SAPIENZA
UNIVERSITÀ DI ROMA

Quantum vertex algebras

Vito Volterra Doctoral School

Dottorato di Ricerca in Matematica – XXXI Ciclo

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October, 31 2018

Thesis defended on January 29, 2019
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Quantum vertex algebras

Ph.D. thesis. Sapienza – University of Rome

This thesis has been typeset by LaTeX and the Sapthesis class.

Version: October, 31 2018

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Acknowledgments

First of all, I am extremely grateful to my thesis advisor, Alberto De Sole. Thank you for your patience and your clarity. Thank you so much for helping me to take the things that I know and helping me to write them down in a clearer way. Thank you for teaching me not to be afraid of computations and for having had trust in me when I haven't and for having been in the department till late with me so many times to work together.

I want to thank Victor Kac who, jointly with Alberto, gave me my research topic. It's completely clear, at least to me, that this work would not have been possible without your intuition and your advices: you saw things that I couldn't see and I worked in the certainty you were right. I am so glad to have had the chance to work with you and to have you as my teacher. Thank you also for having invited me at M.I.T. where I have learnt so much mathematics.

I want to address special thanks to Pavel Etingof who showed me one of the beautiful pages of mathematics that I have ever seen: the Drinfeld's notes on universal triangular R-matrices. Thank you for being so patient when I came repeatedly to your office (some days more than two or three times) full of things unclear to me but extremely clear to you. I am not one of your Ph.D. students but you make me feel one of them. I have no words to thank you in an appropriate way for the passion in doing mathematics that you inspired in me.

I also want to thank Corrado De Concini for all the talks and discussions we had together.

I want also to thank Nicoletta Cantarini. Thank you for introducing me to algebra and for suggesting to me to apply for the Ph.D. in Rome 3 years and a half ago. Without your suggestion, I would have only applied in Bologna and I would have never lived these wonderful three years.

I want to thank my Ph.D. mates: Stefano, Federico, Stefano, Massimo, Erika. And again Francesco, Pier Paolo, Alfie, Marco, Daniele, Lorenzo and Luca. I'll miss you all. As you may see, there are two missing Ph.D. students of the XXXI cycle: Veronica and Luca. I want to thank you out of the the group because we have lived more events together and you deserve a special thank you. I want to start from you Veronica, my academic sister, thank you for all the conferences we have had together (Pisa, Melbourne, Vienna, Split), thank you for waiting for me till late when we were in Bures-sur-Yvette, or for our phone calls from one part of the world (Cambridge, MA, USA where I was) and the other part of the world (Kyoto, Japan where you were), thank you for speaking English when I didn't want to and thank you for all the times we laughed about our misfortunes.
I want also to thank you Luca for all the time we have spent together speaking about Mathematics or Informatics. Thank you also for all the Mickey Mouse's "lasagna classica" which we have eaten in every season and every climatic condition.

I want to thank Laura Fedele: thank you for being my old academic sister and my first friend in Rome. Thank you for helping me in little things, especially where we were abroad and I wasn't able to speak English (the situation is not changed but now I can manage better).

I want to thank Richard: the man with more passports in the world, but the most

American guy I have ever met. Thank you for welcoming me to the U.S.A., for all the Halal Guys' times we had, for that time at the cinema or in front of a beer or a tea. But first of all thank you for your friendship!

I want to thank Daniele Valeri, thank you for explaining to me a lot of things.

I want to thank Paolo and Sofia. Thank you for having been my friends for an infinite time. Thank you because every time we meet it's a day of celebration without any sentence of type "why do you never call or write?".

I want to thank in a particular way my brother Lorenzo, my mother Patrizia and my father Maurizio: thank you for being one pillar of my life. For supporting me every day even though I am far from home.

Last, but not least I want to thank Elisa: thank you for being like you are. You know you are the pillar of my life so I won't say this again. Thank you for every day together in the same place or in completely different places: you in Bologna and me in Rome or you here in Rome and me in Cambridge. Thank you for supporting me every day and for filling my head with other problems so as not to consider mine for a moment. Thank you for your constancy which is an example for me when I have no energy left. But, most importantly, thank you for keeping me with my feet in this earth and at the same time for making me fly. Without you nothing of this could have been and could be possible.

Contents

Introduction

Let $\mathbb{K}$ be a field of characteristic 0. A vertex algebra is a vector space V over the field $\mathbb{K}$ with a non-zero vector $|0\rangle$ (the vacuum vector), a linear map $T : V \to V$ (the translation operator), and a linear map $Y : V \to End_{\mathbb{K}}V[[z, z^{-1}]]$ satisfying the quantum field locality, the translation covariance and the vacuum axioms (cf. [K, Section 1.3]).

By definition, for any $a \in V$, the image under Y of a is a series in z and z^{-1} with coefficients in $End_{\mathbb{K}}V$ which is denoted with $Y(a, z) = \sum_{m\in\mathbb{Z}} a_{(m)}z^{-m-1}$ and which satisfies the property (quantum field axiom) that $Y(a, z)b$ has only a finite number of negative powers of z^{-1} with non-zero coefficient. Therefore $Y(a, z)b \in V((z))$, where $V((z)) = V[[z]][z^{-1}]$ denotes the vector space of Laurent series in z with coefficients in V. An analogue notation is to think of Y as a linear map

$$\begin{aligned} V \otimes V &\to V((z)) \\ a \otimes b &\mapsto Y(z)(a \otimes b) := Y(a, z)b. \end{aligned} \tag{0.1}$$

By thinking of Y as the map $Y : V \to End_{\mathbb{K}}V[[z, z^{-1}]]$, Y has a clear meaning in physical terms, as it is a map between states and quantum fields. On the other hand, thinking of Y as the map $Y : V \otimes V \to V((z))$, Y has an immediate analogue in the quantization of vertex algebras which is the main topic of the present thesis.

The locality axiom, in a vertex algebra, states that, for every $a, b \in V$, there exists $N = N(a, b) \geq 0$ such that

$$\begin{aligned} &(z - w)^N Y(z)(1 \otimes Y(w))(a \otimes b \otimes c) \\ &\qquad = (z - w)^N Y(w)(1 \otimes Y(z))(b \otimes a \otimes c) \end{aligned} \tag{0.2}$$

for any $c \in V$. Following [K, FBZ], one can prove that the following "Associativity Relation" holds for any vertex algebra: for any $a, b, c \in V$, there exists $N \geq 0$ such that

$$\begin{aligned} &(z + w)^N Y(w)(Y(z) \otimes 1)(a \otimes b \otimes c) \\ &\qquad = (z + w)^N i_{z,w} Y(z + w)(1 \otimes Y(w))(a \otimes b \otimes c). \end{aligned} \tag{0.3}$$

Here and below $i_{z,w}$ denotes the geometric series expansion for $|z| > |w|$.
Moreover, one has the "Skew-symmetry Relation":

$$Y(z)(a \otimes b) = e^{zT} Y(-z)(b \otimes a) =: Y^{op}(z)(a \otimes b). \tag{0.4}$$

The following proposition, known as Goddard's Uniqueness Theorem (cf. [G], [K, Theorem 4.4] or [FBZ, Theorem 3.1.1], see also Lemma 1.3.5), holds:

Proposition 0.0.1. *Let V be a vertex algebra and let $a \in V$ and $a(z) : V \to V((z))$ be a quantum field such that:*

1. $a(z)|0\rangle = Y(z)(a \otimes |0\rangle)$*;*
2. $a(z)$ *is local with* $Y(c, z)$ *for all* $c \in V$*.*

Then $a(z) = Y(z)(a \otimes -)$*.*

Following [BK], [FLM] and [LL] we give the following:

Definition 0.0.1. Let V be a vector space and let $|0\rangle \in V$ be a non-zero element (i.e. a pointed vector space [BK]). A *state-field correspondence* on V is a linear map $Y : V \otimes V \to V((z))$ such that

1. (vacuum axioms) $Y(z)(|0\rangle \otimes a) = a$, $Y(z)(a \otimes |0\rangle) = a + T(a)z + \ldots \in V[[z]]$, $(T \in End\ V)$;
2. (Translation Covariance) $TY(z) - Y(z)(1 \otimes T) = \partial_z Y(z)$;
3. $Y(z)(T \otimes 1) = \partial_z Y(z)$.

For $n \in \mathbb{Z}$, the *n-product* of the quantum fields $a(z), b(z) : V \to V((z))$ is:

$$a(z)_{(n)}b(z) = Res_x\big(a(x)b(z)i_{x,z}(x-z)^n - b(z)a(x)i_{z,x}(x-z)^n\big). \tag{0.5}$$

The *Jacobi Identity* for the state-field correspondence Y is the following formula:

$$\begin{aligned} & i_{z,w}\delta(x, z-w)Y(z)\big(1 \otimes Y(w)\big)(a \otimes b \otimes c) \\ & \quad -i_{w,z}\delta(x, z-w)Y(w)\big(1 \otimes Y(z)\big)(b \otimes a \otimes c) \\ & \quad\quad = i_{z,x}\delta(w, z-x)Y(w)\big(Y(x) \otimes 1\big)(a \otimes b \otimes c). \end{aligned} \tag{0.6}$$

(Here and below $\delta(z, w) = \sum_{k \in \mathbb{Z}} z^{-k-1}w^k$).
The *Borcherds Identities* on Y are the following formulas ($n \in \mathbb{Z}$):

$$\begin{aligned} & Y(z)\big(1 \otimes Y(w)\big)(a \otimes b \otimes c)i_{z,w}(z-w)^n \\ & \quad -Y(w)\big(1 \otimes Y(z)\big)(b \otimes a \otimes c)i_{w,z}(z-w)^n \\ & \quad\quad = \sum_{j \geq 0} Y(w)(a_{(n+j)}b \otimes c)\frac{\partial_w^j \delta(z, w)}{j!}. \end{aligned} \tag{0.7}$$

Then, the *n-products Identities* on the state-field correspondence Y are:

$$Y(a, z)_{(n)}Y(b, z) = Y(a_{(n)}b, z). \tag{0.8}$$

Here we switched to the equivalent notation $Y(a, z) \in End\ V[[z, z^{-1}]]$ described at the beginning and we let, as before,

$$a_{(n)}b = Res_z\big(z^n Y(z)(a \otimes b)\big). \tag{0.9}$$

With the previous definitions, one can prove the following characterization theorem collecting results from [BK], [K15] and [L03]:

Theorem 0.0.2. *Let $(V, |0\rangle)$ be a pointed vector space with a state-field correspondence Y. The following statements are equivalent:*

1. *V is a vertex algebra;*

2. *the Jacobi Identity* (0.6) *holds;*

3. *the Associativity Relation* (0.3) *holds (i.e. V is a* field algebra *[BK]) and $Y = Y^{op}$;*

4. *the Borcherds Identities* (0.7) *hold;*

5. *the n-products Identities* (0.8) *holds and $Y = Y^{op}$.*

As far as we know, the equivalences between $(1), (3), (4), (5)$ of the previous theorem are proved in [BK], [K15] and the equivalence between (1) and (2) are proved in [LL].

Let us spend some words on commutative vertex algebras. A commutative vertex algebra is a vertex algebra for which the Locality (0.2) holds for $N = 0$: for any $a, b, c \in V$

$$Y(z)(1 \otimes Y(w))(a \otimes b \otimes c) = Y(w)(1 \otimes Y(z))(b \otimes a \otimes c). \tag{0.10}$$

The following result is proved in [K]:

Theorem 0.0.3. *A vertex algebra is commutative if and only if $Y(z)(a \otimes b) \in V[[z]]$ for any $a, b \in V$.*

Moreover, using the Commutativity (0.10), in [K] it was proved that the -1-product defined in (0.9) gives a structure of unital, associative, commutative algebra with a derivation.
Inspired by this result, we prove in Section 1.3.3 the following generalization:

Theorem 0.0.4. *Let Y be a state-field correspondence on the pointed vector space $(V, |0\rangle)$ (cf. Definition 0.0.1). Assume that $Y(z)(a \otimes b) \in V[[z]]$ for all $a, b \in V$.*

1. *If the Associativity Relation* (0.3) *holds, then V is an associative, unital (with unity $|0\rangle$), differential (with derivation T) algebra with respect to the -1-product.*

2. *If $Y = Y^{op}$, then V is a commutative, unital, differential algebra with respect to the -1-product.*

The notion of *quantum vertex algebra* was introduced in 1998 by P. Etingof and D. Kazhdan in [EK5]. They start with the definition of vertex algebra replacing the underlying vector space over a field $\mathbb{K}$ with a topologically free $\mathbb{K}[[h]]$-module (cf. the definition in Section 2.1). Moreover, they introduce a $\mathbb{K}[[h]]$-linear map

$$\mathcal{S} : V \otimes V \to V \otimes V \otimes \mathbb{K}((z))[[h]], \tag{0.11}$$

which is a shift invariant, unitary solution of the quantum Yang-Baxter equation (cf. Definition 2.2.1). Here and below, the tensor product sign "$\otimes$" denotes the completion of the tensor product in the h-adic topology (cf, Section 2.1). Then, they replace the Locality (0.2) with a deformed version called "$\mathcal{S}$-locality": for any $a, b \in V$ and any $M \in \mathbb{Z}_+$, there exists $N = N(a, b, M) \geq 0$ such that, for any $c \in V$

$$\begin{aligned}(z-w)^N Y(z)\big(1 \otimes Y(w)\big)\big(\mathcal{S}(z-w)(a \otimes b) \otimes c\big)\\ = (z-w)^N Y(w)\big(1 \otimes Y(z)\big)(b \otimes a \otimes c) \mod h^M.\end{aligned} \tag{0.12}$$

Keeping unchanged all the other axioms of a vertex algebra, we get the notion of *braided vertex algebra*. From the definition, it is clear that if V is a braided vertex algebra, V/hV carries a structure of a vertex algebra.

Unfortunately this definition is not enough to have the Associativity Relation (mod h^M). Therefore they impose another axiom, called the *Hexagon Relation* (cf. equation (2.26)), which implies the Associativity Relation. By definition, a *quantum vertex algebra* is a braided vertex algebra for which the Hexagon Relation holds.
Similarly to what happens to a vertex algebra, in a braided vertex algebra one has:

$$Y\mathcal{S} = Y^{op} \tag{0.13}$$

(cf. [EK5, Lemma 1.5], Lemma 2.2.4). It is then natural to ask the following question: let V be a topologically free $\mathbb{K}[[h]]$-module which satisfies the axioms of a braided vertex algebra except maybe for the $\mathcal{S}$-locality; suppose also that the Associativity Relation (0.3) (mod h^M) and $Y\mathcal{S} = Y^{op}$ hold. Is V a quantum vertex algebra?
The answer is given by Theorem 2.4.5 and Remark 2.4.6 which state the following result:

Theorem 0.0.5. *Let V be a topologically free $\mathbb{K}[[h]]$-module endowed with an $\mathcal{S}$-map as in* (0.11) *and a state-field correspondence Y satisfying the Associativity Relation* (0.3) *(mod h^M) and $Y\mathcal{S} = Y^{op}$. Then V is a braided vertex algebra. Moreover, if Y is injective, V is a quantum vertex algebra.*

When we first proved Theorem 0.0.5 we were not aware that the same result already appeared in [L10]. Our proof, which is different from the one in [L10], is a generalization of [BK, Theorem 7.3].

It seems also natural to try to generalize Theorem 0.0.2 to the quantum case. Let us first give the quantum versions of the previous definitions of n-products (0.5), Borcherds identities (0.7) and n-products Identities (0.8). We will use the $\mathcal{S}$-Jacobi Identity in [L10] as a quantum version of the Jacobi Identity.

Definition 0.0.2. Let V be a topologically free $\mathbb{K}[[h]]$-module and let $|0\rangle \in V$ be a non-zero element. Let $Y : V \otimes V \to V((z))$ be a state-field correspondence (cf. Definition 0.0.1) and $\mathcal{S} : V \otimes V \to V \otimes V \otimes \mathbb{K}((z))[[h]]$ a shift invariant, unitary solution of the quantum Yang-Baxter equation (cf. Definition 2.4.2). For any $a, b \in V$ and $n \in \mathbb{Z}$, we define (cf. equation (0.13))

$$a^{\mathcal{S}}_{(n)}b = Res_z\big(z^n Y(z)S(z)(a \otimes b)\big) = Res_z\big(z^n Y^{op}(z)(a \otimes b)\big). \tag{0.14}$$

For $n \in \mathbb{Z}$, the *quantum n-product* of the quantum fields $Y(a, z)$ and $Y(b, z)$ is:

$$\begin{aligned}&\Big(Y(a,w)^{\mathcal{S}}_{(n)}Y(b,w)\Big)(c)\\ &= Res_z\Big(Y(z)\big(1 \otimes Y(w)\big)\big(i_{z,w}\mathcal{S}(z-w)(a \otimes b) \otimes c\big)i_{z,w}(z-w)^n\\ &\quad - Y(w)\big(1 \otimes Y(z)\big)(b \otimes a \otimes c)i_{w,z}(z-w)^n\Big).\end{aligned} \tag{0.15}$$

The *quantum Borcherds Identities* on Y are the following formulas ($n \in \mathbb{Z}$):

$$\begin{aligned} Y(z)\big(1 \otimes Y(w)\big)\big(i_{z,w}\mathcal{S}(z-w)(a \otimes b) \otimes c\big)i_{z,w}(z-w)^n \\ -Y(w)\big(1 \otimes Y(z)\big)(b \otimes a \otimes c)i_{w,z}(z-w)^n \\ = \sum_{j \in \mathbb{Z}_+} Y(w)\big(a^{\mathcal{S}}_{(n+j)}b \otimes c\big)\ \frac{\partial_w^j \delta(z,w)}{j!}. \end{aligned} \tag{0.16}$$

The *quantum n-products Identities* on the state-field correspondence Y are:

$$\Big(Y(a,w)^{\mathcal{S}}_{(n)}Y(b,w)\Big)(c) = Y(w)\big(a^{\mathcal{S}}_{(n)}b \otimes c\big). \tag{0.17}$$

The *$\mathcal{S}$-Jacobi Identity* (cf. [L10]) on Y is:

$$\begin{aligned} i_{z,w}\delta(x, z-w)Y(z)\big(1 \otimes Y(w)\big)(a \otimes b \otimes c) \\ -i_{w,z}\delta(x, z-w)Y(w)\big(1 \otimes Y(z)\big)\big(\mathcal{S}(w-z)(b \otimes a) \otimes c\big) \\ = i_{z,x}\delta(w, z-x)Y(w)\big(Y(x) \otimes 1\big)(a \otimes b \otimes c). \end{aligned} \tag{0.18}$$

We can state the quantum analogue of Theorem 0.0.2 (cf. Theorem 2.4.17):

Theorem 0.0.6. *Let V be a topologically free $\mathbb{K}[[h]]$-module with a non-zero element $|0\rangle \in V$ and a map $\mathcal{S} : V \otimes V \to V \otimes V \otimes \mathbb{K}[[h]]((z))[[h]]$ which is a shift invariant, unitary solution of the quantum Yang-Baxter equation (cf. Definition 0.0.2). The following statements are equivalent:*

1. *V is a braided vertex algebra satisfying the Associativity Relation* (0.3) *(mod h^M);*
2. *the $\mathcal{S}$-Jacobi Identity* (0.18) *holds;*
3. *the Associativity Relation* (0.3) *(mod h^M) holds and $Y\mathcal{S} = Y^{op}$;*
4. *the quantum Borcherds Identities* (0.16) *hold;*
5. *the quantum n-products Identities* (0.17) *and the $\mathcal{S}$-locality hold.*

Unfortunately, there is not an immediate quantum analogue of Goddard's Uniqueness Theorem. Indeed, if the locality can be defined on any couple of quantum fields, the $\mathcal{S}$-locality can be defined only with a state-field correspondence. Nevertheless, we can state the following result (cf. Proposition 2.2.17):

Proposition 0.0.7. *Let V be a braided vertex algebra, let $a \in V$ and let $a(z)$ be a quantum field over $\mathbb{K}[[h]]$ such that*

1. *$a(z)|0\rangle = Y(z)(a \otimes |0\rangle)$;*
2. *$a(z)$ is local (mod h^M for any M) with all quantum fields $Y(z)(c \otimes -)$ on every vector of V.*

Then

$$a(z) = Y(z)\mathcal{S}(z)(a \otimes -) = Y^{op}(z)(a \otimes -). \tag{0.19}$$

Therefore, by [BK, Proposition 4.1], if V is a braided vertex algebra satisfying the Associativity Relation (0.3), one has:

Proposition 0.0.8. *Let V be a braided vertex algebra which satisfies the Associativity Relation* (0.3)*. Let $a \in V$ and let $a(z)$ be a quantum field over $\mathbb{K}[[h]]$ such that $a(z)|0\rangle = Y(z)(a \otimes |0\rangle)$. Then $a(z)$ is local (mod h^M) with all quantum fields $Y(z)(c \otimes -)$ on every vector of V if and only if $a(z) = Y^{op}(z)(a \otimes -)$.*

The quantum analogue of the Commutativity (0.10) is the $\mathcal{S}$-commutativity:

Definition 0.0.3. A braided vertex algebra V satisfies the $\mathcal{S}$-commutativity if the following relation holds:

$$Y(z)\big(1 \otimes Y(w)\big)\big(i_{z,w}\mathcal{S}(z-w)(a \otimes b) \otimes c\big) = Y(w)\big(1 \otimes Y(z)\big)(b \otimes a \otimes c). \quad (0.20)$$

The following theorems are the quantum analogue of Theorem 0.0.3 and Theorem 0.0.4 respectively (cf. Theorems 2.2.23 and 2.2.25).

Theorem 0.0.9. *Let V be a braided vertex algebra. The $\mathcal{S}$-commutativity holds if and only if $Y(z)(a \otimes b) \in V[[z]]$ for any $a, b \in V$.*

Theorem 0.0.10. *Let V be a braided vertex algebra which satisfies*

1. *the $\mathcal{S}$-commutativity* (0.20)*;*
2. *the Associativity Relation* (0.3) *(mod h^M).*

Then V is a unital (with unit $|0\rangle$), associative, differential (with derivation T) $\mathbb{K}[[h]]$-algebra with respect to the -1-product $a_{(-1)}b = Res_z\big(z^{-1}Y(z)(a \otimes b)big\big)$. Moreover the commutation relations are the following:

$$b_{(-1)}a = \big(\, -_{(-1)} -\big)Res_z\big(z^{-1}(e^{zT} \otimes 1)\mathcal{S}(z)(a \otimes b)\big) \quad (0.21)$$

for any $a, b \in V$.

Another interesting problem is to have examples of quantum vertex algebras. In [EK5], P. Etingof and D. Kazhdan define a structure of quantum vertex algebra on the vacuum module of the extended double Yangian of type A (cf. [EK5, Chapter 2], [JKMY, Theorem 4.1]). They name this example "Quantum affine vertex algebra" because its quasiclassical limit is the affine vertex algebra of type A. In Section 2.3.4, we will prove in every detail that the quantum affine vertex algebra satisfies all the axioms of a quantum vertex algebra. We will do the same in Section 2.3.5 for the example of $\mathcal{S}$-commutative quantum vertex algebra given in [JKMY, Proposition 4.2].

An interesting goal could be having new examples associated to any simple Lie algebra. Nonetheless, this is a hard problem because it is related to the universal pseudotriangular R-matrix of the Yangian. Indeed the vacuum module of the extended Double Yangian is a Yetter-Drinfeld module (called also dimodule in [EK5]) of the Yangian. Therefore, one could give the map $\mathcal{S}$ by the action on the Yetter-Drinfeld module of the universal pseudotriangular R-matrix or its inverse (cf. [EK5, Section 3.3]).

V. G. Drinfeld proved that there exists a unique universal pseudotriangular R-matrix of the Yangian, but the proof was never published. We devote the second part of the thesis to the study of some Drinfeld's notes on existence and uniqueness of the universal triangular R-matrices.

A Lie bialgebra $(\mathfrak{g}, [\ ,\ ], \delta)$ over $\mathbb{C}$ is a triple which satisfies:

1. $(\mathfrak{g}, [\ ,\ ])$ is a Lie algebra;
2. δ is a linear map $\delta : \mathfrak{g} \to \Lambda^2\mathfrak{g}$ such that the following *coJacobi Identity* holds:
$$(\delta \wedge 1) \circ \delta = 0; \tag{0.22}$$
3. δ satisfies the 2-cocycle condition
$$\delta([x, y]) = x.\delta(y) - y.\delta(x) \tag{0.23}$$
for any $x, y \in \mathfrak{g}$.

Here we denote with $\delta \wedge 1$ the map in $\mathrm{Hom}\left(\Lambda^2\mathfrak{g}, \Lambda^3\mathfrak{g}\right)$ defined as $(\delta \wedge 1)(a \wedge b) = \delta(a) \wedge b - \delta(b) \wedge a$. And we denote with "." the adjoint action of $\mathfrak{g}$ on $\Lambda^2\mathfrak{g}$:

$$x.(x_1 \wedge x_2) = [x, x_1] \wedge x_2 + x_1 \wedge [x, x_2].$$

In order to construct some Lie bialgebras, one can start with an element $r \in \Lambda^2\mathfrak{g}$ solution of the classical Yang-Baxter equation:

$$\left[r^{12}, r^{13}\right] + \left[r^{12}, r^{23}\right] + \left[r^{13}, r^{23}\right] = 0. \tag{0.24}$$

Then, letting $d_{\mathfrak{g}}r : \mathfrak{g} \to \Lambda^2\mathfrak{g}$ defined as $d_{\mathfrak{g}}r(x) = x.r$ for any $x \in \mathfrak{g}$, it is easy to prove that $(\mathfrak{g}, [\ ,\ ], d_{\mathfrak{g}}r)$ is a Lie bialgebra.
Let H be a topologically free $\mathbb{C}[[h]]$-module which satisfies all the axioms of a Hopf algebra (cf. Section 3.1) with the tensor products understood in the h-adic completed sense. We say that H is a quantization of a Lie bialgebra $(\mathfrak{g}, [\ ,\ ], \delta)$ over $\mathbb{C}$ if the following conditions hold:

1. $H/hH \cong \mathcal{U}(\mathfrak{g})$ as Hopf algebras over $\mathbb{C}$;
2. $\delta(x) \equiv \frac{\Delta(x) - \Delta^{op}(x)}{h} \mod h$.

(Here and below Δ is the coproduct of H and $\Delta^{op} = (1\ 2)\Delta$).
Let H be a quantization of the Lie bialgebra $(\mathfrak{g}, [\ ,\ ], d_{\mathfrak{g}}r)$. The universal triangular R-matrix Σ of H is an element of $H \otimes H$ (here the tensor product is understood in the h-adic completed sense), such that:

(1) $\Sigma \equiv 1 + rh \mod h^2$;

(2) $\Sigma^{12}\Sigma^{21} = 1$;

(3) $(1 \otimes \Delta)(\Sigma) = \Sigma^{12}\Sigma^{13}$;

(4) $\Delta^{op}(a) = \Sigma^{-1}\Delta(a)\Sigma$;

(5) $\Sigma^{12}\Sigma^{13}\Sigma^{23} = \Sigma^{23}\Sigma^{13}\Sigma^{12}$.

The superscripts have the following meaning: for any $a, b \in H$, $(a \otimes b)^{12} = a \otimes b \otimes 1$ and, similarly $(a \otimes b)^{13} = a \otimes 1 \otimes b$ and $(a \otimes b)^{23} = 1 \otimes a \otimes b$.

Recall that the Chevalley-Eilenberg cochain complex of a Lie algebra $\mathfrak{g}$ with coefficients in the module $\Lambda^n\mathfrak{g}$ is given by the following:

$$\Lambda^n\mathfrak{g} \xrightarrow{d_{\mathfrak{g}}} \mathrm{Hom}(\mathfrak{g}, \Lambda^n\mathfrak{g}) \xrightarrow{d_{\mathfrak{g}}} \mathrm{Hom}(\Lambda^2\mathfrak{g}, \Lambda^n\mathfrak{g}) \xrightarrow{d_{\mathfrak{g}}} \cdots \tag{0.25}$$

where $d_{\mathfrak{g}} : \mathrm{Hom}(\Lambda^m\mathfrak{g}, \Lambda^n\mathfrak{g}) \to \mathrm{Hom}(\Lambda^{m+1}\mathfrak{g}, \Lambda^n\mathfrak{g})$ is defined as

$$\begin{aligned} d_{\mathfrak{g}} f(x_1 \wedge \cdots \wedge x_{m+1}) = &\sum_{i=1}^{m+1} (-1)^{i-1} x_i . f(x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x_{m+1}) \\ &+ \sum_{i<j} (-1)^{i+j} f([x_i, x_j] \wedge x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge \widehat{x_j} \wedge \cdots \wedge x_{m+1}), \end{aligned}$$

(here, as before, we denote with "." the adjoint action of $\mathfrak{g}$ on $\Lambda^n\mathfrak{g}$).

One can also define the following cochain complex:

$$\mathrm{Hom}(\Lambda^m\mathfrak{g}, \mathbb{C}) \xrightarrow{d_{\mathfrak{g}^*}} \mathrm{Hom}(\Lambda^m\mathfrak{g}, \mathfrak{g}) \xrightarrow{d_{\mathfrak{g}^*}} \mathrm{Hom}(\Lambda^m\mathfrak{g}, \Lambda^2\mathfrak{g}) \xrightarrow{d_{\mathfrak{g}^*}} \cdots \tag{0.26}$$

where $d_{\mathfrak{g}^*} : \mathrm{Hom}(\Lambda^m\mathfrak{g}, \Lambda^n\mathfrak{g}) \to \mathrm{Hom}(\Lambda^m\mathfrak{g}, \Lambda^{n+1}\mathfrak{g})$ is defined as

$$\begin{aligned} d_{\mathfrak{g}^*} f(x_1 \wedge \cdots \wedge x_m) = \sum_{i=1}^{m} \sum_{(x_i)} (-1)^{i-1} \big(&x_i' \wedge f(x_i'' \wedge x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x_m) \\ &- x_i'' \wedge f(x_i' \wedge x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x_m) \big) \\ - (d_{\mathfrak{g}} r)\big(f(x_1 \wedge \cdots \wedge x_m)\big).& \end{aligned}$$

Here we have

$$\sum_{(x_i)} x_i' \wedge x_i'' = d_{\mathfrak{g}} r(x_i) = x_i . r$$

and, for any $y_1, \ldots, y_n \in \mathfrak{g}$,

$$(d_{\mathfrak{g}} r)(y_1 \wedge \cdots \wedge y_n) = \sum_{j=1}^{n} (-1)^{j-1} y_1 \wedge \cdots \wedge d_{\mathfrak{g}} r(y_j) \wedge \cdots \wedge y_n.$$

Note that, if $\mathfrak{g}$ is finite dimensional, the cochain complex (0.26) is the Chevalley-Eilenberg cochain complex of the lie algebra $\mathfrak{g}^*$ (the bracket is induced by δ) with coefficients in $\Lambda^m\mathfrak{g}^*$ (cf. Appendix B).

Consider now the following bicomplex (cf. bicomplex (3.31)):

$$\begin{array}{ccccccc}
\Lambda^2\mathfrak{g} & \xrightarrow{d_{\mathfrak{g}}} & \mathrm{Hom}(\mathfrak{g},\Lambda^2\mathfrak{g}) & \xrightarrow{d_{\mathfrak{g}}} & \mathrm{Hom}(\Lambda^2\mathfrak{g},\Lambda^2\mathfrak{g}) & \xrightarrow{d_{\mathfrak{g}}} & \cdots \\
\downarrow{\scriptstyle d_{\mathfrak{g}^*}} & & \downarrow{\scriptstyle d_{\mathfrak{g}^*}} & & \downarrow{\scriptstyle d_{\mathfrak{g}^*}} & & \\
\Lambda^3\mathfrak{g} & \xrightarrow{d_{\mathfrak{g}}} & \mathrm{Hom}(\mathfrak{g},\Lambda^3\mathfrak{g}) & \xrightarrow{d_{\mathfrak{g}}} & \mathrm{Hom}(\Lambda^2\mathfrak{g},\Lambda^3\mathfrak{g}) & \xrightarrow{d_{\mathfrak{g}}} & \cdots \\
\downarrow{\scriptstyle d_{\mathfrak{g}^*}} & & \downarrow{\scriptstyle d_{\mathfrak{g}^*}} & & \downarrow{\scriptstyle d_{\mathfrak{g}^*}} & & \\
\vdots & & \vdots & & \vdots & &
\end{array} \tag{0.27}$$

where the rows are cochain complexes (0.25) and the columns are cochain complexes (0.26). One has that $d_{\mathfrak{g}^*} \circ d_{\mathfrak{g}} = d_{\mathfrak{g}} \circ d_{\mathfrak{g}^*}$ (cf. Appendix B). Therefore, one can define the total cochain complex:

$$\Lambda^2\mathfrak{g} \xrightarrow{d} \Lambda^3\mathfrak{g} \oplus \mathrm{Hom}(\mathfrak{g},\Lambda^2\mathfrak{g}) \xrightarrow{d} \cdots \tag{0.28}$$

where d is the total differential

$$d : \bigoplus_{m+n=l} \mathrm{Hom}(\Lambda^m\mathfrak{g},\Lambda^{n+2}\mathfrak{g}) \to \bigoplus_{p+q=l+1} \mathrm{Hom}(\Lambda^p\mathfrak{g},\Lambda^{q+2}\mathfrak{g})$$

defined as $df = d_{\mathfrak{g}}f + (-1)^m d_{\mathfrak{g}^*}f$, for any $f \in \mathrm{Hom}(\Lambda^m\mathfrak{g},\Lambda^{n+2}\mathfrak{g})$.

The following is the theorem proved by V. G. Drinfeld in his notes on existence and uniqueness of the universal triangular R-matrices (cf. Theorem 3.3.11):

Theorem 0.0.11. *Let H be a quantization of the complex Lie bialgebra $(\mathfrak{g},[\ ,\ ],d_{\mathfrak{g}}r)$ where $r \in \Lambda^2\mathfrak{g}$ is a solution of the classical Yang-Baxter equation* (0.24)*. Then the obstructions to the existence and uniqueness of the universal triangular R-matrix lie respectively in the 1^{st} and 0^{th} cohomology groups of the total cochain complex* (0.28)*.*

Note that the 0^{th} and 1^{st} cohomology groups of the total cochain complex (0.28) are almost never both zero, but this does not prevent the universal triangular R-matrix to exist and to be unique. As an example, we prove in Section 3.4 that if $\mathfrak{g}$ is a simple finite dimensional Lie algebra and $r = 0 \in \Lambda^2\mathfrak{g}$, then the first cohomology group of the total cochain complex (0.28) is not zero, but $\Sigma = 1$ is a universal triangular R-matrix.

In conclusion, the existence and uniqueness of the universal R-matrix for a given quantization H of a Lie bialgebra $(\mathfrak{g},[\ ,\ ],d_{\mathfrak{g}}r)$ is highly non trivial and it relies on the understanding of the cohomology of the total complex (0.28).

The thesis is structured as follows.

In Chapter 1 we introduce the notion of pointed vector space, state-field correspondence and field algebra. Then we recall some results which will be used in the section devoted to the study of vertex algebras and in the next chapters. We also introduce the notion of vertex algebra and we provide, as an example, the affine vertex algebra. We recall the Goddard's Uniqueness Theorem and we give some characterizations of vertex algebras. Finally we deal with commutative vertex algebras.

In Chapter 2 we firstly recall some notions about inverse limits and h-adic topology. We introduce the notion of quantum vertex algebra and some useful results. Here is also stated and proved the quantum analogue of Goddard's Uniqueness Theorem. Then, we define $\mathcal{S}$-commutative vertex algebras and we give some results. In the following sections we recall one example of quantum vertex algebra and one example of $\mathcal{S}$-commutative vertex algebra and we check all the axioms of the definition of quantum vertex algebra. In the end of the chapter we give some characterizations of braided vertex algebras which satisfy the Associativity Relation.

In Chapter 3 we recall the definitions of algebra, coalgebra, bialgebra, Hopf algebra and Lie bialgebra. The main theorem in the Drinfeld's notes on existence and uniqueness of the universal triangular R-matrices is stated in Section 3.3 (Theorem 3.3.11) and the proof is in Sections 3.2 and 3.3. In Section 3.5 we slightly modify the arguments in Section 3.3 to be ready to deal with the pseudotriangular case. As a tribute to V. G. Drinfeld, we name Sections 3.2, 3.3 and 3.5 as "Question and Answer 46", "Question and Answer 47" and "Question and Answer 47' ": the same names used in Drinfeld's notes. In Section 3.4 ("Considerations on Question and Answer 47") we give an example of a Lie bialgebra having a non-zero 1^{st} cohomology group of the total complex of (0.27). Then we show that this does not prevent the universal triangular R-matrix to exist.

In Appendix A we prove that for any Lie algebra $\mathfrak{g}$ the cohomology of cochain complex (3.15) is isomorphic to $\Lambda^\bullet\mathfrak{g}$.

In Appendix B we prove that the bicomplex (0.27) is indeed a bicomplex.

Chapter 1

Field algebras and vertex algebras

In Section 1.1 we introduce the notation we use throughout this work. In Section 1.2 we recall the concepts of pointed vector space, state-field correspondence and field algebra and we review some results. In Section 1.3 we recall the concept of vertex algebra, we review some characterizations of vertex algebras (cf. Section 1.3.2) and we deal with commutative vertex algebras (cf. Section (1.3.3)).

1.1 Calculus of formal distributions

Throughout the chapter, we shall follow, unless otherwise stated, the book [K] (see also [K15] for a more recent exposition).
Let $\mathbb{K}$ be a field of characteristic zero and let V be a $\mathbb{K}$-vector space. Let us denote by $V[[z, z^{-1}]]$ the space of formal power series in $z^{\pm 1}$ with coefficients in V which we will also call the space of V-valued formal distributions in $z^{\pm 1}$. A Laurent series is an element $a(z) = \sum_{m\in\mathbb{Z}} a_{(m)} z^{-m-1} \in V[[z, z^{-1}]]$ such that $a_{(m)} = 0$ for $m \gg 0$. We will denote the space of Laurent series by $V((z))$. By definition, the residue of a formal series $a(z) \in V[[z^{\pm 1}]]$ is

$$Res_z\ a(z) = a_{(-1)}. \tag{1.1}$$

Definition 1.1.1. Let V be a $\mathbb{K}$-vector space. A formal $End_{\mathbb{K}}V$-valued distribution $a(z)$ is an $End_{\mathbb{K}}V$-valued *quantum field* if $a(z)b \in V((z))$ for any $b \in V$.

Recall that, for $a(z) \in End_{\mathbb{K}}V[[z, z^{-1}]]$, the following Taylor's formula (see e.g. [K, Prop.2.4]) holds:

$$i_{z,w}a(z+w) = \sum_{j\geq 0} \frac{\partial_z^j a(z)}{j!} w^j = e^{w\partial_z} a(z), \tag{1.2}$$

where $i_{z,w}$ (respectively $i_{w,z}$) denotes the expansion in the domain $|z| > |w|$ (respectively $|w| > |z|$):

$$i_{z,w}(z-w)^n = \sum_{k\geq 0} \binom{n}{k} z^{n-k}(-w)^k, \tag{1.3}$$

$$i_{w,z}(z-w)^n = \sum_{k\geq 0} \binom{n}{k} z^k (-w)^{n-k}. \tag{1.4}$$

Remark 1.1.1. We recall that for any $n \geq 0$ the two previous formulas coincide with the binomial identity.

Definition 1.1.2. The *formal delta distribution* $\delta(z,w)$ is defined as follows:

$$\delta(z,w) = \sum_{m\in\mathbb{Z}} z^{-m-1} w^m. \tag{1.5}$$

Lemma 1.1.2. *The following equalities hold:*

$$(i_{z,w} - i_{w,z})(z-w)^n = \begin{cases} 0 & \text{for } n \geq 0 \\ \frac{1}{(-n-1)!} \partial_w^{-n-1} \delta(z,w) & \text{for } n < 0. \end{cases} \tag{1.6}$$

In particular if $n = -1$ one has

$$\delta(z,w) = i_{z,w} \frac{1}{z-w} - i_{w,z} \frac{1}{z-w}. \tag{1.7}$$

As a consequence $(z-w)\delta(z,w) = 0$.

Proof. If $n \geq 0$, $i_{z,w}(z-w)^n - i_{w,z}(z-w)^n = 0$ by Remark 1.1.1. If $n < 0$ one has

$$\begin{aligned} i_{z,w}(z-w)^n &= \sum_{i\geq 0} \binom{n}{i} z^{n-i} (-w)^i \\ &= \sum_{i\geq 0} \binom{i-n-1}{i} z^{n-i} w^i \\ &= \sum_{j\geq -n-1} \binom{j}{j+n+1} z^{-j-1} w^{j+n+1}. \end{aligned}$$

Since $j \geq -n-1$, one has $j+n+1 \geq 0$ from which $\binom{j}{j+n+1} = \binom{j}{-n-1}$. Moreover $\binom{j}{-n-1} = 0$ for any $j = 0, \ldots, -n-2$. Therefore the following equality holds:

$$i_{z,w}(z-w)^n = \sum_{j\geq 0} \binom{j}{-n-1} z^{-j-1} w^{j+n+1}.$$

Similarly one obtains

$$i_{w,z}(z-w)^n = -\sum_{j<0} \binom{j}{-n-1} z^{-j-1} w^{j+n+1}.$$

It follows that, if $n < 0$, one has

$$i_{z,w}(z-w)^n - i_{w,z}(z-w)^n = \frac{1}{(-n-1)!} \partial_w^{-n-1} \delta(z,w).$$

□

Lemma 1.1.3. *Let $a(z)$ be a V-valued formal distribution. The following equality holds:*

$$Rez_z\big(a(z)\partial_w^n\delta(z,w)\big) = \partial_w^n a(w). \tag{1.8}$$

Proof. First, it is immediate to check that

$$\begin{aligned} Rez_z\big(a(z)\delta(z,w)\big) &= Rez_z\big(a(w)\delta(z,w)\big) \\ &= \big(a(w)Rez_z\delta(z,w)\big) \\ &= a(w). \end{aligned}$$

Then

$$\begin{aligned} Rez_z\big(a(z)\partial_w^n\delta(z,w)\big) &= \partial_w^n Rez_z\big(a(z)\delta(z,w)\big) \\ &= \partial_w^n a(w). \end{aligned}$$

□

Lemma 1.1.4. *Let $a(z) \in V[[z, z^{-1}]]$ be a formal distribution. The following equality holds:*

$$(i_{z,w} - i_{w,z})a(z-w) = Res_x\big(a(x)i_{w,x}\delta(z, w+x)\big). \tag{1.9}$$

Proof. Let $a(z) = \sum_{n\in\mathbb{Z}} a_{(n)}z^{-n-1}$. By Lemma 1.1.2 and Taylor Formula, one has

$$\begin{aligned} (i_{z,w} - i_{w,z})a(z-w) &= \sum_{n\in\mathbb{Z}} a_{(n)}(i_{z,w} - i_{w,z})(z-w)^{-n-1} \\ &= \sum_{n\geq 0} a_{(n)}\frac{\partial_w^n\delta(z,w)}{n!} \\ &= Res_x\big(a(x)e^{x\partial_w}\delta(z,w)\big) \\ &= Res_x\big(a(x)i_{w,x}\delta(z,w+x)\big). \end{aligned}$$

□

Lemma 1.1.5. *If V is a $\mathbb{K}$-algebra, $a(z) \in V((z))$ and $b(z,w) \in V[[z,w]][z^{-1}, w^{-1}]$, then the following equality holds:*

$$Res_z\Big(((i_{z,w} - i_{w,z})a(z-w))b(z,w)\Big) = Res_x\big(a(x)i_{w,x}b(w+x,w)\big). \tag{1.10}$$

Proof. By Lemma 1.1.4, one has

$$\begin{aligned} &Res_z\Big(((i_{z,w} - i_{w,z})a(z-w))b(z,w)\Big) \\ &\quad = Res_z Res_x\big(a(x)i_{w,x}\delta(z,w+x)b(z,w)\big) \\ &\quad = Res_x Res_z\big(a(x)i_{w,x}\delta(z,w+x)b(z,w)\big) \\ &\quad = Res_x\big(a(x)i_{w,x}b(w+x,w)\big). \end{aligned}$$

□

Definition 1.1.3. A formal V-valued distribution $a(z,w) \in V[[z^{\pm 1}, w^{\pm 1}]]$ is said to be local if there exists $N \geq 0$ such that $(z-w)^N a(z,w) = 0$.

Remark 1.1.6. The formal delta distribution is local by Remark 1.1.2.

Theorem 1.1.7 (Decomposition Theorem, [K, Cor.2.2], [K15, Thm.1.2])**.** *Any local formal distribution $a(z,w)$ can be uniquely decomposed as*

$$a(z,w) = \sum_{\substack{j \geq 0 \\ finite}} c^j(w) \frac{\partial_w^j \delta(z,w)}{j!}, \tag{1.11}$$

where

$$c^j(w) = Res_z((z-w)^j a(z,w)).$$

Definition 1.1.4 ([BK])**.** A pair of $End_{\mathbb{K}}V$-valued quantum fields $a(z), b(z)$ is said to be local if there exists $N \geq 0$ such that

$$(z-w)^N [a(z), b(w)] = 0. \tag{1.12}$$

It is called local on $c \in V$ if there exists $N \geq 0$ such that

$$(z-w)^N [a(z), b(w)]c = 0. \tag{1.13}$$

Lemma 1.1.8 ([L03, Lem.2.1])**.** *Let V be a $\mathbb{K}$-vector space and let*

$$\begin{aligned} a(z,w) &\in V((z))((w)) \\ b(z,w) &\in V((w))((z)) \\ c(x,w) &\in V((w))((x)). \end{aligned}$$

Then

$$i_{z,w}\delta(x, z-w)\ a(z,w) - i_{w,z}\delta(x,z-w)b(z,w) = i_{z,x}\delta(w, z-x)\ c(x,w) \tag{1.14}$$

if and only if there exist nonnegative integers k and l such that

$$(z-w)^k\ a(z,w) = (z-w)^k\ b(z,w) \tag{1.15}$$

$$(x+w)^l\ a(x+w,w) = (x+w)^l\ c(x,w). \tag{1.16}$$

1.2 Field algebras

Definition 1.2.1 ([BK])**.** A *pointed vector space* is a couple $(V, |0\rangle)$ where V is a $\mathbb{K}$-vector space and $|0\rangle \in V$ is a fixed non-zero vector. A state-field correspondence on $(V, |0\rangle)$ is a linear map

$$\begin{aligned} Y : V \otimes V &\to V((z)) \\ a \otimes b &\mapsto Y(z)(a \otimes b), \end{aligned} \tag{1.17}$$

such that the following axioms hold:

(i) (Vacuum Axioms) $Y(z)(|0\rangle \otimes a) = a, Y(z)(a \otimes |0\rangle) = a + T(a)z + \ldots \in V[[z]]$ where $T : V \to V$ is called the translation operator;

(ii) (Translation Covariance) $TY(z)(a \otimes b) - Y(z)(a \otimes Tb) = \partial_z Y(z)(a \otimes b)$;

(iii) $Y(z)(Ta \otimes b) = \partial_z Y(z)(a \otimes b)$.

Remark 1.2.1. The name "state-field correspondence" is due to the following: the map Y is equivalent to the $\mathbb{K}$-linear map

$$\begin{aligned} V &\to End_{\mathbb{K}} V[[z, z^{-1}]] \\ a &\mapsto Y(z)(a \otimes -) : V \to V((z)) \end{aligned} \tag{1.18}$$

With an abuse of notation, we denote the map (1.18) with Y and the image of $a \in V$ as $Y(a, z)$. By definition $Y(a, z)$ is a quantum field as it is a map $V \to V((z))$. With this notation the physical meaning of Y as a map between states and quantum fields is clear.

Notation 1. We will use indistinctly the map Y both as defined in (1.17) and in (1.18). In particular, following [K], we will write $Y(z)(a \otimes -) = Y(a, z) = \sum_{k \in \mathbb{Z}} a_{(k)} z^{-k-1}$ where $a_{(k)} \in End_{\mathbb{K}} V$ and $a \in V$.

Remark 1.2.2. Let Y, X be two state-field correspondences. By definition 1.1.4 and our abuse of notation, the locality of two quantum fields $Y(a, z)$ and $X(b, w)$ can be translated as follows: there exists $N = N(a, b) \geq 0$ such that for any $c \in V$ one has

$$(z - w)^N Y(z)(1 \otimes X(w))(a \otimes b \otimes c) = (z - w)^N X(w)(1 \otimes Y(z))(b \otimes a \otimes c).$$

And, similarly, the locality of $Y(a, z)$ and $X(b, w)$ on a vector $b \in V$ can be translated as follows: there exists $N = N(a, b, c) \geq 0$ such that

$$(z - w)^N Y(z)(1 \otimes X(w))(a \otimes b \otimes c) = (z - w)^N X(w)(1 \otimes Y(z))(b \otimes a \otimes c).$$

Proposition 1.2.3 ([BK, Prop.2.7])**.** *If $Y : V \otimes V \to V((z))$ satisfies conditions (i) and (ii) of Definition 1.2.1, then*

1. $Y(z)(a \otimes |0\rangle) = e^{zT} a$;
2. $e^{wT} Y(z) \left(1 \otimes e^{-wT}\right) = i_{z,w} Y(z + w)$.

If, moreover, Y is a state-field correspondence, then

3. $Y(z)(e^{wT} \otimes 1) = i_{z,w} Y(z + w)$

Proof. By the Vacuum Axioms, one has that $T(a) = Res_z(z^{-2} Y(z)(a \otimes |0\rangle))$. As, $Y(z)(|0\rangle \otimes |0\rangle) = |0\rangle$, one has that $T(|0\rangle) = 0$. Using $T(|0\rangle) = 0$ in the Translation Covariance applied to $a \otimes |0\rangle$, one has

$$TY(z)(a \otimes |0\rangle) = \partial_z Y(z)(a \otimes |0\rangle)$$

from which $T(a_{(-m)}|0\rangle) = m a_{(-m-1)}|0\rangle$ for any $m \geq 1$. By the Vacuum Axioms $a_{(-1)}|0\rangle = a$. Therefore, one proves by induction that

$$a_{(-m)}|0\rangle = \frac{T^m}{m!} a.$$

It follows that
$$Y(z)(a\otimes|0\rangle)=e^{zT}a.$$
Point (2) is a consequence of the Translation Covariance. Indeed
$$\begin{aligned}
i_{z,w}Y(z+w)=e^{w\partial_z}Y(z)&=\sum_{k\geq 0}\frac{w^k}{k!}\partial_z^kY(z)\\
&=\sum_{k\geq 0}\frac{w^k}{k!}\sum_{l=0}^{k}\binom{k}{l}T^lY(z)\left(1\otimes T^{k-l}\right)\\
&=\sum_{k\geq 0}\sum_{l\geq 0}\frac{w^l\ T^l}{l!}Y(z)\left(1\otimes\frac{w^{k-l}\ T^{k-l}}{(k-l)!}\right)\\
&=e^{wT}Y(z)\left(1\otimes e^{wT}\right).
\end{aligned}$$
Point (3) is a consequence of the third axiom in the Definition 1.2.1:
$$Y(z)(T\otimes 1)=\partial_zY(z)$$
Indeed
$$\begin{aligned}
i_{z,w}Y(z+w)=e^{w\partial_z}Y(z)&=\sum_{k\geq 0}\frac{w^k}{k!}\partial_z^kY(z)\\
&=\sum_{k\geq 0}\frac{w^k}{k!}Y(z)\left(T^k\otimes 1\right)\\
&=Y(z)\left(e^{wT}\otimes 1\right).
\end{aligned}$$
□

Proposition 1.2.4 ([BK, Prop.2.8])**.** *Given a state-field correspondence Y, define*
$$Y^{op}(z)(a\otimes b)=e^{zT}Y(-z)(b\otimes a). \tag{1.19}$$
Then Y^{op} is also a state-field correspondence.

Lemma 1.2.5 ([BK, Lem.3.8])**.** *Let X and Y be two state-field correspondences, and let $a,b,c\in V$ be such that there exists $N\geq 0$ such that*
$$(z-w)^NY(z)(1\otimes X(w))(a\otimes c\otimes b)=(z-w)^NX(w)(1\otimes Y(z))(c\otimes a\otimes b).$$
Then
$$(z-w)^NY(z)(1\otimes X(w))(a\otimes c\otimes Tb)=(z-w)^NX(w)(1\otimes Y(z))(c\otimes a\otimes Tb).$$

Definition 1.2.2 ([BK])**.** Let $(V,|0\rangle)$ be a pointed vector space with a state-field correspondence Y. Y satisfies the *Associativity Relation* if, for any $a,b,c\in V$, there exists $N\geq 0$
$$(z+w)^NY(w)(Y(z)\otimes 1)(a\otimes b\otimes c)=(z+w)^Ni_{z,w}Y(z+w)(1\otimes Y(w))(a\otimes b\otimes c). \tag{1.20}$$

Proposition 1.2.6 ([BK, Prop.4.1])**.** *Let $(V, |0\rangle)$ be a pointed vector space and with a state-field correspondence Y. Y satisfies the Associativity Relation* (1.20) *if and only if all pairs $\big(Y(a,z), Y^{op}(b,w)\big)$ are local on each $c \in V$.*

Proof. It follows by Proposition 1.2.3. Indeed, on one side, one has

$$\begin{aligned}&(z+w)^N Y(w)\big(Y(z)\otimes 1\big)(a\otimes b\otimes c)\\ &= (z+w)^N e^{wT} Y^{op}(-w)\big(1\otimes Y(z)\big)(c\otimes a\otimes b)\end{aligned} \tag{1.21}$$

and, on the other side, one has

$$\begin{aligned}&(z+w)^N i_{z,w} Y(z+w)\big(Y(w)\otimes 1\big)(a\otimes b\otimes c)\\ &= (z+w)^N e^{wT} Y(z)\big(1\otimes e^{-wT} Y(w)\big)(a\otimes b\otimes c)\\ &= (z+w)^N e^{wT} Y(z)\big(1\otimes Y^{op}(-w)\big)(a\otimes c\otimes b).\end{aligned} \tag{1.22}$$

To conclude, we note that the equality of the left hand sides of equations (1.21) and (1.22) is the Associativity Relation (1.20), while the equality of the right hand sides of equations (1.21) and (1.22) is (up to a factor e^{wT}) the locality of $Y(a,z)$ and $Y^{op}(c,-w)$ on every $b \in V$ (cf. Remark 1.2.2). □

Definition 1.2.3 ([BK])**.** A *field algebra* is a pointed vector space $(V, |0\rangle)$ with a state-field correspondence Y which satisfies the Associativity Relation (1.20).

We will give an example of field algebra in the next section.

1.3 Vertex algebras

Definition 1.3.1. ([K]) A vertex algebra over $\mathbb{K}$ is the following data:

1. a $\mathbb{K}$-vector space V;
2. a vector $|0\rangle \in V$ (vacuum vector);
3. a linear map

   $$\begin{aligned}Y :& V\otimes V \to V((z)),\\ & a\otimes b \mapsto Y(z)(a\otimes b) = \sum_{m\in\mathbb{Z}} a_{(m)} b\ z^{-m-1}\end{aligned}$$

   where $a_{(m)} \in End_{\mathbb{K}} V$;
4. a linear operator $T : V \to V$ (called the translation operator);

which satisfy the following axioms:

(A1) (Locality) for any $a, b \in V$ there exists $N = N(a,b) \geq 0$ such that for any $c \in V$

$$\begin{aligned}&(z-w)^N Y(z)\big(1\otimes Y(w)\big)(a\otimes b\otimes c)\\ &\qquad = (z-w)^N Y(w)\big(1\otimes Y(z)\big)(b\otimes a\otimes c);\end{aligned} \tag{1.23}$$

(A2) $T|0\rangle = 0$ and $\partial_z Y(z) = TY(z) - Y(z)(1 \otimes T)$ (Translation Covariance);

(A3) $Y(z)(|0\rangle \otimes b) = b$ for any $b \in V$; for any $a \in V$, $Y(z)(a \otimes |0\rangle) \in V[[z]]$ and $Y(z)(a \otimes |0\rangle)_{|z=0} = a$.

Remark 1.3.1. By axioms (A2) e (A3) T is determined by Y and the vacuum vector $|0\rangle$: $T(a) = a_{(-2)}|0\rangle$.

Notation 2. With the usual abuse of notation, we also denote with Y the map $V \to End_{\mathbb{K}} V[[z, z^{-1}]], a \mapsto Y(z)(a \otimes -)$. With this map $Y : V \to End_{\mathbb{K}} V[[z, z^{-1}]]$, the vertex algebra axioms of Definition 1.3.1 translate to the axioms of a vertex algebra as defined in [K, Section 1.3] or [K15, Def.1.4].

1.3.1 Affine vertex algebra

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{K}$ with a non-degenerate symmetric invariant bilinear form $(\ , \ )$. Let $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{K}((t)) \oplus \mathbb{K}K$ be the centrally extended loop algebra with the commutation relations given by

$$[K, \ -]_{\widehat{\mathfrak{g}}} = 0 \text{ (i.e. } K \text{ is central)},$$
$$[a \otimes f(t), b \otimes g(t)]_{\widehat{\mathfrak{g}}} = [a, b]_{\mathfrak{g}} \otimes f(t)g(t) - (a, b)K Res_t(fdg)$$

Remark 1.3.2. Recall that, since $\mathfrak{g}$ is finite dimensional, $\mathfrak{g} \otimes \mathbb{K}((t))$ and $\mathfrak{g}((t))$ are isomorphic as vector spaces as well as $\mathfrak{g} \otimes \mathbb{K}[[t]]$ and $\mathfrak{g}[[t]]$.

Remark 1.3.3. $-(a, b)Res_t(fdg)$ is indeed a 2-cocycle of the Chevalley-Eilenberg cochain complex of the Lie algebra $\mathfrak{g}((t))$ with coefficients in the trivial $\mathfrak{g}((t))$-module:

$$\mathbb{K} \xrightarrow{d_{\mathfrak{g}((t))}} \mathrm{Hom}_{\mathbb{K}}\left(\mathfrak{g}((t)), \mathbb{K}\right) \xrightarrow{d_{\mathfrak{g}((t))}} \mathrm{Hom}_{\mathbb{K}}\left(\Lambda^2 \mathfrak{g}((t)), \mathbb{K}\right) \xrightarrow{d_{\mathfrak{g}((t))}} \dots \tag{1.24}$$

with the differential $d_{\mathfrak{g}((t))} : \mathrm{Hom}_{\mathbb{K}}\left(\Lambda^m \mathfrak{g}((t)), \mathbb{K}\right) \to \mathrm{Hom}_{\mathbb{K}}\left(\Lambda^{m+1} \mathfrak{g}((t)), \mathbb{K}\right)$ given by

$$\begin{aligned} &d_{\mathfrak{g}((t))}(f)(x_1 \wedge \dots \wedge x_{m+1}) \\ &\qquad = \sum_{i<j} (-1)^{i+j} f([x_i, x_j] \wedge x_1 \wedge \dots \wedge \widehat{x_i} \wedge \dots \wedge \widehat{x_j} \dots \wedge x_{m+1}) \end{aligned}$$

where $x_1, \dots, x_{m+1} \in \mathfrak{g}((t))$.
Note that, since the action of $\mathfrak{g}((t))$ on $\mathbb{K}$ is zero, the following term of the Chevalley-Eilenberg differential does not appear

$$\sum_{i=1}^{m+1} (-1)^i x_i . f(x_1 \wedge \dots \wedge \widehat{x_i} \wedge \dots \wedge x_{m+1})$$

where "." denotes the action of $\mathfrak{g}((t))$ on $\mathbb{K}$.
Define

$$\begin{aligned} c : \mathfrak{g}((t)) \otimes \mathfrak{g}((t)) &\to \mathbb{K} \\ (a \otimes f(t)) \otimes (b \otimes g(t)) &\mapsto -(a, b) Res_t(fdg). \end{aligned}$$

The map c is skew-symmetric because $(\ ,\ )$ is symmetric and $Res_t(fdg) = -Res_t(gdf)$ (because of the Leibinz Rule and $Res_t(d(fg)) = 0$). Moreover c satisfies the equation

$$\begin{aligned} c((a \otimes f(t))\otimes&[b \otimes g(t), d \otimes h(t)]) \\ &+c((b \otimes g(t)) \otimes [d \otimes h(t), a \otimes f(t)]) \\ &\quad + c((d \otimes h(t)) \otimes [a \otimes f(t), b \otimes g(t)]) = 0 \end{aligned} \tag{1.25}$$

because $(\ ,\ )$ is symmetric and invariant, and $Res_t d(fgh) = 0$. Since c is skew-symmetric, $c \in \mathrm{Hom}_{\mathbb{K}}(\Lambda^2 \mathfrak{g}((t)), \mathbb{K})$. It follows that equation (1.25) implies

$$\begin{aligned} -c([b \otimes g(t),&d \otimes h(t)] \wedge (a \otimes f(t))) \\ &+c([a \otimes f(t), d \otimes h(t)] \wedge (b \otimes g(t))) \\ &\quad - c([a \otimes f(t), b \otimes g(t)] \wedge (d \otimes h(t))) = 0. \end{aligned} \tag{1.26}$$

Let $k \in \mathbb{K}$ and $\mathbb{K}_k$ be the one-dimensional representation of $\mathfrak{g}[[t]] \oplus \mathbb{K}K$ on which K acts as multiplication by the scalar k and $\mathfrak{g}[[t]]$ acts by 0. Then, let us consider the vacuum module of level k of $\widehat{\mathfrak{g}}$:

$$V_k(\mathfrak{g}) = U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[[t]]\oplus\mathbb{K}K)} \mathbb{K}_k.$$

For any $x \in \mathfrak{g}$, let us define the field

$$x(z) = \sum_{i \in \mathbb{Z}} (x \otimes t^i) z^{-i-1} \tag{1.27}$$

and let us define $x_+(z)$ as the regular part of $x(z)$ with respect to z and $x_-(z) = x(z) - x_+(z)$. Let us moreover consider the normally ordered product:

$$: x^1(z)x^2(z) := x^1_+(z)x^2(z) + x^2(z)x^1_-(z)$$

and inductively

$$: x^1(z) \cdots x^m(z) := x^1_+(z) : x^2(z) \cdots x^m(z) : + : x^2(z) \cdots x^m(z) : x^1_-(z).$$

One can define a vertex algebra structure on $V_k(\mathfrak{g})$ as follows:

- the vacuum vector is a highest weight vector of $V_k(\mathfrak{g})$,
- the map Y is given by the following formula
$$Y(x^1_+(u_1) \cdots x^m_+(u_m)|0\rangle, z) = i_{z,u_1} \cdots i_{z,u_m} : x^1(z+u_1) \cdots x^m(z+u_m) :, \tag{1.28}$$
- the translation operator is given by the formula
$$e^{zT} x^1_+(u_1) \cdots x^m_+(u_m)|0\rangle = x^1_+(z+u_1) \cdots x^m_+(z+u_m)|0\rangle \tag{1.29}$$

(see [K] or [FBZ] for the proof that the given structure endows $V_k(\mathfrak{g})$ with a structure of vertex algebra). Note that $V_k(\mathfrak{g})$ is spanned by elements

$$(x^1 \otimes t^{-i_1-1}) \cdots (x^n \otimes t^{-i_n-1})|0\rangle$$

with $i_1, \dots, i_n \geq 0$. In other words $V_k(\mathfrak{g})$ is spanned by the coefficients of

$$x^1_+(u_1) \cdots x^m_+(u_m)|0\rangle.$$

If $\mathfrak{g}$ is a simple Lie algebra and $(\ ,\ )$ is the invariant form normalized to have the squared norm of long roots equal to 2, then $V_k(\mathfrak{g})$ is called the *affine vertex algebra*.

1.3.2 Associativity and other properties

Proposition 1.3.4 ([K15]). *Let $(V, |0\rangle, Y, T)$ be a vertex algebra and let $a \in V$. Then*

$$Y(z)(a \otimes |0\rangle) = e^{zT}a = \sum_{k\geq 0} \frac{T^k a}{k!} z^k. \tag{1.30}$$

Proof. It follows by Proposition 1.2.3 (1). □

Let us consider some statements (borrowed from [FBZ]) which prove that a vertex algebra satisfies the Associativity Relation (1.20):

Lemma 1.3.5 (Goddard's Uniqueness Theorem, [G]). *Let $(V, |0\rangle, Y, T)$ be a vertex algebra and let $a \in V$ and $a(z) \in End_{\mathbb{K}}V[[z, z^{-1}]]$ be an $End_{\mathbb{K}}V$-valued quantum field such that*

$$a(z)|0\rangle = Y(z)(a \otimes |0\rangle)$$

and $a(z)$ is local with $Y(z)(c \otimes -)$ for all $c \in V$. Then

$$a(z) = Y(z)(a \otimes -). \tag{1.31}$$

Proof. Since $a(z)$ and $Y(z)(a \otimes -)$ are local with $Y(z)(c \otimes -)$ for all $c \in V$, there exists $N = N(a, c) \geq 0$ such that

$$\begin{aligned}(z-w)^N a(z)Y(w)(c \otimes |0\rangle) &= (z-w)^N Y(w)\big(c \otimes a(z)|0\rangle\big)\\ &= (z-w)^N Y(w)\big(1 \otimes Y(z)\big)(c \otimes a \otimes |0\rangle)\\ &= (z-w)^N Y(z)\big(1 \otimes Y(w)\big)(a \otimes c \otimes |0\rangle).\end{aligned}$$

By Proposition 1.3.4, it follows that

$$(z-w)^N a(z)e^{wT}c = (z-w)^N Y(z)(a \otimes e^{wT}c). \tag{1.32}$$

Since both sides of equation (1.32) have only non-negative powers on w, one can set $w = 0$ obtaining

$$z^N a(z)c = z^N Y(z)(a \otimes c).$$

Therefore $a(z)c = Y(z)(a \otimes c)$ for any $c \in V$. □

Corollary 1.3.6. *In a vertex algebra $(V, |0\rangle, Y, T)$, the following equality holds*

$$Y(z)(T \otimes 1) = \partial_z Y(z).$$

In particular

$$TY(z) = Y(z)(T \otimes 1) + Y(z)(1 \otimes T).$$

Proof. It's a direct consequence of Lemma 1.3.5. Indeed, by Lemma 1.3.4, we have

$$\partial_z Y(z)(a \otimes |0\rangle) = Y(z)(Ta \otimes |0\rangle)$$

and, if $(z-w)^N Y(z)\big(1 \otimes Y(w)\big)(a \otimes c \otimes -) = (z-w)^N Y(w)\big(1 \otimes Y(z)\big)(c \otimes a \otimes -)$, then

$$(z-w)^{N+1}\partial_z Y(z)\big(1 \otimes Y(w)\big)(a \otimes c \otimes -) = (z-w)^{N+1} Y(w)\big(1 \otimes \partial_z Y(z)\big)(c \otimes a \otimes -).$$

Therefore $\partial_z Y(z)(a \otimes -)$ is local with $Y(w)(c \otimes -)$ for any $c \in V$. Finally, by (A2), one has $\partial_z Y(z) = TY(z) - Y(z)(1 \otimes T)$ therefore $TY(z) = Y(z)(T \otimes 1) + Y(z)(1 \otimes T)$. □

Remark 1.3.7. By the definition of vertex algebra and Corollary 1.3.6, it follows that a vertex algebra is a pointed vector space with a state-field correspondence whose quantum fields are pairwise local.

Lemma 1.3.8 (Skew-symmetry)**.** *Let* $(V, |0\rangle, Y, T)$ *be a vertex algebra. Then the following identity holds in* $V((z))$*:*

$$Y(z)(a \otimes b) = e^{zT} Y(-z)(b \otimes a) \tag{1.33}$$

for any $a, b \in V$*.*

Proof. By Locality (1.23) there exists $N_1 \in \mathbb{Z}_+$ such that

$$(z-w)^N Y(z)(1 \otimes Y(w))(a \otimes b \otimes |0\rangle) = (z-w)^N Y(w)(1 \otimes Y(z))(b \otimes a \otimes |0\rangle).$$

for any $N \geq N_1$. By Proposition 1.3.4, and Proposition 1.2.3, one has:

$$\begin{aligned}(z-w)^N Y(z)(a \otimes e^{wT} b) &= (z-w)^N Y(w)(b \otimes e^{zT} a) \\ &= (z-w)^N e^{zT} e^{-zT} Y(w)(b \otimes e^{zT} a) \\ &= (z-w)^N e^{zT} i_{w,z} Y(w-z)(b \otimes a).\end{aligned}$$

Since $Y(w-z)(b \otimes a) \in V((w-z))$, there exists $N_2 \in \mathbb{Z}_+$ such that $(z-w)^{N_2} Y(w-z)(b \otimes a) \in V[[w-z]]$. Therefore, for any $N \geq max\{N_1, N_2\}$, one can evaluate $(z-w)^N Y(z)(a \otimes e^{wT} b)$ and $(z-w)^N e^{zT} i_{w,z} Y(w-z)(b \otimes a)$ in $w = 0$ obtaining the equation

$$z^N Y(z)(a \otimes b) = z^N e^{zT} Y(-z)(b \otimes a).$$

From which $Y(z)(a \otimes b) = e^{zT} Y(-z)(b \otimes a)$. □

Proposition 1.3.9. *The map* Y *of a vertex algebra satisfies the Associativity Relation* (1.20)*.*

Proof. Using the skew-symmetry (1.33), and Proposition 1.2.3, one has

$$\begin{aligned}&i_{z,w} Y(z+w)(1 \otimes Y(w))(a \otimes b \otimes c) \\ &\quad = i_{z,w} Y(z+w)(1 \otimes Y(w))(a \otimes b \otimes c) \\ &\quad = e^{wT} Y(z)(1 \otimes e^{-wT})(1 \otimes e^{wT} Y(-w))(a \otimes c \otimes b) \\ &\quad = e^{wT} Y(z)(1 \otimes Y(-w))(a \otimes c \otimes b).\end{aligned}$$

On the other hand, using Lemma 1.3.8, one has

$$\begin{aligned}&Y(w)(Y(z) \otimes 1)(a \otimes b \otimes c) \\ &\quad = Y(w)(Y(z) \otimes 1)(a \otimes b \otimes c) \\ &\quad = e^{wT}(e^{-wT} Y(w))(Y(z) \otimes 1)(a \otimes b \otimes c) \\ &\quad = e^{wT} Y(-w)(1 \otimes Y(z))(c \otimes a \otimes b).\end{aligned}$$

Therefore, by Locality (1.23), there exists $N \geq 0$ such that

$$(z+w)^N i_{z,w} Y(z+w)(1 \otimes Y(w))(a \otimes b \otimes c)$$

$$
\begin{aligned}
&= (z+w)^N e^{wT} Y(z)(1 \otimes Y(-w))(a \otimes c \otimes b) \\
&= (z+w)^N e^{wT} Y(-w)(1 \otimes Y(z))(c \otimes a \otimes b) \\
&= (z+w)^N Y(w)(Y(z) \otimes 1)(a \otimes b \otimes c).
\end{aligned}
$$

□

Let us make some comments on Lemma 1.3.8 and Proposition 1.3.9 due to [BK]. First of all, Lemma 1.3.8 can be restated saying that in a vertex algebra the state-fields correspondences Y and Y^{op} coincide; second, we can weaken the hypothesis of Lemma 1.3.8. Indeed the proof only uses that $Y(z)(a \otimes -)$ and $Y(w)(b \otimes -)$ are local on $|0\rangle$. Therefore we can state the following lemma:

Lemma 1.3.10. *Let $(V, |0\rangle)$ be a pointed vector space with a state-field correspondence Y such that $Y(z)(a \otimes -)$ and $Y(w)(b \otimes -)$ are local on $|0\rangle$ for $a, b \in V$. Then the following identity holds in $V((z))$:*

$$Y(z)(a \otimes b) = e^{zT} Y(-z)(b \otimes a). \tag{1.34}$$

Finally we can weaken the hypothesis of Proposition 1.3.9 indeed the proof only uses that $Y(a, z)$ and $Y(c, w)$ are local on b. We can therefore state the following proposition:

Proposition 1.3.11. *Let $(V, |0\rangle)$ be a pointed vector space with a state-field correspondence Y satisfying the locality on any vector of V. Then the map Y satisfies the Associativity Relation* (1.20)*.*

It follows, by Proposition 1.3.9 and Lemma 1.3.8, that a vertex algebra $(V, |0\rangle, Y, T)$ is a field algebra satisfying $Y = Y^{op}$. The converse is also true:

Theorem 1.3.12 ([BK, Thm.7.3])**.** *A vertex algebra is the same as a field algebra $(V, |0\rangle, Y)$ for which $Y = Y^{op}$.*

We omit the proof because the statement will be generalized in Theorem 2.4.5.

Corollary 1.3.13 ([BK])**.** *Let $(V, |0\rangle)$ be a pointed vector space with a state-field correspondence Y. Y is local if and only if it is local on every vector of V.*

Proof. If Y is a state-field correspondence local on every vector of V, by Lemma 1.3.10 and Proposition 1.3.11, $Y = Y^{op}$ and the Associativity Relation (1.20) holds. Therefore, by Theorem 1.3.12, V is a vertex algebra from which the Locality (1.23) follows. □

Switching to the equivalent notation of Y as a map $V \to End_{\mathbb{K}} V[[z, z^{-1}]]$ and using Theorem 1.1.7, the following proposition holds:

Proposition 1.3.14 ([K])**.** *For every elements a, b in a vertex algebra, one has*

$$[Y(a,z), Y(b,w)] = \sum_{\substack{j \geq 0 \\ finite}} c^j(w) \frac{\partial_w^j \delta(z,w)}{j!}, \tag{1.35}$$

where

$$c^j(w) = Res_z (z-w)^j [Y(a,z), Y(b,w)].$$

Let us recall the definition of the n-products:

Definition 1.3.2 ([BK])**.** Let $(V, |0\rangle)$ be a pointed vector space with a state-field correspondence Y (cf. Definition 1.2.1). For any $a, b \in V$ and $n \in \mathbb{Z}$, one defines the *n-product* of the $End_{\mathbb{K}}V$-valued fields $Y(a, z)$ and $Y(b, z)$ by the following formula:

$$\Big(Y(a,z)_{(n)}Y(b,z)\Big)(c) = Res_x\Big(Y(x)\big(1\otimes Y(z)\big)(a\otimes b\otimes c)\ i_{x,z}(x-z)^n \\ - Y(z)\big(1\otimes Y(x)\big)(b\otimes a\otimes c)\ i_{z,x}(x-z)^n\Big). \tag{1.36}$$

Remark 1.3.15. Note that, if $n \geq 0$, $Y(a, z)_{(n)}Y(b, z) = Res_x([Y(a, x), Y(b, z)](x - z)^n)$ and equation (1.35) can be written as

$$[Y(a,z), Y(b,w)] = \sum_{\substack{j\geq 0\\ finite}} Y(a,z)_{(j)}Y(b,z)\frac{\partial_w^j\delta(z,w)}{j!}. \tag{1.37}$$

Theorem 1.3.16. *Let $(V, |0\rangle)$ be a pointed vector space with a state-field correspondence Y. V is a vertex algebra if and only if the following equation, called* Jacobi Identity*, holds:*

$$i_{z,w}\delta(x, z-w)Y(z)\big(1\otimes Y(w)\big)(a\otimes b\otimes c) \\ -i_{w,z}\delta(x, z-w)Y(w)\big(1\otimes Y(z)\big)(b\otimes a\otimes c) \\ = i_{z,x}\delta(w, z-x)Y(w)\big(Y(x)\otimes 1\big)(a\otimes b\otimes c). \tag{1.38}$$

Proof. Let V be a vertex algebra. By Locality (1.23) and Associativity Relation (1.20) with z renamed as $z - w$ one has that there exist N, N' satisfying

$$(z-w)^N Y(z)\big(1\otimes Y(w)\big)(a\otimes b\otimes c) \\ = (z-w)^N Y(w)\big(1\otimes Y(z)\big)(b\otimes a\otimes c)$$

and

$$z^{N'} Y(w)\big(Y(z-w)\otimes 1\big)(a\otimes b\otimes c) \\ = z^{N'} i_{z-w,w} Y(z)\big(1\otimes Y(w)\big)(a\otimes b\otimes c)$$

The Jacobi identity follows by Lemma 1.1.8.
Conversely, if V is a pointed vector space satisfying the Jacoby Identity (1.38), by Lemma 1.1.8, the state-field correspondence satisfies the locality on any vector of V and the Associativity Relation (1.20). In particular, by Corollary 1.3.13, V is a vertex algebra. □

As far as we know, equation (1.38) appeared for the first time in [FLM, Section 8.8] and Theorem 1.3.16 was already proved in [LL, Theorem 3.6.3].

Theorem 1.3.17. *Let $(V, |0\rangle)$ be a pointed vector space with a state-field correspondence Y. If the Jacobi Identity* (1.38) *holds, then Y satisfies the following identities called the* Borcherds Identities*: for any $a, b, c \in V$ and $n \in \mathbb{Z}$*

$$Y(z)\big(1\otimes Y(w)\big)(a\otimes b\otimes c)i_{z,w}(z-w)^n \\ -Y(w)\big(1\otimes Y(z)\big)(b\otimes a\otimes c)i_{w,z}(z-w)^n \\ = \sum_{j\geq 0} Y(w)(a_{(n+j)}b\otimes c)\frac{\partial_w^j\delta(z,w)}{j!}. \tag{1.39}$$

In particular, in a vertex algebra $(V, |0\rangle, Y, T)$*,* Y *satisfies the Borcherds Identities* (1.39)*.*

Proof. Multiplying both sides of the Jacobi Identity (1.38) by x^n and taking the residue Res_x one has, on the left hand side, the expression

$$\begin{aligned} &Y(z)(1 \otimes Y(w))(a \otimes b \otimes c)i_{z,w}(z-w)^n \\ &\quad - Y(w)(1 \otimes Y(z))(b \otimes a \otimes c)i_{w,z}(z-w)^n \end{aligned}$$

and, on the right hand side, the expression

$$\begin{aligned} &Res_x(x^n i_{z,x}\delta(w, z-x)Y(w)(Y(x) \otimes 1)(a \otimes b \otimes c)) \\ &= Y(w)(Res_x(x^n i_{z,x}\delta(w, z-x)Y(x)(a \otimes b)) \otimes c). \end{aligned}$$

On the other hand the following equalities hold:

$$\begin{aligned} &Res_x(x^n i_{z,x}\delta(w, z-x)Y(x)(a \otimes b)) \\ &Res_x(x^n e^{-x\partial_z}\delta(w, z)Y(x)(a \otimes b)) \\ &= \sum_{j \geq 0} (-1)^j a_{(n+j)} b \frac{\partial_z^j}{j!}\delta(z, w) \\ &= \sum_{j \geq 0} a_{(n+j)} b \frac{\partial_w^j}{j!}\delta(z, w). \end{aligned}$$

Therefore, on the right hand side, one has the expression

$$\sum_{j \geq 0} Y(w)(a_{(n+j)} b \otimes c)\frac{\partial_w^j}{j!}\delta(z, w).$$

□

Theorem 1.3.18. *Let* $(V, |0\rangle)$ *be a pointed vector space with a state-field correspondence* Y*. The Borcherds Identities* (1.39) *hold if and only if* Y *satisfies the Locality* (1.23) *and the following identities called the* n-products Identities*: for any* $a, b, c \in V$ *and* $n \in \mathbb{Z}$

$$\Big(Y(a, z)_{(n)}Y(b, z)\Big)(c) = Y(z)(a_{(n)}b \otimes c). \tag{1.40}$$

In particular, in a vertex algebra $(V, |0\rangle, Y, T)$*,* Y *satisfies the* n*-products Identities* (1.40) *by Theorems 1.3.16 and 1.3.17.*

Proof. Taking the residue Res_z on both sides of n-Borcherds Identities (1.39), one has on the left hand side, the n-product

$$\Big(Y(a, z)_{(n)}Y(b, z)\Big)(c)$$

and on the right hand side

$$Res_z \sum_{j \geq 0} Y(w)(a_{(n+j)} b \otimes c)\frac{\partial_w^j}{j!}\delta(z, w)$$

$$
\begin{aligned}
&= \sum_{j\geq 0} \binom{0}{j} Y(w)(a_{(n+j)}b\otimes c)w^{-j} \\
&= Y(w)(a_{(n)}b\otimes c).
\end{aligned}
$$

On the other hand, since $Y(z)\big(1\otimes Y(w)\big)(a\otimes b\otimes c)i_{z,w}(z-w)^n - Y(w)\big(1\otimes Y(z)\big)(b\otimes a\otimes c)i_{w,z}(z-w)^n$ is local, by Theorem 1.1.7, one has that

$$
\begin{aligned}
&Y(z)\big(1\otimes Y(w)\big)(a\otimes b\otimes c)i_{z,w}(z-w)^n \\
&\qquad -Y(w)\big(1\otimes Y(z)\big)(b\otimes a\otimes c)i_{w,z}(z-w)^n \\
&\qquad\qquad = \sum_{\substack{j\geq 0\\ finite}} c^j(w)\ \frac{\partial_w^j \delta(z,w)}{j!},
\end{aligned}
$$

where

$$
\begin{aligned}
c^j(w) &= Res_z\Big(Y(z)\big(1\otimes Y(w)\big)(a\otimes b\otimes c)i_{z,w}(z-w)^{n+j} \\
&\qquad - Y(w)\big(1\otimes Y(z)\big)(b\otimes a\otimes c)i_{w,z}(z-w)^{n+j}\Big) \\
&= \Big(Y(a,z)_{(n+j)}Y(b,w)\Big)(c).
\end{aligned}
$$

As, for any $m\in\mathbb{Z}$, one has

$$
\Big(Y(a,z)_{(m)}Y(b,w)\Big)(c) = Y(w)\big(a_{(m)}b\otimes c\big),
$$

the Borcherds Identities (1.39) follow. □

Remark 1.3.19. In [K15] the same results as above are obtained following a different line of reasoning: first they prove the Extension Theorem [DSK06, Thm.1.5], [K15, Thm.3.1] for a vertex algebra V. This is based on the Dong Lemma [K15, Lem.2.1]. The n-products Identities (1.40) follow as a corollary of the Extension Theorem. The Borcherds Identities (1.39) are then obtained using the n-products Identities (1.40). Unfortunately, the Dong Lemma has no sense for quantum vertex algebras so here we have preferred the previous approach that we will generalize in the next chapter to the quantum case.

Let us consider another lemma:

Lemma 1.3.20 ([BK, Thm.4.1, (a)])**.** *Let $(V,|0\rangle)$ be a pointed vector with a state-field correspondence Y. Then Y satisfies the n-product Identities if and only if*

$$
[Y(a,z), Y^{op}(b,w)] = \sum_{\substack{j\geq 0\\ finite}} Y^{op}(a_{(j)}b,w)\frac{\partial_w^j}{j!}\delta(z,w) \tag{1.41}
$$

for any $a,b\in V$.

We can summarize the results obtained so far in the following theorem which provides several equivalent definitions of a vertex algebra.

Theorem 1.3.21. *Let $(V, |0\rangle)$ be a pointed vector space with a state-field correspondence Y (cf. Definition 1.2.1). The following statements are equivalent:*

1. *V is a vertex algebra;*
2. *the Jacobi Identity* (1.38) *holds;*
3. *V is a field algebra with $Y = Y^{op}$;*
4. *the Borcherds identities* (1.39) *hold;*
5. *the n-products Identities* (1.40) *holds and $Y = Y^{op}$.*

Proof. (1) and (2) are equivalent because of Theorem 1.3.16 and (1) and (3) are equivalent because of Theorem 1.3.12. In particular (3) implies (1) which implies (2). By Theorem 1.3.17, (2) implies (4). By Theorem 1.3.18 if the Borcherds Identity (1.39) hold then the n-products Identities (1.40) are satisfied, moreover the Borcherds Identities imply the locality of any couple of $\big(Y(a,z), Y(b,w)\big)$ with $a, b \in V$ because $a_{(j)}b = 0$ for $j \gg 0$ from which $Y = Y^{op}$. Therefore (4) implies (5). By Lemma 1.3.20, Y satisfies the n-products Identities if and only if equation (1.41) holds. In particular, since $(z-w)^{j+1}\frac{\partial_w^j}{j!}\delta(z,w) = 0$, multiplying by $(z-w)^{max\{j\}+1}$ both sides of equation (1.41), one has the locality between $Y(a,z)$ and $Y^{op}(b,w)$ for any $a, b \in V$ from which the Associativity Relation (1.20) holds because of Lemma 1.2.6. Thus V is a field algebra satisfying $Y = Y^{op}$, i.e. (5) implies (1). □

1.3.3 Commutative vertex algebras

Let $(V, |0\rangle)$ be a pointed vector space with a state-field correspondence Y and let us consider the -1-product:

$$\begin{aligned} -_{(-1)}- : V \otimes V &\to V \\ a \otimes b &\mapsto a_{(-1)}b = Res_z\big(z^{-1}Y(z)(a \otimes b)\big). \end{aligned} \tag{1.42}$$

Lemma 1.3.22. *Let $(V, |0\rangle)$ be a pointed vector space with a state-field correspondence Y. The vacuum vector $|0\rangle \in V$ is the unit of the -1-product and the map T is a derivation.*

Proof. By the vacuum axioms one has

$$|0\rangle_{(-1)}a = Res_z\big(z^{-1}Y(z)(|0\rangle \otimes a)\big) = Res_z\big(z^{-1}a\big) = a$$

and, by Proposition 1.2.3,

$$a_{(-1)}|0\rangle = Res_z\big(z^{-1}Y(z)(a \otimes |0\rangle)\big) = Res_z\big(z^{-1}e^{zT}a\big) = a.$$

Hence $|0\rangle$ is the unit ot the -1-product. Moreover, since $TY(z) - Y(z)(1 \otimes T) = Y(z)(T \otimes 1)$ (cf. Definition 1.2.1), one has

$$TY(a,z)b - Y(a,z)Tb = Y(Ta,z)b.$$

Therefore, multiplying both sides by z^{-1} and taking the residue Res_z, one obtains

$$T\big(a_{(-1)}b\big) - a_{(-1)}Tb = (Ta)_{(-1)}b.$$

□

Lemma 1.3.23. *Let* $(V, |0\rangle)$ *be a pointed vector space with a state-field correspondence* Y *satisfying the Associativity Relation* (1.20) *(i.e.* V *is a field algebra, cf. Definition 1.2.3). Assume moreover that* $Y(z)(a \otimes b) \in V[[z]]$ *for any* $a, b \in V$*. Then the* -1*-product is associative.*

Proof. As $Y(z)(d \otimes e) \in V[[z]]$ for any $d, e \in V$, it follows that $Y(w)(Y(z) \otimes 1)(a \otimes b \otimes c) \in V[[w]][[z]] = V[[z, w]]$ and $i_{z,w}Y(z+w)(1 \otimes Y(w))(a \otimes b \otimes c) \in V[[z]][[w]] = V[[z, w]]$. The algebra of formal power series in z and w has no zero divisors therefore the Associativity Relation (1.20) implies

$$Y(w)\big(1 \otimes Y(z)\big)(a \otimes b \otimes c) = i_{z,w}Y(z+w)\big(1 \otimes Y(w)\big)(a \otimes b \otimes c). \tag{1.43}$$

Multiplying both sides of equation (1.43) by z^{-1} and w^{-1} and taking the residues Res_z and Res_w one has

$$(a_{(-1)}b)_{(-1)}c = a_{(-1)}(b_{(-1)}c)$$

for any $a, b, c \in V$, i.e. the associativity of the -1-product holds. □

Lemma 1.3.24. *Let* $(V, |0\rangle)$ *be a pointed vector space with a state-field correspondence* Y *such that* $Y = Y^{op}$ *and* $Y(z)(a \otimes b) \in V[[z]]$ *for any* $a, b \in V$*. Then the* -1*-product is commutative.*

Proof. Since $Y = Y^{op}$ and $Y^{op}(z)(a \otimes b) = e^{zT}Y(-z)(b \otimes a)$ for any $a, b \in V$, one has

$$Y(z)(a \otimes b) = e^{zT}Y(-z)(b \otimes a). \tag{1.44}$$

Multiplying both sides of equation (1.44) by z^{-1} and taking the residue Res_z, one has

$$a_{(-1)}b = b_{(-1)}a$$

because $Y(z)(a \otimes b) \in V[[z]]$. Therefore, the -1-product is commutative. □

With Lemmas 1.3.22, 1.3.23 and 1.3.24 we can therefore state some results firstly appeared in [K].

Definition 1.3.3 ([K])**.** A vertex algebra $(V, |0\rangle, Y, T)$ is said to be commutative if

$$Y(z)\big(1 \otimes Y(w)\big)(a \otimes b \otimes c) = Y(w)\big(1 \otimes Y(z)\big)(b \otimes a \otimes c) \tag{1.45}$$

for any $a, b, c \in V$.

Note that equation (1.45) is the Locality (1.23) with $N = 0$.

Theorem 1.3.25. *A vertex algebra* $(V, |0\rangle, Y, T)$ *is commutative if and only if* $Y(a, z)b \in V[[z]]$ *for any* $a, b \in V$*.*

Proof. Let us first suppose that $(V, |0\rangle, Y, T)$ is a commutative vertex algebra. Considering $c = |0\rangle$ in equation (1.45) one has

$$Y(z)\big(1 \otimes Y(w)\big)(a \otimes b \otimes |0\rangle) = Y(w)\big(1 \otimes Y(z)\big)(b \otimes a \otimes |0\rangle)$$

from which, using Lemma 1.2.3, one has

$$Y(z)(1\otimes e^{wT})(a\otimes b)=Y(w)(1\otimes e^{zT})(b\otimes a). \tag{1.46}$$

Multiplying both sides of equation (1.46) by w^{-1} and taking the residue Res_w, one has

$$Y(z)(a\otimes b)=b_{-1}e^{zT}a.$$

Hence $Y(z)(a\otimes b)\in V[[z]]$ proving the "only if" part.
On the other hand, if $Y(z)(a\otimes b)\in V[[z]]$ for any $a,b\in V$, then

$$Y(z)(1\otimes Y(w))(a\otimes b\otimes c)\in V[[z,w]],$$
$$Y(w)(1\otimes Y(z))(b\otimes a\otimes c)\in V[[z,w]].$$

Therefore, multiplying both sides of Locality (1.23) by $i_{z,w}(z-w)^{-N}$, the Commutativity (1.45) holds. □

Theorem 1.3.26. *The -1-product of any commutative vertex algebra $(V,|0\rangle,Y,T)$ endows V with a structure of commutative, associative unital algebra with a derivation T. Moreover the state-field correspondence Y is uniquely determined by T and the -1-product by the following formula:*

$$Y(z)(a\otimes b)=(e^{zT}a)_{(-1)}b\in V[[z]]. \tag{1.47}$$

Proof. In the proof of Theorem 1.3.25 we have seen that the state-field correspondence Y of a commutative vertex algebra satisfies the equation $Y(z)(a\otimes b)=b_{(-1)}e^{zT}a$. The statement follows immediately by Lemmas 1.3.22, 1.3.23 and 1.3.24. □

Chapter 2

Quantum vertex algebras

In Section 2.1 we recall the definitions of inverse limit, topologically free $\mathbb{K}[[h]]$-module and topological tensor product. In Section 2.2 we recall the concepts of braided vertex algebra and quantum vertex algebra and we review some results. In Section 2.2.1 we deal with $\mathcal{S}$-commutative braided vertex algebras which satisfy the associativity relation and $\mathcal{S}$-commutative quantum vertex algebras. In Section 2.3 we give a non trivial example of quantum vertex algebra firstly appeared in [EK5] as more recently formulated in [JKMY] and an example of $\mathcal{S}$-commutative quantum vertex algebra firstly appeared in [JKMY] proving in every detail that they satisfy all the axioms of the definition of quantum vertex algebras. In Section 2.4 we give some characterizations of braided vertex algebras which satisfy the associativity relation and of quantum vertex algebras.

2.1 Topologically free $\mathbb{K}[[h]]$-modules

In the following section we shall follow [Kas, Chapter XVI], the proofs omitted in this text can be found there. Let M a left module over the algebra $\mathbb{K}[[h]]$. The family $(h^nM)_{n>0}$ of submodules and the canonical $\mathbb{K}$-linear projections

$$p_n : M/h^nM \to M/h^{n-1}M$$

form an inverse system of $\mathbb{K}$-modules. One can consider the inverse limit

$$\widehat{M} = \varprojlim_n M/h^nM = \left\{ (m_n)_{n>0} \in \prod_{n>0} M/h^nM \;\middle|\; p_n(x_n) = x_{n-1} \text{ for all } n > 1 \right\}$$

which has a natural structure of $\mathbb{K}[[h]]$-module. Moreover it has a natural topology called the inverse limit topology (a family of open neighbourhoods is given by $(h^n\widehat{M})_n$). The module $\widehat{M}$ is called the *h-adic completion* of M. The projections

$$i_n : M \to M/h^nM$$

induce a unique $\mathbb{K}[[h]]$-linear map

$$\begin{aligned} i : M &\to \widehat{M} \\ m &\mapsto \left(i_n(m)\right)_{n>0} \end{aligned} \tag{2.1}$$

such that $\pi_n \circ i = i_n$ for all n where π_n is the projection

$$\begin{aligned}\pi_n : \widehat{M} &\to M/h^n M \\ (m_n)_{n>0} &\mapsto m_n.\end{aligned} \tag{2.2}$$

Note that the kernel of i is given by

$$Ker(i) = \bigcap_{n>0} h^n M \tag{2.3}$$

and that, since the kernel of π_n is $h^n\widehat{M}$, π_n induces an isomorphism

$$\widehat{M}/h^n\widehat{M} \cong M/h^n M. \tag{2.4}$$

Definition 2.1.1. A $\mathbb{K}[[h]]$-module M is *separated* if $Ker(i) = \{0\}$ and it is *complete* if the map i is surjective.

Remark 2.1.1. For any $\mathbb{K}[[h]]$-module M, the completion $\widehat{M}$ is indeed complete because taking the inverse limits on both sides of the isomorphism (2.4) one has $\widehat{\widehat{M}} \cong \widehat{M}$.

Definition 2.1.2. Let W be a $\mathbb{K}$-vector space, any left $\mathbb{K}[[h]]$-module of the form $W[[h]]$ is called a topologically free module.

Remark 2.1.2. If W is a $\mathbb{K}$-vector space, then $W \otimes \mathbb{K}[[h]] \cong W[[h]]$ if and only if W has finite dimension.

Proof. First note that $W \otimes \mathbb{K}[[h]]$ is a $\mathbb{K}[[h]]$-modules with the following action

$$g(h).(w \otimes f(h)) = w \otimes g(h)f(h),$$

where $g(h), f(h) \in \mathbb{K}[[h]]$ and $w \in W$. Note also that $W \otimes \mathbb{K}[[h]]$ is isomorphic to a subspace of $W[[h]]$ by the following $\mathbb{K}[[h]]$-linear map:

$$\begin{aligned}\psi : W \otimes \mathbb{K}[[h]] &\to W[[h]] \\ w \otimes \sum_{n\geq 0} f_n h^n &\mapsto \sum_{n\geq 0} (f_n w) h^n.\end{aligned}$$

To be precise, $\psi(W \otimes \mathbb{K}[[h]])$ is the subspace of $W[[h]]$ of series in h whose coefficients are linear combination of a finite number of vectors of a basis of W. Therefore, if W has finite dimension, $W[[h]]$ is clearly equal to the image of $W \otimes \mathbb{K}[[h]]$ under ψ. Indeed if $\{w_i\}_{i=0,\ldots,m}$ is a basis of W, any element $\sum_{n\geq 0} v_n h^n \in W[[h]]$ can be written as

$$\begin{aligned}\sum_{n\geq 0} v_n h^n &= \sum_{n\geq 0} (a_1^n w_1 + \ldots + a_m^n w_m) h^n \\ &= \left(\sum_{n\geq 0} a_1^n h^n\right) w_1 + \ldots + \left(\sum_{n\geq 0} a_m^n h^n\right) w_m \\ &= \psi\left(w_1 \otimes \sum_{n\geq 0} a_1^n h^n + \ldots w_m \otimes \sum_{n\geq 0} a_m^n h^n\right)\end{aligned}$$

where $a_1^n, \ldots, a_m^n$ are elements of $\mathbb{K}$ for any $n \geq 0$. On the other hand if W has non finite dimension, by the axiom of choice we can extract a countable set of linearly independent vectors of W. Let us denote these vectors as $\{w_i\}_{i\geq 0}$. Thus

$$\sum_{i\geq 0} w_i h^i$$

is an element of $W[[h]]$ which is not in the image under ψ of $W \otimes \mathbb{K}[[h]]$. □

Proposition 2.1.3. *(a) Let $\{e_i\}_{i\in I}$ be a basis of the vector space V. Then the $\mathbb{K}[[h]]$-submodule generated by the set $\{e_i\}_{i\in I}$ is dense in $V[[h]]$ for the h-adic topology.*
(b) For any separated, complete $\mathbb{K}[[h]]$-module N, there is a natural bijection

$$\mathrm{Hom}_{\mathbb{K}[[h]]}\left(V[[h]], N\right) \cong \mathrm{Hom}_{\mathbb{K}}\left(V, N\right).$$

Let M, N be left $\mathbb{K}[[h]]$-modules. Let us recall that we denote with $M \otimes_{\mathbb{K}[[h]]} N$ the $\mathbb{K}[[h]]$-module obtained as the quotient of the vector space $M \otimes N$ by the subspace generated by the elements of the form $fm \otimes n - m \otimes fn$ where f, m and n are respectively elements of $\mathbb{K}[[h]], M$ and N.

Definition 2.1.3. The topological tensor product $M\widehat{\otimes}N$ of the left $\mathbb{K}[[h]]$-modules M and N is the h-adic completion of $M \otimes_{\mathbb{K}[[h]]} N$:

$$M\widehat{\otimes}N = (M \widehat{\otimes_{\mathbb{K}[[h]]}} N) = \varprojlim_{n>0}(M \otimes_{\mathbb{K}[[h]]} N)/h^n(M \otimes_{\mathbb{K}[[h]]} N). \tag{2.5}$$

By definition, the topological tensor product of two left $\mathbb{K}[[h]]$-modules is complete (because it is defined as a completion). Moreover, for any left $\mathbb{K}[[h]]$-modules M, N, P, the following isomorphisms hold:

$$(M\widehat{\otimes}N)\widehat{\otimes}P \cong M\widehat{\otimes}(N\widehat{\otimes}P),$$
$$M\widehat{\otimes}N \cong N\widehat{\otimes}M$$

and

$$\mathbb{K}[[h]]\widehat{\otimes}M \cong \widehat{M} \cong M\widehat{\otimes}\mathbb{K}[[h]].$$

Proposition 2.1.4. *If M and N are topologically free $\mathbb{K}[[h]]$-modules, then so is $M\widehat{\otimes}N$. More precisely, we have*

$$V[[h]]\widehat{\otimes}W[[h]] \cong (V \otimes W)[[h]].$$

Let us give a remark on Proposition 2.1.4:

Remark 2.1.5. Let V, W be two $\mathbb{K}$-vector spaces. The $\mathbb{K}[[h]]$-module $V[[h]] \otimes_{\mathbb{K}[[h]]} W[[h]]$ is isomorphic to $(V \otimes W)[[h]]$ if and only if at least one between V and W has finite dimension.

Proof. Let us first point out that, since the $\mathbb{K}[[h]]$-linear map

$$i : V[[h]] \otimes_{\mathbb{K}[[h]]} W[[h]] \to V[[h]]\widehat{\otimes}W[[h]]$$

defined as in (2.1) is injective, the $\mathbb{K}[[h]]$-linear map j defined as the composition of i and the isomorphism between $V[[h]]\widehat{\otimes}W[[h]]$ and $(V\otimes W)[[h]]$

$$j : V[[h]] \otimes_{\mathbb{K}[[h]]} W[[h]] \to (V\otimes W)[[h]]$$
$$\sum_{i\geq 0} v_i h^i \otimes_{\mathbb{K}[[h]]} \sum_{k\geq 0} w_k h^k \mapsto \sum_{p\geq 0}\left(\sum_{i=0}^{p} v_i \otimes w_{p-i}\right) h^p$$

is injective. In particular $V[[h]] \otimes_{\mathbb{K}[[h]]} W[[h]]$ is isomorphic to a subspace of $(V\otimes W)[[h]]$. Let us, now, suppose that W has finite dimension and let n be its dimension. Any element of $(V\otimes W)[[h]]$ is a linear combination of elements of the form $\sum_{i\geq 0}(v_i\otimes w_i)h^i$. On the other hand, for any basis of $(\bar{w}_k)_{k=1,\dots,n}$ of W, one has $w_i = \sum_{k=1}^{n} a_{ki}\bar{w}_k$ with $a_{ki}\in\mathbb{K}$ for any $k=1,\dots,n$, from which

$$\begin{aligned}\sum_{i\geq 0}(v_i\otimes w_i)h^i &= \sum_{i\geq 0}\left(v_i\otimes \sum_{k=1}^{n} a_{ki}\bar{w}_k\right)h^i\\ &= \sum_{k=1}^{n}\sum_{i\geq 0}\left((a_{ki}v_i)\otimes \bar{w}_k\right)h^i\\ &= j\left(\sum_{k=0}^{n}\left(\sum_{i\geq 0}(a_{ki}v_i)h^i \otimes_{\mathbb{K}[[h]]} \bar{w}_k\right)\right).\end{aligned}$$

It follows that the $\mathbb{K}[[h]]$-linear map j is surjective hence it is an isomorphism. On the other hand if V and W are both non finite dimensional $\mathbb{K}$-vector space, we can extract a countable family of vectors of V, $(v_i)_{i\in\mathbb{N}}$, and a countable family of vectors of W, $(w_i)_{i\in\mathbb{N}}$ and we can consider the following element of $(V\otimes W)[[h]]$:

$$s = \sum_{p\geq 0}\left(\sum_{i=0}^{p(p+1)} v_i\otimes w_i\right)h^p. \tag{2.6}$$

If there exists an element

$$\sum_{r=1}^{m}\sum_{l\geq 0} v_l^r h^l \otimes_{\mathbb{K}[[h]]} \sum_{k\geq 0} w_k^r h^k \in V[[h]] \otimes_{\mathbb{K}[[h]]} W[[h]]$$

such that its image under j is equal to s, we should have

$$\sum_{p\geq 0}\left(\sum_{r=1}^{m}\sum_{l=0}^{p} v_l^r\otimes w_{p-l}^r\right)h^p = \sum_{p\geq 0}\left(\sum_{i=0}^{p(p+1)} v_i\otimes w_i\right)h^p$$

from which

$$\sum_{r=1}^{m}\sum_{l=0}^{p} v_l^r\otimes w_{p-l}^r = \sum_{i=0}^{p(p+1)} v_i\otimes w_i$$

for any $p\geq 0$. The vector space

$$V_L = \langle\sum_{r=1}^{m}\sum_{l=0}^{p} v_l^r\otimes f(w_{p-l}^r) \mid f\in W^*\rangle$$

is a subspace of V with dimension at most $m(p+1)$ where m is fixed. On the other hand the vector space

$$V_R = \langle \sum_{i=0}^{p(p+1)} v_i \otimes f(w_i) \mid f \in W^* \rangle$$

is a subspace of V with dimension $p(p+1)+1$ because the vectors $v_0, \dots, v_{p^2+p}$ are linearly independent and there exist in W^* the elements w_i^* defined as $w_i^*(w_j) = \delta_{ij}$. It follows that, for any $p \in \mathbb{N}$ such that $p \geq m$, $p(p+1)+1 > m(p+1)$, hence the dimension of the vector space V_R is greater than the dimension of the vector space V_L. We have thus proved that j is not an isomorphism because s is not in the image of j. □

2.2 Braided and quantum vertex algebras

Definition 2.2.1 ([EK5]). A *braided vertex algebra* over $\mathbb{K}[[h]]$ is the following data:

1. a topologically free $\mathbb{K}[[h]]$-module V;
2. a vector $|0\rangle \in V$ (called the vacuum vector);
3. a $\mathbb{K}[[h]]$-linear map
$$Y : V \otimes V \to V_h((z)),$$
where $V_h((z)) = \{\sum_{n\in\mathbb{Z}} a_n z^{-n-1} | \ a_n \in V,\ a_n \to 0 \text{ for } n \to \infty\}$, i.e. $V_h((z))$ is the $\mathbb{K}[[h]]$-module of series $a(z) \in V[[z, z^{-1}]]$ such that $a(z) \in V((z)) \bmod h^M$ for any $M \in \mathbb{Z}_+$;
4. a $\mathbb{K}[[h]]$-linear operator $T : V \to V$;
5. a $\mathbb{K}[[h]]$-linear map $\mathcal{S} : V \otimes V \to V \otimes V \otimes \mathbb{K}((z))[[h]]$ such that $\mathcal{S} = 1 + O(h)$ which satisfies the shift condition
$$[T \otimes 1, \mathcal{S}(z)] = -\partial_z \mathcal{S}(z), \tag{2.7}$$
the quantum Yang-Baxter equation
$$\mathcal{S}^{12}(z)\mathcal{S}^{13}(z+w)\mathcal{S}^{23}(w) = \mathcal{S}^{23}(w)\mathcal{S}^{13}(z+w)\mathcal{S}^{12}(z) \tag{2.8}$$
and the unitary condition
$$\mathcal{S}^{21}(z) = \mathcal{S}^{-1}(-z) \tag{2.9}$$
where $\mathcal{S}^{21}(z) := (1\ 2)\mathcal{S}(z)(1\ 2)$ and $(1\ 2)(a \otimes b) := b \otimes a$, for any $a, b \in V$;

subject to the following axioms:

(QA1) ($\mathcal{S}$-locality) for any $a, b \in V$ and any $M \in \mathbb{Z}_+$ there exists $N = N(a, b, M) \geq 0$ such that for any $c \in V$
$$\begin{aligned}(z-w)^N Y(z)(1 \otimes Y(w))(\mathcal{S}(z-w)(a \otimes b) \otimes c)\\ = (z-w)^N Y(w)(1 \otimes Y(z))(b \otimes a \otimes c) \mod h^M;\end{aligned} \tag{2.10}$$

(QA2) (Translation Covariance) $\partial_z Y(z) = TY(z) - Y(z)(1 \otimes T)$ and $T|0\rangle = 0$;

(QA3) (Vacuum Axioms)$Y(z)(|0\rangle \otimes b) = b$ for all $b \in V$; for any $a \in V$, $Y(z)(a \otimes |0\rangle) \in V[[z]]$ and $Y(z)(a \otimes |0\rangle)_{|z=0} = a$.

Remark 2.2.1. Here and below the tensor products are understood in the h-adic completed sense as explained in the following. Since V is a topologically free $\mathbb{K}[[h]]$-module, $W = V/hV$ is a $\mathbb{K}$-vector space such that $V = W[[h]]$. We denote with

$$V \otimes V$$

the tensor product

$$V\widehat{\otimes}V = W[[h]]\widehat{\otimes}W[[h]] \cong (W \otimes W)[[h]], \tag{2.11}$$

where "$\widehat{\otimes}$" is defined as in Definition 2.1.3, the last "$\otimes$" is the tensor product over $\mathbb{K}$ and the isomorphism in equation (2.11) is due to Proposition 2.1.4. Similarly, we denote with

$$V \otimes V \otimes \mathbb{K}((z))[[h]]$$

the tensor product

$$V\widehat{\otimes}V\widehat{\otimes}\mathbb{K}((z))[[h]] \cong (W \otimes W \otimes \mathbb{K}((z)))[[h]], \tag{2.12}$$

again, "$\widehat{\otimes}$" is defined as in Definition 2.1.3, the last "$\otimes$"s are the tensor products over $\mathbb{K}$ and the isomorphism in equation (2.12) is due to Proposition 2.1.4. Let us also point out that, denoting $V = W[[h]]$, we can also write $V_h((z))$ in a more expressive way

$$V_h((z)) = V((z))[[h]] = W[[h]]((z))[[h]] = W((z))[[h]]. \tag{2.13}$$

The only equality to clarify is the last one. On one side one has that $W((z))[[h]] \subseteq W[[h]]((z))[[h]]$ because $W \subseteq W[[h]]$. On the other side, any elements of $W[[h]]((z))[[h]]$ is of the form

$$\sum_{i\geq 0}\sum_{j\geq J_i}\sum_{k\geq 0} w_{ijk}h^k z^j h^i = \sum_{p\geq 0}\sum_{i=0}^{p}\sum_{j\geq J_i} w_{ijp-i}z^j h^p.$$

Defining $\bar{P}_p = min\{J_0, \dots, J_p\}$, one then has

$$\begin{aligned}\sum_{p\geq 0}\sum_{i=0}^{p}\sum_{j\geq J_i} w_{ijp-i}z^j h^p &= \sum_{p\geq 0}\sum_{i=0}^{p}\sum_{j\geq \bar{J}_p} w_{ijp-i}z^j h^p \\ &= \sum_{p\geq 0}\sum_{j\geq \bar{J}_p}\left(\sum_{i=0}^{p} w_{ijp-i}\right) z^j h^p \in W((z))[[h]].\end{aligned}$$

It follows that $W[[h]]((z))[[h]] \subseteq W((z))[[h]]$, hence $W[[h]]((z))[[h]] = W((z))[[h]]$. Therefore, we have that Y is a $\mathbb{K}[[h]]$-linear map defined between the $\mathbb{K}[[h]]$-modules $(W \otimes W)[[h]]$ and $W((z))[[h]]$:

$$Y : (W \otimes W)[[h]] \to W((z))[[h]].$$

And $\mathcal{S}$ is a $\mathbb{K}[[h]]$-linear map defined between the $\mathbb{K}[[h]]$-modules $(W \otimes W)[[h]]$ and $(W \otimes W \otimes \mathbb{K}((z)))[[h]]$:

$$\mathcal{S} : (W \otimes W)[[h]] \to (W \otimes W \otimes \mathbb{K}((z)))[[h]].$$

Finally, the negative powers of $z - w$ in the left hand side of $\mathcal{S}$-locality (2.10) must be expanded where $|z| > |w|$, i.e. we have to think about

$$(z-w)^N Y(z)\big(1 \otimes Y(w)\big)\big(\mathcal{S}(z-w)(a \otimes b) \otimes c\big)$$

as

$$(z-w)^N Y(z)\big(1 \otimes Y(w)\big)\big(i_{z,w}\mathcal{S}(z-w)(a \otimes b) \otimes c\big).$$

This way we have that

$$Y(z)\big(1 \otimes Y(w)\big)\big(i_{z,w}\mathcal{S}(z-w) \otimes 1\big) : (W \otimes W \otimes W)[[h]] \to W((z))((w))[[h]]$$

is a well defined $\mathbb{K}[[h]]$-linear map. Indeed, by Proposition 2.1.4, we have that

$$(W \otimes W \otimes W)[[h]] \cong (W \otimes W)[[h]]\widehat{\otimes}W[[h]].$$

The map $\mathcal{S}(z-w) \otimes 1$ brings

$$(W \otimes W)[[h]]\widehat{\otimes}W[[h]]$$

to

$$\big(W \otimes W \otimes \mathbb{K}((z-w))\big)[[h]]\widehat{\otimes}W[[h]] \cong W[[h]]\widehat{\otimes}(W \otimes W)[[h]]\widehat{\otimes}\mathbb{K}((z-w))[[h]]$$

where we used Proposition 2.1.4, the isomorphism $W \otimes W \otimes \mathbb{K}((z-w)) \otimes W \cong W \otimes W \otimes W \otimes \mathbb{K}((z-w))$ and again Proposition 2.1.4. Applying the map $1 \otimes Y(w)$ we move from

$$W[[h]]\widehat{\otimes}(W \otimes W)[[h]]\widehat{\otimes}\mathbb{K}((z-w))[[h]]$$

to

$$W[[h]]\widehat{\otimes}W((w))[[h]]\widehat{\otimes}\mathbb{K}((z-w))[[h]] \cong \big(W \otimes W((w))\big)[[h]]\widehat{\otimes}\mathbb{K}((z-w))[[h]].$$

Moreover $W \otimes W((w))$ is a $\mathbb{K}[[h]]$-submodule of $(W \otimes W)((w))$ hence we landed in

$$(W \otimes W)((w))[[h]]\widehat{\otimes}\mathbb{K}((z-w))[[h]].$$

The map $Y(z)$ brings $(W \otimes W)((w))[[h]]\widehat{\otimes}\mathbb{K}((z-w))[[h]]$ to

$$W((z))[[h]]((w))[[h]]\widehat{\otimes}\mathbb{K}((z-w))[[h]].$$

Now, since $W((z))[[h]]((w))[[h]] = W((z))((w))[[h]]$ (the strategy is the same used to prove that $W[[h]]((z))[[h]] = W((z))[[h]]$ a few lines above),

$$W((z))[[h]]((w))[[h]]\widehat{\otimes}\mathbb{K}((z-w))[[h]]$$

is brought to

$$W((z))((w))[[h]]\widehat{\otimes}\mathbb{K}((z-w))[[h]] \cong \big(W((z))((w)) \otimes \mathbb{K}((z-w))\big)[[h]]$$

and, since there is a map

$$\mathbb{K}((z-w)) \to \mathbb{K}((z))((w))$$
$$(z-w)^r \mapsto \sum_{l\geq 0} \binom{r}{l} z^{r-l}(-w)^l,$$

given by the expansion in $|z| > |w|$ where $\binom{r}{l}$ is defined as $\frac{r(r-1)\cdots(r-l+1)}{l!}$, it is brought to

$$(W((z))((w)) \otimes \mathbb{K}((z))((w)))[[h]].$$

And, lastly it is brought to

$$W((z))((w))[[h]]$$

by the multiplication of series which is well defined between a series in $W((z))((w))$ and a series in $\mathbb{K}((z))((w))$ as shown in the folllowing. Let

$$\sum_{i\geq I}\sum_{j\geq J_i} w_{ij} z^j w^i \in W((z))((w))$$

and

$$\sum_{k\geq K}\sum_{l\geq L_k} c_{kl} z^l w^k \in \mathbb{K}((z))((w))$$

where $I, J_i, K, L_k \in \mathbb{Z}$. Their multiplication is

$$\sum_{i\geq I}\sum_{j\geq J_i}\sum_{k\geq K}\sum_{l\geq L_k} w_{ij}c_{kl} z^j w^i z^l w^k = \sum_{p\geq I+K}\sum_{i=I}^{p-K}\sum_{m\geq J_i+L_{p-i}}\sum_{j=J_i}^{m-L_{p-i}} w_{ij}c_{p-im-j} z^m w^p.$$

For any fixed p the number of i is finite, therefore we can take $\bar{J}_p = min\{J_I, \ldots, J_{p-K}\}$ and $\bar{L}_p = min\{L_K, \ldots, J_{p-I}\}$ obtaining

$$\begin{aligned}
&\sum_{p\geq I+K}\sum_{i=I}^{p-K}\sum_{m\geq J_i+L_{p-i}}\sum_{j=J_i}^{m-L_{p-i}} w_{ij}c_{p-im-j} z^m w^p \\
&= \sum_{p\geq I+K}\sum_{i=I}^{p-K}\sum_{m\geq \bar{J}_p+\bar{L}_p}\sum_{j=J_i}^{m-L_{p-i}} w_{ij}c_{p-im-j} z^m w^p \\
&= \sum_{p\geq I+K}\sum_{m\geq \bar{J}_p+\bar{L}_p}\left(\sum_{i=I}^{p-K}\sum_{j=J_i}^{m-L_{p-i}} w_{ij}c_{p-im-j}\right) z^m w^p.
\end{aligned}$$

Note that $\sum_{i=I}^{p-K}\sum_{j=J_i}^{m-L_{p-i}} w_{ij}c_{p-im-j}$ is a finite sum for any fixed couple (p,m), from which the well definedness of the multiplication.
The previous argument will be useful when we will deal with $\mathcal{S}$-commutative braided vertex algebras (cf. Section 2.2.1). We have also showed that if we don't expand the negative powers of $z-w$ in $\mathcal{S}(z-w)$, one has

$$Y(z)(1\otimes Y(w))(\mathcal{S}(z-w)(a\otimes b)\otimes c) \in (W((z))((w))\otimes \mathbb{K}((z-w)))[[h]],$$

therefore, mod h^M for any $M \in \mathbb{N}$, we can also find a suitable power $\bar{N}$ of $z-w$ such that

$$(z-w)^{\bar{N}} Y(z)(1 \otimes Y(w))\big(\mathcal{S}(z-w)(a \otimes b) \otimes c\big) \in V((z))((w)).$$

That's why we can avoid the expansion of $\mathcal{S}(z-w)$ in the $\mathcal{S}$-locality (2.10), but we could not in the $\mathcal{S}$-commutative case (cf. Section 2.2.1) and in the definition of the quantum n-products (cf. Definition 2.4.3) where we cannot multiply for an arbitrary power of z-w.

Remark 2.2.2. By axioms (QA2) e (QA3), T is uniquely determined by Y and the vacuum vector $|0\rangle$ indeed it can be shown $T(a) = a_{(-2)}|0\rangle = Res_z(z^{-2}Y(z)(a \otimes |0\rangle))$ (similarly to the case of VAs).

Lemma 2.2.3. *Let $(V, |0\rangle, Y, T, \mathcal{S})$ be a braided vertex algebra. Then*

$$[1 \otimes T, \mathcal{S}(z)] = \partial_z \mathcal{S}(z). \tag{2.14}$$

Proof. From the shift condition (2.7) on $\mathcal{S}$, one has

$$\begin{aligned}
[\mathcal{S}^{-1}(z), T \otimes 1] &= \mathcal{S}^{-1}(z)(T \otimes 1) - (T \otimes 1)\mathcal{S}^{-1}(z) \\
&= \mathcal{S}^{-1}(z)\big((T \otimes 1)\mathcal{S}(z) - \mathcal{S}(z)(T \otimes 1)\big)\mathcal{S}^{-1}(z) \\
&= -\mathcal{S}^{-1}(z)\partial_z\mathcal{S}(z)\mathcal{S}^{-1}(z) \\
&= \partial_z\mathcal{S}^{-1}(z).
\end{aligned}$$

It follows that

$$[T \otimes 1, \mathcal{S}^{-1}(-z)] = \partial_z \mathcal{S}^{-1}(-z)$$

from which, by Unitarity (2.9),

$$[T \otimes 1, \mathcal{S}^{21}(z)] = \partial_z \mathcal{S}^{21}(z).$$

Therefore one has

$$\begin{aligned}
[1 \otimes T, S(z)] &= (1\ 2)[T \otimes 1, \mathcal{S}^{21}(z)](1\ 2) \\
&= (1\ 2)(\partial_z \mathcal{S}^{21}(z))(1\ 2) \\
&= \partial_z((1\ 2)\mathcal{S}^{21}(z)(1\ 2)) \\
&= \partial_z \mathcal{S}(z).
\end{aligned}$$

□

For braided vertex algebras, a "braided version" of the skew-symmetry holds:

Lemma 2.2.4 ([EK5, Lem.1.2])**.** *Let $(V, |0\rangle, Y, T, \mathcal{S})$ be a braided vertex algebra. For any $a, b \in V$ one has*

$$Y(z)\mathcal{S}(z)(a \otimes b) = e^{zT}Y(-z)(b \otimes a)\big(= Y^{op}(z)(a \otimes b)\big) \in V_h((z)). \tag{2.15}$$

Proof. By the $\mathcal{S}$-locality (2.10) with $c = |0\rangle$ and the point (1) of Proposition 1.2.3, one has that, for any $a, b \in V$ and $M \in \mathbb{Z}_+$, there exists $N = N(a, b, M) \geq 0$ such that

$$(z-w)^N Y(z)(1 \otimes e^{wT})\mathcal{S}(z-w)(a \otimes b) = (z-w)^N Y(w)(b \otimes e^{zT} a) \bmod h^M. \quad (2.16)$$

Using the point (2) of Proposition 1.2.3 (which holds because of the Translation Covariance of Definition 2.2.1), equation (2.16) becomes

$$(z-w)^N e^{wT} i_{z,w} Y(z-w)\mathcal{S}(z-w)(a \otimes b)(z-w)^N Y(w)(b \otimes e^{zT} a) \bmod h^M. \quad (2.17)$$

For N big enough, $(z-w)^N Y(z-w)\mathcal{S}(z-w)(a \otimes b) \in V[[z-w]] \bmod h^M$. Therefore both sides of equation (2.17) have only positive powers of z. Evaluate equation (2.17) on $z = 0$ we obtain

$$(-w)^N e^{wT} Y(-w)\mathcal{S}(-w)(a \otimes b) = (-w)^N Y(w)(b \otimes a) \bmod h^M. \quad (2.18)$$

Multiplying both sides of equation (2.18) by $(-w)^{-N} e^{-wT}$ and renaming $-w$ in z, one has

$$Y(z)\mathcal{S}(z)(a \otimes b) = e^{zT} Y(-z)(b \otimes a) \bmod h^M.$$

Since $Y(z)\mathcal{S}(z)(a \otimes b) - e^{zT} Y(-z)(b \otimes a) = 0 \bmod h^M$ for any $M \in \mathbb{Z}_+$ and V is separable as it is a topologically free $\mathbb{K}[[h]]$-module, equation (2.15) follows. $\square$

Remark 2.2.5. The left and right hand sides of equation (2.15) are well defined. Indeed, as shown in Remark 2.2.1, writing $V = W[[h]]$ where $W = V/hV$, we have

$$Y : (W \otimes W)[[h]] \to W((z))[[h]]$$

and

$$\mathcal{S} : (W \otimes W)[[h]] \to (W \otimes W \otimes \mathbb{K}((z)))[[h]] \cong (W \otimes W)\widehat{\otimes}\mathbb{K}((z))[[h]].$$

It follows that the left hand side of equation (2.15) is well defined because

$$Y(z)\mathcal{S}(z) : (W \otimes W)[[h]] \to (W((z)) \otimes \mathbb{K}((z)))[[h]] \cong W((z))[[h]].$$

As for the right hand side, since

$$T : W[[h]] \to W[[h]]$$

is a $\mathbb{K}[[h]]$-linear map,

$$e^{zT} : W[[h]] \to W[[h]][[z]] = W[[z, h]].$$

Thus $Y(-z)$ brings elements of $(W \otimes W)[[h]]$ to elements of $W((z))[[h]]$ and, applying e^{zT}, we land to $W[[h]][[z]]((z))[[h]] = W((z))[[h]]$ obtaining that the right hand side is well defined.

Remark 2.2.6. In the proof of Lemma (2.2.4) we have used a less strong requirement than $\mathcal{S}$-locality (2.10). Indeed we have only used that for any $a, b \in V$ and $M \in \mathbb{Z}_+$, there exists $N = N(a, b, |0\rangle, M) \geq 0$ such that

$$\begin{aligned} &(z-w)^N Y(z)(1 \otimes Y(w))(\mathcal{S}(z-w)(a \otimes b) \otimes |0\rangle) \\ &= (z-w)^N Y(w)(1 \otimes Y(z))(b \otimes a \otimes |0\rangle) \quad \bmod h^M. \end{aligned} \quad (2.19)$$

By Remark 2.2.6, we can state the following result:

Lemma 2.2.7. *Let V be a topologically free $\mathbb{K}[[h]]$-module satisfying all the axioms of a braided vertex algebra except, maybe, the $\mathcal{S}$-locality* (2.10)*. Let us suppose that V satisfies the following axiom: for any $a, b \in V$ and $M \in \mathbb{Z}_+$, there exists $N = N(a, b, |0\rangle, M) \geq 0$ such that*

$$\begin{aligned}&(z-w)^N Y(z)\big(1 \otimes Y(w)\big)\big(\mathcal{S}(z-w)(a \otimes b) \otimes |0\rangle\big)\\&= (z-w)^N Y(w)\big(1 \otimes Y(z)\big)(b \otimes a \otimes |0\rangle) \quad mod\ h^M.\end{aligned} \tag{2.20}$$

Then, for any $a, b \in V$ one has

$$Y(z)\mathcal{S}(z)(a \otimes b) = e^{zT} Y(-z)(b \otimes a)\big(= Y^{op}(z)(a \otimes b)\big) \in V_h((z)). \tag{2.21}$$

The following result still holds:

Corollary 2.2.8 ([EK5, Cor.1.3])**.** *$Y(z)(T \otimes 1) = \partial_z Y(z)$.*

Proof. By the Shift Condition in Definition 2.2.1 and Taylor expansion, one proves that $i_{z,u}\mathcal{S}(z+u) = \left(e^{-uT} \otimes 1\right) \mathcal{S}(z) \left(e^{uT} \otimes 1\right)$. Using Lemma 2.2.4 and Proposition 1.2.3 (2), the following equalities follow:

$$\begin{aligned}Y(z)\mathcal{S}(z)\left(e^{uT} \otimes 1\right) &= e^{zT} Y(-z)\left(1 \otimes e^{uT}\right)(1\,2) = e^{(z+u)T} i_{z,u} Y(-z-u)(1\,2)\\&= i_{z,u}\left(Y(z+u)\mathcal{S}(z+u)\right) = i_{z,u} Y(z+u)\left(e^{-uT} \otimes 1\right)\mathcal{S}(z)\left(e^{uT} \otimes 1\right).\end{aligned}$$

Since $\mathcal{S}(z)\left(e^{uT} \otimes 1\right)$ is invertible, one has

$$Y(z) = i_{z,u} Y(z+u)\left(e^{-uT} \otimes 1\right)$$

from which

$$Y(z)\left(e^{uT} \otimes 1\right) = i_{z,u} Y(z+u) = e^{u\partial_z} Y(z).$$

The lemma is proved taking the coefficient of u in both sides of the above equation. □

Remark 2.2.9. By the definition and Corollary 2.2.8, a braided vertex algebra is a pointed vector space with a state-field correspondence over $\mathbb{K}[[h]]$.

For braided vertex algebras, the Associativity Relation (1.20) does not hold, but the following identity known as Quasi-Associativity holds:

Proposition 2.2.10 ([EK5, Prop.1.1])**.** *The map Y satisfies the Quasi-Associativity relation: for any $a, b, c \in V$ and $M \in \mathbb{Z}_+$, there exists $N \geq 0$ such that*

$$\begin{aligned}&(z+w)^N i_{z,w}\big(Y(z+w)(1 \otimes Y(w))\mathcal{S}^{23}(w)\mathcal{S}^{13}(z+w)(a \otimes b \otimes c)\big)\\&= (z+w)^N Y(w)\mathcal{S}(w)\big(Y(z) \otimes 1\big)(a \otimes b \otimes c)\ mod\ h^M.\end{aligned} \tag{2.22}$$

Proof. Applying e^{-wT} to both sides of the $\mathcal{S}$-locality (2.10), one has

$$\begin{aligned}&(z-w)^N e^{-wT} Y(z)\big(1 \otimes Y(w)\big)\big(\mathcal{S}(z-w)(a \otimes b) \otimes c\big)\\&= (z-w)^N e^{-wT} Y(w)\big(1 \otimes Y(z)\big)(b \otimes a \otimes c) \quad \text{mod } h^M.\end{aligned} \tag{2.23}$$

By Proposition 1.2.3 (2) and Lemma 2.2.4, the left hand side of equation 2.23 is equal to the following:

$$\begin{aligned}&(z-w)^N\iota_{z,w}Y(z-w)\big(1\otimes e^{-wT}Y(w)\big)\big(\mathcal{S}(z-w)(a\otimes b)\otimes c\big)\\&=(z-w)^N\iota_{z,w}Y(z-w)\big(1\otimes Y(-w)\big)\mathcal{S}^{23}(-w)\mathcal{S}^{13}(z-w)(a\otimes c\otimes b).\end{aligned}\tag{2.24}$$

Similarly, using Proposition 1.2.3 (2) on the right hand side of equation 2.23, one has

$$(z-w)^N Y(-w)\mathcal{S}(-w)\big(Y(z)\otimes 1\big)(a\otimes c\otimes b).\tag{2.25}$$

Equation (2.22) is obtained by equating (mod h^M) (2.24) and (2.25), changing the sign of w and switching the letters b and c. □

Definition 2.2.2 ([EK5])**.** A *quantum vertex algebra* is a braided vertex algebra $(V,|0\rangle,Y,T,S)$ which satisfies the following condition

(QA4)

$$\mathcal{S}(w)\big(Y(z)\otimes 1\big)=\big(Y(z)\otimes 1\big)\mathcal{S}^{23}(w)i_{w,z}\mathcal{S}^{13}(z+w),\tag{2.26}$$

called *Hexagon Relation.*

Proposition 2.2.11 ([EK5, Prop.1.4])**.** *In a quantum vertex algebra the Associativity Relation holds: for any $a,b,c\in V$ and $M\in\mathbb{Z}_+$, there exists $N\geq 0$ such that*

$$\begin{aligned}&(z+w)^N i_{z,w}Y(z+w)\big(1\otimes Y(w)\big)(a\otimes b\otimes c)\\&\quad=(z+w)^N Y(w)\big(Y(z)\otimes 1\big)(a\otimes b\otimes c)\ mod\ h^M.\end{aligned}\tag{2.27}$$

Remark 2.2.12. By Corollary 2.2.8 and Proposition 2.2.11, a quantum vertex algebra is a $\mathbb{K}[[h]]$-field algebra.

Proposition 2.2.13 ([EK5, Prop.1.6])**.** *In a quantum vertex algebra, $\mathcal{S}(z)(|0\rangle\otimes b)=|0\rangle\otimes b$.*

Corollary 2.2.14. *In a quantum vertex algebra, $\mathcal{S}(z)(a\otimes|0\rangle)=a\otimes|0\rangle$.*

Proof. By the Proposition 2.2.13 one has

$$\begin{aligned}&\mathcal{S}^{21}(z)(a\otimes|0\rangle)=(1\ 2)\mathcal{S}(z)(1\ 2)(a\otimes|0\rangle)\\&=(1\ 2)\mathcal{S}(z)(|0\rangle\otimes a)=(a\otimes|0\rangle).\end{aligned}$$

Therefore, by Unitarity (2.9), $\mathcal{S}^{-1}(-z)(a\otimes|0\rangle)=(a\otimes|0\rangle)$ from which $\mathcal{S}(z)(a\otimes|0\rangle)=(a\otimes|0\rangle)$. □

Let V be a $\mathbb{K}[[h]]$-module. We denote with $End_{\mathbb{K}[[h]]}V$ the vector space of $\mathbb{K}[[h]]$-linear maps from V to V. We say that an $End_{\mathbb{K}[[h]]}V$-valued formal distribution $a(z)$ is an $End_{\mathbb{K}[[h]]}V$-valued quantum field if $a(z)b\in V_h((z))$ for any $b\in V$.
We say that an $End_{\mathbb{K}[[h]]}V$-valued formal distribution $a(z,w)$ is local if, for any $M\in\mathbb{Z}_+$, there exists $N\geq 0$ such that

$$(z-w)^N a(z,w)=0 \bmod h^M.\tag{2.28}$$

Similarly, we say that $a(z,w)$ is local on the vector $b \in V$ if, for any $M \in \mathbb{Z}_+$, there exists $N \geq 0$, such that

$$(z-w)^N a(z,w)b = 0 \bmod h^M. \tag{2.29}$$

Moreover, we say that two $End_{\mathbb{K}[[h]]}V$-valued quantum fields $a(z)$ and $b(w)$ are local (respectively local on the vector $c \in V$) if $[a(z), b(w)]$ (respectively $[a(z), b(w)]c$) is local.

Replacing the vector space V with a topologically free $\mathbb{K}[[h]]$-module V and thinking the Associativity Relation and the Locality mod h^M for any $M \in \mathbb{Z}_+$, Proposition 1.2.3 and Proposition 1.2.6 still hold.

Lemma 2.2.15. *Let V a topologically free $\mathbb{K}[[h]]$-module V, let $|0\rangle \in V$ and $T : V \to V$ be a linear map such that $T|0\rangle = 0$. Then for any translation covariant (with respect to T) $End_{\mathbb{K}[[h]]}V$-valued quantum field $a(z)$, we have $a(z)|0\rangle \in V[[z]]$.*

Lemma 2.2.16. *Let V a topologically free $\mathbb{K}[[h]]$-module V, let $|0\rangle \in V$ and $T : V \to V$ be a linear map such that $T|0\rangle = 0$. Then for any translation covariant (with respect to T) $End_{\mathbb{K}[[h]]}V$-valued quantum field $a(z) = \sum_{m\in\mathbb{Z}} a_{(m)} z^{-z-1}$, we have*

$$a(z)|0\rangle = e^{zT}a = \sum_{k\geq 0} \frac{T^k a}{k!} z^k, \tag{2.30}$$

where $a = a(z)|0\rangle_{|z=0}$.

The proofs of Lemmas 2.2.15 and 2.2.16 are identical to the one of Proposition 1.2.3 where we replace $Y(z)(a \otimes -)$ with $a(z)$.

Proposition 2.2.17. *Let $(V, |0\rangle, Y, T, \mathcal{S})$ be a braided vertex algebra and let $a \in V$ and $a(z)$ be an $End_{\mathbb{K}[[h]]}V$-valued quantum field such that*

$$a(z)|0\rangle = Y(z)(a \otimes |0\rangle)$$

and $a(z)$ is local on every vector of V with all $Y(z)(c \otimes -)$ with $c \in V$. Then

$$a(z) = Y(z)\mathcal{S}(z)(a \otimes -) = Y^{op}(z)(a \otimes -). \tag{2.31}$$

Proof. For any $c \in V$ and $M \in \mathbb{Z}_+$, by the locality of $a(z)$ on every vector of V with $Y(z)(c \otimes -)$, there exists $N_1 \in \mathbb{Z}_+$ such that

$$(z-w)^{N_1} a(z)Y(w)(c \otimes |0\rangle) = (z-w)^{N_1} Y(w)(c \otimes a(z)|0\rangle) \bmod h^M.$$

By hypothesis, $a(z)|0\rangle = Y(z)(a \otimes |0\rangle)$ from which

$$(z-w)^{N_1} a(z)Y(w)(c\otimes|0\rangle) = (z-w)^{N_1} Y(w)(1\otimes Y(z))(c\otimes a\otimes|0\rangle) \bmod h^M. \tag{2.32}$$

On the other hand, by $\mathcal{S}$-locality (2.10), there exists $N_2 \in \mathbb{Z}_+$ such that

$$\begin{aligned}(z-w)^{N_2} Y(z)(1 \otimes Y(w))(\mathcal{S}(z-w)(a \otimes c) \otimes |0\rangle)\\ = (z-w)^{N_2} Y(w)(1 \otimes Y(z))(c \otimes a \otimes |0\rangle) \bmod h^M.\end{aligned} \tag{2.33}$$

Therefore, using equations (2.32) and (2.33), for any $N \geq max\{N_1, N_2\}$ one has

$$\begin{aligned}(z-w)^N Y(z)(1 \otimes Y(w))(\mathcal{S}(z-w)(a \otimes c) \otimes |0\rangle) \\ = (z-w)^N a(z) Y(w)(c \otimes |0\rangle) \bmod h^M.\end{aligned} \tag{2.34}$$

For N big enough, one can evaluate equation (2.34) in $w = 0$ (because all the powers of w are nonnegative) obtaining that

$$z^N Y(z)\mathcal{S}(z)(a \otimes c) = z^N a(z)c \bmod h^M.$$

It follows that

$$Y(z)\mathcal{S}(z)(a \otimes c) = a(z)c \bmod h^M. \tag{2.35}$$

Equation (2.35) holds for any $M \in \mathbb{Z}_+$ thus $a(z)c = Y(z)\mathcal{S}(z)(a \otimes c)$. □

Corollary 2.2.18. *Let $(V, |0\rangle, Y, T, \mathcal{S})$ be a braided vertex algebra and let $a(z)$ be a translation covariant $End_{\mathbb{K}[[h]]}V$-valued quantum field such that $a(z)$ is local on every vector of V with all $Y(z)(c \otimes -)$ with $c \in V$. Then*

$$a(z) = Y(z)\mathcal{S}(z)(a \otimes -) = Y^{op}(z)(a \otimes -) \tag{2.36}$$

where $a = a(z)|0\rangle_{|z=0}$.

Proof. It's a direct consequence of Lemma 2.2.16 which states that $a(z)|0\rangle = e^{zT}a = Y(z)(a \otimes |0\rangle)$. □

Corollary 2.2.19. *Let $(V, |0\rangle, Y, T, \mathcal{S})$ be a quantum vertex algebra. Then, $a(z)$ is a translation covariant $End_{\mathbb{K}[[h]]}V$-valued quantum field which is local on every vector of V with all $Y(c, z)$ with $c \in V$ if and only if $a(z) = Y(z)\mathcal{S}(z)(a \otimes -) = Y^{op}(a, z)$ where $a = a(z)|0\rangle_{|z=0}$.*

Proof. It follows by Corollary 2.2.18 and Proposition 1.2.6. □

Moreover, from Corollary 2.2.18 and Proposition 1.2.6, the following corollary follows:

Corollary 2.2.20. *Let $(V, |0\rangle, Y, T, \mathcal{S})$ be a braided vertex algebra which satisfies the Associativity Relation (2.27). Let $a(z)$ be a translation covariant $End_{\mathbb{K}[[h]]}V$-valued quantum field. Then $a(z)$ is local (mod h^M) with all quantum fields $Y(z)(c \otimes -)$ on every vector of V if and only if $a(z) = Y^{op}(z)(a \otimes -)$ where $a = a(z)|0\rangle_{|z=0}$.*

Proof. The "if" part follows by Proposition 1.2.6 because V satisfied the Associativity Relation (2.27). The "only if" part is due to Corollary 2.2.18. □

2.2.1 $\mathcal{S}$-commutative quantum vertex algebras

Definition 2.2.3. A braided vertex algebra $(V, |0\rangle, Y, T, \mathcal{S})$ is said to be $\mathcal{S}$-commutative if the following equation holds:

$$\begin{aligned}Y(z)(1 \otimes Y(w))\big(i_{z,w}\mathcal{S}(z-w)(a \otimes b) \otimes c\big) \\ = Y(w)(1 \otimes Y(z))(b \otimes a \otimes c).\end{aligned} \tag{2.37}$$

Remark 2.2.21. The $\mathcal{S}$-commutativity (2.37) is well defined (there are no divergences) as shown in Remark 2.2.1.

Remark 2.2.22. The $\mathcal{S}$-commutativity (2.37) is obtained from the $\mathcal{S}$-locality putting $N = 0$ for any $M \in \mathbb{Z}_+$.

Theorem 2.2.23. *$(V, |0\rangle, Y, T, \mathcal{S})$ is an $\mathcal{S}$-commutative braided vertex algebra if and only if $(V, |0\rangle, Y, T, \mathcal{S})$ is a braided vertex algebra such that $Y(z)(a \otimes b) \in V[[z]]$ for any $a, b \in V$.*

Proof. If $(V, |0\rangle, Y, T, \mathcal{S})$ is an $\mathcal{S}$-commutative braided vertex algebra, considering c of equation (2.37) equal to $|0\rangle$, multiplying both sides by z^{-1} and taking the residue Res_z one has

$$\begin{aligned} Y(w)(b \otimes a) &= Res_z\Big(z^{-1}Y(z)(1 \otimes e^{wT})i_{z,w}\mathcal{S}(z-w)(a \otimes b)\Big) \\ &= Res_z\Big(z^{-1}Y(z)(1 \otimes e^{wT})e^{-w\partial_z}\mathcal{S}(z)(a \otimes b)\Big) \end{aligned} \tag{2.38}$$

from which $Y(w)(b \otimes a) \in V[[w]]$.
On the other hand, if $(V, |0\rangle, Y, T, \mathcal{S})$ is a braided vertex algebra such that $Y(z)(a \otimes b) \in V[[z]]$ for any $a, b \in V$, $Y(w)\big(1 \otimes Y(z)\big)(b \otimes a \otimes c) \in V[[z, w]]$ and $Y(z)\big(1 \otimes Y(w)\big)\big(\mathcal{S}(z-w)(a \otimes b) \otimes c\big)$ mod h^M lies in $V((z))((w))$ for any $M \in \mathbb{Z}_+$. One can therefore multiply both sides of $\mathcal{S}$-locality (2.10) by $i_{z,w}(z-w)^{-N}$ obtaining that

$$\begin{aligned} &Y(z)\big(1 \otimes Y(w)\big)\big(i_{z,w}\mathcal{S}(z-w)(a \otimes b) \otimes c\big) \\ &\qquad = Y(w)\big(1 \otimes Y(z)\big)(b \otimes a \otimes c) \ mod\ h^M. \end{aligned}$$

Since this can be obtained for any $M \in \mathbb{Z}_+$, the $\mathcal{S}$-commutativity (2.37) holds. □

Let us define the -1-product on V as done for the vertex algebras:

Definition 2.2.4. Let $(V, |0\rangle, Y, T, \mathcal{S})$ be a braided vertex algebra, the -1-product is defined as the following $\mathbb{K}[[h]]$-linear map:

$$\begin{aligned} -_{(-1)}- : V \otimes V &\to V \\ a \otimes b &\mapsto Res_z\big(z^{-1}Y(z)(a \otimes b)\big). \end{aligned}$$

Lemma 2.2.24. *Let $(V, |0\rangle, Y, T, \mathcal{S})$ be an $\mathcal{S}$-commutative braided vertex algebra. The state-field correspondence Y can be described in terms of T and the -1-product:*

$$Y(z)(a \otimes b) = \Big(e^{zT}a\Big)_{(-1)} b. \tag{2.39}$$

Proof. Since V is $\mathcal{S}$-commutative, by Theorem 2.2.23, one has $Y(z)(a \otimes b) = \sum_{m\geq 0} a_{(-m-1)}b\, z^m$. By Corollary 2.2.8, one has

$$\left(\frac{T^m a}{m!}\right)_{(-1)} b = a_{(-m-1)}b.$$

As a consequence:

$$Y(z)(a \otimes b) = \sum_{m\geq 0}\left(\frac{T^m a}{m!}\right)_{(-1)} b\, z^m = \Big(e^{zT}a\Big)_{(-1)} b.$$

□

Theorem 2.2.25. *Let* $(V, |0\rangle, Y, T, \mathcal{S})$ *be an* $\mathcal{S}$*-commutative braided vertex algebra satisfying the Associativity Relation* (2.27)*. The* -1*-product endows* V *with a structure of unital, associative, differential (with derivation* T*)* $\mathbb{K}[[h]]$*-algebra and the following commutation relations hold:*

$$b_{(-1)}a = (\, -_{(-1)}\, -)Res_z\left(z^{-1}\left(e^{zT}\otimes 1\right)\mathcal{S}(z)(a\otimes b)\right) \tag{2.40}$$

for any $a, b \in V$.

Proof. By (QA3), the vacuum vector $|0\rangle$ is the unit of the -1-product. By (QA2) and Corollary 2.2.8, T is a derivation. By Theorem 2.2.23, $Y(z)(d\otimes e) \in V[[z]]$ for any $d, e \in V$. As a consequence we can consider $N = 0$ in the Associativity Relation (2.27). Moreover, multiplying the Associativity Relation (2.27) by z^{-1} and w^{-1} and taking the residues Res_z and Res_w, one shows that $(a_{(-1)}b)_{(-1)}c = a_{(-1)}(b_{(-1)}c) \mod h^M$ for any $M \in \mathbb{Z}_+$. Therefore $(a_{(-1)}b)_{(-1)}c = a_{(-1)}(b_{(-1)}c)$ proving the associativity of the -1-product. In the proof of Theorem 2.2.23 we proved that

$$Y(w)(b\otimes a) = Res_z\Big(z^{-1}Y(z)(1\otimes e^{wT})e^{-w\partial_z}\mathcal{S}(z)(a\otimes b)\Big).$$

Multiplying both sides by w^{-1} and taking the residue Res_w, one has

$$b_{(-1)}a = Res_z(z^{-1}Y(z)\mathcal{S}(z)(a\otimes b)).$$

As a consequence, by Lemma 2.2.24, one has

$$\begin{aligned} b_{(-1)}a &= Res_z\left(z^{-1}\left(-_{(-1)}-\right)\left(e^{zT}\otimes 1\right)\mathcal{S}(z)(a\otimes b)\right) \\ &= \left(-_{(-1)}-\right)Res_z\left(z^{-1}\left(e^{zT}\otimes 1\right)\mathcal{S}(z)(a\otimes b)\right). \end{aligned}$$

□

Corollary 2.2.26. *Let* $(V, |0\rangle, Y, T, \mathcal{S})$ *be an* $\mathcal{S}$*-commutative quantum vertex algebra. The* -1*-product endows* V *with a structure of unital, associative, differential (with derivation* T*)* $\mathbb{K}[[h]]$*-algebra satisfying the commutation relations* (2.40)*. Moreover the Hexagon Relation* (2.26) *assumes the following aspect:*

$$\mathcal{S}(w)((-_{(-1)}-)\otimes 1) = ((-_{(-1)}-)\otimes 1)\mathcal{S}^{23}(w)\mathcal{S}^{13}(w).$$

Proof. By Lemma 2.2.24, the Hexagon Relation (2.26) becomes

$$\begin{aligned} &\mathcal{S}(w)((-_{(-1)}-)\otimes 1)\left(e^{zT}\otimes 1\otimes 1\right) \\ &\quad = ((-_{(-1)}-)\otimes 1)\left(e^{zT}\otimes 1\otimes 1\right)\mathcal{S}^{23}(w)i_{w,z}\mathcal{S}^{13}(z+w) \\ &\quad = ((-_{(-1)}-)\otimes 1)\mathcal{S}^{23}(w)\left(e^{zT}\otimes 1\otimes 1\right)i_{w,z}\mathcal{S}^{13}(z+w). \end{aligned}$$

By the Shift Condition in Definition 2.2.1 and Taylor expansion, one proves that $i_{w,z}\mathcal{S}(z+w) = \left(e^{-zT}\otimes 1\right)\mathcal{S}(w)\left(e^{zT}\otimes 1\right)$. As a consequence, one has

$$\begin{aligned} &\mathcal{S}(w)((-_{(-1)}-)\otimes 1)\left(e^{zT}\otimes 1\otimes 1\right) \\ &\quad = ((-_{(-1)}-)\otimes 1)\mathcal{S}^{23}(w)\mathcal{S}^{13}(w)\left(e^{zT}\otimes 1\otimes 1\right). \end{aligned}$$

Since $e^{zT}\otimes 1\otimes 1$ is invertible, the thesis follows. □

2.3 Examples of quantum vertex algebras

Let us give a non trivial example firstly appeared in [EK5] as more recently formulated in [JKMY].

2.3.1 Normalization of the Yang R-matrix

Let $R(u)$ be the Yang R-matrix

$$R(u) = 1 - \frac{\sigma}{u} \in End\ \mathbb{C}^N \otimes End\ \mathbb{C}^N[u^{-1}], \tag{2.41}$$

where σ is the permutation operator in $\mathbb{C}^N \otimes \mathbb{C}^N$, i.e. $\sigma = \sum_{i,j=1}^N e_{ij} \otimes e_{ji}$. Let

$$\overline{R}(u) = g(u)R(u) \in End\ \mathbb{C}^N \otimes End\ \mathbb{C}^N[[u^{-1}]] \tag{2.42}$$

where $g(u) = 1 + \sum_{r\geq 1} g_r u^{-r} \in \mathbb{C}[[u^{-1}]]$ is defined by the equation

$$g(u+N) = g(u)(1-u^{-2}). \tag{2.43}$$

It follows that the coefficients g_r of $g(u)$ are given recursively as

$$g_1 = \frac{1}{N}$$

and

$$g_r = \frac{g_{r-1} + \sum_{k=2}^r (-1)^k \binom{r}{k} g_{r+1-k} N^k}{rN}$$

for any $r \geq 2$.

Remark 2.3.1 ([JKMY]). $g(u)g(-u)(1-u^{-2}) = 1$.

Proof. Replacing u with $-u-N$ in (2.43) one obtains

$$g(-u) = g(-u-N)(1-(-u-N)^{-2}), \tag{2.44}$$

and combining (2.43) and (2.44) one has

$$g(u)g(-u)(1-u^{-2}) = g(u+N)g(-u-N)(1-(u+N)^{-2}). \tag{2.45}$$

Therefore $g(u)g(-u)(1-u^{-2}) = 1 + \sum_{r\geq 1} h_r u^{-r}$ for suitable $h_r \in \mathbb{C}$ and it is invariant under the shift $u \mapsto u+N$. It follows that $g(u)g(-u)(1-u^{-2}) = 1$ indeed if a series $1 + \sum_{r\geq 1} h_r u^{-r}$ is invariant under the shift $u \mapsto u+N$, one has

$$\begin{aligned} 1 + \sum_{r\geq 1} h_r u^{-r} &= 1 + \sum_{r\geq 1}\sum_{k\geq 0} \binom{-r}{k} h_r u^{-r-k} N^k \\ &= 1 + \sum_{r\geq 1}\sum_{k=0}^{r-1} \binom{-r+k}{k} h_{r-k} N^k u^{-r} \end{aligned}$$

and, by induction on the index r, one proves that $h_r = 0$ for any $r \geq 1$. □

Remark 2.3.2 ([JKMY]). $\overline{R}(u)$ is a unitary solution of the quantum Yang-Baxter equation.

Proof. $\overline{R}(u)$ is a solution of the quantum Yang-Baxter equation because $R(u)$ is a solution of the quantum Yang-Baxter equation indeed

$$\begin{aligned}&(R^{12}(u)R^{13}(u+v)R^{23}(v) - R^{23}(v)R^{13}(u+v)R^{12}(u))(a \otimes b \otimes c)\\ &= \frac{1}{u(u+v)v}((b \otimes c \otimes a)v + (c \otimes a \otimes b)(u+v) + (b \otimes c \otimes a)u)\\ &\quad - \frac{1}{u(u+v)v}((c \otimes a \otimes b)v + (b \otimes c \otimes a)(u+v) + (c \otimes a \otimes b)u) = 0.\end{aligned}$$

And it is also unitary indeed

$$\overline{R}^{12}(u)\overline{R}^{21}(-u) = g(u)g(-u)(1-u^{-2}) = 1.$$

□

2.3.2 Double Yangian over $\mathbb{C}[[h]]$

From now on, we will work with algebras and modules over $\mathbb{C}[[h]]$. Therefore we will rescale the spectral parameter $u \mapsto u/h$ in definitions (2.41) and (2.42) of $R(u)$ and $\overline{R}(u)$. In particular $R(u)$ and $\overline{R}(u)$ will be understood in this way:

$$R(u) = 1 - \frac{h\sigma}{u}, \tag{2.46}$$

$$\overline{R}(u) = g(u/h)R(u). \tag{2.47}$$

The double Yangian $DY(\mathfrak{gl}_N)$ over $\mathbb{C}[[h]]$ is the associative algebra over $\mathbb{C}[[h]]$ generated by the central element K and the elements $t_{ij}^{(r)}$ and $t_{ij}^{(-r)}$, where $1 \leq i,\ j \leq N$ and $r \geq 1$, subject to the defining relations

$$R^{12}(u-v)T^{13}(u)T^{23}(v) = T^{23}(v)T^{13}(u)R^{12}(u-v) \tag{2.48}$$

$$R^{12}(u-v)T^{+13}(u)T^{+23}(v) = T^{+23}(v)T^{+13}(u)R^{12}(u-v) \tag{2.49}$$

$$\overline{R}^{12}(u-v+hK/2)T^{13}(u)T^{+23}(v) = T^{+23}(v)T^{13}(u)\overline{R}^{12}(u-v-hK/2). \tag{2.50}$$

The matrices $T(u)$ and $T^+(u)$ are given by

$$T(u) = \sum_{i,j=1}^{N} e_{ij} \otimes t_{ij}(u)$$

with

$$t_{ij}(u) = \delta_{ij} + h\sum_{r\geq 1} t_{ij}^{(r)}\ u^{-r},$$

and

$$T^+(u) = \sum_{i,j=1}^{N} e_{ij} \otimes t_{ij}^+(u)$$

with

$$t_{ij}^+(u) = \delta_{ij} - h\sum_{r\geq 1} t_{ij}^{(-r)}\ u^{r-1}.$$

Notation 3. In the previous lines we have denoted with e_{ij} the matrix units.

Remark 2.3.3. Relations (2.50) imply the following relations

$$\overline{R}^{12}(u-v-hK/2)T^{+13}(u)T^{23}(v) = T^{23}(v)T^{+13}(u)\overline{R}^{12}(u-v+hK/2). \quad (2.51)$$

Proof. Relations (2.50) imply

$$T^{13}(u)T^{+23}(v)\overline{R}^{12}(u-v+hC/2)^{-1} = \overline{R}^{12}(u-v-hC/2)^{-1}T^{+23}(v)T^{13}(u)$$

which, using the unitarity of $\overline{R}$, is equal to

$$T^{13}(u)T^{+23}(v)\overline{R}^{21}(-u+v-hC/2) = \overline{R}^{21}(-u+v+hC/2)T^{+23}(v)T^{13}(u).$$

It follows

$$T^{23}(u)T^{+13}(v)\overline{R}^{12}(-u+v-hC/2) = \overline{R}^{12}(-u+v+hC/2)T^{+13}(v)T^{23}(u)$$

from which, renaming $v \mapsto u$ and $u \mapsto v$ one has relations (2.51). □

The Poincaré-Birkhoff-Witt theorem for the double Yangian extends to the algebra $DY(\mathfrak{gl}_N)$ over $\mathbb{C}[[h]]$. Therefore the subalgebra of $DY(\mathfrak{gl}_N)$ generated by the elements $t_{ij}^{(r)}$, with $i,j = 1,\dots N$ and $r \geq 1$, can be identified with the Yangian $Y(\mathfrak{gl}_N)$ over $\mathbb{C}[[h]]$ defined by relations (2.48). In a similar way the subalgebra of $DY(\mathfrak{gl}_N)$ generated by the elements $t_{ij}^{(-r)}$, with $i,j = 1,\dots N$ and $r \geq 1$, can be identified with the dual Yangian $Y^+(\mathfrak{gl}_N)$ over $\mathbb{C}[[h]]$ defined by the relations (2.49).

2.3.3 The vacuum module of the extended double Yangian

Definition 2.3.1. The double Yangian at the level $c \in \mathbb{C}$ is the quotient $DY_c(\mathfrak{gl}_N)$ of $DY(\mathfrak{gl}_N)$ by the ideal generated by $(K-c)$. The vacuum module $\mathcal{V}_c(\mathfrak{gl}_N)$ at the level c over the double Yangian is the h-adic completion of the quotient

$$\mathcal{V}_c(\mathfrak{gl}_N) = DY_c(\mathfrak{gl}_N)/DY_c(\mathfrak{gl}_N)\langle t_{ij}^{(r)} | \; r \geq 1\rangle. \quad (2.52)$$

Remark 2.3.4. By the Poincaré-Birkhoff-Witt Theorem the vacuum module $\mathcal{V}_c(\mathfrak{gl}_N)$ at the level c over the double Yangian is isomorphic with the dual Yangian $Y^+(\mathfrak{gl}_N)$ as a $\mathbb{C}[[h]]$-module.

As proved in [EK5], the vacuum module $\mathcal{V}_c(\mathfrak{gl}_N)$, as defined in Definition 2.3.1, possesses a quantum vertex algebra structure and its classical limit, obtained for $h \to 0$, is the affine vertex algebra $V_c(\mathfrak{gl}_N)$. Accordingly the quantum vertex algebra on (2.3.1) is called the quantum affine vertex algebra.

Definition 2.3.2. For any nonnegative integer n let us introduce the functions depending on a variable z and a family of variables $u = (u_1, \dots, u_N)$ with values in the space

$$(End\ \mathbb{C}^N)^{\otimes n} \otimes \mathcal{V}_c(\mathfrak{gl}_N) \quad (2.53)$$

by

$$T_n(u) = T^{1n+1}(u_1)\dots T^{nn+1}(u_n), \quad (2.54)$$

$$T_n^+(u) = T^{+1n+1}(u_1)\dots T^{+nn+1}(u_n), \tag{2.55}$$

$$T_n(u|z) = T^{1n+1}(z+u_1)\dots T^{nn+1}(z+u_n), \tag{2.56}$$

$$T_n^+(u|z) = T^{+1n+1}(z+u_1)\dots T^{+nn+1}(z+u_n). \tag{2.57}$$

Equation (2.56) has to be understood in this sense:

$$T_n(u|z) = i_{z,u_1}T^{1n+1}(z+u_1)\dots i_{z,u_n}T^{nn+1}(z+u_n). \tag{2.58}$$

For $n=0$ these products will be considered as being equal to the identity.

Remark 2.3.5. For $T_n^+(u|z)$ it is no necessary to explicit where to expand the power of $z+u_i$ because all the powers are positive.

Definition 2.3.3. For any nonnegative integers m and n, let us introduce the functions depending on a variable z and two families of variables $u=(u_1,\dots,u_n)$ and $v=(v_1,\dots,v_m)$ with values in the space

$$(End\ \mathbb{C}^N)^{\otimes n}\otimes(End\ \mathbb{C}^N)^{\otimes m} \tag{2.59}$$

by

$$R_{nm}^{12}(u|v|z) = \prod_{j=1,\dots,n}^{\rightarrow}\ \prod_{i=n+1,\dots,n+m}^{\leftarrow} R_{nm}^{ji}(z+u_j-v_{i-n}), \tag{2.60}$$

and similarly

$$\overline{R}_{nm}^{12}(u|v|z) = \prod_{j=1,\dots,n}^{\rightarrow}\ \prod_{i=n+1,\dots,n+m}^{\leftarrow} \overline{R}_{nm}^{ji}(z+u_j-v_{i-n}). \tag{2.61}$$

As above empty products will be understood as identities. The superscripts 1 and 2 are meant to indicate the tensor factors in (2.59).

We will also use superscripts for multiple tensor products of the form

$$(End\ \mathbb{C}^N)^{\otimes n}\otimes(End\ \mathbb{C}^N)^{\otimes m}\otimes(End\ \mathbb{C}^N)^{\otimes k}\otimes\mathcal{V}_c(\mathfrak{gl}_N)\otimes\mathcal{V}_c(\mathfrak{gl}_N)\otimes\mathcal{V}_c(\mathfrak{gl}_N). \tag{2.62}$$

Expressions like $T_n^{14}(u)$ or $T_n^{25}(u)$ are meant to be the operators $T_n(u)$ or $T_n(u)$ whose non-identity operator belongs to the corresponding tensor factors. For example the non-identity components of $T_n^{25}(u)$ belong to factors

$$n+1, n+2, \dots, n+m \text{ and } n+m+k+2.$$

Proposition 2.3.6. *Using the above notation, the following relations holds with operators in*

$$(End\ \mathbb{C}^N)^{\otimes n}\otimes(End\ \mathbb{C}^N)^{\otimes m}\otimes\mathcal{V}_c(\mathfrak{gl}_N)\otimes\mathcal{V}_c(\mathfrak{gl}_N):$$

$$\begin{aligned} &R_{nm}^{12}(u|v|z-w)T_n^{13}(u|z)T_m^{23}(v|w) \\ &\quad = T_m^{23}(v|w)T_n^{13}(u|z)R_{nm}^{12}(u|v|z-w) \end{aligned} \tag{2.63}$$

$$\begin{aligned} &R_{nm}^{12}(u|v|z-w)T_n^{+13}(u|z)T_m^{+23}(v|w) \\ &\quad = T_m^{+23}(v|w)T_n^{+13}(u|z)R_{nm}^{12}(u|v|z-w) \end{aligned} \tag{2.64}$$

$$\begin{aligned}&\overline{R}^{12}_{nm}(u|v|z-w+hc/2)T^{13}_{n}(u|z)T^{+23}_{m}(v|w)\\&\quad=T^{+23}_{m}(v|w)T^{13}_{n}(u|z)\overline{R}^{12}_{nm}(u|v|z-w-hc/2),\end{aligned}\tag{2.65}$$

$$\begin{aligned}&\overline{R}^{12}_{nm}(u|v|z-w-hc/2)T^{+13}_{n}(u|z)T^{23}_{m}(v|w)\\&\quad=T^{23}_{m}(v|w)T^{+13}_{n}(u|z)\overline{R}^{12}_{nm}(u|v|z-w+hc/2),\end{aligned}\tag{2.66}$$

Proof. It is proved by induction on relations (2.48), (2.49), (2.50) and (2.51). □

Given two associative algebras $\mathcal{A}$ and $\mathcal{B}$, it can be useful to introduce the following notation used in [JKMY]. Let A_1, A_2 be elements of $\mathcal{A}$, B_1, B_2 be elements of $\mathcal{B}$ and $F = A_1 \otimes B_1$, then we define the following products:

$$\begin{aligned}{}^{ll}F(A_2\otimes B_2)&=A_1A_2\otimes B_1B_2,\\{}^{lr}F(A_2\otimes B_2)&=A_1A_2\otimes B_2B_1,\\{}^{rl}F(A_2\otimes B_2)&=A_2A_1\otimes B_1B_2,\\{}^{rr}F(A_2\otimes B_2)&=A_2A_1\otimes B_2B_1.\end{aligned}\tag{2.67}$$

Moreover, given $\alpha,\beta\in\{l,r\}$ and $F\in\mathcal{A}\otimes\mathcal{B}$, we denote $({}^{\alpha\beta}F)^{-1}$ the element $G\in\mathcal{A}\otimes\mathcal{B}$ such that $({}^{\alpha\beta}G)F=1$. In what follows this notation will be used with $F=R^{12}_{nm}(u|v|z)$ with the first component as element of $\mathcal{A}$ and the second component as element of $\mathcal{B}$. The following results hold:

$$\begin{aligned}\Big({}^{lr}R(u)\Big)^{-1}=\Big({}^{rl}R(u)\Big)^{-1}&=(1-hNu^{-1})^{-1}(R(-u)-hNu^{-1})\\&=(1-hNu^{-1})^{-1}\sigma hu^{-1};\end{aligned}\tag{2.68}$$

$$\Big({}^{ll}R(u)\Big)^{-1}=\Big({}^{rr}R(u)\Big)^{-1}=R(u)^{-1}=(1-h^2u^{-2})^{-1}R(-u);\tag{2.69}$$

from which the inverse operators of $R^{12}_{nm}(u|v|z)$.

Remark 2.3.7. Recall that, since $R(u)$ is defined as

$$R(u)=1-\frac{\sigma h}{u}=\sum_{i,j=1}^{N}e_{ii}\otimes e_{jj}-\frac{h}{u}\sum_{i,j=1}^{N}e_{ij}\otimes e_{ji},$$

one has that

$$R^{21}(u)=(1\ 2)R(u)(1\ 2)=\sum_{i,j=1}^{N}e_{jj}\otimes e_{ii}-\frac{h}{u}\sum_{i,j=1}^{N}e_{ji}\otimes e_{ij}=R(u),$$

from which

$$\Big({}^{lr}R^{(21)}(u)\Big)^{-1}=\Big({}^{rl}R^{(21)}(u)\Big)^{-1}=(1-hNu^{-1})^{-1}\sigma hu^{-1}.\tag{2.70}$$

2.3.4 A quantum vertex algebra structure on the vacuum module of the extended double Yangian

Theorem 2.3.8 ([JKMY, Thm.4.1])**.** *There exists a unique well-defined structure of quantum vertex algebra on the vacuum module $\mathcal{V}_c(\mathfrak{gl}_N)$ with the following data.*

1. *The vacuum vector is*
$$|0\rangle = 1 \in \mathcal{V}_c(\mathfrak{gl}_N); \tag{2.71}$$

2. *the vertex operators are defined by*
$$Y(T_n^+(u)|0\rangle, z) = T_n^+(u|z)T_n(u|z+hc/2)^{-1}; \tag{2.72}$$

3. *the translation operator is defined by*
$$(1^{\otimes n} \otimes e^{zD})T_n^+(u) = T_n^+(u|z)|0\rangle; \tag{2.73}$$

4. *the map $\mathcal{S}$ is defined by the relation*
$$\begin{aligned}&\mathcal{S}^{34}(z)(\overline{R}_{nm}^{12}(u|v|z)^{-1}T_m^{+24}(v)\overline{R}_{nm}^{12}(u|v|z-hc)T_n^{+13}(u)|0\rangle\otimes|0\rangle)\\&=T_n^{+13}(u)\overline{R}_{nm}^{12}(u|v|z+hc)^{-1}T_m^{+24}(v)\overline{R}_{nm}^{12}(u|v|z)|0\rangle\otimes|0\rangle\end{aligned} \tag{2.74}$$

   for operators in
$$(End\ \mathbb{C}^N)^{\otimes n} \otimes (End\ \mathbb{C}^N)^{\otimes m} \otimes \mathcal{V}_c(\mathfrak{gl}_N) \otimes \mathcal{V}_c(\mathfrak{gl}_N). \tag{2.75}$$

Proof. Since the coefficients of $T_n^+(u)|0\rangle$ generate a dense subset of the vacuum module $\mathcal{V}_c(\mathfrak{gl}_N)$, defining $Y, \mathcal{S}, D$ on each element of the dual Yangian $Y^+(\mathfrak{gl}_N)$, one has a definition on the whole space $\mathcal{V}_c(\mathfrak{gl}_N)$. For the same reason it is sufficient to check the axioms on the elements of the dual Yangian $Y^+(\mathfrak{gl}_N)$ to have them satisfied for any element of $\mathcal{V}_c(\mathfrak{gl}_N)$.

Well definedness of Y and $\mathcal{S}$.
Let us prove that the map Y is well defined. It is sufficient to show that $Y(R^{12}(u-v)T^{+14}(u)T^{+24}(v)|0\rangle - T^{+24}(v)T^{+14}(u)R^{12}(u-v)|0\rangle, z)$ lies in the ideal generated by the relations (2.48), (2.49) and (2.50).

$$\begin{aligned}&Y(R^{12}(u-v)T^{+14}(u)T^{+24}(v)|0\rangle - T^{+24}(v)T^{+14}(u)R^{12}(u-v)|0\rangle, z)\\&=Y(R^{12}(u-v)T^{+14}(u)T^{+24}(v)|0\rangle, z) - Y(T^{+24}(v)T^{+14}(u)R^{12}(u-v)|0\rangle, z)\\&=R^{12}(u-v)T^{+14}(z+u)T^{+24}(v)T^{24}(z+v+hc/2)^{-1}T^{14}(z+u+hc/2)^{-1}\\&\quad -T^{+24}(z+u)T^{+14}(z+v)T^{14}(z+u+hc/2)^{-1}T^{24}(z+v+hc/2)^{-1}R^{12}(u-v)\\&\quad -T^{+24}(z+u)T^{+14}(z+v)R^{12}(u-v)T^{24}(z+v+hc/2)^{-1}T^{14}(z+u+hc/2)^{-1}\\&\quad +T^{+24}(z+u)T^{+14}(z+v)R^{12}(u-v)T^{24}(z+v+hc/2)^{-1}T^{14}(z+u+hc/2)^{-1}\\&=(R^{12}(u-v)T^{+14}(z+u)T^{+24}(z+v) - T^{+24}(z+u)T^{+14}(z+v)R^{12}(u-v))\cdot\\&\quad\cdot T^{24}(z+v+hc/2)^{-1}T^{14}(z+u+hc/2)^{-1} + T^{+24}(z+u)T^{+14}(z+v)\cdot\end{aligned}$$

$$\cdot\ (R^{12}(u-v)T^{24}(z+v+hc/2)^{-1}T^{14}(z+u+hc/2)^{-1}$$
$$-\ T^{14}(z+u+hc/2)^{-1}T^{24}(z+v+hc/2)^{-1}R^{12}(u-v)).$$

Therefore $Y\big(R^{12}_{nm}(u-v){T_n^+}^{14}(u){T_m^+}^{24}(v)|0\rangle - {T_m^+}^{24}(v){T_n^+}^{14}(u)R^{12}_{nm}(u-v)|0\rangle, z\big)$ lies in the ideal generated by relations (2.49) and (2.48).
In the same way one proves that

$$Y\big(R^{12}_{nm}(u-v)T_n^{14}(u)T_m^{24}(v){T_k^+}^{34}(\bar v)|0\rangle - T_m^{24}(v)T_n^{14}(u)R^{12}_{nm}(u-v){T_k^+}^{34}(\bar v)|0\rangle, z\big)$$

lies in the ideal generated by relations (2.49), (2.48) and (2.50).
Let us, now, prove that

$$Y(T_n^+(u)|0\rangle, z)T_m^+(v)|0\rangle \in \big(End\ \mathbb{C}^N\big)^{\otimes n}\otimes \big(End\ \mathbb{C}^N\big)^{\otimes m}\otimes$$
$$\otimes\ \big(\mathcal{V}_c(\mathfrak{gl}_N)\big)_h((z))[[u_1,\dots,u_n,v_1,\dots,v_m]]$$

It is sufficient to prove the following:

$$Y(T^+(u)|0\rangle, z)T^+(v)|0\rangle \in End\ \mathbb{C}^N \otimes End\ \mathbb{C}^N \otimes \big(\mathcal{V}_c(\mathfrak{gl}_N)\big)_h((z))[[u,v]].$$

By the definition of the map Y, one has

$$\begin{aligned}
&Y({T^+}^{13}(u)|0\rangle, z)\overline{R}^{12}(u-v+z+hc)^{-1}{T^+}^{23}(v)|0\rangle\\
&\quad = {T^+}^{13}(u+z)T^{13}(u+z+hc/2)^{-1}\overline{R}^{12}(u-v+z+hc)^{-1}{T^+}^{23}(v)|0\rangle\\
&\quad = {T^+}^{13}(u+z){T^+}^{23}(v)\overline{R}^{12}(u-v+z)^{-1}T^{13}(u+z+hc/2)^{-1}|0\rangle\\
&\quad = {T^+}^{13}(u+z){T^+}^{23}(v)\overline{R}^{12}(u-v+z)^{-1}|0\rangle
\end{aligned}$$

where we used the relation (2.65) with $m=n=1$ and the definition of the vacuum module $\mathcal{V}_c(\mathfrak{gl}_N)$. Using the unitarity of $\overline{R}$, one has that $\overline{R}^{12}(u-v+z+hc)^{-1} = \overline{R}^{21}(-u+v-z-hc)$. It follows that

$$\begin{aligned}
&{}^{lr}\overline{R}^{21}(-u+v-z-hc)Y({T^+}^{13}(u)|0\rangle, z){T^+}^{23}(v)|0\rangle\\
&= {T^+}^{13}(u+z){T^+}^{23}(v)\overline{R}^{12}(u-v+z)^{-1}|0\rangle
\end{aligned}$$

from which

$$\begin{aligned}
&Y({T^+}^{13}(u)|0\rangle, z){T^+}^{23}(v)|0\rangle\\
&\quad = {}^{lr}\Big({}^{lr}\overline{R}^{21}(-u+v-z-hc)\Big)^{-1}{T^+}^{13}(u+z){T^+}^{23}(v)\overline{R}^{12}(u-v+z)^{-1}|0\rangle\\
&\quad = g\left(\frac{-u+v-z-hc}{h}\right){}^{lr}\Big({}^{lr}R^{21}(-u+v-z-hc)\Big)^{-1}{T^+}^{13}(u+z)\cdot\\
&\qquad \cdot {T^+}^{23}(v)\overline{R}^{21}(-u+v-z)|0\rangle.
\end{aligned}$$

Recalling that $g(u) = \sum_{r\geq 0} g_r u^{-r}$ where $g_r \in \mathbb{C}$ (cf. Section 2.3.1), using equation (2.70) and the definition of R, one has:

$$Y({T^+}^{13}(u)|0\rangle, z){T^+}^{23}(v)|0\rangle$$

$$
\begin{aligned}
&= g\left(\frac{-u+v-z-hc}{h}\right)\left(1-\frac{hN}{-u+v-z-hc}\right)^{-1}\frac{h}{-u+v-z-hc}\cdot\\
&\quad\cdot {}^{lr}\sigma T^{+13}(u+z)T^{+23}(v)g\left(\frac{-u+v-z}{h}\right)\left(\sigma-\frac{h}{-u+v-z}\right)|0\rangle\\
&= \sum_{r\geq 0}\sum_{s\geq 0}\sum_{k\geq 0} g_r g_k(-u+v-z-hc)^{-r-1-s}(-u+v-z)^{-k}h^{r+1+s+k}N^s\cdot\\
&\quad\cdot {}^{lr}\sigma T^{+13}(u+z)T^{+23}(v)\left(\sigma-\frac{h}{-u+v-z}\right)|0\rangle\\
&= \sum_{r\geq 0}\sum_{s\geq 0}\sum_{k\geq 0}\sum_{l\geq 0}\binom{-r-1-s}{l}(-1)^l c^l N^s g_r g_k(-u+v-z)^{-r-1-s-l-k}h^{r+1+s+k+l}\cdot\\
&\quad\cdot {}^{lr}\sigma T^{+13}(u+z)T^{+23}(v)\left(\sigma-\frac{h}{-u+v-z}\right)|0\rangle.
\end{aligned}
$$

Let us consider the two terms which produce negative powers of z. The first one is the following:

$$
\begin{aligned}
&\sum_{r\geq 0}\sum_{s\geq 0}\sum_{k\geq 0}\sum_{l\geq 0}\binom{-r-1-s}{l}(-1)^l c^l N^s g_r g_k(-u+v-z)^{-r-1-s-l-k}h^{r+1+s+k+l}\\
&= \sum_{p\geq 1}\sum_{\substack{r,s,k,l\geq 0\\ r+s+k+l=p-1}}\binom{-r-1-s}{l}(-1)^l c^l N^s g_r g_k(-u+v-z)^{-p}h^p\\
&= \sum_{p\geq 1}\sum_{\substack{r,s,k,l\geq 0\\ r+s+k+l=p-1}}\sum_{i\geq 0}\sum_{j=0}^{i}\binom{-r-1-s}{l}\binom{-p}{i}\binom{i}{j}(-1)^{l-p-j}c^l N^s g_r g_k u^{i-j}v^j z^{-p-i}h^p\\
&= \sum_{p\geq 1}\sum_{m\geq 0}\sum_{j\geq 0}\gamma_{pmj}z^{-p-m-j}u^m v^j h^p \in \mathbb{C}((z))[[u,v,h]]
\end{aligned}
$$

where

$$
\gamma_{pmj} = \sum_{\substack{r,s,k,l\geq 0\\ r+s+k+l=p-1}}\binom{-r-1-s}{l}\binom{-p}{m+j}\binom{m+j}{j}(-1)^{l-p-j}c^l N^s g_r g_k \in \mathbb{C}.
$$

The only other term which produces negative powers of z is

$$
\begin{aligned}
\frac{h}{-u+v-z} &= \sum_{i\geq 0}\sum_{j=0}^{i}\binom{-1}{i}\binom{i}{j}(-1)^{-1-j}u^{i-j}v^j z^{-1-i}h\\
&= \sum_{m\geq 0}\sum_{j\geq 0}\binom{-1}{m+j}\binom{m+j}{j}(-1)^{-1-j}z^{-1-m-j}u^m v^j h \in \mathbb{C}((z))[[u,v,h]].
\end{aligned}
$$

Since $T^+(u+z)$ has only nonnegative powers of $u+z$, and u,v have only nonnegative powers, we have proved that $Y(T^+(u)|0\rangle,z)T^+(v)|0\rangle \in End\ \mathbb{C}^N\otimes End\ \mathbb{C}^N\otimes (\mathcal{V}_c(\mathfrak{gl}_N))_h((z))[[u,v]]$.

Similarly one proves that $\mathcal{S}$ is well defined.

Axioms on the vacuum vector.

$$Y(|0\rangle, z)T_n^+(u) = T_n^+(u) = Id_{\mathcal{V}_c(\mathfrak{gl}_N)}(T_n^+(u)),$$

$$Y(T_n^+(u)|0\rangle, z)|0\rangle = T_n^+(u|z)T_n(u|z+hc/2)^{-1}|0\rangle = T_n^+(u|z) \in \mathcal{V}_c(\mathfrak{gl}_N)[[z]]$$

and

$$Y(T_n^+(u)|0\rangle, z)|0\rangle_{|z=0} = T_n^+(u)|0\rangle.$$

Translation covariance.
In the following we will make an abuse of notation writing $e^{zD}T_n^+(u)$ instead of $1^{\otimes n} \otimes e^{zD}T_n^+(u)$ and, consequently, $DT_n^+(u)$ instead of $(1^{\otimes n} \otimes D)T_n^+(u)$.
By the definition of the exponential and of the matrix $T_n^+(u)$ one has

$$e^{zD}T_n^+(u) = \sum_{i_1,j_1=1}^{N} \dots \sum_{i_n,j_n=1}^{N} e_{i_1j_1} \otimes \dots \otimes e_{i_nj_n} \otimes \sum_{k\geq 0} \frac{D^k(t^+_{i_1j_1}(u_1)\cdots t^+_{i_nj_n}(u_n))}{k!} z^k$$

and, by the definition of D, one has

$$\begin{aligned} &e^{zD}T_n^+(u) = T_n^+(u|z)|0\rangle \\ &= \sum_{i_1,j_1=1}^{N} \dots \sum_{i_n,j_n=1}^{N} e_{i_1j_1} \otimes \dots \otimes e_{i_nj_n} \otimes t^+_{i_1j_1}(z+u_1)\cdots t^+_{i_nj_n}(z+u_n) \\ &= \sum_{i_1,j_1=1}^{N} \dots \sum_{i_n,j_n=1}^{N} e_{i_1j_1} \otimes \dots \otimes e_{i_nj_n} \otimes \\ &\quad \otimes \sum_{l_1,\dots,l_n\geq 0} \frac{\partial_{u_1}^{l_1} t^+_{i_1j_1}(u_1)}{l_1!} \cdots \frac{\partial_{u_n}^{l_n} t^+_{i_nj_n}(u_n)}{l_n!} z^{l_1+\dots+l_n}, \end{aligned}$$

hence

$$DT_n^+(u) = \Big(\sum_{i=1}^{n} \partial_{u_i}\Big)T_n^+(u). \tag{2.76}$$

It follows that, on one side, one has

$$\begin{aligned} &\partial_z Y(T_n^{+13}(u)|0\rangle, z)T_m^{+23}(v) \\ &= \partial_z(T_n^{+13}(u|z)T_n^{13}(u|z+hc/2)^{-1})T_m^{+23}(v) \\ &= (\partial_z T_n^{+13}(u|z))T_n^{13}(u|z+hc/2)^{-1}T_m^{+23}(v) + \\ &\quad + T_n^{+13}(u|z)(\partial_z T_n^{13}(u|z+hc/2)^{-1})T_m^{+23}(v) \\ &= \Big(\sum_{i=1}^{n} \partial_{u_i} T_n^{+13}(u|z)\Big)T_n^{13}(u|z+hc/2)^{-1}T_m^{+23}(v) + \\ &\quad + T_n^{+13}(u|z)\Big(\sum_{i=1}^{n} \partial_{u_i} T_n^{13}(u|z+hc/2)^{-1}\Big)T_m^{+23}(v) \\ &= \Big(\sum_{i=1}^{n} \partial_{u_i}\Big)T_n^{+13}(u|z)T_n^{13}(u|z+hc/2)^{-1}T_m^{+23}(v). \end{aligned} \tag{2.77}$$

And, on the other side, one has

$$\begin{aligned}
&[D, Y(T_n^{+13}(u)|0\rangle, z)]T_m^{+23}(v) \\
&= DY(T_n^{+13}(u)|0\rangle, z)T_m^{+23}(v) - Y(T_n^{+13}(u)DT_m^{+23}(v) \\
&= D(T_n^{+13}(u|z)T_n^{13}(u|z+hc/2)^{-1}T_m^{+23}(v)) \\
&\quad - T_n^{+13}(u|z)T_n^{13}(u|z+hc/2)^{-1}DT_m^{+23}(v) \\
&= \Big(\sum_{i=1}^{n}\partial_{u_i} + \sum_{k=1}^{m}\partial_{v_k}\Big)T_n^{+13}(u|z)T_n^{13}(u|z+hc/2)^{-1}T_m^{+23}(v) \\
&\quad - T_n^{+13}(u|z)T_n^{13}(u|z+hc/2)^{-1}\Big(\sum_{k=1}^{m}\partial_{v_k}\Big)T_m^{+23}(v) \\
&= \Big(\sum_{i=1}^{n}\partial_{u_i}\Big)T_n^{+13}(u|z)T_n^{13}(u|z+hc/2)^{-1}T_m^{+23}(v).
\end{aligned} \tag{2.78}$$

From which the translation covariance follows.

Invertibility of $\mathcal{S}$.

$$\begin{aligned}
&\mathcal{S}^{34}(z)\Big(\overline{R}_{nm}^{12}(u|v|z)^{-1}T_m^{+24}(v)\overline{R}_{nm}^{12}(u|v|z-hc)T_n^{+13}(u)|0\rangle\otimes|0\rangle\Big) \\
&\quad - \overline{R}_{nm}^{12}(u|v|z)^{-1}T_m^{+24}(v)\overline{R}_{nm}^{12}(u|v|z-hc)T_n^{+13}(u)|0\rangle\otimes|0\rangle \\
&= T_n^{+13}(u)\overline{R}_{nm}^{12}(u|v|z+hc)^{-1}T_m^{+24}(v)\overline{R}_{nm}^{12}(u|v|z) \\
&\quad - \overline{R}_{nm}^{12}(u|v|z)^{-1}T_m^{+24}(v)\overline{R}_{nm}^{12}(u|v|z-hc)T_n^{+13}(u) \\
&= T_n^{+13}(u)T_m^{+24}(v) - T_m^{+24}(v)T_n^{+13}(u) + O(h) \\
&= O(h)
\end{aligned}$$

indeed $\overline{R}(u) = 1 + O(h)$ and $T_n^{+13}(u)$ and $T_m^{+24}(v)$ commute because they insist on different indices.

$\mathcal{S}$ shift invariant.

$$\begin{aligned}
&\Big({}^{ll}\overline{R}_{nm}^{12}(u|v|z)^{-1}\,{}^{lr}\overline{R}_{nm}^{12}(u|v|z-hc)\Big)[D\otimes 1, \mathcal{S}^{34}(z)]\,(T_m^{+24}(v)T_n^{+13}(u)|0\rangle\otimes|0\rangle) \\
&= [D\otimes 1, \mathcal{S}^{34}(z)]\,(\overline{R}_{nm}^{12}(u|v|z)^{-1}T_m^{+24}(v)\overline{R}_{nm}^{12}(u|v|z-hc)T_n^{+13}(u)|0\rangle\otimes|0\rangle) \\
&= (D\otimes 1)\,(T_n^{+13}(u)\overline{R}_{nm}^{12}(u|v|z+hc)^{-1}T_m^{+24}(v)\overline{R}_{nm}^{12}(u|v|z)|0\rangle\otimes|0\rangle) \\
&\quad - \mathcal{S}^{34}(z)(\overline{R}_{nm}^{12}(u|v|z)^{-1}T_m^{+24}(v)\overline{R}_{nm}^{12}(u|v|z-hc)DT_n^{+13}(u)|0\rangle\otimes|0\rangle) \\
&= \bigg(\Big(\sum_{i=1}^{n}\partial_{u_i}\Big)T_n^{+13}(u)\bigg)\overline{R}_{nm}^{12}(u|v|z+hc)^{-1}T_m^{+24}(v)\overline{R}_{nm}^{12}(u|v|z)|0\rangle\otimes|0\rangle \\
&\quad - \mathcal{S}^{34}(z)\bigg(\overline{R}_{nm}^{12}(u|v|z)^{-1}T_m^{+24}(v)\overline{R}_{nm}^{12}(u|v|z-hc)\bigg(\Big(\sum_{i=1}^{n}\partial_{u_i}\Big)T_n^{+13}(u)\bigg)|0\rangle\otimes|0\rangle\bigg) \\
&= \bigg(\sum_{i=1}^{n}\partial_{u_i}\bigg)\Big(T_n^{+13}(u)\overline{R}_{nm}^{12}(u|v|z+hc)^{-1}T_m^{+24}(v)\overline{R}_{nm}^{12}(u|v|z)|0\rangle\otimes|0\rangle\Big)
\end{aligned}$$

$$
\begin{aligned}
&- T_n^{+13}(u)\Big(\Big(\sum_{i=1}^{n}\partial_{u_i}\Big)\overline{R}_{nm}^{12}(u|v|z+hc)^{-1}\Big)T_m^{+24}(v)\overline{R}_{nm}^{12}(u|v|z)|0\rangle\otimes|0\rangle \\
&- T_n^{+13}(u)\overline{R}_{nm}^{12}(u|v|z+hc)^{-1}T_m^{+24}(v)\Big(\Big(\sum_{i=1}^{n}\partial_{u_i}\Big)\overline{R}_{nm}^{12}(u|v|z)|0\rangle\otimes|0\rangle\Big) \\
&- \Big(\sum_{i=1}^{n}\partial_{u_i}\Big)\Big(\mathcal{S}^{34}(z)\Big(\overline{R}_{nm}^{12}(u|v|z)^{-1}T_m^{+24}(v)\overline{R}_{nm}^{12}(u|v|z-hc)T_n^{+13}(u)|0\rangle\otimes|0\rangle\Big)\Big) \\
&+ \mathcal{S}^{34}(z)\left(\left(\Big(\sum_{i=1}^{n}\partial_{u_i}\Big)\overline{R}_{nm}^{12}(u|v|z)^{-1}\right)T_m^{+24}(v)\overline{R}_{nm}^{12}(u|v|z-hc)T_n^{+13}(u)|0\rangle\otimes|0\rangle\right) \\
&+ \mathcal{S}^{34}(z)\Big(\overline{R}_{nm}^{12}(u|v|z)^{-1}T_m^{+24}(v)\Big(\Big(\sum_{i=1}^{n}\partial_{u_i}\Big)\overline{R}_{nm}^{12}(u|v|z-hc)\Big)T_n^{+13}(u)|0\rangle\otimes|0\rangle\Big) \\
=& -\partial_z\Big(T_n^{+13}(u)\overline{R}_{nm}^{12}(u|v|z+hc)^{-1}T_m^{+24}(v)\overline{R}_{nm}^{12}(u|v|z)|0\rangle\otimes|0\rangle\Big) \\
&+ \mathcal{S}^{34}(z)\Big(\partial_z\Big(\overline{R}_{nm}^{12}(u|v|z)^{-1}\,{}^{lr}\overline{R}_{nm}^{12}(u|v|z-hc)\Big)T_m^{+24}(v)T_n^{+13}(u)|0\rangle\otimes|0\rangle\Big) \\
=& -\partial_z\Big(\mathcal{S}^{34}(z)\Big(\overline{R}_{nm}^{12}(u|v|z)^{-1}T_m^{+24}(v)\overline{R}_{nm}^{12}(u|v|z-hc)T_n^{+13}(u)|0\rangle\otimes|0\rangle\Big)\Big) \\
&+ \mathcal{S}^{34}(z)\Big(\partial_z\Big(\overline{R}_{nm}^{12}(u|v|z)^{-1}\,{}^{lr}\overline{R}_{nm}^{12}(u|v|z-hc)\Big)T_m^{+24}(v)T_n^{+13}(u)|0\rangle\otimes|0\rangle\Big) \\
=& -\partial_z\Big({}^{ll}\overline{R}_{nm}^{12}(u|v|z)^{-1}\,{}^{lr}\overline{R}_{nm}^{12}(u|v|z-hc)\mathcal{S}^{34}(z)\Big(T_m^{+24}(v)T_n^{+13}(u)|0\rangle\otimes|0\rangle\Big)\Big) \\
&+ \partial_z\Big({}^{ll}\overline{R}_{nm}^{12}(u|v|z)^{-1}\,{}^{lr}\overline{R}_{nm}^{12}(u|v|z-hc)\Big)\mathcal{S}^{34}(z)\Big(T_m^{+24}(v)T_n^{+13}(u)|0\rangle\otimes|0\rangle\Big) \\
=& -\Big({}^{ll}\overline{R}_{nm}^{12}(u|v|z)^{-1}\,{}^{lr}\overline{R}_{nm}^{12}(u|v|z-hc)\Big)\partial_z\mathcal{S}^{34}(z)\Big(T_m^{+24}(v)T_n^{+13}(u)|0\rangle\otimes|0\rangle\Big).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&[D\otimes 1,\mathcal{S}^{34}(z)]\,(T_m^{+24}(v)T_n^{+13}(u)|0\rangle\otimes|0\rangle) \\
&= -\partial_z\mathcal{S}^{34}(z)\Big(T_m^{+24}(v)T_n^{+13}(u)|0\rangle\otimes|0\rangle\Big),
\end{aligned}
$$

i.e.

$$[D\otimes 1,\mathcal{S}^{34}(z)] = -\partial_z\mathcal{S}^{34}(z).$$

$\mathcal{S}$ solution of the quantum Yang-Baxter equation

On one side one has the following equalities

$$
\begin{aligned}
&\mathcal{S}^{45}(z)\mathcal{S}^{46}(z+w)\mathcal{S}^{56}(w)\Big(\overline{R}_{ml}^{23}(v|\bar v|w)^{-1}\overline{R}_{nl}^{13}(u|\bar v|z+w)^{-1}T_l^{+36}(\bar v)\cdot \\
&\quad\cdot\overline{R}_{nm}^{12}(u|v|z)^{-1}\overline{R}_{ml}^{23}(v|\bar v|w-hc)T_m^{+25}(v)\cdot \\
&\quad\cdot\overline{R}_{nl}^{13}(u|\bar v|z+w-hc)\overline{R}_{nm}^{12}(u|v|z-hc)T_n^{+14}(u)\Big) \\
&= \mathcal{S}^{45}(z)\mathcal{S}^{46}(z+w)\mathcal{S}^{56}(w)\Big(\overline{R}_{ml}^{23}(v|\bar v|w)^{-1}\overline{R}_{nl}^{13}(u|\bar v|z+w)^{-1}\cdot \\
&\quad\cdot\overline{R}_{nm}^{12}(u|v|z)^{-1}T_l^{+36}(\bar v)\overline{R}_{ml}^{23}(v|\bar v|w-hc)T_m^{+25}(v)\cdot \\
&\quad\cdot\overline{R}_{nl}^{13}(u|\bar v|z+w-hc)\overline{R}_{nm}^{12}(u|v|z-hc)T_n^{+14}(u)\Big) \\
&\overset{(e1)}{=} \mathcal{S}^{45}(z)\mathcal{S}^{46}(z+w)\mathcal{S}^{56}(w)\Big(\overline{R}_{nm}^{12}(u|v|z)^{-1}\overline{R}_{nl}^{13}(u|\bar v|z+w)^{-1}\cdot
\end{aligned}
$$

$$
\begin{aligned}
&\cdot \overline{R}^{23}_{ml}(v|\bar v|w)^{-1}T_l^{+36}(\bar v)\overline{R}^{23}_{ml}(v|\bar v|w-hc)^{-1}T_m^{+25}(v)\cdot\\
&\cdot \overline{R}^{13}_{nl}(u|\bar v|z+w-hc)\overline{R}^{12}_{nm}(u|v|z-hc)T_n^{+14}(u)\Big)\\
&= \mathcal{S}^{45}(z)\mathcal{S}^{46}(z+w)\Big(\overline{R}^{12}_{nm}(u|v|z)^{-1}\overline{R}^{13}_{nl}(u|\bar v|z+w)^{-1}\cdot\\
&\cdot \mathcal{S}^{56}(w)\Big(\overline{R}^{23}_{ml}(v|\bar v|w)^{-1}T_l^{+36}(\bar v)\overline{R}^{23}_{ml}(v|\bar v|w-hc)T_m^{+25}(v)\Big)\cdot\\
&\cdot \overline{R}^{13}_{nl}(u|\bar v|z+w-hc)\overline{R}^{12}_{nm}(u|v|z-hc)T_n^{+14}(u)\Big)\\
&= \mathcal{S}^{45}(z)\mathcal{S}^{46}(z+w)\Big(\overline{R}^{12}_{nm}(u|v|z)^{-1}\overline{R}^{13}_{nl}(u|\bar v|z+w)^{-1}\cdot\\
&\cdot T_m^{+25}(v)\overline{R}^{23}_{ml}(v|\bar v|w+hc)^{-1}T_l^{+36}(\bar v)\overline{R}^{23}_{ml}(v|\bar v|w)\cdot\\
&\cdot \overline{R}^{13}_{nl}(u|\bar v|z+w-hc)\overline{R}^{12}_{nm}(v|v|z-hc)T_n^{+14}(u)\Big)\\
&\overset{(e2)}{=} \mathcal{S}^{45}(z)\mathcal{S}^{46}(z+w)\Big(\overline{R}^{12}_{nm}(u|v|z)^{-1}\overline{R}^{13}_{nl}(u|\bar v|z+w)^{-1}\cdot\\
&\cdot T_m^{+25}(v)\overline{R}^{23}_{ml}(v|\bar v|w+hc)^{-1}T_l^{+36}(\bar v)\overline{R}^{12}_{nm}(u|v|z-hc)\cdot\\
&\cdot \overline{R}^{13}_{nl}(u|\bar v|z+w-hc)\overline{R}^{23}_{ml}(v|\bar v|w)T_n^{+14}(u)\Big)\\
&= \mathcal{S}^{45}(z)\mathcal{S}^{46}(z+w)\Big(\overline{R}^{12}_{nm}(u|v|z)^{-1}T_m^{+25}(v)\overline{R}^{13}_{nl}(u|\bar v|z+w)^{-1}\cdot\\
&\cdot \overline{R}^{23}_{ml}(v|\bar v|w+hc)^{-1}\overline{R}^{12}_{nm}(u|v|z-hc)T_l^{+36}(\bar v)\cdot\\
&\cdot \overline{R}^{13}_{nl}(u|\bar v|z+w-hc)T_n^{+14}(u)\overline{R}^{23}_{ml}(v|\bar v|w)\Big)\\
&\overset{(e3)}{=} \mathcal{S}^{45}(z)\Big(\overline{R}^{12}_{nm}(u|v|z)^{-1}T_m^{+25}(v)\overline{R}^{12}_{nm}(u|v|z-hc)\cdot\\
&\cdot \overline{R}^{23}_{ml}(v|\bar v|w+hc)^{-1}\mathcal{S}^{46}(z+w)\Big(\overline{R}^{13}_{nl}(u|\bar v|z+w)^{-1}T_l^{+36}(\bar v)\cdot\\
&\cdot \overline{R}^{13}_{nl}(u|\bar v|z+w-hc)T_n^{+14}(u)\Big)\overline{R}^{23}_{ml}(v|\bar v|w)\Big)\\
&= \mathcal{S}^{45}(z)\Big(\overline{R}^{12}_{nm}(u|v|z)^{-1}T_m^{+25}(v)\overline{R}^{12}_{nm}(u|v|z-hc)\overline{R}^{23}_{ml}(v|\bar v|w+hc)^{-1}\cdot\\
&\cdot T_n^{+14}(u)\overline{R}^{13}_{nl}(u|\bar v|z+w+hc)^{-1}T_l^{+36}(\bar v)\overline{R}^{13}_{nl}(u|\bar v|z+w)\overline{R}^{23}_{ml}(v|\bar v|w)\Big)\\
&= \mathcal{S}^{45}(z)\Big(\overline{R}^{12}_{nm}(u|v|z)^{-1}T_m^{+25}(v)\overline{R}^{12}_{nm}(u|v|z-hc)T_n^{+14}(u)\overline{R}^{23}_{ml}(v|\bar v|w+hc)^{-1}\cdot\\
&\cdot \overline{R}^{13}_{nl}(u|\bar v|z+w+hc)^{-1}T_l^{+36}(\bar v)\overline{R}^{13}_{nl}(u|\bar v|z+w)\overline{R}^{23}_{ml}(v|\bar v|w)\Big)\\
&= T_n^{+14}(u)\overline{R}^{12}_{nm}(u|v|z+hc)^{-1}T_m^{+25}(v)\overline{R}^{12}_{nm}(u|v|z)\overline{R}^{23}_{ml}(v|\bar v|w+hc)^{-1}\cdot\\
&\cdot \overline{R}^{13}_{nl}(u|\bar v|z+w+hc)^{-1}T_l^{+36}(\bar v)\overline{R}^{13}_{nl}(u|\bar v|z+w)\overline{R}^{23}_{ml}(v|\bar v|w).
\end{aligned}
$$

Where we use in $(e1)$, $(e2)$ and $(e3)$ that $\overline{R}$ is a solution of the quantum Yang-Baxter equation.
On the other hand

$$
\begin{aligned}
&\mathcal{S}^{56}(w)\mathcal{S}^{46}(z+w)\mathcal{S}^{45}(z)\Big(\overline{R}^{23}_{ml}(v|\bar v|w)^{-1}\overline{R}^{13}_{nl}(u|\bar v|z+w)^{-1}T_l^{+36}(\bar v)\cdot\\
&\cdot \overline{R}^{12}_{nm}(u|v|z)^{-1}\overline{R}^{23}_{ml}(v|\bar v|w-hc)T_m^{+25}(v)\cdot\\
&\cdot \overline{R}^{13}_{nl}(u|\bar v|z+w-hc)\overline{R}^{12}_{nm}(u|v|z-hc)T_n^{+14}(u)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \mathcal{S}^{56}(w)\mathcal{S}^{46}(z+w)\mathcal{S}^{45}(z)\Big(\overline{R}^{23}_{ml}(v|\bar{v}|w)^{-1}\overline{R}^{13}_{nl}(u|\bar{v}|z+w)^{-1}T^{+36}_{l}(\bar{v})\cdot \\
&\quad\cdot \overline{R}^{12}_{nm}(u|v|z)^{-1}\overline{R}^{23}_{ml}(v|\bar{v}|w-hc)\overline{R}^{13}_{nl}(u|\bar{v}|z+w-hc)\cdot \\
&\quad\cdot T^{+25}_{m}(v)\overline{R}^{12}_{nm}(u|v|z-hc)T^{+14}_{n}(u)\Big) \\
&\overset{(e4)}{=} \mathcal{S}^{56}(w)\mathcal{S}^{46}(z+w)\mathcal{S}^{45}(z)\Big(\overline{R}^{23}_{ml}(v|\bar{v}|w)^{-1}\overline{R}^{13}_{nl}(u|\bar{v}|z+w)^{-1}T^{+36}_{l}(\bar{v})\cdot \\
&\quad\cdot \overline{R}^{13}_{nl}(u|\bar{v}|z+w-hc)\overline{R}^{23}_{ml}(v|\bar{v}|w-hc)\overline{R}^{12}_{nm}(u|v|z)^{-1}\cdot \\
&\quad\cdot T^{+25}_{m}(v)\overline{R}^{12}_{nm}(u|v|z-hc)T^{+14}_{n}(u)\Big) \\
&= \mathcal{S}^{56}(w)\mathcal{S}^{46}(z+w)\Big(\overline{R}^{23}_{ml}(v|\bar{v}|w)^{-1}\overline{R}^{13}_{nl}(u|\bar{v}|z+w)^{-1}T^{+36}_{l}(\bar{v})\cdot \\
&\quad\cdot \overline{R}^{13}_{nl}(u|\bar{v}|z+w-hc)T^{+14}_{n}(u)\overline{R}^{23}_{ml}(v|\bar{v}|w-hc)\overline{R}^{12}_{nm}(u|v|z+hc)^{-1}\cdot \\
&\quad\cdot T^{+25}_{m}(v)\overline{R}^{12}_{nm}(u|v|z)\Big) \\
&= \mathcal{S}^{56}(w)\Big(\overline{R}^{23}_{ml}(v|\bar{v}|w)^{-1}T^{+14}_{n}(u)\overline{R}^{13}_{nl}(u|\bar{v}|z+w+hc)^{-1}T^{+36}_{l}(\bar{v})\cdot \\
&\quad\cdot \overline{R}^{13}_{nl}(u|\bar{v}|z+w)\overline{R}^{23}_{ml}(v|\bar{v}|w-hc)\overline{R}^{12}_{nm}(u|v|z+hc)^{-1}\cdot \\
&\quad\cdot T^{+25}_{m}(v)\overline{R}^{12}_{nm}(u|v|z)) \\
&\overset{(e5)}{=} \mathcal{S}^{56}(w)\Big(\overline{R}^{23}_{ml}(v|\bar{v}|w)^{-1}T^{+14}_{n}(u)\overline{R}^{13}_{nl}(u|\bar{v}|z+w+hc)^{-1}T^{+36}_{l}(\bar{v})\cdot \\
&\quad\cdot \overline{R}^{12}_{nm}(u|v|z+hc)^{-1}\overline{R}^{23}_{ml}(v|\bar{v}|w-hc)\overline{R}^{13}_{nl}(u|\bar{v}|z+w)\cdot \\
&\quad\cdot T^{+25}_{m}(v)\overline{R}^{12}_{nm}(u|v|z)\Big) \\
&= \mathcal{S}^{56}(w)\Big(T^{+14}_{n}(u)\overline{R}^{23}_{ml}(v|\bar{v}|w)^{-1}\overline{R}^{13}_{nl}(u|\bar{v}|z+w+hc)^{-1}\overline{R}^{12}_{nm}(u|v|z+hc)^{-1}\cdot \\
&\quad\cdot T^{+36}_{l}(\bar{v})\overline{R}^{23}_{ml}(v|\bar{v}|w-hc)T^{+25}_{m}(v)\overline{R}^{13}_{nl}(u|\bar{v}|z+w)\overline{R}^{12}_{nm}(u|v|z)\Big) \\
&\overset{(e6)}{=} \mathcal{S}^{56}(w)\Big(T^{+14}_{n}(u)\overline{R}^{12}_{nm}(u|v|z+hc)^{-1}\overline{R}^{13}_{nl}(u|\bar{v}|z+w+hc)^{-1}\overline{R}^{23}_{ml}(v|\bar{v}|w)^{-1}\cdot \\
&\quad\cdot T^{+36}_{l}(\bar{v})\overline{R}^{23}_{ml}(v|\bar{v}|w-hc)T^{+25}_{m}(v)\overline{R}^{13}_{nl}(u|\bar{v}|z+w)\overline{R}^{12}_{nm}(u|v|z)\Big) \\
&= T^{+14}_{n}(u)\overline{R}^{12}_{nm}(u|v|z+hc)^{-1}\overline{R}^{13}_{nl}(u|\bar{v}|z+w+hc)^{-1}\mathcal{S}^{56}(w)\Big(\overline{R}^{23}_{ml}(v|\bar{v}|w)^{-1}\cdot \\
&\quad\cdot T^{+36}_{l}(\bar{v})\overline{R}^{23}_{ml}(v|\bar{v}|w-hc)T^{+25}_{m}(v)\Big)\overline{R}^{13}_{nl}(u|\bar{v}|z+w)\overline{R}^{12}_{nm}(u|v|z) \\
&= T^{+14}_{n}(u)\overline{R}^{12}_{nm}(u|v|z+hc)^{-1}\overline{R}^{13}_{nl}(u|\bar{v}|z+w+hc)^{-1}T^{+25}_{m}(v)\cdot \\
&\quad\cdot \overline{R}^{23}_{ml}(v|\bar{v}|w+hc)^{-1}T^{+36}_{l}(\bar{v})\overline{R}^{23}_{ml}(v|\bar{v}|w)\overline{R}^{13}_{nl}(u|\bar{v}|z+w)\overline{R}^{12}_{nm}(u|v|z) \\
&\overset{(e7)}{=} T^{+14}_{n}(u)\overline{R}^{12}_{nm}(u|v|z+hc)^{-1}\overline{R}^{13}_{nl}(u|\bar{v}|z+w+hc)^{-1}T^{+25}_{m}(v)\cdot \\
&\quad\cdot \overline{R}^{23}_{ml}(v|\bar{v}|w+hc)^{-1}T^{+36}_{l}(\bar{v})\overline{R}^{12}_{nm}(u|v|z)\overline{R}^{13}_{nl}(u|\bar{v}|z+w)\overline{R}^{23}_{ml}(v|\bar{v}|w) \\
&= T^{+14}_{n}(u)\overline{R}^{12}_{nm}(u|v|z+hc)^{-1}T^{+25}_{m}(v)\overline{R}^{13}_{nl}(u|\bar{v}|z+w+hc)^{-1}\cdot \\
&\quad\cdot \overline{R}^{23}_{ml}(v|\bar{v}|w+hc)^{-1}\overline{R}^{12}_{nm}(u|v|z)T^{+36}_{l}(\bar{v})\overline{R}^{13}_{nl}(u|\bar{v}|z+w)\overline{R}^{23}_{ml}(v|\bar{v}|w) \\
&\overset{(e8)}{=} T^{+14}_{n}(u)\overline{R}^{12}_{nm}(u|v|z+hc)^{-1}T^{+25}_{m}(v)\overline{R}^{12}_{nm}(u|v|z)\overline{R}^{23}_{ml}(v|\bar{v}|w+hc)^{-1}\cdot \\
&\quad\cdot \overline{R}^{13}_{nl}(u|\bar{v}|z+w+hc)^{-1}T^{+36}_{l}(\bar{v})\overline{R}^{13}_{nl}(u|\bar{v}|z+w)\overline{R}^{23}_{ml}(v|\bar{v}|w).
\end{aligned}
$$

Where we use in $(e4)$ - $(e8)$ that $\overline{R}$ is a solution of the quantum Yang-Baxter equation.
Therefore one has

$$\begin{aligned}
&\mathcal{S}^{45}(z)\mathcal{S}^{46}(z+w)\mathcal{S}^{56}(w)\Big(\overline{R}^{23}_{ml}(v|\bar{v}|w)^{-1}\overline{R}^{13}_{nl}(u|\bar{v}|z+w)^{-1}T^{+36}_{l}(\bar{v})\cdot\\
&\quad\cdot\overline{R}^{12}_{nm}(u|v|z)^{-1}\overline{R}^{23}_{ml}(v|\bar{v}|w-hc)T^{+25}_{m}(v)\cdot\\
&\quad\cdot\overline{R}^{13}_{nl}(u|\bar{v}|z+w-hc)\overline{R}^{12}_{nm}(u|v|z-hc)T^{+14}_{n}(u)\Big)\\
&=\mathcal{S}^{56}(w)\mathcal{S}^{46}(z+w)\mathcal{S}^{45}(z)\Big(\overline{R}^{23}_{ml}(v|\bar{v}|w)^{-1}\overline{R}^{13}_{nl}(u|\bar{v}|z+w)^{-1}T^{+36}_{l}(\bar{v})\cdot\\
&\quad\cdot\overline{R}^{12}_{nm}(u|v|z)^{-1}\overline{R}^{23}_{ml}(v|\bar{v}|w-hc)T^{+25}_{m}(v)\cdot\\
&\quad\cdot\overline{R}^{13}_{nl}(u|\bar{v}|z+w-hc)\overline{R}^{12}_{nm}(u|v|z-hc)T^{+14}_{n}(u)\Big).
\end{aligned}$$

It follows that $\mathcal{S}$ is a solution of the quantum Yang-Baxter equation.

Unitarity of $\mathcal{S}$.
Using the definition of the map $\mathcal{S}$ and the unitarity of $\overline{R}$ one has the following equalities

$$\begin{aligned}
&\mathcal{S}^{43}(-z)\mathcal{S}^{34}(z)(\overline{R}^{12}_{nm}(u|v|z)^{-1}T^{+24}_{m}(v)\overline{R}^{12}_{nm}(u|v|z-hc)T^{+13}_{n}(u)|0\rangle\otimes|0\rangle)\\
&=\mathcal{S}^{43}(-z)(T^{+13}_{n}(u)\overline{R}^{12}_{nm}(u|v|z+hc)^{-1}T^{+24}_{m}(v)\overline{R}^{12}_{nm}(u|v|z)|0\rangle\otimes|0\rangle)\\
&=\mathcal{S}^{43}(-z)\Bigg(T^{+13}_{n}(u)\left(\overleftarrow{\prod_{j=1,\dots,n}}\ \overrightarrow{\prod_{i=n+1,\dots,n+m}}\overline{R}^{ji}(z+u_j-v_{i-n}+hc)^{-1}\right)\cdot\\
&\quad\cdot T^{+24}_{m}(v)\left(\overrightarrow{\prod_{j=1,\dots,n}}\ \overleftarrow{\prod_{i=n+1,\dots,n+m}}\overline{R}^{ij}(-z-u_j+v_{i-n})^{-1}\right)|0\rangle\otimes|0\rangle\Bigg)\\
&=\mathcal{S}^{43}(-z)\Bigg(T^{+13}_{n}(u)\left(\overleftarrow{\prod_{j=1,\dots,n}}\ \overrightarrow{\prod_{i=n+1,\dots,n+m}}\overline{R}^{ij}(-z-u_j+v_{i-n}-hc)\right)\cdot\\
&\quad\cdot T^{+24}_{m}(v)\left(\overrightarrow{\prod_{j=1,\dots,n}}\ \overleftarrow{\prod_{i=n+1,\dots,n+m}}\overline{R}^{ij}(-z-u_j+v_{i-n})^{-1}\right)|0\rangle\otimes|0\rangle\Bigg)\\
&=\mathcal{S}^{43}(-z)\Bigg(T^{+13}_{n}(u)\left(\overrightarrow{\prod_{i=n+1,\dots,n+m}}\ \overleftarrow{\prod_{j=1,\dots,n}}\overline{R}^{ij}(-z-u_j+v_{i-n}-hc)\right)\cdot\\
&\quad\cdot T^{+24}_{m}(v)\left(\overrightarrow{\prod_{j=1,\dots,n}}\ \overleftarrow{\prod_{i=n+1,\dots,n+m}}\overline{R}^{ij}(-z-u_j+v_{i-n})^{-1}\right)|0\rangle\otimes|0\rangle\Bigg)\\
&=\left(\overrightarrow{\prod_{i=n+1,\dots,n+m}}\ \overleftarrow{\prod_{j=1,\dots,n}}\overline{R}^{ij}(-z-u_j+v_{i-n})\right)T^{+24}_{m}(v)\cdot\\
&\quad\cdot\left(\overrightarrow{\prod_{i=n+1,\dots,n+m}}\ \overleftarrow{\prod_{j=1,\dots,n}}\overline{R}^{ij}(-z-u_j+v_{i-n}+hc)\right)^{-1}\cdot\\
&\quad\cdot T^{+13}_{n}(u)\left(\overrightarrow{\prod_{i=n+1,\dots,n+m}}\ \overleftarrow{\prod_{j=1,\dots,n}}\overline{R}^{ij}(-z-u_j+v_{i-n})\right)\cdot
\end{aligned}$$

$$\cdot \left(\overrightarrow{\prod_{j=1,\dots,n}} \overleftarrow{\prod_{i=n+1,\dots,n+m}} \overline{R}^{ij}(-z-u_j+v_{i-n})^{-1} \right) |0\rangle \otimes |0\rangle$$
$$= \left(\overleftarrow{\prod_{j=1,\dots,n}} \overrightarrow{\prod_{i=n+1,\dots,n+m}} \overline{R}^{ji}(+z+u_j-v_{i-n})^{-1} \right) T_m^{+24}(v) \cdot$$
$$\cdot \left(\overleftarrow{\prod_{i=n+1,\dots,n+m}} \overrightarrow{\prod_{j=1,\dots,n}} \overline{R}^{ij}(-z-u_j+v_{i-n}+hc)^{-1} \right) T_n^{+13}(u)$$
$$= \overline{R}_{nm}^{12}(u|v|z)^{-1} T_m^{+24}(v) \overline{R}_{nm}^{12}(u|v|z-hc) T_n^{+13}(u).$$

Therefore the unitarity of $\mathcal{S}$ follows.

$\mathcal{S}$-locality.
In the following we will write $z-w$ to denote the expansions of the negative powers of $z-w$ where $|z|>|w|$ and $-w+z$ to denote the expansions where $|w|>|z|$. Let us compute $Y(z)(1\otimes Y(w))\mathcal{S}^{45}(z-w)$ on $\overline{R}_{nm}^{12}(u|v|z-w)^{-1}T_m^{+25}(v)\overline{R}_{nm}^{12}(u|v|z-w-hc)T_n^{+14}(u)\overline{R}_{mk}^{23}(v|\bar{v}|w+hc)^{-1}\overline{R}_{nk}^{13}(u|\bar{v}|z+hc)^{-1}T_k^{+36}(\bar{v})|0\rangle\otimes|0\rangle\otimes|0\rangle$.

$$Y(z)(1\otimes Y(w))\mathcal{S}^{45}(z-w)\big(\overline{R}_{nm}^{12}(u|v|z-w)^{-1}T_m^{+25}(v)\overline{R}_{nm}^{12}(u|v|z-w-hc)\cdot$$
$$\cdot T_n^{+14}(u)\overline{R}_{mk}^{23}(v|\bar{v}|w+hc)^{-1}\overline{R}_{nk}^{13}(u|\bar{v}|z+hc)^{-1}T_k^{+36}(\bar{v})|0\rangle\otimes|0\rangle\otimes|0\rangle\big)$$
$$= Y(z)(1\otimes Y(w))\big(T_n^{+14}(u)\overline{R}_{nm}^{12}(u|v|z-w+hc)^{-1}T_m^{+25}(v)T_m^{+25}(v)\cdot$$
$$\cdot \overline{R}_{nm}^{12}(u|v|z-w)\overline{R}_{mk}^{23}(v|\bar{v}|w+hc)^{-1}\overline{R}_{nk}^{13}(u|\bar{v}|z+hc)^{-1}T_k^{+36}(\bar{v})|0\rangle\otimes|0\rangle\otimes|0\rangle\big)$$
$$= T_n^{+14}(u|z)T_n^{14}(u|z+hc/2)^{-1}\overline{R}_{nm}^{12}(u|v|z-w+hc)^{-1}T_m^{+24}(v|w)\cdot$$
$$\cdot T_m^{24}(v|w+hc/2)^{-1}\overline{R}_{nm}^{12}(u|v|z-w)\overline{R}_{mk}^{23}(v|\bar{v}|w+hc)^{-1}\overline{R}_{nk}^{13}(u|\bar{v}|z+hc)^{-1}\cdot$$
$$\cdot T_k^{+34}(\bar{v})|0\rangle$$
$$\stackrel{(e1)}{=} T_n^{+14}(u|z)T_m^{+24}(v|w)\overline{R}_{nm}^{12}(u|v|z-w)^{-1}T_n^{14}(u|z+hc/2)^{-1}T_m^{24}(v|w+hc/2)^{-1}\cdot$$
$$\cdot \overline{R}_{nm}^{12}(u|v|z-w)\overline{R}_{mk}^{23}(v|\bar{v}|w+hc)^{-1}\overline{R}_{nk}^{13}(u|\bar{v}|z+hc)^{-1}T_k^{+34}(\bar{v})|0\rangle$$
$$\stackrel{(e2)}{=} T_n^{+14}(u|z)T_m^{+24}(v|w)\overline{R}_{nm}^{12}(u|v|z-w)^{-1}\overline{R}_{nm}^{12}(u|v|z-w)T_m^{24}(v|w+hc/2)^{-1}\cdot$$
$$\cdot T_n^{14}(u|z+hc/2)^{-1}\overline{R}_{mk}^{23}(v|\bar{v}|w+hc)^{-1}\overline{R}_{nk}^{13}(u|\bar{v}|z+hc)^{-1}T_k^{+34}(\bar{v})|0\rangle$$
$$= T_n^{+14}(u|z)T_m^{+24}(v|w)T_m^{24}(v|w+hc/2)^{-1}\overline{R}_{mk}^{23}(v|\bar{v}|w+hc)^{-1}\cdot$$
$$\cdot T_n^{14}(u|z+hc/2)^{-1}\overline{R}_{nk}^{13}(u|\bar{v}|z+hc)^{-1}T_k^{+34}(\bar{v})|0\rangle$$
$$\stackrel{(e3)}{=} T_n^{+14}(u|z)T_m^{+24}(v|w)T_m^{24}(v|w+hc/2)^{-1}\overline{R}_{mk}^{23}(v|\bar{v}|w+hc)^{-1}T_k^{+34}(\bar{v})\cdot$$
$$\cdot \overline{R}_{nk}^{13}(u|\bar{v}|z)^{-1}|0\rangle$$
$$\stackrel{(e4)}{=} T_n^{+14}(u|z)T_m^{+24}(v|w)T_k^{+34}(\bar{v})\overline{R}_{mk}^{23}(v|\bar{v}|w)^{-1}\overline{R}_{nk}^{13}(u|\bar{v}|z)^{-1}|0\rangle$$

where $(e1)$ is due to relation (2.65), $(e2)$ is due to relation (2.63) and $(e3)$-$(e4)$ are due to relation (2.65) and the definition of $\mathcal{V}_c(\mathfrak{gl}_N)$.

Therefore, one has

$$\begin{aligned}
&Y(z)\big(1\otimes Y(w)\big)\mathcal{S}^{45}(z-w)\big(T_n^{+14}(u)T_m^{+25}(v)T_k^{+36}(\bar v)|0\rangle\otimes|0\rangle\otimes|0\rangle\big)\\
&={}^{lr}\big(\big({}^{lr}\overline{R}^{12}_{nm}(u|v|z-w-hc)\big)^{-1}\big)\,{}^{rl}\big(\big({}^{rl}\overline{R}^{13}_{nk}(u|\bar v|z+hc)\big)^{-1}\big)\cdot\\
&\quad\cdot\,{}^{rl}\big(\big({}^{rl}\overline{R}^{23}_{mk}(v|\bar v|w+hc)\big)^{-1}\big)\overline{R}^{12}_{nm}(u|v|z-w)T_n^{+14}(u|z)T_m^{+24}(v|w)\cdot\\
&\quad\cdot\,T_k^{+34}(\bar v)\overline{R}^{23}_{mk}(v|\bar v|w)^{-1}\overline{R}^{13}_{nk}(u|\bar v|z)^{-1}|0\rangle.
\end{aligned}\tag{2.79}$$

We point out that in equation (2.79), $z-w$ has to be expanded in $|z|>|w|$.
Let us now compute $Y(w)(1\otimes Y(z))(4\ 5)$ on $\overline{R}^{12}_{nm}(u|v|-w+z)^{-1}T_m^{+25}(v)\overline{R}^{12}_{nm}(u|v|-w+z-hc)T_n^{+14}(u)\overline{R}^{23}_{mk}(v|\bar v|w+hc)^{-1}\overline{R}^{13}_{nk}(u|\bar v|z+hc)^{-1}T_k^{+36}(\bar v)|0\rangle\otimes|0\rangle\otimes|0\rangle$:

$$\begin{aligned}
&Y(w)\big(1\otimes Y(z)\big)(4\ 5)\big(\overline{R}^{12}_{nm}(u|v|-w+z)^{-1}T_m^{+25}(v)\overline{R}^{12}_{nm}(u|v|-w+z-hc)\cdot\\
&\quad\cdot\,T_n^{+14}(u)\overline{R}^{23}_{mk}(v|\bar v|w+hc)^{-1}\overline{R}^{13}_{nk}(u|\bar v|z+hc)^{-1}T_k^{+36}(\bar v)|0\rangle\otimes|0\rangle\otimes|0\rangle\big)\\
&=Y(w)\big(1\otimes Y(z)\big)\big(\overline{R}^{12}_{nm}(u|v|-w+z)^{-1}T_m^{+24}(v)\overline{R}^{12}_{nm}(u|v|-w+z-hc)\cdot\\
&\quad\cdot\,T_n^{+15}(u)\overline{R}^{23}_{mk}(v|\bar v|w+hc)^{-1}\overline{R}^{13}_{nk}(u|\bar v|z+hc)^{-1}T_k^{+36}(\bar v)|0\rangle\otimes|0\rangle\otimes|0\rangle\big)\\
&=\overline{R}^{12}_{nm}(u|v|-w+z)^{-1}T_m^{+24}(v|w)T_m^{24}(v|w+hc/2)^{-1}\overline{R}^{12}_{nm}(u|v|-w+z-hc)\cdot\\
&\quad\cdot\,T_n^{+14}(u|z)T_n^{14}(u|z+hc/2)^{-1}\overline{R}^{23}_{mk}(v|\bar v|w+hc)^{-1}\overline{R}^{13}_{nk}(u|\bar v|z+hc)^{-1}T_k^{+34}(\bar v)|0\rangle\\
&\overset{(e5)}{=}\overline{R}^{12}_{nm}(u|v|-w+z)^{-1}T_m^{+24}(v|w)T_n^{+14}(u|z)\overline{R}^{12}_{nm}(u|v|-w+z)\cdot\\
&\quad\cdot\,T_m^{24}(v|w+hc/2)^{-1}\overline{R}^{23}_{mk}(v|\bar v|w+hc)^{-1}T_n^{14}(u|z+hc/2)^{-1}\overline{R}^{13}_{nk}(u|\bar v|z+hc)^{-1}\cdot\\
&\quad\cdot\,T_k^{+34}(\bar v)|0\rangle\\
&\overset{(e6)}{=}\overline{R}^{12}_{nm}(u|v|-w+z)^{-1}T_m^{+24}(v|w)T_n^{+14}(u|z)\overline{R}^{12}_{nm}(u|v|-w+z)\cdot\\
&\quad\cdot\,T_m^{24}(v|w+hc/2)^{-1}\overline{R}^{23}_{mk}(v|\bar v|w+hc)^{-1}T_k^{+34}(\bar v)\overline{R}^{13}_{nk}(u|\bar v|z)^{-1}|0\rangle\\
&\overset{(e7)}{=}\overline{R}^{12}_{nm}(u|v|-w+z)^{-1}T_m^{+24}(v|w)T_n^{+14}(u|z)\overline{R}^{12}_{nm}(u|v|-w+z)T_k^{+34}(\bar v)\cdot\\
&\quad\cdot\,\overline{R}^{23}_{mk}(v|\bar v|w)^{-1}(\bar v)\overline{R}^{13}_{nk}(u|\bar v|z)^{-1}|0\rangle\\
&\overset{(e8)}{=}T_n^{+14}(u|z)T_m^{+24}(v|w)T_k^{+34}(\bar v)\overline{R}^{23}_{mk}(v|\bar v|w)^{-1}(\bar v)\overline{R}^{13}_{nk}(u|\bar v|z)^{-1}|0\rangle
\end{aligned}$$

where $(e5)$-$(e7)$ are due to relation (2.66) and the definition of $\mathcal{V}_c(\mathfrak{gl}_N)$ and $(e8)$ is due to the relation (2.64).
Hence one has

$$\begin{aligned}
&Y(w)\big(1\otimes Y(z)\big)(4\ 5)\big(T_n^{+14}(u)T_m^{+25}(v)T_k^{+36}(\bar v)|0\rangle\otimes|0\rangle\otimes|0\rangle\big)\\
&={}^{lr}\big(\big({}^{lr}\overline{R}^{12}_{nm}(u|v|-w+z-hc)\big)^{-1}\big)\,{}^{rl}\big(\big({}^{rl}\overline{R}^{13}_{nk}(u|\bar v|z+hc)\big)^{-1}\big)\cdot\\
&\quad\cdot\,{}^{rl}\big(\big({}^{rl}\overline{R}^{23}_{mk}(v|\bar v|w+hc)\big)^{-1}\big)\overline{R}^{12}_{nm}(u|v|-w+z)T_n^{+14}(u|z)T_m^{+24}(v|w)\cdot\\
&\quad\cdot\,T_k^{+34}(\bar v)\overline{R}^{23}_{mk}(v|\bar v|w)^{-1}\overline{R}^{13}_{nk}(u|\bar v|z)^{-1}|0\rangle.
\end{aligned}\tag{2.80}$$

Equations (2.79) and (2.80) differ only for the spaces where the powers of $z-w$ are expanded. Nevertheless, for any fixed power of $u_1,\dots,u_n,v_1,\dots,v_m$ (i.e. for any couple of elements in $\mathcal{V}_c(\mathfrak{gl}_N)$) and h, there is a finite number of negative powers of $z-w$ on both equations. Therefore for any couple of elements in $\mathcal{V}_c(\mathfrak{gl}_N)$ and any fixed power of h we can multiply by a suitable nonnegative power of $z-w$ to cancel the negative powers. It follows that the $\mathcal{S}$-locality holds.

Hexagon relation.
On one hand, one has the following equalities

$$\begin{aligned}
&\big(Y(z)\otimes 1\big)\mathcal{S}^{56}(w)\mathcal{S}^{46}(w+z)\big(\overline{R}^{13}_{nk}(u|\bar v|w+z)^{-1}T^{+36}_k(\bar v)\,\cdot\\
&\quad\cdot\,\overline{R}^{13}_{nk}(u|\bar v|w+z-hc)T^{+14}_n(u)\overline{R}^{13}_{nk}(u|\bar v|w+z)^{-1}\overline{R}^{12}_{nm}(u|v|z+hc)^{-1}\,\cdot\\
&\quad\cdot\,\overline{R}^{23}_{mk}(v|\bar v|w-hc)T^{+25}_m(v)|0\rangle\otimes|0\rangle\otimes|0\rangle\big)\\
&=\big(Y(z)\otimes 1\big)\mathcal{S}^{56}(w)\big(T^{+14}_n(u)\overline{R}^{13}_{nk}(u|\bar v|w+z+hc)^{-1}\,\cdot\\
&\quad\cdot\,T^{+36}_k(\bar v)\overline{R}^{12}_{nm}(u|v|z+hc)^{-1}\overline{R}^{23}_{mk}(v|\bar v|w-hc)T^{+25}_m(v)|0\rangle\otimes|0\rangle\otimes|0\rangle\big)\\
&=\big(Y(z)\otimes 1\big)\mathcal{S}^{56}(w)\big(T^{+14}_n(u)\overline{R}^{13}_{nk}(u|\bar v|w+z+hc)^{-1}\,\cdot\\
&\quad\cdot\,\overline{R}^{12}_{nm}(u|v|z+hc)^{-1}T^{+36}_k(\bar v)\overline{R}^{23}_{mk}(v|\bar v|w-hc)T^{+25}_m(v)|0\rangle\otimes|0\rangle\otimes|0\rangle\big)\\
&=\big(Y(z)\otimes 1\big)\big(T^{+14}_n(u)\overline{R}^{13}_{nk}(u|\bar v|w+z+hc)^{-1}\,\cdot\\
&\quad\cdot\,\overline{R}^{12}_{nm}(u|v|z+hc)^{-1}\overline{R}^{23}_{mk}(v|\bar v|w)T^{+25}_m(v)\overline{R}^{23}_{mk}(v|\bar v|w+hc)^{-1}\,\cdot\\
&\quad\cdot\,T^{+36}_k(\bar v)\overline{R}^{23}_{mk}(v|\bar v|w)|0\rangle\otimes|0\rangle\otimes|0\rangle\big)\\
&=T^{+14}_n(u|z)T^{14}_n(u|z+hc/2)^{-1}\overline{R}^{13}_{nk}(u|\bar v|w+z+hc)^{-1}\,\cdot\\
&\quad\cdot\,\overline{R}^{12}_{nm}(u|v|z+hc)^{-1}\overline{R}^{23}_{mk}(v|\bar v|w)T^{+24}_m(v)\overline{R}^{23}_{mk}(v|\bar v|w+hc)^{-1}\,\cdot\\
&\quad\cdot\,T^{+35}_k(\bar v)\overline{R}^{23}_{mk}(v|\bar v|w)|0\rangle\otimes|0\rangle\\
&\overset{(e1)}{=}T^{+14}_n(u|z)T^{14}_n(u|z+hc/2)^{-1}\overline{R}^{23}_{mk}(v|\bar v|w)\,\cdot\\
&\quad\cdot\,\overline{R}^{12}_{nm}(u|v|z+hc)^{-1}\overline{R}^{13}_{nk}(u|\bar v|w+z+hc)^{-1}T^{+24}_m(v)\,\cdot\\
&\quad\cdot\,\overline{R}^{23}_{mk}(v|\bar v|w+hc)^{-1}T^{+35}_k(\bar v)\overline{R}^{23}_{mk}(v|\bar v|w)|0\rangle\otimes|0\rangle\\
&=T^{+14}_n(u|z)\overline{R}^{23}_{mk}(v|\bar v|w)T^{14}_n(u|z+hc/2)^{-1}\,\cdot\\
&\quad\cdot\,\overline{R}^{12}_{nm}(u|v|z+hc)^{-1}T^{+24}_m(v)\overline{R}^{13}_{nk}(u|\bar v|w+z+hc)^{-1}\,\cdot\\
&\quad\cdot\,\overline{R}^{23}_{mk}(v|\bar v|w+hc)^{-1}T^{+35}_k(\bar v)\overline{R}^{23}_{mk}(v|\bar v|w)|0\rangle\otimes|0\rangle\\
&\overset{(e2)}{=}T^{+14}_n(u|z)\overline{R}^{23}_{mk}(v|\bar v|w)T^{+24}_m(v)\overline{R}^{12}_{nm}(u|v|z)^{-1}\,\cdot\\
&\quad\cdot\,\overline{R}^{13}_{nk}(u|\bar v|w+z+hc)^{-1}\overline{R}^{23}_{mk}(v|\bar v|w+hc)^{-1}T^{+35}_k(\bar v)\,\cdot\\
&\quad\cdot\,\overline{R}^{23}_{mk}(v|\bar v|w)|0\rangle\otimes|0\rangle\\
&\overset{(e3)}{=}\overline{R}^{23}_{mk}(v|\bar v|w)T^{+14}_n(u|z)T^{+24}_m(v)\overline{R}^{23}_{mk}(v|\bar v|w+hc)^{-1}\,\cdot\\
&\quad\cdot\,\overline{R}^{13}_{nk}(u|\bar v|w+z+hc)^{-1}\overline{R}^{12}_{nm}(u|v|z)^{-1}T^{+35}_k(\bar v)\,\cdot\\
&\quad\cdot\,\overline{R}^{23}_{mk}(v|\bar v|w)|0\rangle\otimes|0\rangle.
\end{aligned}$$

Where $(e1)$ and $(e3)$ are due to the quantumYang-Baxter equation and $(e2)$ is due to the relation (2.65) and the definition of $\mathcal{V}_c(\mathfrak{gl}_N)$.

On the other hand

$$
\begin{aligned}
&\mathcal{S}^{45}(w)\big(Y(z)\otimes 1\big)\big(\overline{R}^{13}_{nk}(u|\bar{v}|w+z)^{-1}T^{+36}_{k}(\bar{v})\cdot\\
&\quad\cdot\overline{R}^{13}_{nk}(u|\bar{v}|w+z-hc)T^{+14}_{n}(u)\overline{R}^{13}_{nk}(u|\bar{v}|w+z)^{-1}\overline{R}^{12}_{nm}(u|v|z+hc)^{-1}\cdot\\
&\quad\cdot\overline{R}^{23}_{mk}(v|\bar{v}|w-hc)T^{+25}_{m}(v)|0\rangle\otimes|0\rangle\otimes|0\rangle\big)\\
&=\mathcal{S}^{45}(w)\big(\overline{R}^{13}_{nk}(u|\bar{v}|w+z)^{-1}T^{+35}_{k}(\bar{v})\overline{R}^{13}_{nk}(u|\bar{v}|w+z-hc)\cdot\\
&\quad\cdot T^{+14}_{n}(u|z)T^{14}_{n}(u|z+hc/2)^{-1}\overline{R}^{13}_{nk}(u|\bar{v}|w+z)^{-1}\overline{R}^{12}_{nm}(u|v|z+hc)^{-1}\cdot\\
&\quad\cdot\overline{R}^{23}_{mk}(v|\bar{v}|w-hc)T^{+24}_{m}(v)|0\rangle\otimes|0\rangle\big)\\
&\overset{(e4)}{=}\mathcal{S}^{45}(w)\big(\overline{R}^{13}_{nk}(u|\bar{v}|w+z)^{-1}T^{+35}_{k}(\bar{v})\overline{R}^{13}_{nk}(u|\bar{v}|w+z-hc)\cdot\\
&\quad\cdot T^{+14}_{n}(u|z)T^{14}_{n}(u|z+hc/2)^{-1}\overline{R}^{23}_{mk}(v|\bar{v}|w-hc)\overline{R}^{12}_{nm}(u|v|z+hc)^{-1}\cdot\\
&\quad\cdot\overline{R}^{13}_{nk}(u|\bar{v}|w+z)^{-1}T^{+24}_{m}(v)|0\rangle\otimes|0\rangle\big)\\
&=\overline{R}^{23}_{mk}(v|\bar{v}|w)\mathcal{S}^{45}(w)\big(\overline{R}^{23}_{mk}(v|\bar{v}|w)^{-1}\overline{R}^{13}_{nk}(u|\bar{v}|w+z)^{-1}T^{+35}_{k}(\bar{v})\cdot\\
&\quad\cdot\overline{R}^{13}_{nk}(u|\bar{v}|w+z-hc)T^{+14}_{n}(u|z)\overline{R}^{23}_{mk}(v|\bar{v}|w-hc)T^{14}_{n}(u|z+hc/2)^{-1}\cdot\\
&\quad\cdot\overline{R}^{12}_{nm}(u|v|z+hc)^{-1}T^{+24}_{m}(v)\overline{R}^{13}_{nk}(u|\bar{v}|w+z)^{-1}|0\rangle\otimes|0\rangle\big)\\
&\overset{(e5)}{=}\overline{R}^{23}_{mk}(v|\bar{v}|w)\mathcal{S}^{45}(w)\big(\overline{R}^{23}_{mk}(v|\bar{v}|w)^{-1}\overline{R}^{13}_{nk}(u|\bar{v}|w+z)^{-1}T^{+35}_{k}(\bar{v})\cdot\\
&\quad\cdot\overline{R}^{13}_{nk}(u|\bar{v}|w+z-hc)T^{+14}_{n}(u|z)\overline{R}^{23}_{mk}(v|\bar{v}|w-hc)T^{+24}_{m}(v)\cdot\\
&\quad\cdot\overline{R}^{12}_{nm}(u|v|z)^{-1}\overline{R}^{13}_{nk}(u|\bar{v}|w+z)^{-1}|0\rangle\otimes|0\rangle\big)\\
&=\overline{R}^{23}_{mk}(v|\bar{v}|w)\mathcal{S}^{45}(w)\big(\overline{R}^{23}_{mk}(v|\bar{v}|w)^{-1}\overline{R}^{13}_{nk}(u|\bar{v}|w+z)^{-1}T^{+35}_{k}(\bar{v})\cdot\\
&\quad\cdot\overline{R}^{13}_{nk}(u|\bar{v}|w+z-hc)\overline{R}^{23}_{mk}(v|\bar{v}|w-hc)T^{+14}_{n}(u|z)T^{+24}_{m}(v)\cdot\\
&\quad\cdot\overline{R}^{12}_{nm}(u|v|z)^{-1}\overline{R}^{13}_{nk}(u|\bar{v}|w+z)^{-1}|0\rangle\otimes|0\rangle\big)\\
&=\overline{R}^{23}_{mk}(v|\bar{v}|w)T^{+14}_{n}(u|z)T^{+24}_{m}(v)\overline{R}^{23}_{mk}(v|\bar{v}|w+hc)^{-1}\cdot\\
&\quad\cdot\overline{R}^{13}_{nk}(u|\bar{v}|w+z+hc)^{-1}T^{+35}_{k}(\bar{v})\overline{R}^{13}_{nk}(u|\bar{v}|w+z)\overline{R}^{23}_{mk}(v|\bar{v}|w)\cdot\\
&\quad\cdot\overline{R}^{12}_{nm}(u|v|z)^{-1}\overline{R}^{13}_{nk}(u|\bar{v}|w+z)^{-1}|0\rangle\otimes|0\rangle\\
&\overset{(e6)}{=}\overline{R}^{23}_{mk}(v|\bar{v}|w)T^{+14}_{n}(u|z)T^{+24}_{m}(v)\overline{R}^{23}_{mk}(v|\bar{v}|w+hc)^{-1}\cdot\\
&\quad\cdot\overline{R}^{13}_{nk}(u|\bar{v}|w+z+hc)^{-1}T^{+35}_{k}(\bar{v})\overline{R}^{12}_{nm}(u|v|z)^{-1}\cdot\\
&\quad\cdot\overline{R}^{23}_{mk}(v|\bar{v}|w)|0\rangle\otimes|0\rangle\\
&=\overline{R}^{23}_{mk}(v|\bar{v}|w)T^{+14}_{n}(u|z)T^{+24}_{m}(v)\overline{R}^{23}_{mk}(v|\bar{v}|w+hc)^{-1}\cdot\\
&\quad\cdot\overline{R}^{13}_{nk}(u|\bar{v}|w+z+hc)^{-1}\overline{R}^{12}_{nm}(u|v|z)^{-1}T^{+35}_{k}(\bar{v})\cdot\\
&\quad\cdot\overline{R}^{23}_{mk}(v|\bar{v}|w)|0\rangle\otimes|0\rangle.
\end{aligned}
$$

Where $(e4)$ and $(e6)$ are due to the quantumYang-Baxter equation and $(e5)$ is due to the relation 2.65 and the definition of $\mathcal{V}_c(\mathfrak{gl}_N)$.
Therefore one has

$$
\begin{aligned}
&(Y(z)\otimes 1)\mathcal{S}^{56}(w)\mathcal{S}^{46}(w+z)\big(\overline{R}^{13}_{nk}(u|\bar{v}|w+z)^{-1}T^{+36}_{k}(\bar{v})\cdot\\
&\quad\cdot\overline{R}^{13}_{nk}(u|\bar{v}|w+z-hc)T^{+14}_{n}(u)\overline{R}^{13}_{nk}(u|\bar{v}|w+z)^{-1}\overline{R}^{12}_{nm}(u|v|z+hc)^{-1}\cdot
\end{aligned}
$$

$$
\begin{aligned}
&\cdot\ \overline{R}^{23}_{mk}(v|\bar{v}|w-hc)T^{+25}_{m}(v)|0\rangle\otimes|0\rangle\otimes|0\rangle\big)\\
&=\mathcal{S}^{45}(w)(Y(z)\otimes 1)\big(\overline{R}^{13}_{nk}(u|\bar{v}|w+z)^{-1}T^{+36}_{k}(\bar{v})\cdot\\
&\cdot\ \overline{R}^{13}_{nk}(u|\bar{v}|w+z-hc)T^{+14}_{n}(u)\overline{R}^{13}_{nk}(u|\bar{v}|w+z)^{-1}\overline{R}^{12}_{nm}(u|v|z+hc)^{-1}\cdot\\
&\cdot\ \overline{R}^{23}_{mk}(v|\bar{v}|w-hc)T^{+25}_{m}(v)|0\rangle\otimes|0\rangle\otimes|0\rangle\big)
\end{aligned}
$$

from which the Hexagon Relation (2.26) follows. □

Remark 2.3.9. If $N=1$, the quantum vertex algebra of Theorem (2.3.8) is a commutative vertex algebra over $\mathbb{C}[[h]]$.

Proof. If $N=1$, $g(u/h)=\sum_{r\geq 0}\left(\frac{h}{u}\right)^r$ indeed $g_1=1/N=1$ and, by induction, one shows that, if $g_i=1$ for any $1\leq i<r$, then $g_r=1$ indeed

$$
\begin{aligned}
g_r &= \frac{g_{r-1}+\sum_{k=2}^{r}(-1)^k\binom{r}{k}g_{r+1-k}N^k}{rN}\\
&= \frac{1+\sum_{k=2}^{r}(-1)^k\binom{r}{k}}{r}\\
&= \frac{r+\sum_{k=0}^{r}(-1)^k\binom{r}{k}}{r}\\
&= 1.
\end{aligned}
$$

Therefore $g(u/h)=\sum_{r\geq 0}\left(\frac{h}{u}\right)^r=\left(1-\frac{h}{u}\right)^{-1}$. On the other hand, as $N=1$, $R(u)=1-\frac{h}{u}$, from which $\overline{R}(u)=g(u/h)R(u)=1$. By the definition of the map $\mathcal{S}$, it follows that $\mathcal{S}=1$. Since the $\mathcal{S}$-locality (2.10) holds and $\mathcal{S}=1$, V is a vertex algebra over $\mathbb{C}[[h]]$. Moreover, by relation (2.65), it follows that $T^{13}_n(u|z)T^{+23}_m(v|w)=T^{+23}_m(v|w)T^{13}_n(u|z)$, from which

$$
\begin{aligned}
&Y(T^{+13}_n(u)|0\rangle,z)T^{+23}_m(v)|0\rangle\\
&\qquad=T^{+13}_n(u|z)T^{13}_n(u|z+hc/2)^{-1}T^{+23}_m(v)|0\rangle\\
&\qquad=T^{+13}_n(u|z)T^{+23}_m(v)T^{13}_n(u|z+hc/2)^{-1}|0\rangle\\
&\qquad=T^{+13}_n(u|z)T^{+23}_m(v)|0\rangle\in\mathbb{C}^{\otimes n}\otimes\mathbb{C}^{\otimes m}\otimes\mathcal{V}_c(\mathbb{C})[[u,v,z]]
\end{aligned}
$$

for any $n,m\in\mathbb{Z}_+$, thus, by Theorem 1.3.25, $(\mathcal{V}_c(\mathbb{C}),|0\rangle,Y,T)$ of Theorem 2.3.8 is a commutative vertex algebra over $\mathbb{C}[[h]]$. □

2.3.5 An $\mathcal{S}$-commutative quantum vertex algebra

Theorem 2.3.10 ([JKMY, Prop.4.2])**.** *There exists a unique well-defined structure of quantum vertex algebra on the* $\mathbb{C}[[h]]$*-module* $\mathcal{V}_c(\mathfrak{gl}_N)$ *with the following data:*

1. *the vacuum vector is*
$$|0\rangle=1\in\mathcal{V}_c(\mathfrak{gl}_N);\tag{2.81}$$

2. *the vertex operators are defined by*
$$Y(T^+_n(u)|0\rangle,z)=T^+_n(u|z);\tag{2.82}$$

3. the translation operator D is defined by

$$(1^{\otimes n}\otimes e^{zD})T_n^+(u)|0\rangle = T_n^+(u|z)|0\rangle; \tag{2.83}$$

4. the map $\mathcal{S}$ is defined by

$$\begin{aligned}&\mathcal{S}^{34}(z)(T_n^{+13}(u)T_m^{+24}(v)|0\rangle\otimes|0\rangle)\\&\quad=\overline{R}^{12}_{nm}(u|v|z)T_n^{+13}(u)T_m^{+24}(v)\overline{R}^{12}_{nm}(u|v|z)^{-1}|0\rangle\otimes|0\rangle.\end{aligned}\tag{2.84}$$

Proof. Since the coefficients of $T_n^+(u)|0\rangle$ generate a dense subset of the vacuum module $\mathcal{V}_c(\mathfrak{gl}_N)$, defining $Y,\mathcal{S},D$ on each element of the dual Yangian $Y^+(\mathfrak{gl}_N)$, one has a definition on the whole space $\mathcal{V}_c(\mathfrak{gl}_N)$. For the same reason it is sufficient to check the axioms on the elements of the dual Yangian $Y^+(\mathfrak{gl}_N)$ to have them satisfied for any element of $\mathcal{V}_c(\mathfrak{gl}_N)$.

Well definedness of Y and $\mathcal{S}$.
Since $T_n^+(u)\in(End\,\mathbb{C}^N)^{\otimes n}\otimes Y^+(\mathfrak{gl}_N)[[u]]$, one has that $Y(T_n^{+13}(u)|0\rangle,z)T_m^{+23}(v)|0\rangle = T_n^{+13}(u|z)T_m^{+23}(v)|0\rangle\in(End\,\mathbb{C}^N)^{\otimes n}\otimes(End\,\mathbb{C}^N)^{\otimes m}\otimes\mathcal{V}_c(\mathfrak{gl}_N)[[u_1,\dots,u_n,v_1,\dots,v_m,z]]$. Moreover one has

$$\begin{aligned}&Y\Big((R^{12}_{nm}(u|v|0)T_n^{+14}(u)T_m^{+24}(v)|0\rangle\\&\qquad-T_m^{+24}(v)T_n^{+14}(u)R^{12}_{nm}(u|v|0)|0\rangle),z\Big)T_k^{+34}(\bar v)|0\rangle\\&=\Big(R^{12}_{nm}(u|v|0)T_n^{+14}(u|z)T_m^{+24}(v|z)\\&\qquad-T_m^{+24}(v|z)T_n^{+14}(u|z)R^{12}_{nm}(u|v|0)\Big)T_k^{+34}(\bar v)|0\rangle.\end{aligned}$$

Similarly one proves that the map $\mathcal{S}$ is well defined and that $\mathcal{S}(z)(a\otimes b)\in\mathcal{V}_c(\mathfrak{gl}_N)\otimes\mathcal{V}_c(\mathfrak{gl}_N)\otimes\mathbb{C}((z))$ mod h^M for any $M\in\mathbb{Z}_+$ and $a,b\in\mathcal{V}_c(\mathfrak{gl}_N)$. To check the last one, it is sufficient to show that $\mathcal{S}^{34}(z)\big(T^{+13}(u)T^{+24}(v)|0\rangle\otimes|0\rangle\big)\in End\,\mathbb{C}^N\otimes End\,\mathbb{C}^N\otimes\mathcal{V}_c(\mathfrak{gl}_N)\otimes\mathcal{V}_c(\mathfrak{gl}_N)\otimes\mathbb{C}((z))[[u,v]]$ mod h^M for any $M\in\mathbb{Z}_+$. By the definitions of $\mathcal{S}$ and $\overline{R}$, and by equation (2.69), one has

$$\begin{aligned}&\mathcal{S}^{34}(z)(T^{+13}(u)T^{+24}(v)|0\rangle\otimes|0\rangle)\\&\quad=\overline{R}^{12}(u-v+z)T^{+13}(u)T^{+24}(v)\overline{R}^{12}(u-v+z)^{-1}|0\rangle\otimes|0\rangle\\&\quad=R^{12}(u-v+z)T^{+13}(u)T^{+24}(v)R^{12}(u-v+z)^{-1}|0\rangle\otimes|0\rangle\\&\quad=\left(1-\frac{h}{u-v+z}\sigma\right)T^{+13}(u)T^{+24}(v)|0\rangle\otimes|0\rangle\cdot\\&\qquad\cdot\left(1+\frac{h}{u-v+z}\sigma\right)\sum_{k\ge0}(u-v+z)^{-2k}h^{2k}.\end{aligned}$$

On the other hand

$$\sum_{k\ge0}(u-v+z)^{-2k}h^{2k}$$

$$= \sum_{k\geq 0}\sum_{l\geq 0}\sum_{i\geq 0} \binom{-2k}{l}\binom{l}{i}(-1)^i u^{l-i} v^i z^{-2k-l} h^{2k}$$
$$= \sum_{k\geq 0}\sum_{j\geq 0}\sum_{i\geq 0} \binom{-2k}{i+j}\binom{i+j}{i}(-1)^i z^{-2k-i-j} u^j v^i h^{2k} \in \mathbb{C}((z))[[h,u,v]].$$

And the same holds for $\frac{h}{u-v+z}$. In particular, one has that $\mathcal{S}^{34}(z)(T^{+13}(u)T^{+24}(v)|0\rangle\otimes|0\rangle) \in End\,\mathbb{C}^N\otimes End\,\mathbb{C}^N\otimes\mathcal{V}_c(\mathfrak{gl}_N)\otimes\mathcal{V}_c(\mathfrak{gl}_N)\otimes\mathbb{C}((z))[[u,v]] \bmod h^M$ for any $M\in\mathbb{Z}_+$.

Axioms on the vacuum vector.

$$Y(|0\rangle,z)T_n^+(u)|0\rangle = T_n^+(u)|0\rangle = Id_{\mathcal{V}_c(\mathfrak{gl}_N)}(T_n^+(u)|0\rangle).$$

$$Y(T_n^+(u)|0\rangle,z)|0\rangle = T_n^+(u|z)|0\rangle \in (End\,\mathbb{C}^N)^{\otimes n}\otimes\mathcal{V}_c(\mathfrak{gl}_N)[[u,z]]$$

and

$$Y(T_n^+(u)|0\rangle,z)|0\rangle_{|z=0} = T_n^+(u)|0\rangle.$$

Translation covariance.
On the first hand, one has the following equalities

$$\begin{aligned}
&[D, Y(T_n^{+13}(u)|0\rangle,z)]T_m^{+23}(v)|0\rangle\\
&= D\Big(Y(T_n^{+13}(u)|0\rangle,z)T_m^{+23}(v)|0\rangle\Big) - Y(T_n^{+13}(u)|0\rangle,z)D(T_m^{+23}(v)|0\rangle)\\
&= D(T_n^{+13}(u|z)T_m^{+23}(v)|0\rangle) - T_n^{+13}(u|z)D(T_m^{+23}(v)|0\rangle)\\
&= \Big(\sum_{i=1}^n \partial_{u_i} + \sum_{j=1}^m \partial_{v_j}\Big)\Big(T_n^{+13}(u|z)T_m^{+23}(v)|0\rangle\Big) - T_n^{+13}(u|z)\Big(\sum_{j=1}^m \partial_{v_j}\Big)T_m^{+23}(v)|0\rangle\\
&= \Big(\sum_{i=1}^n \partial_{u_i} + \sum_{j=1}^m \partial_{v_j}\Big)\Big(T_n^{+13}(u|z)T_m^{+23}(v)|0\rangle\Big) - \Big(\sum_{j=1}^m \partial_{v_j}\Big)T_n^{+13}(u|z)T_m^{+23}(v)|0\rangle\\
&= \Big(\sum_{i=1}^n \partial_{u_i}\Big)\Big(T_n^{+13}(u|z)T_m^{+23}(v)|0\rangle\Big).
\end{aligned}$$

On the other hand one has the following equalities

$$\begin{aligned}
&\Big(\partial_z Y(T_n^{+13}(u)|0\rangle,z)\Big)T_m^{+23}(v)|0\rangle\\
&= \Big(\Big(\sum_{i=1}^n \partial_{u_i}\Big)Y(T_n^{+13}(u)|0\rangle,z)\Big)T_m^{+23}(v)|0\rangle\\
&= \Big(\Big(\sum_{i=1}^n \partial_{u_i}\Big)T_n^{+13}(u|z)\Big)T_m^{+23}(v)|0\rangle\\
&= \Big(\sum_{i=1}^n \partial_{u_i}\Big)T_n^{+13}(u|z)T_m^{+23}(v)|0\rangle.
\end{aligned}$$

From which the translation covariance holds.

Invertibility of $\mathcal{S}$.
It follows from the fact that both $\overline{R}(u|v|z)$ and $T^+(u)$ are $1+O(h)$.

$\mathcal{S}$ **shift invariant**.

$$
\begin{aligned}
&\Big[D\otimes 1,\mathcal{S}^{34}(z)\Big]\Big(T_n^{+14}(u)T_m^{+25}(v)|0\rangle\otimes|0\rangle\Big)\\
&=(D\otimes 1)\mathcal{S}^{34}(z)\Big(T_n^{+13}(u)T_m^{+24}(v)|0\rangle\otimes|0\rangle\Big)\\
&\quad-\mathcal{S}^{34}(z)(D\otimes 1)\Big(T_n^{+13}(u)T_m^{+24}(v)|0\rangle\otimes|0\rangle\Big)\\
&=\overline{R}^{12}_{nm}(u|v|z)\Big(\Big(\sum_{i=1}^{n}\partial_{u_i}\Big)T_n^{+13}(u)\Big)T_m^{+24}(v)\overline{R}^{12}_{nm}(u|v|z)^{-1}|0\rangle\otimes|0\rangle\\
&\quad-\mathcal{S}^{34}(z)\Big(\Big(\sum_{i=1}^{n}\partial_{u_i}\Big)T_n^{+13}(u)T_m^{+24}(v)|0\rangle\otimes|0\rangle\Big)\\
&=\Big(\sum_{i=1}^{n}\partial_{u_i}\Big)\Big(\overline{R}^{12}_{nm}(u|v|z)T_n^{+13}(u)T_m^{+24}(v)\overline{R}^{12}_{nm}(u|v|z)^{-1}|0\rangle\otimes|0\rangle\Big)\\
&\quad-\Big(\Big(\sum_{i=1}^{n}\partial_{u_i}\Big)\overline{R}^{12}_{nm}(u|v|z)\Big)T_n^{+13}(u)T_m^{+24}(v)\overline{R}^{12}_{nm}(u|v|z)^{-1}|0\rangle\otimes|0\rangle\\
&\quad-\overline{R}^{12}_{nm}(u|v|z)T_n^{+13}(u)T_m^{+24}(v)\Big(\Big(\sum_{i=1}^{n}\partial_{u_i}\Big)\overline{R}^{12}_{nm}(u|v|z)^{-1}|0\rangle\otimes|0\rangle\Big)\\
&\quad-\Big(\sum_{i=1}^{n}\partial_{u_i}\Big)\Big(\overline{R}^{12}_{nm}(u|v|z)T_n^{+13}(u)T_m^{+24}(v)\overline{R}^{12}_{nm}(u|v|z)^{-1}|0\rangle\otimes|0\rangle\Big)\\
&=-\Big(\partial_z\overline{R}^{12}_{nm}(u|v|z)\Big)T_n^{+13}(u)T_m^{+24}(v)\overline{R}^{12}_{nm}(u|v|z)^{-1}|0\rangle\otimes|0\rangle\\
&\quad-\overline{R}^{12}_{nm}(u|v|z)T_n^{+13}(u)T_m^{+24}(v)\Big(\partial_z\overline{R}^{12}_{nm}(u|v|z)^{-1}|0\rangle\otimes|0\rangle\Big)\\
&=-\partial_z(\overline{R}^{12}_{nm}(u|v|z)T_n^{+13}(u)T_m^{+24}(v)\overline{R}^{12}_{nm}(u|v|z)^{-1}|0\rangle\otimes|0\rangle)\\
&=-\partial_z\Big(\mathcal{S}^{34}(z)(T_n^{+13}(u)T_m^{+24}(v)|0\rangle\otimes|0\rangle)\Big)\\
&=-(\partial_z\mathcal{S}^{34}(z))(T_n^{+13}(u)T_m^{+24}(v)|0\rangle\otimes|0\rangle).
\end{aligned}
$$

Therefore one has

$$[D\otimes 1,\mathcal{S}^{34}(z)]=-\partial_z\mathcal{S}^{34}(z).$$

$\mathcal{S}$ **solution of the quantum Yang-Baxter equation**

On the first hand, the following equalities follow

$$
\begin{aligned}
&\mathcal{S}^{45}(z)\mathcal{S}^{46}(z+w)\mathcal{S}^{56}(w)(T_n^{+14}(u)T_m^{+25}(v)T_k^{+36}(\bar v)|0\rangle\otimes|0\rangle\otimes|0\rangle)\\
&=\mathcal{S}^{45}(z)\mathcal{S}^{46}(z+w)\Big(T_n^{+14}(u)|0\rangle\mathcal{S}^{56}(w)(T_m^{+25}(v)T_k^{+36}(\bar v)|0\rangle\otimes|0\rangle)\Big)\\
&=\mathcal{S}^{45}(z)\mathcal{S}^{46}(z+w)\Big(T_n^{+14}(u)\overline{R}^{23}_{mk}(v|\bar v|w)T_m^{+25}(v)T_k^{+36}(\bar v)\cdot\\
&\qquad\qquad\cdot\ \overline{R}^{23}_{mk}(v|\bar v|w)^{-1}|0\rangle\otimes|0\rangle\otimes|0\rangle\Big)\\
&=\overline{R}^{23}_{mk}(v|\bar v|w)\mathcal{S}^{45}(z)\mathcal{S}^{46}(z+w)\Big(T_n^{+14}(u)T_m^{+25}(v)T_k^{+36}(\bar v)|0\rangle\otimes|0\rangle\otimes|0\rangle\Big)\cdot\\
&\quad\cdot\ \overline{R}^{23}_{mk}(v|\bar v|w)^{-1}\\
&=\overline{R}^{23}_{mk}(v|\bar v|w)\mathcal{S}^{45}(z)\Big(\mathcal{S}^{46}(z+w)\Big(T_n^{+14}(u)|0\rangle T_k^{+36}(\bar v)|0\rangle\Big)T_m^{+25}(v)|0\rangle\Big)\cdot
\end{aligned}
$$

$$
\begin{aligned}
&\cdot\ \overline{R}^{23}_{mk}(v|\bar{v}|w)^{-1}\\
&= \overline{R}^{23}_{mk}(v|\bar{v}|w)\mathcal{S}^{45}(z)\Big(\overline{R}^{13}_{nk}(u|\bar{v}|z+w)T^{+\,14}_{n}(u)T^{+\,36}_{k}(\bar{v})\overline{R}^{13}_{nk}(u|\bar{v}|z+w)^{-1}\cdot\\
&\quad\cdot\ T^{+\,25}_{m}(v)|0\rangle\otimes|0\rangle\otimes|0\rangle\Big)\overline{R}^{23}_{mk}(v|\bar{v}|w)^{-1}\\
&= \overline{R}^{23}_{mk}(v|\bar{v}|w)\overline{R}^{13}_{nk}(u|\bar{v}|z+w)\mathcal{S}^{45}(z)\Big(T^{+\,14}_{n}(u)T^{+\,36}_{k}(\bar{v})T^{+\,25}_{m}(v)|0\rangle\otimes|0\rangle\otimes|0\rangle\Big)\cdot\\
&\quad\cdot\ \overline{R}^{13}_{nk}(u|\bar{v}|z+w)^{-1}\overline{R}^{23}_{mk}(v|\bar{v}|w)^{-1}\\
&= \overline{R}^{23}_{mk}(v|\bar{v}|w)\overline{R}^{13}_{nk}(u|\bar{v}|z+w)\mathcal{S}^{45}(z)\Big(T^{+\,14}_{n}(u)T^{+\,25}_{m}(v)|0\rangle\otimes|0\rangle\Big)T^{+\,36}_{k}(\bar{v})|0\rangle\cdot\\
&\quad\cdot\ \overline{R}^{13}_{nk}(u|\bar{v}|z+w)^{-1}\overline{R}^{23}_{mk}(v|\bar{v}|w)^{-1}\\
&= \overline{R}^{23}_{mk}(v|\bar{v}|w)\overline{R}^{13}_{nk}(u|\bar{v}|z+w)\overline{R}^{12}_{nm}(u|v|z)T^{+\,14}_{n}(u)T^{+\,25}_{m}(v)T^{+\,36}_{k}(\bar{v})\cdot\\
&\quad\cdot\ \overline{R}^{12}_{nm}(u|v|z)^{-1}\overline{R}^{13}_{nk}(u|\bar{v}|z+w)^{-1}\overline{R}^{23}_{mk}(v|\bar{v}|w)^{-1}|0\rangle\otimes|0\rangle\otimes|0\rangle.
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
&\mathcal{S}^{56}(w)\mathcal{S}^{46}(z+w)\mathcal{S}^{45}(z)(T^{+\,14}_{n}(u)T^{+\,25}_{m}(v)T^{+\,36}_{k}(\bar{v})|0\rangle\otimes|0\rangle\otimes|0\rangle)\\
&= \mathcal{S}^{56}(w)\mathcal{S}^{46}(z+w)(\overline{R}^{12}_{nm}(u|v|z)T^{+\,14}_{n}(u)T^{+\,25}_{m}(v)\overline{R}^{12}_{nm}(u|v|z)^{-1}T^{+\,36}_{k}(\bar{v})|0\rangle\otimes|0\rangle\otimes|0\rangle)\\
&= \overline{R}^{12}_{nm}(u|v|z)\mathcal{S}^{56}(w)\mathcal{S}^{46}(z+w)(T^{+\,14}_{n}(u)T^{+\,36}_{k}(\bar{v})T^{+\,25}_{m}(v)|0\rangle\otimes|0\rangle\otimes|0\rangle)\overline{R}^{12}_{nm}(u|v|z)^{-1}\\
&= \overline{R}^{12}_{nm}(u|v|z)\mathcal{S}^{56}(w)\Big(\overline{R}^{13}_{nk}(u|\bar{v}|z+w)T^{+\,14}_{n}(u)T^{+\,36}_{k}\overline{R}^{13}_{nk}(u|\bar{v}|z+w)^{-1}\cdot\\
&\quad\cdot\ T^{+\,25}_{m}(v)|0\rangle\otimes|0\rangle\otimes|0\rangle\Big)\overline{R}^{12}_{nm}(u|v|z)^{-1}\\
&= \overline{R}^{12}_{nm}(u|v|z)\overline{R}^{13}_{nk}(u|\bar{v}|z+w)T^{+\,14}_{n}(u)|0\rangle\mathcal{S}^{56}(w)\Big(T^{+\,36}_{k}T^{+\,25}_{m}(v)|0\rangle\otimes|0\rangle\Big)\cdot\\
&\quad\cdot\ \overline{R}^{13}_{nk}(u|\bar{v}|z+w)^{-1}\overline{R}^{12}_{nm}(u|v|z)^{-1}\\
&= \overline{R}^{12}_{nm}(u|v|z)\overline{R}^{13}_{nk}(u|\bar{v}|z+w)T^{+\,14}_{n}(u)|0\rangle\mathcal{S}^{56}(w)\Big(T^{+\,25}_{m}(v)T^{+\,36}_{k}|0\rangle\otimes|0\rangle\Big)\cdot\\
&\quad\cdot\ \overline{R}^{13}_{nk}(u|\bar{v}|z+w)^{-1}\overline{R}^{12}_{nm}(u|v|z)^{-1}\\
&= \overline{R}^{12}_{nm}(u|v|z)\overline{R}^{13}_{nk}(u|\bar{v}|z+w)\overline{R}^{23}_{mk}(v|\bar{v}|w)T^{+\,14}_{n}(u)T^{+\,25}_{m}(v)T^{+\,36}_{k}\cdot\\
&\quad\cdot\ \overline{R}^{23}_{mk}(v|\bar{v}|w)^{-1}\overline{R}^{13}_{nk}(u|\bar{v}|z+w)^{-1}\overline{R}^{12}_{nm}(u|v|z)^{-1}|0\rangle\otimes|0\rangle\otimes|0\rangle\\
&= \overline{R}^{23}_{mk}(v|\bar{v}|w)\overline{R}^{13}_{nk}(u|\bar{v}|z+w)\overline{R}^{12}_{nm}(u|v|z)T^{+\,14}_{n}(u)T^{+\,25}_{m}(v)T^{+\,36}_{k}(\bar{v})\cdot\\
&\quad\cdot\ \overline{R}^{12}_{nm}(u|v|z)^{-1}\overline{R}^{13}_{nk}(u|\bar{v}|z+w)^{-1}\overline{R}^{23}_{mk}(v|\bar{v}|w)^{-1}|0\rangle\otimes|0\rangle\otimes|0\rangle
\end{aligned}
$$

where the last equality is due to the quantum Yang-Baxter equation for $\overline{R}$. Therefore the quantum Yang-Baxter equation for $\mathcal{S}$ follows.

Unitarity of $\mathcal{S}$.

$$
\begin{aligned}
&\mathcal{S}^{43}(-z)\mathcal{S}^{34}(z)\Big(T^{+\,13}_{n}(u)T^{+\,24}_{m}(v)|0\rangle\otimes|0\rangle\Big)\\
&= \mathcal{S}^{43}(-z)\Big(\overline{R}^{12}_{nm}(u|v|z)T^{+\,13}_{n}(u)T^{+\,24}_{m}(v)\overline{R}^{12}_{nm}(u|v|z)^{-1}|0\rangle\otimes|0\rangle\Big)\\
&= (3\ 4)\mathcal{S}^{34}(-z)(3\ 4)(\overline{R}^{12}_{nm}(u|v|z)T^{+\,13}_{n}(u)T^{+\,24}_{m}(v)\overline{R}^{12}_{nm}(u|v|z)^{-1}|0\rangle\otimes|0\rangle)\\
&= (3\ 4)\mathcal{S}^{34}(-z)(\overline{R}^{12}_{nm}(u|v|z)T^{+\,14}_{n}(u)T^{+\,23}_{m}(v)\overline{R}^{12}_{nm}(u|v|z)^{-1}|0\rangle\otimes|0\rangle)
\end{aligned}
$$

$$
\begin{aligned}
&= \overline{R}^{12}_{nm}(u|v|z)(3\ 4)\mathcal{S}^{34}(-z)\big(T^{+\,23}_m(v)T^{+\,14}_n(u)|0\rangle\otimes|0\rangle\big)\overline{R}^{12}_{nm}(u|v|z)^{-1}\\
&= \overline{R}^{12}_{nm}(u|v|z)(3\ 4)\overline{R}^{21}_{mn}(v|u|-z)T^{+\,23}_m(v)T^{+\,14}_n(u)\overline{R}^{21}_{nm}(v|u|-z)^{-1}\cdot\\
&\quad\cdot \overline{R}^{12}_{nm}(u|v|z)^{-1}|0\rangle\otimes|0\rangle\\
&= \overline{R}^{12}_{nm}(u|v|z)\overline{R}^{21}_{mn}(v|u|-z)T^{+\,24}_m(v)T^{+\,13}_n(u)\overline{R}^{21}_{nm}(v|u|-z)^{-1}\cdot\\
&\quad\cdot \overline{R}^{12}_{nm}(u|v|z)^{-1}|0\rangle\otimes|0\rangle\\
&= T^{+\,13}_n(u)T^{+\,24}_m(v)|0\rangle\otimes|0\rangle
\end{aligned}
$$

because of the unitarity of $\overline{R}(u)$.

$\mathcal{S}$**-locality**.

$$
\begin{aligned}
&Y(z)\big(1\otimes Y(w)\big)\mathcal{S}^{45}(z-w)\Big(T^{+\,14}_n(u)T^{+\,25}_m(v)T^{+\,36}_k(\bar v)|0\rangle\otimes|0\rangle\otimes|0\rangle\Big)\\
&= Y(z)\big(1\otimes Y(w)\big)\Big(\overline{R}^{12}_{nm}(u|v|z-w)T^{+\,14}_n(u)T^{+\,25}_m(v)\overline{R}^{12}_{nm}(u|v|z-w)^{-1}\cdot\\
&\qquad\qquad\cdot T^{+\,36}_k(\bar v)|0\rangle\otimes|0\rangle\otimes|0\rangle\Big)\\
&= \overline{R}^{12}_{nm}(u|v|z-w)T^{+\,14}_n(u|z)T^{+\,24}_m(v|w)\overline{R}^{12}_{nm}(u|v|z-w)^{-1}\cdot\\
&\quad\cdot T^{+\,34}_k(\bar v)|0\rangle\otimes|0\rangle\otimes|0\rangle\\
&\overset{(e1)}{=} T^{+\,24}_m(v|w)T^{+\,14}_n(u|z)T^{+\,34}_k(\bar v)|0\rangle.
\end{aligned}
$$

Similarly, one has

$$
\begin{aligned}
&Y(w)\big(1\otimes Y(z)\big)(4\ 5)\Big(T^{+\,14}_n(u)T^{+\,25}_m(v)T^{+\,36}_k(\bar v)|0\rangle\otimes|0\rangle\otimes|0\rangle\Big)\\
&= T^{+\,24}_m(v|w)T^{+\,14}_n(u|z)T^{+\,34}_k(\bar v)|0\rangle.
\end{aligned}
$$

Where $(e1)$ is due to relation (2.64).

Hexagon relation.
On the first hand one has the following equalities

$$
\begin{aligned}
&\big(Y(z)\otimes 1\big)\mathcal{S}^{56}(w)\mathcal{S}^{46}(w+z)\Big(T^{+\,14}_n(u)T^{+\,25}_m(v)T^{+\,36}_k(\bar v)|0\rangle\otimes|0\rangle\otimes|0\rangle\Big)\\
&= \big(Y(z)\otimes 1\big)\mathcal{S}^{56}(w)\mathcal{S}^{46}(w+z)\Big(T^{+\,14}_n(u)T^{+\,36}_k(\bar v)T^{+\,25}_m(v)|0\rangle\otimes|0\rangle\otimes|0\rangle\Big)\\
&= \big(Y(z)\otimes 1\big)\mathcal{S}^{56}(w)\Big(\overline{R}^{13}_{nk}(u|\bar v|w+z)T^{+\,14}_n(u)T^{+\,36}_k(\bar v)\overline{R}^{13}_{nk}(u|\bar v|w+z)^{-1}\cdot\\
&\qquad\qquad\cdot T^{+\,25}_m(v)|0\rangle\otimes|0\rangle\otimes|0\rangle\Big)\\
&= \big(Y(z)\otimes 1\big)\Big(\overline{R}^{13}_{nk}(u|\bar v|w+z)T^{+\,14}_n(u)|0\rangle\mathcal{S}^{56}(w)\big(T^{+\,36}_k(\bar v)T^{+\,25}_m(v)|0\rangle\otimes|0\rangle\big)\cdot\\
&\qquad\cdot \overline{R}^{13}_{nk}(u|\bar v|w+z)^{-1}\Big)\\
&= \big(Y(z)\otimes 1\big)\Big(\overline{R}^{13}_{nk}(u|\bar v|w+z)T^{+\,14}_n(u)\overline{R}^{23}_{mk}(v|\bar v|w)T^{+\,36}_k(\bar v)\cdot\\
&\qquad\cdot T^{+\,25}_m(v)\overline{R}^{23}_{mk}(v|\bar v|w)^{-1}\overline{R}^{13}_{nk}(u|\bar v|w+z)^{-1}|0\rangle\otimes|0\rangle\otimes|0\rangle\Big)
\end{aligned}
$$

$$= \overline{R}^{13}_{nk}(u|\bar{v}|w+z)\overline{R}^{23}_{mk}(v|\bar{v}|w)T^{+\,14}_{n}(u|z)T^{+\,24}_{m}(v)T^{+\,35}_{k}(\bar{v})\cdot$$
$$\cdot\ \overline{R}^{23}_{mk}(v|\bar{v}|w)^{-1}\overline{R}^{13}_{nk}(u|\bar{v}|w+z)^{-1}|0\rangle\otimes|0\rangle.$$

On the other hand one has

$$\mathcal{S}^{45}(w)(Y(z)\otimes 1)\Big(T^{+\,14}_{n}(u)T^{+\,25}_{m}(v)T^{+\,36}_{k}(\bar{v})|0\rangle\otimes|0\rangle\otimes|0\rangle\Big)$$
$$= \mathcal{S}^{45}(w)\Big(T^{+\,14}_{n}(u|z)T^{+\,24}_{m}(v)T^{+\,35}_{k}(\bar{v})|0\rangle\otimes|0\rangle\otimes|0\rangle\Big)$$
$$= \overline{R}^{13}_{nk}(u|\bar{v}|w+z)\overline{R}^{23}_{mk}(v|\bar{v}|w)T^{+\,14}_{n}(u|z)T^{+\,24}_{m}(v)T^{+\,35}_{k}(\bar{v})\cdot$$
$$\cdot\ \overline{R}^{23}_{mk}(v|\bar{v}|w)^{-1}\overline{R}^{13}_{nk}(u|\bar{v}|w+z)^{-1}|0\rangle\otimes|0\rangle.$$

From which the Hexagon Relation (2.26) follows. □

Remark 2.3.11. As shown in Remark 2.3.9, for $N=1$, $\overline{R}(u)=1$. In particular the map $\mathcal{S}$ is the identity. It follows, by the $\mathcal{S}$-commutativity, that $\big(\mathcal{V}_c(\mathbb{C}),|0\rangle,Y,T\big)$ of Theorem 2.3.10 is a commutative vertex algebra over $\mathbb{C}[[h]]$.

2.4 Characterizations of quantum vertex algebras

Definition 2.4.1. A vertex algebra $(V,|0\rangle,Y,T)$ is said to be nondegenerate if the maps:

$$Z_n = Y(z_1)\big(1\otimes Y(z_2)\big)\cdots\big(1^{\otimes n-1}\otimes Y(z_n)\big):$$
$$V^{\otimes n}\to V((z_1))\dots((z_n))$$

are injective for all n.

Proposition 2.4.1 ([EK5, Prop.1.11])**.** *Let* $(V,|0\rangle,Y,T,\mathcal{S})$ *the data satisfying the axioms of a braided vertex algebra, except maybe equations* (2.8) *and* (2.9). *Suppose also that* V/hV *is a nondegenerate vertex algebra. Then:*

1. *equations* (2.8) *and* (2.9) *are automatically satisfied;*

2. *the Hexagon Relation* (2.26) *is equivalent to the Associativity Relation* (2.27).

Proposition 2.4.2 ([EK5, Prop.1.12])**.** *Let* $\mathfrak{g}$ *and* $V=\mathcal{V}_k(\mathfrak{g})$ *like in subsection 1.3.1. If* V *is an irreducible* $\widehat{\mathfrak{g}}$*-module, then* V *is a nondegenerate vertex algebra.*

Proposition 2.4.3 ([EK5, Prop.1.13])**.** *Let the inner product* (,) *on* $\widehat{\mathfrak{g}}$ *be nondegenerate, and* $V=\mathcal{V}_k(\mathfrak{g})$. *Then* V *is nondegenerate for generic* k.

Definition 2.4.2. A *braided topologically free* $\mathbb{K}[[h]]$*-module* $(V,|0\rangle,Y,T,\mathcal{S})$ is the following data:

1. a topologically free $\mathbb{K}[[h]]$-module V;

2. a vector $|0\rangle\in V$;

3. a linear map
$$Y: V\otimes V\to V_h((z));$$

4. a linear operator $T : V \to V$;

5. a linear map $\mathcal{S} : V \otimes V \to V \otimes V \otimes \mathbb{K}((z))[[h]]$ such that $\mathcal{S} = 1 + O(h)$ which satisfies the shift condition $[T \otimes 1, \mathcal{S}(z)] = -\partial_z \mathcal{S}(z)$, the quantum Yang-Baxter equation

$$\mathcal{S}^{12}(z)\mathcal{S}^{13}(z+w)\mathcal{S}^{23}(w) = \mathcal{S}^{23}(w)\mathcal{S}^{13}(z+w)\mathcal{S}^{12}(z) \tag{2.85}$$

and the unitary condition

$$\mathcal{S}^{21}(z) = \mathcal{S}^{-1}(-z); \tag{2.86}$$

subject to the following axioms:

- $T|0\rangle = 0$ and $\partial_z Y(z) = TY(z) - Y(z)(1 \otimes T) = Y(z)(T \otimes 1)$;
- $Y(z)(|0\rangle \otimes b) = b$ for all $b \in V$; for any $a \in V$, $Y(z)(a \otimes |0\rangle)$ is regular at $z = 0$ and its value in $z = 0$ equals a.

As before the tensor products are understood in the h-adic completed sense.

Remark 2.4.4. A braided vertex algebra is a braided topologically free $\mathbb{K}[[h]]$-module which satisfies the $\mathcal{S}$-locality (2.10).

The following theorem is the quantum analogue of Theorem 1.3.12. It first appeared in [L10], Proposition (2.19). Nonetheless, the following proof is more direct than the one in [L10].

Theorem 2.4.5. *A braided vertex algebra which satisfies the Associativity Relation* (2.27) *is the same as a braided topologically free* $\mathbb{K}[[h]]$*-module which satisfies the Associativity Relation* (2.27) *and the equation* $Y\mathcal{S} = Y^{op}$*.*

Proof. The equation $Y\mathcal{S} = Y^{op}$ follows by the $\mathcal{S}$-locality (2.10) by Lemma 2.2.4. Therefore, one only needs to prove that the Associativity Relation (2.27) and $Y\mathcal{S} = Y^{op}$ imply the $\mathcal{S}$-locality (2.10). Let M be an arbitrary positive integer. By Proposition 1.2.6, the Associativity Relation (2.27) implies the following equation: for any $a, b, c \in V$, there exists $L \geq 0$, such that

$$\begin{aligned} &(u-w)^L Y(u)(1 \otimes Y^{op}(w))(b \otimes c \otimes a) \\ &\qquad = (u-w)^L Y^{op}(w)(1 \otimes Y(u))(c \otimes b \otimes a) \ mod\ h^M. \end{aligned} \tag{2.87}$$

Moreover, according to Lemma 1.2.5, the equation still holds if we replace a with Ta. Therefore, one has

$$\begin{aligned} &(u-w)^L Y(u)(1 \otimes Y^{op}(w))(b \otimes c \otimes e^{zT} a) \\ &\qquad = (u-w)^L Y^{op}(w)(1 \otimes Y(u))(c \otimes b \otimes e^{zT} a) \ mod\ h^M. \end{aligned} \tag{2.88}$$

By the definition of Y^{op} and Proposition 1.2.3 we get

$$\begin{aligned} &(u-w)^L\ e^{wT} i_{u,w} Y(u-w)\ (1 \otimes i_{w,z} Y(z-w))(b \otimes a \otimes c) \\ &= (u-w)^L\ e^{wT} i_{w,z} Y(z-w)(i_{u,z} Y(u-z) \otimes 1)(b \otimes a \otimes c) \ mod\ h^M. \end{aligned} \tag{2.89}$$

Since $Y(z-w)(a\otimes c)$ is a Laurent series ($mod\ h^M$) in $z-w$, there exists $P\in\mathbb{Z}_+$ such that $(z-w)^P\ Y(z-w)(a\otimes c)\in V[[z-w]]\ mod\ h^M$. Multiplying both sides of equation (2.89) for $(z-w)^P$ we thus obtain an expression which is regular in w. Putting $w=0$ we obtain

$$Y(b,u)Y(a,z)c = z^{-P}u^{-L}\Big((z-w)^P(u-w)^L e^{wT}\cdot \\ \cdot\ i_{w,z}Y(z-w)\big(i_{u,z}Y(u-z)\otimes 1\big)(b\otimes a\otimes c)\Big)\big|_{w=0}\ mod\ h^M. \tag{2.90}$$

By the Vacuum Axioms of Definition 2.2.1, Corollary 2.2.8, Proposition 1.2.3 and the definition of Y^{op}, we get

$$\begin{aligned} &i_{u,z}Y(u-z)(b\otimes a) = Y(u)(e^{-zT}b\otimes a) = e^{uT}Y^{op}(-u)(a\otimes e^{-zT}b)\\ &= e^{(u-z)T}i_{u,z}Y^{op}(z-u)(a\otimes b) = e^{(u-z)T}i_{u,z}Y(z-u)\mathcal{S}(z-u)(a\otimes b). \end{aligned} \tag{2.91}$$

Inserting equation (2.91) in equation (2.90) and applying again Proposition 1.2.3 we finally obtain the following expression:

$$\begin{aligned} &Y(b,u)Y(a,z)c\\ &= z^{-P}u^{-L}((z-w)^P(u-w)^L e^{wT}\times\\ &\quad\times i_{u,z}i_{w,u}Y\big(Y(z-u)\mathcal{S}(z-u)(a\otimes b),u-w)c)\big|_{w=0}\ mod\ h^M\\ &= z^{-P}u^{-L}((z-w)^P(u-w)^L e^{wT}\times\\ &\quad\times i_{u,z}i_{w,u}Y(u-w)(Y(z-u)\mathcal{S}(z-u)\otimes 1)(a\otimes b\otimes c))\big|_{w=0}\ mod\ h^M. \end{aligned} \tag{2.92}$$

Rewriting formula (2.87) with z instead of u and a, b swapped, one gets:

$$\begin{aligned} &(z-w)^L\ Y(z)(1\otimes Y^{op}(w))(a\otimes c\otimes b)\\ &= (z-w)^L\ Y^{op}(w)(1\otimes Y(z))(1\ 2)(a\otimes c\otimes b)\ mod\ h^M. \end{aligned} \tag{2.93}$$

By definition $\mathcal{S}^{13}(z-u)(a\otimes c\otimes b)=\sum_{i=1}^l a_i\otimes c\otimes b_i f_i(z-u)$ where $f_i(z-u)\in\mathbb{K}((z-u))\ mod\ h^M$. Therefore there is a power Q of $z-u$ (depending only on a and b) such that $(z-u)^Q\ \mathcal{S}^{13}(z-u)(a\otimes c\otimes b)\in\mathbb{K}[[z-u]]\ mod\ h^M$. For every element $a_i\otimes c\otimes b_i\in V^{\otimes 3}$ there exists $L_i\in\mathbb{Z}_+$ such that the following is satisfied

$$\begin{aligned} &(z-w)^{L_i}\ Y(z)(1\otimes Y^{op}(w))(a_i\otimes c\otimes b_i)\\ &= (z-w)^{L_i}\ Y^{op}(w)(1\otimes Y(z))(1\ 2)(a_i\otimes c\otimes b_i)\ mod\ h^M \end{aligned} \tag{2.94}$$

and by Proposition 1.2.5 it is still satisfied

$$\begin{aligned} &(z-w)^{L_i}\ Y(z)(1\otimes Y^{op}(w))(1\otimes 1\otimes e^{uT})(a_i\otimes c\otimes b_i)\\ &= (z-w)^{L_i}\ Y^{op}(w)(1\otimes Y(z))(1\ 2)(1\otimes 1\otimes e^{uT})(a_i\otimes c\otimes b_i)\ mod\ h^M. \end{aligned} \tag{2.95}$$

The expression still holds if we multiply both sides by $(z-u)^Q\ f_i(z-u)$. Therefore if we replace L_i with $L=max\{L_j|\ j=1,\dots,l\}$ and we sum on i we obtain

$$\begin{aligned} &(z-w)^L(z-u)^Q\ Y(z)(1\otimes Y^{op}(w))(1\otimes 1\otimes e^{uT})\mathcal{S}^{13}(a\otimes c\otimes b)\\ &= (z-w)^L(z-u)^Q\ Y^{op}(w)(1\otimes Y(z))(1\ 2)(1\otimes 1\otimes e^{uT})\times\\ &\quad\times\mathcal{S}^{13}(a\otimes c\otimes b)\ mod\ h^M. \end{aligned} \tag{2.96}$$

Applying the definition of Y^{op} and Proposition 1.2.3 one get

$$\begin{aligned}&(z-w)^L(z-u)^Q e^{wT} i_{z,w}Y(z-w)(1\otimes i_{w,u}Y(u-w))\mathcal{S}^{12}(z-u)(a\otimes b\otimes c)\\&=(z-w)^L(z-u)^Q e^{wT} i_{w,u}Y(u-w)\times\\&\quad\times(i_{z,u}Y(z-u)\mathcal{S}(z-u)\otimes 1)(a\otimes b\otimes c)\ mod\ h^M.\end{aligned}\tag{2.97}$$

The expression $(1\otimes i_{w,u}Y(u-w))\mathcal{S}^{12}(z-u)(a\otimes b\otimes c)\in V\otimes V((u-w))$. Thus, there is a power R of $u-w$ such that $(u-w)^R\ (1\otimes i_{w,u}Y(u-w))\mathcal{S}^{12}(z-u)(a\otimes b\otimes c)\in V\otimes V[[u-w]]\ mod\ h^M$, i.e. the first line of equation (2.97) is regular in w. Taking $w=0$ we get

$$\begin{aligned}&(z-u)^Q\ Y(z)(1\otimes Y(u))\mathcal{S}^{12}(z-u)(a\otimes b\otimes c)\\&=u^{-R}\ z^{-L}((u-w)^L(z-w)^L(z-u)^Q\ e^{wT}\\&\quad\times i_{w,u}i_{z,u}Y(u-w)(Y(z-u)\mathcal{S}(z-u)\otimes 1)(a\otimes b\otimes c))\big|_{w=0}\ mod\ h^M.\end{aligned}\tag{2.98}$$

Now, substituting P,r,L,R with their maximum D in equations (2.92) and (2.98), and multiplying equation (2.92) by $(z-u)^Q$ we get

$$\begin{aligned}&(z-u)^Q Y(b,u)Y(a,z)c\\&=z^{-D}u^{-D}((z-w)^D(u-w)^D(z-u)^Q e^{wT}\times\\&\quad\times i_{u,z}i_{w,u}Y(u-w)(Y(z-u)\mathcal{S}(z-u)\otimes 1)(a\otimes b\otimes c)\big|_{w=0}\ mod\ h^M.\end{aligned}\tag{2.99}$$

and

$$\begin{aligned}&(z-u)^Q\ Y(z)(1\otimes Y(u))\mathcal{S}^{12}(z-u)(a\otimes b\otimes c)\\&=u^{-D}\ z^{-D}((u-w)^D(z-w)^D(z-u)^Q\ e^{wT}\times\\&\quad\times i_{w,u}i_{z,u}Y(u-w)(Y(z-u)\mathcal{S}(z-u)\otimes 1)(a\otimes b\otimes c))\big|_{w=0}\ mod\ h^M.\end{aligned}\tag{2.100}$$

The right hand sides of both equations are the same except for the expansions of $z-u$. Nonetheless, $(z-u)^Q\ Y(z-u)\mathcal{S}(z-u)(a\otimes b)\in V((z-u))\ mod\ h^M$. Therefore, multiplying both sides of equations (2.100) and (2.99) for a suitable power H of $z-u$ (H depending only on a and b) such that $(z-u)^{Q+H}\ Y(z-u)\mathcal{S}(z-u)(a\otimes b)\in V[[z-u]]\ mod\ h^M$ and calling $N=H+Q$ we have the $\mathcal{S}$-locality:

$$\begin{aligned}&(z-u)^N\ Y(z)(1\otimes Y(u))\mathcal{S}^{12}(z-u)(a\otimes b\otimes c)\\&=(z-u)^N\ Y(u)(1\otimes Y(z))(b\otimes a\otimes c)\ mod\ h^M.\end{aligned}\tag{2.101}$$

□

Corollary 2.4.6. *A braided topologically free $\mathbb{K}[[h]]$-module which satisfies the Associativity Relation* (2.27)*, the equation $Y\mathcal{S}=Y^{op}$ and which has an injective state-field correspondence Y is a quantum vertex algebra.*

Proof. By the injectivity of Y, the Hexagon Relation (2.26) automatically descends from the $\mathcal{S}$-locality (2.10) and the Associativity Relation (2.27). Indeed, by $\mathcal{S}$-locality (2.10), one has the Quasi-Associativity Relation (2.22) and, by the Associativity

Relation (2.27), one has the following equation: for any $a, b, c \in V$ and $M \in \mathbb{Z}_+$ there exists $L \geq 0$ such that

$$\begin{aligned} &i_{z-w,w}\big(z^L Y(z)(1 \otimes Y(w))\mathcal{S}^{23}(w)\mathcal{S}^{13}(z)(a \otimes b \otimes c)\big) \\ &= z^L Y(w)(Y(z-w) \otimes 1)\mathcal{S}^{23}(w)\mathcal{S}^{13}(z)(a \otimes b \otimes c) \quad mod\ h^M. \end{aligned} \tag{2.102}$$

The left hand sides of the Quasi-Associativity Relation (2.22) and equation (2.102) are the same, therefore the right hand sides are the same. Moreover the expressions

$$Y(w)\mathcal{S}(w)\big(Y(z-w) \otimes 1\big)(a \otimes b \otimes c)$$

and

$$Y(w)(Y(z-w) \otimes 1)\mathcal{S}^{23}(w)\mathcal{S}^{13}(z)(a \otimes b \otimes c)$$

lie in the same space, hence multiplying by $i_{w,z-w}\ z^{-L}$ the right hand sides of the Quasi-Associativity Relation (2.22) and equation (2.102), one has

$$\begin{aligned} &Y(w)\mathcal{S}(w)(Y(z-w)) \otimes 1)(a \otimes b \otimes c) \\ &= Y(w)(Y(z-w) \otimes 1)\mathcal{S}^{23}(w)\mathcal{S}^{13}(z)(a \otimes b \otimes c) \quad mod\ h^M \end{aligned} \tag{2.103}$$

i.e. the equality of the evaluation of both sides of the Hexagon Relation (2.26) through Y. □

Corollary 2.4.7. *Let $(V, |0\rangle, Y, T, \mathcal{S})$ be a braided topologically free $\mathbb{K}[[h]]$-module which satisfies the Associativity Relation (2.27) and the equation $Y\mathcal{S} = Y^{op}$. If V/hV is a nondegenerate vertex algebra, then $(V, |0\rangle, Y, T, \mathcal{S})$ is a quantum vertex algebra.*

Proof. It follows from Theorem 2.4.5 and Proposition 2.4.1. □

Let's now define the n-products of quantum fields in a braided topologically free $\mathbb{K}[[h]]$-module:

Definition 2.4.3. Let $(V, |0\rangle, Y, T, \mathcal{S})$ be a braided topologically free $\mathbb{K}[[h]]$-module and let $a, b \in V$. The quantum n-products of $Y(a, z)$ and $Y(b, z)$ are defined as the following:

$$\begin{aligned} &Y(a,w)^{\mathcal{S}}_{(n)} Y(b, w) \\ &\quad = Res_z\Big(Y(z)(1 \otimes Y(w))\big(i_{z,w}\mathcal{S}(z-w)(a \otimes b) \otimes -\big) i_{z,w}(z-w)^n \\ &\quad - Y(w)\big(1 \otimes Y(z)\big)(b \otimes a \otimes -) i_{w,z}(z-w)^n\Big) \end{aligned} \tag{2.104}$$

for any $n \in \mathbb{Z}$.

Lemma 2.4.8. *The quantum n-products are translation covariant quantum fields satisfying $Y(a, z)^{\mathcal{S}}_{(n)} Y(b, z)|0\rangle \in V[[z]]$.*

Proof. $Y(a, w)^{\mathcal{S}}_{(n)} Y(b, w)$ is an $End_{\mathbb{K}[[h]]}V$-valued quantum field by the definition of the map Y and evaluating the quantum n-products on the vacuum vector $|0\rangle$, one has

$$Y(a,w)^{\mathcal{S}}_{(n)} Y(b, w)|0\rangle$$

$$
\begin{aligned}
&= Res_z\Big(Y(z)(1\otimes e^{wT})i_{z,w}\mathcal{S}(z-w)(a\otimes b)i_{z,w}(z-w)^n \\
&\quad - Y(w)(1\otimes e^{zT})(b\otimes a)i_{w,z}(z-w)^n\Big) \\
&= Res_z\Big(Y(z)(1\otimes e^{wT})i_{z,w}\mathcal{S}(z-w)(a\otimes b)i_{z,w}(z-w)^n\Big) \in V[[w]].
\end{aligned}
$$

Let's now show the translation covariance. On one hand, one has

$$
\begin{aligned}
&[T, Y(a,w)^{\mathcal{S}}_{(n)}Y(b,w)] \\
&= Res_z\Big(TY(z)\big(1\otimes Y(w)\big)\big(\mathcal{S}(z-w)(a\otimes b)\otimes -\big)\ i_{z,w}(z-w)^n \\
&\quad - TY(w)\big(1\otimes Y(z)\big)(b\otimes a\otimes -)\ i_{w,z}(z-w)^n \\
&\quad - Y(z)\big(1\otimes Y(w)\big)\big(\mathcal{S}(z-w)(a\otimes b)\otimes T(-)\big)\ i_{z,w}(z-w)^n \\
&\quad + Y(w)\big(1\otimes Y(z)\big)\big(b\otimes a\otimes T(-)\big)\ i_{w,z}(z-w)^n\Big) \\
&= Res_z\Big(TY(z)\big(1\otimes Y(w)\big)\big(\mathcal{S}(z-w)(a\otimes b)\otimes -\big)\ i_{z,w}(z-w)^n \\
&\quad - Y(z)(1\otimes T)\big(1\otimes Y(w)\big)\big(\mathcal{S}(z-w)(a\otimes b)\otimes -\big)\ i_{z,w}(z-w)^n \\
&\quad + Y(z)(1\otimes T)\big(1\otimes Y(w)\big)\big(\mathcal{S}(z-w)(a\otimes b)\otimes -\big)\ i_{z,w}(z-w)^n \\
&\quad - Y(z)\big(1\otimes Y(w)\big)(1\otimes 1\otimes T)\big(\mathcal{S}(z-w)(a\otimes b)\otimes -\big)\ i_{z,w}(z-w)^n \\
&\quad + Y(z)\big(1\otimes Y(w)\big)(1\otimes 1\otimes T)\big(\mathcal{S}(z-w)(a\otimes b)\otimes -\big)\ i_{z,w}(z-w)^n \\
&\quad - TY(w)\big(1\otimes Y(z)\big)(b\otimes a\otimes -)\ i_{w,z}(z-w)^n \\
&\quad + Y(w)(1\otimes T)\big(1\otimes Y(z)\big)(b\otimes a\otimes -)\ i_{w,z}(z-w)^n \\
&\quad - Y(w)(1\otimes T)\big(1\otimes Y(z)\big)(b\otimes a\otimes -)\ i_{w,z}(z-w)^n \\
&\quad + Y(w)\big(1\otimes Y(z)\big)(1\otimes 1\otimes T)(b\otimes a\otimes -)\ i_{w,z}(z-w)^n \\
&\quad - Y(w)\big(1\otimes Y(z)\big)(1\otimes 1\otimes T)(b\otimes a\otimes -)\ i_{w,z}(z-w)^n \\
&\quad - Y(z)\big(1\otimes Y(w)\big)\big(\mathcal{S}(z-w)(a\otimes b)\otimes T(-)\big)i_{z,w}(z-w)^n \\
&\quad + Y(w)\big(1\otimes Y(z)\big)\big(b\otimes a\otimes T(-)\big)\ i_{w,z}(z-w)^n\Big).
\end{aligned}
$$

Hence, using the translation covariance, one has

$$
\begin{aligned}
&[T, Y(a,w)^{\mathcal{S}}_{(n)}Y(b,w)] \\
&= Res_z\Big(\big(\partial_z Y(z)\big)\big(1\otimes Y(w)\big)\big(\mathcal{S}(z-w)(a\otimes b)\otimes -\big)\ i_{z,w}(z-w)^n \\
&\quad + Y(z)\big(1\otimes \partial_w Y(w)\big)\big(\mathcal{S}(z-w)(a\otimes b)\otimes -\big)\ i_{z,w}(z-w)^n \\
&\quad - \big(\partial_w Y(w)\big)\big(1\otimes Y(z)\big)(b\otimes a\otimes -)\ i_{w,z}(z-w)^n \\
&\quad - Y(w)\big(1\otimes \partial_z Y(z)\big)(b\otimes a\otimes -)\ i_{w,z}(z-w)^n\Big).
\end{aligned}
$$

On the other hand, since

$$
\begin{aligned}
&Res_z\Big(Y(w)\Big(1\otimes \big(\partial_z Y(z)\big)\Big)(b\otimes a\otimes -)\ i_{w,z}(z-w)^n\big) \\
&= Res_z\Big(Y(w)\big(1\otimes Y(z)\big)(b\otimes a\otimes -)\ i_{w,z}\partial_w(z-w)^n\big),
\end{aligned}
$$

and

$$
\begin{aligned}
&Res_z\Big(\big(\partial_z Y(z)\big)\big(1\otimes Y(w)\big)\big(\mathcal{S}(z-w)(a\otimes b)\otimes -\big)\ i_{z,w}(z-w)^n\\
&= Res_z\Big(Y(z)\big(1\otimes Y(w)\big)\big(\partial_w\mathcal{S}(z-w)(a\otimes b)\otimes -\big)\ i_{z,w}(z-w)^n\\
&\quad + Y(z)\big(1\otimes Y(w)\big)\big(\mathcal{S}(z-w)(a\otimes b)\otimes -\big)\ i_{z,w}\partial_w(z-w)^n\big),
\end{aligned}
$$

one has

$$
\begin{aligned}
&Res_z\Big(\big(\partial_z Y(z)\big)\big(1\otimes Y(w)\big)\big(\mathcal{S}(z-w)(a\otimes b)\otimes -\big)\ i_{z,w}(z-w)^n\\
&\quad - Y(w)\Big(1\otimes\big(\partial_z Y(z)\big)\Big)(b\otimes a\otimes -)\ i_{w,z}(z-w)^n\big)\\
&= Res_z\Big(Y(z)\big(1\otimes Y(w)\big)\big(\partial_w\mathcal{S}(z-w)(a\otimes b)\otimes -\big)\ i_{z,w}(z-w)^n\\
&\quad + Y(z)\big(1\otimes Y(w)\big)\big(\mathcal{S}(z-w)(a\otimes b)\otimes -\big)\ i_{z,w}\partial_w(z-w)^n\\
&\quad - Y(w)\big(1\otimes Y(z)\big)(b\otimes a\otimes -)\ i_{w,z}\partial_w(z-w)^n\big).
\end{aligned}
\tag{2.105}
$$

Therefore

$$
\begin{aligned}
&\partial_w\big(Y(a,w)^{\mathcal{S}}_{(n)}Y(b,w)\big)\\
&= Res_z\Big(Y(z)\Big(1\otimes\big(\partial_w Y(w)\big)\Big)\big(\mathcal{S}(z-w)(a\otimes b)\otimes -\big)i_{z,w}(z-w)^n\\
&\quad + Y(z)\big(1\otimes Y(w)\big)\big(\partial_w\mathcal{S}(z-w)(a\otimes b)\otimes -\big)i_{z,w}(z-w)^n\\
&\quad + Y(z)\big(1\otimes Y(w)\big)\big(\mathcal{S}(z-w)(a\otimes b)\otimes -\big)i_{z,w}\partial_w(z-w)^n\\
&\quad - \big(\partial_w Y(w)\big)\big(1\otimes Y(z)\big)(b\otimes a\otimes -)i_{w,z}(z-w)^n\\
&\quad - Y(w)\big(1\otimes Y(z)\big)(b\otimes a\otimes -)i_{w,z}\partial_w(z-w)^n\Big)\\
&= Res_z\Big(Y(z)\Big(1\otimes\big(\partial_w Y(w)\big)\Big)\big(\mathcal{S}(z-w)(a\otimes b)\otimes -\big)i_{z,w}(z-w)^n\\
&\quad - \big(\partial_w Y(w)\big)\big(1\otimes Y(z)\big)(b\otimes a\otimes -)i_{w,z}(z-w)^n\\
&\quad + Res_z\Big(\big(\partial_z Y(z)\big)\big(1\otimes Y(w)\big)\big(\mathcal{S}(z-w)(a\otimes b)\otimes -\big)\ i_{z,w}(z-w)^n\\
&\quad - Y(w)\Big(1\otimes\big(\partial_z Y(z)\big)\Big)(b\otimes a\otimes -)\ i_{w,z}(z-w)^n\Big).
\end{aligned}
$$

□

Definition 2.4.4. Let $(V,|0\rangle,Y,T,\mathcal{S})$ be a braided topologically free $\mathbb{K}[[h]]$-module. For any $a,b\in V$ and $n\in\mathbb{Z}$, we define

$$
a^{\mathcal{S}}_{(n)}b = Res_z\big(z^nY(z)S(z)(a\otimes b)\big) = Res_z\big(z^nY^{op}(z)(a\otimes b)\big) \tag{2.106}
$$

(cf. Lemma 2.2.4).

Lemma 2.4.9. *$Y(a,z)^{\mathcal{S}}_{(n)}Y(b,z)|0\rangle_{|z=0} = a^{\mathcal{S}}_{(n)}b$ for any $a,b\in V$.*

Proof. One has

$$
\begin{aligned}
&Y(a,w)^{\mathcal{S}}_{(n)}Y(b,w)|0\rangle_{|w=0}\\
&= Res_z\Big(Y(z)\big(1\otimes Y(w)\big)\big(\mathcal{S}(z-w)(a\otimes b)\otimes|0\rangle\big)i_{z,w}(z-w)^n
\end{aligned}
$$

$$- Y(w)\big(1\otimes Y(z)\big)(b\otimes a\otimes|0\rangle)i_{w,z}(z-w)^n\big)|_{w=0}$$
$$= Res_z\Big(Y(z)\big(1\otimes Y(w)\big)\big(\mathcal{S}(z-w)(a\otimes b)\otimes|0\rangle\big)i_{z,w}(z-w)^n\Big)|_{w=0}$$

where the last equality holds because there are no negative powers of z in the expression $Y(w)\big(1\otimes Y(z)\big)(b\otimes a\otimes|0\rangle)i_{w,z}(z-w)^n$.
Moreover $Y(z)\big(1\otimes Y(w)\big)\big(\mathcal{S}(z-w)(a\otimes b)\otimes|0\rangle\big)i_{z,w}(z-w)^n$ is regular in w therefore

$$\begin{aligned}
&Res_z\Big(Y(z)\big(1\otimes Y(w)\big)\big(\mathcal{S}(z-w)(a\otimes b)\otimes|0\rangle\big)i_{z,w}(z-w)^n\Big)|_{w=0}\\
&= Res_z\Big(Y(z)\big(1\otimes Y(w)\big)\big(\mathcal{S}(z-w)(a\otimes b)\otimes|0\rangle\big)i_{z,w}(z-w)^n)|_{w=0}\big)\\
&= Res_z\big(Y(z)\mathcal{S}(z)(a\otimes b)z^n\big)\\
&= a^{\mathcal{S}}_{(n)}b.
\end{aligned}$$

□

Definition 2.4.5. Let $(V,|0\rangle,Y,T,\mathcal{S})$ be a braided topologically free $\mathbb{K}[[h]]$-module. The following equalities are called the quantum Borcherds Identities: for any $a,b,c\in V$ and $n\in\mathbb{Z}$,

$$\begin{aligned}
Y(z)\big(1\otimes Y(w)\big)\big(i_{z,w}\mathcal{S}(z-w)(a\otimes b)\otimes c\big)i_{z,w}(z-w)^n&\\
-Y(w)\big(1\otimes Y(z)\big)(b\otimes a\otimes c)i_{w,z}(z-w)^n&\\
=\sum_{j\in\mathbb{Z}_+} Y(w)\big(a^{\mathcal{S}}_{(n+j)}b\otimes c\big)\ \frac{\partial_w^j\delta(z,w)}{j!}&
\end{aligned}\tag{2.107}$$

Definition 2.4.6. Let $(V,|0\rangle,Y,T,\mathcal{S})$ be a braided topologically free $\mathbb{K}[[h]]$-module. The quantum n-products identities are satisfied if, for any $a,b,c\in V$ and $n\in\mathbb{Z}$, one has

$$\Big(Y(a,w)^{\mathcal{S}}_{(n)}Y(b,w)\Big)(c) = Y(w)\big(a^{\mathcal{S}}_{(n)}b\otimes c\big).\tag{2.108}$$

Lemma 2.4.10. *Let $(V,|0\rangle,Y,T,\mathcal{S})$ be a braided topologically free $\mathbb{K}[[h]]$-module. The quantum Borcherds Identities* (2.107) *hold for any $a,b\in V$ and $n\in\mathbb{Z}$ if and only if the n-products identities* (2.108) *and the $\mathcal{S}$-locality* (2.10) *hold.*

Proof. Since the left hand side of equation 2.107 is local (by $\mathcal{S}$-locality 2.10), using the Decomposition Theorem 1.1.7, one has

$$\begin{aligned}
Y(z)\big(1\otimes Y(w)\big)\big(i_{z,w}\mathcal{S}(z-w)(a\otimes b)\otimes c\big)i_{z,w}(z-w)^n&\\
-Y(w)\big(1\otimes Y(z)\big)(b\otimes a\otimes c)i_{w,z}(z-w)^n&\\
=\sum_{j\geq 0} c^j(w)\frac{\partial_w^j\delta(z,w)}{j!}\quad mod\ h^M&
\end{aligned}\tag{2.109}$$

where

$$\begin{aligned}
c^j(w) = Res_z\Big(&Y(z)\big(1\otimes Y(w)\big)\big(i_{z,w}\mathcal{S}(z-w)(a\otimes b)\otimes c\big)i_{z,w}(z-w)^{n+j}\\
&- Y(w)\big(1\otimes Y(z)\big)(b\otimes a\otimes c)i_{w,z}(z-w)^{n+j}\Big)
\end{aligned}$$

$$= \Big(Y(a,w)^{\mathcal{S}}_{(n+j)}Y(b,w)\Big)(c).$$

Moreover, since equation (2.109) holds for any $M \in \mathbb{Z}_+$, one has

$$\begin{aligned} Y(z)(1\otimes Y(w))&(i_{z,w}\mathcal{S}(z-w)(a\otimes b)\otimes c)i_{z,w}(z-w)^n \\ &-Y(w)(\otimes Y(z))(b\otimes a\otimes c)i_{w,z}(z-w)^n \\ &= \sum_{j\geq 0} c_j(w)\frac{\partial_w^j\delta(z,w)}{j!}. \end{aligned}$$

It follows that

$$\begin{aligned} Y(z)(1\otimes Y(w))&(\mathcal{S}(z-w)(a\otimes b)\otimes c)i_{z,w}(z-w)^n \\ &-Y(w)(1\otimes Y(z))(b\otimes a\otimes c)i_{w,z}(z-w)^n \\ &= \sum_{j\in\mathbb{Z}_+}\Big(Y(a,w)^{\mathcal{S}}_{(n+j)}Y(b,w)\Big)(c)\ \frac{\partial_w^j\delta(z,w)}{j!}. \end{aligned} \tag{2.110}$$

If the n-product Identities (2.108) hold, the quantum Borcherds Identities (2.107) hold. Indeed, as $\Big(Y(a,w)^{\mathcal{S}}_{(n+j)}Y(b,w)\Big)(c) = Y(w)(a^{\mathcal{S}}_{(n+j)}b\otimes c)$ for any $j \geq 0$, equation (2.110) becomes

$$\begin{aligned} Y(z)(1\otimes Y(w))&(\mathcal{S}(z-w)(a\otimes b)\otimes c)i_{z,w}(z-w)^n \\ &-Y(w)(1\otimes Y(z))(b\otimes a\otimes c)i_{w,z}(z-w)^n \\ &= \sum_{j\in\mathbb{Z}_+} Y(w)(a^{\mathcal{S}}_{(n+j)}b\otimes c)\ \frac{\partial_w^j\delta(z,w)}{j!}. \end{aligned}$$

On the converse, if the quantum Borcherds Identities (2.107) hold, then the n-product Identities (2.108) hold because taking the residue Res_z on both sides of equation (2.107), the right hand side is

$$\sum_{j\geq 0}\binom{0}{j}Y(w)(a^{\mathcal{S}}_{(n+j)}b\otimes c)w^{-j} = Y(w)(a^{\mathcal{S}}_{(n)}b\otimes c)$$

and the left hand side is $\Big(Y(a,w)^{\mathcal{S}}_{(n+j)}Y(b,w)\Big)(c)$. Moreover, for any $M \in \mathbb{Z}_+$, $a^{\mathcal{S}}_{(n)}b = 0 \bmod h^M$ for $n \gg 0$ because $a^{\mathcal{S}}_{(n)}b = Res_z(z^nY^{op}(z)(a\otimes b))$. Thus the $\mathcal{S}$-locality holds. □

Using Lemma 2.4.10, one shows that the quantum vertex algebras of Theorems 2.3.8 and 2.3.10 satisfy the quantum Borcherds Identities.

Proposition 2.4.11. *The quantum vertex algebra defined in Theorem 2.3.8 satisfies the quantum Borcherds Identities* (2.107).

Proof. By equations (2.79) and (2.80), the quantum n-products are given by the following:

$$\Big(Y(T_n^+(u)|0\rangle,w)^{\mathcal{S}}_{(l)}Y(T_m^+(v)|0\rangle,w)\Big)T_k^+(\bar{v})|0\rangle$$

$$= Res_z\Big(Y(z)(1\otimes Y(w))\mathcal{S}^{45}(z-w)(T_n^{+14}(u)T_m^{+25}(v)T_k^{+36}(\bar v)|0\rangle\otimes|0\rangle\otimes|0\rangle)i_{z,w}(z-w)^l$$
$$-Y(w)(1\otimes Y(z))(4\ 5)(T_n^{+14}(u)T_m^{+25}(v)T_k^{+36}(\bar v)|0\rangle\otimes|0\rangle\otimes|0\rangle)i_{w,z}(z-w)^l\Big)$$
$$= Res_z\Big(i_{z,w}\,{}^{lr}\big(({}^{lr}\overline{R}^{12}_{nm}(u|v|z-w-hc))^{-1}\big)\,{}^{rl}\big(({}^{rl}\overline{R}^{13}_{nk}(u|\bar v|z+hc))^{-1}\big)\cdot$$
$$\cdot\,{}^{rl}\big(({}^{rl}\overline{R}^{23}_{mk}(v|\bar v|w+hc))^{-1}\big)i_{z,w}\overline{R}^{12}_{nm}(u|v|z-w)T_n^{+14}(u|z)T_m^{+24}(v|w)\cdot$$
$$\cdot\,T_k^{+34}(\bar v)\overline{R}^{23}_{mk}(v|\bar v|w)^{-1}\overline{R}^{13}_{nk}(u|\bar v|z)^{-1}|0\rangle i_{z,w}(z-w)^l$$
$$-i_{w,z}\,{}^{lr}\big(({}^{lr}\overline{R}^{12}_{nm}(u|v|z-w-hc))^{-1}\big)\,{}^{rl}\big(({}^{rl}\overline{R}^{13}_{nk}(u|\bar v|z+hc))^{-1}\big)\cdot$$
$$\cdot\,{}^{rl}\big(({}^{rl}\overline{R}^{23}_{mk}(v|\bar v|w+hc))^{-1}\big)i_{w,z}\overline{R}^{12}_{nm}(u|v|z-w)T_n^{+14}(u|z)T_m^{+24}(v|w)\cdot$$
$$\cdot\,T_k^{+34}(\bar v)\overline{R}^{23}_{mk}(v|\bar v|w)^{-1}\overline{R}^{13}_{nk}(u|\bar v|z)^{-1}|0\rangle i_{w,z}(z-w)^l$$
$$= Res_z\bigg((i_{z,w}-i_{w,z})\Big({}^{lr}\big(({}^{lr}\overline{R}^{12}_{nm}(u|v|z-w-hc))^{-1}\big)\overline{R}^{12}_{nm}(u|v|z-w)(z-w)^l\Big)\cdot$$
$$\cdot\,{}^{rl}\big(({}^{rl}\overline{R}^{13}_{nk}(u|\bar v|z+hc))^{-1}\big)\,{}^{rl}\big(({}^{rl}\overline{R}^{23}_{mk}(v|\bar v|w+hc))^{-1}\big)T_n^{+14}(u|z)T_m^{+24}(v|w)\cdot$$
$$\cdot\,T_k^{+34}(\bar v)\overline{R}^{23}_{mk}(v|\bar v|w)^{-1}\overline{R}^{13}_{nk}(u|\bar v|z)^{-1}|0\rangle\bigg).$$

The coefficient of $u_1^{r_1}\cdots u_n^{r_n}v_1^{s_1}\cdots v_m^{s_m}$ of the previous expression is a linear combination of products of two factors: the first one is obtained by

$$ {}^{lr}\big(({}^{lr}\overline{R}^{12}_{nm}(u|v|z-w-hc))^{-1}\big)\overline{R}^{12}_{nm}(u|v|z-w)(z-w)^l \tag{2.111}$$

and the second one is obtained by

$$\begin{aligned}&{}^{rl}\big(({}^{rl}\overline{R}^{13}_{nk}(u|\bar v|z+hc))^{-1}\big)\,{}^{rl}\big(({}^{rl}\overline{R}^{23}_{mk}(v|\bar v|w+hc))^{-1}\big)T_n^{+14}(u|z)\cdot\\ &\cdot\,T_m^{+24}(v|w)T_k^{+34}(\bar v)\overline{R}^{23}_{mk}(v|\bar v|w)^{-1}\overline{R}^{13}_{nk}(u|\bar v|z)^{-1}|0\rangle.\end{aligned}\tag{2.112}$$

Modulo h^M, the terms coming from (2.111) are Laurent series in $z-w$ and the terms coming from (2.112) have a finite number of negative powers of z and w. It follows, using Lemma 1.1.5 on the coefficient of $u_1^{r_1}\cdots u_n^{r_n}v_1^{s_1}\cdots v_m^{s_m}$ for any $r_1,\dots,r_n,s_1,\dots,s_m\in\mathbb{N}$, that

$$\Big(Y(T_n^+(u)|0\rangle,w)^{\mathcal{S}}_{(l)}Y(T_m^+(v)|0\rangle,w)\Big)T_k^+(\bar v)|0\rangle$$
$$= Res_x\bigg({}^{lr}\big(({}^{lr}\overline{R}^{12}_{nm}(u|v|x-hc))^{-1}\big)\overline{R}^{12}_{nm}(u|v|x)x^l\cdot$$
$$\cdot\,i_{w,x}\Big({}^{rl}\big(({}^{rl}\overline{R}^{13}_{nk}(u|\bar v|w+x+hc))^{-1}\big)\,{}^{rl}\big(({}^{rl}\overline{R}^{23}_{mk}(v|\bar v|w+hc))^{-1}\big)\cdot$$
$$\cdot\,T_n^{+14}(u|w+x)T_m^{+24}(v|w)T_k^{+34}(\bar v)\overline{R}^{23}_{mk}(v|\bar v|w)^{-1}\overline{R}^{13}_{nk}(u|\bar v|w+x)^{-1}|0\rangle\Big)\bigg).$$

On the other hand one has

$$T_n^+(u)|0\rangle^{\mathcal{S}}_{(l)}T_m^+(v)|0\rangle$$

$$= Res_z\Big(z^l Y(z)\mathcal{S}^{45}(z)(T_n^{+14}(u)T_m^{+25}(v)|0\rangle\otimes|0\rangle\Big)$$

$$= Res_z\Big(z^l\,{}^{lr}\Big(\Big({}^{lr}\overline{R}_{nm}^{12}(u|v|z-hc)\Big)^{-1}\Big)\overline{R}_{nm}^{12}(u|v|z)T_n^{+14}(u|z)T_n^{14}(u|z+hc/2)^{-1}\cdot$$
$$\cdot\ \overline{R}_{nm}^{12}(u|v|z+hc)^{-1}T_m^{+24}(v)\overline{R}_{nm}^{12}(u|v|z)|0\rangle\Big)$$

$$\stackrel{(e1)}{=} Res_z\Big(z^l\,{}^{lr}\Big(\Big({}^{lr}\overline{R}_{nm}^{12}(u|v|z-hc)\Big)^{-1}\Big)\overline{R}_{nm}^{12}(u|v|z)T_n^{+14}(u|z)T_m^{+24}(v)\overline{R}_{nm}^{12}(u|v|z)^{-1}\cdot$$
$$\cdot\ T_n^{14}(u|z+hc/2)^{-1}\overline{R}_{nm}^{12}(u|v|z)|0\rangle\Big)$$

$$= Res_z\Big(z^l\,{}^{lr}\Big(\Big({}^{lr}\overline{R}_{nm}^{12}(u|v|z-hc)\Big)^{-1}\Big)\overline{R}_{nm}^{12}(u|v|z)T_n^{+14}(u|z)T_m^{+24}(v)\overline{R}_{nm}^{12}(u|v|z)^{-1}\cdot$$
$$\cdot\ \overline{R}_{nm}^{12}(u|v|z)|0\rangle\Big)$$

$$= Res_z\Big(z^l\,{}^{lr}\Big(\Big({}^{lr}\overline{R}_{nm}^{12}(u|v|z-hc)\Big)^{-1}\Big)\overline{R}_{nm}^{12}(u|v|z)T_n^{+14}(u|z)T_m^{+24}(v)|0\rangle\Big).$$

Where $(e1)$ is due to relation (2.65).
Therefore, one has

$$Y\Big(T_n^+(u)|0\rangle^{\mathcal{S}}_{(l)}T_m^+(v)|0\rangle, w\Big)T_k^+(\bar v)|0\rangle$$
$$= Y(w)\Big(Res_z\Big(z^l\,{}^{lr}(({}^{lr}\overline{R}_{nm}^{12}(u|v|z-hc))^{-1})\overline{R}_{nm}^{12}(u|v|z)T_n^{+14}(u|z)T_m^{+24}(v)|0\rangle\cdot$$
$$\cdot\ T_k^{+35}(\bar v)|0\rangle\Big)\Big)$$
$$= Res_z\Big(z^l\,{}^{lr}(({}^{lr}\overline{R}_{nm}^{12}(u|v|z-hc))^{-1})\overline{R}_{nm}^{12}(u|v|z)Y\Big(T_n^{+14}(u|w+z)T_m^{+24}(v|w),w\Big)\cdot$$
$$\cdot\ T_k^{+34}(\bar v)|0\rangle\Big).$$

Moreover,

$$Y\Big(T_n^{+14}(u|z)T_m^{+24}(v),w\Big)T_k^{+34}(\bar v)|0\rangle$$
$$= i_{w,z}\Big({}^{rl}(({}^{rl}\overline{R}_{nk}^{13}(u|\bar v|w+z+hc))^{-1})\,{}^{rl}(({}^{rl}\overline{R}_{mk}^{23}(v|\bar v|w+hc))^{-1})\cdot$$
$$\cdot\ T_n^{+14}(u|w+z)T_m^{+24}(v|w)T_k^{+34}(\bar v)\overline{R}_{mk}^{23}(v|\bar v|w)^{-1}\overline{R}_{nk}^{13}(u|\bar v|w+z)^{-1}|0\rangle\Big).$$

Indeed, $T_n^{+14}(u|w+z) = i_{w,z}T_n^{+14}(u|w+z)$ because $T^+(u)$ has only nonnegative powers of u, and the following equalities hold because of relation (2.65):

$$Y\Big(T_n^{+14}(u|z)T_m^{+24}(v),w\Big)\overline{R}_{mk}^{23}(v|\bar v|w+hc)^{-1}i_{w,z}\overline{R}_{nk}^{13}(u|\bar v|w+z+hc)^{-1}\cdot$$
$$\cdot\ T_k^{+34}(\bar v)|0\rangle$$
$$= T_n^{+14}(u|w+z)T_m^{+24}(v|w)T_m^{24}(v|w+hc/2)^{-1}i_{w,z}T_n^{14}(u|w+z+hc/2)^{-1}\cdot$$
$$\cdot\ \overline{R}_{mk}^{23}(v|\bar v|w+hc)^{-1}i_{w,z}\overline{R}_{nk}^{13}(u|\bar v|w+z+hc)^{-1}T_k^{+34}(\bar v)|0\rangle$$

$$
\begin{aligned}
&= T_n^{+14}(u|w+z)T_m^{+24}(v|w)T_m^{24}(v|w+hc/2)^{-1}\overline{R}_{mk}^{23}(v|\bar{v}|w+hc)^{-1}T_k^{+34}(\bar{v}) \cdot \\
&\quad \cdot\, i_{w,z}\overline{R}_{nk}^{13}(u|\bar{v}|w+z)^{-1}|0\rangle \\
&= T_n^{+14}(u|w+z)T_m^{+24}(v|w)T_k^{+34}(\bar{v})\overline{R}_{mk}^{23}(v|\bar{v}|w)^{-1}i_{w,z}\overline{R}_{nk}^{13}(u|\bar{v}|w+z)^{-1}|0\rangle.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&Y\Big(T_n^{+}(u)|0\rangle, w\Big)\overset{\mathcal{S}}{\underset{(l)}{}}Y\big(T_m^{+}(v)|0\rangle, w\Big)T_k^{+34}(\bar{v})|0\rangle \\
&= Res_z\bigg(z^{l}\,{}^{lr}\big(({}^{lr}\overline{R}_{nm}^{12}(u|v|z-hc))^{-1}\big)\overline{R}_{nm}^{12}(u|v|z) \cdot \\
&\quad \cdot\, i_{w,z}\Big({}^{rl}\big(({}^{rl}\overline{R}_{nk}^{13}(u|\bar{v}|w+z+hc))^{-1}\big)^{rl}\big(({}^{rl}\overline{R}_{mk}^{23}(v|\bar{v}|w+hc))^{-1}\big) \cdot \\
&\quad \cdot\, T_n^{+14}(u|w+z)T_m^{+24}(v|w)T_k^{+34}(\bar{v})\overline{R}_{mk}^{23}(v|\bar{v}|w)^{-1}\overline{R}_{nk}^{13}(u|\bar{v}|w+z)^{-1}|0\rangle\Big)\bigg).
\end{aligned}
$$

Thus, we have proved the following equality:

$$
Y(T_n^{+}(u)|0\rangle, w)\overset{\mathcal{S}}{\underset{(l)}{}}Y(T_m^{+}(v)|0\rangle, w) = Y(T_n^{+}(u)\overset{\mathcal{S}}{\underset{(l)}{}}T_m^{+}(v)|0\rangle, w).
$$

□

Proposition 2.4.12. *The quantum vertex algebra defined in Theorem 2.3.10 satisfies the quantum Borcherds Identities* (2.107)*.*

Proof. Using the computations in the proof of Theorem 2.3.10, the quantum n-products are the following:

$$
\begin{aligned}
&\Big(Y(T_n^{+}(u)|0\rangle, w)\overset{\mathcal{S}}{\underset{(l)}{}}Y(T_m^{+}(v)|0\rangle, w)\Big)T_k^{+}(\bar{v})|0\rangle \\
&= Res_z\Big(Y(z)(1\otimes Y(w))\mathcal{S}^{34}(z-w)(T_n^{+14}(u)T_m^{+25}(v)T_k^{+36}(\bar{v})|0\rangle\otimes|0\rangle\otimes|0\rangle)i_{z,w}(z-w)^{l} \\
&\quad - Y(w)(1\otimes Y(z))(4\ 5)(T_n^{+14}(u)T_m^{+25}(v)T_k^{+36}(\bar{v})|0\rangle\otimes|0\rangle\otimes|0\rangle)i_{w,z}(z-w)^{l}\Big) \\
&= Res_z\Big(T_m^{+24}(v|w)T_n^{+14}(u|z)T_k^{+34}(\bar{v})|0\rangle(i_{z,w}-i_{w,z})(z-w)^{l}\Big).
\end{aligned}
$$

Therefore, by Lemma 1.1.2, one has

$$
Y(T_n^{+}(u)|0\rangle, w)\overset{\mathcal{S}}{\underset{(l)}{}}Y(T_m^{+}(v)|0\rangle, w) = 0.
$$

if $l \geq 0$, and, if $l < 0$, one has

$$
\begin{aligned}
&\Big(Y(T_n^{+}(u)|0\rangle, w)\overset{\mathcal{S}}{\underset{(l)}{}}Y(T_m^{+}(v)|0\rangle, w)\Big)T_k^{+}(\bar{v})|0\rangle \\
&= Res_z\left(T_m^{+24}(v|w)T_n^{+14}(u|z)T_k^{+34}(\bar{v})|0\rangle\frac{\partial_w^{-l-1}\delta(z,w)}{(-l-1)!}\right) \\
&= T_m^{+24}(v|w)Res_z\left(T_n^{+14}(u|z)\frac{\partial_w^{-l-1}\delta(z,w)}{(-l-1)!}\right)T_k^{+34}(\bar{v})|0\rangle \\
&= T_m^{+24}(v|w)\frac{\partial_w^{-l-1}T_n^{+14}(u|w)}{(-l-1)!}T_k^{+34}(\bar{v})|0\rangle.
\end{aligned}
$$

On the other side, one has

$$\begin{aligned}
T_n^+(u)|0\rangle_{(l)}^{\mathcal{S}} T_m^+(v)|0\rangle &= Res_z\Big(z^l Y(z)\mathcal{S}(z)\big(T_n^{+13}(u)T_m^{+24}(v)|0\rangle \otimes |0\rangle\big)\Big)\\
&= Res_z\Big(z^l \overline{R}_{nm}^{12}(u|v|z)T_n^{+13}(u|z)T_m^{+23}(v)\overline{R}_{nm}^{12}(u|v|z)^{-1}|0\rangle\Big)\\
&= Res_z\Big(z^l T_m^{+23}(v)T_n^{+13}(u|z)|0\rangle\Big)
\end{aligned}$$

where the last equality is due to relation (2.64). Note that

$$T_n^+(u)|0\rangle_{(l)}^{\mathcal{S}} T_m^+(v)|0\rangle = 0 \text{ for } l \geq 0$$

because $T_n^{+13}(u|z)$ has only non-negative powers of z.
Thus, we only need to study the case of $l < 0$:

$$\begin{aligned}
Y(T_n^+(u)|0\rangle_{(l)}^{\mathcal{S}} T_m^+(v)|0\rangle, w) &= Y\Big(Res_z(z^l T_m^{+23}(v)T_n^{+13}(u|z)|0\rangle), w\Big)\\
&= Res_z z^l Y(T_m^{+23}(v)T_n^{+13}(u|z)|0\rangle, w)\\
&= Res_z(z^l T_m^{+23}(v|w)T_n^{+13}(u|w+z)|0\rangle)\\
&= T_m^{+23}(v|w)Res_z(z^l T_n^{+13}(u|w+z))|0\rangle\\
&= T_m^{+23}(v|w)\frac{\partial_w^{-l-1}T_n^{+13}(u|w)}{(-l-1)!}|0\rangle.
\end{aligned}$$

Comparing the obtained results, it follows that

$$T_n^+(u|w)|0\rangle_{(l)}^{\mathcal{S}} T_m^+(v|w)|0\rangle = Y(T_n^+(u)|0\rangle_{(l)}^{\mathcal{S}} T_m^+(v)|0\rangle, w).$$

□

The examples defined in Theorems 2.3.8 and 2.3.10 are not isolated cases. Indeed, as shown in the following theorem, the quantum Borcherds Identities hold for any braided vertex algebra which satisfies the Associativity Relation (2.27).

Theorem 2.4.13. *A braided topologically free $\mathbb{K}[[h]]$-module satisfies the $\mathcal{S}$-locality* (2.10) *(i.e it is a braided vertex algebra) and the Associativity Relation* (2.27) *if and only if the quantum Borcherds Identities* (2.107) *hold. In particular the quantum Borcherds Identities* (2.107) *hold in any quantum vertex algebra.*

Lemma 2.4.14. *In a braided topologically free $\mathbb{K}[[h]]$-module satisfying $Y\mathcal{S} = Y^{op}$, the Associativity Relation* (2.27) *holds if and only if, for any $a, b, c \in V$ and $M \in \mathbb{Z}_+$, there exists $L' \geq 0$ such that*

$$\begin{aligned}
&(z+w)^{L'} i_{z,w} Y(z+w)(1 \otimes Y(w))(\mathcal{S}(z)(a \otimes b) \otimes c)\\
&\qquad = (z+w)^{L'} Y(w)(Y^{op}(z) \otimes 1)(a \otimes b \otimes c) \ mod\ h^M.
\end{aligned} \tag{2.113}$$

Proof. By definition of $\mathcal{S}$, there exist $s \in \mathbb{Z}_+$, $K \in \mathbb{Z}$, $a_i, b_i \in V$ and $f_{ik} \in \mathbb{K}$ for $i = 1, \ldots, s$ and $k \geq K$ such that $\mathcal{S}(z)(a \otimes b) = \sum_{i=1}^{s} \sum_{k \geq K} a_i \otimes b_i f_{ik} z^k$ mod h^M.

If the Associativity Relation (2.27) holds, for any $M \in \mathbb{Z}_+$ and $i = 1, \dots, s$, there exists $L_i \geq 0$ such that, for any $c \in V$,

$$\begin{aligned}(z+w)^{L_i} i_{z,w} Y(z+w)\big(1 \otimes Y(w)\big)(a_i \otimes b_i \otimes c) \\ = (z+w)^{L_i} Y(w)\big(Y(z) \otimes 1\big)(a_i \otimes b_i \otimes c) \bmod h^M.\end{aligned} \tag{2.114}$$

In particular, for $L' \geq max\{L_i\}_{i=1,\dots,s}$ one has

$$\begin{aligned}(z+w)^{L'} i_{z,w} Y(z+w)\big(1 \otimes Y(w)\big)(a_i \otimes b_i \otimes c) \\ = (z+w)^{L'} Y(w)\big(Y(z) \otimes 1\big)(a_i \otimes b_i \otimes c) \bmod h^M.\end{aligned} \tag{2.115}$$

Since the left hand side of equation (2.115) lies in $V_h((z))((w))$ and the right hand side lies in $V_h((w))((z))$, one can multiply both sides by $f_{i_k} z^k$ and sum on $i = 1, \dots, s$ and $k \geq K$ obtaining

$$\begin{aligned}&\sum_{i=1}^{s} \sum_{k \geq K} (z+w)^{L'} i_{z,w} Y(z+w)\big(1 \otimes Y(w)\big)(a_i \otimes b_i f_{i_k} z^k \otimes c) \\ &\quad = \sum_{i=1}^{s} \sum_{k \geq K} (z+w)^{L'} Y(w)\big(Y(z) \otimes 1\big)(a_i \otimes b_i f_{i_k} z^k \otimes c) \\ &\quad = \sum_{i=1}^{s} (z+w)^{L'} Y(w)\big(Y(z) \otimes 1\big)\big(\mathcal{S}(z)(a \otimes b) \otimes c\big) \bmod h^M.\end{aligned}$$

Then, as $Y\mathcal{S} = Y^{op}$, one has

$$\begin{aligned}(z+w)^{L'} i_{z,w} Y(z+w)\big(1 \otimes Y(w)\big)\big(\mathcal{S}(z)(a \otimes b) \otimes c\big) \\ = (z+w)^{L'} Y(w)\big(Y^{op}(z) \otimes 1\big)(a \otimes b \otimes c) \bmod h^M.\end{aligned}$$

The converse follows in a similar way because, for suitable $a'_j, b'_j \in V$ and $f'_{j_k} \in \mathbb{K}$, one has

$$\mathcal{S}^{-1}(z)(a \otimes b) = \sum_{j=1}^{s'} \sum_{k \geq K'} a'_j \otimes b'_j f'_{j_k} z^k \bmod h^k.$$

Thus, one can do similar operations with equation (2.113). □

We can now prove Theorem 2.4.13.

Proof. Let $(V, |0\rangle, Y, T, \mathcal{S})$ be a braided vertex algebra which satisfies the Associativity Relation (2.27). By $\mathcal{S}$-locality (2.10), for any $M \in \mathbb{Z}_+$, there exists $N \geq 0$, such that

$$\begin{aligned}(z-w)^N Y(z)\big(1 \otimes Y(w)\big)\big(\mathcal{S}(z-w)(a \otimes b) \otimes c\big) \\ = (z-w)^N Y(w)\big(1 \otimes Y(z)\big)(b \otimes a \otimes c) \quad \bmod h^M.\end{aligned}$$

By the Associativity Relation (2.27) and Lemma 2.4.14, for any $a, b, c \in V$ and $M \in \mathbb{Z}_+$, there exists $L' \geq 0$ such that

$$(z+w)^{L'} i_{z,w} Y(z+w)\big(1 \otimes Y(w)\big)\big(\mathcal{S}(z)(a \otimes b) \otimes c\big)$$

$$= (z+w)^{L'} Y(w)\big(Y^{op}(z)\otimes 1\big)(a\otimes b\otimes c) \bmod h^M.$$

Therefore, by Lemma 1.1.8, the following equation holds:

$$\begin{aligned}
&i_{z,w}\delta(x,z-w)Y(z)\big(1\otimes Y(w)\big)\big(\mathcal{S}(z-w)(a\otimes b)\otimes c\big)\\
&\quad -i_{w,z}\delta(x,z-w)Y(w)\big(1\otimes Y(z)\big)(b\otimes a\otimes c)\\
&\qquad = i_{z,x}\delta(w,z-x)Y(w)\big(Y^{op}(x)\otimes 1\big)(a\otimes b\otimes c) \bmod h^M.
\end{aligned}$$

Since V is a topologically free $\mathbb{K}[[h]]$-module, it is separated (cf. Section 2.1). It follows that

$$\begin{aligned}
&i_{z,w}\delta(x,z-w)Y(z)\big(1\otimes Y(w)\big)\big(\mathcal{S}(z-w)(a\otimes b)\otimes c\big)\\
&\quad -i_{w,z}\delta(x,z-w)Y(w)\big(1\otimes Y(z)\big)(b\otimes a\otimes c)\\
&\qquad = i_{z,x}\delta(w,z-x)Y(w)\big(Y^{op}(x)\otimes 1\big)(a\otimes b\otimes c).
\end{aligned}\tag{2.116}$$

Multiplying equation (2.116) by x^n and taking the residue Res_x, the left hand side is

$$\begin{aligned}
&Y(z)\big(1\otimes Y(w)\big)\big(i_{z,w}\mathcal{S}(z-w)(a\otimes b)\otimes c\big)i_{z,w}(z-w)^n\\
&\quad - Y(w)\big(1\otimes Y(z)\big)(b\otimes a\otimes c)i_{w,z}(z-w)^n
\end{aligned}$$

and the right hand side is

$$\begin{aligned}
&Res_x\Big(x^n i_{z,x}\delta(w,z-x)Y(w)\big(Y^{op}(x)\otimes 1\big)(a\otimes b\otimes c)\Big)\\
&= Res_x\Big(x^n e^{-x\partial_z}\delta(w,z)Y(w)\big(Y^{op}(x)\otimes 1\big)(a\otimes b\otimes c)\Big)\\
&= Res_x\Big(x^n e^{-x\partial_z}\delta(z,w)Y(w)\big(Y^{op}(x)\otimes 1\big)(a\otimes b\otimes c)\Big)\\
&= \sum_{l\geq 0}\frac{(-1)^l\partial_z^l\delta(z,w)}{l!}Y(w)\big(a^{\mathcal{S}}_{(n+l)}b\otimes c\big)\\
&= \sum_{l\geq 0}Y(w)\big(a^{\mathcal{S}}_{(n+l)}b\otimes c\big)\frac{\partial_w^l\delta(z,w)}{l!}.
\end{aligned}$$

Therefore, one has

$$\begin{aligned}
&Y(z)\big(1\otimes Y(w)\big)\big(i_{z,w}\mathcal{S}(z-w)(a\otimes b)\otimes c\big)i_{z,w}(z-w)^n\\
&\quad -Y(w)\big(1\otimes Y(z)\big)(b\otimes a\otimes c)i_{w,z}(z-w)^n\\
&\qquad = \sum_{l\geq 0}Y(w)\big(a^{\mathcal{S}}_{(n+l)}b\otimes c\big)\frac{\partial_w^l\delta(z,w)}{l!}.
\end{aligned}$$

On the other hand if $(V,|0\rangle,Y,T,\mathcal{S})$ is a braided topologically free $\mathbb{K}[[h]]$-module satisfying the quantum Borcherds Identities (2.107), multiplying both sides of the quantum n-Borcherds Identity for by x^{-n-1} and summing over $n\in\mathbb{Z}$, the left hand side becomes

$$\begin{aligned}
&i_{z,w}\delta(x,z-w)Y(z)\big(1\otimes Y(w)\big)\big(\mathcal{S}(z-w)(a\otimes b)\otimes c\big)\\
&\quad -i_{w,z}\delta(x,z-w)Y(w)\big(1\otimes Y(z)\big)(b\otimes a\otimes c)
\end{aligned}$$

and the right hand side becomes

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}} x^{-n-1} \sum_{l\geq 0} Y(w)\big(a^{\mathcal{S}}_{(n+l)}b \otimes c\big)\frac{\partial_w^l \delta(z,w)}{l!} \\
&= \sum_{l\geq 0} Y(w)\Big(\sum_{n\in\mathbb{Z}} a^{\mathcal{S}}_{(n+l)}b\ x^{-n-l-1} \otimes c\Big)\frac{x^l\partial_w^l \delta(z,w)}{l!} \\
&= \sum_{l\geq 0} Y\Big(Y^{op}(x)(a\otimes b), w\Big)c\frac{x^l\partial_w^l \delta(z,w)}{l!} \\
&= i_{z,x}\delta(w,z-x)Y(w)\big(Y^{op}(x)\otimes 1\big)(a\otimes b\otimes c).
\end{aligned}$$

By Lemma 1.1.8, since $Y(z)(a\otimes c) \in V_h((z))$ and $Y(x)(b\otimes a) \in V_h((x))$, the Associativity Relation (2.27) and the $\mathcal{S}$-locality on any vector follow. In particular, it follows the $\mathcal{S}$-locality on the vacuum. Thus $Y\mathcal{S} = Y^{op}$, by Lemma 2.2.7, and the $\mathcal{S}$-locality follows by Theorem 2.4.5. □

Note that, in the proof of Theorem 2.4.13, we have proved the equivalence of equation (2.116) and the Borcherds Identities (2.107). Therefore, one has the following corollary:

Corollary 2.4.15. *The Borcherds Identities* (2.107) *are equivalent to the equation* (2.116).

Moreover, switching in equation (2.116), the variables a and b, the parameters z and w and renaming x as $-x$, the left hand side becomes

$$\begin{aligned}
&i_{w,z}\delta(-x,w-z)Y(w)\big(1\otimes Y(z)\big)\big(\mathcal{S}(w-z)(b\otimes a)\otimes c\big) \\
&\quad - i_{z,w}\delta(-x,w-z)Y(z)\big(1\otimes Y(w)\big)(a\otimes b\otimes c) \\
&= i_{z,w}\delta(x,z-w)Y(z)\big(1\otimes Y(w)\big)(a\otimes b\otimes c) \\
&\quad - i_{w,z}\delta(x,z-w)Y(w)\big(1\otimes Y(z)\big)\big(\mathcal{S}(w-z)(b\otimes a)\otimes c\big)
\end{aligned}$$

and the right hand side becomes

$$\begin{aligned}
&i_{w,x}\delta(z,w+x)Y(z)\big(Y^{op}(-x)\otimes 1\big)(b\otimes a\otimes c) \\
&= i_{z,x}\delta(z-x,w)Y(z)\big(e^{-xT}Y(x)\otimes 1\big)(a\otimes b\otimes c) \\
&= i_{z,x}\delta(w,z-x)Y(z-x)\big(Y(x)\otimes 1\big)(a\otimes b\otimes c) \\
&= i_{z,x}\delta(w,z-x)Y(w)\big(Y(x)\otimes 1\big)(a\otimes b\otimes c)
\end{aligned}$$

where we used the definition of Y^{op} and the property $\partial_z Y(z) = Y(z)(T\otimes 1)$ (cf. Corollary 2.2.8). We have therefore proved the following lemma:

Lemma 2.4.16. *Equation* (2.116) *is equivalent to the following equation:*

$$\begin{aligned}
&i_{z,w}\delta(x,z-w)Y(z)\big(1\otimes Y(w)\big)(a\otimes b\otimes c) \\
&\qquad -i_{w,z}\delta(x,z-w)Y(w)\big(1\otimes Y(z)\big)\big(\mathcal{S}(w-z)(b\otimes a)\otimes c\big) \\
&\qquad\qquad = i_{z,x}\delta(w,z-x)Y(w)\big(Y(x)\otimes 1\big)(a\otimes b\otimes c).
\end{aligned}\tag{2.117}$$

The equation (2.117) first appeared in [L10].

Combining the previous results, we have proved the following theorem:

Theorem 2.4.17. *Let* $(V, |0\rangle, Y, T, \mathcal{S})$ *be a braided topologically free* $\mathbb{K}[[h]]$*-module. The following statements are equivalent:*

1. V *is a braided vertex algebra which satisfies the Associativity Relation* (2.27)*;*
2. *the* $\mathcal{S}$*-Jacobi Identity* (2.117) *holds;*
3. Y *satisfies the Associativity Relation* (2.27) *and the equation* $Y\mathcal{S} = Y^{op}$*;*
4. *the quantum Borcherds Identities* (2.107) *hold;*
5. *the quantum* n*-products Identities* (2.108) *and the* $\mathcal{S}$*-locality hold.*

Chapter 3

Drinfeld's notes on universal triangular R-matrices

In this chapter we present the Drinfeld's notes on existence and uniqueness of universal triangular R-matrices in every detail. In Section 3.1 we review the concepts of algebra, coalgebra, bialgebra, Hopf algebra, topological Hopf algebra and Lie bialgebra. The main body of the notes is in Sections 3.2 and 3.3 in which we deal with the triangular case and in Section 3.5 where we slightly modify the previous arguments to be ready to deal with the pseudotriangular case. In Section 3.4 we give some considerations on Section 3.3. The main theorem proved by V. G. Drinfeld is Theorem 3.3.11 at the end of Section 3.3. As a tribute to V. G. Drinfeld, we name Sections 3.2, 3.3 and 3.5 with the names he gives to these parts in his notes "Question and Answer 46", "Question and Answer 47" and "Question and Answer 47' " and we keep his organization of the sections with a question followed by its answer.

3.1 Hopf algebras

In this section we follow [Kas, Chapter III, Chapter XVI]. Let $\mathbb{K}$ be a field.

Definition 3.1.1. An *algebra* is a triple (A, μ, η), where A is a $\mathbb{K}$-vector space and $\mu : A \otimes A \to A$ and $\eta : \mathbb{K} \to A$ are $\mathbb{K}$-linear maps such that the following diagrams commute:

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\mu\otimes 1} & A \otimes A \\ \Big\downarrow{\scriptstyle 1\otimes\mu} & & \Big\downarrow{\scriptstyle \mu} \\ A \otimes A & \xrightarrow{\quad\mu\quad} & A \end{array} \tag{3.1}$$

and

$$\begin{array}{ccccc} \mathbb{K}\otimes A & \xrightarrow{\eta\otimes 1} & A\otimes A & \xleftarrow{1\otimes\eta} & A\otimes\mathbb{K} \\ & \searrow^{\cong} & \downarrow{\scriptstyle\mu} & {}^{\cong}\swarrow & \\ & & A & & \end{array} \tag{3.2}$$

The map μ is called *product* and the map η is called *unit*.
A homomorphism of algebras $f : (A, \mu, \eta) \to (A', \mu', \eta')$ is a linear map from A to A' such that

$$\mu' \circ (f \otimes f) = f \circ \mu \text{ and } f \circ \eta = \eta'.$$

Definition 3.1.2. A *coalgebra* is a triple (C, Δ, ε), where C is a $\mathbb{K}$-vector space and $\Delta : C \to C \otimes C$ and $\eta : C \to \mathbb{K}$ are $\mathbb{K}$-linear maps such that the following diagrams commute:

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C\otimes C \\ \downarrow{\scriptstyle\Delta} & & \downarrow{\scriptstyle 1\otimes\Delta} \\ C\otimes C & \xrightarrow{\Delta\otimes 1} & C\otimes C\otimes C \end{array} \tag{3.3}$$

and

$$\begin{array}{ccccc} \mathbb{K}\otimes C & \xleftarrow{\varepsilon\otimes 1} & C\otimes C & \xrightarrow{1\otimes\varepsilon} & C\otimes\mathbb{K} \\ & \nwarrow_{\cong} & \uparrow{\scriptstyle\Delta} & {}_{\cong}\nearrow & \\ & & C & & \end{array} \tag{3.4}$$

The map Δ is called *coproduct* and the map ε is called *counit*.
We say that a coalgebra is cocommutative if the following triangle commutes

$$\begin{array}{ccc} & C & \\ {}^{\Delta}\swarrow & & \searrow^{\Delta} \\ C\otimes C & \xrightarrow{\tau_{C,C}} & C\otimes C \end{array} \tag{3.5}$$

where $\tau_{C,C} : C \otimes C \to C \otimes C$ is the linear map defined as $\tau_{C,C}(c \otimes c') = c' \otimes c$ for any $c, c' \in C$.
A homomorphism of coalgebras $f : (C, \Delta, \varepsilon) \to (C', \Delta', \varepsilon')$ is a linear map from C to C' such that

$$(f \otimes f) \circ \Delta = \Delta' \circ f \text{ and } \varepsilon = \varepsilon' \circ f.$$

In the following we will denote $\tau_{C,C}$ also as (1 2) (this notation will be useful when we won't have only swaps).

For any coalgebra (C, Δ, ε), we denote with Δ^{op} the opposite coproduct defined as

$$\Delta^{op} = (1\ 2)\Delta.$$

In particular, note that a coalgebra is cocommutative if $\Delta^{op} = \Delta$.

Theorem 3.1.1. *Let H be a vector space equipped simultaneously with an algebra structure (H, μ, η) and a coalgebra structure (H, Δ, ε). The following statements are equivalent:*

1. *the maps μ and η are homomorphisms of coalgebras;*

2. *the maps Δ and ε are homomorphisms of algebras.*

Definition 3.1.3. A *bialgebra* is a quintuple $(H, \mu, \eta, \Delta, \varepsilon)$ where (H, μ, η) is an algebra, (H, Δ, ε) is a coalgebra verifying the equivalent conditions of Theorem 3.1.1. A homomorphism of bialgebras is a homomorphism for the underlying structures of algebra and coalgebra.

Definition 3.1.4. A *Hopf algebra* $(H, \mu, \eta, \Delta, \varepsilon, S)$ is a bialgebra $(H, \mu, \eta, \Delta, \varepsilon)$ endowed with a $\mathbb{K}$-linear map $S : H \to H$ such that

$$\mu \circ (S \otimes 1) \circ \Delta = \mu \circ (1 \otimes S) \circ \Delta = \eta \circ \varepsilon. \tag{3.6}$$

The map S is called *antipode*.
A homomorphism of Hopf algebras is a homomorphism of bialgebras commuting with the antipodes.

Theorem 3.1.2. *Let $(H, \mu, \eta, \Delta, \varepsilon, S)$ be a Hopf algebra. The antipode S satisfies*

$$S(xy) = S(y)S(x) \text{ and } S(1) = 1$$

for any $x, y \in H$ and

$$(S \otimes S) \circ \Delta = \Delta^{op} \circ S \text{ and } \varepsilon \circ S = \varepsilon.$$

Example 3.1.1. Let $\mathfrak{g}$ be a Lie algebra over the field $\mathbb{K}$. The universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is a Hopf algebra with the following maps: $\Delta : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ and $\varepsilon : \mathcal{U}(\mathfrak{g}) \to \mathbb{K}$ are the homomorphisms of algebras defined on any $x \in \mathfrak{g}$ as

$$\Delta(x) = x \otimes 1 + 1 \otimes x \text{ and } \varepsilon(x) = 0.$$

The antipode $S : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ is defined on any $x \in \mathfrak{g}$ as

$$S(x) = -x$$

and then extended on $\mathcal{U}(\mathfrak{g})$ using Theorem 3.1.2.

Remark 3.1.3. We point out that the Hopf algebra of the previous example is cocommutative.

Similarly, one can define a *topological algebra*, *topological coalgebra*, *topological bialgebra* and *topological Hopf algebra* replacing, in the previous definitions, the underlying spaces with topologically free $\mathbb{K}[[h]]$-modules, the field $\mathbb{K}$ with the ring $\mathbb{K}[[h]]$ and all the tensor products with h-adic completed tensor products (cf. Section 2.1).

Definition 3.1.5. A topological Hopf algebra H is said to be a deformation of the Hopf algebra H' if $H/hH \cong H'$ as Hopf algebras.

Definition 3.1.6. A Lie bialgebra $(\mathfrak{g}, [\ ,\ ], \delta)$ over the field $\mathbb{K}$ is a triple which satisfies:

1. $(\mathfrak{g}, [\ ,\ ])$ is a Lie algebra;
2. δ is a linear map $\delta : \mathfrak{g} \to \Lambda^2\mathfrak{g}$ such that the following *coJacobi Identity* holds:
$$(\delta \wedge 1) \circ \delta = 0; \tag{3.7}$$
3. δ satisfies the 2-cocycle condition
$$\delta([x, y]) = x.\delta(y) - y.\delta(x) \tag{3.8}$$
for any $x, y \in \mathfrak{g}$.

Here we denote with $\delta \wedge 1$ the map in $\mathrm{Hom}\left(\Lambda^2\mathfrak{g}, \Lambda^3\mathfrak{g}\right)$ defined as $(\delta \wedge 1)(a \wedge b) = \delta(a) \wedge b - \delta(b) \wedge a$. And we denote with "." the adjoint action of $\mathfrak{g}$ on $\Lambda^2\mathfrak{g}$:

$$x.(x_1 \wedge x_2) = [x, x_1] \wedge x_2 + x_1 \wedge [x, x_2].$$

Definition 3.1.7. Let H be a topological Hopf algebra. We say that H is a quantization of a Lie bialgebra $(\mathfrak{g}, [\ ,\ ], \delta)$ over $\mathbb{K}$ if the following conditions hold:

1. $H/hH \cong \mathcal{U}(\mathfrak{g})$ as Hopf algebras over $\mathbb{K}$ (i.e. H is a deformation of the Hopf algebra $\mathcal{U}(\mathfrak{g})$);
2. $\delta(x) \equiv \frac{\Delta(x) - \Delta^{op}(x)}{h} \mod h$.

3.2 Question and Answer 46

Let $(\mathfrak{g}, [\ ,\ ], \delta)$ be a complex Lie bialgebra and let H be a quantization of $(\mathfrak{g}, [\ ,\ ], \delta)$ (cf. Section 3.1).

In this section we will give an answer to the following question:
where do the obstructions to the existence and uniqueness of an element $\Sigma \in H\widehat{\otimes}H$ which satisfies the following relations:

(1) $\Sigma \equiv 1 \mod h$;

(2) $\Sigma\Sigma^{21} = 1$;

(3) $(\Delta \otimes 1)(\Sigma)\Sigma^{12} = (1 \otimes \Delta)(\Sigma)\Sigma^{23}$;

(4) $\Delta^{op}(a) = \Sigma^{-1}\Delta(a)\Sigma$;

lie?

We recall that an element $\Sigma \in H\widehat{\otimes}H$ which satisfies Conditions (1) - (4) is called a *coboundary structure* of H.

In the following, for any $A_1 \otimes \cdots \otimes A_n$ in $\mathcal{U}(\mathfrak{g})^{\otimes n}$ and σ in the symmetric group S_n, we denote $\sigma(A_1 \otimes \cdots \otimes A_n) = A_{\sigma^{-1}(1)} \otimes \cdots \otimes A_{\sigma^{-1}(n)}$. Since $H^{\widehat{\otimes} n} \cong \mathcal{U}(\mathfrak{g})^{\otimes n}[[h]]$, we will often think of any element $B \in H^{\widehat{\otimes} n}$ as $B = \sum_{i \geq 0} B_i h^i$ where $B_i \in \mathcal{U}(\mathfrak{g})^{\otimes n}$. Moreover, for any $a \in \mathcal{U}(\mathfrak{g})$, we write $\Delta(a) = \sum_{i \geq 0} \Delta_i(a) h^i$ where $\Delta_i : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$. Similarly we write $\Delta^{op}(a) = \sum_{i \geq 0} \Delta_i^{op}(a) h^i$ where $\Delta_i^{op} : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ is defined as $\Delta_i^{op}(a) = (1\ 2)\Delta_i(a)$. In particular, since $H/hH \cong \mathcal{U}(\mathfrak{g})$ as a Hopf algebra, Δ_0 is the coproduct of $\mathcal{U}(\mathfrak{g})$ and $\Delta_0^{op} = \Delta_0$ because of the cocommutativity of the coproduct of $\mathcal{U}(\mathfrak{g})$ (cf. Section 3.1).

Let n be a positive integer and Σ be an element of $H\widehat{\otimes}H \cong (\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}))[[h]]$ (cf. Section 2.1) satisfying:

(i) $\Sigma \equiv 1 \bmod h$;

(ii) $\Sigma\Sigma^{21} \equiv 1 \bmod h^n$;

(iii) $(\Delta \otimes 1)(\Sigma)\Sigma^{12} \equiv (1 \otimes \Delta)(\Sigma)\Sigma^{23} \bmod h^n$;

(iv) $\Delta^{op}(a) \equiv \Sigma^{-1}\Delta(a)\Sigma \bmod h^n$.

Conditions (ii) - (iv) are satisfied for $n = 1$ because $\Sigma \equiv 1 \bmod h$ and Δ_0 is the coproduct of $\mathcal{U}(\mathfrak{g})$. We want to modify Σ without changing it mod h^n in order to have Conditions (ii) - (iv) satisfied mod h^{n+1}.
First, let us fix Condition (ii). We are looking for an element $f \in \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ such that

$$(\Sigma + f\ h^n)(\Sigma + f\ h^n)^{21} \equiv 1 \bmod h^{n+1}. \tag{3.9}$$

Conditions (i) and (ii) imply

$$(\Sigma + f\ h^n)(\Sigma + f\ h^n)^{21} = 1 + \big((\Sigma\Sigma^{21})_n + f + f^{21}\big)h^n + O(h^{n+1}).$$

It follows that one can modify Σ as $\Sigma' = \Sigma + fh^n$ such that $\Sigma'\Sigma'^{21} \equiv 1 \bmod h^{n+1}$ if and only if $(\Sigma\Sigma^{21})_n$ is symmetric. In that case, we set $f = -\frac{1}{2}(\Sigma\Sigma^{21})_n$. The symmetry of $(\Sigma\Sigma^{21})_n$ is the content of the following proposition:

Proposition 3.2.1. $(\Sigma\Sigma^{21})_n$ *is symmetric.*

Lemma 3.2.2. *If* $\Sigma\Sigma^{21} \equiv 1$ *mod* h^n*, then*

$$\Sigma_l^{21} = \sum_{\substack{i_1 + \ldots + i_k = l \\ i_1, \ldots, i_k \geq 1}} (-1)^k \Sigma_{i_1} \cdots \Sigma_{i_k}$$

for any l *such that* $1 \leq l < n$*. Recall also that* $\Sigma_0^{21} = 1$*.*

Proof. Since $\Sigma \equiv 1 \bmod h$, $\Sigma_0 = 1$ from which $\Sigma_0^{21} = 1$. If $n = 1$ we have no more information about Σ^{21}. So let us suppose $n \geq 2$ and let us proceed by induction on l. If $l = 1$, by $\Sigma\Sigma^{21} \equiv 1 \bmod h^2$ one has $\Sigma_1^{21} = -\Sigma_1$. We must show it is true for l if it is true for any $\tilde{l} < l$. As $l < n$ one has $\Sigma\Sigma^{21} \equiv 1 \bmod h^{l+1}$ from which $\Sigma_l + \Sigma_l^{21} + \sum_{j=1}^{l-1} \Sigma_j \Sigma_{l-j}^{21} = 0$. Then, by induction, one has

$$\begin{aligned}
\Sigma_l^{21} &= -\Sigma_l - \sum_{j=1}^{l-1} \Sigma_j \Sigma_{l-j}^{21} \\
&= -\Sigma_l + \sum_{j=1}^{l-1} \sum_{\substack{i_1+\ldots+i_k=l-j \\ i_1,\ldots,i_k \geq 1}} (-1)^{k+1} \Sigma_j \Sigma_{i_1} \cdots \Sigma_{i_k} \\
&= \sum_{\substack{i_1+\ldots+i_s=l \\ i_1,\ldots,i_s \geq 1}} (-1)^s \Sigma_{i_1} \cdots \Sigma_{i_s}.
\end{aligned}$$

□

Proof. By Lemma 3.2.2 one has

$$\begin{aligned}
(\Sigma\Sigma^{21})_n &= \Sigma_n + \Sigma_n^{21} + \sum_{j=1}^{n-1} \Sigma_j \Sigma_{n-j}^{21} \\
&= \Sigma_n + \Sigma_n^{21} + \sum_{j=1}^{n-1} \sum_{\substack{i_1+\ldots+i_k=n-j \\ i_1,\ldots,i_k \geq 1}} (-1)^k \Sigma_j \Sigma_{i_1} \cdots \Sigma_{i_k} \\
&= \Sigma_n^{21} - \sum_{\substack{i_1+\ldots+i_k=n \\ i_1,\ldots,i_k \geq 1}} (-1)^k \Sigma_{i_1} \cdots \Sigma_{i_k}.
\end{aligned}$$

Expanding $(\Sigma^{21}\Sigma)_n$ in the same way we obtain the same result. □

Therefore $\Sigma - h^n(\Sigma\Sigma^{21})_n/2$ satisfies Condition (ii) mod h^{n+1}. Let us redefine Σ as $\Sigma - h^n(\Sigma\Sigma^{21})_n/2$.

Remark 3.2.3. After this modification the freedom in choosing Σ reduces to elements of $\Lambda^2\mathcal{U}(\mathfrak{g})$

By Condition (iii) there exists an element δ in $\mathcal{U}(\mathfrak{g})^{\otimes 3}$ such that

$$(\Delta \otimes 1)(\Sigma)\Sigma^{12} \equiv (1 \otimes \Delta)(\Sigma)\Sigma^{23} + \delta h^n \bmod h^{n+1}.$$

We want to modify Σ as $\Sigma' = \Sigma + gh^n$ with $g \in \Lambda^2\mathcal{U}(\mathfrak{g})$ such that

$$(\Delta \otimes 1)(\Sigma')\Sigma'^{12} \equiv (1 \otimes \Delta)(\Sigma')\Sigma'^{23} \bmod h^{n+1}. \tag{3.10}$$

This is equivalent to the following equation on g

$$\delta = (1 \otimes \Delta_0)(g) + g^{23} - (\Delta_0 \otimes 1)(g) - g^{12} \tag{3.11}$$

which we want to solve. To prove the equivalence of equations (3.10) and (3.11) just note that

$$\begin{aligned}
&((\Delta \otimes 1)(\Sigma + gh^n))(\Sigma + gh^n)^{12} - ((1 \otimes \Delta)(\Sigma + gh^n))(\Sigma + gh^n)^{23} \\
&\equiv (\delta + (\Delta \otimes 1)(g) + g^{12} - (1 \otimes \Delta)(g) - g^{23})h^n \bmod h^{n+1}.
\end{aligned}$$

Lemma 3.2.4. *The following equation holds:*

$$(1\otimes\Delta_0)(g)+g^{23}-(\Delta_0\otimes 1)(g)-g^{12}=(1\ 3)\big((1\otimes\Delta_0)(g)+g^{23}-(\Delta_0\otimes 1)(g)-g^{12}\big).$$

Proof. Since $g\in\Lambda^2\mathcal{U}(\mathfrak{g})$ and Δ_0 is cocommutative, one has

$$\begin{aligned}
&(1\ 3)\big((1\otimes\Delta_0)(g)+g^{23}-(\Delta_0\otimes 1)(g)-g^{12}\big)\\
&=\big((\Delta_0^{op}\otimes 1)(g^{21})+g^{21}-(1\otimes\Delta_0^{op})(g^{21})-g^{32}\big)\\
&=\big(-(\Delta_0^{op}\otimes 1)(g)-g+(1\otimes\Delta_0^{op})(g)+g^{23}\big)\\
&=\big(-(\Delta_0\otimes 1)(g)-g+(1\otimes\Delta_0)(g)+g^{23}\big).
\end{aligned}$$

□

Therefore a necessary condition for this new modification is δ to satisfy the condition $\delta=(1\ 3)\delta$. This holds by the following proposition:

Proposition 3.2.5. *The element $\delta\in\mathcal{U}(\mathfrak{g})^{\otimes 3}$ satisfies the equation:*

$$\delta=(1\ 3)\delta. \tag{3.12}$$

Lemma 3.2.6. *The following equation holds:*

$$\Sigma^{12}(\Delta^{op}\otimes 1)(\Sigma)\equiv\Sigma^{23}(1\otimes\Delta^{op})(\Sigma)+\delta\ h^n\ \ mod\ h^{n+1}. \tag{3.13}$$

Proof. By Condition (i) one has

$$\begin{aligned}
&\Sigma^{12}(\Delta^{op}\otimes 1)(\Sigma)-\Sigma^{23}(1\otimes\Delta^{op})(\Sigma)\\
&=\Sigma^{12}(\Delta^{op}\otimes 1)\Big(1+\sum_{i\geq 1}\Sigma_i h^i\Big)-\Sigma^{23}(1\otimes\Delta^{op})\Big(1+\sum_{i\geq 1}\Sigma_i h^i\Big)\\
&=\Sigma^{12}+\sum_{i\geq 1}\Sigma^{12}(\Delta^{op}\otimes 1)(\Sigma_i)h^i-\Sigma^{23}-\sum_{i\geq 1}\Sigma^{23}(1\otimes\Delta^{op})(\Sigma_i)h^i.
\end{aligned}$$

Then one concludes using condition (iv) indeed

$$\begin{aligned}
&\Sigma^{12}+\sum_{i\geq 1}\Sigma^{12}(\Delta^{op}\otimes 1)(\Sigma_i)h^i-\Sigma^{23}-\sum_{i\geq 1}\Sigma^{23}(1\otimes\Delta^{op})(\Sigma_i)h^i\\
&=\Sigma^{12}+\sum_{i\geq 1}\big((\Delta\otimes 1)(\Sigma_i)\Sigma^{12}+O(h^n)\big)h^i\\
&\quad-\Sigma^{23}-\sum_{i\geq 1}\big((1\otimes\Delta)(\Sigma_i)\Sigma^{23}+O(h^n)\big)h^i\\
&=\Sigma^{12}+\sum_{i\geq 1}(\Delta\otimes 1)(\Sigma_i)\Sigma^{12}h^i\\
&\quad-\Sigma^{23}-\sum_{i\geq 1}(1\otimes\Delta)(\Sigma_i)\Sigma^{23}h^i+O(h^{n+1})\\
&=(\Delta\otimes 1)(\Sigma)\Sigma^{12}-(1\otimes\Delta)(\Sigma)\Sigma^{23}+O(h^{n+1})\\
&=\delta\ h^n+O(h^{n+1}).
\end{aligned}$$

□

Proof. Taking the inverse of equation (3.13) one has

$$(\Delta^{op}\otimes 1)(\Sigma^{-1})(\Sigma^{12})^{-1}\equiv(1\otimes\Delta^{op})(\Sigma^{-1})(\Sigma^{23})^{-1}-\delta\ h^n \bmod h^{n+1}$$

which is equivalent to

$$(\Delta^{op}\otimes 1)(\Sigma^{21})\Sigma^{21}\equiv(1\otimes\Delta^{op})(\Sigma^{21})\Sigma^{32}-\delta\ h^n \bmod h^{n+1} \tag{3.14}$$

because $\Sigma\Sigma^{21}\equiv 1 \bmod h^{n+1}$. Applying the permutation $(1\ 3)$ to both sides of equation (3.14), it follows

$$(1\otimes\Delta)(\Sigma)\Sigma^{23}\equiv(\Delta\otimes 1)(\Sigma)\Sigma^{12}-(1\ 3)\delta\ h^n \bmod h^{n+1}.$$

Therefore

$$\delta\equiv(1\ 3)\delta \bmod h$$

which implies $\delta=(1\ 3)\delta$ because $\delta\in\mathcal{U}(\mathfrak{g})^{\otimes 3}$. □

Instead of looking for an element $g\in\Lambda^2\mathcal{U}(\mathfrak{g})$ satisfying equation (3.11), we can look for an element $f\in\mathcal{U}(\mathfrak{g})^{\otimes 2}$ satisfying equation (3.11) and then define $g=\frac{f-f^{21}}{2}$ as stated in the following lemma:

Lemma 3.2.7. *If f is an element of $\mathcal{U}(\mathfrak{g})^{\otimes 2}$ satisfying equation* (3.11)*, then $\frac{f-f^{21}}{2}$ is an element of $\Lambda^2\mathcal{U}(\mathfrak{g})$ satisfying equation* (3.11)*.*

Proof.

$$\begin{aligned}
&(1\otimes\Delta_0)\Big(\frac{f-f^{21}}{2}\Big)+\Big(\frac{f-f^{21}}{2}\Big)^{23}-(\Delta_0\otimes 1)\Big(\frac{f-f^{21}}{2}\Big)-\Big(\frac{f-f^{21}}{2}\Big)^{12}\\
&=\frac{1}{2}\delta-\frac{1}{2}((1\otimes\Delta_0)(f^{21})+f^{32}-(\Delta_0\otimes 1)(f^{21})-f^{21})\\
&=\frac{1}{2}\delta-\frac{1}{2}(1\ 3)((\Delta_0^{op}\otimes 1)(f)+f^{12}-(1\otimes\Delta_0^{op})(f)-f^{23})\\
&=\frac{1}{2}\delta+\frac{1}{2}(1\ 3)((1\otimes\Delta_0)(f)+f^{23}-(\Delta_0\otimes 1)(f)-f^{12})\\
&=\frac{1}{2}\delta+\frac{1}{2}(1\ 3)\delta\\
&=\delta.
\end{aligned}$$

□

Let us consider the following cochain complex:

$$\mathbb{C}\xrightarrow{\ d\ }\mathcal{U}(\mathfrak{g})\xrightarrow{\ d\ }\mathcal{U}(\mathfrak{g})^{\otimes 2}\xrightarrow{\ d\ }\dots \tag{3.15}$$

where $d:\mathbb{C}\to\mathcal{U}(\mathfrak{g})$ is the zero map and $d:\mathcal{U}(\mathfrak{g})^{\otimes n}\to\mathcal{U}(\mathfrak{g})^{\otimes n+1}$, for $n>0$, is given by

$$\begin{aligned}
d(x_1\otimes\cdots\otimes x_n)=1\otimes x_1\otimes\cdots\otimes x_n+\sum_{i=1}^{n}(-1)^i x_1\otimes\cdots\otimes\Delta_0(x_i)\otimes\cdots\otimes x_n\\
+(-1)^{n+1}x_1\otimes\cdots\otimes x_n\otimes 1.
\end{aligned}$$

Lemma 3.2.8. *The map d satisfies: $d^2 = 0$ (i.e. it is a differential).*

Proof. Let $x_1, \dots, x_n \in \mathcal{U}(\mathfrak{g})$ with $n > 0$. One has

$$\begin{aligned}
d^2(x_1 \otimes \cdots \otimes x_n) &= 1 \otimes 1 \otimes x_1 \otimes \cdots \otimes x_n \\
&- \Delta_0(1) \otimes x_1 \otimes \cdots \otimes x_n \\
&+ \sum_{i=1}^{n} (-1)^{i+1} 1 \otimes x_1 \otimes \cdots \otimes \Delta_0(x_i) \otimes \cdots \otimes x_n \\
&+ (-1)^{n+2} 1 \otimes x_1 \otimes \cdots \otimes x_n \otimes 1 \\
&+ \sum_{i=1}^{n} (-1)^{i} 1 \otimes x_1 \otimes \cdots \otimes \Delta_0(x_i) \otimes \cdots \otimes x_n \\
&+ \sum_{i=1}^{n} \sum_{j=1}^{n+1} (-1)^{i+j} (1 \otimes \cdots \otimes \underset{j}{\Delta_0} \otimes \cdots \otimes 1) \cdot \\
&\quad \cdot (x_1 \otimes \cdots \otimes \Delta_0(x_i) \otimes \cdots \otimes x_n) \\
&+ \sum_{i=1}^{n} (-1)^{i+n+2} x_1 \otimes \cdots \otimes \Delta_0(x_i) \otimes \cdots \otimes x_n \otimes 1 \\
&+ (-1)^{n+1} 1 \otimes x_1 \otimes \cdots \otimes x_n \otimes 1 \\
&+ \sum_{i=1}^{n} (-1)^{i+n+1} x_1 \otimes \cdots \otimes \Delta_0(x_i) \otimes \cdots \otimes x_n \otimes 1 \\
&+ (-1)^{n+1+n+1} x_1 \otimes \cdots \otimes x_n \otimes \Delta_0(1) \\
&+ (-1)^{n+1+n+2} x_1 \otimes \cdots \otimes x_n \otimes 1 \otimes 1 \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n+1} (-1)^{i+j} (1 \otimes \cdots \otimes \underset{j}{\Delta_0} \otimes \cdots \otimes 1) \cdot \\
&\quad \cdot (x_1 \otimes \cdots \otimes \Delta_0(x_i) \otimes \cdots \otimes x_n) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{i-1} (-1)^{i+j} x_1 \otimes \cdots \otimes \Delta_0(x_j) \otimes \cdots \\
&\quad \cdots \otimes \Delta_0(x_i) \otimes \cdots \otimes x_n \\
&+ \sum_{i=1}^{n} (-1)^{2i} x_1 \otimes \cdots \otimes (\Delta_0 \otimes 1)\Delta_0(x_i) \otimes \cdots \otimes x_n) \\
&+ \sum_{i=1}^{n} (-1)^{2i+1} x_1 \otimes \cdots \otimes (1 \otimes \Delta_0)\Delta_0(x_i) \otimes \cdots \otimes x_n) \\
&+ \sum_{i=1}^{n} \sum_{j=i+1}^{n} (-1)^{i+j+1} x_1 \otimes \cdots \otimes \Delta_0(x_i) \otimes \cdots \\
&\quad \cdots \otimes \Delta_0(x_j) \otimes \cdots \otimes x_n \\
&= \sum_{i=2}^{n} \sum_{j=1}^{i-1} (-1)^{i+j} x_1 \otimes \cdots \otimes \Delta_0(x_j) \otimes \cdots \\
&\quad \cdots \otimes \Delta_0(x_i) \otimes \cdots \otimes x_n
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{n}\sum_{i=j+1}^{n}(-1)^{i+j+1}x_1\otimes\cdots\otimes\Delta_0(x_j)\otimes\cdots\\
&\cdots\otimes\Delta_0(x_i)\otimes\cdots\otimes x_n\\
=&\sum_{i=2}^{n}\sum_{j=1}^{i-1}(-1)^{i+j}x_1\otimes\cdots\otimes\Delta_0(x_j)\otimes\cdots\\
&\cdots\otimes\Delta_0(x_i)\otimes\cdots\otimes x_n\\
&+\sum_{i=2}^{n}\sum_{j=1}^{i-1}(-1)^{i+j+1}x_1\otimes\cdots\otimes\Delta_0(x_j)\otimes\cdots\\
&\cdots\otimes\Delta_0(x_i)\otimes\cdots\otimes x_n\\
=&\ 0
\end{aligned}
$$

from which $d^2 = 0$. □

Lemma 3.2.9. *The image of $g^{\otimes n}$ under the differential d is zero. In particular any element of $\Lambda^n\mathfrak{g}$ is a cocycle.*

Proof. It is a consequence of the definition of the coproduct Δ_0: $\Delta_0(x) = 1\otimes x + x\otimes 1$ for any $x \in \mathfrak{g}$. □

Proposition 3.2.10. *The cohomology of the cochain complex* (3.15) *is isomorphic to $\Lambda^\bullet\mathfrak{g}$.*

Proof. See Appendix A. □

Let us recall the following linear map:

$$
\begin{aligned}
Alt : \mathcal{U}(\mathfrak{g})^{\otimes n} &\to \mathcal{U}(\mathfrak{g})^{\otimes n}\\
x_1\otimes\cdots\otimes x_n &\mapsto \sum_{\sigma\in S_n} sgn(\sigma)\ \sigma(x_1\otimes\cdots\otimes x_n)
\end{aligned}
\tag{3.16}
$$

Remark 3.2.11. $Alt(x - sgn(\sigma)\sigma x) = 0$ for any $x \in \mathcal{U}(\mathfrak{g})^{\otimes n}$ and $\sigma \in S_n$.

Proof. One has the following equations

$$
\begin{aligned}
Alt(x - sgn(\sigma)\sigma x) &= Alt(x) - sgn(\sigma)Alt(\sigma x)\\
&= Alt(x) - sgn(\sigma)\sum_{\mu\in S_n} sgn(\mu\sigma^{-1})\mu\sigma^{-1}(\sigma x)\\
&= Alt(x) - sgn(\sigma)\sum_{\mu\in S_n} sgn(\mu)sgn(\sigma^{-1})\mu x\\
&= Alt(x) - \sum_{\mu\in S_n} sgn(\mu)\mu x\\
&= 0.
\end{aligned}
$$

□

Lemma 3.2.12. *Alt maps coboundaries in* 0 *and cocycles in* $\Lambda^\bullet\mathfrak{g}$.

Proof. Let $\theta \in \mathcal{U}(\mathfrak{g})^{\otimes n}$. Since Δ_0 is cocommutative, one has

$$\begin{aligned}
Alt(d(\theta)) &= Alt\Big(1 \otimes \theta + \sum_{i=1}^{n}(-1)^i(1 \otimes \cdots \otimes \underset{i}{\Delta_0} \otimes \cdots \otimes 1)(\psi) + (-1)^{n+1}\theta \otimes 1\Big) \\
&= Alt\Big(\Big(id - sgn\Big((n\ n+1)\cdots(1\ 2)\Big)(n\ n+1)\cdots(1\ 2)\Big)(1 \otimes \theta) \\
&\quad + \frac{1}{2}\sum_{i=1}^{n}(-1)^i\Big(id - sgn((i\ i+1))(i\ i+1)\Big)(1 \otimes \cdots \otimes \underset{i}{\Delta_0} \otimes \cdots \otimes 1)(\theta)\Big) \\
&= 0.
\end{aligned}$$

Therefore Alt maps coboundaries in 0. Moreover, by Lemma 3.2.9 and since the cohomology of the cochain complex (3.15) is $\Lambda^\bullet\mathfrak{g}$, by Proposition 3.2.10, the kernel of the map $d : \mathcal{U}(\mathfrak{g})^{\otimes n} \to \mathcal{U}(\mathfrak{g})^{\otimes(n+1)}$ is $ker\ d = \Lambda^n\mathfrak{g} \oplus d(\mathcal{U}(\mathfrak{g})^{\otimes n-1})$ for any $n \geq 0$ ($\mathcal{U}(\mathfrak{g})^{\otimes -1}$ is meant to be zero and $\mathcal{U}(\mathfrak{g})^{\otimes 0}$ is meant to be $\mathbb{C}$), from which $Alt(ker\ d) = Alt(\Lambda^n\mathfrak{g}) = \Lambda^n\mathfrak{g}$. □

Proposition 3.2.13. *δ is a 3-cocycle.*

Proof. Since $\Sigma \equiv 1 \bmod h$, one has

$$\begin{aligned}
&(\Delta \otimes 1 \otimes 1)(\Delta \otimes 1)(\Sigma)\delta^{123} \equiv \delta^{123} \bmod h, \\
&(1 \otimes \Delta \otimes 1)(\delta)\Sigma^{23} \equiv (1 \otimes \Delta \otimes 1)(\delta) \bmod h, \\
&(1 \otimes \Delta \otimes 1)(1 \otimes \Delta)(\Sigma)\delta^{234} \equiv \delta^{234} \bmod h.
\end{aligned}$$

It follows

$$\begin{aligned}
&h^n(\delta^{123} + (1 \otimes \Delta \otimes 1)(\delta) + \delta^{234}) \\
&\equiv h^n((\Delta \otimes 1 \otimes 1)(\Delta \otimes 1)(\Sigma)\delta^{123} + (1 \otimes \Delta \otimes 1)(\delta)\Sigma^{23} \\
&\quad + (1 \otimes \Delta \otimes 1)(1 \otimes \Delta)(\Sigma)\delta^{234}) \\
&\equiv (\Delta \otimes 1 \otimes 1)(\Delta \otimes 1)(\Sigma)(\Delta \otimes 1 \otimes 1)(\Sigma \otimes 1)\Sigma^{12} \\
&\quad - (\Delta \otimes 1 \otimes 1)(\Delta \otimes 1)(\Sigma)(1 \otimes \Delta \otimes 1)(\Sigma \otimes 1)\Sigma^{23} \\
&\quad + (1 \otimes \Delta \otimes 1)(\Delta \otimes 1)(\Sigma)(1 \otimes \Delta \otimes 1)(\Sigma \otimes 1)\Sigma^{23} \\
&\quad - (1 \otimes \Delta \otimes 1)(1 \otimes \Delta)(\Sigma)(1 \otimes \Delta \otimes 1)(1 \otimes \Sigma)\Sigma^{23} \\
&\quad + (1 \otimes \Delta \otimes 1)(1 \otimes \Delta)(\Sigma)(1 \otimes \Delta \otimes 1)(1 \otimes \Sigma)\Sigma^{23} \\
&\quad - (1 \otimes \Delta \otimes 1)(1 \otimes \Delta)(\Sigma)(1 \otimes 1 \otimes \Delta)(1 \otimes \Sigma)\Sigma^{34} \\
&= (\Delta \otimes 1 \otimes 1)(\Delta \otimes 1)(\Sigma)(\Delta \otimes 1 \otimes 1)(\Sigma \otimes 1)\Sigma^{12} \\
&\quad - (1 \otimes 1 \otimes \Delta)(1 \otimes \Delta)(\Sigma)(1 \otimes 1 \otimes \Delta)(1 \otimes \Sigma)\Sigma^{34} \bmod h^{n+1}
\end{aligned}$$

because Δ is coassociative and it is a homomorphism of algebras.
On the other hand

$$(\Delta \otimes 1 \otimes 1)(\Delta \otimes 1)(\Sigma)(\Delta \otimes 1 \otimes 1)(\Sigma \otimes 1)\Sigma^{12}$$

$$\begin{aligned}
&- (1 \otimes 1 \otimes \Delta)(1 \otimes \Delta)(\Sigma)(1 \otimes 1 \otimes \Delta)(1 \otimes \Sigma)\Sigma^{34} \\
&= (\Delta \otimes 1 \otimes 1)\big((\Delta \otimes 1)(\Sigma)\Sigma^{12}\big)\Sigma^{12} \\
&\quad - (\Delta \otimes \Delta)(\Sigma)\Sigma^{12}\Sigma^{34} \\
&\quad - (1 \otimes 1 \otimes \Delta)\big((1 \otimes \Delta)(\Sigma)\Sigma^{23}\big)\Sigma^{34} \\
&\quad + (\Delta \otimes \Delta)(\Sigma)\Sigma^{12}\Sigma^{34} \\
&= (\Delta \otimes 1 \otimes 1)\big((\Delta \otimes 1)(\Sigma)\Sigma^{12}\big)\Sigma^{12} \\
&\quad - (\Delta \otimes 1 \otimes 1)\big((1 \otimes \Delta)(\Sigma)\Sigma^{23}\big)\Sigma^{12} \\
&\quad - (1 \otimes 1 \otimes \Delta)\big((1 \otimes \Delta)(\Sigma)\Sigma^{23}\big)\Sigma^{34} \\
&\quad + (1 \otimes 1 \otimes \Delta)\big((\Delta \otimes 1)(\Sigma)\Sigma^{12}\big)\Sigma^{34} \\
&\equiv h^n\big((\Delta \otimes 1 \otimes 1)(\delta)\Sigma^{12} + \\
&\quad + (1 \otimes 1 \otimes \Delta)(\delta)\Sigma^{34}\big) \bmod h^{n+1}
\end{aligned}$$

because $\Sigma^{12}\Sigma^{34} = \Sigma^{34}\Sigma^{12}$, $\Sigma^{34} = (\Delta \otimes 1 \otimes 1)(\Sigma^{23})$, $\Sigma^{12} = (1 \otimes 1 \otimes \Delta)(\Sigma^{12})$, $(\Delta \otimes 1 \otimes 1)(\delta)\Sigma^{12} \equiv (\Delta \otimes 1 \otimes 1)(\delta) \bmod h$ and $(1 \otimes 1 \otimes \Delta)(\delta)\Sigma^{34} \equiv (1 \otimes 1 \otimes \Delta)(\delta) \bmod h$. Therefore one has

$$\delta^{123} + (1 \otimes \Delta \otimes 1)(\delta) + \delta^{234} \equiv (\Delta \otimes 1 \otimes 1)(\delta) + (1 \otimes 1 \otimes \Delta)(\delta) \bmod h$$

which implies

$$\delta^{123} + (1 \otimes \Delta_0 \otimes 1)(\delta) + \delta^{234} = (\Delta_0 \otimes 1 \otimes 1)(\delta) + (1 \otimes 1 \otimes \Delta_0)(\delta)$$

because $\delta \in \mathcal{U}(\mathfrak{g})^{\otimes 3}$. □

Lemma 3.2.14. *If $\omega \in \mathcal{U}(\mathfrak{g})^{\otimes m}$ is an m-cocycle of the cochain complex* (3.15) *satisfying $Alt(\omega) = 0$, then ω is a coboundary.*

Proof. Since ω is an m-cocycle, by Lemma 3.2.9 and since the cohomology of the cochain complex (3.15) is $\Lambda^\bullet\mathfrak{g}$, by Proposition 3.2.10, there exist $\psi \in \Lambda^m\mathfrak{g}$ and $\theta \in \mathcal{U}(\mathfrak{g})^{\otimes m-1}$ such that $\omega = \psi + d(\theta)$. Applying the map Alt to both sides, by Lemma 3.2.12, one has $Alt(\omega) = m!\psi$, from which $\psi = 0$. Thus ω is a coboundary. □

Corollary 3.2.15. *δ is a coboundary.*

Proof. By Proposition 3.2.5 and Remark 3.2.11 one has

$$Alt(\delta) = \frac{1}{2}Alt(\delta + (1\ 3)\delta) = \frac{1}{2}Alt\Big(\delta - sgn((1\ 3))(1\ 3)\delta\Big) = 0.$$

Since δ is a 3-cocycle, by Lemma 3.2.14, δ is a coboundary. □

Since δ is a coboundary, there exists an element $f \in \mathcal{U}(\mathfrak{g})^{\otimes 2}$ satisfying $\delta = df$, i.e.

$$\delta = f^{23} - (\Delta_0 \otimes 1)(f) + (1 \otimes \Delta_0)(f) - f^{12}. \tag{3.17}$$

Remark 3.2.16. The choice of $f \in \mathcal{U}(\mathfrak{g})^{\otimes 2}$ satisfying $\delta = df$ is unique up to an element in $\Lambda^2\mathfrak{g}$. In particular, after this modification, the freedom in choosing Σ is reduced to an element of $\Lambda^2\mathfrak{g}$.

Proof. Let f and $\tilde{f}$ be two elements in $\mathcal{U}(\mathfrak{g})^{\otimes 2}$ satisfying $d(f) = \delta = d(\tilde{f})$. By Lemma 3.2.12, $Alt(f - \tilde{f}) \in \Lambda^2\mathfrak{g}$ because $f - \tilde{f}$ is a cocycle. It follows that $\frac{f-f^{21}}{2} - \frac{\tilde{f}-\tilde{f}^{21}}{2} = \frac{Alt(f)}{2} - \frac{Alt(\tilde{f})}{2} \in \Lambda^2\mathfrak{g}$. □

By Remark 3.2.7, modifying Σ in $\Sigma + \frac{f-f^{21}}{2}h^n$, Σ satisfies $\Sigma\Sigma^{21} \equiv 1 \bmod h^{n+1}$ and $(\Delta \otimes 1)(\Sigma)\Sigma^{12} \equiv (1 \otimes \Delta)(\Sigma)\Sigma^{23} \bmod h^{n+1}$.

Now we need to modify Σ in $\Sigma' = \Sigma + \psi h^n$ satisfying $\Delta^{op}(a) \equiv \Sigma'^{-1}\Delta(a)\Sigma' \bmod h^{n+1}$. Let us set $\varphi : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ as

$$\varphi(a) \equiv \frac{\Delta^{op}(a) - \Sigma^{-1}\Delta(a)\Sigma}{h^n} \bmod h. \tag{3.18}$$

Remark 3.2.17. The equation $\Delta^{op}(a) \equiv \Sigma'^{-1}\Delta(a)\Sigma'$ is equivalent to the equation $[\Delta_0(a), \psi] = \varphi(a)$, indeed since $(\Sigma + \psi h^n)^{-1} \equiv \Sigma^{-1} - \psi h^n \bmod h^{n+1}$, one has

$$\begin{aligned}
&\frac{\Delta^{op}(a) - (\Sigma + \psi h^n)^{-1}\Delta(a)(\Sigma + \psi h^n)}{h^n} \\
&\equiv \frac{\Delta^{op}(a) - \Sigma^{-1}\Delta(a)\Sigma + (\psi\Delta(a) - \Delta(a)\psi)h^n}{h^n} \\
&\equiv \varphi(a) + (\psi\Delta(a) - \Delta(a)\psi) \bmod h.
\end{aligned}$$

Therefore we are looking for an element $\psi \in \Lambda^2\mathfrak{g}$ satisfying $[\Delta_0(a), \psi] = \varphi(a)$ for any $a \in \mathcal{U}(\mathfrak{g})$. As $[\Delta_0(a), \psi] \in \Lambda^2(\mathcal{U}(\mathfrak{g}))$ for any $a \in \mathcal{U}(\mathfrak{g})$, and, $[\Delta_0(a), \psi] \in \Lambda^2\mathfrak{g}$, for any $a \in \mathfrak{g}$, a necessary condition on ψ to exist is that $\varphi(\mathcal{U}(\mathfrak{g})) \subseteq \Lambda^2\mathcal{U}(\mathfrak{g})$ and $\varphi(\mathfrak{g}) \subseteq \Lambda^2\mathfrak{g}$. Let us check that these two conditions are satisfied.

Proposition 3.2.18. $\varphi(\mathcal{U}(\mathfrak{g})) \subseteq \Lambda^2\mathcal{U}(\mathfrak{g})$.

Proof. Since $\Sigma\Sigma^{21} \equiv 1 \bmod h^{n+1}$ and $\Sigma \equiv 1 \bmod h$, one has

$$\begin{aligned}
(1\ 2)\varphi(a) &\equiv \frac{\Delta(a) - (1\ 2)(\Sigma^{-1}\Delta(a)\Sigma)}{h^n} \\
&\equiv \frac{\Delta(a) - (1\ 2)(\Sigma^{21}\Delta(a)\Sigma)}{h^n} \\
&\equiv \frac{\Delta(a) - \Sigma\Delta^{op}(a)\Sigma^{21}}{h^n} \\
&\equiv \frac{\Delta(a) - \Sigma\Delta^{op}(a)\Sigma^{-1}}{h^n} \\
&\equiv \Sigma\frac{\Sigma^{-1}\Delta(a)\Sigma - \Delta^{op}(a)}{h^n}\Sigma^{-1} \\
&\equiv -\varphi(a) \bmod h.
\end{aligned}$$

Therefore $(1\ 2)\varphi(a) = -\varphi(a)$ because $\varphi(a) \in \mathcal{U}(\mathfrak{g})^{\otimes 2}$. □

Lemma 3.2.19. $\varphi(ab) = \varphi(a)\Delta_0(b) + \Delta_0(a)\varphi(b)$, *for any* $a, b \in \mathcal{U}(\mathfrak{g})$.

Proof. Since Δ is a homomorphism of algebras and $\Sigma \equiv 1 \bmod h^{n+1}$, one has

$$\varphi(ab) \equiv \frac{\Delta^{op}(ab) - \Sigma^{-1}\Delta(ab)\Sigma}{h^n}$$

$$
\begin{aligned}
&= \frac{\Delta^{op}(a)\Delta^{op}(b) - \Sigma^{-1}\Delta(a)\Sigma\Sigma^{-1}\Delta(b)\Sigma}{h^n} \\
&= \frac{(\Delta^{op}(a) - \Sigma^{-1}\Delta(a)\Sigma)\Delta^{op}(b) + \Sigma^{-1}\Delta(a)\Sigma(\Delta^{op}(b) - \Sigma^{-1}\Delta(b)\Sigma)}{h^n} \\
&\equiv \varphi(a)\Delta^{op}(b) + \Sigma^{-1}\Delta(a)\Sigma\varphi(b) \\
&\equiv \varphi(a)\Delta_0^{op}(b) + \Delta_0(a)\varphi(b) \bmod h.
\end{aligned}
$$

It follows that $\varphi(ab) = \varphi(a)\Delta_0^{op}(b) + \Delta_0(a)\varphi(b) = \varphi(a)\Delta_0(b) + \Delta_0(a)\varphi(b)$ because Δ_0 is cocommutative. □

Corollary 3.2.20. *Let $\psi \in \Lambda^2\mathfrak{g}$. If $[\Delta_0(a), \psi] = \varphi(a)$ for any $a \in \mathfrak{g}$, then $[\Delta_0(a), \psi] = \varphi(a)$ for any $a \in \mathcal{U}(\mathfrak{g})$.*

Proof. Let $a \in \mathcal{U}(\mathfrak{g})$. If a is a scalar multiple of the unit of $\mathcal{U}(\mathfrak{g})$, there is nothing to prove because both $\varphi(a)$ and $[\Delta_0(a), \psi]$ are zero. So let us suppose $a = x_1 \cdots x_m$, where $x_i \in \mathfrak{g}$, and let us proceed by induction on m. If $m = 1$ there is nothing to prove by hypothesis. On the other hand

$$
\begin{aligned}
[\Delta_0(x_1 \cdots x_m), \psi] &= \Delta_0(x_1)\Delta_0(x_2 \cdots x_m)\psi - \psi\Delta_0(x_1)\Delta_0(x_2 \cdots x_m) \\
&= \Delta_0(x_1)[\Delta_0(x_2 \cdots x_m), \psi] + [\Delta_0(x_1), \psi]\Delta_0(x_2 \cdots x_m).
\end{aligned}
$$

Therefore, by induction and Lemma 3.2.19, one has

$$
[\Delta_0(x_1 \cdots x_m), \psi] = \Delta_0(x_1)\varphi(x_2 \cdots x_m) + \varphi(x_1)\Delta_0(x_2 \cdots x_m) = \varphi(x_1 \cdots x_m).
$$

□

Lemma 3.2.21. *For any $a \in \mathcal{U}(\mathfrak{g})$ the following equation holds:*

$$
(\Delta_0 \otimes 1)(\varphi(a)) + (\varphi \otimes 1)(\Delta_0(a)) = (1 \otimes \Delta_0)(\varphi(a)) + (1 \otimes \varphi)(\Delta_0(a)). \tag{3.19}
$$

Proof. Since $(\Delta \otimes 1)(\Sigma)\Sigma^{12} \equiv (1 \otimes \Delta)(\Sigma)\Sigma^{23} \bmod h^{n+1}$, one has $(1 \otimes \Delta)(\Sigma) \equiv (\Delta \otimes 1)(\Sigma)\Sigma^{12}(\Sigma^{23})^{-1} \bmod h^{n+1}$ and $(1 \otimes \Delta)(\Sigma^{-1}) \equiv \Sigma^{23}(\Sigma^{12})^{-1}(\Delta \otimes 1)(\Sigma^{-1}) \bmod h^{n+1}$. Therefore, one has

$$
\begin{aligned}
&(\Delta_0 \otimes 1)(\varphi(a)) + (\varphi \otimes 1)(\Delta_0(a)) \\
&\quad \equiv \frac{(\Delta \otimes 1)\Delta^{op}(a) - (\Delta \otimes 1)(\Sigma^{-1}\Delta(a)\Sigma)}{h^n} + (\varphi \otimes 1)(\Delta_0(a)) \\
&\quad \equiv \frac{\Sigma^{23}(\Sigma^{12})^{-1}(\Delta \otimes 1)\Delta^{op}(a)\Sigma^{12}(\Sigma^{23})^{-1}}{h^n} \\
&\qquad - \frac{\Sigma^{23}(\Sigma^{12})^{-1}(\Delta \otimes 1)(\Sigma^{-1})(\Delta \otimes 1)\Delta(a)(\Delta \otimes 1)(\Sigma)\Sigma^{12}(\Sigma^{23})^{-1}}{h^n} \\
&\qquad + (\varphi \otimes 1)(\Delta_0(a)) \\
&\quad \equiv \frac{\Sigma^{23}(\Sigma^{12})^{-1}(\Delta \otimes 1)\Delta^{op}(a)\Sigma^{12}(\Sigma^{23})^{-1}}{h^n} \\
&\qquad - \frac{(1 \otimes \Delta)(\Sigma)^{-1}(1 \otimes \Delta)\Delta(a)(1 \otimes \Delta)(\Sigma)}{h^n} \\
&\qquad + (\varphi \otimes 1)(\Delta_0(a))
\end{aligned}
$$

$$
\begin{aligned}
&\equiv \frac{\Sigma^{23}(\Delta^{op}\otimes 1)\Delta^{op}(a)(\Sigma^{23})^{-1} - (1\otimes\Delta)(\Sigma^{-1}\Delta(a)\Sigma)}{h^n} \\
&\quad - \Sigma^{23}(\varphi\otimes 1)(\Delta^{op}(a))(\Sigma^{23})^{-1} + (\varphi\otimes 1)(\Delta_0(a)) \\
&\equiv \frac{\Sigma^{23}(1\otimes\Delta^{op})\Delta^{op}(a)(\Sigma^{23})^{-1} - (1\otimes\Delta)(\Sigma^{-1}\Delta(a)\Sigma)}{h^n} \\
&\quad - (\varphi\otimes 1)(\Delta_0^{op}(a)) + (\varphi\otimes 1)(\Delta_0(a)) \\
&\equiv \frac{(1\otimes\Delta)\Delta^{op}(a) - (1\otimes\Delta)(\Sigma^{-1}\Delta(a)\Sigma)}{h^n} \\
&\quad + \Sigma^{23}(1\otimes\varphi)(\Delta^{op}(a))(\Sigma^{23})^{-1} \\
&\equiv (1\otimes\Delta)(\varphi(a)) + (1\otimes\varphi)(\Delta_0^{op}(a)) \\
&\equiv (1\otimes\Delta_0)(\varphi(a)) + (1\otimes\varphi)(\Delta_0(a)) \mod h.
\end{aligned}
$$

It follows that

$$(\Delta_0\otimes 1)(\varphi(a)) + (\varphi\otimes 1)(\Delta_0(a)) = (1\otimes\Delta_0)(\varphi(a)) + (1\otimes\varphi)(\Delta_0(a))$$

because $\varphi(a)\in\mathcal{U}(\mathfrak{g})^{\otimes 2}$. □

Corollary 3.2.22. *For any $a\in\mathfrak{g}$, equation* (3.19) *becomes:*

$$(\Delta_0\otimes 1)(\varphi(a)) + \varphi(a)\otimes 1 = (1\otimes\Delta_0)(\varphi(a)) + 1\otimes\varphi(a). \tag{3.20}$$

Proof. Since $\varphi(a) = 0$ and $\Delta_0(a) = a\otimes 1 + 1\otimes a$ for any $a\in g$, one has $(\varphi\otimes 1)(\Delta_0(a)) = \varphi(a)\otimes 1$ and $(1\otimes\varphi)(\Delta_0(a)) = 1\otimes\varphi(a)$. The thesis follows by Lemma 3.2.21. □

Corollary 3.2.23. $\varphi(\mathfrak{g})\subseteq\Lambda^2\mathfrak{g}$.

Proof. For any $a\in\mathfrak{g}$, by Proposition 3.2.18, $\varphi(a)\in\Lambda^2(\mathcal{U}(\mathfrak{g}))$ and by Corollary 3.2.22, $\varphi(a)$ is a cocycle of the cochain complex (3.15), therefore $\varphi(a) = \frac{1}{2}Alt(\varphi(a))\in\Lambda^2\mathfrak{g}$ by Lemma 3.2.12. □

Let us consider the Chevalley-Eilenberg cochain complex of the Lie algebra $\mathfrak{g}$ and $\mathfrak{g}$-module $\Lambda^2\mathfrak{g}$:

$$\Lambda^2\mathfrak{g} \xrightarrow{d_\mathfrak{g}} \mathrm{Hom}_\mathbb{C}\left(\mathfrak{g},\Lambda^2\mathfrak{g}\right) \xrightarrow{d_\mathfrak{g}} \mathrm{Hom}_\mathbb{C}\left(\Lambda^2\mathfrak{g},\Lambda^2\mathfrak{g}\right) \xrightarrow{d_\mathfrak{g}} \dots \tag{3.21}$$

with the differential $d_\mathfrak{g} : \mathrm{Hom}_\mathbb{C}\left(\Lambda^m\mathfrak{g},\Lambda^2\mathfrak{g}\right)\to\mathrm{Hom}_\mathbb{C}\left(\Lambda^{m+1}(\mathfrak{g}),\Lambda^2\mathfrak{g}\right)$ given by

$$
\begin{aligned}
d_\mathfrak{g}(f)(x_1\wedge\cdots\wedge x_{m+1}) &= \sum_{i=1}^{m+1}(-1)^i x_i.f(x_1\wedge\cdots\wedge\widehat{x_i}\wedge\cdots\wedge x_{m+1}) \\
&\quad + \sum_{i<j}(-1)^{i+j} f([x_i,x_j]\wedge x_1\wedge\cdots\wedge\widehat{x_i}\wedge\cdots\wedge\widehat{x_j}\cdots\wedge x_{m+1})
\end{aligned}
$$

where "." denotes, as before, the adjoint action of $\mathfrak{g}$ on $\Lambda^2\mathfrak{g}$:

$$
\begin{aligned}
&x_i.f(x_1\wedge\cdots\wedge\widehat{x_i}\wedge\cdots\wedge x_{m+1}) \\
&\qquad = \left[\Delta_0(x_i), f(x_1\wedge\cdots\wedge\widehat{x_i}\wedge\cdots\wedge x_{m+1})\right].
\end{aligned}
$$

(cf. Appendix B for the proof that $d_\mathfrak{g}^2 = 0$).

Proposition 3.2.24. *$\varphi : \mathfrak{g} \to \Lambda^2\mathfrak{g}$ is a 1-cocycle.*

Proof. Let $a, b \in \mathfrak{g}$, then $d_{\mathfrak{g}}(\varphi)(a \wedge b) = a.\varphi(b) - b.\varphi(a) - \varphi([a,b])$. On the other hand $a.\varphi(b) = [\Delta_0(a), \varphi(b)] = \Delta_0(a)\varphi(b) - \varphi(b)\Delta_0(a)$ and $\varphi([a,b]_{\mathfrak{g}}) = \varphi([a,b]_{\mathcal{U}(\mathfrak{g})}) = \varphi(ab) - \varphi(ba)$. It follows that

$$\begin{aligned} d_{\mathfrak{g}}(\varphi)(a \wedge b) = \Delta_0(a)\varphi(b) - \varphi(b)\Delta_0(a) - \Delta_0(b)\varphi(a) + \varphi(a)\Delta_0(b) \\ - \varphi(ab) + \varphi(ba) = 0 \end{aligned}$$

because of Lemma 3.2.19. □

Requiring the existence of $\psi \in \Lambda^2\mathfrak{g}$ such that $[\Delta_0(a), \psi] = \varphi(a)$, for any $a \in \mathfrak{g}$, is the same as requiring φ to be a coboundary of the cochain complex (3.21). Moreover if $\psi, \widetilde{\psi} \in \Lambda^2\mathfrak{g}$ satisfy $d_{\mathfrak{g}}(\psi) = \varphi = d_{\mathfrak{g}}(\widetilde{\psi})$ then $\psi - \widetilde{\psi}$ is a 0-cocycle. Therefore the obstructions to the existence and uniqueness of Σ satisfying conditions (1)-(4) are related to the cohomology of the cochain complex (3.21): the obstruction to the existence is in the group of cohomology $H^1(\mathfrak{g}, \Lambda^2\mathfrak{g})$ and the obstruction to uniqueness is in the group of cohomology $H^0(\mathfrak{g}, \Lambda^2\mathfrak{g})$.

Summarizing, the first theorem that V. G. Drinfeld proves in his notes is the following:

Theorem 3.2.25. *Let H be a quantization of a complex Lie bialgebra $(\mathfrak{g}, [\ ,\ ], \delta)$. Then the obstructions to the existence and uniqueness of a coboundary structure Σ lie respectively in the 1^{st} and 0^{th} cohomology groups of the Chevalley-Eilenberg cochain complex* (3.21)*.*

Let us recall the following theorem:

Theorem 3.2.26 ([J, Theorem 13])**.** *(Whitehead's first lemma) If $\mathfrak{g}$ is a finite-dimensional semisimple Lie algebra over a field of characteristic 0 and M is a finite dimensional $\mathfrak{g}$-module, then $H^1(\mathfrak{g}, M) = 0$ and $H^2(\mathfrak{g}, M) = 0$.*

Corollary 3.2.27. *If H is a quantization of a complex Lie bialgebra $(\mathfrak{g}, [\ ,\ ], \delta)$, where $(\mathfrak{g}, [\ ,\ ])$ is a simple finite dimensional Lie algebra, then there exists a unique coboundary structure such that $\Sigma \equiv 1 \mod h$.*

Proof. Since $\Lambda^2\mathfrak{g}$ is a finite dimensional representation of $\mathfrak{g}$, by Whitehead's first lemma 3.2.26, $H^1(\mathfrak{g}, \Lambda^2\mathfrak{g}) = 0$. Moreover, since the only invariant of $\Lambda^2\mathfrak{g}$ under the action of $\mathfrak{g}$ is 0, $H^0(\mathfrak{g}, \Lambda^2\mathfrak{g}) = 0$. □

3.3 Question and Answer 47

Definition 3.3.1. Let H be a topological bialgebra. We say that there exists a universal triangular R-matrix if there exists an element $\Sigma \in H \widehat{\otimes} H$ such that

(1) $\Sigma \equiv 1 \mod h$;

(2) $\Sigma^{12}\Sigma^{21} = 1$;

(3) $(1 \otimes \Delta)(\Sigma) = \Sigma^{12}\Sigma^{13}$;

(4) $\Delta^{op}(a) = \Sigma^{-1}\Delta(a)\Sigma$;

(5) $\Sigma^{12}\Sigma^{13}\Sigma^{23} = \Sigma^{23}\Sigma^{13}\Sigma^{12}$.

We recall the following definition:

Definition 3.3.2. A triangular Lie bialgebra is a triple $(\mathfrak{g}, [\ ,\ ], r)$ such that $(\mathfrak{g}, [\ ,\ ], d_{\mathfrak{g}}r)$ is a Lie bialgebra (where $d_{\mathfrak{g}}$ is the Chevalley-Eilenberg differential of the cochain complex with Lie algebra $\mathfrak{g}$ and $\mathfrak{g}$-module $\Lambda^2\mathfrak{g}$) and r is an element of $\Lambda^2\mathfrak{g}$ satisfying the classical Yang-Baxter equation:

$$\left[r^{12}, r^{13}\right] + \left[r^{12}, r^{23}\right] + \left[r^{13}, r^{23}\right] = 0. \tag{3.22}$$

In this case r is called a triangular structure.

In this section we will give an answer to the following question:
let $(\mathfrak{g}, [\ ,\ ], r)$ be a triangular Lie bialgebra over $\mathbb{C}$ and let H be a quantization of the Lie bialgebra $(\mathfrak{g}, [\ ,\ ], d_{\mathfrak{g}}r)$ (cf. Section 3.1). Where do the obstructions to the existence and uniqueness of an element $\Sigma \in H\widehat{\otimes}H$ satisfying the following relations

(1) $\Sigma \equiv 1 + rh \mod h^2$;

(2) $\Sigma^{12}\Sigma^{21} = 1$;

(3) $(\Delta \otimes 1)(\Sigma)\Sigma^{12} = (1 \otimes \Delta)(\Sigma)\Sigma^{23}$;

(4) $\Delta^{op}(a) = \Sigma^{-1}\Delta(a)\Sigma$,

(5) $\Sigma^{12}\Sigma^{13}\Sigma^{23} = \Sigma^{23}\Sigma^{13}\Sigma^{12}$;

lie?

Remark 3.3.1. Our question is the same as asking where the obstructions to the existence and uniqueness of a universal triangular R-matrix lie. Indeed, by Lemma 3.3.4, the previous axioms imply that Σ satisfies the condition

$$(1 \otimes \Delta)(\Sigma) = \Sigma^{12}\Sigma^{13}.$$

And the converse is trivial. Indeed

$$\begin{aligned}(\Delta \otimes 1)(\Sigma) &= (1\ 3\ 2)(1 \otimes \Delta)(\Sigma^{21}) \\ &= (1\ 3\ 2)(1 \otimes \Delta)(\Sigma)^{-1}\end{aligned}$$

because $1 \otimes \Delta$ is a homomorphism of algebras and $\Sigma^{12}\Sigma^{21} = 1$. Using $(1 \otimes \Delta)(\Sigma) = \Sigma^{12}\Sigma^{13}$ and $\Sigma^{12}\Sigma^{21} = 1$ one has

$$\begin{aligned}(\Delta \otimes 1)(\Sigma) &= (1\ 3\ 2)\left(\Sigma^{12}\Sigma^{13}\right)^{-1} \\ &= (1\ 3\ 2)(\Sigma^{31}\Sigma^{21}) \\ &= \Sigma^{23}\Sigma^{13}.\end{aligned}$$

Therefore, replacing in $\Sigma^{12}\Sigma^{13}\Sigma^{23} = \Sigma^{23}\Sigma^{13}\Sigma^{12}$, $\Sigma^{23}\Sigma^{13}$ with $(\Delta \otimes 1)(\Sigma)$ and $\Sigma^{12}\Sigma^{13}$ with $(1 \otimes \Delta)(\Sigma)$, one has

$$(1 \otimes \Delta)(\Sigma)\Sigma^{23} = (\Delta \otimes 1)(\Sigma)\Sigma^{12}.$$

It is not the same in the pseudotriangular cases: that's why, in Section 3.5, we slightly modify the arguments in order to prepare to the pseudotriangular cases.

Let us set $\Sigma \equiv 1 + rh \bmod h^2$ with r triangular structure. Since $r \in \Lambda^2\mathfrak{g}$, $\Sigma^{12}\Sigma^{21} \equiv 1 \bmod h^2$, because of the skewsymmetry of r, and $(\Delta \otimes 1)(\Sigma)\Sigma^{12} \equiv (1 \otimes \Delta)(\Sigma)\Sigma^{23} \bmod h^2$, because $r \in \mathfrak{g} \otimes \mathfrak{g}$. Since H is a quantization of the Lie bialgebra $(g, [\ ,\ ], d_{\mathfrak{g}}r)$, $\Delta^{op}(a) \equiv \Sigma^{-1}\Delta(a)\Sigma \bmod h^2$ and since r satisfies the classical Yang-Baxter equation (3.22), $\Sigma^{12}\Sigma^{13}\Sigma^{23} \equiv \Sigma^{23}\Sigma^{13}\Sigma^{12} \bmod h^3$.

Let us consider an integer $n \geq 2$ and an element Σ of $H\widehat{\otimes}H$ satisfying:

(i) $\Sigma \equiv 1 + rh \bmod h^2$,

(ii) $\Sigma\Sigma^{21} \equiv 1 \bmod h^n$,

(iii) $(\Delta \otimes 1)(\Sigma)\Sigma^{12} \equiv (1 \otimes \Delta)(\Sigma)\Sigma^{23} \bmod h^n$,

(iv) $\Delta^{op}(a) \equiv \Sigma^{-1}\Delta(a)\Sigma \bmod h^n$,

(v) $\Sigma^{12}\Sigma^{13}\Sigma^{23} \equiv \Sigma^{23}\Sigma^{13}\Sigma^{12} \bmod h^{n+1}$.

We want to correct Σ without changing it mod h^n such that the relations (ii)-(iv) are satisfied mod h^{n+1} and the relation (v) is satisfied mod h^{n+2}.

First, let us correct Σ (as done in Question and Answer 46) in order to have conditions (ii) and (iii) to be satisfied. After this, by Remark 3.2.16, the freedom in choosing Σ reduces to an element of $\Lambda^2\mathfrak{g}$. Let q be the element in $\mathcal{U}(\mathfrak{g})^{\otimes 3}$ defined by

$$q \equiv \frac{Q-1}{h^{n+1}} \bmod h \tag{3.23}$$

where $Q = \Sigma^{12}\Sigma^{13}\Sigma^{23}(\Sigma^{12})^{-1}(\Sigma^{13})^{-1}(\Sigma^{23})^{-1}$. We have two error terms: $\varphi \in \mathrm{Hom}(\mathfrak{g}, \Lambda^2\mathfrak{g})$ of section 3.2 and the term q. Therefore we would like to find an element $\psi \in \Lambda^2\mathfrak{g}$ such that $\Sigma' = \Sigma + \psi h^n$ satisfies $\Delta^{op}(a) \equiv \Sigma'^{-1}\Delta(a)\Sigma' \bmod h^{n+1}$ and $\Sigma'^{12}\Sigma'^{13}\Sigma'^{23} \equiv \Sigma'^{23}\Sigma'^{13}\Sigma'^{12} \bmod h^{n+2}$. By Remark 3.2.17, we already know that the equation

$$\Delta^{op}(a) \equiv \Sigma'^{-1}\Delta(a)\Sigma' \bmod h^{n+1}$$

is equivalent to the equation

$$[\Delta_0(a), \psi] = \varphi(a).$$

The following proposition gives an equivalent relation to the equation

$$\Sigma'^{12}\Sigma'^{13}\Sigma'^{23} \equiv \Sigma'^{23}\Sigma'^{13}\Sigma'^{12} \bmod h^{n+2}.$$

Proposition 3.3.2. *The equation*

$$\Sigma'^{12}\Sigma'^{13}\Sigma'^{23} \equiv \Sigma'^{23}\Sigma'^{13}\Sigma'^{12} \ \mathit{mod}\ h^{n+2}$$

is equivalent to the condition

$$q = -\frac{1}{2}Alt\Big([r^{12} + r^{13}, \psi^{23}]\Big).$$

Proof.

$$
\begin{aligned}
&\big(\Sigma^{12}+\psi^{12}h^n\big)\big(\Sigma^{13}+\psi^{13}h^n\big)\big(\Sigma^{23}+\psi^{23}h^n\big)\\
&\quad -\big(\Sigma^{23}+\psi^{23}h^n\big)\big(\Sigma^{13}+\psi^{13}h^n\big)\big(\Sigma^{12}+\psi^{12}h^n\big)\\
&\equiv \Sigma^{12}\Sigma^{13}\Sigma^{23}-\Sigma^{23}\Sigma^{13}\Sigma^{12}\\
&\quad +\Big(\big[\psi^{12},r^{13}+r^{23}\big]+\big[\psi^{13},r^{23}-r^{12}\big]+\big[\psi^{23},-r^{12}-r^{13}\big]\Big)h^{n+1} \bmod h^{n+2}.
\end{aligned}
$$

It follows that $\Sigma'^{12}\Sigma'^{13}\Sigma'^{23}\equiv\Sigma'^{23}\Sigma'^{13}\Sigma'^{12} \bmod h^{n+2}$ if and only if

$$
\begin{aligned}
&\Sigma^{12}\Sigma^{13}\Sigma^{23}-\Sigma^{23}\Sigma^{13}\Sigma^{12}\\
&\quad\equiv \big[r^{13}+r^{23},\psi^{12}\big]+\big[r^{23}-r^{12},\psi^{13}\big]+\big[-r^{12}-r^{13},\psi^{23}\big] \bmod h^{n+2}.
\end{aligned}
\tag{3.24}
$$

Multiplying on the right both sides of equation (3.24) by $\big(\Sigma^{23}\Sigma^{13}\Sigma^{12}\big)^{-1}$ and dividing by h^{n+1} one has

$$
\begin{aligned}
q &\equiv \big[r^{13}+r^{23},\psi^{12}\big]+\big[r^{23}-r^{12},\psi^{13}\big]+\big[-r^{12}-r^{13},\psi^{23}\big]\Big)\\
&\equiv -\big((1\ 3\ 2)-(1\ 2)+id\big)\big[r^{12}+r^{13},\psi^{23}\big] \bmod h^{n+2}
\end{aligned}
$$

where we used $r\in\Lambda^2\mathfrak{g}$. On the other hand

$$
\begin{aligned}
-(2\ 3)\big[r^{12}+r^{13},\psi^{23}\big] &= -\big[r^{13}+r^{12},\psi^{32}\big]\\
&= \big[r^{12}+r^{13},\psi^{23}\big]
\end{aligned}
$$

because $\psi\in\Lambda^2\mathfrak{g}$. It follows that

$$
q=-\frac{1}{2}Alt\Big(\big[r^{12}+r^{13},\psi^{23}\big]\Big).
$$

□

Let us prove that $q\in\Lambda^3\mathfrak{g}$.

Proposition 3.3.3. *$q\in\Lambda^3\mathcal{U}(\mathfrak{g})$.*

Proof. One has $\big(\Sigma^{23}\big)^{-1}\big(\Sigma^{13}\big)^{-1}=1+\sum_{m\geq1}c_mh^m$ for suitable $c_m\in\mathcal{U}(\mathfrak{g})^{\otimes3}$, because $\Sigma\equiv1 \bmod h$, and $\big(\Sigma^{12}\big)^{-1}\equiv\Sigma^{21}+dh^{n+1} \bmod h^{n+2}$ for a suitable $d\in\mathcal{U}(\mathfrak{g})^{\otimes2}$, because $\Sigma\Sigma^{21}\equiv1 \bmod h^{n+1}$. It follows that

$$
\begin{aligned}
&\big(\Sigma^{12}\big)^{-1}\big(\Sigma^{21}\big)^{-1}\big(\Sigma^{23}\big)^{-1}\big(\Sigma^{13}\big)^{-1}\\
&\qquad=\big(\Sigma^{12}\big)^{-1}\big(\Sigma^{21}\big)^{-1}+\sum_{m\geq1}\big(\Sigma^{12}\big)^{-1}\big(\Sigma^{21}\big)^{-1}c_mh^m\\
&\qquad\equiv\big(\Sigma^{12}\big)^{-1}\big(\Sigma^{21}\big)^{-1}+\sum_{m\geq1}\big(\Sigma^{21}+dh^{n+1}\big)\big(\Sigma^{21}\big)^{-1}c_mh^m\\
&\qquad\equiv\big(\Sigma^{12}\big)^{-1}\big(\Sigma^{21}\big)^{-1}+\sum_{m\geq1}c_mh^m \bmod h^{n+2}.
\end{aligned}
$$

And, in the same way, one has

$$\begin{aligned}
&(\Sigma^{23})^{-1}(\Sigma^{13})^{-1}(\Sigma^{12})^{-1}(\Sigma^{21})^{-1} \\
&\qquad = (\Sigma^{12})^{-1}(\Sigma^{21})^{-1} + \sum_{m\geq 1} c_m (\Sigma^{12})^{-1}(\Sigma^{21})^{-1}h^m \\
&\qquad \equiv (\Sigma^{12})^{-1}(\Sigma^{21})^{-1} + \sum_{m\geq 1} c_m \Big((\Sigma^{12})^{-1} - dh^{n+1}\Big)(\Sigma^{21})^{-1}h^m \\
&\qquad \equiv (\Sigma^{12})^{-1}(\Sigma^{21})^{-1} + \sum_{m\geq 1} c_m \Sigma^{21}(\Sigma^{21})^{-1}h^m \\
&\qquad = (\Sigma^{12})^{-1}(\Sigma^{21})^{-1} + \sum_{m\geq 1} c_m h^m \mod h^{n+2}.
\end{aligned}$$

Therefore one has

$$\begin{aligned}
&(\Sigma^{12})^{-1}(\Sigma^{21})^{-1}(\Sigma^{23})^{-1}(\Sigma^{13})^{-1} \\
&\equiv (\Sigma^{23})^{-1}(\Sigma^{13})^{-1}(\Sigma^{12})^{-1}(\Sigma^{21})^{-1} \mod h^{n+2}.
\end{aligned} \tag{3.25}$$

Using equation (3.25), one has

$$(1\ 2)Q \equiv \Sigma^{21}Q^{-1}(\Sigma^{21})^{-1} \mod h^{n+2} \tag{3.26}$$

because

$$\begin{aligned}
(1\ 2)Q &= \Sigma^{21}\Sigma^{23}\Sigma^{13}(\Sigma^{21})^{-1}(\Sigma^{23})^{-1}(\Sigma^{13})^{-1} \\
&= \Sigma^{21}\Sigma^{23}\Sigma^{13}\Sigma^{12}(\Sigma^{12})^{-1}(\Sigma^{21})^{-1}(\Sigma^{23})^{-1}(\Sigma^{13})^{-1} \\
&\equiv \Sigma^{21}\Sigma^{23}\Sigma^{13}\Sigma^{12}(\Sigma^{23})^{-1}(\Sigma^{13})^{-1}(\Sigma^{12})^{-1}(\Sigma^{21})^{-1} \\
&= \Sigma^{21}Q^{-1}(\Sigma^{21})^{-1} \mod h^{n+2}.
\end{aligned}$$

Similarly, since $(\Sigma^{23})^{-1}(\Sigma^{13})^{-1}(\Sigma^{12})^{-1} \equiv 1 \mod h$, $(\Sigma^{32})^{-1} \equiv 1 \mod h$ and $\Sigma^{23}\Sigma^{32} \equiv 1 \mod h^{n+1}$, one has

$$(2\ 3)Q \equiv (\Sigma^{23})^{-1}Q^{-1}\Sigma^{23} \mod h^{n+2} \tag{3.27}$$

because

$$\begin{aligned}
(2\ 3)Q &= \Sigma^{13}\Sigma^{12}\Sigma^{32}(\Sigma^{13})^{-1}(\Sigma^{12})^{-1}(\Sigma^{32})^{-1} \\
&= (\Sigma^{23})^{-1}\Sigma^{23}\Sigma^{13}\Sigma^{12}\Sigma^{32}\Sigma^{23}(\Sigma^{23})^{-1}(\Sigma^{13})^{-1}(\Sigma^{12})^{-1}(\Sigma^{32})^{-1} \\
&\equiv (\Sigma^{23})^{-1}\Sigma^{23}\Sigma^{13}\Sigma^{12}(\Sigma^{23})^{-1}(\Sigma^{13})^{-1}(\Sigma^{12})^{-1}\Sigma^{32}\Sigma^{23}(\Sigma^{32})^{-1} \\
&\equiv (\Sigma^{23})^{-1}Q^{-1}\Sigma^{32}\Sigma^{23}(\Sigma^{32})^{-1} \\
&\equiv (\Sigma^{23})^{-1}Q^{-1}\Sigma^{23} \mod h^{n+2}.
\end{aligned}$$

It follows that $(1\ 2)q = -q$ indeed

$$\begin{aligned}(1\ 2)q &\equiv \frac{(1\ 2)Q-1}{h^{n+1}}\\ &\equiv \frac{\Sigma^{21}Q^{-1}\left(\Sigma^{21}\right)^{-1}-1}{h^{n+1}}\\ &= \Sigma^{21}\frac{Q^{-1}-1}{h^{n+1}}\left(\Sigma^{21}\right)^{-1}\\ &\equiv \frac{Q^{-1}-1}{h^{n+1}}\\ &= -Q^{-1}\frac{Q-1}{h^{n+1}}\\ &\equiv -q \bmod h\end{aligned}$$

because $\Sigma^{12}\Sigma^{13}\Sigma^{23} \equiv \Sigma^{23}\Sigma^{13}\Sigma^{12} \bmod h^{n+1}$ implies $Q \equiv 1 \bmod h^{n+1}$. Similarly, by $(2\ 3)Q \equiv \left(\Sigma^{23}\right)^{-1}Q^{-1}\Sigma^{23} \bmod h^{n+2}$, one has $(2\ 3)q = -q$. Thus $q \in \Lambda^3\mathcal{U}(\mathfrak{g})$ because $(1\ 3)q = (1\ 2)(2\ 3)(1\ 2)q = -q$, $(1\ 2\ 3)q = (1\ 2)(2\ 3)q = q$ and $(1\ 3\ 2)q = (2\ 3)(1\ 2)q = q$. □

Lemma 3.3.4. *The following equation holds:*

$$(1\otimes\Delta)(\Sigma) \equiv \Sigma^{12}\Sigma^{13} \mod h^{n+1}. \tag{3.28}$$

Proof. First of all let us remark that $(1\otimes\Delta)(\Sigma) \equiv \Sigma^{12}\Sigma^{13} \bmod h$ because $\Sigma \equiv 1 \bmod h$ and $1\otimes\Delta$ is a homomorphism of algebras. Next let us prove that if $(1\otimes\Delta)(\Sigma) \equiv \Sigma^{12}\Sigma^{13} \bmod h^m$ then $(1\otimes\Delta)(\Sigma) \equiv \Sigma^{12}\Sigma^{13} \bmod h^{m+1}$ for any $m \leq n$. Let us define $a \in \mathcal{U}(\mathfrak{g})^{\otimes 3}$ in the following way:

$$a \equiv \frac{(1\otimes\Delta)(\Sigma)-\Sigma^{12}\Sigma^{13}}{h^m} \bmod h. \tag{3.29}$$

We want to prove that $a = 0$. a is well defined because, by hypothesis, we have $(1\otimes\Delta)(\Sigma) \equiv \Sigma^{12}\Sigma^{13} \bmod h^m$. Since $\Sigma^{23} \equiv 1$ mod h, $(\Delta\otimes 1)(\Sigma)\Sigma^{12} \equiv (1\otimes\Delta)(\Sigma)\Sigma^{23} \bmod h^{n+1}$, $\Sigma^{12}\Sigma^{13}\Sigma^{23} \equiv \Sigma^{23}\Sigma^{13}\Sigma^{12} \bmod h^{n+1}$ and $m \leq n$, one has

$$\begin{aligned}a &\equiv \frac{(1\otimes\Delta)(\Sigma)-\Sigma^{12}\Sigma^{13}}{h^m}\\ &\equiv \frac{(1\otimes\Delta)(\Sigma)-\Sigma^{12}\Sigma^{13}}{h^m}\Sigma^{23}\\ &= \frac{(1\otimes\Delta)(\Sigma)\Sigma^{23}-\Sigma^{12}\Sigma^{13}\Sigma^{23}}{h^m}\\ &\equiv \frac{(\Delta\otimes 1)(\Sigma)\Sigma^{12}-\Sigma^{23}\Sigma^{13}\Sigma^{12}}{h^m}\\ &\equiv \frac{(\Delta\otimes 1)(\Sigma)-\Sigma^{23}\Sigma^{13}}{h^m}\Sigma^{12}\\ &\equiv \frac{(\Delta\otimes 1)(\Sigma)-\Sigma^{23}\Sigma^{13}}{h^m} \bmod h.\end{aligned}$$

On the other hand, since $\Sigma\Sigma^{21} \equiv 1 \bmod h^{n+1}$ and $\Delta \otimes 1$ is a homomorphism of algebras, one has

$$\begin{aligned}
(\Delta \otimes 1)(\Sigma) &= (1\ 3\ 2)(1 \otimes \Delta)(\Sigma^{21}) \\
&\equiv (1\ 3\ 2)\big((1 \otimes \Delta)(\Sigma)\big)^{-1} \\
&\equiv (1\ 3\ 2)\big(\Sigma^{12}\Sigma^{13} + ah^m\big)^{-1} \\
&\equiv (1\ 3\ 2)\big(\Sigma^{31}\Sigma^{21} - ah^m\big) \\
&\equiv \Sigma^{23}\Sigma^{13} - (1\ 3\ 2)ah^m \bmod h^{m+1}.
\end{aligned}$$

Hence $a = -(1\ 3\ 2)a$. Moreover $a = (2\ 3)a$ indeed

$$\begin{aligned}
\big(\Sigma^{23}\big)^{-1}(1 \otimes \Delta)(\Sigma)\Sigma^{23} &= 1 + \sum_{s \geq 1} \big(\Sigma^{23}\big)^{-1}(1 \otimes \Delta)(\Sigma_s)\Sigma^{23}h^s \\
&= 1 + \sum_{s \geq 1} \big((1 \otimes \Delta^{op})(\Sigma_s) + O(h^n)\big)h^s \\
&= (1 \otimes \Delta^{op})(\Sigma) + O(h^{n+1}),
\end{aligned}$$

from which

$$\begin{aligned}
(2\ 3)a &\equiv \frac{(1 \otimes \Delta^{op})(\Sigma) - \Sigma^{13}\Sigma^{12}}{h^m} \\
&\equiv \frac{\big(\Sigma^{23}\big)^{-1}(1 \otimes \Delta)(\Sigma)\Sigma^{23} - \big(\Sigma^{23}\big)^{-1}\Sigma^{23}\Sigma^{13}\Sigma^{12}}{h^m} \\
&\equiv \big(\Sigma^{23}\big)^{-1}\frac{(1 \otimes \Delta)(\Sigma) - \Sigma^{12}\Sigma^{13}}{h^m}\Sigma^{23} \\
&\equiv a \bmod h.
\end{aligned}$$

It follows that

$$a = -(1\ 3\ 2)a = -(1\ 3\ 2)(2\ 3)a = -(1\ 3)a$$

and

$$a = -(1\ 3\ 2)a = (1\ 3\ 2)(1\ 3)a = (1\ 2)a$$

which imply $a = 0$ because

$$a = -(1\ 3\ 2)a = -(1\ 3\ 2)(1\ 2)a = -(2\ 3)a = -a.$$

Thus $(1 \otimes \Delta)(\Sigma) \equiv \Sigma^{12}\Sigma^{13} \bmod h^{m+1}$. Since it is true for any $m \leq n$ one has $(1 \otimes \Delta)(\Sigma) \equiv \Sigma^{12}\Sigma^{13} \bmod h^{n+1}$. □

Lemma 3.3.5. *The following equation holds:*

$$(1 \otimes 1 \otimes \Delta)(Q) \equiv Q^{123}Q^{124} \ \textit{mod}\ h^{n+2}. \tag{3.30}$$

Proof. By Lemma 3.3.4 one has $(1 \otimes \Delta)(\Sigma) \equiv \Sigma^{12}\Sigma^{13} + ah^{n+1} \mod h^{n+2}$. Since $1 \otimes 1 \otimes \Delta$ is a homomorphism of algebras it follows

$$\begin{aligned}
(1 \otimes 1 \otimes \Delta)(Q) &= (1 \otimes 1 \otimes \Delta)\Big(\Sigma^{12}\Sigma^{13}\Sigma^{23}(\Sigma^{12})^{-1}(\Sigma^{13})^{-1}(\Sigma^{23})^{-1}\Big) \\
&= \Sigma^{12}(1 \otimes 1 \otimes \Delta)(\Sigma^{13})(1 \otimes 1 \otimes \Delta)(\Sigma^{23}) \times \\
&\quad \times (\Sigma^{12})^{-1}\Big((1 \otimes 1 \otimes \Delta)(\Sigma^{13})\Big)^{-1}\Big((1 \otimes 1 \otimes \Delta)(\Sigma^{23})\Big)^{-1} \\
&\equiv \Sigma^{12}(\Sigma^{13}\Sigma^{14} + a^{134}h^{n+1})(\Sigma^{23}\Sigma^{24} + a^{234}h^{n+1}) \times \\
&\quad \times (\Sigma^{12})^{-1}\Big((\Sigma^{14})^{-1}(\Sigma^{13})^{-1} - a^{134}h^{n+1}\Big) \times \\
&\quad \times \Big((\Sigma^{24})^{-1}(\Sigma^{23})^{-1} - a^{234}h^{n+1}\Big) \\
&= \Sigma^{12}\Sigma^{13}\Sigma^{14}\Sigma^{23}\Sigma^{24}(\Sigma^{12})^{-1}(\Sigma^{14})^{-1}(\Sigma^{13})^{-1}(\Sigma^{24})^{-1}(\Sigma^{23})^{-1} \\
&\quad + (a^{134} + a^{234} - a^{134} - a^{234})h^{n+1} \\
&= \Sigma^{12}\Sigma^{13}\Sigma^{23}\Sigma^{14}\Sigma^{24}(\Sigma^{12})^{-1}(\Sigma^{14})^{-1}(\Sigma^{24})^{-1}(\Sigma^{13})^{-1}(\Sigma^{23})^{-1} \\
&= Q^{123}\Sigma^{23}\Sigma^{13}\Sigma^{12}\Sigma^{14}\Sigma^{24}(\Sigma^{12})^{-1}(\Sigma^{14})^{-1}(\Sigma^{24})^{-1}(\Sigma^{13})^{-1}(\Sigma^{23})^{-1} \\
&= Q^{123}\Sigma^{23}\Sigma^{13}Q^{124}(\Sigma^{13})^{-1}(\Sigma^{23})^{-1} \mod h^{n+2}.
\end{aligned}$$

Moreover, since $Q \equiv 1 \mod h^{n+1}$ one has $Q^{124} = 1 + \sum_{m \geq n+1} Q_m^{124} h^m$, from which

$$\begin{aligned}
&\Sigma^{23}\Sigma^{13}Q^{124}(\Sigma^{13})^{-1}(\Sigma^{23})^{-1} \\
&\qquad = 1 + \sum_{m \geq n+1} \Sigma^{23}\Sigma^{13}Q_m^{124}(\Sigma^{13})^{-1}(\Sigma^{23})^{-1}h^m \\
&\qquad \equiv Q^{124} \mod h^{n+2}.
\end{aligned}$$

Therefore $(1 \otimes 1 \otimes \Delta)(Q) \equiv Q^{123}Q^{124} \mod h^{n+2}$. □

We can now state the wanted result:

Proposition 3.3.6. $q \in \Lambda^3\mathfrak{g}$.

Proof. By Lemma 3.3.5, one has

$$\begin{aligned}
(1 \otimes 1 \otimes \Delta)(q) &\equiv \frac{(1 \otimes 1 \otimes \Delta)(Q) - 1}{h^{n+1}} \\
&\equiv \frac{Q^{123}Q^{124} - 1}{h^{n+1}} \\
&= \frac{Q^{123}(Q^{124} - 1) + Q^{123} - 1}{h^{n+1}} \\
&\equiv Q^{123}q^{124} + q^{123} \\
&\equiv q^{124} + q^{123} \mod h
\end{aligned}$$

because $Q^{123} \equiv 1 \mod h^{n+1}$. Thus, as $q \in \Lambda^3\mathcal{U}(\mathfrak{g})$, one has $q \in \Lambda^3\mathfrak{g}$. □

Let us consider the following bicomplex:

$$\begin{array}{ccccccc}
\Lambda^2\mathfrak{g} & \xrightarrow{d_{\mathfrak{g}}} & \mathrm{Hom}(\mathfrak{g},\Lambda^2\mathfrak{g}) & \xrightarrow{d_{\mathfrak{g}}} & \mathrm{Hom}(\Lambda^2\mathfrak{g},\Lambda^2\mathfrak{g}) & \xrightarrow{d_{\mathfrak{g}}} & \cdots \\
\big\downarrow{\scriptstyle d_{\mathfrak{g}^*}} & & \big\downarrow{\scriptstyle d_{\mathfrak{g}^*}} & & \big\downarrow{\scriptstyle d_{\mathfrak{g}^*}} & & \\
\Lambda^3\mathfrak{g} & \xrightarrow{d_{\mathfrak{g}}} & \mathrm{Hom}(\mathfrak{g},\Lambda^3\mathfrak{g}) & \xrightarrow{d_{\mathfrak{g}}} & \mathrm{Hom}(\Lambda^2\mathfrak{g},\Lambda^3\mathfrak{g}) & \xrightarrow{d_{\mathfrak{g}}} & \cdots \\
\big\downarrow{\scriptstyle d_{\mathfrak{g}^*}} & & \big\downarrow{\scriptstyle d_{\mathfrak{g}^*}} & & \big\downarrow{\scriptstyle d_{\mathfrak{g}^*}} & & \\
\vdots & & \vdots & & \vdots & &
\end{array} \tag{3.31}$$

where $d_{\mathfrak{g}} : \mathrm{Hom}(\Lambda^m\mathfrak{g},\Lambda^n\mathfrak{g}) \to \mathrm{Hom}(\Lambda^{m+1}\mathfrak{g},\Lambda^n\mathfrak{g})$ is defined as

$$\begin{aligned}
d_{\mathfrak{g}}f(x_1\wedge\cdots\wedge x_{m+1}) = &\sum_{i=1}^{m+1}(-1)^{i-1}x_i.f(x_1\wedge\cdots\wedge\widehat{x_i}\wedge\cdots\wedge x_{m+1}) \\
&+\sum_{i<j}(-1)^{i+j}f([x_i,x_j]\wedge x_1\wedge\cdots\wedge\widehat{x_i}\wedge\cdots\wedge\widehat{x_j}\wedge\cdots\wedge x_{m+1}),
\end{aligned}$$

(here, as before, we denote with "." the adjoint action of $\mathfrak{g}$ on $\Lambda^n\mathfrak{g}$ where $n\geq 2$) and $d_{\mathfrak{g}^*} : \mathrm{Hom}(\Lambda^m\mathfrak{g},\Lambda^n\mathfrak{g}) \to \mathrm{Hom}(\Lambda^m\mathfrak{g},\Lambda^{n+1}\mathfrak{g})$ is defined as

$$\begin{aligned}
d_{\mathfrak{g}^*}f(x_1\wedge\cdots\wedge x_m) = &\sum_{i=1}^{m}\sum_{(x_i)}(-1)^{i-1}\big(x_i'\wedge f(x_i''\wedge x_1\wedge\cdots\wedge\widehat{x_i}\wedge\cdots\wedge x_m) \\
&\qquad - x_i''\wedge f(x_i'\wedge x_1\wedge\cdots\wedge\widehat{x_i}\wedge\cdots\wedge x_m)\big) \\
&- (d_{\mathfrak{g}}r)\big(f(x_1\wedge\cdots\wedge x_m)\big)
\end{aligned}$$

where

$$\sum_{(x_i)} x_i'\wedge x_i'' = d_{\mathfrak{g}}r(x_i) = x_i.r$$

and

$$(d_{\mathfrak{g}}r)(y_1\wedge\cdots\wedge y_n) = \sum_{j=1}^{n}(-1)^{j-1}y_1\wedge\cdots\wedge d_{\mathfrak{g}}r(y_j)\wedge\cdots\wedge y_n$$

for any $y_1,\ldots,y_n\in\mathfrak{g}$.

In the following, for any $\theta\in\mathrm{Hom}(\mathfrak{g},\Lambda^2\mathfrak{g})$, we will denote with $(\theta\wedge 1)$ the linear map of $\mathrm{Hom}(\Lambda^2\mathfrak{g},\Lambda^3\mathfrak{g})$ defined as

$$(\theta\wedge 1)(x\wedge y) = \theta(x)\wedge y - \theta(y)\wedge x$$

for any $x,y\in\mathfrak{g}$.

Proposition 3.3.7. *The maps $d_{\mathfrak{g}}$ and $d_{\mathfrak{g}^*}$ satisfy the following conditions:*

$$d_{\mathfrak{g}}^2 = 0,$$

$$d_{\mathfrak{g}^*}^2 = 0$$

and

$$d_{\mathfrak{g}^*} \circ d_{\mathfrak{g}} = d_{\mathfrak{g}} \circ d_{\mathfrak{g}^*}.$$

The linear maps $d_{\mathfrak{g}^*} : \Lambda^n\mathfrak{g} \to \Lambda^{n+1}\mathfrak{g}$ *and* $d_{\mathfrak{g}^*} : \mathrm{Hom}(\mathfrak{g}, \Lambda^2\mathfrak{g}) \to \mathrm{Hom}(\mathfrak{g}, \Lambda^3\mathfrak{g})$ *can also be written, respectively, in the following ways:*

$$d_{\mathfrak{g}^*}(\psi) = \frac{1}{n!} Alt\Big(\big[r^{12} + r^{13} + \ldots + r^{1n+1}, 1 \otimes \psi\big]\Big), \tag{3.32}$$

where $\psi \in \Lambda^n\mathfrak{g}$*, and*

$$d_{\mathfrak{g}^*}(\theta) = -(\theta \wedge 1) \circ d_{\mathfrak{g}} r - (d_{\mathfrak{g}} r \wedge 1) \circ \theta \tag{3.33}$$

where $\theta \in \mathrm{Hom}(\mathfrak{g}, \Lambda^2\mathfrak{g})$*.*
The total cochain complex of the bicomplex (3.31) *is*

$$\Lambda^2\mathfrak{g} \xrightarrow{\ d\ } \Lambda^3\mathfrak{g} \oplus \mathrm{Hom}(\mathfrak{g}, \Lambda^2\mathfrak{g}) \xrightarrow{\ d\ } \cdots \tag{3.34}$$

with the total differential

$$d : \bigoplus_{m+n=l} \mathrm{Hom}(\Lambda^m\mathfrak{g}, \Lambda^{n+2}\mathfrak{g}) \to \bigoplus_{p+q=l+1} \mathrm{Hom}(\Lambda^p\mathfrak{g}, \Lambda^{q+2}\mathfrak{g})$$

defined as $df = d_{\mathfrak{g}} f + (-1)^m d_{\mathfrak{g}^*} f$*, for any* $f \in \mathrm{Hom}(\Lambda^m\mathfrak{g}, \Lambda^{n+2}\mathfrak{g})$*.*

Proof. See Appendix B. □

Proposition 3.3.8. *If* $\varphi = 0$*, then* q *is a* 0*-cocycle of the Chevalley-Eilenberg cochain complex with Lie algebra* $\mathfrak{g}$ *and* $\mathfrak{g}$*-module* $\Lambda^3\mathfrak{g}$ *(second line of the bicomplex* (3.31)*).*

Proof. Set $\widetilde{\varphi} : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ as the linear map defined in the following way

$$\widetilde{\varphi}(x) \equiv \frac{\Delta^{op}(x) - \Sigma^{-1}\Delta(x)\Sigma h^{n+1}}{h^{n+1}} \mod h.$$

As $\varphi = 0$, the map $\widetilde{\varphi}$ is well defined. Let us first prove that $(\Delta \otimes 1)\Delta(x)$ and Q commute mod h^{n+2} for any $x \in \mathcal{U}(\mathfrak{g})$. Using the coassociativity of Δ and the cocommutativity of Δ_0 one has

$$\begin{aligned}
(\Delta \otimes 1)\Delta(x)Q &= \Sigma^{12}\big(\Sigma^{12}\big)^{-1}(\Delta \otimes 1)\Delta(x)\Sigma^{12}\Sigma^{13}\Sigma^{23}\big(\Sigma^{12}\big)^{-1}\big(\Sigma^{13}\big)^{-1}\big(\Sigma^{23}\big)^{-1} \\
&\equiv \Sigma^{12}\big((\Delta^{op} \otimes 1)\Delta(x) - (\widetilde{\varphi} \otimes 1)\Delta(x)h^{n+1}\big)\Sigma^{13}\Sigma^{23}\big(\Sigma^{12}\big)^{-1}\big(\Sigma^{13}\big)^{-1}\big(\Sigma^{23}\big)^{-1} \\
&\equiv \Sigma^{12}\Sigma^{13}\big(\Sigma^{13}\big)^{-1}\big((1\ 2)(1 \otimes \Delta)\Delta(x)\big)\Sigma^{13}\Sigma^{23}\big(\Sigma^{12}\big)^{-1}\big(\Sigma^{13}\big)^{-1}\big(\Sigma^{23}\big)^{-1} \\
&\quad - (\widetilde{\varphi} \otimes 1)\Delta_0(x)h^{n+1} \\
&= \Sigma^{12}\Sigma^{13}(1\ 2)\Big(\big(\Sigma^{23}\big)^{-1}(1 \otimes \Delta)\Delta(x)\Sigma^{23}\Big)\Sigma^{23}\big(\Sigma^{12}\big)^{-1}\big(\Sigma^{13}\big)^{-1}\big(\Sigma^{23}\big)^{-1}
\end{aligned}$$

$$
\begin{aligned}
&\quad - (\widetilde{\varphi} \otimes 1)\Delta_0(x)h^{n+1} \\
&\equiv \Sigma^{12}\Sigma^{13}\Sigma^{23}(\Sigma^{23})^{-1}\Big((1\ 2)(2\ 3)(1 \otimes \Delta)\Delta(x)\Big)\Sigma^{23}(\Sigma^{12})^{-1}(\Sigma^{13})^{-1}(\Sigma^{23})^{-1} \\
&\quad - ((\widetilde{\varphi} \otimes 1)\Delta_0(x) + (1\ 2)(1 \otimes \widetilde{\varphi})\Delta_0(x))h^{n+1} \\
&= \Sigma^{12}\Sigma^{13}\Sigma^{23}\bigg((1\ 2\ 3)\Big((\Sigma^{12})^{-1}(\Delta \otimes 1)\Delta(x)\Sigma^{12}\Big)\bigg)(\Sigma^{12})^{-1}(\Sigma^{13})^{-1}(\Sigma^{23})^{-1} \\
&\quad - ((\widetilde{\varphi} \otimes 1)\Delta_0(x) + (1\ 2)(1 \otimes \widetilde{\varphi})\Delta_0(x))h^{n+1} \\
&\equiv \Sigma^{12}\Sigma^{13}\Sigma^{23}(\Sigma^{12})^{-1}\Sigma^{12}\Big((1\ 2\ 3)(\Delta^{op} \otimes 1)\Delta(x)\Big)(\Sigma^{12})^{-1}(\Sigma^{13})^{-1}(\Sigma^{23})^{-1} \\
&\quad - ((\widetilde{\varphi} \otimes 1)\Delta_0(x) + (1\ 2)(1 \otimes \widetilde{\varphi})\Delta_0(x) + (1 \otimes \widetilde{\varphi})\Delta_0^{op}(x))h^{n+1} \\
&= \Sigma^{12}\Sigma^{13}\Sigma^{23}(\Sigma^{12})^{-1}\Sigma^{12}((1 \otimes \Delta^{op})\Delta^{op}(x))(\Sigma^{12})^{-1}(\Sigma^{13})^{-1}(\Sigma^{23})^{-1} \\
&\quad - ((\widetilde{\varphi} \otimes 1)\Delta_0(x) + (1\ 2)(1 \otimes \widetilde{\varphi})\Delta_0(x) + (1 \otimes \widetilde{\varphi})\Delta_0^{op}(x))h^{n+1} \\
&= \Sigma^{12}\Sigma^{13}\Sigma^{23}(\Sigma^{12})^{-1}\Big(\Sigma^{12}(\Delta^{op} \otimes 1)\Delta^{op}(x)(\Sigma^{12})^{-1}\Big)(\Sigma^{13})^{-1}(\Sigma^{23})^{-1} \\
&\quad - ((\widetilde{\varphi} \otimes 1)\Delta_0(x) + (1\ 2)(1 \otimes \widetilde{\varphi})\Delta_0(x) + (1 \otimes \widetilde{\varphi})\Delta_0^{op}(x))h^{n+1} \\
&\equiv \Sigma^{12}\Sigma^{13}\Sigma^{23}(\Sigma^{12})^{-1}((\Delta \otimes 1)\Delta^{op}(x))(\Sigma^{13})^{-1}(\Sigma^{23})^{-1} \\
&\quad - ((\widetilde{\varphi} \otimes 1)\Delta_0(x) + (1\ 2)(1 \otimes \widetilde{\varphi})\Delta_0(x) + (1 \otimes \widetilde{\varphi})\Delta_0^{op}(x) \\
&\qquad - (\widetilde{\varphi} \otimes 1)\Delta_0^{op}(x))h^{n+1} \\
&= \Sigma^{12}\Sigma^{13}\Sigma^{23}(\Sigma^{12})^{-1}(\Sigma^{13})^{-1}\Sigma^{13}((1\ 2)(\Delta^{op} \otimes 1)\Delta^{op}(x))(\Sigma^{13})^{-1}(\Sigma^{23})^{-1} \\
&\quad - ((\widetilde{\varphi} \otimes 1)\Delta_0(x) + (1\ 2)(1 \otimes \widetilde{\varphi})\Delta_0(x) + (1 \otimes \widetilde{\varphi})\Delta_0^{op}(x) \\
&\qquad - (\widetilde{\varphi} \otimes 1)\Delta_0^{op}(x))h^{n+1} \\
&= \Sigma^{12}\Sigma^{13}\Sigma^{23}(\Sigma^{12})^{-1}(\Sigma^{13})^{-1}\Big((1\ 2)\Sigma^{23}(1 \otimes \Delta^{op})\Delta^{op}(x)(\Sigma^{23})^{-1}\Big)(\Sigma^{23})^{-1} \\
&\quad - ((\widetilde{\varphi} \otimes 1)\Delta_0(x) + (1\ 2)(1 \otimes \widetilde{\varphi})\Delta_0(x) + (1 \otimes \widetilde{\varphi})\Delta_0^{op}(x) \\
&\qquad - (\widetilde{\varphi} \otimes 1)\Delta_0^{op}(x))h^{n+1} \\
&\equiv \Sigma^{12}\Sigma^{13}\Sigma^{23}(\Sigma^{12})^{-1}(\Sigma^{13})^{-1}\Big((1\ 2)(1 \otimes \Delta)\Delta^{op}(x)\Big)(\Sigma^{23})^{-1} \\
&\quad - ((\widetilde{\varphi} \otimes 1)\Delta_0(x) + (1\ 2)(1 \otimes \widetilde{\varphi})\Delta_0(x) + (1 \otimes \widetilde{\varphi})\Delta_0^{op}(x) \\
&\qquad - (\widetilde{\varphi} \otimes 1)\Delta_0^{op}(x) - (1\ 2)(1 \otimes \widetilde{\varphi})\Delta_0^{op}(x))h^{n+1} \\
&= \Sigma^{12}\Sigma^{13}\Sigma^{23}(\Sigma^{12})^{-1}(\Sigma^{13})^{-1}(\Sigma^{23})^{-1}\Sigma^{23}\Big((1 \otimes \Delta^{op})\Delta(x)\Big)(\Sigma^{23})^{-1} \\
&\quad - ((\widetilde{\varphi} \otimes 1)\Delta_0(x) + (1\ 2)(1 \otimes \widetilde{\varphi})\Delta_0(x) + (1 \otimes \widetilde{\varphi})\Delta_0^{op}(x) \\
&\qquad - (\widetilde{\varphi} \otimes 1)\Delta_0^{op}(x) - (1\ 2)(1 \otimes \widetilde{\varphi})\Delta_0^{op}(x))h^{n+1} \\
&\equiv \Sigma^{12}\Sigma^{13}\Sigma^{23}(\Sigma^{12})^{-1}(\Sigma^{13})^{-1}(\Sigma^{23})^{-1}(1 \otimes \Delta)\Delta(x) \\
&\quad - ((\widetilde{\varphi} \otimes 1)\Delta_0(x) + (1\ 2)(1 \otimes \widetilde{\varphi})\Delta_0(x) + (1 \otimes \widetilde{\varphi})\Delta_0^{op}(x) \\
&\qquad - (\widetilde{\varphi} \otimes 1)\Delta_0^{op}(x) - (1\ 2)(1 \otimes \widetilde{\varphi})\Delta_0^{op}(x) - (1 \otimes \widetilde{\varphi})\Delta_0(x))h^{n+1} \\
&\equiv \Sigma^{12}\Sigma^{13}\Sigma^{23}(\Sigma^{12})^{-1}(\Sigma^{13})^{-1}(\Sigma^{23})^{-1}(\Delta \otimes 1)\Delta(x) \bmod h^{n+2}.
\end{aligned}
$$

It follows that

$$0 \equiv [(\Delta \otimes 1)\Delta(x), Q] \equiv [(\Delta \otimes 1)\Delta(x), 1 + qh^{n+1}]$$

$$\equiv \big[(\Delta_0 \otimes 1)\Delta_0(x), q\big] h^{n+1} \bmod h^{n+2}$$

from which $x.q = \big[(\Delta_0 \otimes 1)\Delta_0(x), q\big] = 0$. □

In general, even though φ is not 0, φ and q have the same image in $\mathrm{Hom}(\mathfrak{g}, \Lambda^3\mathfrak{g})$ up to a scalar:

Proposition 3.3.9. *The images of φ and q in* $\mathrm{Hom}(\mathfrak{g}, \Lambda^3\mathfrak{g})$ *satisfy the following equation:*

$$d_{\mathfrak{g}^*}\varphi = -d_{\mathfrak{g}}q. \tag{3.35}$$

Proof. Let us first compute $d_{\mathfrak{g}}q$. Let x be an element of $\mathcal{U}(\mathfrak{g})$ and let $\widetilde{\varphi}(x)$ be the element in $\mathcal{U}(\mathfrak{g})^{\otimes 2}$ such that

$$\Delta^{op}(x) \equiv \Sigma^{-1}\Delta(x)\Sigma + \varphi(x)h^n + \widetilde{\varphi}(x)h^{n+1} \bmod h^{n+2}$$

and let $\overline{\varphi}(x) = \varphi(x) + \widetilde{\varphi}(x)h$. Repeating the computations in the proof of Proposition 3.3.8 with the needed corrections (due to the fact that φ is not 0 this time, hence we have to deal with $\overline{\varphi}h^n = \varphi h^n + \widetilde{\varphi}h^{n+1}$ instead of only $\widetilde{\varphi}h^{n+1}$), one has

$$\begin{aligned}
&\big[(\Delta \otimes 1)\Delta(x), Q\big] \\
&\equiv \Big(\Sigma^{12}\Sigma^{13}\Sigma^{23}\big(\Sigma^{12}\big)^{-1}\big(\Sigma^{13}\big)^{-1}(1 \otimes \overline{\varphi})\Delta(x)\big(\Sigma^{23}\big)^{-1} \\
&\quad + \Sigma^{12}\Sigma^{13}\Sigma^{23}\big(\Sigma^{12}\big)^{-1}\big((1\ 2)(1 \otimes \overline{\varphi})\Delta^{op}(x)\big)\big(\Sigma^{13}\big)^{-1}\big(\Sigma^{23}\big)^{-1} \\
&\quad + \Sigma^{12}\Sigma^{13}\Sigma^{23}(\overline{\varphi} \otimes 1)\Delta^{op}(x)\big(\Sigma^{12}\big)^{-1}\big(\Sigma^{13}\big)^{-1}\big(\Sigma^{23}\big)^{-1} \\
&\quad - \Sigma^{12}\Sigma^{13}\Sigma^{23}(1 \otimes \overline{\varphi})\Delta^{op}(x)\big(\Sigma^{12}\big)^{-1}\big(\Sigma^{13}\big)^{-1}\big(\Sigma^{23}\big)^{-1} \\
&\quad - \Sigma^{12}\Sigma^{13}\big((1\ 2)(1 \otimes \overline{\varphi})\Delta(x)\big)\Sigma^{23}\big(\Sigma^{12}\big)^{-1}\big(\Sigma^{13}\big)^{-1}\big(\Sigma^{23}\big)^{-1} \\
&\quad - \Sigma^{12}(\widetilde{\varphi} \otimes 1)\Delta(x)\Sigma^{13}\Sigma^{23}\big(\Sigma^{12}\big)^{-1}\big(\Sigma^{13}\big)^{-1}\big(\Sigma^{23}\big)^{-1}\Big)h^n \bmod h^{n+2}.
\end{aligned}$$

From which, using the definition of $\overline{\varphi}$ and recalling that $\Sigma \equiv 1 + rh \bmod h^2$, one has

$$\begin{aligned}
&\big[(\Delta \otimes 1)\Delta(x), Q\big] \\
&\equiv \Big((1 \otimes \varphi)\big(\Delta_1(x) - \Delta_1^{op}(x)\big) - (1\ 2)(1 \otimes \varphi)\big(\Delta_1(x) - \Delta_1^{op}(x)\big) \\
&\quad - (\varphi \otimes 1)\big(\Delta_1(x) - \Delta_1^{op}(x)\big) - \big[r^{12} - r^{23}, (1\ 2)(1 \otimes \varphi)\Delta_0(x)\big] \\
&\quad + \big[r^{13} + r^{23}, (\varphi \otimes 1)\Delta_0(x)\big] - \big[r^{12} + r^{13}, (1 \otimes \varphi)\Delta_0(x)\big]\Big)h^{n+1} \bmod h^{n+2}
\end{aligned}$$

which implies

$$\begin{aligned}
d_{\mathfrak{g}}(q)(x) &= \big[(\Delta_0 \otimes 1)\Delta_0(x), q\big] \\
&= (1 \otimes \varphi)\big(\Delta_1(x) - \Delta_1^{op}(x)\big) - (1\ 2)(1 \otimes \varphi)\big(\Delta_1(x) - \Delta_1^{op}(x)\big) \\
&\quad - (\varphi \otimes 1)\big(\Delta_1(x) - \Delta_1^{op}(x)\big) - \big[r^{12} - r^{23}, (1\ 2)(1 \otimes \varphi)\Delta_0(x)\big] \\
&\quad + \big[r^{13} + r^{23}, (\varphi \otimes 1)\Delta_0(x)\big] - \big[r^{12} + r^{13}, (1 \otimes \varphi)\Delta_0(x)\big]
\end{aligned} \tag{3.36}$$

because

$$\big[(\Delta_0\otimes 1)\Delta_0(x),Q\big]=\big[(\Delta\otimes 1)\Delta(x),1+qh^{n+1}\big]=\big[(\Delta\otimes 1)\Delta(x),q\big]h^{n+1}\mod h^{n+2}.$$

On the other hand, one has

$$\begin{aligned}(1\otimes\varphi)\big(\Delta_1(x)-\Delta_1^{op}(x)\big)-(1\ 2)(1\otimes\varphi)\big(\Delta_1(x)-\Delta_1^{op}(x)\big)\\ -(\varphi\otimes 1)\big(\Delta_1(x)-\Delta_1^{op}(x)\big)=\frac{1}{2}Alt\Big((1\otimes\varphi)\big(\Delta_1(x)-\Delta_1^{op}(x)\big)\Big)\end{aligned}\tag{3.37}$$

indeed, $(2\ 3)\varphi(x)=-\varphi(x)$ because $\varphi\in\mathrm{Hom}(\mathfrak{g},\Lambda^2\mathfrak{g})$, and

$$\begin{aligned}(1\ 3\ 2)(1\otimes\varphi)\big(\Delta_1(x)-\Delta_1^{op}(x)\big)&=(\varphi\otimes 1)\big(\Delta_1^{op}(x)-\Delta_1(x)\big)\\&=-(\varphi\otimes 1)\big(\Delta_1(x)-\Delta_1^{op}(x)\big),\end{aligned}$$

hence, recalling that $(1\ 2\ 3)=(1\ 2)(2\ 3)$ and $(1\ 3\ 2)=(1\ 3)(2\ 3)$, one has

$$\begin{aligned}Alt\Big((1\otimes\varphi)\big(\Delta_1(x)-\Delta_1^{op}(x)\big)\Big)&=\sum_{\sigma\in S_3}sgn(\sigma)\sigma(1\otimes\varphi)\big(\Delta_1(x)-\Delta_1^{op}(x)\big)\\
&=(1\otimes\varphi)\big(\Delta_1(x)-\Delta_1^{op}(x)\big)-(1\ 2)(1\otimes\varphi)\big(\Delta_1(x)-\Delta_1^{op}(x)\big)\\
&\quad-(1\ 3)(1\otimes\varphi)\big(\Delta_1(x)-\Delta_1^{op}(x)\big)-(2\ 3)(1\otimes\varphi)\big(\Delta_1(x)-\Delta_1^{op}(x)\big)\\
&\quad+(1\ 2\ 3)(1\otimes\varphi)\big(\Delta_1(x)-\Delta_1^{op}(x)\big)+(1\ 3\ 2)(1\otimes\varphi)\big(\Delta_1(x)-\Delta_1^{op}(x)\big)\\
&=(1\otimes\varphi)\big(\Delta_1(x)-\Delta_1^{op}(x)\big)-(1\ 2)(1\otimes\varphi)\big(\Delta_1(x)-\Delta_1^{op}(x)\big)\\
&\quad+(1\ 3\ 2)(1\otimes\varphi)\big(\Delta_1(x)-\Delta_1^{op}(x)\big)+(1\otimes\varphi)\big(\Delta_1(x)-\Delta_1^{op}(x)\big)\\
&\quad-(1\ 2)(1\otimes\varphi)\big(\Delta_1(x)-\Delta_1^{op}(x)\big)+(1\ 3\ 2)(1\otimes\varphi)\big(\Delta_1(x)-\Delta_1^{op}(x)\big)\\
&=2\big((1\otimes\varphi)\big(\Delta_1(x)-\Delta_1^{op}(x)\big)-(1\ 2)(1\otimes\varphi)\big(\Delta_1(x)-\Delta_1^{op}(x)\big)\\
&\quad-(\varphi\otimes 1)\big(\Delta_1(x)-\Delta_1^{op}(x)\big)\big).\end{aligned}$$

Using also that $r\in\Lambda^2\mathfrak{g}$, with a similar argument one shows that

$$\begin{aligned}-\big[r^{12}-r^{23},(1\ 2)(1\otimes\varphi)\Delta_0(x)\big]+\big[r^{13}+r^{23},(\varphi\otimes 1)\Delta_0(x)\big]\\ -\big[r^{12}+r^{13},(1\otimes\varphi)\Delta_0(x)\big]=-\frac{1}{2}Alt\Big(\big[r^{12}+r^{13},(1\otimes\varphi)\Delta_0(x)\big]\Big).\end{aligned}\tag{3.38}$$

Moreover, by Condition (iv), since $n\geq 2$, one has $\Delta^{op}(x)\equiv\Sigma^{-1}\Delta(x)\Sigma\mod h^2$ which implies

$$\Delta_1(x)-\Delta_1^{op}(x)=-[\Delta_0(x),r]\tag{3.39}$$

and, since $x\in\mathfrak{g}$ and $\varphi(1)=0$, one has

$$(1\otimes\varphi)\Delta_0(x)=x\otimes\varphi(1)+1\otimes\varphi(x)=1\otimes\varphi(x)\tag{3.40}$$

Therefore, using equations (3.36) - (3.40) and recalling that $[\Delta_0(x),r]=d_{\mathfrak{g}}r(x)$, one has

$$d_{\mathfrak{g}}(q)(x)=-\frac{1}{2}Alt((1\otimes\varphi)d_{\mathfrak{g}}r(x))-\frac{1}{2}Alt\Big(\big[r^{12}+r^{13},1\otimes\varphi(x)\big]\Big).\tag{3.41}$$

On the other hand, since $\varphi \in \mathrm{Hom}(g, \Lambda^2\mathfrak{g})$, for any $z, w \in \mathfrak{g}$ there are $s_1 \in \mathbb{Z}_+$, $w_1^i, w_2^i \in \mathfrak{g}$ with $i = 1, \ldots, s_1$ and $s_2 \in \mathbb{Z}_+$, $z_1^j, z_2^j \in \mathfrak{g}$ with $j = 1, \ldots, s_2$ such that

$$\varphi(w) = \sum_{i=1}^{s_1}(w_1^i \otimes w_2^i - w_2^i \otimes w_1^i) = \sum_{i=1}^{s_1} (id - (1\ 2))(w_1^i \otimes w_2^i)$$

and

$$\varphi(z) = \sum_{j=1}^{s_2}(z_1^j \otimes z_2^j - z_2^j \otimes z_1^j) = \sum_{j=1}^{s_2} (id - (1\ 2))(z_1^j \otimes z_2^j).$$

It follows that

$$\begin{aligned}\frac{1}{2}Alt((1 \otimes \varphi)(z \otimes w - w \otimes z)) &= \frac{1}{2}Alt\Big(\sum_{i=1}^{s_1} (id - (2\ 3))(z \otimes w_1^i \otimes w_2^i) \\ &\quad - \frac{1}{2}\sum_{j=1}^{s_2} (id - (2\ 3))(w \otimes z_1^j \otimes z_2^j)\Big). \\ &= \sum_{i=1}^{s_1} z \wedge w_1^i \wedge w_2^i - \sum_{j=1}^{s_2} w \wedge z_1^j \wedge z_2^j \\ &= \sum_{i=1}^{s_1} w_1^i \wedge w_2^i \wedge z - \sum_{j=1}^{s_2} z_1^j \wedge z_2^j \wedge w \\ &= -(\varphi \wedge 1)(z \wedge w)\end{aligned}$$

from which

$$-\frac{1}{2}Alt((1 \otimes \varphi)d_{\mathfrak{g}}r(x)) = (\varphi \wedge 1)d_{\mathfrak{g}}r(x). \tag{3.42}$$

Similarly, since $d_{\mathfrak{g}}r(x) \in \Lambda^2\mathfrak{g}$, one proves that

$$-\frac{1}{2}Alt\Big([r^{12} + r^{13}, 1 \otimes \varphi(x)]\Big) = (d_{\mathfrak{g}}r \wedge 1)\varphi(x). \tag{3.43}$$

Thus, by equations (3.41) - (3.43) and by Proposition 3.3.7, one has

$$d_{\mathfrak{g}}(q)(x) = (\varphi \wedge 1)d_{\mathfrak{g}}r(x) + (d_{\mathfrak{g}}r \wedge 1)\varphi(x) = -d_{\mathfrak{g}^*}(\varphi)(x). \tag{3.44}$$

□

Let us show that $\varphi - q$ is a cocycle of the total complex (3.34). We must show that $0 = d(\varphi - q) = d_{\mathfrak{g}}(\varphi) - d_{\mathfrak{g}^*}(\varphi) - d_{\mathfrak{g}}(q) - d_{\mathfrak{g}^*}(q)$. On the other hand, we already know that $d_{\mathfrak{g}}(\varphi) = 0$ because of Proposition 3.2.24 and we know that $d_{\mathfrak{g}^*}(\varphi) + d_{\mathfrak{g}}(q) = 0$ because of Proposition 3.3.9. Thus we only need to show that $d_{\mathfrak{g}^*}(q) = 0$.

Lemma 3.3.10. *q is a cocycle of the complex on the first column of bicomplex* (3.31).

Proof. Let us first prove that $[\Sigma^{12}\Sigma^{13}\Sigma^{14}, Q^{234}] \equiv 0 \bmod h^{n+1}$. Using Relation (v), one has

$$\Sigma^{12}\Sigma^{13}\Sigma^{14}Q^{234} = \Sigma^{12}\Sigma^{13}\Sigma^{14}\Sigma^{23}\Sigma^{24}\Sigma^{34}(\Sigma^{23})^{-1}(\Sigma^{24})^{-1}(\Sigma^{34})^{-1}$$

$$
\begin{aligned}
&= \Sigma^{12}\Sigma^{13}\Sigma^{23}\Sigma^{14}\Sigma^{24}\Sigma^{34}(\Sigma^{23})^{-1}(\Sigma^{24})^{-1}(\Sigma^{34})^{-1}\\
&\equiv \Sigma^{23}\Sigma^{13}\Sigma^{12}\Sigma^{14}\Sigma^{24}\Sigma^{34}(\Sigma^{23})^{-1}(\Sigma^{24})^{-1}(\Sigma^{34})^{-1}\\
&\equiv \Sigma^{23}\Sigma^{13}\Sigma^{24}\Sigma^{14}\Sigma^{12}\Sigma^{34}(\Sigma^{23})^{-1}(\Sigma^{24})^{-1}(\Sigma^{34})^{-1}\\
&= \Sigma^{23}\Sigma^{24}\Sigma^{13}\Sigma^{14}\Sigma^{34}\Sigma^{12}(\Sigma^{23})^{-1}(\Sigma^{24})^{-1}(\Sigma^{34})^{-1}\\
&\equiv \Sigma^{23}\Sigma^{24}\Sigma^{34}\Sigma^{14}\Sigma^{13}\Sigma^{12}(\Sigma^{23})^{-1}(\Sigma^{24})^{-1}(\Sigma^{34})^{-1}\\
&\equiv \Sigma^{23}\Sigma^{24}\Sigma^{34}\Sigma^{14}(\Sigma^{23})^{-1}\Sigma^{12}\Sigma^{13}(\Sigma^{24})^{-1}(\Sigma^{34})^{-1}\\
&= \Sigma^{23}\Sigma^{24}\Sigma^{34}(\Sigma^{23})^{-1}\Sigma^{14}\Sigma^{12}(\Sigma^{24})^{-1}\Sigma^{13}(\Sigma^{34})^{-1}\\
&\equiv \Sigma^{23}\Sigma^{24}\Sigma^{34}(\Sigma^{23})^{-1}(\Sigma^{24})^{-1}\Sigma^{12}\Sigma^{14}\Sigma^{13}(\Sigma^{34})^{-1}\\
&\equiv \Sigma^{23}\Sigma^{24}\Sigma^{34}(\Sigma^{23})^{-1}(\Sigma^{24})^{-1}\Sigma^{12}(\Sigma^{34})^{-1}\Sigma^{13}\Sigma^{14}\\
&= \Sigma^{23}\Sigma^{24}\Sigma^{34}(\Sigma^{23})^{-1}(\Sigma^{24})^{-1}(\Sigma^{34})^{-1}\Sigma^{12}\Sigma^{13}\Sigma^{14}\\
&= Q^{234}\Sigma^{12}\Sigma^{13}\Sigma^{14} \mod h^{n+1}.
\end{aligned}
$$

Moreover one has the following equalities

$$
\begin{aligned}
[\Sigma^{12}\Sigma^{13}\Sigma^{14}, Q^{234}-1] &= [\Sigma^{12}\Sigma^{13}\Sigma^{14}, Q^{234}]\\
&= \Sigma^{12}\Sigma^{13}\Sigma^{23}\Sigma^{14}\Sigma^{24}\Sigma^{34}(\Sigma^{23})^{-1}(\Sigma^{24})^{-1}(\Sigma^{34})^{-1}\\
&\quad - \Sigma^{23}\Sigma^{24}\Sigma^{34}(\Sigma^{23})^{-1}(\Sigma^{24})^{-1}(\Sigma^{34})^{-1}\Sigma^{12}\Sigma^{13}\Sigma^{14},
\end{aligned}
\tag{3.45}
$$

$$
\begin{aligned}
&\Sigma^{12}\Sigma^{13}\Sigma^{23}\Sigma^{14}\Sigma^{24}\Sigma^{34}(\Sigma^{23})^{-1}(\Sigma^{24})^{-1}(\Sigma^{34})^{-1}\\
&\quad = (Q^{123}-1)\Sigma^{23}\Sigma^{13}\Sigma^{12}\Sigma^{14}\Sigma^{24}\Sigma^{34}(\Sigma^{23})^{-1}(\Sigma^{24})^{-1}(\Sigma^{34})^{-1}\\
&\qquad + \Sigma^{23}\Sigma^{13}\Sigma^{12}\Sigma^{14}\Sigma^{24}\Sigma^{34}(\Sigma^{23})^{-1}(\Sigma^{24})^{-1}(\Sigma^{34})^{-1},
\end{aligned}
\tag{3.46}
$$

$$
\begin{aligned}
&\Sigma^{23}\Sigma^{13}\Sigma^{12}\Sigma^{14}\Sigma^{24}\Sigma^{34}(\Sigma^{23})^{-1}(\Sigma^{24})^{-1}(\Sigma^{34})^{-1}\\
&\quad = \Sigma^{23}\Sigma^{13}(Q^{124}-1)\Sigma^{24}\Sigma^{14}\Sigma^{12}\Sigma^{34}(\Sigma^{23})^{-1}(\Sigma^{24})^{-1}(\Sigma^{34})^{-1}\\
&\qquad + \Sigma^{23}\Sigma^{24}\Sigma^{13}\Sigma^{14}\Sigma^{34}\Sigma^{12}(\Sigma^{23})^{-1}(\Sigma^{24})^{-1}(\Sigma^{34})^{-1},
\end{aligned}
\tag{3.47}
$$

$$
\begin{aligned}
&\Sigma^{23}\Sigma^{24}\Sigma^{13}\Sigma^{14}\Sigma^{34}\Sigma^{12}(\Sigma^{23})^{-1}(\Sigma^{24})^{-1}(\Sigma^{34})^{-1}\\
&\quad = \Sigma^{23}\Sigma^{24}(Q^{134}-1)\Sigma^{34}\Sigma^{14}\Sigma^{13}\Sigma^{12}(\Sigma^{23})^{-1}(\Sigma^{24})^{-1}(\Sigma^{34})^{-1}\\
&\qquad + \Sigma^{23}\Sigma^{24}\Sigma^{34}\Sigma^{14}\Sigma^{13}\Sigma^{12}(\Sigma^{23})^{-1}(\Sigma^{24})^{-1}(\Sigma^{34})^{-1},
\end{aligned}
\tag{3.48}
$$

$$
\begin{aligned}
&\Sigma^{23}\Sigma^{24}\Sigma^{34}\Sigma^{14}\Sigma^{13}\Sigma^{12}(\Sigma^{23})^{-1}(\Sigma^{24})^{-1}(\Sigma^{34})^{-1}\\
&\quad = \Sigma^{23}\Sigma^{24}\Sigma^{34}\Sigma^{14}(\Sigma^{23})^{-1}\Sigma^{23}\Sigma^{13}\Sigma^{12}(\Sigma^{23})^{-1}(\Sigma^{24})^{-1}(\Sigma^{34})^{-1}\\
&\quad = \Sigma^{23}\Sigma^{24}\Sigma^{34}\Sigma^{14}(\Sigma^{23})^{-1}(1-Q^{123})\Sigma^{23}\Sigma^{13}\Sigma^{12}(\Sigma^{23})^{-1}(\Sigma^{24})^{-1}(\Sigma^{34})^{-1}\\
&\qquad + \Sigma^{23}\Sigma^{24}\Sigma^{34}\Sigma^{14}(\Sigma^{23})^{-1}\Sigma^{12}\Sigma^{13}\Sigma^{23}(\Sigma^{23})^{-1}(\Sigma^{24})^{-1}(\Sigma^{34})^{-1},
\end{aligned}
\tag{3.49}
$$

$$\begin{aligned}
&\Sigma^{23}\Sigma^{24}\Sigma^{34}\Sigma^{14}(\Sigma^{23})^{-1}\Sigma^{12}\Sigma^{13}\Sigma^{23}(\Sigma^{23})^{-1}(\Sigma^{24})^{-1}(\Sigma^{34})^{-1}\\
&\quad=\Sigma^{23}\Sigma^{24}\Sigma^{34}(\Sigma^{23})^{-1}(\Sigma^{24})^{-1}\Sigma^{24}\Sigma^{14}\Sigma^{12}(\Sigma^{24})^{-1}\Sigma^{13}(\Sigma^{34})^{-1}\\
&\quad=\Sigma^{23}\Sigma^{24}\Sigma^{34}(\Sigma^{23})^{-1}(\Sigma^{24})^{-1}(1-Q^{124})\Sigma^{24}\Sigma^{14}\Sigma^{12}(\Sigma^{24})^{-1}\Sigma^{13}(\Sigma^{34})^{-1}\\
&\qquad+\Sigma^{23}\Sigma^{24}\Sigma^{34}(\Sigma^{23})^{-1}(\Sigma^{24})^{-1}\Sigma^{12}\Sigma^{14}\Sigma^{24}(\Sigma^{24})^{-1}\Sigma^{13}(\Sigma^{34})^{-1},
\end{aligned}\tag{3.50}$$

and

$$\begin{aligned}
&\Sigma^{23}\Sigma^{24}\Sigma^{34}(\Sigma^{23})^{-1}(\Sigma^{24})^{-1}\Sigma^{12}\Sigma^{14}\Sigma^{13}(\Sigma^{34})^{-1}\\
&\quad=\Sigma^{23}\Sigma^{24}\Sigma^{34}(\Sigma^{23})^{-1}(\Sigma^{24})^{-1}\Sigma^{12}(\Sigma^{34})^{-1}\Sigma^{34}\Sigma^{14}\Sigma^{13}(\Sigma^{34})^{-1}\\
&\quad=\Sigma^{23}\Sigma^{24}\Sigma^{34}(\Sigma^{23})^{-1}(\Sigma^{24})^{-1}\Sigma^{12}(\Sigma^{34})^{-1}(1-Q^{134})\Sigma^{34}\Sigma^{14}\Sigma^{13}(\Sigma^{34})^{-1}\\
&\qquad+\Sigma^{23}\Sigma^{24}\Sigma^{34}(\Sigma^{23})^{-1}(\Sigma^{24})^{-1}\Sigma^{12}(\Sigma^{34})^{-1}\Sigma^{13}\Sigma^{14}\Sigma^{34}(\Sigma^{34})^{-1}.
\end{aligned}\tag{3.51}$$

Combining equations (3.45) - (3.51) one has

$$\begin{aligned}
&[\Sigma^{12}\Sigma^{13}\Sigma^{14},Q^{234}-1]\\
&\quad=(Q^{123}-1)\Sigma^{23}\Sigma^{13}\Sigma^{12}\Sigma^{14}\Sigma^{24}\Sigma^{34}(\Sigma^{23})^{-1}(\Sigma^{24})^{-1}(\Sigma^{34})^{-1}\\
&\qquad+\Sigma^{23}\Sigma^{13}(Q^{124}-1)\Sigma^{24}\Sigma^{14}\Sigma^{12}\Sigma^{34}(\Sigma^{23})^{-1}(\Sigma^{24})^{-1}(\Sigma^{34})^{-1}\\
&\qquad+\Sigma^{23}\Sigma^{24}(Q^{134}-1)\Sigma^{34}\Sigma^{14}\Sigma^{13}\Sigma^{12}(\Sigma^{23})^{-1}(\Sigma^{24})^{-1}(\Sigma^{34})^{-1}\\
&\qquad+\Sigma^{23}\Sigma^{24}\Sigma^{34}\Sigma^{14}(\Sigma^{23})^{-1}(1-Q^{123})\Sigma^{23}\Sigma^{13}\Sigma^{12}(\Sigma^{23})^{-1}(\Sigma^{24})^{-1}(\Sigma^{34})^{-1}\\
&\qquad+\Sigma^{23}\Sigma^{24}\Sigma^{34}(\Sigma^{23})^{-1}(\Sigma^{24})^{-1}(1-Q^{124})\Sigma^{24}\Sigma^{14}\Sigma^{12}(\Sigma^{24})^{-1}\Sigma^{13}(\Sigma^{34})^{-1}\\
&\qquad+\Sigma^{23}\Sigma^{24}\Sigma^{34}(\Sigma^{23})^{-1}(\Sigma^{24})^{-1}\Sigma^{12}(\Sigma^{34})^{-1}(1-Q^{134})\Sigma^{34}\Sigma^{14}\Sigma^{13}(\Sigma^{34})^{-1}.
\end{aligned}\tag{3.52}$$

It follows

$$\begin{aligned}
&[\Sigma^{12}\Sigma^{13}\Sigma^{14},Q^{234}-1]\\
&\quad=(Q^{123}-1)(1+(r^{13}+r^{12}+r^{14})h)\\
&\qquad-(1+(r^{24}+r^{34}+r^{14})h)(Q^{123}-1)(1+(r^{13}+r^{12}-r^{24}-r^{34})h)\\
&\qquad+(1+(r^{23}+r^{13})h)(Q^{124}-1)(1+(r^{14}+r^{12}-r^{23})h)\\
&\qquad-(1+r^{34}h)(Q^{124}-1)(1+(r^{14}+r^{12}+r^{13}-r^{34})h)\\
&\qquad+(1+(r^{23}+r^{24})h)(Q^{134}-1)(1+(r^{14}+r^{13}+r^{12}-r^{23}-r^{24})h)\\
&\qquad-(1+r^{12}h)(Q^{134}-1)(1+(r^{14}+r^{13})h)+O(h^{n+3})\\
&\quad=\big(-[r^{14}+r^{24}+r^{34},Q^{123}-1]+[r^{13}+r^{23}-r^{34},Q^{124}-1]\\
&\qquad-[r^{12}-r^{23}-r^{24},Q^{134}-1]\big)h+O(h^{n+3}).
\end{aligned}\tag{3.53}$$

On the other hand

$$[\Sigma^{12}\Sigma^{13}\Sigma^{14},Q^{234}-1]=[r^{12}+r^{13}+r^{14},Q^{234}-1]h+O(h^{n+3}).\tag{3.54}$$

Equations (3.53) and (3.54) imply

$$0\equiv[r^{12}+r^{13}+r^{14},Q^{234}-1]+[r^{14}+r^{24}+r^{34},Q^{123}-1]$$

$$+ \left[r^{12} - r^{23} - r^{24}, Q^{134} - 1\right] - \left[r^{13} + r^{23} - r^{34}, Q^{124} - 1\right] \bmod h^{n+2}$$

from which

$$\begin{aligned} 0 =& \left[r^{12} + r^{13} + r^{14}, q^{234}\right] + \left[r^{14} + r^{24} + r^{34}, q^{123}\right] \\ &+ \left[r^{12} - r^{23} - r^{24}, q^{134}\right] - \left[r^{13} + r^{23} - r^{34}, q^{124}\right] \\ &= \big(id - (1\ 2) + (1\ 3\ 2) - (1\ 4\ 3\ 2)\big)\left[r^{12} + r^{13} + r^{14}, q^{234}\right]. \end{aligned} \tag{3.55}$$

Therefore $Alt\Big(\left[r^{12} + r^{13} + r^{14}, q^{234}\right]\Big) = 0$ because

$$\begin{aligned} Alt\Big(\left[r^{12} + r^{13} + r^{14}, q^{234}\right]\Big) &= \sum_{\sigma \in S_4} sgn(\sigma)\sigma\left[r^{12} + r^{13} + r^{14}, q^{234}\right] \\ &= \big(id - (1\ 3) - (2\ 4) + (1\ 2\ 4) + (1\ 3)(2\ 4) + (1\ 4\ 3)\big)\times \\ &\quad \times \big(id - (1\ 2) + (1\ 3\ 2) - (1\ 4\ 3\ 2)\big)\left[r^{12} + r^{13} + r^{14}, q^{234}\right] = 0. \end{aligned}$$

□

If $\varphi - q$ is a coboundary of the total complex (3.34), there exists an element $\psi \in \Lambda^2\mathfrak{g}$ satisfying $\varphi - q = d(\psi) = d_{\mathfrak{g}}(\psi) + d_{\mathfrak{g}^*}(\psi)$ from which $\varphi = d_{\mathfrak{g}}(\psi)$ and $q = -d_{\mathfrak{g}^*}(\psi)$. Hence, since, by Proposition 3.3.7, $-d_{\mathfrak{g}^*}(\psi) = -\frac{1}{2}Alt\Big(\left[r^{12}+r^{13}, \psi^{23}\right]\Big)$, $\Sigma + \psi h^n$ satisfy conditions (ii) - (iv) mod h^{n+1} and (v) mod h^{n+2}. It follows that the obstruction to the existence lies in the cohomology group H^1 of the total cochain complex (3.34) and the obstruction to the uniqueness lies in the cohomology group H^0 of the total cochain complex (3.34).

Summarizing, V. G. Drinfeld in his notes on existence and uniqueness of universal triangular R-matrices proved the following theorem:

Theorem 3.3.11. *Let H be a quantization of a triangular Lie bialgebra $(\mathfrak{g}, [\ ,\ ], r)$ over $\mathbb{C}$ (cf. Definition 3.3.2). Then the obstructions to the existence and uniqueness of the universal triangular R-matrix lie respectively in the 1^{st} and 0^{th} cohomology groups of the total cochain complex* (3.34).

3.4 Considerations on Question and Answer 47

If H^1 is 0 the existence of Σ of section 3.3 is secured, but in general this condition is not satisfied.

Proposition 3.4.1. *Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra and $r = 0 \in \Lambda^2\mathfrak{g}$. Then $H^1 \neq 0$.*

Proof. Let us first point out that, as $r = 0$, the differential of the first column of the bicomplex (3.31) is the 0 map by Proposition 3.3.7. Let $a \in \mathrm{Hom}(\mathfrak{g}, \Lambda^2\mathfrak{g})$ and $b \in \Lambda^3\mathfrak{g}$. $a + b$ is a 1-cocycle of the total complex if $d(a + b) = d_{\mathfrak{g}}a - d_{\mathfrak{g}^*}a + d_{\mathfrak{g}}b + d_{\mathfrak{g}^*}b = 0$. In particular a must be a 1-cocycle of the Chevalley-Eilenberg complex with Lie algebra $\mathfrak{g}$ and $\mathfrak{g}$-module $\Lambda^2\mathfrak{g}$ and $d_{\mathfrak{g}}b - d_{\mathfrak{g}^*}a = 0$ (the condition $d_{\mathfrak{g}^*}b = 0$ gives not additional informations on b because, as already pointed out, $d_{\mathfrak{g}^*}$ of the first column is the 0 map). Since $\mathfrak{g}$ is a simple, finite dimensional Lie algebra and $\Lambda^2\mathfrak{g}$ is finite dimensional, $H^0(\mathfrak{g},\Lambda^2\mathfrak{g}) = 0$ and $H^1(\mathfrak{g},\Lambda^2\mathfrak{g}) = 0$ therefore there exists a unique $\theta \in \Lambda^2\mathfrak{g}$ such that $a = d_\mathfrak{g}\theta$. It follows that condition $d_\mathfrak{g} b - d_{\mathfrak{g}^*} a = 0$ becomes $d_\mathfrak{g} b = 0$ indeed $d_{\mathfrak{g}^*} a = d_{\mathfrak{g}^*} d_\mathfrak{g}\theta = d_\mathfrak{g} d_{\mathfrak{g}^*}\theta = 0$. Therefore a must be equal to $d_\mathfrak{g}\theta$ for $\theta \in \Lambda^2\mathfrak{g}$ and b must be an invariant of $\Lambda^3\mathfrak{g}$ under the adjoint action of $\mathfrak{g}$. On the other hand $a + b = d(\psi)$ for $\psi \in \Lambda^2\mathfrak{g}$ if and only if $a = d_\mathfrak{g}\psi$, which implies that $\psi = \theta$ because $H^0(\mathfrak{g},\Lambda^2\mathfrak{g}) = 0$, and $b = d_{\mathfrak{g}^*}\psi = 0$. It follows that $dim(H^1) = 1$ because $dim(\Lambda^3\mathfrak{g})^\mathfrak{g} = 1$. □

$H^1 \neq 0$ does not prevent Σ from existing as shown in the following example.

Let $\mathfrak{g}$ a simple, finite dimensional Lie algebra and let us consider the following Hopf algebra: $(H, \eta_H, \mu_H, \Delta_H, \varepsilon_H, \mathcal{S}_H)$ where $H = \mathcal{U}(\mathfrak{g})[[h]]$ and η_H, μ_H, Δ_H, ε_H, $\mathcal{S}_H$ are respectively the unit, product, coproduct, counit and antipode of $\mathcal{U}(\mathfrak{g})$ extended to H by $\mathbb{C}[[h]]$-linearity. Let us, then, consider $r = 0 \in \Lambda^2\mathfrak{g}$. r is clearly a solution of the classical Yang-Baxter equation (3.22) and, for any $x \in \mathfrak{g}$, $\Delta(x)^{op} - \Delta(x) = \Delta_0(x)^{op} - \Delta_0(x) = 0$. It follows that H is a quantization of the triangular Lie bialgebra $\big(\mathfrak{g}, [\ ,\ ], 0\big)$. Since $H^0(\mathfrak{g},\Lambda^2\mathfrak{g}) = 0$, because $\mathfrak{g}$ is a simple, finite dimensional Lie algebra, the cohomology group H^0 of the total cochain complex (3.31) is zero from which the uniqueness of Σ (if it exists) follows. Moreover we already know that Σ exists because $\Sigma = 1$ satisfies the conditions (1) - (5). Let us apply anyway the procedure in sections 3.2 and 3.3 to the triangular Lie bialgebra $\big(\mathfrak{g}, [\ ,\ ], 0\big)$ and the quantum group H and let us show that the procedure gives $\Sigma = 1$. We will show that if $\Sigma \equiv 1 \mod h^n$, then the procedure of sections 3.2 and 3.3 modifies Σ as $\Sigma \equiv 1 \mod h^{n+1}$. So let us consider $\Sigma \in H\widehat{\otimes}H$ such that $\Sigma \equiv 1 \mod h^n$ with $n \geq 2$. Clearly $\Sigma \equiv 1 \mod h^n$ satisfies conditions (i)-(v). Let us write $\Sigma \equiv 1 + \Sigma_n h^n \mod h^{n+1}$ with $\Sigma_n \in \mathcal{U}(\mathfrak{g})^{\otimes 2}$. $\Sigma\Sigma^{21} \equiv 1 + (\Sigma_n + \Sigma_n^{21})h^n \mod h^{n+1}$, therefore, after the first correction, Σ becomes $\Sigma - \frac{\Sigma_n + \Sigma_n^{21}}{2}h^n \equiv 1 + \frac{\Sigma_n - \Sigma_n^{21}}{2}h^n \mod h^{n+1}$. As $\frac{\Sigma_n - \Sigma_n^{21}}{2} \in \Lambda^2\mathcal{U}(\mathfrak{g})$, and $\delta = \frac{1}{2}\big((\Delta_0 \otimes 1)(\Sigma_n - \Sigma_n^{21}) + (\Sigma_n^{12} - \Sigma_n^{21}) - (1 \otimes \Delta_0)(\Sigma_n - \Sigma_n^{21}) - (\Sigma_n^{23} - \Sigma_n^{32})\big)$, $-\frac{\Sigma_n - \Sigma_n^{21}}{2}$ is a solution of equation (3.11). As second modification, we set Σ as $\Sigma + \Big(-\frac{\Sigma_n - \Sigma_n^{21}}{2}\Big)h^n \equiv 1 \mod h^{n+1}$. It follows that $\varphi = 0$ and $q = 0$ indeed

$$\varphi(x) \equiv \frac{\Delta^{op}(x) - \Sigma^{-1}\Delta(x)\Sigma}{h^n} \equiv 0 \mod h$$

for any $x \in \mathcal{U}(\mathfrak{g})$, because $\Delta(x) = \Delta_0(x) = \Delta_0^{op}(x) = \Delta^{op}(x)$, and

$$q \equiv \frac{Q - 1}{h^{n+1}} \equiv 0 \mod h$$

because $Q = \Sigma^{12}\Sigma^{13}\Sigma^{23}\big(\Sigma^{12}\big)^{-1}\big(\Sigma^{13}\big)^{-1}\big(\Sigma^{23}\big)^{-1} \equiv 1 + \big(\Sigma_{n+1}^{12} + \Sigma_{n+1}^{13} + \Sigma_{n+1}^{23} - \Sigma_{n+1}^{12} - \Sigma_{n+1}^{13} - \Sigma_{n+1}^{23}\big) = 1 \mod h^{n+2}$. $\psi = 0$ satisfies $d(\psi) = \varphi - q$ from which $\Sigma + \psi h^n = \Sigma \equiv 1 \mod h^{n+1}$ satisfies conditions (i) - (v). Therefore the procedure in sections 3.2 and 3.3 modifies $\Sigma \equiv 1 \mod h^2$ in $\Sigma = 1$.

3.5 Question and Answer 47'

The arguments of section 3.3 don't extend verbatim to the pseudotriangular case so we have to modify the previous arguments. Let Σ be an element of $H\widehat{\otimes}H$ satisfying:

(i) $\Sigma \equiv 1 + rh \bmod h^2$;

(ii) $\Sigma\Sigma^{21} \equiv 1 \bmod h^n$;

(iii) $(1 \otimes \Delta)(\Sigma) \equiv \Sigma^{12}\Sigma^{13} \bmod h^n$;

(iv) $\Delta^{op}(a) \equiv \Sigma^{-1}\Delta(a)\Sigma \bmod h^n$;

(v) $\Sigma^{12}\Sigma^{13}\Sigma^{23} \equiv \Sigma^{23}\Sigma^{13}\Sigma^{12} \bmod h^{n+1}$

where $n \geq 2$.
As before we want to correct Σ without changing it mod h^n such that the relations (ii)-(iv) are satisfied mod h^{n+1} and the relation (v) is satisfied mod h^{n+2}.

First of all let us modify Σ, as done in section 3.2, in order to have $\Sigma\Sigma^{21} \equiv 1 \bmod h^{n+1}$. As shown in section 3.2, after this modification the freedom in choosing Σ reduces to an element of $\Lambda^2\mathcal{U}(\mathfrak{g})$. We want to modify Σ as $\Sigma' = \Sigma - \psi h^n$ with $\psi \in \Lambda^2\mathcal{U}(\mathfrak{g})$ in order to have $(1 \otimes \Delta)(\Sigma') \equiv \Sigma'^{12}\Sigma'^{13} \bmod h^{n+1}$. Let us define the element $a \in \mathcal{U}(\mathfrak{g})^{\otimes 3}$ as

$$a \equiv \frac{(1 \otimes \Delta)(\Sigma) - \Sigma^{12}\Sigma^{23}}{h^n} \bmod h. \tag{3.56}$$

The following proposition gives an equivalent condition to the equation $(1 \otimes \Delta)(\Sigma - \psi h^n) \equiv (\Sigma - \psi h^n)^{12}(\Sigma - \psi h^n)^{13} \bmod h^{n+1}$:

Proposition 3.5.1. *There exists an element $\psi \in \Lambda^2\mathcal{U}(\mathfrak{g})$ satisfying the equation*

$$(1 \otimes \Delta)(\Sigma - \psi h^n) \equiv (\Sigma - \psi h^n)^{12}(\Sigma - \psi h^n)^{13} \mod h^{n+1} \tag{3.57}$$

if and only if

$$a = (1 \otimes \Delta_0)(\psi) - \psi^{12} - \psi^{13}. \tag{3.58}$$

Proof. It is a consequence of the following equations:

$$\begin{aligned}
&(1 \otimes \Delta)(\Sigma - \psi h^n) - (\Sigma - \psi h^n)^{12}(\Sigma - \psi h^n)^{13} \\
&= (1 \otimes \Delta)(\Sigma) - (1 \otimes \Delta)(\psi)h^n - \Sigma^{12}\Sigma^{13} + \psi^{12}\Sigma^{13}h^n + \Sigma^{12}\psi^{13}h^n - \psi^{12}\psi^{13}h^{2n} \\
&\equiv \left(a - (1 \otimes \Delta_0)(\psi) + \psi^{12} + \psi^{13}\right)h^n \bmod h^{n+1}.
\end{aligned}$$

□

Lemma 3.5.2. *The following equalities hold:*

$$(1 \otimes 1 \otimes \Delta)(1 \otimes \Delta)(\Sigma) \equiv \Sigma^{12}(1 \otimes 1 \otimes \Delta)(\Sigma^{13}) \mod h^n, \tag{3.59}$$

$$\Sigma^{12}(1 \otimes 1 \otimes \Delta)(\Sigma^{13}) \equiv \Sigma^{12}\Sigma^{13}\Sigma^{14} \mod h^n, \tag{3.60}$$

$$(1 \otimes \Delta \otimes 1)(1 \otimes \Delta)(\Sigma) \equiv (1 \otimes \Delta \otimes 1)(\Sigma^{12})\Sigma^{14} \mod h^n, \tag{3.61}$$

$$(1 \otimes \Delta \otimes 1)(\Sigma^{12})\Sigma^{14} \equiv \Sigma^{12}\Sigma^{13}\Sigma^{14} \mod h^n. \tag{3.62}$$

Proof. Equations (3.59) and (3.61) follow because Δ is a homomorphism of algebra and Σ satisfies condition (iii), indeed one has

$$(1 \otimes 1 \otimes \Delta)(1 \otimes \Delta)(\Sigma) \equiv (1 \otimes 1 \otimes \Delta)(\Sigma^{12}\Sigma^{13})$$

$$
\begin{aligned}
&= (1\otimes 1\otimes \Delta)(\Sigma^{12})(1\otimes 1\otimes \Delta)(\Sigma^{13})\\
&= \Sigma^{12}(1\otimes 1\otimes \Delta)(\Sigma^{13}) \text{ mod } h^n
\end{aligned}
$$

and

$$
\begin{aligned}
(1\otimes \Delta\otimes 1)(1\otimes \Delta)(\Sigma) &\equiv (1\otimes \Delta\otimes 1)(\Sigma^{12}\Sigma^{13})\\
&= (1\otimes \Delta\otimes 1)(\Sigma^{12})(1\otimes \Delta\otimes 1)(\Sigma^{13})\\
&= (1\otimes \Delta\otimes 1)(\Sigma^{12})\Sigma^{14} \text{ mod } h^n.
\end{aligned}
$$

Equations (3.60) and (3.62) follow directly by condition (iii). □

Lemma 3.5.3. *The following equations hold:*

$$(\Sigma^{23})^{-1}(1\otimes \Delta)(\Sigma)\Sigma^{23} \equiv (1\otimes \Delta^{op})(\Sigma) \ \textit{mod}\ h^{n+1}, \tag{3.63}$$

$$(\Sigma^{23})^{-1}\Sigma^{12}\Sigma^{13}\Sigma^{23} \equiv \Sigma^{13}\Sigma^{12} \ \textit{mod}\ h^{n+1}, \tag{3.64}$$

$$(\Sigma^{23})^{-1}(1\otimes \Delta)(\Sigma)\Sigma^{23} \equiv (\Sigma^{23})^{-1}\Sigma^{12}\Sigma^{13}\Sigma^{23} \ \textit{mod}\ h^{n}, \tag{3.65}$$

$$(1\otimes \Delta^{op})(\Sigma) \equiv \Sigma^{13}\Sigma^{12} \ \textit{mod}\ h^{n}. \tag{3.66}$$

Proof. Equation (3.63) follows from $\Sigma \equiv 1$ mod h indeed

$$
\begin{aligned}
(\Sigma^{23})^{-1}(1\otimes \Delta)(\Sigma)\Sigma^{23} &= (\Sigma^{23})^{-1}(1\otimes \Delta)\Big(1+\sum_{i\geq 1}\Sigma_i h^i\Big)\Sigma^{23}\\
&= 1+\sum_{i\geq 1}(\Sigma^{23})^{-1}(1\otimes \Delta)(\Sigma_i)\Sigma^{23}h^i\\
&= 1+\sum_{i\geq 1}((1\otimes \Delta^{op})(\Sigma_i)+O(h^n))h^i\\
&= 1+\sum_{i\geq 1}(1\otimes \Delta^{op})(\Sigma_i)h^i+O(h^{n+1})\\
&= (1\otimes \Delta^{op})(\Sigma)+O(h^{n+1}).
\end{aligned}
$$

Equation (3.64) follows from $\Sigma^{12}\Sigma^{13}\Sigma^{23} \equiv \Sigma^{23}\Sigma^{13}\Sigma^{12}$ mod h^{n+1} indeed

$$
\begin{aligned}
(\Sigma^{23})^{-1}\Sigma^{12}\Sigma^{13}\Sigma^{23} &\equiv (\Sigma^{23})^{-1}\Sigma^{23}\Sigma^{13}\Sigma^{12}\\
&\equiv \Sigma^{13}\Sigma^{12} \text{ mod } h^{n+1}.
\end{aligned}
$$

Equation (3.65) follows immediately from condition (iii) and, using again condition (iii), equation (3.66) follows indeed

$$
\begin{aligned}
(1\otimes \Delta^{op})(\Sigma) &= (2\ 3)(1\otimes \Delta)(\Sigma)\\
&\equiv (2\ 3)(\Sigma^{12}\Sigma^{13})\\
&\equiv \Sigma^{13}\Sigma^{12} \text{ mod } h^n.
\end{aligned}
$$

□

Proposition 3.5.4. *The following equations hold:*

$$(1 \otimes \Delta_0 \otimes 1)(a) + a^{123} = (1 \otimes 1 \otimes \Delta_0)(a) + a^{134} \tag{3.67}$$

and

$$(2\ 3)a = a. \tag{3.68}$$

Proof. Since Δ is a homomorphism of algebras and $\Sigma \equiv 1 \bmod h$, by Lemma 3.5.2, one has

$$\begin{aligned}
(1 \otimes \Delta_0 \otimes 1)(a) &\equiv (1 \otimes \Delta \otimes 1)(a) \\
&\equiv \frac{1}{h^n}((1 \otimes \Delta \otimes 1)(1 \otimes \Delta)(\Sigma) \\
&\qquad - (1 \otimes \Delta \otimes 1)(\Sigma^{12})(1 \otimes \Delta \otimes 1)(\Sigma^{13})) \\
&\equiv \frac{1}{h^n}\Big((1 \otimes 1 \otimes \Delta)(1 \otimes \Delta)(\Sigma) - \big(\Sigma^{12}\Sigma^{13} + a^{123}h^n\big)\Sigma^{14}\Big) \\
&= \frac{1}{h^n}((1 \otimes 1 \otimes \Delta)(1 \otimes \Delta)(\Sigma) - \Sigma^{12}\Sigma^{13}\Sigma^{14} - a^{123}\Sigma^{14}h^n) \\
&\equiv \frac{1}{h^n}((1 \otimes 1 \otimes \Delta)(1 \otimes \Delta)(\Sigma) - \Sigma^{12}\Sigma^{13}\Sigma^{14} - a^{123}h^n) \\
&= \frac{1}{h^n}((1 \otimes 1 \otimes \Delta)(1 \otimes \Delta)(\Sigma) - \Sigma^{12}\Sigma^{13}\Sigma^{14} - a^{123}h^n \\
&\qquad - \Sigma^{12}a^{134}h^n + \Sigma^{12}a^{134}h^n) \\
&\equiv \frac{1}{h^n}((1 \otimes 1 \otimes \Delta)(1 \otimes \Delta)(\Sigma) - \Sigma^{12}\Sigma^{13}\Sigma^{14} - \Sigma^{12}a^{134}h^n \\
&\qquad - a^{123}h^n + a^{134}h^n) \\
&\equiv \frac{1}{h^n}((1 \otimes 1 \otimes \Delta)(1 \otimes \Delta)(\Sigma) - \Sigma^{12}(1 \otimes 1 \otimes \Delta)(\Sigma^{13}) - a^{123}h^n \\
&\qquad + a^{134}h^n) \\
&= \frac{1}{h^n}((1 \otimes 1 \otimes \Delta)(1 \otimes \Delta)(\Sigma) - (1 \otimes 1 \otimes \Delta)(\Sigma^{12}\Sigma^{13}) - a^{123}h^n \\
&\qquad + a^{134}h^n) \\
&\equiv (1 \otimes 1 \otimes \Delta)(a) - a^{123} + a^{134} \\
&\equiv (1 \otimes 1 \otimes \Delta_0)(a) - a^{123} + a^{134} \bmod h.
\end{aligned}$$

Therefore one has

$$(1 \otimes \Delta_0 \otimes 1)(a) = (1 \otimes 1 \otimes \Delta_0)(a) + a^{134} - a^{123},$$

because $a \in \mathcal{U}(\mathfrak{g})^{\otimes 3}$, from which equation (3.67) follows.
Moreover, using equations (3.63) and (3.64), one has

$$\begin{aligned}
(2\ 3)a &\equiv (2\ 3)\frac{(1 \otimes \Delta)(\Sigma) - \Sigma^{12}\Sigma^{13}}{h^n} \\
&= \frac{(1 \otimes \Delta^{op})(\Sigma) - \Sigma^{13}\Sigma^{12}}{h^n}
\end{aligned}$$

$$\begin{aligned}
&\equiv (\Sigma^{23})^{-1}\frac{(1\otimes\Delta)(\Sigma)-\Sigma^{12}\Sigma^{13}}{h^n}\Sigma^{23}\\
&\equiv (\Sigma^{23})^{-1}a\Sigma^{23}\\
&\equiv a \bmod h
\end{aligned}$$

from which $(2\ 3)a = a$. □

Lemma 3.5.5. *The following equation holds*

$$(\Delta\otimes 1)(\Sigma) \equiv \Sigma^{23}\Sigma^{13} - (1\ 3\ 2)ah^n \ \ mod\ h^{n+1}. \tag{3.69}$$

Proof. Since $\Sigma\Sigma^{21} \equiv 1 \bmod h^{n+1}$, one has

$$\begin{aligned}
(\Delta\otimes 1)(\Sigma) &= (1\ 3\ 2)(1\otimes\Delta)(\Sigma^{21})\\
&\equiv (1\ 3\ 2)(1\otimes\Delta)(\Sigma^{-1})\\
&= (1\ 3\ 2)\big((1\otimes\Delta)(\Sigma)\big)^{-1}\\
&\equiv (1\ 3\ 2)(\Sigma^{12}\Sigma^{13}+ah^n)^{-1}\\
&\equiv (1\ 3\ 2)\Big((\Sigma^{13})^{-1}(\Sigma^{12})^{-1}\Big) - (1\ 3\ 2)ah^n\\
&\equiv (1\ 3\ 2)(\Sigma^{31}\Sigma^{21}) - (1\ 3\ 2)ah^n\\
&\equiv \Sigma^{23}\Sigma^{13} - (1\ 3\ 2)ah^n \bmod h^{n+1}.
\end{aligned}$$

□

Lemma 3.5.6. *The following equations hold:*

$$(\Delta\otimes 1\otimes 1)(1\otimes\Delta)(\Sigma) \equiv (\Delta\otimes 1\otimes 1)\big(\Sigma^{12}\Sigma^{13}\big)\ \ mod\ h^n, \tag{3.70}$$

$$(\Delta\otimes\Delta)(\Sigma) \equiv \Sigma^{23}\Sigma^{13}\Sigma^{24}\Sigma^{14}\ \ mod\ h^n, \tag{3.71}$$

$$(\Delta\otimes 1\otimes 1)(1\otimes\Delta)(\Sigma) \equiv \Sigma^{23}\Sigma^{13}\Sigma^{24}\Sigma^{14}\ \ mod\ h^n, \tag{3.72}$$

$$(\Delta\otimes 1\otimes 1)\big(\Sigma^{12}\Sigma^{13}\big) \equiv \Sigma^{23}\Sigma^{13}\Sigma^{24}\Sigma^{14}\ \ mod\ h^n. \tag{3.73}$$

Proof. By condition (iii) and equation (3.69), one has

$$\begin{aligned}
(\Delta\otimes\Delta)(\Sigma) &= (\Delta\otimes 1\otimes 1)(1\otimes\Delta)(\Sigma)\\
&\equiv (\Delta\otimes 1\otimes 1)(\Sigma^{12}\Sigma^{13})\\
&\equiv (\Delta\otimes 1\otimes 1)(\Sigma^{12}\Sigma^{13})\\
&\equiv \Sigma^{23}\Sigma^{13}\Sigma^{24}\Sigma^{14} \bmod h^n
\end{aligned}$$

and

$$\begin{aligned}
(\Delta\otimes\Delta)(\Sigma) &= (1\otimes 1\otimes\Delta)(\Delta\otimes 1)(\Sigma)\\
&\equiv (1\otimes 1\otimes\Delta)(\Sigma^{23}\Sigma^{13})\\
&\equiv \Sigma^{23}\Sigma^{24}\Sigma^{13}\Sigma^{14}
\end{aligned}$$

$$= \Sigma^{23}\Sigma^{13}\Sigma^{24}\Sigma^{14} \bmod h^n.$$

In particular equations (3.70)-(3.73) follow. □

Lemma 3.5.7. *The following equation holds:*

$$(\Delta_0 \otimes 1 \otimes 1)(a) - a^{312} - a^{412} = -(1\ 3)(2\ 4)(\Delta_0 \otimes 1 \otimes 1)(a) + a^{134} + a^{234}. \quad (3.74)$$

Proof. Using the definition of a one has

$$\begin{aligned}(1 \otimes \Delta)(\Sigma^{21}) &\equiv \left((1 \otimes \Delta)(\Sigma)\right)^{-1} \equiv \left(\Sigma^{12}\Sigma^{13} + ah^n\right)^{-1} \\ &\equiv \Sigma^{31}\Sigma^{21} - ah^n \bmod h^{n+1}.\end{aligned} \quad (3.75)$$

By equation (3.69) one has

$$\begin{aligned}(\Delta \otimes 1)(\Sigma^{21}) &\equiv \left((\Delta \otimes 1)(\Sigma)\right)^{-1} \equiv \left(\Sigma^{23}\Sigma^{13} - a^{312}h^n\right)^{-1} \\ &\equiv \Sigma^{31}\Sigma^{32} + a^{312}h^n \bmod h^{n+1}.\end{aligned} \quad (3.76)$$

Therefore, using Lemma 3.5.6 and equations (3.75) and (3.76), one has

$$\begin{aligned}
(\Delta_0 \otimes 1 \otimes 1)(a) &\equiv (\Delta \otimes 1 \otimes 1)(a) \\
&\equiv \frac{(\Delta \otimes 1 \otimes 1)(1 \otimes \Delta)(\Sigma) - (\Delta \otimes 1 \otimes 1)(\Sigma^{12}\Sigma^{13})}{h^n} \\
&\equiv \frac{(\Delta \otimes \Delta)(\Sigma) - \left(\Sigma^{23}\Sigma^{13} - a^{312}h^n\right)\left(\Sigma^{24}\Sigma^{14} - a^{412}h^n\right)}{h^n} \\
&\equiv \frac{(\Delta \otimes \Delta)(\Sigma) - \Sigma^{23}\Sigma^{13}\Sigma^{24}\Sigma^{14}}{h^n} + a^{312} + a^{412} \\
&\equiv (1\ 3)(2\ 4)\frac{(1\ 3)(2\ 4)(\Delta \otimes \Delta)(\Sigma) - (1\ 3)(2\ 4)\left(\Sigma^{23}\Sigma^{13}\Sigma^{24}\Sigma^{14}\right)}{h^n} \\
&\quad + a^{312} + a^{412} \\
&\equiv (1\ 3)(2\ 4)\frac{(\Delta \otimes \Delta)(\Sigma^{21}) - \Sigma^{41}\Sigma^{31}\Sigma^{42}\Sigma^{32}}{h^n} + a^{312} + a^{412} \\
&\equiv (1\ 3)(2\ 4)\frac{(\Delta \otimes 1 \otimes 1)(1 \otimes \Delta)(\Sigma^{21}) - \Sigma^{41}\Sigma^{42}\Sigma^{31}\Sigma^{32}}{h^n} \\
&\quad + a^{312} + a^{412} \\
&\equiv (1\ 3)(2\ 4)\frac{(\Delta \otimes 1 \otimes 1)\left(\Sigma^{31}\Sigma^{21} - ah^n\right) - \Sigma^{41}\Sigma^{42}\Sigma^{31}\Sigma^{32}}{h^n} \\
&\quad + a^{312} + a^{412} \\
&\equiv (1\ 3)(2\ 4)\frac{(\Delta \otimes 1 \otimes 1)\left(\Sigma^{31}\Sigma^{21}\right) - \Sigma^{41}\Sigma^{42}\Sigma^{31}\Sigma^{32}}{h^n} \\
&\quad + a^{312} + a^{412} - (1\ 3)(2\ 4)(\Delta \otimes 1 \otimes 1)(a) \\
&\equiv (1\ 3)(2\ 4)\frac{\left(\Sigma^{41}\Sigma^{42} + a^{412}h^n\right)\left(\Sigma^{31}\Sigma^{32} + a^{312}h^n\right) - \Sigma^{41}\Sigma^{42}\Sigma^{31}\Sigma^{32}}{h^n} \\
&\quad - (1\ 3)(2\ 4)(\Delta \otimes 1 \otimes 1)(a) + a^{312} + a^{412}
\end{aligned}$$

$$\equiv a^{234} + a^{134} - (1\ 3)(2\ 4)(\Delta \otimes 1 \otimes 1)(a) + a^{312} + a^{412}$$
$$\equiv a^{234} + a^{134} - (1\ 3)(2\ 4)(\Delta_0 \otimes 1 \otimes 1)(a) + a^{312} + a^{412} \bmod h.$$

Since $a \in \mathcal{U}(\mathfrak{g})^{\otimes 3}$, it follows that

$$(\Delta_0 \otimes 1 \otimes 1)(a) = a^{234} + a^{134} - (1\ 3)(2\ 4)(\Delta_0 \otimes 1 \otimes 1)(a) + a^{312} + a^{412}$$

from which one obtains equation (3.74). □

Since $a \in \mathcal{U}(\mathfrak{g})^{\otimes 3}$, we can write $a = \sum_{i=1}^{s} a_1^i \otimes a_2^i$ with $s \in \mathbb{Z}_+$, $a_1^i \in \mathcal{U}(\mathfrak{g})$ elements of a Poincaré-Birkhoff-Witt basis and $a_2^i \in \mathcal{U}(\mathfrak{g})^{\otimes 2}$ for $i = 1, \ldots, s$. By equation (3.67), one has

$$\begin{aligned} 0 &= a^{134} - (1 \otimes \Delta_0 \otimes 1)(a) + (1 \otimes 1 \otimes \Delta_0)(a) - a^{123} \\ &= \sum_{i=1}^{s} a_1^i \otimes \Big(1 \otimes a_2^i - (\Delta_0 \otimes 1)(a_2^i) + (1 \otimes \Delta_0)(a_2^i) - a_2^i \otimes 1\Big). \end{aligned}$$

As $a_1^1, \ldots, a_1^s$ are linearly independent, it follows that

$$1 \otimes a_2^i - (\Delta_0 \otimes 1)(a_2^i) + (1 \otimes \Delta_0)(a_2^i) - a_2^i \otimes 1 = 0$$

for any $i = 1, \ldots, s$, from which a_2^i is a 2-cocycle of complex (3.15) for any $i = 1, \ldots, s$. Moreover, by equation (3.68), one has

$$0 = a - (2\ 3)a = \sum_{i=1}^{s} a_1^i \otimes \big(a_2^i - (1\ 2)a_2^i\big),$$

from which

$$0 = a_2^i - (1\ 2)a_2^i = Alt(a_2^i) \tag{3.77}$$

for any $i = 1, \ldots, s$. By Lemma 3.2.14, a_2^i is a coboundary for any $i = 1, \ldots, s$. Therefore there exist $f_i \in \mathcal{U}(\mathfrak{g})$ (of course not unique) such that $df_i = a_2^i$, or, in other words, such that the following equation is satisfied:

$$1 \otimes f_i - \Delta_0(f_i) + f_i \otimes 1 = a_2^i. \tag{3.78}$$

Let us choose $f_i \in \mathcal{U}(\mathfrak{g})$ satisfying equation (3.78) and let us define

$$f = -\sum_{i=1}^{s} a_1^i \otimes f_i \in \mathcal{U}(\mathfrak{g})^{\otimes 2}. \tag{3.79}$$

It follows that

$$(1 \otimes \Delta_0)(f) - f^{12} - f^{13} = a \tag{3.80}$$

indeed

$$(1 \otimes \Delta_0)(f) - f^{12} - f^{13} = \sum_{i=1}^{s} a_1^i \otimes \big(- \Delta_0(f_i) + f_i \otimes 1 + 1 \otimes f_i\big) = a.$$

Let g an element of $\mathcal{U}(\mathfrak{g})^{\otimes 2}$ defined as $g = f + f^{21}$.

Lemma 3.5.8. *The following equation holds:*

$$(\Delta_0 \otimes \Delta_0)(g) - \big((\Delta_0 \otimes 1)(g)\big)^{123} - \big((\Delta_0 \otimes 1)(g)\big)^{124} - \big((1 \otimes \Delta_0)(g)\big)^{134} \\ - \big((1 \otimes \Delta_0)(g)\big)^{234} + g^{13} + g^{14} + g^{23} + g^{24} = 0. \tag{3.81}$$

Proof. Since one has

$$\begin{aligned}
(\Delta_0 \otimes \Delta_0)(f^{21}) &= (1\ 3)(2\ 4)(\Delta_0 \otimes \Delta_0)(f) \\
\Big((\Delta_0 \otimes 1)(f^{21})\Big)^{123} &= (1\ 3)(2\ 4)(1 \otimes 1 \otimes \Delta_0)(f^{13}) \\
\Big((\Delta_0 \otimes 1)(f^{21})\Big)^{124} &= (1\ 3)(2\ 4)(1 \otimes 1 \otimes \Delta_0)(f^{23})
\end{aligned}$$

$$\begin{aligned}
\Big((1 \otimes \Delta_0)(f^{21})\Big)^{134} &= (1\ 3)(2\ 4)(\Delta_0 \otimes 1 \otimes 1)(f^{12}) \\
\Big((1 \otimes \Delta_0)(f^{21})\Big)^{234} &= (1\ 3)(2\ 4)(\Delta_0 \otimes 1 \otimes 1)(f^{13})
\end{aligned}$$

and equation (3.74), it follows

$$\begin{aligned}
&(\Delta_0 \otimes \Delta_0)(g) - \big((\Delta_0 \otimes 1)(g)\big)^{123} - \big((\Delta_0 \otimes 1)(g)\big)^{124} - \big((1 \otimes \Delta_0)(g)\big)^{134} \\
&\quad - \big((1 \otimes \Delta_0)(g)\big)^{234} + g^{13} + g^{14} + g^{23} + g^{24} \\
&= (\Delta_0 \otimes \Delta_0)(f) - \big((\Delta_0 \otimes 1)(f)\big)^{123} - \big((\Delta_0 \otimes 1)(f)\big)^{124} - \big((1 \otimes \Delta_0)(f)\big)^{134} \\
&\quad - \big((1 \otimes \Delta_0)(f)\big)^{234} + f^{13} + f^{14} + f^{23} + f^{24} \\
&\quad + (\Delta_0 \otimes \Delta_0)(f^{21}) - \Big((\Delta_0 \otimes 1)(f^{21})\Big)^{123} - \Big((\Delta_0 \otimes 1)(f^{21})\Big)^{124} \\
&\quad - \Big((1 \otimes \Delta_0)(f^{21})\Big)^{134} - \Big((1 \otimes \Delta_0)(f^{21})\Big)^{234} + f^{31} + f^{41} + f^{32} + f^{42} \\
&= (\Delta_0 \otimes 1 \otimes 1)\big((1 \otimes \Delta_0)(f) - f^{12} - f^{13}\big) \\
&\quad - \big((1 \otimes 1 \otimes \Delta_0)(f^{13}) - f^{13} - f^{14}\big) - \big((1 \otimes 1 \otimes \Delta_0)(f^{23}) - f^{23} - f^{24}\big) \\
&\quad + (1\ 3)(2\ 4)(\Delta_0 \otimes 1 \otimes 1)\big((1 \otimes \Delta_0)(f) - f^{12} - f^{13}\big) \\
&\quad - (1\ 3)(2\ 4)\big((1 \otimes 1 \otimes \Delta_0)(f^{13}) - f^{13} - f^{14}\big) \\
&\quad - (1\ 3)(2\ 4)\big((1 \otimes 1 \otimes \Delta_0)(f^{23}) - f^{23} - f^{24}\big) \\
&= (\Delta_0 \otimes 1 \otimes 1)(a) - a^{134} - a^{234} + (1\ 3)(2\ 4)(\Delta_0 \otimes 1 \otimes 1)(a) \\
&\quad - (1\ 3)(2\ 4)a^{134} - (1\ 3)(2\ 4)a^{234} \\
&= (\Delta_0 \otimes 1 \otimes 1)(a) - a^{134} - a^{234} + (1\ 3)(2\ 4)(\Delta_0 \otimes 1 \otimes 1)(a) - a^{312} - a^{412} \\
&= 0.
\end{aligned}$$

□

Proposition 3.5.9. *g is an element of $\mathcal{U}(\mathfrak{g}) \otimes \mathfrak{g} + \mathfrak{g} \otimes \mathcal{U}(\mathfrak{g})$ invariant under the action of S_2.*

Proof. As g is defined as $f + f^{21}$, it is obviously invariant under the action of S_2. Let us prove that $g \in \mathcal{U}(\mathfrak{g}) \otimes \mathfrak{g} + \mathfrak{g} \otimes \mathcal{U}(\mathfrak{g})$. Let $\{x_i\}_{i\in I}$, where I is a set of indices, be a basis of the Lie algebra $\mathfrak{g}$. Let $x, y \in \mathcal{U}(\mathfrak{g})$. If $x = \lambda \in \mathbb{C}$ and $y = \mu \in \mathbb{C}$ then

$$\begin{aligned}
&(\Delta_0 \otimes \Delta_0)(x \otimes y) - \big((\Delta_0 \otimes 1)(x \otimes y)\big)^{123} - \big((\Delta_0 \otimes 1)(x \otimes y)\big)^{124} \\
&\quad - \big((1 \otimes \Delta_0)(x \otimes y)\big)^{134} - \big((1 \otimes \Delta_0)(x \otimes y)\big)^{234} + (x \otimes y)^{13} \\
&\quad + (x \otimes y)^{14} + (x \otimes y)^{23} + (x \otimes y)^{24} \\
&= \lambda\mu(1 \otimes 1 \otimes \otimes 1 \otimes 1).
\end{aligned}$$

If $x = \lambda \in \mathbb{C}$, $y = x_{j_1} \cdots x_{j_r}$ where $r \in \mathbb{Z}_+$ and $j_1, \dots, j_r \in I$, then

$$\begin{aligned}
&(\Delta_0 \otimes \Delta_0)(x \otimes y) - \big((\Delta_0 \otimes 1)(x \otimes y)\big)^{123} - \big((\Delta_0 \otimes 1)(x \otimes y)\big)^{124} \\
&\quad - \big((1 \otimes \Delta_0)(x \otimes y)\big)^{134} - \big((1 \otimes \Delta_0)(x \otimes y)\big)^{234} + (x \otimes y)^{13} \\
&\quad + (x \otimes y)^{14} + (x \otimes y)^{23} + (x \otimes y)^{24} \\
&= - \sum_{\substack{\sigma \in S(l, r-l) \\ l=1,\dots,r-1}} \lambda \otimes 1 \otimes x_{j_{\sigma(1)}} \cdots x_{j_{\sigma(l)}} \otimes x_{j_{\sigma(l+1)}} \cdots x_{j_{\sigma(r)}}
\end{aligned}$$

where $S(l, r-l)$ is the set of $(l, r-l)$-shuffles.
If $x = x_{i_1} \cdots x_{i_s}$, $y = \lambda \in \mathbb{C}$ where $s \in \mathbb{Z}_+$ and $i_1, \dots, i_s \in I$, then

$$\begin{aligned}
&(\Delta_0 \otimes \Delta_0)(x \otimes y) - \big((\Delta_0 \otimes 1)(x \otimes y)\big)^{123} - \big((\Delta_0 \otimes 1)(x \otimes y)\big)^{124} \\
&\quad - \big((1 \otimes \Delta_0)(x \otimes y)\big)^{134} - \big((1 \otimes \Delta_0)(x \otimes y)\big)^{234} + (x \otimes y)^{13} \\
&\quad + (x \otimes y)^{14} + (x \otimes y)^{23} + (x \otimes y)^{24} \\
&= - \sum_{\substack{\sigma \in S(m, s-m) \\ m=1,\dots,s-1}} x_{i_{\sigma(1)}} \cdots x_{i_{\sigma(m)}} \otimes x_{i_{\sigma(m+1)}} \cdots x_{i_{\sigma(s)}} \otimes \lambda \otimes 1.
\end{aligned}$$

And if $x = x_{i_1} \cdots x_{i_s}$, $y = x_{j_1} \cdots x_{j_r}$ with $s, r \in \mathbb{Z}_+$ and $i_1, \dots, i_s, j_1, \dots, j_r \in I$, then

$$\begin{aligned}
&(\Delta_0 \otimes \Delta_0)(x \otimes y) - \big((\Delta_0 \otimes 1)(x \otimes y)\big)^{123} - \big((\Delta_0 \otimes 1)(x \otimes y)\big)^{124} \\
&\quad - \big((1 \otimes \Delta_0)(x \otimes y)\big)^{134} - \big((1 \otimes \Delta_0)(x \otimes y)\big)^{234} + (x \otimes y)^{13} \\
&\quad + (x \otimes y)^{14} + (x \otimes y)^{23} + (x \otimes y)^{24}
\end{aligned}$$

$$\begin{aligned}
= \sum_{\substack{\sigma \in S(m, s-m) \\ \nu \in S(l, r-l) \\ m=1,\dots,s-1 \\ l=1,\dots,r-1}} & x_{i_{\sigma(1)}} \cdots x_{i_{\sigma(m)}} \otimes x_{i_{\sigma(m+1)}} \cdots x_{i_{\sigma(s)}} \otimes \\
& \otimes x_{j_{\nu(1)}} \cdots x_{j_{\nu(l)}} \otimes x_{j_{\nu(l+1)}} \cdots x_{j_{\nu(r)}}.
\end{aligned}$$

Therefore $g \in \mathcal{U}(\mathfrak{g}) \otimes \mathfrak{g} + \mathfrak{g} \otimes \mathcal{U}(\mathfrak{g})$. □

We are now ready to modify Σ as $\Sigma - \psi h^n$ with ψ defined in the following proposition:

Proposition 3.5.10. *There exists an element $\widetilde{g} \in \mathcal{U}(\mathfrak{g}) \otimes \mathfrak{g}$, unique up to terms in $\Lambda^2\mathfrak{g}$, satisfying $g = \widetilde{g} + \widetilde{g}^{21}$ such that*

$$\psi = f - \widetilde{g} \tag{3.82}$$

is a solution of equation (3.58) *in* $\Lambda^2\mathcal{U}(\mathfrak{g})$.

Proof. Since the map

$$\begin{aligned} s : \mathcal{U}(\mathfrak{g}) \otimes \mathfrak{g} &\to \big(\mathcal{U}(\mathfrak{g}) \otimes \mathfrak{g} + \mathfrak{g} \otimes \mathcal{U}(\mathfrak{g})\big)^{S_2} \\ h &\mapsto h + h^{21} \end{aligned} \tag{3.83}$$

is surjective, there exists $\widetilde{g} \in \mathcal{U}(\mathfrak{g}) \otimes \mathfrak{g}$ satisfying $g = \widetilde{g} + \widetilde{g}^{21}$. Moreover $\widetilde{g}$ is unique up to elements of $\Lambda^2\mathfrak{g}$ because $\Lambda^2\mathfrak{g}$ is the kernel of the map s. One has that $\psi = f - \widetilde{g} \in \Lambda^2\mathcal{U}(\mathfrak{g})$, indeed

$$\psi + (1\ 2)\psi = f - \widetilde{g} + f^{21} - \widetilde{g}^{21} = (f + f^{21}) - (\widetilde{g} + \widetilde{g}^{21}) = g - g = 0,$$

and, since any element of $x \in \mathfrak{g}$ satisfies $\Delta_0(x) = x + \otimes 1 + 1 \otimes x$ and $\widetilde{g} \in \mathcal{U}(\mathfrak{g}) \otimes \mathfrak{g}$, one has that $(1 \otimes \Delta_0)(\widetilde{g}) - \widetilde{g}^{12} - \widetilde{g}^{13} = 0$ from which ψ is a solution of equation (3.58). □

Let us analyze the freedom of our modification. Instead of $f_i \in \mathcal{U}(\mathfrak{g})$, let us choose another element $f'_i \in \mathcal{U}(\mathfrak{g})$ which satisfies $df'_i = a^i_2$ and let us define $f' = -\sum_{i=1}^s a^i_1 \otimes f'_i \in \mathcal{U}(\mathfrak{g})^{\otimes 2}$. As $f_i - f'_i$ is a 1-cocycle of cochain complex (3.15), $f_i - f'_i = Alt(f_i - f'_i) \in \Lambda^1\mathfrak{g} = \mathfrak{g}$ and $f - f' = -\sum_{i=1}^s a^i_1 \otimes (f_i - f'_i) \in \mathcal{U}(\mathfrak{g}) \otimes \mathfrak{g}$. We define $g' = f' + f'^{21} = g + \sum_{i=1}^s \big(a^i_1 \otimes (f_i - f'_i) + (f_i - f'_i) \otimes a^i_1\big) \in \big(\mathcal{U}(\mathfrak{g}) \otimes \mathfrak{g} + \mathfrak{g} \otimes \mathcal{U}(\mathfrak{g})\big)^{S_2}$ from which there exists an element $\widetilde{g}$ (unique up to an element of $\Lambda^2\mathfrak{g}$) satisfying $\widetilde{g'} + \widetilde{g'}^{21} = g'$. It follows that $\psi - \psi' = f - f' - (\widetilde{g} - \widetilde{g'}) \in \Lambda^2\mathfrak{g}$ indeed $\psi - \psi' \in \Lambda^2\mathcal{U}(\mathfrak{g})$ because $\psi, \psi' \in \Lambda^2\mathcal{U}(\mathfrak{g})$, and $\psi - \psi' \in \mathcal{U}(\mathfrak{g}) \otimes \mathfrak{g}$ because $f - f' \in \mathcal{U}(\mathfrak{g}) \otimes \mathfrak{g}$ and $g, g' \in \mathcal{U}(\mathfrak{g}) \otimes \mathfrak{g}$. Therefore, since $\psi - \psi' \in \Lambda^2\mathfrak{g}$ and $\widetilde{g}$ and $\widetilde{g'}$ are unique up to elements in $\Lambda^2\mathfrak{g}$, the freedom of choosing Σ reduces to an element of $\Lambda^2\mathfrak{g}$.

Now let us define φ as in section 3.2. Proposition 3.2.18, Lemma 3.2.19, Corollary 3.2.20 still hold because their proofs only use that $\Sigma\Sigma^{21} \equiv 1 \mod h^{n+1}$ and $\Sigma \equiv 1 \mod h$. The proof of Lemma 3.2.21 strongly uses the hypothesis $(\Delta \otimes 1)(\Sigma)\Sigma^{12} \equiv (1 \otimes \Delta)(\Sigma)\Sigma^{23} \mod h^{n+1}$ which generates problem in our current case. But the result can also be proved using only $\Sigma \equiv 1 \mod h$, $\Sigma\Sigma^{21} \equiv 1 \mod h^{n+1}$ and $(1 \otimes \Delta)(\Sigma) \equiv \Sigma^{12}\Sigma^{13} \mod h^{n+1}$ as showed in the following proof:

Proof. Since $(1 \otimes \Delta)(\Sigma) \equiv \Sigma^{12}\Sigma^{13} \mod h^{n+1}$ implies

$$(1 \otimes \Delta)(\Sigma^{-1}) \equiv (\Sigma^{13})^{-1}(\Sigma^{12})^{-1} \mod h^{n+1},$$

$$(\Delta \otimes 1)(\Sigma) \equiv \Sigma^{23}\Sigma^{13} \mod h^{n+1}$$

and

$$(\Delta \otimes 1)(\Sigma^{-1}) \equiv (\Sigma^{13})^{-1}(\Sigma^{23})^{-1} \mod h^{n+1},$$

the following equalities follow

$$(\Delta_0 \otimes 1)\varphi(a) + (\varphi \otimes 1)\Delta_0(a)$$

$$
\begin{aligned}
&\equiv (\Delta\otimes 1)\varphi(a) + (\varphi\otimes 1)\Delta(a)\\
&\equiv \frac{(\Delta\otimes 1)\Delta^{op}(a) - (\Delta\otimes 1)(\Sigma^{-1}\Delta(a)\Sigma)}{h^n}\\
&\quad + \frac{(\Delta^{op}\otimes 1)\Delta(a) - (\Sigma^{12})^{-1}(\Delta\otimes 1)\Delta(a)\Sigma^{12}}{h^n}\\
&= \frac{(\Delta\otimes 1)\Delta^{op}(a) - (\Delta\otimes 1)(\Sigma^{-1})(\Delta\otimes 1)\Delta(a)(\Delta\otimes 1)(\Sigma)}{h^n}\\
&\quad + \frac{(\Delta^{op}\otimes 1)\Delta(a) - (\Sigma^{12})^{-1}(\Delta\otimes 1)\Delta(a)\Sigma^{12}}{h^n}\\
&\equiv \frac{(\Delta\otimes 1)\Delta^{op}(a) - (\Sigma^{13})^{-1}(\Sigma^{23})^{-1}(1\otimes\Delta)\Delta(a)\Sigma^{23}\Sigma^{13}}{h^n}\\
&\quad + \frac{(\Sigma^{13})^{-1}(\Delta^{op}\otimes 1)\Delta(a)\Sigma^{13} - (\Sigma^{13})^{-1}(\Sigma^{12})^{-1}(1\otimes\Delta)\Delta(a)\Sigma^{12}\Sigma^{13}}{h^n}\\
&\equiv \frac{(\Delta\otimes 1)\Delta^{op}(a) - (\Sigma^{13})^{-1}((1\otimes\Delta^{op})\Delta(a) - (1\otimes\varphi)\Delta(a)h^n)\Sigma^{13}}{h^n}\\
&\quad + \frac{(\Sigma^{13})^{-1}(\Delta^{op}\otimes 1)\Delta(a)\Sigma^{13} - (1\otimes\Delta)(\Sigma^{-1})(1\otimes\Delta)\Delta(a)(1\otimes\Delta)(\Sigma)}{h^n}\\
&= \frac{1}{h^n}\Big((1\ 3\ 2)(\Delta\otimes 1)\Delta(a) - (\Sigma^{13})^{-1}((2\ 3)(1\otimes\Delta)\Delta(a))\Sigma^{13}\\
&\quad + (\Sigma^{13})^{-1}((1\otimes\varphi)\Delta(a)h^n)\Sigma^{13} + (\Sigma^{13})^{-1}((1\ 2)(\Delta\otimes 1)\Delta(a))\Sigma^{13}\\
&\quad - (1\otimes\Delta)(\Sigma^{-1}\Delta(a)\Sigma)\Big)\\
&\equiv \frac{1}{h^n}\Big((1\ 3\ 2)(\Delta\otimes 1)\Delta(a) - (2\ 3)((\Sigma^{12})^{-1}(\Delta\otimes 1)\Delta(a)\Sigma^{12})\\
&\quad + (\Sigma^{13})^{-1}((1\otimes\varphi)\Delta(a)h^n)\Sigma^{13} + (1\ 2)((\Sigma^{23})^{-1}(1\otimes\Delta)\Delta(a)\Sigma^{23})\\
&\quad - (1\otimes\Delta)(\Delta^{op}(a) - \varphi(a)h^n)\Big)\\
&= \frac{1}{h^n}\Big((1\ 3\ 2)(\Delta\otimes 1)\Delta(a) - (2\ 3)((\Sigma^{12})^{-1}(\Delta\otimes 1)\Delta(a)\Sigma^{12})\\
&\quad + (\Sigma^{13})^{-1}((1\otimes\varphi)\Delta(a)h^n)\Sigma^{13} + (1\ 2)((\Sigma^{23})^{-1}(1\otimes\Delta)\Delta(a)\Sigma^{23})\\
&\quad - (1\ 2\ 3)(1\otimes\Delta)\Delta(a) + (1\otimes\Delta)\varphi(a)h^n\Big)\\
&= \frac{1}{h^n}\Big((2\ 3)\Big((\Delta^{op}\otimes 1)\Delta(a) - (\Sigma^{12})^{-1}(\Delta\otimes 1)\Delta(a)\Sigma^{12}\Big)\\
&\quad - (1\ 2)\Big(-(\Sigma^{23})^{-1}(1\otimes\Delta)\Delta(a)\Sigma^{23}) + (1\otimes\Delta^{op})\Delta(a)\Big)\\
&\quad + (\Sigma^{13})^{-1}(1\otimes\varphi)\Delta(a)\Sigma^{13}h^n + (1\otimes\Delta)\varphi(a)h^n\Big)\\
&\equiv \frac{1}{h^n}\Big((2\ 3)(\varphi\otimes 1)\Delta(a)h^n - (1\ 2)(1\otimes\varphi)\Delta(a)h^n\\
&\quad + (\Sigma^{13})^{-1}(1\otimes\varphi)\Delta(a)\Sigma^{13}h^n + (1\otimes\Delta)\varphi(a)h^n\Big)\\
&= (2\ 3)(\varphi\otimes 1)\Delta(a) - (1\ 2)(1\otimes\varphi)\Delta(a) + (\Sigma^{13})^{-1}(1\otimes\varphi)\Delta(a)\Sigma^{13}\\
&\quad + (1\otimes\Delta)\varphi(a)
\end{aligned}
$$

$$\equiv (2\ 3)(\varphi \otimes 1)\Delta_0(a) - (1\ 2)(1 \otimes \varphi)\Delta_0(a) + (1 \otimes \varphi)\Delta_0(a) + (1 \otimes \Delta_0)\varphi(a) \mod h.$$

On the other hand $(2\ 3)(\varphi\otimes 1)\Delta_0(a) - (1\ 2)(1\otimes\varphi)\Delta_0(a) = 0$ because $\Delta_0(a) = \Delta_0^{op}(a)$ for any $a \in \mathcal{U}(\mathfrak{g})$, therefore

$$(\Delta_0 \otimes 1)\varphi(a) + (\varphi \otimes 1)\Delta_0(a) \equiv (1 \otimes \varphi)\Delta_0(a) + (1 \otimes \Delta_0)\varphi(a) \mod h$$

from which

$$(\Delta_0 \otimes 1)\varphi(a) + (\varphi \otimes 1)\Delta_0(a) = (1 \otimes \varphi)\Delta_0(a) + (1 \otimes \Delta_0)\varphi(a)$$

because $\varphi(a) \in \mathcal{U}(\mathfrak{g})^{\otimes 2}$ for any $a \in \mathcal{U}(\mathfrak{g})$. □

It follows that Lemma 3.2.21 still holds and consequently Corollaries 3.2.22 and 3.2.23 and Proposition 3.2.24 which use in their proof the statement of Lemma 3.2.21 but not the hypothesis $(\Delta \otimes 1)(\Sigma)\Sigma^{12} \equiv (1 \otimes \Delta)(\Sigma)\Sigma^{23} \mod h^{n+1}$. We can now carry on as done in section 3.3.

Appendix A

Appendix to Question and Answer 46

A.1 Hochschild Cohomology

Let A be an algebra over a field $\mathbb{K}$, we recall that the opposite algebra of A, denoted with A^{op}, is the algebra whose underlying vector space is A and whose multiplication is defined by $a \cdot^{op} b = b \cdot a$ where $\cdot$ is the product of A. The enveloping algebra A^e is the algebra whose underlying vector space is $A \otimes A^{op}$ and whose multiplication is given by $(a \otimes c) \cdot^e (b \otimes d) = a \cdot b \otimes c \cdot^{op} d = a \cdot b \otimes d \cdot c$. An A-A-bimodule is an abelian group M with a structure of an A-right module and an A-left module, moreover it satisfies the following condition of compatibility $a(mb) = (am)b$ where $a, b \in A$ and $m \in M$.

Remark A.1.1. An A-A-bimodule M is canonically an A^e module with the following action $(a \otimes b)m = amb$. Viceversa an A^e-module is canonically an A-A-bimodule with the following left action $am = (a \otimes 1)m$ and the following right action $mb = (1 \otimes b)m$.

Definition A.1.1. Let M be an A-A-bimodule. The Hochschild cohomology of M is the vector space $HH^\bullet(A, M) = Ext^\bullet_{A^e}(A, M)$.

By Comparison Theorem [W, Comparison Theorem 2.2.6], the groups $Ext^n_{A^e}(A, M)$ do not depend on the choice of the projective resolution $P_* \longrightarrow A$ therefore we can use the Bar Resolution $B_*(A) \longrightarrow A$:

$$\cdots \xrightarrow{d'} A^{\otimes 3} \xrightarrow{d'} A^{\otimes 2} \xrightarrow{\varepsilon} A \longrightarrow 0 \tag{A.1}$$

where the map $\varepsilon : A^{\otimes 2} \to A$ is the multiplication map of A and the maps $d' : A^{\otimes n+2} \to A^{\otimes n+1}$ are given by

$$d'(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}.$$

The bar resolution is actually a projective resolution indeed $A^{\otimes n+2}$, for any $n \geq 0$, is an A^e-module with the following action $(a \otimes b)(a_0 \otimes \cdots \otimes a_{n+1}) = aa_0 \otimes a_1 \otimes$

$\cdots \otimes a_n \otimes a_{n+1}b$ and it is free because, if $\{e_i\}_{i\in I}$ is a basis of A, then $A^{\otimes n+2} = \bigoplus_{i_1,\dots,i_n\in I} A \otimes \langle e_{i_1} \otimes \cdots \otimes e_{i_n}\rangle \otimes A \cong \bigoplus_{i_1,\dots,i_n\in I} A^e$ where the last isomorphism is given by the isomorphisms of A^e-modules $\varphi_{i_1,\dots,i_n} : A \otimes \langle e_{i_1} \otimes \cdots \otimes e_{i_n}\rangle \otimes A \to A^e$ defined as $\varphi_{i_1,\dots,i_n}(a \otimes e_{i_1} \otimes \cdots \otimes e_{i_n} \otimes b) = a \otimes b$ for any $a, b \in A$. To conclude, by the maps $s_n : A^{\otimes n+2} \to A^{\otimes n+3}$ defined by $s_n(a_0 \otimes \cdots a_{n+1}) = (1 \otimes a_0 \otimes \cdots \otimes a_{n+1})$, one shows that the identity on the bar resolution is chain homotopic to the zero map, therefore the complex is exact.
Applying the functor $\mathrm{Hom}_{A^e}(-, M)$ to the resolution (A.1) one obtains the following complex

$$0 \longrightarrow \mathrm{Hom}_{A^e}(A^{\otimes 2}, M) \longrightarrow \mathrm{Hom}_{A^e}(A^{\otimes 3}, M) \longrightarrow \cdots . \tag{A.2}$$

Moreover the maps $\psi_n : \mathrm{Hom}_{A^e}(A^{\otimes n+2}, M) \to \mathrm{Hom}_{\mathbb{K}}(A^{\otimes n}, M)$ defined by $\psi(f)(a_1 \otimes \cdots \otimes a_n) = f(1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1)$ are isomorphisms of $\mathbb{K}$-vector spaces. Hence the vector space structure of the cohomology of complex (A.2) is the same as the vector space structure of the cohomology of the following complex:

$$0 \longrightarrow M \xrightarrow{d} \mathrm{Hom}_{\mathbb{K}}(A, M) \xrightarrow{d} \mathrm{Hom}_{\mathbb{K}}(A^{\otimes 2}, M) \xrightarrow{d} \cdots \tag{A.3}$$

with the differential $d : \mathrm{Hom}_{\mathbb{K}}(A^n, M) \to \mathrm{Hom}_{\mathbb{K}}(A^{\otimes n+1}, M)$ given by

$$\begin{aligned} d(f)(a_1 \otimes \cdots \otimes a_{n+1}) = a_1 f(a_2 \otimes \cdots \otimes a_{n+1}) \\ + \sum_{i=1}^{n} (-1)^i f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}) \\ + (-1)^{n+1} f(a_1 \otimes \cdots \otimes a_n) a_{n+1}. \end{aligned} \tag{A.4}$$

A.2 Cohomology of the Drinfeld's cochain complex

In this section we will prove that, for any Lie algebra $\mathfrak{g}$, the cohomology of cochain complex (3.15) is isomorphic to $\Lambda^\bullet \mathfrak{g}$.

Let us consider the symmetric algebra $S(\mathfrak{g})$ of $\mathfrak{g}$. $S(\mathfrak{g})$ is a bialgebra with the coproduct Δ_0 and the counit ε_0 defined as follows: Δ_0 is the homomorphism of algebras defined on any $x \in \mathfrak{g}$ as

$$\Delta_0(x) = x \otimes 1 + 1 \otimes x,$$

and ε_0 is the homomorphism of algebras defined on any $x \in \mathfrak{g}$ as

$$\varepsilon_0(x) = 0.$$

By Poincaré-Birkhoff-Witt's theorem, $\mathcal{U}(\mathfrak{g})$ and $S(\mathfrak{g})$ are isomorphic coalgebras. That's why we use, for the coproduct and the counit of $S(\mathfrak{g})$, the same notation which we use for the coproduct and the counit of $\mathcal{U}(\mathfrak{g})$. By the isomorphism of coalgebras between $\mathcal{U}(\mathfrak{g})$ and $S(\mathfrak{g})$, the cochain complex (3.15) is isomorphic to the following one:

$$\mathbb{C} \xrightarrow{d} S(\mathfrak{g}) \xrightarrow{d} S(\mathfrak{g})^{\otimes 2} \xrightarrow{d} \dots \tag{A.5}$$

where $d:\mathbb{C}\to S(\mathfrak{g})$ is the zero map and $d:S(\mathfrak{g})^{\otimes n}\to S(\mathfrak{g})^{\otimes n+1}$, for $n>0$, is given by

$$d(x_1\otimes\cdots\otimes x_n)=1\otimes x_1\otimes\cdots\otimes x_n+\sum_{i=1}^{n}(-1)^i x_1\otimes\cdots\otimes\Delta_0(x_i)\otimes\cdots\otimes x_n$$
$$+(-1)^{n+1}x_1\otimes\cdots\otimes x_n\otimes 1.$$

Therefore we can assume that the Lie algebra $\mathfrak{g}$ is abelian.

We will firstly prove that the cohomology of cochain comples (A.5) is isomorphic to $\Lambda^\bullet\mathfrak{g}$ for any finite dimensional Lie algebra $\mathfrak{g}$. Then we extend the result to any infinite dimensional Lie algebra.

Let $\mathfrak{g}$ a finite dimensional Lie algebra. $S(\mathfrak{g})$ is a graded bialgebra indeed $S(\mathfrak{g})=\bigoplus_{i\geq 0}S(\mathfrak{g})_i$, where $S(\mathfrak{g})_i$ is the vector space generated by all the polynomials of degree i, $S(\mathfrak{g})_i\cdot S(\mathfrak{g})_j\subseteq S(\mathfrak{g})_{i+j}$, $\Delta_{S(\mathfrak{g})}(S(\mathfrak{g})_i)\subseteq\bigoplus_{k+j=i}S(\mathfrak{g})_j\otimes S(\mathfrak{g})_k$ and the counit is zero on $S(\mathfrak{g})_i$ if $i>0$. Moreover, since $\mathfrak{g}$ is finite dimensional $S(\mathfrak{g})_i$ is finite dimensional for any $i\geq 0$. One can consider the following vector space

$$S(\mathfrak{g})^*_{gr}=\bigoplus_{i\geq 0}S(\mathfrak{g})^*_i$$

which is known as the *graded dual* of $S(\mathfrak{g})$. Since $S(\mathfrak{g})_i$ is finite dimensional for any $i\geq 0$, the structure of bialgebra of $S(\mathfrak{g})$ induce a structure of bialgebra on the graded dual $S(\mathfrak{g})^*_{gr}$. In particular $\mathbb{C}$ is a left and right $S(\mathfrak{g})^*_{gr}$-module with the trivial action given by the counit $\varepsilon_{S(\mathfrak{g})^*_{gr}}$ of $S(\mathfrak{g})^*_{gr}$: $\varphi\lambda=\varepsilon_{S(\mathfrak{g})^*_{gr}}(\varphi)\lambda=\lambda\varphi$ for any $\varphi\in S(\mathfrak{g})^*_{gr}$ and $\lambda\in\mathbb{C}$.

Let us consider the Hochschild cohomology $HH^\bullet(S(\mathfrak{g})^*_{gr},\mathbb{C})$ of the algebra $S(\mathfrak{g})^*_{gr}$ and the bimodule $\mathbb{C}$. In Section A.1 we have seen that $HH^\bullet(S(\mathfrak{g})^*_{gr},\mathbb{C})=Ext^\bullet_{(S(\mathfrak{g})^*_{gr})^e}(S(\mathfrak{g})^*_{gr},\mathbb{C})$ is the cohomology of following cochain complex:

$$0\longrightarrow\mathbb{C}\xrightarrow{d}\left(S(\mathfrak{g})^*_{gr}\right)^*\xrightarrow{d}\left(S(\mathfrak{g})^*_{gr}\otimes S(\mathfrak{g})^*_{gr}\right)^*\xrightarrow{d}\cdots\tag{A.6}$$

where the differential $d:\left(\left(S(\mathfrak{g})^*_{gr}\right)^{\otimes n}\right)^*\to\left(\left(S(\mathfrak{g})^*_{gr}\right)^{\otimes n+1}\right)^*$ is given by

$$d(f)(a_1\otimes\cdots\otimes a_{n+1})=\varepsilon_{S(\mathfrak{g})^*_{gr}}(a_1)f(a_2\otimes\cdots\otimes a_{n+1})$$
$$+\sum_{i=1}^{n}(-1)^i f(a_1\otimes\cdots\otimes a_ia_{i+1}\otimes\cdots\otimes a_{n+1})$$
$$+(-1)^{n+1}f(a_1\otimes\cdots\otimes a_n)\varepsilon_{S(\mathfrak{g})^*_{gr}}(a_{n+1}).$$

We want to prove that the vector spaces $HH^\bullet(S(\mathfrak{g})^*_{gr},\mathbb{C})$ and $\Lambda^\bullet\mathfrak{g}$ are isomorphic as stated in the following theorem:

Theorem A.2.1. $HH^\bullet(S(\mathfrak{g})^*_{gr},\mathbb{C})\cong\Lambda^\bullet\mathfrak{g}$ *as vector spaces.*

First, we prove that $Ext^\bullet_{S(\mathfrak{g})^*_{gr}}(\mathbb{C},\mathbb{C})\cong Ext^\bullet_{(S(\mathfrak{g})^*_{gr})^e}(S(\mathfrak{g})^*_{gr},\mathbb{C})$ as vector spaces which follows from the following general result:

Lemma A.2.2 ([M, Theorem VII 3.3]). *Let A be a $\mathbb{K}$-algebra and B, C be two left A-modules. The vector spaces $Ext^\bullet_A(B, C)$ and $Ext^\bullet_{A^e}(A, \mathrm{Hom}_\mathbb{K}(B, C))$ are isomorphic as $\mathbb{K}$-vector spaces.*

Proof. The vector space $\mathrm{Hom}_\mathbb{K}(B, C)$ is an A-A-bimodule with this biaction:

$$(afb)(x) = af(bx)$$

where $a, b \in A$, $f \in \mathrm{Hom}_\mathbb{K}(B, C)$ and $x \in B$. The Bar Resolution (A.1) of A is a projective resolution of A as an A^e-module, as already seen, and it is a projective resolution of A as a right A-module where the action of A on $A^{\otimes n+1}$ is given by $(a_0 \otimes \cdots a_n)a = a_0 \otimes \cdots a_{n-1} \otimes a_n a$ for any $n \geq 0$. $A^{\otimes n+2}$ is a free right A-module because, if $\{e_i\}_{i\in I}$ is a basis of A, then there is an isomorphism of A-modules $\langle e_{i_0} \otimes \cdots \otimes e_{i_n}\rangle \otimes A \cong A$ given by $e_{i_0} \otimes \cdots \otimes e_{i_n} \otimes a \mapsto a$, from which $A^{\otimes n+2} \cong \bigoplus_{i_0,\ldots,i_n \in I} A$ as right A-modules. Applying $\mathrm{Hom}_{A^e}(-, \mathrm{Hom}_\mathbb{K}(B, C))$ to the Bar Resolution, seen as a resolution of A^e-modules, one has the following complex of A^e-modules and homomorphisms of A^e-modules:

$$0 \longrightarrow \mathrm{Hom}_{A^e}\left(A^{\otimes 2}, \mathrm{Hom}_\mathbb{K}(B, C)\right) \xrightarrow{d_1} \mathrm{Hom}_{A^e}\left(A^{\otimes 3}, \mathrm{Hom}_\mathbb{K}(B, C)\right) \xrightarrow{d_1} \cdots \tag{A.7}$$

whose cohomology is $HH^\bullet(A, \mathrm{Hom}_\mathbb{K}(B, C)) = Ext^\bullet_{A^e}(A, \mathrm{Hom}_\mathbb{K}(B, C))$. On the other side, applying the functor $\mathrm{Hom}_A\left(- \otimes_A B, C\right)$ to the Bar Resolution, seen as a resolution of right A-modules, one has the following complex of A-modules and homomorphisms of A-modules:

$$0 \longrightarrow \mathrm{Hom}_A\left(A^{\otimes 2} \otimes_A B, C\right) \xrightarrow{d_2} \mathrm{Hom}_A\left(A^{\otimes 3} \otimes_A B, C\right) \xrightarrow{d_2} \cdots . \tag{A.8}$$

The linear map $\eta : \mathrm{Hom}_A\left(A^{\otimes n+2} \otimes_A B, C\right) \to \mathrm{Hom}_{A^e}\left(A^{\otimes n+2}, \mathrm{Hom}_\mathbb{K}(B, C)\right)$ defined as $\eta(f)(a_0 \otimes \cdots a_{n+1})(-) = f((a_0 \otimes \cdots a_{n+1}) \otimes -)$ is an isomorphism of vector spaces (with inverse $\eta^{-1} : \mathrm{Hom}_{A^e}\left(A^{\otimes n+2}, \mathrm{Hom}_\mathbb{K}(B, C)\right) \to \mathrm{Hom}_A\left(A^{\otimes n+2} \otimes_A B, C\right)$, $\eta^{-1}(g)(a_0 \otimes \cdots \otimes a_{n+1} \otimes_A b) = g(a_0 \otimes \cdots \otimes a_{n+1})(b)$) satisfying $\eta \circ d_1 = d_2 \circ \eta$. It follows that the cohomologies of the complexes (A.7) and (A.8) are isomorphic as vector spaces.
Applying $- \otimes_A B$ to the Bar Resolution, seen as resolution of right A-modules, one has the following complex

$$\cdots \longrightarrow A^{\otimes 3} \otimes_A B \longrightarrow A^{\otimes 2} \otimes_A B \longrightarrow 0$$

whose homology is $Tor^A_\bullet(A, B)$. Since A is a projective A-module $Tor^A_n(A, B) = 0$ for any $n > 0$. Thus the map $\varepsilon \otimes 1 : A^{\otimes 2} \otimes_A B \to A \otimes_A B \cong B$, given by the multiplication map of A, gives a projective resolution of B as a left A-module. Applying the functor $\mathrm{Hom}_A(-, C)$ to the resolution, one obtains the complex (A.8) and its cohomology is $Ext^\bullet_A(B, C)$ by definition. □

In particular, for $\mathbb{K} = \mathbb{C}$, $A = S(\mathfrak{g})^*_{gr}$ and $B = C = \mathbb{C}$, one has the following isomorphism of vector spaces:

$$Ext^\bullet_{S(\mathfrak{g})^*_{gr}}(\mathbb{C}, \mathbb{C}) \cong Ext^\bullet_{(S(\mathfrak{g})^*_{gr})^e}(S(\mathfrak{g})^*_{gr}, \mathbb{C}). \tag{A.9}$$

Second, we compute $Ext^{\bullet}_{S(\mathfrak{g})^*_{gr}}(\mathbb{C},\mathbb{C})$.
The following result holds:

Lemma A.2.3 ([Kas, Lemma XVIII 7.5]). *If V is a finite dimensional vector space over a field $\mathbb{K}$ of characteristic zero, then the graded dual of the graded coalgebra with unit $S(V)$ is the graded algebra with counit $S(V^*)$.*

Remark A.2.4. By Lemma A.2.3, if $\{x_1,\dots,x_n\}$ is a basis of $\mathfrak{g}$ and $\{x_1^*,\dots,x_n^*\}$ is the basis of $\mathfrak{g}^*$ dual to $\{x_1,\dots,x_n\}$, the algebras $S(\mathfrak{g})^*_{gr}$ and $\mathbb{C}[x_1^*,\dots,x_n^*]$ are isomorphic (the commutativity of the product of $S(\mathfrak{g})^*_{gr}$ is due to the cocommutativity of the coproduct of $S(\mathfrak{g})$).

The following theorem holds:

Theorem A.2.5 ([M, Theorem VII 2.2]). *Let $\mathbb{K}$ be a field. $Ext^{\bullet}_{\mathbb{K}[x_1,\dots,x_n]}(\mathbb{K},\mathbb{K}) \cong \Lambda^{\bullet}(\mathbb{K}^n)$ as vector spaces.*

Let R be a ring, (K,∂_K) be a chain complex of right R-modules and (L,∂_L) be a chain complex of left R-modules. Then $(K\otimes_R L, \partial_{K\otimes_R L})$ is a chain complex where

$$K\otimes_R L = \bigoplus_m (K\otimes_R L)_m$$

with

$$(K\otimes_R L)_m = \bigoplus_{i+j=m} K_i \otimes_R L_j$$

and the differential $\partial_{K\otimes_R L}$ is defined as $\partial_{K\otimes_R L}(k\otimes l) = \partial_K(k)\otimes l + (-1)^i k\otimes \partial_L(l)$ for any $k\in K_i$ and $l\in L_{m-i}$.

Lemma A.2.6 ([M, Proposition V 9.1]). *Let $(K,\partial_K),(K',\partial_{K'}),(L,\partial_L),(L,\partial_{L'})$ be chain complexes and $f_1,f_2: K\to K'$, $g_1,g_2: L\to L'$ homomorphisms of chain complexes. If $f_1,f_2: K\to K'$ are chain homotopic and $g_1,g_2: L\to L'$ are chain homotopic, then $f_1\otimes g_1$ and $f_2\otimes g_2$ are chain homotopic. In particular if s is the chain homotopy $f_1\simeq f_2$ and t is the chain homotopy $g_1\simeq g_2$, then the chain homotopy $f_1\otimes g_1\simeq f_2\otimes g_2$ is equal to $u = s\otimes g_1 + f_2\otimes t$, i.e.*

$$u(k\otimes l) = s(k)\otimes g_1(l) + (-1)^i f_2(k)\otimes t(l)$$

for any $k\in K_i$ and $l\in L_j$.

Proof. Since s and t are, respectively, the chain homotopies of $f_1\simeq f_2$ and $g_1\simeq g_2$, it follows that $f_1 - f_2 = \partial_{K'}\circ s + s\circ\partial_K$ and $g_1 - g_2 = \partial_{L'}\circ s + s\circ \partial_L$, and, since f_2 and g_1 are maps of chain complexes, it follows that $f_2\circ\partial_K = \partial_{K'}\circ f_2$ and $g_1\circ\partial_L = \partial_{L'}\circ g_1$. Therefore, for any $k\in K_i$ and $l\in L_j$, one has the following equalities:

$$\begin{aligned}
&(\partial_{K'\otimes_R L'}\circ u + u\circ\partial_{K\otimes_R L})(k\otimes l)\\
&= \partial_{K'\otimes_R L'}\big(s(k)\otimes g_1(l) + (-1)^i f_2(k)\otimes t(l)\big)\\
&\quad + u\big(\partial_K(k)\otimes l + (-1)^i k\otimes\partial_L(l)\big)\\
&= (\partial_{K'}\circ s)(k)\otimes g_1(l) + (-1)^i(\partial_{K'}\circ f_2)(k)\otimes t(l)
\end{aligned}$$

$$
\begin{aligned}
&+ (-1)^{i+1} s(k) \otimes \big(\partial_{L'} \circ g_1\big)(l) + f_2(k) \otimes \big(\partial_{L'} \circ t(l)\big) \\
&+ \big(s \circ \partial_K\big)(k) \otimes g_1(l) + (-1)^{i-1} (f_2 \circ \partial_K)(k) \otimes t(l) \\
&+ (-1)^i s(k) \otimes \big(g_1 \circ \partial_L\big)(l) + f_2(k) \otimes \big(t \circ \partial_L\big)(l) \\
= &\, (f_1 - f_2)(k) \times g_1(l) + f_2(k) \otimes (g_1 - g_2)(l) \\
= &\, (f_1 \otimes g_1 - f_2 \otimes g_2)(k \otimes l).
\end{aligned}
$$

□

Proposition A.2.7 ([M, Proposition VII 2.1])**.** *Let* $P = \mathbb{K}[x_1, \ldots, x_n]$*. The complex of* P*-modules and homomorphisms of* P*-modules*

$$P \otimes \Lambda^{\bullet}(u_1, \ldots, u_n) \longrightarrow \mathbb{K} \longrightarrow 0, \tag{A.10}$$

with the maps $\partial : P \otimes \Lambda^{i+1}(u_1, \ldots, u_n) \to P \otimes \Lambda^i(u_1, \ldots, u_n)$ *defined as*

$$\partial(f \otimes u_{j_1} \wedge \cdots \wedge u_{j_{i+1}}) = \sum_{k=1}^{i} (-i)^{k-1} f x_{j_k} \otimes (u_{j_1} \wedge \cdots \wedge \widehat{u_{j_k}} \wedge \cdots \wedge u_{j_{i+1}}), \tag{A.11}$$

and the augmentation map $\varepsilon : P \to \mathbb{K}$ *defined by* $\varepsilon(1) = 1$*, is a projective resolution of* $\mathbb{K} \cong P/(x_1, \ldots, x_n)$*:*

Proof. Since the action of P on $P \otimes \Lambda^i(u_1, \ldots, u_n)$ is given by $f(g \otimes u_{j_1} \wedge \cdots \wedge u_{j_i}) = fg \otimes u_{j_1} \wedge \cdots \wedge u_{j_i}$ for any $f, g \in P$, the maps ∂ and ε are clearly homomorphisms of P-modules and $P \otimes \Lambda^i(u_1, \ldots, u_n) \cong P \otimes \bigoplus_{j=1}^{\binom{n}{i}} \mathbb{K} \cong \bigoplus_{j=1}^{\binom{n}{i}} P$ as P-modules. Therefore $P \otimes \Lambda^i(u_1, \ldots, u_n)$ is a projective P-module for any $i \geq 0$. Let us prove the exactness of complex (A.10) by induction on n.
If $n = 1$ the complex becomes

$$0 \longrightarrow P \otimes \Lambda^1(u) \longrightarrow P \longrightarrow \mathbb{K} \longrightarrow 0$$

because $\Lambda^2(u) = 0$. The linear maps

$$
\begin{aligned}
s_{-1} : \mathbb{K} &\to P \\
1 &\mapsto 1,
\end{aligned}
$$

$$
\begin{aligned}
s_0 : P &\to P \otimes \Lambda^1(u) \\
\sum_{i=0}^{r} f_i x^i &\mapsto \left(\sum_{i=1}^{r} f_i x^{i-1} \right) \otimes u
\end{aligned}
$$

and

$$
\begin{aligned}
s_1 : P \otimes \Lambda^1(u) &\to 0 \\
f \otimes u &\mapsto 0
\end{aligned}
$$

define a chain homotopy between the identity map and the zero map, from which the exactness.

Setting $P' = \mathbb{K}[x_n]$, $P'' = \mathbb{K}[x_1, \ldots, x_{n-1}]$, $E' = P' \otimes \Lambda^\bullet(u_n)$ and $E'' = P'' \otimes \Lambda^\bullet(u_{u_1,\ldots,n-1})$, by induction we have homotopy maps s'' on the chain complex $(E'', \partial_{E''})$ and s' on $(E', \partial_{E'})$. It follows, by Lemma A.2.6, that the identity map and the zero map are chain homotopic on the complex $P \otimes \Lambda^\bullet(u_1, \ldots, u_n)$ because $E'' \otimes E' \cong P \otimes \Lambda^\bullet(u_1, \ldots, u_n)$ as P-algebras. From which the exactness of the complex $P \otimes \Lambda^\bullet(u_1, \ldots, u_n)$. □

Now we can prove Theorem A.2.5.

Proof. By Proposition A.2.7, the complex $(\mathbb{K}[x_1, \ldots, x_n] \otimes \Lambda^\bullet(u_1, \ldots, u_n), \partial)$ with differential ∂ defined as in Proposition A.2.7, is a projective resolution of $\mathbb{K}[x_1, \ldots, x_n]$-modules of $\mathbb{K}$. Applying the functor $\mathrm{Hom}_{\mathbb{K}[x_1,\ldots,x_n]}(-, \mathbb{K})$ to the resolution $\mathbb{K}[x_1, \ldots, x_n] \otimes \Lambda^\bullet(u_1, \ldots, u_n)$, one obtains the complex

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{K}[x_1,\ldots,x_n]}(\mathbb{K}[x_1, \ldots, x_n], \mathbb{K}) \longrightarrow$$

$$\longrightarrow \mathrm{Hom}_{\mathbb{K}[x_1,\ldots,x_n]}(\mathbb{K}[x_1, \ldots, x_n] \otimes \Lambda^1(u_1, \ldots, u_n), \mathbb{K}) \longrightarrow \cdots$$

whose cohomology is $Ext^\bullet_{\mathbb{K}[x_1,\ldots,x_n]}(\mathbb{K}, \mathbb{K})$. On the other hand, denoting with u_i^* the element of $\langle u_1, \ldots, u_n \rangle^*$ such that $u_i^*(u_j) = \delta_{ij}$, and with ε the augmentation map defined in Proposition A.2.7, the linear map

$$\psi_i : \Lambda^i(u_1, \ldots, u_n) \to \mathrm{Hom}_{\mathbb{K}[x_1,\ldots,x_n]}(\mathbb{K}[x_1, \ldots, x_n] \otimes \Lambda^i(u_1, \ldots, u_n), \mathbb{K}) \quad \text{(A.12)}$$

defined as

$$\psi_i(u_{j_1} \wedge \cdots \wedge u_{j_i})(p \otimes u_{k_1} \wedge \cdots \wedge u_{k_i}) = \varepsilon(p)(u_{j_1}^* \wedge \cdots \wedge u_{j_i}^*)(u_{k_1} \wedge \cdots \wedge u_{k_i}) \quad \text{(A.13)}$$

is an isomorphism of vector spaces. It follows that the cohomology of the complex

$$0 \longrightarrow \mathbb{K} \xrightarrow{\bar{\partial}} \Lambda^1(u_1, \ldots, u_n) \xrightarrow{\bar{\partial}} \Lambda^2(u_1, \ldots, u_n) \xrightarrow{\bar{\partial}}$$

$$\cdots \xrightarrow{\bar{\partial}} \Lambda^n(u_1, \ldots, u_n) \longrightarrow 0,$$

with $\bar{\partial}$ induced by the isomorphism (A.12) is, on one side, isomorphic, as a vector space, to $Ext^\bullet_{\mathbb{K}[x_1,\ldots,x_n]}(\mathbb{K}, \mathbb{K})$ and on the other side is $\Lambda^\bullet(u_1, \ldots, u_n)$ because the maps $\Lambda^i(u_1, \ldots, u_n) \to \Lambda^{i+1}(u_1, \ldots, u_n)$ are given by $u_{j_1} \wedge \ldots \wedge u_{j_i} = \psi_{i+1}^{-1}(\psi_i(u_{j_1} \wedge \ldots \wedge u_{j_i}) \circ \partial)$ which is zero because any non-constant polynomial acts as zero on $\mathbb{K}$. □

Since $S(\mathfrak{g})^*_{gr}$ is a polynomial algebra (cf. Remark A.2.4), by Theorem A.2.5, the following isomorphisms of vector spaces hold:

$$Ext^\bullet_{S(\mathfrak{g})^*_{gr}}(\mathbb{C}, \mathbb{C}) \cong \Lambda^\bullet(\mathbb{C}^{dim(\mathfrak{g})}) \cong \Lambda^\bullet \mathfrak{g}.$$

On the other hand, as $S(\mathfrak{g})_i$ is finite dimensional for any $i \geq 0$, one has the following isomorphism of vector spaces:

$$\left(S(\mathfrak{g})^*_{gr}\right)^{\otimes n} \cong \left(S(\mathfrak{g})^{\otimes n}\right)^*_{gr}. \quad \text{(A.14)}$$

One can consider the graded dual of $\left(S(\mathfrak{g})^{\otimes n}\right)^*_{gr}$:

$$\left(\left(S(\mathfrak{g})^{\otimes n}\right)^*_{gr}\right)^*_{gr} = \bigoplus_{i\geq 0} \bigoplus_{\substack{i_1+\ldots+i_n=i \\ i_1,\ldots,i_n\geq 0}} S(\mathfrak{g})^{**}_{i_1} \otimes \cdots \otimes S(\mathfrak{g})^{**}_{i_n}.$$

Using again the finite dimensionality of $S(\mathfrak{g})_i$ for any $i \geq 0$, one has that the vector spaces $\left(\left(S(\mathfrak{g})^{\otimes n}\right)^*_{gr}\right)^*_{gr}$ and $S(\mathfrak{g})^{\otimes n}$ are isomorphic.

One can define the following decreasing filtration on $\left(\left(S(\mathfrak{g})^{\otimes n}\right)^*_{gr}\right)^*_{gr}$:

$$F^m\left(\left(S(\mathfrak{g})^{\otimes n}\right)^*_{gr}\right)^*_{gr} = \bigoplus_{i\geq m} \bigoplus_{\substack{i_1+\ldots+i_n=i \\ i_1,\ldots,i_n\geq 0}} S(\mathfrak{g})^{**}_{i_i} \otimes \cdots \otimes S(\mathfrak{g})^{**}_{i_n}.$$

The family $\left(\left(\left(S(\mathfrak{g})^{\otimes n}\right)^*_{gr}\right)^*_{gr} / F^m\left(\left(S(\mathfrak{g})^{\otimes n}\right)^*_{gr}\right)^*_{gr}\right)_{m>0}$ and the projections

$$p^n_m : \left(\left(S(\mathfrak{g})^{\otimes n}\right)^*_{gr}\right)^*_{gr} / F^m\left(\left(S(\mathfrak{g})^{\otimes n}\right)^*_{gr}\right)^*_{gr} \to \left(\left(S(\mathfrak{g})^{\otimes n}\right)^*_{gr}\right)^*_{gr} / F^{m-1}\left(\left(S(\mathfrak{g})^{\otimes n}\right)^*_{gr}\right)^*_{gr}$$

form an inverse system of vector spaces. Moreover, the inverse limit and the dual of $\left(S(\mathfrak{g})^{\otimes n}\right)^*_{gr}$ are isomorphic:

$$\varprojlim_n \left(\left(S(\mathfrak{g})^{\otimes n}\right)^*_{gr}\right)^*_{gr} / F^m\left(\left(S(\mathfrak{g})^{\otimes n}\right)^*_{gr}\right)^*_{gr} \cong \left(\left(S(\mathfrak{g})^{\otimes n}\right)^*_{gr}\right)^*.$$

Similarly, one can define the following decreasing filtration on $S(\mathfrak{g})^{\otimes n}$:

$$\mathcal{F}^m S(\mathfrak{g})^{\otimes n} = \bigoplus_{i\geq m} \bigoplus_{\substack{i_1+\ldots+i_n=i \\ i_1,\ldots,i_n\geq 0}} S(\mathfrak{g})_{i_1} \otimes \cdots \otimes S(\mathfrak{g})_{i_n}.$$

The family $\left(S(\mathfrak{g})^{\otimes n}/\mathcal{F}^m S(\mathfrak{g})^{\otimes n}\right)_{m>0}$ and the projections

$$\pi^n_m : S(\mathfrak{g})^{\otimes n}/\mathcal{F}^m S(\mathfrak{g})^{\otimes n} \to S(\mathfrak{g})^{\otimes n}/\mathcal{F}^{m-1} S(\mathfrak{g})^{\otimes n}$$

form an inverse system of vector spaces. Thus, as $\left(\left(S(\mathfrak{g})^{\otimes n}\right)^*_{gr}\right)^*_{gr} \cong S(\mathfrak{g})^{\otimes n}$ and $F^m\left(\left(S(\mathfrak{g})^{\otimes n}\right)^*_{gr}\right)^*_{gr} \cong \mathcal{F}^m S(\mathfrak{g})^{\otimes n}$ for any $m > 0$, the following isomorphisms hold:

$$\left(\left(S(\mathfrak{g})^{\otimes n}\right)^*_{gr}\right)^* \cong \varprojlim_n \left(\left(S(\mathfrak{g})^{\otimes n}\right)^*_{gr}\right)^*_{gr} / F^m\left(\left(S(\mathfrak{g})^{\otimes n}\right)^*_{gr}\right)^*_{gr} \cong \varprojlim_n S(\mathfrak{g})^{\otimes n}/\mathcal{F}^m S(\mathfrak{g})^{\otimes n}. \tag{A.15}$$

In the following, we will denote with $\widehat{S(\mathfrak{g})^{\otimes n}}$ the inverse limit

$$\varprojlim_n S(\mathfrak{g})^{\otimes n}/\mathcal{F}^m S(\mathfrak{g})^{\otimes n}.$$

With the previous notation, by equations (A.14) and (A.15), the following isomorphisms hold:

$$\left(\left(S(\mathfrak{g})^*_{gr}\right)^{\otimes n}\right)^* \cong \left(\left(S(\mathfrak{g})^{\otimes n}\right)^*_{gr}\right)^* \cong \widehat{S(\mathfrak{g})^{\otimes n}}. \tag{A.16}$$

The isomorphism $\left(\left(S(\mathfrak{g})^*_{gr}\right)^{\otimes n}\right)^* \to \widehat{S(\mathfrak{g})^{\otimes n}}$ of equation (A.16) and the differential d of cochain complex (A.6), induce a linear map $d : \widehat{S(\mathfrak{g})^{\otimes n}} \to \widehat{S(\mathfrak{g})^{\otimes n+1}}$ defined on any vector of $S(\mathfrak{g})^{\otimes n}$ as the following:

$$\begin{aligned} d(x_1 \otimes \cdots \otimes x_n) = 1 \otimes x_1 \otimes \cdots \otimes x_n \\ + \sum_{i=1}^{n} (-1)^i x_1 \otimes \cdots \otimes \Delta_0(x_i) \otimes \cdots \otimes x_n \\ + (-1)^{n+1} x_1 \otimes \cdots \otimes x_n \otimes 1. \end{aligned}$$

Note that it's enough to define $d : \widehat{S(\mathfrak{g})^{\otimes n}} \to \widehat{S(\mathfrak{g})^{\otimes n+1}}$ on any element of $S(\mathfrak{g})^{\otimes n}$ to have d defined on any vector of $\widehat{S(\mathfrak{g})^{\otimes n}}$. Indeed, since d is power-preserving, d maps elements of $S(\mathfrak{g})^{\otimes n}/\mathcal{F}^m S(\mathfrak{g})^{\otimes n}$ to elements of $S(\mathfrak{g})^{\otimes n+1}/\mathcal{F}^m S(\mathfrak{g})^{\otimes n+1}$ and $\pi^{n+1}_m \circ d = d \circ \pi^n_m$. Moreover, the inverse limit is functorial (cf. Proposition [Kas, Proposition XVI 9.2]).
It follows that cochain complex (A.6) is isomorphic to the following one:

$$0 \longrightarrow \mathbb{C} \xrightarrow{d} \widehat{S(\mathfrak{g})} \xrightarrow{d} \widehat{S(\mathfrak{g})^{\otimes 2}} \xrightarrow{d} \cdots \tag{A.17}$$

where the differential $d : \widehat{S(\mathfrak{g})^{\otimes n}} \to \widehat{S(\mathfrak{g})^{\otimes n+1}}$ is defined on any vector $x_1 \otimes \cdots \otimes x_n$ of $S(\mathfrak{g})^{\otimes n}$ as

$$\begin{aligned} d(x_1 \otimes \cdots \otimes x_n) = 1 \otimes x_1 \otimes \cdots \otimes x_n \\ + \sum_{i=1}^{n} (-1)^i x_1 \otimes \cdots \otimes \Delta_0(x_i) \otimes \cdots \otimes x_n \\ + (-1)^{n+1} x_1 \otimes \cdots \otimes x_n \otimes 1. \end{aligned}$$

Denoting cochain complex (A.17) with $\widehat{C} = \bigoplus_{i\geq 0} \widehat{S(\mathfrak{g})^{\otimes i}}$ and defining

$$\widehat{C}_k = \bigoplus_{\substack{i\geq 1 \\ j_1+\ldots+j_i=k \\ j_1,\ldots,j_i\geq 0}} S(\mathfrak{g})_{j_1} \otimes \cdots \otimes S(\mathfrak{g})_{j_i},$$

one can rearrange $\widehat{C}$ as $\widehat{C} = \widehat{C}_{\leq N} \oplus \widehat{C}_{>N}$ where $N = dim(\mathfrak{g})$, $\widehat{C}_{\leq N} = \bigoplus_{k=0}^{N} \widehat{C}_k$, $\widehat{C}_{>N} = \prod_{k>N} \widehat{C}_k$.
Since the differential d is power-preserving and the cohomology of $\widehat{C}$, $H^\bullet(\widehat{C})$, is isomorphic to $\Lambda^\bullet \mathfrak{g}$, one has that $\Lambda^\bullet \mathfrak{g} \cong H^\bullet(\widehat{C}) = H^\bullet(\widehat{C}_{\leq N}) \oplus H^\bullet(\widehat{C}_{>N})$ from which $H^\bullet(\widehat{C}_{>N}) = 0$ and $H^\bullet(\widehat{C}_{\leq N}) \cong \Lambda^\bullet(\mathfrak{g})$.

Similarly, the cochain complex (A.5) can be written as $C = C_{\leq N} \oplus C_{>N}$ where

$$C_k = \bigoplus_{\substack{i\geq 1 \\ j_1+\ldots+j_i=k \\ j_1,\ldots,j_i\geq 0}} S(\mathfrak{g})_{j_1} \otimes \cdots \otimes S(\mathfrak{g})_{j_i},$$

$C_{\leq N} = \bigoplus_{k=0}^{N} C_k$ and $C_{>N} = \bigoplus_{k>N} C_k$. One has that $H^\bullet(C) = H^\bullet(C_{\leq N}) \oplus H^\bullet(C_{>N}) = H^\bullet(C_{\leq N})$ because the differential is power-preserving and $H^\bullet(\widehat{C}_{>N}) = 0$.

It follows that $H^\bullet(C) \cong \Lambda^\bullet(\mathfrak{g})$. Indeed $H^\bullet(C_{\leq N}) = H^\bullet(\widehat{C}_{\leq N})$ because $\widehat{C}_{\leq N} = C_{\leq N}$. Therefore one has that, if $\mathfrak{g}$ is a finite dimensional Lie algebra, the cohomology of cochain complex (3.15) is isomorphic to $\Lambda^\bullet\mathfrak{g}$.

Let us now consider an infinite dimensional abelian Lie algebra $\mathfrak{g}$ and let $\{x_i\}_{i\in I}$ be a basis of $\mathfrak{g}$.

Let F be the set of all finite subsets of I. F is a right directed preordered set (cf. [Bou, Section III.2] for the definition) with the inclusion of sets: $\subseteq$. And let us consider the following family of vector spaces indexed by F:

$$\Big(S\big(\langle x_i \mid i \in S\rangle\big)\Big)_{S\in F}.$$

Moreover, let us define, for each pair S_1, S_2 of elements of F such that $S_1 \subseteq S_2$, the inclusion map $i_{S_2\,S_1} : S\big(\langle x_i \mid i \in S_1\rangle\big) \to S\big(\langle x_i \mid i \in S_2\rangle\big)$. It is clear that, for any $S_1, S_2, S_3 \in F$ such that $S_1 \subseteq S_2 \subseteq S_3$, one has $i_{S_3\,S_1} = i_{S_3\,S_2} \circ i_{S_2\,S_1}$ and $i_{S_1\,S_1}$ is the identity map.
The direct limit of the family $\Big(S\big(\langle x_i \mid i \in S\rangle\big)\Big)_{S\in F}$ with respect to the family of mappings $(i_{S_2\,S_1})$ (cf. [Bou, Section III.5] for the definition), which we denote, following [Bou], with

$$\lim_{\rightarrow}\Big(S\big(\langle x_i \mid i \in S\rangle\big)\Big)_{S\in F}$$

is isomorphic to $S\big(\langle x_i \mid i \in I\rangle\big) = S(\mathfrak{g})$. Therefore the n-th cohomology group of cochain complex (A.5), $H^n\big(S(\mathfrak{g})\big)$, is isomorphic to the following

$$H^n\Big(\lim_{\rightarrow}\Big(S\big(\langle x_i \mid i \in S\rangle\big)\Big)_{S\in F}\Big).$$

An element $x \in S(\mathfrak{g})^{\otimes n}$ is a finite sum of tensor products of n polynomials in a finite number of basis vectors x_is. In particular, there exists a finite set, S, of indices such that $x \in S\big(\langle x_i \mid i \in S\rangle\big)^{\otimes n}$. The image of x under the differential d of cochain complex (A.5) lies in $S\big(\langle x_i \mid i \in S\rangle\big)^{\otimes n+1}$ because $\Delta_0(x_i) = x_i \otimes 1 + 1 \otimes x_i$ for any $i \in I$ and $1 \in S\big(\langle x_i \mid i \in S\rangle\big)$. It follows that the direct limit commutes with the cohomology:

$$H^n\Big(\lim_{\rightarrow}\Big(S\big(\langle x_i \mid i \in S\rangle\big)\Big)_{S\in F}\Big) = \lim_{\rightarrow}\Big(H^n\Big(S\big(\langle x_i \mid i \in S\rangle\big)\Big)\Big)_{S\in F}.$$

Since $[x_i, x_j] = 0$ for all $i, j \in S \in F$, the vector space $\langle x_i \mid i \in S\rangle$ is a finite dimensional abelian Lie algebra. We have already proved that the n-th cohomology group of an abelian finite dimensional Lie algebra $\mathfrak{h}$ is isomorphic to $\Lambda^n\mathfrak{h}$. Thus, the following isomorphisms of vector spaces hold:

$$\lim_{\rightarrow}\Big(H^n\Big(S\big(\langle x_i \mid i \in S\rangle\big)\Big)\Big)_{S\in F} \cong \lim_{\rightarrow}\,(\Lambda^n\langle x_i \mid i \in S\rangle)_{S\in F} \cong \Lambda^n\langle x_i \mid i \in I\rangle = \Lambda^n\mathfrak{g}.$$

Therefore, the n-th cohomology group of cochain complex (A.5) is isomorphic to $\Lambda^n\mathfrak{g}$.

By the previous results, we have proved that, for any Lie algebra $\mathfrak{g}$, the cohomology of cochain complex (3.15) is isomorphic to $\Lambda^\bullet\mathfrak{g}$.

Appendix B

Appendix to Question and Answer 47

This second appendix is devoted to the proof of Proposition 3.3.7. First we will prove that $d_{\mathfrak{g}}^2 = 0$, second we will prove that the $d_{\mathfrak{g}^*}^2 = 0$ and, then, we will prove that $d_{\mathfrak{g}} \circ d_{\mathfrak{g}^*} = d_{\mathfrak{g}^*} \circ d_{\mathfrak{g}}$. Finally we will prove that the differential $d_{\mathfrak{g}^*} : \Lambda^n \mathfrak{g} \to \Lambda^{n+1}\mathfrak{g}$ is given by

$$d_{\mathfrak{g}^*}(\psi) = \frac{1}{n!} Alt\Big(\big[r^{12} + r^{13} + \ldots + r^{1n+1}, 1 \otimes \psi\big]\Big) \tag{B.1}$$

and the map $d_{\mathfrak{g}^*} : \mathrm{Hom}(\mathfrak{g}, \Lambda^2\mathfrak{g}) \to \mathrm{Hom}(\mathfrak{g}, \Lambda^3\mathfrak{g})$ is given by

$$d_{\mathfrak{g}^*}(\psi) = -(\psi \wedge 1) \circ d_{\mathfrak{g}} r - (d_{\mathfrak{g}} r \wedge 1) \circ \psi. \tag{B.2}$$

We will prove that $d_{\mathfrak{g}}^2 = 0$, $d_{\mathfrak{g}^*}^2 = 0$ and $d_{\mathfrak{g}} \circ d_{\mathfrak{g}^*} = d_{\mathfrak{g}^*} \circ d_{\mathfrak{g}}$ in a more general setting: instead of considering a triangular Lie bialgebra $(\mathfrak{g}, [\ ,\ ], r)$, we will work on a Lie bialgebra $(\mathfrak{g}, [\ ,\ ], \delta)$ (of course we will obtain the linear maps of the triangular case putting $\delta = d_{\mathfrak{g}} r$). Let us recall that a Lie bialgebra $(\mathfrak{g}, [\ ,\ ], \delta)$ is a triple such that:

1. $(\mathfrak{g}, [\ ,\ ])$ is a Lie algebra,

2. δ is a linear map $\delta : \mathfrak{g} \to \Lambda^2\mathfrak{g}$ such that the coJacobi Identity

$$(\delta \wedge 1) \circ \delta = 0 \tag{B.3}$$

   is satisfied,

3. δ is a 2-cocycle of the Chevalley-Eilenberg's complex, i.e. it satisfies the equation

$$\delta\big([x, y]\big) = x.\delta(y) - y.\delta(x) \tag{B.4}$$

   for any $x, y \in \mathfrak{g}$.

Here, as before, we denoted with "." the adjoint action of $\mathfrak{g}$ on $\Lambda^2\mathfrak{g}$:

$$x.(x_1 \wedge x_2) = [x, x_1] \wedge x_2 + x_1 \wedge [x, x_2].$$

We will use the same notation for the adjoint action of $\mathfrak{g}$ on $\Lambda^p\mathfrak{g}$ for $p \geq 2$:

$$x.(x_1 \wedge \cdots \wedge x_p) = \sum_{i=1}^{p} x_1 \wedge \cdots \wedge [x, x_i] \wedge \cdots \wedge x_p,$$

where $x \in \mathfrak{g}$ and $x_1 \wedge \cdots \wedge x_p \in \Lambda^p\mathfrak{g}$.
Given $x_1 \wedge \cdots \wedge x_p \in \Lambda^p\mathfrak{g}$, we will denote with $\delta(x_1 \wedge \cdots \wedge x_p)$ the element of $\Lambda^{p+1}\mathfrak{g}$ defined as:

$$\delta(x_1 \wedge \cdots \wedge x_p) = \sum_{i=1}^{p} (-1)^{i-1} x_1 \wedge \cdots \wedge \delta(x_i) \wedge \cdots \wedge x_p.$$

Moreover, when needed, we will write

$$\delta(x) = \sum_{(x)} x' \wedge x''$$

where $x, x', x'' \in \mathfrak{g}$. With these notations, for any $f \in \mathrm{Hom}(\Lambda^m\mathfrak{g}, \Lambda^n\mathfrak{g})$ and $x_1, \cdots, x_{m+1} \in \mathfrak{g}$, the maps $d_\mathfrak{g}$ and $d_{\mathfrak{g}^*}$ are the following:

$$d_\mathfrak{g} : \mathrm{Hom}(\Lambda^m\mathfrak{g}, \Lambda^n\mathfrak{g}) \to \mathrm{Hom}(\Lambda^{m+1}\mathfrak{g}, \Lambda^n\mathfrak{g})$$

is defined as

$$\begin{aligned} d_\mathfrak{g} f(x_1 \wedge \cdots \wedge x_{m+1}) = &\sum_{i=1}^{m+1} (-1)^{i-1} x_i.f(x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x_{m+1}) \\ &+ \sum_{i<j} (-1)^{i+j} f([x_i, x_j] \wedge x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge \widehat{x_j} \wedge \cdots \wedge x_{m+1}), \end{aligned} \tag{B.5}$$

and

$$d_{\mathfrak{g}^*} : \mathrm{Hom}(\Lambda^m\mathfrak{g}, \Lambda^n\mathfrak{g}) \to \mathrm{Hom}(\Lambda^m\mathfrak{g}, \Lambda^{n+1}\mathfrak{g})$$

is defined as

$$\begin{aligned} d_{\mathfrak{g}^*} f(x_1 \wedge \cdots \wedge x_m) = &\sum_{i=1}^{m} \sum_{(x_i)} (-1)^{i-1} \big(x_i' \wedge f(x_i'' \wedge x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x_m) \\ &- x_i'' \wedge f(x_i' \wedge x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x_m) \big) \\ &- \delta\big(f(x_1 \wedge \cdots \wedge x_m)\big). \end{aligned} \tag{B.6}$$

Let us introduce some more useful notations: in the following, given $x_1 \wedge \cdots \wedge x_{m+2} \in \Lambda^{m+2}\mathfrak{g}$ with $m \geq 2$, we will denote with x^i the expression $x^i = x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x_{m+2} \in \Lambda^{m+1}\mathfrak{g}$ where $1 \leq i \leq m+2$, with x^{ij} the expression $x^{ij} = x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge \widehat{x_j} \wedge \cdots \wedge \cdots \wedge x_{m+2} \in \Lambda^m\mathfrak{g}$ where $1 \leq i < j \leq m+2$, with x^{ijh} the expression $x^{ijh} = x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge \widehat{x_j} \wedge \cdots \wedge \widehat{x_h} \wedge \cdots \wedge x_{m+2} \in \Lambda^{m-1}\mathfrak{g}$ where $1 \leq i < j < h \leq m+2$ and with x^{ijhk} the expression $x^{ijhk} = x_1 \wedge \cdots \wedge \widehat{x_i} \wedge$

$\cdots \wedge \widehat{x_j} \wedge \cdots \wedge \widehat{x_h} \wedge \cdots \wedge \widehat{x_k} \wedge \cdots \wedge x_{m+2} \in \Lambda^{m-2}\mathfrak{g}$ where $1 \leq i < j < h < k \leq m+2$. Let $f \in \mathrm{Hom}(\Lambda^m \mathfrak{g}, \Lambda^n \mathfrak{g})$,

$$
\begin{aligned}
& d_{\mathfrak{g}}^2(f)(x_1 \wedge \cdots \wedge x_{m+2}) \\
&= \sum_{i=1}^{m+2} (-1)^{i-1} x_i . d_{\mathfrak{g}} f(x^i) + \sum_{i<j} (-1)^{i+j} d_{\mathfrak{g}} f([x_i, x_j] \wedge x^{ij}) \\
&= \sum_{i=1}^{m+2} \Big(\sum_{\substack{k \\ k<i}} (-1)^{i-1} (-1)^{k-1} x_i . x_k . f(x^{ki}) + \sum_{\substack{k \\ k>i}} (-1)^{i-1} (-1)^{k} x_i . x_k . f(x^{ik}) \\
&\qquad + \sum_{\substack{h,k \\ h<k<i}} (-1)^{i-1} (-1)^{h+k} x_i . f([x_h, x_k] \wedge x^{hki}) \\
&\qquad + \sum_{\substack{h,k \\ h<i<k}} (-1)^{i-1} (-1)^{h+k-1} x_i . f([x_h, x_k] \wedge x^{hik}) \\
&\qquad + \sum_{\substack{h,k \\ i<h<k}} (-1)^{i-1} (-1)^{h+k} x_i . f([x_h, x_k] \wedge x^{ihk}) \Big) \\
&\quad + \sum_{i<j} \Big((-1)^{i+j} [x_i, x_j] . f(x^{ij}) + \sum_{\substack{k \\ k<i}} (-1)^{i+j} (-1)^{k} x_k . f([x_i, x_j] \wedge x^{kij}) \\
&\qquad + \sum_{\substack{k \\ i<k<j}} (-1)^{i+j} (-1)^{k-1} x_k . f([x_i, x_j] \wedge x^{ikj}) \\
&\qquad + \sum_{\substack{k \\ i<j<k}} (-1)^{i+j} (-1)^{k} x_k . f([x_i, x_j] \wedge x^{ijk}) \\
&\qquad + \sum_{\substack{k \\ k<i}} (-1)^{i+j} (-1)^{k+2} f([[x_i, x_j], x_k] \wedge x^{kij}) \\
&\qquad + \sum_{\substack{k \\ i<k<j}} (-1)^{i+j} (-1)^{k+1} f([[x_i, x_j], x_k] \wedge x^{ikj}) \\
&\qquad + \sum_{\substack{k \\ i<j<k}} (-1)^{i+j} (-1)^{k} f([[x_i, x_j], x_k] \wedge x^{ijk}) \\
&\qquad + \sum_{\substack{h,k \\ h<k<i<j}} (-1)^{i+j} (-1)^{h+1+k+1} f([x_h, x_k] \wedge [x_i, x_j] \wedge x^{hkij}) \\
&\qquad + \sum_{\substack{h,k \\ h<i<k<j}} (-1)^{i+j} (-1)^{h+1+k} f([x_h, x_k] \wedge [x_i, x_j] \wedge x^{hikj}) \\
&\qquad + \sum_{\substack{h,k \\ h<i<j<k}} (-1)^{i+j} (-1)^{h+1+k-1} f([x_h, x_k] \wedge [x_i, x_j] \wedge x^{hijk})
\end{aligned}
$$

$$+ \sum_{\substack{h,k \\ i<h<k<j}} (-1)^{i+j}(-1)^{h+k} f([x_h, x_k] \wedge [x_i, x_j] \wedge x^{ihkj})$$
$$+ \sum_{\substack{h,k \\ i<h<j<k}} (-1)^{i+j}(-1)^{h+k-1} f([x_h, x_k] \wedge [x_i, x_j] \wedge x^{ihjk})$$
$$+ \sum_{\substack{h,k \\ i<j<h<k}} (-1)^{i+j}(-1)^{h-1+k-1} f([x_h, x_k] \wedge [x_i, x_j] \wedge x^{ijhk})\Big).$$

As $\sum_{i=1}^{m+2} \sum_{\substack{k \\ k<i}} = \sum_{k<i}$ and $\sum_{i=1}^{m+2} \sum_{\substack{k \\ k>i}} = \sum_{i<k}$, renaming k as i and i as j in the first term of the right hand side, and renaming k as j in the second term of the right hand side, they become respectively

$$\sum_{i<j} (-1)^{i+j} x_j . x_i . f(x^{ij})$$

and

$$-\sum_{i<j} (-1)^{i+j} x_i . x_j . f(x^{ij})$$

therefore they cancel with the sixth term of the right hand side

$$\sum_{i<j} (-1)^{i+j} [x_i, x_j] . f(x^{ij}).$$

Renaming h as i, k as j and i as k in the third term of the right hand side, it becomes

$$-\sum_{h<k<i} (-1)^{i+h+k} x_k . f([x_i, x_j] \wedge x^{ijk})$$

therefore it cancels with the ninth term

$$\sum_{i<j} \sum_{\substack{k \\ i<k<j}} (-1)^{i+j+k} x_k . f([x_i, x_j] \wedge x^{ijk})$$

Renaming h as i, i as k and k as j in the fourth term of the right hand side, it becomes

$$\sum_{i<k<j} (-1)^{i+h+k} x_k . f([x_i, x_j] \wedge x^{ikj})$$

therefore it cancels with the eighth term

$$\sum_{i<j} \sum_{\substack{k \\ i<k<j}} (-1)^{i+j+k-1} x_k . f([x_i, x_j] \wedge x^{ikj}).$$

Renaming i as k, h as i and k as j in the fifth term of the right hand side, it becomes

$$-\sum_{k<i<j} (-1)^{i+j+k} x_k . f([x_i, x_j] \wedge x^{kij})$$

therefore it cancels with the seventh term

$$\sum_{i<j} \sum_{\substack{k \\ k<i<j}} (-1)^{i+j+k} x_k . f([x_i, x_j] \wedge x^{kij}).$$

The tenth, eleventh and twelfth terms cancel because of the Jacobi Identity. Finally, renaming the indices of the sums of the last six terms in order to have summations over $i < j < h < k$, one sees that the thirteenth term cancels with the eighteenth, the fourteenth cancels with the seventeenth and the fifteenth cancels with the sixteenth. Thus $d_{\mathfrak{g}}^2 = 0$.

$$
\begin{aligned}
& d_{\mathfrak{g}^*}^2 f(x_1 \wedge \cdots \wedge x_m) \\
& = \sum_{i=1}^{m} \sum_{(x_i)} (-1)^{i-1} (x_i' \wedge d_{\mathfrak{g}^*}(x_i'' \wedge x^i) - x_i'' \wedge d_{\mathfrak{g}^*}(x_i' \wedge x^i)) \\
& \quad - \delta(d_{\mathfrak{g}^*} f(x_1 \wedge \cdots \wedge x_m)) \\
& = \sum_{i=1}^{m} \sum_{(x_i)} \sum_{(x_i'')} (-1)^{i-1} \Big(x_i' \wedge (x_i'')' \wedge f((x_i'')'' \wedge x^i) \\
& \qquad\qquad - x_i' \wedge (x_i'')'' \wedge f((x_i'')' \wedge x^i) \Big) \\
& \quad + \sum_{i=1}^{m} \sum_{\substack{j \\ j<i}} \sum_{(x_i)} \sum_{(x_j)} (-1)^{i-1} (-1)^{j} (x_i' \wedge x_j' \wedge f(x_j'' \wedge x_i'' \wedge x^{ji}) \\
& \qquad\qquad - x_i' \wedge x_j'' \wedge f(x_j' \wedge x_i'' \wedge x^{ji})) \\
& \quad + \sum_{i=1}^{m} \sum_{\substack{j \\ i<j}} \sum_{(x_i)} \sum_{(x_j)} (-1)^{i-1} (-1)^{j-1} (x_i' \wedge x_j' \wedge f(x_j'' \wedge x_i'' \wedge x^{ij}) \\
& \qquad\qquad - x_i' \wedge x_j'' \wedge f(x_j' \wedge x_i'' \wedge x^{ij})) \\
& \quad - \sum_{i=1}^{m} \sum_{(x_i)} \sum_{(x_i')} (-1)^{i-1} \Big(x_i'' \wedge (x_i')' \wedge f((x_i')'' \wedge x^i) \\
& \qquad\qquad - x_i'' \wedge (x_i')'' \wedge f((x_i')' \wedge x^i) \Big) \\
& \quad - \sum_{i=1}^{m} \sum_{\substack{j \\ j<i}} \sum_{(x_i)} \sum_{(x_j)} (-1)^{i-1} (-1)^{j} (x_i'' \wedge x_j' \wedge f(x_j'' \wedge x_i' \wedge x^{ji}) \\
& \qquad\qquad - x_i'' \wedge x_j'' \wedge f(x_j' \wedge x_i' \wedge x^{ji})) \\
& \quad - \sum_{i=1}^{m} \sum_{\substack{j \\ i<j}} \sum_{(x_i)} \sum_{(x_j)} (-1)^{i-1} (-1)^{j-1} (x_i'' \wedge x_j' \wedge f(x_j'' \wedge x_i' \wedge x^{ij})
\end{aligned}
$$

$$- x_i'' \wedge x_j'' \wedge f(x_j' \wedge x_i' \wedge x^{ij}))$$

$$- \sum_{i=1}^{m} \sum_{(x_i)} (-1)^{i-1} \Big(x_i' \wedge \delta\big(f(x_i'' \wedge x^i)\big) - x_i'' \wedge \delta\big(f(x_i' \wedge x^i)\big) \Big)$$
$$- \sum_{i=1}^{m} \sum_{(x_i)} (-1)^{i-1} \big(\delta(x_i') \wedge f(x_i'' \wedge x^i) - \delta(x_i'') \wedge f(x_i' \wedge x^i)\big)$$
$$+ \sum_{i=1}^{m} \sum_{(x_i)} (-1)^{i-1} \Big(x_i' \wedge \delta\big(f(x_i'' \wedge x^i)\big) - x_i'' \wedge \delta\big(f(x_i' \wedge x^i)\big) \Big)$$
$$+ \delta\Big(\delta\big(f(x_1 \wedge \cdots \wedge x_m)\big)\Big).$$

By the coJacobi Identity (B.3), one has

$$\begin{aligned} 0 &= (\delta \wedge 1)\delta(x_i) \\ &= \sum_{(x_i)} \sum_{x_i'} (x_i')' \wedge (x_i')'' \wedge x_i'' - \sum_{(x_i)} \sum_{(x_i'')} (x_i'')' \wedge (x_i'')'' \wedge x_i' \\ &= \sum_{(x_i)} \sum_{x_i'} x_i'' \wedge (x_i')' \wedge (x_i')'' - \sum_{(x_i)} \sum_{(x_i'')} x_i' \wedge (x_i'')' \wedge (x_i'')'' \end{aligned}$$

Therefore the first and the seventh lines of the right hand side cancel. For the same motivation, the second and the eighth lines cancel and the fourteenth line cancels. The thirteenth line cancels the fifteenth because they are the opposite. And, since $\sum_{i=1}^m \sum_{\substack{j \\ j<i}} = \sum_{j<i}$ and $\sum_{i=1}^m \sum_{\substack{j \\ i<j}} = \sum_{i<j}$, renaming the indices of the third to sixth lines and ninth to twelve lines in order to have summation over $i < j$, one has

$$- \sum_{i<j} \sum_{(x_i)} \sum_{(x_j)} (-1)^{i+j} \big(x_j' \wedge x_i' \wedge f(x_i'' \wedge x_j'' \wedge x^{ij})$$
$$- x_j' \wedge x_i'' \wedge f(x_i' \wedge x_j'' \wedge x^{ij})\big)$$

$$+ \sum_{i<j} \sum_{(x_i)} \sum_{(x_j)} (-1)^{i+j} \big(x_i' \wedge x_j' \wedge f(x_j'' \wedge x_i'' \wedge x^{ij})$$
$$- x_i' \wedge x_j'' \wedge f(x_j' \wedge x_i'' \wedge x^{ij})\big)$$

$$+ \sum_{i<j} \sum_{(x_i)} \sum_{(x_j)} (-1)^{i+j} \big(x_j'' \wedge x_i' \wedge f(x_i'' \wedge x_j' \wedge x^{ij})$$
$$- x_j'' \wedge x_i'' \wedge f(x_i' \wedge x_j' \wedge x^{ij})\big)$$

$$- \sum_{i<j} \sum_{(x_i)} \sum_{(x_j)} (-1)^{i+j} \big(x_i'' \wedge x_j' \wedge f(x_j'' \wedge x_i' \wedge x^{ij})$$
$$- x_i'' \wedge x_j'' \wedge f(x_j' \wedge x_i' \wedge x^{ij})\big)$$

which cancel. Finally, writing $f(x_1 \wedge \cdots \wedge x_m) = \sum_{k=1}^{s} f_1^k \wedge \cdots \wedge f_n^k$ where $s \in \mathbb{N}$ and $f_j^k \in \mathfrak{g}$ for $i = 1, \ldots, n$ and $k = 1, \ldots, s$, one has

$$
\begin{aligned}
&\delta\Big(\delta\big(f(x_1 \wedge \cdots \wedge x_m)\big)\Big)\\
&= \sum_{k=1}^{s} \delta\Big(\sum_{i=1}^{n} (-1)^{i-1} f_1^k \wedge \cdots \wedge \delta(f_i^k) \wedge \cdots \wedge f_n^k\Big)\\
&= \sum_{k=1}^{s} \sum_{i=1}^{n} \sum_{\substack{j\\ j<i}} (-1)^{i-1}(-1)^{j-1} f_1^k \wedge \cdots \wedge \delta(f_j^k) \wedge \cdots \wedge \delta(f_i^k) \wedge \cdots \wedge f_n^k\\
&\quad + \sum_{k=1}^{s} \sum_{i=1}^{n} \sum_{\substack{j\\ i<j}} (-1)^{i-1}(-1)^{j} f_1^k \wedge \cdots \wedge \delta(f_i^k) \wedge \cdots \wedge \delta(f_j^k) \wedge \cdots \wedge f_n^k\\
&\quad + \sum_{k=1}^{s} \sum_{i=1}^{n} \sum_{(f_i^k)} (-1)^{i-1} f_1^k \wedge \cdots \wedge \delta\big((f_i^k)'\big) \wedge (f_i^k)'' \wedge \cdots \wedge f_n^k\\
&\quad + \sum_{k=1}^{s} \sum_{i=1}^{n} \sum_{(f_i^k)} (-1)^{i} f_1^k \wedge \cdots \wedge (f_i^k)' \wedge \delta\big((f_i^k)''\big) \wedge \cdots \wedge f_n^k\\
&= \sum_{k=1}^{s} \sum_{j<i} (-1)^{i+j} f_1^k \wedge \cdots \wedge \delta(f_j^k) \wedge \cdots \wedge \delta(f_i^k) \wedge \cdots \wedge f_n^k\\
&\quad - \sum_{k=1}^{s} \sum_{i<j} (-1)^{i+j} f_1^k \wedge \cdots \wedge \delta(f_i^k) \wedge \cdots \wedge \delta(f_j^k) \wedge \cdots \wedge f_n^k\\
&\quad - \sum_{k=1}^{s} \sum_{i=1}^{n} \sum_{(f_i^k)} (-1)^{i} f_1^k \wedge \cdots \wedge \delta\big((f_i^k)'\big) \wedge (f_i^k)'' \wedge \cdots \wedge f_n^k\\
&\quad + \sum_{k=1}^{s} \sum_{i=1}^{n} \sum_{(f_i^k)} (-1)^{i} f_1^k \wedge \cdots \wedge (f_i^k)' \wedge \delta\big((f_i^k)''\big) \wedge \cdots \wedge f_n^k.
\end{aligned}
$$

The third and the fourth line of the right hand side cancel because of coJacobi (B.3) and renaming j as i and i as j in the first line of the right hand side, one sees that the first and second lines cancel. Thus $d_{\mathfrak{g}^*}^2 = 0$.

Let us now prove that $d_{\mathfrak{g}^*} \circ d_{\mathfrak{g}} f = d_{\mathfrak{g}^*} \circ d_{\mathfrak{g}} f$ for any $f \in \mathrm{Hom}(\Lambda^m \mathfrak{g}, \Lambda^n \mathfrak{g})$.

$$
\begin{aligned}
&\big(d_{\mathfrak{g}^*} \circ d_{\mathfrak{g}} f - d_{\mathfrak{g}^*} \circ d_{\mathfrak{g}} f\big)(x_1 \wedge \cdots \wedge x_{m+1})\\
&= \sum_{i=1}^{m+1} \sum_{(x_i)} (-1)^{i-1} \big(x_i' \wedge d_{\mathfrak{g}} f(x_i'' \wedge x^i) - x_i'' \wedge d_{\mathfrak{g}} f(x_i' \wedge x^i)\big)\\
&\quad - \delta\big(d_{\mathfrak{g}} f(x_1 \wedge \cdots \wedge x_{m+1})\big)\\
&\quad - \sum_{i=1}^{m+1} (-1)^{i-1} x_i . d_{\mathfrak{g}^*} f(x^i) - \sum_{i<j} (-1)^{i+j} d_{\mathfrak{g}^*} f([x_i, x_j] \wedge x^{ij})\\
&= \sum_{i=1}^{m+1} \sum_{(x_i)} (-1)^{i-1} \Big(x_i' \wedge \big(x_i''.f(x^i)\big) - x_i'' \wedge \big(x_i'.f(x^i)\big)\Big)
\end{aligned}
$$

$$+ \sum_{i=1}^{m+1} \sum_{(x_i)} \sum_{\substack{j \\ j<i}} (-1)^{i-1}(-1)^{j} \Big(x_i' \wedge \big(x_j . f(x_i'' \wedge x^{ji}) \big) - x_i'' \wedge \big(x_j . f(x_i' \wedge x^{ji}) \big) \Big)$$

$$+ \sum_{i=1}^{m+1} \sum_{(x_i)} \sum_{\substack{j \\ i<j}} (-1)^{i-1}(-1)^{j-1} \Big(x_i' \wedge \big(x_j . f(x_i'' \wedge x^{ij}) \big) - x_i'' \wedge \big(x_j . f(x_i' \wedge x^{ij}) \big) \Big)$$

$$+ \sum_{i=1}^{m+1} \sum_{(x_i)} \sum_{\substack{j \\ j<i}} (-1)^{i-1}(-1)^{j} \big(x_i' \wedge f([x_i'', x_j] \wedge x^{ji}) - x_i'' \wedge f([x_i', x_j] \wedge x^{ji}) \big)$$

$$+ \sum_{i=1}^{m+1} \sum_{(x_i)} \sum_{\substack{j \\ i<j}} (-1)^{i-1}(-1)^{j+1} \big(x_i' \wedge f([x_i'', x_j] \wedge x^{ji}) - x_i'' \wedge f([x_i', x_j] \wedge x^{ij}) \big)$$

$$+ \sum_{i=1}^{m+1} \sum_{(x_i)} \sum_{\substack{j,k \\ j<k<i}} (-1)^{i-1}(-1)^{j+1+k+1} \big(x_i' \wedge f([x_j, x_k] \wedge x_i'' \wedge x^{jki}) - x_i'' \wedge f([x_j, x_k] \wedge x_i' \wedge x^{jki}) \big)$$

$$+ \sum_{i=1}^{m+1} \sum_{(x_i)} \sum_{\substack{j,k \\ j<i<k}} (-1)^{i-1}(-1)^{j+1+k} \big(x_i' \wedge f([x_j, x_k] \wedge x_i'' \wedge x^{jik}) - x_i'' \wedge f([x_j, x_k] \wedge x_i' \wedge x^{jik}) \big)$$

$$+ \sum_{i=1}^{m+1} \sum_{(x_i)} \sum_{\substack{j,k \\ i<j<k}} (-1)^{i-1}(-1)^{j+k} \big(x_i' \wedge f([x_j, x_k] \wedge x_i'' \wedge x^{ijk}) - x_i'' \wedge f([x_j, x_k] \wedge x_i' \wedge x^{ijk}) \big)$$

$$
\begin{aligned}
&- \delta\Big(\sum_{i=1}^{i-1}(-1)^{m+1}x_i.f(x^i)\Big) - \delta\Big(\sum_{i<j}(-1)^{i+j}f([x_i,x_j]\wedge x^{ij})\Big)\\
&- \sum_{i=1}^{m+1}\sum_{\substack{j\\ j<i}}\sum_{(x_j)}(-1)^{i-1}(-1)^{j-1}\Big(x_i.(x_j'\wedge f(x_j''\wedge x^{ji}))\\
&\qquad\qquad\qquad\qquad - x_i.(x_j''\wedge f(x_j'\wedge x^{ji}))\Big)\\
&- \sum_{i=1}^{m+1}\sum_{\substack{j\\ i<j}}\sum_{(x_j)}(-1)^{i-1}(-1)^{j}\Big(x_i.(x_j'\wedge f(x_j''\wedge x^{ij}))\\
&\qquad\qquad\qquad\qquad - x_i.(x_j''\wedge f(x_j'\wedge x^{ij}))\Big)\\
&+ \sum_{i=1}^{m+1}(-1)^{i-1}x_i.(\delta(f(x^i)))\\
&- \sum_{i<j}\sum_{([x_i,x_j])}(-1)^{i+j}\Big([x_i,x_j]'\wedge f([x_i,x_j]''\wedge x^{ij})\\
&\qquad\qquad\qquad - [x_i,x_j]''\wedge f([x_i,x_j]'\wedge x^{ij})\\
&- \sum_{i<j}\sum_{\substack{k\\ k<i<j}}\sum_{(x_k)}(-1)^{i+j}(-1)^{k}(x_k'\wedge f(x_k''\wedge [x_i,x_j]\wedge x^{kij})\\
&\qquad\qquad\qquad\qquad - x_k''\wedge f(x_k'\wedge [x_i,x_j]\wedge x^{kij}))\\
&- \sum_{i<j}\sum_{\substack{k\\ i<k<j}}\sum_{(x_k)}(-1)^{i+j}(-1)^{k-1}(x_k'\wedge f(x_k''\wedge [x_i,x_j]\wedge x^{ikj})\\
&\qquad\qquad\qquad\qquad - x_k''\wedge f(x_k'\wedge [x_i,x_j]\wedge x^{ikj}))\\
&- \sum_{i<j}\sum_{\substack{k\\ i<j<k}}\sum_{(x_k)}(-1)^{i+j}(-1)^{k}(x_k'\wedge f(x_k''\wedge [x_i,x_j]\wedge x^{ijk})\\
&\qquad\qquad\qquad\qquad - x_k''\wedge f(x_k'\wedge [x_i,x_j]\wedge x^{ijk}))\\
&+ \sum_{i<j}(-1)^{i+j}\delta(f([x_i,x_j]\wedge x^{ij}))\\
= &\sum_{i=1}^{m+1}\sum_{(x_i)}(-1)^{i-1}\Big(x_i'\wedge(x_i''.f(x^i)) - x_i''\wedge(x_i'.f(x^i))\Big)
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{i<j}^{m+1}\sum_{(x_j)}(-1)^{i+j}\Big(x_j'\wedge\big(x_i.f(x_j''\wedge x^{ij})\big)-x_j''\wedge\big(x_i.f(x_j'\wedge x^{ij})\big)\Big)\\
&+\sum_{i<j}\sum_{(x_i)}(-1)^{i+j}\Big(x_i'\wedge\big(x_j.f(x_i''\wedge x^{ij})\big)-x_i''\wedge\big(x_j.f(x_i'\wedge x^{ij})\big)\Big)\\
&-\sum_{i<j}\sum_{(x_j)}(-1)^{i+j}\big(x_j'\wedge f([x_j'',x_i]\wedge x^{ij})-x_j''\wedge f([x_j',x_i]\wedge x^{ij})\big)\\
&+\sum_{i<j}\sum_{(x_i)}(-1)^{i+j}\big(x_i'\wedge f([x_i'',x_j]\wedge x^{ij})-x_i''\wedge f([x_i',x_j]\wedge x^{ij})\big)\\
&-\sum_{i<j<k}\sum_{(x_k)}(-1)^{i+j+k}\big(x_k'\wedge f([x_i,x_j]\wedge x_k''\wedge x^{ijk})\\
&\qquad\qquad\qquad -x_k''\wedge f([x_i,x_j]\wedge x_k'\wedge x^{ijk})\big)\\
&+\sum_{i<j<k}\sum_{(x_j)}(-1)^{i+j+k}\big(x_j'\wedge f([x_i,x_k]\wedge x_j''\wedge x^{ijk})\\
&\qquad\qquad\qquad -x_j''\wedge f([x_i,x_k]\wedge x_j'\wedge x^{ijk})\big)\\
&-\sum_{i<j<k}\sum_{(x_i)}(-1)^{i+j+k}\big(x_i'\wedge f([x_j,x_k]\wedge x_i''\wedge x^{ijk})\\
&\qquad\qquad\qquad -x_i''\wedge f([x_j,x_k]\wedge x_i'\wedge x^{ijk})\big)
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{i=1}^{m+1}(-1)^{i-1}\delta\big(x_i.f(x^i)\big)-\sum_{i<j}(-1)^{i+j}\delta\big(f([x_i,x_j]\wedge x^{ij})\big)\\
&-\sum_{i<j}\sum_{(x_i)}(-1)^{i+j}\big([x_j,x_i']\wedge f(x_i''\wedge x^{ij})-[x_j,x_i'']\wedge f(x_i'\wedge x^{ij})\big)\\
&-\sum_{i<j}\sum_{(x_i)}(-1)^{i+j}\Big(x_i'\wedge\big(x_j.f(x_i''\wedge x^{ij})\big)-x_i''\wedge\big(x_j.f(x_i'\wedge x^{ij})\big)\Big)\\
&+\sum_{ß<j}\sum_{(x_j)}(-1)^{i+j}\big([x_i,x_j']\wedge f(x_j''\wedge x^{ij})-[x_i,x_j'']\wedge f(x_j'\wedge x^{ij})\big)\\
&+\sum_{ß<j}\sum_{(x_j)}(-1)^{i+j}\Big(x_j'\wedge\big(x_i.f(x_j''\wedge x^{ij})\big)-x_j''\wedge\big(x_i.f(x_j'\wedge x^{ij})\big)\Big)\\
&+\sum_{i=1}^{m+1}(-1)^{i-1}x_i.\big(\delta(f(x^i))\big)\\
&-\sum_{i<j}\sum_{([x_i,x_j])}(-1)^{i+j}\Big([x_i,x_j]'\wedge f([x_i,x_j]''\wedge x^{ij})\\
&\qquad\qquad\qquad -[x_i,x_j]''\wedge f([x_i,x_j]'\wedge x^{ij})\\
&-\sum_{i<j<k}\sum_{(x_i)}(-1)^{i+j+k}\big(x_i'\wedge f(x_i''\wedge[x_j,x_k]\wedge x^{ijk})
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad - x_i'' \wedge f(x_i' \wedge [x_j, x_k] \wedge x^{ijk})) \\
&+ \sum_{i<j<k} \sum_{(x_j)} (-1)^{i+j+k} \big(x_j' \wedge f(x_j'' \wedge [x_i, x_k] \wedge x^{ijk}) \\
&\qquad\qquad - x_j'' \wedge f(x_j' \wedge [x_i, x_k] \wedge x^{ijk})\big) \\
&- \sum_{i<j<k} \sum_{(x_k)} (-1)^{i+j+k} \big(x_k' \wedge f(x_k'' \wedge [x_i, x_j] \wedge x^{ijk}) \\
&\qquad\qquad - x_k'' \wedge f(x_k' \wedge [x_i, x_j] \wedge x^{ijk})\big) \\
&+ \sum_{i<j} (-1)^{i+j} \delta\big(f([x_i, x_j] \wedge x^{ij})\big).
\end{aligned}
$$

The second line of the right hand side cancels with the sixteenth line, the third line cancels with the fourteenth line, the sixth line cancels with twenty-fourth line, the seventh line cancels with the twenty-fifth line, the eighth line cancels with the twenty-second line, the ninth line cancels with the twenty-third line, the tenth line cancels with the twentieth line, the eleventh line cancels with the twenty-first line and the right term of the twelfth line cancels with the twenty-sixth line. Moreover, by the 2-cocycle condition of δ, one has

$$\delta([x_i, x_j]) = x_i.\delta(x_j) - x_j.\delta(x_i)$$

hence, using the notation $\delta(y) = \sum_{(y)} y' \wedge y''$, one has

$$
\begin{aligned}
\sum_{[x_i,x_j]} [x_i, x_j]' \wedge [x_i, x_j]'' &= \sum_{(x_j)} [x_i, x_j'] \wedge x_j'' + \sum_{(x_j)} x_j' \wedge [x_i, x_j''] \\
&\quad - \sum_{(x_i)} [x_j, x_i'] \wedge x_i'' - \sum_{(x_i)} x_i' \wedge [x_j, x_i''].
\end{aligned}
$$

If follows that the left terms of the fourth, fifth, thirteenth and fifteenth lines and the eighteenth line cancel, and the right terms of fourth, fifth, thirteenth and fifteenth lines and the nineteenth line cancel. Therefore

$$
\begin{aligned}
&\big(d_{\mathfrak{g}^*} \circ d_{\mathfrak{g}} f - d_{\mathfrak{g}^*} \circ d_{\mathfrak{g}} f\big)(x_1 \wedge \cdots \wedge x_{m+1}) \\
&= \sum_{i=1}^{m+1} \sum_{(x_i)} (-1)^{i-1} \Big(x_i' \wedge \big(x_i''.f(x^i)\big) - x_i'' \wedge \big(x_i'.f(x^i)\big)\Big) \\
&\quad - \sum_{i=1}^{m+1} (-1)^{i-1} \delta\big(x_i.f(x^i)\big) + \sum_{i=1}^{m+1} (-1)^{i-1} x_i.\big(\delta(f(x^i))\big).
\end{aligned}
$$

On the other hand, for any $f_1, \ldots, f_n \in \mathfrak{g}$ one has

$$\sum_{i=1}^{m+1} \sum_{(x_i)} (-1)^{i-1} \Big(x_i' \wedge \big(x_i''.(f_1 \wedge \cdots \wedge f_n)\big) - x_i'' \wedge \big(x_i'.(f_1 \wedge \cdots \wedge f_n)\big)\Big)$$

$$
\begin{aligned}
&- \sum_{i=1}^{m+1} (-1)^{i-1} \delta\big(x_i.(f_1 \wedge \cdots \wedge f_n)\big) + \sum_{i=1}^{m+1} (-1)^{i-1} x_i.\big(\delta(f_1 \wedge \cdots \wedge f_n)\big) \\
= &\sum_{i=1}^{m+1} \sum_{j=1}^{n} \sum_{(x_i)} (-1)^{i-1} x_i' \wedge f_1 \wedge \cdots \wedge [x_i'', f_j] \wedge \cdots \wedge f_n \\
&- \sum_{i=1}^{m+1} \sum_{j=1}^{n} \sum_{(x_i)} (-1)^{i-1} x_i'' \wedge f_1 \wedge \cdots \wedge [x_i', f_j] \wedge \cdots \wedge f_n \\
&- \sum_{i=1}^{m+1} \sum_{j=1}^{n} (-1)^{i-1} \delta\big(f_1 \wedge \cdots \wedge [x_i, f_j] \wedge \cdots \wedge f_n\big) \\
&+ \sum_{i=1}^{m+1} \sum_{j=1}^{n} (-1)^{i-1} (-1)^{j-1} x_i.\big(f_1 \wedge \cdots \wedge \delta(f_j) \wedge \cdots \wedge f_n\big) \\
= &- \sum_{i=1}^{m+1} \sum_{j=1}^{n} \sum_{(x_i)} (-1)^{i-1} (-1)^{j-1} f_1 \wedge \cdots \wedge x_i' \wedge [f_j, x_i''] \wedge \cdots \wedge f_n \\
&+ \sum_{i=1}^{m+1} \sum_{j=1}^{n} \sum_{(x_i)} (-1)^{i-1} (-1)^{j} f_1 \wedge \cdots \wedge [f_j, x_i'] \wedge x_i'' \wedge \cdots \wedge f_n \\
&- \sum_{i=1}^{m+1} \sum_{j=1}^{n} \sum_{\substack{k \\ k \neq j}} (-1)^{i-1} (-1)^{k-1} f_1 \wedge \cdots \wedge \delta(f_k) \wedge \cdots \wedge [x_i, f_j] \wedge \cdots \wedge f_n \\
&- \sum_{i=1}^{m+1} \sum_{j=1}^{n} (-1)^{i-1} (-1)^{j-1} f_1 \wedge \cdots \wedge \delta([x_i, f_j]) \wedge \cdots \wedge f_n \\
&+ \sum_{i=1}^{m+1} \sum_{j=1}^{n} \sum_{\substack{k \\ k \neq j}} (-1)^{i-1} (-1)^{j-1} f_1 \wedge \cdots \wedge \delta(f_j) \wedge \cdots \wedge [x_i, f_k] \wedge \cdots \wedge f_n \\
&+ \sum_{i=1}^{m+1} \sum_{j=1}^{n} (-1)^{i-1} (-1)^{j-1} f_1 \wedge \cdots \wedge x_i.\delta(f_j) \wedge \cdots \wedge f_n.
\end{aligned}
$$

The third line of the right hand side is equal to

$$
- \sum_{i=1}^{m+1} \sum_{k \neq j} (-1)^{i-1} (-1)^{k-1} f_1 \wedge \cdots \wedge \delta(f_k) \wedge \cdots \wedge [x_i, f_j] \wedge \cdots \wedge f_n
$$

and the fifth line of the right hand side is equal to

$$
\sum_{i=1}^{m+1} \sum_{k \neq j} (-1)^{i-1} (-1)^{j-1} f_1 \wedge \cdots \wedge \delta(f_j) \wedge \cdots \wedge [x_i, f_k] \wedge \cdots \wedge f_n
$$

hence they cancel. Finally, since the first two lines are equal to

$$
\sum_{i=1}^{m+1} \sum_{j=1}^{n} \sum_{(x_i)} (-1)^{i-1} (-1)^{j} f_1 \wedge \cdots \wedge f_j.\delta(x_i) \wedge \cdots \wedge f_n,
$$

one has

$$\sum_{i=1}^{m+1}\sum_{(x_i)}(-1)^{i-1}\Big(x_i'\wedge(x_i''.(f_1\wedge\cdots\wedge f_n))-x_i''\wedge(x_i'.(f_1\wedge\cdots\wedge f_n))\Big)$$
$$-\sum_{i=1}^{m+1}(-1)^{i-1}\delta(x_i.(f_1\wedge\cdots\wedge f_n))+\sum_{i=1}^{m+1}(-1)^{i-1}x_i.(\delta(f_1\wedge\cdots\wedge f_n))$$
$$=\sum_{i=1}^{m+1}\sum_{j=1}^{n}\sum_{(x_i)}(-1)^{i-1}(-1)^j f_1\wedge\cdots\wedge\big(\delta([x_i,f_j])-x_i.f_j+f_j.\delta(x_i)\big)\wedge\cdots\wedge f_n$$
$$=0$$

because of the 2-cocycle condition on δ. Thus $d_{\mathfrak{g}^*}\circ d_{\mathfrak{g}}=d_{\mathfrak{g}^*}\circ d_{\mathfrak{g}}$. It follows that defining the differential d of the total complex as $df=d_{\mathfrak{g}}f+(-1)^m d_{\mathfrak{g}*}f$ for any $f\in \mathrm{Hom}(\Lambda^m\mathfrak{g},\Lambda^n\mathfrak{g})$, one has $d^2=0$ indeed $d^2f=d\big(d_{\mathfrak{g}}f+(-1)^m d_{\mathfrak{g}^*}f\big)=d_{\mathfrak{g}}^2f+(-1)^{m+1}d_{\mathfrak{g}^*}\circ d_{\mathfrak{g}}f+(-1)^m d_{\mathfrak{g}}\circ d_{\mathfrak{g}^*}f+(-1)^{2m}d_{\mathfrak{g}}^2f=(-1)^m\big(d_{\mathfrak{g}}\circ d_{\mathfrak{g}^*}f-d_{\mathfrak{g}^*}\circ d_{\mathfrak{g}}f\big)=0$.

Let us consider the linear map $d_{\mathfrak{g}^*}:\Lambda^n\mathfrak{g}\to\Lambda^{n+1}\mathfrak{g}$ in the case $\delta=d_{\mathfrak{g}}r$. By definition, for any $f\in\Lambda^n\mathfrak{g}$, one has

$$d_{\mathfrak{g}^*}f=-(d_{\mathfrak{g}}r)(f).$$

By the linearity of $d_{\mathfrak{g}^*}$, we can suppose that $f=x_1\wedge\cdots\wedge x_n$ with $x_1,\ldots,x_n\in\mathfrak{g}$, therefore, writing $r=\sum_{k=1}^s r_1^k\wedge r_2^k$ with $r_1^k,r_2^k\in\mathfrak{g}$ and $s\in\mathbb{N}$, one has

$$\begin{aligned}
d_{\mathfrak{g}^*}(x_1\wedge\cdots\wedge x_n)&=\sum_{i=1}^{n}(-1)^i x_1\wedge\cdots\wedge d_{\mathfrak{g}}r(x_i)\wedge\cdots\wedge x_n\\
&=\sum_{i=1}^{n}\sum_{k=1}^{s}(-1)^i x_1\wedge\cdots\wedge[x_i,r_1^k]\wedge r_2^k\wedge\cdots\wedge x_n\\
&\quad+\sum_{i=1}^{n}\sum_{k=1}^{s}(-1)^i x_1\wedge\cdots\wedge r_1^k\wedge[x_i,r_2^k]\wedge\cdots\wedge x_n\\
&=-\sum_{i=1}^{n}\sum_{k=1}^{s}r_2^k\wedge x_1\wedge\cdots\wedge[r_1^k,x_i]\wedge\cdots\wedge x_n\\
&\quad+\sum_{i=1}^{n}\sum_{k=1}^{s}r_1^k\wedge x_1\wedge\cdots\wedge[r_2^k,x_i]\wedge\cdots\wedge x_n.
\end{aligned}$$

Using the usual identification due to Alt, and recalling that our identification in the case of r becomes $r=\sum_{k=1}^s r_1^k\wedge r_2^k=\sum_{k=1}^s(r_1^k\otimes r_2^k-r_2^k\otimes r_1^k)$, one has

$$d_{\mathfrak{g}^*}(x_1\wedge\cdots\wedge x_n)=Alt\left([r^{12}+r^{13}+\ldots+r^{1n+1},1\otimes x_1\otimes\cdots\otimes x_n]\right),\qquad\text{(B.7)}$$

indeed

$$d_{\mathfrak{g}^*}(x_1\wedge\cdots\wedge x_n)=Alt\Big(-\sum_{i=1}^{n}\sum_{k=1}^{s}r_2^k\otimes x_1\otimes\cdots\otimes[r_1^k,x_i]\otimes\cdots\otimes x_n\Big)$$

$$+ Alt\Big(\sum_{i=1}^{n}\sum_{k=1}^{s} r_1^k \otimes x_1 \otimes \cdots \otimes [r_2^k, x_i] \otimes \cdots \otimes x_n\Big)$$
$$= Alt\left([r^{12} + r^{13} + \ldots + r^{1n+1}, 1 \otimes x_1 \otimes \cdots \otimes x_n]\right).$$

On the other hand, one can embed the symmetric group S_n in the symmetric group S_{n+1} in this way:

$$S_n \to S_{n+1}$$
$$\sigma \mapsto \widetilde{\sigma}$$

where $\widetilde{\sigma}(1) = 1$ and $\widetilde{\sigma}(i+1) = \sigma(i) + 1$. Note that $sgn(\widetilde{\sigma}) = sgn(\sigma)$. It follows that

$$\begin{aligned}
&Alt\left([r^{12} + r^{13} + \ldots + r^{1n+1}, sgn(\sigma) 1 \otimes \sigma(x_1 \otimes \cdots \otimes x_n)]\right)\\
&\quad = sgn(\widetilde{\sigma}) Alt\left([r^{12} + r^{13} + \ldots + r^{1n+1}, \widetilde{\sigma}(1 \otimes x_1 \otimes \cdots \otimes x_n)]\right)\\
&\quad = sgn(\widetilde{\sigma}) Alt\left(\widetilde{\sigma}[r^{12} + r^{13} + \ldots + r^{1n+1}, 1 \otimes x_1 \otimes \cdots \otimes x_n]\right)\\
&\quad = Alt\left([r^{12} + r^{13} + \ldots + r^{1n+1}, 1 \otimes x_1 \otimes \cdots \otimes x_n]\right)
\end{aligned}$$

because $\sum_{\mu \in S_{n+1}} sgn(\mu) sgn(\widetilde{\sigma}) \mu\widetilde{\sigma} = \sum_{\mu \in S_{n+1}} sgn(\mu)\mu$. Therefore, using equation (B.7), one has

$$\begin{aligned}
&Alt\left([r^{12} + r^{13} + \ldots + r^{1n+1}, 1 \otimes Alt(x_1 \otimes \cdots \otimes x_n)]\right)\\
&\quad = \sum_{\sigma \in S_n} Alt\left([r^{12} + r^{13} + \ldots + r^{1n+1}, sgn(\sigma) 1 \otimes \sigma(x_1 \otimes \cdots \otimes x_n)]\right)\\
&\quad = \sum_{\sigma \in S_n} Alt\left([r^{12} + r^{13} + \ldots + r^{1n+1}, 1 \otimes x_1 \otimes \cdots \otimes x_n]\right)\\
&\quad = n! Alt\left([r^{12} + r^{13} + \ldots + r^{1n+1}, 1 \otimes x_1 \otimes \cdots \otimes x_n]\right)\\
&\quad = n! d_{\mathfrak{g}^*}(x_1 \wedge \cdots \wedge x_n).
\end{aligned}$$

From which

$$d_{\mathfrak{g}^*} f = \frac{1}{n!} Alt\Big([r^{12} + \ldots + r^{1n+1}, 1 \otimes f]\Big) \tag{B.8}$$

for any $f \in \Lambda^n \mathfrak{g}$.

It only remains to compute the linear map $d_{\mathfrak{g}^*} : \mathrm{Hom}(\mathfrak{g}, \Lambda^2\mathfrak{g}) \to \mathrm{Hom}(\mathfrak{g}, \Lambda^3\mathfrak{g})$. Let $f \in \mathrm{Hom}(\mathfrak{g}, \Lambda^2\mathfrak{g})$, $x \in \mathfrak{g}$. Writing $f(x) = \sum_{k=1}^{t} f_1^k \wedge f_2^k$ with $f_1^k, f_2^k \in \mathfrak{g}$, $t \in \mathbb{N}$, by the definition of $d_{\mathfrak{g}^*}$, one has

$$\begin{aligned}
d_{\mathfrak{g}^*} f(x) &= \sum_{(x)} \left(x' \wedge f(x'') - x'' \wedge f(x')\right) - \delta(f(x))\\
&= -\sum_{(x)} \left(f(x') \wedge x'' - f(x'') \wedge x'\right)\\
&\quad - \sum_{k=1}^{t} \left(\delta(f_1^k) \wedge f_2^k - \delta(f_2^k) \wedge f_1^k\right)
\end{aligned}$$

$$= -(f \wedge 1) \circ \delta(x) - (\delta \wedge 1) \circ f(x)$$

from which

$$d_{\mathfrak{g}^*} f = -(f \wedge 1) \circ \delta - (\delta \wedge 1) \circ f.$$

Thus, when $\delta = d_{\mathfrak{g}} r$,

$$d_{\mathfrak{g}^*} f = -(f \wedge 1) \circ d_{\mathfrak{g}} r - (d_{\mathfrak{g}} r \wedge 1) \circ f.$$

Let us end this appendix with a remark on why we denoted the vertical maps of the bicomplex (3.31) with $d_{\mathfrak{g}^*}$:

Remark B.0.1. If $(\mathfrak{g}, [\ ,\ ], \delta)$ is a finite dimensional Lie bialgebra, $\left(\mathfrak{g}^*, {}^t\delta, {}^t[\ ,\ ]\right)$ is a finite dimensional Lie bialgebra. Since $\mathfrak{g}$ is finite dimensional, we have an isomorphism of vector spaces from $\mathrm{Hom}(\Lambda^m \mathfrak{g}, \Lambda^n \mathfrak{g})$ to $\mathrm{Hom}(\Lambda^n \mathfrak{g}^*, \Lambda^m \mathfrak{g}^*)$ given by the transpose map:

$$\begin{aligned} \mathrm{Hom}(\Lambda^m \mathfrak{g}, \Lambda^n \mathfrak{g}) &\to \mathrm{Hom}(\Lambda^n \mathfrak{g}^*, \Lambda^m \mathfrak{g}^*) \\ f &\mapsto ({}^t f : \psi \mapsto \psi \circ f). \end{aligned}$$

We could therefore use the Lie algebra structure of $\mathfrak{g}^*$ and the canonical isomorphism between $\mathfrak{g}$ and $\mathfrak{g}^{**}$ to define a linear map

$$\partial : \mathrm{Hom}\left(\Lambda^m \mathfrak{g}, \Lambda^n \mathfrak{g}\right) \to \mathrm{Hom}\left(\Lambda^m \mathfrak{g}, \Lambda^{n+1} \mathfrak{g}\right)$$

in the following way: let $f \in \mathrm{Hom}\left(\Lambda^m \mathfrak{g}, \Lambda^n \mathfrak{g}\right)$, if we denote with $d_{\mathfrak{g}^*}$ the Chevalley-Eilenberg differential of the cochain complex with Lie algebra $(\mathfrak{g}^*, {}^t\delta)$ and $\mathfrak{g}^*$-module $\Lambda^n \mathfrak{g}^*$, $d_{\mathfrak{g}^*}\, {}^t f \in \mathrm{Hom}\left(\Lambda^{n+1} \mathfrak{g}^*, \Lambda^m \mathfrak{g}^*\right)$ from which one has ${}^t d_{\mathfrak{g}^*}\, {}^t f \in \mathrm{Hom}\left(\Lambda^m \mathfrak{g}^{**}, \Lambda^{n+1} \mathfrak{g}^{**}\right)$. Finally, identifying $\mathfrak{g}$ with $\mathfrak{g}^{**}$, we have the wanted linear map from $\mathrm{Hom}\left(\Lambda^m \mathfrak{g}, \Lambda^n \mathfrak{g}\right)$ to $\mathrm{Hom}\left(\Lambda^m \mathfrak{g}, \Lambda^{n+1} \mathfrak{g}\right)$. A straightforward computation shows that ∂ is equal to the map $d_{\mathfrak{g}*}$ of the definition (B.6).

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