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Surface Superconductivity in Presence of Corners

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Contents

Notation

- We denote by C a finite constant which we assume strictly positive and whose value may change from line to line;
- We denote by bold face letters *vectors* in $\mathbb{R}^d$ or $\mathbb{C}^d$, e.g., $\mathbf{x} = (x_1, \cdots, x_n)$,
- For each $z \in \mathbb{C}$ we denote by z^* its complex conjugate;
- We denote the scalar product between two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ by
$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{d} x_i y_i;$$
- We denote the external product between two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ by
$$\mathbf{x} \times \mathbf{y} = \sum_{i,j,k=1}^{d} \epsilon_{ijk} x_j y_k \mathbf{e}_i;$$
- We define the internal product in $\mathbb{C}$, $(\cdot, \cdot) : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ as
$$(z, w) = \frac{1}{2}(zw^* + z^* w).$$
  The scalar product in any Hilbert space $\mathcal{H}$ will be denoted with the bra-ket notation;
- We use the short notation $\partial_s = \frac{\partial}{\partial s}$ for the partial derivative with respect to s;
- We denote the characteristic function of a set D by $\mathbb{1}_D(\cdot)$;
- We use the Landau symbols $\mathcal{O}(\delta)$ and $o(\delta)$ to denote quantities whose absolute value is bounded by $C\delta$ as $\delta \to 0$ or such that
$$\lim_{\delta \to 0} \frac{f(\delta)}{\delta} \to 0$$
  We say that a quantity is of order $\mathcal{O}(\delta^\infty)$ as $\delta \to 0$, if it is $\mathcal{O}(\delta^k)$ for any $k \in \mathbb{R}^+$;
- We write $a \sim b$, if $a/b \to 1$, $a \ll b$, if $a/b \to 0$, $a \propto b$, if $a/b \to C \neq 0, 1$.

Introduction

The aim of this Thesis is to study the response of a type-II superconducting wire with non-smooth cross section to an external time-independent magnetic field h_{ex} parallel to it and with intensity varying in a certain regime.

Superconductivity is a well known quantum critical phenomenon in which the electrons arrange in pairs, known as Cooper pairs, thanks to the relative attraction mediated by the crystal of ions. If all the electrons are paired then superconducting materials exhibit zero electrical resistance in a small temperature range below a critical temperature T_c, which is a characteristic of the material. The response of a superconductor to an external magnetic field is a physically very rich phenomenon. It is well know that when a uniform magnetic field of small intensity is applied to a superconducting sample, this still behaves as a *perfect* superconductor, i.e., it exhibits zero electric resistance. However, if the intensity of the applied magnetic field is large enough, superconductivity breaks down and the sample undergoes a transition to its normal conducting state. Many intermediate regimes can be observed between these two extreme behaviors, in which case the material is said to be in a *mixed* state.

Superconductors can be divided into two types, according to how the breakdown of superconductivity occurs. For type-I, superconductivity is abruptly destroyed via a first order phase transition. In 1957 Abrikosov deduced the existence of a class of materials which exhibit a different behavior, i.e., some of their superconducting properties are preserved when submitted to a suitably large magnetic field. Physically, these two classes can be identified by the value of a parameter κ, also known as the Ginzburg-Landau parameter, which is proportional to the inverse of the penetration depth, a physical quantity typical of the material. This value κ is smaller than $1/\sqrt{2}$ for type-I superconductors and larger than $1/\sqrt{2}$ for the others.

We consider in this work extreme type-II superconductors, i.e., we assume that the Ginzburg-Landau parameter satisfies $\kappa \gg 1$. It is possible to describe the phase transitions in a type-II superconductor by identifying three increasing critical values of the magnetic field. When the first critical value H_{c1} is reached, superconductivity is lost in the bulk of the sample at isolated points. Between the second and third critical fields, i.e., in the regime $H_{c2} \leq h_{\mathrm{ex}} \leq H_{c3}$, superconductivity survives only close to the boundary of the sample, as it was predicted by Saint-James and de Gennes in the 60's and later observed in experiments. Above the third critical field H_{c3}, the sample comes back to its normal state.

In this Thesis we focus on the surface superconductivity regime: in this regime superconductivity is confined near the boundary of the sample. We will perform our investigation in the framework of Ginzburg-Landau (GL) theory. It is indeed well known that close enough to the critical temperature T_c, GL theory provides an accurate description of the physics of superconductivity, despite having been formulated as a phenomenological theory and only later justified in terms of the microscopic BCS theory (BCS stands for Bardeen, Cooper, Schrieffer).

From the mathematical point of view, the free energy of a type-II superconductor confined to an infinite wire of cross section Ω is given by the minimum of the Ginzburg-Landau functional

$$\mathcal{G}_{\kappa}^{\mathrm{GL}}[\Psi, \mathbf{A}] = \int_{\Omega} \mathrm{d}\mathbf{r} \left\{ |(\nabla + i h_{\mathrm{ex}} \mathbf{A})\Psi|^2 - \kappa^2 |\Psi|^2 + \frac{\kappa^2}{2} |\Psi|^4 + h_{\mathrm{ex}} |\mathrm{curl}\mathbf{A} - 1|^2 \right\},$$

where h_{ex} is the applied magnetic field, $h_{\mathrm{ex}}\mathbf{A}$ is the induced vector potential (measured in units h_{ex}) and $|\Psi|^2$ is the density of Cooper pairs, i.e., pairs of superconducting electrons.

Since in the regime of interest superconductivity survives only close to the boundary of the sample, a natural question is: *how does the physics depend on the geometry of the boundary?*

For a type-II superconducting wire with smooth cross-section it has recently been proven [CR14, Pan02a] that the leading order of the energy density does not depend on the shape of the boundary but only on its length. The reason is that, as physically guessed by Saint James and de Gennes, in first approximation the problem can be reduced to a one-dimensional one, depending only on the direction normal to the boundary. On the opposite, the first order correction to the energy density is curvature-dependent [CR16a, CDR17]: regions of larger boundary curvature (counted inwards) attract more superconducting electrons.

It is well known [FH10, SP99] that the value of the third critical field, i.e., the value after which superconductivity is completely lost, depends on the geometrical shape of the mesoscopic superconducting sample. In particular, in presence of corners along the boundary section of the superconductor, we know that the third critical value of the magnetic field can be larger then the one for smooth domains. More precisely, superconductivity survives longer in a corner domain than in a smooth one, if there is at least one corner of angle $0 < \alpha < \pi/2$ along the boundary section [Bon05, Jad01, Pan02b]. It is however conjectured [Bon05] that this should be true for any angle such that $0 < \alpha < \pi$. Physically, this means that, decreasing the intensity of a strong field, superconductivity nucleates first close to the corners and, in particular, around those of smallest opening angles. It is thus to be expected that superconductors with non-smooth cross sections exhibit a richer physics.

The aim of this Thesis is to prove that also in presence of singularities along the boundary, there exists a surface superconductivity regime and that it corresponds to an intensity of the magnetic field in the interval $\kappa^2 < h_{\mathrm{ex}} < \Theta_0^{-1}\kappa^2$. We will prove, indeed, that in this regime, the leading order of the energy density is not affected by the presence of corners and that the density of Cooper pairs in the equilibrium state is approximately constant along the transversal direction. This implies that superconductivity is uniformly distributed near the boundary, at least to leading order.

In addition, we introduce a new effective problem near the corner that allows us to prove a refined asymptotics and to isolate the contributions to the energy density due to the presence of corners. The explicit expression of the effective energy is yet to be found but we formulate a conjecture (Conjecture 5.1) on it based on the behavior for almost flat angles. Indeed, for corners with angles close to π, we are able to explicitly compute the leading order of the corners effective problem and show that it sums up to the smooth boundary contribution to reconstruct the same asymptotics as in smooth domains [CR16a].

The main mathematical tools used in this Thesis cover a wide range of topics in analysis going from Calculus of Variation theory, to nonlinear PDEs: the main question can indeed be naturally formulated as a variational problem, the minimization of the GL functional. The nonlinearity of GL theory has a very important role in the surface superconductivity regime, and, in fact, only genuine nonlinear techniques allow to tackle the main problems. This marks a difference with other regimes (e.g. close to H_{c3}) where perturbation theory around the linear problem is useful. We finally remark that we take the usual mathematical physics point of view and make and effort to estimate as much as possible all the errors in our asymptotic analysis, instead of simply relying on convergence results.

The structure of this Thesis is the following. We start by an introduction to the topic: in Chapter 1, we describe the physical framework of superconductivity; in Chapter 2, we introduce the Ginzburg-Landau theory and we recall some known results about the energy asymptotics in domains with smooth boundary.

In Chapter 3, we focus on the energy density for general domains with corners and we show that to leading order the geometry of the boundary does not affect the asymptotics. In the regime of interest, i.e., the one in which $h_{\mathrm{ex}} = b\kappa^2$ with $1 < b < \Theta_0^{-1}$, it is convenient to change units. We then study the asymptotics of the energy density with respect to a new parameter $\varepsilon := 1/\sqrt{b\kappa^2} \ll 1$. In Theorem 3.1, we prove that, as $\varepsilon \propto \kappa^{-1} \to 0$, the ground state energy is asymptotically equal to

$$E_\varepsilon^{\mathrm{GL}} = \frac{|\partial\Omega| E_0^{1D}}{\varepsilon} + \mathcal{O}(|\log\varepsilon|^2),$$

where $|\partial\Omega|$ is the length of the boundary and E_0^{1D} is the minimum of the same functional one has to take into account in the smooth case to extract the right leading order in the energy asymptotics. A direct consequence of this result is that the modulus of the order parameter is close in L^2-sense to an explicit profile of the form $f_0(\varepsilon^{-1}\mathrm{dist}(\mathbf{r}, \partial\Omega))$, that is the one realizing E_0^{1D}. This shows that the density of Cooper pairs is uniformly distributed along the boundary, at least to leading order, even in presence of corners.

Next, in Chapter 4, we introduce the effective problem useful to describe the corner contributions to the energy density. More precisely, we formulate this new effective problem in order to have the right behavior of the GL minimizer suitably far from the singularity.

Chapter 5 is then devoted to the investigation of the full problem in general domains: in Theorem 5.1 we prove our main result, which is the refined energy asymptotics

$$E_\varepsilon^{GL} = \frac{|\partial\Omega| E_0^{1D}}{\varepsilon} - \mathcal{E}_{\alpha_0}^{\mathrm{corr}}[f_0] \int_0^{|\partial\Omega|} \mathrm{d}\sigma\, \widetilde{k}(\sigma) + \sum_{j\in\Sigma} E_{\mathrm{corner},\alpha_j} + o(1),$$

where $\widetilde{k}(\sigma)$ is the boundary curvature. The quantity $\mathcal{E}_{\alpha_0}^{\mathrm{corr}}$ is a 1D functional which is obtained by retaining only linear terms in εk when expanding the 1D effective problem for the smooth part of the domain.

In Chapter 6, we discuss the particular case of domains with almost flat angles. More precisely, in Theorem 6.1 we prove that the corner contribution to the energy due to an angle equal to $\pi - \delta$ with $\delta \ll 1$ is

$$E_{\mathrm{corner},\alpha} = -\delta \mathcal{E}^{\mathrm{corr}}_{\alpha_0}[f_0] + o(1).$$

Chapter 1

Physics of Superconductivity

This Chapter is devoted to the description of the physical background of superconductivity. We first discuss the main physical features of superconductivity and then we introduce the materials we are interested in, i.e., type-II superconductors. We also give a quick overview of the two main theories commonly used to model superconductivity: the Ginzburg-Landau and the BCS theories, which provide respectively a macroscopic and a microscopic description of the phenomenon. In the last part, we briefly recall the relations between these two theories and comment on the derivation of GL theory from the BCS microscopic model.

1.1 Superconductivity

Superconductivity was discovered in 1911 by H. Kamerlingh Onnes in Leiden. For many years many scientists studied this phenomenon, but only in the 1950s and 1960s a complete theoretical picture of superconductors emerged.

Kamerlingh Onnes observed that the electrical resistance of various metals, such as mercury, disappeared completely below a critical temperature T_c, whose value varies from material to material. Nowadays we know that superconductivity is a quantum critical phenomenon in which the current carriers (electrons) arrange in pairs (Cooper pairs) thanks to the relative attraction which is produced by the displacement of crystal's ions (phonons). Once Cooper pairs are created the resistance to the flowing current drops dramatically. The disappearance of resistance was experimentally observed by showing the never ending flow of currents in a superconducting ring. It has indeed been observed that no changes in the current flow occur for over two years, and the resistivity of some of these materials has been estimated to be not greater than 10^{-23} Ohm/cm.

In addition to this *perfect conductivity* property, superconductors are also characterized by the property of *perfect diamagnetism*, as found out in 1933 by Meissner and Ochsenfeld.[1] They observed that, not only a sufficiently small magnetic field is excluded from entering a superconductor, as one might expect by perfect conductivity, but also that a field inside an originally normal sample is expelled as the material is cooled down through its critical

[1] Actually the diamagnetism is perfect only for bulk samples, since the field penetrates the sample for a finite distance λ, of typically 500 $\mathring{A}$.

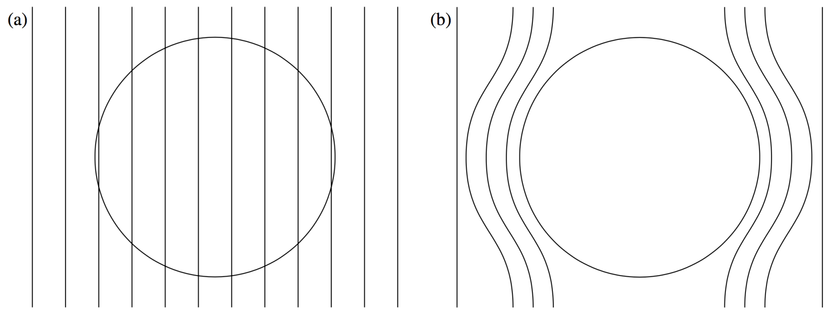

Figure 1.1. A metal sphere in an applied magnetic field (a) above, and (b) below the superconducting transition.

temperature T_c. This phenomenon that consists in the complete ejection of magnetic field lines from the interior of the superconductor is known as *Meissner effect* (see Figure 1.1.)

Sufficiently strong magnetic fields cannot be however excluded from the material, and there exists a critical value of the field, above which the material ceases to be superconducting, even at temperatures below the critical one. This critical field is related to the difference between the free energies of the normal and the superconducting states (denoted by $f_n(T)$ and $f_s(T)$ respectively) in absence of magnetic field. Furthermore, the passage through the critical temperature is reversible. Then, simple thermodynamic arguments can be used (see, e.g., [SJST69]) to show that the transition from the normal to the superconducting state at zero applied magnetic field is not accompanied by any release of latent heat: this describes what is known as a second-order transition. The critical field H_c is determined by equating the energy per unit volume needed to keep the field outside the sample, i.e., $H^2/8\pi$, with the condensation energy:

$$\frac{H_c(T)}{8\pi} = f_n(T) - f_s(T).$$

It was empirically discovered that $H_c(T)$ is quite well approximated by a parabolic law:

$$H_c(T) \sim H_c(0)[1 - (T/T_c)^2].$$

The electrodynamic properties that characterize a superconductor were studied in 1935 by the brothers F. London and H. London, who introduced two equations to describe the microscopic electric and magnetic fields:

$$\mathbf{e} = \frac{\partial}{\partial t}(\Lambda \mathbf{J}_s)$$
$$\mathbf{h} = -c\,\mathrm{curl}(\Lambda \mathbf{J}_s),$$

where $\mathbf{e}$ is the microscopic electric field, $\mathbf{h}$ denotes the value of the flux density of the magnetic field on the microscopic scale and $\mathbf{J}_s$ is the superconducting current. The quantity Λ is a phenomenological parameter related the the number density of superconducting electrons.

The first one of the two equations above describes perfect conductivity: the field accelerates the superconducting electrons rather than simply sustaining their velocity against resistance as it would happen in a normal conductor according to Ohm's law. Furthermore, the second London equation, combined with the Maxwell equation for the magnetic field, leads to another law which describes the Meissner effect:

$$\nabla^2 \mathbf{h} = \frac{\mathbf{h}}{\lambda}.$$

This implies that the magnetic field is exponentially screened from the interior of a sample with penetration depth λ (also known as London penetration depth), which is a parameter typical of the material and which depends on the temperature approximately as

$$\lambda(T) \sim \lambda(0)[1 - (T/T_c)^4]^{-1/2}.$$

Under certain hypothesis it is possible to find a law for the superconducting current that contains both London equations in a compact form. Some years after the discovery of London equations, Pippard suggested a nonlocal generalization of this compact form. However, a completely acceptable microscopic theory was not available until 1957.

1.2 Macroscopic and Microscopic Descriptions of Superconductivity

The first remarkable macroscopic theory was proposed in 1950 by Ginzburg and Landau [GL50]. They introduced a model of superconductivity that has been extremely successful and is widely used in physics, even beyond the theory of superconductivity. Ginzburg and Landau elaborated their model in a phenomenological way, pointing out the macroscopic properties of a superconductor. It is an example of the power of phenomenology even in the absence of a microscopic description. The Ginzburg-Landau (GL) theory is actually a special case of the general Landau–Lifshitz theory of second-order phase transitions, in which one introduces an *order parameter* $\eta(\mathbf{r})$, which is zero above the transition temperature T_c, but takes a finite value for $T < T_c$, and then uses the symmetry of the relevant Hamiltonian to restrict the form of the free energy as a functional of η. In the case of superconductivity, Ginzburg and Landau made the brilliant guess that $\eta(\mathbf{r})$ should have the nature of what they called a *macroscopic wave function* (this is why was denoted as $\psi(\mathbf{r})$). Today we know that $\psi(\mathbf{r})$ is indeed (up to normalization) nothing but the center of mass wave function of the Cooper pairs.

From the microscopic point of view the first completely acceptable microscopic theory for superconductivity was formulated in 1957 by Bardeen, Cooper and Schrieffer [BCS57]. This theory, now known as BCS theory, is built on the existence of an energy gap Δ, of order kT_c, between the ground state and the quasi-particle excitations of the system. The analysis proposed by Bardeen, Cooper and Schrieffer started from a many-body Hamiltonian and their major idea was to describe superconductivity as a pairing mechanism. The basic idea is that even a weak attractive interaction between electrons, such as that caused in second order by the electron-phonon interaction, causes an instability of the ordinary Fermi-sea ground state of the electron gas with respect to the formation of bound pairs of electrons, occupying states with equal and opposite momenta and spins, the so called *Cooper pairs*.

The Ginzburg-Landau theory was not widely accepted immediately, it was considered too phenomenological and its importance was not generally appreciated. However, in 1959, Gor'kov [21] showed that, in the appropriate limit, the macroscopic GL theory can be derived from the microscopic BCS theory near T_c. A simplified version of his explanation was later given by de Gennes [DG89]. Nowadays, the Ginzburg-Landau theory is universally accepted as a valid macroscopic model for low-temperature superconducting effects.

1.3 Type II-Superconductors

The Ginzburg-Landau theory focuses entirely on the superconducting electrons rather then on excitations: Ginzburg and Landau introduced a complex pseudo-wave function ψ (the order parameter) such that $|\psi|^2$ gives the local relative density of the superconducting electrons. This theory introduces also a new characteristic length ξ, known as the *Ginzburg-Landau coherence length*, which characterizes the distance over which ψ can vary without undue energy increase.

The ratio between the Ginzburg-Landau coherence length and the London penetration depth define the so called Ginzburg-Landau parameter:

$$\kappa = \frac{\lambda}{\xi}.$$

Since both the quantities on the r.h.s. of the expression above diverge as $(T_c - T)^{-1/2}$ near T_c, κ is approximately independent on the temperature. For classic pure superconductors $\lambda \sim 500\mathring{A}$ and $\xi \sim 3.000\mathring{A}$, then $\kappa \ll 1$. In this case there is an interfacial layer of thickness $\sim (\xi - \lambda)$ which pays the energetic cost of excluding the magnetic field without enjoying the full condensation energy of the superconducting state. This phenomena implies the existence of a surface energy, so that the minimum energy principle would lead to relatively few such transitions. This was confirmed by some experiments on materials that are nowadays known as type-I superconductors.

Some years later, in 1957 (the same year as the introduction of BCS), Abrikosov published a significant paper [Abr57] in which he investigated what would happen in the Ginzburg-Landau theory if the surface energy accompanying phase transitions was negative, i.e., if κ were large instead of small. He showed that for such materials there is not a discontinuous breakdown of superconductivity in a first-order transition at H_c, but there is a continuous increase in flux penetration starting at a lower critical field H_{c1} and ending at an upper critical field H_{c2}. This kind of materials exhibit a radically different behavior from classic superconductors. For this reason Abrikosov called them *type-II superconductors*. These are then materials of greatest interest, mainly because they can retain superconductivity in presence of large applied magnetic fields.

The existence of type-II superconductors had in a certain sense already predicted by GL theory, some years before Abrikosov discovery. Indeed, Ginzburg and Landau predicted that, in order to minimize the energy, there would be relatively many phase transitions in a material sample, and that, indeed, the normal and superconducting state could coexist in what is known as the mixed state.

1.4 The Ginzburg-Landau Model

In 1950 Ginzburg and Landau postulated that if ψ is small and varies slowly in space, the free energy density can be written as

$$f = f_n + \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{|\mathbf{h}|^2}{8\pi} + \frac{1}{2m_s}\left|\left(-i\hbar\nabla - \frac{e_s\mathbf{A}}{c}\right)\psi\right|^2, \tag{1.1}$$

where f_n is the free energy of the normal (non-superconducting) state in the absence of magnetic field[2], $\mathbf{A}$ is the magnetic potential, $\mathbf{h} = \text{curl}\mathbf{A}$ is the magnetic field, α and β are constants whose values depend on the temperature, c is the speed of light, e_s and m_s are respectively the charge and mass of the superconducting charge-carriers and $2\pi\hbar$ is Planck's constant. In particular, α and β are such that in absence of fields

$$f_s - f_n = \alpha|\psi|^2 + \frac{1}{2}\beta|\psi|^4,$$

which can be viewed as a series expansion in powers of $|\psi|^2$ in which the coefficients are regular functions of the temperature. We now observe that β must be positive, because if it was negative, then the lowest free energy would occur for arbitrarily large values of $|\psi|^2$. On the opposite, α can be positive or negative. If $\alpha > 0$, the minimum free energy occurs at $|\psi|^2 = 0$, corresponding to the normal state. Instead, if $\alpha < 0$, the minimum is reached when

$$|\psi|^2 = |\psi_\infty|^2 \equiv -\frac{\alpha}{\beta},$$

where the notation ψ_∞ is usually used to underline that ψ reaches this valued infinitely deep in the interior of the superconductor.

We now briefly discuss the various terms appearing in the free energy density above. As we recalled at the beginning of this Section, in the Ginzburg-Landau theory the density of superconducting charge-carriers is allowed to be spatially varying. Then, in the free energy density one must take into account the kinetic energy associated with the spatial variation of ψ. For this reason, Ginzburg and Landau postulated that the last term in (1.1) is the energy density, written in a gauge invariant form, due to the spatial variations of ψ. In fact, following [Tin96] we observe the the last term in (1.1) is actually equal to

$$\frac{1}{2m_s}\left[\hbar^2\left|(\nabla|\psi|)\right|^2 + \left|\hbar\nabla\phi - \frac{e_s\mathbf{A}}{c}\right|^2|\psi|^2\right],$$

where ϕ is the phase of ψ, i.e., $\psi = |\psi|e^{i\phi}$. The first term in the expression above modifies the energy when $|\psi|$ varies, whereas the second term can be viewed as a gauge-invariant kinetic energy density associated with currents in the superconductor.

We now simply observe that in presence of an applied magnetic field $\mathbf{H}$, the Gibbs free energy density g differs from f for the work due to the presence of the electromagnetic force induced by the applied field, which is given by $-\mathbf{h}\cdot\mathbf{H}/4\pi$. Then the Gibbs free energy becomes

$$g = f - \frac{\mathbf{h}\cdot\mathbf{H}}{4\pi}.$$

[2] It then follows that $f_n + |\mathbf{h}|^2/8\pi$ is the free energy of the normal state in the presence of the magnetic field $\mathbf{h}$.

If Ω is the region occupied by the superconducting sample, the Gibbs free energy of the sample is then given by

$$\mathcal{G}(\psi, \mathbf{A}) = \int_\Omega \mathrm{d}\Omega \left\{ f_n + \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{1}{2m_s}\left|\left(-i\hbar\nabla - \frac{e_s\mathbf{A}}{c}\right)\psi\right|^2 + \frac{|\mathbf{h}|^2}{8\pi} - \frac{\mathbf{h}\cdot\mathbf{H}}{4\pi}\right\}.$$

The basic thermodynamic postulate of the Ginzburg-Landau theory is that the superconducting sample is in a state such that its Gibbs free energy is minimum. By standard techniques, the minimization of $\mathcal{G}$ with respect to variations in ψ and $\mathbf{A}$ yields the so called Ginzburg-Landau equations:

$$\frac{1}{2m_s}\left(-i\hbar\nabla - \frac{e_s\mathbf{A}}{c}\right)^2\psi + \alpha\psi + \beta|\psi|^2\psi = 0, \qquad \text{in } \Omega,$$

and

$$\nabla^\perp \mathrm{curl}\mathbf{A} + \frac{2\pi i e_s\hbar}{m_s c}(\psi^*\nabla\psi - \psi\nabla\psi^*) + \frac{4\pi e_s^2}{m_s c^2}|\psi|^2\mathbf{A} = \mathrm{curl}\mathbf{H}, \qquad \text{in } \Omega.$$

Sometimes it is convenient to consider the functional $\mathcal{E}$ defined as

$$\mathcal{E}(\psi, \mathbf{A}) = \mathcal{G}(\psi, \mathbf{A}) + \int_\Omega \mathrm{d}\Omega\left(\frac{\alpha^2}{2\beta} + \frac{\mathbf{H}\cdot\mathbf{H}}{8\pi} - f_n\right)$$
$$= \int_\Omega \mathrm{d}\Omega\,\frac{1}{2}\left(\sqrt{\beta}|\psi|^2 + \frac{\alpha}{\sqrt{\beta}}\right)^2 + \int_\Omega \mathrm{d}\Omega\left\{\frac{1}{2m_s}\left|\left(\frac{\hbar\nabla}{i} - \frac{e^*\mathbf{A}}{c}\right)\psi\right|^2 + \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{(\mathbf{h}-H_c)^2}{8\pi}\right\}.$$

The functional above is today known as Ginzburg-Landau Functional.

1.5 Derivation of the Ginzburg-Landau Theory from the BCS Theory

In his paper [Gor59] Gor'kov proved that the Ginzburg-Landau equations follow from the BCS theory of superconductivity. The first fully rigorous mathematical derivation of the Ginzburg-Landau theory from the BCS model [FHSS12] is however very recent. We briefly recall this derivation.

First of all, we note that in the BCS model all the information about the system is encoded in two variables: the reduced one-particle density matrix γ and the pairing density matrix α (which is non-zero only below the critical temperature). Then the state of the system can be described by a 2×2 operator valued matrix[3]

$$\Gamma = \begin{pmatrix} \gamma & \alpha \\ \alpha^* & 1-\gamma^* \end{pmatrix},$$

where Γ is such that $0 \le \Gamma \le 1$, as an operator on $L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d) \simeq L^2(\mathbb{R}^d) \otimes \mathbb{C}^2$. We denote by $\Gamma_T^{\mathrm{normal}}$ the normal state at temperature T, i.e., the state at temperature T without Cooper pairs. We consider a macroscopic sample of a fermionic system in $\mathbb{R}^d$, with $d = 1, 2, 3$. We also suppose that the system is non-translation invariant and also that the external fields are weak and vary only on the macroscopic scale.

[3]Here $(\cdot)^*$ denotes the complex conjugation, i.e., α^* has integral kernel $\alpha^*(x, y)$.

The general form of the BCS functional for the free energy of a fermionic system in $\mathbb{R}^d$ (d=1,2,3) at temperature $T > 0$ (in suitable units) is then,

$$\mathrm{Tr}\big[((-ih\nabla + h\mathbf{A}(x))^2 - \mu + h^2 W(x))\gamma\big] - TS(\Gamma) + \int V(h^{-1}(x-y))|\alpha(x,y)|^2 \mathrm{d}x\mathrm{d}y, \quad (1.2)$$

where $h \ll 1$ is the ratio between the microscopic and the macroscopic scale, μ is the chemical potential, V is a local two-body interacting potential, $\mathbf{A}$ is the magnetic vector potential, W is the external vector potential and $S(\Gamma)$ is the entropy associated to the free state

$$S(\Gamma) = -\mathrm{Tr}[\Gamma \log \Gamma],$$

with the trace both over $\mathbb{C}^2$ and $L^2(\mathbb{R}^3)$.

Before stating the main result, we comment a little bit on where the BCS functional comes from. It is obtained after several approximations and assumptions on the related many-body problem. The first approximation is the restriction of the second quantized many-body Hamiltonian on the BCS type states (quasi-free states that do not have a fixed number of particles). After some assumptions, i.e., translation invariance and $SU(2)$ invariance for rotations of the spin, the state is completely determined by γ and α and then also its energy. For more details we refer to [HHSJ08, Appendix A]. Notice that, since we are interested in slowly varying external fields, $\mathbf{A}$ and W are replaced in (1.2) by $h\mathbf{A}$ and $h^2 W$ respectively.

Let $\psi \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ periodic, i.e., $\psi \in H^1_{\mathrm{per}}(\mathbb{R}^d)$, then the GL functional is defined as

$$\mathcal{E}^{\mathrm{GL}}(\psi, \mathbf{A}) = \int_{[0,1]^d} \mathrm{d}\mathbf{r} \left\{ B_1 |(-i\nabla + 2\mathbf{A}(x))\psi(x)|^2 + B_2 W(x)|\psi(x)|^2 - B_3 D|\psi(x)|^2 + B_4 |\psi(x)|^4 \right\},$$

where $\mathbf{A}$ and W are respectively the vector and the scalar potential and B_1, B_3, $B_4 > 0$, $B_2 \in \mathbb{R}$ and $D \in \mathbb{R}$ are coefficients.[4] We now set

$$E^{\mathrm{GL}} := \inf_{\psi \in H^1_{\mathrm{per}}(\mathbb{R}^d)} \mathcal{E}^{\mathrm{GL}}$$

and

$$F^{\mathrm{BCS}}(T, \mu) = \inf_{\Gamma} \mathcal{F}^{\mathrm{BCS}}(\Gamma),$$

where $\mathcal{F}^{\mathrm{BCS}}$ stands for the BCS functional. The main result in [FHSS12] is

$$F^{\mathrm{BCS}} = \mathcal{F}^{\mathrm{BCS}}(\Gamma_T^{\mathrm{normal}}) + h(E_D^{GL} + o(1)). \quad (1.3)$$

Furthermore, the GL wave function ψ correctly describes the macroscopic behavior of the BCS state: the BCS Cooper pair wave function near the critical temperature T_c equals

$$\alpha(x,y) = \frac{1}{2}(\psi(hx) + h\psi(hy))\alpha_0(x-y)$$

to leading order in h, where α_0 is the unperturbed translation invariant pair function.

The main idea for the proof is that the asymptotic limit may be seen as a semiclassical limit. One of the main difficulties is then to derive a semiclassical expansion with minimal regularity assumptions. For the sake of completeness, we recall the reader that in [FHSS12] authors also make the following technical assumptions:

[4]Notice that here we consider also a scalar potential. This is due the the fact that the system is non-translation invariant.

- the potential V is real-valued, $V(x) = V(-x)$ and $V \in L^{3/2}(\mathbb{R}^3)$,
- the potentials W and $\mathbf{A}$ are periodic and their Fourier coefficients are such that $\widehat{W}(p) \in \ell^1$, $|\widehat{\mathbf{A}}(p)|(1 + |p|) \in \ell^1$.

Chapter 2

Ginzburg-Landau Theory of Superconductivity

In this Chapter we introduce the mathematical framework of the analysis performed in next Chapters. We start by presenting the Ginzburg-Landau theory and its main mathematical properties. After a brief overview on the phase transitions describing the response of a type-II superconductor to an applied magnetic field, both in a smooth domain and in presence of corners, we focus on the regime we are interested in, i.e., the surface superconductivity regime. We then underline the mathematical properties of the physically relevant quantities in this regime. In the last part, we recall some known results about the response of a type-II superconductor with smooth cross-section to an applied magnetic field in the surface superconductivity regime.

2.1 The Ginzburg-Landau Theory

The Ginzburg-Landau (GL) theory is an effective theory used to describe the response of a superconductor to an external magnetic field close to the critical temperature for the superconductivity transition. As already mentioned in the previous Chapter, it was introduced in 1950 by Vitaly Lazarevich Ginzburg and Lev Davidovič Landau [GL50] as a phenomenological macroscopic model to describe the response of a superconductor to an applied magnetic field. It was later justified by Gor'kov [Gor59] as emerging from the microscopic Bardeen-Cooper-Schrieffer (BCS) theory [BCS57] and should thus be thought of as a mean-field/semi-classical approximation of the many-body quantum description of superconducting electrons.

The realistic model is a three-dimensional one. Here we consider cylindrical domains in $\mathbb{R}^3$, then it is natural to work with a two-dimensional model on a cross-section of the cylinder. A natural choice for the external magnetic field is to choose it perpendicular to the cross-section Ω: a non-constant angle between the field and the wire would indeed give rise to genuine three-dimensional effects.

The GL free energy of a type-II superconductor confined to an infinite cylinder of cross section $\Omega \subset \mathbb{R}^2$ is given by

$$\mathcal{G}_\kappa^{\mathrm{GL}}[\psi, \mathbf{A}] = \int_\Omega \mathrm{d}\mathbf{r} \left\{ |(\nabla + i h_{\mathrm{ex}} \mathbf{A})\psi|^2 - \kappa^2 |\psi|^2 + \tfrac{1}{2}\kappa^2 |\psi|^4 \right\} + h_{\mathrm{ex}}^2 \int_{\mathbb{R}^2} \mathrm{d}\mathbf{r} \, |\mathrm{curl}\mathbf{A} - 1|^2, \quad (2.1)$$

where $\psi : \Omega \to \mathbb{C}$ is the order parameter and $h_{\mathrm{ex}}\mathbf{A} : \mathbb{R}^2 \to \mathbb{R}^2$ the vector potential generating the induced magnetic field $h_{\mathrm{ex}}\mathrm{curl}\mathbf{A} = h_{\mathrm{ex}}(\partial_1 \mathbf{A}_2 - \partial_2 \mathbf{A}_1)$. The applied magnetic field is thus of uniform intensity h_{ex} along the superconducting wire and it is assumed to be perpendicular to Ω. The parameter $\kappa > 0$ is the GL parameter (proportional to the inverse of the penetration depth), a physical quantity which is typical of the material and, as we have underlined in the previous Chapter, an extreme type-II superconductor is identified by the condition $\kappa \gg 1$. The limit $\kappa \to \infty$ that we consider is also known as the London limit. We also recall the physical meaning of the order parameter ψ, i.e., $|\psi|^2$ is a measure of the relative density of superconducting Cooper pairs: $|\psi|$ varies between 0 and 1 and $|\psi| = 0$ in a certain region means that there are no Cooper pairs there and thus a loss of superconductivity, whereas if $|\psi| = 1$ somewhere then all the electrons are superconducting in the region. The cases $|\psi| \equiv 0$ and $|\psi| \equiv 1$ everywhere in Ω correspond to the *normal* and to the *perfectly superconducting* states respectively.

Remark 2.1 (Magnetic term). In this Thesis we focus on piecewise smooth domains Ω. Because of the presence of singularities, it is more convenient to consider a GL functional where the last term is integrated on $\mathbb{R}^2$. In the smooth boundary case there is a one-to-one correspondence between minimizers on $\mathbb{R}^2$ and minimizing configurations of the energy restricted to Ω (see, e.g., [SS07, Proposition 3.4]) and therefore the two settings are perfectly equivalent. For non-smooth boundaries such a correspondence is difficult to state because of possible boundary singularities of the solution. However, it is easy to prove [FH10, Lemma 15.3.2] that

$$\mathrm{curl}\mathbf{A} = 1, \qquad \text{in } \mathbb{R}^2 \setminus \overline{\Omega}, \quad (2.2)$$

for any $\mathbf{A}$ weak solution of the GL equations (2.4).

2.1.1 Gauge invariance

The GL functional (2.1) is invariant under gauge transformations.

Proposition 2.1 (Gauge invariance)**.**
Let $f \in H^2_{\mathrm{loc}}(\mathbb{R}^2)$, then

$$\mathcal{G}_\kappa^{\mathrm{GL}}[\psi, \mathbf{A}] = \mathcal{G}_\kappa^{\mathrm{GL}}[\psi e^{-if}, \mathbf{A} + \nabla f].$$

Proof. It is obvious that $|\psi|$ is gauge invariant. The same is true for $\mathrm{curl}\mathbf{A}$ since $\mathrm{curl}\nabla f = \nabla^\perp \cdot \nabla f = 0$, for any $f \in H^2_{\mathrm{loc}}(\mathbb{R}^2)$. We now prove that also the kinetic term is not affected by the gauge transformation

$$\int_\Omega \mathrm{d}\mathbf{r} \left| (\nabla + i\mathbf{A} + i\nabla f)\psi e^{-if} \right|^2 = \int_\Omega \mathrm{d}\mathbf{r} \left| (\nabla - i\nabla f + i\mathbf{A} + i\nabla f)\psi e^{-if} \right|^2 = \int_\Omega \mathrm{d}\mathbf{r} \, |(\nabla + i\mathbf{A})\psi|^2.$$

It then follows that all the terms in (2.1) do not change under gauge transformation. □

Remark 2.2 (Physical meaning of gauge invariance). The most important physical implication of gauge invariance is that configurations which differ only by a gauge transformation describe the same physical state. This implies that the physical quantities associated to a state $(\psi, \mathbf{A})$ must be invariant under gauge transformations. The three main physical observables which can be constructed in a gauge invariant form are $|\psi|^2$, $\mathrm{curl}\mathbf{A}$ and the superconducting current flowing through the wire. Despite its important physical role, gauge invariance poses some mathematical problems to the minimization of (2.1). In particular, if $(\Psi_n, \mathbf{A}_n)$ is a minimizing sequence, then $(\psi_n e^{-if_n}, \mathbf{A}_n + i\nabla f_n)$ must be minimizing too (notice that the gauge phase might depend on n). In order to get rid of this huge freedom in choosing the minimizing configuration one can thus fix the gauge.

Definition 2.1 (Coulomb gauge). *Assume that Ω is bounded and smooth, then we say that $\mathbf{A}$ satisfies the Coulomb gauge if*

$$\begin{cases} \nabla \cdot \mathbf{A} = 0, & \textit{in } \Omega, \\ \boldsymbol{\nu} \cdot \mathbf{A} = 0, & \textit{on } \partial\Omega, \end{cases}$$

where $\boldsymbol{\nu}$ is the unit interior normal vector to $\partial\Omega$.

Proposition 2.2 (Coulomb gauge).
Let Ω be a bounded, simply connected and smooth domain, then for any $\mathbf{A} \in H^1_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ one can always find a gauge transformation, i.e., a function $f \in H^2_{\mathrm{loc}}(\mathbb{R}^2)$, such that $\mathbf{A} + \nabla f$ satisfies the Coulomb gauge.

For the proof of Proposition 2.2 we refer to [FH10, Proposition D.1.1].

Remark 2.3 (Coulomb gauge). From now on when considering a minimizing configuration $(\psi^{\mathrm{GL}}_\kappa, \mathbf{A}^{\mathrm{GL}}_\kappa)$ of (2.1) on a smooth domain, unless stated otherwise, we will always assume that $\mathbf{A}^{\mathrm{GL}}_\kappa$ satisfies the Coulomb gauge. For domains with corners this choice is more delicate: we know that at the singularities the normal to the boundary $\boldsymbol{\nu}$ is not well defined. However, it is always possible to choose a magnetic potential $\mathbf{A}$ such that $\nabla \cdot \mathbf{A} = 0$ [FH10, Lemma D.2.7].

2.1.2 Critical points and Ginzburg-Landau equations.

The equilibrium state of the superconductor minimizes the energy functional $\mathcal{G}^{\mathrm{GL}}_\kappa$. The minimization domain can be taken to be

$$\mathscr{D}^{\mathrm{GL}} = \left\{ (\psi, \mathbf{A}) \in H^1(\Omega) \times H^1_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{R}^2) \mid \mathrm{curl}\mathbf{A} - 1 \in L^2(\mathbb{R}^2) \right\} \tag{2.3}$$

and the minimal energy will be denoted by E^{GL}_κ, while $(\psi^{\mathrm{GL}}_\kappa, \mathbf{A}^{\mathrm{GL}}_\kappa) \in \mathscr{D}^{\mathrm{GL}}$ will stand for any minimizing pair.

Proposition 2.3 (Existence of minimizers).
Let $\Omega \subset \mathbb{R}^2$ be a bounded and simply connected domain. For all $h_{\mathrm{ex}} \in \mathbb{R}^+$ the GL functional (2.1) *has a minimizer in $\mathscr{D}^{\mathrm{GL}}$.*

The existence of such a minimizing pair is a rather standard result, for the proof we refer to [FH10, Chapters 10&15] for both the smooth case and the one with corners.

Definition 2.2 (Critical points). *A configuration $(\psi, \mathbf{A}) \in \mathscr{D}^{\mathrm{GL}}$ is a critical point for $\mathcal{G}_\kappa^{\mathrm{GL}}$ if, for any $(\varphi, \mathbf{B})$ smooth, it holds*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}_\kappa^{\mathrm{GL}}[\psi + t\varphi, \mathbf{A} + t\mathbf{B}]\Big|_{t=0} = 0.$$

Proposition 2.4 (GL equations).
Let Ω be a bounded simply connected domain in $\mathbb{R}^2$, any critical point $(\psi, \mathbf{A})$ of the GL functional solves, at least weakly, the GL equations

$$\begin{cases} -\left(\nabla + ih_{\mathrm{ex}}\mathbf{A}\right)^2 \psi = \kappa^2 \left(1 - |\psi|^2\right) \psi, & \text{in } \Omega, \\ -h_{\mathrm{ex}} \nabla^{\perp} \mathrm{curl} \mathbf{A} = \mathbf{j}_{\mathbf{A}}[\psi] \mathbb{1}_{\Omega}, & \text{in } \mathbb{R}^2, \\ \boldsymbol{\nu} \cdot \left(\nabla + ih_{\mathrm{ex}}\mathbf{A}\right) \psi = 0, & \text{on } \partial\Omega, \\ \mathrm{curl}\mathbf{A} = 1, & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

where the last boundary condition is meant in trace sense and

$$\mathbf{j}_{\mathbf{A}}[\psi] := \tfrac{i}{2}\left[\psi\left(\nabla + ih_{\mathrm{ex}}\mathbf{A}\right)\psi^* - \psi^*\left(\nabla + ih_{\mathrm{ex}}\mathbf{A}\right)\psi\right], \tag{2.5}$$

is the superconducting current. Any minimizing configuration $(\psi_\kappa^{\mathrm{GL}}, \mathbf{A}_\kappa^{\mathrm{GL}})$ is a weak solution of the above system too.

For the proof of Proposition 2.4 we refer to [SS07, Proposition 3.6].

Remark 2.4 (Variational equation for $|\psi|^2$). Sometimes we will use also the variational equation for $|\psi|^2$. In particular, from the variational equation for ψ in (2.4), we derive

$$-(\psi, \Delta\psi) = \kappa^2(1 - |\psi|^2)|\psi|^2 + 2\mathbf{A} \cdot \mathbf{j}[\psi] - |\mathbf{A}|^2|\psi|^2,$$

which yields

$$-\frac{1}{2}\Delta|\psi|^2 = \kappa^2(1 - |\psi|^2)|\psi|^2 - |\nabla_{\mathbf{A}}\psi|^2. \tag{2.6}$$

2.1.3 Useful properties of GL critical points

We now state some useful properties of solutions of the GL equations (2.4).

Proposition 2.5 (Elliptic regularity).
Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply connected and smooth domain, then any solution $(\psi, \mathbf{A})$ of (2.4) *is smooth.*

The proof of Proposition 2.5 is a direct consequence of elliptic regularity theory [Eva97, Sec. 6.3.1, Theorems 2&5]. The idea is to apply elliptic regularity theory to the variational equations for ψ and $\mathbf{A}$ and to perform a bootstrap to get the interior smoothness. However, for the boundary behavior of ψ and $\mathbf{A}$, one has also to use the regularity of $\partial\Omega$. For a detailed proof we refer to [SS07, Proposition 3.8].

Remark 2.5 (Regularity with corners). For domains with corners one can only prove smoothness in the interior of the domain. The singularities along the boundary indeed are an obstruction to the proof of Proposition 2.5.

Proposition 2.6 (A priori bound on ψ).
Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply connected domain, then for any solution $(\psi, \mathbf{A})$ of (2.4), one has

$$\|\psi\|_{L^\infty(\Omega)} \leq 1. \tag{2.7}$$

Proposition 2.7 (A priori bounds).
Let $\Omega \subset \mathbb{R}^2$ be a bounded and simply connected domain, then for any solution $(\psi, \mathbf{A})$ of (2.4), it holds

$$\|(\nabla + ih_{\mathrm{ex}}\mathbf{A})\psi\|_{L^2(\Omega)} \leq \kappa\|\psi\|_{L^2(\Omega)}, \tag{2.8}$$

$$\|\mathrm{curl}\mathbf{A} - 1\|_{L^2(\mathbb{R}^2)} \leq C\kappa\|\psi\|_{L^2(\Omega)}\|\psi\|_{L^4(\Omega)}. \tag{2.9}$$

The estimates in Propositions 2.6 and 2.7 are consequences of the variational equations (2.4) for $(\psi, \mathbf{A})$. For the proofs we refer to [FH10, Proposition 10.3.1 and Lemma 10.3.2] for smooth domains and to [FH10, Chapter 15] for general domains with corners.

2.2 Domain with Corners

Before proceeding further we specify the assumptions on the boundary of the domain: we consider a bounded domain $\Omega \subset \mathbb{R}^2$ open and simply connected with Lipschitz boundary $\partial\Omega$ such that the unit inward normal $\boldsymbol{\nu}$ to the boundary is well defined on $\partial\Omega$ with the possible exception of a finite number of points – the *corners* of Ω. We refer to the monographs [Dau88, Gri11] for a complete discussion of domains with non-smooth boundaries. More precisely we assume that the boundary $\partial\Omega$ is a curvilinear polygon of class C^∞ in the following sense (see also [Gri11, Definition 1.4.5.1]):

Assumption 2.1 (Piecewise smooth boundary).
Let Ω a bounded open subset of $\mathbb{R}^2$, we assume that $\partial\Omega$ is a smooth curvilinear polygon, i.e., for every $\mathbf{r} \in \partial\Omega$ there exists a neighborhood U of $\mathbf{r}$ and a map $\boldsymbol{\Phi} : U \to \mathbb{R}^2$, such that

i) *$\boldsymbol{\Phi}$ is injective;*

ii) *$\boldsymbol{\Phi}$ together with $\boldsymbol{\Phi}^{-1}$ (defined from $\boldsymbol{\Phi}(U)$) are smooth;*

iii) *the region $\Omega \cap U$ coincides with either $\{\mathbf{r} \in \Omega \cap U \mid (\boldsymbol{\Phi}(\mathbf{r}))_1 < 0\}$ or $\{\mathbf{r} \in \Omega \cap U \mid (\boldsymbol{\Phi}(\mathbf{r}))_2 < 0\}$ or $\{\mathbf{r} \in \Omega \cap U \mid (\boldsymbol{\Phi}(\mathbf{r}))_1 < 0, (\boldsymbol{\Phi}(\mathbf{r}))_2 < 0\}$, where $(\boldsymbol{\Phi})_j$ stands for the $j-$th component of $\boldsymbol{\Phi}$.*

Assumption 2.2 (Boundary with corners).
We assume that the set Σ of corners of $\partial\Omega$, i.e., the points where the normal $\boldsymbol{\nu}$ does not exist, is non empty but finite and we denote by β_j the angle of the $j-$th corner (measured towards the interior).

2.3 Critical Fields in a Smooth Domain

As we described above, the GL theory provides the response of a superconductor to an external magnetic field. As the intensity of the applied field increases, one observes three subsequent transitions in an extreme type-II superconductor with smooth boundary and correspondingly three critical values of h_{ex} can be identified, i.e., three critical fields.

Below the first critical field H_{c1} the sample is everywhere in its superconducting state, i.e., $|\psi| \simeq 1$, and the magnetic field does not penetrate the superconductor; this is also known as *Meissner effect* or *Meissner state*. The first critical field $H_{c1} \simeq C_{\Omega} \log \kappa$, where C_{Ω} is a constant depending only on the domain, marks the transition associated to the nucleation of vortices:[1] if $h_{\mathrm{ex}} > H_{c1}$, then the GL minimizer contains at least one vortex, whereas it is vortex-free below H_{c1}. The rigorous mathematical analysis of this regime is summed up in [SS07, Chapter 10]. The larger $h_{\mathrm{ex}} > H_{c1}$ is, the more vortices there are. Vortices repel each other and to minimize their repulsion they arrange in regular configurations, which approach a triangular lattice, also known as Abrikosov lattice, when the number of vortices gets large. These lattice configurations predicted by Abrikosov [Abr57] and later experimentally observed [TE68], survive until the second critical field H_{c2} is reached: see [FK11] for a derivation of the Abrikosov energy. The occurrence of the vortex lattice remains however an open problem.

When $h_{\mathrm{ex}} \simeq \kappa^2$ the vortices density is so large that they overlap. At the second critical field H_{c2}, the transition from bulk to boundary superconductivity occurs, i.e., $|\psi| \simeq 0$ inside the sample. The second critical field is much harder to define, and in fact, no rigorous mathematical definition is given in the literature. The idea confirmed by many experimental results is that in the limit where κ is large, minimizers show a *bulk behavior*, if $h_{\mathrm{ex}} < \kappa^2$, and a *surface concentration*, for $h_{\mathrm{ex}} > \kappa^2$. For this reason, $H_{c2} = \kappa^2$ can be taken as definition. To characterize the physics of superconductivity in this regime, it suffices to work in a thin region of width of order $(\kappa\sigma)^{-\frac{1}{2}}$ from $\partial\Omega$.

Superconductivity survives near the boundary until the third critical field H_{c3}^{smooth} is reached, i.e., for $h_{\mathrm{ex}} \leq H_{c3}^{\mathrm{smooth}}$, with $H_{c3}^{\mathrm{smooth}} = \Theta_0^{-1}\kappa^2 + \mathcal{O}(1)$. The value $\Theta_0^{-1} \simeq 1.6946$ is a sample independent number. More precisely, Θ_0 is the ground state energy of a Schrödinger operator with uniform magnetic field of unit strength, $-(\nabla + \frac{1}{2} i \mathbf{x}^{\perp})^2$, in the half plane. Above H_{c3}^{smooth}, the normal state is the unique minimizer of the GL functional and superconductivity is lost everywhere.

We have sketched above the salient features of the critical fields; for more detailed results we refer to [BBH94, FH06, SS07, Sig13]. From the experimental view point, we mention the direct imaging of Abrikosov lattices (see, e.g., [HRD$^+$89]) and, more recently, of a surface superconductivity state [NSG$^+$09], [SPS$^+$64].

Remark 2.6 (Constant Θ_0). The value Θ_0 is also known as de Gennes's constant. We observe that it can be equivalently defined as the ground state energy of the shifted harmonic oscillator on the half-line, i.e.,

$$\Theta_0 := \min_{\alpha \in \mathbb{R}} \min_{u \in B^1(\mathbb{R}^+)} \int_0^\infty \mathrm{d}t \, \left\{ |\partial_t u|^2 + (t + \alpha)^2 |u|^2 \right\},$$

where $B^1(\mathbb{R}^+) := \{ u \in H^1(\mathbb{R}^+), \, | \, tu \in L^2(\mathbb{R}^+), \, \|u\|_{L^2(\mathbb{R}^+)} = 1 \}$. See [FH10] or [CR14, Section A.1] for more details.

[1] A vortex is an isolated zero of ψ around which the phase of ψ has a non vanishing winding number, the degree of the vortex. When κ is large, $|\psi|$ is close to 1 and differs much from 1 only in regions of characteristic size of order κ^{-1}.

2.4 Critical Fields – Domains with Corners

In presence of corners along the boundary, the above picture requires some modifications. For a discussion of the main physical features we refer to [FDM99, SS07]. While the first transition is untouched, being a bulk phenomena, the two other transitions are expected to depend on the geometrical shape of the mesoscopic superconducting sample. In particular, the third critical field H_{c3} heavily depends on the geometry of the section. We recall that for a sample with smooth cross section the third critical field is completely determined by a linear eigenvalue problem on the half-plane. In [BNF07], it is proven that the value of H_{c3} in a piecewise smooth domain depends on the ground state energy $\mu(\alpha)$ of the same linear problem restricted to an angular sector of angle α. Moreover, under suitable assumptions on $\mu(\alpha)$, it is also proven that the third critical field can be strictly larger than the one for smooth domains, although of the same order.

Before stating the Theorem on the asymptotics of H_{c3}, we recall three conjectures about the ground state energy $\mu(\alpha)$ [Bon05].

Conjecture 2.1. *For any $\alpha \in]0,\pi[$, there exist at least one eigenvalue $\mu(\alpha)$ below Θ_0.*

Conjecture 2.2. *The map $]0,\pi[\ni \alpha \to \mu(\alpha)$ is monotonically increasing.*

Conjecture 2.3. *For $\alpha \in [\pi, 2\pi[$, the infimum of the spectrum is Θ_0.*

Recently it has been proved in [ELPO17] that $\mu(\alpha) < \Theta_0^{-1}$, for any $\alpha \in (0, 0.595\pi)$. Since these conjectures have not yet been proved, although supported by numerical results, the authors assume in [BNF07] that each corner α belongs to $(0,\pi)$ and that $\mu(\alpha) < \Theta_0$. Under these unproven assumptions, the analysis performed in [BNF07] shows that, if at least one corner has an acute angle, the expected value of H_{c3} is

$$\Lambda(\alpha)^{-1}\kappa^2, \tag{2.10}$$

$\Lambda(\alpha) = \min \mu(\alpha)$ with α varying in $0 < \alpha < \pi/2$.

When (and if) $\mu(\alpha) < \Theta_0$, the third critical field thus changes value and, in fact, it is conjectured that another transition might take place below H_{c3}. Indeed, one can distinguish between a state with superconductivity distributed along the boundary and one where it is concentrated near the corner of smallest opening angle. Under the aforementioned conjecture, this is indeed the structure of the GL ground state energy before the transition to the normal state takes place. As strongly suggested by the modified Agmon estimates proven in [BNF07, Theorem 1.6], one should introduce an additional critical field $H_\star$, so that

$$H_{c2} < H_\star \leq H_{c3} \tag{2.11}$$

marking such a phase transition from *boundary* to *corner superconductivity*. The order of magnitude of $H_\star$ is clearly κ^2.

One of the goals of this Thesis is to prove that, if Ω contains finitely many corners, there is uniform surface superconductivity for applied fields satisfying asymptotically

$$1 < \frac{h_{\mathrm{ex}}}{\kappa^2} < \Theta_0^{-1}, \tag{2.12}$$

and the energy expansion is the same as in absence of corners, at least to leading order (Chapter 3). As a consequence, we infer the asymptotic estimate

$$\Theta_0^{-1}\kappa^2 =: H_\star \leq H_{c3}, \tag{2.13}$$

which must hold true also in presence of corners.

In the recent paper [HK17], it is shown that, if $\mu(\alpha) < \mu$ for some $\mu \in (0, \Theta_0)$, then superconductivity is present in a suitable neighborhood of the singularity. Furthermore, if there are two corners with the same angle, then superconductivity is present near both of them. Hence, we can relabel the vertices $\{s_1, \cdots, s_m\}$ in such a way that $\mu(\alpha_1) \geq \cdots \geq \mu(\alpha_m)$. Combining the results in [BNF07], [HK17] with [CG17], we get the existence of the following critical fields:

$$H_{c2} \leq H_\star \leq H_{c3}^1 \leq \cdots H_{c3}^{m-1} \leq H_{c3},$$

where

$$H_{c2} := \kappa^2, \qquad H_{c3,j} = \frac{\kappa^2}{\mu(\alpha_j)}, \quad j \in \{1, \cdots, m-1\}.$$

As explained above, superconductivity is uniformly distributed near the boundary of the sample between H_{c2} and $H_\star$ (so far we have a proof only in the L^2 sense). Furthermore, when $H_{c3,j} \leq h_{\mathrm{ex}} \leq H_{c3,j+1}$ (with $0 \leq j \leq m-1$ and $\mu(\alpha_j) > \mu(\alpha_{j+1})$), superconductivity is confined near the vertex s_j. Above H_{c3} superconductivity disappears everywhere.

Remark 2.7 (Experimental results). In the experimental analysis in [NSG+09] the nucleation of surface superconductivity is observed in a very thin island of material, whose boundary is not completely determined and seems to contain corners. It would then be useful to know that the surface behavior is stable with respect to the presence of mild boundary singularities.

2.5 Surface Superconductivity Regime

We start by first making a change of units, which is mostly convenient in the surface superconductivity regime, i.e., we assume that the applied field h_{ex} is of order κ^2: more precisely

$$h_{\mathrm{ex}} = b\kappa^2, \tag{2.14}$$

for some $0 < b = \mathcal{O}(1)$, and set

$$\varepsilon := b^{-\frac{1}{2}}\kappa^{-1} \ll 1. \tag{2.15}$$

We then study the asymptotics $\varepsilon \to 0$ of the minimization of the GL functional, which in the new units reads

$$\mathcal{G}_\varepsilon^{\mathrm{GL}}[\psi, \mathbf{A}] = \int_\Omega \mathrm{d}\mathbf{r} \left\{ \left| \left(\nabla + i\frac{\mathbf{A}}{\varepsilon^2} \right) \psi \right|^2 - \frac{1}{2b\varepsilon^2}(2|\psi|^2 - |\psi|^4) \right\} + \frac{1}{\varepsilon^4} \int_{\mathbb{R}^2} \mathrm{d}\mathbf{r}\, |\mathrm{curl}\mathbf{A} - 1|^2. \tag{2.16}$$

We also set

$$E_\varepsilon^{\mathrm{GL}} := \min_{(\psi,\mathbf{A}) \in \mathscr{D}^{\mathrm{GL}}} \mathcal{G}_\varepsilon^{\mathrm{GL}}[\psi, \mathbf{A}], \tag{2.17}$$

and denote by $(\psi^{\mathrm{GL}}, \mathbf{A}^{\mathrm{GL}})$ any minimizing pair.

In the new units the surface superconductivity regime coincides with the parameter region

$$1 < b < \Theta_0^{-1}, \tag{2.18}$$

at least for smooth boundaries. The key features of the surface superconductivity phase are listed below:

- the GL order parameter is concentrated in a thin boundary layer of thickness $\simeq \varepsilon$ and exponentially small in ε in the bulk (see next Section 2.6);
- the induced magnetic field is very close to the applied one, i.e.,curl$\mathbf{A} \simeq 1$ [CR14, CR16a, FH10].

Remark 2.8 (Parameter interval). We note that b varies between the lowest eigenvalues of a Schrödinger operator with unit uniform magnetic field in the plane and the inverse of the one in the half-plane respectively.

2.6 Agmon Estimates

In this Section we discuss one of the main features of surface superconductivity: the fact that the superconductivity phase survives only close to the boundary of the sample. Physically, this is due to the penetration inside the sample of the external magnetic field. Mathematically, this phenomenon is expressed by the exponential decay of the order parameter away from the boundary, which is the content of Agmon estimates. We stress that the presence of corners does not influence this behavior [FH10, Section 15.3.1].

Before proceeding further, we underline that Agmon estimates are typically derived for linear models while ψ satisfied a nonlinear differential equation. Moreover, the variational equation for ψ is

$$-(\nabla + ih_{\mathrm{ex}}\mathbf{A})^2\psi + \kappa^2|\psi|^2\psi = \kappa^2\psi.$$

However, the potential $V(\psi) := \kappa^2|\psi|^2$ is positive, which allows us to discard the nonlinear term in what follows. On the other hand, the potential is proportional to $|\psi|^2$, so that where $|\psi|$ is not small (this happens when b is near 1), is more difficult to discard its contribution. Then, for $h_{\mathrm{ex}} \sim \kappa^2$ Agmon estimates are not optimal [FK11], and this is why we use these estimates only for $h_{\mathrm{ex}} > \kappa^2$. In fact, the constant $C(b)$ below tends to 0 as $b \to 1^+$.

Theorem 2.1 (Agmon estimates)**.**
Let $\Omega \subset \mathbb{R}^2$ be a bounded and simply connected domain, if $b > 1$, any configuration $(\psi, \mathbf{A})$ solving the GL equations (2.4) *satisfies the estimate*

$$\int_\Omega \mathrm{d}\boldsymbol{r}\, \exp\left\{\tfrac{C(b)\,\mathrm{dist}(\boldsymbol{r},\partial\Omega)}{\varepsilon}\right\}\left\{|\psi|^2 + \varepsilon^2\left|\left(\nabla + i\tfrac{\boldsymbol{A}}{\varepsilon^2}\right)\psi\right|^2\right\} \le \int_{\{\mathrm{dist}(\boldsymbol{r},\partial\Omega)\le\varepsilon\}} \mathrm{d}\boldsymbol{r}\,|\psi|^2, \tag{2.19}$$

where $C(b) > 0$ depends only on the parameter b.

Remark 2.9 (Conditions on b). In the Theorem above there is no upper bound on b. In particular, we do not require $b < \Theta_0^{-1}$. However, for $b > \Theta_0^{-1}$, the unique ground state is given by the trivial configuration $(0, \mathbf{F})$. Furthermore, in presence of corners, the result might not be optimal for $b > \Theta_0^{-1}$ because of a stronger decay w.r.t. the distance from one or more corners (see Theorem 2.2 below).

To prove Theorem 2.1, one uses the weak formulation of the GL equations (2.4) for $\chi\psi$ where $\chi \in C^\infty(\Omega)$ is a suitable cut-off function. For a detailed proof we refer to [BNF07, Theorem 4.4].

We now observe that (2.19) immediately implies that

$$\int_\Omega \mathrm{d}\mathbf{r}\ \exp\left\{\tfrac{C(b)\,\mathrm{dist}(\mathbf{r},\partial\Omega)}{\varepsilon}\right\}\left\{|\psi|^2+\varepsilon^2\left|\left(\nabla+i\tfrac{\mathbf{A}}{\varepsilon^2}\right)\psi\right|^2\right\}=\mathcal{O}(\varepsilon),$$

thanks to the bound $\|\psi\|_{L^\infty(\Omega)} \leq 1$. Furthermore, because of the diverging exponential factor, (2.19) implies that

$$\int_{\mathrm{dist}(\mathbf{r},\partial\Omega)\geq c_0\varepsilon|\log\varepsilon|}\mathrm{d}\mathbf{r}\ \left\{|\psi|^2+\varepsilon^2\left|\left(\nabla+i\tfrac{\mathbf{A}}{\varepsilon^2}\right)\psi\right|^2\right\}=\mathcal{O}(\varepsilon^{c_0C(b)+1}), \tag{2.20}$$

which can be made smaller than any power of ε by taking the constant c_0 arbitrarily large. Thanks to this fact, one can easily drop energy contributions from regions further away from $\partial\Omega$ than $c_0\varepsilon|\log\varepsilon|$. In other words if we define

$$\mathcal{A}_{\partial\Omega} := \{\mathbf{r}\in\Omega \mid \mathrm{dist}\,(\mathbf{r},\partial\Omega)\leq c_0\varepsilon|\log\varepsilon|\}\,, \tag{2.21}$$

it holds that

$$E_\varepsilon^{\mathrm{GL}} = \mathcal{G}^{\mathrm{GL}}_{\varepsilon,\partial\Omega}[\psi^{\mathrm{GL}},\mathbf{A}^{\mathrm{GL}}]+\mathcal{O}(\varepsilon^\infty), \tag{2.22}$$

where $\mathcal{G}^{\mathrm{GL}}_{\varepsilon,\partial\Omega}$ stands for

$$\mathcal{G}^{\mathrm{GL}}_{\varepsilon,\partial\Omega}[\psi,\mathbf{A}] = \int_{\mathcal{A}_{\partial\Omega}}\mathrm{d}\mathbf{r}\ \left\{\left|\left(\nabla+i\frac{\mathbf{A}}{\varepsilon^2}\right)\psi\right|^2-\frac{1}{2b\varepsilon^2}(2|\psi|^2-|\psi|^4)\right\}+\frac{1}{\varepsilon^4}\int_{\mathbb{R}^2}\mathrm{d}\mathbf{r}\ |\mathrm{curl}\mathbf{A}-1|^2. \tag{2.23}$$

Hence, if $1 < b < \Theta_0^{-1}$, by (2.19), we can restrict the GL functional (2.16) to a suitable boundary layer up to an exponentially small remainder.

For the sake of completeness we now recall that in presence of corners before disappearing superconductivity survives near the corner with smallest opening angle. Mathematically, this takes the form of an exponential decay of the order parameter with respect to the distance from the corners (ordered according to their spectral parameter $\mu(\alpha)$), this is the content of the next Theorem:

Theorem 2.2 (Agmon estimate – corner version)**.**
Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply connected domain satisfying Assumptions 2.1 and 2.2, let[2] *$\tilde{\mu} > 0$ satisfy $\min_{j\in\Sigma}\mu(\alpha_j) < \tilde{\mu} < \Theta_0^{-1}$ and define*

$$\Sigma' := \{s\in\Sigma \mid \mu(\alpha)\leq\tilde{\mu}\}.$$

Then, there exist two constants C', $C(b) > 0$ such that if

$$b > \Theta_0^{-1},$$

and $(\psi,\mathbf{A})$ is a minimizer of the GL functional (2.1)*, then*

$$\int_\Omega \mathrm{d}\boldsymbol{r}\ \exp\left\{\frac{C(b)\,\mathrm{dist}(\boldsymbol{r},\Sigma')}{\varepsilon}\right\}\left\{|\psi|^2+\varepsilon^2|(\nabla+i\frac{\mathbf{A}}{\varepsilon^2}))\psi|^2\right\}$$
$$\leq C'\int_{\{\mathrm{dist}(\boldsymbol{r},\Sigma')\leq\varepsilon\}}\mathrm{d}\boldsymbol{r}\,|\psi|^2, \tag{2.24}$$

For the proof we refer to [BNF07, Theorem 1.6].

[2]Note the request $\mu(\alpha_j) < \Theta_0^{-1}$ for some j.

2.7 Energy Asymptotics in Smooth Domains

We now recall some known results for domains with smooth cross-section. In the last few years many results have been proven on the energy asymptotics of type-II superconductors in the surface superconductivity regime. We start by mentioning the energy asymptotics of [Pa]:

$$E_\varepsilon^{\mathrm{GL}} = \frac{|\partial\Omega| E_b}{\varepsilon} + o(\varepsilon^{-1})$$

where $E_b < 0$ is some constant independent of ε, $|\partial\Omega|$ the length of the boundary of Ω and b is such that $1 < b < \Theta_0^{-1}$. The definition of E_b given in [Pan02a] is somewhat complicated and later works have been devoted to obtaining a simplified expression. In particular, in [AH07] it is proven that, if b is sufficiently close to Θ_0^{-1} (independently of ε), then

$$E_\varepsilon^{\mathrm{GL}} = \frac{|\partial\Omega| E_0^{1D}}{\varepsilon} + \mathcal{O}(1)$$

where E_0^{1D} is obtained by minimizing the following one-dimensional functional both with respect to the function f and to the real number α

$$\mathcal{E}_{0,\alpha}^{1D}[f] := \int_0^{+\infty} \mathrm{d}t \, \left\{ |f'(t)|^2 + (t+\alpha)^2 f^2(t) - \frac{1}{2b}(2f^2(t) - f^4(t)) \right\}. \tag{2.25}$$

In [FHP10], the asymptotics (2.25) was extended to the interval $1.25 \leq b < \Theta_0^{-1}$. More recently, in [CR14], the following Theorem was proven:

Theorem 2.3 (Leading order of the energy and density asymptotics in smooth domains)**.** *Let $\Omega \subset \mathbb{R}^2$ be any smooth, bounded and simply connected domain. For any fixed*

$$1 < b < \Theta_0^{-1}, \tag{2.26}$$

it holds, as $\varepsilon \to 0$,

$$E^{\mathrm{GL}} = \frac{|\partial\Omega| E_0^{1D}}{\varepsilon} + \mathcal{O}(1), \tag{2.27}$$

and

$$\left\| \left| \psi^{\mathrm{GL}}(\mathbf{r}) \right|^2 - f_0^2(\mathrm{dist}(\mathbf{r}, \partial\Omega)/\varepsilon) \right\|_{L^2(\Omega)} \leq C\varepsilon \ll \left\| f_0^2(\mathrm{dist}(\mathbf{r}, \partial\Omega)/\varepsilon) \right\|_{L^2(\Omega)}. \tag{2.28}$$

We now give some ideas about the proof of Theorem 2.3. The first step is the use of Agmon estimates to reduce the GL functional to the boundary layer $\mathcal{A}_{\partial\Omega}$ introduced in (2.21). The second step is to exploit the gauge invariance of the GL functional to replace the magnetic potential with a vector potential with only tangential component. The new vector potential has no normal component and its tangential component depends on the curvature of the boundary. However, to capture the leading order of the energy asymptotics one can neglect the curvature of the boundary.

The main idea in the proof of Theorem 2.3 is that, up to a suitable choice of gauge, any minimizing order parameter for the GL energy has the structure

$$\psi^{\mathrm{GL}}(\mathbf{r}) \simeq f_0(\varepsilon^{-1}\tau) \mathrm{e}^{-i\alpha_0\sigma/\varepsilon} \mathrm{e}^{i\phi_\varepsilon(s,t)} \tag{2.29}$$

where (f_0, α_0) is a minimizing pair for (2.25), (σ, τ) are the boundary coordinates (σ is the tangential length and $\tau = \mathrm{dist}(\mathbf{r}, \partial\Omega)$ for any $\mathbf{r} \in \Omega$) and ϕ_ε is a suitable gauge phase. Once one proves (2.27), the asymptotics of the order parameter (2.28) follows immediately.

In [CR2], it is also stressed that the corrections to the energy asymptotic (2.27) must be curvature-dependent and in [CR16a] this is proven [CR16a, Theorem 2.1]:

Theorem 2.4 (Refined energy asymptotics in smooth domains)**.**
Let $\Omega \subset \mathbb{R}^2$ be any smooth, bounded and simply connected domain. For any fixed

$$1 < b < \Theta_0^{-1}, \tag{2.30}$$

as $\varepsilon \to 0$, it holds

$$E^{\mathrm{GL}} = \frac{1}{\varepsilon} \int_0^{|\partial\Omega|} \mathrm{d}\sigma \, E_\star^{1D}(k(\sigma)) + o(1), \tag{2.31}$$

where $k(\sigma)$ is the boundary curvature and $E_\star(k)$ is the minimum of the following functional

$$\mathcal{E}_{k,\alpha}^{1D}[f] := \int_0^{c_0|\log\varepsilon|} \mathrm{d}t \, (1 - \varepsilon k t) \left\{ |f'(t)|^2 + \frac{(t + \alpha - \frac{1}{2}\varepsilon k t^2)^2}{(1 - \varepsilon k t)^2} f^2(t) - \frac{1}{2b}(2f^2(t) - f^4(t)) \right\}, \tag{2.32}$$

both with respect to the function f and to the real number α.

The main idea for the proof is that the order parameter has the approximate form

$$\Psi^{\mathrm{GL}}(\mathbf{r}) = \Psi^{\mathrm{GL}}(\sigma, \tau) \approx f_{k(\sigma)}\left(\varepsilon^{-1}\tau\right) \mathrm{e}^{-i\alpha(k(\sigma))\sigma/\varepsilon} \mathrm{e}^{i\phi_\varepsilon(\sigma,\tau)} \tag{2.33}$$

with $(f_{k(s)}, \alpha(k(s))$ a minimizing pair for the ε-dependent functional[3] (2.32). The main difference with respect to the proof of Theorem 2.3 is that one has to take into account the curvature corrections. To do this, the boundary layer is split into $N_\varepsilon = \mathcal{O}(\varepsilon^{-1})$ rectangular cells $\{\mathcal{C}_n\}_{n=1,\dots,N_\varepsilon}$ of constant side length $\ell_\varepsilon \propto \varepsilon$ in the tangential direction. We denote $s_n, s_{n+1} = s_n + \ell_\varepsilon$ the s coordinates of the boundaries of the cell $\mathcal{C}_n$. Inside each cell $\mathcal{C}_n$ the boundary curvature $k(\sigma)$ is approximated by its mean value k_n:

$$k_n := \ell_\varepsilon^{-1} \int_{s_n}^{s_{n+1}} \mathrm{d}s \, k(s).$$

We also denote by $f_n := f_{k_n}$ and $\alpha_n := \alpha(k_n)$, the optimal profile and phase associated to k_n, obtained by minimizing (2.32) first with respect to f and then to α.

The strategy is to prove (2.31) by an upper and a lower bound to the reduced GL functional. The crucial point is to estimate the error due to the approximation of the curvature by its mean value in each cell. This requires a detailed analysis of the curvature dependence of the relevant quantities $E_\star^{1D}(k)$, $\alpha(k)$, f_k obtained by minimizing (2.32).

Using the refined energy asymptotics (2.31) one can also proves a refined asymptotics for the density of Cooper pairs [CR16a, Theorem 2.2]:

Theorem 2.5 (Refined density asymptotics in smooth domains)**.**
Let $\Omega \subset \mathbb{R}^2$ be any smooth, bounded and simply connected domain. For any fixed

$$1 < b < \Theta_0^{-1}, \tag{2.34}$$

[3] Notice that $\mathcal{E}_{k,\alpha}^{1D}[f]|_{k=0} = \mathcal{E}_{0,\alpha}^{1D}[f] + \mathcal{O}(\varepsilon^\infty)$.

as $\varepsilon \to 0$, *it holds*

$$\left\| |\psi^{\mathrm{GL}}(\mathbf{r})| - f_0(\varepsilon^{-1}\tau) \right\|_{L^\infty(\mathcal{A}_{\mathrm{bl}})} \leq C\gamma_\varepsilon^{-3/2}\varepsilon^{1/4}|\log\varepsilon|^\infty \ll 1,$$

where $\gamma_\varepsilon \gg \varepsilon^{1/6}|\log\varepsilon|^a$ *(for some constant* $a > 0$ *suitable large) and*

$$\mathcal{A}_{\mathrm{bl}} := \left\{ \mathbf{r} \in \Omega \,|\, f_0(\varepsilon^{-1}\tau) \geq \gamma_\varepsilon \right\} \subset \left\{ \mathrm{dist}(\mathbf{r}, \partial\Omega) \leq \frac{1}{2}\varepsilon\sqrt{|\log\gamma_\varepsilon|} \right\}.$$

In particular, for any $\mathbf{r} \in \partial\Omega$ *we have*

$$\left| |\psi^{\mathrm{GL}}(\mathbf{r})| - f_0(0) \right| \leq C\varepsilon^{1/4}|\log\varepsilon|^\infty \ll 1.$$

The Theorem above tell us that $|\Psi^{\mathrm{GL}}|^2$ is close to $f_0^2(0)$ point wise on $|\partial\Omega|$ in the surface superconductivity regime. This estimate and the fact that f_0 is strictly positive imply that in surface superconductivity regime the density of Cooper pairs never vanish in the boundary layer, there are no vortices there. Superconductivity is thus uniformly distributed along the boundary of the cross-section.

In [CR16b] another expression of $E_\varepsilon^{\mathrm{GL}}$ is derived [CR16b, Lemma 2.1]:

Proposition 2.8.
For any $1 < b < \Theta_0^{-1}$, *as* $\varepsilon \to 0$,

$$E_\star^{1D}(k) = E_0^{1D} - \varepsilon k \mathcal{E}_{\alpha_0}^{\mathrm{corr}}[f_0] + o(1) \tag{2.35}$$

where

$$\mathcal{E}_{\alpha_0}^{\mathrm{corr}}[f_0] := \int_0^{c_0|\log\varepsilon|} \mathrm{d}t\, t \left\{ |f_0(t)|^2 + -\alpha_0(t+\alpha_0)f_0^2(t) - \frac{1}{2b}(2f_0^2(t) - f_0^4(t)) \right\}. \tag{2.36}$$

The expression for $E_\varepsilon^{\mathrm{GL}}$ follows from first order perturbation theory applied to the one dimensional functional (2.32) which allows to expand $E_\star^{1D}(k(\sigma))$ in powers of ε.

Remark 2.10 (First order correction). In the energy expansion (2.35) the second term is independent of the domain. In fact since the Euler characteristic of the domain Ω equals 1, Gauss-Bonnet Theorem yields

$$\int_{\partial\Omega} \mathrm{d}\sigma\, k(\sigma) = 2\pi,$$

so that

$$E_\varepsilon^{\mathrm{GL}} = \frac{|\partial\Omega| E_0^{1D}}{\varepsilon} - 2\pi\mathcal{E}_{\alpha_0}^{\mathrm{corr}} + o(1).$$

From (2.35) it is possible to obtain an estimate of the distribution of $|\psi^{\mathrm{GL}}|^4$, the square of the normalized density of Cooper pairs [CR16b, Theorem 1.1]:

Theorem 2.6 (Curvature dependence of the order parameter)**.**
Let ψ^{GL} *be a GL minimizer and* $D \subset \Omega$ *be a measurable set intersecting* $\partial\Omega$ *with right angles. For any* $1 < b < \Theta_0^{-1}$, *in the limit* $\varepsilon \to 0$,

$$\int_D \mathrm{d}\mathbf{r}\, |\psi^{\mathrm{GL}}|^4 = \varepsilon C_1(b)|\partial\Omega \cap \partial D| + \varepsilon^2 C_2(b) \int_{\partial D \cap \partial\Omega} \mathrm{d}s\, k(s) + o(\varepsilon^2),$$

where

$$C_1(b) = -2bE_0^{1D}, \qquad C_2(b) = 2b\mathcal{E}_{\alpha_0}^{\mathrm{corr}}[f_0].$$

The Theorem above states that, to leading order, $|\psi^{\mathrm{GL}}|$ is concentrated along the boundary $\partial\Omega$ and that the first order correction is proportional to the boundary curvature $k(\sigma)$. Notice that, the estimate for the distribution of $|\psi^{\mathrm{GL}}|^4$ is expected to be wrong for those sets that intersect the boundary with angles different from $\pi/2$: this kind of sets have the same energy to leading order (see [Pan02a]) but it is essential for the correction that ψ^{GL} could be replace with a function with modulus almost constant in the tangential direction. This would obviously produce a different remainder if the angles are not equal to $\pi/2$. For further details see [CR16b].

Remark 2.11 (Sign of $C_2(b)$). It is not easy to determine the sign of $C_2(b)$, but when $|b-\Theta_0^{-1}|$ is small, in which case $C_2(b) \geq 0$. In [CR16b] it is conjectured that it is positive, which would imply that points with larger curvature attract superconductivity. This conjecture is motivated by numerical simulations (see [CDR17]).

Remark 2.12 ($\left|\psi^{\mathrm{GL}}\right|^4$ instead of $\left|\psi^{\mathrm{GL}}\right|^2$). Obviously, it would be preferable to have an estimate of $\left|\psi^{\mathrm{GL}}\right|^2$ but the method used in [CR16b] does not apply to it. In fact, the distribution of $\left|\psi^{\mathrm{GL}}\right|^4$ is more directly linked to the concentration of the energy density.

Remark 2.13 (Limiting regimes.). For the limiting regime $b \to 1^-$, we refer to [FK11] for some results about the boundary behavior of surface superconductivity. On the opposite, the regime $b \to (\Theta_0^{-1})^-$ is studied in [FH06]. In this regime the behavior of the GL functional is approximately linear, which allows to get an estimate for the distribution of $|\psi^{\mathrm{GL}}|^2$ rather than for $|\psi^{\mathrm{GL}}|^4$. In this regime it is also possible to prove that superconductivity can be either uniformly distributed along the boundary or concentrated near the points of maximal curvature.

2.7.1 Useful properties of the 1D effective model

We now recall some properties of the $1D$ effective model. For more details we refer to [CR14, Section 3]. Given the functional (2.25),

$$\mathcal{E}^{1D}_{0,\alpha}[f] := \int_0^{+\infty} \mathrm{d}t \, \left\{ |f'(t)|^2 + (t+\alpha)^2 f^2(t) - \frac{1}{2b}(2f^2(t) - f^4(t)) \right\},$$

we denote by $f_\alpha(t)$ any minimizer for given $\alpha \in \mathbb{R}$ and by E^{1D}_α the corresponding ground state energy,

$$E^{1D}_\alpha := \inf_{f \in H^1(\mathbb{R}^+)} \mathcal{E}^{1D}_\alpha[f],$$

with the convention that $E^{1D}_0 = E^{1D}_{\alpha_0} = \inf_{\alpha\in\mathbb{R}} E^{1D}_\alpha$ and $f_0(t) = f_{\alpha_0}(t)$.

We recall that there exists at least one miniziming value α_0 and that the minimizer f_0 is non-trivial if and only if $b < \Theta_0^{-1}$ [FH10, Proposition 14.2.2]. Furthermore, if $b < \Theta_0^{-1}$, f_0 is unique, it is positive and it is monotonically decreasing for t larger than a given $t_0(b)$ [CR16b, Proposition 3.1]. The minimizer solves the following variational equation [CR14, Proposition 3.1]

$$-f_0''(t) + (t+\alpha_0)^2 f_0(t) = \tfrac{1}{b}(1 - f_0^2(t)) f_0(t), \tag{2.37}$$

with boundary condition $f_0'(0) = 0$. The decay of f_0 can be estimated [CR14, Proposition 3.3]: for any $b < \Theta_0^{-1}$, there exist two constants $0 < c, C < \infty$ such that

$$c \exp\left\{ -\tfrac{1}{2}(t+\sqrt{2})^2 \right\} \leq f_0(t) \leq C \exp\left\{ -\tfrac{1}{2}(t+\alpha)^2 \right\} \tag{2.38}$$

for any $t \in \mathbb{R}^+$. As a direct consequence

$$f_0(t) = \mathcal{O}(\varepsilon^\infty), \qquad \text{for } t \geq c_0 |\log \varepsilon|, \tag{2.39}$$

for any constant $c_0 > 0$. In the next Chapters we will often use the exponential decay of f_0, for this reason we now prove the following Lemma.

Lemma 2.1.
For any $b \leq \Theta_0^{-1}$, and any $n \in \mathbb{N}$, it holds:

$$\int_0^{+\infty} \mathrm{d}t\, t^n f_0^2(t) = \mathcal{O}(1), \qquad \int_{\bar{t}}^{\infty} \mathrm{d}t\, t^n f_0^2(t) = \mathcal{O}(e^{-c_1 \bar{t}^2}),$$

for some constant $c_1 > 0$.

Proof. We recall that

$$f_0^2(t) \leq C e^{-(t+\alpha_0)^2},$$

for some constant $C < \infty$. Then the first estimate easily follows. For the second one we observe that

$$\int_{\bar{t}}^{+\infty} \mathrm{d}t\, t^n f_0^2(t) \leq e^{-\frac{1}{2}(\bar{t}+\alpha_0)^2} \int_{\bar{t}}^{+\infty} \mathrm{d}t\, t^n f_0^2(t) e^{\frac{1}{2}(t+\alpha_0)^2} = \mathcal{O}\big(e^{-C\bar{t}^2}\big).$$

□

Lemma 2.2.
For any $b \in (1, \Theta_0^{-1})$ and ε sufficiently small, there exists a finite constant C such that

$$|f_0'(t)| \leq C\, e^{-\frac{1}{4}t^2}, \tag{2.40}$$

$$|f_0'(t)| \leq C\bar{t}^3 f_0(t) \quad \text{for any } t \in [0, \bar{t}], \tag{2.41}$$

with $\bar{t} \gg 1$.

Proof. For the proof of (2.40) we simply notice that from the variational equation (2.37) for $f_0(t)$, we have

$$|f_0'(t)| \leq C \int_t^{+\infty} \mathrm{d}\eta\, \eta^2 f_0(\eta)$$

Then, via the exponential decay (2.38) of $f_0(t)$, we get

$$\int_t^{+\infty} \mathrm{d}\eta\, \eta^2\, f_0(\eta) \leq e^{-\frac{1}{4}(t+\alpha_0)^2} \int_t^{+\infty} \mathrm{d}\eta\, \eta^2 f_0(\eta) e^{\frac{1}{4}(\eta+\alpha_0)^2} \leq C e^{-\frac{1}{4}t^2}.$$

□

The *potential function* associated to the function f_0 is

$$F_0(t) := 2 \int_0^t \mathrm{d}\xi\; f_0^2(\xi)(\xi + \alpha_0). \tag{2.42}$$

Using that α_0 is strictly negative, it can be shown that $F_0(t)$ is negative and vanishes both at $t = 0$ and $t = \infty$. The fact that $F_0(t)$ vanishes at $t = \infty$ expresses the optimality of α_0.

The *cost function* that will naturally appear in our investigation is

$$K_0(t) := f_0^2(t) + F(t). \tag{2.43}$$

In [CR14, Proposition 3.4] it is proven that

$$K_0(t) \geq 0, \qquad \text{for any } t \in \mathbb{R}^+. \tag{2.44}$$

Remark 2.14 (Cost function). The cost function can be interpreted as a lower bound to the kinetic energy density associated to a vortex at distance t from the boundary. The positivity of $K_0(t)$ implies that in the regime $1 < b < \Theta_0^{-1}$ a vortex would increase the energy.

2.7.2 Dependence on the curvature of the smooth effective problem

We now summarize some useful properties about the dependence on the curvature of the one-dimensional effective problem. We first recall that

$$\mathcal{E}^{1D}_{k,\alpha}[f] := \int_0^{c_0|\log\varepsilon|} \mathrm{d}t\ (1-\varepsilon k t)\left\{|f'(t)|^2 + \frac{(t+\alpha-\frac{1}{2}\varepsilon k t^2)^2}{(1-\varepsilon k t)^2} f^2(t) - \frac{1}{2b}(2f^2(t) - f^4(t))\right\},$$

and we denote by $E_\star(k)$ the the minimum with respect to both the function $f(t)$ and the real number α.

We will use the following estimates.

Proposition 2.9.
Let $k, k' \in \mathbb{R}$ be bounded independently of ε and $1 < b < \Theta_0^{-1}$, then the following holds[4]

$$|E^{1D}_\star(k) - E^{1D}_\star(k')| = \mathcal{O}\left(\varepsilon|k-k'||\log\varepsilon|^\infty\right) \tag{2.45}$$

and

$$|\alpha(k) - \alpha(k')| = \mathcal{O}\left((\varepsilon|k-k'|)^{\frac{1}{2}}|\log\varepsilon|^\infty\right). \tag{2.46}$$

Finally, for all $n \in \mathbb{N}$,

$$\|f_k^{(n)} - f_{k'}^{(n)}\|_{L^\infty([0,c_0|\log\varepsilon|])} = \mathcal{O}\left((\varepsilon|k-k'|)^{\frac{1}{2}}|\log\varepsilon|^\infty\right). \tag{2.47}$$

Remark 2.15. Under the assumptions of Proposition 2.9, it holds

$$\|f_k^2 - f_{k'}^2\|_{L^2([0,c_0|\log\varepsilon|])} = \mathcal{O}\left((\varepsilon|k-k'|)^{\frac{1}{2}}|\log\varepsilon|^\infty\right).$$

Proposition 2.10.
Let $k, k' \in \mathbb{R}$ be bounded independently of ε and $1 < b < \Theta_0^{-1}$, then the following holds

$$\left\|\frac{f_k'}{f_k} - \frac{f_{k'}'}{f_{k'}}\right\|_{L^\infty([0,c_0|\log\varepsilon|])} = \mathcal{O}\left(\varepsilon|k-k'|)^{\frac{1}{2}}|\log\varepsilon|^\infty\right).$$

The equivalent of Lemma 2.2 is the following:

[4] By $\mathcal{O}(|\log\varepsilon|^\infty)$ we mean a quantity which is bounded by a (possibly large) power of $|\log\varepsilon|$.

Lemma 2.3.
For any $b \in (1, \Theta_0^{-1})$, for any $k \in \mathbb{R}$ and ε sufficiently small, there exists a finite constant C such that

$$\|f_k'\|_{L^\infty} = \mathcal{O}(1),$$

$$|f_k'(t)| \leq C\bar{t}^3 f_k(t) \quad \textit{for any } t \in [0, \bar{t}],$$

with $\bar{t} \gg 1$.

The potential function associated to $f_k(t)$ is

$$F_k(t) := 2\int_0^t \mathrm{d}\eta \, \frac{\eta + \alpha(k) - \frac{1}{2}\varepsilon k \eta^2}{1 - \varepsilon k \eta} f_k^2(\eta)$$

and the related cost function is $K_k(t) := f_k^2(t) + F_k(t)$. It is possible to prove that $K_k(t)$ is positive for any $t \leq c_0 |\log \varepsilon|$ [CR16a, Lemma 11].

Lemma 2.4.
Let $k, k' \in \mathbb{R}$ be bounded independently of ε and $1 < b < \Theta_0^{-1}$, then the following holds

$$\sup_{t \in [0, c_0 |\log \varepsilon|]} \left| \frac{F_k}{f_k^2} - \frac{F_{k'}}{f_{k'}^2} \right| = \mathcal{O}\left((\varepsilon |k - k'|)^{\frac{1}{2}} |\log \varepsilon|^\infty \right).$$

For the proof of Proposition 2.9, 2.10 and Lemma 2.4 we refer to [CR16a, Section 3].

Chapter 3

Leading Order of the Energy Asymptotic in a Domains with Corners

In this Chapter we consider an extreme type-II superconducting wire with non-smooth cross section, i.e., with one or more corners at the boundary, and we prove the existence of an interval of applied field, where superconductivity is spread uniformly along the boundary of the sample. More precisely, the energy is not affected to leading order by the presence of corners and the modulus of the Ginzburg-Landau minimizer is approximately constant along the transversal direction. The critical fields delimiting this surface superconductivity regime coincide with the ones in absence of boundary singularities.

3.1 Boundary Coordinates

As we explained in Section 2.6 we can easily drop from the energy the contribution of the region further from $\partial\Omega$ than $c_0\varepsilon|\log\varepsilon|$, then we will work in the region $\mathcal{A}_{\partial\Omega}$ introduced in (2.21)

$$\mathcal{A}_{\partial\Omega} := \{\mathbf{r} \in \Omega \,|\, \mathrm{dist}(\mathbf{r}, \partial\Omega) \le c_0\varepsilon|\log\varepsilon|\}\,.$$

To prove the energy asymptotics, we want to isolate the singular regions around the corners and to use boundary coordinate suitable far from the singularities. We then introduce, as in [FH10, Appendix F], a parametrization of the boundary $\partial\Omega$ denoted by $\boldsymbol{\gamma}(\sigma)$, $\sigma \in [0, \partial\Omega)$, which is piecewise smooth. At any point along the boundary, with the exception of corners Σ, the inward normal to the boundary $\boldsymbol{\nu}(\sigma)$ is well defined and smooth.

The following map however

$$\mathbf{r}(\sigma, \tau) = \boldsymbol{\gamma}(\sigma) + \tau\boldsymbol{\nu}(\sigma), \tag{3.1}$$

with $\tau = \mathrm{dist}(\mathbf{r}, \partial\Omega)$, defines a diffeomorphism only in a thin enough strip along $\partial\Omega$ and far enough from Σ, e.g., in

$$\left\{\mathbf{r} \mid \mathrm{dist}\left(\mathbf{r}, \partial\Omega\right) \le \tau_0, \mathrm{dist}\left(\mathbf{r}, \Sigma\right) \ge \tau_0'\right\},$$

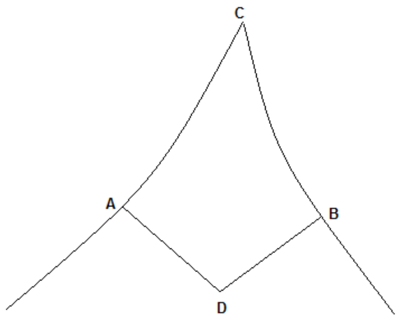

Figure 3.1. Region where boundary coordinates might be ill defined.

with τ_0 small enough and τ_0' suitably chosen (of the same order). In that region we can also define the curvature $\widetilde{k}(\sigma)$ as

$$\boldsymbol{\gamma}''(\sigma) = \widetilde{k}(\sigma)\boldsymbol{\nu}(\sigma).$$

In order to handle the singularities at corners we need to define cells covering the region where tubular coordinates are ill defined. This occurs inside a cell of the form described in Fig. 3.1, where the lengths of the segments AC and BC gets smaller of the quantity τ_0' mentioned above. Since we need to use boundary coordinates in a tubular neighborhood of $\partial\Omega$ of width $c_0\varepsilon|\log\varepsilon|$, all the segments appearing in Fig. 3.1 can be taken to be of order $\mathcal{O}(\varepsilon|\log\varepsilon|)$, thanks to the independence of $\partial\Omega$ of ε.

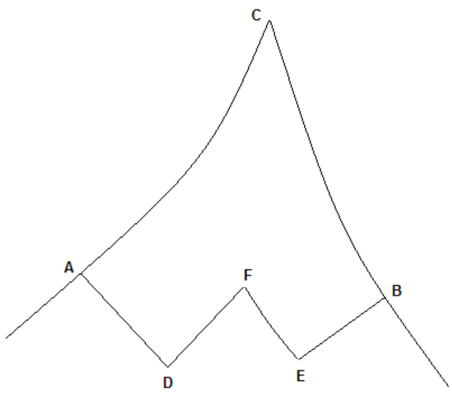

Figure 3.2. Cell $\widetilde{\Gamma}_j$.

For the sake of concreteness we pick a different boundary cell $\widetilde{\Gamma}_j$, as shown in Fig. 3.2, which covers the singular region and meet the following requirements: denoting by C the vertex of the corner, with coordinate σ_j along $\partial\Omega$, we pick two points A and B on $\partial\Omega$, in such a way that the lengths of the curves AC and BC along the boundary are equal and are given by

$$\sigma_j - \sigma_A = \sigma_B - \sigma_j = c_1 \varepsilon |\log \varepsilon|. \tag{3.2}$$

where the constant c_1 is chosen large enough (independent of ε) in order to make the whole construction possible, in particular we choose $c_1 > c_0/\tan(\alpha/2)$. The points D and E are then identified on the straight lines orthogonal to $\partial\Omega$ at A and B respectively, by requiring that the lengths of the segments $\overline{AD}$ and $\overline{BE}$ equal $c_0\varepsilon|\log\varepsilon|$. Finally the point F is the unique one (for ε small enough) at the same distance $c_0\varepsilon|\log\varepsilon|$ from both the left and right portion of $\partial\Omega$. In tubular coordinates we have

$$\begin{aligned}
\overline{AD} &= \{\mathbf{r}(\sigma,\tau) \mid \sigma = \sigma_j - c_1\varepsilon|\log\varepsilon|, t \in [0, c_0\varepsilon|\log\varepsilon|]\}, \\
\overline{BE} &= \{\mathbf{r}(\sigma,\tau) \mid \sigma = \sigma_j + c_1\varepsilon|\log\varepsilon|, t \in [0, c_0\varepsilon|\log\varepsilon|]\},
\end{aligned} \tag{3.3}$$

while

$$\begin{aligned}
\overline{DF} &= \{\mathbf{r}(\sigma,\tau) \mid \sigma \in [\sigma_j - c_1\varepsilon|\log\varepsilon|, \sigma_j - \delta_\pm\varepsilon|\log\varepsilon|], t = c_0\varepsilon|\log\varepsilon|]\}, \\
\overline{EF} &= \{\mathbf{r}(\sigma,\tau) \mid \sigma \in [\sigma_j + \delta_\pm\varepsilon|\log\varepsilon|, \sigma_j + c_1\varepsilon|\log\varepsilon|], t = c_0\varepsilon|\log\varepsilon|]\},
\end{aligned} \tag{3.4}$$

for some $0 < \delta_\pm < c_1$. The final shape of the cell is described in Fig. 3.2 for an acute angle. The definition requires no adaptation however for obtuse angles.

The most important property of the corner cells is that they carry a little amount of energy, which allow us to discard them in the estimate of both the upper and lower bounds to the GL energy. This is directly implied by the smallness of those cells, whose area is $\mathcal{O}(\varepsilon^2|\log\varepsilon|^2)$. At an heuristic level indeed the energy density is of order ε^{-2}, at least close to $\partial\Omega$, and therefore the energy contained in Γ_j is expected to be $\mathcal{O}(|\log\varepsilon|^2)$, i.e., of the same order to the error term in (3.9).

In the rest of the proof we will also use the rescaled boundary coordinates defined in terms of (σ,τ) as

$$s = \frac{\sigma}{\varepsilon}, \qquad t = \frac{\tau}{\varepsilon}, \tag{3.5}$$

we will denote respectively by $\mathcal{A}_\varepsilon$ and Γ_j, the boundary layer $\mathcal{A}_{\partial\Omega}$ and the corner region $\widetilde{\Gamma}_j$ written in the rescaled coordinates (s,t). The boundary layer without the corner cells is called $\mathcal{A}_{\mathrm{cut}}$, i.e.,

$$\mathcal{A}_{\mathrm{cut}} := \mathcal{A}_\varepsilon \setminus \bigcup_{j=1}^{N} \Gamma_j. \tag{3.6}$$

We will denote by $\widetilde{A}_{\mathrm{cut}}$ the non-rescaled boundary layer $\mathcal{A}_{\mathrm{cut}}$. With some abuse of notation we also denote by $\boldsymbol{\nu}(s) := \boldsymbol{\nu}(\varepsilon s)$ and $k(s) := \widetilde{k}(\varepsilon s)$ the normal to $\partial\Omega$ and the curvature in the new coordinates respectively. Before proceeding further we recall that the $\varepsilon-$rescaled GL functional restricted to the boundary layer $\mathcal{A}_\varepsilon$ is

$$\mathcal{G}^{\mathrm{GL}}_{\varepsilon,\mathcal{A}_\varepsilon}[\psi,\mathbf{A}] = \int_{\mathcal{A}_\varepsilon} \mathrm{d}\mathbf{r} \left\{ \left| \left(\nabla + i\frac{\mathbf{A}}{\varepsilon} \right) \psi \right|^2 - \frac{1}{2b}(2|\psi|^2 - |\psi|^4) \right\} + \frac{1}{\varepsilon^4} \int_{\mathbb{R}^2} \mathrm{d}\mathbf{r}\, |\mathrm{curl}\mathbf{A} - 1|^2. \tag{3.7}$$

3.2 Main Result

Before stating the main result we recall that

$$E_\varepsilon^{\mathrm{GL}} = \inf_{(\psi,\mathbf{A})\in\mathscr{D}^{\mathrm{GL}}} \mathcal{G}_\varepsilon^{\mathrm{GL}}[\psi,\mathbf{A}]$$

where

$$\mathcal{G}_\varepsilon^{\mathrm{GL}}[\psi,\mathbf{A}] = \int_\Omega \mathrm{d}\mathbf{r}\, \left\{ \left| \left(\nabla + i\frac{\mathbf{A}}{\varepsilon^2} \right) \psi \right|^2 - \frac{1}{2b\varepsilon^2}(2|\psi|^2 - |\psi|^4) \right\} + \frac{1}{\varepsilon^4}\int_{\mathbb{R}^2} \mathrm{d}\mathbf{r}\, |\mathrm{curl}\mathbf{A} - 1|^2.$$

We also recall that we denote by $(\psi^{\mathrm{GL}}, \mathbf{A}^{\mathrm{GL}})$ any minimizing pair realizing $E_\varepsilon^{\mathrm{GL}}$.

Theorem 3.1 (Leading order asymptotics for a general corner domain)**.**
Let $\Omega \subset \mathbb{R}^2$ be any bounded simply connected domain satisfying the Assumptions 2.1 and 2.2. Then for any fixed

$$1 < b < \Theta_0^{-1}, \tag{3.8}$$

as $\varepsilon \to 0$, it holds

$$E_\varepsilon^{\mathrm{GL}} = \frac{|\partial\Omega| E_0^{1D}}{\varepsilon} + \mathcal{O}(|\log\varepsilon|^2), \tag{3.9}$$

and

$$\left\| \left|\psi^{\mathrm{GL}}(\mathbf{r})\right|^2 - f_0^2(\mathrm{dist}(\mathbf{r},\partial\Omega)/\varepsilon) \right\|_{L^2(\Omega)} = \mathcal{O}(\varepsilon|\log\varepsilon|) \ll \left\| f_0^2(\mathrm{dist}(\mathbf{r},\partial\Omega)/\varepsilon) \right\|_{L^2(\Omega)}. \tag{3.10}$$

Remark 3.1 (Order parameter asymptotics).
The convergence stated in (3.10) implies that f_0 provides a good approximation of $|\psi^{\mathrm{GL}}|$ in the boundary layer, i.e., for $\mathrm{dist}(\mathbf{r},\partial\Omega) \lesssim \varepsilon|\log\varepsilon|$. At larger distance from the boundary both functions are indeed exponentially small in ε and their mass consequently very small. Note also that, if the condition (2.30) is satisfied,

$$\left\| f_0^2(\mathrm{dist}(\mathbf{r},\partial\Omega)/\varepsilon) \right\|_{L^2(\Omega)} \geq c\sqrt{\varepsilon},$$

for some $c > 0$, as it immediately follows by observing that $f_0(t)$ is independent of ε and non-identically zero.

Remark 3.2 (Limiting regimes).
We explicitly chose not to address the limiting cases $b \to 1^+$ or $b \to \Theta_0^{-1}$. In the former case an adaptation of the method might work (see also [CR14, Remark 2.1]), while in the latter the analysis is made much more complicate because of the interplay between corner and boundary confinements. In particular a much more detailed knowledge of the behavior of the linear problem, i.e., the ground state energy of the magnetic Schrödinger operator in a sector of angle α, is needed, e.g., a proof of the conjecture discussed in Section 2.4.

3.3 Strategy of the Proof

The strategy of the proof is very similar to the arguments contained in [CR14] and summarized in Chapter 2. We sketch here the main steps.

The preliminary step, i.e., the *restriction to the boundary layer*, is standard and described in details, e.g., in [FH10, Section 14.4]. The final outcome of this step is a functional restricted to a layer of width $\mathcal{O}(\varepsilon|\log\varepsilon|)$ along the boundary. The main ingredients are as usual Agmon estimates.

Another common step to both the upper and lower bound proofs, although applied to different magnetic potentials, is the *replacement of the magnetic field*, as, e.g., in [FH10, Appendix F]. The presence of corners however calls for suitable modifications, since this step is usually done by exploiting tubular coordinates, which are not defined closed to the boundary singularities. As we are going to see however, such a replacement is needed only in the smooth part of the boundary layer, where it can be done in a rather standard way by making a special choice of the gauge. In fact the only required modification is an adapted definition of the gauge phase close to the corners.

The energy *upper bound* is then trivially obtained by testing the energy on a trial configuration, we will use the same trial configuration as the one in [CR14] suitable far from the corners.

The *lower bound* proof is more involved and requires few more steps.

Since we are interested in the leading term of the energy asymptotics, we want to use the 1D reduced effective problem introduced in (2.25). Being the 1D energy associated to the variation of $|\psi^{\mathrm{GL}}|$ along the normal to the boundary, we need to use the boundary coordinates. At this stage the non-smoothness of the boundary really affects the proof, because the use of tubular coordinates is clearly prevented near the corners. By a simple a priori estimate however we show that one can *drop the energy around corners*.

We are thus left with the energy contributions of the smooth pieces of the boundary layer. There we can pass to boundary coordinates and use the same trick, i.e., a suitable *integration by parts*, involved in the proofs of some earlier results [CR14, CR16a] and inspired by other works on rotating condensates (see, e.g., [CR13, CRY11, CPRY11a, CPRY11b, CPRY12]). Since the region where we perform the integration by parts is not connected however, a naive application of the trick would generate unwanted boundary terms and therefore we will slightly modify the order parameter by introducing a partition of unity around the corners.

Finally the key estimate to complete the lower bound proof is the *positivity of the cost function* (2.44). This step is precisely the same as in [CR14] and is the only point in the proof where the condition $1 < b < \Theta_0^{-1}$ comes explicitly into play. However we recall that the assumption $b > 1$ is required to apply Agmon estimates in the preliminaries, while the condition $b < \Theta_0^{-1}$ is needed in order to ensure that the 1D minimizing profile is non-trivial.

3.4 Proof of the Main Result

To get the desired energy asymptotic we can prove an upper and a lower bound of $E_\varepsilon^{\mathrm{GL}}$.

3.4.1 Upper bound

Proposition 3.1 (Energy upper bound)**.**
Let $1 < b < \Theta_0^{-1}$ and ε be small enough. Then it holds

$$E_\varepsilon^{\mathrm{GL}} \leq \frac{|\partial\Omega| E_0^{1D}}{\varepsilon} + \mathcal{O}(|\log\varepsilon|). \tag{3.11}$$

Proof. As usual we prove the result by evaluating the GL energy on a trial state having the expected physical features. This trial state has to be concentrated near the boundary of the sample and its modulus must be approximately constant in the transversal direction. However since we can not use boundary coordinates at corners, we impose that the function vanishes in a suitable neighborhood of Σ. We will label corners in Σ with their coordinate along the boundary, i.e.,

$$\Sigma = \{(s,0) \mid s = s_j, j = 1, \ldots, N\}. \tag{3.12}$$

We thus introduce a cut-off function $\chi \in C_0^\infty$, such that $0 \leq \chi \leq 1$. Its role is to cut the region close to Σ. For any $(s,t) \in \mathcal{A}_\varepsilon$, we require

$$\chi(s) := \begin{cases} 0, & \text{if } |s - s_j| \leq c_1 |\log\varepsilon|, \text{ for some } j = 1, \ldots, N, \\ 1, & \text{if } |s - s_j| \geq 2c_1 |\log\varepsilon|, \text{ for all } j = 1, \ldots, N. \end{cases} \tag{3.13}$$

The transition from 0 to 1 occurs in a one-dimensional region of length $c_1|\log\varepsilon|$ and therefore we can always assume that

$$|\chi'| = \mathcal{O}(|\log\varepsilon|^{-1}). \tag{3.14}$$

We also define (note that we can not define $\mathcal{D}_{\varepsilon,j}$ using boundary coordinates in the interior because there they are ill defined)

$$\mathcal{D}_{\varepsilon,j} := \{(s,t) \in \mathcal{A}_\varepsilon \mid |s - s_j| \geq 2c_1|\log\varepsilon|\}^c, \tag{3.15}$$

i.e., $(\cup_j \mathcal{D}_{\varepsilon,j})^c$ is the region where $\chi = 1$.

It remains to choose the magnetic potential to complete the test configuration: we thus denote by $\mathbf{F}$ any magnetic potential such that $\nabla \cdot \mathbf{F} = 0$ and $\mathrm{curl}\mathbf{F} = 1$ in $\mathbb{R}^2$. Our trial state is then

$$(\psi_{\mathrm{trial}}, \mathbf{F}), \tag{3.16}$$

where

$$\psi_{\mathrm{trial}}(s,t) = \chi(s) f_0(t) e^{-iS(s)} e^{i\phi_{\mathbf{F}}(s,t)}, \tag{3.17}$$

where $\phi_{\mathbf{F}}$ is a gauge phase and $S(s) := \alpha_0 s - S_{glo}(\varepsilon s)$. In particular for all $(s,t) \in \mathcal{A}_{\mathrm{cut}}$ we define

$$\phi_{\mathbf{F}}(s,t) := -\frac{1}{\varepsilon}\int_0^t \mathrm{d}\eta\, \mathbf{F}(\mathbf{r}(\varepsilon s, \varepsilon\eta)) \cdot \boldsymbol{\nu}(s) - \frac{1}{\varepsilon}\int_0^s \mathrm{d}\xi\, \mathbf{F}(\mathbf{r}(\varepsilon\xi, 0)) \cdot \boldsymbol{\gamma}'(\varepsilon s) + \varepsilon\gamma_\varepsilon s,$$

where

$$\gamma_\varepsilon := \left(\frac{|\Omega|}{|\partial\Omega|\varepsilon^2} - \frac{2\pi}{|\partial\Omega|} \left\lfloor \frac{|\Omega|}{2\pi\varepsilon^2} \right\rfloor \right)$$

is a factor we have to add to have a well globally defined phase.

For $(s,t) \in \mathcal{A}_{\text{cut}}^c \cap \mathcal{A}_\varepsilon$ we can set

$$\phi_{\mathbf{F}}(s,t) := -\frac{1}{\varepsilon}\int_0^t \mathrm{d}\eta\, \varphi(s)\mathbf{F}(\mathbf{r}(\varepsilon s, \varepsilon\eta)) \cdot \boldsymbol{\nu}(s) - \frac{1}{\varepsilon}\int_0^s \mathrm{d}\xi\, \mathbf{F}(\mathbf{r}(\varepsilon\xi, 0)) \cdot \boldsymbol{\gamma}'(\varepsilon s) + \varepsilon\gamma_\varepsilon s$$

where φ is a smooth cut-off function such that for some δ of order $\mathcal{O}(1)$

$$\text{supp}(\varphi') = [s_j - c_1|\log\varepsilon|, s_j - c_1|\log\varepsilon| + \delta] \cup [s_j + c_1|\log\varepsilon| - \delta, s_j + c_1|\log\varepsilon|],$$
$$\varphi(s_j \pm c_1|\log\varepsilon| \mp \delta) = 0,$$
$$\varphi(s_j \pm c_1|\log\varepsilon|) = 1.$$

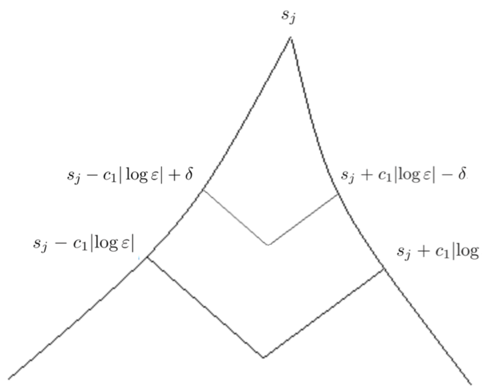

Figure 3.3. The region in which φ' is supported.

Moreover, $S_{glo}(s)$ is a function that ensures that the phase $S(s)$ is well defined. We want to choose $S_{glo}(s)$ in such a way that

$$S\left(\frac{|\partial\Omega|}{\varepsilon}\right) - S(0) = S\left(\frac{|\partial\Omega|}{\varepsilon}\right) \in 2\pi\mathbb{Z}.$$

In particular, we set $S_{glo}(s) := \lambda_\varepsilon s$, then we want that

$$\alpha_0\frac{|\partial\Omega|}{\varepsilon} = 2\pi n + \lambda_\varepsilon\frac{|\partial\Omega|}{\varepsilon}.$$

A possible solution is to pick

$$\lambda_\varepsilon := \left[\alpha_0\frac{|\partial\Omega|}{\varepsilon} - 2\pi\left\lfloor\alpha_0\frac{|\partial\Omega|}{2\pi\varepsilon}\right\rfloor\right]\frac{\varepsilon}{|\partial\Omega|}$$

which implies

$$|S_{glo}(s)| \le C, \qquad |\partial_s S_{glo}(\varepsilon s)| \le C\varepsilon. \tag{3.18}$$

We now observe that the gauge phase defined above is such that the associated vector potential has no normal component, we will denote by $a_{\mathbf{F}}(s,t)$ the non vanishing tangential component, i.e.,

$$a_{\mathbf{F}}(s,t) = (1 - \varepsilon k(s)t)\frac{\boldsymbol{\gamma}'(\varepsilon s) \cdot \mathbf{F}(\mathbf{r}(\varepsilon s, \varepsilon t))}{\varepsilon} + \partial_s\phi_{\mathbf{F}},$$

Before proceeding further we observe that

$$a_{\mathbf{F}}(s,0) = \varepsilon\gamma_\varepsilon, \qquad \partial_t a_{\mathbf{F}}(s,t) = -(1-\varepsilon k(s)t)(\mathrm{curl}\mathbf{F})(\mathbf{r}(\varepsilon s, \varepsilon t)),$$

then

$$a_{\mathbf{F}}(s,t) = \varepsilon\gamma_\varepsilon - t + \mathcal{O}(\varepsilon|\log\varepsilon|^2) = -t + \mathcal{O}(\varepsilon|\log\varepsilon|^2),$$

where the last estimate follows from the fact that $|\gamma_\varepsilon| \leq 1$.

The order parameter decays exponentially as $t \to \infty$, thanks to the point wise estimate of f_0 (2.38), and therefore

$$\psi_{\mathrm{trial}}(\mathbf{r}) = \mathcal{O}(\varepsilon^\infty), \qquad \text{for } \mathbf{r} \in \mathcal{A}_\varepsilon^c.$$

Furthermore, via (3.14), (3.18), the pointwise estimate of f_0 (2.38) and Lemma 2.2, we can discard the contributions of the gradient in the region $\mathcal{A}_\varepsilon^c$. Hence we have

$$\mathcal{G}_\varepsilon^{\mathrm{GL}}[\psi_{\mathrm{trial}}, \mathbf{F}] = \mathcal{G}_{\varepsilon,\mathcal{A}_\varepsilon}^{\mathrm{GL}}[\psi_{\mathrm{trial}}, \mathbf{F}] + \mathcal{O}(\varepsilon^\infty). \tag{3.19}$$

Also, since $\mathrm{curl}\mathbf{F} = 1$, the quantity we have to estimate is actually

$$\begin{aligned}\mathcal{G}_{\varepsilon,\mathcal{A}_\varepsilon}^{\mathrm{GL}}[\psi_{\mathrm{trial}}, \mathbf{F}] &= \int_{\mathcal{A}_\varepsilon} \mathrm{d}\mathbf{r} \, \left\{ \left| \left(\nabla + i\tfrac{\mathbf{F}}{\varepsilon} \right) \psi_{\mathrm{trial}} \right|^2 - \tfrac{1}{2b}(2|\psi_{\mathrm{trial}}|^2 - |\psi_{\mathrm{trial}}|^4) \right\} \\ &= (1 + \mathcal{O}(\varepsilon|\log\varepsilon|)) \int_{\mathcal{A}_\varepsilon} \mathrm{d}s\mathrm{d}t \, \left\{ \chi^2(s) |\partial_t f_0(t)|^2 + f_0^2(t) \, |\partial_s \chi(s)|^2 \right. \\ &\quad \left. + \left| a_{\mathbf{F}}(s,t) - \alpha_0 + \partial_s S_{glo} \right|^2 f_0^2(t) \chi^2(s) - \tfrac{1}{2b} [2\chi^2(s) f_0^2(t) - \chi^4(s) f_0^4(t)] \right\}, \end{aligned} \tag{3.20}$$

where the prefactor $(1 + \mathcal{O}(\varepsilon|\log\varepsilon|))$ comes from an estimate of the Jacobian associated to the the change of coordinates from $\mathbf{r}$ to (s,t), i.e., $(1 - \varepsilon k(s)t)$ (recall that the curvature is uniformly bounded and $t \leq c_0|\log\varepsilon|$ in $\mathcal{A}_\varepsilon$). In what follows we want to prove that

$$\mathcal{G}_{\varepsilon,\mathcal{A}_\varepsilon}^{\mathrm{GL}}[\psi_{\mathrm{trial}}, \mathbf{F}] \leq \int_{\mathcal{A}_\varepsilon \setminus \cup_j \Gamma_j} \mathrm{d}s\mathrm{d}t \, \left\{ f_0'(t)^2 + (t+\alpha_0)^2 f_0^2(t) - \tfrac{1}{2b} [2 f_0^2(t) - f_0^4(t)] \right\} + \mathcal{O}(|\log\varepsilon|), \tag{3.21}$$

where the remainder is due to various factors. We notice in fact that the first term on the r.h.s. of (3.21) equals

$$\begin{aligned} &\int_{\mathcal{A}_\varepsilon \setminus \cup_j \Gamma_j} \mathrm{d}s\mathrm{d}t \, \left\{ f_0'(t)^2 + (t+\alpha_0)^2 f_0^2(t) - \tfrac{1}{2b} [2 f_0^2(t) - f_0^4(t)] \right\} \\ &\qquad = \int_{\mathcal{A}_{\mathrm{cut}}} \mathrm{d}s\mathrm{d}t \, \left\{ f_0'(t)^2 + (t+\alpha_0)^2 f_0^2(t) - \tfrac{1}{2b} [2 f_0^2(t) - f_0^4(t)] \right\} \\ &\qquad = \frac{|\partial\Omega_{\mathrm{cut}}| \, E_0^{1D}}{\varepsilon} = \frac{|\partial\Omega| E_0^{1D}}{\varepsilon} + \mathcal{O}(|\log\varepsilon|), \end{aligned} \tag{3.22}$$

where $\partial\Omega_{\mathrm{cut}} = \partial\Omega \cap \partial\mathcal{A}_{\mathrm{cut}}$, so that $|\partial\Omega_{\mathrm{cut}}| = |\partial\Omega| + \mathcal{O}(\varepsilon|\log\varepsilon|)$ and the prefactor $\mathcal{O}(\varepsilon|\log\varepsilon|)$ in (3.20) produce another error of order $|\log\varepsilon|$. Moreover,

$$\begin{aligned} &-\frac{1}{2b} \int_{\mathcal{A}_\varepsilon} \mathrm{d}s\mathrm{d}t \, [2\chi^2(s) f_0^2(t) - \chi^4(s) f_0^4(t)] \\ &\qquad \leq -\frac{1}{2b} \int_{\mathcal{A}_\varepsilon \setminus \cup_j \Gamma_j} \mathrm{d}s\mathrm{d}t \, [2 f_0^2(t) - f_0^4(t)] + C \int_{\mathcal{A}_\varepsilon} \mathrm{d}s\mathrm{d}t \, (1 - \chi(s)) \, f_0^2(t) \\ &\qquad = -\frac{1}{2b} \int_{\mathcal{A}_\varepsilon \setminus \cup_j \Gamma_j} \mathrm{d}s\mathrm{d}t \, [2 f_0^2(t) - f_0^4(t)] + \mathcal{O}(|\log\varepsilon|), \end{aligned} \tag{3.23}$$

where we used that $\chi \equiv 1$ in $\mathcal{A}_\varepsilon \setminus \cup_j \Gamma_j$ and the fact that f_0 is exponentially decaying (Lemma 2.1). The error then comes from the integration in the s variable. We also get that

$$\int_{\mathcal{A}_\varepsilon} \mathrm{d}s\mathrm{d}t \, \left| a_{\mathbf{F}}(s,t) - \alpha_0 + \partial_s S_{glo}(s) \right|^2 f_0^2(t) \chi^2(s) = \int_{\mathcal{A}_\varepsilon} (t + \alpha_0 - \partial_s S_{glo}(s))^2 f_0^2(t) \chi^2(s)$$
$$+ \int_{\mathcal{A}_\varepsilon} (a_{\mathbf{F}}(s,t) + t)^2 f_0^2(t) \chi^2(s) - 2 \int_{\mathcal{A}_\varepsilon} (a_{\mathbf{F}} + t)(t + \alpha_0 - \partial_s S_{glo}(s)) f_0^2(t) \chi^2(s) \quad (3.24)$$

and via the exponential decay of f_0 (Lemma 2.1), we get

$$\int_{\mathcal{A}_\varepsilon} \mathrm{d}s\mathrm{d}t \, (t + \alpha_0 - \partial_s S_{glo}(s))^2 f_0^2(t) \chi^2(s) \leq \int_{\mathcal{A}_\varepsilon \setminus \cup_j \Gamma_j} \mathrm{d}s\mathrm{d}t \, (t + \alpha_0)^2 f_0^2(t) + \mathcal{O}(1),$$

where we used the estimate in (3.18). Furthermore, since

$$a_{\mathbf{F}}(s,t) + t = \mathcal{O}(\varepsilon t^2),$$

again thanks to the exponential decay of f_0 (Lemma 2.1) we have

$$\int_{\mathcal{A}_\varepsilon} (a_{\mathbf{F}}(s,t) + t)^2 f_0^2(t) \chi^2(s) = \int_{\mathcal{A}_{\mathrm{cut}}} (a_{\mathbf{F}}(s,t) + t)^2 f_0^2(t) \chi^2(s) \leq C\varepsilon^2 \int_{\mathcal{A}_{\mathrm{cut}}} \mathrm{d}s\mathrm{d}t \, t^4 f_0^2(t) = \mathcal{O}(\varepsilon)$$

and

$$2 \left| \int_{\mathcal{A}_\varepsilon} \mathrm{d}s\mathrm{d}t \, (a_{\mathbf{F}}(s,t) + t)(t + \alpha_0 - \partial_s S_{glo}(s)) f_0^2(t) \chi^2(s) \right|$$
$$\leq C\varepsilon \int_{\mathcal{A}_{\mathrm{cut}}} \mathrm{d}s\mathrm{d}t \, t^2 |t + \alpha_0 - \partial_s S_{glo}(s)| f_0^2(t) = \mathcal{O}(1).$$

We then get

$$\int_{\mathcal{A}_\varepsilon} \mathrm{d}s\mathrm{d}t \, \left| a_{\mathbf{F}}(s,t) - \alpha_0 - \partial_s S_{glo}(s) \right|^2 f_0^2 \chi^2 = \int_{\mathcal{A}_\varepsilon \setminus \cup_j \Gamma_j} \mathrm{d}s\mathrm{d}t \, (t + \alpha_0)^2 f_0^2 + \mathcal{O}(1).$$

Finally, the kinetic energy of the cut-off function is bounded as

$$\int_{\mathcal{A}_\varepsilon} \mathrm{d}s\mathrm{d}t \, f_0^2(t) \left| \partial_s \chi(s) \right|^2 \leq C |\log \varepsilon|^{-2} \left| \cup_j \left(\mathcal{D}_{\varepsilon,j} \setminus \mathcal{C}_{\varepsilon,j} \right) \right| = \mathcal{O}(1),$$

thanks to the assumption (3.14). For the first term on the r.h.s. of (3.20) we simply use that $|\chi| \leq 1$ and $\chi = 0$ in Γ_j for all $j \in \Sigma$. Combining (3.21) and (3.22) the energy upper bound is proven. □

3.4.2 Lower bound

Replacement of the vector potential.

In the surface superconductivity regime the induced magnetic field is actually very close to the applied one. This is typical of this regime and is also the reason why the last term in the GL functional is never taken into account, since its contribution is always sub-leading. Hence the induced field is almost uniform and its strength is approximately 1. There are however many magnetic potentials $\mathbf{A}$ generating such a field and it is useful to exploit gauge invariance to select the most convenient one. Here we discuss how it can be done. In particular, we want to prove here that after a change of coordinates and a gauge transformation, for any Ψ and $\mathbf{A}$ solving the GL equations (2.4)

$$\mathcal{G}^{GL}_{\varepsilon,\mathcal{A}_{\mathrm{cut}}}[\Psi,\mathbf{A}] = (1+\mathcal{O}(\varepsilon|\log\varepsilon|))\int_{\mathcal{A}_{\mathrm{cut}}} \mathrm{d}s\mathrm{d}t \, \left\{ |\partial_t \psi|^2 + |(\partial_s + i a_{\mathbf{A}}(s,t))\psi|^2 - \tfrac{1}{2b}[2|\psi|^2 - |\psi|^4] \right\} + \frac{1}{\varepsilon^4}\int_{\mathbb{R}^2} \mathrm{d}\mathbf{r} \, |\mathrm{curl}\mathbf{A} - 1|^2 .$$

To do this we need some assumptions on the vector potential. First of all we recall that in a corner domain we can always pick a magnetic potential $\mathbf{A}$ (see [FH10, Lemma D.2.7]) satisfying the Coulomb gauge, i.e.,

$$\nabla \cdot \mathbf{A} = 0, \qquad \text{in } \Omega. \tag{3.25}$$

In smooth domains (3.25) is accompanied by the boundary condition $\mathbf{A}\cdot\boldsymbol{\nu} = 0$ on $\partial\Omega$. This is clearly not possible in presence of corners, because of the jumps of $\boldsymbol{\nu}$: let us stress that at corners $\boldsymbol{\nu}$ is discontinuous but it remains uniformly bounded all over the boundary, so that, for instance,

$$\int_0^{\frac{|\partial\Omega|}{\varepsilon}} \mathrm{d}s \, \mathbf{A}(\mathbf{r}(\varepsilon s, 0)) \cdot \boldsymbol{\nu}(s) = 0,$$

i.e., the function $\mathbf{A}\cdot\boldsymbol{\nu}$ is integrable. Moreover Stokes formula and elliptic regularity (see, e.g., [Gri11]) implies that the boundary condition is in fact satisfied in trace sense, i.e., almost everywhere.

Under this choices, the following inequality holds true (see [FH10, Eqs. (15.18) and (15.19)]):

$$\|\mathbf{A} - \mathbf{F}\|_{W^{1,2}(\Omega)} \le C \, \|\mathrm{curl}\mathbf{A} - 1\|_{L^2(\mathbb{R}^2)} , \tag{3.26}$$

where $\mathbf{F}(\mathbf{r}) := \frac{1}{2}(-y, x)$. Since $\mathbf{A}$ is defined up to an additive constant, this last inequality holds true only if the value of the constant is suitably chosen. We assume that all the magnetic potentials in this Chapter are taken in such a way that (3.26) applies.

Finally, after another local gauge transformation (see [FH10, Proposition D.1.1]), we can obtain a vector potential $\mathbf{A}$ such that far from corners, where the normal to the boundary is well defined, e.g.,

$$\mathbf{A}(\mathbf{r}(\varepsilon s, 0)) \cdot \boldsymbol{\nu}(s) = 0, \qquad \text{for any } |s - s_j| \ge c_1 |\log\varepsilon|. \tag{3.27}$$

Now the key idea for the replacement, described, e.g., in [FH10, Appendix F] (see also [CR14, Section 4.1] and [CR16a, Proof of Lemma 4]), is that, since at the boundary the normal component of $\mathbf{A}$ vanishes, close enough to $\partial\Omega$, it is possible to find a differentiable function $\phi_{\mathbf{A}}(s,t)$ such that $\mathbf{A} - \nabla\phi_{\mathbf{A}}(s,t) = f(s,t)\mathbf{e}_s$, i.e., the field remains purely tangential close enough to $\partial\Omega$. In addition since $\mathrm{curl}\mathbf{A}$ is approximately 1, the function $f(s,t)$ is close to $-t + o(1)$. Using a gauge transformation built on $\phi_{\mathbf{A}}$, one can then replace $\mathbf{A}$ with a magnetic potential which is of the form described above.

The first step, i.e., the gauge transformation is essentially the same as for smooth domains: thanks to the integrability of both $\boldsymbol{\gamma}'(s)$ and $\mathbf{A}(\mathbf{r}(\varepsilon s, \varepsilon t))$ along $\partial\Omega$, which is guaranteed in the first case by the boundedness of $\boldsymbol{\gamma}'(s)$ and in the second one by Sobolev trace theorem, which gives $\mathbf{A} \in H^{1/2}(\partial\Omega; \mathbb{R}^2)$, we can set for any $(s,t) \in \mathcal{A}_{\mathrm{cut}}$

$$\phi_{\mathbf{A}}(s,t) := -\frac{1}{\varepsilon}\int_0^t \mathrm{d}\eta\, \mathbf{A}(\mathbf{r}(\varepsilon s, \varepsilon\eta)) \cdot \boldsymbol{\nu}(\varepsilon s) - \frac{1}{\varepsilon}\int_0^s \mathrm{d}\xi\, \mathbf{A}(\mathbf{r}(\varepsilon\xi, 0)) \cdot \boldsymbol{\gamma}'(\varepsilon s) + \varepsilon\delta_\varepsilon s, \tag{3.28}$$

with[1]

$$\delta_\varepsilon(s,t) = \frac{1}{\varepsilon^2|\partial\Omega|}\int_\Omega \mathrm{d}\mathbf{r}\, \mathrm{curl}\mathbf{A} - \frac{2\pi}{|\partial\Omega|}\left\lfloor \frac{1}{2\pi\varepsilon^2}\int_\Omega \mathrm{d}\mathbf{r}\, \mathrm{curl}\mathbf{A} \right\rfloor. \tag{3.29}$$

Note that the second term in the expression above is well defined even if s gets close to the corner points Σ, because $\boldsymbol{\gamma}'$ is by assumption an integrable function. On the other hand, for any $(s,t) \in \mathcal{A}_{\mathrm{cut}}^c \cap \mathcal{A}_\varepsilon$, we can set

$$\phi_{\mathbf{A}}(s,t) := -\frac{1}{\varepsilon}\int_0^t \mathrm{d}\eta\, \chi(s)\mathbf{A}(\mathbf{r}(\varepsilon s, \varepsilon\eta)) \cdot \boldsymbol{\nu}(\varepsilon s) - \frac{1}{\varepsilon}\int_0^s \mathrm{d}\xi\, \mathbf{A}(\mathbf{r}(\varepsilon\xi, 0)) \cdot \boldsymbol{\gamma}'(\varepsilon s) + \varepsilon\delta_\varepsilon s, \tag{3.30}$$

where the support of the smooth cut-off function χ is contained in the intervals $[s_j - c_1|\log\varepsilon|, s_j - c_1|\log\varepsilon| + \delta]$ and $[s_j + c_1|\log\varepsilon| - \delta, s_j + c_1|\log\varepsilon|]$, for some δ of order $\mathcal{O}(1)$, and

$$\chi(s_j \pm c_1|\log\varepsilon| \mp \delta) = 0,$$

$$\chi(s_j \pm c_1|\log\varepsilon|) = 1.$$

It is easy to verify that for any $(s,t) \in \mathcal{A}_\varepsilon$, one has

$$\phi_{\mathbf{A}}(s + n|\partial\Omega|, t) = \phi_{\mathbf{A}}(s,t) + 2\pi n, \qquad \text{for any } n \in \mathbb{Z}.$$

In fact for $s = 0$ and $n = 1$, by the divergence Theorem, it holds

$$\begin{aligned} \phi_{\mathbf{A}}(|\partial\Omega|, t) - \phi_{\mathbf{A}}(0,t) &= -\frac{1}{\varepsilon}\int_0^{|\partial\Omega|} \mathrm{d}\xi\, \mathbf{A}(\mathbf{r}(\varepsilon\xi, 0)) \cdot \gamma'(\varepsilon s) + \varepsilon\delta_\varepsilon|\partial\Omega| \\ &= -\frac{1}{\varepsilon}\int_\Omega \mathrm{d}\mathbf{r}\, \mathrm{curl}\mathbf{A}(\mathbf{r}(\varepsilon s, \varepsilon t)) + \varepsilon\delta_\varepsilon|\partial\Omega| \\ &= -2\pi\left\lfloor \frac{1}{2\pi\varepsilon^2}\int_\Omega \mathrm{d}\mathbf{r}\, \mathrm{curl}\mathbf{A} \right\rfloor. \end{aligned}$$

We now use in $\mathcal{A}_{\mathrm{cut}}$ the change of gauge induced by

$$\Psi(\mathbf{r}(\varepsilon s, \varepsilon t)) = \psi(s,t)e^{i\phi_{\mathbf{A}}(s,t)}$$

and since $\frac{\mathbf{A}(\mathbf{r}(\varepsilon s, \varepsilon t)) \cdot \boldsymbol{\nu}(s)}{\varepsilon} + \partial_t\phi_{\mathbf{A}} = 0$, we get

$$\int_{\widetilde{\mathcal{A}}_{\mathrm{cut}}} \mathrm{d}\mathbf{r} \left| \left(\nabla + i\frac{\mathbf{A}}{\varepsilon^2}\right)\Psi \right|^2 = \int_{\mathcal{A}_{\mathrm{cut}}} \mathrm{d}s\mathrm{d}t\, (1 - \varepsilon k(s)t)\left\{ |\partial_t\psi|^2 + \frac{1}{(1 - \varepsilon k(s)t)^2}|(\partial_s + ia_{\mathbf{A}}(s,t))\psi|^2 \right\},$$

where

$$\begin{aligned} a_{\mathbf{A}}(s,t) &= (1 - \varepsilon k(s)t)\frac{\boldsymbol{\gamma}'(\varepsilon s) \cdot \mathbf{A}(\mathbf{r}(\varepsilon s, \varepsilon t))}{\varepsilon} + \partial_s\phi_{\mathbf{A}} \\ &= (1 - \varepsilon k(s)t)\frac{\boldsymbol{\gamma}'(\varepsilon s) \cdot \mathbf{A}(\mathbf{r}(\varepsilon s, \varepsilon t))}{\varepsilon} - \frac{1}{\varepsilon}\int_0^t \mathrm{d}\eta\, \partial_s[\mathbf{A}(\mathbf{r}(\varepsilon s, \varepsilon\eta)) \cdot \boldsymbol{\nu}(s)] \\ &\quad - \frac{\boldsymbol{\gamma}'(\varepsilon s) \cdot \mathbf{A}(\mathbf{r}(\varepsilon s, 0))}{\varepsilon} + \varepsilon\delta_\varepsilon \end{aligned} \tag{3.31}$$

[1] We denote by $\lfloor \cdot \rfloor$ the integer part. Note the missing factors 2π in the definitions [CR14, Eq. (4.8)] and [CR16a, Eq. (5.4)].

The change of coordinates $\mathbf{r} \to (\varepsilon s, \varepsilon t)$ in $\tilde{\mathcal{A}}_{\mathrm{cut}}$ and the simultaneous gauge transformation then yields for any $\Psi \in H^1(\mathcal{A}_\varepsilon)$

$$\mathcal{G}^{GL}_{\varepsilon,\mathcal{A}_{\mathrm{cut}}}[\Psi, \mathbf{A}] = (1 + \mathcal{O}(\varepsilon|\log\varepsilon|)) \int_{\mathcal{A}_{\mathrm{cut}}} \mathrm{d}s\mathrm{d}t \left\{ |\partial_t \psi|^2 + |(\partial_s + i a_{\mathbf{A}}(s,t))\psi|^2 - \tfrac{1}{2b}[2|\psi|^2 - |\psi|^4] \right\} + \frac{1}{\varepsilon^4} \int_{\mathbb{R}^2} \mathrm{d}\mathbf{r} \; |\mathrm{curl}\mathbf{A} - 1|^2 ,$$

where the prefactor $1 + \mathcal{O}(\varepsilon|\log\varepsilon|)$ is due to an estimate of the jacobian $1 - \varepsilon k(s)t$ of the change of coordinates $\mathbf{r} \to (s,t)$ induced by the diffeomorphism $\mathbf{r}(\varepsilon s, \varepsilon t)$. We recall that we denote by $k(s)$ the curvature of the boundary in the rescaled coordinates, of course $k(s)$ is not defined at corners but admits left and right values and

$$\|k\|_{L^\infty(\partial\Omega)} \le C. \tag{3.32}$$

Remark 3.3. Note that we have left untouched the last term of the GL functional, because it will be treated in different ways in the upper and lower bound proofs.

Lemma 3.1.
Let $\mathbf{A}$ be any solution of the GL equations (2.4), then for any $c_0 > 0$,

$$\|a_{\mathbf{A}}(s,t) + t\|^2_{L^2(\mathcal{A}_{\mathrm{cut}})} = \mathcal{O}(\varepsilon|\log\varepsilon|^5). \tag{3.33}$$

Proof. Let $(s,t) \in \mathcal{A}_{\mathrm{cut}}$, we first observe that

$$a_{\mathbf{A}}(s,0) = \varepsilon\delta_\varepsilon = \mathcal{O}(\varepsilon), \tag{3.34}$$

since $|\delta_\varepsilon| \le 1$. The definition of $a_{\mathbf{A}}$ and the vanishing of the normal component of the magnetic potential also implies that

$$\partial_t a_{\mathbf{A}}(s,t) = -(1 - \varepsilon k(s)t)(\mathrm{curl}\mathbf{A})(\mathbf{r}(\varepsilon s, \varepsilon t)). \tag{3.35}$$

In fact

$$\partial_t a_{\mathbf{A}}(s,t) = -(\varepsilon k(s)) \frac{\boldsymbol{\gamma}'(\varepsilon s) \cdot \mathbf{A}(\mathbf{r}(\varepsilon s, \varepsilon t))}{\varepsilon} + (1 - \varepsilon k(s)t) \frac{\boldsymbol{\gamma}'(\varepsilon s) \cdot \partial_t \mathbf{A}(\mathbf{r}(\varepsilon s, \varepsilon t))}{\varepsilon} - \frac{1}{\varepsilon}\partial_s[\mathbf{A}(\mathbf{r}(\varepsilon s, \varepsilon t)) \cdot \boldsymbol{\nu}(\varepsilon s)].$$

We now observe that

$$\boldsymbol{\nu}'(\varepsilon s) = -k(s)\boldsymbol{\gamma}'(\varepsilon s),$$

in fact under our hypothesis $\boldsymbol{\nu} = (-\gamma'_y, \gamma'_x)$. We then get

$$\partial_t a_{\mathbf{A}}(s,t) = (1 - \varepsilon k(s)t) \frac{\boldsymbol{\gamma}'(\varepsilon s) \cdot \partial_t \mathbf{A}(\mathbf{r}(\varepsilon s, \varepsilon t))}{\varepsilon} - \frac{\partial_s \mathbf{A}(\mathbf{r}(\varepsilon s, \varepsilon t)) \cdot \boldsymbol{\nu}(\varepsilon s)}{\varepsilon}.$$

We now analyze separately the two terms in the right hand side above. For the second term it easily follows that

$$\partial_s \mathbf{A}(\mathbf{r}(\varepsilon s, \varepsilon t)) \cdot \boldsymbol{\nu}(\varepsilon s) = \boldsymbol{\gamma}'(\varepsilon s) \cdot \partial_s(A_y(\mathbf{r}(\varepsilon s, \varepsilon t)), -A_x(\mathbf{r}(\varepsilon s, \varepsilon t))).$$

Then

$$\begin{aligned}
&\boldsymbol{\gamma}'(\varepsilon s)\cdot\partial_s(A_y(\mathbf{r}(\varepsilon s,\varepsilon t)),-A_x(\mathbf{r}(\varepsilon s,\varepsilon t)))\\
&\quad=\varepsilon(\gamma_x'(\varepsilon s)+\tau\nu_x'(\varepsilon s))\Big[\gamma_x'(\varepsilon s)\frac{\partial}{\partial x}A_y(\mathbf{r}(\varepsilon s,\varepsilon t))-\gamma_y'(\varepsilon s)\frac{\partial}{\partial x}A_x(\mathbf{r}(\varepsilon s,\varepsilon t))\Big]\\
&\qquad+\varepsilon(\gamma_y'(\varepsilon s)+\tau\nu_y'(\varepsilon s))\Big[\gamma_x'(\varepsilon s)\frac{\partial}{\partial y}A_y(\mathbf{r}(\varepsilon s,\varepsilon t))-\gamma_y'(\varepsilon s)\frac{\partial}{\partial y}A_x(\mathbf{r}(\varepsilon s,\varepsilon t))\Big],
\end{aligned}$$

where we used the fact that $\partial_s x=\varepsilon(\gamma_x'+\tau\nu_x')$. It then follows that

$$\begin{aligned}
\partial_s\mathbf{A}(\mathbf{r}(\varepsilon s,\varepsilon t))\cdot\boldsymbol{\nu}(s)&=\varepsilon(1-\varepsilon k(s)t)\Big((\gamma_x')^2\frac{\partial}{\partial x}A_y(\mathbf{r}(\varepsilon s,\varepsilon t))\\
&\quad-(\gamma_y')^2\frac{\partial}{\partial y}A_x(\mathbf{r}(\varepsilon s,\varepsilon t))-2\gamma_x'\gamma_y'\frac{\partial}{\partial x}A_x(\mathbf{r}(\varepsilon s,\varepsilon t))\Big),
\end{aligned}$$

where we observed that $\boldsymbol{\gamma}'(\varepsilon s)+\tau\boldsymbol{\nu}'(\varepsilon s)=(1-\varepsilon k(s)t)\boldsymbol{\gamma}'(\varepsilon s)$ and that $\nabla\cdot\mathbf{A}=0$. We now consider the other term. i.e. $\boldsymbol{\gamma}'(\varepsilon s)\cdot\partial_t\mathbf{A}(\mathbf{r}(\varepsilon s,\varepsilon t))$. It follows that

$$\begin{aligned}
\boldsymbol{\gamma}'(\varepsilon s)\cdot\partial_t\mathbf{A}(\mathbf{r}(\varepsilon s,\varepsilon t))&=\varepsilon\gamma_x'\Big(\nu_x\frac{\partial}{\partial x}A_x(\mathbf{r}(\varepsilon s,\varepsilon t))+\nu_y\frac{\partial}{\partial y}A_x(\mathbf{r}(\varepsilon s,\varepsilon t))\Big)\\
&\quad+\varepsilon\gamma_y'\Big(\nu_x\frac{\partial}{\partial x}A_y(\mathbf{r}(\varepsilon s,\varepsilon t))+\nu_y\frac{\partial}{\partial y}A_y(\mathbf{r}(\varepsilon s,\varepsilon t))\Big)\\
=\varepsilon\Big[(\gamma_x')^2\frac{\partial}{\partial y}A_x(\mathbf{r}(\varepsilon s,\varepsilon t))&-(\gamma_y')^2\frac{\partial}{\partial x}A_y(\mathbf{r}(\varepsilon s,\varepsilon t))-2\gamma_x'\gamma_y'\frac{\partial}{\partial x}A_x(\mathbf{r}(\varepsilon s,\varepsilon t))\Big].
\end{aligned}$$

Finally we get

$$\begin{aligned}
\partial_t a_{\mathbf{A}}(s,t)&=(1-\varepsilon k(s)t)\frac{\boldsymbol{\gamma}'(\varepsilon s)\cdot\partial_t\mathbf{A}(\mathbf{r}(\varepsilon s,\varepsilon t))}{\varepsilon}-\frac{\partial_s\mathbf{A}(\mathbf{r}(\varepsilon s,\varepsilon t))\cdot\boldsymbol{\nu}(\varepsilon s)}{\varepsilon}\\
&=(1-\varepsilon k(s)t)\Big[|\boldsymbol{\gamma}'(\varepsilon s)|^2\Big(\frac{\partial}{\partial y}A_x-\frac{\partial}{\partial x}A_y\Big)(\mathbf{r}(\varepsilon s,\varepsilon t))\Big]\\
&=-(1-\varepsilon k(s)t)\mathrm{curl}\mathbf{A}(\mathbf{r}(\varepsilon s,\varepsilon t)),
\end{aligned}$$

where we used the fact that $|\boldsymbol{\gamma}'|=1$. From (3.34) and (3.35), we conclude that

$$\begin{aligned}
a_{\mathbf{A}}(s,t)&=\varepsilon\delta_\varepsilon-(1+\mathcal{O}(\varepsilon t)\int_0^t\mathrm{d}\eta\,\mathrm{curl}\mathbf{A}(\mathbf{r}(\varepsilon s,\varepsilon\eta))\\
&=-t-\int_0^t\mathrm{d}\eta\ [\mathrm{curl}\mathbf{A}(\mathbf{r}(\varepsilon s,\varepsilon\eta))-1]+\mathcal{O}(\varepsilon t^2).
\end{aligned}\tag{3.36}$$

Then

$$|a_{\mathbf{A}}+t|^2\le 2\Big(\int_0^t\mathrm{d}t\,|\mathrm{curl}\mathbf{A}-1|\Big)^2+\mathcal{O}(\varepsilon^2t^4).$$

We can also estimate

$$\int_0^t\mathrm{d}\eta\ |\mathrm{curl}\mathbf{A}(\mathbf{r}(\varepsilon s,\varepsilon\eta))-1|\le C|\log\varepsilon|^{\frac12}\Big[\int_0^{c_0|\log\varepsilon|}\mathrm{d}t\ |\mathrm{curl}\mathbf{A}-1|^2\Big]^{1/2},$$

which yields

$$\|a_{\mathbf{A}}+t\|^2_{L^2(\mathcal{A}_{\mathrm{cut}})}\le C|\log\varepsilon|^2\,\|\mathrm{curl}\mathbf{A}-1\|^2_{L^2(\mathcal{A}_{\mathrm{cut}})}+\mathcal{O}(\varepsilon|\log\varepsilon|^5),$$

and therefore the result, via $\|\mathrm{curl}\mathbf{A}-1\|_{L^2(\mathbb{R}^2)}=\mathcal{O}(\varepsilon^{\frac74})$. □

3.4.3 Lower bound and completion of the proof

First of all we recall that thanks to Agmon estimate (Section 2.6) we have that

$$E_\varepsilon^{\mathrm{GL}} = \mathcal{G}_{\varepsilon,\mathcal{A}_\varepsilon}^{\mathrm{GL}}[\psi^{\mathrm{GL}}, \mathbf{A}^{\mathrm{GL}}] + \mathcal{O}(\varepsilon^\infty) \tag{3.37}$$

The next step towards a proof of a suitable lower bound is the control of the energy contributions of corners. This is however rather easy to obtain since

$$\mathcal{G}_{\varepsilon,\mathcal{A}_\varepsilon}^{\mathrm{GL}}[\psi^{\mathrm{GL}}, \mathbf{A}^{\mathrm{GL}}] \geq \mathcal{G}_{\varepsilon,\mathcal{A}_{\mathrm{cut}}}^{\mathrm{GL}}[\psi^{\mathrm{GL}}, \mathbf{A}^{\mathrm{GL}}] + \mathcal{O}(|\log\varepsilon|^2) \geq \mathcal{F}_\varepsilon^{\mathrm{GL}}[\psi^{\mathrm{GL}}, \mathbf{A}^{\mathrm{GL}}] + \mathcal{O}(|\log\varepsilon|^2), \tag{3.38}$$

where $\mathcal{A}_{\mathrm{cut}}$ is given in (3.6),

$$\mathcal{F}_\varepsilon^{\mathrm{GL}}[\psi, \mathbf{A}] := \int_{\mathcal{A}_{\mathrm{cut}}} \mathrm{d}\mathbf{r} \, \left\{ \left| \left(\nabla + i\tfrac{\mathbf{A}}{\varepsilon^2} \right) \psi \right|^2 - \tfrac{1}{2b\varepsilon^2}(2|\psi|^2 - |\psi|^4) \right\} \tag{3.39}$$

and the remainder is produced by the only non-positive term of the GL functional i.e.,

$$-\frac{1}{b\varepsilon^2}\int_{\mathcal{A}_\varepsilon} \mathrm{d}\mathbf{r}\, |\psi^{\mathrm{GL}}|^2 \geq -\frac{1}{b\varepsilon^2}\int_{\mathcal{A}_{\mathrm{cut}}} \mathrm{d}\mathbf{r}\, |\psi^{\mathrm{GL}}|^2 - C|\log\varepsilon|^2,$$

by (2.7) and the area estimate $|\mathcal{C}_j| = \mathcal{O}(\varepsilon^2|\log\varepsilon|^2)$.

The main result concerning the energy lower bound is the following

Proposition 3.2 (Energy lower bound)**.**
If $1 < b < \Theta_0^{-1}$ *as* $\varepsilon \to 0$ *then*

$$E_\varepsilon^{\mathrm{GL}} \geq \frac{|\partial\Omega| E_0^{1D}}{\varepsilon} + \mathcal{O}(|\log\varepsilon|^2). \tag{3.40}$$

The core of the proof is the same argument used in the proof of [CR14, Proposition 4.2], but in order to get to the spot where one can apply the estimate of the cost function, few adjustments are in order. First of all the functional $\mathcal{F}_\varepsilon^{\mathrm{GL}}$ is given on the right domain $\mathcal{A}_{\mathrm{cut}}$, where we can pass to tubular coordinates and replace the vector potential $\mathbf{A}^{\mathrm{GL}}$, but because $\mathcal{A}_{\mathrm{cut}}$ is made of several connected components, we need to suitably modify ψ^{GL} and impose its vanishing at the normal and inner boundaries of those sets. The reason of this will become clear only at a later stage of the proof: thanks to so-imposed Dirichlet boundary conditions, several unwanted boundary terms will vanish when integrating by parts the current term in the functional.

We sum up this preliminary steps in the following

Lemma 3.2.
As $\varepsilon \to 0$

$$\mathcal{F}_\varepsilon^{\mathrm{GL}}[\psi^{\mathrm{GL}}, \mathbf{A}^{\mathrm{GL}}] \geq \mathcal{F}[\psi] + \mathcal{O}(|\log\varepsilon|^2), \tag{3.41}$$

where

$$\mathcal{F}[\psi] := \int_{\mathcal{A}_{\mathrm{cut}}} \mathrm{d}s\mathrm{d}t \, \left\{ |\partial_t\psi|^2 + |(\partial_s - it)\,\psi|^2 - \tfrac{1}{2b}[2|\psi|^2 - |\psi|^4] \right\}, \tag{3.42}$$

and, denoting $\bar{\mathcal{A}}_\varepsilon := \{(s,t) \in \mathcal{A}_\varepsilon \mid t \leq c_0|\log\varepsilon| - \varepsilon\}$,

$$\psi(s,t) := \begin{cases} \psi^{\mathrm{GL}}(\mathbf{r}(\varepsilon s, \varepsilon t)) \, \exp\{-i\phi_{\mathbf{A}^{\mathrm{GL}}}(s,t)\}, & \text{in } \bar{\mathcal{A}}_\varepsilon \setminus \cup_j \mathcal{D}_{\varepsilon,j}, \\ 0, & \text{for } s = s_j \pm c_1|\log\varepsilon|, \\ 0, & \text{for } t = c_0|\log\varepsilon|, \end{cases} \tag{3.43}$$

and $|\psi| \leq |\psi^{\mathrm{GL}}|$ *everywhere.*

Proof. We first pass to boundary coordinates and simultaneously replace the magnetic potential $\mathbf{A}^{\mathrm{GL}}$ as described above: this leads to the lower bound

$$\begin{aligned}\mathcal{F}_{\varepsilon}^{\mathrm{GL}}[\psi^{\mathrm{GL}}, \mathbf{A}^{\mathrm{GL}}] \geq \int_{\mathcal{A}_{\mathrm{cut}}} \mathrm{d}s\mathrm{d}t\ \Big\{ \big|\partial_t \tilde{\psi}\big|^2 + \big| \left(\partial_s + i a_{\mathbf{A}^{\mathrm{GL}}}(s,t)\right) \tilde{\psi} \big|^2 \\ - \tfrac{1}{2b} \Big[2|\tilde{\psi}|^2 - |\tilde{\psi}|^4 \Big] \Big\} + \mathcal{O}(|\log\varepsilon|^2),\end{aligned} \tag{3.44}$$

where $\tilde{\psi}(s,t) = \psi^{\mathrm{GL}}(\mathbf{r}(\varepsilon s, \varepsilon t)) \exp\{-i\phi_{\mathbf{A}^{\mathrm{GL}}}(s,t)\}$ and $a_{\mathbf{A}^{\mathrm{GL}}}$ is given in (3.31). The remainder $\mathcal{O}(|\log\varepsilon|)$ is the product of the prefactor $\mathcal{O}(\varepsilon|\log\varepsilon|)$ due to the jacobian of the coordinate transformation times the negative term proportional to the L^2 norm of $\tilde{\psi}$.

Next acting as in [CR14, Eq. (4.26)] we can estimate for any $\delta > 0$, via Lemma 3.1 and (2.8) and the exponential decay of ψ (Section 2.6)

$$\begin{aligned}&\int_{\mathcal{A}_{\mathrm{cut}}} \mathrm{d}s\mathrm{d}t\ \Big[\big| \left(\partial_s + i a_{\mathbf{A}^{\mathrm{GL}}}(s,t)\right) \tilde{\psi} \big|^2 - \big| \left(\partial_s - it\right) \tilde{\psi} \big|^2 \Big] \\ &= -2\Im \int_{\mathcal{A}_{\mathrm{cut}}} \mathrm{d}s\mathrm{d}t \left[(\partial_s + i a_{\mathbf{A}^{\mathrm{GL}}})\psi \right]^* (a_{\mathbf{A}^{\mathrm{GL}}} + t)\, \psi - \int_{\mathcal{A}_{\mathrm{cut}}} \mathrm{d}s\mathrm{d}t\, |a_{\mathbf{A}^{\mathrm{GL}}} + t|^2 |\psi|^2 \\ &\geq -\delta \int_{\mathcal{A}_{\mathrm{cut}}} |(\partial_s + i a_{\mathbf{A}^{\mathrm{GL}}})\psi|^2 - \left(\frac{1}{\delta} + 1 \right) \int_{\mathcal{A}_{\mathrm{cut}}} \mathrm{d}s\mathrm{d}t\, |a_{\mathbf{A}^{\mathrm{GL}}} + t|^2 |\psi|^2 \\ &\geq -\delta \left\| \left(\nabla + i \tfrac{\mathbf{A}^{\mathrm{GL}}}{\varepsilon^2} \right) \psi^{\mathrm{GL}} \right\|^2_{L^2(\Omega)} - \left(\tfrac{1}{\delta} + 1 \right) \left(|\log\varepsilon|^2 \left\| \mathrm{curl}\mathbf{A} - 1 \right\|^2_{L^2} + C\varepsilon^2 \int_{\mathcal{A}_{\mathrm{cut}}} \mathrm{d}s\mathrm{d}t\, t^4 |\psi|^2 \right) \\ &\geq -C \left[\delta\varepsilon^{-1} + C\delta^{-1}\varepsilon \right] \geq -C,\end{aligned} \tag{3.45}$$

after an optimization over δ. Hence we get from (3.44) and (3.45)

$$\mathcal{F}_{\varepsilon}^{\mathrm{GL}}[\psi^{\mathrm{GL}}, \mathbf{A}^{\mathrm{GL}}] \geq \mathcal{F}[\tilde{\psi}] + \mathcal{O}(|\log\varepsilon|^2). \tag{3.46}$$

To impose the boundary conditions at the normal and inner boundaries of $\mathcal{A}_{\mathrm{cut}}$, we use two different partition of unity, i.e., two pairs of smooth functions $0 \leq \chi_i, \eta_i \leq 1$, $i = 1, 2$, such that $\chi_i^2 + \eta_i^2 = 1$ and

$$\chi_1 = \chi_1(s) = \begin{cases} 1, & \text{in } \mathcal{A}_{\varepsilon} \setminus \cup_j \Gamma_j, \\ 0, & \text{in } \cup_j \Gamma_j, \end{cases} \tag{3.47}$$

$$\chi_2 = \chi_2(t) = \begin{cases} 1, & \text{for } t \in [0, c_0|\log\varepsilon| - \varepsilon], \\ 0, & \text{for } t = c_0|\log\varepsilon|. \end{cases} \tag{3.48}$$

Given the size where χ_i, η_i are not constant, we can assume the following estimates to hold true

$$|\nabla\chi_1| = \mathcal{O}(|\log\varepsilon|^{-1}), \qquad |\nabla\eta_1| = \mathcal{O}(|\log\varepsilon|^{-1}), \tag{3.49}$$

$$|\nabla\chi_2| = \mathcal{O}(\varepsilon^{-1}), \qquad |\nabla\eta_2| = \mathcal{O}(\varepsilon^{-1}). \tag{3.50}$$

The IMS formula then yields

$$\begin{aligned}\mathcal{F}[\tilde{\psi}] \geq \mathcal{F}[\chi_1\chi_2\tilde{\psi}] - \int_{\mathcal{A}_{\varepsilon}} \mathrm{d}s\mathrm{d}t\ \left[{\chi_1'}^2 + {\eta_1'}^2 \right] |\tilde{\psi}|^2 \\ - \int_{\mathcal{A}_{\varepsilon}} \mathrm{d}s\mathrm{d}t\ \left[{\chi_2'}^2 + {\eta_2'}^2 \right] |\tilde{\psi}|^2 + \mathcal{O}(|\log\varepsilon|^2),\end{aligned} \tag{3.51}$$

where we have estimated

$$\mathcal{F}[\eta_1\chi_2\tilde{\psi}] + \mathcal{F}[\eta_2\chi_1\tilde{\psi}] + \mathcal{F}[\eta_1\eta_2\tilde{\psi}] \geq -\frac{1}{b}\int_{\mathcal{A}_\varepsilon} \mathrm{d}s\mathrm{d}t \, \left[\eta_1^2 + \eta_2^2 + \eta_1^2\eta_2^2\right] |\tilde{\psi}|^2 \geq -C|\log\varepsilon|^2.$$

Using (3.49) it is easy to show that the second term in (3.51) can be absorbed in the remainder, in fact

$$\int_{\mathcal{A}_\varepsilon} \mathrm{d}s\mathrm{d}t \, \left[{\chi_1'}^2 + {\eta_1'}^2\right] |\tilde{\psi}|^2 = \int_{c_1|\log\varepsilon|}^{2c_1|\log\varepsilon|}\int_0^{c_0|\log\varepsilon|} \mathrm{d}s\mathrm{d}t \, \left[{\chi_1'}^2 + {\eta_1'}^2\right] |\tilde{\psi}|^2 = \mathcal{O}(1).$$

While, thanks to Agmon estimates,

$$\int_{c_0|\log\varepsilon|-\varepsilon}^{\bar{t}} \mathrm{d}t \int_0^{\frac{|\partial\Omega|}{\varepsilon}} \mathrm{d}s \, |\tilde{\psi}|^2 \leq \int_{\mathrm{dist}(\mathbf{r},\partial\Omega)\geq c_0\varepsilon|\log\varepsilon|-\varepsilon^2} \mathrm{d}\mathbf{r} \, \left|\psi^{\mathrm{GL}}\right|^2 = \mathcal{O}(\varepsilon^{c_0 C_A+1}),$$

i.e., $\tilde{\psi}$ is still smaller than any power of ε in the support of χ_2' and η_2', which implies that the third term in (3.51) can be discarded as well.

In conclusion we obtained

$$\mathcal{F}[\tilde{\psi}] \geq \mathcal{F}[\chi_1\chi_2\tilde{\psi}] + \mathcal{O}(|\log\varepsilon|^2), \tag{3.52}$$

and, setting $\psi := \chi_1\chi_2\tilde{\psi}$, the claim is proven. □

The rest of the lower bound proof is very close to the proof of [CR14, Proposition 4.2]. We sum up the main steps below.

Proof of Proposition 3.2. Combining (3.37) with (3.38) and the result of Lemma 3.2, we have

$$E_\varepsilon^{\mathrm{GL}} \geq \mathcal{F}[\psi] + \mathcal{O}(|\log\varepsilon|^2). \tag{3.53}$$

The next step is thus a lower bound to $\mathcal{F}[\psi]$. First of all we extract from $\mathcal{F}[\psi]$ the desired leading term in the energy asymptotics: by a standard splitting trick, we set

$$\psi(s,t) =: f_0(t)u(s,t)e^{-i\alpha_0 s}, \tag{3.54}$$

which defines a suitable $u \in H^1_{\mathrm{loc}}(\mathcal{A}_{\mathrm{cut}})$. Note that, since α_0 is in general not an integer, u is not periodic and therefore a multi-valued function, but $|u|$ is periodic and this will suffice. Plugging the above ansatz in the functional $\mathcal{F}$, we get

$$\mathcal{F}[\psi] = \frac{|\partial\Omega_{\mathrm{cut}}|E_0^{1D}}{\varepsilon} + \mathcal{E}[u], \tag{3.55}$$

where

$$\mathcal{E}[u] := \int_{\mathcal{A}_{\mathrm{cut}}} \mathrm{d}s\mathrm{d}t \, f_0^2 \left\{ |\partial_t u|^2 + |\partial_s u|^2 - 2(t+\alpha_0)\mathbf{e}_s\cdot\mathbf{j}[u] + \tfrac{1}{2b}f_0^2\left(1-|u|^2\right)^2 \right\}, \tag{3.56}$$

and the superconducting current is given by

$$\mathbf{j}[u] := \tfrac{i}{2}\left(u\nabla u^* - u^*\nabla u\right). \tag{3.57}$$

To prove (3.55) we observe that

$$\begin{aligned}\int_{\mathcal{A}_{\mathrm{cut}}} \mathrm{d}s\mathrm{d}t\, |\partial_t \psi|^2 &= \int_{\mathcal{A}_{\mathrm{cut}}} \mathrm{d}s\mathrm{d}t\, \{f_0^2|\partial_t u|^2 + f_0\partial_t f_0 \partial_t |u|^2 + |u|^2|\partial_t f_0|^2\} \\ &= \int_{\mathcal{A}_{\mathrm{cut}}} \mathrm{d}s\mathrm{d}t\, \{f_0^2|\partial_t u|^2 - f_0|u|^2\partial_t^2 f_0\},\end{aligned} \tag{3.58}$$

where we have integrated by parts in t the term $\int_{\mathcal{A}_{\mathrm{cut}}} \mathrm{d}s\mathrm{d}t\, f_0\partial_t f_0\partial_t|u|^2$ and we have used the fact that the boundary terms vanish because of Neumann conditions satisfied by f_0 at $t = 0$ and Dirichlet conditions satisfied by u at $t = c_0|\log\varepsilon|$ (inherited from the bound satisfied by ψ in (3.43)). We also observe that

$$\int_{\mathcal{A}_{\mathrm{cut}}} \mathrm{d}s\mathrm{d}t\, |(\partial_s - it)\psi|^2 = \int_{\mathcal{A}_{\mathrm{cut}}} \mathrm{d}s\mathrm{d}t\, \{f_0^2|\partial_s u|^2 + (t+\alpha_0)^2 f_0^2 - 2i(t+\alpha_0)(iu, \partial_s u)\}.$$

Inserting the variational equation for f_0 in (3.58), we get

$$\mathcal{F}[\psi] = \frac{|\partial\Omega_{\mathrm{cut}}| E_0^{1D}}{\varepsilon} + \mathcal{E}[u].$$

Since

$$|\partial\Omega_{\mathrm{cut}}| = |\partial\Omega| + \mathcal{O}(\varepsilon|\log\varepsilon|),$$

the lower bound is proven if we can show that $\mathcal{E}[u] \geq 0$. In order to investigate the positivity of $\mathcal{E}[u]$ we use the potential function trick, i.e., we observe that the function $F_0(t)$ defined in (2.42) satisfies

$$F_0'(t) = 2(t+\alpha_0) f_0^2(t), \tag{3.59}$$

and therefore

$$-2\int_{\mathcal{A}_{\mathrm{cut}}} \mathrm{d}s\mathrm{d}t\, (t+\alpha_0) j_s(u) = -\int_{\mathcal{A}_{\mathrm{cut}}} \mathrm{d}s\mathrm{d}t\, \partial_t F_0(t)\, j_s(u) = \int_{\mathcal{A}_{\mathrm{cut}}} \mathrm{d}s\mathrm{d}t\, F_0(t)\, \partial_t j_s[u]$$

where we have denoted by $j_s[u] = \mathbf{e}_s \cdot \mathbf{j}[u]$ the $s-$component of the current. Here the boundary terms vanish because $F_0(0) = 0$ and

$$u(s, c_0|\log\varepsilon|) = 0, \qquad u(s_j \pm c_1|\log\varepsilon|, t) = 0, \tag{3.60}$$

thanks to the boundary conditions inherited from ψ and the strict positivity of f_0.

We now integrate by parts in the s variable the last two terms:

$$\begin{aligned}\int_{\mathcal{A}_{\mathrm{cut}}} \mathrm{d}s\mathrm{d}t\, F_0(t)\, \partial_t j_s[u] &= \tfrac{i}{2}\int_{\mathcal{A}_{\mathrm{cut}}} \mathrm{d}s\mathrm{d}t\, F_0(t)\, \left[\partial_t u\partial_s u^* - \partial_t u^*\partial_s u + u\partial_{s,t}^2 u^* - u^*\partial_{s,t}^2 u\right] \\ &= i\int_{\mathcal{A}_{\mathrm{cut}}} \mathrm{d}s\mathrm{d}t\, F_0(t)\, \left[\partial_t u\partial_s u^* - \partial_t u^*\partial_s u\right],\end{aligned}$$

where again boundary terms are absent thanks to the vanishing of u stated in (3.60). At this stage the non-periodicity of u could affect the result but this is not the case because $u^*\partial_t u$ and its complex conjugate are always periodic. The simple estimate

$$\begin{aligned}i\int_{\mathcal{A}_{\mathrm{cut}}} \mathrm{d}s\mathrm{d}t\, F_0(t)\, \left[\partial_t u\partial_s u^* - \partial_t u^*\partial_s u\right] &\geq -2\int_{\mathcal{A}_{\mathrm{cut}}} \mathrm{d}s\mathrm{d}t\, |F_0(t)||\partial_t u||\partial_s u| \\ &\geq \int_{\mathcal{A}_{\mathrm{cut}}} \mathrm{d}s\mathrm{d}t\, F_0(t)\, [|\partial_t u|^2 + |\partial_s u|^2],\end{aligned}$$

which uses the negativity of $F_0(t)$, leads us to the lower bound for $\mathcal{E}[u]$:

$$\mathcal{E}[u] \geq \int_{\mathcal{A}_{\mathrm{cut}}} \mathrm{d}s\mathrm{d}t \, \left\{ K_0(t) \left(|\partial_t u|^2 + |\partial_s u|^2 \right) + \frac{1}{2b} f_0^4 (1 - |u|^2)^2 \right\}. \tag{3.61}$$

The pointwise positivity (2.44) of $K_0(t)$ for $1 < b < \Theta_0^{-1}$ and the manifest positivity of the second term in the expression above yields the final lower bound

$$\mathcal{E}[u] \geq \frac{1}{2b} \int_{\mathcal{A}_{\mathrm{cut}}} \mathrm{d}s\mathrm{d}t \, f_0^4 (1 - |u|^2)^2 \geq 0. \tag{3.62}$$

□

3.4.4 Proof of Theorem 3.1.

Proof of Theorem 3.1. The combination of the energy upper (Proposition 3.1) and lower (Proposition 3.2) bounds yields the energy asymptotics (3.9). It only remains to prove the estimate on the L^2 norm of the difference $|\psi^{\mathrm{GL}}|^2 - f_0^2$. This is however trivially implied by the lower bound (3.62): if one keeps the positive term appearing on the r.h.s. of the inequality and put it together with (3.53), the splitting (3.55) and the upper bound (3.11), the outcome is

$$\int_{\mathcal{A}_{\mathrm{cut}}} \mathrm{d}s\mathrm{d}t \, f_0^4 (1 - |u|^2)^2 = \mathcal{O}(|\log \varepsilon|^2). \tag{3.63}$$

We now recall that $\psi(s,t) = \chi_1(s)\chi_2(t)\psi^{\mathrm{GL}}(\mathbf{r}(\varepsilon s, \varepsilon t))e^{-i\phi_{\mathbf{A}^{\mathrm{GL}}}(s,t)}$, them $|\psi|$ differs from $|\psi^{\mathrm{GL}}|$ in regions of area $\mathcal{O}(|\log \varepsilon|^2)$ in the rescaled variables, then these regions can be discarded and their contribution be included in the remainder. The same holds true for the corner cells and therefore the final result is (3.10). Note the factor ε^2 appearing on the r.h.s. due to the rescaling $(s,t) \to (\sigma, \tau) = (\varepsilon s, \varepsilon t)$. □

Chapter 4

Corner Effective Problems

This Chapter introduces some new effective problems which we will prove to be useful to derive the first order correction to the energy asymptotics in a general domain with corners. To this purpose, we will work in a different way in the smooth part of the boundary layer and near the singularities: then two different effective problems will appear. To define the corner effective problem, we impose some Dirichlet boundary conditions in a suitable region near each corner. These conditions are motivated from the fact that in the smooth part of the boundary, the minimizing order parameter is such that $\psi^{\mathrm{GL}} \simeq f_0(t)e^{-i\alpha_0 s}$.

In what follows we prove that the new effective problem near the corner is well defined. Since the Dirichlet boundary conditions do not appear naturally in the proof of a lower bound for $E_\varepsilon^{\mathrm{GL}}$, we also compare the ground state energy of the effective problem for the corner region with another variational problem that differs from the effective one by the presence of some suitable boundary terms that will appear naturally in our analysis.

We now stress that since we work with functions in H^1 (thus defined almost everywhere), all the boundary conditions have to be considered in trace sense. This poses no problem with the presence of singularities along the boundary, in fact the trace Theorem is still true in presence of corners. In particular, if we denote by γ_n the operator

$$(\gamma_n u)(x_1, \cdots, x_n) := u(x_1, \cdots, x_{n-1}, 0)$$

for each smooth function u, then it holds [Gri11, Theorem 1.5.1.3]

Theorem 4.1 (Trace).
Let $\Omega \subset \mathbb{R}^n$ be a bounded and open domain with Lipschitz boundary Γ. Then the mapping $u \to \gamma u$ which is defined for each $u \in C^{0,1}(\overline{\Omega})$ has a unique continuous extension as an operator from $W^{1,p}(\Omega)$ to $W_p^{1-\frac{1}{p}}(\Gamma)$.

Before proceeding further, we also recall that, for any minimizing vector potential of the GL functional $\mathbf{A}^{\mathrm{GL}}$, it holds that (see (2.4) and (2.9))

$$\mathrm{curl}\mathbf{A}^{\mathrm{GL}} = 1, \text{ on } \partial\Omega, \qquad \|\mathrm{curl}\mathbf{A}^{\mathrm{GL}} - 1\|_{L^2(\mathbb{R}^2)} = \mathcal{O}(\varepsilon^{7/2}). \tag{4.1}$$

Then by [FH10, Lemma F.1.1] one can prove that there exists a gauge phase ϕ^{GL} defined in (3.28) such that suitably far from the singularities one have

$$\frac{1}{\varepsilon}\tilde{\mathbf{A}}(s,t) = \frac{\mathbf{A}^{\mathrm{GL}}}{\varepsilon}(\mathbf{r}(\varepsilon s, \varepsilon t)) - \nabla\phi_{\mathbf{A}^{\mathrm{GL}}}(s,t) = (-t(1 + \mathcal{O}(\varepsilon|\log\varepsilon|)), 0)$$

For this reason we will study an effective problem in which the vector potential is fixed and equals to $\mathbf{F} := (-t, 0)$ in the region where the boundary coordinates are well defined.

The following Chapter is divided into two main parts:

- In the first part, we consider the two effective problems mentioned above in a rectangular region and we will prove that the two ground states energies are equal up to a small (w.r.t. the side length of the rectangular) remainder term. The reason to consider this kind of domain is that in a generic domain, suitably far from the singularities, the boundary layer that we have to take into account to study surface superconductivity is locally rectangle-like. In this first part we will also prove a result on the asymptotics of the minimizing order parameter;

- In the second part we introduce the corner region we want to consider and we use the results of the first part to prove that also in this domain the ground state energies related to the two variational problems are equal up to small errors.

As we mentioned above, the results of this Chapter will allow us to overcome some problems related to the Dirichlet boundary conditions that will emerge in the analysis of the energy asymptotics for a general domain with corners.

4.1 Effective Problems in a Rectangular Domain

In this Section we consider a rectangular R of side lengths ℓ and $c\ell$, where $c > 0$ is a suitable constant, and $\ell \to +\infty$. We parametrize R as follows:

$$R := \Big\{ (s,t) \,|\, s \in [0,\ell],\, t \in [0, c\ell] \Big\}. \tag{4.2}$$

As we explained above, we will consider different minimization domains: the idea is to prove that the related ground state energies are equal up to a small remainder. We will study the ground state energy of the following functional

$$\mathcal{G}_\ell[\psi] := \int_0^\ell \mathrm{d}s \int_0^{c\ell} \mathrm{d}t \left\{ |\partial_t \psi|^2 + |(\partial_s - it)\psi|^2 - \frac{1}{2b}(2|\psi|^2 - |\psi|^4) \right\}.$$

The functional above is in fact approximately the one that naturally emerge in a region in which boundary coordinates are well defined in the regime of surface superconductivity regime (see Section 2.7).

Before proceeding further, we observe that the variational equation related to $\mathcal{G}_\ell$ is

$$-\partial_t^2 \psi + (t - i\partial_s)^2 \psi = \frac{1}{b}(1 - |\psi|^2)\psi.$$

Then, one can prove that any $\psi \in H^1(R)$ satisfying the variational equation above is exponentially decaying in the $t-$variable. More precisely, it is possible to prove the following Agmon estimate

$$\int_R \mathrm{d}s\mathrm{d}t\; e^{C_A\, t} \left\{ |\psi|^2 + |(\nabla + i(-t,0))\,\psi|^2 \right\} \; \leq \int_0^\ell \mathrm{d}s \int_0^1 \mathrm{d}t\; |\psi|^2 = \mathcal{O}(\ell).$$

Hence, we have

$$\int_0^{\ell} \mathrm{d}s \int_{\bar{\ell}}^{c\ell} \mathrm{d}t\, |\psi|^2 = \mathcal{O}(\ell^{-\infty}), \qquad \int_0^{\ell} \mathrm{d}s \int_{\bar{\ell}}^{c\ell} \mathrm{d}t\, |(\nabla + i(-t,0))\psi|^2 = \mathcal{O}(\ell^{-\infty}) \tag{4.3}$$

for $\bar{\ell} = C\ell + o(1)$ and $C > 0$ suitably large. For the sake of brevity we do not prove these estimates (for further details see [FH10, Chapter 12] or [CR14]). The strategy of the proofs contained in this Section is very similar to the one in [CR14].

4.1.1 Energy asymptotics in a rectangular region with Dirichlet boundary conditions

We now consider the following minimization domain for $\mathcal{G}_\ell$

$$\mathscr{D}_{\mathcal{D}}(R) := \{\psi(s,t) \in H^1(R) \,|\, \psi(s,t) = g(t)e^{-iS(s)}, \text{ for } s = 0, \ell \text{ and } \forall t \in [0, c\ell], \\ \psi(s, c\ell) = 0 \ \ \forall s \in [0,\ell]\}, \tag{4.4}$$

where $S(s)$ is the phase defined as follows

$$S(s) := (\alpha_0 - \delta_\ell)s, \tag{4.5}$$

with $\delta_\ell = o(1)$. The boundary conditions are given in terms of

$$g(t) := \begin{cases} f_0(t), & \text{if } \ t \in [0, \bar{\ell}], \\ \bar{f}_0(t), & \text{if } \ t \in [\bar{\ell}, c\ell], \end{cases} \tag{4.6}$$

where $\bar{\ell} := c\ell(1-\gamma)$ and $\gamma \ll 1$. We pick a function $\bar{f}_0(t)$ such that: $\bar{f}_0(t)$ is monotone, $\bar{f}_0(\bar{\ell}) = f_0(\bar{\ell})$ and $\bar{f}_0(c\ell) = 0$. We can also assume that

$$\|\bar{f}_0(t)\|_{L^\infty[\bar{\ell},c\ell]} \le f_0(\bar{\ell}), \qquad \|\bar{f}_0'(t)\|_{L^\infty[\bar{\ell},c\ell]} = \mathcal{O}(f_0(\bar{\ell})(\ell\gamma)^{-1}).$$

Before proceeding further we justify the choice of the boundary conditions in (4.4). First of all, we know by (2.29) that in the smooth part of the boundary the GL minimizing order parameter is approximately $\psi^{\mathrm{GL}} \approx f_0(t)e^{-i\alpha_0 s}$. Hence, we choose $g(t) \equiv f_0(t)$ in the most part of the region and $S(s)$ equal to $\alpha_0 s$ up to a small remainder. We also know that ψ^{GL} is exponentially small suitably far from the boundary. For this reason we impose vanishing boundary conditions at $t = c\ell$. Finally, the factor $\delta_\ell s$ in the phase (4.5) is the one that we will use in order to have a well defined global phase in general domains. Notice that, for the purpose of this Section, one could also drop this last term.

In this Section we want to study the asymptotics of

$$E_\ell^{\mathcal{D}}(R) := \inf_{\psi \in \mathscr{D}_{\mathcal{D}}(R)} \mathcal{G}_\ell[\psi],$$

as $\ell \to +\infty$. We also denote by $\psi_\ell^{\mathcal{D}} \in \mathscr{D}_{\mathcal{D}}(R)$ any minimizing function.

Theorem 4.2 (Asymptotics of $E_\ell^{\mathcal{D}}(R)$)**.**
Let $R \subset \mathbb{R}^2$ be the rectangular region defined in (4.2)*, then for any fixed*

$$1 < b < \Theta_0^{-1}$$

it holds, as $\ell \to +\infty$,

$$E_\ell^{\mathcal{D}}(R) = \ell E_0^{1D} + \mathcal{O}(\ell\delta_\ell^2). \tag{4.7}$$

We split the proof of Theorem 4.2 in two parts: we have to prove an upper bound and a lower bound to the ground state energy.

Proposition 4.1 (Upper bound)**.**
Let $R \subset \mathbb{R}^2$ *be the rectangular region defined in* (4.2)*, then for any fixed*

$$b < \Theta_0^{-1}$$

it holds, as $\ell \to +\infty$*,*

$$E_\ell^{\mathcal{D}}(R) \le \ell E_0^{1D} + \mathcal{O}(\ell \delta_\ell^2). \tag{4.8}$$

Proof. For the upper bound we just have to choose a suitable trial function: we set

$$\psi_{\mathrm{trial}}(s,t) = g(t) e^{-iS(s)}, \tag{4.9}$$

where $g(t)$ is defined in (4.6) and the phase $S(s)$ in (4.5). We then get

$$\mathcal{G}_\ell[\psi_{\mathrm{trial}}] = \int_0^\ell \mathrm{d}s \int_0^{\bar{\ell}} \mathrm{d}t \left\{ |f_0'(t)|^2 + (t + \alpha_0 - \delta_\ell)^2 f_0^2(t) - \frac{1}{2b}(2 f_0^2(t) - f_0^4(t)) \right\}$$
$$+ \int_0^\ell \mathrm{d}s \int_{\bar{\ell}}^{c\ell} \mathrm{d}t \left\{ |g'(t)|^2 + (t + \alpha_0 - \delta_\ell)^2 g(t)^2 - \frac{1}{2b}(2 g^2(t) - g(t)^4) \right\}.$$

Moreover, we have

$$\int_0^\ell \mathrm{d}s \int_0^{\bar{\ell}} \mathrm{d}t \, (t + \alpha_0 - \delta_\ell)^2 f_0^2(t) = \int_0^\ell \mathrm{d}s \int_0^{\bar{\ell}} \mathrm{d}t \, (t + \alpha_0)^2 f_0^2(t) + \mathcal{O}(\ell \delta_\ell^2).$$

In fact,

$$\int_0^\ell \mathrm{d}s \int_0^{\bar{\ell}} \mathrm{d}t \, (\delta_\ell)^2 f_0^2(t) = \mathcal{O}(\ell \delta_\ell^2)$$

and

$$\int_0^\ell \mathrm{d}s \int_0^{\bar{\ell}} \mathrm{d}t \, \delta_\ell (t + \alpha_0) f_0^2(t) = \ell \delta_\ell \int_0^\infty \mathrm{d}t \, (t + \alpha_0) f_0^2(t) - \ell \delta_\ell \int_{\bar{\ell}}^\infty \mathrm{d}t \, (t + \alpha_0) f_0^2(t)$$
$$= \mathcal{O}\left(\ell \delta_\ell e^{-C\ell^2} \right),$$

where we used the optimality of α_0, i.e., the fact that

$$\int_0^{+\infty} \mathrm{d}t \, f_0^2(t)(t + \alpha_0) = 0,$$

and the fact that $f_0(t)$ is exponentially decaying for large t (Lemma 2.1). We now recall that

$$\|\bar{f}_0'(t)\|_{L^\infty[\bar{\ell}, \ell]} = \mathcal{O}(f_0(\bar{\ell})(\ell\gamma)^{-1}),$$

then, via Lemma 2.1, we get

$$\int_0^\ell \mathrm{d}s \int_{\bar{\ell}}^{c\ell} \mathrm{d}t \, |g'(t)|^2 \le C \int_0^\ell \mathrm{d}s \int_{\bar{\ell}}^{c\ell} \mathrm{d}t \, \frac{f_0^2(\bar{\ell})}{(\ell\gamma)^2} = \mathcal{O}(\gamma^{-1} e^{-C\ell^2}).$$

Furthermore, from the fact that

$$|g(t)| \le C f_0(\bar{\ell}), \qquad \text{for } t \ge \bar{\ell},$$

we can bound also the other terms of the functional:

$$\int_0^{\ell} \mathrm{d}s \int_{\bar{\ell}}^{c\ell} \mathrm{d}t \left\{ (t+\alpha_0-\delta_\ell)^2 g(t)^2 \right\} = \mathcal{O}(\ell e^{-C\ell^2})$$
$$\frac{1}{2b} \int_0^{\ell} \mathrm{d}s \int_{\bar{\ell}}^{c\ell} \mathrm{d}t\, g^4(t) = \mathcal{O}(\ell e^{-C\ell^2}).$$

We then obtain

$$\mathcal{G}_\ell[\psi_{\mathrm{trial}}] = \ell \int_0^{\bar{\ell}} \mathrm{d}t \left\{ |f_0'|^2 + (t+\alpha_0)^2 f_0^2 - \frac{1}{2b}(2f_0^2 - f_0^4) \right\} + \mathcal{O}\left(\left(\ell + \gamma^{-1} \right) e^{-C\ell^2} \right) + \mathcal{O}(\ell \delta_\ell^2).$$

Using again the exponential decay of f_0 (Lemma 2.1) and the variational equation for f_0 (2.37), we also have that

$$\int_0^{\bar{\ell}} \mathrm{d}t \left\{ |f_0'|^2 + (t+\alpha_0-\delta_\ell)^2 f_0^2 - \frac{1}{2b}(2f_0^2 - f_0^4) \right\}$$
$$= \int_0^{\infty} \mathrm{d}t \left\{ |f_0'|^2 + (t+\alpha_0)^2 f_0^2 - \frac{1}{2b}(2f_0^2 - f_0^4) \right\} + \mathcal{O}\left(e^{-C\ell^2} \right) + \mathcal{O}\left(\delta_\ell^2 \right).$$

We then get

$$\mathcal{G}_\ell[\psi_{\mathrm{trial}}] \leq \ell E_0^{1D} + \mathcal{O}\left(\left(\ell + \gamma^{-1} \right) e^{-C\ell^2} \right) + \mathcal{O}(\ell \delta_\ell^2), \tag{4.10}$$

where we recall that E_0^{1D} is the minimum with respect to both f and α of the functional $\mathcal{E}_{0,\alpha}^{1D}$.

If we choose any γ of order $\mathcal{O}(1)$ and the constant $c > 0$ sufficiently large, we finally get

$$\mathcal{G}_\ell[\psi_{\mathrm{trial}}] \leq \ell E_0^{1D} + \mathcal{O}(\ell \delta_\ell^2). \tag{4.11}$$

□

Proposition 4.2 (Lower bound)**.**
Let $R \subset \mathbb{R}^2$ be the rectangular region defined in (4.2)*, then for any fixed*

$$1 < b < \Theta_0^{-1}$$

it holds, as $\ell \to +\infty$,

$$E_\ell^{\mathcal{D}}(R) \geq \ell E_0^{1D} + \mathcal{O}(\ell^{-\infty}). \tag{4.12}$$

For the proof of Proposition 4.2 we use the following Lemma:

Lemma 4.1 (Energy splitting)**.**
Let $R \subset \mathbb{R}^2$ be the rectangular region defined in (4.2)*, we define $u(s,t)$ as*

$$\psi_\ell^{\mathcal{D}}(s,t) =: f_0(t) u(s,t) e^{-i\alpha_0 s}, \tag{4.13}$$

Then for any fixed $1 < b < \Theta_0^{-1}$, as $\ell \to +\infty$ it follows that

$$E_\ell^{\mathcal{D}}(R) = \ell E_0^{1D} + \mathcal{E}_0[u] + \mathcal{O}\left(\ell e^{-C\ell^2} \right), \tag{4.14}$$

where

$$\mathcal{E}_0[u] := \int_0^{\ell} \mathrm{d}s \int_0^{c\ell} \mathrm{d}t\, f_0^2 \left\{ |\partial_s u|^2 + |\partial_t u|^2 - 2(t+\alpha_0) j_s[u] + \frac{f_0^2}{2b}(1-|u|^2)^2 \right\} \tag{4.15}$$

with $j_s[u] := (iu, \partial_s u) = \frac{1}{2}(iu\partial_s u^* - iu^* \partial_s u)$ the tangential component of the superconducting current.

Proof. We first observe that

$$\int_R \mathrm{d}s\mathrm{d}t\, |\partial_t (f_0 u(s,t) e^{-i\alpha_0 s})|^2 = \int_R \mathrm{d}s\mathrm{d}t\, \left\{ |f_0'|^2 |u|^2 + |\partial_t u|^2 f_0^2 + f_0 \partial_t f_0 \partial_t |u|^2 \right\} \tag{4.16}$$

We now integrate by parts in t the last integral in (4.16) and we get

$$\int_0^\ell \mathrm{d}s \int_0^{c\ell} \mathrm{d}t\, f_0 \partial_t f_0 \partial_t |u|^2 = -\int_0^\ell \mathrm{d}s \int_0^{c\ell} \mathrm{d}t \left\{ |\partial_t f_0|^2 |u|^2 + f_0 \partial_t^2 f_0 |u|^2 \right\} \tag{4.17}$$

where the boundary term in $t = 0$ and in $t = c\ell$ vanish because $f_0'(0) = 0$ and $u(s, c\ell) = 0$. We also have that

$$\int_R \mathrm{d}s\mathrm{d}t\, |(\partial_s - it)(f_0 u(s,t) e^{-i\alpha_0 s})|^2$$
$$= \int_R \mathrm{d}s\mathrm{d}t \left\{ f_0^2 |\partial_s u|^2 + (\alpha_0 + t)^2 f_0^2 |u|^2 + f_0^2 (\alpha_0 + t) j_s[u] \right\}.$$

To finish the proof, we use the variational equation (2.37) for f_0

$$-f_0''(t) + (t + \alpha_0)^2 f_0(t) = \frac{1}{b}(1 - f_0^2(t)) f_0(t). \tag{4.18}$$

We then get

$$\mathcal{G}_\ell[\psi] = \int_0^\ell \mathrm{d}s \int_0^{c\ell} \mathrm{d}t\, f_0^2 \left\{ |\partial_s u|^2 + |\partial_t u|^2 - 2(t + \alpha_0) j_s[u] + \frac{f_0^2}{2b}(1 - |u|^2)^2 \right\} - \frac{\ell}{2b} \int_0^{c\ell} \mathrm{d}t\, f_0^4(t).$$

Using again the variational equation for f_0 (2.37) and the fact that it is exponentially decaying (Lemma 2.1), we obtain

$$-\frac{1}{2b} \int_0^{c\ell} \mathrm{d}t\, f_0^4(t) = -\frac{1}{2b} \int_0^\infty \mathrm{d}t\, f_0^4(t) + \frac{1}{2b} \int_{c\ell}^\infty \mathrm{d}t\, f_0^4(t) = E_0^{1D} + \mathcal{O}\left(e^{-C\ell^2}\right), \tag{4.19}$$

which yields the result. □

In order to prove Proposition 4.2, we just need to show that $\mathcal{E}_0[u]$ is positive for any $u = (\psi/f_0) e^{i\alpha_0 s}$ and for any fixed $1 < b < \Theta_0^{-1}$. To show this, following [CR1], we perform an integration by parts which gives rise to some boundary terms: we fixed the boundary conditions precisely in order to make them vanish.

Proposition 4.3.
Let u be the function defined in (4.13), for any fixed $1 < b < \Theta_0^{-1}$, it follows that

$$\mathcal{E}_0[u] \geq 0, \tag{4.20}$$

where $u = (\psi/f_0) e^{i\alpha_0 s}$.

Proof. First of all we observe that the term with the tangential component of the superconducting current is the only term in $\mathcal{E}_0$ with undefined sign. Now we recall that

$$F_0'(t) = 2(t + \alpha_0) f_0^2(t),$$

where $F_0(t)$ is defined in (2.42). It then follows that

$$-2\int_0^{\ell} \mathrm{d}s \int_0^{c\ell} \mathrm{d}t\, f_0^2(t)(t+\alpha_0) j_s[u] = -\int_0^{\ell} \mathrm{d}s \int_0^{c\ell} \mathrm{d}t\, F_0'(t) j_s[u]$$

We now integrate by parts in t the integral above and we get

$$-\int_0^{\ell} \mathrm{d}s \int_0^{c\ell} \mathrm{d}t\, F_0'(t) j_s[u] = \int_0^{\ell} \mathrm{d}s \int_0^{c\ell} \mathrm{d}t\, F_0(t)\partial_t(iu, \partial_s u). \tag{4.21}$$

Note that the boundary terms vanish because $F_0(0) = 0$ and $u(s, c\ell) = 0$. We now perform another integration by parts in s, then

$$\begin{aligned}\int_0^{\ell} \mathrm{d}s \int_0^{c\ell} \mathrm{d}t\, F_0(t)\partial_t(iu, \partial_s u) &= \int_0^{\ell} \mathrm{d}s \int_0^{c\ell} \mathrm{d}t\, 2F_0(t)(i\partial_t u, \partial_s u) \\ &\quad + \int_0^{c\ell} \mathrm{d}t\, F_0(t) j_t[u]\Big|_{s=0}^{s=\ell}.\end{aligned} \tag{4.22}$$

We now consider the first integral on the r.h.s. in (4.22) and we observe that

$$\begin{aligned}\int_0^{\ell} \mathrm{d}s \int_0^{c\ell} \mathrm{d}t\, 2F_0(t)(i\partial_t u, \partial_s u) &\geq -2\int_0^{\ell} \mathrm{d}s \int_0^{c\ell} \mathrm{d}t\, |F_0(t)||\partial_s u||\partial_t u| \\ &\geq \int_0^{\ell} \mathrm{d}s \int_0^{c\ell} \mathrm{d}t\, F_0(t)[|\partial_t u|^2 + |\partial_s u|^2]\end{aligned} \tag{4.23}$$

where we used the fact that $2ab \leq a^2 + b^2$ and that $F_0(t)$ is negative, which yelds $-|F_0(t)| = F_0(t)$. From (4.23) we have the following estimate

$$\begin{aligned}\mathcal{E}_0[u] \geq \int_0^{\ell} \mathrm{d}s \int_0^{c\ell} \mathrm{d}t \left\{ \left[f_0^2(t) + F_0(t) \right] [|\partial_t u|^2 + |\partial_s u|^2] + \frac{f_0^4(t)}{2b}(1-|u|^2)^2 \right\} \\ + \int_0^{c\ell} \mathrm{d}t\, F_0(t) j_t[u]\Big|_{s=0}^{s=\ell}.\end{aligned}$$

We now recall that $K_0(t) := f_0^2(t) + F_0(t) \geq 0$ for $1 < b < \Theta_0^{-1}$, so that

$$\mathcal{E}_0[u] \geq \int_0^{c\ell} \mathrm{d}t\, F_0(t) j_t[u]\,(\ell, t) - \int_0^{c\ell} \mathrm{d}t\, F_0(t) j_t[u](0,t). \tag{4.24}$$

It remains then to control the boundary terms above. First of all we observe that

$$\int_0^{c\ell} \mathrm{d}t\, F_0(t) j_t[u] = \int_0^{\bar{\ell}} \mathrm{d}t\, \frac{F_0(t)}{f_0^2(t)} j_t[\psi] + \int_{\bar{\ell}}^{c\ell} \mathrm{d}t\, \frac{F_0(t)}{f_0^2(t)} j_t[\psi].$$

We first estimate the boundary term for $t \in [0, \bar{\ell}]$. From the boundary conditions we have that $u(0,t) \equiv 1$ for $t \in [0, \bar{\ell}]$, then

$$(iu, \partial_t u)(0,t) = \frac{1}{2}(iu\partial_t u^* - iu^*\partial_t u)(0,t) = 0, \qquad \text{for } t \in [0, \bar{\ell}].$$

The same is true for $u(\ell, t)$ for $t \in [0, \bar{\ell}]$, since $u(\ell, t) = e^{i\delta_\ell \ell}$. In $[\bar{\ell}, \ell]$ we have that $u(0,t) = g(t)/f_0(t)$ and $u(\ell, t) = g(t)/f_0(t)e^{i\delta_\ell \ell}$: being $g(t)/f_0(t)$ a real function, it easily follows that also in this interval the boundary terms vanish.

Then we conclude that

$$\mathcal{E}_0[u] \geq 0, \tag{4.25}$$

and this completes the proof. □

Proof of Proposition 4.2. From Lemma 4.1 and Proposition 4.3, it follows that

$$E_\ell^{\mathcal{D}}(R) \geq \ell E_0^{1D} + \mathcal{O}\left(\ell e^{-C\ell^2}\right).$$

□

Proof of Theorem 4.2. From the upper bound proved in Proposition 4.1 and the lower bound proved in Proposition 4.2, we get the desired energy asymptotics. □

4.1.2 Energy asymptotics in a rectangular region with Neumann boundary conditions.

We now consider the following minimization domain

$$\mathscr{D}_{\mathcal{N}}(R) := \left\{ \psi(s,t) \in H^1(R) \,\middle|\, \psi(s,c\ell) = 0, \ \forall s \in [0,\ell] \right\}. \tag{4.26}$$

In this Section we want to study the energy asymptotics for the following ground state energy

$$\widetilde{E}_\ell^{\mathcal{N}}(R) := \inf_{\psi \in \mathscr{D}_{\mathcal{N}}(R)} \left[\mathcal{G}_\ell[\psi] - \int_0^{c\ell} \mathrm{d}t \, \frac{F_0(t)}{f_0^2(t)} j_t\left[\psi(s,t)\right] \Big|_{s=0}^{s=\ell} \right] \tag{4.27}$$

where $j_t[f]$ is the $t-$component of the superconducting current. We also denote by $\psi_\ell^{\mathcal{N}} \in \mathscr{D}_{\mathcal{N}}(R)$ the minimizing function.

We observe that the boundary terms in (4.27) are the ones that naturally arise when one applies the energy splitting technique to prove a lower bound. For this reason, we will have to impose suitable Dirichlet boundary conditions to make these boundary terms vanish. In what follows we prove that $\widetilde{E}_\ell^{\mathcal{N}}(R)$ equals $E_\ell^{\mathcal{D}}$ to leading order.

Theorem 4.3 (Asymptotics of $\widetilde{E}_\ell^{\mathcal{N}}(R)$)**.**
Let $R \subset \mathbb{R}^2$ be the rectangular region defined in (4.2)*, then for any fixed*

$$1 < b < \Theta_0^{-1}$$

it holds, as $\ell \to +\infty$,

$$\widetilde{E}_\ell^{\mathcal{N}}(R) = \ell E_0^{1D} + \mathcal{O}\left(\ell^{-\infty}\right). \tag{4.28}$$

Proof. We have to prove upper and lower bounds to the energy. For the upper bound we can choose the trial function

$$\psi_{\mathrm{trial}} := g(t) e^{-i\alpha_0 s}.$$

Notice that here we can choose the phase identically equals to $-\alpha_0 s$ because we do not have boundary conditions at $s = 0$ and $s = \ell$. Then, proceeding as in the proof of Proposition 4.1, we get

$$\mathcal{G}_\ell[\psi_{\mathrm{trial}}] \leq \ell E_0^{1D} + \mathcal{O}\left(e^{-C\ell^2}\right). \tag{4.29}$$

For the boundary terms we observe that

$$\int_0^{c\ell} \mathrm{d}t \, \frac{F_0(t)}{f_0^2(t)} j_t\left[\psi(s,t)\right] \Big|_{s=0}^{s=\ell} = 0.$$

In $s=0$ we have $j_t[\psi/f_0] = j_t[1] = 0$, if $t \leq \bar{\ell}$, and $j_t[\psi/f_0] = j_t[\bar{f}_0/f_0] = 0$ in $[\bar{\ell}, c\ell]$, being $\bar{f}_0/f_0$ a real function. The same holds true in $s = \ell$. It then follows that

$$\widetilde{E}_\ell^{\mathcal{N}}(R) \leq \ell E_0^{1D} + \mathcal{O}\left(\ell^{-\infty}\right).$$

We now prove the lower bound. Proceeding as in Lemma 4.1, we define a function u such that

$$\psi_\ell^{\mathcal{N}}(s,t) =: f_0(t)u(s,t)e^{-i\alpha_0 s}.$$

Then, we have

$$\mathcal{G}_R[\psi] \geq \ell E_0^{1D} + \tilde{\mathcal{E}}_0[u] + \int_0^{c\ell} \mathrm{d}t\, F_0(t) j_t[u] \Big|_{s=0}^{s=\ell} + \mathcal{O}\left(e^{-C\ell^2}\right) \tag{4.30}$$

with

$$\tilde{\mathcal{E}}_0[u] := \int_0^\ell \mathrm{d}s \int_0^{c\ell} \mathrm{d}t \left\{ K_0(t)|\nabla u|^2 + \frac{f_0^4(t)}{2b}(1-|u|^2)^2 \right\} \tag{4.31}$$

Then, from the positivity (2.44) of the cost function $K_0(t)$, we deduce that $\tilde{\mathcal{E}}_0[u] \geq 0$ and then

$$\widetilde{E}_\ell^{\mathcal{N}}(R) \geq \ell E_0^{1D} + \mathcal{O}\left(\ell^{-\infty}\right) \tag{4.32}$$

From the upper and the lower bounds we get the desired result. □

4.1.3 Order parameter asymptotics

From the energy asymptotics proved in Theorem 4.2, we can derive a density asymptotics for $|\psi_\ell^{\mathcal{D}}|$. Let us assume that $\delta_\ell \ell^2 \ll 1$ and set

$$\mathcal{A}_{\mathrm{bl}} := \left\{ (s,t) \in R \,|\, f_0(t) \geq \gamma_\ell^{1/4} \right\} \subset \left\{ t \leq C\sqrt{|\log \gamma_\ell|} \right\}, \tag{4.33}$$

where $\gamma_\ell := \left(\delta_\ell \ell^2\right)^{\frac{1}{6}} \ll 1$.

Theorem 4.4.
Under the same assumptions of Theorem 4.2, we have

$$\left\| |\psi_\ell^{\mathcal{D}}|^2 - f_0^2(t) \right\|_{L^2(R)}^2 = \mathcal{O}(\ell \delta_\ell^2), \tag{4.34}$$

$$\left\| |\psi_\ell^{\mathcal{D}}(s,t)| - f_0(t) \right\|_{L^\infty(\mathcal{A}_{\mathrm{bl}})} = \mathcal{O}(\gamma_\ell). \tag{4.35}$$

Remark 4.1 (Pan's conjecture). Observe that the estimate

$$\sup_{s\in[0,\ell]} \left| |\psi_\ell^{\mathcal{D}}(s,0)| - f_0(0) \right| = \mathcal{O}(\gamma_\ell), \tag{4.36}$$

corresponds to Pan's conjecture for $\psi_\ell^{\mathcal{D}}$.

We first need some control on the gradient of $|u|$. In particular following the proof of Lemma 5.3 in [CR14], we can prove the following Lemma.

Lemma 4.2. *Let u be the function such that $u(s,t) = \psi_\ell^{\mathcal{D}}(s,t)e^{i\alpha_0 s}/f_0(t)$, then for any $1 < b < \Theta_0^{-1}$ it holds*

$$|\partial_s|u|| \le Cf_0^{-1}(t), \qquad |\partial_t|u|| \le Cf_0^{-2}(t) \tag{4.37}$$

Proof. We simply observe that

$$|\nabla|u|(s,t)| \le f_0^{-2}(t)|f_0'(t)||\psi_\ell^{\mathcal{D}}(s,t)| + f_0^{-1}(t)|\nabla|\psi_\ell^{\mathcal{D}}(s,t)||.$$

Following the same strategy as in [CR14, Lemma 5.3], we can show that

$$|\nabla|\psi_\ell^{\mathcal{D}}|(s,t)| = \mathcal{O}(1).$$

Then

$$|\partial_s|u|(s,t)| \le Cf_0^{-1}(t)$$

Thanks to the fact that $|f_0'(t)| = \mathcal{O}(1)$ (see Lemma 2.2) and using the control $|\psi_\ell^{\mathcal{D}}| \le 1$ that easily follows from the variational equation for $\psi_\ell^{\mathcal{D}}$ (see Proposition 2.6), we also get

$$|\partial_t|u|(s,t)| \le Cf_0^{-2}(t).$$

□

We now use the Lemma above to prove the density asymptotics.

Proof of Theorem 4.4. We observe that the estimate (4.34) easily follows from the upper bound (4.8) and the lower bound (4.12). One only has to use that

$$\ell E_0^{1D} + \frac{1}{2b}\int_R \mathrm{d}s\mathrm{d}t\, f_0^4(1-|u|^2)^2 + \mathcal{O}(\ell^{-\infty}) \le E_\ell^{\mathcal{D}} \le \ell E_0^{1D} + \mathcal{O}(\ell\delta_\ell^2)$$

and observe that

$$\frac{1}{2b}\int_R \mathrm{d}s\mathrm{d}t\, f_0^4(1-|u|^2)^2 = \frac{1}{2b}\int_R \mathrm{d}s\mathrm{d}t\,(f_0^2 - |\psi_\ell^{\mathcal{D}}|^2)^2.$$

We now prove (4.35). From the energy asymptotics proven in Theorem 4.2, we get

$$\frac{1}{2b}\int_0^\ell \mathrm{d}s \int_0^{c\ell} \mathrm{d}t\, f_0^4(t)(1-|u|^2)^2 = \mathcal{O}(\ell\delta_\ell^2). \tag{4.38}$$

We now prove the desired estimate in $\mathcal{A}_{\mathrm{bl}}$ by contradiction. We suppose that there exists a point $(s_0,t_0) \in \mathcal{A}_{\mathrm{bl}}$ such that

$$|1 - |u|(s_0,t_0)| \ge \gamma_\ell. \tag{4.39}$$

Then thanks to the control on the gradient of $|u|$ (Lemma 4.2) there exists a rectangular region $R_\ell \subset R \cap \mathcal{A}_{\mathrm{bl}}$ of tangential length a of order $\mathcal{O}(\gamma_\ell^{5/4})$ and normal length b of order $\mathcal{O}(\gamma_\ell^{3/2})$, such that it holds

$$|1 - |u|(s,t)| \ge \frac{\gamma_\ell}{2}, \qquad \forall (s,t) \in R_\ell \tag{4.40}$$

By (4.40)

$$\frac{1}{2b}\int_R \mathrm{d}s\mathrm{d}t\, f_0^4(t)(1-|u|^2)^2 \geq \frac{1}{2b}\int_{R\cap R_\ell} \mathrm{d}s\mathrm{d}t\, f_0^4(t)(1-|u|^2)^2$$
$$\geq C\gamma_\ell^3 \cdot \gamma_\ell^{5/4} \cdot \gamma_\ell^{3/2} = \gamma_\ell^{23/4} = (\delta_\ell^2 \ell)^{23/24} \gg \delta_\ell^2 \ell$$

We then get a contradiction in the limit $\ell \to +\infty$. Hence,

$$|1-|u|(s,t)| \leq \gamma_\ell, \qquad \forall (s,t) \in \mathcal{A}_{\mathrm{bl}}. \tag{4.41}$$

We now recall that

$$|\psi_\ell^{\mathcal{D}}| = |u||f_0|,$$

then

$$\Big\| |f_0(t)| - |\psi_\ell^{\mathcal{D}}| \Big\|_{L^\infty(\mathcal{A}_{\mathrm{bl}})} = \| f_0(t)(1-|u|)\|_{L^\infty(\mathcal{A}_\alpha)} = \mathcal{O}(\gamma_\ell) \tag{4.42}$$

From (4.42), it immediately follows that

$$\sup_{s\in[0,\ell]} \Big| |\psi_\ell^{\mathcal{D}}(s,0)| - f_0(0) \Big| = \mathcal{O}(\gamma_\ell). \tag{4.43}$$

□

We can also prove a density asymptotics for $|\psi_\ell^{\mathcal{N}}|$:, we will consider the following region

$$\mathcal{A}_{\mathrm{bl}} := \Big\{ (s,t) \in R \,|\, f_0(t) \geq \ell^{-\beta} \Big\},$$

for any $\beta > 0$.

Theorem 4.5.
Under the same assumptions of Theorem 4.3, we have

$$\Big\| |\psi_\ell^{\mathcal{N}}|^2 - f_0^2(t) \Big\|^2_{L^2(R)} = \mathcal{O}(\ell^{-\infty}), \tag{4.44}$$

$$\Big\| |\psi_\ell^{\mathcal{N}}(s,t)| - f_0(t) \Big\|_{L^\infty(\mathcal{A}_{\mathrm{bl}})} = \mathcal{O}\left(\ell^{-\infty}\right). \tag{4.45}$$

In particular,

Remark 4.2 (Pan's conjecture). Observe that the estimate

$$\sup_{s\in[0,\ell]} \Big| |\psi_\ell^{\mathcal{N}}(s,0)| - f_0(0) \Big| = \mathcal{O}\left(\ell^{-\infty}\right), \tag{4.46}$$

corresponds to Pan's conjecture for $\psi_\ell^{\mathcal{N}}$.

Proof. The proof of the estimate (4.44) immediately follows from the upper bound (4.29) and the lower bound (4.32): the strategy is exactly the same as in the proof of (4.34). We now prove the estimate (4.45). From the energy asymptotics proved in Theorem 4.3, we get

$$\frac{1}{2b}\int_0^\ell \mathrm{d}s \int_0^{c\ell} \mathrm{d}t\, f_0^4(t)(1-|u|^2)^2 = \mathcal{O}(\ell^{-\infty}). \tag{4.47}$$

We now prove by contradiction the estimate in $\mathcal{A}_{\rm bl}$. We suppose that there exist a point $(s_0, t_0) \in \mathcal{A}_{\rm bl}$ such that

$$|1 - |u|(s_0, t_0)| \geq \ell^{-\gamma}, \tag{4.48}$$

for some $\gamma > 0$. Then, thanks to the control on the gradient of $|u|$ (Lemma 4.2), we know that there exists a rectangular region $R_\ell \subset R \cap \mathcal{A}_{\rm bl}$ of tangential length a of order $\ell^{-\beta-\gamma}$ and normal length b of order $\ell^{-2\beta-\gamma}$, such that it holds

$$|1 - |u|(s,t)| \geq \frac{\ell^{-\gamma}}{2}, \qquad \forall (s,t) \in R_\ell. \tag{4.49}$$

From (4.49), it follows that

$$\frac{1}{2b}\int_R \mathrm{d}s\mathrm{d}t\, f_0^4(t)(1-|u|^2)^2 \geq \frac{1}{2b}\int_{R\cap R_\ell} \mathrm{d}s\mathrm{d}t\, f_0^4(t)(1-|u|^2)^2$$
$$\geq C\ell^{-4\beta-2\gamma}|R_\ell| \geq C\ell^{-7\beta-4\gamma}.$$

We then get a contradiction in the limit $\ell \to +\infty$. It then follows that

$$|1 - |u|(s,t)| \leq \ell^{-\gamma}, \qquad \forall (s,t) \in \mathcal{A}_{\rm bl}. \tag{4.50}$$

Proceeding as in the previous Theorem we get the result. □

4.2 Effective Problem near the Corner

We now study the effective problem near the singularities. We denote by Γ_ℓ the angular region with opening angle β represented in Figure 4.1, where

$$\overline{BC} = \overline{CD} = \ell, \qquad \overline{AB} = \overline{DE} = c\ell,$$

with $0 < c < \tan(\alpha/2)$ a given constant.

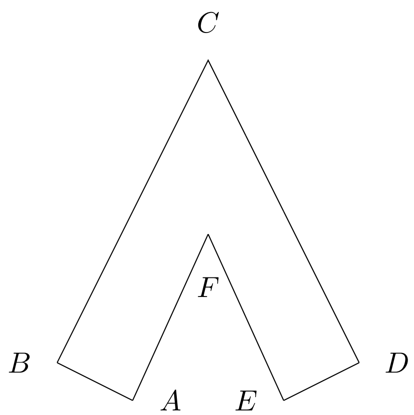

Figure 4.1. The region Γ_ℓ.

We denote the inner boundary by $\partial\Gamma_\ell^{\rm bulk}$, i.e.,

$$\partial\Gamma_\ell^{\rm bulk} := \overline{AF} \cup \overline{EF}.$$

We also set

$$\partial\Gamma_\ell^- := \overline{AB}, \qquad \partial\Gamma_\ell^+ := \overline{DE}, \qquad \partial\Gamma_\ell^{\mathrm{ext}} := \overline{BC} \cup \overline{CD}. \tag{4.51}$$

In what follows we denote by Γ_a the angular region contained in Γ_ℓ and such that any $\mathbf{r} \in \Gamma_a$ has a distance from the vertex at most equal to a, the boundary of this region will be denoted analogously to the one of Γ_ℓ (i.e., $\partial\Gamma_a^\pm$, $\partial\Gamma_a^{\mathrm{ext}}$, $\partial\Gamma_a^{\mathrm{bulk}}$). We now consider the following functional

$$\mathcal{G}_{\Gamma_\ell}[\psi] := \int_{\Gamma_\ell} \mathrm{d}\mathbf{r} \left\{ |(\nabla + i\mathbf{A})\psi|^2 - \frac{1}{2b}(2|\psi|^2 - |\psi|^4) \right\}, \tag{4.52}$$

where $\mathbf{A} \in H^1(\mathbb{R}^2)$ is any fixed vector potential such that $\mathrm{curl}\mathbf{A} = 1$, $\mathrm{div}\mathbf{A} = 0$, $\mathbf{A} = (-t, 0)$ at a distance of order $\mathcal{O}(\ell)$ from the vertex. More precisely, if β is the angle of the corner, then $\mathbf{A} = (-t, 0)$ in $\Gamma_\ell \setminus \Gamma_{\ell - c\ell\tan(\beta/2)}$.

For each $D \subset \Gamma_\ell$ we denote by $\mathcal{G}_D[\psi]$ the restriction of (4.52) to D:

$$\mathcal{G}_D[\psi] := \int_D \mathrm{d}\mathbf{r} \left\{ |(\nabla + i\,\mathbf{A})\psi|^2 - \frac{1}{2b}(2|\psi|^2 - |\psi|^4) \right\}. \tag{4.53}$$

We minimize (4.52) with respect to a function $\psi \in \mathscr{D}_{\mathcal{D}}(\Gamma_\ell)$, where

$$\mathscr{D}_{\mathcal{D}}(\Gamma_\ell) := \left\{ \psi \in H^1(\Gamma_\ell) \,|\, \psi = 0 \ \text{ on } \ \Gamma_\ell^{bulk},\, \psi = g(t)e^{-iS(s)}, \ \text{ if } \ |s - s_\beta| = \ell \right\},$$

where s_β is the tangential coordinate of the vertex and s is the curvilinear coordinate along the exterior boundary. The phase $S(s)$ is defined in (4.5) and the function $g(t)$ is the one defined in (4.6). We now define

$$E_{\mathrm{corner},\beta}(\ell) := -2\ell E_0^{1D} + \inf_{\psi \in \mathscr{D}_{\mathcal{D}}(\Gamma_\ell)} \mathcal{G}_{\Gamma_\ell}[\psi].$$

We also set

$$E^{\mathcal{D}}(\Gamma_\ell) := \inf_{\psi \in \mathscr{D}_{\mathcal{D}}(\Gamma_\ell)} \mathcal{G}_{\Gamma_\ell}[\psi],$$

and we denote by $\psi_\ell \in \mathscr{D}_{\mathcal{D}}(\Gamma_\ell)$ any minimizing function realizing $E^{\mathcal{D}}(\Gamma_\ell)$.

Theorem 4.6 (Existence of minimizers)**.**
Let $\Gamma_\ell \subset \mathbb{R}^2$ be the bounded domain defined above, then the functional

$$\mathcal{G}_{\Gamma_\ell}[\psi] = \int_{\Gamma_\ell} \mathrm{d}\mathbf{r} \left\{ |(\nabla + i\mathbf{A})\psi|^2 - \frac{1}{2b}(2|\psi|^2 - |\psi|^4) \right\}$$

admits at least one minimizer in $\mathscr{D}_{\mathcal{D}}(\Gamma_\ell)$.

Proof. First of all we observe that the set $\mathscr{D}_{\mathcal{D}}(\Gamma_\ell)$ is not empty. We now set

$$m := \inf_{\psi \in \mathscr{D}_{\mathcal{D}}(\Gamma_\ell)} \mathcal{G}_{\Gamma_\ell}, \qquad \text{in } \mathscr{D}_{\mathcal{D}}(\Gamma_\ell).$$

We pick a minimizing sequence $\{\psi_k\}_{k=1}^\infty$ so that

$$\mathcal{G}_{\Gamma_\ell}[\psi_k] \underset{k\to\infty}{\longrightarrow} m.$$

The proof is the same as the one in [FH10, Theorem 10.2.1]. The only difference is due to boundary conditions. We only observe that for each $g \in \mathscr{D}_{\mathcal{D}}(\Gamma_\ell)$, $\psi_k - g \in H_0^1(\Gamma_\ell)$. Now $H_0^1(\Gamma_\ell)$ is a closed, linear subspace of $H^1(\Gamma_\ell)$ and so by Mazur's Theorem is weakly closed. Then $\psi_k - g$ converge weakly in $H_0^1(\Gamma_\ell)$ such that $\psi - g \in H_0^1(\Gamma_\ell)$ and thus $\psi \in \mathscr{D}_{\mathcal{D}}(\Gamma_\ell)$. The convergence of the energy can be shown as in [FH10, Theorem 10.2.1] exploiting the weak semi-continuity of the norms. □

Proposition 4.4 (Existence of $\lim_{\ell\to+\infty} E_{\text{corner},\beta}(\ell)$).
Let $\Gamma_\ell \subset \mathbb{R}^2$ the angular region defined above, then for any fixed

$$1 < b < \Theta_0^{-1},$$

there exists the limit

$$\lim_{\ell\to+\infty} E_{\text{corner},\beta}(\ell) =: E_{\text{corner},\beta}, \tag{4.54}$$

with $|E_{\text{corner},\beta}| < +\infty$.

For the moment we only prove that $E_{\text{corner},\beta}(\ell)$ is bounded from below for any $\ell \in \mathbb{R}^+$ (Proposition 4.5). To prove Proposition 4.4 we need some more information about $E_{\text{corner},\beta}(\ell)$. At the end of the Chapter we prove the existence of the limit (4.54) showing that that $E_{\text{corner},\beta}(\ell)$ is a Cauchy sequence with respect to $\ell \gg 1$ (see Section 4.2.2).

Proposition 4.5 (Lower bound for $E_{\text{corner},\beta}(\ell)$).
Let $\Gamma_\ell \subset \mathbb{R}^2$ be the angular sector defined above, for any fixed

$$1 < b < \Theta_0^{-1}$$

it holds that, as $\ell \to +\infty$,

$$-2\ell E_0^{1D} + E_{\mathcal{D}}(\Gamma_\ell) \geq -C > -\infty.$$

Proof. We first use a partition of unity $\{\chi, \eta\}$ such that $\chi^2 + \eta^2 = 1$ and

$$\operatorname{supp}\chi \subset \Gamma_{\ell_\beta+\delta}, \qquad \operatorname{supp}\eta \subset \Gamma_\ell \setminus \Gamma_{\ell_\beta},$$

where $\ell_\beta = c\ell\tan(\beta/2)$ (with β the angle at the vertex) and δ is of order $\mathcal{O}(1)$. In particular, we assume that

$$\begin{aligned} \chi &\equiv 1, && \text{in } \Gamma_\ell \setminus \Gamma_{\ell_\beta+\delta}, \\ \eta &\equiv 1, && \text{in } \Gamma_{\ell_\beta}, \\ 0 \leq &\chi, \eta \leq 1, && \text{in } \Gamma_{\ell_\beta+\delta} \setminus \Gamma_{\ell_\beta}. \end{aligned}$$

Using the IMS formula [FH10, Section 8.2.2], we get

$$\mathcal{G}_{\Gamma_\ell}[\psi_\ell] = \mathcal{G}_{\Gamma_\ell}[\chi\psi_\ell] + \mathcal{G}_{\Gamma_\ell}[\eta\psi_\ell] - \int_{\Gamma_{\ell_\beta+\delta}\setminus\Gamma_{\ell_\beta}} \mathrm{d}\mathbf{r}\, |\nabla\chi|^2|\psi_\ell|^2 - \int_{\Gamma_{\ell_\beta+\delta}\setminus\Gamma_{\ell_\beta}} |\nabla\eta|^2|\psi_\ell|^2.$$

We now observe that we can also assume

$$|\nabla\chi| \leq \frac{C}{\delta}, \qquad |\nabla\eta| \leq \frac{C}{\delta}.$$

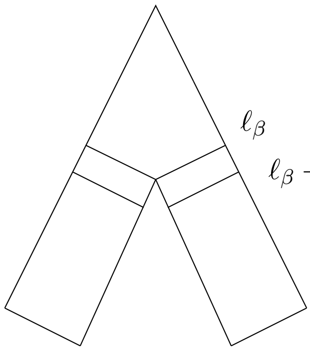

Figure 4.2. The partition of unity $\{\chi, \eta\}$.

Then we have

$$\int_{\Gamma_\ell} \mathrm{d}\mathbf{r}\, |\nabla\chi|^2 |\psi_\ell|^2 = \int_{\Gamma_{2\lambda_\ell} \setminus \Gamma_{\lambda_\ell}} \mathrm{d}\mathbf{r}\, |\nabla\chi|^2 |\psi_\ell|^2 = \mathcal{O}(1). \tag{4.55}$$

The same is true for the integral of $|\nabla\eta|^2|\psi|^2$. We then get

$$\mathcal{G}_{\Gamma_\ell}[\psi_\ell] = \mathcal{G}_{\Gamma_\ell}[\chi\psi_\ell] + \mathcal{G}_{\Gamma_\ell}[\eta\psi_\ell] + \mathcal{O}(1).$$

We now observe that following the same strategy as in Lemma 4.1 and in Proposition 4.3, we get

$$\mathcal{G}_{\Gamma_\ell}[\eta\psi_\ell] \geq 2(\ell - \ell_\beta) E_0^{1D} + \mathcal{O}(e^{-C\ell^2}). \tag{4.56}$$

Here we simply observe that all the boundary terms coming from the integrations by parts vanish because $f_0'(0) = 0$, $\psi|_{\partial\Gamma_\ell^{\mathrm{bulk}}} = 0$, $\psi|_{\partial\Gamma_\ell^{\mathrm{bulk}}} = 0$, $\psi|_{\partial\Gamma_\ell^{\pm}} = g(t)e^{-iS(s)}$, $\eta|_{\partial\Gamma_{\ell_\beta}^{\pm}} \equiv 0$.

We now observe that

$$\mathcal{G}_{\Gamma_\ell}[\chi\psi_\ell] = \mathcal{G}_{\Gamma_{\ell_\beta}}[\psi_\ell] + \mathcal{G}_{\Gamma_{\ell_\beta+\delta} \setminus \Gamma_{\ell_\beta}}[\chi\psi_\ell]. \tag{4.57}$$

In what follows we estimate the term above. For this purpose we use another partition of unity $\{\xi, \zeta\}$ such that $\xi^2 + \zeta^2 = 1$ and such that $\xi \equiv 1$ in $T_\pm$, $\zeta \equiv 1$ in R and $0 \leq \xi, \zeta \leq 1$ otherwise, where the regions $T_\pm$ and R are the ones represented in the Figure 4.3. We recall that $\overline{AB} = \overline{AH} = \ell_\beta + \delta = c\ell\tan(\beta/2) + \delta$. We can also choose the points C and G such that $C\hat{A}G$ is of order $\mathcal{O}(1)$. Using again the IMS formula [FH10, Section 8.2.2], we have

$$\mathcal{G}_{\Gamma_{\ell_\beta+\delta}}[\widetilde{\psi}_\ell] = \mathcal{G}_{\Gamma_{\ell_\beta+\delta}}[\xi\widetilde{\psi}_\ell] + \mathcal{G}_{\Gamma_{\ell_\beta+\delta}}[\zeta\widetilde{\psi}_\ell] - \int_{\Gamma_{\ell_\beta+\delta}} \mathrm{d}\mathbf{r}\, |\nabla\xi|^2 |\widetilde{\psi}_\ell|^2 - \int_{\Gamma_{\ell_\beta+\delta}} \mathrm{d}\mathbf{r}\, |\nabla\zeta|^2 |\widetilde{\psi}_\ell|^2. \tag{4.58}$$

Under our hypothesis we can suppose that

$$|\nabla\xi| = \mathcal{O}(1), \quad |\nabla\zeta| = \mathcal{O}(1).$$

Then via the exponential decay of $\widetilde{\psi}_\ell$, i.e.,

$$|\widetilde{\psi}_\ell(\mathbf{r})| \leq C e^{-\mathrm{dist}(\mathbf{r}, \partial\Gamma_{\ell_\beta+\delta}^{\mathrm{ext}})},$$

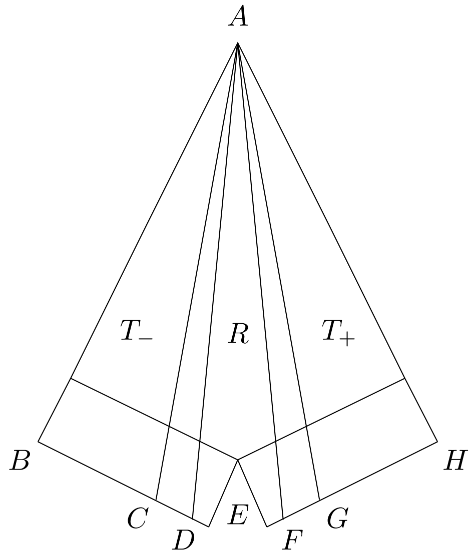

Figure 4.3. The partition of unity $\{\xi, \zeta\}$.

where $\partial\Gamma^{\mathrm{ext}}_{\ell_\beta+\delta} = \overline{AB} \cup \overline{AH}$, we can prove that the last three terms in (4.58) are of order $\mathcal{O}(1)$: up to a small remainder of order $\mathcal{O}(e^{-C\ell^2})$, we can reduce our analysis to the estimate of integrals like the one below

$$\int_0^{C\ell} \mathrm{d}s \int_{s\tan\upsilon}^{s\tan\gamma} \mathrm{d}t\, e^{-2t},$$

where C, υ, γ are suitable quantities of the order $\mathcal{O}(1)$. We then have

$$\int_0^{C\ell} \mathrm{d}s \int_{s\tan\upsilon}^{s\tan\gamma} \mathrm{d}t\, e^{-2t} \leq C \int_0^{C\ell} \mathrm{d}s\, s\, e^{-2s\tan\upsilon} = \mathcal{O}(1),$$

provided $\beta > 0$ uniformly in ℓ. Then

$$\mathcal{G}_{\Gamma_{\ell_\beta+\delta}}[\widetilde{\psi}_\ell] = \mathcal{G}_{\Gamma_{\ell_\beta+\delta}}[\xi\widetilde{\psi}_\ell] + \mathcal{O}(1),$$

We now have to work in the regions $T_\pm$. In T_- the boundary coordinates (s,t) are locally well defined. We then use the energy splitting technique. In particular, we define a function u_- such that

$$\xi\widetilde{\psi}_\ell = f_0(t) u_-(s,t) e^{-i\alpha_0 s}.$$

Following the proofs of Lemma 4.1 and of Proposition 4.3, we get

$$\mathcal{G}_{T_-}[\xi\widetilde{\psi}_\ell] = -\frac{1}{2b}\int_{T_-} \mathrm{d}s\mathrm{d}t\, f_0^4(t) + \mathcal{E}_0[u_-] \geq \frac{1}{2b}\int_{T_-} \mathrm{d}s\mathrm{d}t\, f_0^4(t), \tag{4.59}$$

where the boundary terms coming from the integrations by parts vanish because $f_0'(0) = 0$, $\widetilde{\psi}_\ell|_{\partial\Gamma^-_{\ell_\beta+\delta}} = 0$ and $\xi \equiv 0$ in $\overline{AD}$. Observe that depending on the angle at the vertex one can also have boundary terms along $\overline{DE}$ or $\overline{EF}$ but there $\psi_\ell = 0$. We now denote by R_- the rectangle of side lengths $\overline{AB}$ and $\overline{BC}$ and we observe that

$$-\frac{1}{2b}\int_{T_-} \mathrm{d}s\mathrm{d}t\, f_0^4(t) = -\frac{1}{2b}\int_{R_-} \mathrm{d}s\mathrm{d}t\, f_0^4(t) + \int_{R_-\setminus T_-} \mathrm{d}s\mathrm{d}t\, f_0^4(t)$$
$$= \ell_\beta E_0^{1D} + \frac{1}{2b}\int_{R_-\setminus T_-} \mathrm{d}s\mathrm{d}t\, f_0^4(t) + \mathcal{O}(e^{-C\ell^2}).$$

To finish the proof we observe that from the exponential decay of $f_0(t)$ (Lemma 2.1), it holds

$$\frac{1}{2b}\int_{R_-\setminus T_-} f_0^4(t) = -\frac{1}{2b}\int_0^{c\ell} \mathrm{d}t \int_0^{\frac{t}{\tan\upsilon}} \mathrm{d}s\, f_0^4(t) = -\frac{1}{2b}\int_0^{c\ell} \mathrm{d}t\, \frac{t}{\tan\upsilon} f_0^4(t) = \mathcal{O}(1). \tag{4.60}$$

The same analysis holds in T_+. From (4.55), (4.56), (4.59) and (4.60) we get that there exists a constant $C > 0$ such that

$$E_{\mathrm{corner},\beta}(\ell) = -2\ell E_0^{1D} + \mathcal{G}_{\Gamma_\ell}[\psi_\ell] \geq -2\ell E_0^{1D} + 2(\ell-\ell_\beta)E_0^{1D} + 2\ell_\beta E_0^{1D} - C = -C.$$

□

Boundedness from above is on the other hand easy to get:

Proposition 4.6 (Upper bound for $E_{\mathrm{corner},\beta}(\ell)$).
Let $\Gamma_\ell \subset \mathbb{R}^2$ be the angular sector defined above, for any fixed

$$1 < b < \Theta_0^{-1}$$

it holds that, as $\ell \to +\infty$,

$$-2\ell E_0^{1D} + E_{\mathcal{D}}(\Gamma_\ell) \leq C < +\infty.$$

Proof. The result is obtained by testing the functional on a trial state $g(t)e^{i\alpha_0 s}$. We omit the details for the sake of brevity. □

4.2.1 Analysis of the energy asymptotics of the corner effective problem

In this Section we consider the same variational problem studied in Section 4.1.2 but in the region Γ_ℓ. We prove that the related ground state energy has the same asymptotics, at least to leading order, as $E_{\mathcal{D}}(\Gamma_\ell)$.

We define the following minimization domain

$$\mathscr{D}_{\mathcal{N}}(\Gamma_\ell) := \left\{ \psi \in H^1(\Gamma_\ell) \,\middle|\, \psi|_{\partial\Gamma_\ell^{\mathrm{bulk}}} = 0 \right\}.$$

We are going to study the ground state energy of the following functional

$$\widetilde{\mathcal{G}}_{\Gamma_\ell}[\psi] := \mathcal{G}_{\Gamma_\ell}[\psi] - \int_0^{c\ell} \mathrm{d}t\, \frac{F_0(t)}{f_0^2(t)}\, j_t\left[\psi(\mathbf{r}(s,t))\right]\Big|_{s=s_-}^{s=s_+},$$

where $s_\pm$ are the two points along $\partial\Gamma_\ell^{\mathrm{ext}}$ such that $|s_\pm - s_\beta| = \ell$, with s_β the tangential coordinate of the vertex. We also set

$$\widetilde{E}_{\mathcal{N}}(\Gamma_\ell) := \inf_{\psi\in\mathscr{D}_{\mathcal{N}}(\Gamma_\ell)} \widetilde{\mathcal{G}}_{\Gamma_\ell}[\psi].$$

Theorem 4.7 (Existence of a minimizer)**.**
Let $\Gamma_\ell \subset \mathbb{R}^2$ be the bounded domain defined above, then for any fixed

$$1 < b < \Theta_0^{-1},$$

then the functional $\widetilde{\mathcal{G}}_{\Gamma_\ell}[\psi]$ admits at least one minimizer in $\mathscr{D}_{\mathcal{N}}(\Gamma_\ell)$.

Proof. The proof is the same as the one in [FH10, Theorem 10.2.1], we only have to prove that the boundary terms, i.e.,

$$\int_0^{c\ell} \mathrm{d}t \, \frac{F_0(t)}{f_0^2(t)} j_t[\psi] \bigg|_{s=s_-}^{s=s_+},$$

can be bounded by the rest of the energy so that the whole functional is bounded from below. For this purpose we use a partition of unity $\{\chi, \eta\}$ such that $\chi^2 + \eta^2 = 1$ and

$$\begin{aligned} \chi &\equiv 1, && \text{in } \Gamma_\ell \setminus \Gamma_{\ell_\beta+\delta}, \\ \eta &\equiv 1, && \text{in } \Gamma_{\ell_\beta}, \\ 0 \leq &\chi, \eta \leq 1, && \text{in } \Gamma_{\ell_\beta+\delta} \setminus \Gamma_{\ell_\beta}, \end{aligned}$$

where δ is of order $\mathcal{O}(1)$. We now use the IMS formula [FH10, Section 8.2.2] and we get

$$\begin{aligned} \widetilde{\mathcal{G}}_{\Gamma_\ell}[\psi] = \mathcal{G}_{\Gamma_\ell}[\eta\psi] + \mathcal{G}_{\Gamma_\ell}[\chi\psi] - \int_0^{c\ell} \mathrm{d}t \, \frac{F_0(t)}{f_0^2(t)} j_t[\chi\psi] \bigg|_{s=s_-}^{s=s_+} \\ - \int_{\Gamma_{\ell_\beta} \setminus \Gamma_{\ell_\beta-\delta}} \mathrm{d}\mathbf{r} \, |\nabla\chi|^2 |\psi|^2 - \int_{\Gamma_{\ell_\beta} \setminus \Gamma_{\ell_\beta-\delta}} \mathrm{d}\mathbf{r} \, |\nabla\eta|^2 |\psi|^2 \end{aligned} \quad (4.61)$$

We then use the energy splitting techniques in the region in which $\chi \equiv 1$: for any function $\psi \in \mathscr{D}_{\mathcal{N}}(\Gamma_\ell)$, we define a function $u_\psi(s,t)$ such that for

$$\chi\psi(s,t) = f_0(t) u_\psi(s,t) e^{-i\alpha_0 s}.$$

By the same arguments as in the proof of Lemma 4.1, we have

$$\mathcal{G}_{\Gamma_\ell}[\chi\psi] = 2\delta E_0^{1D} + \mathcal{E}_0[u_\psi],$$

where we recall that

$$\mathcal{E}_0[u_\psi] = \int_{\Gamma_\ell \setminus \Gamma_{\ell_\beta+\delta}} \mathrm{d}s\mathrm{d}t \, f_0^2(t) \left\{ \left|\partial_s u_\psi\right|^2 + \left|\partial_t u_\psi\right|^2 - 2(t+\alpha_0) j_s[u_\psi] + \frac{f_0^2(t)}{2b} (1 - |u_\psi|^2)^2 \right\}.$$

Integrating by parts as in the proof of Lemma 4.3, we get

$$\mathcal{E}_0[u_\psi] \geq \int_{\Gamma_\ell \setminus \Gamma_{\ell_\beta+\delta}} \mathrm{d}s\mathrm{d}t \left\{ K_0(t) |\nabla u_\psi|^2 + \frac{f_0^4(t)}{2b} (1 - |u_\psi|^2)^2 + \int_0^{c\ell} \mathrm{d}t \, F_0(t) j_t[u_\psi] \bigg|_{s=s_-}^{s_+}, \right. \quad (4.62)$$

where we recall that $K_0(t) = f_0^2(t) + F_0(t) \geq 0$. All the boundary terms with the only exception of the one in (4.62) vanish, because $f_0'(0) = 0$, $u_\psi(s, c\ell) = 0$ and $u_\psi(\ell_\beta, t) = 0$ (since $\chi = 0$ at $s = \ell_\beta$). From (4.61) and (4.62) we have that, for any $\psi \in \mathscr{D}_{\mathcal{N}}(\Gamma_\ell)$,

$$\widetilde{\mathcal{G}}_{\Gamma_\ell}[\psi] \geq 2\delta E_0^{1D} + \mathcal{G}_{\Gamma_\ell}[\eta\psi] - \int_{\Gamma_{\ell_\beta} \setminus \Gamma_{\ell_\beta-\delta}} \mathrm{d}\mathbf{r} \, |\nabla\chi|^2 |\psi|^2 - \int_{\Gamma_{\ell_\beta} \setminus \Gamma_{\ell_\beta-\delta}} \mathrm{d}\mathbf{r} \, |\nabla\eta|^2 |\psi|^2.$$

Since we can assume that $|\nabla\chi| = \mathcal{O}(1)$ and $|\nabla\eta| = \mathcal{O}(1)$, we have that

$$\widetilde{\mathcal{G}}_{\Gamma_\ell}[\psi] \geq \mathcal{G}_{\Gamma_\ell}[\eta\psi] + \int_{\Gamma_\ell \setminus \Gamma_{\ell_\beta+\delta}} \mathrm{d}s\mathrm{d}t \left\{ K_0(t)|\nabla u_\psi|^2 + \frac{f_0^4(t)}{2b}(1-|u_\psi|^2)^2 \right\} + \mathcal{O}(1).$$

We now select a minimizing sequence $\{\psi_k\}_{k=1}^\infty$ so that

$$\widetilde{\mathcal{G}}_{\Gamma_\ell}[\psi_k] \underset{k\to\infty}{\longrightarrow} m,$$

where $m := \inf_{\psi \in \mathscr{D}_{\mathcal{N}}(\Gamma_\ell)} \widetilde{G}_{\Gamma_\ell}[\psi]$ and, following the proof in [FH10, Theorem 10.2.1], we can show that the minimizing sequence weakly convergence in $H^1(\Gamma_\ell)$ and it also strongly converges in $L^p(\Gamma_\ell)$. We then deduce the existence of at least one minimizer. □

Theorem 4.8.
Let $\Gamma_\ell \subset \mathbb{R}^2$ the angular region defined above, then for any fixed

$$1 < b < \Theta_0^{-1}$$

it holds that, as $\ell \to +\infty$,

$$E_{\mathcal{D}}(\Gamma_\ell) = \widetilde{E}_{\mathcal{N}}(\Gamma_\ell) + \mathcal{O}(\ell^{-\infty}).$$

To prove Theorem 4.8 we need some preliminary results. In particular we first work in Γ_δ^- represented in Figure 4.4 and defined as

$$\Gamma_\delta^- := \Gamma_\ell \setminus \left\{ (s,t) \in [s_-, s_\delta^-] \times [0, c\ell] \right\},$$

where s_δ^- is the point along $\partial\Gamma_\ell^{\mathrm{ext}}$ such that $|s_- - s_\delta^-| = \delta$.

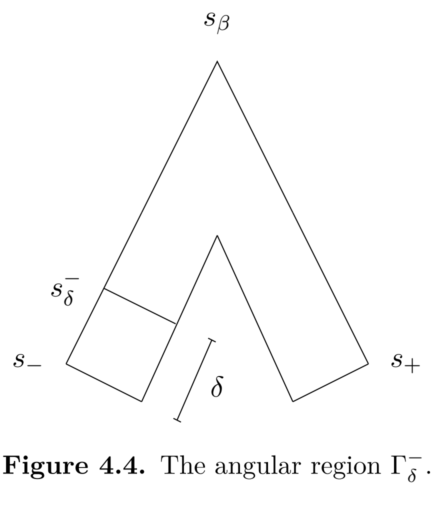

Figure 4.4. The angular region Γ_δ^-.

We now define the following minimization domain

$$\mathscr{D}_{\mathcal{D}}^+(\Gamma_\delta^-) := \left\{ \psi \in H^1(\Gamma_\delta^-) \,\middle|\, \psi|_{\partial\Gamma_\ell^+} = g(t)e^{-S(s)} \,,\, \psi|_{\partial\Gamma_-^{\mathrm{bulk}}} = 0 \right\},$$

where $\partial\Gamma_{-}^{\mathrm{bulk}} := \partial\Gamma_{\ell}^{\mathrm{bulk}} \cap \partial\Gamma_{\delta}^{-}$ and we set

$$\widetilde{E}_{\mathcal{D},+}(\Gamma_{\delta}^{-}) := \inf_{\psi\in\mathscr{D}_{\mathcal{D}}^{+}(\Gamma_{\delta}^{-})} \widetilde{\mathcal{G}}_{\Gamma_{\delta}^{-}}[\psi],$$

where

$$\widetilde{\mathcal{G}}_{\Gamma_{\delta}^{-}}[\psi] := \int_{\Gamma_{\delta}^{-}} \mathrm{d}\mathbf{r} \left\{ |(\nabla + i\mathbf{A})\psi|^2 - \frac{1}{2b}(2|\psi|^2 - |\psi|^4) \right\} - \int_0^{c\ell} \mathrm{d}t\, \frac{F_0(t)}{f_0^2(t)}\, j_t\,[\psi] \Big|_{s=s_{\delta}^{-}}^{s=s_{+}},$$

with the vector potential $\mathbf{A}$ satisfying the same assumptions as before.

Proposition 4.7.
Let Γ_ℓ, $\Gamma_{\delta}^{-} \subset \mathbb{R}^2$ be the two regions defined above, for any fixed

$$1 < b < \Theta_0^{-1}$$

it holds, as $\ell \to +\infty$,

$$E_{\mathcal{D}}(\Gamma_\ell) \le \widetilde{E}_{\mathcal{D},+}(\Gamma_{\delta}^{-}) + C\delta, \tag{4.63}$$

for any $\delta \ll \ell$.

Proof. We first recall that

$$\begin{aligned} E_{\mathcal{D}}(\Gamma_\ell) &= \inf_{\psi\in\mathscr{D}_{\mathcal{D}}(\Gamma_\ell)} \left[\int_{\Gamma_\ell} \mathrm{d}\mathbf{r} \left\{ |(\nabla + i\mathbf{A})\,\psi|^2 - \frac{1}{2b}(2|\psi|^2 - |\psi|^4) \right\} \right] \\ &= \inf_{\psi\in\mathscr{D}_{\mathcal{D}}(\Gamma_\ell)} \mathcal{G}_{\Gamma_\ell}[\psi] \end{aligned}$$

Recalling also the definition (4.53), we have

$$\mathcal{G}_{\Gamma_\ell}[\psi] = \mathcal{G}_{\Gamma_{\delta}^{-}}[\psi] + \mathcal{G}_{\Gamma_\ell\setminus\Gamma_{\delta}^{-}}[\psi] \pm \int_0^{c\ell} \mathrm{d}t\, \frac{F_0(t)}{f_0^2(t)}\, j_t\left[\psi_{\delta}^{-}\right] \Big|_{s=s_{\delta}^{-}}$$

where $\psi_{\delta}^{-} \in \mathscr{D}_{\mathcal{D},+}(\Gamma_{\delta}^{-})$ is a minimizer of $\widetilde{\mathcal{G}}_{\Gamma_{\delta}^{-}}[\psi, \mathbf{A}]$. In order to prove the desired upper bound on $E_{\mathcal{D}}(\Gamma_\ell)$ we have to choose a suitable trial state. The order parameter ψ_{trial} is defined as

$$\psi_{\mathrm{trial}} := \begin{cases} \psi_{\delta}^{-}, & \text{in } \Gamma_{\delta}^{-}, \\ \psi_{R,s_{\delta}^{-}}, & \text{in } \Gamma_\ell \setminus \Gamma_{\delta}^{-}, \end{cases}$$

where $\psi_{R,s_{\delta}^{-}}$ is the minimizer of $\mathcal{G}_{\Gamma_\ell\setminus\Gamma_{\delta}^{-}}[\psi]$ with fixed boundary conditions, i.e.

$$\psi_{R,s_{\delta}^{-}}(s_{-}, t) = g(t)e^{-iS(s_{\ell}^{-})}, \qquad \psi_{R,s_{\delta}^{-}}(s_{\delta}^{-}, t) = \psi_{\delta}^{-}|_{s=s_{\delta}^{-}}.$$

Then we get

$$E^{\mathcal{D}}(\Gamma_\ell) \le \widetilde{E}_{\mathcal{D},+}(\Gamma_{\delta}^{-}) + \mathcal{G}_{\Gamma_\ell\setminus\Gamma_{\delta}^{-}}[\psi_{\mathrm{trial}}] + \int_0^{c\ell} \mathrm{d}t\, \frac{F_0(t)}{f_0^2(t)} J_t[\psi_{\delta}^{-}]\Big|_{s=s_{\delta}^{-}}. \tag{4.64}$$

We now want to prove that

$$E_{\mathcal{D}}(\Gamma_\ell) \le \widetilde{E}_{\mathcal{D},+}(\Gamma_{\delta}^{-}) + o(1). \tag{4.65}$$

Before proceeding further we observe that the vector potential $\mathbf{A}$ is equal to $(-t, 0)$ in the region $[s_-, s_\delta^-] \times [0, c\ell]$, since $\delta \ll \ell$.

To prove (4.65) we observe that, since ψ_{R,s_δ^-} is a minimizer of the functional $\widetilde{\mathcal{G}}_{\Gamma_\ell \setminus \Gamma_\delta^-}$ with fixed vector potential equal to $(-t, 0)$, we get the following variational equation for ψ_{R,s_δ^-} in $\Gamma_\ell \setminus \Gamma_\delta^-$:

$$-\left(\nabla_{s,t} - it\hat{\mathbf{e}}_s\right)^2 \psi_{R,s_\delta^-} = \frac{1}{b}(1 - |\psi_{R,s_\delta^-}|^2)\psi_{R,s_\delta^-}. \tag{4.66}$$

Then for each solutions of the variational equation above, we can use Agmon estimate (4.3) for $|\psi_{R,s_\delta^-}|$ and $|(\nabla_{s,t} - it\hat{\mathbf{e}}_s)\psi_{R,s_\delta^-}|$: since $|s_\delta^- - s_-| = \delta$, we have

$$\int_{s_\ell^-}^{s_\delta^-} \mathrm{d}s \int_0^{c\ell} \mathrm{d}t \left\{ |\partial_t \psi_{R,s_\delta^-}|^2 + |(\partial_s - it)\psi_{R,s_\delta^-}|^2 - \frac{1}{2b}(2|\psi_{R,s_\delta^-}|^2 - |\psi_{R,s_\delta^-}|^4) \right\} = \mathcal{O}(\delta). \tag{4.67}$$

We now estimate the boundary term. Thanks to the boundary condition in s_δ^-, one has

$$\left.\frac{\psi_\delta^-}{f_0}\right|_{s=s_\delta^-} = \left.\frac{\psi_{R,s_\delta^-}}{f_0}\right|_{s=s_\delta^-},$$

and

$$\begin{aligned}
\partial_t(\psi_\delta^-)|_{s=s_\delta^-} &= \left[\lim_{h\to 0} \frac{\psi_\delta^-(s, t+h) - \psi_\delta^-(s,t)}{h}\right]\Big|_{s=s_\delta^-} \\
&= \lim_{h\to 0} \frac{\psi_\delta^-(s_\delta^-, t+h) - \psi_\delta^-(s_\delta^-, t)}{h} \\
&= \lim_{h\to 0} \frac{\psi_{R,s_\delta^-}(s_\delta^-, t+h) - \psi_{R,s_\delta^-}(s_\delta^-, t)}{h} \\
&= \lim_{h\to 0} \left[\frac{\psi_{R,s_\delta^-}(s, t+h) - \psi_{R,s_\delta^-}(s,t)}{h}\right]\Big|_{s=s_\delta^-} = \partial_t(\psi_{R,s_\delta^-})|_{s=s_\delta^-}.
\end{aligned}$$

Then,

$$\int_0^{c\ell} \mathrm{d}t\, \frac{F_0(t)}{f_0^2(t)} j_t\left[\psi_\delta^-\right]\Big|_{s=s_\delta^-} = \int_0^{c\ell} \mathrm{d}t\, \frac{F_0(t)}{f_0^2(t)} j_t\left[\psi_{R,s_\delta^-}\right]\Big|_{s=s_\delta^-}. \tag{4.68}$$

We then estimate the term on the r.h.s. in (4.68). To do this we observe that

$$\begin{aligned}
&\int_0^{c\ell} \mathrm{d}t\, \frac{F_0(t)}{f_0^2(t)} j_t[\psi_{R,s_\delta^-}]\Big|_{s=s_\delta^-} \\
&\qquad = \int_0^{c\ell} \mathrm{d}t\, \frac{F_0(t)}{f_0^2(t)} j_t[\psi_{R,s_\delta^-}]\Big|_{s=s_-} + \int_{s_-}^{s_\delta^-} \mathrm{d}s \int_0^{c\ell} \mathrm{d}t\, \frac{F_0(t)}{f_0^2(t)} \partial_s\left(j_t[\psi_{R,s_\delta^-}]\right) \\
&\qquad = \int_{s_-}^{s_\delta^-} \mathrm{d}s \int_0^{c\ell} \mathrm{d}t\, \frac{F_0(t)}{f_0^2(t)} \partial_s\left(j_t[\psi_{R,s_\delta^-}]\right),
\end{aligned}$$

where we used the fact that the boundary term in $s = s_\ell^-$ vanishes thanks to the Dirichlet conditions. Moreover

$$\partial_s\left(j_t[\psi_{R,s_\delta^-}]\right) = \frac{i}{2}\left[(\partial_s \psi_{R,s_\delta^-})(\partial_t \psi^*_{R,s_\delta^-}) + (\psi_{R,s_\delta^-})(\partial^2_{s,t}\psi^*_{R,s_\delta^-}) + c.c.\right]$$

and integrating by parts the second term on the r.h.s. above, we get

$$\frac{i}{2}\int_{s_-}^{s_\delta^-} \mathrm{d}s \int_0^{c\ell} \mathrm{d}t\, \frac{F_0(t)}{f_0^2(t)} \Big[(\psi_{R,s_\delta^-})(\partial_{s,t}^2 \psi_{R,s_\delta^-}^*) - (\psi_{R,s_\delta^-}^*)(\partial_{s,t}^2 \psi_{R,s_\delta^-}) \Big]$$
$$= \frac{i}{2}\int_{s_-}^{s_\delta^-} \mathrm{d}s\, \frac{F_0(t)}{f_0^2(t)} \Big[(\psi_{R,s_\delta^-})(\partial_s \psi_{R,s_\delta^-}^*) - (\psi_{R,s_\delta^-}^*)(\partial_s \psi_{R,s_\delta^-}) \Big] \Big|_{t=0}^{t=c\ell}$$
$$- \frac{i}{2}\int_{s_-}^{s_\delta^-} \mathrm{d}s \int_0^{c\ell} \mathrm{d}t\, \frac{F_0(t)}{f_0^2(t)} \Big[(\partial_t \psi_{R,s_\delta^-})(\partial_s \psi_{R,s_\delta^-}^*) - (\partial_t \psi_{R,s_\delta^-}^*)(\partial_s \psi_{R,s_\delta^-}) \Big]$$
$$- \frac{i}{2}\int_{s_-}^{s_\delta^-} \mathrm{d}s \int_0^{c\ell} \mathrm{d}t\, \partial_t \left[\frac{F_0(t)}{f_0^2(t)} \right] \Big[(\psi_{R,s_\delta^-})(\partial_s \psi_{R,s_\delta^-}^*) - (\psi_{R,s_\delta^-}^*)(\partial_s \psi_{R,s_\delta^-}) \Big].$$

Before proceeding further we observe that

$$\partial_t \left[\frac{F_0(t)}{f_0^2(t)} \right] = \frac{F_0'(t)}{f_0^2(t)} - 2\frac{F_0(t) f_0'(t)}{f_0^3(t)} = \mathcal{O}(t),$$

by Lemma 2.2 and (2.44). Then using the estimate (4.3) for $|\psi_{R,s_\delta^-}|$ and $|(\nabla_{s,t} - it\hat{\mathbf{e}}_s)\psi_{R,s_\delta^-}|$ in the region $\Gamma_\ell \setminus \Gamma_\delta^-$, we have

$$\frac{i}{2}\int_{s_\ell^-}^{s_\delta^-} \mathrm{d}s \int_0^{c\ell} \mathrm{d}t\, \frac{F_0(t)}{f_0^2(t)} \Big[\psi_{R,s_\delta^-} \partial_{s,t}^2 \psi_{R,s_\delta^-}^* - \psi_{R,s_\delta^-}^* \partial_{s,t}^2 \psi_{R,s_\delta^-} \Big] = \mathcal{O}(\delta)$$

For the same reason

$$\frac{i}{2}\int_{s_\ell^-}^{s_\delta^-} \mathrm{d}s \int_0^{c\ell} \mathrm{d}t\, \frac{F_0(t)}{f_0^2(t)} \Big[(\partial_s \psi_{R,s_\delta^-})(\partial_t \psi_{R,s_\delta^-}^*) - (\partial_s \psi_{R,s_\delta^-}^*)(\partial_t \psi_{R,s_\delta^-}) \Big] = \mathcal{O}(\delta).$$

We then get the desired result

$$E_{\mathcal{D}}(\Gamma_\ell) \le \widetilde{E}_{\mathcal{D},+}(\Gamma_\delta^-) + \mathcal{O}(\delta).$$

□

We now consider the region Γ_δ^+ represented in Figure 4.5 and defined as follows:

$$\Gamma_\delta^+ := \Gamma_\ell \setminus \Big\{ (s,t) \in [s_\delta^+, s_+] \times [0, c\ell] \Big\}$$

As we did before we define

$$\mathscr{D}_{\mathcal{D}}^-(\Gamma_\delta^+) := \left\{ \psi \in H^1(\Gamma_\delta^+) \,\middle|\, \psi|_{\partial\Gamma_\ell^-} = g(t)e^{-S(s)},\ \psi|_{\partial\Gamma_+^{\mathrm{bulk}}} = 0 \right\},$$

where $\partial\Gamma_+^{\mathrm{bulk}} := \partial\Gamma_\ell^{\mathrm{bulk}} \cap \partial\Gamma_\delta^+$ and we set

$$\widetilde{E}_{\mathcal{D},-}(\Gamma_\delta^+) := \inf_{\psi \in \mathscr{D}_{\mathcal{D}}^-(\Gamma_\delta^+)} \widetilde{\mathcal{G}}_{\Gamma_\delta^+}[\psi],$$

where

$$\widetilde{\mathcal{G}}_{\Gamma_\delta^+}[\psi] := \int_{\Gamma_\delta^+} \mathrm{d}\mathbf{r} \left\{ |(\nabla + i\mathbf{A})\psi|^2 - \frac{1}{2b}(2|\psi|^2 - |\psi|^4) \right\} - \int_0^{c\ell} \mathrm{d}t\, \frac{F_0(t)}{f_0^2(t)}\, j_t[\psi] \Big|_{s=s_-}^{s=s_+},$$

with the vector potential $\mathbf{A}$ satisfying the same assumptions as before.

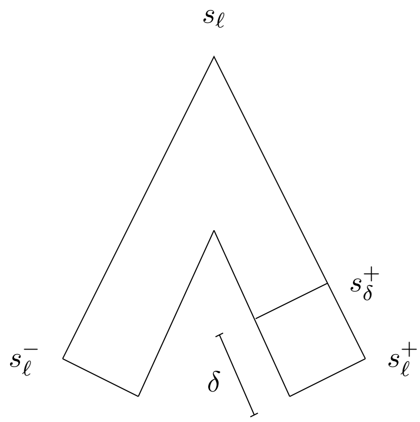

Figure 4.5. The angular region Γ_δ^+

Proposition 4.8.
Let Γ_ℓ, $\Gamma_\delta^+ \subset \mathbb{R}^2$ be the two regions defined above, for any fixed

$$1 < b < \Theta_0^{-1}$$

it holds, as $\ell \to +\infty$,

$$E_{\mathcal{D}}(\Gamma_\ell) \leq \widetilde{E}_{\mathcal{D},-}(\Gamma_\delta^+) + C\delta, \tag{4.69}$$

for any $\delta \ll \ell$.

The proof of the Proposition above is the same of Proposition 4.7 and we omit it for the sake of brevity. Furthermore, if we define

$$\Gamma_\delta := \Gamma_\delta^- \cap \Gamma_\delta^+,$$

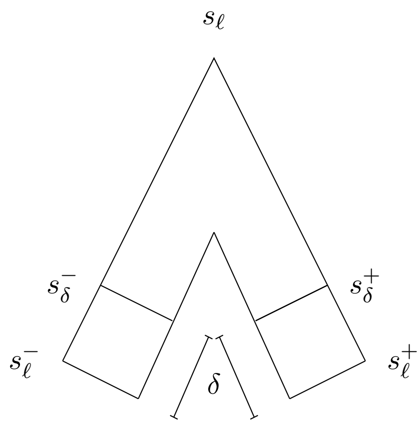

Figure 4.6. The angular region Γ_δ.

$$\mathscr{D}_{\mathcal{N}}(\Gamma_\delta) := \left\{ \psi \in H^1(\Gamma_\delta) \,|\, \psi|_{\partial\Gamma_\ell^{\mathrm{bulk}} \cap \partial\Gamma_\delta} = 0 \right\},$$

$$\widetilde{\mathcal{G}}_{\Gamma_\delta}[\psi] := \int_{\Gamma_\delta^+} \mathrm{d}\mathbf{r} \left\{ |(\nabla + i\mathbf{A})\psi|^2 - \frac{1}{2b}(2|\psi|^2 - |\psi|^4) \right\} - \int_0^{c\ell} \mathrm{d}t \, \frac{F_0(t)}{f_0^2(t)} \, j_t \, [\psi] \Big|_{s=s_\delta^-}^{s=s_\delta^+},$$

and

$$\widetilde{E}_{\mathcal{N}}(\Gamma_\delta) := \inf_{\psi \in \mathscr{D}_{\mathcal{N}}(\Gamma_\delta)} \widetilde{\mathcal{G}}_{\Gamma_\delta}[\psi],$$

then by the same arguments, we can prove the following result

Proposition 4.9.
Let Γ_ℓ, $\Gamma_\delta \subset \mathbb{R}^2$ the two regions defined above, for any fixed

$$1 < b < \Theta_0^{-1}$$

it holds, as $\ell \to +\infty$,

$$E_{\mathcal{D}}(\Gamma_\ell) \leq \widetilde{E}_{\mathcal{N}}(\Gamma_\delta) + C\delta, \tag{4.70}$$

for any $\delta \ll \ell$.

In fact, we can prove a stronger result:

Proposition 4.10.
Let $\Gamma_\ell, \Gamma_\delta \subset \mathbb{R}^2$ be the two regions defined above, then for any fixed

$$1 < b < \Theta_0^{-1}$$

it holds as $\ell \to +\infty$

$$E_{\mathcal{D}}(\Gamma_\ell) \leq \widetilde{E}_{\mathcal{N}}(\Gamma_\ell) + C\delta + \mathcal{O}(\ell^{-\infty}),$$

for any $\delta \ll \ell$.

Proof. From Proposition 4.9, we know that

$$E_{\mathcal{D}}(\Gamma_\ell) \leq \widetilde{E}_{\mathcal{N}}(\Gamma_\delta) + C\delta$$

but we are going to show that

$$\widetilde{E}_{\mathcal{N}}(\Gamma_\delta) \leq \widetilde{E}_{\mathcal{N}}(\Gamma_\ell) + C\delta.$$

We denote by $\widetilde{\psi}_\ell$ any function realizing $\widetilde{E}_{\mathcal{N}}(\Gamma_\ell)$, then

$$\begin{aligned}
\widetilde{E}_{\mathcal{N}}(\Gamma_\ell) &= \mathcal{G}_{\Gamma_\ell}[\widetilde{\psi}_\ell] - \int_0^{c\ell} \mathrm{d}t \, \frac{F_0(t)}{f_0^2(t)} j_t \left[\widetilde{\psi}_\ell\right] \Big|_{s=s_-}^{s=s_+} \\
&= \mathcal{G}_{\Gamma_\delta}[\widetilde{\psi}_\ell] - \int_0^{c\ell} \mathrm{d}t \, \frac{F_0(t)}{f_0^2(t)} j_t \left[\widetilde{\psi}_\ell\right] \Big|_{s=s_\delta^-}^{s=s_\delta^+} \\
&\quad + \mathcal{G}_{\Gamma_\ell \setminus \Gamma_\delta^-}[\widetilde{\psi}_\ell] - \int_0^{c\ell} \mathrm{d}t \, \frac{F_0(t)}{f_0^2(t)} j_t \left[\widetilde{\psi}_\ell\right] \Big|_{s=s_-}^{s=s_\delta^-} \\
&\quad + \mathcal{G}_{\Gamma_\ell \setminus \Gamma_\delta^+}[\widetilde{\psi}_\ell] - \int_0^{c\ell} \mathrm{d}t \, \frac{F_0(t)}{f_0^2(t)} j_t \left[\widetilde{\psi}_\ell\right] \Big|_{s=s_\delta^+}^{s=s_+} \\
&\geq \widetilde{E}_{\mathcal{N}}(\Gamma_\delta) + \mathcal{O}(\delta) + \mathcal{O}(\ell^{-\infty}).
\end{aligned}$$

The remainder in the last estimate above is due to the fact that we know from Theorem 4.3 that the energy in a rectangular region with Neumann boundary conditions is of the order of the length of the tangential boundary up to an error of the order $\mathcal{O}(\ell^{-\infty})$. □

Proof of Theorem 4.8. From Proposition 4.9 and Proposition 4.10, we know that

$$E_{\mathcal{D}}(\Gamma_\ell) \le \widetilde{E}_{\mathcal{N}}(\Gamma_\ell) + C\delta + \mathcal{O}(\ell^{-\infty}). \tag{4.71}$$

However,

$$\mathscr{D}_{\mathcal{D}}(\Gamma_\ell) \subset \mathscr{D}_{\mathcal{N}}(\Gamma_\ell),$$

so that

$$\widetilde{E}_{\mathcal{N}}(\Gamma_\ell) \le E_{\mathcal{D}}(\Gamma_\ell). \tag{4.72}$$

From (4.71) and (4.72) it follows that

$$E_{\mathcal{D}}(\Gamma_\ell) = \widetilde{E}_{\mathcal{N}}(\Gamma_\ell) + \mathcal{O}(\delta) + \mathcal{O}(\ell^{-\infty}),$$

for any $\delta \ll \ell$. Since δ is arbitrary, the result follows. □

4.2.2 Existence of the limit

The goal of this section is to prove Proposition 4.4. We already know from Propositions 4.5 and 4.6 that if the limit exists, then it is finite. We now prove that $E_{\text{corner},\beta}(\ell)$ is a Cauchy sequence with respect to $\ell \gg 1$.

Proposition 4.11 (Upper bound)**.**
Let $0 < \ell_1 - \ell_2 < c\ell_1/\tan(\alpha/2) - \ell_1$ *with* $\ell_1 \to +\infty$*. For any fixed*

$$1 < b < \Theta_0^{-1},$$

it holds that

$$E_{\text{corner},\beta}(\ell_1) - E_{\text{corner},\beta}(\ell_2) \le C\ell_1\delta_{\ell_1} + \mathcal{O}(\ell_2^{-\infty}).$$

Proof. Due to the exponential decay (4.3) of any minimizer, the energy contribution coming from points at a distance larger than $c\ell_2$ from the outer boundary is exponentially small. Therefore we can replace Γ_{ℓ_2} with a new domain $\widetilde{\Gamma}_{\ell_2}$ such that:

$$|\partial\widetilde{\Gamma}^{\pm}_{\ell_2}| = |\partial\Gamma^{\pm}_{\ell_1}| = c\ell_1,$$

up to exponentially small error terms, i.e.,

$$E^{\mathcal{D}}(\Gamma_{\ell_2}) = E^{\mathcal{D}}(\widetilde{\Gamma}_{\ell_2}) + \mathcal{O}(\ell_2^{-\infty}).$$

Note that the domain $\widetilde{\Gamma}_{\ell_2}$ has the same form of Γ_{ℓ_2} thanks to the condition on $\ell_1 - \ell_2$.

We denote by $\psi_{\ell_2} \in \mathscr{D}_{\mathcal{D}}(\widetilde{\Gamma}_{\ell_2})$ a minimizing function of the functional in $\widetilde{\Gamma}_{\ell_2}$. We also denote by R_- and R_+ the rectangular regions such that

$$R_- \cup R_+ = \Gamma_{\ell_1} \setminus \widetilde{\Gamma}_{\ell_2},$$

Then

$$R_- := \left\{ (s,t) \,\middle|\, s \in [s_\beta - \ell_1, s_\beta - \ell_2],\, t \in [0, c\ell_1] \right\},$$

$$R_+ := \left\{ (s,t) \,\middle|\, s \in [s_\beta + \ell_2, s_\beta + \ell_1],\, t \in [0, c\ell_1] \right\}.$$

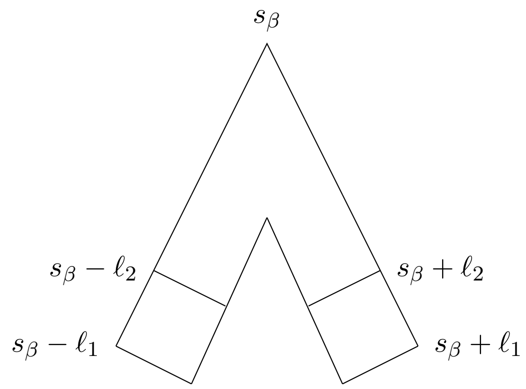

Figure 4.7. The region $\Gamma_{\ell_1} \setminus \Gamma_{\ell_2}$.

We now choose the trial state to test the functional in Γ_{ℓ_1}. For the vector potential we observe that under our hypothesis $\mathbf{A} = (-t, 0)$ in $\Gamma_{\ell_1} \setminus \Gamma_{\ell_2}$. We now define the order parameter $\psi_{\mathrm{trial}} \in \mathscr{D}_{\mathcal{D}}(\Gamma_{\ell_1})$:

$$\psi_{\mathrm{trial}} := \begin{cases} g(t)e^{-i\Phi_-(s)} & \text{in } R_-, \\ \psi_{\ell_2} & \text{in } \widetilde{\Gamma}_{\ell_2}, \\ g(t)e^{-i\Phi_+(s)} & \text{in } R_+, \end{cases}$$

where

$$\Phi_-(s) = -i\alpha_0 s - \delta_{\ell_2}\left(\frac{s_\beta - \ell_2}{\ell_1 - \ell_2}\right)(s - s_\beta + \ell_1) + \delta_{\ell_1}\left(\frac{s_\beta - \ell_1}{\ell_1 - \ell_2}\right)(s - s_\beta + \ell_2),$$

and

$$\Phi_+(s) = -i\alpha_0 s + \delta_{\ell_2}\left(\frac{s_\beta + \ell_2}{\ell_1 - \ell_2}\right)(s - s_\beta - \ell_1) - \delta_{\ell_1}\left(\frac{s_\beta + \ell_1}{\ell_1 - \ell_2}\right)(s - s_\beta - \ell_2).$$

Then in the region R_- the energy equals

$$\int_{R_-} \mathrm{d}s\mathrm{d}t \left\{ |g'(t)|^2 + \left|(\partial_s - it)g(t)e^{-i\Phi_-(s)}\right|^2 - \frac{1}{2b}(2|g(t)|^2 - |g(t)|^4) \right\}$$

We can now estimate the contribution of $\partial_s\Phi_-$ (as in Proposition 4.1) and use the upper bound proved in Proposition 4.1 to deduce that

$$\int_{R_-} \mathrm{d}s\mathrm{d}t \left\{ |g'(t)|^2 + \left|(\partial_s - it)g(t)e^{-i\Phi_-(s)}\right|^2 - \frac{1}{2b}(2|g(t)|^2 - |g(t)|^4) \right\} \\ = (\ell_1 - \ell_2)E_0^{1D} + \mathcal{O}\left(\ell_1\delta_{\ell_1}^2\right) + \mathcal{O}\left(\ell_2\delta_{\ell_2}^2\right) + \mathcal{O}\left(\ell_1^{-\infty}\right). \quad (4.73)$$

The same holds true in R_+ and thus

$$\begin{aligned} E_{\mathrm{corner},\beta}(\ell_1) &\leq -2\ell_1 E_0^{1D} + \int_{\Gamma_{\ell_1}} \mathrm{d}\mathbf{r} \left\{ \left|(\nabla + i\mathbf{A})\psi_{\mathrm{trial}}\right|^2 - \frac{1}{2b}(2|\psi_{\mathrm{trial}}|^2 - |\psi_{\mathrm{trial}}|^4) \right\} \\ &\leq -2\ell_1 E_0^{1D} + E^{\mathcal{D}}(\widetilde{\Gamma}_{\ell_2}) + 2(\ell_1 - \ell_2)E_0^{1D} + C\left(\ell_1\delta_{\ell_1}^2\right) + C\left(\ell_2\delta_{\ell_2}^2\right) + \mathcal{O}\left(\ell_1^{-\infty}\right) \\ &\leq E_{\mathrm{corner},\beta}(\ell_2) + \mathcal{O}(\ell_2^{-\infty}) + C\left(\ell_1\delta_{\ell_1}^2\right) + C\left(\ell_2\delta_{\ell_2}^2\right) + \mathcal{O}\left(\ell_1^{-\infty}\right), \end{aligned}$$

which is the desired result since $\ell_2 \leq \ell_1$. □

Proposition 4.12 (Lower bound)**.**
Let $0 < \ell_1 - \ell_2 < c\ell_1/\tan(\alpha/2) - \ell_1$ *with* $\ell_1 \to +\infty$*. For any fixed*

$$1 < b < \Theta_0^{-1},$$

it holds that

$$E_{\text{corner},\beta}(\ell_2) \leq E_{\text{corner},\beta}(\ell_1) + \mathcal{O}(\ell_2^{-\infty}).$$

Proof. As in Proposition 4.11 we can replace the region Γ_{ℓ_2} with the domain $\widetilde{\Gamma}_{\ell_2}$. We now recall that

$$E_{\text{corner},\beta}(\ell_2) = -2\ell_2 E_0^{1D} + \inf_{\psi \in \mathscr{D}_{\mathcal{D}}(\widetilde{\Gamma}_{\ell_2})} \mathcal{G}_{\widetilde{\Gamma}_{\ell_2}}[\psi] + \mathcal{O}(\ell_2^{-\infty}). \tag{4.74}$$

By Theorem 4.8 we also know that in the limit $\ell \to +\infty$, it holds $E_{\mathcal{D}}(\Gamma_\ell) = \widetilde{E}_{\mathcal{N}}(\Gamma_\ell) + o(1)$, then

$$E_{\text{corner},\beta}(\ell_2) = -2\ell_2 E_0^{1D} + \widetilde{E}_{\mathcal{N}}(\widetilde{\Gamma}_{\ell_2}) + \mathcal{O}(\ell_2^{-\infty}),$$
$$E_{\text{corner},\beta}(\ell_1) = -2\ell_1 E_0^{1D} + \widetilde{E}_{\mathcal{N}}(\Gamma_{\ell_1}) + \mathcal{O}(\ell_1^{-\infty}).$$

We denote by $\widetilde{\psi}_{\ell_1}$ a minimizer of $\widetilde{\mathcal{G}}_{\Gamma_{\ell_1}}$. To prove the desired upper bound on $E_{\text{corner},\beta}(\ell_2)$ we choose as trial function for $\widetilde{\mathcal{G}}_{\widetilde{\Gamma}_{\ell_2}}$

$$\psi_{\text{trial}} := \widetilde{\psi}_{\ell_1}|_{\widetilde{\Gamma}_{\ell_2}}.$$

We then have

$$\widetilde{\mathcal{G}}_{\widetilde{\Gamma}_{\ell_2}}[\psi_{\text{trial}}] = \int_{\widetilde{\Gamma}_{\ell_2}} \mathrm{d}\mathbf{r} \left\{ \left| (\nabla + i\mathbf{A})\psi_{\text{trial}} \right|^2 - \frac{1}{2b}(2|\psi_{\text{trial}}|^2 - |\psi_{\text{trial}}|^4) \right| \right\}$$
$$- \int_0^{c\ell_1} \mathrm{d}t \, \frac{F_0(t)}{f_0^2(t)} \, j_t \left[\psi_{\text{trial}}(\mathbf{r}(s,t)) \right] \Big|_{s=s_\beta-\ell_2}^{s=s_\beta+\ell_2}.$$

It then follows that

$$\widetilde{\mathcal{G}}_{\widetilde{\Gamma}_{\ell_2}}[\psi_{\text{trial}}] = \widetilde{\mathcal{G}}_{\Gamma_{\ell_1}}[\widetilde{\psi}_{\ell_1}] - \int_{\Gamma_{\ell_1} \setminus \widetilde{\Gamma}_{\ell_2}} \mathrm{d}\mathbf{r} \left\{ \left| (\nabla + i\mathbf{A})\widetilde{\psi}_{\ell_1} \right|^2 - \frac{1}{2b}(2|\widetilde{\psi}_{\ell_1}|^2 - |\widetilde{\psi}_{\ell_1}|^4) \right| \right\}$$
$$+ \int_0^{c\ell_1} \mathrm{d}t \, \frac{F_0(t)}{f_0^2(t)} \, j_t \left[\widetilde{\psi}_{\ell_1} \right] \Big|_{s=s_\beta-\ell_1}^{s=s_\beta-\ell_2} + \int_0^{c\ell_1} \mathrm{d}t \, \frac{F_0(t)}{f_0^2(t)} \, j_t \left[\widetilde{\psi}_{\ell_1} \right] \Big|_{s=s_\beta+\ell_2}^{s=s_\beta+\ell_1}.$$

Using the same notation as in Proposition 4.11, we can write

$$\Gamma_{\ell_1} \setminus \widetilde{\Gamma}_{\ell_2} = R_- \cup R_+,$$

so that

$$\widetilde{\mathcal{G}}_{\widetilde{\Gamma}_{\ell_2}}[\psi_{\text{trial}}] = E_{\text{corner},\beta}(\ell_1) + 2\ell_1 E_0^{1D} - \mathcal{G}_{\Gamma_{\ell_1} \setminus \widetilde{\Gamma}_{\ell_2}}[\widetilde{\psi}_{\ell_1}]$$
$$+ \int_0^{c\ell_1} \mathrm{d}t \, \frac{F_0(t)}{f_0^2(t)} \, j_t \left[\widetilde{\psi}_{\ell_1} \right] \Big|_{s=s_\beta-\ell_1}^{s=s_\beta-\ell_2} + \int_0^{c\ell_1} \mathrm{d}t \, \frac{F_0(t)}{f_0^2(t)} \, j_t \left[\widetilde{\psi}_{\ell_1} \right] \Big|_{s=s_\beta+\ell_2}^{s=s_\beta+\ell_1}$$
$$= E_{\text{corner},\beta}(\ell_1) + 2\ell_1 E_0^{1D} - \widetilde{\mathcal{G}}_{R_-}[\widetilde{\psi}_{\ell_1}] - \widetilde{\mathcal{G}}_{R_+}[\widetilde{\psi}_{\ell_1}]$$

since

$$\widetilde{\mathcal{G}}_{\Gamma_{\ell_1}}[\widetilde{\psi}_{\ell_1}] = E_{\mathrm{corner},\beta}(\ell_1) + 2\ell_1 E_0^{1D}. \tag{4.75}$$

Moreover, from Theorem 4.3.

$$-\widetilde{\mathcal{G}}_{R_-}[\widetilde{\psi}_{\ell_1}] - \widetilde{\mathcal{G}}_{R_+}[\widetilde{\psi}_{\ell_1}] \leq -2(\ell_1 - \ell_2)E_0^{1D} + \mathcal{O}(\ell_1^{-\infty}). \tag{4.76}$$

From (4.75) and (4.76), we get

$$\widetilde{\mathcal{G}}_{\widetilde{\Gamma}_{\ell_2}}[\psi_{\mathrm{trial}}] \leq E_{\mathrm{corner},\beta}(\ell_1) + 2\ell_1 E_0^{1D} - 2(\ell_1 - \ell_2)E_0^{1D} + \mathcal{O}(\ell_1^{-\infty})$$

and then, from (4.74), we conclude that

$$E_{\mathrm{corner},\beta}(\ell_2) \leq E_{\mathrm{corner},\beta}(\ell_1) + \mathcal{O}(\ell_2^{-\infty}).$$

□

Proof of Theorem 4.4. By Propositions 4.11 and 4.12, we have that

$$|E_{\mathrm{corner},\beta}(\ell_1) - E_{\mathrm{corner},\beta}(\ell_2)| = \mathcal{O}(\ell_1 \delta_{\ell_1} + \ell_2^{-\infty})$$

for any $\ell_1 > \ell_2 \gg 1$. Since $\ell\delta_\ell = o(1)$ as $\ell \to +\infty$, $E_{\mathrm{corner},\beta}(\ell)$ is Cauchy and therefore admits a limit, uniform boundedness is given by Propositions 4.5 and 4.6. □

Chapter 5

First Order Correction to the Energy Asymptotics

In this Chapter we want to capture the first order correction to the energy asymptotics in a general domain with corners in terms of the the effective problem introduced in the previous Chapter. The goal is to prove that, in a general domain with piecewise smooth boundary, we have

$$E_{\varepsilon}^{GL} = \frac{|\partial\Omega| E_0^{1D}}{\varepsilon} - \mathcal{E}_{\alpha_0}^{\mathrm{corr}}[f_0] \int_0^{|\partial\Omega|} \mathrm{d}\sigma\, \widetilde{k}(\sigma) + \sum_{j \in \Sigma} E_{\mathrm{corner},j} + o(1), \tag{5.1}$$

where $\widetilde{k}(\sigma)$ is the boundary curvature and $E_{\mathrm{corner},j}$ is the energy contribution of the $j-$th corner, $j \in \Sigma$, and it is given by (4.54).

The Chapter is organized as follows. In the first part, after the reduction of the GL functional to a suitable boundary layer, we introduce a local diffeomorphism which allow us to map the corner region on a triangular domain with zero boundary curvature. We then underline how to use the effective problem introduced in Chapter via a substitution of the minimizing vector potential near the singularities and choosing the appropriate coordinates to work in the surface superconductivity regime. We will then prove our main result, i.e., a refined energy asymptotics, first for a polygon and then for a general corner domain with non trivial boundary curvature.

5.1 Parametrization of the Domain

We can restrict the GL functional to the boundary layer $\mathcal{A}_{\partial\Omega}$ using Agmon estimate (Section 2.6), where

$$\mathcal{A}_{\partial\Omega} = \left\{ \mathbf{r} \in \Omega \mid \mathrm{dist}\,(\mathbf{r}, \partial\Omega) \leq c_0 \varepsilon |\log \varepsilon| \right\}.$$

We recall that the boundary coordinates (σ, τ) are defined through the local diffeomorphism (3.1):

$$\mathbf{r} = \boldsymbol{\gamma}(\sigma) + \tau \boldsymbol{\nu}(\sigma),$$

with $\gamma(\sigma) \in \partial\Omega$ and $\nu(\sigma)$ the inward normal to it. We use these coordinates (σ, τ) in the smooth part of the domain, i.e., in the region introduced in Chapter 3

$$\widetilde{A}_{\mathrm{cut}} = \mathcal{A}_{\partial\Omega} \setminus \bigcup_{j\in\Sigma} \widetilde{\Gamma}_j, \tag{5.2}$$

where

$$\widetilde{\Gamma}_j := \{\mathbf{r} \in \Omega \,|\, \mathrm{dist}(\mathbf{r}, \mathbf{r}_j) \le c_1\varepsilon|\log\varepsilon|\}, \tag{5.3}$$

where $\mathbf{r}_j$ is the coordinate of the $j-$th corner, $j \in \Sigma$. We use the same notation as before for $\partial\widetilde{\Gamma}_j$, i.e.,

$$\partial\widetilde{\Gamma}_j = \partial\Gamma_j^+ \cup \partial\Gamma_j^- \cup \partial\Gamma_j^{\mathrm{bulk}} \cup \partial\Gamma_j^{\mathrm{ext}},$$

where (see Figure 5.1)) $\partial\widetilde{\Gamma}_j^- := \overline{AB}$, $\partial\widetilde{\Gamma}_j^- := \overline{DE}$, $\partial\widetilde{\Gamma}_j^{\mathrm{bulk}} := \overline{AF} \cup \overline{EF}$ and $\partial\widetilde{\Gamma}_j^{\mathrm{ext}} := \overline{BC} \cup \overline{CD}$.

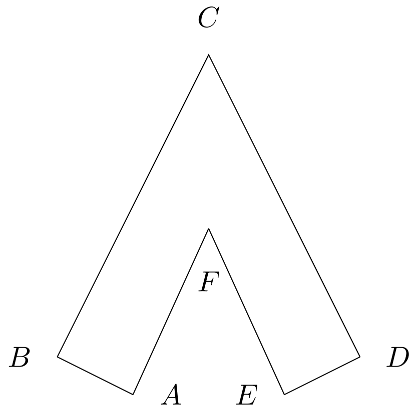

Figure 5.1. The region $\widetilde{\Gamma}_j$.

Because of the presence of singularities we split each cell $\widetilde{\Gamma}_j$, $j \in \Sigma$, into two parts in order to have locally well defined boundary coordinates.

We now observe that, in the region near each corner $j \in \Sigma$ with opening angle β_j, we can define as in [BNF07, Section 6.2] a local diffeomorphism Φ_j of $\mathbb{R}^2$ such that if $\mathbf{r}_j$, $j \in \Sigma$, is the position of the $j-$th vertex, then

$$\Phi_j(\mathbf{r}_j) = 0, \qquad (D\Phi_j)(\mathbf{r}_j) \in SO(2), \qquad \Phi_j(B(\mathbf{r}_j, \rho_j) \cap \Omega) = S_j \cap \Phi_j(B(\mathbf{r}_j, \rho_j)),$$

where we denoted by $D\Phi_j$ the jacobian of Φ_j, S_j stands for an infinite angular sector with opening angle β_j and $B(\mathbf{r}_j, \rho_j)$ denotes a ball centered in $\mathbf{r}_j$ with radius ρ_j.

Lemma 5.1.
Let Φ_j be the diffeomorphism defined above and let Ω_j be the region of all points $\mathbf{r} \in \Omega$ such that $\mathrm{dist}(\mathbf{r}, \mathbf{r}_j) \le d_j$, it holds

$$\int_{\Omega_j} \mathrm{d}\mathbf{r}\, f(\mathbf{r}) = (1 + \mathcal{O}(d_j)) \int_{\widetilde{\Omega}_j} \mathrm{d}\tilde{\mathbf{r}}\, f(\tilde{\mathbf{r}}(\mathbf{r})),$$

where $\tilde{\mathbf{r}} = \Phi_j(\mathbf{r})$ and $\widetilde{\Omega}_j = S_j \cap \Phi_j(B(\mathbf{r}_j, d_j))$.

Proof. The proof simply follows from a Taylor approximation of $|\mathrm{det} D\Phi_j|$. Since $|\mathrm{det} D\Phi_j(\mathbf{r}_j)| = 1$, then by Taylor's formula we get

$$\big||\mathrm{det} D\Phi_j(\mathbf{r})| - 1\big| = \mathcal{O}(d_j).$$

We then have

$$\int_{\Omega_j} \mathrm{d}\mathbf{r}\, f(\mathbf{r}) = \int_{\widetilde{\Omega}_j} \mathrm{d}\tilde{\mathbf{r}} |\mathrm{det} D\Phi_j(\mathbf{r})|\, f(\tilde{\mathbf{r}}(\mathbf{r})) = (1 + C d_j) \int_{\widetilde{\Omega}_j} \mathrm{d}\tilde{\mathbf{r}}\, f(\tilde{\mathbf{r}}(\mathbf{r})).$$

□

We also work with $\varepsilon-$rescaled boundary coordinates (s,t) in the smooth part of the boundary layer:

$$s = \frac{\sigma}{\varepsilon}, \qquad t = \frac{\tau}{\varepsilon}.$$

We recall that we denote by $\mathcal{A}_\varepsilon$ the boundary layer $\mathcal{A}_{\partial\Omega}$ written in the $(s,t)-$coordinates and that the $\varepsilon-$rescaled GL functional restricted to the boundary layer is

$$\mathcal{G}^{\mathrm{GL}}_{\varepsilon,\mathcal{A}_\varepsilon}[\psi,\mathbf{A}] = \int_{\mathcal{A}_\varepsilon} \mathrm{d}s\mathrm{d}t \left\{ \left| \left(\nabla + i\frac{\mathbf{A}}{\varepsilon} \right) \psi \right|^2 - \frac{1}{2b}(2|\psi|^2 - |\psi|^4) \right\} + \frac{1}{\varepsilon^4} \int_{\mathbb{R}^2} \mathrm{d}\mathbf{r}\, |\mathrm{curl}\mathbf{A} - 1|^2.$$

5.2 Main Results

Before stating the main Theorem we anticipate that in the regions $\widetilde{\Gamma}_j$, $j \in \Sigma$, we will use the effective problem defined in Chapter 4 (Section 4.2). To this purpose we recall that in (4.54) we introduced the quantities

$$E_{\mathrm{corner},\beta} = \lim_{\ell \to +\infty} E_{\mathrm{corner},\beta}(\ell)$$

where

$$E_{\mathrm{corner},\beta}(\ell) = -2\ell E_0^{1D} + \inf_{\psi \in \mathscr{D}_\mathcal{D}} \mathcal{G}_{\Gamma_\ell}[\psi]$$

and

$$\mathcal{G}_{\Gamma_\ell}[\psi] = \int_{\Gamma_\ell} \mathrm{d}\mathbf{r} \left\{ \left|(\nabla + i\mathbf{A})\psi\right|^2 - \frac{1}{2b}(2|\psi|^2 - |\psi|^4) \right\},$$

with $\mathbf{A} \in H^1(\mathbb{R}^2)$ such that $\mathrm{curl}\mathbf{A} = 1$, $\mathrm{div}\mathbf{A} = 0$ and $\mathbf{A} = (-t, 0)$ along the boundaries $\partial\Gamma_\ell^\pm$ and

$$\mathscr{D}_\mathcal{D}(\Gamma_\ell) = \left\{ \psi \in H^1(\Gamma_\ell) \,\middle|\, \psi\big|_{\partial\Gamma_\ell^{\mathrm{bulk}}} = 0,\, \psi\big|_{\partial\Gamma_\ell^\pm} = g(t)e^{-iS(s)} \right\}.$$

The function $g(t)$ is defined in (4.6) and the phase $S(s)$ is the same as the one in (4.5).

In this Chapter we consider a domain with $j \in \Sigma$ corners with angle β_j at the vertex. We set for short

$$E_{\mathrm{corner},j} := E_{\mathrm{corner},\beta_j}, \quad E_{\mathrm{corner},j}(\ell) := E_{\mathrm{corner},\beta_j}(\ell).$$

Theorem 5.1 (Refined asymptotics for general corner domains)**.**
Let $\Omega \subset \mathbb{R}^2$ be bounded domain satisfying Assumptions 2.1 and 2.2, then for any fixed

$$1 < b < \Theta_0^{-1}$$

it holds, as $\varepsilon \to 0$*,*

$$E_\varepsilon^{GL} = \frac{|\partial\Omega| E_0^{1D}}{\varepsilon} - \mathcal{E}_{\alpha_0}^{\mathrm{corr}}[f_0] \int_0^{|\partial\Omega|} \mathrm{d}\sigma\, \widetilde{k}(\sigma) + \sum_{j \in \Sigma} E_{\mathrm{corner},j} + o(1). \tag{5.4}$$

In the Theorem above we isolated the contribution to the energy density due to the corners. In order to write the explicit form of this contribution one has to take into account the curvature of the boundary. We recall in fact that for smooth domains the first order correction to the energy asymptotic is proportional to the integral of the curvature along the boundary of the domain and therefore, by Gauss-Bonnet Theorem (see Remark 2.10), it is equal to 2π. Although at the vertex of an angular sector the curvature of the boundary is not well defined, we can think of it in distributional sense. In particular, the Gauss-Bonnet Theorem tell us that for a piecewise smooth domains

$$\int_{\partial\Omega} k = \int_{\partial\Omega_s} k + \sum_j (\pi - \vartheta_j),$$

where $\partial\Omega_s := \partial\Omega \cap \partial\widetilde{\mathcal{A}}_{\mathrm{cut}}$ is the smooth part of the boundary and ϑ_j is the angle at the $j-$th corner. Therefore, the energy asymptotics for smooth domains

$$E_\varepsilon^{\mathrm{GL}} = \frac{|\partial\Omega|}{\varepsilon} E_0^{1D} - \mathcal{E}_{\alpha_0}^{\mathrm{corr}}[f_0] \int_0^{|\partial\Omega|} \mathrm{d}\sigma\, \widetilde{k}(\sigma) + o(1)$$

leads us to the following conjecture:

Conjecture 5.1 (Corner energy)**.**
For any $\vartheta \in [0, 2\pi)$*,*

$$E_{\mathrm{corner}}(\vartheta) = -(\pi - \vartheta)\mathcal{E}_{\alpha_0}^{\mathrm{corr}}[f_0].$$

Remark 5.1 (Pan's conjecture)*.* From the energy asymptotics (5.4), one can deduce an estimate of the order parameter $|\psi^{\mathrm{GL}}|$ in the smooth part of the boundary layer. Indeed, following the same strategies as the in [CR16a, Theorem 2.2], we get

$$\left\| |\psi^{\mathrm{GL}}(\mathbf{r})| - f_0(\mathrm{dist}(\mathbf{r}, \partial\Omega)/\varepsilon) \right\|_{L^2(\widetilde{\mathcal{A}}_{\mathrm{cut}})} = o(\varepsilon)$$

and

$$\sup_{\mathbf{r} \in \partial\Omega_{\mathrm{s}}} \left| |\psi^{\mathrm{GL}}(\mathbf{r})| - f_0(0) \right| = o(1).$$

We prove Theorem 5.1 first for a polygon, i.e., a general corner domain with zero boundary curvature, and then for a general domain with corners. In both cases we work in a different way in the smooth part of the boundary and near the singularities, proving an upper bound and a lower bound to the energy. The two main ingredients of the proof are: the replacement of the minimizing vector potential $\mathbf{A}^{\mathrm{GL}}$ by a new one with zero normal component and the gluing of the corner effective problem with the smooth part of the boundary. In this second step the only non trivial problem occurs for non-zero boundary curvature. This is why we discuss first the simpler case of a polygon.

5.3 Replacement of the Vector Potential

In this Section we want to replace in the GL functional the minimizing vector potential $\mathbf{A}^{\mathrm{GL}}$ with a fixed vector potential. This step in the proof of Theorem 5.1 is performed both in the case of a polygon and of a general domain.

Firs of all, we observe that if $\mathbf{A}$ is a vector potential written with respect to $\mathbf{r}$ then, in the region where (σ,τ) are well defined, we can define the associate vector field $(\widetilde{A}_1,\widetilde{A}_2)$ by setting

$$\widetilde{A}_1(\sigma,\tau) := (1-\widetilde{k}(\sigma)\tau)\mathbf{A}(\mathbf{r}(\sigma,\tau))\cdot\boldsymbol{\gamma}'(\sigma), \qquad \widetilde{A}_2(\sigma,\tau) := \mathbf{A}(\mathbf{r}(\sigma,\tau))\cdot\boldsymbol{\nu}(\sigma),$$

where $\widetilde{k}(\sigma)$ is the curvature of the boundary (recall the definition of tubular coordinates in Section 5.1). It then follows that

$$\partial_\sigma\widetilde{A}_2(\sigma,\tau)-\partial_\tau\widetilde{A}_1(\sigma,\tau) = (1-k(\sigma)\tau)\,\mathrm{curl}\mathbf{A}(\mathbf{r}(\sigma,\tau)). \tag{5.5}$$

Hence, if $\mathbf{F}$ is a vector potential such that $\mathrm{curl}\mathbf{F}=1$ and $\mathbf{F}\cdot\boldsymbol{\nu}=0$, then the corresponding (σ,τ)-components are

$$\widetilde{\mathbf{F}}_1(\sigma,\tau) = -\tau+\frac{1}{2}\widetilde{k}(\sigma)\tau^2, \qquad \widetilde{\mathbf{F}}_2(\sigma,\tau)=0. \tag{5.6}$$

From now on we will denote by $\widetilde{\mathbf{F}}$ a vector potential such that

$$\begin{cases} \mathrm{curl}\widetilde{\mathbf{F}}=1, & \text{in } \mathbb{R}^2,\\ \mathrm{div}\widetilde{\mathbf{F}}=0, & \text{in } \mathbb{R}^2,\\ \widetilde{\mathbf{F}}(\mathbf{r}(\sigma,\tau)) = \big(-\tau+\frac{1}{2}\widetilde{k}(\sigma)\tau^2,0\big), & \text{in } \mathcal{A}_{\partial\Omega}\setminus\bigcup_{j\in\Sigma}\widetilde{\Gamma}_j. \end{cases}$$

Lemma 5.2.
For any vector potential $\mathbf{A}\in H^1(\mathbb{R}^2)$ solving the GL equations (2.4), there exists a constant $C>0$ such that

$$\|\mathbf{A}-\widetilde{\mathbf{F}}\|_{H^1(\mathbb{R}^2)} \le C\|\mathrm{curl}\mathbf{A}-1\|_{L^2(\mathbb{R}^2)}. \tag{5.7}$$

Proof. We first notice that, via the gauge invariance (Proposition 2.1) and Proposition 2.2, we can set $\mathrm{div}\mathbf{A}=0$. This determines $\mathbf{A}$ up to an additive constant. Following [FH10, Lemma D.2.7, Lemma D.2.8 and Section 15.3.1], we can choose this constant in such a way that

$$\|\mathbf{A}-\widetilde{\mathbf{F}}\|_{H^1(\mathbb{R}^2)} \le C\|\mathrm{curl}\mathbf{A}-1\|_{L^2(\mathbb{R}^2)}.$$

□

We now replace the minimizing vector potential in the functional GL in two different way depending on the distance from the corners. We will prove the replacement of the vector potential in the non-rescaled GL functional. We first prove how to work in the regions $\widetilde{\Gamma}_j$ near the corners: we set for short

$$\mathcal{G}^{\mathrm{GL}}_{\varepsilon,j}[\psi,\mathbf{A}] := \int_{\widetilde{\Gamma}_j}\mathrm{d}\mathbf{r}\left\{\left|\left(\nabla+i\frac{\mathbf{A}}{\varepsilon^2}\right)\psi\right|^2-\frac{1}{2b\varepsilon^2}(2|\psi|^2-|\psi|^4)\right\}$$

and

$$\mathcal{G}_{\widetilde{\Gamma}_j}[\psi] := \int_{\widetilde{\Gamma}_j}\mathrm{d}\mathbf{r}\left\{\left|\left(\nabla+i\frac{\widetilde{\mathbf{F}}}{\varepsilon^2}\right)\psi\right|^2-\frac{1}{2b\varepsilon^2}(2|\psi|^2-|\psi|^4)\right\}.$$

Lemma 5.3 (Replacement of the vector potential)**.**
Let $\Omega \subset \mathbb{R}^2$ *be a bounded domain satisfying Assumptions 2.1 and 2.2 and let* $\widetilde{\Gamma}_j$ *be the region defined in* (5.3)*. It holds*

$$\mathcal{G}^{\mathrm{GL}}_{\varepsilon,j}[\psi^{\mathrm{GL}}, \mathbf{A}^{\mathrm{GL}}] \geq \mathcal{G}_{\widetilde{\Gamma}_j}[\psi^{\mathrm{GL}}] + o(1).$$

Proof. We first recall that, thanks to Agmon estimates (2.19), we have

$$E^{\mathrm{GL}}_{\varepsilon} = \mathcal{G}^{\mathrm{GL}}_{\varepsilon,\partial\Omega}[\psi^{\mathrm{GL}}, \mathbf{A}^{\mathrm{GL}}] + \mathcal{O}(\varepsilon^{\infty}).$$

We now observe that, for each $\delta > 0$, it holds

$$\int_{\widetilde{\Gamma}_j} \mathrm{d}\mathbf{r} \left| \left(\nabla + i \frac{\mathbf{A}^{\mathrm{GL}}}{\varepsilon^2} \right) \psi^{\mathrm{GL}} \right|^2 \geq (1-\delta) \int_{\widetilde{\Gamma}_j} \mathrm{d}\mathbf{r} \left| \left(\nabla + i \frac{\widetilde{\mathbf{F}}}{\varepsilon^2} \right) \psi^{\mathrm{GL}} \right|^2 - \frac{1}{\delta} \int_{\widetilde{\Gamma}_j} \mathrm{d}\mathbf{r} \left| \frac{\mathbf{A}^{\mathrm{GL}}}{\varepsilon^2} - \frac{\widetilde{\mathbf{F}}}{\varepsilon^2} \right|^2. \tag{5.8}$$

We now use Lemma 5.2 and (2.9), to get

$$\|\mathbf{A}^{\mathrm{GL}} - \widetilde{\mathbf{F}}\|_{H^1(\mathbb{R}^2)} \leq C \|\mathrm{curl}\mathbf{A}^{\mathrm{GL}} - 1\|_{L^2(\mathbb{R}^2)} = \mathcal{O}(\varepsilon^{7/4}).$$

By Hölder inequality, Sobolev immersion and (2.7), we can control the second term of (5.8) as

$$\begin{aligned} -\frac{1}{\delta} \int_{\widetilde{\Gamma}_j} \mathrm{d}\mathbf{r} \left| \frac{\mathbf{A}^{\mathrm{GL}}}{\varepsilon^2} - \frac{\widetilde{\mathbf{F}}}{\varepsilon^2} \right|^2 |\psi^{\mathrm{GL}}|^2 &\geq -\frac{1}{\delta\varepsilon^4} \left(\int_{\widetilde{\Gamma}_j} \mathrm{d}\mathbf{r} \, |\mathbf{A}^{\mathrm{GL}} - \widetilde{\mathbf{F}}|^{2p} \right)^{\frac{1}{p}} |\widetilde{\Gamma}_j|^{1-\frac{1}{p}} \\ &\geq -C \frac{1}{\delta\varepsilon^4} \|\mathbf{A}^{\mathrm{GL}} - \widetilde{\mathbf{F}}\|^2_{L^{2p}(\mathbb{R}^2)} (\varepsilon^2 |\log\varepsilon|^2)^{1-\frac{1}{p}} = -C\frac{1}{\delta} \varepsilon^{\frac{3}{2}-\frac{2}{p}} |\log\varepsilon|^{2-\frac{2}{p}}. \end{aligned} \tag{5.9}$$

Choosing now p large enough we get

$$-\frac{1}{\delta} \int_{\widetilde{\Gamma}_j} \mathrm{d}\mathbf{r} \left| \frac{\mathbf{A}^{\mathrm{GL}}}{\varepsilon^2} - \frac{\widetilde{\mathbf{F}}}{\varepsilon^2} \right|^2 |\psi^{\mathrm{GL}}|^2 = o(\varepsilon\delta^{-1}).$$

Since by the estimate (2.8) we have

$$\delta \int_{\widetilde{\Gamma}_j} \mathrm{d}\mathbf{r} \left| \left(\nabla + i \frac{\widetilde{\mathbf{F}}}{\varepsilon^2} \right) \psi^{\mathrm{GL}} \right|^2 \leq C \frac{\delta}{\varepsilon},$$

we can choose some $\delta \ll \varepsilon$ to get

$$\mathcal{G}^{\mathrm{GL}}_{\varepsilon,j}[\psi^{\mathrm{GL}}, \mathbf{A}^{\mathrm{GL}}] \geq \int_{\widetilde{\Gamma}_j} \mathrm{d}\mathbf{r} \left\{ \left| \left(\nabla + i \frac{\widetilde{\mathbf{F}}}{\varepsilon^2} \right) \psi^{\mathrm{GL}} \right|^2 - \frac{1}{2b\varepsilon^2} (2|\psi^{\mathrm{GL}}|^2 - |\psi^{\mathrm{GL}}|^4) \right\} + o(1).$$

□

In the smooth part of the boundary layer we replace the minimizing vector potential following [CR14, Proposition 4.1]. In the proof of the main Theorem we then work (recall that (s,t) are the $\varepsilon-$rescaled boundary coordinates) with

$$\mathcal{G}_{\mathrm{cut}}[\psi] := \int_{A_{\mathrm{cut}}} \mathrm{d}s\mathrm{d}t\,(1-\varepsilon k(s)t)\left\{ |\partial_t\psi|^2 + \frac{1}{(1-\varepsilon k(s)t)^2}|(\partial_s + i\widetilde{a}_\varepsilon(s,t))\psi|^2 - \frac{1}{2b}(2|\psi|^2 - |\psi|^4)\right\},$$

where

$$\widetilde{a}_\varepsilon(s,t) = -t + \frac{1}{2}\varepsilon k(s)t^2 + \varepsilon\delta_\varepsilon s$$

and δ_ε is defined in (3.29).

5.4 The Case of a Polygon

We first consider a domain with zero curvature at the boundary (see, e.g., Figure 5.2 for some examples).

Theorem 5.2 (First order energy asymptotics $k=0$)**.**
Let $\mathcal{P} \subset \mathbb{R}^2$ such that $\partial\mathcal{P}$ is a polygon, then for any fixed

$$1 < b < \Theta_0^{-1},$$

it holds, as $\varepsilon \to 0$,

$$E_\varepsilon^{GL} = \frac{|\partial\mathcal{P}|E_0^{1D}}{\varepsilon} + \sum_{j\in\Sigma} E_{\mathrm{corner},j} + o(1). \tag{5.10}$$

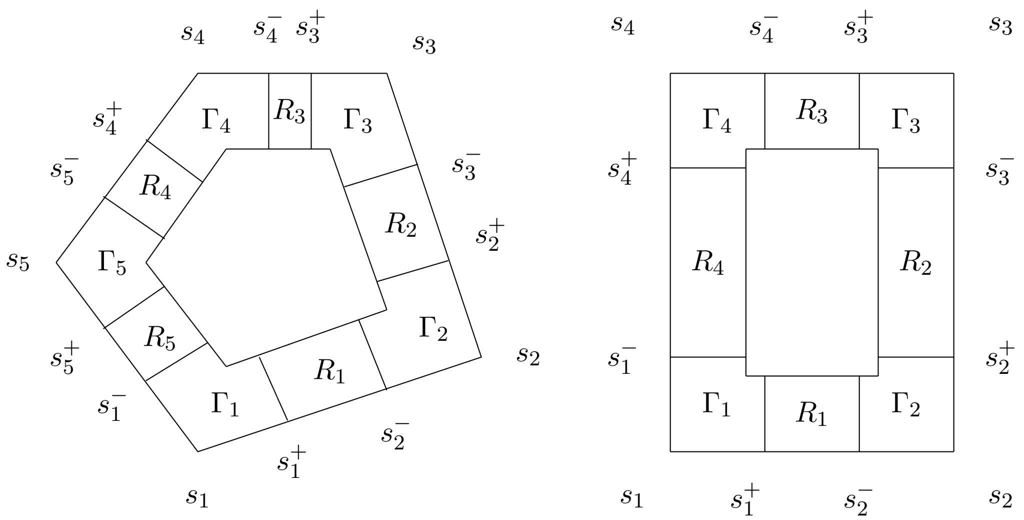

Figure 5.2. Examples of polygons $\mathcal{P}$.

Proof. We split the proof in two parts: we prove an upper bound and a lower bound.

We stress once more that we will work in a different way in the regions near the corner and in the smooth part of the boundary layer. In particular as we explained in Section 5.1 we split the boundary layer as follows

$$\mathcal{A}_\varepsilon = \mathcal{A}_{\mathrm{cut}} \cup \bigcup_{j\in\Sigma} \Gamma_j.$$

We will use the $\varepsilon-$rescaled boundary coordinates (s,t) in the region $\mathcal{A}_\varepsilon \setminus \cup_j \Gamma_j$.

In the regions Γ_j, $j \in \Sigma$, we will use the effective problem introduced in Section 4.2. We then denote by $\psi_j \in \mathscr{D}_{\mathcal{D}}(\Gamma_j)$ any minimizing function in

$$\mathscr{D}_{\mathcal{D}}(\Gamma_j) := \left\{ \psi \in H^1(\Gamma_j) \,|\, \psi = 0 \;\text{ on } \partial\Gamma_j^{\mathrm{bulk}},\, \psi = g(t)e^{-iS(s)} \;\text{ if } |s - s_j| = c_1|\log\varepsilon| \right\}, \tag{5.11}$$

of the functional

$$\mathcal{G}_{\Gamma_j}[\psi] = \int_{\Gamma_j} \mathrm{d}\mathbf{r} \left\{ \left| \left(\nabla + i\frac{\widetilde{\mathbf{F}}}{\varepsilon} \right) \psi_j \right|^2 - \frac{1}{2b}(2|\psi|^2 - |\psi|^4) \right\},$$

where s_j is the tangential coordinate of the vertex, the function $g(t)$ is the one defined in (4.6) and the phase is $S(s) = (\alpha_0 - \delta_{c_1|\log\varepsilon|})s$ with

$$\delta_{c_1|\log\varepsilon|} := \left(\alpha_0 \frac{|\partial\Omega|}{\varepsilon} - 2\pi \left\lfloor \alpha_0 \frac{|\partial\Omega|}{2\pi\varepsilon} \right\rfloor \right) \frac{\varepsilon}{|\partial\Omega|} = \mathcal{O}(\varepsilon). \tag{5.12}$$

For the upper bound we have to chose a suitable trial state $(\psi_{\mathrm{trial}}, \mathbf{A}_{\mathrm{trial}})$. As trial vector potential we choose $\mathbf{A}_{\mathrm{trial}} = \widetilde{\mathbf{F}}$ and we observe that since the boundary curvature is equal to zero, then $\widetilde{\mathbf{F}} = (-\varepsilon t, 0)$ in $\mathcal{A}_{\mathrm{cut}}$. For the trial order parameter, we use a function that is exponentially small suitably far from the boundary of $\mathcal{P}$. In particular, we choose ψ_{trial} in such a way that $\psi_{\mathrm{trial}} \equiv 0$ in $\mathcal{A}_\varepsilon^c$. We now give the precise definition of the trial order parameter in the region $\mathcal{A}_\varepsilon$

$$\psi_{\mathrm{trial}} := \begin{cases} \psi_j, & \text{in } \Gamma_j, \\ g(t)e^{-iS(s)}, & \text{in } \mathcal{A}_\varepsilon \setminus \cup_{j\in\Sigma}\Gamma_j. \end{cases}$$

The choice (5.12) guarantees that $S(s)$ is globally well defined, i.e.,

$$S(s + \varepsilon^{-1}|\partial\Omega|) - S(s) \in 2\pi\mathbb{Z}, \qquad \forall s \in \mathcal{A}_{\mathrm{cut}}.$$

Testing the Ginzburg-Landau functional on the trial configuration $(\psi_{\mathrm{trial}}, \mathbf{A}_{\mathrm{trial}})$, we get

$$\mathcal{G}_\varepsilon^{\mathrm{GL}}[\psi_{\mathrm{trial}}, \mathbf{A}_{\mathrm{trial}}] = \sum_{j\in\Sigma} \mathcal{G}_{\Gamma_j}[\psi_j] + \mathcal{G}^{\mathrm{GL}}_{\varepsilon,\mathcal{A}_\varepsilon}[\psi_{\mathrm{trial}}, \widetilde{\mathbf{F}}]\Big|_{\mathcal{A}_{\mathrm{cut}}}$$

In $\mathcal{A}_{\mathrm{cut}}$ we can use the energy asymptotics proved in a rectangular region in Theorem 4.2 to get

$$\begin{aligned} E_\varepsilon^{GL} &\leq \frac{|\partial\mathcal{P}|E_0^{1D}}{\varepsilon} + \sum_{j\in\Sigma} \big[-2c_1|\log\varepsilon| + E_{\mathcal{D}}(\Gamma_j) \big] + o(1) \\ &= \frac{|\partial\mathcal{P}|E_0^{1D}}{\varepsilon} + \sum_{j\in\Sigma} E_{\mathrm{corner},j}(c_1|\log\varepsilon|) + o(1), \end{aligned} \tag{5.13}$$

In the limit $\varepsilon \to 0$ it then follows that

$$E_\varepsilon^{GL} \leq \frac{|\partial \mathcal{P}| E_0^{1D}}{\varepsilon} + \sum_{j \in \Sigma} E_{\text{corner},j} + o(1). \tag{5.14}$$

For the lower bound we first observe that by Agmon estimates (Section 2.6), we can reduce the GL functional to the boundary layer $\mathcal{A}_\varepsilon$ [FH10, 14.4.1]

$$\mathcal{G}_\varepsilon^{\mathrm{GL}}[\psi^{\mathrm{GL}}, \mathbf{A}^{\mathrm{GL}}] \geq \int_{\mathcal{A}_\varepsilon} \mathrm{d}\mathbf{r} \left\{ \left| \left(\nabla + i \frac{\mathbf{A}^{\mathrm{GL}}}{\varepsilon} \right) \psi^{\mathrm{GL}} \right|^2 - \frac{1}{2b} (2|\psi^{\mathrm{GL}}|^2 - |\psi^{\mathrm{GL}}|^4) \right\} + \mathcal{O}(\varepsilon^\infty).$$

Now we replace the vector potential as described in Section 5.3 and get

$$\mathcal{G}_\varepsilon^{\mathrm{GL}}[\psi^{\mathrm{GL}}, \mathbf{A}^{\mathrm{GL}}] \geq \sum_{j \in \Sigma} \mathcal{G}_{\Gamma_j}[\psi^{\mathrm{GL}}] + \mathcal{G}_{\mathrm{cut}}[\psi^{\mathrm{GL}}] + o(1). \tag{5.15}$$

We now work in $\mathcal{A}_{\mathrm{cut}}$. First of all we write $\mathcal{A}_{\mathrm{cut}}$ as union of rectangular region R_j as follows

$$\mathcal{A}_{\mathrm{cut}} =: \bigcup_{j \in \Sigma} R_j.$$

In each rectangular region R_j we define a new function $u_j(s,t)$ as

$$\psi^{\mathrm{GL}}(s,t) =: f_0(t) u_j(s,t) e^{-i\alpha_0 s}, \qquad \forall (s,t) \in R_j.$$

Proceeding as in the proof of Theorem 4.2 we get

$$E_\varepsilon^{\mathrm{GL}} \geq \frac{|\partial \mathcal{P}| E_0^{1D}}{\varepsilon} - 2c_1 \sum_{j \in \Sigma} |\log \varepsilon| E_0^{1D} + \sum_{j \in \Sigma} \mathcal{E}_0[u_j] \tag{5.16}$$

where the functional $\mathcal{E}_0[u_j]$ is defined analogously to the functional (4.15), i.e.,

$$\mathcal{E}_0[u_j] = \int_{R_j} \mathrm{d}s\mathrm{d}t \, f_0^2(t) \left\{ |\nabla u_j|^2 - 2(t + \alpha_0) j_s[u_j] - \frac{f_0^2(t)}{2b} (1 - |u_j|^2)^2 \right\}.$$

As in Proposition 4.3, by integrating by parts, we obtain

$$\mathcal{E}_0[u_j] \geq \int_{R_k} \mathrm{d}s\mathrm{d}t \left\{ K_0(t) |\nabla u_j|^2 + \frac{f_0^4(t)}{2b} (1 - |u_j|^2)^2 \right\} + \sum_{j \in \Sigma} \int_0^{t_\varepsilon} \mathrm{d}t \, F_0(t) j_t[u_j] \Big|_{s = s_j^+}^{s = s_j^-},$$

where $t_\varepsilon := c_0 |\log \varepsilon|$. Furthermore, thanks to the positivity of the cost function (2.44), we have

$$\mathcal{E}_0[u_j] \geq \sum_{j \in \Sigma} \int_0^{t_\varepsilon} \mathrm{d}t \, F_0(t) j_t[u_j] \Big|_{s = s_j^+}^{s = s_j^-} \tag{5.17}$$

From (5.15), (5.16) and (5.17), we get

$$E_\varepsilon^{\mathrm{GL}} \geq \frac{|\partial \mathcal{P}| E_0^{1D}}{\varepsilon} + \sum_{j \in \Sigma} \left[-2c_1 |\log \varepsilon| + \mathcal{G}_{\Gamma_j}[\psi^{\mathrm{GL}}] - \int_0^{t_\varepsilon} \mathrm{d}t \, F_0(t) j_t[u_j] \Big|_{s = s_j^-}^{s = s_j^+} \right] + o(1). \tag{5.18}$$

To complete the proof, it suffices to remark that

$$\sum_{j\in\Sigma}\left[\mathcal{G}_{\Gamma_j}[\psi^{\mathrm{GL}}]-\int_0^{t_\varepsilon}\mathrm{d}t\,F_0(t)j_t[u_j]\Big|_{s=s_j^-}^{s=s_j^+}\right]\geq\sum_{j\in\Sigma}\widetilde{E}_{\mathcal{N}}(\Gamma_j),\tag{5.19}$$

where we recall that

$$\widetilde{E}_{\mathcal{N}}(\Gamma_j)=\inf_{\psi\in\mathscr{D}_{\mathcal{N}}(\Gamma_j)}\left[\mathcal{G}_{\Gamma_j}[\psi^{\mathrm{GL}}]-\int_0^{t_\varepsilon}\mathrm{d}t\,\frac{F_0(t)}{f_0^2(t)}j_t[\psi]\Big|_{s=s_j^-}^{s=s_j^+}\right]$$

with

$$\mathscr{D}_{\mathcal{N}}(\Gamma_j)=\left\{\psi\in H^1(\Gamma_j)\,|\,\psi|_{\partial\Gamma_j^{\mathrm{bulk}}}=0\right\}.$$

Putting together (5.18) and (5.19), we are left with

$$E_\varepsilon^{GL}\geq\frac{|\partial\mathcal{P}|E_0^{1D}}{\varepsilon}+\sum_{j\in\Sigma}\left[-2c_1|\log\varepsilon|+\widetilde{E}_{\mathcal{N}}(\Gamma_j)\right]+o(1).\tag{5.20}$$

Now we can use Theorem 4.8, to get

$$\sum_{j\in\Sigma}\left[-2c_1|\log\varepsilon|+\widetilde{E}_{\mathcal{N}}(\Gamma_j)\right]\geq\sum_{j\in\Sigma}\left[-2c_1|\log\varepsilon|+E_{\mathcal{D}}(\Gamma_j)\right]+o(1)=\sum_{j\in\Sigma}E_{\mathrm{corner},j}(c_1|\log\varepsilon|)+o(1).\tag{5.21}$$

Taking the limit $\varepsilon\to 0$ in (5.20) and using (5.21), we finally get

$$E_\varepsilon^{\mathrm{GL}}\geq\frac{|\partial\mathcal{P}|E_0^{1D}}{\varepsilon}+\sum_{j\in\Sigma}E_{\mathrm{corner},j}+o(1).\tag{5.22}$$

The upper bound (5.14) and the lower bound (5.22) together yield that as $\varepsilon\to 0$

$$E_\varepsilon^{GL}=\frac{|\partial\mathcal{P}|E_0^{1D}}{\varepsilon}+\sum_{j\in\Sigma}E_{\mathrm{corner},j}+o(1).$$

□

5.5 The Case of a General Domain

We now consider a general domain with at most a finite number of corners along its boundary. We want to extend Theorem 5.2 to a generic domain, the only difference is that we have now to take into account the curvature of the smooth part of the boundary.

5.5.1 Setting

As usual we can restrict our analysis to a suitable boundary layer via Agmon estimates (Section 2.6). As in the proof of Theorem 5.2 we work in a different way near the corner (for the moment we suppose there is only one) and in the smooth part of the boundary layer.

Near the corner we will consider a region Γ with the same properties as the one represented in Figure 4.1 and with $\ell=c_1|\log\varepsilon|$. We will denote by s_β the coordinate at the vertex and by $s_\pm$ the two points of $\partial\Gamma^{\mathrm{ext}}$ such that $|s_\pm-s_\beta|=c_1|\log\varepsilon|$.

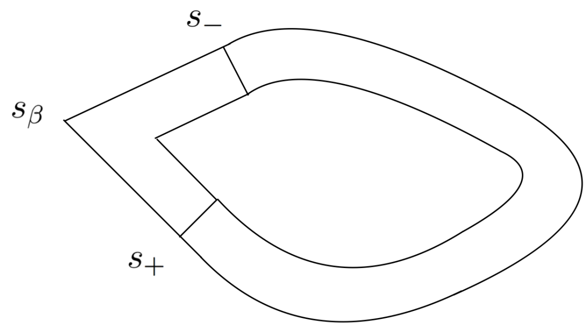

Figure 5.3. An example of a generic domain Ω with only one corner along $\partial\Omega$.

The main difference with respect to the case of a polygon is that here the boundary curvature is not trivial. We then need some transition regions to glue together the effective problem in Γ with the one in the smooth part of the boundary layer. We use the following regions

$$R_\delta^- := \left\{(s,t) \,|\, s \in [s_\delta^-, s_-],\, t \in [0, t_\varepsilon]\right\}, \tag{5.23}$$

and

$$R_\delta^+ := \left\{(s,t) \,|\, s \in [s_+, s_\delta^+],\, t \in [0, t_\varepsilon]\right\}, \tag{5.24}$$

where $t_\varepsilon := c_0|\log\varepsilon|$, $\delta = o(1)$ and $s_\delta^\pm$ are the two points of $\partial\Omega$ such that $|s_\delta^\pm - s_\pm| = \delta$. We also set

$$\Gamma_\delta := R_\delta^- \cup \Gamma \cup R_\delta^+.$$

For simplicity we fix the origin of the rescaled tangential coordinate s in s_δ^-, then

$$s_\delta^- = 0, \qquad s_- = \delta, \qquad s_\beta = c_1|\log\varepsilon| + \delta, \qquad s_+ = 2c_1|\log\varepsilon| + \delta, \qquad s_\delta^+ = 2c_1|\log\varepsilon| + 2\delta.$$

In the smooth part of the boundary layer, we introduce a cell decomposition as in [CR16a]:

$$\mathcal{A}_\varepsilon \setminus \Gamma_\delta = \bigcup_{j=1}^{N_\varepsilon} \mathscr{C}_j,$$

where

$$\mathscr{C}_j := [s_j, s_{j+1}] \times [0, c_0|\log\varepsilon|], \qquad \text{with } |s_{j+1} - s_j| = 1$$

and

$$N_\varepsilon = \frac{|\partial\Omega \setminus \partial\Gamma_\delta|}{\varepsilon} \propto \varepsilon^{-1}$$

We will approximate the curvature $\widetilde{k}(\sigma)$ of the boundary in each cell by its mean value

$$k_j := \int_{s_j}^{s_{j+1}} \mathrm{d}s\, k(s),$$

recall that $k(s) = \widetilde{k}(\sigma) = \widetilde{k}(\varepsilon s)$. Since the boundary curvature is not trivial we can not neglect its contributions in the 1D-effective problem. For this reason we define

$$\alpha_j := \alpha(k_j), \qquad f_j(t) := f_{k_j}(t),$$

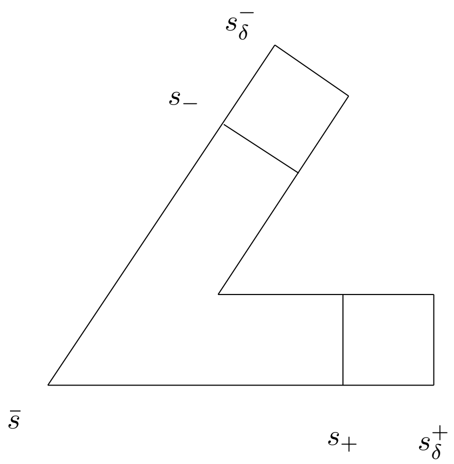

Figure 5.4. The boundary layer near the corner.

and we recall that $(\alpha(k_j), f_{k_j})$ is the minimizing pair of the functional $\mathcal{E}^{1D}_{k_j,\alpha}[f]$ defined in (2.32), i.e.,

$$\mathcal{E}^{1D}_{k_j,\alpha}[f] := \int_0^{c_0|\log\varepsilon|} \mathrm{d}t\,(1-\varepsilon k_j t)\left\{|\partial_t f|^2 + \frac{(t+\alpha-\frac{1}{2}\varepsilon k_j t^2)^2}{(1-\varepsilon k_j t)^2} f^2 - \frac{1}{2b}(2f^2 - f^4)\right\}.$$

In the proof of Theorem 5.1, we also use the following function

$$g_j(t) := \begin{cases} f_j(t) & \text{if } \ t \in [0, \bar{t}] \\ \bar{f}_j(t) & \text{if } \ t \in [\bar{t}, t_\varepsilon], \end{cases}$$

where $\bar{t} := t_\varepsilon(1-\gamma)$ and $\gamma \ll 1$. We pick a function $\bar{f}_j(t)$ such that: $\bar{f}_j(t)$ is monotone, $\bar{f}_j(\bar{t}) = f_j(\bar{t})$ and $\bar{f}_j(t_\varepsilon) = 0$. We also assume that

$$\left\|\bar{f}_j(t)\right\|_{L^\infty[\bar{t},t_\varepsilon]} \leq f_j(\bar{t}), \qquad \|\bar{f}_j'(t)\|_{L^\infty[\bar{t},t_\varepsilon]} = \mathcal{O}\left(f_0(\bar{t})\,(t_\varepsilon\gamma)^{-1}\right.$$

In first approximation, we can consider the simpler case of a domain such that:

(a) the boundary curvature of $\partial\Omega \cap \partial\Gamma$ is equal to zero,

(b) there is only one corner along the boundary of $\partial\Omega$.

At the end of the Chapter we will underline how the proof can be adapted to a general bounded domain with non trivial boundary curvature and with at most a finite number of corners along $\partial\Omega$.

5.5.2 Upper bound

For simplicity we only prove the upper bound for a corner domain satisfying assumptions (a) and (b).

Proposition 5.1 (Upper bound)**.**
Let $\Omega \subset \mathbb{R}^2$ be bounded domain satisfying assumptions (a) and (b) then for any fixed

$$b < \Theta_0^{-1},$$

it holds, as $\varepsilon \to 0$,

$$E_\varepsilon^{GL} \leq \frac{|\partial\Omega| E_0^{1D}}{\varepsilon} - \mathcal{E}_{\alpha_0}^{\mathrm{corr}}[f_0] \int_0^{|\partial\Omega|} \mathrm{d}\sigma\, \tilde{k}(\sigma) + E_{\mathrm{corner}} + o(1). \tag{5.25}$$

Proof. We split the proof in several steps.

Step 1 (*Trial State*). As usual to prove the upper bound, we choose a suitable trial state $(\psi_{\mathrm{trial}}, \mathbf{A}_{\mathrm{trial}})$. As before we choose $\mathbf{A}_{\mathrm{trial}} = \widetilde{\mathbf{F}}$ and $\psi_{\mathrm{trial}} \equiv 0$ in $\mathcal{A}_\varepsilon^c$. In $\mathcal{A}_\varepsilon$, we set

$$\psi_{\mathrm{trial}} := \begin{cases} \psi_\Gamma, & \text{in } \Gamma, \\ (g(t) + \chi_-) e^{-i(S_-(s) - \omega_\varepsilon s)}, & \text{in } \mathcal{R}_\delta^-, \\ (g(t) + \chi_+) e^{-i(S_+(s) - \omega_\varepsilon s)}, & \text{in } \mathcal{R}_\delta^+, \\ g_j\, e^{-i(\alpha_j - \omega_\varepsilon) s}, & \text{in } \mathscr{C}_j. \end{cases} \tag{5.26}$$

We denoted by $\psi_\Gamma \in \mathscr{D}_{\mathcal{D}}(\Gamma)$ the minimizing function of the effective problem $\mathcal{G}_\Gamma[\psi]$ with fixed vector potential equal to $\widetilde{\mathbf{F}}$. We recall that

$$\mathcal{G}_\Gamma[\psi] = \int_\Gamma \mathrm{d}\mathbf{r} \left\{ \left|(\nabla + i\widetilde{\mathbf{F}})\psi\right|^2 - \frac{1}{2b}(2|\psi|^2 - |\psi|^4) \right\}$$

and

$$\mathscr{D}_{\mathcal{D}}(\Gamma) = \left\{ \psi \in H^1(\Gamma) \,|\, \psi_{\partial\Gamma^{\mathrm{bulk}}} = 0,\ \psi|_{\partial\Gamma^\pm} = g(t) e^{-iS(s)} \right\},$$

with $g(t)$ and $S(s)$ defined exactly as in Theorem 5.2. We now define the trial order parameter in $\mathcal{A}_\varepsilon \setminus \Gamma$ as in [CR16a] inside the cell $\mathscr{C}_j$: we set

$$\psi_j(s,t) := \left[g_j(t) + \chi_j(s,t)\right],$$

with

$$\chi_j(s,t) := \left[g_{j+1}(t) - g_j(t)\right] \left(1 - \frac{s - s_{j+1}}{s_j - s_{j+1}}\right).$$

In the transition regions $R_\delta^\pm$ we have to choose suitable functions in order to have a continuous trial order parameter ψ_{trial} . We denote by $k_- := k_1$ and by $k_+ := k_{N_\varepsilon}$, the mean values of the curvature in the cell $\mathscr{C}_1$ and in the cell $\mathscr{C}_{N_\varepsilon}$ respectively, and we define:

$$\chi_-(s,t) := \left(g_{k_-}(t) - g(t)\right) \left(1 - \frac{s - s_\delta^-}{s_- - s_\delta^-}\right),$$

$$\chi_+(s,t) := \left(g_{k_+}(t) - g(t)\right) \left(1 - \frac{s - s_\delta^+}{s_+ - s_\delta^+}\right).$$

For the phase of ψ_{trial} in $R_\delta^{\pm}$, we set

$$S_-(s) := \alpha_0 \frac{s_-}{\gamma}(s - s_\delta^-) - \alpha_{k_-} \frac{s_\delta^-}{\gamma}(s - s_-),$$
$$S_+(s) := -\alpha_0 \frac{s_+}{\gamma}(s - s_\delta^+) + \alpha_{k_+} \frac{s_\delta^+}{\gamma}(s - s_+).$$

For simplicity we denote by ω_ε the factor $\omega_\varepsilon := \delta_{c_1|\log\varepsilon|}$ defined in (5.12),i.e., the factor that ensures that the phase of the trial state is globally well defined. With our choices we have a continuous trial function such that

$$\psi_{\mathrm{trial}}(s_-, t) = g(t) e^{-i(\alpha_0 - \omega_\varepsilon)s_-}, \qquad \psi_{\mathrm{trial}}(s_\delta^-, t) = g_{k_-}(t) e^{-i(\alpha_{k_-} - \omega_\varepsilon)s_\delta^-},$$

and

$$\psi_{\mathrm{trial}}(s_+, t) = g(t) e^{-i(\alpha_0 - \omega_\varepsilon)s_+}, \qquad \psi_{\mathrm{trial}}(s_\delta^+, t) = g_{k_+}(t) e^{-i(\alpha_{k_+} - \omega_\varepsilon)s_\delta^+}.$$

Step 2 (*Upper bound near the corner*). Testing the GL functional on the trial state $(\psi_{\mathrm{trial}}, \mathbf{A}_{\mathrm{trial}})$ in the region Γ, we have

$$\mathcal{G}_\varepsilon^{\mathrm{GL}}[\psi_{\mathrm{trial}}, \mathbf{A}_{\mathrm{trial}}]\Big|_{\Gamma} = \int_\Gamma \mathrm{d}\mathbf{r} \left\{ \left|(\nabla + i\widetilde{\mathbf{F}})\psi_\Gamma\right|^2 - \frac{1}{2b}(2|\psi_\Gamma|^2 - |\psi_\Gamma|^4) \right\} = E_{\mathcal{D}}(\Gamma) \quad (5.27)$$

Step 3 (*Upper bound in $\mathcal{A}_\varepsilon \setminus \Gamma_\delta$*). We now observe that proceeding as in [CR16b, Section 4] in the region $\mathcal{A}_\varepsilon \setminus \Gamma_\delta$ we have that

$$\mathcal{G}_\varepsilon^{\mathrm{GL}}[\psi_{\mathrm{trial}}, \mathbf{A}_{\mathrm{trial}}]\Big|_{\mathcal{A}_\varepsilon \setminus \Gamma_\delta} = \frac{|\partial\Omega \setminus \partial\Gamma_\delta| E_0^{1D}}{\varepsilon} - \varepsilon \mathcal{E}_{\alpha_0}^{\mathrm{corr}}[f_0] \int_{s_\delta^+}^{s_\delta^-} \mathrm{d}s\, k(s) + o(1), \quad (5.28)$$

where we used also Theorem 2.8. Then from (5.28) we get

$$\mathcal{G}_\varepsilon^{GL}[\psi_{\mathrm{trial}}, \mathbf{A}_{\mathrm{trial}}]\Big|_{\mathcal{A}_\varepsilon \setminus \Gamma} = \left[\frac{|\partial\Omega|}{\varepsilon} - 2c_1|\log\varepsilon| - 2\delta \right] E_0^{1D} - \varepsilon \mathcal{E}_{\alpha_0}^{\mathrm{corr}}[f_0] \int_0^{\frac{|\partial\Omega|}{\varepsilon}} \mathrm{d}s\, k(s) + o(1), \quad (5.29)$$

where we used the fact that (recall $\delta \ll 1$)

$$\int_{s_\delta^+}^{s_\delta^-} \mathrm{d}s\, k(s) = \int_0^{\frac{|\partial\Omega|}{\varepsilon}} \mathrm{d}s\, k(s) + o(1).$$

Step 4 (*Upper bound in the transition regions*). We now work in the region $R_\delta\pm$. We have

$$\begin{aligned} &\mathcal{G}_\varepsilon^{\mathrm{GL}}[\psi_{\mathrm{trial}}, \mathbf{A}_{\mathrm{trial}}]\Big|_{\Gamma_\delta \setminus \Gamma} \\ &= (1 + o(1)) \int_{R_\delta^-} \mathrm{d}s\mathrm{d}t \left\{ |\partial_t \psi_{\mathrm{trial}}|^2 + (\partial_s - it)\psi_{\mathrm{trial}}|^2 - \frac{1}{2b}(2|\psi_{\mathrm{trial}}|^2 - |\psi_{\mathrm{trial}}|^4) \right\} \\ &\quad + (1 + o(1)) \int_{R_\delta^+} \mathrm{d}s\mathrm{d}t \left\{ |\partial_t \psi_{\mathrm{trial}}|^2 + (\partial_s - it)\psi_{\mathrm{trial}}|^2 - \frac{1}{2b}(2|\psi_{\mathrm{trial}}|^2 - |\psi_{\mathrm{trial}}|^4) \right\}, \end{aligned}$$

where the prefactor is due to the estimate of the Jacobian of the change to tubular coordinates. Here we use again that $\delta \ll 1$.

We now estimate the energy contribution in R_δ^+. First of all, proceeding as in Proposition 4.1 we can discard the energy contribution in the region $R_\delta^+ \cap \{ t \geq \bar{t} \}$ thanks to the exponential decay of f_0 (Lemma 2.1), then

$$\mathcal{G}_\varepsilon^{\mathrm{GL}}[\psi_{\mathrm{trial}}, \mathbf{A}_{\mathrm{trial}}]\Big|_{R_\delta^+} = \int_{s_+}^{s_\delta^+} \mathrm{d}s \int_0^{\bar{t}} \mathrm{d}t \left\{ \left| \left(\nabla + i \frac{\widetilde{\mathbf{F}}}{\varepsilon} \right) \psi_{\mathrm{trial}} \right|^2 - \frac{1}{2b} (2|\psi_{\mathrm{trial}}|^2 - |\psi_{\mathrm{trial}}|^4) \right\} + \mathcal{O}(\varepsilon^\infty).$$

From now on we will often use the estimates (2.46) and (2.47), i.e.,

$$|\alpha_k - \alpha_0| = \mathcal{O}(\varepsilon^{\frac{1}{2}} |\log \varepsilon|^\infty) \qquad \text{for } k \in \mathbb{R}^+ \tag{5.30}$$

$$|f_0(t) - f_k(t)| = \mathcal{O}(\varepsilon^{\frac{1}{2}} |\log \varepsilon|^\infty) \qquad \text{for } k \in \mathbb{R}^+. \tag{5.31}$$

We now split the proof of the upper bound in R_δ^+ in some more steps.

- *Kinetic energy $t-$component.* We first consider the $t-$derivative of the trial order parameter in R_δ^+ and we observe that

$$\partial_t [(f_0(t) + \chi_+) e^{-i(S_+(s) - \omega_\varepsilon s)}] = \left[f_0'(t) + [f_{k_+}'(t) - f_0'(t)] \left(1 - \frac{s - s_\delta^+}{s_+ - s_\delta^+} \right) \right] e^{-i(S_+(s) - \omega_\varepsilon s)}.$$

Then via (5.31) we get

$$\begin{aligned} \left| \partial_t [(f_0(t) + \chi_+) e^{-i(S_+(s) - \omega_\varepsilon s)}] \right|^2 &= \left| f_0' + (f_k' - f_0') \left(1 - \frac{s - s_\delta^-}{s_- - s_\delta^-} \right) \right|^2 \\ &= |f_0'|^2 + \mathcal{O}(\varepsilon^{\frac{1}{2}} |\log \varepsilon|^\infty). \end{aligned} \tag{5.32}$$

- *Kinetic energy $s-$component.* For the $s-$component of the kinetic energy in the region R_δ^+ we have that

$$|(\partial_s - it)\psi_{\mathrm{trial}}|^2 = \left| [t + S_+'(s) - \omega_\varepsilon] \left[-i f_0(t) - i \left(f_k(t) - f_0(t) \right) \left(1 - \frac{s - s_\delta^+}{s_+ - s_\delta^+} \right) \right] + \frac{f_k(t) - f_0(t)}{\delta} \right|^2.$$

We now observe that

$$t + S_+'(s) - \omega_\varepsilon = t - \frac{s_+}{\delta} \alpha_0 + \frac{s_\delta^+}{\delta} \alpha_{k_+} - \omega_\varepsilon,$$

and simply using

$$s_+ = s_\delta^+ - \delta,$$

we get

$$t - \frac{s_+}{\delta}\alpha_0 + \frac{s_\delta^+}{\delta}\alpha_{k_+} - \omega_\varepsilon = t + \alpha_0 + \frac{s_\delta^+}{\delta}(\alpha_{k_+} - \alpha_0) - \omega_\varepsilon$$

Then,

$$\begin{aligned}|(\partial_s - it)\psi_{\text{trial}}|^2 &= \frac{1}{\delta^2}(f_{k_+} - f_0)^2 \\ &+ \left|\left[t + \alpha_0 + \frac{s_\delta^+}{\delta}(\alpha_{k_+} - \alpha_0) - \omega_\varepsilon\right]\left[f_0(t) + (f_{k_+}(t) - f_0(t))\Big(1 - \frac{s - s_\delta^+}{s_+ - s_\delta^+}\Big)\right]\right|^2.\end{aligned}$$

By straightforward calculations, we thus get

$$\begin{aligned}&\left|(\partial_s - it)\psi_{\text{trial}}\right|^2 \\ &\le f_0^2(t)(t+\alpha_0)^2 + \frac{1}{\delta^2}(f_{k_+}(t) - f_0(t))^2 + \left|f_0(t)\left[\frac{s_\delta^+}{\delta}(\alpha_{k_+} - \alpha_0)\right]\right|^2 + |\omega_\varepsilon f_0(t)|^2 \\ &+ \left|(f_{k_+}(t) - f_0(t))(t+\alpha_0)\Big(1 - \frac{s - s_\delta^+}{s_+ - s_\delta^+}\Big)\right|^2 \\ &+ \left|\omega_\varepsilon(f_{k_+}(t) - f_0(t))\Big(1 - \frac{s - s_\delta^+}{s_+ - s_\delta^+}\Big)\right|^2 \\ &+ \left|\frac{s_\delta^+}{\delta}(f_{k_+}(t) - f_0(t))(\alpha_{k_+} - \alpha_0)\Big(1 - \frac{s - s_\delta^+}{s_+ - s_\delta^+}\Big)\right|^2 \\ &+ 2f_0(t)(t+\alpha_0)\left[(f_{k_+}(t) - f_0(t))\left[t + \alpha_0 + \frac{s_\delta^+}{\delta}(\alpha_{k_+} - \alpha_0) - \omega_\varepsilon\right]\Big(1 - \frac{s - s_\delta^+}{s_+ - s_\delta^+}\Big)\right] \\ &+ 2f_0^2(t)(t+\alpha_0)\left[\frac{s_\delta^+}{\delta}(\alpha_{k_+} - \alpha_0) - \omega_\varepsilon\right]\end{aligned}$$

In what follows we estimate each term on the r.h.s. of the above expression, with the only exception of the term $f_0^2(t)(t+\alpha_0)^2$. Via (5.30) and (5.31) and the fact that

$$\frac{(s_\delta^+)^2}{\delta^2} = \mathcal{O}(\delta^{-2}|\log\varepsilon|^2), \qquad \omega_\varepsilon = \mathcal{O}(\varepsilon),$$

we have

$$\frac{1}{\delta^2}\int_0^{t_\varepsilon} \mathrm{d}t \int_{s_+}^{s_\delta^+} \mathrm{d}s\,(f_{k_+}(t) - f_0(t))^2 = \mathcal{O}(\varepsilon\delta^{-1}|\log\varepsilon|^\infty),$$

$$\frac{(s_\delta^+)^2}{\delta^2}\int_0^{t_\varepsilon} \mathrm{d}t \int_{s_+}^{s_\delta^+} \mathrm{d}s\, f_0^2(t)(\alpha_{k_+} - \alpha_0)^2 = \mathcal{O}(\varepsilon\delta^{-1}|\log\varepsilon|^\infty),$$

$$\int_0^{t_\varepsilon} \mathrm{d}t \int_{s_+}^{s_\delta^+} \mathrm{d}s\, f_0^2(t)\omega_\varepsilon = \mathcal{O}(\varepsilon\delta),$$

$$\int_0^{t_\varepsilon} \mathrm{d}t \int_{s_+}^{s_\delta^+} \mathrm{d}s\,(f_{k_+}(t) - f_0(t))^2(t+\alpha_0)^2\Big(1 - \frac{s - s_\delta^+}{s_+ - s_\delta^+}\Big)^2 = \mathcal{O}\left(\varepsilon\delta|\log\varepsilon|^\infty\right),$$

$$\frac{(s_\delta^+)^2}{\delta^2}\int_0^{t_\varepsilon}\mathrm{d}t\int_{s_+}^{s_\delta^+}\mathrm{d}s\,(f_{k_+}(t)-f_0(t))^2(\alpha_{k_+}-\alpha_0)^2\Big(1-\frac{s-s_\delta^+}{s_+-s_\delta^+}\Big)^2 = \mathcal{O}\left(\varepsilon^2\delta^{-1}|\log\varepsilon|^\infty\right),$$

$$\int_0^{t_\varepsilon}\mathrm{d}t\int_{s_+}^{s_\delta^+}\mathrm{d}s\left|\omega_\varepsilon(f_{k_+}(t)-f_0(t))\left(1-\frac{s-s_\delta^+}{s_+-s_\delta^+}\right)\right|^2 = \mathcal{O}(\varepsilon^3\delta|\log\varepsilon|^\infty),$$

$$2\int_0^{t_\varepsilon}\mathrm{d}t\int_{s_+}^{s_\delta^+}\mathrm{d}s\,f_0(t)(t+\alpha_0)^2(f_{k_+}(t)-f_0(t))\Big(1-\frac{s-s_\delta^+}{s_+-s_\delta^+}\Big) = \mathcal{O}(\varepsilon^{\frac12}\delta|\log\varepsilon|^\infty),$$

$$\frac{2s_\delta^+}{\delta}\int_0^{t_\varepsilon}\mathrm{d}t\int_{s_-}^{s_\delta^-}\mathrm{d}s\,f_0(t)(t+\alpha_0)(f_{k_+}(t)-f_0(t))(\alpha_{k_+}-\alpha_0)\Big(1-\frac{s-s_\delta^+}{s_+-s_\delta^+}\Big) = \mathcal{O}(\varepsilon|\log\varepsilon|^\infty),$$

$$-2\omega_\varepsilon\int_0^{t_\varepsilon}\mathrm{d}t\int_{s_+}^{s_\delta^+}\mathrm{d}s\,f_0(t)(t+\alpha_0)(f_{k_+}(t)-f_0(t))\Big(1-\frac{s-s_\delta^+}{s_+-s_\delta^+}\Big) = \mathcal{O}(\varepsilon^{\frac32}\delta|\log\varepsilon|^\infty),$$

$$\frac{2s_\delta^+}{\gamma}\int_0^{t_\varepsilon}\mathrm{d}t\int_{s_+}^{s_\delta^+}\mathrm{d}s\,f_0^2(t)(t+\alpha_0)(\alpha_{k_+}-\alpha_0) = \mathcal{O}(\varepsilon^\infty),$$

$$-2\omega_\varepsilon\int_0^{t_\varepsilon}\mathrm{d}t\int_{s_+}^{s_\delta^+}\mathrm{d}s\,f_0^2(t)(t+\alpha_0) = \mathcal{O}(\varepsilon^\infty),$$

where the last two estimates follow from the optimality of α_0, i.e.,

$$\int_0^\infty \mathrm{d}t\,f_0^2(t)(t+\alpha_0) = 0$$

If we optimize over δ, we get $\delta = \varepsilon^{\frac14}$. Hence, in $[s_+, s_\delta^+]\times[0,t_\varepsilon]$, we conclude that

$$\int_{R_\delta^+}\mathrm{d}s\mathrm{d}t\,|(\partial_s - it)\psi_{\mathrm{trial}}|^2 \le \int_{R_\delta^+}\mathrm{d}s\mathrm{d}t\,f_0^2(t+\alpha_0)^2 + \mathcal{O}(\varepsilon^{\frac34}|\log\varepsilon|^\infty) \tag{5.33}$$

- *Order parameter.* We now observe that in R_δ^+,

$$\begin{aligned}-|\psi_{\mathrm{trial}}|^2 &= -\left|f_0(t) + (f_{k_+}(t)-f_0(t))\left(1-\frac{s-s_\delta^+}{s_+-s_\delta^+}\right)\right|^2 = \\ &= -f_0^2(t) - \left|(f_{k_+}(t)-f_0(t))\left(1-\frac{s-s_\delta^+}{s_+-s_\delta^+}\right)\right|^2 \\ &\quad - 2f_0(t)(f_{k_+}(t)-f_0(t))\left(1-\frac{s-s_\delta^+}{s_+-s_\delta^+}\right) \\ &= -f_0^2(t) + \mathcal{O}(\varepsilon^{\frac12}|\log\varepsilon|^\infty).\end{aligned} \tag{5.34}$$

and

$$|\psi_{\mathrm{trial}}|^4 = (f_0^2(t) + \mathcal{O}(\varepsilon^{\frac12}|\log\varepsilon|^\infty))^2 = f_0^4(t) + \mathcal{O}(\varepsilon^{\frac12}|\log\varepsilon|^\infty) \tag{5.35}$$

- *Energy bound.* From the estimates (5.32), (5.33), (5.34), (5.35) we get

$$\begin{aligned}&\mathcal{G}_{\varepsilon}^{\mathrm{GL}}[\psi_{\mathrm{trial}}, \mathbf{A}_{\mathrm{trial}}]\Big|_{R_\delta^+}\\ &= \int_{R_\delta^+} \mathrm{d}s\mathrm{d}t \left\{ |f_0'(t)|^2 + (t+\alpha_0)^2 f_0^2(t) - \frac{1}{2b}(2f_0^2(t) - f_0^4(t)) \right\} + \mathcal{O}(\varepsilon^{\frac{3}{4}}|\log\varepsilon|^\infty)\\ &= \delta E_0^{1D} + \mathcal{O}(\varepsilon^{\frac{3}{4}}|\log\varepsilon|^\infty).\end{aligned} \tag{5.36}$$

The same estimate holds true in R_δ^-, the only difference is that now we have to use the fact that

$$s_- = s_\delta^- + \delta$$

to get

$$t + S_-'(s) - \omega_\varepsilon = t + \frac{s_-}{\delta}\alpha_0 - \frac{s_\delta^-}{\delta}\alpha_{k_-} - \omega_\varepsilon = t + \alpha_0 + \frac{s_\delta^-}{\delta}(\alpha_0 - \alpha_{k_-}) + o(1)$$

Then,

$$\begin{aligned}&\mathcal{G}_{\Gamma_\delta\setminus\Gamma}[\psi_{\mathrm{trial}}, \mathbf{A}_{\mathrm{trial}}]\\ &= \int_{\Gamma_\delta\setminus\Gamma} \mathrm{d}s\mathrm{d}t \left\{ |f_0'(t)|^2 + (t+\alpha_0)^2 f_0^2(t) - \frac{1}{2b}(2f_0^2(t) - f_0^4(t)) \right\} + \mathcal{O}(\varepsilon^{\frac{3}{4}}|\log\varepsilon|^\infty)\\ &= 2\delta E_0^{1D} + \mathcal{O}(\varepsilon^{\frac{3}{4}}|\log\varepsilon|^\infty).\end{aligned} \tag{5.37}$$

Step 5 (*Final step*). From (5.27), (5.29), (5.37), we conclude that

$$\begin{aligned}E_\varepsilon^{GL} &\le \left(\frac{|\partial\Omega|}{\varepsilon} - 2\delta - 2c_1|\log\varepsilon| + 2\delta \right) E_0^{1D} + E_{\mathcal{D}}(\Gamma) - \mathcal{E}_{\alpha_0}^{\mathrm{corr}}[f_0] \int_0^{|\partial\Omega|} \mathrm{d}\sigma\, \tilde{k}(\sigma) + o(1)\\ &\le \frac{|\partial\Omega| E_0^{1D}}{\varepsilon} - \mathcal{E}_{\alpha_0}^{\mathrm{corr}}[f_0] \int_0^{|\partial\Omega|} \mathrm{d}\sigma\, \tilde{k}(\sigma) + E_{\mathrm{corner}}(c_1|\log\varepsilon|) + o(1),\end{aligned}$$

where we recall that $\tilde{k}(\sigma) = k(s)$ and that $E_{\mathrm{corner}}(c_1|\log\varepsilon|) = -2c_1|\log\varepsilon| + E_{\mathcal{D}}(c_1|\log\varepsilon|)$. Then, in the limit $\varepsilon \to 0$, we finally obtain,

$$E_\varepsilon^{GL} \le \frac{|\partial\Omega| E_0^{1D}}{\varepsilon} - \mathcal{E}_{\alpha_0}^{\mathrm{corr}}[f_0] \int_0^{|\partial\Omega|} \mathrm{d}\sigma\, \tilde{k}(\sigma) + E_{\mathrm{corner}} + o(1).$$

□

5.5.3 Lower bound

We prove the lower bound for a corner domain satisfying hypothesis (a) and (b).

Proposition 5.2 (Lower bound)**.** *Let $\Omega \subset \mathbb{R}^2$ be bounded domain satisfying assumptions (a) and (b) then for any fixed,*

$$1 < b < \Theta_0^{-1},$$

it holds, as $\varepsilon \to 0$,

$$E_\varepsilon^{GL} \ge \frac{|\partial\Omega| E_0^{1D}}{\varepsilon} - \mathcal{E}_{\alpha_0}^{\mathrm{corr}}[f_0] \int_0^{|\partial\Omega|} \mathrm{d}\sigma\, \tilde{k}(\sigma) + E_{\mathrm{corner}} + o(1). \tag{5.38}$$

Proof. We split the proof in several steps.

Step 1 (*Agmon estimates*). As in the proof of Theorem 5.2, we have

$$\mathcal{G}_{\varepsilon}^{\mathrm{GL}}[\psi^{\mathrm{GL}}, \mathbf{A}^{\mathrm{GL}}] \geq \int_{\mathcal{A}_{\varepsilon}} \mathrm{d}\mathbf{r} \left\{ \left| \left(\nabla + i \frac{\widetilde{\mathbf{F}}}{\varepsilon} \right) \psi^{\mathrm{GL}} \right|^2 - \frac{1}{2b} (|\psi^{\mathrm{GL}}|^2 - |\psi^{\mathrm{GL}}|^4) \right\} + \mathcal{O}(\varepsilon^{\infty}).$$

Step 2 (*Lower bound near the corner*). We now want to use the corner effective problem in the region Γ. First of all we notice that

$$\begin{aligned}\mathcal{G}_{\varepsilon}^{\mathrm{GL}}[\psi^{\mathrm{GL}}, \mathbf{A}^{\mathrm{GL}}] \geq & \int_{\mathcal{A}_{\varepsilon}} \mathrm{d}\mathbf{r} \left\{ \left| \left(\nabla + i \frac{\widetilde{\mathbf{F}}}{\varepsilon} \right) \psi^{\mathrm{GL}} \right|^2 - \frac{1}{2b} (|\psi^{\mathrm{GL}}|^2 - |\psi^{\mathrm{GL}}|^4) \right\} + \mathcal{O}(\varepsilon^{\infty}) \\ & \pm \int_0^{t_\varepsilon} \mathrm{d}t \, \frac{F_0(t)}{f_0^2(t)} j_t[\psi^{\mathrm{GL}}(s,t)] \bigg|_{s=s_-} \pm \int_0^{t_\varepsilon} \mathrm{d}t \, \frac{F_0(t)}{f_0^2(t)} j_t[\psi^{\mathrm{GL}}(s,t)] \bigg|_{s=s_+}.\end{aligned}$$

Via Lemma 5.3 we replace the vector potential $\mathbf{A}^{GL}$ with $\widetilde{\mathbf{F}}$ in the region Γ and we observe that the definition of $\widetilde{E}_{\mathcal{N}}(\Gamma)$ implies

$$\int_{\Gamma} \mathrm{d}\mathbf{r} \left\{ \left| \left(\nabla + i \frac{\mathbf{A}^{\mathrm{GL}}}{\varepsilon} \right) \psi^{\mathrm{GL}} \right|^2 - \frac{1}{2b} (|\psi^{\mathrm{GL}}|^2 - |\psi^{\mathrm{GL}}|^4) \right\} - \int_0^{t_\varepsilon} \mathrm{d}t \, \frac{F_0(t)}{f_0^2(t)} j_t[\psi^{\mathrm{GL}}] \bigg|_{s=s_-}^{s=s_+} \geq \widetilde{E}_{\mathcal{N}}(\Gamma).$$

Via Theorem 4.8 we also know that

$$\widetilde{E}_{\mathcal{N}}(\Gamma) = E_{\mathcal{D}}(\Gamma) + o(1).$$

Then, we get

$$\begin{aligned}\mathcal{G}_{\varepsilon}^{\mathrm{GL}}[\psi^{\mathrm{GL}}, \mathbf{A}^{\mathrm{GL}}] \geq & \, E_{\mathcal{D}}(\Gamma) \\ & + \mathcal{G}_{\varepsilon, \mathcal{A}_\varepsilon}^{\mathrm{GL}}[\psi^{\mathrm{GL}}, \mathbf{A}^{\mathrm{GL}}] \bigg|_{\mathcal{A}_\varepsilon \setminus \Gamma} + \int_0^{t_\varepsilon} \mathrm{d}t \, \frac{F_0(t)}{f_0^2(t)} j_t[\psi^{\mathrm{GL}}(s,t)] \bigg|_{s=s_-}^{s=s_+} + o(1).\end{aligned} \tag{5.39}$$

Step 3 (*Lower bound in $\mathcal{A}_\varepsilon \setminus \Gamma$*). We now work in $\mathcal{A}_\varepsilon \setminus \Gamma$: via the replacement of the vector potential $\mathbf{A}^{\mathrm{GL}}$ (see Section 5.3) and proceeding as in [CR16a, Section 5], we define in each cell $\mathscr{C}_j \subset (\mathcal{A}_\varepsilon \setminus \Gamma)$ a function $u_j(s,t)$ such that

$$\psi^{\mathrm{GL}}(s,t) =: f_j(t) u_j(s,t) e^{-i(\alpha_{k_j} + \omega_\varepsilon)s} \quad \text{in } \mathscr{C}_j,$$

As in [CR16a, Lemma 5.3] and via Theorem 2.8, we get[1]

$$\begin{aligned}\mathcal{G}_{\varepsilon, \mathcal{A}_\varepsilon}^{\mathrm{GL}}[\psi^{\mathrm{GL}}, \mathbf{A}^{\mathrm{GL}}] \bigg|_{\mathcal{A}_\varepsilon \setminus \Gamma} & \\ = \left[\frac{|\partial\Omega|}{\varepsilon} - 2c_1 |\log \varepsilon| \right] E_0^{1D} - \mathcal{E}_{\alpha_0}^{\mathrm{corr}}[f_0] \int_0^{|\partial\Omega|} \mathrm{d}\sigma \, \widetilde{k}(\sigma) + \sum_{j=1}^{N_\varepsilon} \mathcal{E}_j[u_j] + o(1),\end{aligned} \tag{5.40}$$

[1] Recall that in [CR16a] the tangential coordinate is not rescaled.

where

$$\mathcal{E}_j[u_j] := \int_{\mathscr{C}_j} \mathrm{d}s\mathrm{d}t\,(1-\varepsilon k_j t) f_j^2 \Big\{ |\partial_t u_j|^2 + \frac{1}{(1-\varepsilon k_j t)^2}|\partial_s u_j|^2 - 2\frac{t+\alpha_j - \frac{1}{2}k_j t^2}{(1-\varepsilon k_j t)^2} J_s[u_j] + \frac{1}{2b} f_j (1-|u_j|^2)^2 \Big\}.$$

We have now to prove that $\mathcal{E}_j[u_j] \geq 0$. Proceeding as in [CR16a, Lemma 5.4], we integrate by parts in s and t. Consequently, some boundary terms appears: we can control them as in [CR16a, Lemma 5.4] with the only exception of the ones in $s = s_+$ and in $s = s_-$. Then, we get

$$\mathcal{E}_j[u_j] \geq \int_0^{t_\varepsilon} \mathrm{d}t\, F_{k_+}(t) j_t[u_{k_+}] \Big|_{s=s_-} - \int_0^{t_\varepsilon} \mathrm{d}t\, F_{k_-}(t) j_t[u_{k_-}] \Big|_{s=s_+} + o(1). \tag{5.41}$$

From (5.40) and (5.41), we have

$$\mathcal{G}_{\mathcal{A}_\varepsilon}[\psi^{\mathrm{GL}}, \mathbf{A}^{\mathrm{GL}}] \Big|_{\mathcal{A}_\varepsilon \setminus \Gamma} \geq \left[\frac{|\partial\Omega|}{\varepsilon} - 2c_1 |\log\varepsilon| \right] E_0^{1D} - \mathcal{E}_{\alpha_0}^{\mathrm{corr}}[f_0] \int_0^{|\partial\Omega|} \mathrm{d}\sigma\, \widetilde{k}(\sigma) + \int_0^{t_\varepsilon} \mathrm{d}t\, F_{k_+}(t) j_t[u_{k_+}] \Big|_{s=s_-} - \int_0^{t_\varepsilon} \mathrm{d}t\, F_{k_-}(t) j_t[u_{k_-}] \Big|_{s=s_+} + o(1). \tag{5.42}$$

Step 4 (*Final step*). Putting together (5.39) and (5.42), we obtain

$$E_\varepsilon^{\mathrm{GL}} \geq \frac{|\partial\Omega| E_0^{1D}}{\varepsilon} - \mathcal{E}_{\alpha_0}^{\mathrm{corr}}[f_0] \int_0^{|\partial\Omega|} \mathrm{d}\sigma\, \widetilde{k}(\sigma) + E_{\mathrm{corner}}(c_1|\log\varepsilon|) + \int_0^{t_\varepsilon} \mathrm{d}t\, F_{k_+}(t) j_t[u_{k_+}] \Big|_{s=s_-} - \int_0^{t_\varepsilon} \mathrm{d}t\, F_{k_-}(t) j_t[u_{k_-}] \Big|_{s=s_+} + \int_0^{t_\varepsilon} \mathrm{d}t\, \frac{F_0(t)}{f_0^2(t)} j_t[\psi^{\mathrm{GL}}(s,t)] \Big|_{s=s_-}^{s=s_+} + o(1). \tag{5.43}$$

To conclude the proof of the lower bound we simply have to estimate all the boundary terms in (5.43). We first consider the boundary term in $s = s_+$. We have

$$\int_0^{t_\varepsilon} \mathrm{d}t\, \frac{F_0(t)}{f_0^2(t)} j_t[\psi^{\mathrm{GL}}(s,t)] \Big|_{s=s_+} - \int_0^{t_\varepsilon} \mathrm{d}t\, F_{k_+}(t) J_t \left[\frac{\psi^{\mathrm{GL}}(s,t)}{f_{k_+}(t)} e^{-i(\alpha_{k_+} s + \omega\varepsilon s)} \right] \Big|_{s=s_+}$$
$$= \int_0^{t_\varepsilon} \mathrm{d}t\, \frac{F_0(t)}{f_0^2(t)} j_t \left[\psi^{\mathrm{GL}}(s,t) \right] \Big|_{s=s_+} - \int_0^{t_\varepsilon} \mathrm{d}t\, \frac{F_{k_+}(t)}{f_{k_+}^2(t)} j_t \left[\psi^{\mathrm{GL}}(s,t) \right] \Big|_{s=s_+}$$
$$= \int_0^{t_\varepsilon} \mathrm{d}t \left[\frac{F_0(t)}{f_0^2(t)} - \frac{F_{k_+}(t)}{f_{k_+}^2(t)} \right] j_t \left[\psi^{\mathrm{GL}}(s,t) \right] \Big|_{s=s_+}.$$

Lemma 2.4 yields

$$\sup_{t\in[0,t_\varepsilon]} \left| \frac{F_0}{f_0^2} - \frac{F_{k_+}}{f_{k_+}} \right| \leq C(\varepsilon k_+)^{\frac{1}{2}} |\log\varepsilon|^\infty.$$

We also recall that

$$\|\psi^{\mathrm{GL}}\|_{L^\infty} \leq 1, \qquad \|\partial_t \psi^{\mathrm{GL}}\|_{L^\infty} \leq 1,$$

(see (2.7) and [CR16a, Section 5.1]). It then follows that

$$\int_0^{t_\varepsilon} \mathrm{d}t \left[\frac{F_0(t)}{f_0^2(t)} - \frac{F_{k_+}(t)}{f_{k_+}^2(t)} \right] j_t \left[\psi^{\mathrm{GL}}(s,t) \right] \Big|_{s=s_+} = \mathcal{O}\left((\varepsilon k_+)^{\frac{1}{2}} |\log \varepsilon|^\infty \right). \tag{5.44}$$

The same holds true for the boundary term in $s = s_-$. From (5.44) and (5.43), we conclude that

$$E_\varepsilon^{GL} \geq \frac{|\partial\Omega| E_0^{1D}}{\varepsilon} - \mathcal{E}_{\alpha_0}^{\mathrm{corr}}[f_0] \int_0^{|\partial\Omega|} \mathrm{d}\sigma\, \widetilde{k}(\sigma) + E_{\mathrm{corner}}(c_1 |\log \varepsilon|) + o(1),$$

and, in the limit $\varepsilon \to 0$, we get the desired lower bound

$$E_\varepsilon^{GL} \geq \frac{|\partial\Omega| E_0^{1D}}{\varepsilon} - \mathcal{E}_{\alpha_0}^{\mathrm{corr}}[f_0] \int_0^{|\partial\Omega|} \mathrm{d}s\, \widetilde{k}(\sigma) + E_{\mathrm{corner}} + o(1).$$

□

5.5.4 Proof of Theorem 5.1

Proof of Theorem 5.1. If Ω is a corner domain satisfying assumptions (a) and (b) then, from the upper bound in Proposition 5.1 and the lower bound in Proposition 5.2, we have that

$$E_\varepsilon^{GL} = \frac{|\partial\Omega| E_0^{1D}}{\varepsilon} - \mathcal{E}_{\alpha_0}^{\mathrm{corr}}[f_0] \int_0^{|\partial\Omega|} \mathrm{d}s\, \widetilde{k}(\varepsilon s) + E_{\mathrm{corner}} + o(1). \tag{5.45}$$

In what follows we explain how to adapt the proof for a more general domain. We work in the non-rescaled boundary coordinates (σ, τ) in the region

$$\widetilde{\mathcal{A}}_{\mathrm{cut}} = \mathcal{A}_{\partial\Omega} \setminus \widetilde{\Gamma}.$$

To remove hypothesis (a) we can use the local diffeomorphism Φ introduced in Section 5.1 in the region $\widetilde{\Gamma}$. From Lemma 5.1, we have that for any integrable function f, it holds

$$\int_{\left\{ \mathbf{r} \in \mathcal{A}_{\partial\Omega} \,|\, \mathrm{dist}(\mathbf{r}, \mathbf{r}_\beta) \leq c_1 \varepsilon |\log \varepsilon| \right\}} \mathrm{d}\mathbf{r}\, f(\mathbf{r}) = (1 + \mathcal{O}(\varepsilon |\log \varepsilon|)) \int_{\widetilde{\Gamma}} \mathrm{d}\widetilde{\mathbf{r}}\, f(\widetilde{\mathbf{r}}(\mathbf{r})),$$

where $\mathbf{r}_\beta$ is the coordinate of the vertex and $\widetilde{\mathbf{r}} = \Phi(\mathbf{r})$. From the definition of Φ, it also follows that

$$\widetilde{\mathbf{F}}(\Phi(\mathbf{r})) \Big|_{\left\{ \Phi(\mathbf{r}) \in \partial\widetilde{\Gamma}^\pm \right\}} = (-\tau, 0).$$

After the extension of the diffeomorphism Φ to a global one equal to the identity suitably far from the corner, we can proceed as in Propositions 5.1 and 5.2 to prove the asymptotics (5.45) for a general corner domain with non trivial boundary curvature near the corner.

We now discuss how to prove the energy asymptotics in a domain with more than one corner (at most a finite number) along the boundary section. We first observe that for each corner j, $j \in \Sigma$, we need two transition regions like the one defined in (5.23) and (5.24). Then proceeding as in Proposition 5.1, we easily get the desired upper bound. For the lower bound we only point out that, after the energy splitting (5.40), we have to estimate for each corner j, $j \in \Sigma$, some boundary terms like the ones in (5.43). Since the number of corners along the boundary of the domain is finite, this does not affect the final estimate.

In conclusion, we proved that in a general corner domains satisfying Assumptions 2.1 and 2.2, it holds, in the limit $\varepsilon \to 0$,

$$E_\varepsilon^{GL} = \frac{|\partial\Omega| E_0^{1D}}{\varepsilon} - \mathcal{E}_{\alpha_0}^{\mathrm{corr}}[f_0] \int_0^{|\partial\Omega|} \mathrm{d}\sigma\, \widetilde{k}(\sigma) + \sum_{j\in\Sigma} E_{\mathrm{corner},j} + o(1).$$

□

Chapter 6

Almost Flat Corners

We conjectured in Chapter 5 that the corner contribution to the energy in the surface superconductivity regime is

$$E_{\mathrm{corner}}(\vartheta) = -(\pi - \vartheta)\mathcal{E}^{\mathrm{corr}}_{\alpha_0}[f_0].$$

In this Chapter we estimate the energy contribution due to a corner with opening angle near π and prove that

$$E_{\mathrm{corner}}(\pi - \delta) \underset{\delta \simeq 0}{=} -\delta\mathcal{E}^{\mathrm{corr}}_{\alpha_0}[f_0] + o(\delta)$$

This Chapter is organized as follows: we first introduce the domain we want to consider and discuss some preliminary results useful to study the same effective problem as in Chapter 4, but in an almost flat angular region. In the last part, we underline how this allows us to prove a more refined asymptotics in a general domain with one corner (or at most a finite number of corners) with angle close to π.

6.1 Main Result

Before stating the main result, we recall the effective problem we are going to use. We

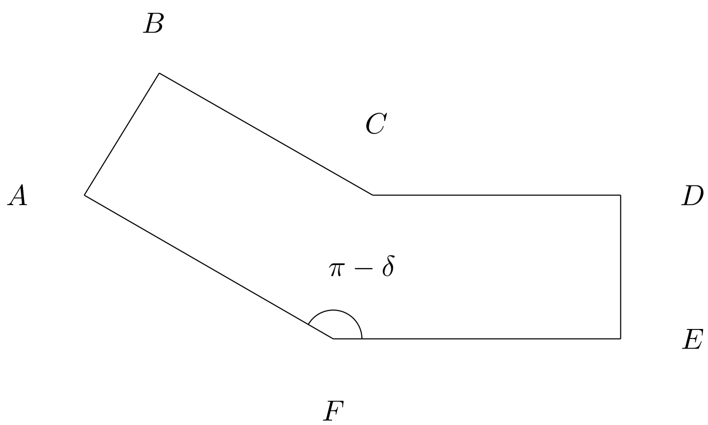

Figure 6.1. The angular region Γ.

consider the region Γ represented in Figure 6.1, in particular, we set $\overline{AF} = \overline{EF} = \ell$ and $\overline{AB} = \overline{DE} = c\ell$, with $\ell \to +\infty$ and $c > 0$ a fixed constant such that $c < \tan((\pi - \delta)/2)$. The effective problem in the angular region Γ is then

$$\mathcal{G}_\Gamma[\psi] = \int_\Gamma \mathrm{d}\mathbf{r} \left\{ \left|(\nabla + i\mathbf{A})\psi\right|^2 - \frac{1}{2b}(2|\psi|^2 - |\psi|^4) \right\}$$

and

$$E_{\mathrm{corner}} = \liminf_{\ell \to +\infty} \left[-2\ell E_0^{1D} + \inf_{\psi \in \mathscr{D}_\mathcal{D}} \mathcal{G}_\Gamma[\psi] \right]$$

where $\mathbf{A} \in H^1(\mathbb{R}^2)$ is any vector potential such that $\mathrm{curl}\mathbf{A} = 1$, $\mathrm{div}\mathbf{A} = 0$ and $\mathbf{A}\big|_{\partial\Gamma_\pm^{\mathrm{bd}}} = (-t, 0)$ and the domain $\mathscr{D}_\mathcal{D}(\Gamma)$ is defined as in (5.11). We now state the main result:

Theorem 6.1.
Let $\Gamma \subset \mathbb{R}^2$ be the angular region introduced above, then for any fixed

$$1 < b < \Theta_0^{-1},$$

it holds that, as $\ell \to +\infty$ and $\delta \to 0$,

$$E_{\mathrm{corner}} = -\delta \mathcal{E}^{\mathrm{corr}}_{\alpha_0}[f_0] + \mathcal{O}(\delta^{4/3}) + \mathcal{O}\big(e^{-C\ell^2}\big), \tag{6.1}$$

where $\mathcal{E}^{\mathrm{corr}}_\alpha[f_0]$ is the functional defined in (2.36).

Remark 6.1 (Correction Energy). We recall that the functional $\mathcal{E}^{\mathrm{corr}}_{\alpha_0}$ is defined as in (2.36), i.e.

$$\mathcal{E}^{\mathrm{corr}}_{\alpha_0}[f_0] := \int_0^{c_0|\log\varepsilon|} \mathrm{d}t\, t \left\{ |f_0(t)|^2 + -\alpha_0(t + \alpha_0) f_0^2(t) - \frac{1}{2b}(2f_0^2(t) - f_0^4(t)) \right\}.$$

As we already discussed in Chapter 2 (see Remark 2.11), although some numerical evidences suggest that this contribution to the energy is positive, there are no rigorous proofs about that. The positivity of $\mathcal{E}^{\mathrm{corr}}_{\alpha_0}[f_0]$ would imply that superconductivity concentrates more where the curvature is large.

We now observe that, proceeding as in Chapter 5 and using Theorem 6.1, we can prove the following result

Corollary 6.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a finite number of corners along $\partial\Omega$ with almost flat angles β_j, $j \in \Sigma$, i.e., such that $|\beta_j - \pi| \ll 1$, then for any fixed*

$$1 < b < \Theta_0^{-1}$$

it holds, as $\varepsilon \to 0$,

$$\left| E_\varepsilon^{GL} - \frac{|\partial\Omega| E_0^{1D}}{\varepsilon} + 2\pi \mathcal{E}^{\mathrm{corr}}_{\alpha_0}[f_0] \right| \leq C\delta^{4/3} + o(1). \tag{6.2}$$

Remark 6.2 (Energy correction). We recall that the integral of the curvature along the smooth part of the boundary of Γ is equal to $2\pi - \delta$ by Gauss-Bonnet Theorem. If we use the asymptotics (6.1) for E_{corner} together with the energy asymptotics (5.4), we get (6.2).

6.2 Systems of Coordinates

Looking at the Figure 6.1, we observe that the tangential coordinate along $\overline{AE} \cup \overline{EF}$ is not well-defined near the vertex. For this reason, we split the angular sector into two subregions, in which we are allowed to use the tangential length along the boundary and the distance from the outer boundary as coordinates. We call these subregions $\Gamma_\pm$ (see Figure 6.2 below). We also define

$$\begin{aligned} &\partial\Gamma_-^{\mathrm{ext}} := \overline{EF}, && \partial\Gamma_+^{\mathrm{ext}} := \overline{AF}, && \partial\Gamma^{\mathrm{ext}} := \partial\Gamma_-^{\mathrm{ext}} \cup \partial\Gamma_+^{\mathrm{ext}}, \\ &\partial\Gamma_-^{\mathrm{bd}} := \overline{DE}, && \partial\Gamma_+^{\mathrm{bd}} := \overline{AB}, && \partial\Gamma^{\mathrm{bd}} := \partial\Gamma_-^{\mathrm{bd}} \cup \partial\Gamma_+^{\mathrm{bd}}, \\ &\partial\Gamma_-^{\mathrm{bulk}} := \overline{CD}, && \partial\Gamma_+^{\mathrm{bulk}} := \overline{BC}, && \partial\Gamma^{\mathrm{bulk}} := \partial\Gamma_-^{\mathrm{bulk}} \cup \partial\Gamma_+^{\mathrm{bulk}}. \end{aligned}$$

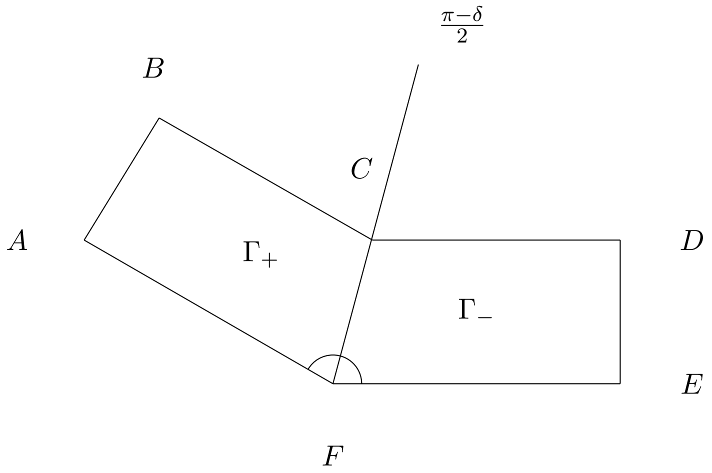

Figure 6.2. The two subregions $\Gamma_\pm$.

We now introduce the set of coordinates that we are going to use in $\Gamma_\pm$. If $\mathbf{r} = (x, y)$ are the cartesian coordinates centered at the vertex with the $x-$axis parallel to $\overline{EF}$ and the $y-$axis parallel to $\overline{DE}$, then

$$\begin{aligned} t_- &:= \mathrm{dist}(\mathbf{r}, \partial\Gamma_-^{\mathrm{ext}}) \equiv y, && \forall \mathbf{r} \in \Gamma_-, \\ t_+ &:= \mathrm{dist}(\mathbf{r}, \partial\Gamma_+^{\mathrm{ext}}) \equiv x \sin\delta + y\cos\delta, && \forall \mathbf{r} \in \Gamma_+. \end{aligned}$$

Remark 6.3 (Continuity of the normal). The values $t_\pm$ coincide on the common boundary of Γ_- and Γ_+, i.e., on the line $y = \tan\left(\frac{\pi-\delta}{2}\right) x$. Indeed, we have

$$t_- = \tan\left(\frac{\pi-\delta}{2}\right)x, \qquad t_+ = \left[\sin\delta + \tan\left(\frac{\pi-\delta}{2}\right)\cos\delta\right]x = \tan\left(\frac{\pi-\delta}{2}\right)x.$$

If we denote by s_- and s_+ the tangential variables in Γ_- and Γ_+ and we center the coordinates at the vertex of the corner, it follows that

$$\Gamma_- := \left\{ (t_-, s_-) \,\middle|\, |\, 0 \leq t_- \leq c\ell,\ \tan\frac{\delta}{2} t_- \leq s_- \leq \ell \right\},$$

$$\Gamma_+ := \left\{(t_+, s_+) \,\middle|\, 0 \le t_+ \le c\ell,\ -\ell \le s_+ \le -\tan\frac{\delta}{2} t_+\right\}.$$

Whenever there is no confusion, we will denote by (s,t) the appropriate boundary coordinates. We will work in a different way in the regions near the vertex and therefore we split the region Γ in the four subregions defined below and represented in Figure 6.3:

$$\begin{aligned}\Gamma_{-,\delta} &:= \Gamma \cap \left\{\vartheta \in \left[0, \frac{\pi-\delta-\gamma}{2}\right]\right\}, & \Gamma_{-,\gamma} &:= \Gamma \cap \left\{\vartheta \in \left[\frac{\pi-\delta-\gamma}{2}, \frac{\pi-\delta}{2}\right]\right\},\\ \Gamma_{+,\gamma} &:= \Gamma \cap \left\{\vartheta \in \left[\frac{\pi-\delta}{2}, \frac{\pi-\delta+\gamma}{2}\right]\right\}, & \Gamma_{+,\delta} &:= \Gamma \cap \left\{\vartheta \in \left[\frac{\pi-\delta+\gamma}{2}, \pi-\delta\right]\right\}.\end{aligned} \tag{6.3}$$

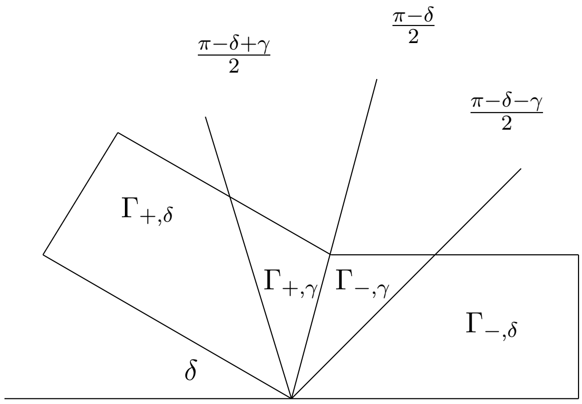

Figure 6.3. The four regions $\Gamma_{-,\delta}$, $\Gamma_{-,\gamma}$, $\Gamma_{+,\delta}$,$\Gamma_{+,\gamma}$.

In the regions $\Gamma_{\pm,\gamma}$ near the corners it is more convenient to work with polar coordinates. In particular, in the region $\Gamma_{-,\gamma}$ we use the coordinates $(\rho,\vartheta) \in [0,\ell] \times \left[\frac{\pi-\delta-\gamma}{2}, \frac{\pi-\delta}{2}\right]$, such that

$$\begin{cases} t_- = \rho \sin\vartheta, \\ s_- = \rho\cos\vartheta. \end{cases} \tag{6.4}$$

Instead in $\Gamma_{+,\gamma}$ we use $(\rho,\vartheta) \in [0,\ell] \times \left[\frac{\pi-\delta}{2}, \frac{\pi-\delta+\gamma}{2}\right]$ with

$$\begin{cases} t_+ = \rho\sin(\pi-\delta-\vartheta), \\ s_+ = -\rho\cos(\pi-\delta-\vartheta). \end{cases} \tag{6.5}$$

Before proceeding further, we observe that, using the polar coordinates just introduced, one has

$$\Gamma_{-,\gamma} = \left\{(\rho,\vartheta) \,|\, 0 \le \rho \le \frac{c\ell}{\sin\vartheta}, \frac{\pi-\delta-\gamma}{2} \le \vartheta \le \frac{\pi-\delta}{2}\right\},$$

and

$$\Gamma_{-,\gamma} = \left\{(\rho,\vartheta) \,|\, 0 \le \rho \le \frac{c\ell}{\sin\vartheta}, \frac{\pi-\delta}{2} \le \vartheta \le \frac{\pi-\delta+\gamma}{2}\right\}.$$

Remark 6.4 (Systems of Coordinates). We underline that the systems of coordinates introduced above are well defined in a corner region with a general opening angle. However they are particularly useful only in this Chapter. To prove the main result of the Chapter we often use some approximations on the coordinates coming mainly from the fact that $(\pi-\delta)/2 \simeq \pi/2$. The same strategy does not apply to a general corner region, since in this case the approximations produce errors of the order $\mathcal{O}(1)$.

6.3 Useful Estimates

We prove here some useful bounds in the regions $\Gamma_{\pm,\gamma}$. We first observe that in these regions, it holds

$$\sin\vartheta = 1+\mathcal{O}(\delta^2+\gamma^2), \qquad \cos\vartheta = \mathcal{O}(\delta+\gamma),$$

and this leads us to some other useful approximations.

Lemma 6.1.
Let $\delta \ll \gamma \ll 1$*, then has for any* $\vartheta \in \left[\frac{\pi-\delta-\gamma}{2}, \frac{\pi-\delta+\gamma}{2}\right]$*, there exists a constant* $C>0$ *such that*

$$\left|f_0(\rho\sin\vartheta) - f_0(\rho)\right| \le C\gamma e^{-C\rho^2}, \tag{6.6}$$

$$\left|\partial_\rho f_0(\rho\sin\vartheta) - f_0'(\rho)\right| \le C\gamma^2 e^{-C\rho^2} \tag{6.7}$$

$$\left|\frac{1}{\rho}\partial_\vartheta f_0(\rho\sin\vartheta)\right| \le C\gamma e^{-C\rho^2}. \tag{6.8}$$

Proof. For the proof of (6.6) we simply notice that

$$f_0(\rho\sin\vartheta) = f_0(\rho) + f_0'(\rho\sin\tilde{\vartheta})\left(\vartheta - \frac{\pi}{2}\right),$$

for some $\tilde{\vartheta} \in \left[\frac{\pi}{2}-\frac{\delta}{2}-\frac{\gamma}{2}, \frac{\pi}{2}-\frac{\delta}{2}+\frac{\gamma}{2}\right]$. We now use Lemma 2.2 to get

$$|f_0'(\rho\sin\tilde{\vartheta})| \le Ce^{-\frac{1}{4}\rho^2\sin^2\tilde{\vartheta}} \le Ce^{-C\rho^2},$$

for some $c>0$. The estimate (6.6) now easily follows, since

$$\left(\vartheta - \frac{\pi}{2}\right) = \mathcal{O}(\gamma).$$

For the proof of (6.7) and (6.8) we have to use that

$$\frac{1}{\rho}\partial_\vartheta\partial_\rho(f_0(\rho\sin\vartheta)) = \frac{1}{\rho}f_0'(\rho\sin\vartheta)\cos\vartheta + f_0''(\rho\sin\vartheta)\sin\vartheta\cos\vartheta,$$

$$\frac{1}{\rho^2}\partial_\vartheta^2(f_0(\rho\sin\vartheta)) = f_0''(\rho\sin\vartheta)\cos^2\vartheta - \frac{1}{\rho}f_0'(\rho\sin\vartheta)\sin\vartheta.$$

Following now the same strategy as before and using the variational equation of f_0 to control f_0'', i.e.,

$$-f_0'' + (t+\alpha_0)^2 f_0 = \frac{1}{b}(1-f_0^2)f_0,$$

we get the estimates (6.7) and (6.8). ☐

6.4 Proof of the Main Result

First of all we prove that up to a change of gauge we can fix in $\mathcal{G}_\Gamma$ the vector potential equal to $\mathbf{F}(\mathbf{r}) = (-y, 0)$.

Proposition 6.1.
Let $\Gamma \subset \mathbb{R}^2$ be the angular region defined above and let $\mathbf{F}(\mathbf{r}) := (-y, 0)$, then there exists a gauge phase $\Phi(\mathbf{r})$ such that

$$\mathbf{F}(\mathbf{r}) + \nabla\Phi(\mathbf{r}) = \mathbf{A}(\mathbf{r}),$$

where $\mathbf{A} \in H^1(\mathbb{R}^2)$ is any vector potential such that $\mathrm{curl}\mathbf{A} = 1$, $\mathrm{div}\mathbf{A} = 0$ *and* $\mathbf{A}|_{\partial\Gamma_\pm^{\mathrm{bd}}} = (-t, 0)$.

Proof. We first observe that in $\Gamma_{-,\delta}$ we have $\mathbf{F}(\mathbf{r}(s,t)) = (-t, 0)$. Furthermore, in $\Gamma_{+,\delta}$ it holds

$$\mathbf{F}(\mathbf{r}(s,t)) = (s \sin\delta \cos\delta - t(\cos\delta)^2, s(\sin\delta)^2 - t \sin\delta\cos\delta).$$

We now define $\Phi(\mathbf{r})$ such that

$$\Phi(\mathbf{r}(s,t)) = 0, \qquad \text{in } \Gamma_{-,\delta},$$

and

$$\Phi(\mathbf{r}(s,t)) = -(\sin\delta)^2 st - \frac{1}{2}\sin\delta\cos\delta(s^2 - t^2), \qquad \text{in } \Gamma_{+,\delta}.$$

We can then define Φ in $\Gamma_{-,\gamma} \cup \Gamma_{+,\gamma}$ in such a way that $\Phi \in C^1(\mathbb{R}^2)$ and

$$\mathbf{F}(\mathbf{r}) + \nabla\Phi(\mathbf{r}) = \mathbf{A}(\mathbf{r}).$$

□

For simplicity we then consider the functional $\mathcal{G}_\Gamma[\psi]$ with a fixed vector potential equal to $\mathbf{F}(\mathbf{r}) = (-y, 0)$. With some abuse of notation we still denote this functional by $\mathcal{G}_\Gamma$, i.e.

$$\mathcal{G}_\Gamma[\psi] = \int_\Gamma \mathrm{d}\mathbf{r} \left\{ \left|(\nabla + i\mathbf{F})\psi\right|^2 - \frac{1}{2b}(2|\psi|^2 - |\psi|^4) \right\}.$$

Thanks to Proposition 6.1 it suffices to show that as $\ell \to +\infty$ and $\delta \to 0$, it holds

$$E_\Gamma = 2\ell E_0^{1D} - \delta \mathcal{E}_{\alpha_0}^{\mathrm{corr}}[f_0] + \mathcal{O}(\delta^{4/3}) + \mathcal{O}(e^{-C\ell^2}).$$

As usual, to derive the energy asymptotics, we prove an upper and a lower bound for the ground state energy of $\mathcal{G}_\Gamma$. Before proceeding further, we underline that to take into account the jump of the tangential coordinate at the vertex we have to choose a suitable trial phase for the trial order parameter. This is because the minimizing order parameter depends on the tangential coordinate only in the phase. We then define the phase Φ of the trial function in the upper bound, which will also be involved in the energy splitting in the lower bound We set

$$\Phi := \begin{cases} \Phi_-(s_-, t_-), & \text{in } \Gamma_{-,\delta}, \\ \Delta\Phi(\rho, \vartheta), & \text{in } \Gamma_{\pm,\gamma}, \\ \Phi_+(s_+, t_+), & \text{in } \Gamma_{+,\delta}. \end{cases}$$

The two phases $\Phi_{\pm}$ are defined as follows

$$\Phi_-(s_-, t_-) := -\alpha_0 s_-, \tag{6.9}$$

$$\Phi_+(s_+, t_+) := -\alpha_0 s_+ - (\sin^2\delta) t_+ s_+ - \frac{1}{4}\sin(2\delta)(s_+^2 - t_+^2). \tag{6.10}$$

We now use polar coordinates to define $\Delta\Phi$. To this purpose it is convenient to rewrite the phases $\Phi_{\pm}$ in polar coordinates (ρ, ϑ) in (6.4) and (6.5):

$$\Phi_-(\rho, \vartheta) := -\alpha_0 \rho \cos\vartheta, \tag{6.11}$$

$$\Phi_+(\rho, \vartheta) := \alpha_0 \rho \cos(\pi-\delta-\vartheta) - \frac{\rho^2}{2}\sin^2\delta \sin(2(\pi-\delta-\vartheta)) + \frac{\rho^2}{4}\sin(2\delta)\cos(2(\pi-\delta-\vartheta)). \tag{6.12}$$

Then we define the phase $\Delta\Phi$ as

$$\begin{aligned}\Delta\Phi(\rho, \vartheta) := \Phi_-\left(\rho, \frac{\pi-\delta-\gamma}{2}\right)\left(\frac{\pi-\delta+\gamma}{2} - \vartheta\right)\frac{1}{\gamma} + \\ + \Phi_+\left(\rho, \frac{\pi-\delta+\gamma}{2}\right)\left(\vartheta - \frac{\pi-\delta-\gamma}{2}\right)\frac{1}{\gamma},\end{aligned}$$

or equivalently

$$\Delta\Phi(\rho, \vartheta) = \alpha_0 \rho \sin\left(\frac{\delta+\gamma}{2}\right)\left(\frac{2\vartheta-\pi+\delta}{\gamma}\right) + \frac{\rho^2}{2}\sin\delta\cos\gamma\left(\vartheta - \frac{\pi-\delta-\gamma}{2}\right)\frac{1}{\gamma}. \tag{6.13}$$

Remark 6.5 (Phase factor). Notice that we have defined $\Delta\Phi$ in such a way that

$$\begin{aligned}\Delta\Phi\left(\rho, \frac{\pi-\delta-\gamma}{2}\right) &= \Phi_-\left(\rho, \frac{\pi-\delta-\gamma}{2}\right), \\ \Delta\Phi\left(\rho, \frac{\pi-\delta+\gamma}{2}\right) &= \Phi_+\left(\rho, \frac{\pi-\delta+\gamma}{2}\right),\end{aligned}$$

which ensures that the phase is continuous. Notice also the two jumps of order $\mathcal{O}(\delta+\gamma)$ (first term) and $\mathcal{O}(\delta)$ (second term) in (6.13) in the region $\Gamma_{-,\gamma} \cup \Gamma_{+,\gamma}$. This is indeed required to make the phase continuous and will be a key ingredient of the proof. In fact, the precise form of $\Delta\Phi$ might not be so relevant but the jumps are crucial to get the desired correction.

We denote by E_Γ the ground state energy

$$E_\Gamma := \inf_{\psi \in \mathscr{D}_{\mathcal{D}}(\Gamma)} \mathcal{G}_\Gamma[\psi],$$

and by $\psi_\Gamma \in \mathscr{D}_{\mathcal{D}}(\Gamma)$ any corresponding minimizer, where

$$\mathscr{D}_{\mathcal{D}}(\Gamma) := \left\{\psi \in H^1(\Gamma) \,|\, \psi = g(t)e^{-i\Phi_{\pm}(s,t)} \text{ on } \partial\Gamma_{\pm}^{\mathrm{vert}},\ \psi = 0 \text{ on } \partial\Gamma_{\pm}^{\mathrm{bulk}}\right\}. \tag{6.14}$$

Remark 6.6 (Boundary coordinates). The function $g(t)$ in (6.14) is defined exactly as in (4.6). Notice that we have chosen for simplicity not to add a global factor in the boundary coordinates, unlike in Chapter 5, since we think of Γ as the full domain. However, if Γ is the corner part of a simply connected domain, one can add the phase δ_ℓ defined in (5.12) and get the same asympotics up to small remainder terms.

6.4.1 Upper bound

Proposition 6.2.
Let $\Gamma \subset \mathbb{R}^2$ *be the angular region introduced above, then for any fixed*

$$b < \Theta_0^{-1},$$

it holds that, as $\ell \to +\infty$*,*

$$E_\Gamma \leq 2\ell E_0^{1D} - \delta \mathcal{E}_{\alpha_0}^{\mathrm{corr}}[f_0] + \mathcal{O}(\delta^{4/3}) + \mathcal{O}(e^{-C\ell^2}).$$

Proof. To prove the upper bound we have to choose a suitable trial function ψ_{trial}, which has to be supported in the boundary layer and decay exponentially suitably far from it. We set

$$\psi_{\mathrm{trial}} := \begin{cases} g(t_-)e^{i\Phi_-(t_-,s_-)}, & \text{in } \Gamma_{-,\delta}, \\ g(t_-)e^{i\Delta\Phi(t_-,s_-)}, & \text{in } \Gamma_{-,\gamma}, \\ g(t_+)e^{i\Delta\Phi(t_+,s_+)}, & \text{in } \Gamma_{+,\gamma}, \\ g(t_+)e^{i\Phi_+(t_+,s_+)}, & \text{in } \Gamma_{+,\delta}, \end{cases} \tag{6.15}$$

where we recall that the function $g(t)$ is given by

$$g(t_\pm) = \begin{cases} f_0(t_\pm), & \text{if } t_\pm \in [0, \bar{\ell}\,], \\ \bar{f}_0(t_\pm), & \text{if } t_\pm \in [\bar{\ell}, c\ell], \end{cases}$$

with $\bar{\ell} := c\ell(1-\eta)$ and $\eta \ll 1$. We also require that: $\bar{f}_0(t)$ is monotone, $\bar{f}_0(\bar{\ell}) = f_0(\bar{\ell})$ and $\bar{f}_0(c\ell) = 0$. We can also assume that

$$\|\bar{f}_0(t)\|_{L^\infty[\bar{\ell},c\ell]} \leq f_0(\bar{\ell}), \quad \|\bar{f}_0'(t)\|_{L^\infty[\bar{\ell},c\ell]} \leq C f_0(\bar{\ell})\eta^{-1}.$$

We now work in the region $\Gamma_{-,\delta}$ and we define

$$\overline{\Gamma}_{-,\delta} := \Gamma_{-,\delta}\big|_{\{t_- \leq \bar{\ell}\}}.$$

Proceeding as in Proposition 4.1, we have[1]

$$\mathcal{G}_\Gamma[\psi_{\mathrm{trial}}]\Big|_{\Gamma_{-,\delta}} = \mathcal{G}_\Gamma\big[f_0 e^{i\Phi_-}\big]\Big|_{\overline{\Gamma}_{-,\delta}} + \mathcal{O}(e^{-C\ell^2}).$$

We denote by R the rectangle obtained by completing $\Gamma_{-,\delta}$, i.e.,

$$R := \Big\{(s_-, t_-) \in [0,\ell] \times [0, c\ell]\Big\}$$

and we observe that

$$\mathcal{G}_\Gamma\big[f_0 e^{i\Phi_-}\big]\Big|_{\overline{\Gamma}_{-,\delta}} = \mathcal{G}_\Gamma\big[f_0 e^{i\Phi_-}\big]\Big|_{R} - \mathcal{G}_\Gamma\big[f_0 e^{i\Phi_-}\big]\Big|_{R\setminus\overline{\Gamma}_{-,\delta}} + \mathcal{O}(e^{-C\ell^2}).$$

[1] Notice that here we have a smaller remainder than the one in Proposition 4.1: this is due to the absence of the correction δ_ℓ in the phase factor.

As in Proposition 4.1 we can prove that

$$\mathcal{G}_\Gamma\left[f_0 e^{i\Phi_-}\right]\Big|_R = \ell E_0^{1D} + \mathcal{O}(e^{-C\ell^2}),$$

then

$$\mathcal{G}_\Gamma[\psi_{\mathrm{trial}}]\Big|_{\Gamma_{-,\delta}} = \ell E_0^{1D} - \mathcal{G}_\Gamma[f_0 e^{i\Phi_-}]\Big|_{R\setminus\Gamma_{-,\delta}} + \mathcal{O}(e^{-C\ell^2}).$$

It also follows that

$$\begin{aligned}\mathcal{G}_\Gamma[f_0 e^{i\Phi_-}]\big|_{R\setminus\Gamma_{-,\delta}} &= \int_0^{\bar\ell} \mathrm{d}t \int_0^{\tan\left(\frac{\delta+\gamma}{2}\right)t} \mathrm{d}s\, \left\{|f_0'|^2 + (t+\alpha_0)^2 f_0^2 - \frac{1}{2b}(2f_0^2 - f_0^4)\right\} \\ &= \frac{\delta+\gamma}{2}\int_0^{c\ell} t\mathrm{d}t \left\{|f_0'|^2 + (t+\alpha_0)^2 f_0^2 - \frac{1}{2b}(2f_0^2 - f_0^4)\right\} + \mathcal{O}(\gamma^3) + \mathcal{O}(e^{-C\ell^2}),\end{aligned}$$

where we used the fact that f_0 is exponentially decaying (Lemma 2.1) and a Taylor expansion to approximate the tangent. We also assumed that $\gamma \gg \delta$ for simplicity.

We now work in the region $\Gamma_{+,\delta}$. If $\bar\Gamma_{+,\delta} := \Gamma_{+,\delta}|_{\{t\le\bar\ell\}}$, we have

$$\mathcal{G}_\Gamma[\psi_{\mathrm{trial}}]\big|_{\Gamma_{+,\delta}} = \mathcal{G}_\Gamma\left[f_0 e^{i\Phi_+}\right]\big|_{\bar\Gamma_{+,\delta}} + \mathcal{O}(e^{-C\ell^2}).$$

We now observe that

$$\mathcal{G}_\Gamma\left[f_0 e^{i\Phi_+}\right]\Big|_{\bar\Gamma_{+,\delta}} = \int_{\bar\Gamma_{+,\delta}} \mathrm{d}s\mathrm{d}t \left\{|(\nabla + i\mathbf{F}(s,t))(f_0 e^{i\Phi_+})|^2 - \frac{1}{2b}(2|f_0|^2 - |f_0|^4)\right\},$$

where, denoting by $\hat{\mathbf{e}}_s$ and $\hat{\mathbf{e}}_t$ the unit vectors along the exterior boundary and its normal respectively,

$$\begin{aligned}\mathbf{F}(s,t)\cdot\hat{\mathbf{e}}_s &= s\sin\delta\cos\delta - t(\cos^2\delta), \\ \mathbf{F}(s,t)\cdot\hat{\mathbf{e}}_t &= s(\sin^2\delta) - t\sin\delta\cos\delta.\end{aligned}$$

Then

$$|(\nabla + i\mathbf{F}(s,t))(f_0 e^{i\Phi_+})| = |\partial_s(f_0 e^{i\Phi_+}) + i(\mathbf{F}(s,t)\cdot\hat{\mathbf{e}}_s)f_0|^2 + |\partial_t(f_0 e^{i\Phi_+}) + i(\mathbf{F}(s,t)\cdot\hat{\mathbf{e}}_t)f_0|^2.$$

Since $\partial_s\Phi_+ = -\alpha_0 - \sin^2\delta t - \sin\delta\cos\delta s$, we have

$$\left|\partial_s(f_0 e^{i\Phi_+}) + i(\mathbf{F}(s,t)\cdot\hat{\mathbf{e}}_s)f_0\right|^2 = f_0^2\left|\partial_s\Phi_+ + \sin\delta\cos\delta s - (\cos^2\delta)t\right|^2 = f_0^2(t+\alpha_0)^2,$$

and, being $\partial_t\Phi_+ = -\sin^2\delta s + \sin\delta\cos\delta t$, we also have

$$\left|\partial_t(f_0 e^{i\Phi_+}) + i(\mathbf{F}(s,t)\cdot\hat{\mathbf{e}}_t)f_0\right|^2 = \left|f_0'(t) + i\partial_t\Phi_+ + i(\sin^2\delta)s - i\sin\delta\cos\delta t\right|^2 = \left|f_0'(t)\right|^2.$$

From the previous calculations we thus have

$$\mathcal{G}_\Gamma[f_0 e^{i\Phi_+}]\Big|_{\bar\Gamma_{+,\delta}} = \int_{\bar\Gamma_{+,\delta}} \mathrm{d}s\mathrm{d}t \left\{|f_0'(t)|^2 + (t+\alpha_0)^2 f_0^2(t) - \frac{1}{2b}(2f_0^2(t) - f_0^4(t))\right\}$$

Proceeding as in $\Gamma_{-,\delta}$, we get

$$\begin{aligned}\mathcal{G}_\Gamma[\psi_{\mathrm{trial}}]\Big|_{\Gamma_{-,\delta}\cup\Gamma_{+,\delta}} &= 2\ell E_0^{1D} \\ &- (\delta+\gamma)\int_0^{c\ell} t\mathrm{d}t\,\left\{|f_0'|^2 + (t+\alpha_0)^2 f_0^2 - \frac{1}{2b}(2f_0^2 - f_0^4)\right\} + \mathcal{O}(\gamma^3) + \mathcal{O}(e^{-C\ell^2}).\end{aligned}$$

We now work in the other two regions $\Gamma_{\pm,\gamma}$ close to the bisector. We recall that

$$\Gamma_{-,\gamma} = \left\{(\rho,\vartheta)\,\middle|\, 0\le\rho\le\frac{c\ell}{\sin\vartheta},\ \frac{\pi-\delta-\gamma}{2}\le\vartheta\le\frac{\pi-\delta}{2}\right\}.$$

By the same arguments as before, if we denote by $\overline{\Gamma}_{-,\gamma} := \Gamma_{-,\gamma}|_{\{t\le\bar{t}\}}$, we have

$$\mathcal{G}_\Gamma[\psi_{\mathrm{trial}}]\Big|_{\Gamma_{-,\gamma}} = \int_{\overline{\Gamma}_{-,\gamma}} \rho\mathrm{d}\vartheta\mathrm{d}\rho\,\left\{\left|(\nabla + i\mathbf{F}(\rho,\vartheta))f_0 e^{i\Delta\Phi}\right|^2 - \frac{1}{2b}(2f_0^2 - f_0^4)\right\} + \mathcal{O}(e^{-C\ell^2}),$$

where we have omitted for short the dependence of f_0 on ρ, ϑ, i.e., $f_0(\rho\sin\vartheta)$. In the following we are going to commit the same abuse of notation. Since $\bar{\ell}/\sin\vartheta \ge \bar{\ell}$, we can replace in the integrals below $\bar{\ell}/\sin\vartheta$ with $\bar{\ell}$ up to an error $\mathcal{O}(e^{-C\ell^2})$ thanks to the exponential decay of f_0. Before proceeding further, we observe that being $y = \rho\sin\vartheta$ in $\Gamma_{-,\gamma}$, we have that

$$\begin{aligned}\mathbf{F}\cdot\hat{\mathbf{e}}_\rho &:= (-\rho\sin\vartheta, 0)\cdot(\cos\vartheta,\sin\vartheta) = -\rho\sin\vartheta\cos\vartheta,\\ \mathbf{F}\cdot\hat{\mathbf{e}}_\vartheta &:= (-\rho\sin\vartheta, 0)\cdot(-\sin\vartheta,\cos\vartheta) = \rho\sin^2\vartheta.\end{aligned} \tag{6.16}$$

From (6.16), the density of the kinetic energy in $\overline{\Gamma}_{-,\gamma}$ equals

$$\begin{aligned}&\left|(\nabla + i\mathbf{F})f_0 e^{i\Delta\Phi}\right|^2 \\ &\quad= \left|\partial_\rho(f_0 e^{i\Delta\Phi}) + i(\mathbf{F}\cdot\hat{\mathbf{e}}_\rho)f_0 e^{i\Delta\Phi}\right|^2 + \left|\frac{1}{\rho}\partial_\vartheta(f_0 e^{i\Delta\Phi}) + i(\mathbf{F}\cdot\hat{\mathbf{e}}_\vartheta)f_0 e^{i\Delta\Phi}\right|^2.\end{aligned} \tag{6.17}$$

We now consider the first term in (6.17):

$$\begin{aligned}\left|\partial_\rho(f_0 e^{i\Delta\Phi}) + i(\mathbf{F}\cdot\hat{\mathbf{e}}_\rho)f_0 e^{i\Delta\Phi}\right|^2 &= |\partial_\rho f_0 + i\partial_\rho(\Delta\Phi) + i(\mathbf{F}\cdot\hat{\mathbf{e}}_\rho)|^2 \\ &= |\partial_\rho f_0|^2 + |\partial_\rho(\Delta\Phi)f_0 + (\mathbf{F}\cdot\hat{\mathbf{e}}_\rho)f_0|^2 \\ &\le |\partial_\rho f_0|^2 + 2|\partial_\rho(\Delta\Phi)f_0|^2 + 2|(\mathbf{F}\cdot\hat{\mathbf{e}}_\rho)f_0|^2.\end{aligned} \tag{6.18}$$

Since

$$\partial_\rho(\Delta\Phi) = \alpha_0\sin\left(\frac{\delta+\gamma}{2}\right)\left(\frac{2\vartheta-(\pi-\delta)}{\gamma}\right) + \rho\sin\delta\cos\gamma\left(\vartheta - \frac{\pi-\delta-\gamma}{2}\right)\frac{1}{\gamma},$$

we have that

$$|\partial_\rho(\Delta\Phi)|^2 \le C\left|\sin\left(\frac{\delta+\gamma}{2}\right)\left(\frac{2\vartheta-(\pi-\delta)}{\gamma}\right)\right|^2 + C\rho^2\left|\sin\delta\cos\gamma\left(\vartheta-\frac{\pi-\delta-\gamma}{2}\right)\frac{1}{\gamma}\right|^2. \tag{6.19}$$

By the following estimates (recall that $\gamma \gg \delta$)

$$\sin\left(\frac{\delta+\gamma}{2}\right) = \frac{\delta+\gamma}{2} + \mathcal{O}(\gamma^3), \quad \sin\delta = \delta + \mathcal{O}(\delta^3), \quad \cos\gamma = 1 + \mathcal{O}(\gamma^2), \quad \sin^2(2\vartheta) = \mathcal{O}(\gamma^2),$$

(6.19) and the exponential decay of f_0 (Lemma 2.1), we get

$$|\partial_\rho \Delta\Phi|^2 = \mathcal{O}(\gamma^2) + \mathcal{O}(\rho^2\delta^2), \qquad |\mathbf{F}\cdot\hat{\mathbf{e}}_\rho|^2 = \mathcal{O}(\rho^2\gamma^2)$$

and thus

$$\int_{\overline{\Gamma}_{-,\gamma}} \rho \mathrm{d}\rho \mathrm{d}\vartheta \, |\partial_\rho(f_0 e^{i\Delta\Phi}) + i(\mathbf{F}\cdot\hat{\mathbf{e}}_\rho) f_0 e^{i\Delta\Phi}|^2 \leq \int_{\overline{\Gamma}_{-,\gamma}} \rho \mathrm{d}\rho \mathrm{d}\vartheta \, |\partial_\rho f_0|^2 + \mathcal{O}(\gamma^2). \tag{6.20}$$

We now estimate the angular component of the kinetic energy in $\overline{\Gamma}_{-,\gamma}$:

$$\left|\frac{1}{\rho}\partial_\vartheta(f_0 e^{i\Delta\Phi}) + i(\mathbf{F}\cdot\hat{\mathbf{e}}_\vartheta) f_0 e^{i\Delta\Phi}\right|^2 = \frac{1}{\rho^2}|\partial_\vartheta f_0|^2 + \left|\frac{1}{\rho} f_0 \partial_\vartheta \Delta\Phi + (\mathbf{F}\cdot\hat{\mathbf{e}}_\vartheta) f_0 e^{i\Delta\Phi}\right|^2$$

We then compute

$$\begin{aligned}
\frac{1}{\rho}\partial_\vartheta(\Delta\Phi) &= \frac{1}{\gamma}\left(2\alpha_0 \sin\left(\frac{\delta+\gamma}{2}\right) + \frac{\rho}{2}\sin\delta\cos\gamma\right) \\
&= \left(\alpha_0 + \frac{\delta}{\gamma}\left(\frac{\rho}{2} + \alpha_0\right) + \mathcal{O}(\gamma^2) + \mathcal{O}(\rho\delta\gamma)\right)
\end{aligned}$$

and being $\sin^2\vartheta = 1 + \mathcal{O}(\gamma^2)$ in $\Gamma_{-,\gamma}$, we get

$$\begin{aligned}
\frac{1}{\rho}\partial_\vartheta(\Delta\Phi) + (\mathbf{F}\cdot\hat{\mathbf{e}}_\vartheta) = \frac{1}{\rho}\partial_\vartheta(\Delta\Phi) + \rho\sin^2\vartheta &= \frac{1}{\rho}\partial_\vartheta(\Delta\Phi) + \rho(1 + \mathcal{O}(\gamma^2)) \\
&= \left[\rho + \alpha_0 + \frac{\delta}{\gamma}\left(\frac{\rho}{2} + \alpha_0\right)\right] + (1+\rho)\,\mathcal{O}(\gamma^2).
\end{aligned}$$

It then follows that

$$\begin{aligned}
&\int_{\overline{\Gamma}_{-,\gamma}} \rho \mathrm{d}\rho \mathrm{d}\vartheta \, f_0^2 \left|\frac{1}{\rho}\partial_\vartheta(\Delta\Phi) + (\mathbf{F}\cdot\hat{\mathbf{e}}_\vartheta)\right|^2 \\
&\qquad = \int_{\overline{\Gamma}_{-,\gamma}} \rho \mathrm{d}\rho \mathrm{d}\vartheta \, f_0^2 \left[(\rho + \alpha_0) + \frac{\delta}{\gamma}\left(\frac{\rho}{2} + \alpha_0\right)\right]^2 + \mathcal{O}(\gamma^2).
\end{aligned}$$

For the first term on the r.h.s. of the expression above, we observe that

$$\begin{aligned}
\int_{\Gamma_{-,\gamma}} \rho \mathrm{d}\rho \mathrm{d}\vartheta \, f_0^2 \left[(\rho + \alpha_0) + \frac{\delta}{\gamma}\left(\frac{\rho}{2} + \alpha_0\right)\right]^2 &= \frac{2\delta}{\gamma}\int_{\frac{\pi-\delta-\gamma}{2}}^{\frac{\pi-\delta}{2}} \mathrm{d}\vartheta \int_0^{c\ell} \rho \mathrm{d}\rho \, f_0^2 (\rho + \alpha_0)\left(\frac{\rho}{2} + \alpha_0\right) \\
&+ \frac{\delta^2}{\gamma^2}\int_{\frac{\pi-\delta-\gamma}{2}}^{\frac{\pi-\delta}{2}} \mathrm{d}\vartheta \int_0^{c\ell} \rho \mathrm{d}\rho \, f_0^2 \left(\frac{\rho}{2} + \alpha_0\right)^2 \\
&+ \int_{\frac{\pi-\delta-\gamma}{2}}^{\frac{\pi-\delta}{2}} \mathrm{d}\vartheta \int_0^{c\ell} \rho \mathrm{d}\rho \, f_0^2 (\rho + \alpha_0)^2 + \mathcal{O}(e^{-C\ell^2}).
\end{aligned} \tag{6.21}$$

Therefore, using the estimates proven in Lemma 6.1, we obtain

$$\int_{\frac{\pi-\delta-\gamma}{2}}^{\frac{\pi-\delta}{2}} \mathrm{d}\vartheta \int_0^{c\ell} \rho \mathrm{d}\rho\, f_0^2(\rho\sin\vartheta)\,(\rho+\alpha_0)^2 = \frac{\gamma}{2}\int_0^{c\ell} \rho\mathrm{d}\rho\, f_0^2(\rho)(\rho+\alpha_0)^2 + \mathcal{O}(\gamma^2), \tag{6.22}$$

$$\frac{\delta^2}{\gamma^2}\int_{\frac{\pi-\delta-\gamma}{2}}^{\frac{\pi-\delta}{2}} \mathrm{d}\vartheta \int_0^{c\ell} \rho \mathrm{d}\rho\, f_0^2(\rho\sin\vartheta)\left(\frac{\rho}{2}+\alpha_0\right)^2 = \\ = \frac{\delta^2}{2\gamma}\int_0^{c\ell} \rho\mathrm{d}\rho\, f_0^2(\rho)\left(\frac{\rho}{2}+\alpha_0\right)^2 + \mathcal{O}\left(\delta^2\right) = \mathcal{O}(\delta^2\gamma^{-1}),$$

$$\frac{2\delta}{\gamma}\int_{\frac{\pi-\delta-\gamma}{2}}^{\frac{\pi-\delta}{2}} \mathrm{d}\vartheta \int_0^{c\ell} \rho \mathrm{d}\rho\, f_0^2(\rho\sin\vartheta)(\rho+\alpha_0)\left(\frac{\rho}{2}+\alpha_0\right) = \\ = \delta\int_0^{c\ell} \rho\mathrm{d}\rho\, f_0^2(\rho)(\rho+\alpha_0)\left(\frac{\rho}{2}+\alpha_0\right) + \mathcal{O}(\delta\gamma). \tag{6.23}$$

We now consider the terms involving the derivative of f_0 and using again Lemma 6.1, it follows that

$$\mathcal{G}_\Gamma[f_0 e^{i\Delta\Phi}]\Big|_{\Gamma_{-,\gamma}} = \frac{\gamma}{2}\int_0^{c\ell}\rho\mathrm{d}\rho\left\{|f_0'(\rho)|^2 + f_0^2(\rho)(\rho+\alpha_0)^2 - \frac{1}{2b}(2f_0^2(\rho)-f_0^4(\rho))\right\} \\ + \delta\int_0^{c\ell}\rho\mathrm{d}\rho\, f_0^2(\rho)(\rho+\alpha_0)\left(\frac{\rho}{2}+\alpha_0\right) + \mathcal{O}(\delta^2\gamma^{-1}+\gamma^2) + \mathcal{O}(e^{-C\ell^2}). \tag{6.24}$$

If we now consider the last region $\Gamma_{+,\gamma}$, we get the same contributions as in (6.24). Then putting all together, we finally have

$$\begin{aligned}\mathcal{G}_\Gamma[\psi_{\mathrm{trial}}] &= 2\ell E_0^{1D} \\ &- (\delta+\gamma)\int_0^{c\ell}\mathrm{d}t\, t\left\{|f_0'(t)|^2 + (t+\alpha_0)^2 f_0^2(t) - \frac{1}{2b}(2f_0^2(t)-f_0^4(t))\right\} + \\ &+ \gamma\int_0^{c\ell}\mathrm{d}t\, t\left\{|f_0'(t)|^2 + (t+\alpha_0)^2 f_0^2(t) - \frac{1}{2b}(2f_0^2(t)-f_0^4(t))\right\} + \\ &+ 2\delta\int_0^{c\ell}\mathrm{d}t\, t f_0^2(t)(t+\alpha_0)\left(\frac{t}{2}+\alpha_0\right) + \mathcal{O}(\delta^2\gamma^{-1}+\gamma^2) + \mathcal{O}(e^{-C\ell^2})\end{aligned}$$

To conclude the proof we have only to use the identity

$$-(t+\alpha_0)^2 + 2(t+\alpha_0)\left(\frac{t}{2}+\alpha_0\right) = (t+\alpha_0)(-t-\alpha_0+t+2\alpha_0) = \alpha_0(t+\alpha_0). \tag{6.25}$$

In fact, by (6.25), we get

$$E_\Gamma \le 2\ell E_0^{1D} - \delta\int_0^{c\ell}\mathrm{d}t\, t\left\{|f_0'(t)|^2 - \alpha_0(t+\alpha_0)f_0^2 - \frac{1}{2b}(2f_0^2(t)-f_0^4(t))\right\} \\ + \mathcal{O}(\delta^2\gamma^{-1}+\gamma^2) + \mathcal{O}(e^{-C\ell^2}).$$

Then,

$$E_\Gamma \le 2\ell E_0^{1D} - \delta\mathcal{E}_{\alpha_0}^{\mathrm{corr}}[f_0] + \mathcal{O}(\delta^2\gamma^{-1}+\gamma^2) + \mathcal{O}(e^{-C\ell^2}).$$

Choosing now $\gamma = \mathcal{O}(\delta^{2/3})$, we get the result (notice that this also meet the request $\gamma \gg \delta$). □

6.4.2 Lower bound

Proposition 6.3 (Lower bound)**.**
Let $\Gamma \subset \mathbb{R}^2$ be the angular region introduced above, then for any fixed

$$1 < b < \Theta_0^{-1},$$

it holds that as $\ell \to +\infty$ and $\delta \to 0$

$$E_\Gamma \geq 2\ell E_0^{1D} - \delta \mathcal{E}_{\alpha_0}^{\mathrm{corr}}[f_0] + \mathcal{O}(\delta^{4/3}) + \mathcal{O}(e^{-C\ell^2}).$$

A key ingredient of the lower bound is a suitable energy splitting that we state in next

Proposition 6.4 (Energy splitting)**.**
Let $\Gamma \subset \mathbb{R}^2$ be the angular region introduced above, then for any fixed

$$1 < b < \Theta_0^{-1},$$

we define the functions u_1, u_2, u_3, u_4 in such a way that

$$\psi_\Gamma =: \begin{cases} u_1 f_0 e^{i\Phi_-}, & in\, \Gamma_{-,\delta}, \\ u_2 f_0 e^{i\Delta\Phi}, & in\, \Gamma_{-,\gamma}, \\ u_3 f_0 e^{i\Delta\Phi}, & in\, \Gamma_{+,\gamma}, \\ u_4 f_0 e^{i\Phi_+}, & in\, \Gamma_{+,\delta}, \end{cases} \tag{6.26}$$

with $\delta \ll \gamma \ll 1$. Then, as $\ell \to +\infty$,

$$E_\Gamma \geq 2\ell E_0^{1D} - \delta \mathcal{E}_{\alpha_0}^{\mathrm{corr}}[f_0] + \sum_{j=1}^{4} \mathcal{E}_0[u_j] - C\frac{\delta}{\gamma} \sum_{j=2}^{3} \int_{\Gamma_j} \mathrm{d}s\mathrm{d}t \left\{ f_0^2 |\nabla u_j|^2 + [f_0^4 (1 - |u_j|^2)^2] \right\} \\ + \mathcal{O}(\delta^{4/3}) + \mathcal{O}(e^{-C\ell^2}),$$

where

$$\mathcal{E}_0[u_j] := \int_{\Gamma_j} \mathrm{d}s\mathrm{d}t\, f_0^2 \left\{ |\partial_t u_j|^2 + |\partial_s u_j|^2 - 2(t + \alpha_0)\,(iu_j, \partial_s u_j) + \frac{f_0^2}{2b}(1 - |u_j|^2)^2 \right\} \tag{6.27}$$

and Γ_j the region where u_j is supported.

Before proving the Proposition above, we prove a useful Lemma. More precisely we now prove some estimates which allow us to replace $|u|$ with 1 in the region where the angle is near $\pi/2$.

Lemma 6.2.
Let $\delta \ll \gamma \ll 1$ and let $L > 1$, for any $\vartheta \in \left[\frac{\pi-\delta-\gamma}{2}, \frac{\pi-\delta+\gamma}{2}\right]$ and any function $u(\rho, \vartheta)$, it holds that

$$\int_{\Gamma_{\pm,\gamma}} \mathrm{d}\rho\mathrm{d}\vartheta\, \rho^\beta f_0^2 |u|^2 \geq \gamma \int_0^{c\ell} \mathrm{d}\rho\, \rho^\beta f_0^2(\rho) - \frac{C}{L} \int_{\Gamma_{\pm,\gamma}} \rho\mathrm{d}\rho\mathrm{d}\vartheta\, f_0^4(\rho \sin\vartheta)(1 - |u|^2)^2 \\ + \mathcal{O}(\gamma^2) + \mathcal{O}(\gamma L^{2\beta+1}) + \mathcal{O}(e^{-C\ell^2}), \tag{6.28}$$

for any $\beta \geq 0$.

Proof. We observe that

$$\int_{\Gamma_{\pm,\gamma}} \mathrm{d}\rho \mathrm{d}\vartheta \rho^{\beta} f_0^2(\rho\sin\vartheta)|u|^2 = \int_{\Gamma_{\pm,\gamma}} \mathrm{d}\rho \mathrm{d}\vartheta \rho^{\beta} f_0^2(\rho\sin\vartheta) + \int_{\Gamma_{\pm,\gamma}} \mathrm{d}\rho \mathrm{d}\vartheta \rho^{\beta} f_0^2(\rho\sin\vartheta)(|u|^2-1). \tag{6.29}$$

We now use the approximation for $f_0(\rho\sin\vartheta)$ proved in Lemma 6.1, i.e.,

$$\left| f_0^2(\rho\sin\vartheta) - f_0^2(\rho) \right| \le C\gamma e^{-C\rho^2}. \tag{6.30}$$

Since $c\ell/\sin\vartheta \ge c\ell$, in the integrals below we can replace $c\ell/\sin\vartheta$ with $c\ell$ up to a small error $\mathcal{O}(e^{-C\ell^2})$. By (6.30), we have

$$\begin{aligned}\int_{\Gamma_{\pm,\gamma}} \mathrm{d}\rho \mathrm{d}\vartheta \rho^{\beta} f_0^2(\rho\sin\vartheta) &= \gamma \int_0^{c\ell} \mathrm{d}\rho\, \rho^{\beta} f_0^2(\rho) + \mathcal{O}(\gamma) \int_{\Gamma_{\pm,\gamma}} \mathrm{d}\rho \mathrm{d}\vartheta \rho^{\beta} e^{-C\rho^2} + \mathcal{O}(e^{-C\ell^2}) \\ &= \gamma \int_0^{c\ell} \mathrm{d}\rho\, \rho^{\beta} f_0^2(\rho) + \mathcal{O}(\gamma^2) + \mathcal{O}(e^{-C\ell^2}).\end{aligned} \tag{6.31}$$

We now estimate the second term in (6.29), first of all we observe that

$$\begin{aligned}\int_{\Gamma_{-,\gamma}} \mathrm{d}\rho \mathrm{d}\vartheta\, \rho^{\beta}\, f_0^2(\rho\sin\vartheta)(|u|^2-1) &= \int_{\frac{\pi-\delta-\gamma}{2}}^{\frac{\pi-\delta}{2}} \int_0^L \mathrm{d}\rho\rho^{\beta}\, f_0^2(\rho\sin\vartheta)(|u|^2-1) \\ &+ \int_{\frac{\pi-\delta-\gamma}{2}}^{\frac{\pi-\delta}{2}} \int_L^{c\ell} \mathrm{d}\rho\rho^{\beta}\, f_0^2(\rho\sin\vartheta)(|u|^2-1) + \mathcal{O}(e^{-C\ell^2}).\end{aligned} \tag{6.32}$$

We estimate the first term on the r.h.s. of (6.32) as

$$\begin{aligned}&\int_{\frac{\pi-\delta-\gamma}{2}}^{\frac{\pi-\delta}{2}} \int_0^L \mathrm{d}\rho\rho^{\beta}\, f_0^2(\rho\sin\vartheta)(|u|^2-1) \\ &\ge -C\xi\gamma \int_0^L \mathrm{d}\rho\, \rho^{2\beta-1} - \frac{C}{\xi} \int_{\Gamma_{-,\gamma}} \mathrm{d}\rho \mathrm{d}\vartheta\, \rho\, f_0^4(\rho)(1-|u|^2)^2 + \mathcal{O}(e^{-C\ell^2}) \\ &\ge -CL^{2\beta}\gamma\xi - \frac{C}{\xi} \int_{\Gamma_{-,\gamma}} \mathrm{d}\rho \mathrm{d}\vartheta\, \rho\, f_0^4(t)(1-|u|^2)^2 + \mathcal{O}(e^{-C\ell^2})\end{aligned}$$

where the remainder is due to the exponential decay of f_0 (Lemma 2.1). Choosing now $\xi = L$, we then get

$$\begin{aligned}\int_{\frac{\pi-\delta-\gamma}{2}}^{\frac{\pi-\delta}{2}} \int_0^L \mathrm{d}\rho\rho^{\beta}\, f_0^2(\rho\sin\vartheta)(|u|^2-1) &\ge -\frac{C}{L} \int_{\Gamma_{-,\gamma}} \mathrm{d}\rho \mathrm{d}\vartheta\, \rho^{\beta}\, f_0^2(1-|u|^2)^2 \\ &- C\gamma L^{2\beta+1} + \mathcal{O}(e^{-C\ell^2}).\end{aligned} \tag{6.33}$$

On the other hand, the second term on the r.h.s. of (6.32) is exponentially small in L: in fact

$$\begin{aligned}\int_{\frac{\pi-\delta-\gamma}{2}}^{\frac{\pi-\delta}{2}} \int_L^{c\ell} \mathrm{d}\rho\rho^{\beta}\, f_0^2(\rho\sin\vartheta)||u|^2-1| &\le \int_{\frac{\pi-\delta-\gamma}{2}}^{\frac{\pi-\delta}{2}} \int_L^{c\ell} \mathrm{d}\rho\rho^{\beta}\, f_0^2(\rho) + C\gamma e^{-CL^2} \\ &\le C\gamma e^{-CL^2}.\end{aligned} \tag{6.34}$$

From (6.31), (6.33), (6.34), we conclude that

$$\int_{\Gamma_{\pm,\gamma}} \mathrm{d}\rho \mathrm{d}\vartheta\, \rho^\beta f_0^2 |u|^2 \geq \gamma \int_0^{c\ell} \mathrm{d}\rho\, \rho^\beta f_0^2(\rho) - \frac{C}{L} \int_{\Gamma_{\pm,\gamma}} \rho \mathrm{d}\rho \mathrm{d}\vartheta\, f_0^4(\rho \sin\vartheta)(1-|u|^2)^2 + \mathcal{O}(\gamma^2) + \mathcal{O}(\gamma L^{2\beta+1}) + \mathcal{O}(e^{-C\ell^2}).$$

□

Proposition 6.4. We first consider the region $\Gamma_{-,\delta}$, here $\psi_\Gamma = u_1 f_0 e^{i\Phi_-}$, then

$$\mathcal{G}_\Gamma[\psi_\Gamma]\Big|_{\Gamma_{-,\delta}} = \int_{\Gamma_{-,\delta}} \mathrm{d}s\mathrm{d}t \left\{ \left|(\nabla + i(-t,0)) f_0 u_1 e^{i\Phi_1(s,t)}\right|^2 - \frac{1}{2b}(2f_0^2|u_1|^2 - f_0^4|u_1|^4) \right\}.$$

Since

$$|(\nabla + i(-t,0)) f_0 u_1 e^{i\Phi_1}|^2 = |f_0'|^2|u_1|^2 + |\partial_s u_1|^2 f_0^2 + |\partial_t u_1|^2 f_0^2 + (t+\alpha_0)^2 f_0^2 |u_1|^2 + - 2(t+\alpha_0) f_0^2\, j_s[u_1] + f_0 \partial_t f_0 \partial_t |u_1|^2,$$

where $j_s[u_1] = \mathbf{j}[u_1] \cdot \hat{\mathbf{e}}_s$, we get

$$\mathcal{G}_\Gamma[\psi_\Gamma]\Big|_{\Gamma_{-,\delta}} = \int_{\Gamma_{-,\delta}} \mathrm{d}s\mathrm{d}t \Big\{ |f_0'|^2|u_1|^2 + |\partial_s u_1|^2 f_0^2 + |\partial_t u_1|^2 f_0^2 + (t+\alpha_0)^2 f_0^2|u_1|^2 - 2(t+\alpha_0) f_0^2\, j_s[u_1] + \partial_t f_0 f_0 \partial_t |u_1|^2 + \frac{1}{2b}(2f_0^2 - f_0^4) \Big\}. \tag{6.35}$$

We now observe that

$$\int_{\Gamma_{-,\delta}} \mathrm{d}s\mathrm{d}t\, f_0(t) \partial_t f_0 \partial_t |u_1|^2 = \int_{\Gamma_{-,\delta}} \mathrm{d}s\mathrm{d}t\, f_0(t) \nabla f_0 \cdot \nabla |u_1|^2 = \int_{\partial\Gamma_{-,\delta}} \mathrm{d}\sigma\, \hat{\mathbf{n}}_- \cdot \nabla f_0 f_0 |u_1|^2 - \int_{\Gamma_{-,\delta}} \mathrm{d}s\mathrm{d}t\, \{|f_0'|^2|u_1|^2 - f_0 f_0''|u_1|^2\}, \tag{6.36}$$

where $\hat{\mathbf{n}}_-$ is the outward normal unit vector along $\partial\Gamma_{-,\delta}$. Before taking into account the boundary terms in (6.36), we introduce some notation: with respect to the Figure 6.4, we set

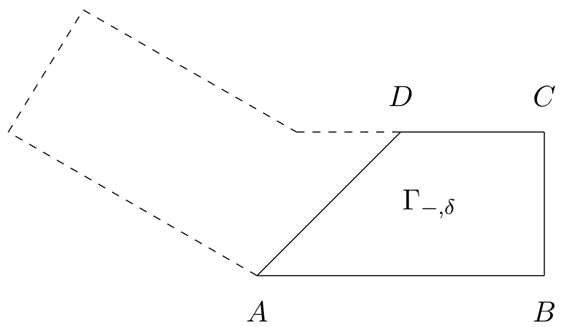

Figure 6.4. The region $\Gamma_{-,\delta}$.

$$\overline{AB} =: \partial\Gamma^{\mathrm{ext}}_{-,\delta}, \qquad \overline{BC} =: \partial\Gamma^{\mathrm{bd}}_{-,\delta}, \qquad \overline{CD} =: \partial\Gamma^{\mathrm{bulk}}_{-,\delta}, \qquad \overline{AD} =: \partial\Gamma^{\mathrm{obl}}_{-,\delta}.$$

We then have

$$\int_{\partial\Gamma_{-,\delta}} \mathrm{d}\sigma\, \hat{\mathbf{n}}_- \cdot \nabla f_0 f_0 |u_1|^2 = \int_{\partial\Gamma^{\mathrm{obl}}_{-,\delta}} \mathrm{d}\sigma\, \hat{\mathbf{n}}_- \cdot \nabla f_0 f_0 |u_1|^2, \tag{6.37}$$

In fact, the boundary term along $\partial\Gamma^{\mathrm{ext}}_{-,\delta}$ vanishes because $f_0'(0) = 0$, the one along $\partial\Gamma^{\mathrm{bulk}}_{-,\delta}$ is equal to zero because $u_1(s, c\ell) = 0$, and on $\partial\Gamma^{\mathrm{bd}}_{-,\delta}$ we simply use that $\hat{\mathbf{n}}_- \cdot \nabla f_0 = 0$ there. From (6.35), (6.36) and (6.37), we thus get

$$\begin{aligned} \mathcal{G}_\Gamma[\psi_\Gamma]\Big|_{\Gamma_{-,\delta}} = \int_{\Gamma_{-,\delta}} \mathrm{d}s\mathrm{d}t \Big\{ &- f_0 f_0'' ||u_1|^2 + |\partial_s u_1|^2 f_0^2 + |\partial_t u_1|^2 f_0^2 + (t+\alpha_0)^2 f_0^2 |u_1|^2 \\ &- 2(t+\alpha_0) f_0^2\, j_s[u_1] + \frac{1}{2b}(2f_0^2 - f_0^4) \Big\} + \int_{\partial\Gamma^{\mathrm{obl}}_{-,\delta}} \mathrm{d}\sigma\, \hat{\mathbf{n}}_- \cdot \nabla f_0 f_0 |u_1|^2. \end{aligned}$$

Then, using the variational equations for f_0 we have

$$\begin{aligned} \mathcal{G}_\Gamma[\psi_\Gamma]\Big|_{\Gamma_{-,\delta}} &= \int_{\Gamma_{-,\delta}} \mathrm{d}s\mathrm{d}t\, f_0^2 \Big\{ |\partial_s u_1|^2 + |\partial_t u_1|^2 - 2(t+\alpha_0)\, j_s[u_1] + \frac{f_0^2}{2b}(1 - |u_1|^2)^2 \Big\} \\ &\quad - \frac{1}{2b} \int_{\Gamma_{-,\delta}} \mathrm{d}s\mathrm{d}t f_0^4(t) + \int_{\partial\Gamma^{\mathrm{obl}}_{-,\delta}} \mathrm{d}\sigma\, \hat{\mathbf{n}}_- \cdot \nabla f_0 \cdot f_0 \cdot |u_1|^2. \end{aligned} \tag{6.38}$$

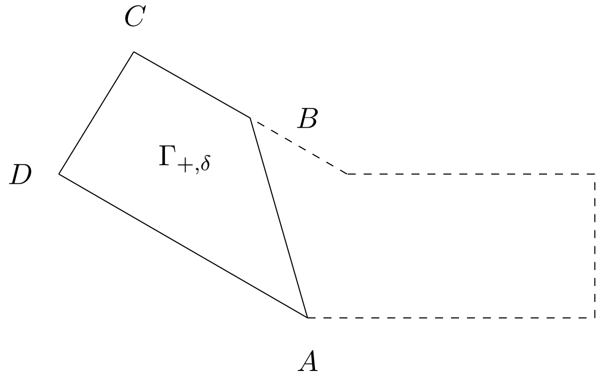

Figure 6.5. The region $\Gamma_{+,\delta}$.

We can perform a similar analysis in $\Gamma_{+,\delta}$, to get

$$\begin{aligned} \mathcal{G}_\Gamma[\psi_\Gamma]\Big|_{\Gamma_{+,\delta}} &= \int_{\Gamma_{+,\delta}} \mathrm{d}s\mathrm{d}t\, f_0^2 \Big\{ |\partial_s u_4|^2 + |\partial_t u_4|^2 - 2(t+\alpha_0)\, j_s[u_4] + \frac{f_0^2}{2b}(1 - |u_4|^2)^2 \Big\} \\ &\quad - \frac{1}{2b} \int_{\Gamma_{+,\delta}} \mathrm{d}s\mathrm{d}t f_0^4(t) + \int_{\partial\Gamma^{\mathrm{obl}}_{+,\delta}} \mathrm{d}\sigma\, \hat{\mathbf{n}}_+ \cdot \nabla f_0 f_0 |u_4|^2, \end{aligned} \tag{6.39}$$

where, looking at Figure 6.5, $\overline{AB} =: \partial\Gamma^{\mathrm{obl}}_{+,\delta}$. We also denote by $\hat{\mathbf{n}}_+$ the outer normal unit vector to $\partial\Gamma_{+,\delta}$.

We now consider $\Gamma_{-,\gamma}$. As in the upper bound we can work in this region using polar coordinates. Before proceeding further, we recall that

$$\mathbf{F}(\rho, \vartheta) := \rho(-\sin\vartheta\cos\vartheta, \sin^2\vartheta).$$

and

$$\mathcal{G}_\Gamma[\psi_\Gamma]\Big|_{\Gamma_{-,\gamma}} = \int_{\frac{\pi-\delta-\gamma}{2}}^{\frac{\pi-\delta}{2}} \mathrm{d}\vartheta \int_0^{\frac{c\ell}{\sin\vartheta}} \rho \mathrm{d}\rho \left\{ |(\nabla + i\mathbf{F}) f_0 u_2 e^{i\Delta\Phi(\rho,\vartheta)}|^2 - \frac{1}{2b}(2f_0^2|u_2|^2 - f_0^4|u_2|^4) \right\}.$$

Since

$$(\nabla + i\mathbf{F}(\rho,\vartheta)) f_0 u_2 e^{i\Delta\Phi} = \left[\partial_\rho(f_0 u_2 e^{i\Delta\Phi}) - i\rho\sin\vartheta\cos\vartheta(f_0 u_2 e^{i\Delta\Phi})\right]\hat{\mathbf{e}}_\rho + \left[\frac{1}{\rho}\partial_\vartheta(f_0 u_2 e^{i\Delta\Phi}) + i\rho\sin^2\vartheta(f_0 u_2 e^{i\Delta\Phi})\right]\hat{\mathbf{e}}_\vartheta,$$

then

$$\left|(\nabla + i\mathbf{F}) f_0 u_2 e^{i\Delta\Phi}\right|^2 = \left[\partial_\rho(f_0 u_2 e^{i\Delta\Phi}) - i\rho\sin\vartheta\cos\vartheta(f_0 u_2 e^{i\Delta\Phi})\right]^2 + \left[\frac{1}{\rho}\partial_\vartheta(f_0 u_2 e^{i\Delta\Phi}) + i\rho\sin^2\vartheta(f_0 u_2 e^{i\Delta\Phi})\right]^2. \tag{6.40}$$

Recall that $\delta \ll \gamma \ll 1$. We first consider the $\hat{\mathbf{e}}_\rho$−component:

$$\begin{aligned}\left[\partial_\rho(f_0 u_2 e^{i\Delta\Phi}) - i\frac{\rho}{2}\sin(2\vartheta)(f_0 u_2 e^{i\Delta\Phi})\right]^2 &= |\partial_\rho f_0|^2|u_2|^2 + |\partial_\rho u_2|^2 f_0^2 + |\partial_\rho(\Delta\Phi)|^2 f_0^2|u_2|^2 \\ &+ \frac{\rho^2}{4}\sin^2(2\vartheta) f_0^2|u_2|^2 - \rho\sin(2\vartheta) f_0^2|u_2|^2\partial_\rho(\Delta\Phi) \\ &+ \left[2\partial_\rho(\Delta\Phi) - \rho\sin(2\vartheta)\right] j_\rho[u_2] f_0^2 \\ &+ f_0(\partial_\rho f_0)\partial_\rho|u_2|^2.\end{aligned} \tag{6.41}$$

Since, obviously,

$$\int_{\Gamma_{-,\gamma}} \rho\mathrm{d}\rho\mathrm{d}\vartheta \left\{ |\partial_\rho(\Delta\Phi)|^2 f_0^2|u_2|^2 + \frac{\rho^2}{4}\sin^2(2\vartheta) f_0^2|u_2|^2 \right\} \geq 0,$$

we are going to drop the corresponding term. Recalling that

$$\partial_\rho(\Delta\Phi) = \alpha_0 \sin\left(\frac{\delta+\gamma}{2}\right)\left(\frac{2\vartheta - \pi + \delta}{\gamma}\right) + \rho\sin\delta\cos\gamma\left(\vartheta - \frac{\pi-\delta-\gamma}{2}\right)\frac{1}{\gamma},$$

we compute

$$\begin{aligned}\int_{\Gamma_{-,\gamma}} \rho\mathrm{d}\rho\mathrm{d}\vartheta\, 2\partial_\rho(\Delta\Phi)\, j_\rho[u_2] f_0^2 = \int_{\Gamma_{-,\gamma}} \rho\mathrm{d}\rho\mathrm{d}\vartheta \Bigg[&\alpha_0\sin\left(\frac{\delta+\gamma}{2}\right)\left(\frac{2\vartheta-\pi+\delta}{\gamma}\right) \\ &+ \rho\sin\delta\cos\gamma\left(\vartheta - \frac{\pi-\delta-\gamma}{2}\right)\frac{1}{\gamma}\Bigg] j_\rho[u_2] f_0^2,\end{aligned} \tag{6.42}$$

where again f_0 is short for $f_0(\rho\sin\vartheta)$ and $j_\rho[u_2] = \mathbf{j}[u_2]\cdot\hat{\mathbf{e}}_\rho$. In what follows we often use the estimates

$$\sin\left(\frac{\delta+\gamma}{2}\right) = \frac{\delta+\gamma}{2} + \mathcal{O}(\gamma^3), \qquad \sin\delta\cos\gamma = \delta + \mathcal{O}(\delta\gamma^2).$$

The $\rho-$component of the current in (6.42) is not expected to contribute much to the energy, since as usual, it is the tangential component $j_s[u_2]$ which matters and

$$\frac{\partial}{\partial s} = \cos\vartheta \frac{\partial}{\partial \rho} - \frac{\sin\vartheta}{\rho}\frac{\partial}{\partial \vartheta},$$

which inside $\Gamma_{-,\gamma}$ implies

$$\frac{\partial}{\partial s} = -\frac{1}{\rho}\frac{\partial}{\partial \vartheta} + o(1)\left(\frac{\partial}{\partial \rho} + \frac{\partial}{\partial \vartheta}\right).$$

Indeed, we prove that the component of the current along $\hat{\mathbf{e}}_\rho$ yields a small contribution: we estimate the firs term on the r.h.s. of (6.42) as

$$\begin{aligned}\int_{\Gamma_{-,\gamma}} \rho \mathrm{d}\rho \mathrm{d}\vartheta\, 2\alpha_0 \, \sin\left(\frac{\delta+\gamma}{2}\right)\left(\frac{2\vartheta-\pi+\delta}{\gamma}\right) j_\rho[u_2] f_0^2 (\rho\sin\vartheta) \\ \geq -C\gamma \int_{\Gamma_{-,\gamma}} \rho \mathrm{d}\rho \mathrm{d}\vartheta\, f_0^2 |u_2|^2 - C\gamma \int_{\Gamma_{-,\gamma}} \rho \mathrm{d}\rho \mathrm{d}\vartheta\, f_0^2 |\partial_\rho u_2|^2\end{aligned} \tag{6.43}$$

By Lemma 6.2 with $L = 1$, we get

$$\begin{aligned}\int_{\Gamma_{-,\gamma}} \rho \mathrm{d}\rho \mathrm{d}\vartheta\, 2\alpha_0 \, \sin\left(\frac{\delta+\gamma}{2}\right)\left(\frac{2\vartheta-\pi+\delta}{\gamma}\right) j_\rho[u_2] f_0^2 (\rho\sin\vartheta) \\ \geq -C\gamma \int_{\Gamma_{-,\gamma}} \rho \mathrm{d}\rho \mathrm{d}\vartheta\, f_0^4 (1-|u_2|^2)^2 - C\gamma \int_{\Gamma_{-,\gamma}} \rho \mathrm{d}\rho \mathrm{d}\vartheta\, f_0^2 |\partial_\rho u_2|^2 \\ + \mathcal{O}(\gamma^2) + \mathcal{O}(e^{-C\ell^2}).\end{aligned} \tag{6.44}$$

For the other term in (6.42), we can proceed in a similar way and, using that $\sin\delta\cos\gamma = \mathcal{O}(\delta)$, we get

$$\begin{aligned}\int_{\Gamma_{-,\gamma}} \rho \mathrm{d}\rho \mathrm{d}\vartheta\, 2\partial_\rho(\Delta\Phi)\, j_\rho[u_2] f_0^2 \\ \geq -C\delta \int_{\Gamma_{-,\gamma}} \rho \mathrm{d}\rho \mathrm{d}\vartheta\, f_0^4 (1-|u_2|^2)^2 - C\delta \int_{\Gamma_{-,\gamma}} \rho \mathrm{d}\rho \mathrm{d}\vartheta\, f_0^2 |\partial_\rho u_2|^2 \\ + \mathcal{O}(\gamma\delta) + \mathcal{O}(e^{-C\ell^2}).\end{aligned} \tag{6.45}$$

Since $\sin(2\vartheta) = \mathcal{O}(\gamma)$ in $\Gamma_{-,\gamma}$, we also get the following estimate

$$\begin{aligned}-\int_{\Gamma_{-,\gamma}} \rho \mathrm{d}\rho \mathrm{d}\vartheta\, \rho\sin(2\vartheta)\, j_\rho[u_2] f_0^2 \\ \geq -C\gamma \int_{\Gamma_{-,\gamma}} \rho \mathrm{d}\rho \mathrm{d}\vartheta\, f_0^4 (1-|u_2|^2)^2 - C\gamma \int_{\Gamma_{-,\gamma}} \rho \mathrm{d}\rho \mathrm{d}\vartheta\, f_0^2 |\partial_\rho u_2|^2 \\ + \mathcal{O}(\gamma^2) + \mathcal{O}(e^{-C\ell^2}).\end{aligned} \tag{6.46}$$

From (6.41), (6.44), (6.45) and (6.46), we get

$$\begin{aligned}\int_{\Gamma_{-,\gamma}} \rho \mathrm{d}\rho \mathrm{d}\vartheta \left[\partial_\rho (f_0 u_2 e^{i\Delta\Phi}) - i\frac{\rho}{2}\sin(2\vartheta)(f_0 u_2 e^{i\Delta\Phi})\right]^2 \\ \geq \int_{\Gamma_{-,\gamma}} \rho \mathrm{d}\rho \mathrm{d}\vartheta \left\{ |\partial_\rho f_0|^2 |u_2|^2 + (1+\mathcal{O}(\gamma))|\partial_\rho u_2|^2 f_0^2 + f_0(\partial_\rho f_0)\partial_\rho |u_2|^2 \right\} \\ - C\gamma \int_{\Gamma_{-,\gamma}} \rho \mathrm{d}\rho \mathrm{d}\vartheta\, f_0^4 (1-|u_2|^2)^2 + \mathcal{O}(\gamma^2) + \mathcal{O}(e^{-C\ell^2}).\end{aligned} \tag{6.47}$$

We now consider the component along $\hat{\mathbf{e}}_\vartheta$ of the kinetic energy, i.e., the second term on the r.h.s of (6.40). We observe that

$$\left[\frac{1}{\rho}\partial_\vartheta(f_0 u_2 e^{i\Delta\Phi}) + i\rho\sin^2\vartheta(f_0 u_2 e^{i\Delta\Phi})\right]^2 = \frac{1}{\rho^2}|\partial_\vartheta f_0|^2|u_2|^2 + \frac{1}{\rho^2}|\partial_\vartheta u_2|^2 f_0^2 + \frac{1}{\rho^2} f_0(\partial_\vartheta f_0)(\partial_\vartheta |u_2|^2) + \left(\frac{1}{\rho}\partial_\vartheta(\Delta\Phi) + \rho\sin^2\vartheta\right)^2 f_0^2|u_2|^2 + 2\left(\frac{1}{\rho}\partial_\vartheta\Delta\Phi + \rho\sin^2\vartheta\right) j_\vartheta[u_2] f_0^2, \tag{6.48}$$

where $j_\vartheta[u_2] = \mathbf{j}[u_2]\cdot\hat{\mathbf{e}}_\vartheta$. Let us first estimate

$$\int_{\Gamma_{-,\gamma}} \rho \mathrm{d}\rho\mathrm{d}\vartheta \left(\frac{1}{\rho}\partial_\vartheta(\Delta\Phi) + \rho\sin^2\vartheta\right)^2 f_0^2|u_2|^2.$$

Recalling that

$$\frac{1}{\rho}\partial_\vartheta(\Delta\Phi) = \frac{1}{\gamma}\left(2\alpha_0\sin\left(\frac{\delta+\gamma}{2}\right) + \frac{\rho}{2}\sin\delta\cos\gamma\right) = \left(\alpha_0 + \frac{\delta}{\gamma}\left(\frac{\rho}{2}+\alpha_0\right) + \mathcal{O}(\gamma^2) + \mathcal{O}(\rho\delta\gamma)\right), \tag{6.49}$$

and

$$\mathbf{F}\cdot\hat{\mathbf{e}}_\vartheta = \rho\,(1+\mathcal{O}(\gamma^2)),$$

we obtain

$$\left(\frac{1}{\rho}\partial_\vartheta(\Delta\Phi) + \rho\sin^2\vartheta\right) = \left(\rho + \alpha_0 + \frac{\delta}{\gamma}\left(\frac{\rho}{2}+\alpha_0\right) + \mathcal{O}(\gamma^2) + \mathcal{O}(\rho\delta\gamma)\right). \tag{6.50}$$

Since $f_0|u_2| = |\psi_\Gamma|$ in $\Gamma_{-,\gamma}$, the exponential decay (4.3) of $|\psi_\Gamma|$ yields

$$\int_{\Gamma_{-,\gamma}} \rho\mathrm{d}\rho\mathrm{d}\vartheta \left(\frac{1}{\rho}\partial_\vartheta(\Delta\Phi) + \rho\sin^2\vartheta\right)^2 f_0^2|u_2|^2 = \int_{\Gamma_{-,\gamma}} \rho\mathrm{d}\rho\mathrm{d}\vartheta \left[\rho + \alpha_0 + \frac{\delta}{\gamma}\left(\frac{\rho}{2}+\alpha_0\right)\right]^2 f_0^2|u_2|^2 + \mathcal{O}(\gamma^2). \tag{6.51}$$

We now consider the first term on the r.h.s. of the expression above, using again Lemma 6.2, we have

$$\frac{2\delta}{\gamma}\int_{\Gamma_{-,\gamma}} \rho\mathrm{d}\rho\mathrm{d}\vartheta\,(\rho+\alpha_0)^2\left(\frac{\rho}{2}+\alpha_0\right) f_0^2|u_2|^2 \geq \delta\int_0^{c\ell}\rho\mathrm{d}\rho\,(\rho+\alpha_0)\left(\frac{\rho}{2}+\alpha_0\right) f_0^2(\rho) - \frac{C\delta}{\gamma}\int_{\Gamma_{-,\gamma}}\rho\mathrm{d}\rho\mathrm{d}\vartheta\, f_0^4(1-|u_2|^2)^2 + \mathcal{O}(\delta\gamma) + \mathcal{O}(e^{-C\ell^2}), \tag{6.52}$$

$$\frac{\delta^2}{\gamma^2}\int_{\Gamma_{-,\gamma}} \rho\mathrm{d}\rho\mathrm{d}\vartheta \left(\frac{\rho}{2}+\alpha_0\right)^2 f_0^2|u_2|^2 \geq -\frac{C\delta^2}{\gamma^2}\int_{\Gamma_{-,\gamma}}\rho\mathrm{d}\rho\mathrm{d}\vartheta\, f_0^4(1-|u_2|^2)^2 + \mathcal{O}(\delta^2\gamma^{-1}) + \mathcal{O}(e^{-C\ell^2}), \tag{6.53}$$

for some $L \gg 1$ to be chosen later.

Putting together (6.51), (6.52), (6.53), we get

$$\begin{aligned}
\int_{\Gamma_{-,\gamma}} \rho \mathrm{d}\rho \mathrm{d}\vartheta \left(\frac{1}{\rho} \partial_\vartheta (\Delta \Phi) + \rho \sin^2 \vartheta \right)^2 f_0^2 |u_2|^2 \\
\geq \int_{\Gamma_{-,\gamma}} \rho \mathrm{d}\rho \mathrm{d}\vartheta \, (\rho + \alpha_0)^2 f_0^2 + \delta \int_0^{c\ell} \rho \mathrm{d}\rho \, (\rho + \alpha_0) \left(\frac{\rho}{2} + \alpha_0 \right) f_0^2(\rho) \\
- \frac{\delta}{\gamma} \int_{\Gamma_{-,\gamma}} \rho \mathrm{d}\rho \mathrm{d}\vartheta \, f_0^4 (1 - |u_2|^2)^2 \\
+ \mathcal{O}(\gamma^2) + \mathcal{O}(\delta^2 \gamma^{-1}) + \mathcal{O}(e^{-C\ell^2}). \quad (6.54)
\end{aligned}$$

We now estimate the term in (6.48) involving the current, i.e.,

$$2 \int_{\Gamma_{-,\gamma}} \rho \mathrm{d}\rho \mathrm{d}\vartheta \left(\frac{1}{\rho} \partial_\vartheta (\Delta \Phi) + \rho \sin^2 \vartheta \right) j_\vartheta [u_2] f_0^2.$$

We recall that

$$\frac{\partial}{\partial s} = -\frac{1}{\rho} \frac{\partial}{\partial \vartheta} + o(1) \left(\frac{\partial}{\partial \rho} + \frac{\partial}{\partial \vartheta} \right),$$

for this reason we now want to prove that

$$\begin{aligned}
2 \int_{\Gamma_{-,\gamma}} \rho \mathrm{d}\rho \mathrm{d}\vartheta \left\{ \left(\frac{1}{\rho} \partial_\vartheta (\Delta \Phi) + \rho \sin^2 \vartheta \right) j_\vartheta [u_2] f_0^2 \right\} \\
= 2 \int_{\Gamma_{-,\gamma}} \rho \mathrm{d}\rho \mathrm{d}\vartheta \, (\rho + \alpha_0) f_0^2 \left(i u_2, \frac{1}{\rho} u_2 \right) + o(1)
\end{aligned}$$

By (6.49), we have

$$\begin{aligned}
2 \int_{\Gamma_{-,\gamma}} \rho \mathrm{d}\rho \mathrm{d}\vartheta \left(\frac{1}{\rho} \partial_\vartheta (\Delta \Phi) + \rho \sin^2 \vartheta \right) f_0^2 \, j_\vartheta [u_2] \\
= 2 \int_{\Gamma_{-,\gamma}} \rho \mathrm{d}\rho \mathrm{d}\vartheta \, (\rho + \alpha_0) f_0^2 \, j_\vartheta [u_2] + \frac{2\delta}{\gamma} \int_{\Gamma_{-,\gamma}} \rho \mathrm{d}\rho \mathrm{d}\vartheta \left(\frac{\rho}{2} + \alpha_0 \right) f_0^2 \, j_\vartheta [u_2] \\
+ C(\gamma^2) \int_{\Gamma_{-,\gamma}} \rho \mathrm{d}\rho \mathrm{d}\vartheta \, f_0^2 \, j_\vartheta [u_2] \\
+ C(\delta\gamma) \int_{\Gamma_{-,\gamma}} \rho \mathrm{d}\rho \mathrm{d}\vartheta \, \rho \, f_0^2 \, j_\vartheta [u_2]. \quad (6.55)
\end{aligned}$$

We estimate the second term on the r.h.s. of (6.55) as

$$\begin{aligned}
\frac{2\delta}{\gamma} \int_{\Gamma_{-,\gamma}} \rho \mathrm{d}\rho \mathrm{d}\vartheta \left(\frac{\rho}{2} + \alpha_0 \right) f_0^2 \left(i u_2, \frac{1}{\rho} \partial_\vartheta u_2 \right) \\
\geq -C \frac{\delta^2 L}{\gamma^2} \int_{\Gamma_{-,\gamma}} \rho \mathrm{d}\rho \mathrm{d}\vartheta \left(\frac{\rho}{2} + \alpha_0 \right)^2 f_0^2 |u_2|^2 - \frac{C}{L} \int_{\Gamma_{-,\gamma}} \rho \mathrm{d}\rho \mathrm{d}\vartheta \, \frac{1}{\rho^2} |\partial_\vartheta u_2|^2 f_0^2.
\end{aligned}$$

and by using Lemma 6.2, we get

$$\begin{aligned}
\frac{2\delta}{\gamma} \int_{\Gamma_{-,\gamma}} \rho \mathrm{d}\rho \mathrm{d}\vartheta \left(\frac{\rho}{2} + \alpha_0 \right) \left(i u_2, \frac{1}{\rho} \partial_\vartheta u_2 \right) f_0^2 \\
\geq -\frac{C}{L} \int_{\Gamma_{-,\gamma}} \rho \mathrm{d}\rho \mathrm{d}\vartheta \, \frac{1}{\rho^2} |\partial_\vartheta u_2|^2 f_0^2 - \frac{\delta^2}{\gamma^2} L \int_{\Gamma_{-,\gamma}} \rho \mathrm{d}\rho \mathrm{d}\vartheta \, f_0^4 (1 - |u_2|^2)^2 + \mathcal{O}(\delta^2 \gamma^{-1}). \quad (6.56)
\end{aligned}$$

Estimating also the last two terms on the r.h.s. of (6.55) and choosing $L = \gamma\delta^{-1}$, we obtain

$$
\begin{aligned}
2\int_{\Gamma_{-,\gamma}} \rho \mathrm{d}\rho\mathrm{d}\vartheta \left(\frac{1}{\rho}\partial_\vartheta(\Delta\Phi) + \rho\sin^2\vartheta\right)\left(iu_2, \frac{1}{\rho}\partial_\vartheta u_2\right) f_0^2 \\
\geq 2\int_{\Gamma_{-,\gamma}} \rho \mathrm{d}\rho\mathrm{d}\vartheta\,(\rho+\alpha_0)\left(iu_2, \frac{1}{\rho}\partial_\vartheta u_2\right) f_0^2 - \frac{C\delta}{\gamma}\int_{\Gamma_{-\gamma}} \rho \mathrm{d}\rho\mathrm{d}\vartheta\, f_0^4(1-|u_2|^2)^2 \\
- \frac{C\delta}{\gamma}\int_{\Gamma_{-,\gamma}} \rho \mathrm{d}\rho\mathrm{d}\vartheta\,\frac{1}{\rho^2}|\partial_\vartheta u_2|^2 f_0^2 \\
+ \mathcal{O}(\gamma^2) + \mathcal{O}(\delta^2\gamma^{-1}) + \mathcal{O}(e^{-C\ell^2}). \quad (6.57)
\end{aligned}
$$

From (6.54) and (6.57) we estimate

$$
\begin{aligned}
\int_{\Gamma_{-,\gamma}} \rho \mathrm{d}\rho\mathrm{d}\vartheta \left[\frac{1}{\rho}\partial_\vartheta(f_0 u_2 e^{i\Delta\Phi}) + i\rho\sin^2\vartheta(f_0u_2e^{i\Delta\Phi})\right]^2 \\
\geq \int_{\Gamma_{-,\gamma}} \rho \mathrm{d}\rho\mathrm{d}\vartheta\,\frac{1}{\rho^2}|\partial_\vartheta f_0|^2|u_2|^2 + \left(1-\frac{C\delta}{\gamma}\right)\frac{1}{\rho^2}|\partial_\vartheta u_2|^2 f_0^2 + \frac{1}{\rho^2} f_0(\partial_\vartheta f_0)(\partial_\vartheta|u_2|^2) \\
+ \int_{\Gamma_{-,\Gamma}} \rho \mathrm{d}\rho\mathrm{d}\vartheta\,(\rho+\alpha_0)^2 f_0^2 + 2\int_{\Gamma_{-,\gamma}} \rho \mathrm{d}\rho\mathrm{d}\vartheta\,(\rho+\alpha_0)\left(iu_2, \frac{1}{\rho}\partial_\vartheta u_2\right) f_0^2 \\
+ \delta\int_0^{c\ell} \rho \mathrm{d}\rho\,(\rho+\alpha_0)\left(\frac{\rho}{2}+\alpha_0\right) f_0^2(\rho) - \frac{C\delta}{\gamma}\int_{\Gamma_{-,\gamma}} \rho \mathrm{d}\rho\mathrm{d}\vartheta\, f_0^4(1-|u_2|^2)^2 \\
+ \mathcal{O}(\gamma^2 + \delta^2\gamma^{-1}) + \mathcal{O}(e^{-C\ell^2}). \quad (6.58)
\end{aligned}
$$

Thanks to the previous estimates in (6.47) and (6.58),

$$
\begin{aligned}
&\int_{\Gamma_{-,\gamma}} \rho \mathrm{d}\rho\mathrm{d}\vartheta\,|(\nabla + i\mathbf{F}(\rho,\vartheta))f_0 e_2 e^{i\Delta\Phi}|^2 \\
&\geq \int_{\Gamma_{-,\gamma}} \rho \mathrm{d}\rho\mathrm{d}\vartheta \left\{\left[|\partial_\rho f_0|^2 + \frac{1}{\rho^2}|\partial_\vartheta f_0|^2\right]|u_2|^2 + \left(1-\frac{C\delta}{\gamma}\right)\left[|\partial_\rho u_2|^2 + \frac{1}{\rho^2}|\partial_\vartheta u_2|^2\right] f_0^2 + (\rho+\alpha_0)^2 f_0^2 \right. \\
&\qquad \left. + 2(\rho+\alpha_0)\left(iu_2, \frac{1}{\rho}\partial_\vartheta u_2\right) f_0^2 + f_0(\partial_\rho f_0)\partial_\rho|u_2|^2 + \frac{1}{\rho^2} f_0(\partial_\vartheta f_0)(\partial_\vartheta|u_2|^2)\right\} \\
&\qquad + \delta\int_0^{c\ell} \rho \mathrm{d}\rho\,(\rho+\alpha_0)\left(\frac{\rho}{2}+\alpha_0\right) f_0^2(\rho) - \frac{C\delta}{\gamma}\int_{\Gamma_{-,\gamma}} \rho \mathrm{d}\rho\mathrm{d}\vartheta\, f_0^4(1-|u_2|^2)^2 \\
&\qquad + \mathcal{O}(\gamma^2 + \delta^2\gamma^{-1}) + \mathcal{O}(e^{-C\ell^2}). \quad (6.59)
\end{aligned}
$$

We now consider the following two terms of (6.59):

$$
f_0(\partial_\rho f_0)\partial_\rho|u_2|^2 + \frac{1}{\rho^2} f_0(\partial_\vartheta f_0)\partial_\vartheta|u_2|^2 = f_0\left[\left(\partial_\rho\hat{\mathbf{e}}_\rho + \frac{1}{\rho}\partial_\vartheta\hat{\mathbf{e}}_\vartheta\right) f_0\right]\cdot\left[\left(\partial_\rho\hat{\mathbf{e}}_\rho + \frac{1}{\rho}\partial_\vartheta\hat{\mathbf{e}}_\vartheta\right)|u_2|^2\right],
$$

which implies

$$
\begin{aligned}
\int_{\Gamma_{-,\gamma}} \rho \mathrm{d}\rho\mathrm{d}\vartheta \left\{ f_0\left[\left(\hat{\mathbf{e}}_\rho\partial_\rho + \hat{\mathbf{e}}_\vartheta\frac{1}{\rho}\partial_\vartheta\right) f_0\right]\cdot\left[\left(\partial_\rho\hat{\mathbf{e}}_\rho + \frac{1}{\rho}\partial_\vartheta\hat{\mathbf{e}}_\vartheta\right)|u_2|^2\right]\right\} \\
= \int_0^{c\ell}\int_{t\tan\left(\frac{\delta}{2}\right)}^{t\tan\left(\frac{\delta+\gamma}{2}\right)} \mathrm{d}s\mathrm{d}t\, f_0\nabla f_0\cdot\nabla|u_2|^2 + \mathcal{O}(e^{-C\ell^2}).
\end{aligned}
$$

Integrating by parts, we get

$$\int_0^{c\ell}\int_{t\tan\left(\frac{\delta}{2}\right)}^{t\tan\left(\frac{\delta+\gamma}{2}\right)} \mathrm{d}s\mathrm{d}t\, f_0\nabla f_0\cdot\nabla|u_2|^2$$
$$= -\int_0^{c\ell}\int_{t\tan\left(\frac{\delta}{2}\right)}^{t\tan\left(\frac{\delta+\gamma}{2}\right)} \mathrm{d}s\mathrm{d}t\,\left\{|f_0'|^2|u_2|^2 + f_0 f_0''|u_2|^2\right\} + \int_{\partial\Gamma_{-,\gamma}} \mathrm{d}\sigma\,\hat{\mathbf{n}}_{-,\gamma}\cdot(\nabla f_0)f_0|u_2|^2, \quad (6.60)$$

where $\hat{\mathbf{n}}_{-,\gamma}$ is the outer unit normal vector to $\partial\Gamma_{-,\gamma}$. Before estimate the boundary terms in (6.36), we introduce some notations for the boundary of $\Gamma_{-,\gamma}$. As in Figure 6.6, we have that $\overline{AB}\equiv\partial\Gamma_{-,\delta}^{\mathrm{obl}}$. Moreover, we set $\overline{AC}:=\partial\Gamma_{\gamma}^{\mathrm{bis}}$ and $\overline{BC}:=\partial\Gamma_{-,\gamma}^{\mathrm{bulk}}$. Then the boundary term in

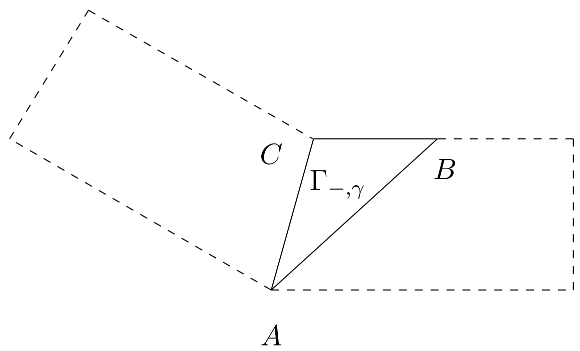

Figure 6.6. The region $\Gamma_{-,\gamma}$.

(6.60) equals

$$\begin{aligned}\int_{\partial\Gamma_{-,\gamma}} \mathrm{d}\sigma\,\hat{\mathbf{n}}_{-,\gamma}\cdot(\nabla f_0)f_0|u_2|^2 &= \int_{\partial\Gamma_{-,\delta}^{\mathrm{obl}}} \mathrm{d}\sigma\,\hat{\mathbf{n}}_{-,\gamma}\cdot(\nabla f_0)f_0|u_2|^2 \\ &\quad + \int_{\partial\Gamma_{-,\gamma}^{\mathrm{bulk}}} \mathrm{d}\sigma\,\hat{\mathbf{n}}_{-,\gamma}\cdot(\nabla f_0)f_0|u_2|^2 + \int_{\partial\Gamma_{+,\delta}^{\mathrm{bis}}} \mathrm{d}\sigma\,\hat{\mathbf{n}}_{-,\gamma}\cdot(\nabla f_0)f_0|u_2|^2 \\ &= \int_{\partial\Gamma_{-,\delta}^{\mathrm{obl}}} \mathrm{d}\sigma\,\hat{\mathbf{n}}_{-,\gamma}\cdot(\nabla f_0)f_0|u_2|^2 + \int_{\partial\Gamma_{\gamma}^{\mathrm{bis}}} \mathrm{d}\sigma\,\hat{\mathbf{n}}_{-,\gamma}\cdot(\nabla f_0)f_0|u_2|^2.\end{aligned}$$

In fact, the boundary term on $\partial\Gamma_{-,\gamma}^{\mathrm{bulk}}$ vanishes because $u_2\equiv 0$ there. It then follows that

$$\begin{aligned}\int_{\Gamma_{-,\gamma}} \rho\mathrm{d}\rho\mathrm{d}\vartheta\Big\{f_0\Big[\Big(\partial_\rho\hat{\mathbf{e}}_\rho+\frac{1}{\rho}\partial_\vartheta\hat{\mathbf{e}}_\vartheta\Big)f_0\Big]\cdot\Big[\Big(\partial_\rho\hat{\mathbf{e}}_\rho+\frac{1}{\rho}\partial_\vartheta\hat{\mathbf{e}}_\vartheta\Big)|u_2|^2\Big]\Big\} \\ = -\int_0^{c\ell}\int_{t\tan\left(\frac{\delta}{2}\right)}^{t\tan\left(\frac{\delta+\gamma}{2}\right)} \mathrm{d}s\mathrm{d}t\,\left\{|f_0'|^2|u_2|^2 + f_0 f_0''|u_2|^2\right\} \\ + \int_{\partial\Gamma_{-,\delta}^{\mathrm{obl}}} \mathrm{d}\sigma\,\hat{\mathbf{n}}_{-,\gamma}\cdot(\nabla f_0)f_0|u_2|^2 + \int_{\partial\Gamma_{\gamma}^{\mathrm{bis}}} \mathrm{d}\sigma\,\hat{\mathbf{n}}_{-,\gamma}\cdot(\nabla f_0)f_0|u_2|^2. \quad (6.61)\end{aligned}$$

If we change now coordinates and use (6.59) and (6.61) and the variational equation (2.37) for f_0, we get

$$\begin{aligned}
\mathcal{G}_\Gamma[\psi_\Gamma]\Big|_{\Gamma_{-,\gamma}} \geq & -\frac{1}{2b}\int_{\Gamma_{-,\gamma}} \mathrm{d}s\mathrm{d}t\, f_0^4(t) \\
& + \int_{\Gamma_{-,\gamma}} \mathrm{d}s\mathrm{d}t\, f_0^2(t)\Big\{ \left[1-\frac{C\delta}{\gamma}\right]|\nabla u_2|^2 - 2(t+\alpha_0)(iu_2,\partial_s u_2) \\
& + \left(\frac{1}{2b}-\frac{C\delta}{\gamma}\right) f_0^2(1-|u_2|^2)^2\Big\} \\
& + \delta\int_0^{c\ell} t\mathrm{d}t\,\left(\frac{t}{2}+\alpha_0\right)(t+\alpha_0)f_0^2(t) \\
& + \int_{\partial\Gamma^{\mathrm{obl}}_{-,\delta}} \mathrm{d}\sigma\,\hat{\mathbf{n}}_{-,\gamma}\cdot(\nabla f_0)f_0|u_2|^2 + \int_{\partial\Gamma^{\mathrm{bis}}_{\gamma}} \mathrm{d}\sigma\,\hat{\mathbf{n}}_{-,\gamma}\cdot(\nabla f_0)f_0|u_2|^2 \\
& + \mathcal{O}(\gamma^2+\delta^2\gamma^{-1}) + \mathcal{O}(e^{-C\ell^2}).
\end{aligned} \tag{6.62}$$

We now consider the region $\Gamma_{+,\gamma}$ represented in Figure 6.7, where $\overline{AB} = \partial\Gamma^{\mathrm{bis}}_{\gamma}$ and $\overline{AC} = \partial\Gamma^{\mathrm{obl}}_{+,\delta}$. Performing a similar analysis as the one in $\Gamma_{-,\gamma}$, we get

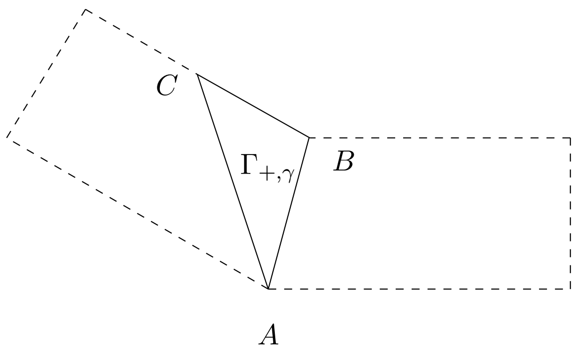

Figure 6.7. The region $\Gamma_{+,\gamma}$.

$$\begin{aligned}
\mathcal{G}_\Gamma[\psi_\Gamma]\Big|_{\Gamma_{+,\gamma}} \geq & -\frac{1}{2b}\int_{\Gamma_{+,\gamma}} \mathrm{d}s\mathrm{d}t\, f_0^4(t) \\
& + \int_{\Gamma_{+,\gamma}} \mathrm{d}s\mathrm{d}t\, f_0^2(t)\Big\{ \left[1-\left(\frac{C\delta}{\gamma}\right)\right]|\nabla u_3|^2 - 2(t+\alpha_0)(iu_3,\partial_s u_3) \\
& + \left(\frac{1}{2b}-\frac{C\delta}{\gamma}\right) f_0^2(1-|u_3|^2)^2\Big\} \\
& + \delta\int_0^{c\ell} t\mathrm{d}t\,\left(\frac{t}{2}+\alpha_0\right)(t+\alpha_0)f_0^2(t) \\
& + \int_{\partial\Gamma^{\mathrm{obl}}_{+,\delta}} \mathrm{d}\sigma\,\hat{\mathbf{n}}_{+,\gamma}\cdot(\nabla f_0)f_0|u_2|^2 + \int_{\partial\Gamma^{\mathrm{bis}}_{\gamma}} \mathrm{d}\sigma\,\hat{\mathbf{n}}_{+,\gamma}\cdot(\nabla f_0)f_0|u_2|^2 \\
& + \mathcal{O}(\gamma^2+\delta^2\gamma^{-1}) + \mathcal{O}(e^{-C\ell^2}),
\end{aligned} \tag{6.63}$$

where $\hat{\mathbf{n}}_{+,\gamma}$ is the outward unit normal vector to $\partial\Gamma_{+,\gamma}$.

To conclude the proof we have to take into account the four boundary terms appearing in (6.38), (6.62), (6.63) and (6.39). All these contributions sum up to zero since

- from the definition of u_1 and u_2 and the continuity of ψ, it follows that
$$u_1\big|_{\partial\Gamma^{\mathrm{obl}}_{-,\delta}} = u_2\big|_{\partial\Gamma^{\mathrm{obl}}_{-,\delta}};$$
then the boundary term vanishes because $\hat{\mathbf{n}}_- = -\hat{\mathbf{n}}_{-,\gamma}$.
- From the definition of u_3 and u_4 and the continuity of ψ, it follows that
$$u_3\big|_{\partial\Gamma^{\mathrm{obl}}_{+,\delta}} = u_4\big|_{\partial\Gamma^{\mathrm{obl}}_{+,\delta}};$$
then the boundary term vanishes because $\hat{\mathbf{n}}_{+,\gamma} = -\hat{\mathbf{n}}_+$.
- From the definition of u_2 and u_3 and the continuity of ψ, it follows that
$$u_2\big|_{\partial\Gamma^{\mathrm{bis}}_{\gamma}} = u_3\big|_{\partial\Gamma^{\mathrm{bis}}_{\gamma}};$$
then the boundary term vanishes because $\hat{\mathbf{n}}_{-,\gamma} = -\hat{\mathbf{n}}_{+,\gamma}$.

In conclusion,
$$\begin{aligned}\mathcal{G}_\Gamma[\psi_\Gamma] \geq & -\frac{1}{2b}\left[\int_{\Gamma_-} \mathrm{d}s\mathrm{d}t\, f_0^4(t) + \int_{\Gamma_+} \mathrm{d}s\mathrm{d}t\, f_0^4(t)\right] + \sum_{j=1}^{4} \mathcal{E}_0[u_j] \\ & + 2\delta \int_0^{c\ell} t\mathrm{d}t \left(\frac{t}{2}+\alpha_0\right)(t+\alpha_0) f_0^2(t) \\ & - \frac{C\delta}{\gamma} \sum_{j=2,3} \int_{\Gamma_j} \mathrm{d}s\mathrm{d}t \left\{ f_0^2 |\nabla u_j|^2 + f_0^4 (1-|u_j|^2)^2 \right\} \\ & + \mathcal{O}(\gamma^2 + \delta^2\gamma^{-1}) + \mathcal{O}(e^{-C\ell^2}).\end{aligned}$$
From the variational equation of f_0, we deduce that
$$-\frac{1}{2b}\int_{\Gamma_-} \mathrm{d}s\mathrm{d}t\, f_0^4 = \int_{\Gamma_-} \mathrm{d}s\mathrm{d}t \left\{ |f_0'|^2 + (t+\alpha_0)^2 - \frac{1}{2b}(2f_0^2 - f_0^4) \right\}$$
and, by the same arguments used in the upper bound, we can show that
$$\begin{aligned}&\int_{\Gamma_-} \mathrm{d}s\mathrm{d}t \left\{ |f_0'|^2 + (t+\alpha_0)^2 - \frac{1}{2b}(2f_0^2 - f_0^4) \right\} \\ &\qquad = \ell E_0^{1D} - \frac{\delta}{2}\int_0^{c\ell} t\mathrm{d}t \left\{ |f_0'|^2 + (t+\alpha_0)^2 - \frac{1}{2b}(2f_0^2 - f_0^4) \right\} + \mathcal{O}(\delta^3).\end{aligned}$$
The same result holds in Γ_+ and therefore
$$\begin{aligned}&-\frac{1}{2b}\left[\int_{\Gamma_-} \mathrm{d}s\mathrm{d}t\, f_0^4(t) + \int_{\Gamma_+} \mathrm{d}s\mathrm{d}t\, f_0^4(t)\right] + 2\delta \int_0^{c\ell} t\mathrm{d}t \left(\frac{t}{2}+\alpha_0\right)(t+\alpha_0) f_0^2(t) = \\ &\qquad 2\ell E_0^{1D} - \delta \int_0^{c\ell} t\mathrm{d}t \left\{ |f_0'|^2 - \alpha_0(t+\alpha_0) f_0^2 - \frac{1}{2b}(2f_0^2 - f_0^4) \right\} + \mathcal{O}(\delta^3).\end{aligned}$$
It then follows that
$$\begin{aligned}\mathcal{G}_\Gamma[\psi_\Gamma] \geq 2\ell E_0^{1D} - \delta \mathcal{E}^{\mathrm{corr}}_{\alpha_0}[f_0] + \sum_{i=1}^{4} \mathcal{E}_0[u_i] - \frac{C\delta}{\gamma} \sum_{j=2,3} \int_{\Gamma_j} \mathrm{d}s\mathrm{d}t \left\{ f_0^2 |\nabla u_j|^2 + f_0^4 (1-|u_j|^2)^2 \right\} \\ + \mathcal{O}(\gamma^2 + \delta^2\gamma^{-1}) + \mathcal{O}(e^{-C\ell^2}).\end{aligned}$$
Optimizing the remainder, i.e., choosing $\gamma = \delta^{\frac{2}{3}}$, we get the desired lower bound. □

To complete the proof of the lower bound we have only to prove the following Proposition

Proposition 6.5.
Under the same hypothesis of Proposition 6.4, it holds that

$$\sum_{j=1}^{4}\mathcal{E}_0[u_j]-\frac{C\delta}{\gamma}\sum_{j=2}^{3}\int_{\Gamma_j}\mathrm{d}s\mathrm{d}t\left\{f_0^2|\nabla u_j|^2+f_0^4(1-|u_j|^2)^2\right\}\geq 0. \tag{6.64}$$

Proof. We first prove the positivity of $\mathcal{E}_0[u_1]$ and $\mathcal{E}_0[u_4]$. We know that

$$\mathcal{E}_0[u_1]=\int_{\Gamma_{-,\delta}}\mathrm{d}s\mathrm{d}t\,f_0^2\left\{|\partial_t u_1|^2+|\partial_s u_1|^2-2(t+\alpha_0)(iu_1,\partial_s u_1)+\frac{f_0^2}{2b}(1-|u_1|^2)^2\right\}. \tag{6.65}$$

Recalling that

$$F_0(t):=2\int_0^t\mathrm{d}\eta\,(\eta+\alpha_0)f_0^2$$

and

$$F_0'(t)=2(t+\alpha_0)f_0^2(t),$$

we have

$$-2\int_{\Gamma_{-,\delta}}\mathrm{d}s\mathrm{d}t\,(t+\alpha_0)f_0^2(iu_1,\partial_s u_1)=-\int_{\Gamma_{-,\delta}}\mathrm{d}s\mathrm{d}t\,F_0'(t)(iu_1,\partial_s u_1)$$

We now denote by $\mathbf{j}[u_1]=(j_s[u_1],j_t[u_1])$ and we observe that

$$\int_{\Gamma_{-,\delta}}\mathrm{d}s\mathrm{d}t\,\,-F_0'(t)j_s[u_1]=\int_{\Gamma_{-,\delta}}\mathrm{d}s\mathrm{d}t\,\nabla^{\perp}F_0(t)\cdot\mathbf{j}[u_1] \tag{6.66}$$

We now integrate by parts and get

$$\int_{\Gamma_{-,\delta}}\mathrm{d}s\mathrm{d}t\,\nabla^{\perp}F_0\cdot j_s[u_1]=-\int_{\Gamma_{-,\delta}}\mathrm{d}s\mathrm{d}t\,F_0\nabla^{\perp}\cdot\mathbf{j}[u_1]+\int_{\partial\Gamma_{-,\delta}}\mathrm{d}\sigma\,F_0\,\hat{\boldsymbol{\tau}}_-\cdot\mathbf{j}[u_1], \tag{6.67}$$

where $\hat{\boldsymbol{\tau}}_-$ is the unit tangential vector to $\partial\Gamma_{-,\delta}$. We first consider the first term on the r.h.s of (6.67), we observe that

$$\begin{aligned}\nabla^{\perp}\cdot\mathbf{j}[u_1]&=-\partial_t(iu_1,\partial_s u_1)+\partial_s(iu_1,\partial_t u_1)\\&=\frac{1}{2}\Big[-\partial_t(iu_1\partial_s u_1^*-iu_1^*\partial_s u_1)+\partial_s(iu_1\partial_t u_1^*-iu_1^*\partial_t u_1)\Big]\\&=\frac{1}{2}\Big[-i\partial_t u_1\partial_s u_1^*-iu_1\partial_t\partial_s u_1^*+i\partial_t u_1^*\partial_s u_1+iu_1^*\partial_t\partial_s u_1\Big]\\&+\frac{1}{2}\Big[i\partial_s u_1\partial_t u_1^*+iu_1\partial_s\partial_t u_1^*-i\partial_s u_1^*\partial_t u_1-iu_1\partial_s\partial_t u_1\Big]\\&=\frac{1}{2}\Big[2i\partial_s u_1\partial_t u_1^*-2i\partial_s u_1^*\partial_t u_1\Big]=2(i\partial_s u_1,\partial_t u_1).\end{aligned}$$

We now consider the boundary term, it follows that

$$\begin{aligned}\int_{\partial\Gamma_{-,\delta}}\mathrm{d}\sigma\,F_0\,\hat{\boldsymbol{\tau}}_-\cdot\mathbf{j}[u_1]&=\int_{\partial\Gamma_{-,\delta}^{\mathrm{ext}}}\mathrm{d}\sigma\,F_0\,\hat{\boldsymbol{\tau}}_-\cdot\mathbf{j}[u_1]+\int_{\partial\Gamma_{-,\delta}^{\mathrm{vert}}}\mathrm{d}\sigma\,F_0\,\hat{\boldsymbol{\tau}}_-\cdot\mathbf{j}[u_1]\\&+\int_{\partial\Gamma_{-,\delta}^{\mathrm{bulk}}}\mathrm{d}\sigma\,F_0\,\hat{\boldsymbol{\tau}}_-\cdot\mathbf{j}[u_1]+\int_{\partial\Gamma_{-,\delta}^{\mathrm{obl}}}\mathrm{d}\sigma\,F_0\,\hat{\boldsymbol{\tau}}_-\cdot\mathbf{j}[u_1]\\&=\int_{\partial\Gamma_{-,\delta}^{\mathrm{obl}}}\mathrm{d}\sigma\,F_0\,\hat{\boldsymbol{\tau}}_-\cdot\mathbf{j}[u_1]+\mathcal{O}(e^{-C\ell^2}),\end{aligned}$$

where we used the fact that $F_0(0) = 0$, $u_1\big|_{\Gamma^{\mathrm{bulk}}_{-,\delta}} = 0$ and that $u_1\big|_{\Gamma^{\mathrm{vert}}_{-,\delta}} = \mathcal{O}(e^{-C\ell^2})$ thanks to the fact that $u_1\big|^{\mathrm{bd}}_{\Gamma_{-,\delta}} = g(t)/f_0(t)e^{iS(s)}$ and to the exponential decay of f_0. We also observe that

$$-2\int_{\Gamma_{-,\gamma}} \mathrm{d}s\mathrm{d}t\, F_0(t)(i\partial_s u_1, \partial_t u_1) \geq -\int_{\Gamma_{-,\gamma}} \mathrm{d}s\mathrm{d}t\, |F_0(t)|(|\partial_s u_1|^2 + |\partial_t u_1|^2)$$
$$= \int_{\Gamma_{-,\gamma}} \mathrm{d}s\mathrm{d}t\, F_0(t)(|\partial_s u_1|^2 + |\partial_t u_1|^2)$$

and thus

$$\mathcal{E}_0[u_1] \geq \int_{\Gamma_{-,\delta}} \mathrm{d}s\mathrm{d}t \left\{ (f_0^2 + F_0(t))[|\partial_t u_1|^2 + |\partial_s u_1|^2] + \frac{f_0^4}{2b}(1 - |u_1|^2)^2 \right\}$$
$$+ \int_{\partial\Gamma^{\mathrm{obl}}_{-,\delta}} \mathrm{d}\sigma\, F_0\, \hat{\boldsymbol{\tau}}_- \cdot \mathbf{j}[u_1] + \mathcal{O}(e^{-C\ell^2}).$$

Using now the positivity of the cost function $K_0(t) = f_0^2(t) + F_0(t)$, we finally get

$$\mathcal{E}_0[u_1] \geq \int_{\partial\Gamma^{\mathrm{obl}}_{-,\delta}} \mathrm{d}\sigma\, F_0\, \hat{\boldsymbol{\tau}}_- \cdot \mathbf{j}[u_1] + \mathcal{O}(e^{-C\ell^2}). \tag{6.68}$$

With similar techniques we can prove that

$$\mathcal{E}_0[u_4] \geq \int_{\partial\Gamma^{\mathrm{obl}}_{+,\delta}} \mathrm{d}\sigma\, F_0\, \hat{\boldsymbol{\tau}}_+ \cdot \mathbf{j}[u_4] + \mathcal{O}(e^{-C\ell^2}), \tag{6.69}$$

where $\hat{\boldsymbol{\tau}}_+$ is the unit tangential vector to $\partial\Gamma_{+,\delta}$.

We now consider the term related to u_2, following the same strategy of before we get

$$\mathcal{E}_0[u_2] - \frac{C\delta}{\gamma}\int_{\Gamma_{-,\gamma}} \mathrm{d}s\mathrm{d}t \left\{ f_0^2(|\partial_s u_2|^2 + |\partial_t u_2|^2) + f_0^4(1 - |u_2|^2)^2 \right\}$$
$$\geq \int_{\Gamma_{-,\gamma}} \mathrm{d}s\mathrm{d}t \left\{ \left(K_0(t) - \frac{C\delta}{\gamma}f_0^2\right)[|\partial_t u_2|^2 + |\partial_s u_2|^2] + f_0^4\left(\frac{1}{2b} - \frac{C\delta}{\gamma}\right)(1 - |u_2|^2)^2 \right\}$$
$$+ \int_{\partial\Gamma^{\mathrm{obl}}_{-,\delta}} \mathrm{d}\sigma\, F_0\, \hat{\boldsymbol{\tau}}_{-,\gamma} \cdot \mathbf{j}[u_2] + \int_{\partial\Gamma^{\mathrm{bis}}_{\gamma}} \mathrm{d}\sigma\, F_0\, \hat{\boldsymbol{\tau}}_{-,\gamma} \cdot \mathbf{j}[u_2] + \mathcal{O}(e^{-C\ell^2}),$$

where $\hat{\boldsymbol{\tau}}_-$ is the unit tangential vector to $\partial\Gamma_{-,\gamma}$. Before proceeding further we now simply observe that $K_0(t) \geq cf_0^2(t)$ for some $c > 0$ whenever $b > 1$, which also implies that $K_0(t) - \frac{\delta}{\gamma}f_0^2 \geq 0$ for any $\gamma \gg \delta$. We therefore obtain

$$\mathcal{E}_0[u_2] \geq \int_{\partial\Gamma^{\mathrm{obl}}_{-,\delta}} \mathrm{d}\sigma\, F_0\, \hat{\boldsymbol{\tau}}_{-,\gamma} \cdot \mathbf{j}[u_2] + \int_{\partial\Gamma^{\mathrm{bis}}_{\gamma}} \mathrm{d}\sigma\, F_0\, \hat{\boldsymbol{\tau}}_{-,\gamma} \cdot \mathbf{j}[u_2] + \mathcal{O}(e^{-C\ell^2}). \tag{6.70}$$

In the very same way we also have

$$\mathcal{E}_0[u_3] \geq \int_{\partial\Gamma^{\mathrm{bis}}_{\gamma}} \mathrm{d}\sigma\, F_0\, \hat{\boldsymbol{\tau}}_{+,\gamma} \cdot \mathbf{j}[u_3] + \int_{\partial\Gamma^{\mathrm{obl}}_{+,\delta}} \mathrm{d}\sigma\, F_0\, \hat{\boldsymbol{\tau}}_{+,\gamma} \cdot \mathbf{j}[u_3] + \mathcal{O}(e^{-C\ell^2}), \tag{6.71}$$

where $\hat{\boldsymbol{\tau}}_{+,\gamma}$ is the unit tangential vector to $\partial\Gamma_{+,\gamma}$.

From (6.68), (6.69), (6.70) and (6.71) we finally get

$$\begin{aligned}
\sum_{i=1}^{4} \mathcal{E}_0[u_i] - \sum_{i=2}^{3} \int_{\Gamma_{\ell,\delta}} \mathrm{d}s\mathrm{d}t \, \frac{C}{|\log\delta|} \left\{ f_0^2 |\nabla u_i|^2 + f_0^4 (1-|u_i|^2)^2 \right\} \\
\geq \int_{\partial\Gamma^{\mathrm{obl}}_{-,\delta}} \mathrm{d}\sigma \, F_0 \, \hat{\boldsymbol{\tau}}_- \cdot \mathbf{j}[u_1] + \int_{\partial\Gamma^{\mathrm{obl}}_{-,\delta}} \mathrm{d}\sigma \, F_0 \, \hat{\boldsymbol{\tau}}_{-,\gamma} \cdot \mathbf{j}[u_2] + \int_{\partial\Gamma^{\mathrm{bis}}_{\gamma}} \mathrm{d}\sigma \, F_0 \, \hat{\boldsymbol{\tau}}_{-,\gamma} \cdot \mathbf{j}[u_2] \\
+ \int_{\partial\Gamma^{\mathrm{bis}}_{\gamma}} \mathrm{d}\sigma \, F_0 \, \hat{\boldsymbol{\tau}}_{+,\gamma} \cdot \mathbf{j}[u_3] + \int_{\partial\Gamma^{\mathrm{obl}}_{+,\delta}} \mathrm{d}\sigma \, F_0 \, \hat{\boldsymbol{\tau}}_{+,\gamma} \cdot \mathbf{j}[u_3] + \int_{\partial\Gamma^{\mathrm{obl}}_{+,\delta}} \mathrm{d}\sigma \, F_0 \, \hat{\boldsymbol{\tau}}_+ \cdot \mathbf{j}[u_4] \\
+ \mathcal{O}(e^{-C\ell^2}).
\end{aligned}$$

But all the boundary terms vanish by construction, since the various splitting were done with respect to a continuous reference function. □

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