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Covariant quantum mechanics as a tool for quantum-gravity phenomenology

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Abstract

Covariant quantum mechanics (CQM) is a background-indipendent reformulation of quantum mechanics in which the time coordinate is treated as a dynamical degree of freedom, rather than an external evolution parameter. CQM was originally conceived and developed as a basic relational quantum formalism for discussing some conceptual issues associated with quantum gravity in a simplified setting. However, in recent times, a few papers have fruitfully employed it to incorporate quantum-gravity effects such as spacetime noncommutativity into simple phenomenological models. In this Ph.D thesis, I explore in more detail this possibility, providing a first systematic investigation of CQM as a tool for quantum-gravity phenomenology, rather than a toy model of full quantum gravity. In particular, starting from the ordinary CQM of a single relativistic particle, I build generalized models for the description of free quantum particles propagating on noncommutative or curved spacetimes. The present work is theoretical in character. I mainly focus on the development and characterization of a CQM-based framework suitable for dealing with generic spacetime noncommutativity or metric, and study simple examples.

Contents

Introduction

§ The lack of a satisfactory quantum theory of gravity has been one of the major open problems in theoretical physics for almost a century. Developing a consistent model of gravitational phenomena in the quantum regime has proved very difficult for both phenomenological and theoretical reasons. On the phenomenological side, the new theory is expected to become relevant at scales of the order of the Planck length

$$L_P = \sqrt{\frac{G\hbar}{c^3}} \sim 10^{-35}\,\mathrm{m}, \tag{0.1}$$

and this makes it very hard to detect any genuine quantum-gravity effects. As a matter of fact, none has been observed yet and theoretical progress is severely hindered by the absence of observational evidence. On the theoretical side, the main problem is that our best classical theory of gravity, general relativity, cannot be subjected to any standard quantization procedure. Every attempt at reconciling it with quantum mechanics has invariably led to formidable technical and conceptual challenges, which suggest that quantum gravity could be very different from standard quantum theories.

§ Much of the incompatibility between quantum mechanics and general relativity can be traced back to their very different notions of dynamics. In quantum field theory (QFT), dynamical variables like quantum fields evolve within a spacetime background and their behaviour in space and time is determined by some local evolution equations. The properties of background coordinates and their mutual relations, which encode the causal structure of spacetime, are instead postulated at the beginning and held fixed, whatever is happening to the dynamical variables. This fundamental dychotomy, somewhat simplified, is replicated in finite-dimensional quantum mechanics (QM), with the time parameter serving as the fixed background and canonical variables being the dynamical quantities. Quantum dynamics is thus basically structured as a description of quantum degrees of freedom evolving within a classical nondynamical background. In general relativity, on the contrary, every physical quantity is dynamical and there is no fixed background in which the evolution of the remaining variables takes place. The theory directly accounts for the relative change of any variable with respect to any other and all of them, including spacetime determinations like durations or distances, participate equally in the dynamics. The adoption of such a relational notion of dynamics has been crucial for successfully incorporating gravitational interactions into classical mechanics. The same notion, though, cannot be implemented in the context of ordinary quantum theory, whose very structure and interpretation depend on the existence of a fixed background.

§ In the last decades, consideration of this crucial issue has stimulated the search for a generalized, background-independent quantum formalism. In the early 2000s, a series of papers by Rovelli [1–3] emphasized the relational character of general-relativistic dynamics in classical mechanics and discussed the general structure of a relational quantum formalism, thereby renewing interest in this problem. At the same time, several studies [4–6] were developing and investigating covariant quantum mechanics (CQM), a background-indipendent reformulation of QM in which the time coordinate is treated as a dynamical degree of freedom. Shortly afterward, a generalization of standard QFT which allowed for background-indipendence, called general boundary formalism, was proposed by Oeckl [7]. This was the first of a series of papers on the new framework which culminated in the report [8], where a more sophisticated and conceptually sound version of the formalism, deeply influenced by ideas coming from quantum information, is discussed in detail and comprehensively related to previous and alternative approaches. In the meantime, general boundary formalism has become a cornerstone of loop quantum gravity [9, 10], one of the most popular candidate theories of quantum gravity.

§ The last twenty years have also witnessed significant progress on the phenomenological side. The main problem with quantum-gravity phenomenology has always been that fundamental theories of quantum gravity, such as loop quantum gravity, are not yet sufficiently developed to provide testable predictions in realistic scenarios. Without precise, quantitative indications on where to look and what to look for, detecting Planck-scale-suppressed quantum-gravity effects is an exceedingly difficult task. In the late 1990s, trying to address this problem, phenomenology-oriented reserchers started to bypass fundamental theories and adopt a more heuristic approach. The idea, well illustrated in [11], is to neglect virtually every aspect of quantum gravity and develop a simplified model in which a single quantum-gravity effect, which is both reasonably expected and well-understood on a qualitative or semi-quantitative level, can be quantitatively described and meaningfully compared with observations. The choice of the candidate effects is usually suggested by either a fundamental theory or a careful analysis of the quantum-gravity problem, but the associated models (or test theories, as they are called in [11]) are then studied by themselves, without reference to the original suggestions. As a result, quantum-gravity phenomenology has progressively become a research field on its own, focused on the study of simplified models meant to guide the observational efforts toward the detection of the first quantum-gravity effects. The new heuristic approach, combined with the steady improvement of observational accuracy, has born fruit. The observational contexts in which genuine Planck-scale sensitivity is found to be realistically within reach has been continuously growing since the late 1990s and a few test theories have even been observationally excluded in some region of their parameter space [11].

§ These two lines of research, one foundational and one phenomenological, have currently little overlap. The problem of background independence in quantum mechanics is mostly a concern of researchers interested in fundamental theories of quantum gravity, whereas phenomenologists tend to work on comparatively slight modifications of standard classical or quantum models, which are more easily related to known physics. However, in recent years, a few papers [12–14] have fruitfully employed CQM to study the propagation of free quantum particles on noncommutative spacetimes, thereby building phenomenological models into a framework coming from the foundational camp. This is not surprising. CQM has been originally developed as a basic relational formalism for discussing several conceptual issues associated with quantum gravity without them getting obscured by the technical complications necessary to manage actual gravitational interactions. It is therefore ideally suited to the implementation of phenomenological test theories which, for all their simplicity, must address the issue of background-independence. The propagation of quantum particles on noncommutative spacetimes is the perfect example of a simple scenario in which ordinary QM, with its background time parameter, is of no use. In fact, nontrivial commutation relations involving the time coordinate cannot be made sense of unless time is treated as a dynamical variable like the other coordinates. What is actually surprising is that the potential usefulness of CQM for quantum-gravity phenomenology has been so far almost ignored by most phenomenologists. The promising seminal papers [12–14] have failed to make an impact on the literature and research in that direction has stagnated. Moreover, the applicability of CQM to other interesting phenomenological scenarios, such as the propagation of quantum particles on curved spacetimes, has never been explored.

§ In this Ph.D thesis, I provide a first systematic investigation of CQM as a tool for quantum-gravity phenomenology, rather than a toy model of full quantum gravity. In particular, starting from the ordinary CQM of a single relativistic particle, I build generalized models for the description of free quantum particles propagating on noncommutative or curved spacetimes. The present work is theoretical in character. I mainly focus on the development and characterization of a CQM-based framework suitable for dealing with generic spacetime noncommutativity or metric, and study simple examples. In the near future, building on the results reported here, I plan to perform a full phenomenological analysis of these models along the lines of the pioneering [13]. In the remainder of this introduction, I give a brief overview of the structure of the thesis.

Outline of the thesis

Covariant quantum mechanics

§ The starting point of my research is standard CQM, whose mathematical and conceptual foundations I review and discuss in Chapter 1. As said before, CQM is a background-independent formulation of QM in which standard canonical coordinates, time x^0 and its conjugate momentum $-p_0$ are all considered dynamical variables and treated on equal footing. The usual Heisenberg algebra of observables and space of states are obtained from such an extended canonical algebra via the imposition of an appropriately chosen hamiltonian constraint H. This is exactly analogous to what happens in classical constrained theories [15], whose formal structure was actually taken as a model for the development of CQM. Let us illustrate these points in some more detail for a single quantum particle, the system which will actually be studied in this work. In this case, the extended canonical algebra $\mathcal{V}$ is generated by the particle's 4-momenta p_μ and spacetime coordinates x^ν. Upon quantization, the canonical coordinates p_μ and x^ν are required to satisfy canonical commutation relations

$$\begin{aligned} [p_\mu, p_\tau] &= 0, && (0.2)\\ [p_\mu, x^\nu] &= i\hbar\delta_\mu{}^\nu, && (0.3)\\ \left[x^\nu, x^\lambda\right] &= 0. && (0.4) \end{aligned}$$

This enlarged Heisenberg algebra admits a unique representation on the space $\mathcal{K}$ of square-integrable functions of 4-momentum, given by the self-adjoint operators

$$\begin{aligned} \widehat{p}_\mu\psi(p) &= p_\mu\psi(p), && (0.5)\\ \widehat{x}^\nu\psi(p) &= -i\hbar\partial_{p_\nu}\psi(p). && (0.6) \end{aligned}$$

The space $\mathcal{K}$ is called kinematical Hilbert space and is the quantum analogue of the extended phase space of constrained classical mechanics. In particular, it must not be confused with the Hilbert space $\mathcal{P}$ of ordinary QM, which contains the physical states of the system. Starting from $\mathcal{V}$ and $\mathcal{K}$, quantum dynamics in the Heisenberg picture is then recovered specifying a self-adjoint hamiltonian constraint $H(\widehat{p}_\mu, \widehat{x}^\nu) \in \mathcal{V}$, without reference to any background. The physical Hilbert space $\mathcal{P}$ is given by the (improper) kernel of H equipped with a suitably modified scalar product, while the algebra $\mathcal{O}$ of the conserved quantities, which contains the usual Heisenberg observables, is obtained requiring that it commutes with H. In practice, classical Hamilton-Jacobi theory directly provides a set of independent generators of $\mathcal{O}$, apart from minor operator-ordering issues. The special constraint

$$H_{nr}(\widehat{p}_\mu, \widehat{x}^\nu) = \widehat{p}_0 - H_0(\widehat{p}^i, \widehat{x}^j) \qquad (0.7)$$

reproduces standard nonrelativistic QM with hamiltonian H_0, but the covariant formalism is obviously more powerful. For example, the dynamics of a free relativistic scalar particle of mass m is described by the quadratic constraint

$$H_r(\widehat{p}_\mu, \widehat{x}^\nu) = \widehat{p}^\mu\widehat{p}_\mu - m^2c^2. \qquad (0.8)$$

§ One of the most delicate interpretive issues related to CQM concerns the extraction of dynamical information from the algebra $\mathcal{O}$ of the conserved quantities. In fact, the covariant criterion given above is sufficient to find the Heisenberg observables, but not to identify them. Take for instance the x^i coordinate at time t in single-particle nonrelativistic QM. Among the various constants of motion $\widehat{O} \in \mathcal{O}$ there is also $\widehat{O}' = \widehat{x}^i(t)$, but in CQM there is no Heisenberg equation to tell us that

$\widehat{O}'$ is the time evolution of $\widehat{O}'' = \widehat{x}^i(0)$, nor even any obvious way to understand that $\widehat{O}'$ and $\widehat{O}''$ are actually associated with the x^i coordinate at some fixed times. This problem can be solved, at least in principle, looking at the dependence of $\widehat{O}'$ and $\widehat{O}''$ on the canonical operators $\widehat{x}^i$ and $\widehat{x}^0$, as argued for the first time in [16]. A sizable portion of Chapter 1 is devoted to a detailed discussion of the identification of the observables, based on the almost perfect correspondence between CQM and classical constrained theories in this regard.

CQM and curved spacetimes

§ In Chapter 2, I present an original generalization of CQM suitable for describing quantum particles propagating on curved backgrounds. Strictly speaking, this is not a problem of quantum gravity, since the gravitational field is held fixed and treated classically. However, cosmological signals, which are our most promising observational window on quantum gravity [11], travel for billions of years on a curved spacetime before being detected on Earth. An understanding of the dynamics of quantum particles on curved backgrounds, other than being interesting *per se*, is then essential for building realistic phenomenological test theories. This problem has been traditionally studied in the context of QFT on curved manifolds, where a satisfactory formulation of the theory of a free scalar field is now available and significant progress has been made toward the perturbative treatment of interactions [17, 18]. Nevertheless, the formal complexity and abstract character of this framework make it difficult to introduce quantum-gravity effects such as spacetime noncommutativity into the picture. It is therefore worthwhile to investigate alternative, simpler models for phenomenological purposes.

§ The idea of using CQM to study particle propagation on a curved background came from some recent results about standard nonrelativistic QM on an arbitrary configuration manifold $\mathcal{M}$ [19, 20], which I review at the beginning of Chapter 2. In the first paper [19], the authors identify a unique diffeomorphism-invariant generalization of the algebra of Heisenberg observables of a quantum system living on $\mathcal{M}$ and provide a complete classification of all its regular representations. In the subsequent work [20], the costruction is refined and a noncommutative Poisson algebra associated with the configuration manifold is understood as the geometric source of both classical and quantum mechanics on $\mathcal{M}$. In the first part of Chapter 2, I show that the application of the same concepts to a generic spacetime manifold $\mathcal{S}$ naturally yields a generalization of the extended canonical algebra $\mathcal{V}$ of single-particle CQM. Generalized momenta $p_{\mathbf{v}}$ and coordinates x^f are respectively associated with smooth vector fields $\mathbf{v} \in \mathrm{Vect}(\mathcal{S})$ and arbitrary functions $f \in C^\infty(\mathcal{S})$, and the redundant degrees of freedom are eliminated by the prescription

$$p_{f\mathbf{v}} = \frac{1}{2}(x^f p_{\mathbf{v}} + p_{\mathbf{v}} x^f), \tag{0.9}$$

which algebraically implements the multiplication of a vector field $\mathbf{v}$ by a smooth function f. The usual commutation relations (0.2)-(0.4) are then replaced by

$$\left[x^f, x^g\right] = 0, \tag{0.10}$$

$$\left[p_{\mathbf{v}}, x^f\right] = i\hbar\langle \mathbf{v}, f\rangle, \tag{0.11}$$

$$\left[p_{\mathbf{v}}, p_{\mathbf{w}}\right] = i\hbar p_{\langle \mathbf{v}, \mathbf{w}\rangle}, \tag{0.12}$$

where $f, g \in C^\infty(\mathcal{S})$, $\mathbf{v}, \mathbf{w} \in \mathrm{Vect}(\mathcal{S})$ and $\langle \mathbf{v}, \cdot\rangle$ denotes the Lie derivative along $\mathbf{v}$. Starting from the extended canonical algebra $\mathcal{V}_\mathcal{S}$ generated by coordinates x^f and momenta $p_{\mathbf{v}}$, or rather its representation on the associated kinematical Hilbert space $\mathcal{K}_\mathcal{S}$, the specification of a self-adjoint hamiltonian constraint determines the quantum dynamics, exactly as in standard CQM. The natural choice is the straightforward generalization of (0.8) to arbitrary spacetime manifolds, which in some local canonical coordinates $\widehat{x}^\nu = \widehat{x}^{x^\nu}$ and $\widehat{p}_\mu = \widehat{p}_{\partial/\partial x^\mu}$ reads

$$H_g(\widehat{p}_\mu, \widehat{x}^\nu) = g^{\mu\tau}(\widehat{x}^\nu)\widehat{p}_\mu\widehat{p}_\tau - m^2c^2 + \text{h.c.} \tag{0.13}$$

The resulting framework is the quantum version of the geodesic hamiltonian formalism employed to study the motion of free bodies in general relativity.

§ In the second part of Chapter 2, the general framework described above is specialized to analyze the propagation of a quantum particle on a de Sitter background. Being maximally symmetric, de Sitter spacetime is probably the simplest manifold to which the previous formalism can be applied. In fact, the resulting model can be completely characterized and physical amplitudes for various quantities of interest can be computed. The analysis of the de Sitter model is more than just an illustrative application of the general theory, it is also of phenomenological interest in its own right. Since our universe is apparently dominated by dark energy, de Sitter spacetime is actually the first nontrivial approximation of our expanding cosmological background. The model is also ideally suited to the discussion of the much debated and still unsettled issue of the generalization of Heisenberg's uncertainty principle in the presence of curvature [21].

CQM and noncommutative spacetimes

§ The last chapter, Chapter 3, is devoted to the study of the interplay between CQM and spacetime noncommutativity. Robust heuristic arguments [22, 23], as well as most fundamental approaches to quantum gravity, strongly suggest that sharp spacetime localization of a particle could be impossible at scales of the order of the Planck length. The resulting spacetime fuzziness is reminiscent of the phase-space fuzziness of nonrelativistic QM, and can therefore be effectively modelled via the introduction of nontrivial commutation rules among spacetime coordinates. In the last decades, the idea of spacetime noncommutativity has attracted the interest of quantum-gravity theorists and phenomenologists alike and a sizeable literature have been devoted to the study of noncommutative spacetime models. On the theoretical side, noncommutative geometry is regarded as the quantum analogue of ordinary differential geometry [24]. Given the importance of the latter for the formulation of general relativity, it could be the most appropriate language in which to express and address the quantum-gravity problem. On the phenomenological side, test theories based on noncommutative spacetimes have been developed to characterize potentially observable effects of spacetime fuzziness independently of the details of quantum gravity, in the spirit of the heuristic approach described above.

§ These phenomenological efforts have mostly focused on the study of the breaking or deformation of Poincaré symmetries induced by spacetime noncommutativity [12, 13, 25–27]. For the sake of definiteness, let us consider the much-studied κ-Minkowski spacetime [28], defined by the commutation relations

$$\left[x^\nu, x^\lambda\right] \;=\; i\ell(x^\nu {\delta_0}^\lambda - x^\lambda {\delta_0}^\nu), \tag{0.14}$$

where ℓ is a fundamental length of the order of the Planck scale (0.1). It is easily checked that (0.14) does not transform covariantly under a generic Poincaré transformation. This lack of covariance can mean either that the principle of relativity breaks down at the Planck scale or that the Poincaré transformations are inadequate to describe relativistic symmetries in this regime, and must be deformed to accomodate the fundamental length constant ℓ in the same way Galileo transformations had to be deformed to accomodate the universal speed constant c. The latter point of view, which is adopted in this thesis work, is usually referred to as DSR in the literature [29]. In either case, observables depending on exact Poincaré covariance, such as the form of the relativistic dispersion relation, get modified and the resulting Planck-scale corrections are among the main targets of present-day quantum-gravity phenomenology [11].

§ So far, spacetime noncommutativity has been usually introduced into classical or quantum field theory, replacing the ordinary Minkowski background with some noncommutative algebra of coordinates [25–27, 30]. In this context, ordinary Lie algebras have proved ill-suited to deal with deformed relativistic symmetries and have been replaced by more general structures called Hopf algebras [31]. However, while this generalization makes sense and is indeed quite natural at the level of symmetry generators, it leads to puzzling results about infinitesimal symmetry transformations, such as the noncommutativity of transformation parameters and the impossibility of arbitrarily assigning their values [27]. Moreover, the whole field-theoretical approach appears somewhat removed from the heuristic analogy with QM which actually motivated spacetime noncommutativity in the first place. In fact, in field theory, background coordinates are not physical observables, but just convenient labels encoding the mutual spacetime relations of field variables. Therefore, their nontrivial commutation properties do not reflect the incompatibility of physical measurements, like in the case of noncommutative quantum observables, and apparently do not admit any other straightforward physical interpretation. Ultimately, while there are good reasons to adopt a field-theoretical approach, such as its actual relevance in 3D quantum gravity [32], there seems to be also sufficient motivation to explore other possibilities.

§ Since both time and spatial coordinates are represented by self-adjoint operators at the kinematical level, CQM is the ideal tool for introducing arbitrary commutation relations among spacetime coordinates into a single-particle setting. In Chapter 3, I describe an original CQM-based approach to generic spacetime noncommutativity of the form

$$\left[x^{\nu}, x^{\lambda}\right] = i\ell\Gamma^{\nu\lambda}{}_{\alpha}x^{\alpha} + i\ell^{2}\Theta^{\nu\lambda}, \tag{0.15}$$

which was inspired to some extent by the pioneering papers [12, 13]. The idea is to start from single-particle CQM and replace the canonical commutation relations (0.2)-(0.4) with

$$\begin{aligned} [p_{\mu}, p_{\tau}] &= 0, \qquad &(0.16)\\ [p_{\mu}, x^{\nu}] &= i\hbar[\Delta(\ell p)]_{\mu}{}^{\nu}, \qquad &(0.17)\\ \left[x^{\nu}, x^{\lambda}\right] &= i\ell\Gamma^{\nu\lambda}{}_{\alpha}x^{\alpha} + i\ell^{2}\Theta^{\nu\lambda}, \qquad &(0.18) \end{aligned}$$

where a momentum-dependent deformation of the Heisenberg relations must be introduced to preserve the Jacobi identities. In this context, deformed relativistic symmetries can be treated like ordinary quantum-mechanical symmetries and there is no need for Hopf-algebraic concepts, thereby avoiding problems with infinitesimal symmetry transformations. A complete characterization of all the possible CQM-based models of this kind and their deformed relativistic symmetries is provided in the first part of Chapter 3, where it is proved that they can all be obtained from standard CQM and its usual Poincaré transformations via a suitable change of variables. In the second part of the chapter, the new framework is applied to some much-studied and phenomenologically interesting noncommutative spacetimes, including κ-Minkowski (0.14), in order to illustrate its versatility. The usual Hopf-algebraic treatment of their symmetries is also reviewed and compared with the CQM-based one.

Chapter 1

Covariant quantum mechanics

§ In this chapter, I introduce the framework of covariant quantum mechanics, the starting point of the original work reported in Chapters 2 and 3. As anticipated in the Introduction, this basic background-independent quantum formalism has been developed by several authors [4–6] to discuss some conceptual issues associated to quantum gravity in a simplified context and has recently been employed for phenomenological purposes in [12–14]. In my presentation, which is somewhat original, I loosely follow [4] and Rovelli's early seminal paper [16], rather than his more recent proposal [6]. This choice lets me stress and fully exploit the analogy with classical finite-dimensional constrained hamiltonian mechanics [15], which will play a crucial rôle also in the next chapter.

1.1 Relational dynamics

§ In hamiltonian mechanics (HM) [33], the dynamics of a classical system with N degrees of freedom is formulated in terms of N coordinates Q^j and the corresponding N conjugate momenta P_i. Any property of the system which can be quantified and experimentally measured is represented by an element of the canonical algebra $\mathcal{O}_c$ generated by P_i and Q^j, i.e. a function $O(P_k, Q^k)$ of the canonical coordinates. Given a hamiltonian function $H_0(P_k, Q^k) \in \mathcal{O}_c$, Hamilton's equation of motion

$$\frac{dO}{dt} = \{H_0, O\}_0 = \frac{\partial H_0}{\partial P_k}\frac{\partial O}{\partial Q^k} - \frac{\partial O}{\partial P_k}\frac{\partial H_0}{\partial Q^k} \tag{1.1}$$

determines the evolution of any observable $O \in \mathcal{O}_c$ in time t, which is modelled as a background evolution parameter.

§ The same notion of dynamics underlies Heisenberg quantum mechanics (QM) [34]. In this context, a quantum system with N degrees of freedom is described by a noncommutative canonical *-algebra $\mathcal{O}$ generated by N coordinates Q^j and the corresponding N conjugate momenta P_i, which satisfy the canonical commutation relations

$$
\begin{aligned}
[P_i, P_l] &= 0, && (1.2)\\
\left[P_i, Q^j\right] &= i\hbar\delta_i^{\ j}, && (1.3)\\
\left[Q^j, Q^m\right] &= 0. && (1.4)
\end{aligned}
$$

Any physical property of the system is represented by some function $O(P_k, Q^k) \in \mathcal{O}$ of the noncommutative canonical coordinates. Given a quantum hamiltonian $H_0(P_k, Q^k) \in \mathcal{O}$, Heisenberg's equation of motion

$$
i\hbar\frac{dO}{dt} = [H_0, O] \qquad (1.5)
$$

determines the evolution of any observable $O \in \mathcal{O}$ in time t, which again is regarded as a nondynamical evolution parameter.

§ This clear-cut formal and conceptual separation between time and the other physical variables does not reflect any fundamental difference in their operational definition. In the laboratory time is not known *a priori*, but must be read off an instrument (a clock) like any other quantity. From a strictly operational point of view, then, dynamics is just the study of the correlation between the physical properties of some system of interest and a particular physical property of another system taken as a clock. Redefining the system so as to include the clock, we can generalize and regard dynamics as the study of the correlation between different properties of some physical system [1, 3].

§ A closer correspondence with experimental practice is not the only motivation supporting the adoption of such a relational notion of dynamics. Ultimately, the abstract time of HM and QM is an idealization based on the possibility of decoupling the dynamics of the clock from that of the system under investigation. The idea is that we can first characterize once and for all the operation of our clock, and then forget about it and use its calibrated readings to infer the values of the absolute background parameter t appearing in the description of the system of interest. If we neglect gravity, the decoupling assumption is not problematic and this procedure can be carried on. When the gravitational interaction enters the picture, though, the operation of any clock, which is a material system, gets necessarily dependent on what is going on in the rest of the universe. In general relativity, for example, background spacetime coordinates have no physical meaning and each clock ticks at its own rate, determined by the solution of Einstein's field equations. When dealing with gravitational phenomena, then, the usual notion of dynamics as evolution in (space)time breaks down and we are forced to adopt a relational, background-independent point of view [1, 2].

§ In the light of the above considerations, we can appreciate the conceptual and practical importance of a background-independent generalization of both HM and QM, which puts time and other physical quantities on the same footing. At the classical level, such a generalized framework, called parametrized or constrained hamiltonian mechanics (CHM), has been known for decades. Its infinite-dimensional counterpart has even been successfully applied to general relativity [35], which cannot be dealt with by means of ordinary hamiltonian methods. At the quantum level, due to the intrinsically probabilistic nature of the theory, the implementation of the relational paradigm has turned out to be more problematic and the development of a background-independent generalization of QM is still an active field of research.

§ Among the relational quantum formalisms which have been proposed so far, covariant quantum mechanics (CQM) [4–6] is probably the most straightforward. CQM has been originally obtained by directly quantizing CHM and finding a sensible interpretation of the results [16]. It turns out that the quantization procedure is far from being trivial [36] and the resulting framework suffers from serious limitations when compared to its classical mold [37]. However, within its limits, CQM is still a well-defined background-independent quantum formalism, which can adequately model a few interesting quantum systems out of the scope of standard QM. In the next chapters, CQM will actually be put to good use in the field of quantum-gravity phenomenology. In preparation for that, the rest of this chapter contains a detailed introduction to the CQM framework.

1.2 Constrained hamiltonian mechanics

§ Since CQM is basically a straightforward quantization of CHM, it is convenient to start our analysis with a brief, self-contained review of CHM. The reader looking for a more extensive treatment of the classical theory is referred to Sundermeyer's beautiful monograph on the subject [15].

1.2.1 Kinematics

§ In CHM the kinematics of a classical system with $N+1$ degrees of freedom (including any clock variables) is completely encoded in an extended canonical algebra $\mathcal{V}_c$ of real functions defined on a $(2N+2)$-dimensional manifold $\mathcal{K}_c$ called kinematical phase space. The algebra $\mathcal{V}_c$ is generated by $N+1$ coordinates Q^ν and the corresponding $N+1$ conjugate momenta P_μ. Any physical variable which can be experimentally measured on the system, including any clock reading, is represented by a real function $V(P_\alpha, Q^\alpha) \in \mathcal{V}_c$ of the canonical coordinates. The extended canonical algebra comes naturally equipped with the generalized Poisson bracket

$$\{V, W\} = \frac{\partial V}{\partial P_\alpha}\frac{\partial W}{\partial Q^\alpha} - \frac{\partial W}{\partial P_\alpha}\frac{\partial V}{\partial Q^\alpha}, \tag{1.6}$$

and the canonical coordinates P_μ and Q^ν satisfy the canonical relations

$$\{P_\mu, P_\tau\} = 0, \tag{1.7}$$

$$\{P_\mu, Q^\nu\} = \delta_\mu{}^\nu, \tag{1.8}$$

$$\left\{Q^\nu, Q^\lambda\right\} = 0. \tag{1.9}$$

§ Points in the kinematical phase space $\mathcal{K}_c$ are called kinematical states. Evaluating any variable $V : \mathcal{K}_c \longrightarrow \mathbb{R}$ on a kinematical state $\psi \in \mathcal{K}_c$ results in a real number $V[\psi]$, the value of V at ψ. If we perform a simultaneous measurement of all the canonical coordinates of the system, we obtain $2N+2$ outcomes P_μ^0 and Q_0^ν which univocally identify a kinematical state ψ, the actual state of the system, via

$$P_\mu[\psi] = P_\mu^0, \tag{1.10}$$
$$Q^\nu[\psi] = Q_0^\nu. \tag{1.11}$$

Not all kinematical states can be actual states of the system. Experience tells us that a classical system with $N+1$ degrees of freedom only allows for $2N+1$ independent physical variables. This means that physically accessible states must be contained in a $(2N+1)$-dimensional submanifold $\mathcal{A} \subset \mathcal{K}_c$. In CHM, the set of accessible states $\mathcal{A}$ is implicitly determined by the equation

$$H(P_\alpha, Q^\alpha)[\phi] = 0. \tag{1.12}$$

The variable $H(P_\alpha, Q^\alpha) \in \mathcal{V}_c$ is called hamiltonian constraint and plays a crucial rôle also in the description of the generalized dynamics of the system.

§ It is worth explicitly pointing out that accessible states are still kinematical objects, that is, they merely summarize the outcome of a complete measurement on the system and cannot be used to make predictions without additional dynamical information. Our dynamical knowledge of the system is instead directly and completely encoded in its so-called physical states, which will be defined below. This distinction is foreign to standard HM, where initial conditions, which are particular accessible states, provide the only available notion of state. In the relational framework of both CHM and CQM, however, it becomes very important. In fact, as we will see below, the failure to identify physical states as the primary dynamical objects and the consequent reliance on accessible states in their stead are the very reasons why time plays so special a rôle in standard HM. Moreover, the notion of accessible state, whose very definition depends on physical variables having definite values, does not survive the transition to an intrinsically probabilistic theory such as CQM, whereas physical states can be naturally defined even in this context.

1.2.2 Dynamics

§ In CHM the dynamical problem amounts to finding the value $\mathbf{E}_W[V|W_0,\Psi]$ of any physical quantity V when the value of another variable W is found to be W_0 and the system is in a given physical state Ψ. The latter, as anticipated above, contains all the information about the actual dynamics of the system and can be univocally inferred from its experimentally determined actual state. If W is a clock reading, the problem reduces to finding the ordinary evolution in physical time. However, since the formalism puts all the physical variables on the same footing, it must be able to yield arbitrarily conditioned predictions such as $\mathbf{E}_W[V|W_0,\Psi]$.

§ The physical states of the system are determined by the hamiltonian constraint $H(P_\alpha, Q^\alpha)$ introduced above, or rather by the associated Hamilton's equation of motion

$$\frac{dV}{d\lambda} = \{H, V\}\,. \tag{1.13}$$

Equation (1.13) describes the evolution of any physical variable $V = V(0)$ with respect to an unphysical parameter λ or, equivalently, the evolution $\alpha_\lambda(\phi)$ of the actual state ϕ of the system. The 1-parameter group of diffeomorphisms α_λ is obtained by requiring that

$$V(0)[\alpha_\lambda(\phi)] = V(\lambda)[\phi] \tag{1.14}$$

for every $V \in \mathcal{V}_c$ and $\phi \in \mathcal{A}$. This condition, when $V = H$, specializes to

$$H[\alpha_\lambda(\phi)] = H(\lambda)[\phi] = H[\phi] = 0, \tag{1.15}$$

so that α_λ actually maps the space of accessible states $\mathcal{A}$ onto itself. The orbits of the group α_λ are the physical states of the system and completely encode its relational dynamics. If $\phi \in \mathcal{A}$ is the actual state of the system, the accessible states lying on the corresponding orbit $\Psi(\lambda) = \alpha_\lambda(\phi)$ are the only ones which the system can actually access in the given experimental conditions. In terms of standard HM, as we will see in more detail below, the orbit $\Psi(\lambda)$ represents the actual motion of the system as a function of the unphysical parameter λ. The $2N$-dimensional space $\mathcal{P}_c$ of physical states, which is technically the quotient of the action of α_λ on $\mathcal{A}$, is called physical phase space.

§ Given a physical state $\Psi \in \mathcal{P}_c$, it is easy to compute any prediction $\mathbf{E}_W[V|W_0, \Psi]$ of the theory. In fact, it is sufficient to solve the equation

$$W[\Psi(\lambda_0)] = W_0 \tag{1.16}$$

for λ_0, i.e. to identify among the possible actual states of the system those compatible with the condition $W = W_0$. The corresponding predictions, i.e. the values of V on such states, are then given by

$$\mathbf{E}_W[V|W_0, \Psi] = V[\Psi(\lambda_0)]. \tag{1.17}$$

If W is not a clock reading or another natural independent variable, the first equation may have more than one solutions, or none at all. In the first case, the prediction will be multi-valued. In the second case, it will be ill-conditioned and therefore ill-defined. These are unavoidable consequences of allowing for general independent variables W and should not be considered deficiencies of the framework.

1.2.3 Standard hamiltonian mechanics

§ In order to better understand the formal structure introduced above and its close connection with standard HM, it is instructive to recover the familiar concepts of the latter in the context of CHM. Let us take a clock reading T as the first canonical coordinate Q^0 and let us suppose that the remaining N coordinates Q^j describe the configuration of a standard classical system with N degrees of freedom. Then, the particular hamiltonian constraint

$$H_{nr}(P_0, P_k, Q^0, Q^k) = P_0 + H_0(P_k, Q^k) \tag{1.18}$$

determines the same dynamics with respect to the physical time $T = Q^0$ in CHM as the hamiltonian function $H_0(P_k, Q^k)$ does with respect to the abstract time parameter t in HM. To see that this is the case, let us take a generic observable $O(P_k, Q^k) \in \mathcal{O}_c$ of standard HM and compute the predictions $\mathbf{E}_T[O|T_0, \Psi]$ associated to its evolution in physical time, with the system in physical state $\Psi(\lambda) = \alpha_\lambda(\phi)$. First of all, we must solve equation (1.16), which now reads

$$T[\Psi(\lambda_0)] = T_0. \tag{1.19}$$

Since

$$\frac{dT}{d\lambda} = \{H_{nr}, T\} = \{P_0, T\} = 1, \tag{1.20}$$

we have

$$T[\Psi(\lambda)] = T(\lambda)[\phi] = T[\phi] + \lambda. \tag{1.21}$$

If we choose the starting point ϕ of the orbit Ψ so that $T[\phi] = 0$, i.e. if we take for ϕ the only accessible state in Ψ which yields the value zero for the clock reading, we find that $T[\Psi(\lambda)] = \lambda$ and thus $\lambda_0 = T_0$. As a result, we have

$$\mathbf{E}_T[O|T_0, \Psi] = O[\Psi(\lambda_0)] = O(\lambda_0)[\phi] = O(T_0)[\phi], \tag{1.22}$$

and the dynamics of O in physical time T is directly encoded in its dependence on the parameter λ. But this is given by

$$\frac{dO}{d\lambda} = \{H_{nr}, O\} = \frac{\partial H_0}{\partial P_k}\frac{\partial O}{\partial Q^k} - \frac{\partial O}{\partial P_k}\frac{\partial H_0}{\partial Q^k} = \{H_0, O\}_0\,, \tag{1.23}$$

which is exactly (1.1) with the evolution parameter t replaced by λ. In addition, we see that all the information about the physical state Ψ of the system is contained in the accessible state ϕ placed at the intersection between Ψ and the hypersurface $\mathcal{T}$ defined by $T[\psi] = 0$. The set of these initial states, which represents the physical phase space $\mathcal{P}_c$, is the $(2N)$-dimensional intersection between $\mathcal{A}$ and $\mathcal{T}$. Since it can be coordinatized by the $2N$ canonical coordinates P_i and Q^j, it is easily recognized as ordinary phase space.

§ We can conclude that standard HM is just a reformulation of CHM in the particular case in which the hamiltonian constraint is of the form (1.18). We also understand exactly how CHM generalizes the old framework. The motions of old hamiltonian mechanics, which were obtained in parametric form, are now represented explicitly as curves in the accessible space $\mathcal{A}$. Their initial conditions, which used to represent physical states, are now recognized as just convenient labels for the motions themselves, which are identified as the correct, timeless physical states. This shift in perspective makes time lose its special rôle and allows CHM to deal with scenarios where a standard clock variable cannot be defined or its dynamics depends on what is happening to the rest of the system.

1.2.4 Conserved quantities and predictions

§ The formulation of CHM given in the previous paragraphs is suitable for introducing the formalism and highlighting its kinship to ordinary hamiltonian mechanics, but does not admit a sensible quantum translation. In fact, the solution of equation (1.16) for λ_0 and its substitution in (1.17) is impossible in the context of an intrinsically probabilistic theory where physical variables have no well-defined values. On a formal level, it does not make sense to consider an equation like (1.16) when W is a linear operator on a Hilbert space instead of a numerical function. Therefore, we will now discuss an alternative formulation of CHM, which focuses on conserved quantities and does away with the parametric evolution in λ. This way of looking at CHM, while somewhat abstract and implicit, is ideal for explaining the crucial relation between conserved quantities and predictions, and naturally leads to the quantization of the theory.

§ In standard HM an observable C is conserved if its time evolution is trivial, i.e. if $C(t) = C(0) = C$. In CHM, where every $W \in \mathcal{V}_c$ can be used to track the dynamics, a variable C is conserved when all the associated predictions are trivial, i.e. when $\mathbf{E}_W[C|W_0, \Psi] = C[\phi]$ for every $W \in \mathcal{V}_c$, $W_0 \in \mathbb{R}$ and $\Psi(\lambda) = \alpha_\lambda(\phi) \in \mathcal{P}_c$. Since predictions are computed plugging particular values of the parameter λ into (1.17), this can only happen if $C[\Psi(\lambda)] = C(\lambda)[\phi] = C[\phi]$ for every $\lambda \in \mathbb{R}$, i.e. if

$$\frac{dC}{d\lambda} = \{H, C\} = 0. \tag{1.24}$$

The vanishing of the Poisson bracket $\{H, C\}$ is a differential condition on the function $C(P_\alpha, Q^\alpha)$ which can be checked without knowing the evolution α_λ of the system with respect to λ. The nontrivial solutions of equation (1.24), i.e. those which are independent of H, form a $2N$-dimensional subalgebra of conserved quantities $\mathcal{C}_c \subset \mathcal{V}_c$.

§ The conserved algebra $\mathcal{C}_c$ implicitly encodes all the information about the dynamics of the system and provides an alternative characterization of its physical states. Let us suppose to know $2N$ independent generators C_l of $\mathcal{C}_c$, i.e. $2N$ independent constants of motion. Since every $C \in \mathcal{C}_c$ is conserved along any orbit $\Psi \in \mathcal{P}_c$, $C[\Psi(\lambda)]$ is independent of λ and it is possible to regard C as a function $C[\Psi]$ of the physical state of the system. If we view $\mathcal{C}_c$ as an algebra of functions defined on the physical phase space $\mathcal{P}_c$, every $\Psi \in \mathcal{P}_c$ is univocally identified by the values $C_l[\Psi]$ of the $2N$ conserved generators. We can then forget that physical states are orbits in kinematical phase space and treat $\mathcal{P}_c$ as just an abstract manifold coordinatized by C_l, exactly like we defined $\mathcal{K}_c$ as just an abstract manifold coordinatized by the canonical coordinates P_μ and Q^ν. Adopting this perspective, physical states can be characterized without any explicit knowledge of the evolution α_λ of the system. The group α_λ is not even necessary to associate the physical state $\Psi(\lambda) = \alpha_\lambda(\phi)$ to some actual state $\phi \in \mathcal{A}$. In fact, since

$$C[\Psi] = C[\phi] \tag{1.25}$$

for every $C \in \mathcal{C}_c$, the actual state ϕ directly determines the $2N$ values $C_l[\Psi]$ which identify the physical state Ψ.

§ Knowing the conserved algebra $\mathcal{C}_c$, or rather a complete set of conserved generators C_l, it is also possible to compute any prediction $\mathbf{E}_W[V|W_0, \Psi]$ of the theory. Let us first consider two special cases. If $V \in \mathcal{C}_c$, then $\mathbf{E}_W[V|W_0, \Psi] = V[\Psi]$ for every $W_0 \in \mathbb{R}$ and the prediction is trivial. If $V \notin \mathcal{C}_c$ but $W \in \mathcal{C}_c$, we can have either $W[\Psi] = W_0$ or $W[\Psi] \neq W_0$. In any case the prediction is ill-defined, because it is either indeterminate or ill-conditioned. The interesting situation is when neither V nor W are conserved, i.e. when they are independent of C_l. In this case, the hamiltonian constraint H and the $2N+1$ variables W, C_l are independent and span the extended canonical algebra $\mathcal{V}_c$. This means that, at least locally, we can write V as some function $V(H, W, C_l)$. If we now restrict V to the accessible space $\mathcal{A}$, the hamiltonian constraint vanishes identically and we find

$$V|_{\mathcal{A}} = V(0, W, C_l) = \widetilde{V}(W; C_l). \tag{1.26}$$

The function $\widetilde{V}$ encodes all the information about the dynamics of V with respect to W and can be used to find all the associated predictions. In fact, setting $W = W_0$ in $\widetilde{V}$ yields a conserved quantity $\widetilde{V}(W_0; C_l) \in \mathcal{C}_c$ which gives the value of V when $W = W_0$ as a function of the conserved generators C_l. If we know the physical state Ψ of the system, we can determine the constant values of C_l and finally obtain the prediction $\mathbf{E}_W[V|W_0, \Psi]$. In other words, we have

$$\mathbf{E}_W[V|W_0, \Psi] = \widetilde{V}(W_0; C_l)[\Psi]. \tag{1.27}$$

Parametrized constants of motion such as $\widetilde{V}(W_0; C_l)$ are called evolving constants [4, 16]. They are characterized by the property that their values on any $\Psi \in \mathcal{P}_c$ are the predictions for V when $W = W_0$ and the system is in physical state Ψ. In general, the function $\widetilde{V}$ will not be defined for all possible $W_0, C_l[\Psi] \in \mathbb{R}$, and may also be globally multi-valued. As explained above, this is not a problem. On the contrary, the resulting ill-conditioned or multi-valued predictions are expected and unavoidable consequences of dealing with general independent variables.

§ The above considerations suffice to demonstrate that physical states are univocally identified by the possible values of the conserved generators C_l and that the dynamics of any variable V with respect to any other variable W is completely contained in some evolving constant $\widetilde{V}(W_0; C_l)$, i.e. in the functional dependence of V on H, W and C_l. This means that the theoretical framework of CHM can be fully developed in terms of conserved quantities, without even introducing the unphysical parameter λ.

1.2.5 Hamilton-Jacobi theory

§ The implicit formulation of CHM discussed above reduces the dynamical problem to the determination of the conserved algebra $\mathcal{C}_c$, or rather of its generators C_l. However, in order for it to be practically relevant, there must be a systematic way of finding a set of $2N$ independent C_l. In fact, it is clear that equation (1.24) is of little use in this regard. A well-definite general solution to this problem is provided by Hamilton-Jacobi theory.

§ Given any hamiltonian constraint $H(P_\alpha, Q^\alpha) \in \mathcal{V}_c$, the corrisponding Hamilton-Jacobi (H-J) equation is the partial differential equation

$$H\left(\frac{\partial \Sigma}{\partial Q^\alpha}, Q^\alpha\right) = 0, \tag{1.28}$$

where the unknown Σ is a function of the $N+1$ coordinates Q^ν. If we neglect an overall additive constant, the general solution $\Sigma(Q^\alpha; \beta_k)$, called Hamilton's principal function, depends on N arbitrary real parameters β_i. The main result of Hamilton-Jacobi theory [33] is that there exist $2N$ independent constants of motion $B_i, D^i \in \mathcal{C}_c$ such that

$$P_\mu = \frac{\partial \Sigma}{\partial Q^\mu}(Q^\alpha; B_k), \tag{1.29}$$

$$D^i = \frac{\partial \Sigma}{\partial \beta_i}(Q^\alpha; B_k). \tag{1.30}$$

These $2N+1$ equations, when restricted to the accessible space $\mathcal{A}$, are not all independent, because they are related by the vanishing of the hamiltonian constraint. As a result, the system (1.29)-(1.30) can in general be solved for the $2N$ constants of motion $B_i(P_\alpha, Q^\alpha)$ and $D^i(P_\alpha, Q^\alpha)$.

§ The H-J equation (1.28) for a generic constraint is very difficult to solve. One usually assumes that there is a solution of the form

$$\Sigma(Q^\alpha; \beta_k) = \sum_{\nu=1}^{N+1} \Sigma_\nu(Q^\nu; \beta_k), \tag{1.31}$$

plugs it in (1.28) and hopes that the resulting equation can be solved by separation of variables. When this is the case, (1.28) is said to be Jacobi-integrable and the constants B_i and D^i can be readily computed. If (1.28) is not Jacobi-integrable, the Hamilton-Jacobi framework described above is of little practical value. This may seem a severe limitation of the applicability of the theory, but there are a lot of interesting models described by Jacobi-integrable H-J equations. All the single-particle examples discussed in this thesis work are actually of this kind.

§ With Hamilton-Jacobi theory, we have now at our disposal a handy recipe to specify and fully characterize any finite-dimensional mechanical system in the context of CHM without appealing to its evolution in λ. First of all, we define the system of interest by specifying a $(2N+2)$-dimensional extended canonical algebra of real functions $\mathcal{V}_c$ and a hamiltonian constraint $H \in \mathcal{V}_c$. The kinematical phase space $\mathcal{K}_c$ is the $(2N+2)$-dimensional manifold on which the elements of $\mathcal{V}_c$ are defined. Then, we solve the H-J equation associated to H and invert (1.29)-(1.30) to find $2N$ independent generators C_l of the conserved algebra $\mathcal{C}_c$. The physical phase space is the $2N$-dimensional manifold coordinatized by the conserved generators C_l. Finally, we compute all the relevant functions $\widetilde{V}$ and identify the corresponding conserved quantities $\widetilde{V}(W_0; C_l)$ as evolving constants. In this way, their values on any physical state Ψ get associated via equation (1.27) to physical predictions about the correlation of some variables V and W. As we will see in the next section, most of this classical recipe can be straightforwardly carried over to the quantum context, resulting in a powerful relational reformulation of QM.

1.3 Covariant quantum mechanics

§ In his classic monograph [34], Dirac proposed a method to derive Heisenberg QM from standard HM. According to this prescription, called classical analogy, the quantum canonical algebra $\mathcal{O}$ associated to any mechanical system is obtained from its classical counterpart $\mathcal{O}_c$ by replacing the Poisson bracket with the commutator. More precisely, if $O, O' \in \mathcal{O}_c$ are two classical observables and $q(O), q(O') \in \mathcal{O}$ are the corresponding quantum observables, the commutator $[q(O), q(O')]$ and the Poisson bracket $\{O, O'\}_0$ are required to satisfy

$$[q(O), q(O')] = i\hbar \{O, O'\}_0 \,. \tag{1.32}$$

When taken seriously, the classical analogy has been shown to yield contradictory results [38], so that it cannot be regarded as a rigorous quantization prescription. However, Dirac's heuristic correctly points to a close connection between QM and HM. In fact, equation (1.32) does actually hold when applied to the cartesian coordinates of some euclidean configuration space and their conjugate momenta, resulting in the so-called canonical quantization of the classical model. At the beginning of the next chapter, we will return to the classical analogy and discuss in more detail the relationship between QM and HM for generic configuration manifolds. In the following, instead, we assume that Q^ν are cartesian coordinates of some flat configuration space and apply the canonical quantization prescription to CHM. In this way, we straightforwardly obtain CQM, a generalization of Heisenberg QM in which time is regarded as a quantum dynamical variable.

1.3.1 Kinematics

§ In CQM the kinematics of a quantum system with $N+1$ degrees of freedom (including any clock variables) is described by an abstract extended canonical *-algebra $\mathcal{V}$. This is a complex associative algebra with involution generated by $N+1$ coordinates Q^ν and the corresponding $N+1$ conjugate momenta P_μ. The canonical coordinates P_μ and Q^ν are hermitian, i.e. satisfy

$$P_\mu = (P_\mu)^*, \tag{1.33}$$
$$Q^\nu = (Q^\nu)^*, \tag{1.34}$$

and obey the canonical commutation relations

$$[P_\mu, P_\tau] = 0, \tag{1.35}$$
$$[P_\mu, Q^\nu] = i\hbar\delta_\mu{}^\nu, \tag{1.36}$$
$$\left[Q^\nu, Q^\lambda\right] = 0. \tag{1.37}$$

Any physical variable which can be experimentally measured on the system, including any clock reading, is represented by a hermitian element $V(P_\alpha, Q^\alpha) = V(P_\alpha, Q^\alpha)^* \in \mathcal{V}$.

§ The abstract *-algebra $\mathcal{V}$, by itself, is not sufficient to completely characterize the kinematics of the system. In order to get a definite quantum model, it is also necessary to specify an irreducible representation of $\mathcal{V}$ in terms of densely defined linear operators on a Hilbert space. However, under mild regularity assumptions, all such representations are unitarily equivalent to the so-called Schrödinger representation [39]. As a result, a complete kinematical description of the system is given by the Schrödinger representation of $\mathcal{V}$. The Hilbert space of this representation is the set $\mathcal{K} = L^2(\mathbb{R}^{N+1}, d^{N+1}p)$ of square-integrable complex-valued functions of $N+1$ real variables, equipped with the L^2 scalar product

$$\langle\psi_1|\psi_2\rangle = \int \overline{\psi_1(p_\alpha)}\psi_2(p_\alpha)d^{N+1}p, \tag{1.38}$$

and any variable $V \in \mathcal{V}$ is associated to a linear operator $\widehat{V}$ defined on some dense domain $\mathcal{D}(\widehat{V}) \subseteq \mathcal{K}$. In particular, the operators representing the canonical coordinates are naturally defined on the space $\mathcal{D} = C_0^\infty(\mathbb{R}^{N+1}) \subset \mathcal{K}$ of smooth functions with compact support, and their action on any $\psi \in \mathcal{D}$ is given by

$$\widehat{P}_\mu\psi(p_\alpha) = p_\mu\psi(p_\alpha), \tag{1.39}$$
$$\widehat{Q}^\nu\psi(p_\alpha) = -i\hbar\frac{\partial\psi(p_\alpha)}{\partial p_\nu} = -i\hbar\partial^\nu\psi(p_\alpha). \tag{1.40}$$

Any other variable $V(P_\alpha, Q^\alpha) \in \mathcal{V}$ is then represented by the operator $\widehat{V} = V(\widehat{P}_\alpha, \widehat{Q}^\alpha)$, which is generally well-defined on some dense domain $\mathcal{D}(\widehat{V}) \subseteq \mathcal{D}$. Since $\widehat{P}_\mu$ and $\widehat{Q}^\nu$ satisfy the canonical commutation relations (1.35)-(1.37) and are hermitian operators with respect to the scalar product (1.38), the resulting operator algebra is actually a representation of $\mathcal{V}$ on $\mathcal{K}$. In the following, for notational convenience, we will systematically identify $\mathcal{V}$ with its Schrödinger representation and write $\widehat{V} \in \mathcal{V}$.

§ In the Schrödinger representation, physical variables are represented by hermitian operators. This means that any physical operator $\widehat{V} \in \mathcal{V}$ is equal to its hermitian adjoint $\widehat{V}^\dagger$ on its domain of definition $\mathcal{D}(\widehat{V})$, i.e. that

$$\langle\psi_1|\widehat{V}\psi_2\rangle = \langle\widehat{V}\psi_1|\psi_2\rangle = \langle\widehat{V}^\dagger\psi_1|\psi_2\rangle \tag{1.41}$$

for every $\psi_1, \psi_2 \in \mathcal{D}(\widehat{V})$. It is customary to express the hermiticity of $\widehat{V}$ by writing $\widehat{V} = \widehat{V}^\dagger$, but it is important to understand that $\widehat{V}$ and $\widehat{V}^\dagger$, unlike the corresponding abstract algebra elements V and V^*, are not identical in general. In fact, it may happen that the adjoint operator $\widehat{V}^\dagger$, which is obviously given by (1.41) on $\mathcal{D}(\widehat{V})$, is actually well-defined on a bigger domain $\mathcal{D}(\widehat{V}^\dagger) \supset \mathcal{D}(\widehat{V})$, i.e. that there exists some $\psi' \notin \mathcal{D}(\widehat{V})$ such that

$$\langle\psi'|\widehat{V}\psi\rangle = \langle\psi''|\psi\rangle = \langle\widehat{V}^\dagger\psi'|\psi\rangle \tag{1.42}$$

for some $\psi'' \in \mathcal{K}$ and every $\psi \in \mathcal{D}(\widehat{V})$. In this case, the hermitian adjoint $\widehat{V}^\dagger$ is said to be an extension of the original operator $\widehat{V}$. If instead $\mathcal{D}(\widehat{V}^\dagger) = \mathcal{D}(\widehat{V})$, $\widehat{V}$ and $\widehat{V}^\dagger$ are truly identical and the operator $\widehat{V}$ is called self-adjoint. When $\widehat{V}$ is not self-adjoint, it may happen that it admits a unique self-adjoint extension, i.e. that there exists a unique self-adjoint operator $\widehat{V}'$ such that $\mathcal{D}(\widehat{V}') \supseteq \mathcal{D}(\widehat{V})$ and $\widehat{V}' = \widehat{V}$ on $\mathcal{D}(\widehat{V})$. In this case, the operator $\widehat{V}$ is called essentially self-adjoint on its domain $\mathcal{D}(\widehat{V})$. All these concepts are best illustrated by considering a specific example. As claimed above, the canonical coordinates $\widehat{P}_\mu$ and $\widehat{Q}^\nu$ are hermitian operators. In fact, we have

$$\begin{aligned}\langle\psi_1|\widehat{P}_\mu\psi_2\rangle &= \int \overline{\psi_1(p_\alpha)} p_\mu \psi_2(p_\alpha) d^{N+1}p = \\ &= \int \overline{p_\mu\psi_1(p_\alpha)}\psi_2(p_\alpha) d^{N+1}p = \langle\widehat{P}_\mu\psi_1|\psi_2\rangle\end{aligned} \tag{1.43}$$

and, integrating by parts,

$$\begin{aligned}\langle\psi_1|\widehat{Q}^\nu\psi_2\rangle &= -i\hbar\int \overline{\psi_1(p_\alpha)}\partial^\nu\psi_2(p_\alpha) d^{N+1}p = \\ &= i\hbar\int \overline{\partial^\nu\psi_1(p_\alpha)}\psi_2(p_\alpha) d^{N+1}p = \langle\widehat{Q}^\nu\psi_1|\psi_2\rangle\end{aligned} \tag{1.44}$$

for every $\psi_1, \psi_2 \in \mathcal{D}$. However, they are not self-adjoint, because their hermitian adjoints are defined on a much bigger domain than $\mathcal{D}$. For example, let $\psi'(p_\alpha) \in \mathcal{K}$ be any differentiable L^2 function such that both $\psi''(p_\alpha) = p_\alpha\psi'(p_\alpha)$ and $\psi'''(p_\alpha) = -i\hbar\partial^\nu\psi'(p_\alpha)$ are still in $\mathcal{K}$ but the second derivatives $\partial^\nu\partial^\sigma\psi'(p_\alpha)$ do not exist. Then, $\psi' \notin \mathcal{D}$ but we have

$$\langle\psi'|\widehat{P}_\mu\psi\rangle = \int \overline{p_\mu\psi'(p_\alpha)}\psi(p_\alpha) d^{N+1}p = \langle\psi''|\psi\rangle, \tag{1.45}$$

$$\langle\psi'|\widehat{Q}^\nu\psi\rangle = i\hbar\int \overline{\partial^\nu\psi'(p_\alpha)}\psi(p_\alpha) d^{N+1}p = \langle\psi'''|\psi\rangle \tag{1.46}$$

for every $\psi \in \mathcal{D}$, so that the adjoints $(\widehat{P}_\mu)^\dagger$ and $(\widehat{Q}^\nu)^\dagger$ are both well-defined on ψ'. Even if they fail to be self-adjoint, $\widehat{P}_\mu$ and $\widehat{Q}^\nu$ are essentially self-adjoint on $\mathcal{D}$. In fact, the operators

$$(\widehat{P}_\mu)'\psi(p_\alpha) = p_\mu\psi(p_\alpha), \tag{1.47}$$

$$(\widehat{Q}^\nu)'\psi(p_\alpha) = -i\hbar\partial^\nu\psi(p_\alpha), \tag{1.48}$$

defined on the domains

$$\mathcal{D}((\widehat{P}_\mu)') = \{\psi(p_\alpha) \in \mathcal{K} \,:\, p_\mu\psi(p_\alpha) \in \mathcal{K}\}, \tag{1.49}$$

$$\mathcal{D}((\widehat{Q}^\nu)') = \{\psi(p_\alpha) \in \mathcal{K} \,:\, \partial^\nu\psi(p_\alpha) \in \mathcal{K}\}, \tag{1.50}$$

respectively, can be proved to be the only self-adjoint operators which reduce to $\widehat{P}_\mu$ and $\widehat{Q}^\nu$ when restricted to $\mathcal{D}$ [40]. From this example we see that an essentially self-adjoint operator is just the restriction of an actually self-adjoint operator to a smaller domain. We also understand why it is often convenient, in practice, to work with such restrictions. In this way, in fact, it is often possible to simultaneously define all the relevant operators, as well as their sufficiently regular functions, on a single domain. Had we defined the canonical coordinates on their separate self-adjointness domains $\mathcal{D}((\widehat{P}_\mu)')$ and $\mathcal{D}((\widehat{Q}^\nu)')$, we would need to carefully discuss the domain of any function $V(\widehat{P}_\mu, \widehat{Q}^\nu)$ we might be interested in, even the simplest ones. On the restricted domain $\mathcal{D}$, instead, both the canonical coordinates and all their smooth functions are simultaneously well-defined.

§ The distinction between hermiticity and self-adjointness may seem overly technical, but it is actually very important from a quantum-theoretical point of view. The celebrated spectral theorem [40] guarantees that the spectrum $\sigma(\widehat{V})$ of a self-adjoint operator $\widehat{V}$ is a subset of the real line, and that, given any vector $\psi \in \mathcal{D}(\widehat{V})$, it is possible to find a unique probability measure $\mu(\widehat{V}, \psi)$ with support $\sigma(\widehat{V})$ such that

$$\int_{\mathbb{R}} f(x)\, d\mu(\widehat{V}, \psi) = \int_{\sigma(\widehat{V})} f(x)\, d\mu(\widehat{V}, \psi) = \langle \psi | f(\widehat{V}) \psi \rangle \tag{1.51}$$

for every sufficiently regular $f(\widehat{V}) \in \mathcal{V}$. This result, which does not hold for generic hermitian operators, is the mathematical foundation of the usual probabilistic interpretation of Heisenberg QM. In fact, the spectral theorem makes it possible to identify the spectrum $\sigma(\widehat{O})$ of any self-adjoint operator $\widehat{O}$ with the set of possible values of the corresponding observable $O \in \mathcal{O}$, which are always real. At the same time, it univocally associates a spectral measure $\mu(\widehat{O}, \Psi)$ to every quantum state $\Psi \in \mathcal{D}(\widehat{O})$, thereby associating a probability

$$\mathbf{P}(\rho|\Psi) = \int_\rho d\mu(\widehat{O}, \Psi) \tag{1.52}$$

to every measurable $\rho \subset \sigma(\widehat{O})$. This $\mathbf{P}(\rho|\Psi)$ is then naturally identified with the probability of obtaining a value of O within the spectral region ρ when the system is in state Ψ. And the quadratic form $\langle \Psi | \widehat{O} \Psi \rangle$, which satisfies

$$\langle \Psi | \widehat{O} \Psi \rangle = \int_{\mathbb{R}} x\, d\mu(\widehat{O}, \Psi), \tag{1.53}$$

is consequently interpreted as the expected value $\mathbf{E}[O|\Psi]$ of the observable O in state Ψ. The same results obviously apply to any essentially self-adjoint operator, which is just a restriction of its unique self-adjoint extension. Since essentially self-adjoint operators are the only hermitian operators which admit a consistent probabilistic interpretation, the apparently subtle mathematical distinction between self-adjointness and hermiticity is actually of great physical importance. In particular, it is clear that all quantum operators required to yield physical predictions, such as observable operators in standard QM, must be essentially self-adjoint. This is true also in CQM, where constants of motion, to be discussed below, are indeed required to be essentially self-adjoint. At the kinematical level, though, we can safely allow for generic hermitian operators. In fact, physical variables, which represent indeterminate measurable quantities, are not directly related to the predictions of the theory and do not need any probabilistic interpretation.

§ The adjoint map †, being the Schrödinger representation of the abstract involution *, satisfies all the algebraic properties of an involution. More explicitly, we have

$$(\widehat{V}^\dagger)^\dagger = \widehat{V}, \tag{1.54}$$
$$(\widehat{V}\widehat{W})^\dagger = \widehat{W}^\dagger\widehat{V}^\dagger, \tag{1.55}$$
$$(\widehat{V}+z\widehat{W})^\dagger = \widehat{V}^\dagger+\overline{z}\widehat{W}^\dagger \tag{1.56}$$

for every $\widehat{V},\widehat{W}\in\mathcal{V}$ and $z\in\mathbb{C}$. Equations (1.54)-(1.56), like the hermiticity condition $\widehat{V}=\widehat{V}^\dagger$, cannot be regarded as identities, because the domains of the left and right sides are generally different. Their actual meaning is that both sides give the same result when applied to any vector in their common domain of definition. However, these relations can be effectively used to check whether some $\widehat{V}=V(\widehat{P}_\alpha,\widehat{Q}^\alpha)\in\mathcal{V}$ is hermitian. In fact, let us suppose that $V(\widehat{P}_\alpha,\widehat{Q}^\alpha)$ is a symmetric function of the canonical coordinates $\widehat{P}_\mu$ and $\widehat{Q}^\nu$, i.e. that, through the algebraic manipulations (1.54)-(1.56), we find

$$V(\widehat{P}_\alpha,\widehat{Q}^\alpha)^\dagger = V((\widehat{P}_\alpha)^\dagger,(\widehat{Q}^\alpha)^\dagger). \tag{1.57}$$

Then, since $\widehat{P}_\mu$ and $\widehat{Q}^\nu$ are hermitian, we have

$$V(\widehat{P}_\alpha,\widehat{Q}^\alpha)^\dagger = V(\widehat{P}_\alpha,\widehat{Q}^\alpha) \tag{1.58}$$

on their common domain of definition. But this is obviously $\mathcal{D}(\widehat{V})$ and therefore $\widehat{V}$ is hermitian. We can conclude that the hermiticity of any $\widehat{V}\in\mathcal{V}$, being equivalent to the symmetry of the function $V(\widehat{P}_\alpha,\widehat{Q}^\alpha)$, is a purely algebraic property, exactly like the hermiticity of the corresponding abstract variable V. In particular, it is independent of $\widehat{V}$ actually being a partially defined operator on the Hilbert space $\mathcal{K}$, even if the very definition (1.42) of the hermitian adjoint $\widehat{V}^\dagger$ heavily relies on the Hilbert structure of $\mathcal{K}$. Unfortunately, the formal properties of the adjoint map do not provide any equally simple criterion for self-adjointness, which is a matter of domains of definition. Even if two operators $\widehat{V}$ and $\widehat{W}$ are separately self-adjoint on their respective domains $\mathcal{D}(\widehat{V})$ and $\mathcal{D}(\widehat{W})$, there is no guarantee that their sum $\widehat{V}+\widehat{W}$ nor their symmetric product $\widehat{V}\widehat{W}+\widehat{W}\widehat{V}$ are also self-adjoint. The sum $\widehat{V}+\widehat{W}$, for example, is naturally defined on the common domain $\mathcal{D}(\widehat{V})\cap\mathcal{D}(\widehat{W})$ and is identical to $\widehat{V}^\dagger+\widehat{W}^\dagger$, but nothing prevents its hermitian adjoint $(\widehat{V}+\widehat{W})^\dagger$ to have a bigger domain, thereby spoiling its self-adjointness. This would not be a problem if $\widehat{V}+\widehat{W}$ were essentially self-adjoint on $\mathcal{D}(\widehat{V})\cap\mathcal{D}(\widehat{W})$, but even this is not guaranteed in general [40]. As a consequence, symmetric functions of essentially self-adjoint operators, while always being hermitian, usually fail to be essentially self-adjoint on their domains, and additional, non-algebraic arguments are always needed to prove the contrary. The non-algebraic character of self-adjointness is a major issue in Heisenberg QM, where several functions of the canonical coordinates, contrary to their classical counterparts, are not associated to any observables because they fail to be essentially self-adjoint. As we will see below, it is an even more serious problem in CQM, where the requirement of self-adjointness entails severe restrictions on the choice of the independent variables, thereby impairing the relational character of the framework and limiting its applicability.

§ Vectors in $\mathcal{K}$ are called kinematical states and $\mathcal{K}$ itself is known as the kinematical Hilbert space of the system. Contrary to their classical counterparts, all kinematical states are purely formal objects devoid of any physical meaning. Drawing on the classical analogy, one might think that some kinematical states represent actual quantum states of the system associated with the simultaneous sharp measurement of a maximal set of compatible dynamical variables. After all, given any $\psi \in \mathcal{K}$, it is always possible to find $N+1$ independent commuting self-adjoint operators $\widehat{V}_\alpha \in \mathcal{V}$ such that the joint spectral measure

$$d\mu(\psi) = \prod_{\alpha=1}^{N+1} d\mu(\widehat{V}_\alpha, \psi) \tag{1.59}$$

is arbitrarily concentrated around a point in the joint spectrum $\sigma = \sigma(\widehat{V}_1) \times \cdots \times \sigma(\widehat{V}_{N+1})$. And such a $d\mu(\psi)$ could very well represent the output of a simultaneous measurement of the commuting variables $\widehat{V}_\alpha$ and the associated uncertainty. The problem is that the state ψ associates a probability distribution $d\mu(\widehat{V}, \psi)$ to every self-adjoint operator $\widehat{V} \in \mathcal{V}$, and it is very difficult to make sense of it when $\widehat{V}$ is functionally independent of $\widehat{V}_\alpha$. In fact, after a measurement, there are no probability distributions, just more or less uncertain values which have been actually measured for the chosen quantities V_α. Probabilistic concepts only belong to the predictive aspect of a theory. The usual quantum states of Heisenberg QM, for example, encode our dynamical knowledge of the system and allow us to make probabilistic predictions for the values of all the observables we might decide to measure at any time t. After measuring some $O \in \mathcal{O}$ and getting, say, a definite value O_0, we obtain a new state which is an eigenvector of the corresponding operator $\widehat{O}$ with eigenvalue O_0. But this is not an actual state in the classical sense. It is just the updated summary of our dynamical information about the system, which again yields probabilistic predictions for the outcomes of all possible subsequent measurements. These genuine quantum states will be recovered below as the physical states of the system and retain their usual interpretation. On the other hand, kinematical states, as well as the spectrum of any essentially self-adjoint variable $\widehat{V} \in \mathcal{V}$, cannot be given any sensible physical meaning and must therefore be regarded as just instrumental for the definition of the quantum dynamics.

1.3.2 Physical states

§ Quantum physical states, much like their classical counterparts, completely encode our dynamical knowledge of the system and represent all its possible quantum motions. Of course, due to the intrinsic uncertainty of quantum phenomena, they can no more be associated with definite orbits in some phase space. Instead, they are identified with vectors in some physical Hilbert space. Like Heisenberg states in standard QM, such vectors contain full information about the whole dynamical history of the system, and are therefore the natural quantum analogues of classical motions. The reference to Heisenberg QM is not accidental. In fact, as we will see below, physical states actually reduce to Heisenberg states when recovering standard QM from CQM.

§ In CQM the physical states of the system are determined by a hamiltonian constraint $\widehat{H} = H(\widehat{P}_\alpha, \widehat{Q}^\alpha) \in \mathcal{V}$, like in the classical case. The operator $\widehat{H}$ is required to be essentially self-adjoint and its spectrum must contain the value zero. Intuitively, any physical state Ψ of the system is represented by a solution of the hamiltonian constraint $\widehat{H}$, i.e. by a (possibly generalized) function $\Psi(p_\alpha)$ such that

$$H(\widehat{P}_\alpha, \widehat{Q}^\alpha)\Psi(p_\alpha) = 0. \tag{1.60}$$

This condition is the natural quantum analogue of the classical vanishing of H on the accessible space $\mathcal{A}$ and, *a fortiori*, on all the orbits $\Psi \in \mathcal{P}_c$. However, in order for the solutions of (1.60), which form a complex vector space $\mathcal{P}'$, to represent genuine physical states, a positive definite scalar product must be defined on some $\mathcal{P} \subseteq \mathcal{P}'$ so as to obtain a Hilbert space. Such a Hilbert structure is necessary because, in its absence, it would be impossible to give linear operators defined on $\mathcal{P}$ any probabilistic interpretation. When zero is a proper eigenvalue of the operator $\widehat{H}$, the solution space $\mathcal{P}'$ is just the kernel of $\widehat{H}$, i.e. a subspace of the kinematical Hilbert space $\mathcal{K}$. This means that $\mathcal{P}'$ is itself a Hilbert space with respect to the kinematical scalar product (1.38), and we can directly identify it as the physical Hilbert space $\mathcal{P}$ of the system. When zero lies in the continuous spectrum of $\widehat{H}$, though, the solutions of (1.60) are not L^2 functions belonging to $\mathcal{D}(\widehat{H}) \subseteq \mathcal{K}$ anymore, but rather distributions, i.e. linear functionals defined on some dense, $\widehat{H}$-invariant subspace $\mathcal{F} \subset \mathcal{D}(\widehat{H})$ of sufficiently regular test functions. If $\langle \Phi, \varphi \rangle$ denotes the natural pairing between any distribution Φ and test function $\varphi \in \mathcal{F}$, the action of the hermitian operator $\widehat{H}$ on Φ is defined by setting

$$\langle \widehat{H}\Phi, \varphi \rangle = \langle \Phi, \widehat{H}\varphi \rangle \tag{1.61}$$

for every $\varphi \in \mathcal{F}$, and distributional solutions Ψ of (1.60) are identified by the condition

$$\langle \widehat{H}\Psi, \varphi \rangle = \langle \Psi, \widehat{H}\varphi \rangle = 0 \tag{1.62}$$

for every $\varphi \in \mathcal{F}$. In this case, which is the usual one, the solution space $\mathcal{P}'$ is not a subspace of $\mathcal{K}$ and does not inherit any natural Hilbert structure. Nevertheless, a positive definite scalar product on a suitably chosen subset $\mathcal{P}_0 \subseteq \mathcal{P}'$ can still be defined starting from the pairing between distributions and test functions. Let $\Psi_\mathbf{k}$ be a basis of the solution space $\mathcal{P}'$ and let us assume, as it is usually the case, that the generalized sum

$$P_0(\varphi) = \sum_\mathbf{k} \overline{\langle \Psi_\mathbf{k}, \varphi \rangle} \Psi_\mathbf{k} \tag{1.63}$$

is convergent for every $\varphi \in \mathcal{F}$. The antilinear operator $P_0 : \mathcal{F} \longrightarrow \mathcal{P}'$ can be interpreted as a sort of dual projector onto the improper kernel of $\widehat{H}$. In fact, if $\Psi_\mathbf{k}$ were dual to some actual null eigenvectors of $\widehat{H}$, i.e. if there were $\psi_\mathbf{k} \in \mathcal{K}$ such that

$$\langle \Psi_\mathbf{k}, \varphi \rangle = \langle \psi_\mathbf{k} | \varphi \rangle \tag{1.64}$$

for every $\varphi \in \mathcal{F}$, then the generalized sum (1.63) would be dual to the vector

$$\widehat{P}_0\varphi = \sum_{\mathbf{k}} \langle \psi_{\mathbf{k}} | \varphi \rangle \psi_{\mathbf{k}}, \tag{1.65}$$

and the operator P_0 would correspond via duality to the standard projector $\widehat{P}_0$ onto the proper kernel of $\widehat{H}$. With the help of P_0, we can define a degenerate, positive definite scalar product between any test functions $\varphi_1, \varphi_2 \in \mathcal{F}$, the so-called physical scalar product

$$\langle \varphi_1 | \varphi_2 \rangle_H = \langle P_0(\varphi_1), \varphi_2 \rangle = \sum_{\mathbf{k}} \overline{\langle \Psi_{\mathbf{k}}, \varphi_1 \rangle} \langle \Psi_{\mathbf{k}}, \varphi_2 \rangle. \tag{1.66}$$

Let now $\mathcal{P}_0$ denote the image of P_0, i.e. the set of all $\Psi \in \mathcal{P}'$ such that

$$\Psi = P_0(\varphi_\Psi) \tag{1.67}$$

for some $\varphi_\Psi \in \mathcal{F}$. It is easy to check that the equation

$$\langle \Psi_1 | \Psi_2 \rangle = \langle \Psi_1, \varphi_{\Psi_2} \rangle \tag{1.68}$$

defines a nondegenerate, positive definite scalar product on $\mathcal{P}_0$. In fact, since

$$\langle \Psi_1, \varphi_{\Psi_2} \rangle = \langle \varphi_{\Psi_1} | \varphi_{\Psi_2} \rangle_H = \overline{\langle \varphi_{\Psi_2} | \varphi_{\Psi_1} \rangle_H} = \overline{\langle \Psi_2, \varphi_{\Psi_1} \rangle}, \tag{1.69}$$

the product (1.68) is actually hermitian, positive-definite and independent of the particular choice of φ_{Ψ_1} or φ_{Ψ_2}. And it is also nondegenerate because we can have $\langle \Psi | \Psi' \rangle = 0$ for every $\Psi' \in \mathcal{P}_0$ only if $\langle \Psi, \varphi \rangle = 0$ for every $\varphi \in \mathcal{F}$, and this in turn implies that $\Psi = 0$. The completion $\mathcal{P}$ of the vector space $\mathcal{P}_0$ equipped with the scalar product (1.68) can then be identified as the physical Hilbert space of the system. The same Hilbert structure can be obtained starting from the test space $\mathcal{F}$ equipped with the physical scalar product (1.66) and identifying all the vectors $\varphi, \varphi' \in \mathcal{F}$ such that

$$\|\varphi - \varphi'\|_H^2 = \langle \varphi - \varphi' | \varphi - \varphi' \rangle_H = 0. \tag{1.70}$$

In this way, the product (1.66) becomes nondegenerate and the completion of $\mathcal{F}$ is made into a proper Hilbert space $\mathcal{P}$. A physical state Ψ of the system is now represented by an equivalence class of test functions under the relation (1.70). However, since the condition

$$\|\varphi - \varphi'\|_H^2 = 0 \tag{1.71}$$

can be rewritten as

$$P_0(\varphi) = P_0(\varphi'), \tag{1.72}$$

the equivalence class of any $\varphi \in \mathcal{F}$ is univocally determined by its image $P_0(\varphi)$ under P_0, and we get back the same physical Hilbert space as before.

§ Apart from the mathematical subtleties needed to deal with continuous spectra, it is clear from the above construction that the physical Hilbert space $\mathcal{P}$ of the system is ultimately the kernel of the hamiltonian constraint, even when zero is not a proper eigenvalue of $\widehat{H}$. However, while it is usually fair to neglect all the above technicalities and always treat $\mathcal{P}$ as if it were the actual kernel of $\widehat{H}$, there are two main complications which arise when zero lies in the continuous spectrum of $\widehat{H}$. First of all, the construction of $\mathcal{P}$ in this case generally depends on the choice of the test space $\mathcal{F} \subset \mathcal{D}(\widehat{H})$. That different choices can actually lead to different physical Hilbert spaces was explicitly demonstrated in [36], where the above method of constructing physical states and the associated ambiguities were rigorously discussed in a more general setting than CQM. This means that, in order to get a definite quantum model, we have to univocally specify the test space $\mathcal{F}$ in a physically motivated way. While this is a highly nontrivial issue in quantum gravity, where models are infinite-dimensional and there are multiple constraints, in the simplified context of CQM it is natural to identify $\mathcal{F}$ with the common domain $\mathcal{D}$ of the canonical coordinates. In fact, the space $\mathcal{D}$ of smooth functions with compact support is both sufficiently small for its dual to contain nontrivial solutions of the hamiltonian constraint and sufficiently large to be left invariant by the action of any smooth function of the canonical coordinates. As we will see below, the latter property guarantees the existence of a wealth of conserved operators acting on the physical Hilbert space. Moreover, most interesting hamiltonian constraints, including all the single-particle examples discussed in this thesis work, are themselves smooth functions of $\widehat{P}_\mu$ and $\widehat{Q}^\nu$, and this makes $\mathcal{D}$ automatically $\widehat{H}$-invariant, as required of the test space. For all these reasons, in the following we will always use $\mathcal{D}$ as the test space $\mathcal{F}$ for the definition of physical states. The second complication arising when zero lies in the continuous spectrum of $\widehat{H}$ is related to the actual implementation of the above construction. Although the physical scalar product can be abstractly defined via the generalized projector P_0, finding its actual expression for a generic hamiltonian constraint is usually very difficult. This is a serious obstacle to the practical application of the CQM framework to the analysis of a wide range of systems. A possible way of dealing with this problem is discussed below and successfully applied in Chapter 2 to work out the de Sitter model.

1.3.3 Conserved quantities and predictions

§ In the framework of CQM, the dynamical problem amounts to finding the expected value $\mathbf{E}_W[V|W_0;\Psi]$ of any physical quantity V when the value of another variable W is found to be W_0 and the system is in a given physical state $\Psi \in \mathcal{P}$. If W is a clock reading, the problem reduces to finding the ordinary quantum evolution in physical time, but the formalism, which puts all physical variables on the same footing, should be able to yield arbitrarily conditioned predictions such as $\mathbf{E}_W[V|W_0;\Psi]$, at least in principle. Like in the implicit formulation of CHM, the predictions of the theory are obtained from a $2N$-dimensional conserved *-algebra $\mathcal{C}$, which in the quantum case consists of densely defined operators on $\mathcal{P}$. However, both the determination of $\mathcal{C}$ and the identification of the evolving constants turn out to be considerably more problematic than in the classical context.

§ Intuitively, a variable $\widehat{C} \in \mathcal{V}$ is conserved, i.e. belongs to the conserved algebra

$\mathcal{C}$, when it is independent of $\widehat{H}$ and satisfies

$$[\widehat{H}, \widehat{C}] = 0, \tag{1.73}$$

which is the natural quantum analogue of the classical condition (1.24). Since $\widehat{H}\widehat{C}$ and $\widehat{C}\widehat{H}$ are generally defined on different domains, one should not regard (1.73) as an identity, but rather as the statement that $\widehat{H}$ and $\widehat{C}$ commute when applied to any vector $\psi \in \mathcal{D}([\widehat{H}, \widehat{C}]) = \mathcal{D}(\widehat{H}\widehat{C}) \cap \mathcal{D}(\widehat{C}\widehat{H})$. This condition, however, would be almost meaningless without some additional characterization of $\mathcal{D}([\widehat{H}, \widehat{C}])$. Since we are interested in the physical Hilbert space $\mathcal{P}$, which is the kernel of the hamiltonian constraint, it is natural to assume that $\mathcal{D}([\widehat{H}, \widehat{C}])$ contains all null eigenvectors of $\widehat{H}$. If zero is a proper eigenvalue of $\widehat{H}$, i.e. if $\mathcal{P}$ is the actual kernel of the hamiltonian constraint, the assumption makes sense and the resulting characterization of $\mathcal{C}$ as a subalgebra of $\mathcal{V}$ does not need any refinements. In this case, in fact, condition (1.73) implies that $\widehat{C}$ is well-defined on $\mathcal{P}$ and leaves it invariant. As a result, it is possible to regard $\mathcal{C}$ as a *-algebra of densely defined operators on the physical Hilbert space by just restricting any conserved operator $\widehat{C}$ to $\mathcal{P} \subset \mathcal{K}$. When zero belongs to the continuous spectrum of $\widehat{H}$, though, things get considerably more complicated. As discussed above, the physical Hilbert space is not a subset of $\mathcal{K}$ in this case, and a generic variable $\widehat{V} \in \mathcal{V}$, which is a well-defined operator on $\mathcal{D}(\widehat{V}) \subseteq \mathcal{K}$, may fail to be a well-defined operator on $\mathcal{P}$. Since conserved quantities must be represented by densely defined operators on $\mathcal{P}$, it is clear that condition (1.73) is no more sufficient to identify an element $\widehat{C}$ of the conserved algebra $\mathcal{C}$, but we also have to check that $\widehat{C}$ can be meaningfully regarded as an operator on $\mathcal{P}$. Moreover, in the absence of null eigenvectors of $\widehat{H}$, our assumption about $\mathcal{D}([\widehat{H}, \widehat{C}])$ does not make sense anymore and must be reformulated. Thankfully, both problems can be addressed and their solution leads to a refined definition of $\mathcal{C}$ which works even when zero is not an eigenvalue of $\widehat{H}$.

§ When looking for distributional solutions to the hamiltonian constraint, we defined the action of $\widehat{H}$ on any distribution Φ by setting

$$\langle \widehat{H}\Phi, \varphi \rangle = \langle \Phi, \widehat{H}\varphi \rangle \tag{1.74}$$

for every $\varphi \in \mathcal{D}$. The idea underlying this definition is that the complex number $\langle \Phi, \widehat{H}\varphi \rangle$, being linearly dependent on φ, identifies a unique distribution $\Phi' : \mathcal{D} \longrightarrow \mathbb{C}$ such that

$$\langle \Phi', \varphi \rangle = \langle \Phi, \widehat{H}\varphi \rangle \tag{1.75}$$

for every $\varphi \in \mathcal{D}$. When both Φ and Φ' happen to be dual to some test functions φ_1 and φ_2, i.e. when

$$\langle \Phi', \varphi \rangle = \langle \varphi_2 | \varphi \rangle = \langle \varphi_1 | \widehat{H}\varphi \rangle = \langle \Phi, \widehat{H}\varphi \rangle \tag{1.76}$$

for every $\varphi \in \mathcal{D}$, we have $\varphi_2 = \widehat{H}^\dagger \varphi_1$ by the definition (1.42) of the hermitian adjoint $\widehat{H}^\dagger$. It is then natural to regard (1.75) as defining a sort of distributional extension of $\widehat{H}^\dagger = \widehat{H}$, and write $\Phi' = \widehat{H}^\dagger \Phi = \widehat{H}\Phi$ even when Φ and Φ' are not dual to any test functions. In the same way, we can define the action of a generic variable $\widehat{V}$ and its adjoint $\widehat{V}^\dagger$ on any distribution Φ by setting

$$\begin{aligned} \langle \widehat{V}\Phi, \varphi\rangle &= \langle \Phi, \widehat{V}^\dagger \varphi\rangle, && (1.77)\\ \langle \widehat{V}^\dagger\Phi, \varphi\rangle &= \langle \Phi, \widehat{V} \varphi\rangle && (1.78) \end{aligned}$$

for every $\varphi \in \mathcal{D}$. In order for (1.77) and (1.78) to make sense, it is necessary and sufficient that $\widehat{V}\varphi$ and $\widehat{V}^\dagger\varphi$ are test functions for every $\varphi \in \mathcal{D}$, i.e. that both the original variable $\widehat{V}$ and its adjoint $\widehat{V}^\dagger$ are defined on $\mathcal{D}$ and leave it invariant. Any $\widehat{V} \in \mathcal{V}$ satisfying this condition is called a distributional variable, as it univocally identifies an operator acting on distributions. Let us now consider any distributional variable $\widehat{C} \in \mathcal{V}$ satisfying (1.73). Its adjoint $\widehat{C}^\dagger$ also satisfies (1.73) because

$$[\widehat{H}, \widehat{C}^\dagger] = [\widehat{C}, \widehat{H}]^\dagger = 0 \tag{1.79}$$

on the appropriate domain. Therefore, since $\mathcal{D} \subset \mathcal{D}([\widehat{H}, \widehat{C}]) \cap \mathcal{D}([\widehat{H}, \widehat{C}^\dagger])$, we have

$$\begin{aligned} \widehat{H}\widehat{C}\varphi &= \widehat{C}\widehat{H}\varphi, && (1.80)\\ \widehat{H}\widehat{C}^\dagger\varphi &= \widehat{C}^\dagger\widehat{H}\varphi && (1.81) \end{aligned}$$

for every $\varphi \in \mathcal{D}$. If $\Psi \in \mathcal{P}'$ is any distributional solution of the hamiltonian constraint, it is easy to verify that both $\widehat{C}\Psi$ and $\widehat{C}^\dagger\Psi$ are solutions too. In fact, if $\widehat{H}\Psi = 0$, we have

$$\begin{aligned} \langle \widehat{H}\widehat{C}\Psi, \varphi\rangle &= \langle \Psi, \widehat{C}^\dagger\widehat{H}\varphi\rangle = \langle \Psi, \widehat{H}\widehat{C}^\dagger\varphi\rangle = \langle \widehat{H}\Psi, \widehat{C}^\dagger\varphi\rangle = 0, && (1.82)\\ \langle \widehat{H}\widehat{C}^\dagger\Psi, \varphi\rangle &= \langle \Psi, \widehat{C}\widehat{H}\varphi\rangle = \langle \Psi, \widehat{H}\widehat{C}\varphi\rangle = \langle \widehat{H}\Psi, \widehat{C}\varphi\rangle = 0 && (1.83) \end{aligned}$$

for every $\varphi \in \mathcal{D}$, and this in turn implies that $\widehat{H}\widehat{C}\Psi = \widehat{H}\widehat{C}^\dagger\Psi = 0$. We can conclude that distributional variables satisfying condition (1.73) are univocally associated to well-defined linear operators on the solution space $\mathcal{P}'$. Unfortunately, this is not sufficient to identify the set of all such variables, which is obviosly a *-algebra, as the conserved algebra $\mathcal{C}$. In fact, while the action of $\widehat{C}$ on any physical state $\Psi \in \mathcal{P} \subseteq \mathcal{P}'$ yields another distributional solution $\widehat{C}\Psi$ of the hamiltonian constraint, nothing guarantees that $\widehat{C}\Psi$ is also a physical state. As a consequence, $\widehat{C}$ might still fail to be a well-defined operator on the physical Hilbert space $\mathcal{P}$. However, we can prove that $\widehat{C}$ actually maps sufficiently many physical states to other physical states by refining our previous assumption on $\mathcal{D}([\widehat{H}, \widehat{C}])$.

§ When zero lies in the continuous spectrum of the hamiltonian constraint, there are no null eigenvectors of $\widehat{H}$ and the projector $\widehat{P}_0$ onto the proper kernel of $\widehat{H}$ is the zero operator. As a result, our former straightforward assumption that $\widehat{P}_0\widehat{C} = \widehat{C}\widehat{P}_0$ on $\mathcal{D}(\widehat{C})$ is now empty. However, we can recover what essentially amounts to the same physical characterization of $\widehat{C}$ by employing a slightly more sofisticated formalism. As discussed above, the spectral theorem guarantees the existence of a spectral measure $\mu(\widehat{H}, \psi)$ with support $\sigma(\widehat{H})$ such that

$$\int_{\mathbb{R}} f(x)\, d\mu(\widehat{H}, \psi) = \int_{\sigma(\widehat{H})} f(x)\, d\mu(\widehat{H}, \psi) = \langle \psi | f(\widehat{H})\psi\rangle \tag{1.84}$$

for every sufficiently regular $f(\widehat{H}) \in \mathcal{V}$ and $\psi \in \mathcal{K}$. If we substitute the characteristic function χ_ρ of some measurable set $\rho \subseteq \mathbb{R}$ for f in relation (1.84), we implicitly define a self-adjoint projection operator $\widehat{P}_\rho = \chi_\rho(\widehat{H})$. The map $\Pi(\widehat{H})$ given by $\rho \mapsto \widehat{P}_\rho$ is the so-called projection-valued measure associated to $\widehat{H}$. It is univocally identified by the requirement that

$$\langle\psi|\widehat{P}_\rho\psi\rangle = \langle\psi|\left(\int_\rho d\Pi(\widehat{H})\right)\psi\rangle = \int_\rho d\langle\psi|\Pi(\widehat{H})\psi\rangle = \int_\rho d\mu(\widehat{H},\psi) \tag{1.85}$$

for every measurable $\rho \subseteq \mathbb{R}$ and $\psi \in \mathcal{K}$. In terms of $\Pi(\widehat{H})$, the spectral theorem states that

$$f(\widehat{H}) = \int_{\sigma(\widehat{H})} f(x)\, d\Pi(\widehat{H}) = \int_{\mathbb{R}} f(x)\, d\Pi(\widehat{H}) \tag{1.86}$$

for every measurable function f. Let now χ_a with $a \geq 0$ denote the characteristic function of the real interval $(-a, a] \subset \mathbb{R}$ and let us consider the projector

$$\widehat{P}_a = \int_{\mathbb{R}} \chi_a(x)\, d\Pi(\widehat{H}). \tag{1.87}$$

Since zero is not a proper eigenvalue of $\widehat{H}$, the set $\{0\} \subset \mathbb{R}$ has spectral measure zero and thus $\widehat{P}_0 = 0$, as we already observed above. However, since zero lies in the support $\sigma(\widehat{H})$ of any $\mu(\widehat{H},\psi)$, every interval $(-a, a]$ with $a > 0$ has positive spectral measure and the corresponding $\widehat{P}_a$ is not trivial. We can then regard the family of projectors $\widehat{P}_a$ with $a > 0$ as a continuous nontrivial approximation to $\widehat{P}_0$ and rewrite our previous assumption on $\mathcal{D}([\widehat{H},\widehat{C}])$ in terms of $\widehat{P}_a$. Let $\mathcal{P}_a \subset \mathcal{K}$ with $a > 0$ denote the images of the projectors $\widehat{P}_a$. Even if the actual kernel of $\widehat{H}$ is trivial, the subspaces $\mathcal{P}_a$ are not and, for small a, can be regarded as approximate kernels of $\widehat{H}$. In fact, if $\psi_a = \widehat{P}_a\psi$ with $\psi \in \mathcal{K}$ is any vector in $\mathcal{P}_a$, we have

$$\begin{aligned} \|\widehat{H}\psi_a\|^2 &= \langle\widehat{H}\widehat{P}_a\psi|\widehat{H}\widehat{P}_a\psi\rangle = \langle\psi|\widehat{H}^2\widehat{P}_a\psi\rangle = \\ &= \int_{\mathbb{R}} x^2\chi_a(x)\, d\mu(\widehat{H},\psi) \leq a^2 \int_{\mathbb{R}} \chi_a(x)\, d\mu(\widehat{H},\psi) = \\ &= a^2\langle\psi|\widehat{P}_a\psi\rangle = a^2\langle\widehat{P}_a\psi|\widehat{P}_a\psi\rangle = a^2\|\psi_a\|^2. \end{aligned} \tag{1.88}$$

And this in turn implies that $\|\widehat{H}\| \leq a$ on $\mathcal{P}_a$ or, equivalently, that we can make $\widehat{H}$ arbitrarily close to the zero operator on $\mathcal{P}_a$ by choosing a sufficiently small value for the parameter a. Therefore, it is natural to slightly strengthen our former assumption on $\mathcal{D}([\widehat{H},\widehat{C}])$ and require that $\mathcal{P}_c \subset \mathcal{D}([\widehat{H},\widehat{C}])$ for some $c > 0$. Together with this condition, equation (1.73) implies that

$$\widehat{P}_a\widehat{C} = \widehat{C}\widehat{P}_a \tag{1.89}$$

on $\mathcal{D}(\widehat{C})$ for every $a \leq c$. If $\widehat{P}_0$ is not trivial, (1.89) yields $\widehat{P}_0\widehat{C} = \widehat{C}\widehat{P}_0$ on $\mathcal{D}(\widehat{C})$ and we just recover our previous characterization of $\widehat{C}$. If $\widehat{P}_0$ is trivial, though, our new assumption on $\mathcal{D}([\widehat{H},\widehat{C}])$ translates into an analogous relation between $\widehat{C}$ and the dual projector P_0 onto the improper kernel of $\widehat{H}$. To see that this is the case, let $P_a(\varphi)$ with $a > 0$ denote the distribution dual to the vector $\widehat{P}_a\varphi \in \mathcal{K}$, i.e. let us define

$$\langle P_a(\varphi_1), \varphi_2\rangle = \langle\widehat{P}_a\varphi_1|\varphi_2\rangle \tag{1.90}$$

for every $\varphi_1, \varphi_2 \in \mathcal{D}$. The antilinear distribution-valued operators P_a are the dual projectors onto the spectral intervals $(-a, a]$ of $\widehat{H}$. Since $\widehat{C}$ is a distributional variable satisfying (1.89), we have

$$\begin{aligned} \langle P_a(\widehat{C}\varphi_1), \varphi_2\rangle &= \langle \widehat{P}_a\widehat{C}\varphi_1|\varphi_2\rangle = \langle \widehat{C}\widehat{P}_a\varphi_1|\varphi_2\rangle = \langle \widehat{P}_a\varphi_1|\widehat{C}^\dagger\varphi_2\rangle = \\ &= \langle P_a(\varphi_1), \widehat{C}^\dagger\varphi_2\rangle = \langle \widehat{C}P_a(\varphi_1), \varphi_2\rangle \end{aligned} \tag{1.91}$$

for every $\varphi_1, \varphi_2 \in \mathcal{D}$. Therefore, condition (1.89) yields

$$P_a\widehat{C} = \widehat{C}P_a, \tag{1.92}$$

where $P_a\widehat{C}$ and $\widehat{C}P_a$ denote the distribution-valued operators $\varphi \mapsto P_a(\widehat{C}\varphi)$ and $\varphi \mapsto \widehat{C}P_a(\varphi)$, respectively. Let us now consider the dual projector P_0 onto the improper kernel of $\widehat{H}$. It is intuitively clear that P_a should somehow converge to P_0 when $a \to 0$. However, we cannot prove this fact starting from our somewhat loose definition (1.63). Instead, we will use the family of dual projectors P_a to provide an alternative, rigorous definition of P_0, thereby clarifying the relation between P_a and P_0. Given any test functions φ_1 and φ_2, the limit

$$\begin{aligned} \lim_{a\to 0} \langle (2a)^{-1}P_a(\varphi_1), \varphi_2\rangle &= \lim_{a\to 0} (2a)^{-1}\langle \widehat{P}_a\varphi_1|\varphi_2\rangle = \\ &= \lim_{a\to 0} \int_{\mathbb{R}} (2a)^{-1}\chi_a(x)\, d\langle\varphi_1|\Pi(\widehat{H})\varphi_2\rangle = \\ &= \int_{\mathbb{R}} \delta(x)\, d\langle\varphi_1|\Pi(\widehat{H})\varphi_2\rangle \end{aligned} \tag{1.93}$$

exists whenever the complex measure $\langle\varphi_1|\Pi(\widehat{H})\varphi_2\rangle$ is absolutely continuous in some neighbourhood of $x = 0$, i.e. whenever zero belongs to the continuous spectrum of $\widehat{H}$. This means that

$$P_0(\varphi) = \lim_{a\to 0} (2a)^{-1}P_a(\varphi) \tag{1.94}$$

is a well-defined distribution for every $\varphi \in \mathcal{D}$. Moreover, since

$$\begin{aligned} \langle \widehat{H}P_0(\varphi_1), \varphi_2\rangle &= \lim_{a\to 0} (2a)^{-1}\langle \widehat{H}\widehat{P}_a\varphi_1|\varphi_2\rangle = \\ &= \lim_{a\to 0} \int_{\mathbb{R}} (2a)^{-1}x\,\chi_a(x)\, d\langle\varphi_1|\Pi(\widehat{H})\varphi_2\rangle = \\ &= \int_{\mathbb{R}} x\delta(x)\, d\langle\varphi_1|\Pi(\widehat{H})\varphi_2\rangle = 0 \end{aligned} \tag{1.95}$$

for every $\varphi_1, \varphi_2 \in \mathcal{D}$, $P_0(\varphi)$ is always a generalized solution of the hamiltonian constraint. It is then natural to regard the antilinear operator $P_0 : \mathcal{D} \longrightarrow \mathcal{P}'$ as the generalized projector onto the improper kernel of $\widehat{H}$ and to replace our former definition (1.63) with (1.94). Multiplying both sides of (1.92) by $(2a)^{-1}$ and taking the limit as $a \to 0$, we finally obtain

$$P_0\widehat{C} = \widehat{C}P_0, \tag{1.96}$$

which is the ultimate implication of (1.73) and our strengthened assumption on $\mathcal{D}([\widehat{H}, \widehat{C}])$.

§ Let us now return to the conserved algebra $\mathcal{C}$ and suppose that zero is not a proper eigenvalue of $\widehat{H}$. Under the assumption that $\mathcal{P}_c \subset \mathcal{D}([\widehat{H}, \widehat{C}])$ for some $c > 0$, we can finally prove that $\mathcal{C}$ can be actually identified with the *-algebra of all distributional variables satisfying (1.73). In fact, let $\widehat{C}$ be such a variable and let $\Psi = P_0(\varphi_\Psi)$ be any physical state in the image $\mathcal{P}_0$ of P_0. Since $\widehat{C}$ satisfies (1.96), we have

$$\widehat{C}\Psi = \widehat{C}P_0(\varphi_\Psi) = P_0(\widehat{C}\varphi_\Psi). \tag{1.97}$$

But this means that $\widehat{C}\Psi \in \mathcal{P}_0 \subseteq \mathcal{P}$ or, equivalently, that $\widehat{C}$ maps physical states in $\mathcal{P}_0$ to other physical states. Since $\mathcal{P}_0$ is dense in the physical Hilbert space $\mathcal{P}$, we can conclude that $\widehat{C}$ is at least densely defined on $\mathcal{P}$ and is therefore a conserved operator. Relation (1.96) is also crucial in providing the involution † in $\mathcal{C}$ with a concrete interpretation. Given any variable $\widehat{C} \in \mathcal{C}$, its hermitian adjoint $\widehat{C}^\dagger$ also belongs to the conserved algebra $\mathcal{C}$, by the very definition of distributional variables. The map $\widehat{C} \mapsto \widehat{C}^\dagger$ is therefore a well-defined involution on $\mathcal{C}$. However, it may very well have nothing to do with the Hilbert structure of $\mathcal{P}$. In particular, $\widehat{C}^\dagger$ may not be the adjoint of $\widehat{C}$ with respect to the scalar product (1.68) on the physical Hilbert space. Since there is no direct relation between the kinematical and physical scalar products, the fact that

$$\langle \psi' | \widehat{C}\psi \rangle = \langle \widehat{C}^\dagger \psi' | \psi \rangle \tag{1.98}$$

for every $\psi \in \mathcal{D}(\widehat{C})$ and $\psi' \in \mathcal{D}(\widehat{C}^\dagger)$ does not imply, by itself, that

$$\langle \Psi' | \widehat{C}\Psi \rangle = \langle \widehat{C}^\dagger \Psi' | \Psi \rangle \tag{1.99}$$

for every $\Psi, \Psi' \in \mathcal{P}_0$. Nevertheless, the last equation is a direct consequence of (1.96), because we can write

$$\langle \Psi' | \widehat{C}\Psi \rangle = \langle \Psi', \varphi_{\widehat{C}\Psi} \rangle = \langle \Psi', \widehat{C}\varphi_\Psi \rangle = \langle \widehat{C}^\dagger \Psi' | \Psi \rangle \tag{1.100}$$

for every $\Psi, \Psi' \in \mathcal{P}_0$. As a consequence, $\widehat{C}^\dagger$ is the hermitian adjoint of $\widehat{C}$ even if we forget about their original definition on $\mathcal{K}$ and regard both as just densely defined operators on $\mathcal{P}$. This seemingly technical result is actually very important from a physical point of view. In fact, while it would be exceedingly difficult to discuss the hermiticity of any operator $\widehat{C} \in \mathcal{C}$ directly on $\mathcal{P}$, it is an almost trivial matter to do the same on $\mathcal{K}$, where one has only to check whether $\widehat{C}$ is a symmetric function of the canonical coordinates.

§ Not every conserved operator $\widehat{C} \in \mathcal{C}$ represents an actually measurable conserved quantity, i.e. a constant of motion. For this to be the case, $\widehat{C}$ must yield the expected value of the corresponding physical quantity for every possible physical state in its dense domain of definition $\mathcal{P}_0 \subseteq \mathcal{P}$. As discussed above, this is possible only if $\widehat{C}$ is essentially self-adjoint on $\mathcal{P}_0$, its expected value on any $\Psi \in \mathcal{P}_0$ being given by

$$\mathbf{E}[C|\Psi] = \langle \Psi | \widehat{C}\Psi \rangle. \tag{1.101}$$

We have just seen that $\widehat{C}$ is hermitian as an operator on $\mathcal{P}$ whenever it is hermitian as an operator on $\mathcal{K}$, i.e. whenever $\widehat{C} = \widehat{C}^\dagger$ on the dense domain $\mathcal{D} \subset \mathcal{K}$. It is then natural to ask whether the same is true about essential self-adjointness. As briefly but convincingly argued in [36], the answer is in the affirmative. Under quite general assumptions, a conserved operator $\widehat{C}$ is essentially self-adjoint on $\mathcal{P}_0$ whenever it is essentially self-adjoint on $\mathcal{D}$. The reason is that the self-adjointness of any operator $\widehat{C} \in \mathcal{C}$ is equivalent to the existence of sufficiently many bounded real functions of $\widehat{C}$, such as its spectral projections $\chi_\rho(\widehat{C})$, on the appropriate domain. When $\widehat{C}$ is essentially self-adjoint on $\mathcal{D}$, all its bounded real functions are well-defined operators on $\mathcal{K}$. Let us now suppose that, as it is usually the case, a sufficient number of such operators leave the test space $\mathcal{D}$ invariant. Then, it is clear that the corresponding functions of $\widehat{C}$ also exist as well-defined operators on $\mathcal{P}$. Moreover, they are still real and bounded, because, as discussed above, the hermiticity of some conserved operator on $\mathcal{K}$ implies its hermiticity on $\mathcal{P}$, and the same is true for boundedness [36]. But this implies that, as claimed above, $\widehat{C}$ is essentially self-adjoint on $\mathcal{P}_0$. The requirement that constants of motion be represented by essentially self-adjoint operators breaks the analogy between CHM and CQM, which would otherwise be almost perfect. In fact, several conserved quantities which are well-defined in the classical regime cannot be recovered in the quantum case because the corresponding operators are not essentially self-adjoint on $\mathcal{D}$. This is not surprising. Even in ordinary QM, some perfectly acceptable classical observables do not translate into self-adjoint operators upon quantization and must therefore be discarded. However, in the case of CQM, this phenomenon has much deeper physical implications, which will be discussed at the end of this section.

§ The predictions of the theory are obtained from the conserved *-algebra $\mathcal{C}$ in the very same way as in the implicit formulation of CHM described above. Self-adjoint operators defined on $\mathcal{P}$, i.e. constants of motion, encode all the dynamical information about the corresponding conserved physical quantities and can be used to make probabilistic predictions about their values in the same way as in ordinary QM. Upon measuring some conserved quantity C, we can only obtain a value belonging to the spectrum of $\widehat{C}$. If C_0 is an (improper) eigenvalue of $\widehat{C}$ and Ψ_{C_0} is the associated (improper) eigenvector, the probability (density) $P(C_0)$ of actually getting the value C_0 for C when the system is in state $\Psi \in \mathcal{P}$ is given by

$$P(C_0) = |\langle \Psi_{C_0} | \Psi \rangle|^2. \tag{1.102}$$

After the measurement, the system is left in the physical state Ψ_{C_0}. The expected value $\mathbf{E}_C(\Psi)$ of C on state Ψ is given by

$$\mathbf{E}_C(\Psi) = \langle \Psi | \widehat{C} | \Psi \rangle. \tag{1.103}$$

This is actually the prediction $\mathbf{E}_{C|W}(W_0; \Psi)$, which is independent of the choice of the clock variable W and its value W_0 because C is conserved. A generic prediction $\mathbf{E}_{V|W}(W_0; \Psi)$, expressing the correlation between two independent non-conserved variables $V, W \in \mathcal{V}$, can be recovered from an appropriately chosen evolving constant, as in the classical case. Let us suppose to know $2N$ self-adjoint generators $\widehat{C}_l$ of the conserved algebra $\mathcal{C}$. If we could write $\widehat{V}$ as a function $V(\widehat{H}, \widehat{W}, \widehat{C}_l)$ and then substitute $W_0 \in \mathbb{R}$ for $\widehat{W}$ and zero for $\widehat{H}$, we would obtain a conserved quantity $\widetilde{V}(W_0; \widehat{C}_l) = V(0, W_0, \widehat{C}_l) \in \mathcal{C}$. Unfortunately, this straightforward recipe, which we successfully carried through in the classical case, does not even make sense in the quantum regime. In fact, since the operators $\widehat{H}$, $\widehat{W}$ and $\widehat{C}_l$ do not commute, one can usually write $\widehat{V}$ as a function of them in infinitely many different ways and the functional dependence $V(\widehat{H}, \widehat{W}, \widehat{C}_l)$ is not univocally defined to begin with. It is sufficient to take any such expression, rearrange it so as to make it display some nontrivial commutators and then evaluate them using the canonical commutation rules. As a result, the previous prescription actually yields infinitely many different conserved quantities like $V(0, W_0, \widehat{C}_l)$ and we need some additional conditions to univocally identify a specific function of W_0. In general, the problem is almost intractable, but we can find a unique prescription whenever $\widehat{W}$ is a canonical coordinate. Let $\widehat{W}_c$ be the variable canonically conjugate to $\widehat{W}$ and let $\widehat{P}_i$ and $\widehat{Q}^j$ denote the remaining $2N$ canonical coordinates. By eliminating $\widehat{W}_c$ with the help of the hamiltonian constraint, the conserved generators $\widehat{C}_l$ can be expressed as functions $C_l(\widehat{W}, \widehat{P}_k, \widehat{Q}^k)$ of $\widehat{W}$, $\widehat{P}_i$ and $\widehat{Q}^j$ alone. The evolving constant $\widetilde{V}(W_0; \widehat{C}_l)$ yielding the value of V when $W = W_0$ is then univocally identified by requiring that

$$\widetilde{V}(W_0; C_l(W_0, \widehat{P}_k, \widehat{Q}^k)) = \widehat{V}. \tag{1.104}$$

This implicit characterization of evolving constants avoids the aforementioned problems because the reduced conserved generators $C_l(\widehat{W}, \widehat{P}_k, \widehat{Q}^k)$ are by construction well-defined functions of $\widehat{W}$, which commutes with both $\widehat{P}_i$ and $\widehat{Q}^j$. As a result, the functional dependence of $\widetilde{V}$ on W_0 is unambiguously determined. In the following, we will always assume that the independent variable is one of the canonical coordinates and use (1.104) to identify the evolving constant which gives the values of V when $W = W_0$. The associated prediction will be then obviously given by

$$\mathbf{E}_W[V|W_0; \Psi] = \langle \Psi | \widetilde{V}(W_0; \widehat{C}_l) | \Psi \rangle. \tag{1.105}$$

1.3.4 Heisenberg quantum mechanics

§ The $2N$-dimensional *-algebra $\mathcal{C}$ and the physical Hilbert space $\mathcal{P}$ are just the canonical algebra $\mathcal{O}$ and the usual Hilbert space of ordinary Heisenberg QM. To see that this is actually the case, let us take the clock reading T as the first canonical coordinate Q^0 and let us suppose that the remaining N coordinates Q^j describe the configuration of a quantum system with N degrees of freedom. We will now show that the special hamiltonian constraint

$$H_{nr}(\widehat{P}_0, \widehat{P}_k, \widehat{T}, \widehat{Q}^k) = \widehat{P}_0 + H_0(\widehat{P}_k, \widehat{Q}^k) \tag{1.106}$$

determines the same dynamics with respect to the physical time $T = Q^0$ in CQM as the hamiltonian operator $H_0(\widehat{P}_k, \widehat{Q}^k)$ does with respect to the abstract time parameter t in ordinary Heisenberg QM. More specifically, we will prove that the Heisenberg operator $\widehat{O}(t)$ representing the time evolution of a generic observable $O(\widehat{P}_k, \widehat{Q}^k)$ can be identified with the evolving constant $\widetilde{O}(T_0; \widehat{C}_{P_k}, \widehat{C}_{Q^k})$ for $T_0 = t$, where the $2N$ independent constants $\widehat{C}_{P_i}$ and $\widehat{C}_{Q^j}$ satisfy

$$\widehat{C}_{P_i} = \widetilde{P}_i(0; \widehat{C}_{P_k}, \widehat{C}_{Q^k}), \tag{1.107}$$
$$\widehat{C}_{Q^j} = \widetilde{Q}^j(0; \widehat{C}_{P_k}, \widehat{C}_{Q^k}). \tag{1.108}$$

§ Let us suppose that $\widehat{O}(t) = \bar{O}(t; \widehat{P}_k, \widehat{Q}^k)$ obeys the Heisenberg evolution equation

$$i\hbar \frac{\partial \bar{O}}{\partial t}(t, \widehat{P}_k, \widehat{Q}^k) - [H_0(\widehat{P}_k, \widehat{Q}^k), \bar{O}(t, \widehat{P}_k, \widehat{Q}^k)] = 0, \tag{1.109}$$

and consider the variable

$$\widehat{C}_O = \bar{O}(-\widehat{T}, \widehat{P}_k, \widehat{Q}^k). \tag{1.110}$$

Since $\widehat{T}$ commutes with the canonical coordinates $\widehat{P}_i$ and $\widehat{Q}^j$, we have

$$[\widehat{H}_{nr}, \widehat{C}_O] = -i\hbar \frac{\partial \bar{O}}{\partial t}(-\widehat{T}, \widehat{P}_k, \widehat{Q}^k) + [H_0(\widehat{P}_k, \widehat{Q}^k), \bar{O}(-\widehat{T}, \widehat{P}_k, \widehat{Q}^k)] = 0, \tag{1.111}$$

and $\widehat{C}_O$ is a constant of motion. Therefore, the map $\widehat{O}_t \mapsto \widehat{C}_O$ associates a conserved quantity $\widehat{C}_O \in \mathcal{C}$ to any Heisenberg evolution $\widehat{O}(t)$. In particular, it identifies $2N$ independent generators $\widehat{C}_{P_k}$ and $\widehat{C}_{Q^k}$ of the conserved algebra $\mathcal{C}$ via

$$\begin{aligned} \widehat{C}_{P_i} &= \bar{P}_i(-\widehat{T}, \widehat{P}_k, \widehat{Q}^k), & (1.112)\\ \widehat{C}_{Q^j} &= \bar{Q}^j(-\widehat{T}, \widehat{P}_k, \widehat{Q}^k). & (1.113) \end{aligned}$$

Let us now define the parametrized variable

$$\widehat{V}_O(\lambda) = \bar{O}(\lambda; \bar{P}_i(-\lambda, \widehat{P}_k, \widehat{Q}^k), \bar{Q}^j(-\lambda, \widehat{P}_k, \widehat{Q}^k)) = \bar{O}(\lambda; \widehat{P}_i(-\lambda), \widehat{Q}^j(-\lambda)). \tag{1.114}$$

Deriving $\widehat{V}_O$ with respect to λ, we obtain

$$\begin{aligned} i\hbar \frac{d\widehat{V}_O}{d\lambda} &= i\hbar \frac{\partial \bar{O}}{\partial t}(\lambda, \widehat{P}_k, \widehat{Q}^k) - i\hbar \frac{\partial \bar{O}}{\partial P_k}(\lambda, \widehat{P}_k, \widehat{Q}^k) \star \frac{d\widehat{P}_k}{dt} - i\hbar \frac{\partial \bar{O}}{\partial Q^k}(\lambda, \widehat{P}_k, \widehat{Q}^k) \star \frac{d\widehat{Q}^k}{dt} = \\ &= [H_0(\widehat{P}_k, \widehat{Q}^k), C_O(\lambda, \widehat{P}_k, \widehat{Q}^k)] - \frac{\partial \bar{O}}{\partial P_k}(\lambda, \widehat{P}_k, \widehat{Q}^k) \star [H_0(\widehat{P}_k, \widehat{Q}^k), \widehat{P}_k] - \\ &\quad - \frac{\partial \bar{O}}{\partial Q^k}(\lambda, \widehat{P}_k, \widehat{Q}^k) \star [H_0(\widehat{P}_k, \widehat{Q}^k), \widehat{Q}^k] = 0, \end{aligned} \tag{1.115}$$

where for brevity we have omitted the argument of $\widehat{P}_i(-\lambda)$ and $\widehat{Q}^j(-\lambda)$ and we have symbolically summarized with a $\star$ the complicated noncommutative expansion of the variable $\widehat{V}_O(\lambda)$ in terms of $\widehat{P}_i(-\lambda)$ and $\widehat{Q}^j(-\lambda)$. Since $\widehat{V}_O(\lambda)$ is actually independent of λ, we have

$$\widehat{V}_O(\lambda) = \widehat{V}_O(0) = \bar{O}(0, \widehat{P}_k(0), \widehat{Q}^k(0)) = \bar{O}(0, \widehat{P}_k, \widehat{Q}^k) = \widehat{O}(0) = \widehat{O}, \tag{1.116}$$

and we can write

$$\widehat{O} = \bar{O}(\lambda; \bar{P}_i(-\lambda, \widehat{P}_k, \widehat{Q}^k), \bar{Q}^j(-\lambda, \widehat{P}_k, \widehat{Q}^k)). \tag{1.117}$$

This equation is still satisfied if we substitute the variable $\widehat{T}$, which commutes with $\widehat{P}_i$ and $\widehat{Q}^j$, for the parameter t:

$$\widehat{O} = \bar{O}(\widehat{T}; \bar{P}_k(-\widehat{T}, \widehat{P}_k, \widehat{Q}^k), \bar{Q}^k(-\widehat{T}, \widehat{P}_k, \widehat{Q}^k)) = \bar{O}(\widehat{T}; \widehat{C}_{P_k}, \widehat{C}_{Q^k}). \tag{1.118}$$

But this is just the expression of the variable $\widehat{O}$ as a function of $\widehat{T}$ and the $2N$ independent conserved quantities $\widehat{C}_{P_i}$ and $\widehat{C}_{Q^j}$. The evolving constant yielding the expected value of O when $T = T_0$ is thus given by

$$\widetilde{O}(T_0; \widehat{C}_{P_k}, \widehat{C}_{Q^k}) = \bar{O}(T_0; \widehat{C}_{P_k}, \widehat{C}_{Q^k}). \tag{1.119}$$

In particular, we have

$$\widetilde{P}_i(0; \widehat{C}_{P_k}, \widehat{C}_{Q^k}) = \bar{P}_i(0; \widehat{C}_{P_k}, \widehat{C}_{Q^k}) = \widehat{C}_{P_i}, \tag{1.120}$$
$$\widetilde{Q}^j(0; \widehat{C}_{P_k}, \widehat{C}_{Q^k}) = \bar{Q}^j(0; \widehat{C}_{P_k}, \widehat{C}_{Q^k}) = \widehat{C}_{Q^j}. \tag{1.121}$$

So far, we have proved that the function $\widetilde{O}$, encoding the relational evolution of any observable O with respect to the clock variable T, and the function $\bar{O}$, encoding the standard Heisenberg evolution of O in parameter time t, are the same, but there is more. For any physical state $\Psi \in \mathcal{P}$, the prediction $\mathbf{E}_{O|T}(T_0; \Psi)$, i.e. the expected value of O when $T = T_0$ and the system is in state Ψ, is given by

$$\mathbf{E}_{O|T}(T_0; \Psi) = \langle \Psi | \widetilde{O}(T_0; \widehat{C}_{P_k}, \widehat{C}_{Q^k}) | \Psi \rangle. \tag{1.122}$$

In Heisenberg QM, the same prediction, with the time parameter t instead of the clock variable T, is given by

$$\mathbf{E}_{O|t}(T_0; \Psi) = \langle \eta | \bar{O}(T_0; \widehat{P}_k, \widehat{Q}^k) | \eta \rangle_h, \tag{1.123}$$

where η is the Heisenberg state of the system and the subscript h denotes the Heisenberg scalar product. As both $(\widehat{C}_{P_i}, \widehat{C}_{Q^j})$ and $(\widehat{P}_i, \widehat{Q}^j)$ satisfy canonical commutation relations and $\widetilde{O} = \bar{O}$, the mapping $\widetilde{O}(T_0; \widehat{C}_{P_k}, \widehat{C}_{Q^k}) \mapsto \bar{O}(T_0; \widehat{P}_k, \widehat{Q}^k)$ is an isomorphism between the operator algebras $\mathcal{C}$ and $\mathcal{O}$. But this means that, for any physical state Ψ, we can find a Heisenberg state η_Ψ such that

$$\langle \eta_\Psi | \bar{O}(T_0; \widehat{P}_k, \widehat{Q}^k) | \eta_\Psi \rangle_H = \langle \Psi | \widetilde{O}(T_0; \widehat{C}_{P_k}, \widehat{C}_{Q^k}) | \Psi \rangle, \tag{1.124}$$

because both $\mathcal{C}$ and $\mathcal{O}$ have the same Schrödinger action on their respective Hilbert spaces. In other words, as claimed above, the Heisenberg algebra of observables $\mathcal{O}$ and the associated Hilbert space can be identified with the conserved algebra $\mathcal{C}$ and the physical Hilbert space $\mathcal{P}$.

§ We can conclude that standard Heisenberg QM is just a reformulation of CQM with a hamiltonian constraint of the form (1.106). It is worth to underline that the fundamental dynamical objects of CQM, i.e. $\mathcal{C}$ and $\mathcal{P}$, were already correctly identified in Heisenberg QM. The special rôle of time was entirely due to $\mathcal{O} = \mathcal{C}$ not being considered in its relation to the extended canonical algebra $\mathcal{V}$. In fact, let us suppose that we are directly given $\mathcal{C}$ and its representation on $\mathcal{P}$, without obtaining them from $\mathcal{V}$ and its representation on $\mathcal{K}$ via a hamiltonian constraint $\widehat{H}$. Then, there is no way in which we can extract any predictions from the model, because the relations between the conserved quantities and the variables, which let us identify any $\widehat{C} \in \mathcal{C}$ as an evolving constant encoding some dynamical correlation, are not known. Even if we are told that a set of generators $\widehat{C}_l$ are actually the evolving constants $\widetilde{V}_l(W_0;\widehat{C}_n)$ yielding the predictions for some variables V_l when another variable $W = W_0$, there is in general no way of identifying $\widetilde{V}_l(W_1;\widehat{C}_n)$ for $W_1 \neq W_0$ without knowing the function $\widetilde{V}$. But this function can only be obtained going back to $\mathcal{V}$ and writing V in terms of H, W and $\widehat{C}_l$ viewed as dynamical variables. The only exception is when the constraint is of the special form (1.106). In this case, the dynamics with respect to the singled-out variable $\widehat{T}$ is completely encoded into the conserved quantity $\widehat{H}_0$, which can be directly specified on $\mathcal{C}$. The evolution in $\widehat{T}$ is then recovered as a parametric evolution governed by the Heisenberg equation of motion: starting from $\widehat{C}_{P_i}$ and $\widehat{C}_{Q^j}$, already identified as $\widetilde{P}_i(0;\widehat{C}_n)$ and $\widetilde{Q}^j(0;\widehat{C}_n)$, all the evolving constants of the form $\widetilde{O}(T_0;\widehat{C}_n)$ are determined. However, in more general situations, there may be no clock independent of the dynamics of the rest of the system and we may be interested in more general predictions than $\mathbf{E}_{O|T}$. CQM, by illuminating the relation between $\mathcal{C}$ and $\mathcal{V}$ and identifying the hamiltonian constraint as the actual source of the generalized dynamics, is capable of dealing with such non-standard scenarios.

1.3.5 Conserved quantities and physical Hilbert space

§ In the previous sections we have illustrated the fundamental concepts of CQM in such a way as to highlight its similarity to CHM. This was not done just to streamline the presentation of the covariant quantum formalism. In fact, the close connection between CQM and CHM plays a crucial rôle in the actual characterization of the quantum dynamics. Both in CHM and in CQM, we can compute any prediction from the knowledge of $2N$ independent generators C_l of the conserved algebra $\mathcal{C}$. The dynamical problem is therefore reduced to the determination of a conserved basis C_l. In the classical case, Hamilton-Jacobi theory provides us with a systematic way of finding C_l as functions of P_μ and Q^ν, subject to the only condition that the H-J equation be Jacobi-integrable. In the quantum case, there is no Hamilton-Jacobi theory, but we can make up for it making good use of the classical analogy. Let us consider two covariant models, one classical and one quantum, with the same degrees of freedom and the same hamiltonian constraint $H(P_\alpha, Q^\alpha)$. If we associate any classical variable $V(P_\alpha, Q^\alpha) \in \mathcal{V}_c$ to its quantization $V(\widehat{P}_\alpha, \widehat{Q}^\alpha) \in \mathcal{V}$, the close formal analogy between CHM and CQM strongly suggests that classical conserved quantities $C_l(P_\alpha, Q^\alpha) \in \mathcal{C}_c$, which can be found solving the H-J equation for H, get mapped to conserved operators $\widehat{C}_l = C_l(\widehat{P}_\alpha, \widehat{Q}^\alpha) \in \mathcal{C}$, apart from minor ordering issues. This prescription does actually work and is the most straightforward way of explicitly obtaining the conserved operators $\widehat{C}_l$ in any CQM-based model. In Chapter 2, while revisiting the classical analogy, we provide a rigorous proof of the validity of this method and successfully apply it to the identification of the conserved generators in the de Sitter model.

§ With this last observation, we are now ready to sketch the quantum version of the recipe we gave above for characterizing a mechanical system in CHM. First of all, we define the system specifying a $2N+2$-dimensional extended canonical algebra $\mathcal{V}$ and a hamiltonian constraint $H \in \mathcal{V}$. The Schrödinger representation of this algebra univocally identifies a kinematical Hilbert space $\mathcal{K}$ on which any physical variable

V acts as a hermitian operator $\widehat{V}$. Secondly, we solve the classical H-J equation associated with H and invert equations (1.29)-(1.30) to find the $2N$ independent conserved functions $C_l(P_\alpha, Q^\alpha)$ and the corresponding generators $\widehat{C}_l = C_l(\widehat{P}_\alpha, \widehat{Q}^\alpha)$ of the quantum conserved algebra $\mathcal{C}$. Finally, we use condition (1.104) to identify the evolving constants $\widetilde{V}(W_0; \widehat{C}_l)$ encoding the dynamical correlation between any variables of interest V and W. In order to complete the characterization of the system, it is now sufficient to find the physical Hilbert space $\mathcal{P}$ and turn evolving constants into predictions via equation (1.105).

§ This last step, as we already remarked above, is the most critical one. In fact, the definition of the physical scalar product via the generalized projector P_0 onto the improper kernel of $\widehat{H}$, while theoretically satisfying, results almost always exceedingly difficult in practice. First of all, in order to determine P_0, we have to explicitly find all the distributional solutions of the eigenvalue equation

$$H(\widehat{P}_\alpha, \widehat{Q}^\alpha)\Psi(p_\alpha) = 0, \tag{1.125}$$

which is a formidable task by itself for generic constraints. But even if we know a N-dimensional complete set of generalized null eigenvectors $\Psi_{\mathbf{k}}(p_\alpha)$ of $\widehat{H}$, the actual computation of the physical scalar product

$$\begin{aligned} \langle \varphi_1 | \varphi_2 \rangle_H &= \sum_{\mathbf{k}} \overline{\langle \Psi_{\mathbf{k}}, \varphi_1 \rangle} \langle \Psi_{\mathbf{k}}, \varphi_2 \rangle = \\ &= \int d^N k \int \overline{\varphi_1(p_\alpha)} \Psi_{\mathbf{k}}(p_\alpha) \overline{\Psi_{\mathbf{k}}(q_\alpha)} \varphi_2(q_\alpha)\, d^{N+1}p\, d^{N+1}q \end{aligned} \tag{1.126}$$

for generic $\varphi_1, \varphi_2 \in \mathcal{D}$ is still very difficult, because it involves defining a generalized Fourier transform and computing all its coefficients. Another possibility is to write symbolically

$$\begin{aligned} \langle \varphi_1 | \varphi_2 \rangle_H &= \int \overline{\varphi_1(p_\alpha)} \delta(H(p_\alpha, -i\hbar\partial^\alpha)) \varphi_2(p_\alpha) d^{N+1}p = \\ &= \int_{-\infty}^{\infty} d\tau \int \overline{\varphi_1(p_\alpha)} e^{i\tau H(p_\alpha, -i\hbar\partial^\alpha)} \varphi_2(p_\alpha) d^{N+1}p, \end{aligned} \tag{1.127}$$

and to develop techniques to directly evaluate this expression. This is the strategy advocated in [6], but it does not make explicitly carrying out the actual computations for arbitrary constraints any simpler. All these difficulties are the finite-dimensional counterparts of the infamous Hilbert space problem [41], which has systematically hampered every attempt at the canonical quantization of general relativity. Thankfully, in our finite dimensional context, there is a way of bypassing the obstacle altogether.

§ In CHM, as explained above, there are two distinct characterizations of any physical state $\Psi \in \mathcal{P}_c$. It can either be obtained as an orbit induced by the action of the hamiltonian flow α_λ or be identified by the values $C_l[\Psi]$ of the $2N$ generators of the conserved algebra $\mathcal{C}_c$. In the first case we are viewing Ψ as an object defined on the kinematical phase space $\mathcal{K}_c$ by means of some additional structure, specifically the diffeomorphism group α_λ, whereas in the second case Ψ is directly given at the dynamical level, with hardly any reference to kinematical concepts. The possible values of the constants of motion C_l are obviously determined regarding them as dynamical variables, i.e. functions on $\mathcal{K}_c$, but this is the only way in which kinematics affects the characterization of Ψ in terms of $C_l[\Psi]$. The definition of quantum physical states in terms of a modified scalar product on the kinematical Hilbert space $\mathcal{K}$ is clearly reminiscent of the first classical characterization of $\mathcal{P}_c$. In order to avoid the problems associated with the physical scalar product, we should somehow do away with $\mathcal{K}$ and directly obtain $\mathcal{P}$ from the conserved algebra $\mathcal{C}$, as we did with classical physical states. This is actually possible whenever the algebra $\mathcal{C}$ admits a unique irreducible representation. If it is a $2N$-dimensional canonical algebra, for example, its only irreducible representation is the usual Schrödinger representation and $\mathcal{P}$ is some $L^2(\mathbb{R}^N, d^N x)$. This is precisely what happens in standard Heisenberg QM. In the previous section, we have remarked that Heisenberg QM is a reformulation of some special CQM-based models in which the algebra of observables $\mathcal{O}$, i.e. the conserved algebra $\mathcal{C}$, is considered by itself rather than in its relation with the extended canonical algebra $\mathcal{V}$. In particular, the usual Heisenberg states, which are just the physical states, are not obtained through a modification of the kinematical scalar product, but directly as vectors in the Hilbert space representation of $\mathcal{O}$, which is unique because $\mathcal{O}$ is a canonical algebra generated by $\widehat{C}_{P_i}$ and $\widehat{C}_{Q^j}$. This example suggests a strategy for finding the physical Hilbert space in the general case: if we can find some basis $\widehat{C}_l$ of $\mathcal{C}$ which satisfies canonical commutation relations, the physical Hilbert space can be determined forgetting about the kinematical space. While this may be not always the case, it is true in many interesting examples. In the next chapter, the method will be successfully employed to bypass the problems associated with the modified scalar product in the de Sitter model.

1.3.6 Symmetries

§ In standard classical and quantum mechanics, the symmetries of a mechanical system play an important rôle in its dynamical characterization. The same is true in CQM, where symmetry transformations are handled in almost the same way as in standard QM. Any algebraic symmetry of the system is described by an (anti)automorphism σ of the extended canonical *-algebra $\mathcal{V}$ which maps the hamiltonian constraint H to itself. Such a σ is called a Wigner symmetry on the kinematical Hilbert space $\mathcal{K}$ if it can be unitarily implemented on $\mathcal{K}$, i.e. if it is possible to find a (anti)unitary operator $\widehat{U}_\sigma$ on $\mathcal{K}$ such that

$$\sigma(\widehat{V}) = \widehat{U}_\sigma \widehat{V} \widehat{U}_\sigma^\dagger \tag{1.128}$$

for every $\widehat{V} \in \mathcal{V}$. If this is not the case, the symmetry σ is said to be spontaneously broken on $\mathcal{K}$. Whenever the kinematical Hilbert space is unique, i.e. $\mathcal{V}$ admits only one irreducible representation, it is easy to prove that all algebraic symmetries are also Wigner symmetries and there can be no spontaneous symmetry breaking [42]. This is precisely what happens in CQM when the configuration space of the system is flat, because the only irreducible representation of $\mathcal{V}$ is then the Schrödinger representation. In Chapter 2, where we deal with the extension of CQM to generic configuration manifolds, we will also discuss symmetries in the general case. For now, as we did above, we will just assume that the configuration manifold is flat, so that all mechanical symmetries of the system are represented by (anti)unitary operators $\widehat{U}_\sigma$ acting as in (1.128).

§ The physical motivations behind the above definitions are easily explained. Let us suppose that σ is an algebraic symmetry and let $V' = \sigma(V)$ for any $V \in \mathcal{V}$. Since $H' = H$ and σ is an (anti)automorphism, we have

$$[\widehat{H}, \widehat{V}'] = [\widehat{H}', \widehat{V}'] = [\widehat{H}, \widehat{V}]' \tag{1.129}$$

for every $V \in \mathcal{V}$. As a result, a distributional variable $C \in \mathcal{V}$ is a constant of motion, i.e.

$$[\widehat{H}, \widehat{C}] = 0, \tag{1.130}$$

if and only if C' is a constant too. This means that an algebraic symmetry σ is automatically an (anti)automorphism of the conserved algebra $\mathcal{C}$. Moreover, let us consider any pair of independent, nonconstant, physical variables $V, W \in \mathcal{V}$. Acting with σ on the associated evolving constant $\widetilde{V}(W_0; \widehat{C}_l)$, we obtain

$$\sigma(\widetilde{V}(W_0; \widehat{C}_l)) = \widetilde{V}(W_0; \sigma(\widehat{C}_l)) = \widetilde{V}(W_0; C_l(\sigma(\widehat{W}), \sigma(\widehat{P}_i), \sigma(\widehat{Q}^j))). \tag{1.131}$$

If we now substitute W_0 for $\sigma(\widehat{W})$ in the above expression, we find

$$\widetilde{V}(W_0; C_l(W_0, \sigma(\widehat{P}_i), \sigma(\widehat{Q}^j))) = \sigma(\widetilde{V}(W_0; C_l(W_0, \widehat{P}_i, \widehat{Q}^j)) = \sigma(\widehat{V}), \tag{1.132}$$

where in the last step we used the defining property (1.104) of $\widetilde{V}(W_0; \widehat{C}_l)$. But this is precisely the condition identifying the evolving constant $\widetilde{V}'(W_0'; \widehat{C}_l)$ which encodes the correlation between the variables $V' = \sigma(V)$ and $W' = \sigma(W)$. We can conclude that

$$\widetilde{V}'(W_0'; C_l) = \sigma(\widetilde{V}(W_0'; C_l)). \tag{1.133}$$

From a physical point of view, (1.133) states that if we first perform the transformation σ on the extended canonical algebra $\mathcal{V}$ and then find the evolution of V' with respect to W', we obtain the same result as if we first find the evolution of V with respect to W and then perform the transformation σ on the resulting evolving constant $\widetilde{V}(W_0'; C_l)$. This is actually the usual physical definition of a dynamical symmetry, just so generalized as to meaningfully apply in a relational context. In fact, if we replaced the variable W with an invariant time parameter t, equation (1.133) would just mean that the evolution $V'(t)$ of the transformed variable V' can be obtained transforming the time evolution $V(t)$ of the original variable, i.e. that $V'(t) = \sigma(V(t))$. We can conclude that an algebraic symmetry σ of the system is a transformation which preserves the functional form of its relational dynamics. If σ is a Wigner symmetry implemented by some (anti)unitary operator $\widehat{U}_\sigma$, it also preserves the numerical values of all the predictions $\mathbf{E}_{V|W}(W_0; \Psi)$ of the model. In fact, since

$$\widehat{H}' = \widehat{U}_\sigma \widehat{H} \widehat{U}_\sigma^\dagger = \widehat{H}, \tag{1.134}$$

the operator $\widehat{U}_\sigma$ commutes with the hamiltonian constraint and is therefore a well-defined (anti)unitary operator on the physical Hilbert space $\mathcal{P}$. Let us then define $\Psi' = \widehat{U}_\sigma \Psi$ for every physical state $\Psi \in \mathcal{P}$. If we compute the prediction $\mathbf{E}_{V'|W'}(W_0'; \Psi')$, using (1.133) we find

$$\begin{aligned}\mathbf{E}_{V'|W'}(W_0'; \Psi') &= \langle \Psi' | \widetilde{V}'(W_0'; C_l') | \Psi' \rangle = \langle \Psi' | \sigma(\widetilde{V}(W_0'; C_l)) | \Psi' \rangle = \\ &= \langle \Psi | \widehat{U}_\sigma^\dagger \widehat{U}_\sigma \widetilde{V}(W_0'; C_l) \widehat{U}_\sigma^\dagger \widehat{U}_\sigma | \Psi \rangle = \langle \Psi | \widetilde{V}(W_0'; C_l) | \Psi \rangle = \\ &= \mathbf{E}_{V|W}(W_0'; \Psi). \end{aligned} \tag{1.135}$$

In other words, given any $\Psi \in \mathcal{P}$, we can always find another physical state Ψ' such that the expected values of the transformed variable V' when $W' = W_0'$ and the system is in state Ψ' are equal to the expected values of the original variable V when $W = W_0'$ and the system is in the original state Ψ. Since predictions encode the possible outcomes and probabilities of all the measurements we can perform on the system, this means that there is no experimental way of distinguishing the transformed system from the original one, i.e. that σ is a fully-fledged physical symmetry of the model.

§ Any continuous group $\mathcal{G}$ of symmetry transformations of the system is represented by a continuous group of (anti)automorphisms of the extended canonical algebra $\mathcal{V}$ leaving invariant the hamiltonian constraint H. Given any $g, g', gg' \in \mathcal{G}$, the corresponding (anti)automorphisms σ_g, $\sigma_{g'}$ and $\sigma_{gg'}$ satisfy the honomorphism relation

$$\sigma_{gg'} = \sigma_g \circ \sigma_{g'}. \tag{1.136}$$

In particular, if e is the identity element in $\mathcal{G}$, we have $\sigma_e = \mathrm{id}$. If the group $\mathcal{G}$ is connected, then it is possible to continuously deform any σ_g into the identity $\sigma_e = \mathrm{id}$, which is trivially an automorphism. In this case, for every $g \in \mathcal{G}$, σ_g must be a linear automorphism too. Let us now assume for simplicity that $\mathcal{G}$ is connected and suppose that σ_g is a Wigner symmetry on $\mathcal{K}$ for every $g \in \mathcal{G}$. Then, if $\widehat{U}_g$ is the unitary operator implementing σ_g, we must have

$$\widehat{U}_{gg'} = e^{ic(g,g')} \widehat{U}_g \widehat{U}_{g'} \tag{1.137}$$

for some phase $c(g, g') \in \mathbb{R}$ satisfying

$$c(g, g') + c(gg', g'') = c(g', g'') + c(g, g'g'') \tag{1.138}$$

and for every $g, g', g'' \in \mathcal{G}$. In fact, it is easy to check that $\widehat{U}_g$ satisfying (1.137) and (1.138) is equivalent to σ_g obeying the homomorphism relation (1.136). In most physically interesting cases, such as the Poincaré group $\mathcal{G}_M$, it is possible to redefine the operators $\widehat{U}_g$, which are only determined up to a phase by equation (1.128), so as to remove the factor $e^{ic(g,g')}$ in (1.137) and simply obtain

$$\widehat{U}_{gg'} = \widehat{U}_g \widehat{U}_{g'}. \tag{1.139}$$

Technically, this means that the operators $\widehat{U}_g$ form a unitary representation of the symmetry group $\mathcal{G}$. In the following, for notational convenience, we will systematically identify the abstract group $\mathcal{G}$ with its realizations in terms of automorphisms and unitary operators, and write $\sigma_g \in \mathcal{G}$ and $\widehat{U}_g \in \mathcal{G}$.

§ If the unitary group $\mathcal{G}$ is an M-dimensional Lie group, then by Stone's theorem [39] there exist M independent self-adjoint operators $\widehat{G}_r = \widehat{G}_r^\dagger$ such that, for every $\widehat{U}_g \in \mathcal{G}$,

$$\widehat{U}_g = e^{i\kappa^r(g)\widehat{G}_r} = \widehat{U}(\kappa^u) \tag{1.140}$$

for some $\kappa^r(g) \in \mathbb{R}$. The operators $\widehat{G}_r$ are called the generators of the symmetry group $\mathcal{G}$ and completely characterize its local structure. The vector space spanned by the $\widehat{G}_r$ equipped with the commutator is a representation of the group's Lie algebra. In particular, the generators satisfy

$$[\widehat{G}_r, \widehat{G}_s] = i{g_{rs}}^u \widehat{G}_u, \tag{1.141}$$

where the real parameters ${g_{rs}}^u$ are the structure constants of the group. Since every $\widehat{U}_g \in \mathcal{G}$ commutes with the hamiltonian constraint H, all the generators $\widehat{G}_r$ commute with H too. As a result, they are self-adjoint elements of the conserved algebra $\mathcal{C}$, i.e. measurable constants of motion. Since the dynamics of the model is completely determined once we know sufficiently many independent conserved quantities, the analysis of the symmetries of the system provides us with crucial dynamical information. Vice versa, any self-adjoint conserved quantity $\widehat{C} \in \mathcal{C}$ is associated to the 1-parameter unitary group

$$\widehat{U}(\kappa) = e^{i\kappa\widehat{C}}, \tag{1.142}$$

whose action obviously leaves H invariant and thus defines a continuous group of Wigner symmetries of the system. We can conclude that, in CQM, the study of the dynamics of a mechanical system and the characterization of its symmetries are actually the same thing.

§ When the constraint is of the particular form (1.18), we recover the usual notions of symmetry of standard Heisenberg QM. Let us suppose that ρ is an (anti)automorphism of the extended canonical algebra $\mathcal{V}$ which leaves invariant P_0 and the subalgebra $\mathcal{T} \subset \mathcal{V}$ generated by the time variable $Q^0 = T$. It is easy to verify that the most general action of ρ on T compatible with the previous requirements is given by $\rho(T) = \pm T - t$ for some $t \in \mathbb{R}$. Such a transformation ρ represents a dynamical symmetry if and only if it leaves invariant the standard hamiltonian $H_0(P_k, Q^k)$. In this case, ρ is a well-defined (anti)automorphism also on the conserved algebra $\mathcal{C}$, i.e. the Heisenberg algebra of observables, and from (1.133) we find

$$\rho(P_i)|_{\rho(T)=0} \;=\; \rho(P_i)(\pm t) \;=\; \rho(P_i(0)), \tag{1.143}$$

$$\rho(Q^j)|_{\rho(T)=0} \;=\; \rho(Q^j)(\pm t) \;=\; \rho(Q^j(0)), \tag{1.144}$$

for all the canonical variables P_i and Q^j. This means that a symmetry like ρ can be directly defined, without reference to $\mathcal{V}$, as an (anti)automorphism of the algebra of observables $\mathcal{C}$ which leaves invariant the ordinary hamiltonian H_0, because its action on $\mathcal{V}$ can then be univocally recovered by (1.143)-(1.144). But this is precisely the usual definition of an algebraic symmetry in Heisenberg QM. We also automatically recover the well-known result that a Heisenberg symmetry ρ is an antiautomorphism if and only if it reverses the sign of time. In fact, when viewed as a transformation of the extended canonical algebra $\mathcal{V}$, ρ must preserve the canonical commutator

$$[P_0, T] = i\hbar \tag{1.145}$$

while leaving P_0 invariant, so that we can have $\rho(i) = -i$ if and only if $\rho(T) = -T - t$. If ρ is a Wigner symmetry, then the corresponding (anti)unitary operator $\widehat{U}_\rho$ is well-defined on the algebra of observables $\mathcal{C}$ and commutes with $\widehat{H}_0$, so that it is a Wigner symmetry also in the sense of ordinary QM. Given a (connected) continuous group of Wigner symmetries ρ_g, the corresponding self-adjoint generators commute with the standard hamiltonian operator $\widehat{H}_0$ and determine the degeneracies of the energy spectrum of the system. Vice versa, any self-adjoint observable $\widehat{C}$ which commutes with $\widehat{H}_0$ generates a 1-parameter group of ordinary Wigner symmetries via (1.142). In particular, $\widehat{H}_0$ itself generates the group of time translations $T \mapsto T - t$.

§ It is worth to explicitly point out that, when the constraint is of the form (1.18), covariant quantum symmetries do not simply reduce to ordinary quantum symmetries. Instead, the latter are recovered as special covariant symmetry transformations which act trivially on the clock time and its conjugate momentum. This explains why the rôle of symmetries, expecially continuous ones, is so different in the two frameworks. In CQM, as discussed above, any system always features a wealth of continuous symmetry transformations, and finding a complete set of independent self-adjoint generators is the very solution to the generalized dynamical problem. In ordinary QM, on the contrary, a system can feature a lot of symmetry transformations, a few ones or just time translations. The dynamics of any time-dependent observable $\widehat{V} \in \mathcal{C}$ is always directly given by the latter, i.e. by

$$\widehat{V}(t) = e^{-i\widehat{H}_0 t}\widehat{V}(0)e^{i\widehat{H}_0 t}, \tag{1.146}$$

and the analysis of other symmetries, when they exist, is just useful for finding the spectrum of $\widehat{H}_0$. Once again, the difference is due to ordinary QM being directly defined in terms of the algebra of observables $\mathcal{C}$, without considering it in its relation with the extended canonical algebra $\mathcal{V}$. When viewed as elements of $\mathcal{V}$, all observables in $\mathcal{C}$ generate unitary automorphisms of $\mathcal{V}$ preserving the hamiltonian constraint $\widehat{H}$. However, if one ignores the embedding of $\mathcal{C}$ in $\mathcal{V}$, such transformations just appear as automorphisms of $\mathcal{C}$, most of which map the evolution $\widehat{O}(t)$ of any observable $\widehat{O} \in \mathcal{C}$ with respect to clock time $\widehat{T}$ to the evolution of a transformed variable $\widehat{O}'$ with respect to some $\widehat{T}' \neq \widehat{T}$. Since Heisenberg QM can only deal with evolution in $\widehat{T}$, the only distinguished unitary automorphisms of $\mathcal{C}$ are those which leave invariant the ordinary hamiltonian $\widehat{H}_0$, because they map time evolutions to time evolutions. By shifting the focus from $\mathcal{C}$ to $\mathcal{V}$, CQM elucidates the deep dynamical meaning of all the others, thereby providing an even stronger connection between symmetries and dynamics.

1.4 Free quantum particle on Minkowski spacetime

§ In order to better illustrate the rather abstract points discussed in the previous sections, we will now apply CQM to the simplest system which cannot be described in terms of a constraint of the special form (1.106): a free relativistic spinless particle. The quantum dynamics of this system is well-known from the study of the free Klein-Gordon wave equation and the associated quantum field theory, but it cannot be satisfactorily dealt with in the context of ordinary QM. Therefore, it is the perfect example to demonstrate the consistency and the flexibility of the CQM framework, as well as concretely illustrate its features. Furthermore, in the next chapters we will be concerned with free quantum particles propagating on curved or noncommutative spacetime manifolds. Since the CQM-based treatment of those systems yields a natural generalization of the covariant quantum model of an ordinary relativistic particle, a detailed study of the latter will also serve as a template for the subsequent analysis of more exotic scenarios.

1.4.1 Specification of the model

§ The eight-dimensional extended canonical *-algebra $\mathcal{V}_M$ encoding the kinematics of a particle on a Minkowski background is generated by the spacetime coordinates X^ν of the particle and the corresponding conjugate momenta P_μ. Since Minkowski spacetime is flat, we can follow the quantization scheme presented above and require that P_μ and X^ν satisfy the canonical commutation relations

$$[P_\mu, P_\tau] = 0, \tag{1.147}$$

$$[P_\mu, X^\nu] = i\hbar\delta_\mu{}^\nu, \tag{1.148}$$

$$\left[X^\nu, X^\lambda\right] = 0, \tag{1.149}$$

and are hrrmitian:

$$P_\mu = (P_\mu)^*, \tag{1.150}$$

$$X^\nu = (X^\nu)^*. \tag{1.151}$$

If $\eta^{\mu\nu} = \mathrm{diag}\{1, -1, -1, -1\}$ denotes the Minkowski metric, the hermitian variables X^ν and $P^\mu = \eta^{\mu\alpha} P_\alpha$ physically represent measurements of the spacetime position and the 4-momentum of the particle with respect to some inertial reference frame. The extended canonical *-algebra $\mathcal{V}_M$ admits a unique irreducible representation in terms of linear operators on the space $\mathcal{K} = L^2(\mathbb{R}^4, d^4p)$ of square-integrable functions on $\mathbb{R}^4$, equipped with the scalar product

$$\langle\psi_1|\psi_2\rangle = \int \overline{\psi}(p_\alpha)\phi(p_\alpha)d^4p. \tag{1.152}$$

The action of the canonical coordinates P_μ and X^ν on any wavefunction $\psi \in \mathcal{D}_M$ is given by

$$\widehat{P}_\mu\psi(p_\alpha) = p_\mu\psi(p_\alpha), \tag{1.153}$$

$$\widehat{X}^\nu\psi(p_\alpha) = -i\hbar\frac{\partial\psi(p_\alpha)}{\partial p_\nu} = -i\hbar\partial^\nu\psi(p_\alpha). \tag{1.154}$$

The dynamics of the model is specified by requiring that the particle obeys the relativistic dispersion relation. This condition is enforced by the relativistic hamiltonian constraint

$$H_r(\widehat{P}_\alpha, \widehat{X}^\alpha) = \widehat{P}^\alpha \widehat{P}_\alpha - m^2c^2, \tag{1.155}$$

where m is the rest mass of the particle.

1.4.2 Conserved quantities

§ The conserved algebra $\mathcal{C}_M$ is identified requiring that any $\widehat{C} \in \mathcal{C}_M$ is independent of $\widehat{H}_r$ and satisfies

$$[\widehat{H}_r, \widehat{C}] = 0. \tag{1.156}$$

As discussed in Section 1.3.5, we can obtain six independent generators solving the classical H-J equation associated to the constraint (1.155), which reads

$$H_r\left(\frac{\partial \Sigma}{\partial X^\alpha}, X^\alpha\right) = \eta^{\alpha\beta}\frac{\partial \Sigma}{\partial X^\alpha}\frac{\partial \Sigma}{\partial X^\beta} - m^2c^2 = 0. \tag{1.157}$$

If we assume a general solution $\Sigma(X^\alpha; \pi_k)$ of the form

$$\Sigma(X^\alpha; \pi_k) = \sum_{\nu=1}^{4} \Sigma_\nu(X^\nu; \pi_k), \tag{1.158}$$

equation (1.157) separates into the uncoupled differential system

$$\frac{\partial \Sigma_j}{\partial X^j} = \pi_j, \tag{1.159}$$

$$\frac{\partial \Sigma_0}{\partial X^0} = \pm\sqrt{\pi_k\pi_k + m^2c^2} = \pm\pi_0(\pi_k), \tag{1.160}$$

and Σ is trivially given by

$$\Sigma(X^\alpha; \pi_k) = \pi_k X^k \pm \pi_0(\pi_k) X^0. \tag{1.161}$$

The associated H-J system (1.29)-(1.30) reads

$$P_i = \frac{\partial \Sigma}{\partial X^i}(X^\alpha; \Pi_k) = \Pi_i, \tag{1.162}$$

$$P_0 = \frac{\partial \Sigma}{\partial X^0}(X^\alpha; \Pi_k) = \pm\pi_0(\Pi_k) = \pm\pi_0(P_k), \tag{1.163}$$

$$D^j = \frac{\partial \Sigma}{\partial \pi_j}(X^\alpha; \Pi_k) = X^j \pm \frac{\Pi_j}{\pi_0(\Pi_k)} X^0 = X^j + \frac{P_j}{P_0} X^0. \tag{1.164}$$

As anticipated in Section 1.2.5, one equation, in this case equation (1.163), is functionally dependent on the others through the enforcement of the hamiltonian constraint H_r, and we immediately find that the covariant momenta P_i and the variables

$$D^j = X^j + \frac{P_j}{P_0} X^0 = X^j - \frac{P^j}{P_0} X^0 \tag{1.165}$$

are six independent constants of motion for the system. This result, when $P_0 = \pi_0(P_k)$, is perfectly consistent with the well-known classical dynamics of a free relativistic point particle. Since the particle's motion is uniform and rectilinear, its spatial momenta P^i are conserved. The particle's energy is also constant and equal to $E = \pi_0(P_k)c = P_0 c$, so as to satisfy the dispersion relation (1.155). Its velocity components V^i are then given by

$$V^i = \frac{\partial E}{\partial P^i} = \frac{P^i c^2}{E} = \frac{P^i c}{P_0}, \tag{1.166}$$

and the conserved quantities D^j represent its spatial coordinates X^j at time $T = c^{-1}X^0 = 0$:

$$D^j = X^j - \frac{P^j}{P_0}X^0 = X^j - V^j T = X^j(0). \tag{1.167}$$

However, let us forget about our prior knowledge of the classical dynamics of the system and carry on our analysis of the quantum model following the general principles stated in the previous sections. We expect that some quantization of the classical H-J constants of motion yields the generators of the quantum conserved algebra $\mathcal{C}_M$. It is easily checked that this is actually the case, because the self-adjoint variables $\widehat{P}_i$ and

$$\widehat{D}^j = \widehat{X}^j - \frac{1}{2}[\widehat{P}_0^{-1}\widehat{P}^j c, \widehat{T}]_+ = \widehat{X}^j - \widehat{P}_0^{-1}\widehat{P}^j c\widehat{T} + \frac{i\hbar}{2}\widehat{P}_0^{-2}\widehat{P}^j, \tag{1.168}$$

actually commute with the hamiltonian constraint $\widehat{H}_r$.

1.4.3 Evolving constants

§ Having found the generators of $\mathcal{C}_M$, we can identify all the evolving constants of the model. For example, let us suppose to be interested in the particle's X^j coordinate at time $T = T_0$. The corresponding evolving constant $\widetilde{X}(T_0; \widehat{P}_k, \widehat{D}^k)$ is obtained expressing $\widehat{X}^j$ in terms of $\widehat{T}$ and the conserved generators $\widehat{P}_i$, $\widehat{D}^j$, and then substituting T_0 for $\widehat{T}$. From (1.168), we have

$$\widehat{X}^j = \widehat{D}^j + \frac{\widehat{P}^j c}{\pi_0(\widehat{P}_k)}\widehat{T}, \tag{1.169}$$

and thus find

$$\widetilde{X}^j(T_0; \widehat{P}_k, \widehat{D}^k) = \widehat{D}^j + \frac{\widehat{P}^j c T_0}{\pi_0(\widehat{P}_k)}. \tag{1.170}$$

In particular, we correctly identify $\widehat{D}^j$ as the evolving constants $\widetilde{X}^j(0; \widehat{P}_k, \widehat{D}^k)$ yielding the particle's position at time $T = 0$. As emphasized in the preceding sections, CQM is a relational framework. This means that the inertial time T is just a convenient choice for the clock variable and any other physical quantity should equally well serve as a clock. It is then interesting to find some nonstandard evolving constants in our simple model. Let us focus for definiteness on $\widetilde{X}(Y_0; \widehat{P}_k, \widehat{D}^k)$ and $\widetilde{T}(Y_0; \widehat{P}_k, \widehat{D}^k)$, i.e. on the evolving constants yielding the expected values of T and $X = X^1$ when $Y = X^2$ is equal to Y_0. Using again (1.168), we find

$$\widehat{T} = \frac{\pi_0(\widehat{P}_k)}{\widehat{P}^2 c}(\widehat{Y} - \widehat{D}^2) \tag{1.171}$$

and

$$\widehat{X} = \widehat{D}^1 + \frac{\widehat{P}^1 c}{\pi_0(\widehat{P}_k)}\widehat{T} = \widehat{D}^1 + \frac{\widehat{P}^1}{\widehat{P}^2}(\widehat{Y} - \widehat{D}^2). \tag{1.172}$$

The evolving constants we are interested in are then given by

$$\widetilde{T}(Y_0; \widehat{P}_k, \widehat{D}^k) =: \frac{\pi_0(\widehat{P}_k)}{\widehat{P}^2 c}(Y_0 - \widehat{D}^2) : \tag{1.173}$$

and

$$\widetilde{X}(Y_0; \widehat{P}_k, \widehat{D}^k) = \widehat{D}^1 + : \frac{\widehat{P}^1}{\widehat{P}^2}(Y_0 - \widehat{D}^2) :, \tag{1.174}$$

respectively, where the : indicates some normal ordering of the enclosed factors. Unfortunately, it is easy to prove that both these evolving constants are not essentially self-adjoint on $\mathcal{D}_M$, independent of the choice of the normal product. As remarked above, this means that they do not represent physical constants of motion and must therefore be discarded. As discussed in detail in [37], this problem is shared by all covariant quantum models, even by the simplest single-particle examples, and severely limits the applicability of the CQM framework. In the rest of this thesis work, we will just accept this limitation and restrict our choice of the independent variables to those few which actually yield self-adjoint evolving constants.

1.4.4 Physical Hilbert space

§ In order to extract physical predictions from the model and complete its dynamical characterization, we are left with the task of identifying the physical Hilbert space of the system. As discussed above, this is the most problematic issue with CQM, as the determination of the physical scalar product is in general very difficult. Thankfully, our model is so simple that we can obtain the physical Hilbert space directly from the definition. In fact, in the Schrödinger representation (1.153)-(1.154), the hamiltonian constraint $\widehat{H}_r$ is just the multiplication operator

$$\widehat{H}_r \psi(p_\alpha) = (p^\alpha p_\alpha - m^2 c^2)\psi(p_\alpha), \tag{1.175}$$

and the action of the generalized projector P_0 onto the improper kernel of $\widehat{H}_r$ on any $\varphi \in \mathcal{D}_M$ is given by

$$P_0(\varphi) = \delta(p^\alpha p_\alpha - m^2 c^2)\varphi(p_\alpha). \tag{1.176}$$

As a result, the physical scalar product between any $\varphi_1, \varphi_2 \in \mathcal{D}_M$ reads

$$\begin{aligned}
\langle \varphi_1 | \varphi_2 \rangle_{H_r} &= \int \overline{\varphi}_1(p_\alpha)\delta(p^\alpha p_\alpha - m^2c^2)\varphi_2(p_\alpha) d^4p = \\
&= \int \overline{\varphi}_1(\pi_0(p_k), p_k)\varphi_2(\pi_0(p_k), p_k)\frac{d^3p}{2\pi_0(p_k)} + \\
&\quad + \int \overline{\varphi}_1(-\pi_0(p_k), p_k)\varphi_2(-\pi_0(p_k), p_k)\frac{d^3p}{2\pi_0(p_k)} = \\
&= \int \left\{ \overline{\Psi}_{1,+}(p_k)\Psi_{2,+}(p_k) + \overline{\Psi}_{1,-}(p_k)\Psi_{2,-}(p_k) \right\} \frac{d^3p}{2\pi_0(p_k)}, \quad (1.177)
\end{aligned}$$

where we have set

$$\Psi_+(p_k) = \varphi(\pi(p_k), p_k), \quad (1.178)$$

$$\Psi_-(p_k) = \varphi(-\pi(p_k), p_k), \quad (1.179)$$

for every $\psi \in \mathcal{D}_M$. It is clear that two test functions $\varphi, \varphi' \in \mathcal{D}_M$ are equivalent under (1.70), i.e. their difference has a vanishing physical norm, if and only if $\Psi_+ = \Psi'_+$ and $\Psi_- = \Psi'_-$. This means that a physical state is unambigously identified by a couple of wavefunctions $\Psi_+, \Psi_- \in L^2(\mathbb{R}^3, d^3p/2\pi_0)$. The physical Hilbert space of the model $\mathcal{P}_M$ is then isomorphic to the vector sum of two copies of $L^2(\mathbb{R}^3, d^3p/2\pi_0)$ equipped with the scalar product (1.177), and we can represent any physical state $\Psi \in \mathcal{P}_M$ as some couple $(\Psi_+, \Psi_-) \in L^2(\mathbb{R}^3, d^3p/2\pi_0) \oplus L^2(\mathbb{R}^3, d^3p/2\pi_0)$.

Chapter 2

Covariant quantum mechanics and curved spacetimes

§ In the first part of this chapter, the CQM framework introduced in Chapter 1 is generalized and adapted to the description of free quantum particles propagating on curved spacetime manifolds. As anticipated in the Introduction, this is done carrying over some recent results about QM on arbitrary configuration manifolds [19, 20] into the CQM framework. The resulting theory is the quantum analogue of the classical hamiltonian description of geodesic motion. In the second part of the chapter, the general formalism is applied to the special case of a free spinless particle travelling on a de Sitter spacetime. Thanks to the simplicity of the de Sitter metric, all the relevant computations can be explicitly carried out and yield a predictive, quantitative model of some phenomenological interest.

2.1 The classical analogy revisited

§ In Chapter 1, Dirac's famous classical analogy and its success in deriving Heisenberg QM from classical hamiltonian mechanics were mentioned as arguments supporting the canonical quantization procedure used to obtain CQM from classical CHM. At the same time, the heuristic nature of the prescription and its mathematical inconsistency were also touched upon, but not elaborated in detail. The reason was that, in the case of flat configuration manifolds, canonical quantization of the cartesian coordinates and their conjugate momenta yields a well-defined and unambiguous result, i.e. standard QM. Therefore, assuming flatness of the extended configuration space, we were able to disregard the question of the actual import of the classical analogy and introduce the CQM framework in the most intuitive possible way. However, the additional assumption entails a serious limitation of the applicability of the CQM formalism with respect to its classical counterpart. In fact, the same is true about standard QM. In order to obtain a full correspondence between CQM and CHM, it is then necessary to investigate the mathematical foundations of the classical analogy. This has been done by Morchio and Strocchi in the recent papers [19, 20], where the source of the classical analogy is convicingly identified and standard single-particle QM is accordingly extended to arbitrary spatial manifolds. In this section their work is briefly summarized in preparation for its subsequent application to CQM.

§ As already recalled in the previous chapter, the classical analogy is the assumption that there exists a 1-to-1 correspondence q between the classical algebra of observables $\mathcal{O}_c$ of a mechanical system and its quantum counterpart $\mathcal{O}$ such that

$$[q(O), q(O')] = i\hbar \{O, O'\}_0 , \tag{2.1}$$

for any $O, O' \in \mathcal{O}_c$. Unfortunately, such a correspondence cannot hold if we simultaneously require q to satisfy other natural conditions, such as $q(f(O)) = f(q(O))$ for any sufficiently regular function f [38]. A way out of these difficulties is provided by the canonical quantization prescription. It amounts to requiring that (2.1) is satisfied just for some canonical basis P_i and Q^j of the classical algebra of observables $\mathcal{O}_c$, i.e. that the corresponding self-adjoint quantum variables satisfy the canonical commutation relations

$$[P_i, P_j] = 0, \tag{2.2}$$

$$\left[P_i, Q^j\right] = i\hbar \delta_i{}^j, \tag{2.3}$$

$$\left[Q^i, Q^j\right] = 0. \tag{2.4}$$

The quantum algebra of observables $\mathcal{O}$ is then defined as the *-algebra generated by such P_i and Q^j, without further reference to the classical one. If Q^j are cartesian coordinates on some euclidean space, the canonical quantization prescription correctly identifies the Heisenberg algebra of observables of standard QM. Due to the noncommutativity of the quantum canonical coordinates, different quantum observables, for example $P_i Q^j$ and $Q^j P_i$, may correspond to the same classical phase-space function, and this is the reason why the naïve classical analogy (2.1) cannot be implemented.

§ The problem with canonical quantization is that it is a coordinate-dependent prescription. The fundamental observables P_i and Q^j are not just a convenient coordinatization of an unambiguous, already identified algebra of observables, but rather a means for its very definition. As a consequence, the resulting $\mathcal{O}$ could very well depend on the specific choice of the canonical basis. When the configuration space $\mathcal{M}$ of the system is euclidean, cartesian flat coordinates are geometrically distinguished from all the others and can be used to uniquely identify the Heisenberg algebra as the observable algebra $\mathcal{O}$, but for general manifolds all coordinate systems are equivalent and the coordinate-dependence of the canonical quantization procedure seriously undermines its applicability. Interestingly, this problem does not affect classical hamiltonian mechanics. In fact, the classical algebra of observables $\mathcal{O}_c$ contains all the sufficiently regular functions defined on phase space, i.e. the cotangent bundle $\mathcal{P} = T^*(\mathcal{M})$, and the classical Poisson bracket

$$\{O, O'\}_0 = \frac{\partial O}{\partial P_k}\frac{\partial O'}{\partial Q^k} - \frac{\partial O'}{\partial P_k}\frac{\partial O}{\partial Q^k} \tag{2.5}$$

is actually invariant under canonical transformations, i.e. independent of the choice of the canonical basis P_i and Q^j. We can now understand the appeal of the naïve classical analogy (2.1) for Dirac and the other pioneers, who viewed it as a means of making canonical quantization depend only on the classical, geometrically defined construction. However, as said before, (2.1) can be shown to be mathematically inconsistent and the problem of the definition of QM on arbitrary configuration manifolds remained open until a few years ago, when the above-mentioned contributions by Morchio and Strocchi shed light on the issue.

§ The starting points of the construction reported in [19, 20] are the associative algebra $C^\infty(\mathcal{M})$ of smooth real fuctions on $\mathcal{M}$ and the Lie algebra $\mathrm{Vect}(\mathcal{M})$ of smooth vector fields with compact support in $\mathcal{M}$, which together completely encode the geometry of the configuration manifold $\mathcal{M}$. A vector field $\mathbf{v} \in \mathrm{Vect}(\mathcal{M})$ is a derivation on $C^\infty(\mathcal{M})$, i.e. a linear map $\mathbf{v} : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ which satisfies the Leibniz rule

$$\mathbf{v}(fg) = \mathbf{v}(f)g + f\mathbf{v}(g) \tag{2.6}$$

for every $f, g \in C^\infty(\mathcal{M})$. The Lie product $\langle \mathbf{v}, \mathbf{w} \rangle$ between any $\mathbf{v}, \mathbf{w} \in \mathrm{Vect}(\mathcal{M})$ is defined by

$$\langle \mathbf{v}, \mathbf{w} \rangle (f) = \mathbf{v}(\mathbf{w}(f)) - \mathbf{w}(\mathbf{v}(f)). \tag{2.7}$$

Since the action of vector fields on functions satisfies (2.6), the Lie structure of $\mathrm{Vect}(\mathcal{M})$ can be extended to the direct sum $C^\infty(\mathcal{M}) + \mathrm{Vect}(\mathcal{M})$ by setting

$$\begin{aligned} \langle \mathbf{v}, f \rangle &= \mathbf{v}(f), & (2.8)\\ \langle f, g \rangle &= 0, & (2.9) \end{aligned}$$

for every $\mathbf{v} \in \mathrm{Vect}(\mathcal{M})$ and $f, g \in C^\infty(\mathcal{M})$. The Lie algebra $\mathrm{Vect}(\mathcal{M})$ is infinite-dimensional as a vector space over $\mathbb{R}$, but it is not if we consider that different vector fields may be functionally dependent through multiplication by functions in $C^\infty(\mathcal{M})$. The rigorous way of taking this fact into account is to regard $\mathrm{Vect}(\mathcal{M})$ as a module over $C^\infty(\mathcal{M})$. This means that there exist a product $\circ : C^\infty(\mathcal{M}) \times \mathrm{Vect}(\mathcal{M}) \to \mathrm{Vect}(\mathcal{M})$ which is distributive in both factors and associative in the first, i.e. such that

$$\begin{aligned} (f+g)\circ \mathbf{v} &= f\circ \mathbf{v} + g \circ \mathbf{v}, & (2.10)\\ f \circ (\mathbf{v}+\mathbf{w}) &= f \circ \mathbf{v} + f \circ \mathbf{w}, & (2.11)\\ (fg)\circ \mathbf{v} &= f \circ (g \circ \mathbf{v}), & (2.12) \end{aligned}$$

for every $f, g \in C^\infty(\mathcal{M})$ and $\mathbf{v}, \mathbf{w} \in \mathrm{Vect}(\mathcal{M})$. The module product $\circ$ interacts in a very nice way with the Lie structure of $C^\infty(\mathcal{M}) + \mathrm{Vect}(\mathcal{M})$. In fact, it is easy to verify that it naturally extends the Leibniz rule (2.6) via

$$\begin{aligned} \langle \mathbf{v}, f \circ \mathbf{w} \rangle &= \langle \mathbf{v}, f \rangle \circ \mathbf{w} + f \circ \langle \mathbf{v}, \mathbf{w} \rangle, & (2.13)\\ \langle f \circ \mathbf{v}, g \rangle &= f \langle \mathbf{v}, g \rangle. & (2.14) \end{aligned}$$

When a Lie algebra of derivations $\mathcal{L}$ acting on some commutative algebra $\mathcal{F}$ is also a module over $\mathcal{F}$ and the product $\circ$, in addition, satisfies (2.13)-(2.14), then the couple $(\mathcal{L},\mathcal{F})$ equipped with the Lie and module products is called a Lie-Rinehart (LR) algebra. We can then summarize all the relations between $C^\infty(\mathcal{M})$ and $\mathrm{Vect}(\mathcal{M})$ discussed above by saying that $L_R(\mathcal{M}) = (C^\infty(\mathcal{M}), \mathrm{Vect}(\mathcal{M}))$ is a LR algebra. The LR structure $L_R(\mathcal{M})$ compactly encodes all the geometrical information about the configuration space, but it is not very easy to deal with from a mathematical point of view, because it is not an associative algebra. Therefore, it is convenient to embed it in its unique universal enveloping Poisson algebra $P_R(\mathcal{M})$. This is obtained taking the free polynomial algebra generated by all $\mathbf{v} \in \mathrm{Vect}(\mathcal{M})$ and $f \in C^\infty(\mathcal{M})$, extending the Lie product from $L_R(\mathcal{M})$ exclusively via the application of the Leibniz rule in both factors and implementing the module product through the symmetric part of the associative product, i.e. requiring that

$$f \circ \mathbf{v} = \frac{1}{2}(f\mathbf{v} + \mathbf{v}f) \tag{2.15}$$

for every $f \in C^\infty(\mathcal{M})$ and $\mathbf{v} \in \mathrm{Vect}(\mathcal{M})$. It is important to explicitly point out that $P_R(\mathcal{M})$ is not the universal enveloping algebra of $L_R(\mathcal{M})$ in the sense of the standard theory of Lie algebras. In particular, the Lie product is not implemented in $P_R(\mathcal{M})$ through the commutator, but just extended from $L_R(\mathcal{M})$ via the Leibniz rule. Let * be the unique real linear involution which leaves $L_R(\mathcal{M})$ pointwise invariant, i.e. such that $f^* = f$ and $\mathbf{v}^* = \mathbf{v}$ for every $f, \mathbf{v} \in L_R(\mathcal{M})$, and extends to $P_R(\mathcal{M})$ through $(ab)^* = b^*a^*$. It is easy to verify that

$$\langle a, b\rangle^* = \langle a^*, b^*\rangle \tag{2.16}$$

for every $a, b \in P_R(\mathcal{M})$, so that $P_R(\mathcal{M})$ is actually a Poisson *-algebra.

§ The previous construction identifies a non-commutative Poisson *-algebra $P_R(\mathcal{M})$ which is coordinate-independent and univocally determined by the geometry of the configuration manifold $\mathcal{M}$. Moreover, if $\mathcal{M}$ is N-dimensional, the corresponding $P_R(\mathcal{M})$ is $2N$-dimensional, being generated by N independent functions and N functionally independent vector fields. All these features strongly suggest that it could be this structure, rather than the classical commutative Poisson algebra $\mathcal{O}_c$, the true source of the classical analogy and the starting point of a coordinate-indipendent quantization procedure. In [19, 20], the authors prove that this is exactly the case, thereby vindicating the pioneering intuition of Dirac. More precisely, they show that it is always possible to find an antihermitian element $z = -z^* \in P_R(\mathcal{M})$ such that the commutator and the Lie product satisfy

$$\begin{aligned} \langle a, z\rangle &= [a, z] = 0, & (2.17)\\ [a, b] &= z\langle a, b\rangle, & (2.18) \end{aligned}$$

for every $a, b \in P_R(\mathcal{M})$. If one now requires the central variable z to be trivial, i.e. a multiple of the identity, there are only two possibile realizations of a given $P_R(\mathcal{M})$. Either $z = 0$, in which case $P_R(\mathcal{M})$ becomes commutative and can be easily identified as the classical algebra of observables $\mathcal{O}_c$ with the usual Poisson bracket, or $z = iz_0$ with $z_0 \in \mathbb{R}$. In this second case, the Lie product conflates with the commutator and $P_R(\mathcal{M})$ becomes isomorphic to the complex *-algebra $\mathcal{A}(\mathcal{M})$ generated by generalized self-adjoint coordinates and momenta Q^f and $P_{\mathbf{v}}$ satisfying the homomorphism relations

$$\begin{aligned} P_{r\mathbf{v}+\mathbf{w}} &= rP_{\mathbf{v}} + P_{\mathbf{w}}, && (2.19)\\ Q^{rf+g} &= rQ^f + Q^g, && (2.20)\\ Q^{fg} &= Q^f Q^g, && (2.21)\end{aligned}$$

the commutation rules

$$\begin{aligned} [P_{\mathbf{v}}, P_{\mathbf{w}}] &= i\hbar P_{\langle \mathbf{v},\mathbf{w}\rangle}, && (2.22)\\ \left[P_{\mathbf{v}}, Q^f\right] &= i\hbar Q^{\langle \mathbf{v},f\rangle}, && (2.23)\\ \left[Q^f, Q^g\right] &= 0, && (2.24)\end{aligned}$$

and the condition

$$P_{f\circ\mathbf{v}} = \frac{1}{2}(Q^f P_{\mathbf{v}} + P_{\mathbf{v}} Q^f), \qquad (2.25)$$

for every $r \in \mathbb{R}$, $f, g \in C^\infty(\mathcal{M})$ and $\mathbf{v}, \mathbf{w} \in \mathrm{Vect}(\mathcal{M})$. Once given a configuration manifold $\mathcal{M}$, this *-algebra is univocally determined and reduces to the usual Heisenberg algebra of observables when $\mathcal{M} = \mathbb{R}^N$. Therefore, it is natural to identify $\mathcal{A}(\mathcal{M})$ as the quantum observable algebra $\mathcal{O}$ also in the general case. It is also worth to point out that, since every 1-parameter group of diffeomorphisms α_λ on $\mathcal{M}$ leaves invariant the algebric structure of $C^\infty(\mathcal{S}) + \mathrm{Vect}(\mathcal{S})$, the maps

$$\begin{aligned} P_{\mathbf{v}} &\mapsto P_{\alpha_\lambda(\mathbf{v})}, && (2.26)\\ Q^f &\mapsto Q^{\alpha_\lambda(f)}, && (2.27)\end{aligned}$$

preserve the relations (2.19)-(2.25) and are therefore automorphisms of $\mathcal{A}(\mathcal{M})$, i.e. kinematical symmetries of the associated generalized QM. This important property, which is obviously shared with the classical algebra of observables $\mathcal{O}_c$, is a further compelling reason for identifying its quantum counterpart $\mathcal{O}$ with $\mathcal{A}(\mathcal{M})$. These results show that the naïve classical analogy is definitely just an accidental and imperfect consequence of both the classical and the quantum algebras of observables being alternative realizations of the same, purely geometrical structure $P_R(\mathcal{M})$. The antihermitian character of the element z also explains why in the quantum case $z \neq 0$ a complex structure is needed even for real $\mathcal{M}$.

§ In [19], other than identifying the above $\mathcal{A}(\mathcal{M})$ as the correct generalization of the Heisenberg algebra for arbitrary configuration manifolds, the authors provide a complete classification of its irreducible representations on a Hilbert space, proving that, locally, they are all unitarily equivalent to the Schrödinger representation. More precisely, let $\mathcal{D} \subset \mathcal{M}$ be a region diffeomorphic to an open hypersphere and let us consider the restriction $\mathcal{A}(\mathcal{D})$ of $\mathcal{A}(\mathcal{M})$. Then, choosing some smooth local coordinates x^j in $\mathcal{D}$ and setting $P_i = P_{\partial/\partial x^i}$, $Q^j = Q^{x^j}$, all irreducible representations of $\mathcal{A}(\mathcal{D})$ are unitarily equivalent to the Schrödinger representation on $L^2(\mathcal{D}, d^N p)$, which is defined by the following action of the fundamental observables P_i and Q^j on any wavefunction $\Psi \in L^2(\mathcal{D}, d^N p)$:

$$\widehat{P}_i \Psi(p_k) = p_i \Psi(p_k), \tag{2.28}$$

$$\widehat{Q}^j \Psi(p_k) = -i\hbar \frac{\partial \Psi(p_k)}{\partial p_j} = -i\hbar \partial^j \Psi(p_k). \tag{2.29}$$

In this representation, any automorphism (2.26)-(2.27) corresponding to a 1-parameter group of diffeomorphisms generated by some vector field $\mathbf{a} \in \mathrm{Vect}(\mathcal{M})$ is unitarily implemented by the 1-parameter group of operators $\widehat{U}_{\mathbf{a}}(\lambda) = \exp(i\lambda \widehat{P}_{\mathbf{a}})$. At the global level, topological effects may arise if the configuration manifold $\mathcal{M}$ is not simply connected. In this case, as shown in detail in [19], the irriducible representations of $\mathcal{A}(\mathcal{M})$, which are all locally Schrödinger, are in 1-to-1 correspondence with the irreducible unitary representations of the first homotopy group $\pi_1(\mathcal{M})$ of the manifold.

2.2 Covariant quantum mechanics revisited

§ The construction of the previous section was originally developed in the context of ordinary QM. However, being purely kinematical in character, it can be carried over to CQM with very slight modifications. This way, other than providing the framework described in Chapter 1 with a rigorous and fully geometrical foundation, we can extend the CQM formalism so as to deal with non-flat (extended) configuration spaces. From a phenomenological point of view, this is very interesting because it makes it possible to study the motion of a free quantum particle on a curved background in a simple single-particle setting, avoiding the complications arising from the definition of quantum field theories on curved spacetimes. In this section, we present such a generalization of CQM. For the sake of concreteness, and without loss of generality, we restrict our treatment to the physically relevant scenario of a spinless particle on a curved spacetime.

§ Let $\mathcal{S}$ be an arbitrary 4-dimensional spacetime manifold. In order to define single-particle CQM on $\mathcal{S}$, we take the *-algebra $\mathcal{A}(\mathcal{S})$ introduced above as the extended canonical *-algebra $\mathcal{V}_{\mathcal{S}}$. Given some smooth global coordinates x^ν on $\mathcal{S}$ and the corresponding basis vector fields $\mathbf{e}_\mu = \partial/\partial x^\mu$, $\mathcal{V}_{\mathcal{S}}$ is generated by eight self-adjoint variables $P_\mu = P_{\mathbf{e}_\mu}$ and $X^\nu = Q^{x^\nu}$, which satisfy the canonical commutation relations

$$[P_\mu, P_\nu] = 0, \tag{2.30}$$

$$[P_\mu, X^\nu] = i\hbar \delta_\mu{}^\nu, \tag{2.31}$$

$$\left[X^\nu, X^\lambda\right] = 0. \tag{2.32}$$

The homomorphism relations (2.19)-(2.21) and module product condition (2.25) univocally associate a self-adjoint variable $V \in \mathcal{V}_{\mathcal{S}}$ to any $f, \mathbf{v} \in C^\infty(\mathcal{S}) + \mathrm{Vect}(\mathcal{S})$. More explicitly, given any $f(x^\alpha) \in C^\infty(\mathcal{S})$ and $\mathbf{v} = v^\beta(x^\alpha)\mathbf{e}_\beta \in C^\infty(\mathcal{S})$, the corresponding variables are

$$Q^f = f(X^\alpha), \tag{2.33}$$
$$P_{\mathbf{v}} = \frac{1}{2}(v^\beta(X^\alpha)P_\beta + P_\beta v^\beta(X^\alpha)). \tag{2.34}$$

Therefore, any physical variable which can be experimentally measured, which must be a function of the particle's spacetime position and 4-velocity with respect to some reference frame, is represented by a self-adjoint element $V \in \mathcal{V}_\mathcal{S}$. Any 1-parameter group α_λ of diffeomorphisms on $\mathcal{S}$ induces a corresponding 1-parameter group of automorphisms of $\mathcal{V}_\mathcal{S}$. If α_λ is generated by a vector field $\mathbf{a} \in \mathcal{S}$, the corresponding automorphisms are implemented by the adjoint action of the unitary variables $U_\mathbf{a}(\lambda) = \exp(i\lambda P_\mathbf{a})$. If $\mathcal{S}$ is simply connected, or if we restrict to a simply connected domain $\mathcal{D} \subset \mathcal{S}$, the extended canonical algebra $\mathcal{V}_\mathcal{S}$ admits a unique irreducible representation, the above-mentioned Schrödinger representation, on the kinematical Hilbert space $\mathcal{K}_\mathcal{S} = L^2(\mathbb{R}^4, d^4p)$ of the square-integrable functions of four real variables, equipped with the L^2 scalar product

$$\langle\psi|\phi\rangle = \int \overline{\psi}(p_\alpha)\phi(p_\alpha)d^4p. \tag{2.35}$$

In this representation, the action of the canonical coordinates on a wavefunction $\psi \in \mathcal{K}_\mathcal{S}$ is given by

$$\widehat{P}_\mu\psi(p_\alpha) = p_\mu\psi(p_\alpha), \tag{2.36}$$
$$\widehat{X}^\nu\psi(p_\alpha) = -i\hbar\frac{\partial\psi(p_\alpha)}{\partial p_\nu} = -i\hbar\partial^\nu\psi(p_\alpha). \tag{2.37}$$

Starting from the generalized extended canonical *-algebra $\mathcal{V}_\mathcal{S}$ and the associated kinematical Hilbert space $\mathcal{K}_\mathcal{S}$, the construction of the CQM framework goes on exactly as explained in the previous chapter. Both the physical Hilbert space $\mathcal{P}_\mathcal{S}$ and the conserved algebra $\mathcal{C}_\mathcal{S}$ are determined by a self-adjoint hamiltonian constraint $H(\widehat{P}_\alpha, \widehat{X}^\alpha) \in \mathcal{V}_\mathcal{S}$ via the modified scalar product

$$\langle\psi_1|\psi_2\rangle_H = \langle\psi_1|\widehat{P}_H|\psi_2\rangle \tag{2.38}$$

and the condition

$$[\widehat{H}, \widehat{C}] = 0, \tag{2.39}$$

respectively, as seen before. The evolving constants encoding the relational dynamics are then obtained exactly as in Chapter 1.

§ Having completely characterized the generalized CQM framework, we just have to find a hamiltonian constraint suitable for describing the physical scenario we are interested in, i.e. the propagation of a free spinless particle on a curved background. In the classical case, the correct geodesic equations can be derived from the so-called geodesic hamiltonian constraint

$$G_g(P_\alpha, X^\alpha) = g^{\alpha\beta}(X^\gamma)P_\alpha P_\beta - m^2c^2, \tag{2.40}$$

where $g^{\mu\nu}(x^\alpha)$ is the spacetime metric in the coordinates x^ν and m is the rest mass of the particle. In fact, setting $P^\nu = g^{\nu\beta}(X^\alpha)P_\beta$, Hamilton's equations of motion are

$$\frac{dP_\mu}{d\lambda} = -\frac{\partial G_g}{\partial X^\mu} = -\frac{\partial g^{\alpha\beta}}{\partial x^\mu}(X^\gamma)P_\alpha P_\beta = \frac{\partial g_{\alpha\beta}}{\partial x^\mu}(X^\gamma)P^\alpha P^\beta, \tag{2.41}$$

$$\frac{dX^\nu}{d\lambda} = \frac{\partial G_g}{\partial P_\nu} = 2P^\nu, \tag{2.42}$$

and substitution of (2.42) into (2.41) yields

$$\frac{d^2X^\nu}{d\lambda^2} + \Gamma^\nu{}_{\alpha\beta}(X^\gamma)\frac{dX^\alpha}{d\lambda}\frac{dX^\beta}{d\lambda} = 0, \tag{2.43}$$

where the functions

$$\Gamma^\nu{}_{\alpha\beta} = \frac{1}{2}g^{\nu\gamma}\left(\frac{\partial g_{\beta\gamma}}{\partial x^\alpha} + \frac{\partial g_{\alpha\gamma}}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\right) \tag{2.44}$$

are the coefficients of the Levi-Civita connection. It is then natural to assume that the quantum dynamics is given by the corresponding quantum constraint $G_g(\widehat{P}_\alpha, \widehat{Q}^\alpha)$. However, the operator $G_g(\widehat{P}_\alpha, \widehat{Q}^\alpha)$ is not self-adjoint in general. We can try to solve this issue by reordering the operators P_μ and X^ν in the expression of G_g, but there is not a unique way of doing it. Just to mention the two simplest possibilities, both

$$\widehat{H}_g^{(1)} = \widehat{P}_\alpha g^{\alpha\beta}(\widehat{X}^\gamma)\widehat{P}_\beta - m^2c^2 \tag{2.45}$$

and

$$\widehat{H}_g^{(2)} = \frac{1}{2}\left\{g^{\alpha\beta}(\widehat{X}^\gamma)\widehat{P}_\alpha\widehat{P}_\beta + \widehat{P}_\alpha\widehat{P}_\beta g^{\alpha\beta}(\widehat{X}^\gamma)\right\} - m^2c^2 \tag{2.46}$$

are self-adjoint and reduce to the classical expression (2.40) for commutative $\widehat{P}_\mu$ and $\widehat{X}^\nu$, but there are actually infinitely many other quantum constraints with the same properties. It is a typical instance of the ordering problem which affects all attempts at guessing quantum hamiltonians from the corresponding classical ones. Thankfully, having set up a coordinate-independent quantum framework, we can avoid the problem altogether and determine the correct quantum constraint directly appealing to the geometrical meaning of (2.40). Let $\mathbf{e}_a = e_a{}^\mu(x^\alpha)\mathbf{e}_\mu$, with $a \in \{0,1,2,3\}$, be an orthonormal tetrad in $\mathcal{S}$. As it is well known, the contravariant components $g^{\mu\nu}(x^\alpha)$ of the metric can be expressed in terms of the tetrad $\mathbf{e}_a$ as

$$g^{\mu\nu}(x^\alpha) = \eta^{ab}e_a{}^\mu(x^\alpha)e_b{}^\nu(x^\alpha), \tag{2.47}$$

where $\eta^{ab} = \mathrm{diag}\{1,-1,-1,-1\}$ is the Minkowski metric. This means that the classical constraint (2.40) can be rewritten as

$$G_g(P_\alpha, X^\alpha) = \eta^{ab}e_a{}^\alpha(X^\gamma)e_b{}^\beta(X^\gamma)P_\alpha P_\beta - m^2c^2 = \eta^{ab}P_{\mathbf{e}_a}P_{\mathbf{e}_b} - m^2c^2, \tag{2.48}$$

where $P_{\mathbf{e}_a}$ and $P_{\mathbf{e}_b}$ are the momenta corresponding to the vector fields $\mathbf{e}_a$ and $\mathbf{e}_b$ in the algebra $P_R(\mathcal{S})$. Since the tetrad identifies a local inertial frame, expression (2.48) makes it clear that G_g enforces the relativistic on-shell condition for all inertial observers. Other than clarifying the physical import of the constraint, formula (2.48) allows for an unambiguous identification of its quantum version. In fact, the only quantum constraint bearing the same geometrical and physical meaning is clearly

$$\widehat{H}_g = \eta^{ab}\widehat{P}_{\mathbf{e}_a}\widehat{P}_{\mathbf{e}_b} - m^2c^2, \tag{2.49}$$

i.e., in local coordinates,

$$H_g(\widehat{P}_\alpha, \widehat{X}^\alpha) = \frac{1}{4}\eta^{ab}[e_a{}^\alpha(\widehat{X}^\gamma), \widehat{P}_\alpha]_+[e_b{}^\beta(\widehat{X}^\gamma), \widehat{P}_\beta]_+ - m^2c^2. \tag{2.50}$$

We see that $\widehat{H}_g$ is in general different from both $\widehat{H}_g^{(1)}$ and $\widehat{H}_g^{(2)}$, and that it would have been very difficult to identify without the help of geometrical considerations. This neat result clearly shows the superiority of the generalized, fully geometrical approach to (C)QM advocated in this chapter over the standard treatment.

2.3 Free quantum particle on de Sitter spacetime

§ In the following, a CQM-based model for a free spinless particle moving on a de Sitter spacetime will be explicitly developed in all details. For later convenience, we will start from a review of the symmetry algebra of the de Sitter spacetime. To avoid inessential complications, we will discuss the model in 2+1 spacetime dimensions, the extension to arbitrary dimensions being pretty straightforward.

2.3.1 The algebra $so(1,3)$

§ The self-adjoint generators $M_{\mu\nu}$ of the Lie algebra $so(1,3)$ obey the following commutation rules:

$$[M_{\mu\nu}, M_{\rho\sigma}] = i\hbar\left(\eta_{\rho\nu}M_{\mu\sigma} - \eta_{\mu\rho}M_{\nu\sigma} + \eta_{\sigma\nu}M_{\rho\mu} - \eta_{\mu\sigma}M_{\rho\nu}\right). \tag{2.51}$$

It is convenient to define vector generators via

$$N_i = M_{0i}, \tag{2.52}$$
$$L_i = \frac{1}{2}\varepsilon_{ijk}M_{jk}, \tag{2.53}$$

with $i,j,k \in \{1,2,3\}$ and to rewrite the previous commutation relations as

$$[L_i, L_j] = i\hbar\varepsilon_{ijk}L_k, \tag{2.54}$$
$$[N_i, L_j] = i\hbar\varepsilon_{ijk}N_k, \tag{2.55}$$
$$[N_i, N_j] = -i\hbar\varepsilon_{ijk}L_k. \tag{2.56}$$

If we perform the basis transformation

$$P_i = \frac{1}{2}(L_i + iN_i), \tag{2.57}$$
$$Q_i = \frac{1}{2}(L_i - iN_i), \tag{2.58}$$

the commutation rules become

$$\begin{aligned} [P_i, P_j] &= i\hbar\varepsilon_{ijk}P_k, && (2.59)\\ [Q_i, Q_j] &= i\hbar\varepsilon_{ijk}Q_k, && (2.60)\\ [P_i, Q_j] &= 0, && (2.61)\end{aligned}$$

and we see that the Lie algebra $so(1,3)$ is the product of two $su(2)$ subalgebras generated by P_i and Q_i. Therefore, $so(1,3)$ admits exactly two Casimir elements: they are

$$\begin{aligned} P^2 &= P_jP_j = \frac{1}{4}[L_jL_j - N_jN_j + 2iL_jN_j], && (2.62)\\ Q^2 &= Q_jQ_j = \frac{1}{4}[L_jL_j - N_jN_j - 2iL_jN_j], && (2.63)\end{aligned}$$

or, equivalently, the self-adjoint combinations

$$\begin{aligned} C_1 &= 2(P^2 + Q^2) = L_jL_j - N_jN_j = L^2 - N^2, && (2.64)\\ C_2 &= i(Q^2 - P^2) = L_jN_j. && (2.65)\end{aligned}$$

In fact, we have

$$\begin{aligned} [C_1, L_i] &= [L_jL_j, L_i] - [N_jN_j, L_i] = \\ &= i\hbar\varepsilon_{jik}(L_kL_j + L_jL_k - N_kN_j - N_jN_k) = 0, && (2.66)\\ [C_1, N_i] &= [L_jL_j, N_i] - [N_jN_j, N_i] = \\ &= i\hbar\varepsilon_{jik}(N_kL_j + L_jN_k + L_kN_j + N_jL_k) = 0, && (2.67)\\ [C_2, L_i] &= [L_jN_j, L_i] = i\hbar\varepsilon_{jik}(L_kN_j + L_jN_k) = 0, && (2.68)\\ [C_2, N_i] &= [L_jN_j, N_i] = i\hbar\varepsilon_{jik}(N_kN_j - L_jL_k) = \\ &= \frac{1}{2}i\hbar\varepsilon_{jik}([N_k, N_j] - [L_j, L_k]) = \frac{1}{2}\hbar^2\varepsilon_{jik}(\varepsilon_{kjl} + \varepsilon_{jkl})L_l = 0. && (2.69)\end{aligned}$$

The generators H, P_m, K_m and J of the most general symmetry group for a homogeneous and isotropic 2+1 dimensional spacetime satisfy

$$\begin{aligned} [J, K_m] &= i\hbar\varepsilon_{mn}K_n, && (2.70)\\ [J, P_m] &= i\hbar\varepsilon_{mn}P_n, && (2.71)\\ [J, H] &= 0, && (2.72)\\ [H, K_m] &= i\hbar P_m, && (2.73)\\ [H, P_m] &= \frac{i\hbar c^2}{R^2}K_m, && (2.74)\\ [P_m, K_n] &= \frac{i\hbar}{c^2}\delta_{mn}H, && (2.75)\\ [P_m, P_n] &= \frac{i\hbar}{R^2}\varepsilon_{mn}J, && (2.76)\\ [K_m, K_n] &= -\frac{i\hbar}{c^2}\varepsilon_{mn}J, && (2.77)\end{aligned}$$

where the speed c and the length R are fundamental constants and $m, n \in \{1, 2\}$. In the limit $R \to \infty$ we recover the 2+1 Poincaré Lie algebra, whence the labels

H, P_m, K_m and J for the generators. The commutation rules (2.70)-(2.77) define a $so(1,3)$ Lie algebra: in fact they can be obtained from (2.54)-(2.56) or directly from (2.51) by setting

$$\begin{aligned} H &= \frac{c}{R}N_3 = \frac{c}{R}M_{03}, && (2.78)\\ P_m &= -\frac{1}{R}\varepsilon_{mn}L_n = -\frac{1}{2R}\varepsilon_{mn}\varepsilon_{njk}M_{jk} = \frac{1}{R}M_{m3}, && (2.79)\\ K_m &= \frac{1}{c}N_m = \frac{1}{c}M_{0m}, && (2.80)\\ J &= L_3 = \frac{1}{2}\varepsilon_{3jk}M_{jk} = M_{12}. && (2.81) \end{aligned}$$

In the new notation, the Casimirs read

$$\begin{aligned} C_1 &= L^2 - N^2 = R^2P^2 + J^2 - \frac{R^2}{c^2}H^2 - c^2K^2, && (2.82)\\ C_2 &= L_jN_j = RcP_2K_1 - RcP_1K_2 + \frac{R}{c}JH. && (2.83) \end{aligned}$$

It is convenient to work with the renormalized quantities

$$\begin{aligned} c^4M^2 &= -\frac{c^2}{R^2}C_1 = H^2 - c^2P^2 - \frac{c^2}{R^2}(J^2 - c^2K^2), && (2.84)\\ W &= \frac{1}{Rc}C_2 = P_2K_1 - P_1K_2 + \frac{1}{c^2}JH, && (2.85) \end{aligned}$$

because, taking the limit $R \to \infty$, M^2 and W reduce to the familiar mass and spin Casimir elements of the 2+1 Poincaré Lie algebra.

2.3.2 Irreducible unitary representations

§ Since $so(1,3) \simeq so(3) \oplus so(3)$, every irreducible finite-dimensional representation of $SO(1,3)$ is isomorphic to a tensor product representation $j_1 \otimes j_2$ of $SO(3)$ for some choice of $j_1, j_2 \in \{0, 1/2, 1, \dots\}$. These representations, however, cannot be unitary, because the generators P_i and Q_i of the two $SO(3)$ subgroups are represented by hermitian matrices and this results in N_i being anti-hermitian. In fact, we have

$$N_i^\dagger = (iQ_i - iP_i)^\dagger = iP_i^\dagger - iQ_i^\dagger = iP_i - iQ_i = -N_i. \qquad (2.86)$$

We can conclude that irreducible unitary representations of the Lorentz group $SO(1,3)$ must be infinite-dimensional. This means, in particular, that they cannot be obtained from the $so(1,3) \simeq so(3) \oplus so(3)$ isomorphism. Nevertheless, it is still possible to use our knowledge of the unitary finite-dimensional irreps of $SO(3)$ to find them. Every irreducible unitary representation α of the Lorentz group must contain a unitary representation of its $SO(3)$ subgroup generated by L_i. This implies that the representation space $\mathcal{H}_\alpha$ must be the direct sum of $SO(3)$-invariant subspaces $\mathcal{H}_j$ labelled by half-integer numbers $j \in \{0, 1/2, 1, \dots\}$. We can then choose the standard basis $|j, m\rangle$, $m \in \{-j, -j+1, \dots, j\}$ in each of these subspaces and thus obtain a complete orthonormal basis $|\alpha; j, m\rangle$ for $\mathcal{H}_\alpha$. These basis vectors are eigenstates of L^2 and L_3 with eigenvalues

$$
\begin{aligned}
L^2|\alpha; j, m\rangle &= \hbar^2 j(j+1)|\alpha; j, m\rangle, && (2.87)\\
L_3|\alpha; j, m\rangle &= \hbar m|\alpha; j, m\rangle. && (2.88)
\end{aligned}
$$

The action of the operators L_1 and L_2 is given instead by

$$
\begin{aligned}
L_1|\alpha; j, m\rangle &= \frac{\hbar}{2}\sqrt{j(j+1)-m(m-1)}|\alpha; j, m-1\rangle + \\
&\quad + \frac{\hbar}{2}\sqrt{j(j+1)-m(m+1)}|\alpha; j, m+1\rangle, && (2.89)\\
L_2|\alpha; j, m\rangle &= \frac{i\hbar}{2}\sqrt{j(j+1)-m(m-1)}|\alpha; j, m-1\rangle - \\
&\quad - \frac{i\hbar}{2}\sqrt{j(j+1)-m(m+1)}|\alpha; j, m+1\rangle, && (2.90)
\end{aligned}
$$

or, introducing the ladder operators $L_\pm = L_1 \pm iL_2$, by

$$
L_\pm|\alpha; j, m\rangle = \hbar\sqrt{j(j+1)-m(m\pm 1)}|\alpha; j, m\pm 1\rangle. \qquad (2.91)
$$

In order to complete the representation of $SO(1,3)$, it is now sufficient to find the action of N_i on the basis $|\alpha; j, m\rangle$. Since the operators N_i transform like the components of a 3-vector under the action of $SO(3)$, we can use the Wigner-Eckart theorem [43] to simplify the relevant matrix elements. Introducing the ladder operators $N_\pm = N_1 \pm iN_2$ we have

$$
\begin{aligned}
\langle\alpha; j', m'|N_\pm|\alpha; j, m\rangle &= \mp\sqrt{2}\langle j, m; 1, \pm 1|j', m'\rangle\langle\alpha; j'||N||\alpha; j\rangle, && (2.92)\\
\langle\alpha; j', m'|N_3|\alpha; j, m\rangle &= \langle j, m; 1, 0|j', m'\rangle\langle\alpha; j'||N||\alpha; j\rangle. && (2.93)
\end{aligned}
$$

The nonvanishing Clebsch-Gordan coefficients are

$$
\begin{aligned}
\langle j, m; 1, \pm 1|j', m\pm 1\rangle &= \sqrt{\frac{(j\mp m)(j\mp m-1)}{2j(2j+1)}}\delta_{j-1,j'} \mp \sqrt{\frac{(j\mp m)(j\pm m+1)}{2j(j+1)}}\delta_{j,j'} + \\
&\quad + \sqrt{\frac{(j\pm m+2)(j\pm m+1)}{(2j+2)(2j+1)}}\delta_{j+1,j'}, && (2.94)\\
\langle j, m; 1, 0|j', m\rangle &= -\sqrt{\frac{(j-m)(j+m)}{j(2j+1)}}\delta_{j-1,j'} + \frac{m}{\sqrt{j(j+1)}}\delta_{j,j'} + \\
&\quad + \sqrt{\frac{(j-m+1)(j+m+1)}{(2j+1)(j+1)}}\delta_{j+1,j'}, && (2.95)
\end{aligned}
$$

so that we can write

$$
\begin{aligned}
\langle\alpha; j'||N||\alpha; j\rangle &= \hbar n_j^- \sqrt{j(2j+1)}\delta_{j',j-1} + \hbar n_j\sqrt{(j(j+1)}\delta_{j',j} + \\
&\quad + \hbar n_j^+\sqrt{(2j+1)(j+1)}\delta_{j',j+1} && (2.96)
\end{aligned}
$$

for some complex coefficients n_j^-, n_j and n_j^+. The action of the operators $N_\pm$ and N_3 is then given by

$$\begin{aligned}
N_\pm|\alpha;j,m\rangle &= \mp\hbar n_j^-\sqrt{(j\mp m)(j\mp m-1)}|\alpha;j-1,m\pm 1\rangle + \\
&+\hbar n_j\sqrt{(j\mp m)(j\pm m+1)}|\alpha;j,m\pm 1\rangle \mp \\
&\mp\hbar n_j^+\sqrt{(j\pm m+2)(j\pm m+1)}|\alpha;j+1,m\pm 1\rangle, \qquad (2.97)\\
N_3|\alpha;j,m\rangle &= -\hbar n_j^-\sqrt{(j-m)(j+m)}|\alpha;j-1,m\rangle + \hbar n_j m|\alpha;j,m\rangle + \\
&+\hbar n_j^+\sqrt{(j-m+1)(j+m+1)}|\alpha;j+1,m\rangle, \qquad (2.98)
\end{aligned}$$

Requiring that $N_\pm$ and N_3 satisfy the commutation relations

$$\begin{aligned}
[N_\pm,N_3] &= \pm\hbar L_\pm, \qquad (2.99)\\
[N_+,N_-] &= -2\hbar L_3, \qquad (2.100)
\end{aligned}$$

we find the following set of conditions on n_j^-, n_j and n_j^+:

$$\begin{aligned}
\{(j+1)n_j-(j-1)n_{j-1}\}\,n_j^- &= 0, \qquad (2.101)\\
\{(j+1)n_{j+1}-jn_j\}\,n_j^+ &= 0, \qquad (2.102)\\
(2j-1)n_j^-n_{j-1}^+-n_j^2-(2j+3)n_j^+n_{j+1}^- &= 1. \qquad (2.103)
\end{aligned}$$

Since the indices j are positive numbers, there must be a minimum j in the representation α. Let us denote this minimum value by j_α. We must obviously have $n_{j_\alpha}^-=0$, because j cannot become smaller than j_α. The condition (2.101) is then automatically satisfied for $j=j_\alpha$. When $j>j_\alpha$, instead, $n_j^-\neq 0$ and from (2.101) we can derive a recurrence relation for the coefficients n_j:

$$n_{j+1}=\frac{jn_j}{j+2}. \qquad (2.104)$$

The general solution is

$$n_j=\frac{j_\alpha n_\alpha}{j(j+1)}, \qquad (2.105)$$

where n_α is a complex number independent of j. The second condition is now automatically satisfied and we are left with the third. Substituting n_j into (2.103) and setting $m_j=-n_j^-n_{j-1}^+$ we obtain a recurrence relation for m_j:

$$m_{j+1}=\frac{2j-1}{2j+3}m_j+\frac{1}{2j+3}\left(1+\frac{j_\alpha^2n_\alpha^2}{j^2(j+1)^2}\right). \qquad (2.106)$$

Since $m_{j_\alpha}=0$, for $j>j_\alpha$ we find

$$m_j=\frac{1}{4j^2-1}\sum_{k=j_\alpha}^{j-1}(2k+1)\left(1+\frac{j_\alpha^2n_\alpha^2}{k^2(k+1)^2}\right)=\frac{(j^2-j_\alpha^2)(j^2+n_\alpha^2)}{j^2(4j^2-1)}, \qquad (2.107)$$

and this determines $n_j^- = \sqrt{m_j} e_j$ and $n_{j-1}^+ = -\sqrt{m_j} e_j^{-1}$ up to a complex-valued sequence e_j. Finally, in order for N_i to be hermitian, we must require that

$$\begin{aligned} n_j &= n_j^*, && (2.108)\\ n_j^- &= -(n_{j-1}^+)^*. && (2.109) \end{aligned}$$

Multiplying the second condition by $-n_{j-1}^+$ we obtain

$$m_j = |m_j||e_j|^{-2}, \tag{2.110}$$

from which we can deduce that m_j must be nonnegative and e_j must be a complex phase. By suitably adjusting the relative phases between basis vectors $|\alpha; j, m\rangle$ corresponding to different values of j, we can set $e_j = 1$ for every j and get $n_j^- = -n_{j-1}^+ = \sqrt{m_j}$. The last condition (2.108) can be rewritten as

$$j_\alpha(n_\alpha - n_\alpha^*) = 0 \tag{2.111}$$

and it is satisfied if either $n_\alpha \in \mathbb{R}$ or $j_\alpha = 0$. In the first case, m_j is automatically nonnegative and we find no further restriction on n_α and j_α. In the second, for $j \in \{1, 2, \dots\}$ we have

$$m_j = \frac{j^2 + n_\alpha^2}{4j^2 - 1}, \tag{2.112}$$

which is obviously nonnegative when $n_\alpha \in \mathbb{R}$ but also when $n_\alpha = i\nu_\alpha$ with $\nu_\alpha \in (-1, 1)$.

§ In conclusion, we can now characterize all the irreducible unitary representations of the Lorentz group $SO(1,3)$. There are two distinct classes of irreps, the so-called principal and complementary series. The representations in the principal series are labelled by a semi-integer nonnegative number j_0 and an arbitrary real number n. The representation space is

$$\mathcal{H}_{j_0,n} = \bigoplus_{k=0}^{\infty} \mathcal{H}_{j_0+k} \tag{2.113}$$

and the action of the operators $N_\pm$ and N_3 is given by

$$\begin{aligned} N_\pm|j_0, n; j, m\rangle &= \mp\hbar\sqrt{m_j(j_0,n)(j \mp m)(j \mp m - 1)}|j_0, n; j-1, m \pm 1\rangle + \\ &\quad + \hbar n_j(j_0,n)\sqrt{(j \mp m)(j \pm m + 1)}|j_0, n; j, m \pm 1\rangle \pm \\ &\quad \pm \hbar\sqrt{m_{j+1}(j_0,n)(j \pm m + 2)(j \pm m + 1)}|j_0, n; j+1, m \pm 1\rangle, && (2.114)\\ N_3|j_0, n; j, m\rangle &= -\hbar\sqrt{m_j(j_0,n)(j - m)(j + m)}|j_0, n; j-1, m\rangle + \\ &\quad + \hbar n_j(j_0,n) m|j_0, n; j, m\rangle - \\ &\quad - \hbar\sqrt{m_{j+1}(j_0,n)(j - m + 1)(j + m + 1)}|j_0, n; j+1, m\rangle, && (2.115) \end{aligned}$$

where

$$\begin{aligned} n_j(j_0,n) &= \frac{j_0 n}{j(j+1)}, && (2.116)\\ m_j(j_0,n) &= \frac{(j^2-j_0^2)(j^2+n^2)}{j^2(4j^2-1)}(1-\delta_{j,j_0}). && (2.117)\end{aligned}$$

The representations in the complementary series, instead, are labelled by a single real number $\nu \in [-1,1]$. The representation space is

$$\mathcal{H}_\nu = \bigoplus_{k=0}^{\infty} \mathcal{H}_k \tag{2.118}$$

and the action of the operators $N_\pm$ and N_3 is given by

$$\begin{aligned} N_\pm|\nu;j,m\rangle &= \mp\hbar\sqrt{m_j(\nu)(j\mp m)(j\mp m-1)}|\nu;j-1,m\pm 1\rangle \pm \\ &\pm \hbar\sqrt{m_{j+1}(\nu)(j\pm m+2)(j\pm m+1)}|\nu;j+1,m\pm 1\rangle, && (2.119)\\ N_3|\nu;j,m\rangle &= -\hbar\sqrt{m_j(\nu)(j-m)(j+m)}|\nu;j-1,m\rangle - \\ &- \hbar\sqrt{m_{j+1}(\nu)(j-m+1)(j+m+1)}|\nu;j+1,m\rangle, && (2.120)\end{aligned}$$

where

$$m_j(\nu) = \frac{j^2-\nu^2}{4j^2-1}(1-\delta_{j,0}). \tag{2.121}$$

By Schur's lemma, the Casimir elements M^2 and W must be multiple of the identity in every irreducible representation (j_0,n) or ν of the Lorentz group. Therefore, computing their expectation values on states $|\nu;j,m\rangle$ and $|j_0,n;j,m\rangle$, we can express their eigenvalues m^2 and w in terms of the representation labels n, j_0, ν. Using the identity

$$M^2 = \frac{1}{R^2c^2}(N^2-L^2) = \frac{1}{R^2c^2}(N_+N_- + N_3^2 + \hbar L_3 - L^2), \tag{2.122}$$

we find

$$\begin{aligned} m^2 &= \frac{\hbar^2}{R^2c^2}(1+n^2-j_0^2), && (2.123)\\ w &= \frac{\hbar^2}{Rc} n j_0, && (2.124)\end{aligned}$$

for the principal series and

$$\begin{aligned} m^2 &= \frac{\hbar^2}{R^2c^2}(1-\nu^2), && (2.125)\\ w &= 0, && (2.126)\end{aligned}$$

for the complementary one.

2.3.3 General kinematics

§ In order to describe a spinless quantum particle on a 2+1 de Sitter spacetime, it is necessary to identify the relevant algebra of dynamical variables and find its irreducible representations. The algebra of dynamical variables is generated by coordinate functions $f \in C^\infty(dS)$, giving the spacetime position of the particle, and translation generators p_v, which are associated with vector fields $v \in \mathcal{V}(dS)$ and give the generalized momenta of the particle. Both f and p_v are self-adjoint and the canonical commutation relations

$$\begin{aligned} [f, g] &= 0, && (2.127)\\ [p_v, f] &= i\hbar\{v, f\}, && (2.128)\\ [p_v, p_w] &= i\hbar p_{\{v,w\}}, && (2.129) \end{aligned}$$

are satisfied for every $f, g \in C^\infty$ and $v, w \in \mathcal{V}(dS)$. A 2+1 de Sitter spacetime dS, once embedded in a 3+1 Minkowskian spacetime, is a hyperboloid of the form

$$X_\mu X^\mu = R^2. \tag{2.130}$$

It is convenient to parametrize this manifold via the so-called flat coordinates $t, x, y \in \mathbb{R}$ defined by

$$\begin{aligned} X^0 &= R\sinh\frac{ct}{R} + \frac{x^2+y^2}{2R}e^{\frac{ct}{R}}, && (2.131)\\ X^1 &= e^{\frac{ct}{R}}x, && (2.132)\\ X^2 &= e^{\frac{ct}{R}}y, && (2.133)\\ X^3 &= R\cosh\frac{ct}{R} - \frac{x^2+y^2}{2R}e^{\frac{ct}{R}}. && (2.134) \end{aligned}$$

The Lebesgue measure $d\mu_L(t, x, y)$ on a 2+1 de Sitter spacetime in this coordinate system is given by

$$d\mu_L(t, x, y) = \delta(X_\mu X^\mu - R^2)d^4X = ce^{2\frac{ct}{R}}dt\,dx\,dy = ce^{2\frac{ct}{R}}dt\,dx\,dy. \tag{2.135}$$

If we take t, x and y as our fundamental position variables, it is natural to take the corresponding translation generators $p_t = p_{\partial_t}$, $p_x = p_{\partial_x}$ and $p_y = p_{\partial_y}$ as fundamental momentum variables, so that the algebra of dynamical variables is generated by six basis elements t, x, y, p_t, p_x and p_y satisfying

$$\begin{aligned} [t, x] &= [t, y] = [x, y] = 0, && (2.136)\\ [p_t, p_x] &= [p_t, p_y] = [p_x, p_y] = 0, && (2.137)\\ [p_t, x] &= [p_t, y] = 0, && (2.138)\\ [p_x, t] &= [p_x, y] = 0, && (2.139)\\ [p_y, t] &= [p_y, x] = 0, && (2.140)\\ [p_t, t] &= [p_x, x] = [p_y, y] = i\hbar. && (2.141) \end{aligned}$$

The unique irreducible representation of this algebra is the Schrödinger representation on $L^2(dS, d\mu_L)$ defined by

$$\begin{aligned}
\widehat{t}\psi(t,x,y) &= t\psi(t,x,y), && (2.142)\\
\widehat{x}\psi(t,x,y) &= x\psi(t,x,y), && (2.143)\\
\widehat{y}\psi(t,x,y) &= y\psi(t,x,y), && (2.144)\\
\widehat{p}_t\psi(t,x,y) &= i\hbar\left(\partial_t + \frac{c}{R}\right)\psi(t,x,y), && (2.145)\\
\widehat{p}_x\psi(t,x,y) &= i\hbar\partial_x\psi(t,x,y), && (2.146)\\
\widehat{p}_y\psi(t,x,y) &= i\hbar\partial_y\psi(t,x,y). && (2.147)
\end{aligned}$$

§ It is worth explicitly pointing out that this representation carries no information about the differential structure (covariant derivative, curvature, etc.) of the de Sitter manifold. In fact it is obviously unitarily equivalent to the Schrödinger representation for a quantum particle moving in standard Minkowski spacetime. This makes perfect sense from an operational point of view. In the real world we cannot tell apart a de Sitter spacetime from a Minkowskian one until we see some matter in motion, for example a bunch of free particles travelling along geodesics. The differential structure of spacetime depends on the (inertial) dynamics of matter and, therefore, we should not expect it to show up at a purely kinematical level. To make this clear, suppose that we are provided with a list of all possible instantaneous states of a certain material system, say the already mentioned bunch of free particles, complete with the expectation values of all the dynamical variables on those states. This is by definition a full solution to the kinematical problem. It is easy to see that we have no way of determining the curvature or other differential properties of the embedding spacetime from this information alone. In order to do that, we would instead need to compare two consecutive states of the system, necessarily bringing dynamical considerations into the picture.

2.3.4 Symmetries

§ The dynamics of a free particle on a 2+1 de Sitter spacetime must be covariant under the action of the Lorentz group $SO(1,3)$. This requirement is actually the dynamical definition of the system. We therefore face the task of identifying the self-adjoint generators of $SO(1,3)$ within the Schrödinger representation described before. The Lorentz group $SO(1,3)$ is the group of all linear transformations of 3+1 Minkowski spacetime which leave invariant the hyperbolic constraint (2.130) defining de Sitter manifolds. It is generated by the vector fields

$$m_{\mu\nu} = X_\mu \frac{\partial}{\partial X^\nu} - X_\nu \frac{\partial}{\partial X^\mu} = X_\mu\partial_\nu - X_\nu\partial_\mu. \tag{2.148}$$

From the Lie brackets

$$\{m_{\mu\nu}, m_{\rho\sigma}\} = \eta_{\rho\nu}m_{\mu\sigma} - \eta_{\mu\rho}m_{\nu\sigma} + \eta_{\sigma\nu}m_{\rho\mu} - \eta_{\mu\sigma}m_{\rho\nu} \tag{2.149}$$

we know that the corresponding translation generators $p_{\mu\nu} = p_{m_{\mu\nu}}$ must satisfy

$$[p_{\mu\nu}, p_{\rho\sigma}] = i\hbar(\eta_{\rho\nu}p_{\mu\sigma} - \eta_{\mu\rho}p_{\nu\sigma} + \eta_{\sigma\nu}p_{\rho\mu} - \eta_{\mu\sigma}p_{\rho\nu}). \tag{2.150}$$

Comparing these commutation relations with (2.51), we see that, for this particular system, $p_{\mu\nu}$ are the self-adjoint $SO(1,3)$ generators $M_{\mu\nu}$ introduced in the first section. Therefore, the redefined generators H, P_m, K_m and J are given by

$$\begin{aligned} H &= \frac{c}{R}p_{03}, & (2.151)\\ P_m &= \frac{1}{R}p_{m3}, & (2.152)\\ K_m &= \frac{1}{c}p_{0m}, & (2.153)\\ J &= p_{12}. & (2.154) \end{aligned}$$

In order to find their Schrödinger representation, we start expressing ∂_μ (actually its tangential component) in terms of ∂_t, ∂_x and ∂_y. Inverting the coordinate relations (2.131)-(2.134) we obtain

$$\begin{aligned} t &= \frac{R}{c}\log\left(\frac{X^0+X^3}{R}\right), & (2.155)\\ x &= \frac{RX^1}{X^0+X^3}, & (2.156)\\ y &= \frac{RX^2}{X^0+X^3}. & (2.157) \end{aligned}$$

Therefore, we have

$$\begin{aligned} \partial_0 &= \partial_0 t\,\partial_t + \partial_0 x\,\partial_x + \partial_0 y\,\partial_y = e^{-\frac{ct}{R}}\left(\frac{1}{c}\partial_t - \frac{x}{R}\partial_x - \frac{y}{R}\partial_y\right), & (2.158)\\ \partial_1 &= \partial_1 t\,\partial_t + \partial_1 x\,\partial_x + \partial_1 y\,\partial_y = e^{-\frac{ct}{R}}\partial_x, & (2.159)\\ \partial_2 &= \partial_2 t\,\partial_t + \partial_2 x\,\partial_x + \partial_2 y\,\partial_y = e^{-\frac{ct}{R}}\partial_y, & (2.160)\\ \partial_3 &= \partial_3 t\,\partial_t + \partial_3 x\,\partial_x + \partial_3 y\,\partial_y = e^{-\frac{ct}{R}}\left(\frac{1}{c}\partial_t - \frac{x}{R}\partial_x - \frac{y}{R}\partial_y\right). & (2.161) \end{aligned}$$

Now we can write $m_{\mu\nu}$ in terms of ∂_t, ∂_x and ∂_y

$$
\begin{aligned}
m_{01} &= \left(R\sinh\frac{ct}{R}+\frac{x^2+y^2}{2R}e^{\frac{ct}{R}}\right)\partial_1+e^{\frac{ct}{R}}x\,\partial_0= \\
&= \left(Re^{-\frac{ct}{R}}\sinh\frac{ct}{R}-\frac{x^2-y^2}{2R}\right)\partial_x-\frac{xy}{R}\,\partial_y+\frac{x}{c}\,\partial_t, \qquad (2.162)\\
m_{02} &= \left(R\sinh\frac{ct}{R}+\frac{x^2+y^2}{2R}e^{\frac{ct}{R}}\right)\partial_2+e^{\frac{ct}{R}}y\,\partial_0= \\
&= \left(Re^{-\frac{ct}{R}}\sinh\frac{ct}{R}-\frac{y^2-x^2}{2R}\right)\partial_y-\frac{xy}{R}\,\partial_x+\frac{y}{c}\,\partial_t, \qquad (2.163)\\
m_{03} &= \left(R\sinh\frac{ct}{R}+\frac{x^2+y^2}{2R}e^{\frac{ct}{R}}\right)\partial_3+\left(R\cosh\frac{ct}{R}-\frac{x^2+y^2}{2R}e^{\frac{ct}{R}}\right)\partial_0= \\
&= \frac{R}{c}\,\partial_t-x\,\partial_x-y\,\partial_y, \qquad (2.164)\\
m_{12} &= e^{\frac{ct}{R}}y\,\partial_1-e^{\frac{ct}{R}}x\,\partial_2=y\,\partial_x-x\,\partial_y, \qquad (2.165)\\
m_{13} &= \left(R\cosh\frac{ct}{R}-\frac{x^2+y^2}{2R}e^{\frac{ct}{R}}\right)\partial_1-e^{\frac{ct}{R}}x\,\partial_3= \\
&= \left(Re^{-\frac{ct}{R}}\cosh\frac{ct}{R}+\frac{x^2-y^2}{2R}\right)\partial_x+\frac{xy}{R}\,\partial_y-\frac{x}{c}\,\partial_t, \qquad (2.166)\\
m_{23} &= \left(R\cosh\frac{ct}{R}-\frac{x^2+y^2}{2R}e^{\frac{ct}{R}}\right)\partial_2-e^{\frac{ct}{R}}y\,\partial_3= \\
&= \left(Re^{-\frac{ct}{R}}\cosh\frac{ct}{R}+\frac{y^2-x^2}{2R}\right)\partial_y+\frac{xy}{R}\,\partial_x-\frac{y}{c}\,\partial_t, \qquad (2.167)
\end{aligned}
$$

and therefore $p_{\mu\nu}$ in terms of p_t, p_x and p_y

$$
\begin{aligned}
p_{01} &= \left(Re^{-\frac{ct}{R}}\sinh\frac{ct}{R} - \frac{x^2-y^2}{2R}\right)p_x - \frac{xy}{R}\,p_y + \frac{x}{c}\,p_t + \frac{1}{4R}[x^2,p_x] + \frac{x}{2R}[y,p_y] = \\
&= \left(Re^{-\frac{ct}{R}}\sinh\frac{ct}{R} - \frac{x^2-y^2}{2R}\right)p_x - \frac{xy}{R}\,p_y + \frac{x}{c}\left(p_t - i\hbar\frac{c}{R}\right), && (2.168)\\
p_{02} &= \left(Re^{-\frac{ct}{R}}\sinh\frac{ct}{R} - \frac{y^2-x^2}{2R}\right)p_y - \frac{xy}{R}\,p_x + \frac{y}{c}\,p_t + \frac{1}{4R}[y^2,p_y] + \frac{y}{2R}[x,p_x] = \\
&= \left(Re^{-\frac{ct}{R}}\sinh\frac{ct}{R} - \frac{y^2-x^2}{2R}\right)p_y - \frac{xy}{R}\,p_x + \frac{y}{c}\left(p_t - i\hbar\frac{c}{R}\right), && (2.169)\\
p_{03} &= \frac{R}{c}\,p_t - x\,p_x - y\,p_y + \frac{1}{2}[x,p_x] + \frac{1}{2}[y,p_y] = \frac{R}{c}\left(p_t - i\hbar\frac{c}{R}\right) - x\,p_x - y\,p_y, && (2.170)\\
p_{12} &= y\,p_x - x\,p_y, && (2.171)\\
p_{13} &= \left(Re^{-\frac{ct}{R}}\cosh\frac{ct}{R} + \frac{x^2-y^2}{2R}\right)p_x + \frac{xy}{R}\,p_y - \frac{x}{c}\,p_t - \frac{1}{4R}[x^2,p_x] - \frac{x}{2R}[y,p_y] = \\
&= \left(Re^{-\frac{ct}{R}}\cosh\frac{ct}{R} + \frac{x^2-y^2}{2R}\right)p_x + \frac{xy}{R}\,p_y - \frac{x}{c}\left(p_t + i\hbar\frac{c}{R}\right), && (2.172)\\
p_{23} &= \left(Re^{-\frac{ct}{R}}\cosh\frac{ct}{R} + \frac{y^2-x^2}{2R}\right)p_y + \frac{xy}{R}\,p_x - \frac{y}{c}\,p_t - \frac{1}{4R}[y^2,p_y] - \frac{y}{2R}[x,p_x] = \\
&= \left(Re^{-\frac{ct}{R}}\cosh\frac{ct}{R} + \frac{y^2-x^2}{2R}\right)p_y + \frac{xy}{R}\,p_x - \frac{y}{c}\left(p_t + i\hbar\frac{c}{R}\right). && (2.173)
\end{aligned}
$$

Using the known representation of p_t, p_x and p_y we finally find

$$
\begin{aligned}
\widehat{H}\psi &= \frac{c}{R}\widehat{p}_{03}\psi = i\hbar\left(\partial_t - c\frac{x}{R}\,\partial_x - c\frac{y}{R}\,\partial_y\right)\psi, && (2.174)\\
\widehat{P}_1\psi &= \frac{1}{R}\widehat{p}_{13}\psi = i\hbar\left\{\left(e^{-\frac{ct}{R}}\cosh\frac{ct}{R} + \frac{x^2-y^2}{2R^2}\right)\partial_x + \frac{xy}{R^2}\,\partial_y - \frac{x}{Rc}\partial_t\right\}\psi, && (2.175)\\
\widehat{P}_2\psi &= \frac{1}{R}\widehat{p}_{23}\psi = i\hbar\left\{\left(e^{-\frac{ct}{R}}\cosh\frac{ct}{R} + \frac{y^2-x^2}{2R^2}\right)\partial_y + \frac{xy}{R^2}\,\partial_x - \frac{y}{Rc}\partial_t\right\}\psi, && (2.176)\\
\widehat{K}_1\psi &= \frac{1}{c}\widehat{p}_{01}\psi = \frac{i\hbar}{c}\left\{\left(Re^{-\frac{ct}{R}}\sinh\frac{ct}{R} - \frac{x^2-y^2}{2R}\right)\partial_x - \frac{xy}{R}\,\partial_y + \frac{x}{c}\partial_t\right\}\psi, && (2.177)\\
\widehat{K}_2\psi &= \frac{1}{c}\widehat{p}_{02}\psi = \frac{i\hbar}{c}\left\{\left(Re^{-\frac{ct}{R}}\sinh\frac{ct}{R} - \frac{y^2-x^2}{2R}\right)\partial_y - \frac{xy}{R}\,\partial_x + \frac{y}{c}\partial_t\right\}\psi, && (2.178)\\
\widehat{J}\psi &= \widehat{p}_{12}\psi = i\hbar(y\,\partial_x - x\,\partial_y)\psi. && (2.179)
\end{aligned}
$$

Taking the limit $R \to \infty$ we recover, as we should, the standard Schrödinger representation of the generators of the 2+1 Poincaré group

$$\widehat{H}\psi \simeq i\hbar\partial_t\psi, \tag{2.180}$$
$$\widehat{P}_1\psi \simeq i\hbar\,\partial_x\psi, \tag{2.181}$$
$$\widehat{P}_2\psi \simeq i\hbar\,\partial_y\psi, \tag{2.182}$$
$$\widehat{K}_1\psi \simeq \frac{i\hbar}{c^2}\left(c^2 t\,\partial_x + x\,\partial_t\right)\psi, \tag{2.183}$$
$$\widehat{K}_2\psi \simeq \frac{i\hbar}{c^2}\left(c^2 t\,\partial_y + y\,\partial_t\right)\psi, \tag{2.184}$$
$$\widehat{J}\psi = i\hbar(y\,\partial_x - x\,\partial_y)\psi. \tag{2.185}$$

Finally, we can now find the representation of the two Casimir elements M^2 and W. Since we are modelling a spinless particle, we expect the spin operator $\widehat{W}$ to vanish identically. A tedious but straightforward computation actually gives

$$\begin{aligned} c^4\widehat{M}^2\psi &= \left\{\widehat{H}^2 - c^2\widehat{P}^2 - \frac{c^2}{R^2}(\widehat{J}^2 - c^2\widehat{K}^2)\right\}\psi = \\ &= -\hbar^2\left\{\partial_t^2 + 2\frac{c}{R}\,\partial_t - c^2 e^{-2\frac{ct}{R}}(\partial_x^2 + \partial_y^2)\right\}\psi, \end{aligned} \tag{2.186}$$
$$\widehat{W}\psi = \left(\widehat{P}_2\widehat{K}_1 - \widehat{P}_1\widehat{K}_2 + \frac{1}{c^2}\widehat{J}\widehat{H}\right)\psi = 0. \tag{2.187}$$

2.3.5 General dynamics

§ Having obtained the Schrödinger representation of the generators H, P_m, K_m and J of the $SO(1,3)$ symmetry group, we are ready to characterize the quantum dynamics of the system. In order to do that, we must specify a hamiltonian constraint H_{dS}. The algebra of observables will then be identified as the set of dynamical variables commuting with H_{dS} and the physical Hilbert space as the (improper) kernel of H_{dS} equipped with a suitably modified scalar product. Almost by definition, the hamiltonian constraint for a free particle moving on a de Sitter spacetime must be invariant under the action of the kinematical symmetry group $SO(1,3)$. Equivalently, it must commute with all the generators and therefore be a function of the nontrivial Casimir element M^2. Since we want to recover the standard Poincaré constraint

$$H_r(m_P) = H^2 - c^2P^2 - m_P^2c^4 \tag{2.188}$$

in the limit $R \to \infty$, the only natural choice for H_{dS} is

$$H_{dS}(m_{dS}) = M^2c^4 - m_{dS}^2c^4 = H^2 - c^2P^2 - \frac{c^2}{R^2}(J^2 - c^2K^2) - m_{dS}^2c^4, \tag{2.189}$$

where m_P^2 is the ordinary squared mass of the particle and m_{dS}^2 is such that $m_{dS}^2 \to m_P^2$ when $R \to \infty$. Being an eigenspace of both M^2 and W with eigenvalues m_{dS}^2 and 0 respectively, the physical Hilbert space of the system must be one of the $\mathcal{H}_\nu$, $\mathcal{H}_{0,n}$ or $\mathcal{H}_{j_0,0}$ analyzed before, the carriers of the irreducible representations of

$SO(1,3)$ with $w=0$. Recalling the expressions (2.123) and (2.125) for the Casimir element M^2, we see that the choice of $\mathcal{H}_\nu$, $\mathcal{H}_{0,n}$ or $\mathcal{H}_{j_0,0}$ depends on whether $\mu^2 \le 0$, $0<\mu^2<1$ or $\mu^2 \ge 1$, where

$$\mu^2 = \frac{m_{dS}^2}{m_*^2} \tag{2.190}$$

and

$$m_* = \frac{\hbar}{Rc}. \tag{2.191}$$

If $\mu^2 \le 0$, the physical Hilbert space is $\mathcal{H}_{j_0,0}$ with

$$j_0 = \sqrt{1-\mu^2}; \tag{2.192}$$

if $0<\mu^2<1$, it is $\mathcal{H}_\nu$ with

$$\nu = \sqrt{1-\mu^2}; \tag{2.193}$$

and finally, if $\mu^2 \ge 1$, it is $\mathcal{H}_{0,n}$ with

$$n = \sqrt{\mu^2-1}. \tag{2.194}$$

In the last two cases we could arbitrarily choose the positive roots because representations labelled by opposite values of n or ν are equivalent when $j_0=0$. Let us now suppose that the particle is massive, that is $m_P^2>0$. Since $m_{dS}^2 \to m_P^2$ when $R\to\infty$, we must also have that $\mu^2 \to +\infty$ in the same limit. In fact $m_{dS}^2 = \mu^2 m_*^2$ and $m_* \propto 1/R \to 0^+$ when $R\to\infty$. It is then clear that the only physically relevant representations are those of the type $(0,n)$, because the other ones cannot accomodate the Poincaré limit $R\to\infty$. In the following we will therefore assume that $\mu^2 \ge 1$. This assumption will also naturally suggest how to deal with the massless case $m_P^2=0$.

2.3.6 Physical Hilbert space I

§ In order to find the actual form of the physical states in the Schrödinger representation, it is necessary to find the kernel of the hamiltonian constraint operator

$$\widehat{H}_{dS}(m_{dS})\psi \quad = \quad -m_*^2c^4\left(\partial_\tau^2 + 2\partial_\tau - R^2e^{-2\tau}\nabla^2 + \mu^2\right)\psi, \tag{2.195}$$

where we have set

$$\tau = \frac{ct}{R} \tag{2.196}$$

and

$$\nabla^2 = \partial_x^2 + \partial_y^2. \tag{2.197}$$

We expect physical states to be eigenfunctions of the laplacian and therefore of the form

$$\psi_{\mu,k}(\tau,x,y) = f_{\mu,k}(\tau)e^{ik_x x}e^{ik_y y}. \tag{2.198}$$

Acting on $\psi_{\mu,k}$ with the hamiltonian constraint, we obtain the following differential equation for $f_{\mu,k}$:

$$\left(\partial_\tau^2 + 2\partial_\tau + R^2k^2e^{-2\tau} + \mu^2\right) f_{\mu,k}(\tau) = 0. \tag{2.199}$$

Changing variable to

$$\eta = e^{-\tau} \tag{2.200}$$

we get

$$\left(\eta^2\,\partial_\eta^2 - \eta\,\partial_\eta + R^2k^2\eta^2 + \mu^2\right) f_{\mu,k}(\eta) = 0, \tag{2.201}$$

and writing

$$f_{\mu,k}(\eta) = \eta g_{\mu,k}(\eta) \tag{2.202}$$

we finally find for $g_{\mu,k}$

$$\left(\eta^2\,\partial_\eta^2 + \eta\,\partial_\eta + R^2k^2\eta^2 + \mu^2 - 1\right) g_{\mu,k}(\eta) = 0. \tag{2.203}$$

This is the Bessel differential equation of order $\sqrt{1-\mu^2} = in$ in the variable $Rk\eta$. The general solution can be conveniently written as a linear combination of $H_{in}^{(1)}$ and $H_{in}^{(2)}$, the so-called Hankel functions of the first and second kind:

$$g_{\mu,k}(\eta) = C_1H_{in}^{(1)}(Rk\eta) + C_2H_{in}^{(2)}(Rk\eta). \tag{2.204}$$

We can conclude that the physical Hilbert space is spanned by wavefunctions of the form

$$\psi_{\mu,k}(t,x,y) = \left\{C_1H_{in}^{(1)}(Rke^{-\frac{ct}{R}}) + C_2H_{in}^{(2)}(Rke^{-\frac{ct}{R}})\right\} e^{-\frac{ct}{R}}e^{ik_x x}e^{ik_y y}. \tag{2.205}$$

§ In order to fix the arbitrary constants C_1 and C_2, we must require that we recover the usual Poincaré positive-energy physical states

$$\psi_{m_P,k}(t,x,y) \propto e^{-i\varepsilon(m_P,k)ct}e^{ik_x x}e^{ik_y y} \tag{2.206}$$

with

$$\varepsilon(m_P,k) = \sqrt{k^2 + \frac{m_P^2c^2}{\hbar^2}} \tag{2.207}$$

in the limit $R \to \infty$. For large positive β the following asymptotic relations hold for Hankel functions of imaginary order [44]:

$$H^{(1)}_{i\beta}(\beta z) = \sqrt{\frac{2}{\pi\beta}} e^{\frac{\pi\beta}{2}} e^{-\frac{i\pi}{4}} \frac{1}{\sqrt[4]{1+z^2}} \left\{ e^{i\beta\xi(z)} + \mathcal{O}(\beta^{-1}) \right\}, \tag{2.208}$$

$$H^{(2)}_{i\beta}(\beta z) = \sqrt{\frac{2}{\pi\beta}} e^{-\frac{\pi\beta}{2}} e^{\frac{i\pi}{4}} \frac{1}{\sqrt[4]{1+z^2}} \left\{ e^{-i\beta\xi(z)} + \mathcal{O}(\beta^{-1}) \right\}, \tag{2.209}$$

where

$$\xi(z) = \sqrt{1+z^2} + \log\left(\frac{z}{1+\sqrt{1+z^2}}\right). \tag{2.210}$$

If we set

$$m = nm_* = \sqrt{m_{dS}^2 - m_*^2} \tag{2.211}$$

and

$$z = \frac{Rk}{n} e^{-\frac{ct}{R}} = \frac{\hbar k}{mc} e^{-\frac{mc^2 t}{n\hbar}}, \tag{2.212}$$

for large $n \propto R$ we find

$$H^{(1)}_{in}(Rke^{-\frac{ct}{R}}) = H^{(1)}_{in}(nz) \simeq K^{(1)}_n \frac{1}{\sqrt[4]{1+z^2}} e^{in\xi(z)}, \tag{2.213}$$

$$H^{(2)}_{in}(Rke^{-\frac{ct}{R}}) = H^{(2)}_{in}(nz) \simeq K^{(2)}_n \frac{1}{\sqrt[4]{1+z^2}} e^{-in\xi(z)}, \tag{2.214}$$

up to terms of order n^{-1}. At first order in n^{-1} we have

$$z \simeq \frac{\hbar k}{mc} - \frac{kct}{n}, \tag{2.215}$$

$$\sqrt{1+z^2} \simeq \frac{\hbar\varepsilon(m,k)}{mc} - \frac{k^2 ct}{n\varepsilon(m,k)}, \tag{2.216}$$

$$\log\left(\frac{z}{1+\sqrt{1+z^2}}\right) \simeq \log\left(\frac{\hbar k}{mc+\hbar\varepsilon(m,k)}\right) - \frac{m^2c^3 t}{n\hbar^2\varepsilon(m,k)}, \tag{2.217}$$

$$\xi(z) \simeq \frac{\hbar\varepsilon(m,k)}{mc} + \log\left(\frac{\hbar k}{mc+\hbar\varepsilon(m,k)}\right) - \frac{\varepsilon(m,k)ct}{n} \tag{2.218}$$

Therefore, for large R,

$$H^{(1)}_{in}(Rke^{-\frac{ct}{R}}) \simeq C^{(1)} e^{-i\varepsilon(m,k)ct}, \tag{2.219}$$

$$H^{(2)}_{in}(Rke^{-\frac{ct}{R}}) \simeq C^{(2)} e^{i\varepsilon(m,k)ct}, \tag{2.220}$$

up to terms in ct/R. Choosing as physical states

$$\psi_{n,k}(t,x,y) = CH^{(1)}_{in}(Rke^{-\frac{ct}{R}}) e^{-\frac{ct}{R}} e^{ik_x x} e^{ik_y y}, \tag{2.221}$$

we recover the usual Poincaré states (2.206) with $m_P = m$. We can conclude that the relation between de Sitter and Poincaré masses is given by

$$m_{dS}^2 = m^2 + m_*^2 = m_P^2 + m_*^2 \tag{2.222}$$

and the physical Hilbert space is $\mathcal{H}_{0,n}$ with $n = m_P/m_*$.

§ Equipped with formula (2.222), we can now discuss the massless case. If $m_P^2 = 0$ we have $m_{dS}^2 = m_*^2$, so that n vanishes for all values of R. The physical Hilbert space $\mathcal{H}_{0,0}$ is then spanned by wavefunctions of the form

$$\psi_{0,k}(t,x,y) = \left\{ C_1 H_0^{(1)}(Rke^{-\frac{ct}{R}}) + C_2 H_0^{(2)}(Rke^{-\frac{ct}{R}}) \right\} e^{-\frac{ct}{R}} e^{ik_x x} e^{ik_y y}. \tag{2.223}$$

For large positive z the following asymptotic relations hold for $H_0^{(1)}$ and $H_0^{(2)}$ [44]:

$$\begin{aligned} H_0^{(1)}(z) &= \sqrt{\frac{2}{\pi z}} e^{-\frac{i\pi}{4}} \left\{ e^{iz} + \mathcal{O}(z^{-1}) \right\}, && (2.224) \\ H_0^{(2)}(z) &= \sqrt{\frac{2}{\pi z}} e^{\frac{i\pi}{4}} \left\{ e^{-iz} + \mathcal{O}(z^{-1}) \right\}. && (2.225) \end{aligned}$$

Therefore, for large R,

$$\begin{aligned} H_0^{(1)}(Rke^{-\frac{ct}{R}}) &\simeq C^{(1)} e^{-ikct} = C^{(1)} e^{-i\varepsilon(0,k)ct}, && (2.226) \\ H_0^{(2)}(Rke^{-\frac{ct}{R}}) &\simeq C^{(2)} e^{ikct} = C^{(2)} e^{i\varepsilon(0,k)ct}, && (2.227) \end{aligned}$$

up to terms in ct/R. Choosing as physical states

$$\psi_{0,k}(t,x,y) = CH_0^{(1)}(Rke^{-\frac{ct}{R}}) e^{-\frac{ct}{R}} e^{ik_x x} e^{ik_y y}, \tag{2.228}$$

we recover the usual massless Poincaré states in the limit $R \to \infty$. In conclusion, we have found that the physical Hilbert space for both massive and massless particles is $\mathcal{H}_{0,n}$ with $n = m_P/m_*$. In the Schrödinger representation an orthogonal basis is given by the wavefunctions

$$\psi_{n,k}(t,x,y) = CH_{in}^{(1)}(Rke^{-\frac{ct}{R}}) e^{-\frac{ct}{R}} e^{ik_x x} e^{ik_y y}. \tag{2.229}$$

The standard $L^2(dS, d\mu_L)$ scalar product

$$(f|g) = \int \overline{f}(t,x,y) g(t,x,y) c e^{2\frac{ct}{R}} dt\, dx\, dy \tag{2.230}$$

is obviously ill-defined on the $\psi_{n,k}(t,x,y)$. This is not surprising, since we are restricting ourselves to the physical Hilbert space, which is the kernel of the hamiltonian constraint operator $\widehat{H}_{dS}(n)$. The well-behaved, physical scalar product on $\mathcal{H}_{0,n}$ is actually

$$\langle f|g \rangle = \int \overline{f}(t,x,y) g(t,x,y) \delta\left(\widehat{H}_{dS}(n)\right) c e^{2\frac{ct}{R}} dt\, dx\, dy. \tag{2.231}$$

If we try to evaluate

$$\begin{aligned} \langle \psi_{n,k} | \psi_{n,k'} \rangle &= \int \overline{\psi_{n,k}}(t,x,y) \psi_{n,k'}(t,x,y) \delta\left(\widehat{H}_{dS}(n)\right) c e^{2\frac{ct}{R}} dt\, dx\, dy = \\ &= 4\pi^2 |C|^2 \delta^{(2)}(\mathbf{k} - \mathbf{k}') \int |H_{in}^{(1)}(Rke^{-\frac{ct}{R}})|^2 \delta\left(\widehat{H}_{dS}(n)\right) c dt \qquad (2.232) \end{aligned}$$

however, we are stuck with the complicated delta function in the integrand, which prevents us from finding the value of the normalization constant C. Thankfully, we do not have to solve this problem the hard way. In the following, we will arrive at the physical Hilbert space and the associated scalar product through independent arguments, thereby avoiding this obstruction altogether.

2.3.7 Observables

§ In the context of covariant quantum mechanics, the algebra of observables is the set of dynamical variables commuting with the hamiltonian constraint. The corresponding operators in the Schrödinger representation automatically admit a self-adjoint restriction to the physical Hilbert space and the dynamics of the theory can be extracted from their expectation values on physical states. In the de Sitter case, by construction, the generators of the kinematical symmetry group $SO(1,3)$ are the only independent variables commuting with $H_{dS}(m_{dS})$, and the algebra of observables is generated by H, P_m, K_m and J. However, since both $\widehat{H}_{dS}(m_{dS})$ and $\widehat{W}$ vanish identically on the physical Hilbert space, two of the operators $\widehat{H}$, $\widehat{P}_m$, $\widehat{K}_m$ and $\widehat{J}$, when restricted to $\mathcal{H}_{0,n}$, can be expressed in terms of the other. Therefore, the algebra of observables is actually generated by only four independent observables, as it should. Unfortunately, this abstract characterization of the observables does not say anything about the physical meaning of any given operator. For example, let us suppose to be interested in the particle's position at coordinate time t_0. All we know is that the observables $\widehat{x}(t_0)$ and $\widehat{y}(t_0)$ are some functions of the operators $\widehat{H}$, $\widehat{P}_m$, $\widehat{K}_m$ and $\widehat{J}$: we cannot pick the correct functions without additional information. In order to shed light on this issue, it is useful to review the dynamics of classical particles on a de Sitter spacetime.

2.3.8 Classical observables

§ Classical free point-particles on curved spacetimes travel along geodesics. Therefore, a covariant way of describing the dynamics is provided by the geodetic hamiltonian formalism. In this context, parametrized geodesics $x^\mu(s)$ are obtained as solutions of the canonical equations associated with the geodetic hamiltonian constraint

$$G(m_P) = \frac{1}{2}\left\{ g^{\mu\nu}(x) p_\mu p_\nu - m_P^2 c^2 \right\}, \tag{2.233}$$

where $g^{\mu\nu}(x)$ is the inverse metric of the spacetime manifold and m_P^2 the proper (Poincaré) mass of the particle. The constraint $G(m_P)$ is then required to identically vanish in order to enforce the mass-shell condition. Let us now specify this general theory to 2+1 de Sitter spacetime. In this case, the invariant interval in flat coordinates is given by

$$ds^2 = c^2 dt^2 - e^{2\frac{ct}{R}}(dx^2 + dy^2) \tag{2.234}$$

and the corresponding geodetic constraint is

$$G_{dS}(m_P) = \frac{1}{2}\left\{ \frac{1}{c^2} p_t^2 - e^{-2\frac{ct}{R}}(p_x^2 + p_y^2) - m_P^2 c^2 \right\}. \tag{2.235}$$

§ Before proceeding, it is worth pointing out that direct quantization of $2c^2 G_{dS}(m_P)$ yields immediately the covariant hamiltonian constraint $\widehat{H}_{dS}(m_{dS})$ obtained before.

In fact we have

$$\begin{aligned}
2c^2\widehat{G}_{dS}(m_P) &= \widehat{p}_t^2 - c^2\widehat{e^{-2\frac{ct}{R}}(\widehat{p}_x^2+\widehat{p}_y^2)} - m_P^2c^4 = \\
&= -\hbar^2\left\{\left(\partial_t + \frac{c}{R}\right)^2 - c^2e^{-2\frac{ct}{R}}(\partial_x^2+\partial_y^2) + \frac{m_P^2c^4}{\hbar^2}\right\} = \\
&= -m_*^2c^4\left\{\partial_\tau^2 + 2\partial_\tau - R^2e^{-2\tau}(\partial_x^2+\partial_y^2) + \frac{m_P^2}{m_*^2} + 1\right\}, \quad (2.236)
\end{aligned}$$

and the last expression is identical to $\widehat{H}_{dS}(m_{dS})$ if the relation between de Sitter and Poincaré masses is that given by (2.222). In a sense, this is actually another derivation of formula (2.222) for the de Sitter mass m_{dS}. The possibility of obtaining the quantum dynamics directly from the geodetic constraint will be crucial when we apply our formalism to spacetimes which are not maximally symmetric. The fact that in the simple case of a de Sitter spacetime the more fundamental symmetry-based approach led to the same result is an important consistency check for the more general procedure.

§ Returning to the classical theory, the canonical equations associated with $G_{dS}(m_P)$ are

$$k\dot{t} = \frac{1}{c^2}p_t, \tag{2.237}$$

$$k\dot{x} = -e^{-2\frac{ct}{R}}p_x, \tag{2.238}$$

$$k\dot{y} = -e^{-2\frac{ct}{R}}p_y, \tag{2.239}$$

$$k\dot{p}_x = 0, \tag{2.240}$$

$$k\dot{p}_y = 0, \tag{2.241}$$

where the dot stands for the derivative with respect to the affine parameter s and k is an arbitrary momentum scale. Since the momenta p_x and p_y are constants of motion, dividing the second and third equations by the first we obtain the following uncoupled differential equations for $x(t)$ and $y(t)$:

$$\frac{dx(t)}{dt} = -c^2e^{-2\frac{ct}{R}}\frac{p_x}{p_t} = -cp_xe^{-2\frac{ct}{R}}\frac{1}{\sqrt{e^{-2\frac{ct}{R}}p^2 + m_P^2c^2}}, \tag{2.242}$$

$$\frac{dy(t)}{dt} = -c^2e^{-2\frac{ct}{R}}\frac{p_y}{p_t} = -cp_ye^{-2\frac{ct}{R}}\frac{1}{\sqrt{e^{-2\frac{ct}{R}}p^2 + m_P^2c^2}}, \tag{2.243}$$

where in the last equalities we have used the mass-shell condition $G_{dS}(m_P) = 0$ and set $p^2 = p_x^2 + p_y^2$. Let us focus for definiteness on the equation for $x(t)$ (the other is formally identical). After the usual change of variable

$$\eta = e^{\frac{ct}{R}}, \tag{2.244}$$

it can be rewritten as

$$\frac{dx(\eta)}{d\eta} = -Rp_x\frac{1}{\eta^2\sqrt{p^2 + m_P^2c^2\eta^2}} \tag{2.245}$$

and the further substitution

$$m_P c\eta = p\sinh\chi \tag{2.246}$$

simplifies it to

$$\frac{dx(\chi)}{d\chi} = -\frac{Rm_P cp_x}{p^2}\frac{1}{\sinh^2\chi}. \tag{2.247}$$

The solution of the latter equation is trivially

$$x(\chi) = \frac{Rm_P cp_x}{p^2}\coth\chi + K, \tag{2.248}$$

and tracing back all the substitutions we finally find that

$$x(t) = \frac{Rp_x}{p^2}\sqrt{p^2 e^{-2\frac{ct}{R}} + m_P^2 c^2} + C_x, \tag{2.249}$$

or, equivalently, that

$$C_x = x - \frac{Rp_x}{p^2}\sqrt{p^2 e^{-2\frac{ct}{R}} + m_P^2 c^2} \tag{2.250}$$

is a constant of motion. The conserved quantities p_x, p_y and C_x contain all the information about the x-trajectory of the particle, i.e. about the correlation between dynamical variables x and t: in fact, we can compute the particle x coordinate at time t_0, which is another constant of motion, as

$$x(t_0) = C_x + \frac{Rp_x}{p^2}\sqrt{p^2 e^{-2\frac{ct_0}{R}} + m_P^2 c^2}. \tag{2.251}$$

The previous expression can be rewritten in terms of the basis variables and reads

$$x(t_0) = x - \frac{Rp_x}{p^2}\sqrt{p^2 e^{-2\frac{ct}{R}} + m_P^2 c^2} + \frac{Rp_x}{p^2}\sqrt{p^2 e^{-2\frac{ct_0}{R}} + m_P^2 c^2}. \tag{2.252}$$

In the limit $R \to \infty$ we have

$$\sqrt{p^2 e^{-2\frac{ct}{R}} + m_P^2 c^2} \simeq E(m_P, p)\left(\frac{1}{c} - \frac{p^2 c^2 t}{RE(m_P, p)^2}\right) \tag{2.253}$$

with

$$E(m_P, p) = \sqrt{p^2 c^2 + m_P^2 c^4}, \tag{2.254}$$

and $x(t_0)$ correctly assumes its familiar Poincaré form

$$x(t_0) \simeq x + \frac{p_x c^2}{E(m_P, p)}(t - t_0). \tag{2.255}$$

The four independent constants of motion p_x, p_y, C_x and C_y, obviously given by

$$C_y = y - \frac{Rp_y}{p^2}\sqrt{p^2 e^{-2\frac{ct}{R}} + m_P^2 c^2}, \tag{2.256}$$

do not just describe trajectories, but actually characterize the correlation of any dynamical variable with any other. In fact, let us suppose to be interested in the value of some variable a when another independent variable $b = b_0$. If both a and b do not depend on the constants, i.e. if they are not conserved quantities, we can use a, b, p_x, p_y, C_x and C_y to coordinatize the algebra of dynamical variables instead of the standard basis x, y, t, p_x, p_y and p_t. After the enforcement of the hamiltonian constraint (2.235), the six generators are not independent anymore. This means that it is possible, in general, to write any one of them, say a, as a function of the other five, obtaining

$$a = f_{a|b}(b, p_x, p_y, C_x, C_y). \tag{2.257}$$

Knowing the function $f_{a|b}$, we can finally compute the value of a when $b = b_0$, which is the constant of motion

$$a(b_0) = f_{a|b}(b_0, p_x, p_y, C_x, C_y). \tag{2.258}$$

This procedure is a straightforward generalization of what we have done before to find $x(t_0)$. This example clearly shows how, given any well-posed dynamical question, it is possible to identify a conserved quantity encoding the answer. In the next paragraph we will apply this technique to our quantum model.

2.3.9 Quantum observables

§ As discussed before, the algebra of observables of a quantum particle propagating on a de Sitter spacetime is generated by four independent functions of the six $SO(1,3)$ generators H, P_m, K_m and J. Since

$$p_m = P_m + \frac{c}{R} K_m \tag{2.259}$$

are constants of motion, it is natural to choose H, p_m and J as basis observables. In fact, the momentum of the particle at time t_0 measured by static observers with respect to our flat coordinate system is given by

$$p^m(t_0) = -e^{-2\frac{ct_0}{R}} p_m, \tag{2.260}$$

so that $-p_m$ can be given a direct physical interpretation as the particle's momentum at time 0. Moreover, p_m are the best observables in terms of which to express other physically relevant constants of motion, such as the positions $x(t_0)$ and $y(t_0)$ or

$$E(t_0) = p_t(t_0) = c\sqrt{e^{-2\frac{ct_0}{R}}(p_x^2 + p_y^2) + m_P^2 c^2} = \sqrt{e^{-2\frac{ct_0}{R}} p^2 c^2 + m_P^2 c^4}, \tag{2.261}$$

that is the particle's energy measured by static observers in flat coordinates. Having chosen our basis observables, we are left with the task of finding functions $f_{a|b}(H, p_m, J)$ corresponding to interesting dynamical questions, such as positions and momenta at time t_0. Simple expressions for the latter were readily obtained in the previous section, where we were also able to express $x(t_0)$ and $y(t_0)$ in terms of C_x and C_y as

$$\begin{aligned} x(t_0) &= C_x + \frac{Rp_x}{p^2}\sqrt{p^2 e^{-2\frac{ct_0}{R}} + m_P^2 c^2} = C_x + \frac{Rp_x E(t_0)}{cp^2}, && (2.262)\\ y(t_0) &= C_y + \frac{Rp_y}{p^2}\sqrt{p^2 e^{-2\frac{ct_0}{R}} + m_P^2 c^2} = C_y + \frac{Rp_y E(t_0)}{cp^2}. && (2.263)\end{aligned}$$

In order to find $f_{x|t}(H, p_m, J)$ and $f_{y|t}(H, p_m, J)$, it is therefore sufficient to write C_x and C_y as functions of H, p_m and J. Since

$$\begin{aligned} H &= p_t - \frac{c}{R}(xp_x + yp_y + i\hbar), && (2.264)\\ J &= yp_x - xp_y, && (2.265)\end{aligned}$$

we easily obtain

$$\begin{aligned} C_x &= x - \frac{Rp_x p_t}{cp^2} = -\left(\frac{R}{c}H + i\hbar\right)\frac{p_x}{p^2} - J\frac{p_y}{p^2}, && (2.266)\\ C_y &= y - \frac{Rp_y p_t}{cp^2} = -\left(\frac{R}{c}H + i\hbar\right)\frac{p_y}{p^2} + J\frac{p_x}{p^2}, && (2.267)\end{aligned}$$

so that

$$\begin{aligned} x(t_0) &= C_x + \frac{Rp_x E(t_0)}{cp^2} = \left(E(t_0) - H - i\hbar\frac{c}{R}\right)\frac{Rp_x}{cp^2} - J\frac{p_y}{p^2}, && (2.268)\\ y(t_0) &= C_y + \frac{Rp_y E(t_0)}{cp^2} = \left(E(t_0) - H - i\hbar\frac{c}{R}\right)\frac{Rp_y}{cp^2} + J\frac{p_x}{p^2}. && (2.269)\end{aligned}$$

We can check the correctness of this result by directly computing the commutators between p_m, $x(t_0)$ and $y(t_0)$. Making use of the relations

$$\begin{aligned} [H, p_m] &= [H, P_m] + \frac{c}{R}[H, K_m] = \frac{i\hbar c^2}{R^2}K_m + \frac{i\hbar c}{R}P_m = \frac{i\hbar c}{R}p_m, && (2.270)\\ [J, p_m] &= [J, P_m] + \frac{c}{R}[J, K_m] = i\hbar\varepsilon_{mn}P_n + \frac{c}{R}i\hbar\varepsilon_{mn}K_m = i\hbar\varepsilon_{mn}p_n, && (2.271)\end{aligned}$$

we find

$$\begin{aligned} [p_m, x(t_0)] &= [p_m, C_x] = -[p_m, H]\frac{Rp_x}{cp^2} - [p_m, J]\frac{p_y}{p^2} = \\ &= i\hbar\frac{p_x p_m}{p^2} + i\hbar\varepsilon_{mn}\frac{p_y p_n}{p^2} = i\hbar\delta_{m1}, && (2.272)\\ [p_m, y(t_0)] &= [p_m, C_y] = -[p_m, H]\frac{Rp_y}{cp^2} + [p_m, J]\frac{p_x}{p^2} = \\ &= i\hbar\frac{p_y p_m}{p^2} - i\hbar\varepsilon_{mn}\frac{p_x p_n}{p^2} = i\hbar\delta_{m2}, && (2.273)\end{aligned}$$

and finally

$$\begin{aligned} [x(t_0), y(t_0)] &= [C_x, C_y] + \left[C_x, \frac{Rp_y E(t_0)}{cp^2}\right] + \left[\frac{Rp_x E(t_0)}{cp^2}, C_y\right] = \\ &= [C_x, C_y] + \frac{Re^{-2\frac{ct_0}{R}}p_x}{cE(t_0)p^2}p_y - \frac{2RE(t_0)p_x}{cp^4}p_y - \\ &\quad - \frac{Re^{-2\frac{ct_0}{R}}p_y}{cE(t_0)p^2}p_x + \frac{2RE(t_0)p_y}{cp^4}p_x = [C_x, C_y] = 0, && (2.274)\end{aligned}$$

where in the last step we have

$$\begin{aligned}
[C_x, C_y] &= \left[-\left(\frac{R}{c}H + i\hbar\right)\frac{p_x}{p^2} - J\frac{p_y}{p^2}, -\left(\frac{R}{c}H + i\hbar\right)\frac{p_y}{p^2} + J\frac{p_x}{p^2}\right] = \\
&= \left(\frac{R}{c}H + i\hbar\right)\left\{\frac{R}{c}\left[\frac{p_x}{p^2}, H\right]\frac{p_y}{p^2} + \frac{R}{c}\left[H, \frac{p_y}{p^2}\right]\frac{p_x}{p^2} + \left[J, \frac{p_y}{p^2}\right]\frac{p_y}{p^2} - \left[\frac{p_x}{p^2}, J\right]\frac{p_x}{p^2}\right\} + \\
&\quad + J\left\{\frac{R}{c}\left[\frac{p_y}{p^2}, H\right]\frac{p_y}{p^2} - \frac{R}{c}\left[H, \frac{p_x}{p^2}\right]\frac{p_x}{p^2} - \left[\frac{p_y}{p^2}, J\right]\frac{p_x}{p^2} - \left[J, \frac{p_x}{p^2}\right]\frac{p_y}{p^2}\right\} = 0.
\end{aligned}$$

Since $x(t_0)$, $y(t_0)$ and p_m satisfy canonical commutation relations, the identification of $x(t_0)$ and $y(t_0)$ as flat coordinates at time t_0 is justified.

2.3.10 Physical Hilbert space II

§ We are now ready to tackle the problem of the physical Hilbert space and the associated scalar product. In the previous paragraph we saw that the algebra of observables is generated by four independent elements $p^m(0) = -p_m$, $x(0)$ and $y(0)$ which satisfy the canonical commutation relations

$$[x(0), y(0)] = [p^m(0), p^n(0)] = 0, \tag{2.275}$$
$$[x(0), p^m(0)] = i\hbar\delta_1{}^m, \tag{2.276}$$
$$[y(0), p^m(0)] = i\hbar\delta_2{}^m. \tag{2.277}$$

This means that its unique irreducible representation is the usual Schrödinger representation on $L^2(\mathbb{R}^2, d\chi\, d\gamma)$, which is given by

$$\widehat{x}(0)\Psi(\chi,\gamma) = i\hbar\partial_\chi\Psi(\chi,\gamma), \tag{2.278}$$
$$\widehat{y}(0)\Psi(\chi,\gamma) = i\hbar\partial_\gamma\Psi(\chi,\gamma), \tag{2.279}$$
$$\widehat{p}^x(0)\Psi(\chi,\gamma) = \chi\Psi(\chi,\gamma), \tag{2.280}$$
$$\widehat{p}^y(0)\Psi(\chi,\gamma) = \gamma\Psi(\chi,\gamma). \tag{2.281}$$

We can now find the operators corresponding to other interesting observables by simply expressing them as functions of $p^m(0)$, $x(0)$ and $y(0)$. For $p^m(t_0)$, $E(t_0)$, $x(t_0)$, $y(t_0)$, H and J we have

$$p^m(t_0) = e^{-2\frac{ct_0}{R}}p^m(0), \tag{2.282}$$
$$E(t_0) = \sqrt{e^{-2\frac{ct_0}{R}}p^2(0)c^2 + m_P^2c^4}, \tag{2.283}$$
$$x(t_0) = x(0) + \frac{Rp^x(0)}{cp^2(0)}\left\{E(0) - E(t_0)\right\}, \tag{2.284}$$
$$y(t_0) = y(0) + \frac{Rp^y(0)}{cp^2(0)}\left\{E(0) - E(t_0)\right\}, \tag{2.285}$$
$$H = E(0) + \frac{c}{R}\left\{p^x(0)x(0) + p^y(0)y(0) + i\hbar\right\}, \tag{2.286}$$
$$J = p^y(0)x(0) - p^x(0)y(0), \tag{2.287}$$

and therefore obtain

$$\widehat{p}^x(t_0)\Psi(\chi,\gamma) = e^{-2\frac{ct_0}{R}}\chi\Psi(\chi,\gamma), \tag{2.288}$$

$$\widehat{p}^y(t_0)\Psi(\chi,\gamma) = e^{-2\frac{ct_0}{R}}\gamma\Psi(\chi,\gamma), \tag{2.289}$$

$$\widehat{E}(t_0)\Psi(\chi,\gamma) = \sqrt{e^{-2\frac{ct_0}{R}}\Pi^2c^2 + m_P^2c^4}\,\Psi(\chi,\gamma), \tag{2.290}$$

$$\widehat{x}(t_0)\Psi(\chi,\gamma) = \left(i\hbar\partial_\chi + \frac{R\chi}{c\Pi^2}\left\{\widehat{E}(0) - \widehat{E}(t_0)\right\}\right)\Psi(\chi,\gamma), \tag{2.291}$$

$$\widehat{y}(t_0)\Psi(\chi,\gamma) = \left(i\hbar\partial_\gamma + \frac{R\gamma}{c\Pi^2}\left\{\widehat{E}(0) - \widehat{E}(t_0)\right\}\right)\Psi(\chi,\gamma), \tag{2.292}$$

$$\widehat{H}\Psi(\chi,\gamma) = \left(\widehat{E}(0) + \frac{i\hbar c}{R}\left\{1 + \chi\partial_\chi + \gamma\partial_\gamma\right\}\right)\Psi(\chi,\gamma), \tag{2.293}$$

$$\widehat{J}\Psi(\chi,\gamma) = i\hbar(\gamma\partial_\chi - \chi\partial_\gamma)\Psi(\chi,\gamma), \tag{2.294}$$

where $\Pi^2 = \chi^2 + \gamma^2$. All these operators are easily shown to be self-adjoint with respect to the standard $L^2(\mathbb{R}^2, d\chi\, d\gamma)$ scalar product

$$\langle\Psi|\Phi\rangle = \int \overline{\Psi}(\chi,\gamma)\Phi(\chi,\gamma)d\chi\, d\gamma, \tag{2.295}$$

which must then be the correct, physical one. This completes the definition of our quantum model.

Chapter 3

Covariant quantum mechanics and noncommutative spacetimes

§ In this chapter, I introduce spacetime noncommutativity and characterize all possible CQM-based models of noncommutative spacetime together with their symmetries. After some preliminary general remarks about the concept of spacetime noncommutativity, I review the Hopf-algebraic treatment of noncommutative spacetime symmetries and then explain how to use CQM to build single-particle noncommutative spacetime models whose deformed relativistic symmetries admit a standard quantum-mechanical description. Part of this chapter has already been published in [45]. At the end, as an illustrative example, a model of Θ-Minkowski noncommutativity is investigated in some detail.

3.1 Spacetime noncommutativity

§ In recent years a sizeable literature has been devoted to the study of noncommutative spacetimes such as Θ-Minkowski

$$\left[x^\nu, x^\lambda\right] = i\Theta^{\nu\lambda}, \tag{3.1}$$

or κ-Minkowski

$$\left[x^\nu, x^i\right] = i\ell{\delta_0}^\nu x^i, \tag{3.2}$$

where the antisymmetric matrix $\Theta^{\nu\lambda}$ and the parameter ℓ are assumed to be relativistic invariants of the order of the Planck area and the Planck length respectively. The general idea is to explore the physical consequences of assuming nontrivial commutation relations among spacetime coordinates as in the above examples. But what does this exactly mean?

§ It is well known that the geometric properties of an ordinary manifold are completely encoded in the commutative algebra of coordinate functions defined on it.

Therefore, in a somewhat abstract but definite sense, ordinary geometry is the study of commutative algebras looked at as the algebras of coordinate functions on some manifold. Even though noncommutative algebras cannot be interpreted as algebras of coordinate functions on any ordinary manifold, the algebraic translation of the standard geometric concepts, suitably generalized, might make sense even in this more general context. If this were the case, it would be possible to define noncommutative manifolds as geometric objects identified by noncommutative algebras of coordinate functions and in turn to look at noncommutative algebras from a geometric point of view. Noncommutative geometry would be a major breakthrough in the field of pure mathematics but could also have far-reaching consequences in the domain of theoretical physics. For example it could lead to a coordinate-free quantum mechanics in the same way symplectic geometry led to a coordinate-free classical mechanics. In the last decades an increasing number of mathematicians and mathematical physicists have been working on noncommutative geometry and significant progress has been made on several issues, even though a general theory is still lacking.

§ In general relativity spacetime is regarded as a riemannian manifold whose geometric properties are dynamically tied with its energy-momentum content. It is reasonable to expect that quantum gravity effects should modify this picture since the gravitational field, which determines the geometric properties of spacetime, should become somehow quantized in this regime. The spacetime noncommutativity approach to quantum gravity is based on the idea that, regardless of the specific form a fully-fledged theory of quantum gravity may take, spacetime should become as a result a noncommutative manifold in the sense of the previous paragraph [24]. If this were the case, noncommutative geometry would be the correct language in which to formulate and address the quantum gravity problem. In addition, even before addressing the problem of spacetime dynamics, by merely adapting existing physical theories (such as classical or quantum field theory) to the new noncommutative framework it should be possible to obtain corrections to known physics which, being independent of the details of quantum gravity, would be very important from a phenomenological perspective [11]. This intriguing possibility has stimulated a search for potentially observable physical effects associated to the simplest examples of noncommutative spacetimes, such as those mentioned at the beginning. In particular, new field-theoretical techniques have been developed to deal with fields propagating on a noncommutative background manifold [25–27, 30].

3.2 Symmetries of noncommutative spacetimes

§ Continuous symmetries play a very important rôle in the characterization of spacetime manifolds. For example, the standard Minkowski spacetime is uniquely identified requiring that it is left invariant by any Poincaré transformation. In the case of an ordinary spacetime manifold, continuous symmetry transformations form a Lie group of diffeomorphisms, which is generated by a Lie algebra of independent derivation operators acting on the coordinates. In order to fully characterize, at least locally, the symmetry group, it is sufficient to specify the action of the Lie generators on an arbitrary coordinate basis. A simple illustration is provided once again by the Poincaré transformations of standard Minkowski spacetime, which are generated by the following ten derivations:

$$P_\mu \triangleright f(x) = i\partial_\mu f(x) \tag{3.3}$$
$$M_{\rho\sigma} \triangleright f(x) = i(x_\rho\partial_\sigma - x_\sigma\partial_\rho)f(x). \tag{3.4}$$

Given their action on the standard basis x^ν

$$P_\mu \triangleright x^\nu = i\delta_\mu^{\ \nu}, \tag{3.5}$$
$$M_{\rho\sigma} \triangleright x^\nu = i(x_\rho\delta_\sigma^{\ \nu} - x_\sigma\delta_\rho^{\ \nu}), \tag{3.6}$$

the generators L_j are uniquely determined on the whole algebra of coordinate functions via linearity

$$L_j \triangleright f + g = L_j \triangleright f + L_j \triangleright g, \tag{3.7}$$

the Leibniz rule

$$L_j \triangleright fg = (L_j \triangleright f)g + f(L_j \triangleright g), \tag{3.8}$$

and their action on the constant function **1**

$$L_j \triangleright \mathbf{1} = 0. \tag{3.9}$$

This minimal characterization of symmetries is purely algebraic and could in principle make sense even in the noncommutative case. Unfortunately, this simple Lie structure is too limited to account for the geometry of noncommutative spacetimes. In fact, the application of an ordinary Lie generator to the commutator

$$[f, g] = fg - gf = h \neq 0$$

would yield

$$L_j \triangleright [f, g] = [L_j \triangleright f, g] + [f, L_j \triangleright g] = L_j \triangleright h, \tag{3.10}$$

which is too stringent a requirement to provide interesting symmetry groups for general noncommutative spacetimes.

§ As an example, let us consider Θ-Minkowski spacetime. We would like to find deformed generators $P_\mu^{(\Theta)}$ and $M_{\rho\sigma}^{(\Theta)}$ whose action reduces to the Minkowski one (3.5)-(3.6) in the limit $\Theta^{\nu\lambda} \to 0$. In this case, condition (3.10) translates into

$$L_j \triangleright \left[x^\nu, x^\lambda\right] = \left[L_j \triangleright x^\nu, x^\lambda\right] + \left[x^\nu, L_j \triangleright x^\lambda\right] = \tag{3.11}$$
$$= L_j \triangleright i\Theta^{\nu\lambda} = 0, \tag{3.12}$$

and it is easy to see that $P_\mu^{(\Theta)} = P_\mu$, the usual translation generators, satisfy it. However, introducing the shorthand notation

$$\Upsilon_{\rho\ \sigma}^{\ \alpha\ \beta} = \left(\Theta_\rho^{\ \alpha}\delta_\sigma^{\ \beta} - \Theta_\sigma^{\ \alpha}\delta_\rho^{\ \beta}\right) - \left(\Theta_\rho^{\ \beta}\delta_\sigma^{\ \alpha} - \Theta_\sigma^{\ \beta}\delta_\rho^{\ \alpha}\right), \tag{3.13}$$

we have in general that

$$\begin{aligned} M_{\rho\sigma} \triangleright \left[x^\nu, x^\lambda\right] &= \left[M_{\rho\sigma} \triangleright x^\nu, x^\lambda\right] + \left[x^\nu, M_{\rho\sigma} \triangleright x^\lambda\right] = \\ &= -\Theta_\rho{}^\lambda \delta_\sigma{}^\nu + \Theta_\sigma{}^\lambda \delta_\rho{}^\nu + \Theta_\rho{}^\nu \delta_\sigma{}^\lambda - \Theta_\sigma{}^\nu \delta_\rho{}^\lambda = \Upsilon_\rho{}^\nu{}_\sigma{}^\lambda \neq 0, \end{aligned} \quad (3.14)$$

and a generic Θ-deformation of the form

$$M^{(\Theta)}_{\rho\sigma} \triangleright x^\nu = i(x_\rho g_{\sigma\nu} - x_\sigma g_{\rho\nu} + \Theta_\alpha{}^\beta F_\beta{}^\alpha(x)) \quad (3.15)$$

cannot compensate in any way a non-vanishing $\Upsilon_\rho{}^\nu{}_\sigma{}^\lambda$ of order Θ, since it would just introduce into (3.14) terms of order Θ^2. As claimed before, we see that, even for simple noncommutative spacetimes, it may be impossible to find a physically interesting deformation of Poincaré symmetries without relaxing the ordinary Lie-algebraic structure of the generators.

3.2.1 Hopf algebras

§ In pure mathematics, a Hopf algebra or quantum group is a unital, associative algebra $\mathcal{H}$ over the complex field equipped with three linear maps $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$, $\kappa : \mathcal{H} \to \mathbb{C}$ and $\Sigma : \mathcal{H} \to \mathcal{H}$. Let us write the action of Δ on a generic algebra element $H \in \mathcal{H}$ as

$$\Delta H = \sum_{k=1}^{n(H)} H_k^{(1)} \otimes H_k^{(2)}. \quad (3.16)$$

Then, for the quadruple $(\mathcal{H}, \Delta, \Gamma, \Sigma)$ to define a Hopf algebra, the following properties must hold for every $H, L \in \mathcal{H}$:

- both Δ and κ must be algebra homomorphisms, that is

$$\kappa(HL) = \kappa(H)\kappa(L), \quad (3.17)$$
$$\Delta(HL) = \Delta H \Delta L; \quad (3.18)$$

- the map Δ (the so-called coproduct) must be coassociative, that is

$$\sum_{k=1}^{n(H)} \Delta H_k^{(1)} \otimes H_k^{(2)} = \sum_{k=1}^{n(H)} H_k^{(1)} \otimes \Delta H_k^{(2)}; \quad (3.19)$$

- the map κ (the so-called counit) must satisfy

$$\sum_{k=1}^{n(H)} \kappa(H_k^{(1)}) H_k^{(2)} = \sum_{k=1}^{n(H)} H_k^{(1)} \kappa(H_k^{(2)}) = H; \quad (3.20)$$

- the map Σ (the so-called antipode) must satisfy

$$\sum_{k=1}^{n(H)} \Sigma(H_k^{(1)}) H_k^{(2)} = \sum_{k=1}^{n(H)} H_k^{(1)} \Sigma(H_k^{(2)}) = \kappa(H)\mathbb{1}, \quad (3.21)$$

  where $\mathbb{1}$ is the identity in $\mathcal{H}$.

Starting from these axioms, one can prove that the antipode is an antihomomorphism, that is

$$\Sigma(HL) \;=\; \Sigma(L)\Sigma(H), \tag{3.22}$$

for every $H, L \in \mathcal{H}$ [31]. In the following we will be concerned with finitely generated Hopf algebras, and this theorem makes their characterization very easy. In that case, in fact, the action of the three maps Δ, κ and Σ on a generic $H \in \mathcal{H}$ is univocally determined via properties (3.17), (3.18) and (3.22) by their action on the generators $H_j \in \mathcal{H}$ and their trivial action on the identity

$$\Delta(\mathbb{1}) \;=\; \mathbb{1} \otimes \mathbb{1}, \tag{3.23}$$
$$\kappa(\mathbb{1}) \;=\; 1, \tag{3.24}$$
$$\Sigma(\mathbb{1}) \;=\; \mathbb{1}. \tag{3.25}$$

§ Hopf algebras can seem quite abstract objects, but their definition is actually motivated by a very important example. Let us consider a unital, associative algebra $\mathcal{X}$ with unity $\mathbf{1} \in \mathcal{X}$ and product $\mu : \mathcal{X} \otimes \mathcal{X} \to \mathcal{X}$ generated by a finite number of elements $x_i \in \mathcal{X}$. The set of linear transformations from $\mathcal{X}$ to itself is another unital, associative algebra $\mathcal{H}_{\mathcal{X}}$: the identity is given by the trivial transformation

$$\mathbb{1} \triangleright f = f \tag{3.26}$$

and the associative product of $H, L \in \mathcal{H}_{\mathcal{X}}$ by the composition

$$HL \triangleright f = H \triangleright (L \triangleright f). \tag{3.27}$$

Let us now suppose that, for some algebra $\mathcal{H} \subseteq \mathcal{H}_{\mathcal{X}}$, we know the action of every $H \in \mathcal{H}$ on the generators $x_i \in \mathcal{X}$ and there exist two linear maps $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ and $\kappa : \mathcal{H} \to \mathbb{C}$ such that

$$H \triangleright fg \;=\; H \triangleright \mu(f \otimes g) \;=\; \mu(\Delta H \triangleright f \otimes g), \tag{3.28}$$
$$H \triangleright \mathbf{1} \;=\; \kappa(H)\mathbf{1}, \tag{3.29}$$

for every $f, g \in \mathcal{X}$ and $H \in \mathcal{H}$. In this case, the action of $\mathcal{H}$ on the whole of $\mathcal{X}$ is univocally determined by its action on the generators x_i. In fact, using the notation (3.16) for ΔH, we can express the action of a generic $H \in \mathcal{H}$ on a product fg in terms of the action of $\mathcal{H}$ on the factors,

$$H \triangleright fg \;=\; \mu(\Delta H \triangleright f \otimes g) \;=\; \sum_{k=1}^{n(H)} (H_k^{(1)} \triangleright f)(H_k^{(2)} \triangleright g), \tag{3.30}$$

and (3.29) directly gives the action of H on the empty product, i.e. on the identity $\mathbf{1}$. From the definition of the product on $\mathcal{H}$, we have

$$\begin{aligned}
\kappa(HL)\mathbf{1} \;&=\; HL \triangleright \mathbf{1} = H \triangleright (L \triangleright \mathbf{1}) = H \triangleright \kappa(L)\mathbf{1} = \kappa(H)\kappa(L)\mathbf{1}, \\
m(\Delta(HL) \triangleright f \otimes g) \;&=\; HL \triangleright fg = H \triangleright (L \triangleright fg) = H \triangleright m(\Delta L \triangleright f \otimes g) = \\
&=\; m(\Delta H \triangleright (\Delta L \triangleright f \otimes g)) = m(\Delta H \Delta L \triangleright f \otimes g)),
\end{aligned}$$

for every $f, g \in \mathcal{X}$ and $H, L \in \mathcal{H}$. This means that Δ and κ are homomorphisms or, equivalently, that they satisfy properties (3.17) and (3.18). Moreover, the associativity of $\mathcal{X}$ requires that

$$\begin{aligned} H \triangleright (fg)h &= \mu(\Delta H \triangleright fg \otimes h) = \mu\left(\sum_{k=1}^{n(H)} H_k^{(1)} \otimes H_k^{(2)} \triangleright fg \otimes h\right) = \\ &= \mu\left(\sum_{k=1}^{n(H)} \Delta H_k^{(1)} \otimes H_k^{(2)} \triangleright f \otimes g \otimes h\right) \end{aligned}$$

be equal to

$$\begin{aligned} H \triangleright f(gh) &= \mu(\Delta H \triangleright f \otimes gh) = \mu\left(\sum_{k=1}^{n(H)} H_k^{(1)} \otimes H_k^{(2)} \triangleright f \otimes gh\right) = \\ &= \mu\left(\sum_{k=1}^{n(H)} H_k^{(1)} \otimes \Delta H_k^{(2)} \triangleright f \otimes g \otimes h\right) \end{aligned}$$

for every $f, g, h \in \mathcal{X}$ and $H \in \mathcal{H}$, so that Δ satisfies property (3.19). Finally, since

$$\sum_{k=1}^{n(H)} \kappa(H_k^{(1)}) H_k^{(2)} \triangleright f = H \triangleright \mathbf{1} f = H \triangleright f = H \triangleright f \mathbf{1} = \sum_{k=1}^{n(H)} H_k^{(1)} \kappa(H_k^{(2)}) \triangleright f$$

for every $f \in \mathcal{X}$, κ must satisfy (3.20). We can conclude that a Hopf algebra $\mathcal{H}$ is, first and foremost, an algebra of linear transformations of some coordinate algebra $\mathcal{X}$. Its defining property is that, given its action on a coordinate basis x_i, its action on every other coordinate function $f \in \mathcal{X}$ is encoded into two linear maps Δ and κ via (3.28) and (3.29). Continuous symmetry groups of ordinary spaces (spatial rotations, say) and the corresponding Lie algebras of derivations are structures of this kind. As we have seen before, the action of a Lie symmetry generator on a generic function can be obtained from its action on some coordinate basis via the Leibniz rule and its trivial action on constant functions. Equivalently, we can say that a Lie algebra $\mathcal{L}$ with generators $L_j \in \mathcal{L}$ (its universal enveloping algebra, to be precise) is a Hopf algebra with coproduct

$$\Delta_{\mathcal{L}} L_i = L_i \otimes \mathbb{1} + \mathbb{1} \otimes L_i, \tag{3.31}$$

and counity

$$\kappa_{\mathcal{L}}(L_i) = 0. \tag{3.32}$$

The elements G of the corresponding Lie group $\mathcal{G}_{\mathcal{L}}$ act on product functions as

$$G \triangleright fg = (G \triangleright f)(G \triangleright g) \tag{3.33}$$

and on constant functions as

$$G \triangleright \mathbf{1} = 1. \tag{3.34}$$

Therefore, the group $\mathcal{G}$ can also be seen as a Hopf algebra with coproduct

$$\Delta_{\mathcal{G}} G = G \otimes G \tag{3.35}$$

and counity

$$\kappa_{\mathcal{G}}(G) = 1. \tag{3.36}$$

In these examples there is a notion of inverse (respectively infinitesimal or finite) transformation implemented by the operators

$$\begin{aligned} \Sigma_{\mathcal{L}}(L_j) &= -L_j, && (3.37)\\ \Sigma_{\mathcal{G}}(G) &= G^{-1}. && (3.38) \end{aligned}$$

Since

$$(\Sigma_{\mathcal{L}} \otimes 1\!\!1)\Delta_{\mathcal{L}} L_j = \Sigma_{\mathcal{L}}(L_j)1\!\!1 + \Sigma_{\mathcal{L}}(1\!\!1)L_j = 0 = \kappa(L_j)1\!\!1, \tag{3.39}$$

and

$$(\Sigma_{\mathcal{G}} \otimes 1\!\!1)\Delta_{\mathcal{G}} G = \Sigma_{\mathcal{G}}(G)G = 1\!\!1 = \kappa(G)1\!\!1, \tag{3.40}$$

$\Sigma_{\mathcal{L}}$ and $\Sigma_{\mathcal{G}}$ satisfy property (3.21) and are therefore the antipodes of the respective algebras. This can serve as a motivation for the last Hopf algebra axiom. In conclusion, Hopf algebras are particular algebras of linear transformations of some coordinate algebra $\mathcal{X}$. Their action on coordinate products is fully specified by a coproduct Δ and a counity κ, and a notion of inverse transformation is provided by an antipode Σ. As we have argued before, these properties make them natural generalizations of ordinary Lie groups and Lie algebras.

§ In the last two decades, Hopf-algebraic techniques have been successfully employed to characterize the symmetries of various noncommutative spacetimes. Acting on the coproduct, i.e. relaxing the Leibniz rule, it is possible to circumvent the difficulties mentioned at the beginning of this section and write deformed Poincaré transformations for a wide range of noncommutative models. Among them, Θ-Minkowski and κ-Minkowski spacetimes were the focus of most studies, both for their simplicity and their phenomenological relevance. In the following paragraphs, we will characterize the corresponding Hopf algebras, verifying that a deformation of the standard Lie coproduct is sufficient to make them compatible with the nontrivial commutation relations of the coordinates.

3.2.2 The Poincaré algebra

§ First, it is instructive to translate the standard description of relativistic symmetries in ordinary Minkowski spacetime into Hopf-algebraic language. Let $\mathcal{X}_0$ denote the algebra of coordinates on Minkowski spacetime, that is the free polynomial algebra generated by the usual, commuting x^ν. The Minkowski Hopf algebra or

Poincaré algebra $\mathcal{H}_0$ is generated by the following linear operators P_μ and $M_{\rho\sigma}$ acting on $\mathcal{X}_0$:

$$\begin{aligned} P_\mu \triangleright x^\nu &= i\delta_\mu^{\ \nu}, && (3.41)\\ M_{\rho\sigma} \triangleright x^\nu &= i(x_\rho \delta_\sigma^{\ \nu} - x_\sigma \delta_\rho^{\ \nu}). && (3.42)\end{aligned}$$

Introducing the standard definitions

$$\begin{aligned} K_j &= M_{0j}, && (3.43)\\ R_j &= \frac{1}{2}\varepsilon_j^{\ lm} M_{lm}, && (3.44)\end{aligned}$$

the action of $M_{\rho\sigma}$ can also be written as

$$\begin{aligned} K_j \triangleright x^0 &= -ix_j, && (3.45)\\ K_j \triangleright x^i &= ix_0\delta_j^{\ i}, && (3.46)\\ R_j \triangleright x^0 &= 0, && (3.47)\\ R_j \triangleright x^i &= ix_k \varepsilon_j^{\ ik}. && (3.48)\end{aligned}$$

The product is given by composition and counit and coproduct are trivial, that is

$$\begin{aligned} \kappa(P_\mu) &= 0, && (3.49)\\ \kappa(M_{\rho\sigma}) &= 0, && (3.50)\\ \Delta P_\mu &= P_\mu \otimes 1\!\!1 + 1\!\!1 \otimes P_\mu, && (3.51)\\ \Delta M_{\rho\sigma} &= M_{\rho\sigma} \otimes 1\!\!1 + 1\!\!1 \otimes M_{\rho\sigma}. && (3.52)\end{aligned}$$

It can be straightforwardly checked that $\mathcal{H}_0$ is indeed a Hopf algebra with antipode

$$\begin{aligned} \Sigma(P_\mu) &= -P_\mu, && (3.53)\\ \Sigma(M_{\rho\sigma}) &= -M_{\rho\sigma}. && (3.54)\end{aligned}$$

The standard commutation relations among the Poincaré generators P_μ and $M_{\rho\sigma}$ are implicitly contained in this minimal characterization. In fact, since

$$\begin{aligned} P_\tau P_\mu \triangleright x^\nu &= P_\tau \triangleright i\delta_\mu^{\ \nu} = 0, && (3.55)\\ M_{\rho\sigma} P_\mu \triangleright x^\nu &= M_{\rho\sigma} \triangleright i\delta_\mu^{\ \nu} = 0, && (3.56)\\ P_\mu M_{\rho\sigma} \triangleright x^\nu &= P_\mu \triangleright i(x_\rho \delta_\sigma^{\ \nu} - x_\sigma \delta_\rho^{\ \nu}) = i(ig_{\mu\rho}\delta_\sigma^{\ \nu} - ig_{\mu\sigma}\delta_\rho^{\ \nu}), && (3.57)\\ M_{\tau\upsilon} M_{\rho\sigma} \triangleright x^\nu &= M_{\tau\upsilon} \triangleright i(x_\rho \delta_\sigma^{\ \nu} - x_\sigma \delta_\rho^{\ \nu}) = i\left\{ i(x_\tau g_{\upsilon\rho} - x_\upsilon g_{\tau\rho})\delta_\sigma^{\ \nu} - \right.\\ &\quad \left. - i(x_\tau g_{\upsilon\sigma} - x_\upsilon g_{\tau\sigma})\delta_\rho^{\ \nu} \right\}, && (3.58)\end{aligned}$$

we have

$$\begin{aligned} [P_\mu, P_\tau] \triangleright x^\nu &= 0, && (3.59)\\ [P_\mu, M_{\rho\sigma}] \triangleright x^\nu &= i(ig_{\mu\rho}\delta_\sigma^{\ \nu} - ig_{\mu\sigma}\delta_\rho^{\ \nu}) = i(g_{\mu\rho}P_\sigma - g_{\mu\sigma}P_\rho) \triangleright x^\nu, && (3.60)\\ [M_{\tau\upsilon}, M_{\rho\sigma}] \triangleright x^\nu &= i\left\{ i(x_\tau g_{\upsilon\rho} - x_\upsilon g_{\tau\rho})\delta_\sigma^{\ \nu} - i(x_\tau g_{\upsilon\sigma} - x_\upsilon g_{\tau\sigma})\delta_\rho^{\ \nu} \right\} - \\ &\quad - i\left\{ i(x_\rho g_{\sigma\tau} - x_\sigma g_{\rho\tau})\delta_\upsilon^{\ \nu} - i(x_\rho g_{\sigma\upsilon} - x_\sigma g_{\rho\upsilon})\delta_\tau^{\ \nu} \right\} = \\ &= i(g_{\tau\rho}M_{\sigma\upsilon} - g_{\tau\sigma}M_{\rho\upsilon} + g_{\upsilon\rho}M_{\tau\sigma} - g_{\upsilon\sigma}M_{\tau\rho}) \triangleright x^\nu. && (3.61)\end{aligned}$$

We can then conclude that

$$
\begin{aligned}
[P_\mu, P_\tau] &= 0, && (3.62)\\
[P_\mu, M_{\rho\sigma}] &= i(g_{\mu\rho}P_\sigma - g_{\mu\sigma}P_\rho), && (3.63)\\
[M_{\tau\upsilon}, M_{\rho\sigma}] &= i(g_{\tau\rho}M_{\sigma\upsilon} - g_{\tau\sigma}M_{\rho\upsilon} + g_{\upsilon\rho}M_{\tau\sigma} - g_{\upsilon\sigma}M_{\tau\rho}), && (3.64)
\end{aligned}
$$

because the coproducts and the counits of both sides are the same[1]. We can rewrite the commutation rules (3.63) and (3.64) in terms of the generators (3.43) and (3.44) as

$$
\begin{aligned}
[P_0, K_j] &= iP_j, && (3.65)\\
[P_j, K_k] &= i\delta_{jk}P_0, && (3.66)\\
[P_0, R_j] &= 0, && (3.67)\\
[P_j, R_k] &= i\varepsilon_{jk}{}^{l}P_l, && (3.68)\\
[K_j, K_k] &= -i\varepsilon_{jk}{}^{l}R_l, && (3.69)\\
[K_j, R_k] &= i\varepsilon_{jk}{}^{l}K_l, && (3.70)\\
[R_j, R_k] &= i\varepsilon_{jk}{}^{l}R_l. && (3.71)
\end{aligned}
$$

3.2.3 The Θ-Minkowski Hopf algebra

§ The noncommutative Θ-Minkowski spacetime is defined by the deformed commutation relations

$$\left[x^\nu, x^\lambda\right] = i\Theta^{\nu\lambda}, \qquad (3.72)$$

where $\Theta^{\nu\lambda}$ is a relativistically invariant antisymmetric matrix of the order of the Planck area. Let $\mathcal{X}_\Theta$ denote the algebra of coordinates on Θ-Minkowski spacetime, that is the free polynomial algebra generated by the x^ν. The Θ-Minkowski Hopf algebra $\mathcal{H}_\Theta$ is generated by the following linear operators $P_\mu^{(\Theta)}$ and $M_{\rho\sigma}^{(\Theta)}$ acting on $\mathcal{X}_\Theta$:

$$
\begin{aligned}
P_\mu^{(\Theta)} \triangleright x^\nu &= i\delta_\mu{}^\nu, && (3.73)\\
M_{\rho\sigma}^{(\Theta)} \triangleright x^\nu &= i(x_\rho\delta_\sigma{}^\nu - x_\sigma\delta_\rho{}^\nu). && (3.74)
\end{aligned}
$$

As in the Poincaré case, the product is given by composition and the counit is trivial. The coproduct is deformed, though, and reads

$$
\begin{aligned}
\Delta P_\mu^{(\Theta)} &= P_\mu^{(\Theta)} \otimes 1\!\!1 + 1\!\!1 \otimes P_\mu^{(\Theta)}, && (3.75)\\
\Delta M_{\rho\sigma}^{(\Theta)} &= M_{\rho\sigma}^{(\Theta)} \otimes 1\!\!1 + 1\!\!1 \otimes M_{\rho\sigma}^{(\Theta)} + \frac{1}{2}\Upsilon_\rho{}^\alpha{}_\sigma{}^\beta P_\alpha^{(\Theta)} \otimes P_\beta^{(\Theta)}, && (3.76)
\end{aligned}
$$

[1] For example, we have

$$
\begin{aligned}
\Delta\,[P_\mu, M_{\rho\sigma}] &= [\Delta P_\mu, \Delta M_{\rho\sigma}] = [P_\mu, M_{\rho\sigma}] \otimes 1\!\!1 + 1\!\!1 \otimes [P_\mu, M_{\rho\sigma}] = \\
&= i(g_{\mu\rho}P_\sigma - g_{\mu\sigma}P_\rho) \otimes 1\!\!1 + 1\!\!1 \otimes i(g_{\mu\rho}P_\sigma - g_{\mu\sigma}P_\rho) = \Delta\ i(g_{\mu\rho}P_\sigma - g_{\mu\sigma}P_\rho).
\end{aligned}
$$

with $\Upsilon_{\rho\ \sigma}^{\ \alpha\ \beta}$ defined as in (3.13). It can be easily checked that $\mathcal{H}_\Theta$ is indeed a Hopf algebra with antipode

$$\Sigma(P_\mu^{(\Theta)}) = -P_\mu^{(\Theta)}, \tag{3.77}$$

$$\Sigma(M_{\rho\sigma}^{(\Theta)}) = -M_{\rho\sigma}^{(\Theta)} + \frac{1}{2}\Upsilon_{\rho\ \sigma}^{\ \alpha\ \beta} P_\alpha^{(\Theta)} P_\beta^{(\Theta)}. \tag{3.78}$$

Thanks to the modified coproduct, the action of $M_{\rho\sigma}^{(\Theta)}$ is compatible with the Θ-Minkowski commutation rules (3.72), even if $\mathcal{H}_\Theta$ reduces to the standard Poincaré algebra for $\Theta \to 0$. In fact, we have[2]

$$\begin{aligned} M_{\rho\sigma}^{(\Theta)} \triangleright [x^\nu, x^\lambda] &= \Delta M_{\rho\sigma}^{(\Theta)} \triangleright (x^\nu \otimes x^\lambda - x^\lambda \otimes x^\nu) = \left[M_{\rho\sigma}^{(\Theta)} \triangleright x^\nu, x^\lambda\right] + \left[x^\nu, M_{\rho\sigma}^{(\Theta)} \triangleright x^\lambda\right] + \\ &+ \frac{1}{2}\Upsilon_{\rho\ \sigma}^{\ \alpha\ \beta} \left\{(P_\alpha^{(\Theta)} \triangleright x^\nu)(P_\beta^{(\Theta)} \triangleright x^\lambda) - (P_\alpha^{(\Theta)} \triangleright x^\lambda)(P_\beta^{(\Theta)} \triangleright x^\nu)\right\} = \\ &= \Upsilon_{\rho\ \sigma}^{\ \nu\ \lambda} - \frac{1}{2}\Upsilon_{\rho\ \sigma}^{\ \alpha\ \beta}(\delta_\alpha^{\ \nu}\delta_\beta^{\ \lambda} - \delta_\alpha^{\ \lambda}\delta_\beta^{\ \nu}) = 0 = M_{\rho\sigma}^{(\Theta)} \triangleright i\Theta^{\nu\lambda}. \end{aligned} \tag{3.79}$$

This simple example should clarify how a general coproduct can easily allow for physically interesting deformations of the Poincaré algebra when nontrivial commutation relations among spacetime coordinates are assumed. In the special case of Θ-Minkowski, the nontrivial coproduct of $M_{\rho\sigma}^{(\Theta)}$ does not lead to a change in the standard Poincaré commutation rules (3.63) and (3.64). In fact, applying the coproduct to both sides of those equations and performing some tedious but straightforward computations, we obtain perfectly consistent results:

$$\begin{aligned} \Delta\left[P_\mu^{(\Theta)}, M_{\rho\sigma}^{(\Theta)}\right] &= \left[\Delta P_\mu^{(\Theta)}, \Delta M_{\rho\sigma}^{(\Theta)}\right] = \left[P_\mu^{(\Theta)}, M_{\rho\sigma}^{(\Theta)}\right] \otimes \mathbb{1} + \mathbb{1} \otimes \left[P_\mu^{(\Theta)}, M_{\rho\sigma}^{(\Theta)}\right] = \\ &= i(g_{\mu\rho}P_\sigma^{(\Theta)} - g_{\mu\sigma}P_\rho^{(\Theta)}) \otimes \mathbb{1} + \mathbb{1} \otimes i(g_{\mu\rho}P_\sigma^{(\Theta)} - g_{\mu\sigma}P_\rho^{(\Theta)}) = \\ &= \Delta\ i(g_{\mu\rho}P_\sigma^{(\Theta)} - g_{\mu\sigma}P_\rho^{(\Theta)}), \end{aligned} \tag{3.80}$$

$$\begin{aligned} \Delta\left[M_{\tau\upsilon}^{(\Theta)}, M_{\rho\sigma}^{(\Theta)}\right] &= \left[\Delta M_{\tau\upsilon}^{(\Theta)}, \Delta M_{\rho\sigma}^{(\Theta)}\right] = \left[M_{\tau\upsilon}^{(\Theta)}, M_{\rho\sigma}^{(\Theta)}\right] \otimes \mathbb{1} + \mathbb{1} \otimes \left[M_{\tau\upsilon}^{(\Theta)}, M_{\rho\sigma}^{(\Theta)}\right] + \\ &+ \frac{1}{2}\Upsilon_{\rho\ \sigma}^{\ \alpha\ \beta}\left[M_{\tau\upsilon}^{(\Theta)}, P_\alpha^{(\Theta)}\right] \otimes P_\beta^{(\Theta)} + \frac{1}{2}\Upsilon_{\rho\ \sigma}^{\ \alpha\ \beta} P_\alpha^{(\Theta)} \otimes \left[M_{\tau\upsilon}^{(\Theta)}, P_\beta^{(\Theta)}\right] + \\ &+ \frac{1}{2}\Upsilon_{\tau\ \upsilon}^{\ \alpha\ \beta}\left[P_\alpha^{(\Theta)}, M_{\rho\sigma}^{(\Theta)}\right] \otimes P_\beta^{(\Theta)} + \frac{1}{2}\Upsilon_{\tau\ \upsilon}^{\ \alpha\ \beta} P_\alpha^{(\Theta)} \otimes \left[P_\beta^{(\Theta)}, M_{\rho\sigma}^{(\Theta)}\right] = \\ &= \left[M_{\tau\upsilon}^{(\Theta)}, M_{\rho\sigma}^{(\Theta)}\right] \otimes \mathbb{1} + \mathbb{1} \otimes \left[M_{\tau\upsilon}^{(\Theta)}, M_{\rho\sigma}^{(\Theta)}\right] + \\ &+ \frac{i}{2}(g_{\tau\rho}\Upsilon_{\sigma\ \upsilon}^{\ \alpha\ \beta} - g_{\tau\sigma}\Upsilon_{\rho\ \upsilon}^{\ \alpha\ \beta} + g_{\upsilon\rho}\Upsilon_{\tau\ \sigma}^{\ \alpha\ \beta} - g_{\upsilon\sigma}\Upsilon_{\tau\ \rho}^{\ \alpha\ \beta})P_\alpha^{(\Theta)} \otimes P_\beta^{(\Theta)} = \\ &= \Delta\ i(g_{\tau\rho}M_{\sigma\upsilon}^{(\Theta)} - g_{\tau\sigma}M_{\rho\upsilon}^{(\Theta)} + g_{\upsilon\rho}M_{\tau\sigma}^{(\Theta)} - g_{\upsilon\sigma}M_{\tau\rho}^{(\Theta)}). \end{aligned} \tag{3.81}$$

3.2.4 The κ-Minkowski Hopf algebra

§ The noncommutative κ-Minkowski spacetime is defined by the deformed commutation relations

$$\left[x^\nu, x^i\right] = i\ell\delta_0^{\ \nu} x^i, \tag{3.82}$$

[2]In the following equation, as well as in the rest of the chapter, tensor products of the form $\Delta H \triangleright f \otimes g$ will be implicitly contracted. For example, $\Delta M_{\rho\sigma}^{(\Theta)} \triangleright x^\nu \otimes x^\lambda$ will stand for $\mu\left(\Delta M_{\rho\sigma}^{(\Theta)} \triangleright x^\nu \otimes x^\lambda\right)$.

where ℓ is a relativistically invariant parameter of the order of the Planck length. Let $\mathcal{X}_\kappa$ denote the algebra of coordinates on κ-Minkowski spacetime. The κ-Minkowski Hopf algebra $\mathcal{H}_\kappa$ is generated by the following linear operators $P_\mu^{(\kappa)}$, $K_j^{(\kappa)}$ and $R_j^{(\kappa)}$ acting on $\mathcal{X}_\kappa$:

$$\begin{aligned}
P_\mu^{(\kappa)} \triangleright x^\nu &= i\delta_\mu{}^\nu, && (3.83)\\
K_j^{(\kappa)} \triangleright x^0 &= -ix_j, && (3.84)\\
K_j^{(\kappa)} \triangleright x^i &= ix_0\delta_j{}^i, && (3.85)\\
R_j^{(\kappa)} \triangleright x^0 &= 0, && (3.86)\\
R_j^{(\kappa)} \triangleright x^i &= ix_k\varepsilon_j{}^{ik}. && (3.87)
\end{aligned}$$

As before, the product is given by composition and the counit is trivial. The coproduct, instead, is given by

$$\begin{aligned}
\Delta P_0^{(\kappa)} &= P_0^{(\kappa)} \otimes 1\!\!1 + 1\!\!1 \otimes P_0^{(\kappa)}, && (3.88)\\
\Delta P_j^{(\kappa)} &= P_j^{(\kappa)} \otimes 1\!\!1 + e^{-\ell P_0^{(\kappa)}} \otimes P_j^{(\kappa)}, && (3.89)\\
\Delta K_j^{(\kappa)} &= K_j^{(\kappa)} \otimes 1\!\!1 + e^{-\ell P_0^{(\kappa)}} \otimes K_j^{(\kappa)} - \ell\varepsilon_j{}^{lm} P_l^{(\kappa)} \otimes R_m^{(\kappa)}, && (3.90)\\
\Delta R_j^{(\kappa)} &= R_j^{(\kappa)} \otimes 1\!\!1 + 1\!\!1 \otimes R_j^{(\kappa)}. && (3.91)
\end{aligned}$$

It can be straightforwardly checked that $\mathcal{H}_\kappa$ is indeed a Hopf algebra with antipode

$$\begin{aligned}
\Sigma(P_0^{(\kappa)}) &= -P_0^{(\kappa)}, && (3.92)\\
\Sigma(P_j^{(\kappa)}) &= -e^{\ell P_0^{(\kappa)}} P_j^{(\kappa)}, && (3.93)\\
\Sigma(K_j^{(\kappa)}) &= -e^{\ell P_0^{(\kappa)}} K_j^{(\kappa)} - \ell\varepsilon_j{}^{lm} e^{\ell P_0^{(\kappa)}} P_l^{(\kappa)} R_m^{(\kappa)}, && (3.94)\\
\Sigma(R_j^{(\kappa)}) &= -R_j^{(\kappa)}. && (3.95)
\end{aligned}$$

Thanks to the modified coproduct, the action of $\mathcal{H}_\kappa$ is compatible with the κ-Minkowski commutation rules (3.82). For example, we have

$$\begin{aligned}
P_j^{(\kappa)} \triangleright [x^\nu, x^i] &= \Delta P_j^{(\kappa)} \triangleright (x^\nu \otimes x^i - x^i \otimes x^\nu) = \left[P_j^{(\kappa)} \triangleright x^\nu, x^i\right] + \left[x^\nu, P_j^{(\kappa)} \triangleright x^i\right] + \\
&\quad + \ell(P_0^{(\kappa)} \triangleright x^\nu)(P_j^{(\kappa)} \triangleright x^i) - \ell(P_0^{(\kappa)} \triangleright x^i)(P_j^{(\kappa)} \triangleright x^\nu) = \\
&= -\ell\delta_0{}^\nu\delta_j{}^i = P_j^{(\kappa)} \triangleright i\ell\delta_0{}^\nu x^i. && (3.96)
\end{aligned}$$

As in the Θ-Minkowski case, one can verify that the usual Poincaré commutation rules are compatible with the new coproduct. There is an exception, though: applying the coproduct to the left-hand side of equation (3.66), we would obtain

$$\begin{aligned}
\Delta\left[P_j^{(\kappa)}, K_k^{(\kappa)}\right] &= \left[\Delta P_j^{(\kappa)}, \Delta K_k^{(\kappa)}\right] = \left[P_j^{(\kappa)}, K_k^{(\kappa)}\right] \otimes 1\!\!1 + \left[e^{-\ell P_0^{(\kappa)}}, K_k^{(\kappa)}\right] \otimes P_j^{(\kappa)} + \\
&\quad + e^{-2\ell P_0^{(\kappa)}} \otimes \left[P_j^{(\kappa)}, K_k^{(\kappa)}\right] - \ell\varepsilon_k{}^{lm} e^{-\ell P_0^{(\kappa)}} P_l^{(\kappa)} \otimes \left[P_j^{(\kappa)}, R_m^{(\kappa)}\right]. && (3.97)
\end{aligned}$$

but this is obviously different from $\Delta\, i\delta_{jk}P_0^{(\kappa)}$. In order to make everything consistent, it is necessary and sufficient to deform (3.66) into

$$\left[P_j^{(\kappa)}, K_k^{(\kappa)}\right] = i\delta_{jk}\left(\frac{1\!\!1 - e^{-2\ell P_0^{(\kappa)}}}{2\ell} + \frac{\ell}{2}P_l^{(\kappa)}P_l^{(\kappa)}\right) - i\ell P_j^{(\kappa)}P_k^{(\kappa)}. \qquad (3.98)$$

It is easy to check that these modified commutation relations are compatible with the coproduct and that the standard formula is recovered in the commutative limit $\ell \to 0$.

3.2.5 Symmetry transformations

§ In the previous paragraphs, we have seen that, equipping the ordinary Poincaré symmetry generators with a nontrivial coproduct, we can have them act consistently on noncommutative coordinate functions. The infinitesimal symmetry transformation of parameters $\omega^{\rho\sigma}$ and ϵ^μ, formally described by the infinitesimal variation operator

$$\delta_{\epsilon,\omega} = i\epsilon^\mu P_\mu + i\omega^{\rho\sigma}M_{\rho\sigma}, \qquad (3.99)$$

is then well defined. If ϵ^μ and $\omega^{\rho\sigma}$ are ordinary real numbers, though, we find that $\delta_{\epsilon,\omega}$ does no more satisfy the Leibniz rule and cannot be regarded as a differential operator. From a physical point of view, this is a serious problem. In order to generalize the concepts of differential geometry and general relativity to noncommutative manifolds, it is important that infinitesimal coordinate transformations are described by derivative vector fields of some kind. This property is also crucial for the validity of Noether's theorem and the existence of conserved charges in field theories defined on noncommutative spacetimes. This difficulty can be circumvented by assuming that transformation parameters are not numbers but algebraically nontrivial objects which do not generally commute with spacetime coordinates[3]. Let us see how this strategy works for Θ-Minkowski and κ-Minkowski symmetries. In the first case, it is easy to verify that we can actually have $\delta^{(\Theta)}_{\epsilon,\omega}$ satisfy the Leibniz rule, that is

$$\begin{aligned}
\delta^{(\Theta)}_{\epsilon,\omega} \triangleright x^\nu x^\lambda &= i\epsilon^\mu_{(\Theta)}(\Delta P^{(\Theta)}_\mu \triangleright x^\nu \otimes x^\lambda) + i\omega^{\rho\sigma}(\Delta M^{(\Theta)}_{\rho\sigma} \triangleright x^\nu \otimes x^\lambda) = \\
&= i\epsilon^\mu_{(\Theta)}\left\{(P^{(\Theta)}_\mu \triangleright x^\nu)x^\lambda + x^\nu(P^{(\Theta)}_\mu \triangleright x^\lambda)\right\} + \\
&\quad + i\omega^{\rho\sigma}\left\{(M^{(\Theta)}_{\rho\sigma} \triangleright x^\nu)x^\lambda + x^\nu(M^{(\Theta)}_{\rho\sigma} \triangleright x^\lambda)\right\} + \\
&\quad + \frac{i}{2}\omega^{\rho\sigma}\Upsilon_{\rho\ \sigma}^{\ \alpha\ \beta}(P^{(\Theta)}_\alpha \triangleright x^\nu)(P^{(\Theta)}_\beta \triangleright x^\lambda)
\end{aligned}$$

[3]To be precise, that some parameter multiplication α, which is a linear operator on the algebra of coordinates, may not commute with left multiplication by x^ν:

$$\alpha(x^\nu f(x)) - x^\nu(\alpha f(x)) = [\alpha, x^\nu]\, f(x) \neq 0.$$

By definition, parameter multiplication satisfies the so-called mixed associative property

$$(\alpha f(x))g(x) = \alpha(f(x)g(x))$$

for every coordinate functions $f(x)$ and $g(x)$, so that it must in any case commute with right multiplication by x^ν.

equal

$$\begin{aligned}(\delta^{(\Theta)}_{\epsilon,\omega} \triangleright x^\nu)x^\lambda + x^\nu(\delta^{(\Theta)}_{\epsilon,\omega} \triangleright x^\lambda) &= i\left\{\epsilon^\mu_{(\Theta)}(P^{(\Theta)}_\mu \triangleright x^\nu)x^\lambda + x^\nu\epsilon^\mu_{(\Theta)}(P^{(\Theta)}_\mu \triangleright x^\lambda)\right\} + \\ &+ i\left\{\omega^{\rho\sigma}(M^{(\Theta)}_{\rho\sigma} \triangleright x^\nu)x^\lambda + x^\nu\omega^{\rho\sigma}(M^{(\Theta)}_{\rho\sigma} \triangleright x^\lambda)\right\},\end{aligned}$$

provided that

$$\left[\epsilon^\mu_{(\Theta)}, x^\nu\right] = \frac{i}{2}\omega^{\rho\sigma}\Upsilon_{\rho\ \sigma}^{\ \mu\ \nu}, \tag{3.100}$$

$$[\omega^{\rho\sigma}, x^\nu] = 0. \tag{3.101}$$

In the κ-Minkowski case, it is convenient to write the infinitesimal variation in terms of the redefined generators (3.43) and (3.44) as

$$\delta^{(\kappa)}_{\epsilon,\phi,\theta} = i\epsilon^\mu_{(\kappa)}P^{(\kappa)}_\mu + i\phi^j K^{(\kappa)}_j + i\theta^j R^{(\kappa)}_j. \tag{3.102}$$

Requiring as before that $\delta^{(\kappa)}_{\epsilon,\phi,\theta}$ satisfy the Leibniz rule

$$\delta^{(\kappa)}_{\epsilon,\phi,\theta} \triangleright x^\nu x^\lambda = (\delta^{(\kappa)}_{\epsilon,\phi,\theta} \triangleright x^\nu)x^\lambda + x^\nu(\delta^{(\kappa)}_{\epsilon,\phi,\theta} \triangleright x^\lambda),$$

we immediately find

$$\left[\epsilon^0_{(\kappa)}, x^\nu\right] = 0, \tag{3.103}$$

$$\left[\epsilon^j_{(\kappa)}, x^\nu\right] = i\ell\delta_0^{\ \nu}\epsilon^j_{(\kappa)}, \tag{3.104}$$

$$\left[\phi^j, x^\nu\right] = i\ell\delta_0^{\ \nu}\phi^j, \tag{3.105}$$

$$\left[\theta^j, x^\nu\right] = i\ell\delta_l^{\ \nu}\varepsilon^{lj}_{\ \ m}\phi^m. \tag{3.106}$$

§ From a purely mathematical point of view, we have seen that nontrivial transformation parameters can make infinitesimal symmetries behave like differential operators. Nevertheless, the physical interpretation of such noncommuting transformation parameters is obscure. We have assumed from the beginning that spacetime coordinates obey nontrivial commutation relations, but it can be proved that parameters such as $\epsilon^\mu_{(\Theta)}$ or $\epsilon^\mu_{(\kappa)}$ cannot be coordinate functions. Let us check this explicitly in the case of $\epsilon^\mu_{(\Theta)}$. If

$$\epsilon^\mu_{(\Theta)} = f^\mu(x), \tag{3.107}$$

we would find

$$\left[\epsilon^\mu_{(\Theta)}, x^\nu\right] = [f^\mu(x), x^\nu] = i\Theta^{\alpha\nu}\partial_\alpha f^\mu(x). \tag{3.108}$$

For this to equal (3.100), that is

$$\left[\epsilon^\mu_{(\Theta)}, x^\nu\right] = \frac{i}{2}\omega^{\rho\sigma}\Upsilon_{\rho\ \sigma}^{\ \mu\ \nu} = i\omega^{\rho\sigma}\Theta^{\alpha\beta}(g_{\alpha\rho}\delta_\beta^{\ \mu}\delta_\sigma^{\ \nu} + g_{\alpha\sigma}\delta_\beta^{\ \nu}\delta_\rho^{\ \mu}), \tag{3.109}$$

we should have

$$\delta_\beta{}^\nu \partial_\alpha f^\mu(x) = \omega^{\rho\sigma}(g_{\alpha\rho}\delta_\beta{}^\mu \delta_\sigma{}^\nu + g_{\alpha\sigma}\delta_\beta{}^\nu \delta_\rho{}^\mu). \tag{3.110}$$

Taking $\beta = \mu \neq \nu$, we see that $\omega^{\rho\sigma}$ should vanish, and thus $f^\mu(x)$ should be a constant. We can conclude that there is no function of the noncommutative coordinates x^ν which obeys (3.100) when some $\omega^{\rho\sigma}$ are nonvanishing. We are then left with the problem of finding a physical interpretation for $\epsilon^\mu_{(\Theta)}$. There is another weird consequence of relation (3.100). No matter what kind of object $\epsilon^\mu_{(\Theta)}$ may turn out to be, it is clear that it cannot vanish whenever some $\omega^{\rho\sigma} \neq 0$. This result has been taken as an indication that, on Θ-Minkowski, pure Lorentz transformations may not exist [27]. The same reasoning applies to relation (3.106) in the κ-Minkowski case. When some $\phi_j \neq 0$, θ_j cannot vanish, and this seems to imply that, on κ-Minkowski, boosts must always be accompanied by rotations [27]. These results are very strange and call for a better understanding of the physical nature of the nontrivial transformation parameters. In the next sections, we will argue that all these problems simply disappear if we adopt the point of view of CQM. In fact, the new approach provides a natural explanation for the puzzling features of the Hopf-algebraic treatment of symmetries in terms of ordinary quantum mechanical concepts.

3.3 Which noncommutative coordinates?

§ In theoretical physics, we can refer to two distinct concepts when talking about spacetime coordinates. On the one hand, we can mean arbitrary real functions defined on a spacetime manifold, like in differential geometry. These coordinates are just mathematical labels used to distinguish spacetime points and are not physical observables. A good example is given by spacetime coordinates in quantum field theory. In this context, observables are smeared field operators and coordinates only serve as a means of describing their relationships and tracing their dynamics. In fact, we can write quantum field theories with respect to arbitrary coordinate systems by changing variables in the equations of motion. The same can be said of classical general relativity, where the equations of motion are even covariant under general coordinate transformations. Let us call this first kind of coordinates 'background coordinates'. On the other hand, we can refer to the observable spacetime position of some actual event with respect to some physical reference frame. In this case, coordinates are genuine dynamical quantities whose values can be theoretically computed and experimentally measured. A good example is given by the inertial cartesian spatial coordinates of a point particle at time t_0 in standard QM. It is clear that formal changes of variables can have no effects on such objects. Let us call this second kind of coordinates 'particle coordinates'.

§ Schrödinger quantum field theory, i.e. second-quantized nonrelativistic quantum mechanics, provides us with an explicit expression of inertial particle spatial coordinates x^i at time t_0 in terms of inertial background spatial coordinates z^i, thereby making manifest the deep conceptual difference between the two. Let a quantum field $\widehat{\psi}(z,t)$ be a solution of the Schrödinger equation

$$i\hbar\partial_t\widehat{\psi}(z,t) = \left(-\frac{\hbar^2}{2m}\nabla_z^2 + V(z,t)\right)\widehat{\psi}(z,t), \tag{3.111}$$

and let $\mathcal{H}^1$ be the Hilbert space of one-particle states $|1;\alpha\rangle$, defined by

$$\widehat{N}|1;\alpha\rangle = \int \widehat{\psi}(z,t)^\dagger\widehat{\psi}(z,t)d^3z|1;\alpha\rangle = |1;\alpha\rangle. \tag{3.112}$$

Then it is easy to verify that the observable

$$\widehat{x}^i(t_0) = \int z^i\widehat{\psi}(z,t_0)^\dagger\widehat{\psi}(z,t_0)d^3z, \tag{3.113}$$

when restricted to $\mathcal{H}^1$, is the i-th particle coordinate at time t_0 of standard Heisenberg QM. In this simple example, we clearly see that particle coordinates, being smeared field operators, are genuine observables, whereas background coordinates are mathematical labels devoid of direct physical meaning. In particular, we could write the theory in any background coordinate system and get back the same observable $\widehat{x}^i(t_0)$ by changing variables in the integral.

§ Most papers about noncommutative spacetime physics are based on the assumption of noncommutative background coordinates [25–27, 30]. In these studies, classical or quantum field theories are defined and characterized after replacing the ordinary Minkowski background with some noncommutative algebra of coordinates. Unlike nontrivial commutation properties of quantum observables, which reflect the incompatibility of the corresponding physical quantities, this kind of background noncommutativity does not admit a straightforward physical interpretation and appears therefore somewhat removed from the naïve intuition described above. The obvious analogy with standard QM and the well-established physical interpretation of noncommuting quantum observables make the assumption of noncommutative particle coordinates more natural and straightforward. Since both time and particle spatial coordinates are represented by hermitian operators at the kinematical level, CQM is the ideal tool for introducing arbitrary commutation relations among spacetime coordinates into a single-particle setting. It is sufficient to replace the trivial commutation rules (1.36) with more general ones. This realization, in the special case of κ-Minkowski spacetime, was the main ingredient of a pioneering but somewhat underappreciated work by Amelino-Camelia, Astuti and Rosati [12] and is in fact the starting point of the present analysis. In the following, we will study commutation rules of the form

$$\left[\widehat{x}^\nu, \widehat{x}^\lambda\right] = i\ell\Gamma^{\nu\lambda}{}_\alpha\widehat{x}^\alpha + i\ell^2\Theta^{\nu\lambda}, \tag{3.114}$$

where ℓ is a fundamental length of the order of the Planck scale, while $\Gamma^{\nu\lambda}{}_\alpha$ and $\Theta^{\nu\lambda}$ are dimensionless matrices antisymmetric in the indices ν and λ. These are the most general commutation relations which trivialize in the limit $\ell \to 0$ and are analytic in both the spacetime coordinates $\widehat{x}^\nu$ and the deformation parameter ℓ. They include the popular Θ-Minkowski (3.72) and κ-Minkowski (3.82) as particular cases.

3.4 Deformed canonical algebra

§ First of all, we must redefine the extended canonical algebra $\mathcal{V}$ in the noncommutative models. Taking the canonical commutation relations (1.35)-(1.36) and replacing (1.36) with (3.114), we obtain

$$
\begin{aligned}
[\widehat{p}_\mu, \widehat{p}_\tau] &= 0, && (3.115)\\
[\widehat{p}_\mu, \widehat{x}^\nu] &= i\hbar\delta_\mu^{\ \nu}, && (3.116)\\
\left[\widehat{x}^\nu, \widehat{x}^\lambda\right] &= i\ell\Gamma^{\nu\lambda}{}_\alpha \widehat{x}^\alpha + i\ell^2\Theta^{\nu\lambda}. && (3.117)
\end{aligned}
$$

These commutation rules are not consistent in general, because the Jacobi identities for $\widehat{p}_\mu$, $\widehat{x}^\nu$ and $\widehat{x}^\lambda$ are violated. To take care of this problem, we allow for a momentum-dependent deformation of the Heisenberg relations (3.116) and write

$$
\begin{aligned}
[\widehat{p}_\mu, \widehat{p}_\tau] &= 0, && (3.118)\\
[\widehat{p}_\mu, \widehat{x}^\nu] &= i\hbar[\Delta(\ell\widehat{p})]_\mu^{\ \nu}, && (3.119)\\
\left[\widehat{x}^\nu, \widehat{x}^\lambda\right] &= i\ell\Gamma^{\nu\lambda}{}_\alpha \widehat{x}^\alpha + i\ell^2\Theta^{\nu\lambda}. && (3.120)
\end{aligned}
$$

A priori, the dimensionless matrix $\Delta_\mu^{\ \nu}$ could also depend on $\ell^{-1}\widehat{x}^\nu$, but we are ruling out this possibility to avoid non-analyticity in either the spacetime coordinates $\widehat{x}^\nu$ or the deformation parameter ℓ. We are also leaving (3.118) undeformed because we are assuming that gravity is negligible and spacetime is flat. In order for (3.118)-(3.120) to be consistent and reduce to (1.35)-(1.36) in the commutative limit, $\Delta_\mu^{\ \nu}$ must satisfy $[\Delta(0)]_\mu^{\ \nu} = \delta_\mu^{\ \nu}$ and

$$
\ell\Gamma^{\nu\lambda}{}_\alpha\Delta_\mu^{\ \alpha} + \hbar\Delta_\alpha^{\ \nu}\partial_\alpha\Delta_\mu^{\ \lambda} - \hbar\Delta_\alpha^{\ \lambda}\partial_\alpha\Delta_\mu^{\ \nu} = 0, \tag{3.121}
$$

where we have set $\partial_\alpha = \partial/\partial p_\alpha$. Since these conditions do not determine $\Delta_\mu^{\ \nu}$ uniquely, we must conclude that the modified commutation relations (3.114) are not sufficient to fully characterize spacetime noncommutativity in our covariant single-particle setting, but must be complemented by a compatible deformation of the Heisenberg relations. It is worth explicitly pointing out that different choices of $\Delta_\mu^{\ \nu}$ are not physically equivalent, because they determine different uncertainty relations between particle coordinates and momenta.

§ Assuming that $\Delta_\mu^{\ \nu}$ is an invertible matrix, we can define a set of deformed hermitian coordinates $\widehat{q}^\nu$ as

$$
\begin{aligned}
\widehat{q}^\nu &= \frac{1}{2}\left\{[\widehat{x}^\alpha - \ell\Sigma^\alpha(\ell\widehat{p})][\Delta^{-1}(\ell\widehat{p})]_\alpha^{\ \nu} + \text{h.c.}\right\} =\\
&= \widehat{x}^\alpha[\Delta^{-1}(\ell\widehat{p})]_\alpha^{\ \nu} + \frac{1}{2}i\hbar[\Delta(\ell\widehat{p})]_\gamma^{\ \alpha}\partial_\gamma[\Delta^{-1}(\ell\widehat{p})]_\alpha^{\ \nu} - \ell\Sigma^\alpha(\ell\widehat{p})[\Delta^{-1}(\ell\widehat{p})]_\alpha^{\ \nu} =\\
&= [\widehat{x}^\alpha - \ell\Sigma^\alpha(\ell\widehat{p}) - i\ell\Omega^\alpha(\ell\widehat{p})][\Delta^{-1}(\ell\widehat{p})]_\alpha^{\ \nu}, \qquad (3.122)
\end{aligned}
$$

where Σ^α is a still unspecified vector depending on the momenta and we have introduced the shorthand notation

$$
\Omega^\alpha(\ell\widehat{p}) = \frac{\hbar}{2\ell}\partial_\gamma[\Delta(\ell\widehat{p})]_\gamma^{\ \alpha}. \tag{3.123}
$$

Inverting the previous relations, we can express $\widehat{x}^\nu$ as functions of $\widehat{q}^\nu$ and $\widehat{p}_\mu$:

$$\widehat{x}^\nu = \frac{1}{2}\left\{\widehat{q}^\alpha[\Delta(\ell\widehat{p})]_\alpha^{\ \nu} + \ell\Sigma^\nu(\ell\widehat{p}) + \text{h.c.}\right\} = \widehat{q}^\alpha[\Delta(\ell\widehat{p})]_\alpha^{\ \nu} + \ell\Sigma^\nu(\ell\widehat{p}) + i\ell\Omega^\nu(\ell\widehat{p}) \quad (3.124)$$

This change of variables is useful because we can have $\widehat{p}_\mu$ and $\widehat{q}^\nu$ satisfy canonical commutation relations by appropriately choosing Σ^α. In fact, computing the relevant commutators, we find

$$[\widehat{p}_\mu, \widehat{q}^\nu] = i\hbar\delta_\mu^{\ \nu}, \quad (3.125)$$

irrespectively of Σ^α, and

$$\begin{aligned}\left[\widehat{q}^\nu, \widehat{q}^\lambda\right] &= \frac{1}{2}\Big\{(\Delta^{-1})_\alpha^{\ \nu}(\Delta^{-1})_\beta^{\ \lambda}\Big[(i\ell\Gamma^{\alpha\beta}_{\ \ \gamma}\Delta_\mu^{\ \gamma} + i\hbar\Delta_\gamma^{\ \alpha}\partial_\gamma\Delta_\mu^{\ \beta} - i\hbar\Delta_\gamma^{\ \beta}\partial_\gamma\Delta_\mu^{\ \alpha})\widehat{q}^\mu + \\ &\quad + (i\hbar\ell\Delta_\gamma^{\ \alpha}\partial_\gamma\Sigma^\beta - i\hbar\ell\Delta_\gamma^{\ \beta}\partial_\gamma\Sigma^\alpha + i\ell^2\Gamma^{\alpha\beta}_{\ \ \gamma}\Sigma^\gamma + i\ell^2\Theta^{\alpha\beta})\Big] + \text{h.c.}\Big\} \quad (3.126)\end{aligned}$$

The first term in square brackets vanishes because of the identities (3.121), while the second can be put to zero by choosing Σ^α such that

$$i\hbar\Delta_\gamma^{\ \alpha}\partial_\gamma\Sigma^\beta - i\hbar\Delta_\gamma^{\ \beta}\partial_\gamma\Sigma^\alpha + i\ell\Gamma^{\alpha\beta}_{\ \ \gamma}\Sigma^\gamma + i\ell\Theta^{\alpha\beta} = 0. \quad (3.127)$$

Therefore, we can describe our deformed canonical algebra $(\widehat{p}_\mu, \widehat{x}^\nu)$ as just a standard canonical algebra $(\widehat{p}_\mu, \widehat{q}^\nu)$ equipped with momentum-dependent functions $\Delta_\mu^{\ \nu}$ and Σ^α satisfying (3.121) and (3.127), respectively. Noncommutative coordinates $\widehat{x}^\nu$ are then given by (3.124). This description of $(\widehat{p}_\mu, \widehat{x}^\nu)$ is a generalization of the concept of pre-geometric representation introduced and developed in [25] and [12], respectively. Since conditions (3.127) are not sufficient to univocally determine Σ^α, different choices of Σ^α are associated with different, physically equivalent representations of the deformed canonical algebra. Commutative coordinates $\widehat{q}^\nu_\Sigma$ and $\widehat{q}^\nu_{\Sigma+\delta\Sigma}$ corresponding to representations Σ^α and $\Sigma^\alpha + \delta\Sigma^\alpha$ via (3.122) are related by

$$\widehat{q}^\nu_{\Sigma+\delta\Sigma} = \widehat{q}^\nu_\Sigma - \ell\delta\Sigma^\alpha[\Delta^{-1}]_\alpha^{\ \nu}. \quad (3.128)$$

3.5 Deformed Poincaré symmetries

§ We are now ready to discuss deformed relativistic symmetries in our single-particle framework. As discussed in detail in Chapter 1, in CQM continuous groups of kinematical symmetries are described by continuous groups of automorphisms of the extended canonical *-algebra $\mathcal{V}$. Such transformations can always be unitarily implemented on $\mathcal{V}$ and are fully characterized by a set of essentially self-adjoint generators $\widehat{g}_i \in \mathcal{V}$. A Poincaré transformation (Λ, a), for example, is described by the following map:

$$\begin{aligned}\widehat{x}^\nu &\mapsto \Lambda^\nu_{\ \alpha}\widehat{x}^\alpha + a^\nu, \quad (3.129)\\ \widehat{p}_\mu &\mapsto \Lambda_\mu^{\ \beta}\widehat{p}_\beta, \quad (3.130)\end{aligned}$$

which obviously preserves canonical commutation relations (1.35)-(1.36). This group of automorphisms is generated by the self-adjoint operators $\widehat{p}_\mu$ and $\widehat{m}_{\rho\sigma}$, where

$$\widehat{m}_{\rho\sigma} = \widehat{x}_\rho \widehat{p}_\sigma - \widehat{x}_\sigma \widehat{p}_\rho. \tag{3.131}$$

If we impose the Poincaré-invariant constraint

$$\widehat{H}_r = \widehat{P}^\alpha \widehat{P}_\alpha - m^2 c^2, \tag{3.132}$$

$\widehat{p}_\mu$ and $\widehat{m}_{\rho\sigma}$ become constants of motion and the resulting model, describing a free relativistic scalar particle, is Poincaré-symmetric.

§ When we introduce spacetime noncommutativity, Poincaré transformations are not kinematical symmetries anymore. In fact, the modified commutation rules (3.114) are not preserved in general by the action (3.129)-(3.130). There are two alternative attitudes we can take towards this breaking of standard relativistic symmetries. On the one hand, we can view it as evidence of the failure of the relativity principle and the existence of preferred reference frames. On the other hand, we can just take it as an indication that ordinary Poincaré transformations are inadequate to describe relativistic symmetries in this regime and must be deformed to accomodate the new fundamental scale ℓ, in the same way Galileo transformations had to be deformed to accomodate the universal speed constant c. We adopt this second perspective, usually referred to as DSR in the literature [29], and assume that our noncommutative models admit deformed relativistic symmetries which reduce to Poincaré transformations in the limit $\ell \to 0$.

§ A priori, the kinematical symmetry group could be discontinuous in the presence of spacetime noncommutativity. If this were the case, though, we could not even speak of symmetry generators and the commutative limit would be exceedingly singular to be dealt with. Therefore, we will rule out this possibility and assume that deformed relativistic symmetries are described by a 10-dimensional Lie group of automorphisms of the deformed canonical algebra (3.118)-(3.120), like in the commutative case. Even requiring that these transformations reduce to standard Poincaré symmetries in the limit $\ell \to 0$, the problem is obviously underconstrained and we expect to find many possible deformations of the usual relativistic symmetries for any given noncommutative single-particle model (3.118)-(3.120). Our aim is to characterize as sharply as we can these possibilities.

§ First of all, we observe that deformed translations must be generated by $\widehat{p}_\mu$, because momenta are physically defined as the generators of spacetime translations. Let us then denote the deformed Lorentz generators with $\widehat{m}_{\rho\sigma}$, like in the commutative case. The deformed symmetry algebra generated by ($\widehat{p}_\mu$,$\widehat{m}_{\rho\sigma}$) must contract to the Poincaré algebra in the limit $\ell \to 0$. However, it is a well-known result by Levy-Nahas [46] that the only Lie algebra deformations of the Poincaré algebra are the de Sitter and anti de Sitter algebras. Since we are assuming that spacetime is flat, we can conclude that $\widehat{p}_\mu$ and $\widehat{m}_{\rho\sigma}$ must satisfy their usual commutation relations, that is

$$
\begin{aligned}
[\widehat{p}_\mu, \widehat{p}_\tau] &= 0, && (3.133)\\
[\widehat{p}_\mu, \widehat{m}_{\rho\sigma}] &= i\hbar(g_{\rho\mu}\widehat{p}_\sigma - g_{\mu\sigma}\widehat{p}_\rho), && (3.134)\\
[\widehat{m}_{\mu\nu}, \widehat{m}_{\rho\sigma}] &= i\hbar\left(g_{\rho\nu}\widehat{m}_{\mu\sigma} - g_{\mu\rho}\widehat{m}_{\nu\sigma} + g_{\sigma\nu}\widehat{m}_{\rho\mu} - g_{\mu\sigma}\widehat{m}_{\rho\nu}\right), && (3.135)
\end{aligned}
$$

even in the noncommutative case. This means that deformed Poincaré symmetries have their usual action (3.130) on momenta, with the deformation only affecting the transformation (3.129) of spacetime coordinates.

§ To complete our analysis, it is convenient to choose a representation Σ^α and express $\widehat{m}_{\rho\sigma}$ as functions of the canonical variables $(\widehat{p}_\mu, \widehat{q}^\nu_\Sigma)$. It now follows from (3.134) that $\widehat{m}_{\rho\sigma}$ must be linear in $\widehat{q}^\nu_\Sigma$, so that we can generically write

$$
\widehat{m}_{\rho\sigma} = \left[\widehat{q}^\Sigma_\rho + \ell\Phi_\rho(\ell\widehat{p})\right]\widehat{p}_\sigma - \left[\widehat{q}^\Sigma_\sigma + \ell\Phi_\sigma(\ell\widehat{p})\right]\widehat{p}_\rho. \tag{3.136}
$$

Requiring that $\widehat{m}_{\rho\sigma}$ satisfy the last commutation rules (3.135), we obtain the following conditions on Φ_ρ:

$$
g_{\sigma\gamma}\partial_\gamma\Phi_\rho - g_{\rho\gamma}\partial_\gamma\Phi_\sigma = 0. \tag{3.137}
$$

If we make the substitution

$$
\Phi^\rho = \delta\Sigma^\alpha(\Delta^{-1})_\alpha^{\ \rho}, \tag{3.138}
$$

a tedious but straightforward calculation shows that Φ_ρ satisfy (3.137) if and only if $\delta\Sigma^\alpha$ satisfy the homogeneous version of (3.127). As a consequence, $\overline{\Sigma}^\alpha = \Sigma^\alpha - \delta\Sigma^\alpha$ defines another representation of the deformed canonical algebra. Writing $\widehat{m}_{\rho\sigma}$ in terms of the canonical variables $(\widehat{p}_\mu, \widehat{q}^\nu_{\overline{\Sigma}})$, we obtain at last

$$
\widehat{m}_{\rho\sigma} = \widehat{q}^{\overline{\Sigma}}_\rho\widehat{p}_\sigma - \widehat{q}^{\overline{\Sigma}}_\sigma\widehat{p}_\rho. \tag{3.139}
$$

In other words, we have proved that it is always possible to find a unique set of commutative coordinates $\widehat{q}^\nu_{\overline{\Sigma}}$ which transform like standard 4-vectors under the action of the deformed Lorentz symmetries. This means that the corresponding representation $\overline{\Sigma}^\alpha$ univocally determines the action of the deformed Lorentz transformations on the deformed canonical algebra and is therefore physically distinguished from the others.

§ We can finally provide a complete and very compact characterization of the possible single-particle quantum models of spacetime noncommutativity (3.114) and their deformed relativistic symmetries. They are all obtained from a standard canonical algebra $(\widehat{p}_\mu, \widehat{q}^\nu)$ by specifying an invertible matrix $[\Delta(\ell\widehat{p})]_\mu^{\ \nu}$ and a vector $\Sigma^\alpha(\ell\widehat{p})$ satisfying (3.121), (3.127) and the boundary conditions $[\Delta(0)]_\mu^{\ \nu} = \delta_\mu^{\ \nu}$. Noncommutative spacetime coordinates $\widehat{x}^\nu$ are defined via (3.124) and the action of deformed relativistic symmetries (Λ, a) is given by the ordinary Poincaré action on the standard canonical coordinates $(\widehat{p}_\mu, \widehat{q}^\nu)$:

$$
\begin{aligned}
\widehat{q}^\nu &\mapsto \Lambda^\nu_{\ \alpha}\widehat{q}^\alpha + a^\nu, && (3.140)\\
\widehat{p}_\mu &\mapsto \Lambda_\mu^{\ \beta}\widehat{p}_\beta. && (3.141)
\end{aligned}
$$

This results in a deformed action on spacetime coordinates $\widehat{x}^\nu$, given by

$$\begin{aligned}\widehat{x}^\nu \quad\mapsto\quad & \widehat{x}^\gamma[\Delta^{-1}(\ell\widehat{p})]_\gamma^{\ \beta}\Lambda^\alpha_{\ \beta}[\Delta(\ell\Lambda\widehat{p})]_\alpha^{\ \nu}+a^\alpha[\Delta(\ell\Lambda\widehat{p})]_\alpha^{\ \nu}+\ell\,[\Sigma^\nu(\ell\Lambda\widehat{p})+i\Omega^\nu(\ell\Lambda\widehat{p})]- \\ & -\ell\,[\Sigma^\gamma(\ell\widehat{p})+i\Omega^\gamma(\ell\widehat{p})]\,[\Delta^{-1}(\ell\widehat{p})]_\gamma^{\ \beta}\Lambda^\alpha_{\ \beta}[\Delta(\ell\Lambda\widehat{p})]_\alpha^{\ \nu}. \qquad (3.142)\end{aligned}$$

The deformed symmetry group is generated by the momenta $\widehat{p}_\mu$ and the self-adjoint operators

$$\widehat{m}_{\rho\sigma}=\widehat{q}_\rho\widehat{p}_\sigma-\widehat{q}_\sigma\widehat{p}_\rho, \qquad (3.143)$$

and the corresponding infinitesimal variations of spacetime coordinates $\widehat{x}^\nu$ are given by

$$\begin{aligned}\delta_\varepsilon\widehat{x}^\nu &= \frac{1}{i\hbar}\varepsilon^\mu[\widehat{p}_\mu,\widehat{x}^\nu]=\varepsilon^\mu[\Delta(\ell\widehat{p})]_\mu^{\ \nu}, \qquad (3.144)\\ \delta_\varphi\widehat{x}^\nu &= \frac{1}{i\hbar}\varphi^{\rho\sigma}[\widehat{m}_{\rho\sigma},\widehat{x}^\nu]= \\ &= 2\varphi^{\rho\sigma}\Big\{\widehat{x}^\gamma[\Delta^{-1}(\ell\widehat{p})]_\gamma^{\ \alpha}\Big(g_{\alpha[\rho}[\Delta(\ell\widehat{p})]_{\sigma]}^{\ \nu}+\widehat{p}_{[\rho}\partial^{\sigma]}[\Delta(\ell\widehat{p})]_\alpha^{\ \nu}\Big)- \\ &\quad-\ell\,[\Sigma^\gamma(\ell\widehat{p})+i\Omega^\gamma(\ell\widehat{p})]\,[\Delta^{-1}(\ell\widehat{p})]_\gamma^{\ \alpha}\Big(g_{\alpha[\rho}[\Delta(\ell\widehat{p})]_{\sigma]}^{\ \nu}+\widehat{p}_{[\rho}\partial^{\sigma]}[\Delta(\ell\widehat{p})]_\alpha^{\ \nu}\Big)+ \\ &\quad+\ell\widehat{p}_{[\rho}\partial^{\sigma]}\,[\Sigma^\gamma(\ell\widehat{p})+i\Omega^\gamma(\ell\widehat{p})]\Big\}, \qquad (3.145)\end{aligned}$$

where little square brackets denote antisymmetrization.

§ In order to obtain a complete covariant quantum-mechanical model, we must still specify a hamiltonian constraint which is invariant under deformed Poincaré symmetries and reduces to the usual one in the commutative limit. In the light of our previous findings, however, the problem is trivial. In fact, since momenta have their usual transformation properties, the undeformed relativistic contraint

$$\widehat{H}_r=\widehat{P}^\alpha\widehat{P}_\alpha-m^2c^2 \qquad (3.146)$$

is invariant under deformed symmetries and is therefore the only natural choice.

3.6 Discussion and comparison with other approaches

§ Having obtained a precise characterization of the possible single-particle quantum models of spacetime noncommutativity (3.114), we can now discuss their general features and comment on other approaches.

§ Our main result is that spacetime noncommutativity has the only effect of deforming the relation between the particle spacetime coordinates $\widehat{x}^\nu$ and the conjugate variables $\widehat{q}^\nu$ of the corresponding momenta, replacing the simple identification

$$\widehat{x}^\nu=\widehat{q}^\nu \qquad (3.147)$$

with the general momentum-dependent formula

$$\widehat{x}^\nu=\frac{1}{2}\left\{\widehat{q}^\alpha[\Delta(\ell\widehat{p})]_\alpha^{\ \nu}+\ell\Sigma^\nu(\ell\widehat{p})+\text{h.c.}\right\}. \qquad (3.148)$$

In particular, the momentum space of our models is not affected by the noncommutativity and the dispersion relation is undeformed. This negative result is quite relevant from a phenomenological point of view, because modified relativistic dispersion relations have been the main target of recent searches for observable quantum gravity effects [11]. It is also at odds with what was proposed for κ-Minkowski in the pioneering papers [12, 13], which actually motivated our study. On the positive side, our models exhibit deformed Heisenberg relations, i.e. nontrivial $\Delta_\mu{}^\nu$, whenever $\Gamma^{\nu\lambda}{}_\alpha \neq 0$. This deformation generally affects the translation properties of both the particle's physical position and the associated quantum uncertainty. We expect the resulting effects to include features of relative locality [47] such as those reported in [13, 48], which could provide interesting targets for quantum-gravity phenomenology. Even when $\Gamma^{\nu\lambda}{}_\alpha = 0$, the deformed commutation rules among $\widehat{m}_{\rho\sigma}$ and $\widehat{x}^\nu$ should give rise to relative locality effects under Lorentz transformations. Since our framework allows for physical amplitudes to be computed at any order in the deformation parameter ℓ, this phenomenology can be quantitatively characterized beyond the previous qualitative remarks.

§ As discussed above, spacetime noncommutativity is usually introduced in a field-theoretical setting, replacing the usual Minkowski background with a noncommutative algebra of coordinates. In this context, it is necessary to introduce noncommutative transformation parameters and the nontrivial coproduct can also affect the commutation properties of symmetry generators, thus leading to deformed relativistic wave equations and dispersion relations. In the special case of κ-Minkowski, for instance, the coproduct induces nonlinear commutation rules among momenta and boost generators and consequently determines a deformed Casimir operator. This description of the κ-Poincaré Hopf algebra, reviewed above, is actually the starting point of our main reference [12] and the source of our disagreement. All the difference between our treatment and the usual approach can be traced back to our assumption that deformed Poincaré symmetries are standard quantum symmetries described by a Lie algebra of generators. Without this hypothesis, we would have obtained a much wider class of models, including those explored in the seminal works [12, 13], and we too would probably have relied on Hopf-algebraic considerations to identify the relevant ones. A general classification of this kind is actually available in the literature. In a recent paper [49], a variety of noncommutative spacetimes as well as the corresponding Hopf symmetries have been characterized in terms of nonlinear realizations of the Heisenberg algebra. Even if the authors adopt a field-theoretical point of view based on Hopf algebras, it is straightforward to recast their results in our single-particle framework and obtain a formal generalization of our models. The problem with these generalized models is that their symmetry generators do not admit a clear physical interpretation. In fact, nonlinear commutation relations among the generators, as found in [12], cannot hold if the group of finite symmetry transformations is continuous. But if it were discontinuous, then there would be no well-defined infinitesimal transformations to begin with, and symmetry generators would lose their usual physical meaning. By assuming standard quantum symmetries, we found all covariant single-particle models which are not affected by such interpretive difficulties and can directly provide interesting phenomenology. Our approach has the additional advantage of being entirely independent of Hopf-algebraic concepts, thereby avoiding all the problems associated with noncommutative transformation parameters.

§ We have considered for simplicity momentum-independent deformation matrices ${\Gamma^{\nu\lambda}}_\alpha$ and $\Theta^{\nu\lambda}$, but our analysis also applies with little changes to momentum-dependent commutation relations

$$\left[\widehat{x}^\nu, \widehat{x}^\lambda\right] = \frac{1}{2} i \left(\ell {[\Gamma(\ell\widehat{p})]^{\nu\lambda}}_\alpha \widehat{x}^\alpha + \ell^2 [\Theta(\ell\widehat{p})]^{\nu\lambda} + \text{h.c.}\right). \tag{3.149}$$

Therefore, our results can be readily extended to include the much-studied Snyder noncommutative spacetime [50] and similar examples.

3.7 Quantum particle on a Θ-Minkowski background

§ The algebra of dynamical variables $\mathcal{V}$ associated with a quantum particle is generated by the particle's coordinates x^ν and momenta p_μ. We will model a quantum particle moving on a Θ-Minkowski background requiring that the hermitian generators $p_\mu^{(\Theta)}, x^\nu \in \mathcal{V}_\Theta$ satisfy the following commutation relations:

$$\begin{aligned} \left[p_\mu^{(\Theta)}, p_\tau^{(\Theta)}\right] &= 0, && (3.150) \\ \left[p_\mu^{(\Theta)}, x^\nu\right] &= i{\delta_\mu}^\nu, && (3.151) \\ \left[x^\nu, x^\lambda\right] &= i\Theta^{\nu\lambda}. && (3.152) \end{aligned}$$

These are the usual canonical commutation rules with the exception of the last ones, which establish the Θ-Minkowski noncommutativity of spacetime coordinates. The algebra $\mathcal{V}_\Theta$ is actually an ordinary Heisenberg algebra in the variables $p_\mu^{(\Theta)}$ and

$$q^\nu_{(\Theta)} = x^\nu - \frac{1}{2}\Theta^{\nu\alpha} p_\alpha^{(\Theta)}. \tag{3.153}$$

In fact, we have

$$\begin{aligned} \left[p_\mu^{(\Theta)}, q^\nu_{(\Theta)}\right] &= i{\delta_\mu}^\nu, && (3.154) \\ \left[q^\nu_{(\Theta)}, q^\lambda_{(\Theta)}\right] &= \left[x^\nu - \frac{1}{2}\Theta^{\nu\alpha} p_\alpha^{(\Theta)}, x^\lambda - \frac{1}{2}\Theta^{\lambda\beta} p_\beta^{(\Theta)}\right] = 0. && (3.155) \end{aligned}$$

3.7.1 Kinematical symmetries of $\mathcal{V}_\Theta$

§ A continuous kinematical symmetry of a quantum system is a one-parameter group of automorphisms of its algebra of dynamical variables $\mathcal{V}$. For finite-dimensional quantum systems, it is always generated by the adjoint action of a self-adjoint operator a, so that the corrisponding infinitesimal variation of parameter α reads

$$\delta_\alpha \triangleright v = i\alpha \left[a, v\right], \tag{3.156}$$

for every $v \in \mathcal{V}$. In the case of a quantum particle on ordinary Minkowski spacetime, kinematical symmetries are the standard translations and Lorentz transformations, which are generated by the momenta p_μ and the self-adjoint variables

$$m_{\rho\sigma} = x_\rho p_\sigma - x_\sigma p_\rho, \tag{3.157}$$

respectively. In fact, the adjoint action of p_μ and $m_{\rho\sigma}$, restricted to the particle's coordinates x^ν, is precisely that of the Poincaré generators P_μ and $M_{\rho\sigma}$ seen before:

$$[p_\mu, x^\nu] = i\delta_\mu{}^\nu = P_\mu \triangleright x^\nu, \tag{3.158}$$
$$[m_{\rho\sigma}, x^\nu] = i(x_\rho \delta_\sigma{}^\nu - x_\sigma \delta_\rho{}^\nu) = M_{\rho\sigma} \triangleright x^\nu. \tag{3.159}$$

In the case of a quantum particle on Θ-Minkowski spacetime, we would like to find a ten-dimensional algebra of symmetry generators which reduce to the standard ones for $\Theta \to 0$. Ordinary translations, generated by the momenta $p_\mu^{(\Theta)}$, are still symmetries of the system. Lorentz transformations are not, and we expect the corresponding generator $m_{\rho\sigma}$ to be deformed. There are of course infinitely many self-adjoint Θ-deformations of $m_{\rho\sigma}$, but we must impose that their commutators are elements of the symmetry algebra and this restriction severely limits our possibilities. The most natural way to comply with this requirement is to take

$$\begin{aligned} m_{\rho\sigma}^{(\Theta)} &= q_\rho^{(\Theta)} p_\sigma^{(\Theta)} - q_\sigma^{(\Theta)} p_\rho^{(\Theta)} = \\ &= x_\rho p_\sigma^{(\Theta)} - x_\sigma p_\rho^{(\Theta)} - \frac{1}{2}\left\{\Theta_\rho{}^\alpha p_\alpha^{(\Theta)} p_\sigma^{(\Theta)} - \Theta_\rho{}^\alpha p_\alpha^{(\Theta)} p_\sigma^{(\Theta)}\right\} \end{aligned} \tag{3.160}$$

as deformed Lorentz generators. In terms of the canonical variables $p_\mu^{(\Theta)}$ and $q^\nu_{(\Theta)}$, the $m_{\rho\sigma}^{(\Theta)}$ are just the usual Lorentz generators. As a consequence, they satisfy the usual commutation relations among themselves and $p_\mu^{(\Theta)}$, namely

$$[p_\mu, m_{\rho\sigma}] = i(g_{\mu\rho} p_\sigma - g_{\mu\sigma} p_\rho), \tag{3.161}$$
$$[m_{\tau\upsilon}, m_{\rho\sigma}] = i(g_{\tau\rho} m_{\sigma\upsilon} - g_{\tau\sigma} m_{\rho\upsilon} + g_{\upsilon\rho} m_{\tau\sigma} - g_{\upsilon\sigma} m_{\tau\rho}). \tag{3.162}$$

The only difference with the standard case is that the action of $m_{\rho\sigma}^{(\Theta)}$ on the non-commutative spacetime coordinates x^ν reads

$$\begin{aligned} \left[m_{\rho\sigma}^{(\Theta)}, x^\nu\right] &= \left[m_{\rho\sigma}^{(\Theta)}, q^\nu_{(\Theta)} + \frac{1}{2}\Theta^{\nu\alpha} p_\alpha^{(\Theta)}\right] = \\ &= i\left\{q_\rho^{(\Theta)}\delta_\sigma{}^\nu - q_\sigma^{(\Theta)}\delta_\rho{}^\nu + \frac{1}{2}(\Theta_\rho{}^\nu \delta_\sigma{}^\alpha - \Theta_\sigma{}^\nu \delta_\rho{}^\alpha) p_\alpha^{(\Theta)}\right\} = \\ &= i\left\{x_\rho \delta_\sigma{}^\nu - x_\sigma \delta_\rho{}^\nu + \frac{1}{2}\Upsilon_\rho{}^\nu{}_\sigma{}^\alpha p_\alpha^{(\Theta)}\right\} \end{aligned} \tag{3.163}$$

instead of (3.159). We see that deformed boosts mix coordinates and momenta. Since the system features an intrinsic length scale (the matrix Θ has the dimension of a length squared), this is not surprising. In the next paragraph we will show that this circumstance is the very source of the problems we encountered when dealing with the Hopf description of Θ-Minkowski symmetries.

3.7.2 Explicit representation of $\mathcal{H}_\Theta$ on $\mathcal{V}_\Theta$

§ Infinitesimal kinematical symmetries of $\mathcal{V}_\Theta$ of parameters $\varepsilon^\mu_{(\Theta)}$ and $\omega^{\rho\sigma}$ are described by the infinitesimal variation operator

$$\delta_{\varepsilon,\omega} \triangleright v = i\varepsilon^\mu_{(\Theta)} \left[p^{(\Theta)}_\mu, v\right] + i\omega^{\rho\sigma} \left[m^{(\Theta)}_{\rho\sigma}, v\right]. \tag{3.164}$$

These transformations obviously satisfy the Leibniz rule and the parameters $\varepsilon^\mu_{(\Theta)}$ and $\omega^{\rho\sigma}$ are ordinary real numbers. Let us now consider the following linear operators on $\mathcal{V}_\Theta$:

$$P^{(\Theta)}_\mu \triangleright v = \left[p^{(\Theta)}_\mu, v\right], \tag{3.165}$$

$$M^{(\Theta)}_{\rho\sigma} \triangleright v = \left[m^{(\Theta)}_{\rho\sigma}, v\right] + \frac{1}{2}\Upsilon_{\rho\ \sigma}^{\ \alpha\ \beta} p^{(\Theta)}_\alpha \left[p^{(\Theta)}_\beta, v\right]. \tag{3.166}$$

It is straightforward to check that

$$P^{(\Theta)}_\mu \triangleright x^\nu = \left[p^{(\Theta)}_\mu, x^\nu\right] = i\delta_\mu^{\ \nu}, \tag{3.167}$$

$$M^{(\Theta)}_{\rho\sigma} \triangleright x^\nu = \left[m^{(\Theta)}_{\rho\sigma}, x^\nu\right] + \frac{1}{2}\Upsilon_{\rho\ \sigma}^{\ \alpha\ \beta} p^{(\Theta)}_\alpha \left[p^{(\Theta)}_\beta, x^\nu\right] = i(x_\rho\delta_\sigma^{\ \nu} - x_\sigma\delta_\rho^{\ \nu}), \tag{3.168}$$

and also that

$$\begin{aligned} P^{(\Theta)}_\mu \triangleright x^\nu x^\lambda &= \left[p^{(\Theta)}_\mu, x^\nu x^\lambda\right] = \left[p^{(\Theta)}_\mu, x^\nu\right] x^\lambda + x^\nu \left[p^{(\Theta)}_\mu, x^\lambda\right] \\ &= \left(P^{(\Theta)}_\mu \otimes 1\!\!1 + 1\!\!1 \otimes P^{(\Theta)}_\mu\right) \triangleright x^\nu \otimes x^\lambda, \end{aligned} \tag{3.169}$$

$$\begin{aligned} M_{\rho\sigma} \triangleright x^\nu x^\lambda &= \left[m^{(\Theta)}_{\rho\sigma}, x^\nu x^\lambda\right] + \frac{1}{2}\Upsilon_{\rho\ \sigma}^{\ \alpha\ \beta} p^{(\Theta)}_\alpha \left[p^{(\Theta)}_\beta, x^\nu x^\lambda\right] = \\ &= (M^{(\Theta)}_{\rho\sigma} \triangleright x^\nu)x^\lambda + x^\nu(M^{(\Theta)}_{\rho\sigma} \triangleright x^\lambda) + \frac{1}{2}\Upsilon_{\rho\ \sigma}^{\ \alpha\ \beta} \left[p^{(\Theta)}_\alpha, x^\nu\right]\left[p^{(\Theta)}_\beta, x^\lambda\right] = \\ &= \left(M^{(\Theta)}_{\rho\sigma} \otimes 1\!\!1 + 1\!\!1 \otimes M^{(\Theta)}_{\rho\sigma} + \frac{1}{2}\Upsilon_{\rho\ \sigma}^{\ \alpha\ \beta} P^{(\Theta)}_\alpha \otimes P^{(\Theta)}_\beta\right) \triangleright x^\nu \otimes x^\lambda. \end{aligned} \tag{3.170}$$

Therefore, $P^{(\Theta)}_\mu$ and $M^{(\Theta)}_{\rho\sigma}$ can be identified with the generators of the Θ-Minkowski Hopf algebra $\mathcal{H}_\Theta$. Rewriting the infinitesimal variation operator $\delta_{\varepsilon,\omega}$ in terms of $P^{(\Theta)}_\mu$ and $M^{(\Theta)}_{\rho\sigma}$, we obtain

$$\delta_{\omega,\varepsilon} v = i\varepsilon^\mu_{(\Theta)} \left[p^{(\Theta)}_\mu, v\right] + i\omega^{\rho\sigma} \left[m^{(\Theta)}_{\rho\sigma}, v\right] = \tag{3.171}$$

$$= i\varepsilon^\mu \left[p^{(\Theta)}_\mu, v\right] + i\omega^{\rho\sigma} \left(M^{(\Theta)}_{\rho\sigma} \triangleright v - \frac{1}{2}\Upsilon_{\rho\ \sigma}^{\ \alpha\ \mu} p^{(\Theta)}_\alpha \left[p^{(\Theta)}_\mu, v\right]\right) = \tag{3.172}$$

$$= \left(i\left\{\varepsilon^\mu - \frac{1}{2}\omega^{\rho\sigma}\Upsilon_{\rho\ \sigma}^{\ \alpha\ \mu} p^{(\Theta)}_\alpha\right\} P^{(\Theta)}_\mu + i\omega^{\rho\sigma} M^{(\Theta)}_{\rho\sigma}\right) \triangleright v,$$

which is identical to the Hopf infinitesimal variation operator $\delta^{(\Theta)}_{\epsilon,\omega}$ of translation parameters

$$\epsilon^\mu_{(\Theta)} = \varepsilon^\mu_{(\Theta)} - \frac{1}{2}\omega^{\rho\sigma}\Upsilon_{\rho\ \sigma}^{\ \alpha\ \mu} p^{(\Theta)}_\alpha. \tag{3.173}$$

Being functions of the momenta, the $\epsilon^\mu_{(\Theta)}$ automatically satisfy the commutation relations (3.100):

$$\left[\epsilon^\mu_{(\Theta)}, x^\nu\right] = -\frac{1}{2}\omega^{\rho\sigma}\Upsilon_{\rho\ \sigma}^{\ \alpha\ \mu}\left[p^{(\Theta)}_\alpha, x^\nu\right] = \frac{i}{2}\omega^{\rho\sigma}\Upsilon_{\rho\ \sigma}^{\ \mu\ \nu}. \tag{3.174}$$

We can conclude that the weird commutation properties of the Hopf-algebraic transformation parameters admit a natural explanation in a single-particle quantum mechanical setting.

Conclusions and outlook

§ In this Ph.D thesis, I have explored the potential of the CQM framework as a tool for quantum-gravity phenomenology. After a review of the basic concepts and methods of CQM (Chapter 1), I presented two original applications of the formalism of potential phenomenological relevance. First, I have provided CQM with a rigorous geometric foundation and extended it to generic configuration manifolds (Chapter 2). This let me build single-particle models for free spinless quanta propagating on a curved spacetime, which could be useful in the analysis of astrophysical and cosmological signals. In particular, I have explicitly worked out such a model in the case of the cosmologically relevant de Sitter spacetime, verifying the effectiveness of this approach. Second, I have used CQM to implement nontrivial commutation relations among spacetime coordinates in a single-particle context and found a complete characterization of all the possible quantum models of this kind in terms of the Minkowski model (Chapter 3). These could yield interesting quantum-gravity phenomenology if applied to the analysis of cosmological signals coming from very far away.

§ The results reported here are mostly theoretical and have many possible applications. Curved spacetimes other than de Sitter could be modelled and analyzed with the help of our coordinate-independent, generalized CQM. A study of the scattering of quantum particles by a Schwarzschild blackhole is a particularly interesting possibility in this regard. Noncommutative spacetimes other than Θ-Minkowski and κ-Minkowski could be quantitatively studied in our general framework along the lines of [13], computing physical amplitudes at first order in the deformation parameter ℓ. From a purely theoretical perspective, the next step would be the analysis of the propagation of free quantum particles on spacetimes which are both curved and noncommutative. Such models would be very interesting for phenomenology, because they could be realistically applied to the analysis of cosmological signals. In order to get there, one should somehow merge the results of Chapter 2 and Chapter 3. However, at the present stage of development of the theory, this extension is apparently out of reach. Some further technical progress is probably needed before attempting such a generalization.

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