b^2}}$. For more information about scaling and the physical parameters, see [47]. Figures from the preprint of [47] available at `http://icm.mcs.kent.edu/reports/2000/ICM-200002-0001.pdf` — Reproduced with permission of E.C. Gartland.



(a) *Biaxial torus solution.* (b) *Split core solution.*

Figure 2.7: Prototypical eigenvalue plots for biaxial torus solutions and and split core solutions. Lenghts are expressed in units of $\xi_0 := \sqrt{\frac{27cL}{b^2}}$. For more information about scaling and the physical parameters, see [47]. Figures from the preprint of [47] available at `http://icm.mcs.kent.edu/reports/2000/ICM-200002-0001.pdf` — Reproduced with permission of E.C. Gartland.

a closed curve. One may prefer to ask, instead, the existence in $\overline{\Omega} \setminus \left\{\tilde{\beta} = 1\right\}$ of a non-contractible closed curve $c \in \pi_1(\Omega)$ such that

$$c : S^1 \to \left\{\tilde{\beta} = -1\right\}.$$

Although clearer, this position may be source of problems. To see this, observe that $\tilde{\beta}(Q)$ has the same regularity as Q. In particular, it is real-analytic whenever Q is. Thus, for any regular value $c \in [-1, +1]$, $\tilde{\beta}(Q)^{-1}\{c\}$ is a smooth submanifold of B_1 [108, Lemma 3.1]. By Sard's theorem, a.e. $c \in [-1, +1]$ is a regular value for $\tilde{\beta}(Q)$. The main problem here is that we are not assured at all that $c = -1$ is actually a regular value for $\tilde{\beta}(Q)$. Thus, $\tilde{\beta}(Q)^{-1}\{-1\}$ may be not a manifold, for instance it may be a union of crossing curves, and in general it has multiple connected components. A way to overcome this obstacle, suggested by the physical view-point, may be substituting the sharp value $c = -1$ with a narrow range I of values and then looking for closed curves in $\tilde{\beta}(Q)^{-1}\{I\}$. However, the consequent picture looks difficult to handle and it would deserve further analysis. Thus, requiring the existence of a curve as in the above seems a little more demanding than our request of two linking compact sets of opposite uniaxiality.

Remark 2.5.13. With reference to the Introduction, one may want to be able to exhibit biaxial torus solutions even without the symmetry constraint, of course. In this respect, Definition 1.2, although still making sense, looks a little too demanding w.r.t. that one may reasonably hope to prove, because of the sharpness of the requirements. A more suitable definition would involve a fattening of the levels $\tilde{\beta} = -1$ and $\tilde{\beta} = +1$ and the existence of two linking compact sets $\mathcal{U}_+$ and $\mathcal{U}_-$ such that

$$\max_{\mathcal{U}_-} \tilde{\beta}(Q) < \min_{\mathcal{U}_+} \tilde{\beta}(Q).$$

Thus, in a more general setting (but always thinking to homeotropic-like boundary conditions) one may consider the following

Definition 2.1 (Biaxial torus solution, revised). Suppose Ω is a bounded simply connected domain in $\mathbb{R}^3$ with smooth boundary. A smooth critical point $Q \in W^{1,2}_{Q_b}(\Omega, \mathcal{S}_0) \cap C^0(\overline{\Omega}, \mathcal{S}_0)$ of the LdG energy $E(\cdot; \Omega)$ w.r.t. some assigned Dirichlet boundary condition Q_b is called a *biaxial torus solution in* Ω if there exist in $\overline{\Omega}$ two linking compact sets $\mathcal{U}_-$, $\mathcal{U}_+$ so that:

(i) $Q \neq 0$ in $\overline{\Omega}$;

(ii) We have

$$-1 \leq \max_{\mathcal{U}_-} \tilde{\beta}(Q) < \min_{\mathcal{U}_+} \tilde{\beta}(Q) \leq 1. \tag{2.5.38}$$

Chapter 3

Harmonic maps

Synopsis. In this Chapter we recall basic facts about harmonic maps. In this work harmonic maps appear as asymptotic objects in the blow-up analysis, more precisely as *tangent maps* from $\mathbb{R}^3$ into S^4, see below and Chapters 4, 7. Thus, the present exposition is geared toward the regularity theory (§3.3) and peculiar properties of harmonic maps between spheres (§3.6).

3.1 Weakly harmonic maps

Let (M^m, g) and (N^n, h) be Riemannian manifolds, M possibly with boundary but not N. For our purposes, assuming that (M, g) and (N, h) are smooth is not a restriction. If N is compact, there exist, by the Nash-Moser embedding theorem, $k \in \mathbb{N}$ and an isometric embedding $\mathcal{J} : N \to (\mathbb{R}^k, \langle\cdot,\cdot\rangle)$, where $\langle\cdot,\cdot\rangle$ denotes the standard Euclidean inner product. One can then consider the space

$$W^{1,2}_{\mathcal{J}}(M, N) = \left\{ u \in W^{1,2}(M, \mathbb{R}^k) : u(x) \in \mathcal{J}(N) \text{ a.e.} \right\},$$

on which the *Dirichlet integral*

$$E(u) = \frac{1}{2} \int_M |du|^2_{T^*M\otimes\mathbb{R}^k} \,\mathrm{dvol}_M$$

makes sense. In local coordinates,

$$|du|^2_{T^*M\otimes\mathbb{R}^k} = \sum_{i,j,\alpha,\beta} g^{ij} \frac{\partial u^\alpha}{\partial x^i} \frac{\partial u^\beta}{\partial x^j} h_{\alpha\beta} \quad \text{and} \quad \mathrm{dvol}_M = \sqrt{\det g_{ij}}\,\mathrm{d}x.$$

For brevity, we shall often write $|du|^2_{T^*M\otimes\mathbb{R}^k} \equiv |\nabla u|^2$ in the sequel.

Note that, generally, the isometric embedding $\mathcal{J}$ is not unique, so that the definition of $W^{1,2}_{\mathcal{J}}(M, N)$ (and so the value of $E(u)$) actually depends on the choice of the embedding. However, assuming M is also compact, then all spaces $W^{1,2}_{\mathcal{J}}(M, N)$ are homeomorphic and $E(\mathcal{J}_2 \circ \mathcal{J}_1^{-1} \circ u) = E(u)$ for each pair of isometric embeddings $\mathcal{J}_1, \mathcal{J}_2$ [69]. Taking M compact, we shall then simply write $W^{1,2}(M, N)$ without specifying the embedding. Further, to simplify notations, we can think of M as an open domain in $\mathbb{R}^d$ (this is not a restriction because our main issues will be of local nature) and write more succinctly

$$E(u) = \frac{1}{2} \int_M |\nabla u|^2 \,\mathrm{d}x.$$

Let $\mathcal{O}$ be a neighborhood of N in $\mathbb{R}^k$ and let $\Pi_N : \mathcal{O} \to N$ the nearest point projection (well-defined and smooth, because N is smooth and compact). Let $u \in W^{1,2}(M,N)$. For any map $v \in W^{1,2}(M,\mathbb{R}^k) \cap L^\infty(M,\mathbb{R}^k)$ and any ε sufficiently small, we have $u + \varepsilon v \in \mathcal{O}$ so that the map

$$u_\varepsilon^v := \Pi_N(u + \varepsilon v) \tag{3.1.1}$$

is well-defined and $u_\varepsilon^v \in W^{1,2}(M,N)$. The family of maps $\{u_\varepsilon^v\}$ is called an *outer variation* of u.

Definition 3.1 (Weakly harmonic map)**.** Let $u \in W^{1,2}(M,N)$, $v \in W^{1,2}(M,\mathbb{R}^k) \cap L^\infty(M,\mathbb{R}^k)$ and let u_ε^v be defined as in (3.1.1). The map u is called *weakly harmonic* iff it holds

$$\left.\frac{dE(u_\varepsilon^v)}{d\varepsilon}\right|_{\varepsilon=0} = 0$$

for any $v \in W^{1,2}(M,\mathbb{R}^k) \cap L^\infty(M,\mathbb{R}^k)$.

A direct computation (detailed, e.g., in [111] or in [134]) shows that the above property is equivalent to say that u solves, in the sense of distributions, the *Euler-Lagrange equations*

$$\Delta_g u + A_u(\nabla u, \nabla u)u = 0, \tag{3.1.2}$$

where Δ_g is the Laplace-Beltrami operator on (M,g) and $A_u(\cdot,\cdot)$ the second fundamental form of the immersion $N \to \mathbb{R}^k$ evaluated at u. Equations (3.1.2) are a system of coupled *quasilinear* elliptic partial differential equations of second order.

A natural question is that of regularity of weakly harmonic maps. If $m = 1$, (3.1.2) is the geodesics ODE, so that in this case harmonic maps are geodesics and hence they are smooth (as the domain and the target manifolds permit). For $m = 2$, weakly harmonic maps are again smooth by Hélein's theorem (see next sections). In general, because of the quasilinearity of (3.1.2), the regularity of weakly harmonic maps is a hard problem. Indeed, in general they are not regular at all, as shown by Rivière [123, 124]. Partial regularity and even full regularity are possible with extra assumptions, as we shall see later. In particular, any *continuous* weakly harmonic map is smooth by standard bootstrap arguments (see, for instance, [70, 86, 129] and Chapter 4). However, continuity is usually very difficult to prove. The *singular set* of a weakly harmonic map u is defined as the set of points in M at which u is discontinuous.

Besides outer variations, one can consider also *inner variations*: given a 1-parameter family of diffeomorphisms $\phi_t : M \to M$ such that $\phi_0 = id$, one can consider the family of maps

$$u_t := u \circ \phi_t, \tag{3.1.3}$$

for $u \in W^{1,2}(M,N)$. The family $\{u_t\}$ is called an *inner variation*. We state the following definition.

Definition 3.2 (Weakly stationary harmonic map)**.** Let $u \in W^{1,2}(M,N)$ and let u_t be defined as in (3.1.3). The map u is called a *weakly stationary harmonic map* if (i) it is weakly harmonic and (ii) it holds

$$\left.\frac{dE(u_t)}{dt}\right|_{t=0} = 0$$

for any inner variation u_t.

If u is a smooth harmonic map (that is, a continuous weakly harmonic map), then it is also stationary, as can be easily seen integrating by parts the Euler-Lagrange equations. In general, there is no connection between being weakly harmonic and being weakly stationary harmonic (that is, satisfying (ii); see counterexample in [66, Example 1.4.19][1]). A detailed analysis of stationary harmonic maps is given in [91].

Stationarity is a very important property in the quest for regularity. Indeed, weakly stationary harmonic maps enjoy the following *monotonicity formula*, firstly proven by Price [121].

Proposition 3.1 (Monotonicity formula for weakly stationary harmonic maps, [121])**.** *Let $u \in W^{1,2}(M, N)$ be a weakly stationary harmonic map. Then we have*

$$\begin{aligned} &R_2^{2-m} \int_{B_{R_2}(x_0)} |\nabla u|^2 \, dx - R_1^{2-m} \int_{B_{R_1}(x_0)} |\nabla u|^2 \, dx \\ &= 2 \int_{B_{R_2}(x_0) \setminus B_{R_1}(x_0)} |x|^{2-m} \left| \frac{\partial u}{\partial r} \right|^2 \, dx, \end{aligned} \tag{3.1.4}$$

where $B_{R_1}(x_0) \subset B_{R_2}(x_0) \subset\subset M$ and $\frac{\partial}{\partial r}$ denotes the radial derivative.

The quantity

$$\mathscr{E}_{R,x_0}(u) := R^{2-m} \int_{B_R(x_0)} |\nabla u|^2 \, \mathrm{d}x \tag{3.1.5}$$

is called the *rescaled energy* of u in $B_R(x_0)$; its relevance will become clear in the sequel.

To end this section, we establish the fundamental definition of *minimizing harmonic map.*

Definition 3.3 (Minimizing map)**.** A map $u \in W^{1,2}(M, N)$ is *(energy-)minimizing* iff for each $x \in M$ there exists a compact neighborhood K_x such that for each $w \in W^{1,2}(M, N)$ with $u = w$ a.e. in $M \setminus K_x$ we have $E(u) \leq E(w)$.

If u is minimizing, it is a weakly stationary harmonic map. Schoen & Uhlenbeck built in [130, 131, 132] an elegant partial regularity theory for minimizing harmonic map which has been extended to weakly stationary harmonic maps by Evans [40] and Bethuel [11]. We shall review this theory below.

3.2 Examples of harmonic maps

Here we collect some examples of harmonic mappings which are particularly useful to know. For more examples, see, e.g. [34, 35, 69, 141]. As before, M and N are smooth compact Riemannian manifolds.

- **Constant maps** $u : M \to N$ and the **identity map** $id : M \to M$ are obviously harmonic maps.
- **Isometries** are harmonic maps. Further, composition of harmonic maps with isometries preserves harmonicity.

[1] In Hélein's terminology, a map $u \in W^{1,2}(M, N)$ having the property (ii) is called *weakly Noether harmonic.*

- If $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$, the harmonic maps are mappings $u : \mathbb{R}^m \to \mathbb{R}^n$ whose components are harmonic functions on $\mathbb{R}^m$. Thus, any harmonic map is completely smooth in this case.

- If $u : M \to \mathbb{R}^n$, u is a harmonic map iff its components are harmonic functions on M.

- **Geodesics** a smooth curve $\gamma : I \subset \mathbb{R} \to N$, I an open interval, is harmonic iff it defines a geodesics (i.e., iff it is parametrized by a multiple of the arc-length).

- The following is perhaps the most important example of harmonic map. Let $M = B^m$ and $N = S^{m-1}$. The map $\frac{x}{|x|} : B_1^m \to S^{m-1}$ belongs to $W^{1,2}(B_1^m, S^{m-1})$ and it clearly satisfies the harmonic map equation for maps into spheres

$$\Delta u = - |\nabla u|^2 \, u,$$

in the weak sense, therefore it is weakly harmonic. Actually, this map is minimizing in any dimension $m \geq 3$, by results of Brezis, Coron & Lieb [21] (for $m = 3$) and Lin [93] (for the general case). It is worth remarking explicitly that this example shows that minimizing harmonic map need not to be smooth nor even continuous.

- Let $M = B_1^m$ and $N = S^m$. The map $\omega = B_1^m \to S^n$ defined by $\omega(x) := \left(0, \frac{x}{|x|}\right)$, called *the equator map* [79], belongs to $W^{1,2}(B_1^m, S^m)$ and is weakly harmonic. Further, it is minimizing iff $m \geq 7$ by results of Jäger & Kaul [79].

3.3 Partial interior regularity

In their pioneering paper [36], Eells & Sampson proved the existence of smooth harmonic maps into manifolds of nonpositive sectional curvature. When the curvature hypothesis is not satisfied, the situation becomes much more involved. Existence usually comes from the direct method in the calculus of variations [69, Section 5]; for example, this is the case when the class of maps $\mathcal{E} \subset W^{1,2}(M, N)$, $\mathcal{E} \neq \varnothing$, of interest is defined by some Dirichlet boundary condition. Indeed, under this circumstance $\mathcal{E}$ is closed with respect to the weak topology of $W^{1,2}(M, N)$. Let $(u_n)_n \subset \mathcal{E}$ be a minimizing sequence. Note that minimizing sequences have bounded energy. Since the Dirichlet energy is weakly lower-semicontinuous, it thus follows from the direct method (e.g., [40, Theorem 8.21]) that there exist $u^\star \in \mathcal{E}$ such that

(i) $u_n \rightharpoonup u^\star$ in $W^{1,2}(M, N)$;

(ii) $u_n \to u^\star$ in $L^2(M, \mathbb{R}^k)$ by the Rellich-Kondrachov compactness theorem and hence, up to subsequences, pointwise a.e. (thus, $u^\star(x) \in N$ for a.e. $x \in M$);

(iii) $u^\star$ is a minimizing map, and so it is weakly stationary harmonic.

We now ask about the regularity of minimizing harmonic maps. The beautiful theory by Schoen&Uhlenbeck [130] shows that they are *partially regular* and in general no more than that. More precisely, they proved[2]

[2]We borrow the formulation of most of the following results from the nice report of Hardt [59].

Theorem 3.2 (Schoen&Uhlenbeck's regularity theorem)**.** *An energy-minimizing map $u : M \to N$ is smooth away from a closed (singular) subset of M that is discrete in M if $m = 3$ and has Hausdorff dimension $\leq m - 3$ if $m \geq 4$.*

This theorem is optimal, as the celebrated example of the map $\frac{x}{|x|} : B_1^m \to S^{m-1}$ shows. The key ingredient in their proof is the following gradient estimate.

Theorem 3.3 (Schoen&Uhlenbeck's ε-regularity theorem)**.** *There are positive constants $\varepsilon_0 = \varepsilon_0(m, N)$, $C = C(m, N)$ so that if $u : B_1^m \to N$ is an energy-minimizing map with $\varepsilon = \int_{B_1} |\nabla u|^2 \, dx \leq C\varepsilon_0$, then*

$$\sup_{B_{1/2}} |\nabla u|^2 \leq C\varepsilon.$$

In [27], Chen and Lin give a terse quick proof of this result involving a Ginzburg-Landau approximation.

Liao verified in [90] that ε-regularity does not hold for general weakly harmonic maps. Indeed, looking at the proof of the ε-regularity theorem, it is easy realized that the key property is the monotonicity formula. Evans [38] established ε-regularity for stationary harmonic maps into spheres by exploiting the special structure of the Euler-Lagrange equations due to the geometry of the sphere. This particular structure was previously revealed by Hélein [67] who in fact successfully derived in that paper the smoothness of weakly harmonic maps from a Riemannian surface into spheres. In another fundamental paper, Hélein [68] developed a moving frame technique for weakly harmonic maps (a *moving frame* is a special frame for the tangent bundle along the image of weakly harmonic maps) and used it to prove smoothness for general weakly harmonic maps from a surface. Bethuel [11] then used Hélein's tecnique and the *compensated compactness* phenomen discovered in [28] to extend the ε-regularity theorem of Evans to general target manifolds. He also proved a partial regularity theorem for weakly stationary harmonic maps; before stating it, we need to introduce some useful tools.

Observe that, by the monotonocity formula, the rescaled energy of u, defined in (3.1.5), i.e.,

$$\mathscr{E}_{R,x_0}(u) = R^{2-m} \int_{B_R(x_0)} |\nabla u|^2 \, \mathrm{d}x,$$

is nondecreasing with R, so it has a limit as $R \to 0$ at each point x_0, which is called the *density* of u at the point $x_0 \in M$, often denoted $\Theta_u(x_0)$:

$$\Theta_u(x_0) := \lim_{R \to 0} R^{2-m} \int_{B_R(x_0)} |\nabla u|^2 \, \mathrm{d}x.$$

The important property of the function Θ_u is being upper-semicontinuous [134, Section 2.5], that is,

$$x_j \to x_0 \in M \implies \Theta_u(x_0) \geq \limsup_{j \to \infty} \Theta_u(x_j).$$

Note that, due to Lebesgue differentiation theorem, Θ_u vanishes at regular points.

Next, we define the *concentration set* Z_u of $u \in W^{1,2}(M, N)$ as

$$Z_u := \left\{ x_0 \in M : \lim_{R \to \infty} R^{2-m} \int_{B_R(x_0)} |\nabla u|^2 \, \mathrm{d}x > 0 \right\}. \tag{3.3.1}$$

Then we state

Theorem 3.4 (Bethuel's regularity theorem)**.** *Any stationary harmonic map $u : M \to N$ is smooth on $M \setminus Z_u$, where Z_u, defined in* (3.3.1)*, is closed in M and has $\mathcal{H}^{m-2}(Z_u) = 0$.*

Remark 3.3.1. According to [59], although all known examples of stationary harmonic maps u have $\mathcal{H}^{m-3}(Z_u) = 0$, this fact has not yet proven in full generality.

Note that Bethuel's theorem is not as good as Schoen&Uhlenbeck's theorem. To understand the reason, let us first simplify the notation by taking $M = \Omega$, $\Omega \subset \mathbb{R}^m$ an open set (since regularity is a local issue, this is not a restriction). Then, let us fix $a \in \Omega$ and consider the family of *rescaled maps* $\{u_r\}_{r>0}$, where

$$u_r(x) := u(a + rx), \quad x \in \mathbb{R}^m. \tag{3.3.2}$$

The thing making the difference between minimizing and weakly stationary harmonic maps is that [98] for the first ones there is a sequence $r_i \to 0$ such that the sequence $(u_{r_i})_{r_i}$ converges **strongly** in $W^{1,2}_{\text{loc}}(\Omega, N)$ to a limiting harmonic map $u_\infty \in W^{1,2}_{\text{loc}}(\Omega, N)$ as $i \to \infty$. The map u_∞ is called a *tangent map* at x. Minimality passes to strong limits, so that u_∞ is minimizing (hence called a *minimizing tangent map*, MTM). Observe that the existence of a tangent map at each point follows by the monotonicity formula but for weakly stationary (nonminimizing) harmonic maps the convergence **fails** to be strong in general [73, Example 3.2].

Moreover, along with the monotonicity formula, the strong convergence implies that the density function Θ_u is upper-semicontinuous with the respect to the *joint variables* u_{r_i} and x_{r_i}, meaning that (see [134, § 2.11], [53, Proposition 10.26] and Chapter 5)

$$x_{r_i} \to x_0 \in \Omega \implies \Theta_u(x_0) \geq \limsup_{i\to\infty} \Theta_{u_{r_i}}(x_{r_i}).$$

In particular, it easy proved that

$$\Theta_{u_\infty}(0) = \Theta_u(a),$$

i.e., that u_∞ is homogeneous of degree 0 (see, e.g., [134, Chapter 3] for a detailed proof of this assertion).

The strong convergence to a minimizing tangent map and the joint upper-semicontinuity of the density are the keys of the so-called *dimension reduction* argument (see [53, Theorem 10.18] its application in this context and [42] for the original idea) that yields the improvement from Bethuel's theorem to Schoen&Uhlenbeck's theorem.

The result in [98], due to Luckhaus, plays a very important rôle in the theory of minimizing harmonic maps and we will be concerned in finding analogues for the situations considered in this work. The one in the above is actually an application of a more general result, customary called *Luckhaus' compactness theorem* that is worth recalling:

Theorem 3.5 (Luckhaus' compactness theorem, [98])**.** *Let u_j be a sequence of minimizing harmonic maps in $W^{1,2}(\Omega, N)$ with locally equibounded energies, that is,*

$$\sup_j \int_{B_\rho(y)} |\nabla u_j|^2 \, dx < +\infty$$

for each ball $\overline{B_\rho(y)} \subset \Omega$. Then there is subsequence $(u_{j_k})_{j_k}$ converging strongly in $W^{1,2}_{loc}(\Omega, N)$ as $k \to \infty$ to a minimizing tangent map $u_\infty \in W^{1,2}_{loc}(M, N)$.

We now ask about complete regularity. The already-mentioned example of the map $\omega(x) = \frac{x}{|x|} : B_1^m \to S^{m-1}$ shows that, in general, minimizing harmonic maps need not to be smooth. This example also shows that the dimensions of the domain and of the target manifold are important. In general, in order to have complete regularity, we need to rule out all possible nonconstant minimizing tangent maps at any point. A theorem establishing that all minimizing tangent maps have to be constant is customary called a *Liouville-type theorem*.

Usually, a Liouville theorem is a very difficult property to get, see, for example, [119] for a case where much work is needed. Schoen&Uhlenbeck proved a Liouville-type theorem for minimizing tangent maps with values into spheres in [132] (see also [95, 115] for various refinements of their result). The main point is that a minimizing map is *stable*, meaning that its second variation is a positive-definite quadratic form (i.e., the map cannot loose energy in any direction), thus giving an integral inequality (called a *stability inequality*). Precisely,

Definition 3.4 (Stable harmonic map)**.** Let $u : M \to N$ be a weakly harmonic map. Then u is said to be *stable* if it holds

$$\left.\frac{d^2E(u_\varepsilon)}{d\varepsilon^2}\right|_{\varepsilon=0} = \left.\frac{d^2}{d\varepsilon^2}\left(\int_M \frac{1}{2}\,|\nabla u_\varepsilon|^2 \;\mathrm{dvol}_M\right)\right|_{\varepsilon=0} \geq 0 \tag{3.3.3}$$

for any admissible outer variation $\{u_\varepsilon\}_{\varepsilon>0}$.

Interestingly, at least in the case of maps with values into spheres, the stability inequality depends on the dimension of the target but it is independent of that of the domain. Another integral inequality, in the inverse direction and involving the dimension of the domain but not that of the target, can be derived by integration of the *Bochner identity* recalled in Appendix B (see [95, Section 2] for optimal results when the target manifold is a sphere). Comparing the above-mentioned inequalities, one deduces that, for appropriate values of the dimensions of the domain and of the target manifolds, the only possibility for the map to be stable is to be constant. Hence minimizing tangent maps have to be constant in these cases. In particular, stable tangent maps from $\mathbb{R}^3$ into S^4 are all constant.

Remark 3.3.2. One of the main difficulties in the present work is that the available stability/instability theorems (recalled in Section 7.1.1) **does not hold** when S^1-equivariance is imposed. Indeed, their proofs rely on arguments that are generally incompatible with equivariance and so they need to be proved again with the aid of *ad hoc* constructions that preserve equivariance at each step. It could be, and it actually happens in our context (see Chapter 7), that equivariance is so strong that it prevents us from destabilizing some tangent maps (even those that, without equivariance, would be unstable).

Remark 3.3.3. In connection also with the previous remark, besides that as a matter of principle, let us observe explicitly that what we really need is not proving the instability of all nonconstant tangent maps, which is a sufficient condition, but proving that nonconstant tangent maps are not minimizing. Stability implies local minimality but not global minimality, in general. So, when stable nonconstant tangent maps are found, one has to test whether they are really minimizing or not. We prove that, in the S^1-equivariant context, there are nonconstant minimizing tangent maps (see Theorem 7.6).

Finally, it may be interesting to recall the relationship between stability and regularity in the general setting. In this direction, the simplest thing to do is dealing with weakly stationary harmonic maps satisfying a stability inequality[3]. This has been done in [72] and [73], where such maps are called *stable-stationary harmonic maps*, yielding the following important result, which can be regarded as an improvement of the Bethuel's theorem when the additional assumption of stability is made.

Theorem 3.6 (Hong&Wang, [73]). *Let $u \in W^{1,2}(M,N)$ be a stationary harmonic map which satisfies the following stability inequality: there exists a positive constant A (depending possibly on u) such that*

$$\int_M |\nabla V|^2 \geq A \int_M |\nabla u|^2 |V|^2 \qquad \forall V \in C_c^1(M, \mathbb{R}^k).$$

Then there exists a closed set $\Sigma \subset M$, whose Hausdorff dimension is at most $m-3$, such that $u \in C^\infty(M \setminus \Sigma, N)$.

3.4 The Dirichlet problem for harmonic maps

The Dirichlet problem for harmonic maps from a Riemannian manifold M with boundary ∂M into a Riemannian manifold (without boundary) N consists in the assignment of a boundary datum φ on ∂M and then in looking for solutions u of the Euler-Lagrange equations such that $u = \varphi$ in the trace sense on ∂M. For technical reasons (i.e., in order to have sensible estimates), a certain amount of regularity is required on the manifolds and on the boundary datum. In their important work [131] concerning the Dirichlet problem for minimizing harmonic maps, Schoen and Uhlenbeck assume M is C^2, ∂M of class $C^{2,\alpha}$ and $\phi \in C^{2,\alpha}(\partial M, N)$. Similar hypotheses are assumed also in [10, 64, 97].

Having already proved partial interior regularity, the main point in the Dirichlet problem for harmonic maps is proving some boundary regularity. The path is analogous to that for the interior regularity. One needs to prove:

(i) A boundary monotonicity formula;

(ii) a boundary ε-regularity theorem;

(iii) a boundary strong compactness theorem for rescaled maps;

(iv) that any minimizing tangent map has to be constant.

Due to topological reasons (cfr. [131], p. 253), one has to expect that boundary regularity is in fact stronger than interior regularity. Indeed, for smooth boundary data, any minimizing harmonic map is completely smooth near the boundary.

Theorem 3.7 ([131, Theorem 2.7]). *Let M be a compact manifold with $C^{2,\alpha}$ boundary. Suppose $\phi \in C^{2,\alpha}(\partial M, N)$ and $u \in W^{1,2}(M,N)$ is a minimizing map with $u = \varphi$ on ∂M. Then there exists $\delta > 0$ such that u is $C^{2,\alpha}$ in a full δ-neighborhood of ∂M. Moreover, if M, N are C^∞ and $\varphi \in C^\infty(M,N)$, then u is also C^∞ in a full neighborhood of ∂M.*

We do not enter into greater details here because we shall do so when dealing the boundary regularity in our specific cases, in Chapters 4 and 7. However, let us mention that, when $M = B_1 \subset \mathbb{R}^3$ and $N = S^2$, fundamental results in particular on

[3]The stereographic projection provides us an example of a stable-stationary harmonic map which is not minimizing, so that stable-stationarity is a weaker property than minimality [73].

the number of singularities have been obtained by Almgren and Lieb [2]. In particular, they show that there exists a universal constant C_{AL} such that the number $\mathcal{N}$ of interior singularity of an energy minimizing map $u \in W^{1,2}_{\varphi}(B_1, S^2)$, where $\varphi : S^2 \to S^2$ is smooth, is bounded by

$$\mathcal{N} \leq C_{\mathrm{AL}} \int_{S^2} |\nabla_T \varphi|^2 \,\mathrm{dvol}_{S^2}. \tag{3.4.1}$$

Remark 3.4.1. Note that the situation is quite different for weakly stationary (non-minimizing) harmonic maps. Indeed, in this case, a boundary monotonicity formula *does not follow* from stationarity. For more details, cfr. [91] and [97, Section 4.4].

3.4.1 Nonuniqueness

Naturally related to the solvability of the Dirichlet problem is the question of the uniqueness of solutions. Although a general result seems to be lacking, it is however true that in some important cases nonuniqueness occurs (even in spectacular ways). In particular, it is known

Theorem 3.8 (Benci & Coron, [10])**.** *Let $D = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\right\}$ and let $\gamma : \partial D \to S^2$ be a $C^{2,\delta}(\partial D)$ for some $\delta \in (0,1)$. If γ is not constant then there exist at least two functions in $C^{2,\delta}(\overline{D}, S^n)$ which are solutions of the Dirichlet problem*

$$\begin{cases} -\Delta u = |\nabla u|^2 u & \text{in } D, \\ u = \gamma & \text{on } \partial D. \end{cases}$$

Moreover, Rivière proved

Theorem 3.9 (Rivière, [122])**.** *Let $\Omega \subset \mathbb{R}^3$ be a smooth bounded domain, N a surface diffeomorphic to $S^2 \subset \mathbb{R}^3$, $\varphi : \partial\Omega \to N$ be a given smooth map. If φ is non-constant, there exists infinitely many weakly harmonic maps from Ω into N equal to φ at the boundary.*

Some complementary results were given by Isobe [78]. Pakzad [116] later extended Rivière's theorem to cover the case of weakly harmonic mappings from a regular bounded domain in $\mathbb{R}^n$ into, again, S^2. Precisely,

Theorem 3.10 (Pakzad, [116])**.** *Let Ω be a regular domain in $\mathbb{R}^n$, $n \geq 3$, and φ a non-constant smooth map from $\partial\Omega$ into S^2. Then φ admits infinitely many weakly harmonic extensions.*

On the other hand, Almgren & Lieb proved that the set of boundary data having unique minimizers is dense in $H^{\frac{1}{2}}(S^2, S^2)$, a result known as "generic uniqueness" of minimizers [2, Theorem 4.1].

3.4.2 Lavrentiev gap phenomenon

The nonuniqueness phenomenon is often related to another one, the so-called *Lavrentiev gap phenomenon.* Let M, N be Riemannian manifold, $\partial M \neq \varnothing$, N without boundary and let $\varphi : \partial M \to N$. Let us set

$$\mu_\varphi := \inf_{u \in W^{1,2}_{\varphi}(M,N)} \int_M |\nabla u|^2 \,\mathrm{dvol}_M, \quad \mu^*_\varphi := \inf_{u \in C^\infty_\varphi(M,N)x} \int_M |\nabla u|^2 \,\mathrm{dvol}_M.$$

It is plain that $\mu_\varphi \leq \mu^*_\varphi$. The point is that the inequality can actually be *strict*, as firstly shown by Hardt & Lin[4] [64]. This occurrence is called the *Lavrentiev gap phenomenon*. Moreover, the construction of Hardt and Lin allows for prescribing the number of singularieties *a priori* [64]. Precisely, they proved

Theorem 3.11 (Hardt & Lin, [64])**.** *For any positive integer N, there exists a smooth function $g : S^2 \to S^2$ that has degree zero so that any energy-minimizing map $v \in W^{1,2}_g(B_1, S^2)$ must have at least N singularities.*

They proved also

Theorem 3.12 (Hardt & Lin, [64])**.** *For any positive integer N, there exists a smooth function $g : S^2 \to S^2$ that has degree zero for which there is the gap*

$$\inf_{u\in W^{1,2}_g(B_1,S^2)} E(u) \leq \frac{1}{2N} < \frac{1}{2} \leq \inf_{u\in W^{1,2}_g(B_1,S^2)\cap C^0(\overline{B_1})} E(u)$$

Another important result, due to Bethuel, Brezis & Coron [12], also shows the relation with the nonuniqueness phenomenon:

Theorem 3.13 (Bethuel, Brezis & Coron, [12])**.** *Let $\Omega \subset \mathbb{R}^3$ be a smooth bounded domain, $\varphi : \partial\Omega \to S^2$ be a given smooth map. Suppose φ satisfies*

(i) $\deg \varphi = 0$ *and* $\mu_\varphi < \mu^*_\varphi$ *or*

(ii) $\deg \varphi \neq 0$.

Then there exists infinitely many weakly harmonic maps with boundary value φ.

Recently, Mazowiecka and Strzelecki [107], elaborating on the Hardt and Lin results, exploited a modification of the installing singularities trick by Almgren & Lieb [2, Theorem 4.3] to prove that Lavrentiev gap phenomenon holds for a dense subset of the set of smooth degree zero boundary data $\varphi : S^2 \to S^2$, where "dense" is referred to the $W^{1,p}$-topology, $1 \leq p < 2$. Their result is sharp w.r.t. p: it fails for $p = 2$.

Remark 3.4.2. Sometimes, especially in applications in mathematical physics, and mostly in the theories of liquid crystals and elasticity, the Lavrentiev gap phenomenon is interpreted by saying that *the fuction space is part of the model* [6, 7]. It occurs also when the function spaces are not $W^{1,2}$ and C^∞ but, for instance, when one is SBV and the other a $W^{1,2}$ space [6]. Although we shall not purse this point of view here, larger function spaces can accommodate wilder behavior and this sometimes leads to lower energies and other advantages (along with some disadvantages).

3.4.3 Stability of singularities

One may wonder whether singularities of minimizing harmonic maps are stable w.r.t. perturbations of the boundary datum. In [65], Hardt & Lin proved

Theorem 3.14 (Stability of singularities)**.** *Suppose $\Omega \subset \mathbb{R}^3$ is a smooth bounded domain in $\mathbb{R}^3$, $\psi \in \mathrm{Lip}(\partial\Omega, S^2)$, and v is the unique energy-minimizing map from Ω to S^2 with $v|\partial\Omega \equiv \psi$. There exists a positive number β and, for any positive ε, a*

[4]The theorem by Hardt and Lin has receveid much attention and some generalizations over the years, notably that due Giaquinta, Modica & Soucek [54] (which we will not state here).

positive number δ so that for any $\varphi \in C^{1,\alpha}(\partial\Omega, S^2)$ with $\|\varphi - \psi\|_{Lip} \leq \delta$ and for any energy-minimizing $u \in W^{1,2}(\Omega, S^2)$ with $u|\partial\Omega \equiv \varphi$, one has

$$\|u - v \circ \eta\|_{C^\beta} \leq \varepsilon$$

for some bi-lipschitz transformation η of Ω with $\|\eta - id_\Omega\|_{Lip} \leq \varepsilon$. In particular, η maps the singularities of u onto the singularities of v.

3.5 Harmonic maps from surfaces

Let $u : (M, g) \to (N, h)$ be a smooth map and $\Psi : M \to M$ be a conformal diffeomorphism, i.e., $\Psi^* g = \lambda^2 g$, with λ a smooth function. Then it can be checked (see, for instance, [66, Chapter 1] or [141, Section 1.2.1]) that

$$E(u \circ \Psi) = \int_M \lambda^{2-m} |\nabla u|^2 \, \mathrm{dvol}_M.$$

Let (M^2, g) be a two-dimensional Riemannian manifold. Then the Dirichlet energy is *conformally invariant*. In particular, any smooth harmonic map is weakly conformal. Other special things happen: as for the case of harmonic functions, the property of being a harmonic map depends only on the conformal structure of the surface [3]. We have a special way of writing locally the tension field in terms of the isothermal coordinates on open sets of M; further, if M is oriented, we can choose an atlas of oriented isothermal coordinates (x, y) on M and, setting, $z = x + iy$, we can endow M with a complex structure so that it becomes a Riemann surface. Any holomorphic or antiholomorphic map is then harmonic with respect to any Hermitian metric. In particular, due to the Liouville theorem for holomorphic functions on $\mathbb{C}$, if $M = S^2$, then any harmonic map from S^2 to any Riemannian manifold is weakly conformal and a map from S^2 to S^2 is harmonic iff it is holomorphic or antiholomorphic [88].

For the regularity of weakly harmonic maps from a surface, note that the monotonicity formula (3.1.4) becomes an identity. Using this fact, the conformality of the energy integral and a special moving frame technique, Hélein [67, 68] proved the following fundamental theorem.

Theorem 3.15 (Hélein)**.** *Any weakly harmonic map from a two-dimensional compact Riemannian manifold to any compact Riemannian manifold is smooth.*

3.6 Harmonic maps into spheres

Harmonic maps into euclidean spheres have been and are extensively studied because the geometry of the sphere gives rise to a particularly simple situation. Let (M, g) be a Riemannian manifold, S^n the euclidean n-sphere endowed with the standard metric. The harmonic maps equation then reads

$$\Delta_g u = - |\nabla u|^2 u. \tag{3.6.1}$$

This is a system of quasilinear coupled elliptic partial differential equations of second order. Regularity of solutions depends on both M and n and we have already seen that it is known when M is a Riemannian surface, by a result of Hélein [67]. On the other hand, when $\dim M \geq 3$, there exist weakly harmonic maps discontinuous at every point [123, 124]. Stationary harmonic maps behave better, as shown by Evans

[40], who extended Hélein's technique to higher-dimensional domains and obtained a partial regularity result. The case of minimizing harmonic maps into spheres has been treated by Schoen & Uhlenbeck, with later refinements especially by Okayasu [115] and Lin & Wang [95]; some of these results are of direct interest for us and recalled in Chapter 7. Also minimizing maps need not to be smooth when $\dim M \geq 3$.

There would be many topics to be discussed about harmonic maps into spheres but we shall limit ourselves only to those that are directly relevant to the sequel: some useful consequences of harmonicity; the classification of harmonic maps $S^2 \to S^2$ and the result of Brezis, Coron & Lieb on the structure of minimizing tangent maps $\mathbb{R}^3 \to S^2$. Another very significant topic, that of axially symmetric harmonic maps from the ball $B_1 \subset \mathbb{R}^3$ into S^2, has been already discussed at end of Section 2.3.

3.6.1 Useful identities

Let us note some useful identities holding for harmonic maps from S^2 into the sphere S^n. Let $u : S^2 \to S^n$ be harmonic, think the sphere S^n as a submanifold of $\mathbb{R}^{n+1}$ and let $(e_i)_{i=1}^{n+1}$ be the canonical basis of $\mathbb{R}^{n+1}$. We can then write $u = \sum_{i=1}^{n+1} u_i e_i$. Note that, by Hélein's theorem, each component u_i is a smooth function real-valued function on the sphere. Thus, we can use the harmonic map equation pointwise.

Take the scalar product of the harmonic map equation with e_i and then integrate both members on S^2. By the divergence theorem we have

$$\int_{S^2} u_i \, |\nabla u|^2 \, \mathrm{dvol}_{S^2} = 0. \tag{3.6.2}$$

By a similar device, other identities can be derived for any integer power p of u_i. In particular, for $p = 2$,

$$\boxed{\int_{S^2} u_i^2 \, |\nabla u|^2 \, \mathrm{dvol}_{S^2} = \int_{S^2} |\nabla u_i|^2 \, \mathrm{dvol}_{S^2}.} \tag{3.6.3}$$

Of course, the above identities hold also for tangent maps $\omega : \mathbb{R}^3 \to S^n$, because of harmonicity and degree-zero homogeneity.

3.6.2 Classification of harmonic maps $S^2 \to S^2$

Harmonic maps from S^2 into S^2 can be classified by their topological degree as a consequence of a result of Hopf (see [88, § 8]). The classification, due to Lemaire [88], exploits the standard identification of the sphere S^2 with the extended complex plane $\mathbb{C} \cup \{\infty\}$. More specifically, Lemaire first shows that any harmonic map between a Riemannian suface of genus 0 and a Riemannian surface must holomorphic or antiholomorphic [88, Corollaire 2.9]. Next, let $f : S^2 \to S^2$ be harmonic; without loss of generality, we can suppose f holomorphic. When the target is the sphere, one can view f as a meromorphic function and it is well-known that such functions are rational. Thus, denoting $\pi : S^2 \to \mathbb{C}$ the stereographic projection and $g = \pi \circ f \circ \pi^{-1}$, one can write

$$g = \frac{\sum_{i=0}^{r} a_i z^i}{\sum_{j=0}^{s} b_j z^j}, \quad a_i, b_j \in \mathbb{C}.$$

The degree d of f is thus the maximum between r and s. There are harmonic maps of any degree and their energy is

$$E(f) = 4\pi |d| .$$

In particular, harmonic maps of degree 1 are of the form

$$\pi \circ f \circ \pi^{-1} = \frac{az+b}{cz+d},$$

with $a, b, c, d \in \mathbb{C}$. As observed in [21, p. 678], up to fixed rotations of the domain and of the target, one can assume $c = 0, d = 1, a = \gamma b$, with $\gamma > 0$, thus reducing to

$$\pi \circ f \circ \pi^{-1} = b(z+\gamma), \quad b \neq 0.$$

In Chapter 6 we will go into further details because we will need to classify the S^1-equivariant harmonic maps $S^2 \to S^2$.

3.6.3 Minimizing harmonic maps from $\mathbb{R}^3$ into S^2 and the Brezis-Coron-Lieb theorem

In the fundamental paper [21], Brezis, Coron & Lieb addressed two relevant problems concerning harmonic maps from a domain Ω in $\mathbb{R}^3$ into S^2: (a) determining the minimum energy when the location and the topological degree of the singularities are prescribed, and (b) assigned a boundary condition g on $\partial\Omega$, determining in what cases $g(x/|x|)$, i.e., the homogeneous degree zero extension of the boundary datum, minimizes the energy. This last problem has, evidently, a direct application in the study of tangent maps and hence on the regularity theory for harmonic maps into S^2.

For our purposes, the main result of [21] can be cast in following form found in [30].

Theorem 3.16 (Brezis, Coron & Lieb)**.** *The map $u : \overline{B_1} \to S^2$ is a minimizing tangent map if there exists $\mathcal{R} \in \mathrm{SO}(3)$ such that $u(x) = \pm\mathcal{R}x/|x|$.*

In particular, $\frac{x}{|x|}$ is minimizing in its own class. The proof relies on the classification in the previous subsection and on some ingredients introduced in [21]. The first is the *center-of-mass condition* (see also Chapter 6), that is, any weakly stationary harmonic map $u : \overline{B_1} \to S^2$ must satisfy

$$\int_{S^2} x_i |\nabla u|^2 \,\mathrm{dvol}_{S^2} = 0, \quad i = 1, 2, 3.$$

The second ingredient is the fact that for any tangent map with degree $|d| \geq 2$ the energy may be decresead by splitting the singularity $\{0\}$ into $|d|$ distinct points (by means of the so-called *dipole construction*, also introduced in [21]). The final step is showing that $\frac{x}{|x|}$ is minimizing, a fact proven in [21, Theorem 7.3] and later generalized, with a much quicker argument, by Lin [93].

Remark 3.6.1. The dipole construction has also another important application in the construction of Rivière's pathological examples [123, 124] of weakly harmonic maps discontinuous everywhere. It is worth highlighting that the map constructed by Rivière in [123] is axially symmetric.

Chapter 4

Landau-de Gennes theory with norm-constraint and without symmetry

Synopsis. In this Chapter we prove existence and full regularity of minimizers of the LdG energy (1.1.3) in the class $\mathcal{A}_{Q_b}$ defined in (1.1.14), with Q_b given in (1.1.13). Existence is a straightforward consequence of the direct method of the Calculus of Variations, regularity requires more work and it is subdived into interior regularity and boundary regularity. The main difficulty here is that the Euler-Lagrange equations (4.2.1) are quasilinear. We follow an approach typical of geometric measure theory: we first derive a monotonicity formula, Theorem 4.3, and then we use it to prove an ε-regularity theorem (Theorem 4.6) and a compactness theorem for blow-ups in the strong topology of $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^3, \mathcal{S}_0)$ (Theorem 4.11). Blowing-up around putative singularities gives tangent maps which are locally minimizing harmonic maps from $\mathbb{R}^3$ to S^4. The Liouville theorem of Schoen&Uhlenbeck [132, Corollary 2.8] implies that all minimizing tangent maps (MTM) are constant in this case, so that, by ε-regularity theorem, we have Hölder continuity around each interior point. Higher regularity follows by bootstrap arguments in Section 4.5. Then we prove boundary regularity. The approach is analogous to that for the interior regularity, the main steps being obtaining a boundary monotonicity formula (formula (4.8.3)) and finding a suitable way of extending the map across the boundary. Once we got these, the proofs of ε-regularity theorem and of compactness theorem are readily adapted. Then the conclusion is achieved by ruling out all possible nonconstant MTM at boundary points, exploiting [97, Theorem 2.4.3].

4.1 Existence of minimizers

We prove the existence of minimizers of the LdG energy in the class (1.1.14). This serves also as an occasion for introduce the important map

$$H(x) = \sqrt{\frac{3}{2}}\left(\frac{x}{|x|} \otimes \frac{x}{|x|} - \frac{I}{3}\right). \tag{4.1.1}$$

Note that H is S^4-valued and that it is the homogeneous degree-zero extension to B_1 of the boundary datum Q_b in (1.1.13).

Proposition 4.1. *Let $E(\cdot; B_1)$ be the LdG energy, defined as in* (1.1.3) *and considered over the class $\mathcal{A}_{Q_b}$, where the boundary condition Q_b is given in* (1.1.13)*. Then there exists at least one minimizer of $E(\cdot; B_1)$ in the class $\mathcal{A}_{Q_b}$.*

Proof. The proof is a simple application of the direct method. We have to prove that $\mathcal{A}_{Q_b}$ is nonempty and that $E(\cdot; B_1)$ is lower semicontinuous with respect to the weak topology of $\mathcal{A}_{Q_b}$.

To see that $\mathcal{A}_{Q_b} \neq \varnothing$, it suffices to note that the map H given in (4.1.1) belongs to $\mathcal{A}_{Q_b}$.

Next, observe that, since Q_b is a Dirichlet boundary condition, $\mathcal{A}_{Q_b}$ is weakly sequentially closed in $W^{1,2}(B_1, S^4)$ (see, for example, [69, Section 5.2]), hence it inherits the weak sequential topology by that of $W^{1,2}(B_1, S^4)$ and, in turn, by that of $W^{1,2}(B_1, \mathcal{S}_0)$ (since $W^{1,2}(B_1, S^4)$ is strongly and weakly closed in $W^{1,2}(B_1, \mathcal{S}_0)$ [35]).

Lastly, we see that $E(\cdot; B_1)$ is bounded below and lower semicontinuous in the weak topology of $W^{1,2}(B_1, \mathcal{S}_0)$. Indeed, by writing $E(\cdot; B_1) \equiv E(\nabla Q, Q, x)$, we have that its energy density is convex in ∇Q and that $E(\nabla Q, Q, x)$ is coercive in Q. Then [40, Theorem 8.2] applies and thus $E(\cdot; B_1)$ is lower semicontinuous with respect to the weak topology of $W^{1,2}(B_1, \mathcal{S}_0)$ and, in turn, with respect to the weak topology of $\mathcal{A}_{Q_b}$.

In view of the above considerations, the direct method applies and the conclusion follows. □

4.2 Euler-Lagrange equations

In this section we find out the Euler-Lagrange equations associated to the LdG energy functional considered over the class $\mathcal{A}_{Q_b}$. As we shall see, they form a system of quasilinear elliptic equations of the second-order.

Proposition 4.2 (Euler-Lagrange equations)**.** *Let $E(\cdot; B_1)$ the LdG energy functional defined in* (1.1.3) *over the class $\mathcal{A}_{Q_b}$ given in* (1.1.14) *with Q_b as in* (1.1.13) *and let $Q \in \mathcal{A}_{Q_b}$ be a critical point of $E(\cdot; B_1)$. Then Q is a solution in the sense of distributions of the following boundary value problem:*

$$\begin{cases} L\Delta Q_{ij} + L\,|\nabla Q|^2\, Q_{ij} = b\left(Q_{ij}\operatorname{Tr}(Q^3) - Q_{ik}Q_{kj} + \frac{1}{3}\delta_{ij}\right) \text{ on } B_1 \\ Q_{ij} = (Q_b)_{ij} \text{ in the trace sense on } \partial B_1. \end{cases} \tag{4.2.1}$$

Proof. The derivation is standard, the only two points to which pay attention being the traceless constraint and the norm constraint. For notational convenience, we set $L = 1$ within this proof.

Let $\phi \in C_c^\infty(B_1, \mathcal{S}_0)$. Since $|Q| = 1$ a.e., we have $|Q + t\phi| \neq 0$ a.e. for sufficiently small $t > 0$. To take the tracelessness constraint into account, we add a term $\lambda \operatorname{Tr} Q$ to the energy density, where λ is a Lagrange multiplier to be found. We calculate

$$\begin{aligned} \frac{d}{dt}\bigg|_{t=0} E\left(\frac{Q + t\phi}{|Q + t\phi|}\right) &= \int_{B_1} Q_{ij,k}\left(\phi_{ij,k} - \frac{\partial}{\partial x_k}(Q_{ij}Q_{lm}\phi_{lm})\right) \\ &+ \frac{\partial F}{\partial Q_{ij}}(\phi_{ij} - Q_{ij}Q_{lm}\phi_{lm}) + \lambda\delta_{ij}(\phi_{ij} - Q_{ij}Q_{lm}\phi_{lm})\,\mathrm{d}x. \end{aligned}$$

Since $|Q|^2 = 1$, we have $Q_{ij}Q_{ij,k} = 0$ for all k, so that

$$\int_{B_1} Q_{ij,k} \frac{\partial}{\partial x_k} (Q_{ij} Q_{lm} \phi_{lm})\, \mathrm{d}x = \int_{B_1} |\nabla Q|^2 Q_{ij} \phi_{ij}\, \mathrm{d}x,$$

where we relabeled the dummy indexes. Further, we have

$$\delta_{ij} Q_{ij} Q_{lm} \phi_{lm} = Q_{ii} Q_{lm} \phi_{lm} = 0$$

because of the tracelessness constraint. Next, we have

$$\frac{\partial F}{\partial Q_{ij}} = (c-a) Q_{ij} - b Q_{ik} Q_{kj},$$

hence

$$\frac{\partial F}{\partial Q_{ij}} \phi_{ij} = [(c-a) Q_{ij} - b Q_{ik} Q_{kj}] \phi_{ij}$$

and

$$\frac{\partial F}{\partial Q_{ij}} (Q_{ij} Q_{lm} \phi_{lm}) = [(c-a) Q_{ij} - b Q_{ij} \operatorname{Tr}(Q^3)] \phi_{ij},$$

where again we relabeled saturated indexes. Hence we get

$$\begin{aligned} &\left.\frac{d}{dt}\right|_{t=0} E\left(\frac{Q+t\phi}{|Q+t\phi|}\right) \\ &= \int_{B_1} Q_{ij,k} \phi_{ij,k} + \left[-|\nabla Q|^2 Q_{ij} + b\left(Q_{ij} \operatorname{Tr}(Q^3) - Q_{ik} Q_{kj}\right) + \lambda \delta_{ij} \right] \phi_{ij}\, \mathrm{d}x. \end{aligned}$$

Integrating $\int_{B_1} Q_{ij,k} \phi_{ij,k}\, \mathrm{d}x$ by parts, we obtain

$$\begin{aligned} &\left.\frac{d}{dt}\right|_{t=0} E\left(\frac{Q+t\phi}{|Q+t\phi|}\right) \\ &= \int_{B_1} \left[-Q_{ij,kk} - |\nabla Q|^2 Q_{ij} + b\left(Q_{ij} \operatorname{Tr}(Q^3) - Q_{ik} Q_{kj}\right) + \lambda \delta_{ij} \right] \phi_{ij}\, \mathrm{d}x. \end{aligned}$$

Requiring $\left.\frac{d}{dt}\right|_{t=0} E\left(\frac{Q+t\phi}{|Q+t\phi|}\right) = 0$, we have

$$\Delta Q_{ij} + |\nabla Q|^2 Q_{ij} - b\left(Q_{ij} \operatorname{Tr}(Q^3) - Q_{ik} Q_{kj}\right) - \lambda \delta_{ij} = 0 \tag{4.2.2}$$

in the sense of distributions, by the arbitrariness of ϕ.

In order to determine λ, we multiply both sides of (4.2.2) by δ_{ij},

$$\Delta Q_{ii} + |\nabla Q|^2 Q_{ii} - b(Q_{ii} \operatorname{Tr}(Q^3) - Q_{ik} Q_{ik}) - 3\lambda = b Q_{ik} Q_{ik} - 3\lambda = 0$$

which implies

$$\lambda = \frac{b}{3}.$$

Finally, in the sense of distributions, we have

$$\Delta Q_{ij} + |\nabla Q|^2 Q_{ij} - b\left(Q_{ij} \operatorname{Tr}(Q^3) - Q_{ik} Q_{kj} + \frac{1}{3} \delta_{ij}\right) = 0,$$

hence the conclusion. □

4.3 Monotonicity formula

In this section we obtain the monotonicity formula for the minimizers of the LdG energy in the class $\mathcal{A}_{Q_b}$. The proof closely follows that of [119, Proposition 4.3]. We shall also see some useful corollaries.

Theorem 4.3 (Monotonicity formula). *Let $Q \in W^{1,2}(B_1, S^4)$ be a stationary point for $E(Q; B_1)$ defined* (1.1.3) *with respect to inner variations (see* (4.3.3)*), let $x_0 \in B_1$ and suppose that $B_{R_1}(x_0) \subset B_{R_2}(x_0) \subset B_1$. Then Q satisfies the following* monotonicity formula*:*

$$\begin{aligned}\frac{1}{R_2}\int_{B_{R_2}(x_0)} e(\nabla Q, Q) - \frac{1}{R_1}\int_{B_{R_1}(x_0)} e(\nabla Q, Q)\\ = L\int_{B_{R_2}(x_0)\setminus B_{R_1}(x_0)} \frac{1}{|x-x_0|}\left|\frac{\partial Q}{\partial r}\right|^2 + 2\int_{R_1}^{R_2}\frac{dR}{R^2}\int_{B_R(x_0)} F(Q),\end{aligned} \quad (4.3.1)$$

where $r = (x-x_0)/|x-x_0|$ and $\frac{\partial}{\partial r}$ denotes the derivative in the direction of r.

Proof. Pick $\varepsilon > 0$. If ε is sufficiently small, the mappings

$$x \mapsto x + \varepsilon\phi(x), \quad \phi \in C_c^\infty(B_1, \mathbb{R}^3), \quad (4.3.2)$$

are a one-parameter family of diffeomorphisms of B_1. Define

$$\Psi_\varepsilon(x) = x + \varepsilon\phi(x)$$

and set

$$Q_\varepsilon(x) := (Q \circ \Psi_\varepsilon)(x) = Q(x + \varepsilon\phi(x)). \quad (4.3.3)$$

The mapping $\varepsilon \mapsto E(Q_\varepsilon; B_1)$ is C^1 (see, e.g., [129] or [66, Chapter 1]) so it makes sense to calculate

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} E(Q_\varepsilon; B_1). \quad (4.3.4)$$

Let's calculate the expression in (4.3.4) and then put the variation equal to zero.

Following [119, Lemma 4.2], we first note that, for $\varepsilon > 0$ sufficiently small, it holds

$$D\Psi_\varepsilon^{-1}(x) = 1 - \varepsilon \operatorname{div}\phi(x) + \mathcal{O}(\varepsilon)$$

uniformly on B_1. Furthermore, for all $f \in L^1$ and all $g \in C^1$,

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\int f(\Psi_\varepsilon(x))g(x)\,\mathrm{d}x = -\int f(x)g(x)\operatorname{div}\phi(x)\,\mathrm{d}x - \sum_c\int f(x)\frac{\partial g}{\partial x_c}\phi_c(x)\,\mathrm{d}x. \quad (4.3.5)$$

For Q_ε as in (4.3.3),

$$\begin{aligned}\frac{\partial Q_\varepsilon}{\partial x_j}(x) &= \sum_c \frac{\partial Q}{\partial x_c}(\Psi_\varepsilon(x))\frac{\partial(\Psi_\varepsilon)_c(x)}{\partial x_j}\\ &= \sum_c \frac{\partial Q}{\partial x_c}(\Psi_\varepsilon(x))\left(\delta_{cj} + \varepsilon\frac{\partial\phi_c}{\partial x_j}\right)\end{aligned} \quad (4.3.6)$$

Applying repeatedly equations (4.3.5), (4.3.6), we have

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}\sum_j\int\frac{\partial Q_\varepsilon}{\partial x_j}\frac{\partial Q_\varepsilon}{\partial x_j}=-\sum_j\int \operatorname{div}\phi(x)\frac{\partial Q}{\partial x_j}\frac{\partial Q}{\partial x_j}+2\sum_{j,c}\int\frac{\partial \phi_c}{\partial x_j}(x)\frac{\partial Q}{\partial x_j}\frac{\partial Q}{\partial x_c}.$$

Set

$$y=x+\varepsilon\phi(x);$$

then $Q_\varepsilon(x)=Q(y)$ and

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}Q(y)=\sum_j\frac{\partial Q(x)}{\partial x_j}\;\frac{dy_j}{d\varepsilon}\bigg|_{\varepsilon=0}. \tag{4.3.7}$$

By the chain rule, it follows

$$\frac{\partial Q(y)}{\partial y_j}=\sum_k\frac{\partial Q(y)}{\partial x_k}\frac{\partial x_k}{\partial y_j}. \tag{4.3.8}$$

Clearly

$$x_k=y_k-\varepsilon\phi_k(x), \tag{4.3.9}$$

hence

$$\frac{\partial x_k}{\partial y_j}=\delta_{kj}-\varepsilon\frac{\partial \phi_k}{\partial y_j}(x). \tag{4.3.10}$$

Applying the chain rule and retaining only those terms that injected into (4.3.10) give a contribution of order at most ε,

$$\frac{\partial \phi_k}{\partial y_j}=\sum_a\frac{\partial \phi_k}{\partial x_a}\frac{\partial x_a}{\partial y_j}=\sum_a\frac{\partial \phi_k}{\partial x_a}\left(\delta_{aj}-\varepsilon\frac{\partial \phi_k}{\partial y_j}\right)\approx\frac{\partial \phi_k}{\partial x_j}$$

hence we have

$$\frac{\partial x_k}{\partial y_j}=\delta_{kj}-\varepsilon\frac{\partial \phi_k}{\partial x_j},$$

and

$$\frac{\partial Q}{\partial y_j}(y)=\sum_k\frac{\partial Q}{\partial x_k}(y)\left(\delta^{kj}-\varepsilon\frac{\partial \phi_k}{\partial x_j}\right),$$

and

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}Q_\varepsilon(x)=\sum_j\frac{\partial Q}{\partial x_j}(x)\phi_j(x)\equiv(\phi\cdot\nabla Q)(x). \tag{4.3.11}$$

Since the integrand in $E(Q;B_1)$ satisfies (with respect to ε) the hypotheses of the theorem for interchanging the integral sign and derivative, we can differentiate with respect to ε under the integral sign.

By translational invariance of the formula (4.3.1), it suffices to prove it for $x_0=0$. Imitating [119, Proposition 4.3], we take $h\in C^\infty(\mathbb{R})$ increasing with $h(t)\equiv 0$ for $t<0$ and $h(t)\equiv 1$ for $t\geq 1$ and we select

$$\phi(x)=h(R-|x|)x,$$

for $R > 0$ such that $\overline{B_R} \subset B_1$. Then

$$\operatorname{div} \phi(x) = 3h(R - |x|) - |x| \, h'(R - |x|), \tag{4.3.12}$$

where the factor 3 is due to the dimension of the domain (in general, for an m-dimensional domain $\Omega \subset \mathbb{R}^m$, the constant factor would be m). We then have

$$\frac{\partial \phi_c}{\partial x_j} = \delta_{cj} - \frac{x_c x_j}{|x|} h', \tag{4.3.13}$$

$$\sum_c \frac{\partial Q}{\partial x_c} \frac{\partial \phi_c}{\partial x_j} = \frac{\partial Q}{\partial x_j} h - x_j \frac{\partial Q}{\partial r} h', \tag{4.3.14}$$

where $\frac{\partial}{\partial r}$ stands for the directional derivative in the radial direction $x/|x|$ and $r = |x|$.

Put (4.3.12) and (4.3.13) into (4.3.7) and, in turn, into (4.3.4), then set the variation equal to zero:

$$\begin{aligned} 0 = \sum_j \int_{B_R} -(3h - |x| \, h') \frac{L}{2} \frac{\partial Q}{\partial x_j} \frac{\partial Q}{\partial x_j} + 2 \sum_{j,c} \int_{B_R} \left(\delta_j^c h - \frac{x_c x_j}{|x|} h' \right) \frac{L}{2} \frac{\partial Q}{\partial x_c} \frac{\partial Q}{\partial x_j} \\ + \int_{B_1} \frac{d}{d\varepsilon} \bigg|_{\varepsilon = 0} F(Q_\varepsilon) \, \mathrm{d}x. \end{aligned} \tag{4.3.15}$$

By (4.3.11) and the fact that $F(Q)$ is a polynomial in Q, it follows

$$\frac{d}{d\varepsilon} \bigg|_{\varepsilon = 0} F(Q_\varepsilon) = \phi \cdot \nabla F(Q). \tag{4.3.16}$$

Hence

$$\int_{B_1} \frac{d}{d\varepsilon} \bigg|_{\varepsilon = 0} F(Q_\varepsilon) \, \mathrm{d}x = \int_{B_1} \phi \cdot \nabla F(Q) \, \mathrm{d}x. \tag{4.3.17}$$

After an integration by parts,

$$\int_{B_R} \phi \cdot \nabla F(Q) = -\int_{B_R} \operatorname{div} \phi F(Q) \, \mathrm{d}x = -3 \int_{B_R} h F(Q) + \int_{B_R} |x| \, h' F(Q) \, \mathrm{d}x, \tag{4.3.18}$$

because h is compactly supported in B_R.

We recall that

$$\sum_c \frac{\partial Q}{\partial x_c} \phi_c = |x| \frac{\partial Q}{\partial r} h \tag{4.3.19}$$

and we insert (4.3.18) and (4.3.19) into (4.3.15), so that we get

$$\begin{aligned} 0 = \int_{B_R} -h \frac{L}{2} |\nabla Q|^2 + \sum_j \int_{B_R} |x| \, h' \frac{\partial Q}{\partial x_j} \frac{\partial Q}{\partial x_j} \, \mathrm{d}x - L \int_{B_R} h' \left| \frac{\partial Q}{\partial r} \right|^2 \\ - 3 \int_{B_R} h F(Q) + \int_{B_R} |x| \, h' F(Q) \end{aligned} \tag{4.3.20}$$

Letting $h \to \chi_{\{t>0\}}$ (i.e., $h\phi$ to the characteristic function of B_R and hence $h'\phi$ to measure concentrated on ∂B_R) gives

$$\begin{aligned} 0 = &- \int_{B_R} \frac{L}{2} |\nabla Q|^2 + R \int_{\partial B_R} \frac{L}{2} |\nabla Q|^2 - 3 \int_{B_R} F(Q) + R \int_{\partial B_R} F(Q) \\ &- LR \int_{\partial B_R} \left| \frac{\partial Q}{\partial r} \right|^2 . \end{aligned} \tag{4.3.21}$$

Keeping in mind the definition of the energy density $e(\nabla Q, Q)$, we rewrite (4.3.21) in the following fashion:

$$0 = - \int_{B_R} e(\nabla Q, Q) + R \int_{\partial B_R} e(\nabla Q, Q) - LR \int_{\partial B_R} \left| \frac{\partial Q}{\partial r} \right|^2 - 2 \int_{B_R} F(Q). \tag{4.3.22}$$

Recall that, for all integrable functions f and for a.e. $R > 0$,

$$R^{m-1} \frac{d}{dr} \left(R^{2-m} \int_{B_R} f \right) = (2 - m) \int_{B_R} f - R \int_{\partial B_R} f, \tag{4.3.23}$$

where m is the dimension of the domain. Divide both members of (4.3.22) by R^2 and use (4.3.23):

$$0 = \frac{d}{dR} \left(\frac{1}{R} \int_{B_R} e(\nabla Q, Q) \right) - L \int_{\partial B_R} \frac{1}{R} \left| \frac{\partial Q}{\partial r} \right|^2 - \frac{2}{R^2} \int_{B_R} F(Q). \tag{4.3.24}$$

Now integrate with respect to R from R_1 to R_2 and use

$$\frac{d}{dR} \int_{B_R} f = \int_{\partial B_R} f \tag{4.3.25}$$

which holds for all integrable functions f and almost every $R > 0$. We then have the monotonicity formula:

$$\begin{aligned} &\frac{1}{R_2} \int_{B_{R_2}} e(\nabla Q, Q) - \frac{1}{R_1} \int_{B_{R_1}} e(\nabla Q, Q) \\ &= L \int_{B_{R_2} \setminus B_{R_1}} \frac{1}{|x|} \left| \frac{\partial Q}{\partial r} \right|^2 + 2 \int_{R_1}^{R_2} \frac{\mathrm{d}R}{R^2} \int_{B_R} F(Q). \end{aligned} \tag{4.3.26}$$

Since $F(Q) \geq 0$, the right hand side is nonnegative. This concludes the proof. $\square$

The following corollaries are straightforward consequences of the monotonicity formula.

Corollary 4.4. *For all $R_1 \leq R_2$ such that $\overline{B_{R_1}}(x_0) \subset \overline{B_{R_2}}(x_0) \subset B_1$,*

$$\frac{1}{R_2} \int_{B_{R_2}(x_0)} e(\nabla Q, Q) \geq \frac{1}{R_1} \int_{B_{R_1}(x_0)} e(\nabla Q, Q), \tag{4.3.27}$$

i.e., the rescaled energy $E_{R,x_0} := \frac{1}{R} \int_{B_R(x_0)} e(\nabla Q, Q)$ is nondecreasing for all $0 < R <$ $\mathrm{dist}(\partial B_1, B_R(x_0))$.

Corollary 4.5. *The limit*

$$\lim_{R\to 0} \frac{1}{R} \int_{B_R(x_0)} e(\nabla Q, Q)$$

exists for all $x_0 \in B_1$.

4.4 ε-regularity theorem

Hoping that any weak solution to the Euler-Lagrange equations (4.2.1) is smooth is too much: indeed, counterexamples to regularity and even to partial regularity are obtained by factorization to $\mathbb{R}P^2$ of Rivière's pathological examples of completely discontinuous weakly harmonic maps from B_1 into S^2. Thus, we have to ask something more to weak solutions in order to prove their regularity. As in the harmonic map theory, the crucial additional property is a monotonicity formula.

Remark 4.4.1. As in the case of harmonic maps into spheres [40, 68], Eq. (4.3.1) and the possibility of recasting the Euler-Lagrange equations in the form (4.4.11) is all that we need in our arguments. Thus, we do not need to ask to weak solutions to be stationary but only to satisfy (4.3.1). This fact will be very important when we will deal with the symmetric case. Indeed, it is not easy to write down a condition of stationarity w.r.t. inner variations for S^1-equivariant maps while it is not difficult to derive a monotonicity formula for minimizers in the symmetric class, see Section 5.7.

The aim of this section is to prove the following theorem.

Theorem 4.6 (ε-regularity)**.** *Let* $Q \in W^{1,2}(B_1, S^4)$ *be a weak solution of the Euler-Lagrange equations* (4.2.1) *satisfying* (4.3.1) *and let* $x_0 \in B_1$. *There exist* $\varepsilon > 0$ *and* $\beta > 0$ *such that, if* $\overline{B_{R_0}(x_0)} \subset B_1$ *and*

$$\frac{1}{R_0} \int_{B_{R_0}(x_0)} \frac{1}{2} |\nabla Q|^2 \, dx \leq \varepsilon, \tag{4.4.1}$$

then $Q \in C^{0,\beta}(\overline{B_{R_0/2}(x_0)}, S^4)$.

Theorem 4.6 is very similar to [119, Proposition 4.5]. In fact, it is an elaboration of that result in a slightly different context.

We now come to explaining the strategy of the proof. First of all, we point out a remark.

Remark 4.4.2. Condition (4.4.1) readily implies, by the monotonicity formula, that

$$\sup_{\bar{x}\in B_{R_0}(x_0)} \sup_{0<r\leq R_0} \frac{1}{r} \int_{B_r(\bar{x})} \frac{1}{2} |\nabla Q|^2 \, \mathrm{d}x \leq 2\varepsilon. \tag{4.4.2}$$

In other terms, *at sufficiently small scales the scaled energy is locally uniformly bounded by its limit at any point.*

The main idea of the proof, tracing back to [26], is to exploit the integral characterization of Hölder continuity for functions in $L^2(B_{R_0/2}(x_0), \mathbb{R}^5)$ given by Campanato. To this end, we show that Q belongs to $\mathrm{BMO}(B_{R_0}(x_0))$ and that there is a *quantitive* decay of the BMO norm at smaller and smaller scales. Up to choose R_0 sufficiently small, this implies, via the John-Nirenberg inequality, that Q belongs to the Campanato space $\mathcal{L}^{2,\eta}(B_{R_0/2}(x_0), \mathbb{R}^5)$ for some $\eta \in (3, 5]$. Now, Campanato's theorem (see, for instance, [53, Theorem 5.5] or [134, Lemma 1.1]) implies that

$$\mathcal{L}^{2,\eta}(B_{R_0/2}(x_0), \mathbb{R}^5) \simeq C^{0,\beta}(\overline{B_{R_0/2}(x_0)}, \mathbb{R}^5),$$

with $\beta = \frac{\eta-3}{2}$, i.e., Q is Hölder continuous with exponent β in $\overline{B_{R_0/2}(x_0)}$. To say it more precisely, the L^2-class of Q contains a Hölder continuous representative. Remembering that Q is S^4-valued, the thesis follows.

We now recall the definition of Campanato and BMO spaces. Let $\Omega \subset \mathbb{R}^m$ be a domain with the following property [53, Section 5.1] : *there exists a constant $A > 0$ such that for all $x_0 \in \Omega$, $\rho < \operatorname{diam}\Omega$, we have*

$$|B_\rho(x_0) \cap \Omega| \geq A\rho^m.$$

This property is sometimes called *extension property.* Every Lipschitz domain enjoys this property.

Take $1 \leq p \leq +\infty$, $x_0 \in \Omega$ and set $\Omega(x_0, \rho) := \Omega \cap B_\rho(x_0)$. For each $\lambda \geq 0$, the *Campanato space* $\mathcal{L}^{p,\lambda}(\Omega)$ is defined as the space

$$\mathcal{L}^{p,\lambda}(\Omega) := \left\{ u \in L^p(\Omega) : [u]^p_{p,\lambda} := \sup_{x_0\in\Omega} \sup_{\rho>0} \rho^{-\lambda} \int_{\Omega(x_0,\rho)} \left| u - \fint_{\Omega(x_0,\rho)} u \right|^p < +\infty \right\}.$$

$[u]_{p,\lambda}$ is the Campanato (p, λ)-seminorm of u. Fixed a $\rho_0 > 0$, $[u]^p_{p,\lambda}$ is equivalent to

$$\sup_{x_0\in\Omega} \sup_{0<\rho\leq\rho_0} \rho^{-\lambda} \int_{\Omega(x_0,\rho)} \left| u - \fint_{\Omega(x_0,\rho)} u \right|^p \tag{4.4.3}$$

which says that only small radii are relevant in proving that a function belongs to a Campanato space. This fact will be very useful in the proof.

Theorem 4.7 (Campanato)**.** *Let $\Omega \subset \mathbb{R}^m$ be a domain with the extension property. For $m < \lambda \leq m + p$ and $\alpha = \frac{\lambda-m}{p}$ we have*

$$\mathcal{L}^{p,\lambda}(\Omega) \simeq C^{0,\alpha}(\overline{\Omega}).$$

Moreover, the seminorms $[\cdot]_{C^{0,\alpha}}$ and $[\cdot]_{\mathcal{L}^{p,\lambda}}$ are equivalent.

We further recall that, if $\Omega \subset \mathbb{R}^m$ is open and $u \in L^1_{\text{loc}}(\Omega)$, $u \in \text{BMO}(\Omega)$ if

$$\|u\|_{\text{BMO}(\Omega)} = \sup_D \fint_D |u - u_D| < +\infty,$$

where $u_D = \fint_D u$ and the supremum is taken over all balls $D \subset \Omega$. It is evident from the definition that, if $A \subset B$, then $\|u\|_{\text{BMO}(A)} \leq \|u\|_{\text{BMO}(B)}$. Indeed, the sup in the right hand side is taken over a family of balls which surely includes the family of all balls contained in A. In other words, the BMO norm is monotonic with respect to the inclusion of domains.

Remark 4.4.3. There is no universal (i.e., function independent) way to estimate how much the BMO norm in A is smaller than BMO norm in B, even if one knows how much A is smaller than B. To estimate the difference without giving explicitly a function, further information is required. This is why we require the condition of ε-smallness (4.4.2), as we shall see.

The following result is of capital importance to our strategy.

Theorem 4.8 (John-Nirenberg inequality)**.** *Let $u \in \text{BMO}(\Omega)$. For each $p \in (1, \infty)$, there exists $C_p > 0$ (depending only on p and m) such that*

$$\|u\|_{\text{BMO}(\Omega)} \leq \sup_D \left(\fint_D |u - u_D|^p \right)^{1/p} \leq C_p \|u\|_{\text{BMO}(\Omega)} < +\infty, \tag{4.4.4}$$

with the sup taken over all balls $D \subset \Omega$.

John-Nirenberg inequality is exploited to link between BMO and Campanato spaces when a quantitative decay of the BMO norm happens. Note that John-Nirenberg inequality implies that, if $u \in \mathrm{BMO}(\Omega)$, then

$$\int_D \left| u - \fint_D u \right|^p \le \tilde{C}_{p,m}^p R_D^m \tag{4.4.5}$$

for each m-ball D (whose radius is R_D) contained in Ω, where $\tilde{C}_{p,m} > 0$ is a constant which accounts for the p-th John-Nirenberg constant and for the volume factor.

We shall make use of the following lemmas. The first one will allow to control the decay of the BMO-norm with a power of the scale, that is the property needed in the definition of Campanato spaces.

Lemma 4.1. *Let $R > 0$ be fixed and let $\varphi : (0, R] \to \mathbb{R}$ be monotonically increasing. Suppose there exists $\sigma \in (0, 1)$, $\gamma > \log_\sigma \left(\frac{1}{2}\right)$ and a constant $C_1 > 0$ such that*

$$\varphi(\sigma t) \le \frac{1}{2}\varphi(t) + C_1 t^\gamma \tag{4.4.6}$$

for all $t \in (0, R]$ and that there exists $C_2 > 0$ a (finite) constant such that

$$\varphi(R) \le C_2. \tag{4.4.7}$$

Then

$$\varphi(t) \le C \left(\frac{t}{R}\right)^\alpha, \tag{4.4.8}$$

where $\alpha = \log_\sigma \left(\frac{1}{2}\right)$ and $C = C_2 + 2C_1 R^\gamma$.

Proof. Since (4.4.6) holds for each $t \in (0, R]$, picking $t' = \sigma^{\tilde{k}-1} t$, $\tilde{k} \ge 1$ an integer, we get

$$\varphi(\sigma^{\tilde{k}} t) \le \frac{1}{2}\varphi(\sigma^{\tilde{k}-1} t) + C_1 (\sigma^{\tilde{k}-1})^\gamma t^\gamma.$$

Again, (4.4.6) implies

$$\varphi(\sigma^{\tilde{k}-1} t) \le \frac{1}{2}\varphi(\sigma^{\tilde{k}-2} t) + C_1 (\sigma^{\tilde{k}-1})^\gamma t^\gamma.$$

Repeating $(\tilde{k} - 1)$-times, we find

$$\varphi(\sigma^{\tilde{k} t}) \le \left(\frac{1}{2}\right)^{\tilde{k}} \varphi(t) + C_1 t^\gamma \sum_{j=0}^{\tilde{k}-1} \sigma^{\gamma(\tilde{k}-1-j)} 2^{-j}. \tag{4.4.9}$$

Since $\sigma < 1$, each term in the sum is smaller than the corresponding term in the geometric sum $\sum_{j=0}^{\tilde{k}-1} \left(\frac{1}{2}\right)^j$ which in turn is obviously smaller than the sum of the corresponding series. Hence we have

$$\varphi(\sigma^{\tilde{k} t}) \le \left(\frac{1}{2}\right)^{\tilde{k}} \varphi(t) + 2C_1 t^\gamma. \tag{4.4.10}$$

Now, choose $k \ge 1$ an integer such that $t \in [\sigma^k R, \sigma^{k-1} R)$. By the monotonicity of φ in $(0, R]$, (4.4.10) and (4.4.7), it follows

$$\varphi(t) \le \varphi(\sigma^k R) \le \left(\frac{1}{2}\right)^k \varphi(R) \le \left(\frac{1}{2}\right)^k C_2.$$

Set $\alpha = \log_\sigma\left(\frac{1}{2}\right)$. Then $\left(\frac{1}{2}\right)^k = \sigma^{\alpha k}$. Since $t \geq \sigma^k R$, we have

$$\sigma^{k\alpha} \leq \left(\frac{t}{R}\right)^\alpha.$$

Hence,

$$\varphi(t) \leq C_2 \left(\frac{t}{R}\right)^\alpha + 2C_1 t^\gamma.$$

Since $\gamma > \alpha$ and $t/R < 1$, we have

$$t^\gamma = R^\gamma \left(\frac{t}{R}\right)^\gamma \leq R^\gamma \left(\frac{t}{R}\right)^\alpha$$

from which (4.4.8) follows. □

Note that $\log_\sigma\left(\frac{1}{2}\right) = \frac{\ln\left(\frac{1}{2}\right)}{\ln \sigma}$, from which it is easily seen that α is positive and it is an increasing function with σ.

The second lemma is a standard *a priori* estimate for the gradient of a solution of certain elliptic systems.

Lemma 4.2. *Let $m \geq 3$, $B_{\bar{R}} \subset \mathbb{R}^m$ an open ball of radius $\bar{R} > 0$. Let $q \in \left(\frac{m}{m-1}, 2\right)$ and let $s = \frac{qm}{q+m}$. Then there exists $C > 0$, depending only on q, such that, if $A \in L^2(B_{\bar{R}}, \mathbb{R}^n)$, $g \in L^2(B_{\bar{R}})$ and $u \in W^{1,2}_0(B_{\bar{R}})$ is a weak solution to*

$$\Delta u = \operatorname{div} A + g, \tag{4.4.11}$$

then

$$\|\nabla u\|_{L^q(B_{\bar{R}})} \leq C\left(\|A\|_{L^q(B_{\bar{R}})} + \|g\|_{L^s(B_{\bar{R}})}\right). \tag{4.4.12}$$

Proof. Calculations in [53, §7.1.2] may be readily adapted to yield the claim. □

We now prove Theorem 4.6.

Proof of the ε-regularity theorem. To begin with, note that, if Q satisfies (4.4.2), then $Q \in \mathrm{BMO}(B_{R_0}(x_0))$ and $\|Q\|_{\mathrm{BMO}(B_{R_0}(x_0))} \leq C\sqrt{\varepsilon}$. Indeed, by Cauchy-Schwarz and Poincaré inequalities,

$$\|Q\|_{\mathrm{BMO}(B_{R_0}(x_0))} \leq \sup_{D_r \subset B_{R_0}} \sqrt{\fint_{D_r} \left|Q - \fint_{D_r} Q\right|^2} \leq C \sup_{D_r \subset B_{R_0}} \sqrt{\frac{1}{r}\fint_{D_r} |\nabla Q|^2} \leq C\sqrt{\varepsilon}. \tag{4.4.13}$$

The goal is now to obtain a quantitative decay for $\|Q\|_{\mathrm{BMO}(D)}$, for all balls D contained in $B_{R_0/2}(x_0)$, so that we can exploit the John-Nirenberg inequality to conclude that Q belongs to a Campanato space isomorphic to a Hölder space.

Let $\sigma \in (0, 1/8]$ be fixed but to be specified later. For each $\hat{x} \in B_{R_0/2}$ and for each $t \in (0, R_0/2]$, let $D_t = D_t(\hat{x}) \subset B_{R_0}$ be an open ball of radius t. For each $\bar{x} \in D_{\sigma t} = D_{\sigma t}(\hat{x})$, let $r \in (0, t)$ be such that $B_{\sigma r}(\bar{x}) \subset D_{\sigma t}$. Then $B_r(\bar{x}) \subset D_t \subset B_{R_0}$ and, for $\bar{r} \in (r/2, r)$, the bound

$$\frac{1}{\bar{r}} \int_{B_{\bar{r}}(\bar{x})} |\nabla Q|^2 \,\mathrm{d}x \leq C\varepsilon \tag{4.4.14}$$

still holds.

Within the range of this demonstration, it is often useful to explicitly consider that each element of $W^{1,2}(B_1, S^4)$ is, in particular, an element of $W^{1,2}(B_1, \mathcal{S}_0) \simeq W^{1,2}(B_1, \mathbb{R}^5)$. Let $T_0 \in \mathcal{S}_0$ be a constant matrix. We show that there exists $\tilde{r} \in (r/2, r)$ such that

$$\int_{\partial B_{\tilde{r}}(\bar{x})} |Q - T_0| \leq \frac{8}{\tilde{r}} \int_{B_{\tilde{r}}(\bar{x})} |Q - T_0| \, . \tag{4.4.15}$$

More precisely, we show that (4.4.15) holds in $(r/2, r)$ up to a set Σ of measure at most $r/8$. Remember that for all $g \in L^1$ the identity

$$\int_{r/2}^{r} \int_{\partial B_\sigma(x_0)} g \, \mathrm{d}\sigma = \int_{B_r(x_0) \setminus B_{r/2}(x_0)} g \, \mathrm{d}x$$

holds. Now suppose, for the sake of a contradiction, that it holds

$$\int_{\partial B_{\tilde{r}}(\bar{x})} |Q - T_0| > \frac{8}{\tilde{r}} \int_{B_{\tilde{r}}(\bar{x})} |Q - T_0|$$

on a set $\Sigma' \subset (r/2, r)$ of measure bigger than $r/8$. Using the previous identity, we get

$$\int_{B_r(\bar{x}) \setminus B_{r/2}(\bar{x})} |Q - T_0| = \int_{\Sigma'} \int_{\partial B_{\tilde{r}}(\bar{x})} |Q - T_0| \, \mathrm{d}r + \int_{I \setminus \Sigma'} \int_{\partial B_{\tilde{r}}(\bar{x})} |Q - T_0| \, ,$$

where we set $I := (r/2, r)$. The last addendum at second member is surely nonnegative, while the first is, by hypothesis, greater than $|\Sigma'| \, 8r^{-1} \int_{B_r(x_0) \setminus B_{r/2}(x_0)} |Q - T_0|$. Hence $|\Sigma'| > r/8$ leads to a contradiction.

After having verified the existence of $\tilde{r}$ as above, pick one of them and set $\bar{r} = \tilde{r}$.

We are going to obtain the BMO norm decay by an iterative procedure, whose starting point is Lemma 4.2. However, to apply the lemma, we have to cast the system in the form

$$\Delta u = \operatorname{div} A + g,$$

for $u \in W_0^{1,2}(B_{\bar{r}}, \mathbb{R}^5)$. Let's start by rewriting the Euler-Lagrange equations (4.2.1) in the form

$$\Delta Q_{ij} = - |\nabla Q|^2 \, Q_{ij} + f_{ij},$$

with

$$f_{ij} = -\frac{1}{L} b \left[Q_{ij} \operatorname{Tr}(Q^3) - Q_{ik} Q_{kj} + \frac{1}{3} \delta_{ij} \right] . \tag{4.4.16}$$

Clearly, $f \in L^\infty(B_1, \mathbb{R}^5)$ and, since B_1 is bounded, we have $f \in L^p$ for all $p \geq 1$.

We now claim that there exists a harmonic extension h of $Q|_{\partial B_{\bar{r}}}$ in $B_{\bar{r}}$. More precisely, $h \in W^{1,2}(B_{\bar{r}}, \mathbb{R}^5)$, $h = Q|_{\partial B_{\bar{r}}}$ on $\partial B_{\bar{r}}$ and $h \in C^\infty$ in the interior.

Indeed, since $Q \in W^{1,2}(B_1, \mathbb{R}^5)$, asking $h = Q|_{\partial B_{\bar{r}}}$ on $\partial B_{\bar{r}}$, we have $h - Q \in W_0^{1,2}(B_{\bar{r}}, \mathbb{R}^5)$ and it is well known [55, Theorem 8.9] that a solution of the Dirichlet problem

$$\begin{cases} \Delta h = 0 & \text{in } B_{\bar{r}}, \\ h = Q|_{\partial B_{\bar{r}}} & \text{on } \partial B_{\bar{r}}, \end{cases}$$

exists, it is unique and it enjoys the desired properties (C^∞-regularity in the interior follows by Weyl's lemma, see, for instance, [134, Section 1.5]).

Because of the harmonicity of h, $Q - h \in W_0^{1,2}(B_{\bar{r}}, \mathbb{R}^5)$ is still a weak solution to the system

$$\Delta(Q - h) = -|\nabla Q|^2 Q + f.$$

To cast the system in the appropriate form for the application of Lemma 4.2, we use a trick by Hélein, that allows to rewrite the nonlinearity in a more convenient fashion. Since Q is S^4-valued, $|Q|^2 = Q_{ij}Q_{ij} = 1$, that implies

$$\frac{1}{2}\nabla(|Q|^2) = Q_{kl}\nabla(Q_{kl}) = 0,$$

hence

$$Q_{kl}\nabla(Q_{ij}) \cdot \nabla(Q_{kl}) = 0.$$

On the other hand, $|\nabla Q^2|\, Q_{ij} = (\nabla(Q_{kl}) \cdot \nabla(Q_{kl}))Q_{ij}$. Subtracting this equation to the previous one,

$$|\nabla Q|^2 Q_{ij} = (\nabla(Q_{kl})Q_{ij} - \nabla(Q_{ij})Q_{kl}) \cdot \nabla(Q_{kl}).$$

Define

$$A_{ij}^{kl} = -\left(\nabla(Q_{kl})Q_{ij} - \nabla(Q_{ij})Q_{kl}\right)$$

and write

$$-|\nabla Q|^2 Q_{ij} = A_{ij}^{kl} \cdot \nabla(Q_{kl}).$$

Remark 4.4.4. In these calculations, the indexes i, j are fixed, while the indexes k, l are summed. For i, j fixed, A_{ij}^{kl} is a vector field.

We now try to get a divergence in the second member (the raising of other terms without derivatives is allowed[1]). We note that, in the sense of distributions,

$$\operatorname{div}(A_{ij}^{kl}Q_{kl}) = \operatorname{div}(A_{ij}^{kl})Q_{kl} + A_{ij}^{kl} \cdot \nabla(Q_{kl}).$$

Let's calculate $\operatorname{div}(A_{ij}^{kl})$. Let $\varphi \in C_c^\infty(B_{\bar{r}})$.

Remark 4.4.5. According to Remark 4.4.4, $\operatorname{div}(A_{ij}^{kl})$ is a scalar field and hence has to be tested against a scalar field.

Thus,

$$\begin{aligned}\int_{B_{\bar{r}}} \operatorname{div}(A_{ij}^{kl})\varphi \,\mathrm{d}x &= -\int_{B_{\bar{r}}} A_{ij}^{kl} \cdot \nabla\varphi \,\mathrm{d}x \\ &= -\int_{B_{\bar{r}}} (\nabla(Q_{kl})Q_{ij} - \nabla(Q_{ij})Q_{kl}) \cdot \nabla\varphi \,\mathrm{d}x.\end{aligned}$$

Observe that

$$Q_{ij}\nabla\varphi = \nabla(Q_{ij}\varphi) - (\nabla(Q_{ij}))\varphi, \tag{4.4.17}$$

$$Q_{kl}\nabla\varphi = \nabla(Q_{kl}\varphi) - (\nabla(Q_{kl}))\varphi. \tag{4.4.18}$$

[1]Even terms growing as $|\nabla Q|$ are acceptable, see the proof of [119, Proposition 4.5].

hence

$$\begin{aligned}-A_{ij}^{kl}\cdot\nabla\varphi &= \nabla(Q_{kl})\cdot[\nabla(Q_{ij}\varphi)-\varphi\nabla(Q_{ij})]-\nabla(Q_{ij})\cdot[\nabla(Q_{kl}\varphi)-\varphi\nabla(Q_{kl})]\\ &= \nabla(Q_{kl})\cdot\nabla(Q_{ij}\varphi)-\nabla(Q_{ij})\cdot\nabla(Q_{kl}\varphi).\end{aligned}$$

Using the Euler-Lagrange equations,

$$\begin{aligned}&\int_{B_{\bar r}}\nabla(Q_{kl})\cdot\nabla(Q_{ij}\varphi)-\nabla(Q_{ij})\cdot\nabla(Q_{kl}\varphi)\,\mathrm{d}x\\ &=-\int_{B_{\bar r}}(\Delta Q_{kl})Q_{ij}\varphi\,\mathrm{d}x+\int_{B_{\bar r}}(\Delta Q_{ij})Q_{kl}\varphi\,\mathrm{d}x\\ &=\int_{B_{\bar r}}\left(|\nabla Q|^2 Q_{kl}Q_{ij}+f_{kl}Q_{ij}\right)\varphi\,\mathrm{d}x-\int_{B_{\bar r}}\left(|\nabla Q|^2 Q_{ij}Q_{kl}+f_{ij}Q_{kl}\right)\varphi\,\mathrm{d}x\\ &=-\int_{B_{\bar r}}(f_{kl}Q_{ij}-f_{ij}Q_{kl})\varphi\,\mathrm{d}x.\end{aligned}$$

Hence, in the sense of distributions,

$$\operatorname{div}(A_{ij}^{kl})=f_{ij}Q_{kl}-f_{kl}Q_{ij}.$$

Substituting back,

$$\begin{aligned}A_{ij}^{kl}\nabla(Q_{kl}) &= \operatorname{div}(A_{ij}^{kl}Q_{kl})+(f_{kl}Q_{ij}-f_{ij}Q_{kl})Q_{kl}\\ &= \operatorname{div}(A_{ij}^{kl}Q_{kl})-\underbrace{f_{ij}+f_{kl}Q_{kl}Q_{ij}}_{:=g_{ij}}\end{aligned}$$

Note that, if $T_0\in\mathcal{S}_0$ is a constant matrix, then

$$\operatorname{div}(A_{ij}^{kl}(Q_{kl}-(T_0)_{kl}))=\operatorname{div}(A_{ij}^{kl})(Q_{kl}-(T_0)_{kl})+A_{ij}^{kl}\cdot\nabla(Q_{kl}),$$

so that we have

$$\Delta Q=\operatorname{div}(A\cdot(Q-T_0))+g. \tag{4.4.19}$$

Before going on, we remark that for ∇h the following estimate holds:

$$|\nabla h(x)|^p\le C\bar r^{-p}⨍_{B_{\bar r}(\bar x)}|Q-T_0|^p\,, \tag{4.4.20}$$

with $T_0\in\mathcal{S}_0$ the aforementioned constant matrix. Indeed, h is a harmonic function, so $h-T_0$ is harmonic as well and hence, thanks to the mean property of harmonic functions, we have

$$\sup_{B_{\theta\bar r}(\bar x)}\bar r\,|\nabla h|\le C⨍_{\partial B_{\bar r}(\bar x)}|h-T_0|$$

for each $\theta\in(0,1)$. Since h and Q agree on the boundary of $B_{\bar r}(\bar x)$, $h-T_0$ and $Q-T_0$ also agree on $\partial B_{\bar r}(\bar x)$. Thus, by estimate (4.4.15) and by Jensen inequality, (4.4.20) follows.

Before we can apply Lemma 4.2, we have to prove the estimates on the norms of $A\cdot(Q-T_0)$ and g required in its statement. We prove that

$$\|A\cdot(Q-T_0)\|_{L^q(B_{\bar r}(\bar x))}\le C\sqrt{\varepsilon}\sqrt{\bar r}\|Q-T_0\|_{L^{\frac{2q}{2-q}}(B_{\bar r}(\bar x))}, \tag{4.4.21}$$

and that

$$\|g\|_{L^s(B_{\bar r}(\bar x))}\le C\bar r^{\frac{3}{s}}. \tag{4.4.22}$$

The second one is straightforward, since $g\in L^\infty(B_{\bar r},\mathbb{R}^5)$. For the first one, observe that $|A(x)|\le C\,|\nabla Q(x)|$ for each $x\in B_{\bar R}$. Hence, by Hölder inequality with conjugate exponents $\frac{2}{q}$ and $\frac{2}{2-q}$ by (4.4.2), (4.4.21) follows.

Estimates (4.4.21) and (4.4.22) contribute to give the following:

$$⨍_{B_{\bar r}(\bar x)}|\nabla(Q-h)|^q\le C\varepsilon^{q/2}\bar r^{-q}\left(⨍_{B_{\bar r}(\bar x)}|Q-T_0|^{\frac{2q}{2-q}}\right)^{\frac{2-q}{2}}+C\bar r^{-q}\bar r^{2q}. \tag{4.4.23}$$

Indeed, thanks to (4.4.21) and (4.4.22), we can apply Lemma 4.2, that gives (4.4.23), once one has gathered a factor $\bar r^3$ to have the volume mean.

Now, let

$$p=q^*=\frac{3q}{3-q}>q$$

and let

$$\rho=\sigma r<\bar r<r.$$

We now prove the central estimate

$$\left(⨍_{B_{\sigma r}(\bar x)}|Q-h(\bar x)|^p\right)^{1/p}\le(\sigma^{-d/p}\varepsilon^{1/2}+\sigma)\|Q\|_{\mathrm{BMO}(D_t)}+C\sigma^{-d/p}t^2, \tag{4.4.24}$$

Since $\rho<\bar r$, clearly

$$⨍_{B_{\sigma r}(\bar x)}|Q-h(\bar x)|^p\le\frac{C}{\rho^3}\int_{B_{\bar r}(\bar x)}|Q-h|^p+\frac{C}{\rho^3}\int_{B_{\sigma r}(\bar x)}|h-h(\bar x)|^p\,. \tag{4.4.25}$$

By the mean value theorem, for all $y\in B_{\sigma r}(\bar x)$,

$$|h(y)-h(\bar x)|\le|\nabla h(\xi)|\,|y-\bar x|\,,$$

where ξ is the unknown point of Lagrange on the segment $[y,\bar x]$ joining y and $\bar x$; since its length is surely at most diam $B_{\sigma r}(\bar x)$, being $\sigma\le 1/8$, we have

$$\frac{C}{\rho^3}\int_{B_{\sigma r}(\bar x)}|h-h(\bar x)|^p\le C\rho^p\sup_{B_{\bar r/4}(\bar x)}|\nabla h|^p\,.$$

Recalling (4.4.20), we estimate the second term in (4.4.25) as follows:

$$\frac{C}{\rho^3}\int_{B_{\sigma r}(\bar x)}|h-h(\bar x)|^p\le C\sigma^p⨍_{B_{\bar r}(\bar x)}|Q-T_0|^p\,. \tag{4.4.26}$$

We now deal with the first term in (4.4.25). Since $Q-h\in W^{1,2}_0(B_{\bar r},\mathbb{R}^5)$ and $p=q^*$, Sobolev-Gagliardo-Nirenberg inequality gives

$$\begin{aligned}\frac{C}{\rho^3}\int_{B_{\bar r}(\bar x)}|Q-h|^p &\le \frac{C}{\rho^3}\left(\int_{B_{\bar r}(\bar x)}|\nabla(Q-h)|^q\right)^{p/q}\\ &= C\frac{\bar r^{3p/q}}{\rho^3}\left(\fint_{B_{\bar r}(\bar x)}|\nabla(Q-h)|^q\right)^{p/q}\\ &\le \frac{C}{\rho^3}\bar r^{3p/q}\left(\bar r^{-p}\varepsilon^{p/2}\left(\fint_{B_{\bar r}(\bar x)}|Q-T_0|^{\frac{2q}{2-q}}\right)^{p\frac{2-q}{2q}}+C\bar r^p\right),\end{aligned}$$

the last inequality follows by (4.4.23). Being $\sigma=\frac{\rho}{r}$ and $\bar r<r$, we have $\sigma^{-1}>\frac{\bar r}{\rho}$. Write

$$\begin{aligned}\bar r^{3p/q-p} &= \bar r^3\bar r^{3p/q-p-3} &&=: \bar r^3\bar r^c,\\ \bar r^{3p/q+p} &= \bar r^3\bar r^{3p/q+p-3} &&=: \bar r^3\bar r^b.\end{aligned}$$

By the definition of p, we see that $c\equiv 0$ and $b=2p$, thus

$$\frac{C}{\rho^3}\int_{B_{\bar r}(\bar x)}|Q-h|^p \le C\sigma^{-3}\left(\bar\varepsilon^{p/2}\left(\fint_{B_{\bar r}(\bar x)}|Q-T_0|^{\frac{2q}{2-q}}\right)^{p\frac{2-q}{q}}+\bar r^{2p}\right). \tag{4.4.27}$$

By (4.4.27), (4.4.26) and the definition of σ, we have

$$\begin{aligned}\fint_{B_{\sigma r}(\bar x)}|Q-h(\bar x)|^p \le C\sigma^{-3}&\left(\bar\varepsilon^{p/2}\left(\fint_{B_{\bar r}(\bar x)}|Q-T_0|^{\frac{2q}{2-q}}\right)^{p\frac{2-q}{2q}}+\bar r^{2p}\right)\\ &+C\sigma^p\fint_{B_{\bar r}(\bar x)}|Q-T_0|^p,\end{aligned} \tag{4.4.28}$$

so

$$\begin{aligned}\left(\fint_{B_{\sigma r}(\bar x)}|Q-h(\bar x)|^p\right)^{1/p} \le C\sigma^{-3/p}&\left(\bar\varepsilon^{1/2}\left(\fint_{B_{\bar r}(\bar x)}|Q-T_0|^{\frac{2q}{2-q}}\right)^{\frac{2-q}{2q}}+\bar r^2\right)^{1/p}\\ &+C\sigma\left(\fint_{B_{\bar r}(\bar x)}|Q-T_0|^p\right)^{1/p}.\end{aligned} \tag{4.4.29}$$

Fix $T_0=\fint_{B_{\bar r}(\bar x)}Q$. Noting that $\bar r<r$, so that $B_{\bar r}(\bar x)\subset B_r(\bar x)\subset D_t$, John-Nirenberg inequality gives

$$\left(\fint_{B_{\sigma r}(\bar x)}|Q-h(\bar x)|^p\right)^{1/p} \le C\left(\sigma^{-3/p}\varepsilon^{1/2}+\sigma\right)\|Q\|_{\mathrm{BMO}(D_t)}+C\sigma^{-3}t^2 \tag{4.4.30}$$

Since constant matrices coincide with their means, we can bring them in and out the means when convenient. Then Hölder inequality and (4.4.30) imply

$$\fint_{B_{\sigma r}(\bar{x})} \left| Q - \fint_{B_{\sigma r}(\bar{x})} Q \right| \leq \left(\fint_{B_{\sigma r}(\bar{x})} |Q - h(\bar{x})|^2 \right)^{1/2}$$
$$\leq \left(\fint_{B_{\sigma r}(\bar{x})} |Q - h(\bar{x})|^p \right)^{1/p}$$
$$\leq C \left(\sigma^{-3/p} \varepsilon^{1/2} + \sigma \right) \|Q\|_{\mathrm{BMO}(D_t)} + C\sigma^{-3/p} t^2.$$

Since $\bar{x}$ was arbitrary among points in $D_{\sigma t}$ such that the ball with radius σr and center $\bar{x}$ was contained $D_{\sigma t}$, we can take the supremum over the balls $B_{\sigma r} \subset D_{\sigma t}$, so that

$$\|Q\|_{\mathrm{BMO}(D_{\sigma t})} \leq C \left(\sigma^{-3/p} \varepsilon^{1/2} + \sigma \right) \|Q\|_{\mathrm{BMO}(D_t)} + C\sigma^{-3/p} t^2. \tag{4.4.31}$$

Now, choose $\sigma \in (0, 1/8]$ and $\varepsilon > 0$ so small that $C(\sigma^{-3/p}\varepsilon^{1/2} + \sigma) < \frac{1}{2}$. Hence, for some sufficiently large constant $C(q, \sigma, \varepsilon) > 0$ (independent of Q, $\hat{x}$ and t) and each $t \in (0, R_0/2]$, we have

$$\|Q\|_{\mathrm{BMO}(D_{\sigma t})} \leq \frac{1}{2} \|Q\|_{\mathrm{BMO}(D_t)} + C(q, \sigma, \varepsilon) t^2. \tag{4.4.32}$$

Set $\alpha = \log_\sigma \left(\frac{1}{2}\right)$ and note that, for $\sigma \in (0, 1/8]$, we have $\alpha \in (0, 1/3]$. Hence, in virtue of the monotonicity of the BMO-norm and (4.4.13), by Lemma 4.1 with $\varphi(t) = \|Q\|_{\mathrm{BMO}(D_t)}$, $R = R_0/2$ and $\gamma = 2$ and by (4.4.32), it follows

$$\|Q\|_{\mathrm{BMO}(D_t)} \leq \left(2^\alpha \sigma^{-\alpha} \sqrt{\varepsilon} C' + 2R_0^2 \right) \left(\frac{t}{R_0} \right)^\alpha = C'' \left(\frac{t}{R_0} \right)^\alpha \tag{4.4.33}$$

where C' denotes the constant appearing in (4.4.13).

Now, by definition of BMO norm and the John-Nirenberg inequality, it follows

$$\fint_{D_t(\hat{x})} \left| Q - \fint_{D_t(\hat{x})} Q \right|^2 \leq C_2 \|Q\|^2_{\mathrm{BMO}(D_t(\hat{x}))},$$

with C_2 the John-Nirenberg constant for $p = 2$. By (4.4.33),

$$\fint_{D_t(\hat{x})} \left| Q - \fint_{D_t(\hat{x})} Q \right|^2 \leq \tilde{C} \left(\frac{t}{R_0} \right)^{2\alpha},$$

where $\tilde{C}$ is constant accounting for the John-Nirenberg constant and for the square of C''. Set

$$\beta = 2\alpha.$$

According to (4.4.5), we have

$$\int_{D_t(\hat{x})} \left| Q - \fint_{D_t(\hat{x})} Q \right|^2 \leq \tilde{C} \left(\frac{t}{R_0} \right)^{\beta+3}. \tag{4.4.34}$$

Since $\hat{x}$ was arbitrary in $B_{R_0/2}(x_0)$, (4.4.34) says exactly that Q belongs to the Campanato space $\mathcal{L}^{2,\beta+3}(B_{R_0/2}(x_0), \mathbb{R}^5)$. From Campanato's theorem, we have $Q \in C^{0,\beta}(\overline{B_{R_0/2}(x_0)}, \mathbb{R}^5)$. Since Q is S^4-valued, the thesis follows. □

Remark 4.4.6. Theorem 4.6 is *qualitatively different* from the analogous theorem for harmonic maps by Schoen & Uhlenbeck. Indeed, Schoen & Uhlenbeck theorem [130, 134] apply to *minimizing* harmonic maps, while here we ask only stationarity. Moreover, Schoen-Uhlenbeck theorem provides also an estimate for the gradient in terms of the rescaled energy. However, in low dimensions, precisely at most 3, continuity allows to get L^4-integrability of the gradient (cfr. Proposition 4.9) that is the threshold at which it is possible bootstrapping as in proof of Theorem 4.10.

4.5 Higher regularity

For higher regularity we use a bootstrap argument, as in [119, Proposition 5.2]. In order to start the iteration, it is however necessary to show that if Q is a weak solutions of the Euler-Lagrange equations and it is continuous, then it is actually a *strong solution*. More precisely, we prove the following.

Proposition 4.9. *Let $Q \in W^{1,2}(B_1, S^4)$ be a weak solutions of the Euler-Lagrange equations* (4.2.1)*. If $Q \in C^0(B_1, S^4)$, then $Q \in W^{1,4}(B_1, S^4) \cap W^{2,2}(B_1, S^4)$.*

Proof. We follow the proof of [119, Proposition 1]. Rewrite the Euler-Lagrange equations in the form

$$\Delta Q = \mathcal{F}(f(Q(x)), Q, \nabla Q) = \mathscr{F}(x, Q, \nabla Q), \tag{4.5.1}$$

where $\mathcal{F} : (\mathcal{S}_0)^3 \times \mathcal{S}_0 \times (\mathcal{S}_0)^3 \to \mathcal{S}_0$ is a real-analytic map (indeed, it is polynomial in its arguments). Since $f(Q(\cdot))$ (defined in (4.4.16)) is smooth and Q is S^4-valued, $\mathscr{F}(x, s, p)$ satisfies (by construction) on the image of Q the structure hypothesis

$$|\mathscr{F}(x, s, p)| + |\nabla_s \mathscr{F}(x, s, p)| \le c_0(1 + |p|^2), \tag{4.5.2}$$
$$|\nabla \mathscr{F}(x, s, p)| + |\nabla_p \mathscr{F}(x, s, p)| \le c_1(1 + |p|^2), \tag{4.5.3}$$

on $B_1 \times S^4 \times (\mathcal{S}_0)^3$. Using now [80, Lemma 8.5.1] and [80, Lemma 8.5.3], each continuous solution of (4.5.1) is locally $W^{1,4} \cap W^{2,2}$ and the conclusion follows by taking a finite covering of B_1. ☐

Theorem 4.10 (Higher regularity)**.** *Let $Q \in W^{1,2}(B_1, S^4)$ be a weak solution of the Euler-Lagrange equations* (4.2.1)*. If $Q \in C^0(B_1, S^4)$, then Q is real-analytic.*

Proof. By Proposition 4.9, $Q \in W^{1,4}(B_1, S^4) \cap W^{2,2}(B_1, S^4)$. By Sobolev embedding theorem, $\nabla Q \in L^6$ and by (4.5.2) it follows $\mathscr{F}(x, Q, \nabla Q) \in L^3$. Linear elliptic regularity for (4.5.1) gives $Q \in W^{2,3}$ and in turn $\nabla Q \in L^p$ for all $p < \infty$ again by Sobolev embedding. Now, if $\nabla Q \in L^p$ for all $p < \infty$, then the same is true (because of (4.5.2)) also for $\mathscr{F}(x, Q, \nabla Q)$ and the linear elliptic regularity for Q gives $Q \in W^{2,p}$ for all $p < \infty$. By Sobolev-Morrey embedding, it then follows $Q \in C^{1,\alpha}$ for all $\alpha \in (0, 1)$. Going back to Euler-Lagrange equations, a bootstrap argument in the Hölder spaces $C^{l,\alpha}$, $l \ge 1$ leads us to the chain of implications

$$Q \in C^{l,\alpha} \implies \Delta Q \in C^{l-1,\alpha} \implies Q \in C^{l+1,\alpha},$$

so that $Q \in C^\infty(B_1, S^4)$. Since $f(\cdot)$ is smooth, the results in [110, Chapter VI] imply that each smooth solution of the Euler-Lagrange equations is in fact real-analytic. ☐

4.6 The compactness theorem

Let $Q \in \mathcal{A}_{Q_b}$ be a minimizer of the LdG energy in the class $\mathcal{A}_{Q_b}$. Let $x_0 \in B_1$ and denote

$$R_0 = \max\left\{ r > 0 : \overline{B_r(x_0)} \subset B_1 \right\}. \tag{4.6.1}$$

Clearly, $0 < R_0 \leq 1$. Pick $R > 0$. We set

$$Q_R(x) := Q(x_0 + Rx) \tag{4.6.2}$$

for all $x \in \mathbb{R}^3$ such that the right hand side makes sense. As R varies in $(0, R_0]$, we get a family of maps $\{Q_R\}_R$. We call each map as in (4.6.2) a *scaled map* or a *blow-up* of Q with center x_0. We note few facts.

First, note that, for R fixed, Q_R is well-defined on each ball $B_{\frac{\rho R_0}{R}}$ with $\rho \in (0,1]$. Second, the balls $B_{\frac{\cdot}{R}}$ become bigger and bigger as R decreases.

We now prove that $\{Q\}_R$ is locally equibounded in $W^{1,2}_{\text{loc}}(\mathbb{R}^3, S^4)$.

Lemma 4.3. *Let $Q \in \mathcal{A}_{Q_b}$ be a minimizer of the LdG energy* (1.1.3)*. Pick $\sigma > 0$ and, for $R < R_0/\sigma$, define scaled maps Q_R as in* (4.6.2)*. Then*

$$\limsup_{R \to 0} \int_{B_\sigma} |\nabla Q_R|^2 \, dx < +\infty \tag{4.6.3}$$

for each $\sigma > 0$. In other words, the family $\{Q_R\}_R$ is locally equibounded in $W^{1,2}_{loc}(\mathbb{R}^3, S^4)$.

Proof. Let $\sigma > 0$ be arbitrary. For all $R < R_0/\sigma$, by the monotonicity formula (4.3.1) we have

$$\begin{aligned}
\frac{1}{\sigma} \int_{B_\sigma} |\nabla Q_R|^2 \, \mathrm{d}x &\leq \frac{1}{\sigma} \int_{B_\sigma} |\nabla Q_R|^2 + 2R^2 F(Q_R) \, \mathrm{d}x \\
&\leq \frac{2}{\sigma R} \int_{B_{\sigma R}(x_0)} \frac{1}{2} |\nabla Q|^2 + F(Q) \, \mathrm{d}x \\
&\leq \frac{2}{R_0} E(Q; B_{R_0}(x_0)),
\end{aligned}$$

hence

$$\limsup_{R \to 0} \int_{B_\sigma} |\nabla Q_R|^2 \, \mathrm{d}x < +\infty$$

for each $\sigma > 0$.

By the very definition of Q_R, we clearly have that $\{Q\}_R$ is locally equibounded in $L^2_{\text{loc}}(\mathbb{R}^3, S^4)$. Since each compact set in $\mathbb{R}^3$ can be enclosed in a sufficiently large ball centered at the origin, the second claim follows. ☐

Remark 4.6.1. We observe that, in particular, each Q_R is well-defined on B_{R_0}. This means that, up to a fixed translation and a fixed dilation, we can assume $x_0 = 0$ and $R_0 = 1$.

For the sake of a lighter notation, from now on we take Remark 4.6.1 into account.

Following e.g. [130, 39, 60, 98], we next define scaled energy functionals. Set

$$E_R\left(\tilde{Q}; B_{\frac{1}{R}}\right) = \int_{B_{\frac{1}{R}}} \frac{1}{2}\left|\nabla\tilde{Q}\right|^2 + R^2 F(\tilde{Q})\,\mathrm{d}x. \tag{4.6.4}$$

$E_R\left(\cdot; B_{\frac{1}{R}}\right)$ is well defined for $\tilde{Q} \in W^{1,2}\left(B_{\frac{1}{R}}, \mathcal{S}_0\right)$. Note that

$$E_R\left(Q_R; B_1\right) = \frac{1}{R}E(Q; B_R),$$

so that $E_R\left(Q_R; B_1\right)$ increases as $R \nearrow 1$ by the monotonicity formula (4.3.1).

We now prove a lemma analogous to [119, Lemma 4.6].

Lemma 4.4. *Let $Q \in \mathcal{A}_{Q_b}$ be a minimizer of the LdG energy in $\mathcal{A}_{Q_b}$. Let R, Q_R, E_R as above. Let $\rho \in (0,1)$ and let $\{v_R\}_R \subset W^{1,2}(B_1, S^4)$ be a family of mappings such that $v_R = Q_R$ on ∂B_ρ. Then*

$$\liminf_{R\to 0}\int_{B_\rho} \left|\nabla Q_R\right|^2\, dx \le \liminf_{R\to 0}\int_{B_\rho} \left|\nabla v_R\right|^2\, dx. \tag{4.6.5}$$

Proof. Define $\tilde{v}_R(x) = v_R(R^{-1}x)$ so that $\tilde{v}_R \in W^{1,2}(B_{\rho R}, S^4)$. Since $\tilde{v}_R = Q$ on $\partial B_{\rho R}$ we can extend $\tilde{v}_R$ as Q on the whole $B_1 \setminus B_{\rho R}$. We have

$$\begin{aligned} E(\tilde{v}_R; B_1) - E(Q; B_1) &= E(\tilde{v}_R; B_{\rho R}) - E(Q; B_{\rho R}) \\ &= \int_{B_{\rho R}} \frac{1}{2}\left|\nabla \tilde{v}_R\right|^2 + F(\tilde{v}_R)\,\mathrm{d}x - \int_{B_{\rho R}} \frac{1}{2}\left|\nabla Q\right|^2 + F(Q)\,\mathrm{d}x \\ &\ge 0, \end{aligned}$$

because Q minimizes $E(\cdot; B_1)$ in $\mathcal{A}_{Q_b}$.

Since $F \in L^\infty$, it follows

$$\left|\int_{B_{\rho R}} F(\tilde{v}_R)\,\mathrm{d}x - \int_{B_{\rho R}} F(Q)\,\mathrm{d}x\right| \le C\rho^3 R^3,$$

for some constant $C > 0$. Scaling back gives

$$\int_{B_\rho} \frac{1}{2}\left|\nabla Q_R\right|^2\,\mathrm{d}x + \mathcal{O}(R) \le \int_{B_\rho} \frac{1}{2}\left|\nabla v_R\right|^2\,\mathrm{d}x,$$

so we can pass to the limit inferior on both sides as $R \to 0$ and then we get (4.6.5). ☐

Before stating the compactness theorem, we recall the Luckhaus' lemma [98, 134] whose statement below is directly written for our specific context.

Lemma 4.5 (Luckhaus)**.** *Let $u, v \in W^{1,2}(S^2, S^4)$. Then, for each $\lambda \in (0,1)$ there is $w \in W^{1,2}(S^2 \times (1-\lambda, 1), \mathcal{S}_0)$ such that $w|_{S^2\times\{1\}} = u, w|_{S^2\times 1-\lambda} = v$,*

$$\int_{S^2\times(1-\lambda,1)} |\nabla w|^2 \le C\lambda\int_{S^2}\left(|\nabla_T u|^2 + |\nabla_T v|^2\right) + C\lambda^{-1}\int_{S^2}|u-v|^2 \tag{4.6.6}$$

and

$$\begin{aligned} \operatorname{dist}^2(w(x), S^4) \le C\lambda^{-2} \left(\int_{S^2} \left(|\nabla_T u|^2 + |\nabla_T v|^2 \right) \right)^{\frac{1}{2}} \left(\int_{S^2} |u - v|^2 \right)^{\frac{1}{2}} \\ + C\lambda^{-3} \int_{S^2} |u - v|^2 \end{aligned} \tag{4.6.7}$$

for a.e. $x \in S^2 \times (1 - \lambda, 1)$*. Here* ∇_T *is the gradient on* S^2*.*

The proof can be found, along with some useful corollaries, for instance in [134, Chapter 2].

Theorem 4.11 (Compactness theorem in the nonsymmetric case)**.** *Let* $Q \in \mathcal{A}_{Q_b}$ *be a minimizer of* $E(\cdot, B_1)$ *in the class* $\mathcal{A}_{Q_b}$*. Let* $R \in (0, 1]$ *and let* Q_R *be as in* (4.6.2) *(with Remark 4.6.1 understood). Then there is* $Q_0 \in W^{1,2}_{loc}(\mathbb{R}^3, S^4)$ *and there is a sequence* $(Q_{R_j})_{R_j}$*,* $R_j \to 0$ *as* $j \to \infty$*, which converges to* Q_0 *in the strong topology of* $W^{1,2}_{loc}(\mathbb{R}^3, \mathcal{S}_0)$ *as* $j \to \infty$*. In addition,* Q_0 *is a locally minimizing harmonic map and it is degree-zero homogeneous.*

Proof. We essentially follow the proof of [119, Proposition 4.4] (which in turn retraces that by Lin and Wang [97, Lemma 2.2.13]) up to minor modifications.

By Lemma 4.3, $\{Q_R\}_R$ is locally equibounded in $W^{1,2}_{\text{loc}}(\mathbb{R}^3, S^4)$ and so it is each sequence $(Q_{R_j})_{R_j}$, with $R_j \to 0$ as $j \to \infty$, extracted from it. By the Rellich-Kondrachov theorem there exists $Q_0 \in W^{1,2}_{\text{loc}}(\mathbb{R}^3, \mathcal{S}_0)$ so that, up to subsequences, we have $Q_{R_j} \rightharpoonup Q_0$ (weakly) as $j \to \infty$ in $W^{1,2}_{\text{loc}}(\mathbb{R}^3, \mathcal{S}_0)$ and $Q_{R_j} \to Q_0$ strongly in $L^2_{\text{loc}}(\mathbb{R}^3, S^4)$. Thus, in particular, $Q_0(x) \in S^4$ a.e., i.e., $Q_0 \in W^{1,2}_{\text{loc}}(\mathbb{R}^3, S^4)$. By the monotonicity formula (4.3.1) and the equiboundedness of the potential (so that it disappears in the limit $R \to 0$), it easily follows (mimicking, for instance, [130, Lemma 2.6] or the reasoning in [134, Section 3.2]) that Q_0 is degree-zero homogeneous. Thus, it is enough to show strong convergence and minimality in some ball $B_\rho \subset B_1$ to get the same properties on any $B_\rho \subset \mathbb{R}^3$ for any $\rho > 0$, by the scale invariance of Q_0 and the existence of the full limit of $\frac{1}{R} \int_{B_R} |\nabla Q|^2 \, \mathrm{d}x$ as $R \to 0$.

Let $\delta \in (0, 1)$ be a fixed number and let $\bar{w} \in W^{1,2}(B_1, S^4)$ be such that $\bar{w} \equiv Q_0$ a.e. on $B_1 \setminus B_{1-\delta}$. By Fatou's lemma and Fubini's theorem, there exists $\rho \in (1 - \delta, 1)$ such that

$$\lim_{j \to \infty} \int_{\partial B_\rho} \left| Q_{R_j} - Q_0 \right|^2 \mathrm{d}\mathcal{H}^2 = 0,$$

and

$$\int_{\partial B_\rho} \left(\left| \nabla Q_{R_j} \right|^2 + |\nabla Q_0|^2 \right) \mathrm{d}\mathcal{H}^2 \le C < +\infty.$$

Applying Lemma 4.5 to

$$\lambda = \lambda_{R_j} < \delta, \quad u = Q_{R_j}(\rho \cdot), \quad v = \bar{w}(\rho \cdot) \equiv Q_0(\rho \cdot),$$

for a decreasing sequence of numbers $\lambda_{R_j} \to 0$, we conclude that there exists a sequence $w_{R_j} \in W^{1,2}(B_\rho, \mathcal{S}_0)$ such that if we choose e.g.

$$\lambda_{R_j} = \left(\int_{\partial B_\rho} \left| Q_{R_j} - Q_0 \right|^2 \mathrm{d}H^2 \right)^{1/6} < \delta,$$

then we have

$$w_{R_j}(x) = \begin{cases} \bar{w}\left(\frac{x}{1-\lambda_{R_j}}\right), & |x| \le \rho(1-\lambda_{R_j}), \\ Q_{R_j}(x), & |x| = \rho. \end{cases} \tag{4.6.8}$$

$$\int_{B_\rho \setminus B_{\rho(1-\lambda_{R_j})}} \left|\nabla w_{R_j}\right|^2 \le C\left[\lambda_{R_j}\int_{\partial B_\rho}\left(\left|\nabla_T Q_{R_j}\right|^2 + |\nabla_T Q_0|^2\right) + \lambda_{R_j}^{-1}\int_{\partial B_\rho}\left|Q_{R_j} - Q_0\right|^2\right] \overset{j\to\infty}{\longrightarrow} 0. \tag{4.6.9}$$

$$\operatorname{dist}(w_{R_j}, S^4) \overset{j\to\infty}{\longrightarrow} 0 \text{ uniformly on } B_\rho \setminus B_{\rho(1-\lambda_{R_j})}. \tag{4.6.10}$$

Define comparison maps $(v_{R_j})_{R_j} \subset W^{1,2}(B_\rho, S^4)$

$$v_{R_j}(x) = \begin{cases} \overline{w}\left(\frac{x}{1-\lambda_{R_j}}\right), & |x| \le \rho(1-\lambda_{R_j}) \\ \Pi(w_{R_j}(x)), & \rho(1-\lambda_{R_j}) \le |x| \le \rho, \end{cases}$$

where $\Pi : \mathcal{O} \to S^4$ is the nearest point projection ($\mathcal{O}$ a sufficiently narrow neighborhood of S^4 so that Π is well-defined and smooth as needed). Then, by Lemma 4.4, (4.6.9) and (4.6.10),

$$\begin{aligned} \int_{B_\rho} |\nabla Q_0|^2 &\le \liminf_{j\to\infty} \int_{B_\rho} \left|Q_{R_j}\right|^2 \\ &\le \liminf_{j\to\infty} \int_{B_\rho} \left|\nabla v_{R_j}\right|^2 \\ &= \lim_{j\to\infty}\left[\int_{B_{\rho(1-\lambda_{R_j})}} \left|\nabla\overline{w}\left(\frac{\cdot}{1-\lambda_{R_j}}\right)\right|^2 + \int_{B_\rho\setminus B_{\rho(1-\lambda_{R_j})}} \left|\nabla(\Pi\circ w_{R_j})\right|^2\right] \\ &\le \lim_{j\to\infty}\left[(1-\lambda_{R_j})\int_{B_\rho}|\nabla\overline{w}|^2 + C\,\mathrm{Lip}(\Pi)^2\int_{B_\rho\setminus B_{\rho(1-\lambda_{R_j})}}\left|\nabla w_{R_j}\right|^2\right] \\ &= \int_{B_\rho} |\nabla\overline{w}|^2. \end{aligned} \tag{4.6.11}$$

Since $\overline{w}$ is arbitrary, inequality (4.6.11) implies both minimality of Q_0 and strong convergence $Q_{R_j} \to Q_0$ in $W^{1,2}(B_\rho, S^4)$ as $j \to \infty$. □

An easy consequence of Theorem 4.11 is the (joint) upper semicontinuity of the density of Q_0. Together with Theorem 4.6, this readily implies that the singular set of a minimizer $Q \in \mathcal{A}_{Q_b}$ of the LdG energy in the class $\mathcal{A}_{Q_b}$ is a finite set of isolated points. We shall give explicit proofs for the analogous results in the equaviarant case. In fact, the proofs are exactly the same in the two cases, up to typographical modifications which make the equivariant case a little more delicate (compare Corollaries 5.15, 5.16).

4.7 Liouville theorem and global interior regularity

In the nonsymmetric case, the Liouville theorem of Schoen-Uhlenbeck [132, Corollary 2.8] applies, hence we have that minimizing tangent maps are constant and from this global regularity for minimizers of the LdG energy follows.

To summarize, we state

Theorem 4.12 (Global interior regularity of nonsymmetric minimizers)**.** *Let $Q \in \mathcal{A}_{Q_b}$ be a minimizer for the LdG energy* (1.1.3) *within the class $\mathcal{A}_{Q_b}$, defined in* (1.1.14)*, with Q_b as in* (1.1.13)*. Then Q is real-analytic in the interior of B_1; i.e., $Q \in C^\omega(B_1, S^4)$.*

Proof. By the ε-regularity theorem, Theorem 4.6, Q is (Hölder-)continuous in a neighboorhood of each point at which there is no concentration of energy. By the monotonicity formula, the singular set of Q coincides with its concentration set, hence a point where Q is not singular is in fact a regular point. Moreover, the singular set is a closed subset of B_1 of Hausdorff dimension strictly less than one (see, for instance, [53, Proposition 9.21]). To decide whether a point $x_0 \in B_1$ is singular or not, blow-up Q around it. The compactness theorem, Theorem 4.11, implies that, whatever x_0, blown-up maps around it converges in the strong $W^{1,2}_{\text{loc}}(\mathbb{R}^3, S^4)$-topology to a minimizing tangent map $Q_0 \in W^{1,2}_{\text{loc}}(\mathbb{R}^3, S^4)$. Q_0 is a homogeneous-degree zero harmonic map from $\mathbb{R}^3$ into S^4, so it is smooth on $\mathbb{R}^3 \setminus \{0\}$ by the Hélein's theorem [68], in fact constant by the Liouville theorem of Schoen& Uhlenbeck [132, Corollary 2.8]. Then Q is Hölder continuous in a neighborhood of x_0. Since x_0 is arbitrary, Q_0 is Hölder continuous everywhere. Hence Q_0 is a strong solution of the Euler-Lagrange equations (4.2.1), so that it fits hypotheses of the higher-regularity theorems, Proposition 4.9 and Theorem 4.10. Thus we have the conclusion. □

Note that the Liouville theorem of Schoen&Uhlenbeck *does not* hold in the symmetric case. Its lack is the main difficulty in the study of the S^1-equivariant problem.

4.8 Boundary regularity

We now extend the regularity results up to the boundary. More precisely, we prove that there exists a full neighboorhood of the boundary S^2 of B_1 on which Q is Hölder continuous (hence smooth as the datum and the boundary allow by higher-regularity theorems). The argument is similar to that for interior regularity. Hence, following [131] and [97, Section 2.4], we need a boundary monotonicity formula, a boundary ε-regularity theorem, a boundary compactness theorem and we have to prove the nonexistence of nonconstant minimizing tangent maps at any boundary point.

Actually, the main step towards boundary regularity is the boundary monotonicity formula. Indeed, once we got this, we can use it to prove the boundary versions of ε-regularity theorem and of the compactness of blow-ups with a small modification of the argument that we shall indicate afterwards. Moreover, the last step can be accomplished by [97, Theorem 2.4.3] (which we state below for our specific case).

Theorem 4.13 ([97, Theorem 2.4.3])**.** *Any minimizing harmonic map $u_0 \in W^{1,2}(B_1^+, S^4)$ that is homogeneous of degree zero and that is constant on $B_1 \cap \{x_n = 0\}$ must be constant.*

Before stating the result, we introduce a bit of notation. Let $x_0 \in \partial B_1 = S^2$. We denote

$$\Omega_r := B_1 \cap B_r(x_0) \tag{4.8.1}$$

and we let n be the outward unit normal to $\partial\Omega_r$. The area element on $\partial\Omega_r$ is denoted $d\sigma$. Throughout this section, we set $L = 1$ for ease.

We now state the result.

Theorem 4.14 (Boundary monotonicity formula)**.** *Let $Q \in \mathcal{A}_{Q_b}$ be a minimizer of the LdG energy in the class $\mathcal{A}_{Q_b}$, where Q_b is given in* (1.1.13) *and let $x_0 \in \partial B_1$. Define*

$$\mathcal{E}_r = \frac{1}{r}\int_{\Omega_r} \frac{1}{2}|\nabla Q|^2 + F(Q)\, dx. \tag{4.8.2}$$

Then there exist $R_0 > 0$ and a constant $C = C(a,b,c,Q_b,R_0)$, $C > 0$, so that

$$\frac{d\mathcal{E}_r}{dr} \geq -C(a,b,c,Q_b,R_0), \quad \forall 0 < r < R_0. \tag{4.8.3}$$

Proof. The main idea is to follow the proof of [106, Lemma 9]. The only problem with this is that calculations in it require $Q \in W^{3,2}(B_1, S^4) \cap C^{1,\alpha}(\overline{B_1}, S^4)$ for all $\alpha \in (0,1)$, and we do not known if this is the case. To overcome the problem, we use the following trick: for any $\varepsilon > 0$, define

$$E_\varepsilon(P; B_1) := E(P; B_1) + \frac{1}{4\varepsilon^2}\int_{B_1}\left(1 - |P|^2\right)^2 \mathrm{d}x + \frac{1}{2}\int_{B_1}|P - Q|^2\, \mathrm{d}x,$$

on

$$\mathcal{A}_{Q_b}(\mathcal{S}_0) := \left\{P \in W^{1,2}(B_1, S^4) : P = Q_b \text{ on } S^2\right\}.$$

Since the norm constraint has been removed, the Euler-Lagrange equations corresponding to $E_\varepsilon(\cdot; B_1)$ are semilinear with a polynomial nonlinearity. Thus, any critical point of $E_\varepsilon(\cdot; B_1)$ is completely smooth up to the boundary and hence we get a boundary monotonicity formula as in [106] (of course we will have also some extra terms generated by the two penalization we introduced). If we now consider a family $\{P^\varepsilon\}_{\varepsilon>0}$ of minimizers of $E_\varepsilon(\cdot; B_1)$, then it is easy to prove[2] that $P^\varepsilon \to Q$ strongly in $W^{1,2}(B_1, S^4)$ and hence we obtain a boundary monotonicity formula for Q passing to the limit in the boundary monotonicity formulae for the P^εs. Since the extra terms vanish in the limit, we get Eq. (4.3.1). □

Once the boundary monotonicity formula is given, we can extend the ε-regularity theorem also to balls intersecting the boundary. In order to do this, a suitable reflection argument of the map across the boundary is needed. Indeed, in order the argument *via* John-Nirenberg inequality works, we have to define the map in a full ball and to control the energy of the extension with the energy of the original map, *via* the monotonicity formula. On the other hand, we also need that the extended map satisfies an appropriate version of Lemma 4.2, thus the extension must be constructed with care. A suitable way is the following: write $x' = \frac{x}{|x|}$ and define

[2] The proof of this statement is completely analogous to the one we shall give in Chapter 5 for S^1-equivariant minimizers. Since the symmetric case is less explored, we feel more convenient to give full details for that case and being more concise here, in order to contain the volume of this thesis.

$$\hat{Q}(x) := \begin{cases} Q(x), & \text{if } x \in B_1, \\ Q\left(\frac{x}{|x|^2}\right), & \text{if } x \in B_2 \setminus B_1. \end{cases} \tag{4.8.4}$$

Writing the system of the Euler-Lagrange equations in the shortened form

$$\begin{cases} -\Delta Q = |\nabla Q|^2 Q + f(Q), & \text{in } B_1, \\ Q = Q_b, & \text{on } \partial B_1, \end{cases}$$

and recalling that, if $u : B_1 \to \mathbb{R}$ and $v(x) = u\left(\frac{x}{|x|^2}\right)$ for $x \in B_2 \setminus B_1$ it holds

$$\int_{B_2 \setminus B_1} \frac{1}{|x|^2} |\nabla v|^2 \, \mathrm{d}x = \int_{B_1 \setminus B_{1/2}} |\nabla u|^2 \, \mathrm{d}x,$$

we are led to

$$\begin{cases} -\operatorname{div}\left(\frac{1}{|x|^2}\nabla \hat{Q}\right) = \frac{1}{|x|^2}\left|\nabla \hat{Q}\right|^2 \hat{Q} + \frac{1}{|x|^6} f(\hat{Q}), & \text{in } B_2 \setminus B_1, \\ \hat{Q} = Q_b, & \text{on } \partial B_1. \end{cases} \tag{4.8.5}$$

Therefore

$$-\Delta \hat{Q} = \left|\nabla \hat{Q}\right|^2 \hat{Q} - \left(\frac{x}{|x|^2} \cdot \nabla\right) \hat{Q} + \frac{1}{|x|^4} f(\hat{Q}), \quad \text{in } B_2 \setminus B_1 \tag{4.8.6}$$

in the weak sense.

Notation. For $P, N \in \mathcal{S}_0$, we let $P \otimes N$ the linear mapping $\mathcal{S}_0 \to \mathcal{S}_0$ given by

$$(P \otimes N)Q := \langle N, Q \rangle \, P. \tag{4.8.7}$$

The map $\otimes$ is bilinear on $\mathcal{S}_0 \times \mathcal{S}_0$.

For $x \in B_2$, we define

$$\tilde{Q}(x) = \begin{cases} (2Q_b(x') \otimes Q_b(x') - I)\, \hat{Q}(x), & \text{if } 1 < |x| < 2, \\ Q(x), & \text{if } |x| < 1. \end{cases} \tag{4.8.8}$$

Observe that for $|x| > 1$

$$\begin{aligned} \frac{\partial \tilde{Q}}{\partial r}(x) &= (2Q_b(x') \otimes Q_b(x') - I)\, \frac{\partial \hat{Q}(x)}{\partial r} \\ &= (I - 2Q_b(x') \otimes Q_b(x'))\, \frac{1}{|x|^2}\frac{\partial Q}{\partial r}\left(\frac{x}{|x|^2}\right) \\ &= \frac{1}{|x|^2}\frac{\partial Q}{\partial r}\left(\frac{x}{|x|^2}\right) - 2\left\langle Q_b(x'), \frac{\partial Q}{\partial r}\left(\frac{x}{|x|^2}\right)\right\rangle Q_b(x') \\ &= \frac{1}{|x|^2}\frac{\partial Q}{\partial r}\left(\frac{x}{|x|^2}\right) - 2\left\langle Q_b(x') - Q\left(\frac{x}{|x|^2}\right), \frac{\partial Q}{\partial r}\left(\frac{x}{|x|^2}\right)\right\rangle Q_b(x'), \end{aligned}$$

so that $\frac{\partial \tilde{Q}}{\partial r} = \frac{\partial Q}{\partial r}$ on ∂B_1, which leads to

$$\int_{B_2} \left\langle \tilde{Q}, \varphi \right\rangle \mathrm{d}x = \int_{B_1} \langle -\Delta Q, \varphi \rangle \, \mathrm{d}x + \int_{B_2 \setminus B_1} \left\langle -\Delta \tilde{Q}, \varphi \right\rangle \mathrm{d}x \quad \forall \varphi \in C_c^\infty(B_2, \mathbb{R}^5).$$

The fact that the previous equality holds for *any* $\varphi \in C_c^\infty(B_2, \mathbb{R}^5)$ and not only for those vanishing on ∂B_1 follows from arguments in [127, 128], since the reflection map satisfies the same relevant properties.

Remark 4.8.1. For $N \in \mathcal{S}_0$, $|N| = 1$, the mapping $N \mapsto (2N \otimes N - I)$ is isometric and $(2N \otimes N - I)(2N \otimes N - I) = I$.

Easy computations give

$$\begin{aligned} -\Delta \tilde{Q} &= (2Q_b \otimes Q_b - I)(-\Delta \hat{Q}) + (\text{terms of order } \leq 1) \\ &= \left|\nabla \hat{Q}\right|^2 \tilde{Q} + (\text{terms of order } \leq 1) \\ &= \left|\nabla \tilde{Q}\right|^2 \tilde{Q} + (\text{terms of order } \leq 1), \end{aligned}$$

where we used (4.8.6), in the weak sense in $B_2 \setminus B_1$. Thus,

$$-\Delta \tilde{Q} = \left|\nabla \tilde{Q}\right|^2 \tilde{Q} + g(x, \tilde{Q}, \nabla \tilde{Q}), \quad \text{in } B_2 \setminus B_1, \tag{4.8.9}$$

with $\left|g(x, \tilde{Q}, \nabla \tilde{Q})\right| \leq C\left(1 + \left|\nabla \tilde{Q}\right|\right)$. We can now state

Theorem 4.15 (Boundary ε-regularity theorem)**.** *Let $x_0 \in \partial B_1$ and let $Q \in \mathcal{A}_{Q_b}$ be a weak solution of the Euler-Lagrange equations* (4.2.1) *satisfying* (4.8.3)*. There exist $R > 0$ and $\varepsilon > 0$ such that, if*

$$\frac{1}{2R} \int_{\Omega_{2R}} |\nabla Q|^2 \, dx \leq \varepsilon,$$

and if $Q_b \in C^\infty(\partial B_1, \mathcal{S}_0)$, then $Q \in C^\infty\left(\overline{B_1} \cap B_{R/4}(x_0), \mathcal{S}_0\right)$.

Proof. The small energy assumption on Q implies, by means of the boundary monotonicity formula, small energy for $\tilde{Q}$ in $B_{2R}(x_0)$. Now, we would like to apply Theorem 4.6 to $\tilde{Q}$ and prove Hölder continuity in the ball $B_R(x_0) \subset B_2$, and thus the Hölder continuity of $\tilde{Q}$ near $B_R(x_0) \cap \partial B_1$. In order to do this, it suffices to observe that the additional terms due to the reflection procedure can still be accommodated within the proof of Theorem 4.6, see the proof of [119, Proposition 4.5] for details.

Established the Hölder continuity of $\tilde{Q}$ in $B_R(x_0)$, we write (4.8.9) as

$$-\Delta \tilde{Q} = H(x, \tilde{Q}, \nabla \tilde{Q}),$$

where $\left|H(x, \tilde{Q}, \nabla \tilde{Q})\right| \leq C\left(1 + \left|\nabla \tilde{Q}\right|^2\right)$. As in [129] (one could equally well appeal to results in Section 4.5), we get

$$\|\nabla \tilde{Q}\|_{L^\infty(B_{R/2}(x_0))} \leq C$$

and from here the linear theory gives $Q \in C^\infty\left(\overline{B_1} \cap B_{R/4}(x_0), \mathcal{S}_0\right)$. □

Remark 4.8.2. In the case of a more general domain, another issue arises and must be taken into account, that is, the deformation of the metric on the domain due to how the reflection is made. Let η denote the deformed metric. In this respect, one must verify that the system can be cast in the form (4.4.11), with $\left|g(x, \tilde{Q}, \nabla_\eta \tilde{Q})\right| \leq C\left(1 + \left|\nabla_\eta \tilde{Q}\right|\right)$, and div and Δ to be understood as the divergence and the Laplace-Beltrami operators w.r.t. the metric η. Observe that η may not be smooth (and, in general, will not). Next, harmonicity of functions must be understood w.r.t. the metric η and thus some care is needed to obtain gradient estimates analogous to Eq. (4.4.20). In the case above, the transformation involved by the reflection is the Kelvin transform, which is conformal and, because of this, it is quite harmless in both directions, as we saw above.

As a consequence of Theorem 4.15, the singular set and the concentration set of Q coincide up to the boundary. Thus, it suffices to show strong convergence of blow-ups to minimizing tangent maps constant on the boundary of $\partial\mathbb{R}^3_+$ to conclude, in view of Theorem 4.13. We have the following theorem.

Theorem 4.16 (Boundary strong compactness theorem)**.** *Let $x_0 \in \partial B_1$ and let $Q \in \mathcal{A}_{Q_b}$ be a minimizer of the $E(\cdot; B_1)$ in the class $\mathcal{A}_{Q_B}$. Let $R \in (0,1]$ and define $Q_{R,x_0} := Q(x_0 + Rx)$, where $x \in R^{-1}(B_1 \setminus \{x_0\})$. Then there exist a sequence $(R_j)_j$, with $R_j \to 0$ as $j \to \infty$, and $Q_0 \in W^{1,2}_{loc}(\mathbb{R}^3_+, S^4)$ so that $Q_{R_j,x_0} \to Q_0$ strongly in $W^{1,2}_{loc}(\mathbb{R}^3_+, \mathcal{S}_0)$. In addition, Q_0 is a locally minimizing harmonic map into S^4 with $Q_0|_{\partial\mathbb{R}^3_+} = const.$. Moreover, Q_0 is degree-zero homogeneous.*

Proof. Clearly, $\{Q_{R,x_0}\}_R$ is bounded in norm and hence there exist a sequence $(R_j)_j$, $R_j \to 0$ as $j \to \infty$, and a weak limit $Q_0 \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^3_+, \mathcal{S}_0)$. By Rellich-Kondrachov theorem, up to subsequences, the convergence is strong in L^2_{loc} and hence we can assume $Q_0(x) \in S^4$ a.e.. Since Q agrees with Q_b on ∂B_1 (in the trace sense) and Q_b is smooth, by weak convergence and the continuity of the trace operator, we have $Q_0|_{\partial\mathbb{R}^3_+} = \mathrm{const.}$ on the boundary. From now on, owing to the boundary monotonicity formula, the proof goes exactly as in the interior case but considering the domains Ω_r and their suitable homothetic restrictions instead of full balls. □

Thus, an application of Therem 4.13 gives

Theorem 4.17 (Boundary regularity)**.** *Let $Q \in \mathcal{A}_{Q_b}$ be a minimizer of $E(\cdot; B_1)$ in the class $\mathcal{A}_{Q_b}$, with Q_b as in* (1.1.13)*. Then there exist a $\delta > 0$ and a neighborhood $\mathcal{O}_\delta$ of ∂B_1 such that $Q \in C^\omega(\mathcal{O}_\delta, S^4)$. Thus, $Q \in C^\omega(\overline{B_1}, S^4)$.*

Proof. Due to Theorem 4.13, any minimizing tangent map at the boundary is constant. Thus, the concentration set of Q is empty in a full neighborhood of the boundary. By Theorem 4.15, Q is completely smooth around any boundary point (and also real-analytic, by the linear theory, since Q_b is such). By covering, we have both assertions. □

Remark 4.8.3. Nothing here really depends on the specific form of the boundary datum Q_b. Smoothness up to the boundary and interior real-analyticity will hold true also for data like those considered in [106], i.e. $Q_b = \left(n_b \otimes n_b - \frac{1}{3}I\right)$, where $n_b \in W^{1,2}(S^2, S^2)$.

Chapter 5

Landau-de Gennes theory with norm-constraint and with symmetry, I

Synopsis. In this Chapter we add another constraint to the norm constraint already considered in Chapter 4, that is we require the Q-tensors to be also S^1-equivariant, see Eq. (1.1.16). The reason for doing this is provided by Theorem 5.1, which essentially converts smooth S^1-equivariant solutions into biaxial torus solution, in the sense of Definition 1.2. Although such a conversion is logically the final step of the process, we place Theorem 5.1 at the very beginning of our discussion here to motivate our approach to the problem. Note that, in principle, there is no relation between S^1-equivariant minimizers and critical points of the nonsymmetric problem, see Sections 5.2, 5.3. Since symmetry is plugged by hand, we have to provide a bridge between the two things in order that our approach makes sense. This is done in Section 5.5, where an *ad hoc* version of Palais' Principle of Symmetric Criticality [117] is proven, ensuring that a S^1-equivariant minimizer is indeed a critical point also of the nonsymmetric problem. We then follow the same steps as in Chapter 4: we obtain a monotonicity formula in Section 5.7 and the strong compactness of blow-ups in Section 5.8. To step further, we need a classification of all possible tangent maps, an issue deserving an its own chapter (also because of the fact that it is of its own interest) and developed in Chapter 6. Stability of the possible tangent maps will be studied in Chapter 7.

5.1 The semidisk argument

In this section we prove a topological result ensuring that smooth S^1-equivariant minimizers of the LdG energy functional (1.1.3) with respect the boundary condition (1.1.13) are indeed biaxial torus solutions. We shall also give some slight generalizations.

Theorem 5.1 (The semidisk argument)**.** *Let $Q \in \mathcal{A}^{ax}_{Q_b}$ be a S^1-equivariant minimizer of the LdG energy functional* (1.1.3)*, with Q_b as in* (1.1.13)*. If Q is smooth in $\overline{B_1}$, Q is a biaxial torus solution in B_1, in the sense of Definition 1.2.*

Proof. Take any plane Π_z containing the z-axis. Let D be the disk be the disk obtained intersecting B_1 and Π_z and let D^+ be one of the two congruent semidisks in which D splits. Let us write $\partial D^+ = A \cup C$, where A is the segment $[-1, +1]$ on the z-axis and C the semicircle connecting the end-points of A.

Let Q be a minimizer of $E(Q, B_1)$ as in the statement. Let $\lambda_1 \leq \lambda_2 \leq \lambda_3$ denote the eigenvalues of Q. Clearly, $Q = Q_b$ on C and thus $\lambda_1 = \lambda_2$ on C; moreover, by equivariance, $\lambda_1 = \lambda_2$ on A. If we follow the change of direction of the eigenvector of the highest eigenvalue λ_3 of Q_b along ∂D, we get the nontrivial path in $\mathbb{R}P^2$.

We now claim that there exists $\bar{x} \in D$ so that $\lambda_2(\bar{x}) = \lambda_3(\bar{x})$. Indeed, suppose that such a point does not exists. Then λ_3 would a simple eigenvalue in the whole $\overline{D}$, hence the corresponding eigenspace would be well-defined and continuous in $\overline{D}$. Thus, we would have a map from $\overline{D}$ into $\mathbb{R}P^2$ that is homotopic to a constant and, at the same time, nontrivial on the boundary. Since Q is smooth and nonvanishing, this is impossible and a point $\bar{x} \in B_1$ with the claimed property must exists.

Now, by equivariance we get a whole circle of such points, this circle lying in a plane perpendicular to the z-axis and linking the boundary of D. Hence, Q is a biaxial torus solution, in the sense of Definition 1.2. □

Remark 5.1.1. It is crucial in the above argument that $Q \neq 0$ everywhere in $\overline{D}$ (and hence in $\overline{B_1}$). Proving that $\{x \in \overline{B_1} : Q(x) = 0\} \neq \varnothing$ promises to be very difficult; in the nonsymmetric case, this is has been proven by Contreras & Lamy [29] assuming reduced temperature (see §2.5.4) sufficiently large. Assuring the nonvanishing of Q is precisely why we assume the Lyuksyutov constraint, which is approximately valid deep in the nematic phase [100, 120]. The Lyuksyutov constraint is assumed in some simulations (such as [84, 85, 137]) but not in all (e.g., [47, 32, 75]). In any case, deep in the nematic phase the biaxial torus solutions turn out to be preferred versus the hedgehog.

Notice that, in the above proof, it is not really important that Q_b is uniaxial with identical lowest eigenvalues; in this respect, all we need to conclude is in fact that the highest eigenvalue remains always simple on the boundary. This observation immediately yields

Corollary 5.2. *Let $Q \in \mathcal{A}^{ax}_{Q_b}$ be a S^1-equivariant minimizer of the LdG energy functional* (1.1.3)*, with $Q_b \in C^\infty(\partial B_1, S^4)$ a smooth S^1-equivariant boundary datum such that its highest λ_3 is simple everywhere on the boundary. If Q is smooth in $\overline{B_1}$, then Q is a biaxial torus solution in B_1, in the sense of Definition 1.2.*

5.2 Transformation groups and equivariance

Let M, N be smooth manifolds, G a group acting on M and N by diffeomorphism by means of representations

$$\pi^M : G \to \operatorname{Diff} M, \tag{5.2.1}$$

$$\pi^N : G \to \operatorname{Diff} N, \tag{5.2.2}$$

and let $u : M \to N$; we say that u is (π^M, π^N)-*equivariant* if and only if the diagram:

$$\begin{array}{ccc} M & \xrightarrow{u} & N \\ {\scriptstyle \pi^M}\downarrow & & \downarrow{\scriptstyle \pi^N} \\ M & \xrightarrow{u} & N \end{array}$$

commutes, namely, iff the following intertwining relation is satisfied:

$$u(\pi^M(g)x) = \pi^N(g)u(x), \quad \forall g \in G, \forall x \in M. \tag{5.2.3}$$

Informally, this is the same to say that acting on $x \in M$ and then applying u is the same than acting on $u(x) \in N$.

Notation. Sometimes, for the sake of a lighter notation, we write $g\cdot$ to mean the action of the element $g \in G$ by means of an already-specified action on objects following $g\cdot$.

Triples (M, G, π^M) are said *transformation groups* when the action of G on M is compatible with the structure on M. In case M is a smooth manifold, requiring G be a Lie group acting by diffeomorphisms (i.e., $\pi^M(g) : M \to M$ is a diffeomorphism for each $g \in G$) ensures that the group action is compatible with the differentiable structure on M. We then say that M is a G-*manifold* and that π^M is a G-*action* (of G on M).

Suppose we are interested in a class $\mathcal{A}$ of maps $u : M \to N$. We can use (5.2.3) to induce an action of G on $\mathcal{A}$ setting

$$g \cdot u \equiv \pi^N(g^{-1})u(\pi^M(g)\cdot). \tag{5.2.4}$$

Equivariant maps are then the fixed points of this twisted action.

The interpretation in this case requires little more care. For instance, when $\mathcal{A} \equiv C^\infty(M, N)$, then (5.2.4) defines in fact a G-action in the above sense. The same when $\mathcal{A} = W^{1,2}(M, V)$, with V a linear space. When $\mathcal{A} = W^{1,2}(M, N)$, then[1] $g\cdot u \in W^{1,2}(M, N)$ if $u \in W^{1,2}(M, N)$, so (5.2.4) is meaningful. However, $W^{1,2}(M, N)$ is *not* a manifold when $\dim M > 1$ [66], so there is no smooth structure to preserve. Nevertheless, if G is a topological group, then the action is continuous; said another way, the function

$$\Psi(g, u) = g \cdot u$$

is continuous with respect to g as a function on $G \times W^{1,2}(M, N)$ (endowed with the product topology).

Some standard references for the theory of transformation groups are [82, 19, 83]. Below, we recall some result of interest for us in the sequel.

If M is a smooth manifold, G a compact Lie group and (M, G, π^M) is a transformation group, for any $x \in M$ the orbit

$$Gx := \left\{\pi^M(g)(x) : g \in G\right\}$$

is an embedded submanifold of M. As a consequence of the Tubular Neighborhood Theorem [19, Theorem IV.2.2], an entire tubular neighborhood of Gx will have orbits of at least the same dimension. Thus, the function $x \mapsto \dim Gx$ is lower semicontinuous.

Now, let M, N be compact Riemannian manifolds (M with or without boundary, N without boundary) and let G act by isometries on M, N. Since G acts by isometries on M and N, it is possible pulling-back vector fields on M and N by $\pi^M(g)$, resp., $\pi^N(g)$ for all $g \in G$. Let X be a vector field on M. We write

[1] The claim is true without modifications if G acts by isometries on M and N. If G acted by diffeomorphism, we had have to consider pullback metrics on M. We do not stress this point in our notations because we deal mainly with isometries.

$$g \cdot X = \pi^M(g)^* X, \tag{5.2.5}$$

meaning

$$(\pi^M(g)^* X)(p) := \mathrm{d}\pi^M(g^{-1})_{\pi^M(g)(p)}(X(\pi^M(g)(p))), \quad \forall p \in M. \tag{5.2.6}$$

We use analogous notations for vector fields on N.

Now, suppose $u : M \to N$ is equivariant (for the time being, assume u is smooth; later we will indicate how to generalize to the Sobolev case) and let Z be a section of the pull-back bundle u^*TN (i.e., a vector field along u). Then

$$(g \cdot Z)(p) = \mathrm{d}\pi^N(g^{-1})_{u(\pi^M(g)(p))} Z(\pi^M(g)p), \quad \forall p \in M. \tag{5.2.7}$$

For $u : M \to N$ a smooth map, the differential $\mathrm{d}u$ of u can be interpreted as a u^*TN-valued one-form on M; i.e., as a section of the bundle $T^*M \otimes u^*TN$. Let $\tilde{D}$ denote the covariant derivative induced by the Levi-Civita connection on the product bundle $\odot^2 T^*M \otimes u^*TN$. The trace (with respect to the metric on M) of the covariant differential $\tilde{D}(\mathrm{d}u)$ of u is called the *tension field* of u and it is denoted $\tau(u)$ [34]. It is a vector field along u, to which (5.2.7) applies when u is equivariant (in fact, more is true, see below).

When $u \in W^{1,2}(M, N)$, we have to renounce the intrinsic view; however, the above discussion still makes sense if we embed N in $\mathbb{R}^n$ via Nash-Moser theorem and view all the objects extrinsically (i.e., $\mathbb{R}^n$-valued), interpreting all the equalities in the sense of distributions. With some care, it is then possible to prove the following lemma [74, Lemma 6] (we refer to [74] for a proof).

Lemma 5.1 ([74, Lemma 6])**.** *Suppose that $u \in W^{1,2}(M, N)$ is equivariant. Then the following identities hold in the sense of distributions for every $g \in G$.*

(i) $g \cdot du(X) = du(g \cdot X)$ for every smooth tangent vector field X on M.

(ii) $g \cdot D_X Z = D_{g \cdot X}(g \cdot Z)$ for every $Z \in L^2(u^*TN)$ and every smooth tangent vector field X on M (here D_X denotes the covariant derivative induced by the Levi-Civita connection on u^*TN).

(iii) $g \cdot \tau(u) = \tau(u)$.

Remark 5.2.1. When $u \in W^{1,2}(M, N)$, $\tau(u)$ is a little better than a distribution: it belongs to $H^{-1} + L^1$ (with gain of integrability L^2 via Sobolev embedding). Usually [111, Chapter 2] it is given the name *tension field* only when it belongs to $L^1_{\mathrm{loc}}(M, \mathbb{R}^n)$. These subtleties are immaterial for our purposes: we shall content ourselves to consider $\tau(u)$ as a $\mathbb{R}^n$-valued distribution on M.

5.3 Minimization in a symmetric class

Let $\mathscr{M}$ be a set, G a group acting on $\mathscr{M}$ and $\mathscr{E} : \mathscr{M} \to \mathbb{R}$ be a functional. Let Σ denote the set of symmetric (equivariant) points of $\mathscr{E}$, i.e.,

$$\Sigma = \{u \in \mathscr{M} : g \cdot u = u \quad \forall g \in G\},$$

where $g \cdot u$ is defined in (5.2.4).

Definition 5.1 (Critical equivariant point)**.** Let $\mathscr{M}$ be a set, G a group acting on $\mathscr{M}$, Σ the set of equivariant points of $\mathscr{M}$ and let $\mathscr{E} : \mathscr{M} \to \mathbb{R}$ be a functional. We say that $u \in \Sigma$ is a *critical equivariant point* of $\mathscr{E}$ if it is an extremal of $\mathscr{E}$ with respect all allowed[2] equivariant variations.

Clearly, u is a critical equivariant point of $\mathscr{E}$ iff it is a critical point of $\mathscr{E}|_\Sigma$ (since this restricts admissible variations to those belonging to Σ).

The following question arises naturally.

Problem. Under what conditions on $\mathscr{M}$, G and $\mathscr{E}$ it turns out that a critical equivariant point of $\mathscr{E}$ is in fact a critical point of $\mathscr{E}$ in $\mathscr{M}$?

Restrictions are needed. Palais has shown this in [117] by counterexamples. In the same remarkable paper, Palais has also given sufficient conditions on $\mathscr{M}$, G and $\mathscr{E}$ so that it is in fact true that critical equivariant points are equivariant critical points. This last statement is usually known as the *Principle of symmetric criticality* (abbreviated herein to "the Principle").

In particular, the Principle holds when $\mathscr{M}$ is a Riemannian manifold (of finite or infinite dimension), G a group acting on $\mathscr{M}$ by isometries and $\mathscr{E}$ a smooth G-invariant functional. This is the simplest setting because of the comfortable framework provided by Riemannian geometry, helping in both calculations and interpretation.

One can weaken the hypotheses on $\mathscr{M}$ at price of strengthening those on G. In particular, one can ask $\mathscr{M}$ be a smooth Banach G-manifold and then the Principle continues holding true if G is assumed to be a compact Lie group. In the case G is a semisimple Lie group, the Principle is true for finite dimensional real-analytic G-manifolds. In any case, however, a smooth structure on $\mathscr{M}$ is required, in order to give a precise meaning to the objects involved.

We now consider the problem of minimizing $E(\cdot; B_1)$ in $\mathcal{A}^{\mathrm{ax}}_{Q_b}$. We observe that $\mathcal{A}^{\mathrm{ax}}_{Q_b}$ is weakly closed in $\mathcal{A}_{Q_b} = W^{1,2}_{Q_b}(B_1, S^4)$. However, $W^{1,2}_{Q_b}(B_1, S^4)$ is *not* a manifold [66]. This is definitely a problem, since none of Palais' results applies. One can think to get round the problem considering that $W^{1,2}_{Q_b}(B_1, \mathcal{S}_0)$ is in fact a Hilbert space, so that the Principle applies and a critical point of $E(\cdot; B_1)$ restricted to

$$\Sigma = \left\{ Q \in W^{1,2}_{Q_b}(B_1, \mathcal{S}_0) : Q = g^{-1} \cdot Q(g\cdot) \, \forall g \in S^1 \right\}$$

is in fact a critical point for $E(\cdot; B_1)$ in $W^{1,2}_{Q_b}(B_1, \mathcal{S}_0)$. However, here the point is that the norm constraint generates a curvature term in the Euler-Lagrange equations (compare (5.6.1) and [106, Eq. (14)], so that a critical point of $E(\cdot; B_1)$ in $\mathcal{A}^{\mathrm{ax}}_{Q_b}$ satisfies a *different* system of equations than a critical point of $E(\cdot; B_1)$ in Σ. In other words, we cannot avoid facing directly the lacking of a smooth structure on $\mathcal{A}_{Q_b}$.

Fortunately, the main idea of Palais still holds. We describe it in few words below. Suppose $u \in \Sigma$ is a critical point of $\mathscr{E}|_\Sigma$. Then $\nabla \mathscr{E}_u$ ($\nabla \mathscr{E}$ is the gradient vector field associated to $\mathscr{E}$) is orthogonal to $T_u\Sigma$ (the tangent space to Σ at u). To show that u is a critical point of $\mathscr{E}$ in $\mathscr{M}$, then it is sufficient to prove that $\nabla \mathscr{E}_u \in T_u\Sigma$.

Of course, we cannot speak of tangent spaces, gradients and so on in our setting but we can substitute $\nabla \mathscr{E}_u$ with the Euler-Lagrange operator, seen as a distribution, and we can replace the membership to $T_u\Sigma$ (i.e., the vanishing of the differential $\mathrm{d}E_u$) with the vanishing in the sense of distributions. If Q is a critical point of $E(\cdot; B_1)$ in $\mathcal{A}^{\mathrm{ax}}_{Q_b}$, we then have the vanishing of the Euler-Lagrange operator $\mathscr{E}(Q)$ in the sense of

[2]Depending on the problem, the set of allowed variations can be smaller than Σ.

distributions when tested against equivariant variations. We then test $\mathscr{E}(Q)$ against arbitrary test functions. Using the fact that S^1 is a compact topological group, we have an invariant Haar integral on S^1 and we can use this invariance to transfer the equivariance of Q to the test function. Hence we are led again to testing $\mathscr{E}(Q)$ against equivariant variations and so it vanishes in the sense of distribution. We shall see details in Section 5.5.

5.4 Existence of minimizers

Here we prove the existence of minimizers of the LdG energy in the class $\mathcal{A}_{Q_b}$ given in (1.1.15) of the S^1-equivariant Q-tensors.

Proposition 5.3 (Existence of minimizers)**.** *Let $E(\cdot;B_1)$ the LdG energy, defined as in* (1.1.3) *and considered over the class $\mathcal{A}^{ax}_{Q_b}$ given* (1.1.15)*, with Q_b as in* (1.1.13)*. Then there exists at least one minimizer of $E(\cdot;B_1)$ in the class* (1.1.15)*.*

Proof. As we saw in the proof of Proposition 4.1, the map $H(x) = \sqrt{\frac{3}{2}}\left(\frac{x}{|x|}\otimes\frac{x}{|x|}-\frac{I}{3}\right)$ belongs to $\mathcal{A}_{Q_b}$ and so it will suffice to observe that it is S^1-equivariant (in fact, SO(3)-equivariant) to conclude that $\mathcal{A}^{\mathrm{ax}}_{Q_b}$ is nonempty.

We already know from Proposition 4.1 that $E(\cdot;B_1)$ is bounded below and lower semicontinuous with respect to the weak topology on $\mathcal{A}_{Q_b}$. We now take a minimizing sequence $(v_k)_k \subset \mathcal{A}^{\mathrm{ax}}_{Q_b}$ and we show that it converges to a limit in $\mathcal{A}^{\mathrm{ax}}_{Q_b}$. Indeed, as in the proof of [40, Theorem 2 in 8.2], by convexity, coercivity and Poincaré inequality, $(v_k)_k$ is bounded in $W^{1,2}(B_1,S^4)$. Then we can extract a subsequence weakly convergent in $W^{1,2}(B_1,S^4)$, hence in $\mathcal{A}_{Q_b}$ (because $\mathcal{A}_{Q_b}$ is weakly closed). Let v denote its weak limit. By the Rellich-Kondrachov theorem, we can extract a further subsequence $(v_{k_{j'}})_{k_{j'}}$ strongly convergent in $L^2(B_1,S^4)$ to v and hence we pick up another subsequence converging pointwise a.e.. Since S^1-equivariance is a pointwise-property, it then follows that v is S^1-equivariant. Hence $v\in\mathcal{A}^{\mathrm{ax}}_{Q_b}$. □

5.5 Symmetric criticality for S^1-equivariance of Q-tensors

As we already seen in Section 5.3, it is not immediate that a critical point of $E(\cdot;B_1)|_{\mathcal{A}^{\mathrm{ax}}_{Q_b}}$ is a critical point of $E(\cdot;B_1)|_{\mathcal{A}_{Q_b}}$. Here we show that this is indeed the case.

Remembering Definition 5.1, a critical equivariant point $Q\in\mathcal{A}^{\mathrm{ax}}_{Q_b}$ solves the Euler-Lagrange (4.2.1) in the sense of distributions when we permit only S^1-equivariant variations. Our aim is to show that we can allow arbitrary variations and the same continues holding true.

Since $\mathcal{A}_{Q_b}$ is *not* a manifold [66], Palais' results do not apply. However, we can borrow the same principle, as explained at the end of Section 5.3. We supply the lacking of a differentiable structure on $\mathcal{A}_{Q_b}$ by exploiting the Haar invariant integral on S^1 in such a way to transfer the equivariance from a critical point $Q\in\mathcal{A}^{\mathrm{ax}}_{Q_b}$ to variantions against it is tested. This technique is inspired by the proof of [49, Theorem 1] and by arguments in [74].

Notation. In what follows, $-f(Q)$ denotes the right-hand side of (4.2.1). We set

$$\tau(Q) = \Delta Q + |\nabla Q|^2\, Q. \tag{5.5.1}$$

h_{S^1} denotes the Haar measure on S^1. We recall that h_{S^1} a regular biinvariant Borel probability measure; it is the unique Borel probability measure on S^1 with this property [126, Theorem 5.14].

Before stating the theorem, we recall from Section 5.2 (cfr. [74, Lemma 6(c)]) that $\tau(Q)$ is equivariant (i.e., a fixed point of the S^1(-twisted)-action on maps):

$$\tau(Q) = g \cdot \tau(Q).$$

We further observe that also $f(Q)$ is equivariant:

$$f(Q) = g \cdot f(Q).$$

Theorem 5.4. *Let $Q \in \mathcal{A}_{Q_b}^{ax}$ a critical point of $E(\cdot; B_1)|_{\mathcal{A}_{Q_b}^{ax}}$. Then Q is a critical point of $E(\cdot; B_1)$ in the class $\mathcal{A}_{Q_b}$.*

Proof. Let $\phi^{S^1} \in C_c^\infty(B_1, \mathcal{S}_0)$ be a S^1-equivariant variation. Since $|Q| = 1$ a.e., we have $\left|Q + t\phi^{S^1}\right| \neq 0$ for sufficiently small $t > 0$. Then, by hypothesis,

$$\frac{d}{dt}\bigg|_{t=0} E\left(\frac{Q + t\phi^{S^1}}{|Q + t\phi^{S^1}|}\right) = 0.$$

Now let $\phi \in C_c^\infty(B_1, \mathcal{S}_0)$ be arbitrary. Using Haar's theorem [126, Theorem 5.14] and the equivariance of $\tau(Q)$, $f(Q)$, we have

$$\begin{aligned}
\frac{d}{dt}\bigg|_{t=0} E\left(\frac{Q + t\phi}{|Q + t\phi|}\right) &= \int_{B_1} [\tau(Q) + f(Q)]\,\phi\,\mathrm{d}x = \int_{B_1}\int_{S^1} [\tau(Q) + f(Q)]\,\phi\,\mathrm{d}h_{S^1}\,\mathrm{d}x \\
&= \int_{B_1}\int_{S^1} [g\cdot\tau(Q) + g\cdot f(Q)]\,\phi\,\mathrm{d}h_{S^1}\,\mathrm{d}x = \int_{B_1}\int_{S^1} g\cdot[\tau(Q) + f(Q)]\,\phi\,\mathrm{d}h_{S^1}\,\mathrm{d}x \\
&= \int_{B_1} [\tau(Q) + f(Q)] \int_{S^1} (g^{-1}\cdot\phi)\,\mathrm{d}h_{S^1}\,\mathrm{d}x = \int_{B_1} [\tau(Q) + f(Q)]\,\varphi^{S^1}\,\mathrm{d}x,
\end{aligned}$$

where $\varphi^{S^1} = \int_{S^1} g^{-1}\cdot\phi\,\mathrm{d}h_{S^1}$. We note that φ^{S^1} is S^1-equivariant. Indeed,

$$\begin{aligned}
\pi^{S^4}(m)\varphi^{S^1}(x) &= \int_{S^1} \pi^{S^4}(m)\pi^{S^4}(g)\phi(\pi^{B_1}(g^{-1}x))\,\mathrm{d}h_{S^1} \\
&= \int_{S^1} \pi^{S^4}(g)\phi(\pi^{B_1}(g^{-1}m)x)\,\mathrm{d}h_{S^1} \\
&= \varphi^{S^1}(\pi^{B_1}(m)x), \quad \forall m \in S^1,
\end{aligned}$$

where we repeatedly used Haar's theorem.

We now have

$$\frac{d}{dt}\bigg|_{t=0} E\left(\frac{Q + t\phi}{|Q + t\phi|}\right) = \int_{B_1} [\tau(Q) + f(Q)]\,\varphi^{S^1} = 0,$$

hence the conclusion follows by the initial remark. □

5.6 Euler-Lagrange equations

As a consequence of Theorem 5.4, the Euler-Lagrange equations (4.2.1) carry over to the S^1-equivariant case with no modifications. We report them below for convenience.

Proposition 5.5 (Euler-Lagrange equations)**.** *Let $E(\cdot; B_1)$ the LdG energy functional defined in* (1.1.3) *over the class $\mathcal{A}^{ax}_{Q_b}$ given in* (1.1.15)*, with Q_b as in* (1.1.13)*, and let $Q \in \mathcal{A}^{ax}_{Q_b}$ be a critical point of $E(\cdot; B_1)$. Then Q is a solution in the sense of distributions of the following boundary value problem:*

$$\begin{cases} L\Delta Q_{ij} + L\,|\nabla Q|^2\, Q_{ij} = b\left(Q_{ij}\operatorname{Tr}(Q^3) - Q_{ik}Q_{kj} + \frac{1}{3}\delta_{ij}\right) \text{ in } B_1, \\ Q_{ij} = (Q_b)_{ij} \text{ in the trace sense on } \partial B_1. \end{cases} \tag{5.6.1}$$

5.7 Monotonicity formula

Our strategy for obtaining a monotonicity formula for minimizers in the class $\mathcal{A}^{\mathrm{ax}}_{Q_b}$ consist in getting it by taking the limit in the monotonicity formulae for approximate minimizers. To state this program more precisely, we start by fixing the minimizer[3] $Q^\star \in \mathcal{A}^{\mathrm{ax}}_{Q_b}$ for which we want to derive the monotonicity formula and picking arbitrarily $\varepsilon > 0$. Then we define the energy functional

$$E_\varepsilon(\tilde{Q}; B_1) = \int_{B_1} \frac{1}{2}\left|\nabla\tilde{Q}\right|^2 + F(\tilde{Q})\,\mathrm{d}x + \frac{1}{4\varepsilon^2}\int_{B_1}\left(1 - \left|\tilde{Q}\right|^2\right)^2 \mathrm{d}x + \frac{1}{2}\int_{B_1} |Q - Q^\star|^2\,\mathrm{d}x, \tag{5.7.1}$$

where $F(\cdot)$ is the potential appearing in the LdG energy and we consider $E_\varepsilon(\cdot; B_1)$ defined over the class

$$\mathcal{A}^{\mathrm{ax}}_{Q_b}(\mathcal{S}_0) := \left\{\tilde{Q} \in W^{1,2}_{Q_b}(B_1, \mathcal{S}_0) : \tilde{Q} \text{ is } S^1\text{-equivariant}\right\}. \tag{5.7.2}$$

Note that we already know, by Proposition 5.3, that $\mathcal{A}^{\mathrm{ax}}_{Q_b}(\mathcal{S}_0)$ is nonempty.

We call a functional of this kind a *penalized energy functional* for the LdG energy. Note that, if $\tilde{Q} = Q^\star$, then it makes sense to evaluate $E_\varepsilon(Q^\star; B_1)$ and in fact we have $E_\varepsilon(Q^\star; B_1) = E(Q^\star; B_1)$ for any $\varepsilon > 0$.

The reason for introducing penalized functionals lies in the following observations. First of all, we note that

Proposition 5.6. *The Euler-Lagrange equations for the energy functional $E_\varepsilon(\cdot; B_1)$, defined as in* (5.7.1) *over the class* (5.7.2)*, are*

$$\Delta\tilde{Q}_{ij} = -a\tilde{Q}_{ij} - b\left(\tilde{Q}_{ik}\tilde{Q}_{kj} - \frac{1}{3}\delta_{ij}\right) + c\tilde{Q}_{ij}\operatorname{Tr}(\tilde{Q})^2 + \frac{1}{\varepsilon^2}\tilde{Q}_{ij}\left(\left|\tilde{Q}\right|^2 - 1\right) + \left(\tilde{Q}_{ij} - Q^\star_{ij}\right). \tag{5.7.3}$$

The derivation is standard, the only detail to which pay attention being the tracelessness constraint. This is taken into account by adding a Lagrange multiplier to the energy density as in proof of Theorem 4.2 (calculations are much easier now because there is no norm constraint).

[3]The trick of adding a penalization for being far from the fixed minimizer has been suggested by V. Millot (private communication).

We point out that the system (5.7.3) is *semilinear*, with a polynomial nonlinearity. Regularity is then trivial in this case. Hence, any solution of the boundary-value problem formed by (5.7.3) and the boundary condition in (5.7.2) is real-analytic in the interior and smooth as the boundary datum and the boundary permit in a neighborhood of the boundary.

Next, observe that solutions to (5.7.3) do exist; indeed, we have

Proposition 5.7. *Let $\varepsilon > 0$ and let $E_\varepsilon(\cdot; B_1)$ be defined as in* (5.7.1) *over the class* (5.7.2)*. Then there exists a minimizer $\tilde{Q}^\varepsilon$ for $E_\varepsilon(\cdot; B_1)$ in the class $\mathcal{A}^{ax}_{Q_b}(\mathcal{S}_0)$.*

Proof. We already observed that the admissible class is not empty. Obviously, $E_\varepsilon(\cdot; B_1)$ is bounded below. Being Q_b a Dirichlet boundary condition, $W^{1,2}_{Q_b}(B_1, \mathcal{S}_0)$ is closed with respect to the weak topology of $W^{1,2}(B_1, \mathcal{S}_0)$. To show that $E_\varepsilon(\cdot; B_1)$ is weakly lower semicontinuous in the weak topology of $W^{1,2}_{Q_b}(B_1, \mathcal{S}_0)$, we note that the corresponding energy density is convex in $\nabla(\cdot)$ and that it is coercive in $(\cdot)$. Indeed, the first integral in (5.7.1) is exactly the LdG energy which we have already shown to be weakly lower semicontinuous with respect to the weak topology of $W^{1,2}(B_1, \mathcal{S}_0)$, while the second integrand is clearly coercive in $(\cdot)$. To conclude, we take a minimizing sequence $(v_k)_k$ in $\mathcal{A}^{\mathrm{ax}}_{Q_b}(\mathcal{S}_0)$. The convexity of $E(\cdot; B_1)$ in $\nabla(\cdot)$ and its coercivity in $(\cdot)$, together with Poincaré inequality, imply that $(v_k)_k$ is bounded in $W^{1,2}_{Q_b}(B_1, \mathcal{S}_0)$, hence it converges to a limit $\tilde{Q}^\varepsilon \in W^{1,2}_{Q_b}(B_1, \mathcal{S}_0)$ (because $W^{1,2}_{Q_b}(B_1, \mathcal{S}_0)$ is weakly closed). By Rellich-Kondrachov theorem, we can extract a subsequence strongly convergent in $L^2(B_1, \mathcal{S}_0)$ to $\tilde{Q}^\varepsilon$ and then a further subsequence converging pointwise a.e. to $\tilde{Q}^\varepsilon$. Then $\tilde{Q}^\varepsilon$ must be S^1-equivariant because S^1-equivariance passes to pointwise limits. □

By the principle of symmetric criticality, cfr. Section 5.3, a minimizer of $E_\varepsilon(\cdot; B_1)$ is a solution in the sense of distributions of the Euler-Lagrange equations (5.7.3); in fact, it is in particular a smooth classical solution, as we remarked.

Now, let $\varepsilon > 0$ and let $\tilde{Q}^\varepsilon$ a minimizer for $E_\varepsilon(\cdot; B_1)$. Being a smooth critical point, $\tilde{Q}^\varepsilon$ is stationary with respect both external variations and internal variations, so that a monotonicity formula for $\tilde{Q}^\varepsilon$ is simply derived by multiplying (5.7.1) by $x_k \partial_k Q_{ij}$ and then integrating by parts. We have the following result.

Proposition 5.8 (Monotonicity formula for minimizers of penalized functionals)**.** *Fix a minimizer $Q^\star$ of the LdG energy, $Q^\star \in \mathcal{A}^{ax}_{Q_b}$. Let $\varepsilon > 0$ and let $E_\varepsilon(\cdot; B_1)$ defined as in* (5.7.1) *over the class* (5.7.2)*. Suppose $\tilde{Q}^\varepsilon$ is a minimizer of $E_\varepsilon(\cdot; B_1)$ in the class* (5.7.2)*. Then, for every $x_0 \in B_1$ and every $0 < R_1 < R_2$ such that $B_{R_2}(x_0) \subset\subset B_1$, the following monotonicty formula holds:*

$$
\begin{aligned}
\frac{1}{R_2}\int_{B_{R_2}(x_0)} \mathcal{E}_\varepsilon(\nabla\tilde{Q}^\varepsilon, \tilde{Q}^\varepsilon) - \frac{1}{R_1}\int_{B_{R_1}(x_0)} \mathcal{E}_\varepsilon(\nabla\tilde{Q}^\varepsilon, \tilde{Q}^\varepsilon) \int_{B_{R_2}(x_0)\setminus B_{R_1}(x_0)} \frac{1}{|x-x_0|}\left|\frac{\partial \tilde{Q}^\varepsilon}{\partial r}\right|^2 \\
+ 2\int_{R_1}^{R_2} \frac{dR}{R^2}\int_{B_R(x_0)} F(\tilde{Q}^\varepsilon) + 2\int_{R_1}^{R_2} \frac{dR}{R^2}\int_{B_R(x_0)} \frac{\left(1 - \left|\tilde{Q}^\varepsilon\right|^2\right)^2}{4\varepsilon^2} \\
+ 2\int_{R_1}^{R_2} \frac{dR}{R^2}\int_{B_R(x_0)} \frac{\left|Q^\star - \tilde{Q}^\varepsilon\right|^2}{2} + \int_{R_1}^{R_2} \frac{dR}{R^2}\int_{B_R(x_0)} \left\langle \tilde{Q}^\varepsilon - Q^\star, x\cdot\nabla Q^\star\right\rangle, \quad (5.7.4)
\end{aligned}
$$

where

$$\mathcal{E}_\varepsilon(\nabla\tilde{Q}^\varepsilon, \tilde{Q}^\varepsilon) = \frac{1}{2}\left|\nabla\tilde{Q}^\varepsilon\right|^2 + F(\tilde{Q}^\varepsilon) + \frac{1}{4\varepsilon^2}\left(1 - \left|\tilde{Q}^\varepsilon\right|^2\right)^2 + \frac{1}{2}\left|\tilde{Q}^\varepsilon - Q^\star\right|^2$$

and $\frac{\partial}{\partial r}$ *means the directional derivative in the radial direction* $(x - x_0)/|x - x_0|$.

Proof. By translational invariance, it suffices to prove (5.7.4) for $x_0 = 0$. We will henceforth drop the specification of the center of the balls.

Multiply (5.7.3) by $x_k\partial_k\tilde{Q}^\varepsilon$ and then integrate by parts on a ball B_R such that $\overline{B_R} \subset B_1$. We are readily led to

$$\begin{aligned}
0 = &\int_{B_R} x_k\partial_k\tilde{Q}^\varepsilon_{ij}(x)\Big\{\Delta\tilde{Q}^\varepsilon_{ij}(x) \\
&- \left[\frac{\partial F(\tilde{Q}^\varepsilon(x))}{\partial\tilde{Q}^\varepsilon_{ij}} + \frac{1 - \left|\tilde{Q}^\varepsilon(x)\right|^2}{\varepsilon^2}\tilde{Q}^\varepsilon_{ij}(x) + (\tilde{Q}^\varepsilon_{ij} - Q^\star_{ij}) - \frac{1}{3}\delta_{ij}\operatorname{Tr}[(\tilde{Q}^\varepsilon(x))^2]\right]\Big\}\,\mathrm{d}x.
\end{aligned}$$

Since $\operatorname{Tr}\tilde{Q}^\varepsilon = 0$,

$$\int_{B_R} x_k\partial_k\tilde{Q}^\varepsilon_{ij}(x)\left(\frac{1}{3}\delta_{ij}\operatorname{Tr}[(\tilde{Q}^\varepsilon(x))^2]\right)\mathrm{d}x = \frac{1}{3}\int_{B_R} x_k\partial_k\operatorname{Tr}\tilde{Q}^\varepsilon(x)\operatorname{Tr}[(\tilde{Q}^\varepsilon(x))^2]\,\mathrm{d}x \equiv 0.$$

Hence

$$\begin{aligned}
0 = &\int_{B_R} x_k\partial_k\tilde{Q}^\varepsilon_{ij}(x)\left\{\Delta\tilde{Q}^\varepsilon_{ij}(x)\left[\frac{\partial F(\tilde{Q}^\varepsilon(x))}{\partial\tilde{Q}^\varepsilon_{ij}} + \frac{1 - \left|\tilde{Q}^\varepsilon(x)\right|^2}{\varepsilon^2}\tilde{Q}^\varepsilon_{ij}(x)\right] + (\tilde{Q}^\varepsilon_{ij} - Q^\star_{ij})\right\} \\
= &\int_{B_R}\Delta\tilde{Q}^\varepsilon_{ij}(x)x_k\partial_k\tilde{Q}^\varepsilon_{ij}(x) - \int_{B_R} x_k\partial_k\tilde{Q}^\varepsilon_{ij}(x)\frac{\partial F(\tilde{Q}^\varepsilon(x))}{\partial\tilde{Q}^\varepsilon_{ij}} \\
- &\int_{B_R} x_k\partial_k\tilde{Q}^\varepsilon_{ij}\frac{1 - \left|\tilde{Q}^\varepsilon(x)\right|^2}{\varepsilon^2}\tilde{Q}^\varepsilon_{ij}(x) + (\tilde{Q}^\varepsilon_{ij}(x) - Q^\star_{ij}(x))\tilde{Q}^\varepsilon_{ij}(x) \\
\equiv &\, I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

We have

$$\begin{aligned}
I_1 &= \int_{B_R} x_k\partial_k\tilde{Q}^\varepsilon_{ij}(x)\Delta\tilde{Q}^\varepsilon_{ij}(x)\,\mathrm{d}x \\
&\equiv \int_{B_R} x_k\partial_k\tilde{Q}^\varepsilon_{ij}(x)\tilde{Q}^\varepsilon_{ij,ll}(x)\,\mathrm{d}x \\
&= -\int_{B_R}\tilde{Q}^\varepsilon_{ij,l}(x)(\delta_{lk}\tilde{Q}^\varepsilon_{ij,l}(x) + x_k\tilde{Q}^\varepsilon_{ij,kl}(x))\,\mathrm{d}x + \int_{\partial B_R}\tilde{Q}^\varepsilon_{ij,l}(x)x_k\tilde{Q}^\varepsilon_{ij,k}(x)\frac{x_l}{R}\,\mathrm{d}\sigma \\
&= -\int_{B_R}\tilde{Q}^\varepsilon_{ij,l}(x)\tilde{Q}^\varepsilon_{ij,l}(x)\,\mathrm{d}x + 3\int_{B_R}\frac{1}{2}\tilde{Q}^\varepsilon_{ij,l}(x)\tilde{Q}^\varepsilon_{ij,l}(x)\,\mathrm{d}x \\
&\quad - \int_{\partial B_R}\frac{\tilde{Q}^\varepsilon_{ij,l}(x)\tilde{Q}^\varepsilon_{ij,l}(x)}{2}\frac{x_kx_k}{R}\,\mathrm{d}\sigma + \int_{\partial B_R}\frac{(\tilde{Q}^\varepsilon_{ij,k}(x)x_k)^2}{R}\,\mathrm{d}\sigma.
\end{aligned} \tag{5.7.5}$$

Similarly,

$$\begin{aligned} I_2 &= -\int_{B_R} x_k \partial_k \tilde{Q}^{\varepsilon}_{ij}(x) \frac{\partial F(\tilde{Q}^{\varepsilon}(x))}{\partial Q_{ij}} \, \mathrm{d}x \\ &= -\int_{B_R} x_k \partial_k F(\tilde{Q}^{\varepsilon}(x)) \, \mathrm{d}x \\ &= 3\int_{B_R} F(\tilde{Q}^{\varepsilon}(x)) \, \mathrm{d}x - \int_{\partial B_R} F(\tilde{Q}^{\varepsilon}(x)) \frac{x_k x_k}{R} \, \mathrm{d}\sigma. \end{aligned} \tag{5.7.6}$$

Observe that

$$\tilde{Q}^{\varepsilon}_{ij}(x) \partial_k \tilde{Q}^{\varepsilon}_{ij}(x) \left(\frac{1 - \left| \tilde{Q}^{\varepsilon}(x) \right|^2}{\varepsilon^2} \right) = \partial_k \left[\frac{\left(1 - \left| \tilde{Q}^{\varepsilon}(x) \right|^2 \right)^2}{4\varepsilon^2} \right],$$

so that it follows

$$\begin{aligned} I_3 &= -\int_{B_R} x_k \partial_k \tilde{Q}^{\varepsilon}_{ij}(x) \frac{1 - \left| \tilde{Q}^{\varepsilon}(x) \right|^2}{\varepsilon^2} \tilde{Q}^{\varepsilon}_{ij}(x) \, \mathrm{d}x \\ &= -\int_{B_R} x_k \partial_k \left[\frac{\left(1 - \left| \tilde{Q}^{\varepsilon}(x) \right|^2 \right)^2}{4\varepsilon^2} \right] \mathrm{d}x \\ &= 3\int_{B_R} \frac{\left(1 - \left| \tilde{Q}^{\varepsilon}(x) \right|^2 \right)^2}{4\varepsilon^2} \, \mathrm{d}x - \int_{\partial B_R} \frac{\left(1 - \left| \tilde{Q}^{\varepsilon}(x) \right|^2 \right)^2}{4\varepsilon^2} \frac{x_k x_k}{R} \, \mathrm{d}\sigma. \end{aligned} \tag{5.7.7}$$

Analogously,

$$I_4 = 3\int_{B_R} \frac{\left| \tilde{Q}^{\varepsilon} - Q^{\star} \right|^2}{2} \, \mathrm{d}x - \int_{\partial B_R} \frac{\left| \tilde{Q}^{\varepsilon} - Q^{\star} \right|^2}{2} \frac{x_k x_k}{R} \, \mathrm{d}\sigma - \int_{B_R} \left\langle \tilde{Q}^{\varepsilon} - Q^{\star}, x \cdot \nabla Q^{\star} \right\rangle \mathrm{d}x. \tag{5.7.8}$$

Joining (5.7.5), (5.7.6), (5.7.7), (5.7.8) and recalling the definition of radial derivative, we have

$$\begin{aligned} 0 = &\int_{B_R} \frac{1}{2} \left| \nabla \tilde{Q}^{\varepsilon} \right|^2 - R\int_{\partial B_R} \left| \nabla \tilde{Q}^{\varepsilon} \right|^2 + R\int_{\partial B_R} \left| \frac{\partial \tilde{Q}^{\varepsilon}}{\partial r} \right|^2 + 3\int_{B_R} F(\tilde{Q}^{\varepsilon}) - R\int_{\partial B_R} F(\tilde{Q}^{\varepsilon}) \\ &+ 3\int_{B_R} \frac{\left(1 - \left| \tilde{Q}^{\varepsilon} \right|^2 \right)^2}{4\varepsilon^2} - R\int_{\partial B_R} \frac{\left(1 - \left| \tilde{Q}^{\varepsilon} \right|^2 \right)^2}{4\varepsilon^2} \\ &+ 3\int_{B_R} \frac{\left| \tilde{Q}^{\varepsilon} - Q^{\star} \right|}{2} - R\int_{\partial B_R} \frac{\left| \tilde{Q}^{\varepsilon} - Q^{\star} \right|^2}{2} - \int_{B_R} \left\langle \tilde{Q}^{\varepsilon} - Q^{\star}, x \cdot \nabla Q^{\star} \right\rangle. \end{aligned}$$

Comparing to the expression for the energy density $\mathcal{E}_{\varepsilon}$ given in the statement,

$$
\begin{aligned}
0 = &\int_{B_R} \mathcal{E}_\varepsilon(\nabla \tilde{Q}^\varepsilon, \tilde{Q}^\varepsilon) - R\int_{\partial B_R} \mathcal{E}^\varepsilon(\nabla \tilde{Q}^\varepsilon, \tilde{Q}^\varepsilon) + R\int_{\partial B_R} \left|\frac{\partial \tilde{Q}^\varepsilon}{\partial r}\right|^2 \\
&+ 2\int_{B_R} F(\tilde{Q}^\varepsilon) + 2\int_{B_R} \frac{\left(1 - \left|\tilde{Q}^\varepsilon\right|^2\right)^2}{4\varepsilon^2} \\
&+ 2\int_{B_R} \frac{\left|\tilde{Q}^\varepsilon - Q^\star\right|^2}{2} + \int_{B_R} \left\langle Q^\star - \tilde{Q}^\varepsilon, x \cdot \nabla Q^\star \right\rangle.
\end{aligned}
$$

Divide both members by R^2 and use (4.3.23),

$$
\begin{aligned}
\frac{d}{dR}\left(\frac{1}{R}\int_{B_R} \mathcal{E}^\varepsilon(\nabla \tilde{Q}^\varepsilon, \tilde{Q}^\varepsilon)\right) &= \frac{1}{R}\int_{\partial B_R} \left|\frac{\partial Q}{\partial r}\right|^2 + \frac{2}{R^2}\int_{B_R} F(\tilde{Q}^\varepsilon) \\
+ \frac{2}{R^2}\int_{B_R} \frac{\left(1 - \left|\tilde{Q}^\varepsilon\right|^2\right)^2}{4\varepsilon^2} &+ \frac{2}{R^2}\int_{B_R} \frac{\left|\tilde{Q}^\varepsilon - Q^\star\right|^2}{2} + \frac{1}{R^2}\int_{B_R} \left\langle Q^\star - \tilde{Q}^\varepsilon, x \cdot \nabla Q^\star \right\rangle.
\end{aligned}
$$

Now integrate on $[R_1, R_2]$. It follows:

$$
\begin{aligned}
\frac{1}{R_2}\int_{B_{R_2}} \mathcal{E}_\varepsilon(\nabla \tilde{Q}^\varepsilon, \tilde{Q}^\varepsilon) - \frac{1}{R_1}\int_{B_{R_1}} \mathcal{E}_\varepsilon(\nabla \tilde{Q}^\varepsilon, \tilde{Q}^\varepsilon) &= \int_{B_{R_2}\setminus B_{R_1}} \frac{1}{|x|}\left|\frac{\partial \tilde{Q}^\varepsilon}{\partial r}\right|^2 \\
+ 2\int_{R_1}^{R_2} \frac{\mathrm{d}R}{R^2}\int_{B_R} F(\tilde{Q}^\varepsilon) &+ 2\int_{R_1}^{R_2} \frac{\mathrm{d}R}{R^2}\int_{B_R} \frac{\left(1 - \left|\tilde{Q}^\varepsilon\right|^2\right)^2}{4\varepsilon^2} \\
+ 2\int_{R_1}^{R_2} \frac{\mathrm{d}R}{R^2}\int_{B_R} \frac{\left|\tilde{Q}^\varepsilon - Q^\star\right|^2}{2} &+ \int_{R_1}^{R_2} \frac{\mathrm{d}R}{R^2}\int_{B_R} \left\langle Q^\star - \tilde{Q}^\varepsilon, x \cdot \nabla Q^\star \right\rangle.
\end{aligned}
$$

□

So far we have proved that, for each $\varepsilon > 0$, the functional $E_\varepsilon(\cdot; B_1)$ is well-defined and have minimizers $\tilde{Q}^\varepsilon$ in the class $\mathcal{A}^{\mathrm{ax}}_{Q_b}(\mathcal{S}_0)$ which satisfy the monotonicity formula (5.7.4). Therefore, as $\varepsilon > 0$ runs, we get families

$$\{E_\varepsilon(\tilde{Q}^\varepsilon; B_1)\}_{\varepsilon>0}, \quad \{\tilde{Q}^\varepsilon\}_{\varepsilon>0}.$$

We now show that they are both equibounded, the first one in $\mathbb{R}$ and the second one in $\mathcal{A}^{\mathrm{ax}}_{Q_b}(\mathcal{S}_0)$.

Lemma 5.2. *Fix a minimizer $Q^\star$ of the LdG energy, $Q^\star \in \mathcal{A}^{ax}_{Q_b}$. For each $\varepsilon > 0$, let $E_\varepsilon(\cdot; B_1)$ be defined as in (5.7.1) over the class (5.7.2) and let $\tilde{Q}^\varepsilon$ be a minimizer of $E_\varepsilon(\cdot; B_1)$ in the class (5.7.2). Then the family $\{E_\varepsilon(\tilde{Q}^\varepsilon; B_1)\}_{\varepsilon>0}$ is equibounded in $\mathbb{R}$; i.e., there exists some finite constant $C > 0$ such that*

$$\sup_{\varepsilon>0} E_\varepsilon(\tilde{Q}^\varepsilon; B_1) \le C$$

Proof. As we already pointed out, it makes sense to evaluate $E_\varepsilon(\cdot; B_1)$ for the fixed LdG minimizers $Q^\star \in \mathcal{A}^{\mathrm{ax}}_{Q_b}$. For each $\varepsilon > 0$ we have

$$E_\varepsilon(\tilde{Q}^\varepsilon; B_1) \leq E_\varepsilon(Q^\star; B_1).$$

because $\tilde{Q}^\varepsilon$ is a minimizer of $E_\varepsilon(\cdot; B_1)$. Since LdG minimizers in $\mathcal{A}^{\mathrm{ax}}_{Q_b}$ have finite LdG energy, there exists $C > 0$ such that $E(Q; B_1) = C$, so that

$$\sup_{\varepsilon>0} E_\varepsilon(\tilde{Q}^\varepsilon; B_1) \leq C.$$

□

The equiboundedness of $\{\tilde{Q}^\varepsilon\}_{\varepsilon>0}$ requires more work. Before proving it, we observe the following simple fact.

Lemma 5.3. *Fix a minimizer $Q^\star$ of the LdG energy, $Q^\star \in \mathcal{A}^{ax}_{Q_b}$. For each $\varepsilon > 0$, let $E_\varepsilon(\cdot; B_1)$ be defined as in* (5.7.1) *over the class* (5.7.2) *and let $\tilde{Q}^\varepsilon$ be a minimizer of $E_\varepsilon(\cdot; B_1)$ in the class* (5.7.2)*. Pick a sequence $(\tilde{Q}^{\varepsilon_j})_{\varepsilon_j}$, $\varepsilon_j \to 0$ as $j \to \infty$, from the family $\{\tilde{Q}^\varepsilon\}_{\varepsilon>0}$ and suppose it has a weak limit Q^∞ (i.e., a limit in the weak topology of $W^{1,2}(B_1; \mathcal{S}_0)$) as $j \to \infty$. Then Q^∞ is S^1-equivariant and it is S^4-valued.*

Proof. By the equiboundedness of $\{E_\varepsilon(\cdot; B_1)\}_{\varepsilon>0}$, there is some finite constant $C > 0$ such that

$$\frac{1}{4\varepsilon^2} \int_{B_1} \left(1 - \left|\tilde{Q}^\varepsilon\right|^2\right)^2 \mathrm{d}x \leq C \implies \int_{B_1} \left(1 - \left|\tilde{Q}^\varepsilon\right|^2\right)^2 \mathrm{d}x \leq C_1 \varepsilon^2 \tag{5.7.9}$$

for each $\varepsilon > 0$. Note also that, since $(\tilde{Q}^{\varepsilon_j})_{\varepsilon_j}$ converges weakly, it is bounded in the $W^{1,2}$-norm, so Rellich-Kondrachov theorem implies that, upon passing to a subsequence (which we not relabel) if necessary, $\tilde{Q}^{\varepsilon_j} \to Q^\infty$ strongly in $L^2(B_1; \mathcal{S}_0)$. This ensures that there exists another subsequence (not relabeled) on which $\tilde{Q}^{\varepsilon_j}(x) \to Q^\infty(x)$ for a.e. $x \in B_1$ as $j \to \infty$. Pointwise convergence a.e. implies in turn that Q^∞ is S^1-equivariant, since the $\tilde{Q}^{\varepsilon_j}$-s are. By Fatou's lemma and (5.7.9),

$$\int_{B_1} \left(1 - |Q^\infty|^2\right)^2 = \int_{B_1} \liminf_{j\to\infty} \left(1 - \left|\tilde{Q}^{\varepsilon_j}\right|^2\right)^2 \leq \liminf_{j\to\infty} \int_{B_1} \left(1 - \left|\tilde{Q}^{\varepsilon_j}\right|^2\right)^2 \leq 0.$$

Since $\left(1 - |Q^\infty|^2\right)^2 \geq 0$, this implies $|Q^\infty| = 1$ a.e. and we are done. □

We now recall from [106, Lemma 1] that for each $\hat{Q} \in \mathcal{S}_0$ we have

$$\mathrm{Tr}(\hat{Q}^3) \leq \frac{\left|\hat{Q}\right|^3}{\sqrt{6}}. \tag{5.7.10}$$

Lemma 5.4. *Fix a minimizer $Q^\star$ of the LdG energy, $Q^\star \in \mathcal{A}^{ax}_{Q_b}$. For $\varepsilon > 0$, let $E_\varepsilon(\cdot; B_1)$ be defined as in* (5.7.1) *over the class* (5.7.2) *and let $\tilde{Q}^\varepsilon$ be a critical point for $E_\varepsilon(\cdot; B_1)$ in the class* (5.7.2)*. Then*

$$\|\tilde{Q}^\varepsilon\|_{L^\infty(B_1, \mathcal{S}_0)} \leq 1. \tag{5.7.11}$$

Proof. We closely follow the argument of [106, Proposition 3]. Suppose, for the sake of a contradiction, that there exists $x^* \in B_1$ such that $\left|\tilde{Q}^\varepsilon\right|(x^*) > 1$. If $\tilde{Q}^\varepsilon$ is a critical point for $E_\varepsilon(\cdot; B_1)$ in the class (5.7.2), it is actually a free critical point by the results quoted in Section 5.3, and it solves the boundary-value problem

$$\begin{cases} \Delta\tilde{Q}^\varepsilon_{ij} = -a\tilde{Q}^\varepsilon_{ij} - b\left(\tilde{Q}^\varepsilon_{ik}\tilde{Q}^\varepsilon_{kj} - \frac{1}{3}\delta_{ij}\right) + c\tilde{Q}^\varepsilon_{ij}\operatorname{Tr}(\tilde{Q}^\varepsilon)^2 \\ + \frac{1}{\varepsilon^2}\tilde{Q}^\varepsilon_{ij}\left(\left|\tilde{Q}^\varepsilon\right|^2 - 1\right) + (\tilde{Q}^\varepsilon_{ij} - Q^\star_{ij}) \text{ in } B_1, \\ \tilde{Q}^\varepsilon = Q_b \text{ on } \partial B_1. \end{cases}$$

By our choice of the boundary condition, $\left|\tilde{Q}^\varepsilon\right| = 1$ on the boundary. Since the function $\left|\tilde{Q}^\varepsilon\right|^2 : \overline{B_1} \to \mathbb{R}$ must attain its maximum at $x^* \in B_1$, we necessarily have that

$$\Delta\left(\frac{1}{2}\left|\tilde{Q}^\varepsilon\right|^2\right)(x^*) \leq 0. \tag{5.7.12}$$

Multiplying both sides of the Euler-Lagrange equations by $\tilde{Q}^\varepsilon_{ij}$, we obtain

$$\begin{aligned} \Delta\left(\frac{1}{2}\left|\tilde{Q}^\varepsilon\right|^2\right)(x^*) &= -a\operatorname{Tr}((\tilde{Q}^\varepsilon)^2) - b\operatorname{Tr}((\tilde{Q}^\varepsilon)^3) + c\left(\operatorname{Tr}(\tilde{Q}^\varepsilon)^2\right)^2 \\ &\quad + \left|\nabla\tilde{Q}^\varepsilon\right|^2 + \frac{1}{\varepsilon^2}\left|\tilde{Q}^\varepsilon\right|^2\left(\left|\tilde{Q}^\varepsilon\right|^2 - 1\right) + \frac{1}{\varepsilon^2}\tilde{Q}^\varepsilon_{ij}\left(\tilde{Q}^\varepsilon_{ij} - Q^\star_{ij}\right). \end{aligned} \tag{5.7.13}$$

We note that

$$-a\operatorname{Tr}((\tilde{Q}^\varepsilon)^2) - b\operatorname{Tr}((\tilde{Q}^\varepsilon)^3) + c\left(\operatorname{Tr}(\tilde{Q}^\varepsilon)^2\right)^2 \geq f\left(\left|\tilde{Q}^\varepsilon\right|\right), \tag{5.7.14}$$

where

$$f\left(\left|\tilde{Q}^\varepsilon\right|\right) = -a\left|\tilde{Q}^\varepsilon\right|^2 - \frac{b}{\sqrt{6}}\left|\tilde{Q}^\varepsilon\right|^3 + c\left|\tilde{Q}^\varepsilon\right|^4$$

by (5.7.10). Next we note that

$$f\left(\left|\tilde{Q}^\varepsilon\right|\right) > 0 \text{ for } \left|\tilde{Q}^\varepsilon\right| > 1,$$

and that

$$\left|\tilde{Q}^\varepsilon\right|^2\left(\left|\tilde{Q}^\varepsilon\right|^2 - 1\right) > 0 \text{ for } \left|\tilde{Q}^\varepsilon\right| > 1,$$

and

$$\tilde{Q}^\varepsilon_{ij}(\tilde{Q}^\varepsilon_{ij} - Q^\star_{ij}) \geq \left|\tilde{Q}^\varepsilon\right|\left(\left|\tilde{Q}^\varepsilon\right| - 1\right) > 0 \text{ for } \left|\tilde{Q}^\varepsilon\right| > 1,$$

which together (5.7.13) and (5.7.14) imply

$$\Delta\left(\frac{1}{2}\left|\tilde{Q}^\varepsilon\right|^2\right)(x) > 0$$

for all interior points $x \in B_1$ with $\left|\tilde{Q}^\varepsilon(x)\right| > 1$. This contradicts (5.7.12) and thus gives the conclusion. □

Corollary 5.9. *Fix a minimizer $Q^\star$ of the LdG energy, $Q^\star \in \mathcal{A}^{ax}_{Q_b}$. For each $\varepsilon > 0$, let $E_\varepsilon(\cdot; B_1)$ be defined as in (5.7.1) over the class (5.7.2) and let $\tilde{Q}^\varepsilon$ be a minimizer of $E_\varepsilon(\cdot; B_1)$ in the class (5.7.2). Then the family $\{\tilde{Q}^\varepsilon\}_{\varepsilon>0}$ is equibounded in the $W^{1,2}$-norm, i.e., there exists a constant $C > 0$ such that*

$$\sup_{\varepsilon>0} \|\tilde{Q}^\varepsilon\|_{W^{1,2}(B_1,\mathcal{S}_0)} \leq C.$$

Proof. Since $\{E_\varepsilon\}_{\varepsilon>0}$ is equibounded by Lemma 5.2, it follows that there exists a constant $C_1 > 0$ such that

$$\sup_{\varepsilon>0} \|\nabla Q\|^2_{L^2(B_1,\mathcal{S}_0)} \leq C_1.$$

By Lemma 5.4 and by the boundedness of B_1, it follows that

$$\sup_{\varepsilon>0} \|\tilde{Q}^\varepsilon\|^2_{L^2(B_1,\mathcal{S}_0)} \leq C_2$$

for some constant C_2. Then

$$\sup_{\varepsilon>0} \|\tilde{Q}^\varepsilon\|^2_{W^{1,2}(B_1,\mathcal{S}_0)} \leq C,$$

where $C = C_1 + C_2$. □

Then we have

Proposition 5.10. *Fix a minimizer $Q^\star$ of the LdG energy, $Q^\star \in \mathcal{A}^{ax}_{Q_b}$. For each $\varepsilon > 0$, let $E_\varepsilon(\cdot; B_1)$ be defined as in (5.7.1) over the class (5.7.2) and let $\tilde{Q}^\varepsilon$ be a minimizer of $E_\varepsilon(\cdot; B_1)$ in the class (5.7.2). Then, for each sequence $(Q^{\varepsilon_j})_{\varepsilon_j}$, where $\varepsilon_j \to 0$ as $j \to \infty$, picked up from the family $\{\tilde{Q}^\varepsilon\}_{\varepsilon>0}$, there is a subsequence (not relabeled) such that:*

(a) $Q^{\varepsilon_j} \rightharpoonup Q^\star$ in $W^{1,2}(B_1, \mathcal{S}_0)$ as $j \to \infty$;

(b) $Q^{\varepsilon_j} \to Q^\star$ in $L^2(B_1, \mathcal{S}_0)$ as $j \to \infty$;

Proof. By Corollary 5.9, $\{Q^\varepsilon\}_{\varepsilon>0}$ is equibounded and so is every sequence picked up from it. Thus, for any such sequence there exists a weak limit $Q^\infty \in W^{1,2}(B_1, \mathcal{S}_0)$. By Rellich-Kondrachov theorem and by Lemma 5.3, both conclusions in the statement hold for Q^∞. We now show that $Q^\infty = Q^\star$ in $L^2(B_1, \mathcal{S}_0)$. Since $E_\varepsilon(Q^\star; B_1) = E(Q^\star; B_1)$, by the weak lower semicontinuity of the first and third integrals in $E_\varepsilon(\cdot; B_1)$ (which do not depend on ε), equiboundedness of $\{E_\varepsilon(\cdot; B_1)\}_{\varepsilon>0}$ and Fatou's lemma, we have

$$\int_{B_1} \frac{1}{2} |\nabla Q^\infty|^2 + F(Q^\infty)\, \mathrm{d}x + \frac{1}{2} \int_{B_1} |Q^\infty - Q^\star|^2\, \mathrm{d}x \\ \leq \liminf_{j\to\infty} E_\varepsilon(\tilde{Q}^{\varepsilon_j}; B_1) \leq \limsup_{j\to\infty} E_{\varepsilon_j}(\tilde{Q}^{\varepsilon_j}; B_1) \leq E(Q^\star; B_1).$$

Since $Q^\star$ is a minimizer of the LdG w.r.t. its boundary condition and we have already proven that $Q^\infty \in \mathcal{A}^{\mathrm{ax}}_{Q_b}$, we must have

$$E(Q^\infty; B_1) \geq E(Q^\star; B_1).$$

Thus, the above inequalities clearly imply

$$\int_{B_1} |Q^\infty - Q^\star|^2 \,\mathrm{d}x = 0,$$

i.e., $Q^\infty = Q^\star$ in $L^2(B_1, \mathcal{S}_0)$ and thus both conclusions follow at once. □

The $W^{1,2}$-convergence in the above result is actually strong.

Corollary 5.11. *Under the hypotheses of Proposition 5.10 (we retain the same notations), we have $\tilde{Q}^{\varepsilon_j} \to Q^\star$ strongly in $W^{1,2}(B_1, \mathcal{S}_0)$ as $j \to \infty$.*

Proof. Since we already have the weak convergence $\tilde{Q}^{\varepsilon_j} \rightharpoonup Q^\star$ as $j \to \infty$ and since $W^{1,2}(B_1, \mathcal{S}_0)$ is a Hilbert space, it suffices to prove that

$$\int_{B_1} \frac{1}{2} \left|\nabla \tilde{Q}^{\varepsilon_j}\right|^2 \mathrm{d}x \to \int_{B_1} \frac{1}{2} |\nabla Q^\star|^2 \,\mathrm{d}x \tag{5.7.15}$$

as $j \to \infty$.

Now, $Q^\star$ is a minimizer of the LdG energy. On the other hand, it is also the weak sequential limit of $(\tilde{Q}^{\varepsilon_j})_{\varepsilon_j}$ and hence

$$E(Q^\star; B_1) \leq \liminf_{j\to\infty} E_{\varepsilon_j}(\tilde{Q}^{\varepsilon_j}; B_1) \leq \limsup_{j\to\infty} E_{\varepsilon_j}(\tilde{Q}^{\varepsilon_j}; B_1) \leq E(Q^\star; B_1), \tag{5.7.16}$$

which implies

$$\lim_{j\to\infty} E(\tilde{Q}^{\varepsilon_j}; B_1) = E(Q^\star; B_1) \tag{5.7.17}$$

which forces

$$\lim_{j\to\infty} \frac{1}{4\varepsilon_j^2} \int_{B_1} \left(1 - \left|\tilde{Q}^{\varepsilon_j}\right|^2\right)^2 \mathrm{d}x = 0 \tag{5.7.18}$$

Now, observe that

$$F(\tilde{Q}^{\varepsilon_j}) \to F(Q^\star) \text{ strongly in } L^2(B_1, \mathcal{S}_0), \tag{5.7.19}$$

since $\tilde{Q}^{\varepsilon_j} \to Q^\star$ strongly in $L^2(B_1, \mathcal{S}_0)$ and $F(\cdot)$ is a polynomial function. Therefore we have

$$\lim_{j\to\infty} \int_{B_1} F(\tilde{Q}^{\varepsilon_j}) \,\mathrm{d}x = \int_{B_1} F(Q^\star) \,\mathrm{d}x \tag{5.7.20}$$

which together with (5.7.16) and (5.7.18) enforces (5.7.15), thus the conclusion follows. □

Corollary 5.11 qualifies in a precise sense the minimizers $\tilde{Q}^\varepsilon$s as *approximate minimizers* for the LdG energy. We can now state the main result of this section. In fact, it is a simple consequence of Corollary 5.11. Nevertheless, in view of its global importance in the present work, we prefer to state it as a theorem.

Theorem 5.12 (Monotonicity formula)**.** *Fix a minimizer $Q^\star$ of the LdG energy, $Q^\star \in \mathcal{A}^{ax}_{Q_b}$. For each $\varepsilon > 0$, let $E_\varepsilon(\cdot; B_1)$ be defined as in* (5.7.1) *over the class* (5.7.2) *and let $\tilde{Q}^\varepsilon$ be a minimizer of $E_\varepsilon(\cdot; B_1)$ in the class* (5.7.2)*. Pick a sequence $(\tilde{Q}^{\varepsilon_j})_{\varepsilon_j}$, $\varepsilon_j \to 0$ as $j \to \infty$, from the family $\{\tilde{Q}^\varepsilon\}_{\varepsilon>0}$ and let $Q^\star \in \mathcal{A}^{ax}_{Q_b}$ its strong $W^{1,2}$-limit (see Corollary 5.11). Then $Q^\star$ satisfies the following monotonicity formula:*

$$\frac{1}{R_2}\int_{B_{R_2}(x_0)} e(\nabla Q^\star, Q^\star) - \frac{1}{R_1}\int_{B_{R_1}(x_0)} e(\nabla Q^\star, Q^\star)$$
$$= \int_{B_{R_2}(x_0)\setminus B_{R_1}(x_0)} \frac{1}{|x-x_0|}\left|\frac{\partial Q^\star}{\partial r}\right|^2 + 2\int_{R_1}^{R_2}\frac{dR}{R^2}\int_{B_R(x_0)} F(Q^\star), \quad (5.7.21)$$

where $x_0 \in B_1$, $R_1 < R_2$ are such that $B_{R_2}(x_0) \subset\subset B_1$ and $e(\nabla Q, Q)$ is the LdG energy density as defined in (1.1.4)*. As always, $\frac{\partial}{\partial r}$ means the directional derivative in the radial direction $(x-x_0)/|x-x_0|$.*

Proof. Write (5.7.4) for $\tilde{Q}^{\varepsilon_j}$ and take the limit $j \to \infty$ on both sides. Then

$$\lim_{j\to\infty}\left(\frac{1}{R_2}\int_{B_{R_2}(x_0)} \mathcal{E}^{\varepsilon_j}(\nabla\tilde{Q}^{\varepsilon_j}, \tilde{Q}^{\varepsilon_j}) - \frac{1}{R_1}\int_{B_{R_1}(x_0)} \mathcal{E}^{\varepsilon_j}(\nabla\tilde{Q}^{\varepsilon_j}, \tilde{Q}^{\varepsilon_j})\right)$$
$$= \lim_{j\to\infty}\left(\int_{B_{R_2}(x_0)\setminus B_{R_1}(x_0)} \frac{1}{|x-x_0|}\left|\frac{\partial\tilde{Q}^{\varepsilon_j}}{\partial r}\right|^2 + 2\int_{R_1}^{R_2}\frac{\mathrm{d}R}{R^2}\int_{B_R(x_0)} F(\tilde{Q}^{\varepsilon_j})\right.$$
$$+ 2\int_{R_1}^{R_2}\frac{\mathrm{d}R}{R^2}\int_{B_R(x_0)} \frac{\left(1-\left|\tilde{Q}^{\varepsilon_j}\right|^2\right)^2}{4\varepsilon^2} + 2\int_{R_1}^{R_2}\frac{\mathrm{d}R}{R^2}\int_{B_R(x_0)} \frac{\left|\tilde{Q}^{\varepsilon} - Q^\star\right|^2}{2}$$
$$\left. + \int_{R_1}^{R_2}\frac{\mathrm{d}R}{R^2}\int_{B_R(x_0)} \left\langle \tilde{Q}^{\varepsilon} - Q^\star, x\cdot\nabla Q^\star\right\rangle\right), \quad (5.7.22)$$

First, note that the strong convergence $\nabla\tilde{Q}^{\varepsilon_j} \to \nabla Q^\star$ as $j\to\infty$ implies the strong convergence

$$\frac{\partial\tilde{Q}^{\varepsilon_j}}{\partial r} \to \frac{\partial Q^\star}{\partial r} \text{ as } j\to\infty \text{ in } L^2(B_1, \mathcal{S}_0).$$

Indeed, by definition of directional derivative,

$$0 \le \int_{B_1}\left|\frac{\partial\tilde{Q}^{\varepsilon_j}}{\partial r} - \frac{\partial Q^\star}{\partial r}\right|^2 \mathrm{d}x \le \int_{B_1}\left|\nabla\tilde{Q}^{\varepsilon_j} - \nabla Q^\star\right|^2 \mathrm{d}x$$

and the last member tends to 0 as $j\to\infty$. Next, observe that (5.7.18) implies

$$\lim_{j\to\infty}\frac{1}{4\varepsilon_j^2}\int_{B_R}\left(1-\left|\tilde{Q}^{\varepsilon_j}\right|^2\right)^2 = 0, \quad \forall 0<R<1,$$

since $\left(1-\left|\tilde{Q}^{\varepsilon_j}\right|^2\right)^2 \ge 0$. In view of the strong $W^{1,2}$-convergence of Q^{ε_j} towards $Q^\star$, (5.7.19), (5.7.18) and the above remark, all the limits involved in (5.7.22) do exist, so that the limit of the sum becomes the sum of the limits and we get (5.7.21). ☐

Remark 5.7.1. This argument actually holds entirely also in the nonsymmetric case. In a previous version of this work, the trick of fixing a minimizer and then penalizing being distant from it was not recognized. Because of this, the approximation argument was less powerful and gave a monotonicity formula only for a class of minimizers, those that can be obtained as strong limits of minimizers of Ginzburg-Landau-type penalizations of the LdG energy. Thus, even if all the computations were valid also without symmetry, there was a substantial reason to not extend the argument also in the nonsymmetric setting. In this version of the work, we retained the inner-variations method in §4.3 for a threefold reason: to not change too much the new version w.r.t. the one approved by the referees (even if one of them envisaged the possibility of extension to all minimizers); to detail a second way to get the monotonicity formula (which avoids requiring explicitly stationarity) and to highlight an important difference between the symmetric and the nonsymmetric case.

5.8 The compactness theorem

In this section we prove a strong compactness theorem for blow-ups of S^1-minimizers Q of $E(\cdot;B_1)$ in the class $\mathcal{A}^{\mathrm{ax}}_{Q_b}$.

The main difficulty in the proof is the lack of the Luckhaus' lemma (Lemma 4.5) whose proof is indeed nonequivariant, in the sense that comparison maps constructed in it are generally *not* equivariant. Looking at the proof of Theorem 4.11 to understand the problem, we see that it would not be legitimate to invoke the minimality of scaled maps Q_R, since they are minimizing only among equivariant maps. We shall remedy this inconvenient by constructing by hand comparisons maps having the same properties as those given by the Luckhaus' lemma but that are also equivariant. We note that there is in literature an equivariant version of the Luckhaus' lemma, proved by Gastel [49]. However, it cannot be used in this context without some adaptations because of the fact that our energy functional is not the Dirichlet energy but the LdG energy. Moreover, we do not need the full power of the Luckhaus' lemma and we do not aim to full generality, so we preferred to make a more explicit construction modeled over the specific case at hand. For the construction, we followed the line of earlier results, limited to the context of axially symmetric harmonic maps from B_1 into S^2, due to Grotowski [56] who in turn detailed a sketchy argument of Hardt, Kinderlehrer & Lin [61].

We point out three important remarks. First, let $x_0 \in B_1$ and consider the family of blown-up maps $\{Q_R\}_R$ (of course, the center of blowing-up is understood to be x_0). Then maps Q_R-s are generally *not* equivariant. Indeed, blowing-up is easily seen not to preserve equivariance in general. However, if $x_0 \in \{z\text{-axis}\} \cap B_1$, then blow-ups *are* equivariant.

Secondly, the ε-regularity theorem, Theorem 4.6, and higher-regularity theorems hold without modifications even for $Q \in \mathcal{A}^{\mathrm{ax}}_{Q_b}$ minimizing the LdG energy. Indeed, Theorem 5.4 maps Q into a critical point of the free problem. Criticality, along with a monotonicity formula, is all that is needed in the proof of Theorem 4.6.

Thirdly, S^1-minimizers Q of $E(\cdot;B_1)$ *cannot* have singularities located off the z-axis. Indeed, any $Q \in W^{1,2}(B_1,S^4)$ has $\mathcal{H}^1$-null concentration set [53, Proposition 9.21] and, by the monotonicity formula and the ε-regularity theorem, $\operatorname{sing} Q = \Sigma(Q)$. Hence, all possible singularities of Q are contained into the z-axis. Since the rôle of blowing-up is allowing to decide whether a point is singular or not, we see that we need to blow-up Q only around points belonging to the z-axis.

In view of the previous remarks, definitions and auxiliary results in Section 4.6 carry over without modifications, except for the fact that blowing-up centers are always understood as points belonging to the z-axis and that all the statements must be understood in the S^1-equivariant setting. Keeping this in mind, we now restate, for the sake of convenience, some of that results. Proofs are exactly the same and we shall not rewrite them here. Plainly, when the monotonicity formula is evoked, we now mean formula (5.7.21).

Let $x_0 \in \{z\text{-axis}\} \cap B_1$ and let $Q \in \mathcal{A}^{\mathrm{ax}}_{Q_b}$ be a minimizer of the LdG energy (1.1.3) in the class $\mathcal{A}^{\mathrm{ax}}_{Q_b}$. Let R_0 as in (4.6.1). We define maps Q_R as in (4.6.2). In view of the previous remarks, each Q_R is S^1-equivariant. We have

Lemma 5.5. *Let $Q \in \mathcal{A}^{ax}_{Q_b}$ be a minimizer of the LdG energy* (1.1.3) *in the class $\mathcal{A}^{ax}_{Q_b}$. Fixed $x_0 \in \{z\text{-axis}\} \cap B_1$, pick arbitrarily $\sigma > 0$ and, for $R < R_0/\sigma$, define scaled maps Q_R as in* (4.6.2)*. Then*

$$\limsup_{R\to 0} \int_{B_\sigma} |\nabla Q_R|^2 \, dx < +\infty \tag{5.8.1}$$

for each $\sigma > 0$. In other words, the family $\{Q_R\}_R$ is locally equibounded in $W^{1,2}_{loc}(\mathbb{R}^3, S^4)$.

From now on, we take Remark 4.6.1 into account. So, all Q_R-s are well-defined on B_1. Scaled energy functionals are defined exactly as in (4.6.4) but, of course, restricted to S^1-equivariant maps.

It still holds, with the same proof, the following lemma.

Lemma 5.6. *Let $Q \in \mathcal{A}^{ax}_{Q_b}$ be a minimizer of the LdG energy in $\mathcal{A}^{ax}_{Q_b}$. Let R, Q_R be as above. Let $\rho \in (0,1)$ and let $\{v_R\}_R \subset W^{1,2}(B_1, S^4)$ be a family of S^1-equivariant mappings such that $v_R = Q_R$ on ∂B_ρ. Then*

$$\liminf_{R\to 0} \int_{B_\rho} |\nabla Q_R|^2 \, dx \le \liminf_{R\to 0} \int_{B_\rho} |\nabla v_R|^2 \, dx. \tag{5.8.2}$$

We can now prove

Theorem 5.13 (Compactness theorem in the equivariant case)**.** *Let $Q \in \mathcal{A}^{ax}_{Q_b}$ be a minimizer of the LdG energy in $\mathcal{A}^{ax}_{Q_b}$. Fix $x_0 \in \{z\text{-axis}\} \cap B_1$ and let R, Q_R, E_R be as above and consider the family $\{Q_R\}_R$. Then there is $Q_0 \in W^{1,2}_{loc}(\mathbb{R}^3, S^4)$ and there is a sequence $(Q_{R_j})_{R_j}$, $R_j \to 0$ as $j \to \infty$, which converges to Q_0 in the strong topology of $W^{1,2}_{loc}(\mathbb{R}^3, \mathcal{S}_0)$. In addition, Q_0 is a S^1-equivariant locally minimizing harmonic map and it is degree-zero homogeneous.*

Proof. As we said, the proof must be different from that of Theorem 4.11 because of the fact that Luckhaus' lemma does not apply. On the other hand, by results above, this is the only actual difference and we shall remedy the lack by constructing S^1-equivariant competitors that satisfy all needed properties by hand. The rest of the proof goes essentially like that of Theorem 4.11.

By Lemma 5.5, $\{Q_R\}_R$ is locally equibounded in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^3, S^4)$ and so it is each sequence $(Q_{R_j})_{R_j}$, with $R_j \to 0$ as $j \to \infty$, extracted from it. By the Rellich-Kondrachov theorem there exists $Q_0 \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^3, \mathcal{S}_0)$ so that, up to subsequences, we have $Q_{R_j} \rightharpoonup Q_0$ (weakly) as $j \to \infty$ in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^3, \mathcal{S}_0)$ and $Q_{R_j} \to Q_0$ strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^3, S^4)$. Thus, in particular, $Q_0(x) \in S^4$ a.e., i.e., $Q_0 \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^3, S^4)$. Moreover, Q_0 is S^1-equivariant, because the pointwise limit of a sequence of S^1-equivariant maps is itself S^1-equivariant. By the monotonicity formula (5.7.21) and the equiboundedness of the potential (so that the potential disappears in the limit $R \to 0$), it easily follows (mimicking, for instance, [130, Lemma 2.6] or the reasoning in [134, Section 3.2]) that Q_0 is degree-zero homogeneous. Thus, it is enough to show strong convergence and minimality in some ball $B_\rho \subset B_1$ to get the same properties on any $B_\rho \subset \mathbb{R}^3$ for any $\rho > 0$, by scale invariance of Q_0 and the existence of the full limit of $\frac{1}{R}\int_{B_R} |\nabla Q|^2 \, dx$ as $R \to 0$.

Let $\delta \in (0,1)$ be a fixed number and let $w \in W^{1,2}(B_1, S^4)$ be such that $w \equiv Q_0$ a.e. on $B_1 \setminus B_{1-\delta}$. By Fatou's lemma and Fubini's theorem, there exists $\rho \in (1-\delta, 1)$ such that

$$\lim_{j\to\infty} \int_{\partial B_\rho} \left|Q_{R_j} - Q_0\right|^2 \, d\mathcal{H}^2 = 0, \tag{5.8.3}$$

and

$$\int_{\partial B_\rho} \left(\left|\nabla Q_{R_j}\right|^2 + |\nabla Q_0|^2\right) d\mathcal{H}^2 \le C < +\infty. \tag{5.8.4}$$

Let us choose

$$\lambda_{R_j} = \left(\int_{B_\rho} \left|Q_{R_j} - Q_0\right|^2 \, d\mathcal{H}^2\right)^{1/6}.$$

Then $\lambda_{R_j} < \delta$ for j large enough and $\lambda_{R_j} \to 0$ as $j \to \infty$.

Before introducing comparision maps, we slice B_ρ in a convenient way, following ideas in [61]. We set

$$B_\rho = B_{\left(1-\lambda_{R_j}\right)\rho} \cup E_{\rho,j} \cup F_{\rho,j}, \tag{5.8.5}$$

where

$$F_{\rho,j} = \left\{x \in B_\rho : (1-\lambda_{R_j})\rho \le |x| \le \rho,\ |(x_1,x_2)| \le \lambda_{R_j}\rho\,|x|\right\} \tag{5.8.6}$$

and

$$E_{\rho,j} = \left(B_\rho \setminus B_{\left(1-\lambda_{R_j}\right)\rho}\right) \setminus F_{\rho,j}. \tag{5.8.7}$$

Of course,

$$F_{\rho,j} = F^+_{\rho,j} \cup F^-_{\rho,j},$$

with $F^\pm_{\rho,j}$ congruent caps, one in the northen hemisphere (+) and the other in the southern hemisphere (-).

Next, we define comparison maps following [61]. We set

$$v_{R_j} = \begin{cases} w\left(\frac{\cdot}{1-\lambda_{R_j}}\right), & \text{on } B_{\left(1-\lambda_{R_j}\right)\rho}, \\ J_{R_j}, & \text{on } E_{\rho,j}, \\ \hat{J}_{R_j}, & \text{on } F^\pm_{\rho,j}. \end{cases} \tag{5.8.8}$$

Above, $\hat{J}_{R_j}$ denotes the homogeneous degree-zero extension with center at $z^\pm_{\rho,j} = \left(0,0,\pm\left(1-\frac{1}{2}\lambda_{R_j}\right)\rho\right)$ of the boundary trace on $\partial F^\pm_{\rho,j}$ to the interior of $F^\pm_{\rho,j}$.

The definition of J_{R_j} requires a little of care. We would like to define

$$J_{R_j}(x) = \Pi(\gamma_{R_j}(x)) \text{ for } (1-\lambda_{R_j})\rho \le |x| \le \rho,$$

where $\Pi : \mathcal{O} \to S^4$ is the nearest point projection ($\mathcal{O}$ a sufficiently narrow neighborhood of S^4 so that Π is well-defined and smooth as needed) and $\gamma_{R_j}(x)$ indicates the linear interpolation between

$$w\left(\frac{x}{1-\lambda_{R_j}}\right) = Q_0\left(\frac{x}{1-\lambda_{R_j}}\right) = Q_0\left(\rho\frac{x}{|x|}\right), \qquad x \in S^2_{(1-\lambda_{R_j})\rho},$$

(the first equality above follows from these facts: when $(1-\lambda_{R_j})\rho \le |x| \le \rho$, we have $|x|/(1-\lambda_{R_j}) \ge \rho$; being $\rho \in (1-\delta, 1)$, we have $\frac{x}{1-\lambda_{R_j}} \in B_1 \setminus B_\rho \subset B_1 \setminus B_{1-\delta}$, on which $Q \equiv w$ by the choice of w) and $Q_{R_j}(\rho x/|x|)$; precisely, we set

$$\gamma_{R_j}(x) = (\rho\lambda_R)^{-1}(\rho - |x|)Q_0(\rho x/|x|) + (\rho\lambda_R)^{-1}(|x| - \rho(1-\lambda_R))Q_{R_j}(\rho x/|x|). \quad (5.8.9)$$

However, two problems can arise: first, when $\gamma_{R_j}(x) = 0$, J_{R_j} is not even defined; second, when $Q_0(\rho x/|x|)$ and $Q_{R_j}(\rho x/|x|)$ are antipodal points on S^4 the projection is ambiguous. Both of these are solved by taking j sufficiently large. Indeed, since $Q_{R_j} \to Q_0$ strongly in $L^2(B_\rho, S^4)$, if j is sufficiently large, then almost every point $\gamma_{R_j}(x)$ lies in a fixed neighborhood of S^4 (we choose j so large that such a neighborhood is away from 0) and almost every $Q_{R_j}(x)$ lies sufficiently close to $Q_0(x)$ to avoid ambiguities. So, by taking j sufficiently large, J_{R_j} is well-defined and also Lipschitz.

Looking at (5.8.8), we see that

$$v_{R_j}|_{\partial B_{\left(1-\lambda_{R_j}\right)\rho}} = Q_0(\rho\cdot), \qquad v_{R_j}|_{\partial B_\rho} = Q_{R_j}(\rho\cdot).$$

Moreover, the v_{R_j}-s are S^4-valued. We now need Luckhaus-type estimates to conclude.

As we pointed out, for j large enough, J_{R_j} is Lipschitz, hence

$$\left|\nabla J_{R_j}\right|^2 \le \mathrm{Lip}(\Pi)^2 \left|\nabla \gamma_{R_j}\right|^2 \quad (5.8.10)$$

almost everywhere.

Putting a spherical coordinate system (r, ω) on $\mathbb{R}^3$ (of course, $r = |x|$), we have

$$\rho\frac{x}{|x|} = \rho\frac{r\omega}{r} = \rho\omega,$$

thus

$$\begin{aligned}
&\int_{E_{\rho,j}} |\nabla\gamma_{\rho,j}(x)|^2 \, \mathrm{d}x \le \int_{B_\rho \setminus B_{(1-\lambda_{R_j})\rho}} |\nabla\gamma_{\rho,j}(x)|^2 \, \mathrm{d}x \\
&= \int_{(1-\lambda_{R_j})\rho}^{\rho} \int_{\partial B_\sigma} \left(\left|\frac{\partial\gamma_{\rho,j}(r,\omega)}{\partial r}\right|^2 + \frac{1}{r^2}\left|\nabla_T \gamma_{R_j}(r,\omega)\right|^2 \right) r^2 \mathrm{d}r \, \mathrm{d}\omega \\
&\le C\left(\int_{(1-\lambda_{R_j})\rho}^{\rho} \int_{\partial B_\sigma} (\rho\lambda_{R_j})^{-2} \left|Q_{R_j}(\rho\omega) - Q_0(\rho\omega)\right|^2 r^2 \mathrm{d}r \, \mathrm{d}\omega \right. \qquad (5.8.11) \\
&\left. + \lambda_{R_j} \int_{\partial B_\rho} |\nabla_T Q_0(\rho\omega)|^2 + \left|\nabla_T Q_{R_j}(\rho\omega)\right|^2 \mathrm{d}\mathcal{H}^2 \right) \\
&\le C_1 \left(\lambda_{R_j}^{-1} \int_{\partial B_\rho} \left|Q_{R_j} - Q_0\right|^2 \mathrm{d}\mathcal{H}^2 + \lambda_{R_j} \int_{\partial B_\rho} |\nabla_T Q_0|^2 + \left|\nabla_T Q_{R_j}\right|^2 \mathrm{d}\mathcal{H}^2 \right)
\end{aligned}$$

Note that the r.h.s. of (5.8.11) approaches zero as $j \to \infty$ by the choice of λ_{R_j} and the strong convergence $Q_{R_j} \to Q_0$ in L^2.

By definition of degree-zero homogeneous extension, the Fubini's theorem and the same calculations above, we get also

$$\begin{aligned}&\int_{F_{\rho,j}} \left|\nabla v_{R_j}\right|^2 \mathrm{d}x = \int_{F_{\rho,j}} \left|\nabla \hat{J}_{R_j}\right|^2 \mathrm{d}x \le C\lambda_{R_j} \int_{\partial B_\rho} \left|\nabla_T J_{R_j}\right|^2 \mathrm{d}\mathcal{H}^2 \\ &\le C\lambda_{R_j} \int_{\partial B_\rho} \left|\nabla_T \gamma_{R_j}\right|^2 \mathrm{d}\mathcal{H}^2 \le C_2\lambda_{R_j} \int_{\partial B_\rho} |\nabla_T Q_0|^2 + \left|\nabla_T Q_{R_j}\right|^2 \mathrm{d}\mathcal{H}^2\end{aligned} \tag{5.8.12}$$

whose r.h.s. tends to zero as $j \to \infty$ because of (5.8.4).

Summarizing, we proved

$$\begin{aligned}&\int_{B_\rho} \left|\nabla v_{R_j}\right|^2 \mathrm{d}x \le \int_{B_{(1-\lambda_{R_j})\rho}} \left|\nabla w\left(\frac{\cdot}{1-\lambda_{R_j}}\right)\right|^2 \mathrm{d}x \\ &+ C_1\left(\lambda_{R_j}^{-1}\int_{\partial B_\rho} \left|Q_{R_j} - Q_0\right|^2 \mathrm{d}\mathcal{H}^2 + \lambda_{R_j}\int_{\partial B_\rho} |\nabla_T Q_0|^2 + \left|\nabla_T Q_{R_j}\right|^2 \mathrm{d}\mathcal{H}^2\right) \\ &+ C_2\lambda_{R_j}\int_{\partial B_\rho} |\nabla_T Q_0|^2 + \left|\nabla_T Q_{R_j}\right|^2 \mathrm{d}\mathcal{H}^2\end{aligned} \tag{5.8.13}$$

We are now close to the conclusion. Indeed, since $F(\cdot)$ is essentially bounded, we have

$$\liminf_{j\to\infty} E_{R_j}(Q_{R_j}; B_\rho) = \liminf_{j\to\infty} \int_{B_\rho} \left|\nabla Q_{R_j}\right|^2 \mathrm{d}x$$

and

$$\liminf_{j\to\infty} E_{R_j}(v_{R_j}; B_\rho) = \liminf_{j\to\infty} \int_{B_\rho} \left|\nabla v_{R_j}\right|^2 \mathrm{d}x.$$

By Lemma 5.6 and (5.8.13) we get

$$\begin{aligned}\int_{B_\rho} |\nabla Q_0|^2 &\le \liminf_{j\to\infty} \int_{B_\rho} \left|\nabla Q_{R_j}\right|^2 \le \liminf_{j\to\infty} \int_{B_\rho} \left|\nabla v_{R_j}\right|^2 \\ &= \lim_{j\to\infty} \left[\int_{B_{(1-\lambda_{R_j})\rho}} \left|\nabla w\left(\frac{\cdot}{1-\lambda_{R_j}}\right)\right|^2 + \int_{B_\rho\setminus B_{\rho(1-\lambda_{R_j})}} \left|\nabla v_{R_j}\right|^2\right] \\ &\le \lim_{j\to\infty} \left[(1-\lambda_{R_j})\int_{B_\rho} |\nabla w|^2 \right. \\ &+ C_1\left(\lambda_{R_j}^{-1}\int_{\partial B_\rho} \left|Q_{R_j} - Q_0\right|^2 \mathrm{d}\mathcal{H}^2 + \lambda_{R_j}\int_{\partial B_\rho} |\nabla_T Q_0|^2 + \left|\nabla_T Q_{R_j}\right|^2 \mathrm{d}\mathcal{H}^2\right) \\ &\left. + C_2\lambda_{R_j}\int_{\partial B_\rho} |\nabla_T Q_0|^2 + \left|\nabla_T Q_{R_j}\right|^2 \mathrm{d}\mathcal{H}^2\right] \\ &= \int_{B_\rho} |\nabla w|^2 .\end{aligned} \tag{5.8.14}$$

Since w is arbitrary, inequality (5.8.14) implies both minimality of Q_0 and strong convergence $Q_{R_j} \to Q_0$ in $W^{1,2}(B_\rho, S^4)$ as $j \to \infty$. □

We now prove some easy corollaries of Theorem 5.13.

Corollary 5.14. *Let $Q_0 \in W^{1,2}_{loc}(\mathbb{R}^3, S^4)$ the S^1-equivariant degree-zero homogeneous harmonic map given in Theorem 5.13. Then $Q_0 \in C^\infty(\mathbb{R}^3 \setminus \{0\}, S^4)$.*

Proof. Since Q_0 is degree-zero homogeneous, it is in particular a harmonic map from S^2 into S^4. Hence, by Hélein's theorem [68], it is smooth as a map $S^2 \to S^4$. By degree-zero homogeneity, it is smooth in $\mathbb{R}^3 \setminus \{0\}$. □

Exactly as in [134, Sections 2.5, 2.11], [53, Proposition 10.26], we can prove joint upper semicontinuity (with respect to joint variables Q_0 and y, see below) of the density. The strong convergence of the blown-up maps is essential to the argument.

Corollary 5.15. *Let Q, $(Q_{R_j})_{R_j}$ and Q_0 as in the statement of Theorem 5.13, let $y \in \{z\text{-axis}\} \cap B_1$ and let $(y_{R_j})_{R_j} \subset \{z\text{-axis}\} \cap B_1$ be a sequence converging to y. The density function of Q_0, Θ_{Q_0}, is jointly upper semicontinuous with respect to Q_0 and y, meaning that*

$$\Theta_{Q_0}(y) \geq \limsup_{j\to\infty} \Theta_{Q_{R_j}}(y_{R_j}).$$

Proof. By Theorem 5.13, $Q_{R_j} \to Q_0$ strongly in $W^{1,2}_{\text{loc}}$. Let $(y_{R_j})_{R_j}$, y, $y_{R_j} \to y$ as in the statement and let us fix $\rho, \varepsilon > 0$ such that $\overline{B_{\rho+\varepsilon}(y)} \subset B_1$. For j large enough, $\left|y_{R_j} - y\right| < \varepsilon$, hence $B_\rho(y_{R_j}) \subset B_{\rho+\varepsilon}(y)$, implying

$$\Theta_{Q_{R_j}}(y_{R_j}) \leq \frac{1}{\rho}\int_{B_\rho(y_{R_j})} \left|\nabla Q_{R_j}\right|^2 \,\mathrm{d}x \leq \frac{1}{\rho}\int_{B_{\rho+\varepsilon}(y)} \left|\nabla Q_{R_j}\right|^2 \,\mathrm{d}x.$$

By the $W^{1,2}_{\text{loc}}$-convergence, for j large enough,

$$\frac{1}{\rho}\int_{B_{\rho+\varepsilon}(y)} \left|\nabla Q_{R_j}\right|^2 \,\mathrm{d}x \leq \frac{1}{\rho}\int_{B_{\rho+\varepsilon}(y)} |\nabla Q_0|^2 \,\mathrm{d}x + \varepsilon.$$

Letting $\varepsilon \to 0$ first and then $\rho \to 0$ completes the proof. □

Corollary 5.15 plays a fundamental role in the proof of the next result about the reduction of the dimension of the singular set. Here, we closely follow the line of the corresponding theorem for (nonsymmentric) minimizing harmonic maps [53, Theorem 10.18].

Corollary 5.16. *Let $Q \in \mathcal{A}^{ax}_{Q_b}$ be a minimizer of the LdG energy* (1.1.3) *in the class $\mathcal{A}^{ax}_{Q_b}$. Then its singular set* $\operatorname{sing}(Q)$ *is a finite set of isolated points located on the z-axis.*

Proof. We recall that the singular set $\operatorname{sing}(Q)$ coincides with the concentration set $\Sigma(Q)$ in view of ε-regularity theorem and higer-regularity theorems.

Let $(x_\nu)_\nu$ be a sequence of singular points converging to x_0 (of course, all these points lie on the z-axis). Up to translations, we can assume $x_0 = 0$. For each ν, let $Q^{(\nu)} := Q(2\,|x_\nu|\,x)$; the maps $Q^{(\nu)}$ are an equibounded sequence of blown-up maps with singularities $y_\nu = \frac{x_\nu}{2|x_\nu|}$, $|y_\nu| = \frac{1}{2}$. Up to subsequences, by Theorem 5.13 we can assume $Q^{(\nu)} \to v$ strongly $W^{1,2}_{\text{loc}}(\mathbb{R}^3, \mathcal{S}_0)$, where v is energy minimizing. We can also assume $y_\nu \to y_0$, with $|y_0| = \frac{1}{2}$. By Corollary 5.15, y_0 is a concentration point, hence a singular point by Theorem 4.6.

We now claim that v is (positively) degree-zero homogeneous: by the $W^{1,2}_{\text{loc}}$-convergence, we have

$$
\begin{aligned}
\frac{1}{\rho}\int_{B_\rho}|\nabla v|^2\,\mathrm{d}x &= \lim_{\nu\to\infty}\frac{1}{\rho}\int_{B_\rho}\left|\nabla Q^{(\nu)}\right|^2\,\mathrm{d}x\\
&= \lim_{\nu\to\infty}\frac{1}{2\rho\,|x_\nu|}\int_{B_{2\rho|x_\nu|}(0)}|\nabla Q|^2\,\mathrm{d}x\\
&= \Theta_Q(0),
\end{aligned}
$$

hence the left-hand side does not depends on ρ. Then, by the monotonicity formula (5.7.21), $\left|\frac{\partial v}{\partial R}\right| = 0$ a.e. and the claim follows. We now have that the whole segment $\{\lambda y_0 : \lambda > 0\} \cap B_1$ is singular, hence $\mathcal{H}^1(\Sigma(Q)) > 0$ and this is absurd. □

Chapter 6

Classification of S^1-equivariant harmonic spheres from S^2 into S^4

Synopsis. In this Chapter we classify all S^1-equivariant harmonic spheres from S^2 into S^4. As a preliminary step, we first classify all S^1-equivariant harmonic spheres from S^2 into S^2 (§6.1). Then we note that it follows from a theorem of Almgren [1] and Calabi [23] that only two cases are possible for harmonic maps $S^2 \to S^4$: either the span of the range of the map is a three dimensional linear subspace of $\mathbb{R}^5 \simeq \mathcal{S}_0$, or it is *linearly full*; i.e., the span of its range is a five dimensional linear subspace of $\mathbb{R}^5 \simeq \mathcal{S}_0$. We decompose $\mathcal{S}_0$ into subspaces invariant under the S^1-action (§6.2.1) and we find that $\mathcal{S}_0$ can be written as the direct sum $L_2 \oplus L_1 \oplus L_0$, where L_ks are invariant subspaces on which S^1 acts by rotations of degree k. It turns out that the only three dimensional invariant subspaces are $L_1 \oplus L_0$ and $L_2 \oplus L_0$, thus harmonic spheres from S^2 into the unit sphere in $L_d \oplus L_0$ ($d = 1, 2$) must have degree d and hence they arise as special cases of the more general classification in §6.1. In particular, they must have energy 4π and 8π, respectively. For the general case, we exploit the properties of the *twistor fibration* $\tau : \mathbb{C}P^3 \to S^4$ (§ 6.2.3) to reduce the study of S^1-equivariant harmonic spheres $S^2 \to S^4$ to the much simpler study of *horizontal* algebraic curves $\mathbb{C}P^1 \to \mathbb{C}P^3$ of degree 3 that are equivariant w.r.t. the appropriate lifting of the S^1-action. The result (Theorem 6.2) is that all S^1-equivariant harmonic spheres $S^2 \to S^4$ can be classified in terms of two complex parameters. It also follows that linearly full maps must have energy 12π.

6.1 S^1-equivariant harmonic maps $S^2 \to S^2$

We now classify all S^1-equivariant harmonic maps $S^2 \to S^2$. In this case, the classification is particularly easy because it is possible to give a very explicit description of the maps involved. Moreover, as we shall see, S^1-equivariant harmonic maps $S^2 \to S^2$ are those S^1-equivariant harmonic maps $S^2 \to S^4$ that are not linearly full.

Let us start looking at S^2 as an embedded submanifold of $\mathbb{R}^3$. Every nonconstant harmonic map $\omega : S^2 \to S^2$ has degree $\deg \omega \neq 0$ and it is holomorphic or antiholomorphic; this fact is linked to the synergy between the structure of the energy and the geometry of the problem. In particular, it can be shown that

$$E(\omega) = \int_{S^2} \frac{1}{2} |\nabla\omega|^2 \, \mathrm{d}\,\mathrm{vol}_{S^2} = 4\pi \, |\deg \omega| \, .$$

Let us denote $\pi : S^2 \setminus \{S\} \to \mathbb{C}$ the stereographic projection from the south pole and let

$$\pi^{-1} : \mathbb{C} \to S^2 \setminus \{S\} : z \mapsto \left(\frac{2z}{1 + |z|^2}, \frac{1 - |z|^2}{1 + |z|^2} \right)$$

the inverse map. The projection π is a homeomorphism between $S^2 \setminus \{S\}$ and $\mathbb{C}$ which extends to an homeomorphism between S^2 and $\mathbb{C} \cup \{\infty\}$ via one-point compactification. We can regard S^2 as $\mathbb{C} \cup \{\infty\}$ and the application between spheres $x \mapsto \omega(x)$ as an application $\mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ whose restriction between $\mathbb{C}$ and $\mathbb{C}$ is a ratio of polynomials in z or in $\bar{z}$ (not both[1]), $f(z) = \frac{P(z)}{Q(z)}$, with $P, Q \in \mathbb{C}[z]$ coprime, whose topological degree is $d = \max\{\deg P, \deg Q\}$. The poles of Q coincide with the points that are mapped to the south pole. In this sense, the singularities of the (anti)meromorphic map f are artificial; i.e., they are due to the inadequacy of the coordinate system induced by the stereographic projection from the south pole in describing the behavior of the function near the south pole.

Thus, ω is harmonic iff $f = \pi \circ \omega \circ \pi^{-1}$ is a rational function in z or in $\bar{z}$. We explicitly remark that, for each polynomial f in z or in $\bar{z}$ (with positive degree), the application $\pi^{-1} \circ f \circ \pi$ is a harmonic map. In particular, all harmonic maps $S^2 \to S^2$ having topological degree 1 are obtained from polynomials of the form $f(z) = \frac{az+b}{cz+d}$, $\left| \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right| \neq 0$ (analogously for polynomials in $\bar{z}$).

We note that

$$\omega^{-1}(\{N\}) = \{P = 0\}, \quad \omega^{-1}(\{S\}) = \{Q = 0\}.$$

Dealing with the point at infinity separately is particularly uncomfortable when S^1-equivariance is taken into account. Then, it is convenient looking at S^2 as $\mathbb{C}P^1$ and describing maps in terms of homogeneous coordinates on $\mathbb{C}P^1$. Then we have:

$$\begin{aligned} & x \in S^2 \leftrightarrow [z_0, z_1] \in \mathbb{C}P^1, \\ & (N \neq) x \mapsto [1, \pi^{-1}(x)], \\ & N \mapsto [0, 1], \end{aligned}$$

so we can look at ω as the application

$$[z_0, z_1] \overset{\omega}{\longmapsto} [\mathcal{D}(z_0, z_1), \mathcal{N}(z_0, z_1)],$$

with $\mathcal{N}$, $\mathcal{D}$ coprime polynomials of the same degree, obtained by homogeneization of P and Q respectively:

$$\mathcal{N}(z_0, z_1) = z_0^d P\left(\frac{z_1}{z_0}\right), \quad \mathcal{D}(z_0, z_1) = z_0^d Q\left(\frac{z_1}{z_0}\right).$$

We now take S^1-equivariance into account, where S^1-*equivariance* means S^1-equivariance with respect to S^1-actions on S^2 specified by representations $\rho : S^2 \to \mathrm{Diff}(S^2)$ on the

[1] Say, in z, to fix notations.

domain and $\rho' : S^2 \to \mathrm{Diff}(S^2)$ on the target. A map $\omega : S^2 \to S^2$ is (ρ, ρ')-equivariant iff intertwines ρ e ρ':

$$\omega \circ \rho = \rho' \circ \omega.$$

From the above, $S^2(\subset \mathbb{R}^3) \simeq \mathbb{C} \cup \{\infty\} \simeq \mathbb{C}P^1$. We write $x = (x', x_3)$, $x' \in \mathbb{R}^2$, for points of S^2 seen into $\mathbb{R}^3$, and we identify $g \in S^1$ as a rotation R around the axis of equivariance when we look S^2 as a submanifold of $\mathbb{R}^3$ and we indentify $g = e^{i\phi}$ in the remaining cases. We set up the ρ-action as follows:

$$gx = (Rx', x_3) \leftrightarrow gz = e^{i\phi} z \leftrightarrow \rho(g)([z_0, z_1]) = [z_0, e^{i\phi} z_1].$$

Similarly, we let S^1 act on S^2 as a rotation of fixed degree $k \in \mathbb{Z} \setminus \{0\}$.

By comparison of explicit representations of ω in terms of homogeneous coordinates and observing that the (ρ, ρ')-action of S^1 preserves the degree and the property of two polynomials of being coprime, we have that the harmonicity of ω implies that $\omega \circ \rho$ and $\rho' \circ \omega$ are also harmonic. Thus we can ask

Problem. What are all (ρ, ρ')-equivariant harmonic maps?

We note that N and S are left fixed by every $\rho(g)$, $g \in S^1$, so that

$$\omega(N) = \rho'(g) \circ \omega(N), \quad \omega(S) = \rho'(g) \circ \omega(S).$$

On the other hand, $\omega(N)$ and $\omega(S)$ are left fixed by $\rho'(g)$ for all $g \in S^1$. Then we have

$$\{\omega(N), \omega(S)\} = \{N, S\},$$

so that only two cases are possible: either ω fixes the poles or ω exchanges the poles. Furthermore, there are no other points left fixed by the (ρ, ρ')-action of S^1. These facts allow to deduce the explicit form of all (ρ, ρ')-equivariant harmonic maps.

Suppose that the north pole is left fixed. The counterimages of the north pole are the zeros of the denominator of the polynomial representation of ω. Hence, there is only one possible zero, namely, $z = 0$. Since S is also fixed, the only point mapped into S is S itself. Thus, there is no point (other than the point at infinity) at which Q vanishes, so that Q must be constant. A similar argument holds when ω exchanges the poles.

If ω has degree d, 4 possibilities are left:

$$f_1(z) = \lambda z^d, \quad f_2(z) = \lambda z^{-d}, \quad f_3(z) = \lambda \bar{z}^{-d}, \quad f_4(z) = \lambda \bar{z}^d.$$

Among these, f_1 and f_4 fix the poles while f_2 and f_3 exchange the poles. Further, only f_1 and f_3 are compatible with the (ρ, ρ')-equivariance respect to the S^1-actions previously specified. Hence, as $\lambda \in \mathbb{C} \setminus \{0\}$ varies[2], the maps of the form λz^d or $\lambda \bar{z}^{-d}$ provide, by pre-composition with the stereographic projection π and post-composition with π^{-1}, all (ρ, ρ')-equivariant harmonic maps $S^2 \to S^2$ of degree d.

[2]We shall see that for a class of minimizing maps in the equivariant class equivariance and stationarity together imply $|\lambda| = 1$.

6.2 Classification of S^1-equivariant linearly full harmonic spheres $S^2 \to S^4$

6.2.1 Decomposition of $\mathcal{S}_0$ into invariant subspaces

Notation. Throughout this section $E(\omega)$ will denote the Dirichlet energy of the map $\omega : S^2 \to S^4$. Bold letters denote vectors in $\mathbb{R}^3$ (seen as column vectors).

Let $\omega : S^2 \to S^4$ be harmonic, ω noncostant. Then $E(\omega) = 4\pi\,|d|$, $d \in \mathbb{Z} \setminus \{0\}$, see [138]. Denote $V = \operatorname{span}\operatorname{Ran}\omega$; two cases are possible: either

$$\dim V \leq 3,$$

or

$$\dim V = 5;$$

the case $\dim V = 4$ reduces to $\dim V = 3$ because of the theorems of Almgren [1] and Calabi [23] (compare also [132, Lemma 1.1]). We note that, because of equivariance, the images of the maps are invariant subsets of $\mathcal{S}_0$, hence V is also invariant in any case.

When we see S^2 as a submanifold of $\mathbb{R}^3$, we look at $R \in S^1$ as a (2×2)-matrix (a rotation in the plane orthogonal to the axis of equivariance) and we denote

$$\pi(R) = \tilde{R} = \begin{pmatrix} R & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix}$$

the representation of elements in S^1 in terms of automorphisms of $\mathcal{S}_0$, where R is the (2×2)-block mentioned above.

For elements in $\mathcal{S}_0$, we employ the notation

$$Q = \begin{pmatrix} Q_0 & \mathbf{q} \\ \mathbf{q^t} & q_0 \end{pmatrix},$$

where q_0 is a scalar and Q_0 a 2×2 matrix. The S^1-action on $\mathcal{S}_0$ by conjugation can be written explicitly, in terms of automorphisms, as

$$\tilde{R}^t Q \tilde{R} = \begin{pmatrix} R^t Q_0 R & \tilde{R}\mathbf{q} \\ (\tilde{R}\mathbf{q})^t & q_0 \end{pmatrix}. \tag{6.2.1}$$

We decompose $\mathcal{S}_0$ as a direct sum in a fashion that will allow us to describe easily all its invariant subspace. To do this, it is enough to determine a basis for each block, paying attention to the fact that basis matrices must be traceless and symmetric.

Remark 6.2.1. The decomposition we are going to introduce is not new. It appeared, for instance, in [137], although in a nonequivariant context.

We start with

$$e_0 = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \tag{6.2.2}$$

Looking at (6.2.1), it is straightforward to deduce that the subspace

$$L_0 = \mathbb{R}e_0 \tag{6.2.3}$$

is invariant.

The remaining matrices will have the component $(3,3)$ equals to zero. Let's start by considering the "vector" blocks, in the sense of (6.2.1). We choose first

$$e_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{6.2.4}$$

and then

$$e_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{6.2.5}$$

The subspace

$$L_1 = \mathbb{R}e_1 \oplus \mathbb{R}e_2 \tag{6.2.6}$$

is invariant (we remark explicitly that none of the blocks are, if taken separately, since the S^1-action by conjungation maps each one onto the other).

It remains the (2×2)-block. We select first

$$e_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{6.2.7}$$

so that the matrix

$$e_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{6.2.8}$$

completes the basis. The subspace

$$L_2 = \mathbb{R}e_3 \oplus \mathbb{R}e_4 \tag{6.2.9}$$

is invariant.

Clearly,

$$\mathcal{S}_0 = L_2 \oplus L_1 \oplus L_0$$

is a decomposition of $\mathcal{S}_0$ into invariant subspaces. We note that the invariant subspaces of dimension 3 are

$$L_{1,0} = L_1 \oplus L_0 \tag{6.2.10}$$

and

$$L_{2,0} = L_2 \oplus L_0. \tag{6.2.11}$$

Thus, all invariant (proper) subspaces of $\mathcal{S}_0$ are $L_0, L_1, L_2, L_1 \oplus L_0, L_2 \oplus L_0$.

With the exception of constant maps, we have that

$$V = L_1 \oplus L_0$$

or

$$V = L_2 \oplus L_0$$

or otherwise $\dim V = 5$.

Let us study L_2 more closely. It is plain that

$$L_2 \simeq \left\{ \begin{pmatrix} a & b \\ b & -a \end{pmatrix}, a, b \in \mathbb{R} \right\} \subset M_{2\times 2}(\mathbb{R}).$$

The application $C : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ mapping $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ -c & -d \end{pmatrix}$ squares to the identity; it can be represented by the matrix

$$C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and it establishes an isomorphism between L_2 and $\mathbb{C}$. Indeed,

$$CL_2 = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, a, b \in \mathbb{R} \right\} \simeq \mathbb{C}.$$

Thus we can see C as an application $L_2 \to \mathbb{C}$, more precisely, as an isometric isomorphism which associate to each $A \in L_2$ a unique complex number $z = CA$. The first observation we make is that the S^1-action by conjungation corresponds to the S^1-action on $\mathbb{C}$ by rotations of degree 2. Precisely,

Lemma 6.1. *Let $A \in L_2$ and let $R \in S^1$. Then*

$$R^t AR = CR^2CA$$

Proof. We have

$$CR^2CA = CR^2z = CR(Rz) = CR(zR) = (CR)(zR) = (CR)^t zR = R^tC^2AR = R^tAR,$$

where we used $CA = z$ for some $z \in C$ and the fact that, if we look at R as a rotation in the complex plane, $R = e^{i\phi}$, then we have $Rz = zR$ since the complex multiplication in commutative. Finally, CR is a symmetric matrix. □

Corollary 6.1. *Let $\pi : S^1 \to \mathrm{Aut}(L_2)$ denote the S^1-action on L_2 by conjugation and let $\tilde{\pi} : S^1 \to \mathrm{Aut}(\mathbb{C})$ the action by rotations of degree 2: $S^1 \ni R \mapsto \{z \mapsto R^2z\} \in \mathbb{C}$. Then $\pi = C\tilde{\pi}$.*

Proof. Obvious from the lemma. □

Noting that $L_1 = \mathbb{R}e_1 \oplus \mathbb{R}e_2$, we see that $L_1 \simeq \mathbb{C}$, where the isomorphism is provided by the identity and, in analogy with the above discussion, that the S^1-action on $\mathbb{C}$ induced by the S^1-action on L_1 is by rotations of degree 1.

To recapitulate, if $\dim V = 3$, we have either

$$V = L_1 \oplus L_0 \overset{I}{\simeq} \mathbb{C} \oplus \mathbb{R},$$

or

$$V = L_2 \oplus L_0 \overset{C}{\simeq} \mathbb{C} \oplus \mathbb{R}.$$

In the first case, if $\omega : S^2 \subset \mathbb{R}^3 \to S^2 \subset L_1 \oplus L_0$ is a harmonic map, S^1-equivariance means

$$\omega(\tilde{R}p) = \tilde{R}\omega(p),$$

where $p = (p_0, p_1, p_2) \in S^2 \subset \mathbb{R}^3$ and $\tilde{R}p = (p_0, e^{i\theta}(p_1 + ip_2))$. This implies

$$E(\omega) = 4\pi.$$

Looking at the classification in Section 6.1, this means that all equivariant harmonic maps $\omega : S^2 \to S^2 \subset L_1 \oplus L_0$ are of the form $\lambda z \in \mathbb{C}$ or $\frac{\lambda}{\bar{z}}$ with $\lambda \in \mathbb{C} \setminus \{0\}$. Actually, we can restrict ourselves to consider only the first case because the second reduces to the first one up to a fixed isometry (i.e., the antipodal map). **We shall do so from now on without explicit mention**.

In the second case, if $\omega : S^2 \subset \mathbb{R}^3 \to S^2 \subset L_2 \oplus L_0$ is harmonic, S^1-equivariance means that

$$\omega(\tilde{R}p) = \tilde{R}^2\omega(p),$$

with p as above and hence

$$E(\omega) = 8\pi.$$

Each $\omega : S^2 \to S^2 \subset L_2 \oplus L_1$ equivariant is of the form λz^2, with $\lambda \in \mathbb{C} \setminus \{0\}$.

Remark 6.2.2. We remark that equivariance firmly fixes the energy of the harmonic maps into the two classes. Indeed, harmonic maps $S^2 \to S^2$ can have, *a priori*, arbitrarly high energy, i.e., arbitrarily high degree. On the other hand their explicit description in terms of polynomials in z implies that the degree have to be the maximum between the degree of the denominator and that of the numerator. The request of equivariance fixes the degree of admissible polynomials, hence the topological degree and, in turn, the energy. This implies that, in low image dimensions ($= 3$), the energy is also low ($\leq 8\pi$). On the contrary, a map having "high" energy ($\geq 12\pi$) is *necessarily* linearly full (i.e., $\dim V = 5$) and vice versa.

In Section 6.1 we exploited the fact that S^2 and $\mathbb{C}P^1$ are isomorphic to get a very explicit description of harmonic maps $\omega : S^2 \to S^2$. Similarly, we can observe that the *twistor fibration*[3] τ maps $\mathbb{C}P^3$ onto S^4. We can thus think to replicate the previous strategy and thus study algebraic curves $\mathbb{C}P^1 \to \mathbb{C}P^3$ of degree 3, and then projecting them on S^4 by means of τ. Actually, we start with linearly full harmonic maps $\omega : S^2 \to S^4$, so, first of all, we have to check that each linearly full harmonic map $\omega : S^2 \to S^4$ has a lifting $\tilde{\omega} : \mathbb{C}P^1 \to \mathbb{C}P^3$ and that this lifting is an algebraic curve, i.e., we have to prove that the following diagram:

$$\begin{array}{ccc} & & \mathbb{C}P^3 \\ & \overset{\tilde{\omega}}{\nearrow} & \downarrow \tau \\ S^2 & \xrightarrow{\ \omega\ } & S^4 \end{array}$$

commutes, that is, $\omega = \tau \circ \tilde{\omega}$. In order to treat the problem systematically, we introduce the twistor fibration following [16, 17, 18].

[3] Bolton and Woodward [16, 17, 18] make use of the symbol π which can be cause of confusion in this work.

6.2.2 The twistor fibration $\tau : \mathbb{C}P^3 \to S^4$

Let Σ be a Riemann surface. The fiber bundle $\tau : \mathrm{SO}(2m+1)/\mathrm{U}(m) \to S^{2m}$ is called *twistor fibration* in the case τ is a Riemannian submersion with the further property that the composition with τ sets up a bijection between holomorphic *horizontal* maps $\Sigma \to \mathrm{SO}(2m+1)/\mathrm{U}(m)$ and *superminimal* maps $\Sigma \to \mathbb{R}P^{2m}$.

Remark 6.2.3. We shall not introduce the notion of superminimality, mentioned only for the sake of completeness, since we are interested only to the case $\Sigma = S^2$, in which case any harmonic map is superminimal [22]; the interested reader can consult the paper by Bryant [22] for further information. We will make clear in a moment what we mean by *horizontality*, restricting to our concrete case for simplicity.

Let $m = 2$. Then $\mathrm{SO}(5)/\mathrm{U}(2) \simeq \mathbb{C}P^3$ and the twistor fibration $\tau : \mathbb{C}P^3 \to S^4$ is the composition of the Hopf map[4] $\mathbb{C}P^3 \to \mathbb{H}P^1$ with the standard identification $\mathbb{H}P^1 \simeq S^4 \subset \mathbb{H} \oplus \mathbb{R} \simeq \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{R}$ given by the stereographic projection from the south pole into the 4-equatorial plane $\mathbb{H}$ in $\mathbb{H}P^1$. Explicitly, $\mathbb{H} \ni q \mapsto [1, q] \in \mathbb{H}P^1$ and

$$[q_1, q_2] \in \mathbb{H}P^1 \leftrightarrow \frac{(2\overline{q}_1 q_2, |q_1|^2 - |q_2|^2)}{|q_1|^2 + |q_2|^2} \in S^4.$$

Looking at $\mathbb{R}^5$ as $\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{R}$, the twistor fibration is given by the explicit formula [16]

$$\tau([z_1, z_2, z_3, z_4]) = \frac{(2(\overline{z}_1 z_3 + z_2 \overline{z}_4), 2(\overline{z}_1 z_4 - z_2 \overline{z}_3), |z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2)}{|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2}.$$

If $\mathbb{C}P^3$ is endowed with the Fubini-Study metric of constant sectional holomorphic curvature $+1$, τ becomes a Riemannian submersion [22, 16]. The *horizontal distribution* on $\mathbb{C}P^3$ consists of those tangent vectors to $\mathbb{C}P^3$ that are orthogonal to the fibers of τ. If we see $\tilde{\omega} : \mathbb{C}P^1 \to \mathbb{C}P^3$ as a curve from $\mathbb{C} \oplus \mathbb{C}$ into $\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$ and if we define the 1-form over $\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$

$$\eta = z_2 \, \mathrm{d}z_1 - z_1 \, \mathrm{d}z_2 + z_4 \, \mathrm{d}z_3 - z_3 \, \mathrm{d}z_4,$$

the horizontality condition can be written

$$\tilde{\omega}^* \eta = 0. \tag{6.2.12}$$

Since S^2 is compact and it is isomorphic to $\mathbb{C}P^1$ (via the identification built in Section 6.1), each $\tilde{\omega} : \mathbb{C}P^1 \to \mathbb{C}P^3$ holomorphic horizontal curve is an algebraic horizontal curve [18] and $\omega = \tau \circ \tilde{\omega}$ is harmonic [16, 17]. Conversely, each linearly full harmonic map $\omega : S^2 \to S^4$ is of the form $\pm\tau \circ \tilde{\omega}$ for some unique linearly full horizontal algebraic curve $\tilde{\omega}$, called the *twistor lifting* of ω. The symbol $\pm$ denotes the possible application of the antipodal map. If we denote d the algebraic degree of $\tilde{\omega}$, its energy is $4\pi d$. Summing up, after having made precise sense of the objects appearing in the diagram at end of Section 6.2.1, we can assert that it commutes.

[4]Identyfing $\mathbb{H}^2 \simeq \mathbb{C}^4$ through $(z_1, z_2, z_3, z_4) \leftrightarrow (z_1 + z_2 j, z_3 + z_4 j)$, the *Hopf map* ρ takes the complex line $\mathbb{C}\mathbf{v}$ in $\mathbb{C}^4$ and associate with it the quaternionic line $\mathbb{H}\mathbf{v}$. Explicitly, $\rho([z_1, z_2, z_3, z_4]) = [z_1 + z_2 j, z_3 + z_4 j]$.

6.2.3 S^1-equivariance of the twistor fibration.

We are concerned with maps $S^2 \to S^4$ that are S^1-equivariant. We now introduce a S^1-action on $\mathbb{C}P^3$ which is compatible with the S^1-actions on $\mathbb{C}P^1$ and on S^4. Put $g = e^{i\phi} \in S^1$ and

$$[z_1, z_2, z_3, z_4] \overset{\rho''(g)}{\longmapsto} [z_1, e^{3i\phi} z_2, e^{2i\phi} z_3, e^{i\phi} z_4]. \tag{6.2.13}$$

Such an action is well-defined: indeed, we get $[0, 0, 0, 0]$ if and only if we start from $[0, 0, 0, 0]$. Furthermore, each $g \in S^1$, $\rho''(g)$ is the representation of an automorphism of $\mathbb{C}P^3$, i.e., a matrix belonging to $\mathrm{PGL}_4(\mathbb{C})$.

Lemma 6.2 (Equivariance of the twistor fibration)**.** *Let $\rho'' : \mathbb{C}P^3 \to \mathrm{PGL}_4(\mathbb{C})$ be the representation of the S^1-action on $\mathbb{C}P^3$ defined in* (6.2.13)*. Then*

$$\tau(\rho''([z_1, z_2, z_3, z_4])) = \tilde{R}^t \tau([z_1, z_2, z_3, z_4]) \tilde{R}, \tag{6.2.14}$$

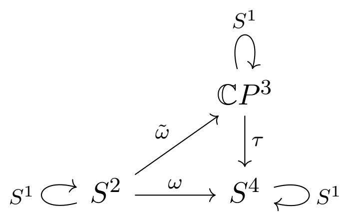
with $\tilde{R} = \begin{pmatrix} e^{i\phi} & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix}$.

Proof. Plainly,

$$\begin{aligned} \tau(\rho''([z_1, z_2, z_3, z_4])) &= \tau([z_1, e^{3i\phi} z_2, e^{2i\phi} z_3, e^{i\phi} z_4]) \\ &= \frac{(2e^{2i\phi}(\overline{z}_1 z_3 + z_2 \overline{z}_4), 2e^{i\phi}(\overline{z}_1 z_4 - z_2 \overline{z}_3), |z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2)}{|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2} \\ &\in L_2 \oplus L_1 \oplus L_0, \end{aligned}$$

hence the conclusion. □

Since τ preserves S^1-equivariance, the diagram



commutes, so that we can use the twistor fibration to study linearly full equivariant harmonic maps from S^2 to S^4. We aim to adapt the strategy already exploited for maps $S^2 \to S^2$ and so trying to carry out an explicit classification.

Remark 6.2.4. We already noticed in Section 6.2.1 that the physics of the problem supports linearly full maps having the lowest energy, i.e, 12π. On the other hand, without some extra hypotheses, we cannot exclude the possibility of higher energy maps. The further request of compatibility between all S^1-actions implies (6.2.13) and fixes to 12π the energy of S^1-equivariant linearly full harmonic maps from S^2 into S^4.

6.2.4 Classification theorem

As we already remarked in Section 6.2.3, the relevance of twistor fibration to the theory of harmonic maps from S^2 to S^4 comes from the fact that [16, 17] the τ-projection of any horizontal linearly full algebraic curve ψ into $\mathbb{C}P^3$ is a linearly full harmonic map and, conversely, for each linearly full harmonic map $\omega : S^2 \to S^4$ there exists a unique linearly full algebraic curve $\tilde{\omega}$ into $\mathbb{C}P^3$ which is its twistor lift. Moreover, equivariance for ω implies equivariance for $\tilde{\omega}$. Indeed, by equivariance, $\omega \cdot \rho_g = \pi_g \cdot \omega$, so that the twistor lifts coincide too and the claim follows.

The request of compatibility between the ρ-action on $\mathbb{C}P^1$ and the π-action on S^4 fixes ρ'' as the S^1-action on $\mathbb{C}P^3$ (with our conventions about the relevant variable on $\mathbb{C}P^1$). Hence, very restrictive conditions are imposed on the homogeneous coordinates of $\tilde{\omega}([\lambda_0, \lambda_1])$.

If $\tilde{\omega}$ is an algebraic curve having degree 3, the homogeneous coordinates of an image point are necessarily coprime homogeneous polynomials of degree 3. Equivariance with respect to the ρ''-action implies that the polynomials P_i in

$$\tilde{\omega}([\lambda_0, \lambda_1]) = [P_1(\lambda_0, \lambda_1), P_2(\lambda_0, \lambda_1), P_3(\lambda_0, \lambda_1), P_4(\lambda_0, \lambda_1)],$$

are the following:

$$P_1 = \mu_1 \lambda_0^3, \quad P_2 = \mu_2 \lambda_1^3, \quad P_3 = \mu_3 \lambda_1^2 \lambda_0, \quad P_4 = \mu_4 \lambda_0^2 \lambda_1,$$

the coefficients μ_i are complex numbers.

Up to now, the μ_i are arbitrary. However, since $\tilde{\omega}$ must be linearly full, $\mu_1 \neq 0$ so that, without loss of generality, we may set $\mu_1 = 1$. Next, horizontality sets up a relation between the other variables, so that one is a function the other two. By (6.2.12), we have that horizontality yields

$$(3\mu_2 - \mu_3\mu_4)\lambda_0^2\lambda_1^2(\lambda_1 \,\mathrm{d}\lambda_0 - \lambda_0 \mathrm{d}\lambda_1) = 0,$$

and, by arbitrariness of λ_0, λ_1, we get

$$3\mu_2 = \mu_3\mu_4. \tag{6.2.15}$$

Hence, every horizontal linearly full algebraic curves of degree 3 is of the form

$$\tilde{\omega}([\lambda_0, \lambda_1]) = \left[\lambda_0^3, \frac{\mu_3\mu_4}{3}\lambda_1^3, \mu_3\lambda_1^2\lambda_0, \mu_4\lambda_0^2\lambda_1\right] \tag{6.2.16}$$

where $(\mu_3, \mu_4) \in \mathbb{C} \oplus \mathbb{C}$, hence

$$\begin{aligned}(\tau \circ \tilde{\omega})([\lambda_0, \lambda_1]) = \frac{1}{\mathscr{D}} \Bigg(& 2\mu_3 \overline{\lambda_0^2} \lambda_1^2 \left(|\lambda_0|^2 + \frac{|\mu_4|^2 |\lambda_1|^2}{3} \right), \\ & 2\mu_4 \overline{\lambda_0} \lambda_1 \left(|\lambda_0|^4 - \frac{|\mu_3|^2 |\lambda_1|^4}{3} \right), \\ & |\lambda_0|^4 (|\lambda_0|^2 - |\mu_4|^2 |\lambda_1|^2) + |\lambda_1|^4 \left(\frac{|\mu_3|^2 |\mu_4|^2 |\lambda_1|^2}{9} - |\mu_3|^2 |\lambda_0|^2 \right) \Bigg)\end{aligned} \tag{6.2.17}$$

with

$$\mathscr{D} = |\lambda_0|^4 (|\lambda_0|^2 + |\mu_4|^2 |\lambda_1|^2) + |\lambda_1|^4 \left(\frac{|\mu_3|^2 |\mu_4|^2 |\lambda_1|^2}{9} + |\mu_3|^2 |\lambda_0|^2 \right).$$

Note, in particular, that setting $\mu_3 = \mu_4 = \sqrt{3}$ gives back the hedgehog.

Summarizing, we have proven the following classification theorem.

Theorem 6.2. *Each horizontal (ρ, ρ'')-equivariant linearly full algebraic curve $\tilde{\omega} : \mathbb{C}P^1 \to \mathbb{C}P^3$ of degree 3 has the form* (6.2.16) *for some $(\mu_3, \mu_4) \in \mathbb{C} \oplus \mathbb{C}$, hence, each (ρ, π)-equivariant linearly full harmonic map $\omega : S^2 \simeq \mathbb{C}P^1 \to S^4 \subset \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{R}$ having energy 12π has the form* (6.2.17) *for some $(\mu_3, \mu_4) \in \mathbb{C} \oplus \mathbb{C}$.*

In terms of coordinates (θ, ϕ) on the sphere[5], we can write

$$\boxed{\omega = \left(\omega_2(\theta)e^{2i\phi}, \omega_1(\theta)e^{i\phi}, \omega_0(\theta)\right).} \tag{6.2.18}$$

The expression of the components of ω in spherical coordinates, in terms of the parameters μ_3, μ_4, is given by (A.1). Note that, by Hélein's theorem, the components ω_i are smooth functions on S^2. Observe also that S^1-equivariance forces

$$\omega_2(0) = \omega_2(\pi) \equiv 0 \text{ and } \omega_1(0) = \omega_1(\pi) \equiv 0. \tag{6.2.19}$$

Remark 6.2.5. Observe that the function ω_0 is always real-valued, while ω_1, ω_2 are complex valued whenever the parameters μ_3, μ_4 are such and real-valued otherwise.

Notation. In the following, when recurring to (6.2.18), we will often drop the argument θ of the functions ω_i.

6.3 The Center-of-Mass condition

We recall that it is a general property of stationary harmonic maps with values into spheres, firstly highlighted in [21, Remark 7.6] for harmonics $S^2 \to S^2$ and then generalized by [96, Theorem C], that the center-of-mass (CoM) of the measure $|\nabla\omega|^2 \, dx$ must be placed at the origin. In the case of S^2-valued maps, this fact reduces the arbitrariness in the parameter of the maps to a fixed phase, which is however inessential for what concerns stability. In the case of linearly full maps, it links the two arbitrary parameters, so that only one remains actually independent.

6.3.1 The CoM condition for S^2-valued maps

In the case of maps with values in S^2, the CoM condition suffices to state that the parameter λ of Section 6.1 has to be a fixed phase [21].

6.3.2 The CoM condition for S^4-valued maps

Linearly full maps are classified up to *two* arbitrary complex parameters. For each map, up to a fixed rotation of the domain S^2, one of them (say, μ_4) can always be thought of as real. We now show that the CoM condition determines one parameter as a function of the other. To this end, it is found convenient rescaling μ_4, setting

$$\mu_4 := \frac{\tilde{\mu}_4}{|\mu_3|^2} \quad \text{with } \tilde{\mu}_4 \in \mathbb{R} \setminus \{0\}.$$

Write $x = (x_1, x_2, x_3)$; S^1-equivariance implies that

$$\int_{S^2} x \left|\nabla\omega\left(\frac{x}{|x|}\right)\right|^2 \mathrm{dvol}_{S^2} = 0 \iff \int_{S^2} x_3 \left|\nabla\omega\left(\frac{x}{|x|}\right)\right|^2 \mathrm{dvol}_{S^2} = 0.$$

[5]We follow the convention more frequently adopted in physics, according to which θ is the latitude and ϕ the colatitude.

In terms of the complex variable z (by means of the stereographic projection $\pi : S^2 \to \mathbb{C} \cup \{\infty\}$), we have

$$x_3 = \frac{1 - |z|^2}{1 + |z|^2} \quad \text{and} \quad \mathrm{dvol}_{S^2} = \frac{4\,\mathrm{d}z}{\left(1 + |z|^2\right)^2}.$$

Then we have

$$\int_{S^2} x_3 \left| \nabla \omega \left(\frac{x}{|x|} \right) \right|^2 \mathrm{dvol}_{S^2} = \int_{\mathbb{C}} \frac{1 - |z|^2}{1 + |z|^2} \left| \nabla_T \omega(z) \right|^2_{S^2} \frac{4\mathrm{d}z}{\left(1 + |z|^2\right)^2}.$$

Here ∇_T denotes the gradient on S^2 and $|\cdot|_{S^2}$ the norm with respect to the canonical metric on S^2. Denoting ∇_z the gradient on $\mathbb{C}$ w.r.t. the complex variable z, we have

$$\int_{\mathbb{C}} \frac{1 - |z|^2}{1 + |z|^2} \left| \nabla_T \omega(z) \right|^2_{S^2} \frac{4\mathrm{d}z}{\left(1 + |z|^2\right)^2} = \int_{\mathbb{C}} \frac{1 - |z|^2}{1 + |z|^2} \left| \nabla_z \omega(z) \right|^2 \mathrm{d}z.$$

Let $\xi = \mu_4 z$. Since every harmonic map from S^2 into a compact Riemannian manifold is conformal, we find that

$$\int_{\mathbb{C}} \frac{1 - |z|^2}{1 + |z|^2} \left| \nabla_z \omega(z) \right|^2 \mathrm{d}z = \int_{\mathbb{C}} \frac{1 - \frac{|\xi|^2}{|\mu_4|^2}}{1 + \frac{|\xi|^2}{|\mu_4|^2}} \left| \nabla_z \omega^{(\mu_3)}(\xi) \right|^2 \mathrm{d}\xi.$$

The index (μ_3) remarks that μ_3 is held fixed. Let us consider the second member above as a function of $|\mu_4|$:

$$h(|\mu_4|) = \int_{\mathbb{C}} \frac{1 - \frac{|\xi|^2}{|\mu_4|^2}}{1 + \frac{|\xi|^2}{|\mu_4|^2}} \left| \nabla_z \omega^{(\mu_3)}(\xi) \right|^2 \mathrm{d}\xi.$$

Recall that $|\mu_4| \in (0, +\infty)$. Define

$$g_\xi(|\mu_4|) = \frac{|\mu_4|^2 - |\xi|^2}{|\mu_4|^2 + |\xi|^2}, \quad |\mu_4| \in (0, +\infty).$$

Note that $g_\xi(\cdot)$ is an increasing continuous function with

$$g_\xi(0^+) = -1 \quad \text{and} \quad g_\xi(+\infty) = +1.$$

It then follows that h is also increasing and continuous, with

$$h(0) = -E(\omega) \quad \text{and} \quad h(\infty) = E(\omega).$$

The intermediate value theorem then implies that, for each $\mu_3 \in \mathbb{C}^*$, there exists a unique $|\hat{\mu}_4|$ such that $h(|\hat{\mu}_4|) = 0$. So, for stationary maps, $|\mu_4|$ is a continuous function of μ_3, actually a smooth function by the implicit function theorem.

Chapter 7

Landau-de Gennes theory with norm-constraint and with symmetry, II

Synopsis. In this Chapter we prove stability and instability results for the S^1-equivariant tangent maps that can appear as strong limits of blow-ups (centered at points of the z-axis) of S^1-equivariant minimizers of the LdG energy in the class $\mathcal{A}^{\mathrm{ax}}_{Q_b}$. Of course, in principle we aimed at ruling out all nonconstant tangent maps, thus getting a Liouville theorem and, in turn, global interior regularity for minimizers. Recall that in the previous Chapter we classified all possible nonconstant tangent maps and we learned, in particular, that — due to equivariance — their restriction to the sphere can only have energy 4π, 8π or 12π. The span of their ranges is included, respectively, into $L_1 \oplus L_0$, $L_2 \oplus L_0$ or into a 5-dimensional linear subspace of $L_2 \oplus L_1 \oplus L_0$. Unfortunately, and quite surprisingly in view of the explicit classification, we did not succeed in completing the program, because tangent maps with values into $L_1 \oplus L_0$ turned out to be minimizing in their class (Theorem 7.6) and we have no reasons for be assured that singularities associated to this kind of tangent maps is actually avoided by minimizers of the LdG energy in the S^1-equivariant class $\mathcal{A}^{\mathrm{ax}}_{Q_b}$. As expected, tangent maps with values into $L_2 \oplus L_0$ and linearly full tangent maps are instead unstable, see Theorem 7.8 (or Section 7.1.4 for more detailed calculations for maps into $L_2 \oplus L_0$). Thus, we have only a partial regularity theorem, Theorem 7.10, but with the further knowledge of the possible singularities and their structure. Such a knowledge will allow us to explore interesting behaviors of minimizers w.r.t. to special boundary data in Chapter 8.

In the following, we shall carry out explicit computations whenever possibile, even if this may produce some repetitions. In fact, this way we are able to investigate more deeply the structure of S^1-equivariant harmonic maps between spheres, a topic of its own interest, see e.g. [48, 50].

7.1 Results on the stability and the instability of S^1-equivariant tangent maps

7.1.1 Approach to instability

In this section, we shall prove stability and instability results for the S^1-equivariant tangent maps given by Theorem 5.13. We recall that a harmonic map $\omega : M \to N$ is said to be *stable* if it happens

$$\left.\frac{d^2E(\omega_t)}{dt^2}\right|_{t=0} = \frac{d^2}{dt^2}\left(\int_M \frac{1}{2}|\nabla\omega_t|^2 \,\mathrm{dvol}_M\right)\Bigg|_{t=0} \geq 0 \tag{7.1.1}$$

for all admissible outer variations ω_t. Otherwise, ω is called *unstable*. We note that $\left.\frac{d^2E(\omega_t)}{dt^2}\right|_{t=0}$ is a quadratic form, firstly computed by Smith [135]. Important results are known about the (in)stability of nonsymmetric harmonic maps when the domain and/or the target are special manifolds. In particular, we recall the following theorems.

Theorem 7.1 (Schoen&Uhlenbeck, [132])**.** *For $k \geq 3$, define $d(3) = 3$ and $d(k) = \lfloor \min\{k/2+1, 6\}\rfloor$ for $k \geq 4$. Then, for $k \geq 3$ and $n \leq d(k)$, there is no noncostant minimizing map from $\mathbb{R}^n$ to S^k.*

Theorem 7.2 (Lin&Wang, [95])**.** *For $m \geq 2$ and $k \geq 3$, if $\omega \in C^\infty(\mathbb{R}^{m+1}, S^k)$ is a stable tangent map, then $\frac{(m-1)^2}{4m} \geq \frac{k-2}{k}$.*

As Y. Xin remarks in [141, Section 6.3], stability in the equivariant case is much different from nonsymmetric stability. Indeed, he says, it could actually happen that an unstable harmonic map is a stable equivariant harmonic map.

Xin's prophecy actually comes true in our case, because, as we shall see, tangent maps with values in $S^2 \subset L_1 \oplus L_0$ are stable. In fact, we shall prove the stronger statement that these maps are even minimizing.

On the contrary, tangent maps with values in $S^2 \subset L_2 \oplus L_0$ and linearly full tangent maps are unstable.

For further reference, we recall below the sharp Hardy inequality in $\mathbb{R}^n$.

Theorem 7.3 (Sharp Hardy inequality)**.** *Let $n \geq 3$ and let $\psi \in W^{1,2}(\mathbb{R}^n)$. Then*

$$\left(\frac{n-2}{2}\right)^2 \int_{\mathbb{R}^n} \frac{|\psi|^2}{|x|^2}\, dx \leq \int_{\mathbb{R}^n} |\nabla\psi|^2 \, dx. \tag{7.1.2}$$

The constant $\left(\frac{n-2}{2}\right)^2$ is sharp.

7.1.2 Second variation formula

Let ω be a tangent map given by Theorem 5.13. Let $V \in C_c^\infty(\mathbb{R}^3, \mathbb{R}^5)$ be S^1-equivariant and define

$$\omega_t(x) := \frac{\omega(x) + tV(x)}{|\omega(x) + tV(x)|} \quad (x \in \mathbb{R}^3).$$

Since ω is S^4-valued, ω_t is well-defined for sufficiently small t, because in this case $\omega(x) + tV(x) \neq 0$ for all $x \in \mathbb{R}^3$. We are really interested only in small t, so we will henceforth drop this specification. The family $\{\omega_t\}$ is an admissible outer variation of ω. Indeed, $\omega_t \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^3, S^4)$ for all t and ω_t is S^1-equivariant:

$$\begin{aligned}
\omega_t(Rx) &= \frac{\omega(Rx) + tV(Rx)}{|\omega(Rx) + tV(Rx)|} = \frac{R\omega(x)R^{-1} + tRV(x)R^{-1}}{|R\omega(x)R^{-1} + tRV(x)R^{-1}|} \\
&= \frac{R(\omega(x) + tV(x))R^{-1}}{|R(\omega(x) + tV(x))R^{-1}|} = R\left(\frac{\omega(x) + tV(x)}{|\omega(x) + tV(x)|}\right)R^{-1} \\
&= R\omega_t(x)R^{-1} \qquad \forall x \in \mathbb{R}^3,\ \forall R \in S^1,
\end{aligned}$$

because of the linearity of the S^1-action by conjugation on S^4 and the fact that this action is by isometries.

Let us set

$$\mathcal{Q}(V;\omega) := \left.\frac{d^2E(\omega_t)}{dt^2}\right|_{t=0} = \left[\frac{d^2}{dt^2}\left(\int_{\mathbb{R}^3}\frac{1}{2}\left|\nabla\omega_t\right|^2\,\mathrm{d}x\right)\right]_{t=0}. \tag{7.1.3}$$

Proposition 7.4 (Second variation formula)**.** *Let ω, V and $\mathcal{Q}(V;\omega)$ as above. Then*

$$\begin{aligned}\mathcal{Q}(V;\omega) &= \int_{\mathbb{R}^3}\left\{(4(\omega\cdot V)^2-|V|^2)\,|\nabla\omega|^2-|\nabla(\omega\cdot V)|^2-4(\omega\cdot V)\,\langle\nabla V,\nabla\omega\rangle+|\nabla V|^2\right\}\,dx\\ &= \int_{\mathbb{R}^3}\left\{-\left|V_T\right|^2|\nabla\omega|^2+|\nabla V_T|^2\right\}\,dx,\end{aligned} \tag{7.1.4}$$

where

$$V_T = V-(\omega\cdot V)\omega,$$

is the tangential part of V to the image of ω.

Proof. For brevity, we will hencefort omit the argument x. As in [132], we calculate

$$\begin{aligned}\frac{d(|\nabla\omega_t|^2)}{dt} &= \frac{d\,\langle\nabla\omega_t,\nabla\omega_t\rangle}{dt} = 2\left\langle\nabla\left(\frac{d\omega_t}{dt}\right),\nabla\omega_t\right\rangle,\\ \frac{d^2(|\nabla\omega_t|^2)}{dt^2} &= 2\frac{d}{dt}\left\langle\nabla\left(\frac{d\omega_t}{dt}\right),\nabla\omega_t\right\rangle\\ &= 2\left\langle\nabla\left(\frac{d^2\omega_t}{dt^2}\right),\nabla\omega_t\right\rangle+2\left\langle\nabla\left(\frac{d\omega_t}{dt}\right),\nabla\left(\frac{d\omega_t}{dt}\right)\right\rangle.\end{aligned}$$

We set $\dot\omega = \left.\frac{d\omega_t}{dt}\right|_{t=0}$. Then,

$$\left.\frac{d(|\nabla\omega_t|^2)}{dt}\right|_{t=0} = 2\,\langle\nabla\dot\omega,\nabla\omega\rangle\,, \tag{7.1.5}$$

$$\left.\frac{d^2(|\nabla\omega_t|^2)}{dt^2}\right|_{t=0} = 2\,\langle\nabla\ddot\omega,\nabla\omega\rangle+2\,\langle\nabla\dot\omega,\nabla\dot\omega\rangle\,. \tag{7.1.6}$$

Being

$$\begin{aligned}\frac{d\omega_t}{dt} &= \frac{d}{dt}\left(\frac{\omega+tV}{|\omega+tV|}\right)\\ &= \frac{V}{|\omega+tV|}-\frac{\omega+tV}{|\omega+tV|^3}\cdot(V\cdot\omega+tV\cdot V),\end{aligned}$$

we have

$$\dot\omega = V-(\omega\cdot V)\omega$$

and

$$\ddot{\omega} = 3(\omega \cdot V)^2\omega - (V \cdot V)\omega - 2(\omega \cdot V)V.$$

Then

$$\begin{aligned}
\frac{1}{2}\frac{d^2}{dt^2}(|\nabla\omega_t|^2)\Big|_{t=0} &= \langle\nabla\ddot{\omega}, \nabla\omega\rangle + \langle\nabla\dot{\omega}, \nabla\dot{\omega}\rangle \\
&= \Big\langle\nabla[3(\omega \cdot V)^2\omega - (V \cdot V)\omega - 2(\omega \cdot V)V], \nabla\omega\Big\rangle \\
&\quad + \langle\nabla[V - (\omega \cdot V)\omega], \nabla[V - (\omega \cdot V)\omega]\rangle \\
&= 3\Big\langle\nabla[(\omega \cdot V)^2\omega], \nabla\omega\Big\rangle - \langle\nabla[(V \cdot V)\omega], \nabla\omega\rangle - 2\langle\nabla[(\omega \cdot V)V, \nabla\omega\rangle \\
&\quad + \langle\nabla V, \nabla V\rangle - 2\langle\nabla V, \nabla[(\omega \cdot V)\omega]\rangle + \langle\nabla[(\omega \cdot \omega)\omega], \nabla[(\omega \cdot V)\omega]\rangle .
\end{aligned}$$

Easy calculations in components (taking advantage of the fact that $\omega_i\partial_k\omega_i = 0$ because ω is S^4-valued) lead to the first line in (7.1.4). The second line follows as in the proof of [97, Proposition 1.6.1] □

Recall that we have classified S^1-equivariant harmonic spheres $S^2 \to S^4$ in terms of two complex parameters. We now observe that the number of free parameters can be reduced when studying stability. Indeed, trivially, one of the two parameters, say μ_4, can always be taken real and positive, up to a fixed rotation of S^2. Thus, we are left with a free real parameter and a free complex parameter. However, we claim that, for what concerns the stability issue, both μ_3 and μ_4 can be taken real and, moreover, positive. More precisely, it is trivial to prove the following

Lemma 7.1. *Let $\mu_3 \in \mathbb{C}$, $\mu_4 \in \mathbb{R}$, $\omega : S^2 \to S^4$, $\omega = (\omega_2 e^{2i\phi}, \omega_1 e^{i\phi}, \omega_0)$, be the harmonic sphere whose parameters are μ_3, μ_4. Define $\omega^{(\alpha)} : S^2 \to S^4$ setting*

$$\omega^{(\alpha)} = (\omega_2 e^{i\alpha} e^{2i\phi}, \omega_1 e^{i\phi}, \omega_0)$$

for any $\alpha \in \mathbb{R}$. Then, for any $\alpha \in \mathbb{R}$,

(i) $\omega^{(\alpha)}$ is harmonic;

(ii) ω satisfies the CoM condition if and only if $\omega^{(\alpha)}$ satisfies the CoM condition;

(iii) ω is stable if and only if $\omega^{(\alpha)}$ is stable;

Proof. (i) Indeed, for any $\alpha \in \mathbb{R}$, $\omega^{(\alpha)}$ we clearly have $\left|\nabla\omega^{(\alpha)}\right|^2 = |\nabla\omega|^2$. By the definition of α and the harmonicity of ω, $\omega^{(\alpha)}$ satisfies

$$\Delta\omega^{(\alpha)} = -\left|\nabla\omega^{(\alpha)}\right|^2 \omega^{(\alpha)},$$

thus $\omega^{(\alpha)}$ is a weakly harmonic map.

(ii) It follows trivially from the equality $|\nabla\omega|^2 = \left|\nabla\omega^{(\alpha)}\right|^2$.

(iii) We prove that ω is unstable if and only if $\omega^{(\alpha)}$ is unstable. Suppose ω is unstable and let $V \in C_c^\infty(\mathbb{R}^3, \mathbb{R}^5)$ be a S^1-equivariant vector field such that $\mathcal{Q}(V; \omega) < 0$, where $\mathcal{Q}(V; \omega)$ is given by (7.1.4). Let $V_T = V - (\omega \cdot V)\omega$. Then, by (7.1.4),

$$\mathcal{Q}(V;\omega) = \mathcal{Q}(V_T;\omega) = \int_{\mathbb{R}^3} \left\{ |\nabla V_T|^2 - |V_T|^2 |\nabla \omega|^2 \right\}.$$

Since $|\nabla \omega|^2 = \left|\nabla \omega^{(\alpha)}\right|^2$, we have

$$\mathcal{Q}(V;\omega) = \mathcal{Q}(V_T;\omega) = \mathcal{Q}(V_T;\omega^{(\alpha)}) = \mathcal{Q}(V;\omega^{(\alpha)}) < 0,$$

so that $\omega^{(\alpha)}$ is unstable. The converse follows exchanging the rôle of ω and $\omega^{(\alpha)}$. □

It is then clear that, written $\mu_3 = |\mu_3| e^{i\alpha}$, taking $\omega^{(-\alpha)}$ proves our claim. In particular, **for what concerns the stability issue, the functions ω_1, ω_2 can always be thought of as real-valued**.

7.1.3 Stability and minimality of $L_1 \oplus L_0$-valued tangent maps

Due to the classification result in [21], all minimizing maps $u \in W^{1,2}$ from $\mathbb{R}^3$ into $S^2 \simeq L_1 \oplus L_0$ such that $u(x) = x$ on S^2 are obtained from $\frac{x}{|x|}$ (sometimes called *the radial projection*, cfr. [3]) by rotations and traslations of the origin, i.e., they are of the form $\mathcal{R}\left(\frac{x-a}{|x-a|}\right)$, for $a \in \mathbb{R}^3$ and $\mathcal{R}$ a rotation in $\mathbb{R}^3$. Since these transformations are isometries, it is sufficient to our purposes to consider the case $\mathcal{R} =$ identity and $a = 0$. The corresponding $L_1 \oplus L_0$-valued tangent map (often called *the equator map*, cfr. [4, 76, 79]) in our framework can be written

$$\omega = \left(0, \frac{x}{|x|}\right) \in L_2 \oplus (L_1 \oplus L_0). \tag{7.1.7}$$

Let z denote the variable in $\mathbb{C}$. The most general deformation vector field $V \in C_c^\infty(\mathbb{R}^3; L_2 \oplus L_1 \oplus L_0)$ compatible with the S^1-equivariance is of the form

$$V(r,z) = \Big(f_2(r,|z|)e^{2i\phi}, f_1(r,|z|)e^{i\phi}, f_0(r,|z|)\Big),$$

with f_2, f_1, f_0 of compact support, f_2, f_1 complex-valued functions vanishing at poles. For such a V, the second variation of the energy of ω is of course given by (7.1.4). However, since ω has no components along L_2, such a variation splits into the sum of two pieces, namely

$$\mathcal{Q}_2 = \mathcal{Q}(f_2(r,|z|)e^{2i\phi}, 0, 0); \omega) \quad \text{and} \quad \mathcal{Q}_{1,0} = \mathcal{Q}(0, f_1(r,|z|)e^{i\phi}, f_0(r,|z|); \omega). \tag{7.1.8}$$

Since ω is minimizing as a map into $L_1 \oplus L_0$, $\mathcal{Q}_{1,0}$ is nonnegative definite, so that it suffices to consider $\mathcal{Q}_2$. In this case, the second variation formula reduces directly to the second line of (7.1.4).

We now prove the following stability result.

Theorem 7.5 (Equivariant stability of the equator map.)**.** *The tangent map $\omega : \mathbb{R}^3 \to S^2 \subset L_1 \oplus L_0 \subset L_2 \oplus L_1 \oplus L_0$ defined by $\omega(x) = \left(0, \frac{x}{|x|}\right)$ is stable w.r.t. S^1-equivariant variations, meaning that, for any S^1-equivariant $V \in C_c^\infty(\mathbb{R}^3; L_2 \oplus L_1 \oplus L_0)$, the second variation $\mathcal{Q}(V;\omega)$, computed as in (7.1.4), is nonnegative.*

Proof. The above remarks show that it suffices to prove that $\mathcal{Q}_2 \geq 0$, where $\mathcal{Q}_2$ is as in (7.1.8), only for all S^1-equivariant $V \in C_c^\infty(\mathbb{R}^3; L_2 \oplus L_1 \oplus L_0)$ of the special form

$$V(r,z) = (f_2(r,|z|)e^{2i\phi}, 0, 0) \in L_2 \oplus L_1 \oplus L_0,$$

with f_2 a smooth function of compact support. Note that, in this case, the second variation formula reduces directly to the second line of (7.1.4). Recall that, *a priori*, f_2 is complex-valued. The argument below will show that actually it suffice to consider f_2 real-valued.

Suppose first f_2 is real-valued. Let us start by rewriting $\mathcal{Q}_2$ in terms of integrals in polar coordinates in the complex plane. To this end, first observe that, switching from cartesian to spherical coordinates in $\mathbb{R}^3$, we have (recall that ω is homogeneous of degree zero)

$$\begin{aligned}
\mathcal{Q}_2 &= \int_{\mathbb{R}^3} \left\{ -|f_2|^2 |\nabla\omega|^2 + \left|\nabla(f_2 e^{2\phi})\right|^2 \right\} \mathrm{d}x \\
&= \int_{S^2\times(0,+\infty)} \left\{ -\frac{|f_2|^2}{r^2} |\nabla_T\omega|^2 + \left|\frac{\partial f_2}{\partial r}\right|^2 + \frac{1}{r^2}\left|\nabla_T(f_2 e^{2i\phi})\right|^2 \right\} r^2 \mathrm{d}r \, \mathrm{dvol}_{S^2}.
\end{aligned}$$

By the Fubini's theorem, we can integrate firstly over r and then over S^2. Recall that S^2 and $\mathbb{C} \cup \{\infty\}$ are diffeomorphic through the stereographic projection; the change of coordinates theorem implies that

$$|\nabla_T\omega|^2 \, \mathrm{dvol}_{S^2} = |\nabla_z\omega|^2 \, \mathrm{d}z,$$

where ∇_z denotes the gradient w.r.t. the complex coordinate z on $\mathbb{C}$. Similarly,

$$\left|\nabla_T(f_2 e^{2i\phi})\right|^2 \mathrm{dvol}_{S^2} = \left|\nabla_z(f_2 e^{2i\phi})\right|^2 \mathrm{d}z$$

and, moreover,

$$\mathrm{dvol}_{S^2} = \frac{4}{\left(1+|z|^2\right)^2} \mathrm{d}z.$$

Recalling that $|\nabla_T\omega|^2 = 2$, from the above we have

$$\mathcal{Q}_2 = \int_{\mathbb{C}} \mathrm{d}z \int_0^{+\infty} \left\{ -\frac{8}{\left(1+|z|^2\right)^2}\frac{|f_2|^2}{r^2} + \frac{4}{\left(1+|z|^2\right)^2}\left|\frac{\partial f_2}{\partial r}\right|^2 + \frac{1}{r^2}\left|\nabla_z\left(f_2 e^{2i\phi}\right)\right|^2 \right\} r^2 \, \mathrm{d}r.$$

Now, by the sharp Hardy inequality we have

$$\int_{\mathbb{C}}\int_0^{+\infty} \left|\frac{\partial f_2}{\partial r}\right|^2 r^2 \mathrm{d}r \, \mathrm{d}z \geq \int_{\mathbb{C}}\int_0^{+\infty} \frac{1}{4}\frac{|f_2|^2}{r^2} r^2 \mathrm{d}r \, \mathrm{d}z,$$

hence,

$$\mathcal{Q}_2 \geq \int_{\mathbb{C}} \mathrm{d}z \int_0^{+\infty} \left\{ -\frac{8}{\left(1+|z|^2\right)^2}\frac{|f_2|^2}{r^2} + \frac{1}{\left(1+|z|^2\right)^2}\frac{|f_2|^2}{r^2} + \frac{1}{r^2}\left|\nabla_z\left(f_2 e^{2i\phi}\right)\right|^2 \right\} r^2 \, \mathrm{d}r.$$

We now use again the Fubini's theorem to revert the order of integration. Thus, if the following integral:

$$\int_{\mathbb{C}} \left\{ -\frac{7}{\left(1+|z|^2\right)^2} |f_2|^2 + \frac{1}{r^2} \left|\nabla_z \left(f_2 e^{2i\phi}\right)\right|^2 \right\} \mathrm{d}z$$

is nonnegative for any choice of f_2, then the equator map ω is stable.

To show that this is indeed the case, it is convenient to switch from the complex coordinate z to polar coordinates $\rho = |z|$ and $\phi \in [0, 2\pi)$ in the complex plane. Denote the partial derivative w.r.t. ρ with a prime. Observe that

$$\left|\nabla_z \left(f_2 e^{2i\phi}\right)\right|^2 = |f_2'|^2 + \frac{4}{\rho^2} |f_2|^2 .$$

Being ρ the Jacobian of the coordinates transformation, we have

$$\mathcal{Q}_2 = 2\pi \int_0^{+\infty} r^2 \, \mathrm{d}r \int_0^{\infty} \left\{ -\frac{7\rho}{(1+\rho^2)^2} \frac{|f_2|^2}{r^2} + \frac{\rho}{r^2} |f_2'|^2 + \frac{4}{r^2 \rho} |f_2|^2 \right\} \mathrm{d}\rho.$$

(Note that nothing depends on ϕ, so that the ϕ-integral factors out by the Fubini's theorem.)

Thus, we have further reduced our question to show that

$$\int_0^{\infty} \left\{ -\frac{7\rho}{(1+\rho^2)^2} |f_2|^2 + \rho |f_2'|^2 + \frac{4}{\rho} |f_2|^2 \right\} \mathrm{d}\rho \geq 0 \quad \text{for any choice of } f_2,$$

i.e, that

$$\int_0^{\infty} \left\{ \frac{-7\rho^2 + 4(1+\rho^2)^2}{\rho (1+\rho^2)^2} |f_2|^2 + \rho |f_2'|^2 \right\} \mathrm{d}\rho \geq 0 \quad \text{for any choice of } f_2. \tag{7.1.9}$$

But we have

$$-7\rho^2 + 4\left(1+\rho^2\right)^2 = 4 + \rho^2 + 4\rho^4,$$

which is strictly positive for all $\rho \in (0, +\infty)$. Thus, the integral (7.1.9) is positive for all f_2 not identically zero, and obviously zero otherwise. Hence, ω is stable.

Now suppose f_2 is complex-valued. In this case it is slightly more transparent writing V as $\mathbb{R}^5$-valued:

$$V = (g_1 \cos(2\phi), g_2 \sin(2\phi), 0, 0, 0),$$

where g_1, g_2 are smooth real-valued functions of compact support such that

$$f_2 e^{2i\phi} = (g_1 \cos(2\phi), g_2 \sin(2\phi)).$$

Quick calculations show that

$$|\nabla V|^2 = |\partial_r g_1|^2 \cos^2(2\phi) + |\partial_r g_2|^2 \sin^2(2\phi) + \frac{1}{r^2} \left(|\partial_\theta g_1|^2 \cos^2(2\phi) + |\partial_\theta g_2|^2 \sin^2(2\phi)\right) + \frac{4\left(|g_1|^2 + |g_2|^2\right)}{r^2 \sin^2 \theta}$$

and

$$|V|^2 = g_1^2 \cos^2(2\phi) + g_2^2 \sin^2(2\phi).$$

Substituting $|\nabla V|^2$ and $|V|^2$ into the second variation formula and using Fubini's theorem to perform an integration w.r.t. ϕ, we are led to the sum of two pieces, one containing only g_1 and the other only g_2, both of which with the same structure as in the previous part. Hence, the argument above gives that both contributions are nonnegative and, again, the stability of ω. □

We now prove a much stronger statement, that is, the equator map is actually minimizing among S^1-equivariant S^4-valued harmonic maps with the same boundary condition. Clearly, equivariance will be **crucial** for the argument; without it, the stament would be *false*, by well-known results by Jäger and Kaul [79].

Before stating our theorem, some important remarks are in order.

The previous equivariant stability theorem stimulates the natural question whether this new piece of information could be used to deduce minimality. We remark that a similar question[1] is investigated in [76] (cfr. Proposition 5.2 there) and it is there proven that stability in a **fixed** direction not explored by the map in the target space implies minimality. One may wonder whether the same continues holding true in the S^1-equivariant case for the homogeneous degree-zero extension of the equator map, also in view of its minimality as a map from $\mathbb{R}^3$ into $S^2 \subset L_1 \oplus L_0$. We must be aware that the situation is not exactly the same: on the one hand, there is no *fixed* direction in the target space not explored by the equator map, because of the fact that equivariance intertwines the directions within invariant subspaces. On the other hand, the equivariant stability with respect L_2-valued perturbation is much *weaker* than the stability required in the statement of [76, Proposition 5.2], because it holds only for $W_0^{1,2} \cap L^\infty(B_1, \mathbb{R})$ functions w of the special form $w(r,\theta,\phi) = v(r,\theta)e^{2i\phi}$. So, we should expect that complications will arise. If one tries to adapt the proof, it turns out that the main trouble is the weakness of the equivariant stability; this makes the elegant method of [76] very difficult to pursue. In particular, also upon using other useful inequalities such as [77, Eq. (2.10)] to step forward and gain control on as more terms as possible, a term whose sign is not clear remains and apparently one is left without further weapons at disposal. For these reasons, we approached the problem in a different way. It remains an open question whether equivariant stability with respect general equivariant perturbation actually implies equivariant minimality or not.

Notation. For the sake of a lighter notation, we write ω also for the homogeneous degree-zero extension of the equator map on to the whole unit ball B_1.

Theorem 7.6 (S^1-equivariant minimality of the equator map)**.** *The tangent map $\omega : \mathbb{R}^3 \to S^2 \subset L_1 \oplus L_0 \subset L_2 \oplus L_1 \oplus L_0$ defined by $\omega(x) = \left(0, \frac{x}{|x|}\right)$ is minimizing among S^1-equivariant S^4-valued harmonic maps $\mathbb{R}^3 \to S^4$ with the same boundary condition.*

Proof. Let $u : \mathbb{R}^3 \to S^4$ be a S^1-equivariant harmonic map such that $u|_{S^2} = \omega|_{S^2}$. Since u and ω coincide on $\partial B_1 = S^2$, let us extend u to ω outside B_1 and let us still call, with a slight abuse of notation, this extension u. We write

$$u = (u_2 e^{2i\phi}, u_1 e^{i\phi}, u_0) \in L_2 \oplus L_1 \oplus L_0,$$

[1] In a non-equivariant context.

where $u_i = u_i(r,\theta)$, $i = 0,1,2$. Note that equivariance forces u_1, u_2 to be vanishing at poles and that the boundary condition implies, in particular, that u_2 must vanish in a neighbourhood of S^2 (while u_1 coincides in the trace sense with $\sin\theta$ and u_0 with $\cos\theta$ there).

Now we use a trick similar to that exploited by Ignat, Nguyen, Slastikov & Zarnescu in the proof of [76, Theorem 1.3], that is, from u we construct the map

$$\tilde{u} = \left(0, \sqrt{u_2^2 + u_1^2}e^{i\phi}, u_0\right).$$

Then $\tilde{u} \in W^{1,2}(B_1, S^2)$, $S^2 \subset L_1 \oplus L_0$, $\tilde{u}$ is S^1-equivariant, $\tilde{u}|_{S^2} = \omega|_{S^2}$ and we claim that

$$E(\tilde{u}) \leq E(u).$$

Indeed, we have

$$\begin{aligned} |\nabla\tilde{u}|^2 &= \left|\frac{\partial\tilde{u}_1}{\partial r}\right|^2 + \frac{1}{r^2}\left|\frac{\partial\tilde{u}_1}{\partial\theta}\right|^2 + \frac{1}{r^2}\left|\frac{\partial u_0}{\partial\theta}\right|^2 + \left|\frac{\partial u_0}{\partial r}\right|^2 + \frac{1}{r^2\sin^2(\theta)}\left|\frac{\partial\tilde{u}_1}{\partial\phi}\right|^2 \\ &= |\nabla_{r,\theta}\tilde{u}_1|^2 + |\nabla_{r,\theta}u_0|^2 + \frac{u_2^2 + u_1^2}{r^2\sin^2(\theta)}, \end{aligned}$$

where $|\nabla_{r,\theta}\tilde{u}_1|^2$, $|\nabla_{r,\theta}u_0|^2$ are defined in an obvious way from the first line above. Now, note that

$$\begin{aligned} |\nabla_{r,\theta}\tilde{u}_1|^2 &= \frac{1}{u_2^2 + u_1^2}|u_2\nabla_{r,\theta}u_2 + u_1\nabla_{r,\theta}u_1|^2 \\ &= \frac{1}{u_2^2 + u_1^2}\left(u_2^2|\nabla_{r,\theta}u_2|^2 + u_1^2|\nabla_{r,\theta}u_1|^2 + 2u_2u_1\nabla_{r,\theta}u_2\cdot\nabla_{r,\theta}u_1\right) \\ &\leq |\nabla_{r,\theta}u_2|^2 + |\nabla_{r,\theta}u_1|^2, \end{aligned}$$

where the last inequality follows since, by Cauchy and Cauchy-Schwartz inequalities,

$$\begin{aligned} (u_2^2 + u_1^2)(|\nabla_{r,\theta}u_2|^2 + |\nabla_{r,\theta}u_1|^2) &= u_2^2|\nabla_{r,\theta}u_2|^2 + u_1^2|\nabla_{r,\theta}u_2|^2 + u_2^2|\nabla_{r,\theta}u_1|^2 + u_1^2|\nabla_{r,\theta}u_1|^2 \\ &\geq u_2^2|\nabla_{r,\theta}u_2|^2 + u_1^2|\nabla_{r,\theta}u_1|^2 + 2u_2u_1\nabla_{r,\theta}u_2\cdot\nabla_{r,\theta}u_1. \end{aligned}$$

To conclude that $E(\tilde{u}) \leq E(u)$, it now suffices to observe that, from the above,

$$|\nabla u|^2 = |\nabla_{r,\theta}u_2|^2 + |\nabla_{r,\theta}u_1|^2 + |\nabla_{r,\theta}u_0|^2 + \frac{4u_2^2 + u_1^2}{r^2\sin^2(\theta)} \geq |\nabla\tilde{u}|^2.$$

Thus, $\tilde{u}$ is a map in $W^{1,2}(B_1, S^2)$, $S^2 \subset L_1 \oplus L_0$, that coincides with ω on $\partial B_1 = S^2$, having lower energy than u. But ω is minimizing among $W^{1,2}(B_1, S^2)$ maps subject to its own boundary condition, hence

$$E(\omega) \leq E(\tilde{u}) \leq E(u),$$

and this concludes the proof. □

Remark 7.1.1. Note that, as we already remarked, equivariance is essential for the argument, because it yields $E(\tilde{u}) \leq E(u)$. If we do not require equivariance (or, more mildly, if we do not know that the ϕ-part of the gradient of $\tilde{u}$ is smaller than that of

u), then there is no reason for $E(\tilde{u})$ to be smaller than $E(u)$. Indeed, competitors u with component u_2 having arbitrary dependence on ϕ (here including no dependence at all) must be taken into account. This provides enough space to make the conclusion false (see, for instance, [79]; compare also [76, Example 1.6]).

Remark 7.1.2. Let us explicitly remark that the trick used in the proof is not in contrast with the result of Jäger & Kaul, in the sense that it cannot be used to try to invalidate their instability theorem, as one might think at first sight. Indeed, let $u : \mathbb{R}^3 \to S^4$ be any harmonic map that coincides with ω on S^2. Then, one may think to mimic the above argument, "bringing u_1 into u_2" this time, thus getting $\tilde{u} = \left(\sqrt{u_2^2 + u_1^2}, 0, u_0\right)$ (here there is no request of equivariance, so indexes are merely labels and $u_i = u_i(r, \theta, \phi)$). Then $|\tilde{u}| = 1$ but it is clear that $\tilde{u}$ **does not** agree with $\omega = (0, \sin(\theta)e^{i\phi}, \cos(\theta))$ on S^2, so this procedure does not make any sense. For the same reason, it makes no sense to consider $\hat{u} = \left(u_2, 0, \sqrt{u_1^2 + u_0^2}\right)$.

7.1.4 Instability of $L_2 \oplus L_0$-valued tangent maps

We now prove that tangent maps with values into $L_2 \oplus L_0$ are unstable. Here, we shall prove this by direct computation. The technique we shall use for linearly full maps extends also to the present case; however, we believe that it may be interesting to carry out explicit computations, where possible, for the sake of a deeper control of the objects under studying.

Theorem 7.7 (Instability of $L_2 \oplus L_0$-valued tangent maps)**.** *Let $\omega : \mathbb{R}^3 \to S^2 \subset L_2 \oplus L_0 \subset L_2 \oplus L_1 \oplus L_0$ be a S^1-equivariant minimizing tangent map given by Theorem 5.13. Then ω is constant.*

Proof. By the classification in Section 6.1, it suffices to consider the map ω corresponding to $\pi^{-1}\left(\left(\pi\left(\frac{x}{|x|}\right)\right)^2\right)$. We recall that this map is minimizing among the $L_2 \oplus L_0$-valued harmonic maps with the same degree. It then follows that, to make ω unstable, we have to push it in the L_1-direction.

As a function from $S^2 \simeq \mathbb{C} \cup \{\infty\}$ into $\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{R} \simeq L_2 \oplus L_1 \oplus L_0$, ω can be written

$$\omega(z) = \left(\omega_2(|z|)e^{2i\phi}, 0, \omega_0(|z|)\right),$$

with

$$\omega_2(|z|)e^{2i\phi} = \frac{2z^2}{1 + |z|^4} \quad \text{and} \quad \omega_0(|z|) = \frac{1 - |z|^4}{1 + |z|^4}.$$

In what follows, working in spherical coordinates will be more convenient, because it will be more apparent how to choose the deformation field. Let us then set

$$V(r, \theta, \phi) = g(r, \theta)\left(0, \psi(\theta)e^{i\phi}, 0\right),$$

with $g : (0, \infty) \times [0, \pi] \to \mathbb{R}$ and $\psi : [0, \pi] \to \mathbb{C} \simeq \mathbb{R}^2$ smooth functions of compact support and such that $g(0)\psi(0) = g(\pi)\psi(\pi) = 0$ to be chosen. Then the second variation formula becomes

$$\mathcal{Q}_\psi(g;\omega) = \int_{\mathbb{R}^3} \left\{ -\frac{g^2}{r^2} \left(|\psi|^2 |\nabla_T \omega|^2 - \frac{1}{\sin^2(\theta)} |\partial_\phi V|^2 - |\partial_\theta \psi|^2 \right) + |\psi|^2 |\partial_r g|^2 + \frac{|\psi|^2}{r^2} |\nabla_T g|^2 \right\} \mathrm{d}x.$$

We clearly have $|\partial_\phi V|^2 = g^2 |\psi|^2$. A quick computation shows that

$$|\nabla_T \omega|^2 = \frac{8 \sin^2(\theta)}{(1+\cos^2(\theta))^2}.$$

A careful look at these formulae gives insight on the choice of $\psi(\theta)$. Let us take g a radial function. Setting

$$\psi(\theta) e^{i\phi} = \frac{1}{\sqrt{2}} (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi)),$$

we get

$$|\psi|^2 = \sin^2(\theta) \quad \text{and} \quad |\partial_\theta \psi|^2 = \cos^2(\theta),$$

so that

$$|\psi|^2 |\nabla_T \omega|^2 - \frac{1}{\sin^2(\theta)} |\partial_\phi V|^2 - |\partial_\theta \psi|^2 = \frac{8 \sin^4(\theta)}{(1+\cos^2(\theta))^2} - (1+\cos^2(\theta)).$$

Observe that nothing depends on ϕ. Switching to spherical coordinates in $\mathbb{R}^3$ in the integrals (using the Fubini's theorem), we then need only to calculate

$$\int_0^\pi \frac{8\sin^4(\theta)}{(1+\cos^2(\theta))^2} \sin(\theta) \mathrm{d}\theta \quad \text{and} \quad \int_0^\pi (1+\cos^2(\theta)) \sin(\theta) \mathrm{d}\theta.$$

Both integrals are elementary. The second one reduces to

$$\int_{-1}^1 (1+t^2)\,\mathrm{d}t = \frac{8}{3}$$

and the first one gives

$$\begin{aligned}
\int_0^\pi \frac{8\sin^4(\theta)}{(1+\cos^2(\theta))^2} \sin(\theta)\mathrm{d}\theta &= 8 \int_0^\pi \left(\frac{1-\cos^2(\theta)}{1+\cos^2(\theta)} \right)^2 \sin(\theta)\mathrm{d}\theta \\
&= 8 \int_{-1}^1 \left(\frac{1-t^2}{1+t^2} \right)^2 \mathrm{d}t \\
&= 8 \left[-2\arctan(t) + \frac{2t}{1+t^2} + t + C \right]_{-1}^1 \\
&> 6.8.
\end{aligned}$$

We then have $6.8 - \frac{8}{3} \geq 4$. Turning back to $\mathcal{Q}_\psi(g;\omega)$, we have

$$\frac{4}{3} \mathcal{Q}_\psi(g;\omega) \leq \int_0^\infty \left(-3\frac{g^2}{r^2} + |\partial_r g|^2 \right) r^2 \mathrm{d}r. \tag{7.1.10}$$

Since $3 > \frac{1}{4}$, the sharp Hardy inequality (7.1.2) is violated and we can find $g \in C_c^\infty((0,\infty))$ so that the r.h.s. of (7.1.10) is negative. Hence, the conclusion follows. □

7.1.5 Instability of linearly full tangent maps

We now prove that any S^1-equivariant tangent map $\omega : \mathbb{R}^3 \to S^4$ with not identically zero component along L_2 is unstable. Note that explicit calculations are out of question for general linearly full maps, because of overwhelming difficulties due to the amazing complexity of the expression for $|\nabla\omega|^2$, see (A.3). Our proof proceeds by contradiction and relies on the identification of a suitable stability inequality, given by the following lemma.

Lemma 7.2. *Let $\omega : \mathbb{R}^3 \to S^4$ be a stable S^1-equivariant tangent map. Then, ω satisfies the following stability inequality:*

$$\int_{S^2} g^2 |\nabla\omega|^2 \, dvol_{S^2} \leq \int_{S^2} \left\{ \frac{1}{4} g^2 + |\partial_\theta g|^2 + \frac{g^2}{\sin^2\theta} \right\} dvol_{S^2}, \tag{7.1.11}$$

for all $g \in C^\infty([0,\pi])$ vanishing at poles, that is: $g(0) = g(\pi) \equiv 0$.

Proof. Let $V \in C_c^\infty(\mathbb{R}^3, \mathbb{R}^5)$ a deformation vector field of the form

$$V = \left(0, \psi(r,\theta) i e^{i\phi}, 0\right), \tag{7.1.12}$$

with $\psi \in C_c^\infty((0,+\infty) \times [0,\pi], \mathbb{R})$ vanishing at poles:

$$\psi(\cdot, 0) = \psi(\cdot, \pi) \equiv 0. \tag{7.1.13}$$

Then, $\omega \cdot V \equiv 0$, so that the second variation $\mathcal{Q}(V;\omega)$ is given directly by the second line in (7.1.4). We have (recall that ω is degree-zero homogeneous)

$$\mathcal{Q}(V;\omega) = \int_{\mathbb{R}^3} \left\{ -\frac{\psi^2}{r^2} |\nabla_T \omega|^2 + |\partial_r \psi|^2 + \frac{1}{r^2} \left(|\partial_\theta \psi|^2 + \frac{\psi^2}{\sin^2\theta} \right) \right\} \mathrm{d}x.$$

We now decompose

$$\psi(r,\theta) = \varphi(r) g(\theta),$$

with

$$\varphi \in C_c^\infty((0,+\infty)) \text{ and } g \in C^\infty([0,\pi]) : g(0) = g(\pi) \equiv 0.$$

Hence,

$$\mathcal{Q}(V;\omega) = \int_{\mathbb{R}^3} \left\{ -\frac{\varphi^2 g^2}{r^2} |\nabla_T \omega|^2 + g^2 |\partial_r \varphi|^2 + \frac{\varphi^2}{r^2} \left(|\partial_\theta g|^2 + \frac{g^2}{\sin^2\theta} \right) \right\} \mathrm{d}x.$$

We now optimize w.r.t. φ using the sharp Hardy inequality (7.1.2), as in [95, Proposition 2.1]. Claiming stability, we must have that quadratic form in g:

$$\mathscr{Q}(g;\omega) = \int_{S^2} \left\{ -g^2 |\nabla\omega|^2 + \frac{1}{4} g^2 + |\partial_\theta g|^2 + \frac{g^2}{\sin^2\theta} \right\} \mathrm{dvol}_{S^2}$$

is nonnegative definite, i.e., $\mathscr{Q}(g;\omega) \geq 0$ for any $g \in C^\infty([0,\pi])$ vanishing at 0 and at π. This yields (7.1.11).

□

We can now state the main result.

Theorem 7.8 (Instability of tangent maps that are not into $L_1 \oplus L_0$)**.** *Any S^1-equivariant tangent map $\omega : \mathbb{R}^3 \to S^4$ with a non-identically vanishing component along L_2 is unstable.*

Proof. Let us write ω in the form

$$\omega = \left(\omega_2(\theta)e^{2i\phi}, \omega_1(\theta)e^{i\phi}, \omega_0(\theta)\right).$$

Let $\omega_2 \not\equiv 0$ and suppose, for a contradiction, that ω is stable. Observe that ω_2 fits the same hypotheses as g in the statement of Lemma 7.2, so that it can be plugged[2] into (7.1.11). Doing this gives:

$$\int_{S^2} \omega_2^2 \left|\nabla\omega\right|^2 \mathrm{dvol}_{S^2} \le \int_{S^2} \left\{\frac{1}{4}\omega_2^2 + \left|\partial_\theta\omega_2\right|^2 + \frac{\omega_2^2}{\sin^2\theta}\right\} \mathrm{dvol}_{S^2}. \tag{7.1.14}$$

On the other hand, by (3.6.3) the l.h.s. above is given by

$$\int_{S^2} \omega_2^2 \left|\nabla\omega\right|^2 \mathrm{dvol}_{S^2} = \int_{S^2} \left|\nabla\left(\omega_2 e^{2i\phi}\right)\right|^2 = \int_{S^2} \left\{\left|\partial_\theta\omega_2\right|^2 + \frac{4\omega_2^2}{\sin^2\theta}\right\} \mathrm{dvol}_{S^2}.$$

Comparing to (7.1.14), we see that stability forces

$$\int_{S^2} \frac{3\omega_2^2}{\sin^2\theta} \mathrm{dvol}_{S^2} \le \int_{S^2} \frac{1}{4}\omega_2^2 \, \mathrm{dvol}_{S^2},$$

which is clearly impossible, unless $\omega_2 \equiv 0$. Thus, ω cannot be stable. ☐

As a particular case, we have recovered instability of tangent maps into $L_2 \oplus L_0$, already proven by explicit calculation in Theorem 7.7.

Particular case: instability of the hedgehog

We have two other proofs for the instability of the hedgehog. The first proof takes advantage of the fact that the hedgehog actually enjoys the full SO(3)-equivariance. The second proof is achieved showing that, assuming stability, the sharp Hardy inequality is violated.

Lemma 7.3. *Let ω, V and $\mathcal{Q}(V;\omega)$ as in Proposition 7.4. Set $V = \varphi e_0$, with $\varphi = \varphi(r,\theta)$. Then*

$$\mathcal{Q}(\varphi e_0;\omega) = \int_{\mathbb{R}^3} \left\{\left|\nabla\varphi\right|^2(1-\omega_0^2) + \varphi^2\left[2\left|\nabla\omega_0\right|^2 - (1-\omega_0^2)\left|\nabla\omega\right|^2\right]\right\}. \tag{7.1.15}$$

Proof. Setting $V = \varphi e_0$ in $\mathcal{Q}(V;\omega)$, given by (7.1.4), we find

$$\mathcal{Q}(\varphi e_0;\omega) = \int_{\mathbb{R}^3} \Big\{\varphi^2(4(\omega\cdot e_0)^2\left|\nabla\omega\right|^2 - \left|\nabla\omega\right|^2 - \left|\nabla(\omega\cdot e_0)\right|^2 + \left|\nabla\varphi\right|^2(1-(\omega\cdot e_0)^2) \\ -6\varphi(\omega\cdot e_0)\nabla\varphi\cdot\nabla(\omega\cdot e_0)\Big\}. \tag{7.1.16}$$

[2]Recall that ω_2 can always be thought of as real-valued when studying stability, see Lemma 7.1.

We note that $6\varphi\omega_0\nabla\varphi\cdot\nabla\omega_0 = \frac{3}{2}\nabla\varphi^2\cdot\nabla\omega_0^2$. As in [132], we integrate by parts, using

$$\Delta\omega_0 = -\omega_0 \left|\nabla\omega\right|^2,$$

which follows from the harmonic maps equations because e_0 is constant. This yields the conclusion. □

Theorem 7.9. *The tangent map given by the hedgehog,* $H(x) = \sqrt{\frac{3}{2}}\left(\frac{x}{|x|}\otimes\frac{x}{|x|} - \frac{1}{3}\right)$, *is unstable.*

First proof. Since the hedgehog is SO(3)-equivariant, it is S^1-equivariant with respect *all* axes through the origin. The equivalence of all directions allows to proceed exactly as in the Schoen-Uhlenbeck proof, choosing an orthonormal basis in $\mathbb{R}^5$ and then summing all the corresponding contributions. This removes the dependence on the direction in the quadratic form and, on the other hand, all the contributions are equal. Then the argument of Schoen-Uhlenbeck in the last part of the proof of [132, Theorem 2.7] implies that the quadratic form $\mathcal{Q}(\varphi e_0; H)$ is not positive-definite ($\varphi\in C_c^\infty(\mathbb{R}^3)$ radial). □

Second proof. Recalling the decomposition

$$\omega = \begin{pmatrix} \Omega_0 & \mathbf{v} \\ \mathbf{v}^t & w_0 \end{pmatrix}$$

and the form of the matrices e_i of the basis we selected for $L_2\oplus L_1\oplus L_0$, we have

$$\omega\cdot e_0 = \omega_0 = \sqrt{\frac{3}{2}}w_0.$$

In particular, for the hedgehog we have

$$\omega_0 = \sqrt{\frac{3}{2}}h_0 = \frac{3}{2}\left(\frac{x_3^2}{r^2} - \frac{1}{3}\right) = \frac{3}{2}\left(\cos^2\theta - \frac{1}{3}\right),$$

where h_0 is w_0 evaluated for the hedgehog.

Pick $\varphi_r \in C_c^\infty(\mathbb{R}^3)$ a radial function. Then we have

$$\begin{aligned}\mathcal{Q}(\varphi_r e_0; H) = \int_0^\infty\int_0^\pi\int_0^{2\pi} &\left\{ |\partial_r\varphi_r|^2\left(\frac{3}{4} - \frac{9}{4}\cos^4\theta + \frac{3}{2}\cos^2\theta\right)\right.\\ &\left.+\varphi_r^2\left(18\cos^2\theta - 18\cos^4\theta - \left(\frac{3}{4} - \frac{9}{4}\cos^4\theta + \frac{3}{2}\cos^2\theta\right)|\nabla H|^2\right)\right\} r^2\sin\theta \mathrm{d}\phi\,\mathrm{d}\theta\,\mathrm{d}r.\end{aligned}$$

Recalling that $|\nabla H|^2 = \frac{6}{r^2}$, we reduce to

$$\begin{aligned}\mathcal{Q}(\varphi_r e_0; H) =& \frac{3}{2}\pi\int_0^\infty\int_0^\pi\left\{\left(\frac{3}{4} - \frac{9}{4}\cos^4\theta + \frac{3}{2}\cos^2\theta\right)|\partial_r\varphi_r|^2\right.\\ &+\frac{\varphi_r^2}{r^2}\left(-\frac{9}{2}\cos^4\theta + 9\cos^2\theta - \frac{9}{2}\right) r^2\sin\theta\mathrm{d}\theta, \mathrm{d}r.\end{aligned}$$

The θ-integrals are elementary and we get

$$\mathcal{Q}(\varphi_r e_0; H) = \frac{5}{4}\pi\times\int_0^\infty\left(|\partial_r\varphi_r|^2 - 3\frac{\varphi_r^2}{r^2}\right) r^2\mathrm{d}r. \tag{7.1.17}$$

Since $3 > \frac{1}{4}$, the sharp Hardy inequality (7.1.2) is violated and hence the conclusion follows. □

7.2 Partial interior regularity

The results in Section 7.1 can be summarized to give the following partial interior regularity theorem.

Theorem 7.10 (Partial interior regularity theorem for S^1-equivariant minimizers)**.** *Let $Q \in W^{1,2}(B_1, S^4)$ be a minimizer of the LdG energy* (1.1.3) *in the class* (1.1.15)*, with Q_b as in* (1.1.13)*. Then $Q \in C^\omega(B_1 \setminus \Sigma(Q), S^4)$, where $\Sigma(Q) \subset \{z\text{-axis}\}$ is a finite set of isolated points, possibly empty, of even cardinality. Moreover, if $a \in \Sigma(Q)$, then the tangent map of Q at a is the equator map, up to an isometry.*

Proof. Q cannot have singularities off the z-axis, thus it is continuous in $B_1 \setminus (B_1 \cap$ z-axis) and hence real-analytic by higher regularity theorems in Chapter 4. By Theorem 5.13, $\Sigma(Q)$ is a finite set of isolated points located on the z-axis.

Next, by Theorem 7.6 and Theorem 7.8, every nonconstant tangent map, if any, must be obtained by the equator map $\omega = \left(0, \frac{x}{|x|}\right)$ with an isometry. We have only to prove that the number of singular points must be even.

Let $a \in \Sigma(Q)$. Then, passing through a along the z-axis, Q suffers a discontinuity and its leading eigenvector, which must be directed along the z-axis because of equivariance, changes sign. If there are n discontinuities, then the sign changes n times. But, denoting H the hedgehog map, at the poles we have

$$Q(N) = H(N) = H(S) = Q(S),$$

hence n must be even. □

Remark 7.2.1. More succintly, $\Sigma(Q)$ consists of a finite number of *dipoles*, in the sense of [21].

Remark 7.2.2. In order to achieve complete regularity, we have to rule out all the possible dipoles. This appears to be a very difficult task, for the following reason. In [47], Gartland & Mkaddem numerically showed the existence of a metastable solution, by them called the *split core solution*, bearing a remarkable resemblance with singular single-dipole solutions allowed by Theorem 7.10. This **strongly** suggests that the removal of potential dipoles, whenever possible, must be a **highly nonlinear** effect due to minimality. Said another way, this issue **cannot** be addressed looking only at the linearized problem.

Remark 7.2.3. Note that here nothing really depends on the specific form of the boundary condition Q_b, provided it is S^1-equivariant and takes the same values at the poles (otherwise singularities are unavoidable).

The following dichotomy is an easy consequence of the above partial regularity result.

Corollary 7.11. *Let $Q \in \mathcal{A}^{ax}_{Q_b}$ be a minimizer of the LdG energy* (1.1.3) *in the class* (1.1.15)*, with Q_b as in* (1.1.13)*. Then either Q is singular with a finite number of dipoles or Q is a biaxial torus solution in B_1, in the sense of Definition 1.2.*

Proof. If Q has singularities, then they must form a finite number of dipoles by Theorem 7.10. Otherwise, Q is a smooth S^1-equivariant minimizer of the LdG energy (1.1.3) and then it is a biaxial solution in B_1 by Theorem 5.1. □

Remark 7.2.4. Corollary 7.11 extends to the case of any smooth S^1-equivariant boundary condition Q_b that satisfies $\tilde{\beta}(Q_b) \equiv +1$, $Q_b(N) = Q_b(S)$ and whose leading eigenspace perform the nontrivial path in $\mathbb{R}P^2$ on S^2.

7.3 Boundary regularity

Due to symmetry, boundary regularity is somewhat simpler in the equivariant case but, nonetheless, it deserves some special care.

The first step is a boundary monotonicity formula. The argument giving the interior boundary monotonicity formula actually holds up to the boundary and hence we have

Theorem 7.12 (Boundary monotonicity formula). *Let $Q \in \mathcal{A}^{ax}_{Q_b}$ be a minimizer of $E(\cdot; B_1)$ in the class $\mathcal{A}^{ax}_{Q_b}$, with Q_b as in* (1.1.13) *and let $x_0 \in \partial B_1$. Define*

$$\mathcal{E}_r = \frac{1}{r}\int_{\Omega_r} \frac{1}{2}\left|\nabla Q\right|^2 + F(Q)\, dx. \tag{7.3.1}$$

Then there exist $R_0 > 0$ and a constant $C = C(a, b, c, Q_b, R_0)$, $C > 0$, so that

$$\frac{d\mathcal{E}_r}{dr} \geq -C(a, b, c, Q_b, R_0), \quad \forall 0 < r < R_0. \tag{7.3.2}$$

Proof. The proof exploits the approximation trick used for the interior monotonicity formula and goes exactly as in the proof of Theorem 4.14. □

Once the boundary monotonicity formula is given, we can use it to extend the ε-regularity theorem and the strong compactness theorem for rescaled maps. Note that the extension argument in Section 4.8 preserves equivariance, so it can be used also in this case and hence the ε-regularity theorem still holds, since it is valid for any weak solution of the Euler-Lagrange equations (4.2.1) satisfying also (4.8.3), and thus in particular for minimizers in the symmetric class, by the theorem above.

For what concerns the strong compactness theorem, differently from the non-symmetric case, we now need to show compactness for rescaled maps only at poles. Indeed, off the z-axis we cannot have concentration and the boundary monotonicity formula implies strong convergence to a constant around any point off the z-axis. With a slight modification of Theorem 5.13, we get

Theorem 7.13 (Boundary strong compactness theorem in the symmetric case). *Let $Q \in \mathcal{A}^{ax}_{Q_b}$ be a minimizer of $E(\cdot; B_1)$ in the class $\mathcal{A}^{ax}_{Q_b}$, with Q_b as in* (1.1.13). *Let $R \in (0, 1]$ and define $Q_{R,N} := Q(N + Rx)$, where $x \in R^{-1}(B_1 \setminus \{N\})$ and N is the north pole of B_1 (analogous definitions for the south pole S). Then there exist a sequence $(R_j)_j$, with $R_j \to 0$ as $j \to \infty$, and $Q_0 \in W^{1,2}_{loc}(\mathbb{R}^3_+, S^4)$ so that $Q_{R_j,x_0} \to Q_0$ strongly in $W^{1,2}_{loc}(\mathbb{R}^3_+, \mathcal{S}_0)$. In addition, Q_0 is a locally minimizing harmonic map into S^4 with $Q_0|_{\partial\mathbb{R}^3_+} = const.$. Moreover, Q_0 is degree-zero homogeneous.*

Proof. Thanks to the boundary monotonicity formula, the proof is similar to that of Theorem 5.13. The only difference is that we now have to construct competitors in the region(s) $\Omega_r = B_r(N) \cap \partial B_1$ (analogous for S instead of N) and consider their suitable homothetic restrictions instead of full balls. To this end, we slice Ω_r in a similar way as in Theorem 5.13 and hence we define comparison maps in the same way. Analogous calculations then lead to the conclusion. □

Since we are in dimensions 3, full regularity now follows from Wood's theorem [131] and the fact that singularities form at most a discrete set. Thus,

Theorem 7.14 (Boundary regularity in the symmetric case)**.** *Let* $Q \in \mathcal{A}^{ax}_{Q_b}$ *be a minimizer of* $E(\cdot; B_1)$ *in the class* $\mathcal{A}^{ax}_{Q_b}$*, with* Q_b *as in* (1.1.13)*. Then there exist a* $\delta > 0$ *and a neighborhood* $\mathcal{O}_\delta$ *of* ∂B_1 *such that* $Q \in C^\omega(\mathcal{O}_\delta, S^4)$*.*

Proof. Theorem 7.13 implies that, for both N and S, $Q_0 : S^2_+ \to S^4$ is constant on ∂S^2_+ and hence [131, Lemma 2.5] implies that Q_0 is constant in both cases. Thus, the energy of Q cannot concentrate in N and S and hence, by Theorem 4.15, Q is completely smooth there. Since the singular set of Q is at most discrete, then one can walk upward from the south pole or downward from the north pole along the z-axis for at least a distance $\delta > 0$ without encountering singularities. □

Chapter 8

Biaxial torus solutions and singular minimizers for special boundary data

Synopsis. Here we produce boundary data so that the corresponding minimizers are biaxial torus solutions in B_1 (§8.2); boundary data so that the corresponding minimizers have singularities (§8.3) and boundary data so that the corresponding minimizers are biaxial torus solutions in a subregion of B_1 and have singularities outside (§8.4). We next (§8.5) exploit the classification of all possibile tangent maps carried out in Chapter 7 to extend well-known theorems in harmonic maps, such as generic uniqueness and convergence of singularities to singularities, in order to use them to investigate to some extent the question whether there are boundary conditions that are critical in the sense they have at least two minimizers of different character: a biaxial torus solution and a split solution.

8.1 A S^1-equivariant version of Luckhaus' compactness theorem

In what follows, we shall need the result below, which is an S^1-equivariant version of the Luckhaus' compactness theorem. Actually, its proof will involve exactly the same estimates as in the proof of Theorem 5.13, which is indeed a particular instance of this more general statement.

Theorem 8.1 (S^1-equivariant Luckhaus' compactness theorem). *Let $\{Q^s\}_{s\in[0,1]}$ be a family of minimizers (with respect their own boundary data) of the LdG energy with locally equibounded energies, i.e. there is $C > 0$, independent of s, such that*

$$\sup_s E(Q^s; B_\rho(x)) \leq C < +\infty,$$

for any ball $\overline{B_\rho(x)} \subset B_1$. Then there exist $Q^* \in W^{1,2}_{loc}(B_1, S^4)$ and a sequence $(s_j)_j$, $s_j \to 0$ as $j \to +\infty$, such that $Q^{s_j} \to Q^*$ strongly in $W^{1,2}_{loc}(B_1, S^4)$. In particular, Q^* is a S^1-equivariant local minimizer of the LdG energy (1.1.3) *w.r.t. its own boundary condition.*

Proof. We sketch here only the main points, referring to the proof of Theorem 5.13 for explicit calculations.

The equiboundedness condition on the energy actually implies, in view of the fact that $|Q^s| = 1$, the equiboundedness of the family $\{Q^s\}_{s\in[0,1]}$ in $W^{1,2}(B_1, S^4)$. Thus, there is $Q^* \in W^{1,2}(B_1, \mathbb{R}^5)$ and there is a sequence $(s_j)_j$, $s_j \to 0$ as $j \to +\infty$, such that $Q^{s_j} \rightharpoonup Q^*$ as $j \to \infty$. By the Rellich-Kondrachov theorem, there is a subsequence (not relabeled) on which $Q_j \to Q^*$ strongly in L^2 and hence a further subsequence (again, not relabeled) on which $Q_j(x) \to Q^*(x)$ pointwise a.e.. Thus, we have that Q^* belongs to $W^{1,2}(B_1, S^4)$ and that Q^* is S^1-equivariant (since S^1-equivariance is a pointwise property).

It remains to prove that Q^* is locally minimizing w.r.t its own boundary condition. By the weak lower semicontinuity of the Landau-de Gennes energy, weak convergence $W^{1,2}$, and the fact the $F(Q)$ is a polynomial in Q, plus the strong L^2-convergence of Q^{s_j} to Q^*, this would follow automatically if the convergence was strong in $W^{1,2}_{\text{loc}}$ (i.e., if also the elastic energies of the Q^s's converge to the elastic energy of Q^*). In order to prove strong convergence, we first observe that, by Fatou's lemma and Fubini's theorem, we can fix $\delta \in (0, 1)$ and find $\rho \in (1 - \delta, 1)$ so that

$$\lim_{j\to+\infty} \int_{\partial B_\rho} |Q^{s_j} - Q^*|^2 \, \mathrm{d}\mathcal{H}^2 = 0$$

and

$$\int_{B_\rho} \left(|\nabla_T Q^{s_j}|^2 + |\nabla_T Q^*|^2\right) \mathrm{d}\mathcal{H}^2 \leq C < +\infty,$$

which are exactly (5.8.3) and (5.8.4). We then choose λ_j as below (5.8.4) and slice B_ρ as explained in (5.8.5), (5.8.6), (5.8.7). Then we take $w \in W^{1,2}(B_1, S^4)$ such that $w = Q^*$ a.e on $B_1 \setminus B_{1-\delta}$ and we construct comparison maps $(v_j)_j$ as in (5.8.8). The subsequent estimates then follow at once, leading to the conclusion. □

8.2 A family of boundary data whose minimizers are biaxial torus solutions

In this section we shall prove the following theorem.

Theorem 8.2 (Biaxial torus solutions for special boundary data)**.** *There exist non-trivial S^1-equivariant boundary data $Q_b \in C^\infty(S^2, S^4)$ such that the corresponding S^1-equivariant minimizer of the LdG energy* (1.1.3) *are biaxial torus solutions in B_1.*

The proof will be constructive: we shall craft a family of boundary data $\{Q^s_b\}_{s\in[0,1]}$ such that the family $\{Q^s\}_{s\in[0,1]}$ of corresponding minimizers satisfy, for small s, the conclusion of the theorem.

Proof. As anticipated above, we construct a family of special boundary data and, correspondingly, a family of minimizers of the LdG energy that exhibit the required property. The idea behind the construction is the following. Suppose that we have a family $\{Q^s\}$ of minimizers (with respect to their own boundary condition) of the LdG energy, labeled by a parameter $s \in [0, 1]$, and that this family converges, as $s \to 0$, to a minimizing map, which is also smooth in $\overline{B_1}$. If the convergence is strong enough, then all minimizers sufficiently near to the limiting smooth minimizer inherit smoothness from it (because a strong enough convergence will keep the rescaled energy small and hence Theorem 4.6 applies). If now the minimizers fullfill the conditions for applying the semidisk argument in Section 5.1, then they are biaxial torus solutions in B_1, at least for s sufficiently small.

In order to craft the desired family of minimizers, we shall assign appropriately the family of their boundary data. To do this, we observe that, from the point of view of the semidisk argument, a good behavior is that of the hedgehog map H while, on the side of the regularity issue, a good behavior is that of constant maps. Noting that at the poles of S^2 we have $H = e_0$, we may think to do our construction by spreading, step-by-step, this constant value along the boundary while concentrating the rest of the hedgehog far from the z-axis, so that the limiting datum, for $s \to 0$, will be the constant vector field e_0, but, for any $s > 0$, the hedgehog-like behavior is preserved. Clearly, the minimizer with respect to the limiting datum e_0 is e_0 itself. Provided that we showed that the minimizers are actually near, in the $W^{1,2}$-norm, to e_0 for $s \ll 1$, they must be smooth and have, by construction, suitable boundary data for the semidisk argument. In terms of $s \in [0,1]$, we then want

$$Q_b^0 \equiv e_0 \quad \text{and} \quad Q_b^1 \equiv H,$$

with $Q_b^s \in C^\infty(S^2, S^4)$, $0 < s < 1$, S^1-equivariant maps interpolating between e_0 and H in the sense above. If we are able to assign the data so that the corresponding minimizers $\{Q^s\}_{s\in[0,1]}$ will have equibounded LdG energies, then we can assure $W^{1,2}_{\mathrm{loc}}$-convergence in the interior of B_1 by means of Theorem 8.1. Adding some uniform boundary regularity (for s small) to this, we can be sure that the corresponding minimizers have all (for s sufficiently small) the required properties to conclude. We shall work details in several steps below.

Step 1. Construction of $\{Q_b^s\}_{s\in[0,1]}$ *and* $\{Q^s\}_{s\in[0,1]}$. Let $s \in [0,1]$. Define

$$Q_b^s(\theta,\phi) = \begin{cases} e_0, & \text{if } 0 \le \theta \le \frac{\pi(1-s)}{2} \text{ or } \frac{\pi(1+s)}{2} \le \theta \le \pi, \\ H\left(\frac{1}{s}\left(\theta - \frac{\pi(1-s)}{2}\right), \phi\right), & \text{if } \frac{\pi(1-s)}{2} < \theta < \frac{\pi(1+s)}{2}, \end{cases} \tag{8.2.1}$$

where H denotes, as always, the hedgehog map.

Note that $Q_b^s \in H^{\frac{1}{2}}(S^2, S^4)$ for any $s \in [0,1]$. In particular, we have

$$Q_b^0 \equiv e_0 \quad \text{and} \quad Q_b^1 \equiv H.$$

Further, each Q_b^s is S^1-equivariant. For each s, let Q^s a minimizer of the LdG energy (1.1.3) in the class $\mathcal{A}^{\mathrm{ax}}_{Q_b^s}$.

Step 2. Equiboundedness of $\{Q^s\}_{s\in[0,1]}$. We claim that the family $\{Q^s\}_{s\in[0,1]}$ has equibounded energy. To show this, we need to control the Dirichlet energy $E_D(Q^s; B_1)$ uniformly w.r.t. s. By the equiboundedness of the potential, such a control is actually all that we need. In order to get it, let us note that, for any $s \in [0,1]$,

$$E_D(Q^s; B_1) \le C_1 \left| Q_b^s - \fint Q_b^s \right|^2_{\dot{H}^{\frac{1}{2}}(S^2,S^4)}, \tag{8.2.2}$$

where $C_1 > 0$ is a constant. On the other hand, we also have

$$\left|Q_b^s - \fint Q_b^s\right|^2_{\dot{H}^{\frac{1}{2}}(S^2,S^4)} \leq C_2 \left\|Q_b^s - \fint Q_b^s\right\|_{L^2(S^2,S^4)} \|\nabla Q_b^s\|_{L^2(S^2,S^4)}, \qquad (8.2.3)$$

where $C_2 > 0$ is another constant.

It is easy realized that

$$\left\|Q_b^s - \fint Q_b^s\right\|_{L^2(S^2,S^4)} \leq C_3 s, \qquad (8.2.4)$$

because of $|Q_b^s| = 1$ and the fact that Q_b^s differs from its mean only on that strip on which Q_b^s behaves like H, which corresponds to an interval of values of θ having length s.

We now estimate $\|\nabla Q_b^s\|_{L^2(S^2,S^4)}$. It is *a priori* clear that this term cannot be bounded, since we are concentrating a finite change in the datum in a smaller and smaller region. For convenience, let us set

$$\tilde{\theta}_s = \frac{1}{s}\left(\theta - \frac{\pi(1-s)}{2}\right), \quad 0 < s \leq 1. \qquad (8.2.5)$$

Note that $\tilde{\theta}_1 = \theta$.

Clearly,

$$E_D(Q_b^s) = \int_{S^2} |\nabla Q_b^s|^2 \, \mathrm{dvol}_{S^2} = 2\pi \int_{\frac{\pi(1-s)}{2}}^{\frac{\pi(1-s)}{2}} \left|\nabla H\left(\tilde{\theta}_s, \phi\right)\right|^2 \sin(\theta)\, \mathrm{d}\theta.$$

We have

$$\left|\nabla H\left(\tilde{\theta}_s, \phi\right)\right|^2 = \frac{1}{s^2}\left|\partial_{\tilde{\theta}_s} H\left(\tilde{\theta}_s, \phi\right)\right|^2 + \frac{1}{\sin^2(\theta)}\left|\partial_\phi H\left(\tilde{\theta}_s, \phi\right)\right|^2.$$

Recalling that

$$|\partial_\theta H(\theta,\phi)|^2 = \frac{1}{\sin^2(\theta)}\,|\partial_\phi H(\theta,\phi)|^2 = 3,$$

we see that

$$\left|\nabla H\left(\tilde{\theta}_s, \phi\right)\right|^2 = \frac{3}{s^2} + 3\frac{\sin^2(\tilde{\theta}_s)}{\sin^2(\theta)}$$

Observe that $\sin(\theta) > \sin\left(\frac{\pi(1-s)}{2}\right)$ for $\frac{\pi(1-s)}{2} < \theta < \frac{\pi(1+s)}{2}$. Thus,

$$\int_{\frac{\pi(1-s)}{2}}^{\frac{\pi(1+s)}{2}} \frac{\sin^2(\tilde{\theta}_s)}{\sin^2(\theta)}(\sin(\theta))\,\mathrm{d}\theta \leq \frac{1}{\sin\left(\frac{\pi(1-s)}{2}\right)} \int_{\frac{\pi(1-s)}{2}}^{\frac{\pi(1+s)}{2}} 1\,\mathrm{d}\theta = \frac{\pi s}{\sin\left(\frac{\pi(1-s)}{2}\right)}, \qquad (8.2.6)$$

whose last member clearly tends to zero as $s \to 0$. On the other hand, the integrand at l.h.s. above is a smooth function of s for $s \in (0,1]$. Defining

$$G(s) := \begin{cases} \int_{\frac{\pi(1-s)}{2}}^{\frac{\pi(1+s)}{2}} \frac{\sin^2(\tilde{\theta}_s)}{\sin^2(\theta)}(\sin(\theta))\,\mathrm{d}\theta, & s \in (0,1], \\ 0, & s = 0, \end{cases}$$

we have that G is a continuous function of $s \in [0,1]$, hence G is bounded for $s \in [0,1]$. This readily yields

$$\int_{\frac{\pi(1-s)}{2}}^{\frac{\pi(1+s)}{2}} \frac{\sin^2(\tilde{\theta}_s)}{\sin^2(\theta)}(\sin(\theta))\,\mathrm{d}\theta \leq C_4$$

Then, we have the estimate

$$E_D(Q_b^s) \leq C_5 \frac{\sin\left(\frac{\pi s}{2}\right)}{s^2} + C_6. \tag{8.2.7}$$

By taking the square roots of (8.2.4) and (8.2.7) and inserting them into (8.2.3), we get

$$\sup_{s\in[0,1]} E(Q^s, B_1) \leq C, \tag{8.2.8}$$

where $C > 0$ is a constant which does not depends on s. We can now also smoothing a bit the dependence on θ in Q_b^s while keeping equiboundedness (possibly, with a larger constant). This returns us a family of *smooth* boundary data satisfying the desired properties. With slight abuse of notation, we shall replace each Q_b^s above with its smooth counterpart without relabeling. We then consider the associated minimizers of the LdG energy, again denoted Q^s, and these will satisfy again (8.2.8) (with, possibly, a different constant).

Step 3. Strong compactness and uniform boundary regularity for s small. Thus, we have a family of S^1-equivariant minimizers of the LdG energy with equibounded energies. In particular, this family is equibounded in $W^{1,2}(B_1, S^4)$, and this means that there is $Q^* \in W^{1,2}(B_1, S^4)$ and there is a sequence $(s_j)_j$, $s_j \to 0$ as $j \to \infty$, such that $Q^{s_j} \rightharpoonup Q^*$ as $j \to \infty$. Theorem 8.1 then shows that this convergence in actually strong in $W^{1,2}_{\mathrm{loc}}(B_1, S^4)$, so that Q^* is actually a local minimizer of the LdG energy with respect to its own boundary trace. But $\mathrm{tr}(Q^*) = e_0$, because the trace operator commutes with weak limits, and hence $Q^* \equiv e_0$, because Q^* is minimizing but e_0 is clearly the unique minimizer with respect to its own boundary trace. Further, locally strong converge implies small rescaled energy in compact sets, hence we can fix a small $\delta > 0$ and, for j sufficiently large, Q^{s_j} must be smooth at least in any compact set $K \subset B_1$ such that $\mathrm{dist}(K, N), \mathrm{dist}(K, S) > \delta$, where N and S denote the north and south pole of S^2, respectively.

Moreover, for j sufficiently large we have uniform boundary regularity. Indeed, S^1-equivariance excludes concentration points near the boundary out of the z-axis and, for sufficiently large j, $Q_b^{s_j}$ is uniformly constant near the poles. This means that we can fix $\eta > 0$ and then find $J \in \mathbb{N}$ such that Q^{s_j} is smooth in a η-neighborhood of S^2 for all $j > J$.

Combining this with locally strong convergence, we conclude that Q^{s_j} has to be smooth in $\overline{B_1}$ for all j sufficiently large (that is, so large that we can take $\eta > \delta$), i.e, for s_j sufficiently small.

To conclude, it now suffices to observe that such minimizers fulfill the hypotheses of Corollary 5.2 and hence they are biaxial torus solutions in B_1, in the sense of Definition 1.2. □

8.3 A family of boundary data whose minimizers have singularities

In this section we prove a complementary result w.r.t. that we proved in Section 8.2, that is, we can arrange the boundary datum so that the appearance of singularities is enforced.

Theorem 8.3. *There exist nontrivial S^1-equivariant boundary data $Q_b \in C^\infty(S^2, S^4)$ such that the corresponding S^1-equivariant minimizers of the LdG energy* (1.1.3) *necessarily have singularities.*

Proof. As that of Theorem 8.2, the proof will be constructive, the difference being that this time we shall take a map different from the hedgehog as starting map of our construction. Indeed, we can force the appearance of a singularity on the z-axis by selecting a boundary datum whose eigenframes at the poles have opposite orientations. A map with this property is the equator map (7.1.7). Apart from this and the fact that we now want to concentrate the data near a pole, the construction goes exactly as before. We work out details below.

Step 1. Construction of $\{Q_b^s\}_{s\in[0,1]}$ and $\{Q^s\}_{s\in[0,1]}$. Let $s \in [0,1]$. Define

$$Q_b^s(\theta,\phi) = \begin{cases} \omega\left(\pi - \frac{\theta}{s}, \phi\right), & \text{if } 0 \le \theta < \pi s, \\ e_0, & \text{if } \pi s \le \theta \le \pi, \end{cases} \tag{8.3.1}$$

where ω denotes the equator map.

Note that $Q_b^s \in H^{\frac{1}{2}}(S^2, S^4)$ for any $s \in [0,1]$. In particular, we have

$$Q_b^0 \equiv e_0 \quad \text{and} \quad Q_b^1 \equiv \omega.$$

Further, each Q_b^s is S^1-equivariant. For each s, let Q^s a minimizers of the LdG energy (1.1.3) in the class $\mathcal{A}^{\mathrm{ax}}_{Q_b^s}$..

Step 2. Equiboundedness of $\{Q^s\}_{s\in[0,1]}$. We claim that the family $\{Q^s\}_{s\in[0,1]}$ has equibounded energy. As in proof of Theorem 8.2, in order to show this, we need to control the Dirichlet energy $E_D(Q^s; B_1)$ uniformly w.r.t. s. By the equiboundedness of the potential, such a control is actually all that we need. Recall that, for any $s \in [0,1]$, Eqs. (8.2.2) and (8.2.3) hold. Thus, we need to estimate

$$\left\| Q_b^s - \fint Q_b^s \right\|_{L^2(S^2,S^4)} \quad \text{and} \quad \|\nabla Q_b^s\|_{L^2(S^2,S^4)}.$$

It is clear that the first one behaves like s for the same reason explained below (8.2.4). Moreover, it is plain that they are both zero for $s = 0$. We then need to estimate the norm of the gradient only for $s > 0$. To this end, let us set

$$\tilde{\theta}_s = \pi - \frac{\theta}{s}, \quad 0 < s \leq 1.$$

Recalling that for the equator map it holds

$$|\partial_\theta \omega(\theta, \phi)|^2 = \frac{1}{\sin^2(\theta)} \, |\partial_\phi \omega(\theta, \phi)|^2 = 1,$$

we see that

$$\left|\nabla \omega(\tilde{\theta}_s, \phi)\right|^2 = \frac{1}{s^2} + \frac{\sin^2(\tilde{\theta}_s)}{\sin^2(\theta)}.$$

Since $\sin\left(\pi - \frac{\theta}{s}\right) = \sin\left(\frac{\theta}{s}\right)$, let us redefine $\tilde{\theta}_s$ as $\frac{\theta}{s}$. Let J be the greatest integer contained in $1/s$ and let $0 \leq r \leq 1$ so that $1/s = J + r$. Notice that

$$\begin{aligned}
\int_0^{\pi s} \frac{\sin^2(\tilde{\theta}_s)}{\sin^2(\theta)} \sin(\theta) \, \mathrm{d}\theta &= \int_0^{\pi s} \frac{\sin^2\left(\frac{\theta}{s}\right)}{\sin(\theta)} \, \mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{J+r}} \frac{\sin^2((J+r)\theta)}{\sin(\theta)} \, \mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{J+r}} \frac{(\sin(J\theta)\cos(r\theta) + \cos(J\theta)\sin(r\theta))^2}{\sin(\theta)} \, \mathrm{d}\theta \\
&\leq 2 \int_0^{\frac{\pi}{J+r}} \left\{ \frac{|\sin(J\theta)|}{\sin(\theta)} + C_1 \right\} \mathrm{d}\theta \\
&\leq C_2,
\end{aligned}$$

where we used the elementary limit $\lim_{x \to 0} \frac{\sin(ax)}{\sin(x)} = a$ for any $a \in \mathbb{R}$. Thus,

$$\begin{aligned}
E_D(Q_b^s) &= \int_0^{\pi s} \left\{ \frac{1}{s^2} + \frac{\sin^2(\tilde{\theta}_s)}{\sin^2(\theta)} \right\} \sin(\theta) \, \mathrm{d}\theta \\
&\leq \frac{1 - \cos(\pi s)}{s^2} + C_2.
\end{aligned}$$

Hence,

$$\sup_{s \in [0,1]} E_D(Q_b^s) \leq C_3. \tag{8.3.2}$$

Further, since $1 - \cos(\pi s) \approx \frac{1}{2}\pi^2 s^2$ for $s \ll 1$, we also have that

$$\lim_{s \to 0} E_D(Q_b^s) = C_4, \tag{8.3.3}$$

where $C_3 > 0$ is a constant, which in turn yields

$$\lim_{s \to 0} E_D(Q^s) = 0. \tag{8.3.4}$$

We can now smoothing a bit the dependence on θ in Q_b^s, thus we can replace each Q_b^s with a smooth counterpart, while keeping equiboundedness and (8.3.4) for the corresponding minimizers.

Step 3. Strong compactness. Due to Eq. (8.3.2), we have a family of S^1-equivariant minimizers of the LdG energy with equibounded energies. In particular, this family is equibounded in $W^{1,2}(B_1, S^4)$, and this means that there is $Q^* \in W^{1,2}(B_1, S^4)$ and there is a sequence $(s_j)_j$, $s_j \to 0$ as $j \to \infty$, such that $Q^{s_j} \rightharpoonup Q^*$ as $j \to \infty$. Differently from the previous case, the stronger information carried by (8.3.4) gives the strong convergence to Q^*, and hence its minimality, even without passing through Theorem 8.1. By the weak convergence of the boundary traces to e_0, it then follows $Q^* = e_0$.

Thus, for s sufficiently small, the minimizer Q^s must be smooth in a compact set $K_s \subset B_1$, with K_s invading the whole ball as s tends to zero. However, in the present situation we cannot have uniform boundary regularity because the boundary datum clearly forces the arising of at least one singularity on the z-axis (such singularities escape towards the boundary as $s \to 0$). Thus, there is $\bar{s} > 0$ such that for $s < \bar{s}$ the minimizers Q^s satisfy the conclusion of the theorem. □

Remark 8.3.1. The kind of singularity induced by the boundary data considered in the theorem is a bit artificial from the physical point of view. In fact, we are not aware of the appearance of such type of singularities in physical significant situations. Nevertheless, our construction is certainly mathematically legitimate and leads to a conclusion that may be considered significant, at least as a matter of principle.

Remark 8.3.2. In connection to the previous remark, it would be more interesting to consider a boundary datum in $\mathcal{Q}_{\min}$, where $\mathcal{Q}_{\min} \simeq \mathbb{R}P^2$ is defined in (1.1.9), which do not induced singularities *a priori* for topological reasons. Such boundary data look also more physical, being at the bottom of the potential well. In this case, split solutions are analogous to the *large solutions* considered in [20].

8.4 A family of boundary data whose minimizers are biaxial torus solutions with singularities

Matching Theorem 8.2 and Theorem 8.3 it is easy to prove that one can construct boundary data whose minimizers are biaxial torus solutions in a subregion of B_1 and have singularities outside.

Theorem 8.4. *There exists S^1-equivariant boundary data $Q_b \in C^\infty(S^2, S^4)$ whose corresponding S^1-equivariant minimizers of the LdG energy* (1.1.3) *are biaxial torus solutions in a subregion of B_1 and have singularities outside.*

Proof. Let $s \in [0,1]$. Define

$$Q_b^s(\theta,\phi) = \begin{cases} \omega\left(\pi - \frac{\theta}{s}, \phi\right), & \text{if } 0 \le \theta < \pi s, \\ e_0, & \text{if } \pi s \le \theta \le \frac{\pi(1-s)}{2} \\ H\left(\frac{1}{s}\left(\theta - \frac{\pi(1-s)}{2}\right), \phi\right), & \text{if } \frac{\pi(1-s)}{2} < \theta < \frac{\pi(1+s)}{2}, \\ e_0, & \text{if } \frac{\pi(1+s)}{2} \le \theta \le \pi. \end{cases} \tag{8.4.1}$$

Then $Q_b^s \in H^{\frac{1}{2}}(S^2, S^4)$ for any $s \in [0,1]$, $Q_b^0 \equiv e_0$ and $Q_b^1 \equiv H$. Further, $Q_b^s \rightharpoonup e_0$ as $s \to 0$.

For each $s \in [0,1]$, let Q^s be a minimizer of the LdG energy (1.1.3) w.r.t. to the boundary datum Q_b^s. The estimates in Theorem 8.2 and Theorem 8.3 show that $\{Q_s\}_{s\in[0,1]}$ is equibounded[1] in $W^{1,2}(B_1, S^4)$ and thus, by Theorem 8.1, there are $Q^* \in W^{1,2}(B_1, S^4)$ and a sequence s_j, $s_j \to 0$ as $j \to \infty$, such that $Q^{s_j} \to Q^*$ strongly in $W^{1,2}_{\rm loc}(B_1, S^4)$. By strong convergence, Q^* is minimizing. On the other hand, since the trace operator is weakly continuous, $\mathrm{tr}(Q^*) = e_0$ on S^2. Thus $Q^* = e_0$, because e_0 is clearly the unique minimizer w.r.t. its own boundary condition. Next, strong convergence yields small rescaled energy well in the interior of B_1, and thus, for j sufficiently large, the minimizers Q^{s_j} have only a finite number (at least one) of singularities near the poles.

Let D be the disk obtained slicing the ball with the plane $\{\phi = 0\}$, D^+ its portion on the right, say, of the z-axis. From the above, for sufficiently large j we can select a finite portion $\mathscr{A}^j$ of the z-axis which is free from singularities and a finite portion $\mathscr{S}^j$ of the semicircle closing D^+ on which $Q_b^{s_j}$ is uniaxial with identical lowest eigenvalues. By the strong L^2-convergence $Q^{s_j} \to e_0$, enlarging j if necessary, we can assume that the leading eigenvalue of Q^{s_j} is simple everywhere far away from the poles. Thus, we can connect the end-points $\mathscr{A}^j$ and $\mathscr{S}^j$ with two arcs so that, restricted to the curve $\mathscr{C}^j$ obtained this way, Q^{s_j} is uniaxial with identical lowest eigenvalues. Let $\mathscr{D}^j$ the domain encircled by $\mathscr{C}^j$. We can apply the semidisk argument of Section 5.1 to the restriction of Q^{s_j} to $\mathscr{D}^j$ thus getting the Q^{s_j} is also a biaxial torus solution in $\mathscr{D}^j$, in the sense of Definition 1.2. □

Let us comment on this result. Its interest relies mainly in the fact that it shows that biaxial torus solutions are compatible with singularities, although in numerical simulations [47, 84, 85, 137] they look as smooth solutions. In the paper [47], a special kind of singular solution (which, however, is only metastable) has been detected. This solution, called the *split core solution*, is S^1-equivariant with a (uniaxial) line of disclination on the z-axis with isotropic end-points. See also Section 2.5.7 for a quick description of this kind of solution. Thus, it resembles, at least qualitatively, the picture that may arise with the hedgehog boundary condition (remember that we have not been able to exclude $L_1 \oplus L_0$-valued tangent maps, which are also minimizing), hence giving a hint of the fact the singularities may be unavoidable. Our theorem may be superficially interpreted as a strengthening of the numerical suggestion. However, it must be remarked that the type of singularities that are generated in our context are **very different** from those of the split core solution or, more generally, from those that S^1-equivariant minimizers w.r.t. the hedgehog boundary condition may have.

To understand the difference, recall that by Theorem 7.10 the only allowed singularities for S^1-equivariant minimizers subject to the hedgehog boundary condition are dipoles. Thus, singularities have to come out in even number and have alternating degrees. On the contrary, in the present setting we can have both a even and an odd number of singularities and they need not, *a priori*, to have alternating degrees.

8.5 Generic uniqueness, nonuniqueness and related results

In this Section we derive some easy consequences of boundary regularity and the facts that singularities are isolated and their tangent approximations well-known. The

[1] At this point, we can smoothing a bit the θ-dependence in Q_b^s, thus getting smooth boundary data while keeping equiboundedness.

following results are counterparts of established theorems in the context of harmonic maps from B_1 into S^2.

We begin by showing that the set of S^1-equivariant boundary data having unique S^1-equivariant minimizer is dense in $H^{\frac{1}{2}}(S^2, S^4)$, a fact proven for (nonsymmetric) harmonic maps from B_1 into S^2 in [2, Theorem 4.1].

Theorem 8.5 (Generic uniqueness)**.** *Let $Q_b \in C^\infty(S^2, S^4)$ be S^1-equivariant and let $Q \in \mathcal{A}^{ax}_{Q_b}$ be a minimizer of $E(\cdot; B_1)$ in $\mathcal{A}^{ax}_{Q_b}$. For any $0 < \lambda < 1$, define $Q^\lambda : B_1 \to S^4$ and $Q^\lambda_b : S^2 \to S^4$ setting*

$$\begin{aligned} Q^\lambda(x) &:= Q(\lambda x), \quad x \in B_1, \\ Q^\lambda_b(y) &:= Q(\lambda y), \quad y \in S^2. \end{aligned}$$

Define also scaled functionals $\mathcal{E}^\lambda(\cdot; B_1)$ setting, for $P \in \mathcal{A}^{ax}_{Q^\lambda_b}$,

$$\mathcal{E}^\lambda(P; B_1) := \int_{B_1} L\,|\nabla P|^2 + \lambda^2 F(P)\, dx. \tag{8.5.1}$$

Then, for any $\lambda \in (0,1)$, Q^λ is the unique minimizer of $\mathcal{E}^\lambda(\cdot; B_1)$ in the class $\mathcal{A}^{ax}_{Q^\lambda_b}$. Moreover, $Q^\lambda_b \to Q_b$ strongly in $H^{\frac{1}{2}}(S^2, S^4)$.

Proof. By definition of $\mathcal{E}^\lambda(\cdot; B_1)$ and Q^λ, we have

$$\mathcal{E}^\lambda(Q^\lambda; B_1) = \frac{1}{\lambda} E(Q; B_\lambda).$$

The key point is now that Q is the unique minimizer of $E(Q; B_\lambda)$ subject to $Q^\lambda_b = Q|_{S^2_\lambda}$. Indeed, suppose $\tilde{Q}$ is another minimizer in B_λ such that $\tilde{Q}|_{S^2_\lambda} = Q^\lambda_b$. Define

$$Q^* = \begin{cases} \tilde{Q} & \text{in } B_\lambda, \\ Q & \text{in } B_1 \setminus B_\lambda. \end{cases}$$

Then Q^* is a minimizer of $E(\cdot; B_1)$ in $\mathcal{A}^{\mathrm{ax}}_{Q_b}$. Indeed,

$$E(Q^*; B_1) = E(\tilde{Q}; B_\lambda) + E(Q; B_1 \setminus B_\lambda) = E(Q; B_\lambda) + E(Q; B_1 \setminus B_\lambda) = E(Q; B_1).$$

Being Q, Q^* minimizers of $E(\cdot; B_1)$, they must be real-analytic in the interior of B_1 out of a finite set of isolated singular points, by results in Chapter 7. On the other hand, we have $Q = Q^*$ in $B_1 \setminus B_\lambda$ and hence everywhere by analytic continuation. Turning back to Q^λ, we then see that it must be the unique minimizer of $\mathcal{E}^\lambda(\cdot; B_1)$ w.r.t. Q^λ_b.

Regard to the second assertion, observe that, being $Q_b \in C^\infty(S^2, S^4)$ (C^3 would be enough actually, as in [2]), we have that there is $\delta > 0$ so that $Q \in C^\infty(\mathcal{O}_\delta, S^4)$, where $\mathcal{O}_\delta$ is a δ-neighborhood of S^2. Taking $\lambda \in (1-\delta, 1)$, we have that the family of boundary traces $\{Q^\lambda_b\}_\lambda$ depends in a C^1-way on λ and hence the conclusion follows immediately. ☐

Thanks to the fact that we know that the only nonconstant minimizing tangent maps are of the type $\left(0_{L_2}, \frac{x}{|x|}\right)$, we can rephrase almost word-by-word the argument by Almgren & Lieb [2, Theorem 1.8] to prove

Theorem 8.6 (Singular points converge to singular points)**.** *Suppose that* $(Q_i)_i \subset W^{1,2}(B_1, S^4)$ *is a sequence of* S^1*-equivariant minimizers of* $E(\cdot; B_1)$ *in their own class converging strongly in the* $W^{1,2}$*-sense to* $Q \in \mathcal{A}^{ax}_{Q_b}$*. Then*

(a) If y_i *is a singular point for* Q_i *and we have* $y_i \to y$ *in* B_1*, then* y *is a singular point for* Q*.*

(b) If $y \in B_1$ *is a singular point for* Q *then, for all sufficiently large* i*,* Q_i *has a singular point at some* y_i*, with* $y_i \to y$ *in* B_1*.*

Proof. Let $\mathcal{E}_{r,y}(Q) := \frac{1}{r}\int_{B_r(y)} e(\nabla Q, Q)\,\mathrm{d}x$, with $e(\nabla Q, Q)$ as in (1.1.4), the rescaled energy of Q in $B_r(y)$. If y is not a singular point of Q, then $\mathcal{E}_{r,y}(Q) < r$ for r sufficiently small. On the other hand, by monotonicity, $\mathcal{E}_{r,y}(Q_i) \geq r$. Since the convergence $Q_i \to Q$ is strong, we get a contradiction.

To prove the second assertion, observe that we know that all possible nonconstant tangent maps are of the form $\left(0_{L_2}, \frac{x}{|x|}\right)$, up to an isometry, and hence there is a smooth retraction Π on S^2 so that $\Pi \circ Q$ maps small spheres centered at y onto S^2 with topological degree ± 1. Since the same property must hold for Q_i for all sufficiently large i (by strong convergence), this implies the existence of discontinuities for the Q_i's inside the same spheres. □

We can also extend the theorem on uniform distance between singularities, [2, Theorem 2.1], to get

Theorem 8.7 (Uniform distance between singularities)**.** *Let* $Q_b \in H^{\frac{1}{2}}(S^2, S^4)$ *and let* $Q \in \mathcal{A}^{ax}_{Q_b}$ *be a minimizer in of the LdG energy* (1.1.3) *in the class* $\mathcal{A}^{ax}_{Q_b}$*. Suppose* $y \in B_1$ *is a singular point of* Q *and let* d *denote the distance from* y *to* S^2*. Then there is a universal constant* C *such that there is no other singularity within distance* Cd *of* y*.*

Proof. The argument goes exactly as in [2, Theorem 2.1]. We can assume $y = 0$. Suppose, for a contradiction, that we can find a sequence of minimizers Q^j with singularities at 0 and at x^j, with $x^j \to 0$. Dilate the balls so that $|x^j| = 1$. In the new ball of radius 2, in dilate coordinates, the energy is uniformly bigger than 8π. Indeed, if uniformity were false, by compactness we would have equality. This means there is no energy outside the two balls of radius 1 centered at 0 and at x^j, and hence Q^j would be constant by monotonicity. Again by monotonicity, the energy is greater than $4\pi r$ for all r bigger than 2. We now choose a strongly convergent subsequence of the Q^j's in the original ball to a minimizer $Q \in \mathcal{A}^{\mathrm{ax}}_{Q_b}$ with a singularity at y. Obviously, y is isolated and its corresponding tangent map is of the form $\left(0_{L_2}, \frac{x-y}{|x-y|}\right)$ by results in Chapter 7. By strong convergence, we can find a sufficiently large j and a suitable smooth retraction Π so that incoming maps Q^j retract on S^2 by means of Π and thus for the maps $\Pi \circ Q^j$ the origin is a singularity of degree 1. It then follows that in small balls around the origin the energy is close as we please to $4\pi r$. But, undoing the above dilation, we see that the energy must be uniformly bigger than $4\pi r$ for small r. This contradiction proves the theorem. □

Before going on, let us establish, for the sake of clarity, the following definition, already mentioned in the Introduction.

Definition 8.1 (Split solution)**.** Let $Q_b \in C^\infty(S^2, S^4)$ be S^1-equivariant and let Q be a minimizer of the LdG energy (1.1.3) in the class $\mathcal{A}^{\mathrm{ax}}_{Q_b}$. We say that Q is a *split* *solution* if the singular set of Q, $\operatorname{sing} Q$, consists of a finite number of dipoles; i.e., it contains an even number of singularities of the type $\left(0_{L_2}, \pm\frac{x}{|x|}\right)$ so that the sum of their degrees (as maps from S^2 into $S^2 \subset L_1 \oplus L_0$) is zero.

Next, we prove that along smooth curves connecting boundary data whose minimizers are all biaxial torus solutions at one end and all split solutions at the other end there exists a boundary data having both types as minimizer. We follow the line of [107], whose authors extend an argument due to Hardt & Lin [65].

Theorem 8.8. *Suppose that $\{Q_b^s\}_{s\in[0,1]}$ is a smooth curve of S^1-equivariant boundary data $Q_b^s \in C^\infty(S^2, S^4)$ so that all S^1-equivariant minimizers subject to the boundary condition Q_b^0 are biaxial torus solutions in B_1 and all S^1-equivariant minimizers subject to Q_b^1 are split solutions. Then there exists $\sigma \in (0,1)$ so that Q_b^σ serves as a boundary condition for at least two S^1-equivariant minimizers in $\mathcal{A}^{ax}_{Q_b^\sigma}$, one of which is a smooth biaxial torus solution in B_1 and the other a split solution.*

Proof. Let $\{Q_b^s\}_{s\in[0,1]}$ as in the statement. Let

$$\begin{aligned}\sigma = \sup\{s \in [0,1] :\ & \text{each LdG energy minimizer} \\ & \text{with boundary data } Q_b^s \text{ is a biaxial torus solution in } B_1\}.\end{aligned}$$

We may choose a sequence $s_i \nearrow \sigma$ and a sequence of S^1-equivariant minimizers Q^{s_i} that are biaxial torus solutions in B_1 such that $Q^{s_i}|_{S^2} = Q_b^{s_i}$. Similarly, we choose a sequence $t_i \searrow \sigma$ and a sequence $V^{t_i} \in W^{1,2}(B_1, S^4)$ of minimizers having at least two singularities, with $V^{t_i}|_{S^2} = Q_b^{t_i}$. Passing to subsequences (not relabeled), we have that there exist $Q \in W^{1,2}(B_1, S^4)$ and $V \in W^{1,2}(B_1, S^4)$ so that $Q^{s_i} \to Q$ strongly in $W^{1,2}$ and $V^{t_i} \to V$ again strongly in $W^{1,2}$. Moreover, we have $Q|_{S^2} = V|_{S^2} = Q_b^\sigma$.

Note that $0 < \sigma < 1$. Indeed, in view of strong convergence, if $\sigma = 0$, then Q_b^0 must have a minimizer of split type, a contradiction. In a similar fashion, if $\sigma = 1$, then Q_b^1 must have a minimizer which is a biaxial torus solution in B_1 and this is again in contrast with the fact that all minimizers associated to Q_b^1 are split.

By the very definition of σ, Q is a biaxial torus solution. In particular, Q cannot have singularities. Indeed, assume that Q has singularities. Then, by strong convergence, in arbitrary small balls around each singularity of Q there would be a singularity of Q^{s_i} for i sufficiently large, a contradiction.

We now prove that the map V has at least two singularities. Indeed, any V^{t_i} has at least two singularities and we know, by Theorem 8.6, that singular points converge to singular points. We need only to check that singularities cannot merge in the limit nor escape to the boundary. To see this, we first observe that they cannot merge in the interior because they must stay at uniform distance by Theorem 8.7. Next, we note that they cannot escape to the boundary: indeed, since $Q_b^{t_i}$ and Q_b^σ are close to each other in C^∞ for i sufficiently large, we can find a uniform neighborhood of the boundary which contains no singularities of V and of all V^{t_i} sufficiently close to V. ☐

Remark 8.5.1. We are not assured of the existence of a smooth curve as in the statement of Theorem 8.8 when the domain is B_1. The family of boundary data crafted in Theorem 8.4 is not suitable to the purpose (though it shows that boundary data with the properties required to Q_b^0 actually exist), because it is not smooth enough. Smoothness w.r.t. to the parameter ensures enough uniform boundary regularity in order to prevent the escaping of singularities to the boundary as $t_i \searrow \sigma$ and hence to conclude that the map V is again a split solution.

With reference to Remark 8.5.1, there are two main difficulties in building a smooth curve of boundary data with the required properties on the sphere. The first one is connected with the geometry of the sphere, which makes it difficult to understand how to manage the few boundary data whose minimizers have a known character (i.e., biaxial or split) to produce the desired behavior. The second difficulty, actually intertwined with the first one, is due to the competition between the elastic and the bulk term in the LdG energy; specifically, it not clear whether one of the two terms should be dominant looking only at the boundary condition and hence if the onset of singularities would be a way to save energy.

Anyway, choosing suitably an axially symmetric domain and the material constants, the picture simplifies considerably. We now explain how to do this so that a smooth curve as in Theorem 8.8 exists. The construction is analogous to that leading to well-known gap phenomena in harmonic maps [64, 63]. To better understand how the material parameters should be related to the purpose, let us return to the LdG energy (1.1.3) and, for a moment, discard the constraint $|Q|^2 = 1$ as well as the convention of having subtracted to $F(Q)$ its infimum value (see the Introduction). The norm of the Q-tensors belonging to $\mathcal{Q}_{\min}$ is then $|Q|^2 = \frac{2s_+^2}{3}$, where s_+ is given in (1.1.8). Scaling Q so that $Q = \sqrt{\frac{2}{3}}s_+\bar{Q}$ and subtracting $\inf F(Q)$ to the energy density (1.1.4), we have

$$E(Q;\Omega) - E_{\mathrm{eq}} = s_+^2 \int_\Omega \frac{L}{2}\left|\nabla\bar{Q}\right|^2 + \frac{a}{6}\left(1-\left|\bar{Q}\right|^2\right)^2 + \frac{bs_+}{54}\left(1+3\left|\bar{Q}\right|^4 - 4\sqrt{6}\,\mathrm{Tr}\,\bar{Q}^3\right)\mathrm{d}x.$$

Let $D = \operatorname{diam}\Omega$ and rescale also the domain so that we have new variables $\bar{x} = \frac{x}{D}$ on $\bar{\Omega}$, with $\operatorname{diam}\bar{\Omega} = 1$. Then, we further impose the norm constraint $\left|\bar{Q}\right| = 1$ and hence we reduce to

$$I(Q;\Omega) := \frac{E(Q;\Omega) - E_{\mathrm{eq}}}{LDs_+} = \int_{\bar{\Omega}} \frac{1}{2}\left|\nabla\bar{Q}\right|^2 + \frac{2bs_+}{27L}\left(1-\sqrt{6}\,\mathrm{Tr}\,\bar{Q}^3\right)\mathrm{d}x. \tag{8.5.2}$$

Set

$$\kappa := \frac{2bs_+D^2}{27L}. \tag{8.5.3}$$

We will be interested in the regime κ small, as we shall see below.

Theorem 8.9. *There exists an axially symmetric domain $\Omega \subset \mathbb{R}^3$ and constants $L > 0, b > 0$ so that a smooth curve of boundary functions $[0,1] \ni s \mapsto Q_b^s \in C^\infty(\Omega, S^4)$, having the properties listed in the statement of Theorem 8.8 can be found.*

Proof. Let us consider a capsule-shaped open domain Ω, coaxial with the z-axis, whose horizontal section has radius ε and whose height is $2N$, where $N \gg \varepsilon$ will be determined later. Let's place the barycenter of Ω at the origin.

To construct the required family of boundary functions, let us first write $\Omega = \Omega^+ \cup \Omega^-$, where $\Omega^+ = \{(x,y,z) \in \Omega : z > 0\}$ and $\Omega^- = \{(x,y,z) \in \Omega : z \leq 0\}$. For $s = 0$, on $\partial\Omega^+ \setminus \{(x,y,z) \in \Omega : z = 0\}$ we put $Q_b^0 = e_0$. Let P a plane through the z-axis and, for instance, the point $(\varepsilon, 0, -N/2) \in \partial\Omega^-$. Along $(P \cap \partial\Omega^-) \setminus \{(x,y,z) \in \Omega : z = 0\}$, in a small region of size λ around the plane $\{z = -N/2\}$, we let the leading eigenvalue of Q_b^s performing the nontrivial path in $\mathbb{R}P^2$. We then take $Q_b^s \equiv e_0$ on the remaining part of $(P \cap \partial\Omega^-) \setminus \{(x,y,z) \in \Omega : z = 0\}$ and then we define Q_b^s everywhere on $\partial\Omega^- \setminus \{(x,y,z) \in \Omega : z = 0\}$ by imposing S^1-equivariance. Taking λ sufficiently close to zero, we can reason as in Theorem 8.2 and then conclude that any S^1-equivariant minimizers of $E(\cdot;\Omega)$ subject to Q_b^0 must be a biaxial torus solution in Ω.

Now, for $s > 0$, we let Q_b^s as in the above in the bottom-half of $\partial\Omega$. Let P the same plane as before. In the upper-half of $P \cap \partial\Omega$, choose A, B points away from the plane $\{z = 0\}$ and from $(0,0,N)$, with $\mathrm{dist}_{P\cap\partial\Omega}(A,B) = \Lambda$. We let Q_b^s start twisting smoothly at A; at any step in s, Q_b^s twists a little more along the path from A to B, in order to become $-e_0$ in the limit $s \nearrow 1$. At the point $A'(s)$, Q_b^s stop twisting and then we keep the twisted value from $A'(s)$ to a point $B'(s)$, where it begin twisting back to return e_0 in B. We choose N and Λ so large that we can choose $A'(s)$ and $B'(s)$ so that twisting requires only a controlled energy, less than some fixed constant. We then define Q_b^s on the whole of $\partial\Omega^+$ by requiring S^1-equivariance.

Suppose s is close to 1 and let Q^s be a S^1-equivariant minimizer of $E(\cdot;\Omega)$ subject to Q_b^s. If Q^s was smooth, then it would be e_0 everywhere along the z-axis, due to our choice of the boundary condition. But the boundary condition also forces Q^s to be $-e_0$ on a large portion of size Λ of the upper boundary $\partial\Omega^+$. Because of this, the I-energy of Q^s should roughly pay[2] more than $C\Lambda\varepsilon^{-1}/N$, with $C > 0$ a constant, which can be approximated by $C\varepsilon^{-1}$, up to enlarging and adjusting N and Λ, if necessary. Since the potential is bounded because of the Lyuksyutov constraint, if we take κ sufficiently small, then the I-energy of Q is a small perturbation of its (rescaled) Dirichlet energy. Then, scaling back, one can follow the line of [63] and construct a map $\tilde{Q}$ which is identically $-e_0$ on the large central region and that, when N is sufficiently larger than ε, pays less energy than any continuous mapping with the same boundary trace. Thus, if Q^s is a minimizer, it has to be discontinuous, and hence it must be a split solution, because the boundary condition agrees with e_0 at $(0,0,N)$ and at $(0,0,-N)$.

Thus, the curve $\{Q_b^s\}_{s\in[0,1]}$ has the required properties. □

Remark 8.5.2. We are not claiming anything about the physical significance of Theorem 8.9, in the sense that we do not know whether materials with suitable values of the constant exist so that the phenomenon described above actually happens.

Remark 8.5.3. A key point in the construction in Theorem 8.9 is the fact the bulk potential is bounded and this allows for tuning the parameters in order to reduce to a situation analogous to that of axially symmetric harmonic maps (cfr. [63, Theorem 3.2]). The rôle played by $\frac{bD^2}{L}$ and the fact that the boundary datum goes also outside $\mathbb{R}P^2$ arise the question whether the same phenomenon continues to happen, at least under the Lyuksyutov constraint, in the more interesting situation when boundary datum is all $\mathbb{R}P^2$-valued. Keeping in mind how $\mathbb{R}P^2$ embeds into S^4, it is easily realized that this should be the case for boundary data — on domains such as Ω in Theorem 8.9 — that are closer to $-e_0$ than to e_0 on a large part of $\partial\Omega$. In this case, smooth solutions, being everywhere e_0 on the z-axis, must be *large* as the onset of singularities becomes energetically convenient when $\frac{bD^2}{L}$ is small enough. Anyway, details are more delicate and will be developed elsewhere.

Remark 8.5.4. To be rigorous, we need boundary regularity and, to state it in the case of a capsule-like domain, we should take Remark 4.8.2 into account. The claimed

[2]The important point here is the linear growth with Λ, cfr. [57].

issues can be still handled quite easily but details will be worked out in a future publication.

Remark 8.5.5. In connection to the above remarks, nothing excludes, at the moment being, that the hedgehog boundary condition on the sphere is critical, in the sense of having both biaxial torus minimizers and split minimizers.

Appendix A

The norm of the gradient of a S^1-equivariant linearly full harmonic sphere $S^2 \to S^4$

Thanks to our classification theorem, Theorem 6.2, we know how all S^1-equivariant linearly full harmonic spheres $\omega : S^2 \to S^4$ look like in terms of the complex parameters μ_3, μ_4. This allows us to calculate $|\nabla_T \omega|^2$ in terms of μ_3, μ_4.

We start noting that, in terms of coordinates (θ, ϕ) (as usual, θ is the polar angle and ϕ the azimuthal angle) on S^2, we have from (6.2.17)

$$
\begin{aligned}
(\tau \circ \tilde{\omega})(\theta, \phi) = \frac{1}{\mathscr{D}} \Bigg(& 2\mu_3 \tan^2\left(\frac{\theta}{2}\right) e^{2i\phi} \left(1 + \frac{|\mu_4|^2}{3} \tan^2\left(\frac{\theta}{2}\right)\right), \\
& 2\mu_4 \tan\left(\frac{\theta}{2}\right) e^{i\phi} \left(1 - \frac{|\mu_3|^2}{3} \tan^4\left(\frac{\theta}{2}\right)\right), \\
& 1 + \frac{|\mu_3|^2 |\mu_4|^2}{9} \tan^6\left(\frac{\theta}{2}\right) - |\mu_3|^2 \tan^4\left(\frac{\theta}{2}\right) - |\mu_4|^2 \tan^2\left(\frac{\theta}{2}\right) \Bigg)
\end{aligned}
\tag{A.1}
$$

and

$$
\mathscr{D} = 1 + \frac{|\mu_3|^2 |\mu_4|^2}{9} \tan^6\left(\frac{\theta}{2}\right) + |\mu_3|^2 \tan^4\left(\frac{\theta}{2}\right) + |\mu_4|^2 \tan^2\left(\frac{\theta}{2}\right)
$$

Then, long but elementary calculations show that

$$\partial_\theta \omega_0 = -\frac{\tan\left(\frac{\theta}{2}\right)}{\mathscr{D}^2\cos^2\left(\frac{\theta}{2}\right)}\left(2\left|\mu_4\right|^2 + 4\left|\mu_3\right|^2\tan^2\left(\frac{\theta}{2}\right) - \frac{4}{9}\left|\mu_3\right|^2\left|\mu_4\right|^4\tan^6\left(\frac{\theta}{2}\right) - \frac{2}{9}\left|\mu_3\right|^4\left|\mu_4\right|^2\tan^8\left(\frac{\theta}{2}\right)\right)$$

$$\partial_\theta \omega_1 = \frac{\mu_4 e^{i\phi}}{\mathscr{D}^2\cos^2\left(\frac{\theta}{2}\right)}\left(1 - \left|\mu_4\right|^2\tan^2\left(\frac{\theta}{2}\right) - \frac{14}{3}\left|\mu_3\right|^2\tan^4\left(\frac{\theta}{2}\right) - \frac{14}{9}\left|\mu_3\right|^2\left|\mu_4\right|^2\tan^6\left(\frac{\theta}{6}\right)\right.$$
$$\left. -\frac{1}{3}\left|\mu_3\right|^4\tan^8\left(\frac{\theta}{2}\right) + \frac{1}{27}\left|\mu_3\right|^4\left|\mu_4\right|^2\tan^{10}\left(\frac{\theta}{2}\right)\right)$$

$$\partial_\theta \omega_2 = \frac{\mu_3 e^{2i\phi}\tan\left(\frac{\theta}{2}\right)}{\mathscr{D}^2\cos^2\left(\frac{\theta}{2}\right)}\left(2 + \frac{4}{3}\left|\mu_4\right|^2\tan^2\left(\frac{\theta}{2}\right) - 2\left(\left|\mu_3\right|^2 - \frac{1}{3}\left|\mu_4\right|^4\right)\tan^4\left(\frac{\theta}{2}\right) - \frac{4}{9}\left|\mu_4\right|^2\tan^6\left(\frac{\theta}{2}\right)\right.$$
$$\left. -\frac{2}{27}\left|\mu_3\right|^2\left|\mu_4\right|^4\tan^8\left(\frac{\theta}{2}\right)\right).$$

We also have

$$\left|\partial_\phi \omega\right|^2 = 4\left|\omega_2\right|^2 + \left|\omega_1\right|^2 \tag{A.2}$$

so that

$$\begin{aligned}
\left|\nabla_T \omega\right|^2 = \frac{2}{\mathscr{D}^4\cos^4\left(\frac{\theta}{2}\right)} & \Big\{\left|\mu_4\right|^2 \\
& + 2\left(\left|\mu_4\right|^4 + 2\left|\mu_3\right|^2\right)\tan^2\left(\frac{\theta}{2}\right) \\
& + \left|\mu_4\right|^2\left(12\left|\mu_3\right|^2 + \left|\mu_4\right|^4\right)\tan^4\left(\frac{\theta}{2}\right) \\
& + 8\left|\mu_3\right|^2\left(\left|\mu_3\right|^2 + \frac{4}{3}\left|\mu_4\right|^4\right)\tan^6\left(\frac{\theta}{2}\right) \\
& + 14\left|\mu_3\right|^2\left|\mu_4\right|^2\left(\left|\mu_3\right|^2 + \frac{2}{9}\left|\mu_4\right|^4\right)\tan^8\left(\frac{\theta}{2}\right) \\
& + 4\left|\mu_3\right|^2\left(\frac{5}{3}\left|\mu_3\right|^2\left|\mu_4\right|^4 + \left|\mu_3\right|^4 + \frac{1}{9}\left|\mu_4\right|^8\right)\tan^{10}\left(\frac{\theta}{2}\right) \\
& + \frac{14}{9}\left|\mu_3\right|^4\left|\mu_4\right|^2\left(2\left|\mu_3\right|^2 + \left|\mu_4\right|^4\right)\tan^{12}\left(\frac{\theta}{2}\right) \\
& + \frac{8}{81}\left|\mu_3\right|^4\left|\mu_4\right|^4\left(12\left|\mu_3\right|^2 + \left|\mu_4\right|^4\right)\tan^{14}\left(\frac{\theta}{2}\right) \\
& + \frac{1}{9}\left|\mu_3\right|^6\left|\mu_4\right|^2\left(\left|\mu_3\right|^2 + \frac{4}{3}\left|\mu_4\right|^4\right)\tan^{16}\left(\frac{\theta}{2}\right) \\
& + \frac{2}{81}\left|\mu_3\right|^6\left|\mu_4\right|^4\left(\left|\mu_3\right|^2 + \frac{2}{9}\left|\mu_4\right|^4\right)\tan^{18}\left(\frac{\theta}{2}\right) \\
& + \frac{1}{27^2}\left|\mu_3\right|^8\left|\mu_4\right|^6\tan^{20}\left(\frac{\theta}{2}\right)\Big\}.
\end{aligned} \tag{A.3}$$

Note that $\frac{1}{\cos^4\left(\frac{\theta}{2}\right)} = \left(1 + \tan^2\left(\frac{\theta}{2}\right)\right)^2$. In particular, taking the hedgehog, i.e., setting $\mu_3 = \mu_4 = \sqrt{3}$, an amazing simplification happens and we are left with $\left|\nabla_T H\right|^2 = 6$ (of course, this what we expected and knew from the SO(3)-equivariance of the hedgehog).

Remarkably, the following equipartition-property, verified by direct calculation, holds, due to the S^1-equivariance:

$$|\partial_\theta \omega|^2 = \frac{1}{\sin^2(\theta)} \, |\partial_\phi \omega|^2 \tag{A.4}$$

Appendix B

General properties of S^1-equivariant harmonic spheres

S^1-equivariant harmonic spheres, arising in this work in connection to S^1-equivariant tangent maps from $\mathbb{R}^3$ into S^4, are actually a topic of independent interest, see for instance [48, 50]. Thus, a detailed knowledge of their structure may be worth also beyond the scope of the present work. Below we expose some general features we have found in our study of the stability/instability issue of S^1-equivariant tangent maps. Some of them are reported even if not used, at last, to derive our stability/instability results because expressions of a somewhat surprisingly peculiar behaviour induced by S^1-equivariance.

Our classification theorem, Theorem 6.2, allow us to retrieve interesting pieces of information about S^1-equivariant harmonic spheres. Of course the most important one for our purposes is the form of the energy density $|\nabla\omega|^2$. This may be calculated in terms of the parameters μ_3, μ_4 of Theorem 6.2; the expression we have found is given by Eq. (A.3). It is remarkable that a unexpected **equipartition property holds**: indeed, we have

$$\boxed{|\partial_\theta \omega|^2 = \frac{1}{\sin^2\theta}\, |\partial_\phi \omega|^2 \,.} \tag{A.4}$$

In particular, we can write

$$|\nabla\omega|^2 = \frac{8\omega_2^2 + 2\omega_1^2}{\sin^2\theta}. \tag{B.1}$$

Note also that the equipartition property implies

$$|\nabla\omega_0|^2 = |\partial_\theta \omega_0|^2 \leq \frac{1}{2}\, |\nabla\omega|^2 \tag{B.2}$$

and, integrating both sides and using (3.6.2),

$$\int_{S^2} |\nabla\omega_0|^2 \,\mathrm{dvol}_{S^2} = \int_{S^2} \omega_0^2\, |\nabla\omega|^2 \,\mathrm{dvol}_{S^2} < \frac{1}{2}\int_{S^2} |\nabla\omega|^2 \,\mathrm{dvol}_{S^2}. \tag{B.3}$$

Observe that equality is never achieved in (B.3).

Looking at the explicit expressions in Appendix A, it is easily realized that the above relations, although unexpected and apparently helpful, are too cumbersome to be really useful, unless in special cases. Among these, there are maps with values into $L_1 \oplus L_0$ or into $L_2 \oplus L_0$, respectively, and the linearly full map given by the hedgehog.

Recall that we are really interested only in stationary harmonic spheres. Thus, the first two of them are completely characterized by fundamental results in [21] and hence calculations are really explicit, see Section 7.1.3 and Section 7.1.4. The case of the hedgehog is instead tractable because of the additional SO(3)-equivariance, which implies that $|\nabla_T H|^2$ is constant and hence necessarily $|\nabla_T H|^2 = 6$. Alternatively, an easy check shows that the parameters of the hedgehog are $\mu_3 = \mu_4 = \sqrt{3}$ and from here the constancy of $|\nabla_T H|^2$ follows by comparison to (A.3). Moreover, the hedeghog is the only linearly full map, up to rotations, having constant energy density.

The above-quoted special cases may be developed in full details, although two of them are instances of the more general phenomenon in Theorem 7.8, because explicit computations may reveal details of independent interest.

The linearly full case is instead out of range of explicit calculations. For the argument of Theorem 7.8, there is no need to explore further features of S^1-equivariant harmonic spheres. Anyway, with an eye towards other possible applications, it may be interesting to recognize the following properties.

Having the literature in mind, especially [36, 95, 132], we see that the Bochner identity for harmonic maps between spheres, reported in (B.5), often plays an important rôle. Let us recall that, if $u \in C^\infty(M,N)$ is a smooth harmonic map, then we have the identity (see [34])

$$\Delta_M(|\nabla u|^2) = |\nabla(du)|^2 + \langle \mathrm{Ric}_M \nabla u, \nabla u\rangle + \langle \mathrm{Riem}_N(u)(\nabla u, \nabla u)\nabla u, \nabla u\rangle \,, \tag{B.4}$$

called the *Bochner identity* (or, sometimes, *Bochner formula*).

Now, let $m, k \geq 2$, $(e_\alpha)_{\alpha=1}^m$ be any local orthonormal frame of the sphere S^m and $\omega \in C^\infty(S^m, S^k)$. Then (B.4) specializes to (see [95])

$$\Delta\left(\frac{1}{2}|\nabla\omega|^2\right) = \left|\nabla^2\omega\right|^2 + (m-1)\,|\nabla\omega|^2 - \sum_{\alpha,\beta=1}^{m}\left\{|\nabla_{e_\alpha}\omega|^2\left|\nabla_{e_\beta}\omega\right|^2 - \left\langle\nabla_{e_\alpha}, \nabla_{e_\beta}\omega\right\rangle^2\right\}. \tag{B.5}$$

For S^1-equivariant harmonic spheres $S^2 \to S^4$, (B.5) simplifies to (B.6) in the Lemma below.

Lemma B.1. *Let $\omega \in C^\infty(S^2, S^4)$ a smooth S^1-equivariant harmonic map. Then ω satisfies the following Bochner identity*

$$\Delta\left(\frac{1}{2}|\nabla\omega|^2\right) = \left|\nabla^2\omega\right|^2 + |\nabla\omega|^2 - \frac{1}{2}|\nabla\omega|^4 \,. \tag{B.6}$$

Proof. The couple $\left\{\frac{\partial}{\partial\theta}, \frac{1}{\sin(\theta)}\frac{\partial}{\partial\phi}\right\}$ is a local orthonormal frame on S^2 and note that

$$\left\langle \partial_\theta\omega, \frac{1}{\sin\theta}\partial_\phi\omega\right\rangle = \frac{1}{\sin\theta}\left((\partial_\theta\omega_2)e^{2i\phi}, (\partial_\theta\omega_1)e^{i\phi}, \partial\omega_0\right)\cdot(2\omega_2 i e^{2i\phi}, \omega_1 i e^{i\phi}, 0) \equiv 0.$$

Thus, using the equipartition property, the last term at the r.h.s. in (B.5) reduces to $-\frac{1}{2}|\nabla\omega|^4$ (which is its minimum value, compare [95, Eq. (2.13)]). □

Remark. It may be interesting also to know explicitly the form of the terms involved in the r.h.s. of Eq. (B.6). We start noting that, since $|\nabla\omega|^2$, $|\nabla\omega|^4$ are only semi-explicit, we might think to compute $\left|\nabla^2\omega\right|^2$ and then to use it to simplify the r.h.s. of (B.6).

However, in doing this — in order to make the computation affordable — we used equipartition and (B.1) and the resulting equation turned out to be an identity, thus carrying no further information. A more direct approach looks not viable.

Because of the previous Lemma and the above remark, we did not succeed in improving the Bochner inequality of Lin & Wang [95, Eq. (2.11)], recalled below:

$$\Delta\left(\frac{1}{2}\left|\nabla\omega\right|^2\right) \geq 2\left|\nabla\left|\nabla\omega\right|\right|^2 + \left|\nabla\omega\right|^2 - \frac{1}{2}\left|\nabla\omega\right|^4, \tag{B.7}$$

for any nonconstant harmonic map $\omega \in C^\infty(S^2, S^4)$.

Next, we observe an elementary but possibly useful fact.

Lemma B.2. *Let $\omega : S^2 \to S^4$ be a S^1-equivariant harmonic sphere. Then*

$$\int_{S^2} \left|\nabla\omega\right|^2 \, dvol_{S^2} < \int_{S^2} \left|\nabla\omega\right|^4 \sin^2\theta \, dvol_{S^2}. \tag{B.8}$$

Proof. Since $|\omega|^2 = \omega_2^2 + \omega_1^2 + \omega_0^2 \equiv 1$, we have

$$\begin{aligned} \left|\nabla\omega\right|^2 &= (\omega_2^2 + \omega_1^2 + \omega_0^2)\left|\nabla\omega\right|^2 \\ &= (4\omega_2^2 + \omega_1^2)\left|\nabla\omega\right|^2 + (-3\omega_2^2 + \omega_0^2)\left|\nabla\omega\right|^2. \end{aligned}$$

By (A.4),

$$\left|\nabla\omega\right|^2 = \frac{1}{2}\left|\nabla\omega\right|^4 \sin^2\theta - 3\omega_2^2\left|\nabla\omega\right|^2 + \omega_0^2\left|\nabla\omega\right|^2.$$

Integrating both sides and using (3.6.3),

$$\begin{aligned} \int_{S^2} \left|\nabla\omega\right|^2 \mathrm{dvol}_{S^2} &= \frac{1}{2}\int_{S^2} \left|\nabla\omega\right|^4 \sin^2\theta \, \mathrm{dvol}_{S^2} + \int_{S^2} \left\{-3\left|\nabla\omega_2\right|^2 + \left|\nabla\omega_0\right|^2\right\} \mathrm{dvol}_{S^2} \\ &\leq \frac{1}{2}\int_{S^2} \left|\nabla\omega\right|^4 \sin^2\theta \, \mathrm{dvol}_{S^2} + \int_{S^2} \left|\nabla\omega_0\right|^2 \mathrm{dvol}_{S^2} \end{aligned}$$

and from here the conclusion follows using (B.3). □

Remark. In the spirit of being always explicit whenever possible, we made many attempts to derive instability of linearly full maps appealing to equipartition, (B.3), (B.8), (B.6) and to relations (3.6.2), (3.6.3) (etc. . .). Essentially, the line of reasoning was trying to replicate the proof of [95, Proposition 2.5], taking care of the fact that equivariance requires the use of appropriately weighted versions of (B.6), in order to get rid of divergent terms near the poles (whose appearance is forced by the equivariance itself, see e.g. Eq. (7.1.11)). In this direction, we obtained some hints but not decisive arguments. This study generated the results reported above.

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