Reduction of discrete-time two-channel delayed systems

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***Abstract*—In this paper, the reduction method is extended to time-delay systems affected by two mismatched input delays. To this end, the intrinsic feedback structure of the retarded dynamics is exploited to deduce a reduced dynamics which is free of delays. Moreover, among other possibilities, an Immersion and Invariance feedback over the reduced dynamics is designed for achieving stabilization of the original dynamics. A chained sampled-data dynamics is used to show the effectiveness of the proposed control strategy through simulations.**

***Index Terms*—Delay systems, Sampled-data control, Stability of nonlinear systems**

I. Introduction

NONLINEAR discrete-time dynamics with input delays exhibit a strict feedback form over an extended state space [1]–[7]. Taking advantage of this peculiar cascade structure, several stabilizing controllers have been proposed. In [8], Immersion and Invariance (I&I) based controllers have been designed for discrete-time nonlinear dynamics with input delays while improving prediction-based methodologies. Those invariance-based strategies are generally easier to deduce than predictor-based ones. Indeed, the necessity of computing a prediction of the state trajectories over the delayed window is weakened by requiring convergence to some suitably shaped set over which the closed-loop dynamics recovers the ideal delay-free one.

Those methodologies apply to sampled-data dynamics as well when affected by a constant input delay. In this scenario, the continuous-time dynamics is controlled through piecewise constant input signals while measures of the state are available only at the sampling instants. Accordingly, the stabilizing sampled-data feedback can be designed over an extended discrete-time equivalent dynamics which exhibits a strict-feedback form, too. In this scenario, predictor and I&I-based control laws are discussed and compared in [9], [10]. Truncated expansions in powers of the sampling period δ are also proposed to approximate the exact solutions which are difficult to compute in practice.

A more recent approach concerns reduction-based methods aimed at reducing the input delayed dynamics to a delay free one that is equivalent (from the point of view of stability) to the original one [11]. Since reduction implicitly relies upon prediction, stabilization of the reduced dynamics implies stabilization of the delayed one. However, an interesting feature of reduction stands in the simplification of the design because the reduced dynamics is by construction free of delays. Moreover, contrarily to prediction dynamics, the reduced model is not a delay-free copy of the system dynamics but differs in the controlled vector fields that come to be explicitly parametrized by the delay-length so leaving space for a further redesign.

Up to now, the discussion has been referring to single-input dynamics though extensions to the case of multiple inputs is straightforward whenever the input channels are uniformly delayed (i.e., affected by the same delay) as developed in [12]. In continuous time, predictor-based techniques for multi-input linear time-invariant systems affected by distinct input delays have been proposed in [13]–[15] with extensions to nonlinear dynamics in [16].

The aim of this paper is to address this problem in the nonlinear context when considering dynamics affected by two distinct input delays. The contribution relies upon the possibility of extending the reduction method [12] to this class of dynamics by taking advantage of the feedback structure underlying the evolutions of the retarded system. First, a state augmentation is used to make the delayed dynamics uniform in the action of the delays (i.e., the extended system is affected by the same delay); then, a modified reduction variable is exhibited so to transform the input-delayed dynamics into a delay-free one over the extended state-space. Finally, among other possibilities, an I&I design procedure is worked out for stabilizing the reduced dynamics. The cascade structure allows to conclude that stabilization of the reduced dynamics implies stabilization of the input-delayed one.

The paper is organized as follows. In Section II, recalls on the discrete-time reduction method are provided when the inputs are affected by the same delay. The case of two-channel time delays systems is studied in Section III by exhibiting a reduced dynamics which is free of delays. An I&I-based design procedure over the reduced model is then presented in Section IV. In Section V, the case of a chained dynamics is considered as a case of study while Section VI concludes the paper.

Notations: $\mathbf{0}_{i\times j}$ denotes the $i\times j$-dimensional matrix whose entries are zero, I_N stands for the N dimensional identity matrix while $\mathbf{1}_j$ the column vector whose entries are all ones. Maps and vector fields are assumed smooth. Given $i, j \in \mathbb{N}$

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such that $j < i$, $u_{[k-i,k-j[}$ denotes the history of the discrete variable u over the window $[k-i,k-j[$ (i.e., $u_{[k-i,k-j[} = \{u(k-i),\dots,u(k-j-1)\}$). The symbol "$\circ$" denotes the composition of functions. I and Id denote respectively the identity matrix and the identity operator. Given a vector field f, L_f denotes the Lie derivative operator, $\mathrm{L}_f = \sum_{i=1}^{n} f_i(\cdot)\nabla_{x_i}$ with $\nabla_{x_i} := \frac{\partial}{\partial x_i}$ while $\nabla = (\nabla_{x_1},\dots,\nabla_{x_n})$. $e^{\mathrm{L}_f}\mathrm{Id}$ denotes the associated Lie series operator, $e^{\mathrm{L}_f} := \mathrm{Id} + \sum_{i\geq 1} \frac{\mathrm{L}_f^i}{i!}$

II. Recalls on discrete-time reduction

Consider the nonlinear discrete-time system

$$x(k+1) = F(x(k), u(k-N)) \tag{1}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$, possessing an equilibrium at the origin and affected by a discrete delay $N \geq 0$ uniformly acting over each input channel. Invertibility of the function $F_0(\cdot) = F(\cdot, 0)$ with respect to the state vector x is assumed.

In [12], it was proven that the problem of finding a stabilizer for (1) can be settled over a new dynamics that is equivalent to the original one as far as stability properties are concerned. For, we introduce the so-called *reduction* variable

$$\eta(k) := F_0^{-N}(\cdot) \circ F^N(x(k), u_{[k-N,k[}) \tag{2}$$

where $u_{[k-N,k[}$ denotes the history of the control signal and

$$F_0^N(x) = \underbrace{F_0(\cdot)\circ\cdots\circ F_0(x)}_{\text{N times}},\; F_0^{-N}(x) = \underbrace{F_0^{-1}(\cdot)\circ\cdots\circ F_0^{-1}(x)}_{\text{N times}}$$

represent the usual N-times composition of the drift term and the corresponding inverse. Composing N steps ahead the full dynamics (1), one computes for $N \geq 1$

$$\begin{aligned} F^N(x(k), u_{[k-N,k[}) :=& F^{N-1}(\cdot, u_{[k-N+1,k[}) \circ F(x(k), u(k-N)) \\ :=& F(\cdot, u(k-1)) \circ \cdots \circ F(x(k), u(k-N)) \end{aligned}$$

with $F^1(x(k), u_{[k-1,k[}) := F(x(k), u(k-1))$. It is a matter of computations to verify that (2) evolves according to the *reduced dynamics*

$$\eta(k+1) = F_r(\eta(k), u(k)) \tag{3}$$

with $F_r(\eta, u) := F_0^{-N}(\cdot) \circ F(\cdot, u) \circ F_0^N(\eta)$.

The reduced dynamics (3) is delay free with the same drift as (1) but modified controlled vector field. More precisely, when assuming $p = 1$ for the sake of simplicity, (3) rewrites as the N-depending dynamics

$$\begin{aligned} \eta(k+1) =& F_0(\eta(k)) \\ &+ \int_0^{u(k)} \nabla_u \big(F_0^{-N}(\cdot) \circ F(\cdot, u) \circ F_0^N(\eta(k))\big)\mathrm{d}u. \end{aligned} \tag{4}$$

Hence, the problem stands in finding a feedback $u(k) = \alpha(\eta(k))$ stabilizing the equilibrium of (3) so getting in turn stabilization of (1) in closed loop as, by construction, $x(k+N) = F_0^N(\eta(k))$. In fact, one gets in closed loop the cascade

$$\begin{aligned} x(k+1) =& F_0^N(\eta_1(k)) \\ \eta_1(k+1) =& \eta_2(k), \quad \dots, \quad \eta_{N-1}(k+1) = \eta(k) \\ \eta(k+1) =& F_r(\eta(k), \alpha(\eta(k))) \end{aligned}$$

with $\eta_1(k) = F(x(k), \alpha(F_0^{-N}(x(k))))$.

Several strategies aimed at computing the reduction-based feedback have been discussed in [12] by exploiting the properties of the time-delay system (1) in free evolution.

III. Two-channel time-delay systems

In the sequel we address the problem of stabilizing the time-delay system

$$x(k+1) = F(x(k), u_1(k-N_1), u_2(k-N_2)) \tag{5}$$

whose input channels $u_i \in \mathbb{R}^{p_i}$, $i = 1,2$ with $p = p_1 + p_2$, are affected by different time delays verifying, after possible index sorting, $N_2 - N_1 = N > 0$; $F_0(\cdot) := F(\cdot, 0, 0)$ is assumed to be invertible over $\mathbb{R}^n$.

The design we propose is based on three steps: first, we introduce a dynamical extension over the control u_2 so to compensate the mismatch among the two input delays; then, we extend the reduction method as recalled in Section II over the extended dynamics; finally, we design one possible reduction-based feedback based on Immersion and Invariance.

Remark 3.1: The presented results apply to sampled-data systems affected by entire delays; namely, one considers continuous-time dynamics of the form

$$\dot{x}(t) = f(x(t)) + g_1(x(t))u_1(t-\tau_1) + g_2(x(t))u_2(t-\tau_2)$$

with $u_i(t) = u_i(k\delta) = u_i(k)$ for $t \in [k\delta, (k+1)\delta[$ and $\tau_i = N_i\delta$ for some $N_i \in \mathbb{N}$ $(i = 1,2)$. In that case, for $x(k) = x(k\delta)$, the discrete-time equivalent model gets the δ-dependent form

$$\begin{aligned} x(k+1) =& F^{\delta}(x(k), u_1(k-N_1), u_2(k-N_2)) \\ =& e^{\delta(\mathrm{L}_f + \mathrm{L}_{g_1 u_1(k-N_1)} + \mathrm{L}_{g_2 u_2(k-N_2)})} x\big|_{x(k)}. \end{aligned}$$

A. The dynamical extension

Let us introduce the new state $\xi := (\xi_1,\dots,\xi_N)^\top \in \mathbb{R}^{p_2 N}$ (with $N = N_2 - N_1$ being the mismatch between the two delays) evolving as the linear dynamics

$$\xi(k+1) = A\xi(k) + Bu_2(k)$$

with $\xi_i(k) = u_2(k-N+i-1)$ for $i = 1,\dots,N$ and

$$A = \begin{pmatrix} \mathbf{0}_{p_2(N-1)\times p_2} & \mathrm{I}_{p_2(N-1)} \\ \mathbf{0}_{p_2\times p_2} & \mathbf{0}^\top_{p_2(N-1)\times p_2} \end{pmatrix}, \quad B = \begin{pmatrix} \mathbf{0}_{p_2(N-1)\times p_2} \\ \mathrm{I}_{p_2} \end{pmatrix}.$$

Accordingly, the extended delayed system exhibits a cascade structure affected by both state and input delays of the same length N_1; i.e., when setting $\xi_1(k-N_1) = u_2(k-N_2)$

$$x(k+1) = F(x(k), u_1(k-N_1), \xi_1(k-N_1)) \tag{6a}$$

$$\xi(k+1) = A\xi(k) + Bu_2(k) \tag{6b}$$

with $\xi_N(k+1) = \xi_1(k+N) = u_2(k)$ for $k \geq 0$.

B. The reduced dynamics

Because of the cascade structure of (6), we introduce the extended reduction variable $\eta_e^\top = (\eta^\top,\ \xi^\top)$ composed of two components: the usual one defined for the x-dynamics (6a) over the N_1-steps delay; a mere copy of the state extension ξ. Accordingly, one gets

$$\begin{aligned}\eta(k) :=& F_0^{-N_1}(\cdot)\circ F^{N_1}(x(k), u_{1[k-N_1,k[}, \xi_{1[k-N_1,k[})\\ =& F_0^{-N_1}(\cdot)\circ F(\cdot, u_1(k-1), \xi_1(k-1))\circ\ldots\\ &\circ F(x(k), u_1(k-N_1), \xi_1(k-N_1)).\end{aligned}$$

By construction, the η_e-dynamics is delay free with respect to the control variables $u=(u_1,u_2)$. As a matter of fact, one computes the *extended reduced dynamics* as

$$\eta(k+1) = F_r(\eta(k), u_1(k), \xi_1(k)) \tag{7a}$$
$$\xi(k+1) = A\xi(k) + Bu_2(k) \tag{7b}$$

with $F_r(\eta,u_1,\xi_1) = F_0^{-N_1}(\cdot)\circ F(\cdot,u_1,\xi_1)\circ F_0^{N_1}(\eta)$ and a copy of (6b) which is free of delays itself. Moreover, (7) exhibits a cascade structure with connection variable ξ_1 and unchanged drift term $F_r(\eta,0,0) = F_0(\eta)$.

The following result can be thus given while the proof is omitted as it follows the lines of [12] by exploiting the strict-feedforward cascade structure of (7) when suitably interconnected to the original dynamics (5) in closed loop [17].

Theorem 3.1: Consider the two-channel input delayed dynamics (5) with invertible drift term $F_0(\cdot)$. Any feedback $u=(\alpha_1(\eta,\xi),\alpha_2(\eta,\xi))$ achieving Global Asymptotic Stability (GAS) of the equilibrium of the reduced model (7) ensures GAS of the equilibrium of (5).

The cascade structure is the core of the stabilizing design over the reduced dynamics we shall discuss on in Section IV among other possibilities.

Remark 3.2: As an alternative reduction design, one might introduce an artificial delay over the less retarded input channel u_1 so to directly compensate the delay mismatch and then apply the standard methodology in [12]. However, this approach induces a dynamical feedback over u_1.

C. An alternative differential/difference representation

Assuming for the sake of simplicity $p_1 = p_2 = 1$ and following [18], one can equivalently describe the dynamics (7) via the so-called (F_0, G)-representation. Denoting by $\eta_e^+(u_1,u_2)$ any curve in $\mathbb{R}^{n+N}$ parametrized by (u_1,u_2), an equivalent representation of (7) is provided through two coupled difference-differential equations over $\mathbb{R}^{n+N}$ as

$$\eta^+ = F_0(\eta), \quad \eta^+ := \eta^+(0,0) \tag{8a}$$
$$\xi^+ = A\xi, \quad \xi^+ := \xi^+(0,0) \tag{8b}$$
$$\frac{\partial\eta^+(u_1,u_2)}{\partial u_1} = G_1(\eta^+(u_1,u_2), \xi_1^+(u_1,u_2), u_1, u_2) \tag{8c}$$
$$\frac{\partial\xi^+(u_1,u_2)}{\partial u_1} = 0 \tag{8d}$$
$$\frac{\partial\eta^+(u_1,u_2)}{\partial u_2} = 0, \quad \frac{\partial\xi^+(u_1,u_2)}{\partial u_2} = B \tag{8e}$$

with $G_1(\eta,\xi_1,u_1,u_2)$ being a vector field over $\mathbb{R}^{n+N}$, parameterized by (u_1,u_2) and verifying [1]

$$G_1(F_r(\eta,u_1,\xi_1),\xi_2,u_1,u_2) := \nabla_{u_1}F_r(\eta,\xi_1,u_1). \tag{9}$$

Thus, for any $(k,\eta_e(k),u_1(k),u_2(k))$, one recovers (7) by integrating (8c)-(8d) over the interval $[0,u_1(k)[$ and (8e) over the interval $[0,u_2(k)[$ and initial condition (8a)-(8b) with $\eta_e = \eta_e(k)$; i.e. $\eta_e(k+1) = \eta_e^+(u_1(k),u_2(k))$ with

$$\begin{aligned}\eta_e(k+1) =& \eta_e^+(0,0) + \int_0^{u_1(k)} G_{e1}(\eta_e^+(u_1,0),u_1,0)\mathrm{d}u_1\\ &+ \int_0^{u_2(k)} G_{e2}(\eta_e^+(u_1(k),u_2),u_1(k),u_2)\mathrm{d}u_2\end{aligned} \tag{10}$$

with $G_{e1} = (G_1, 0)$, $G_{e2} = (0, B)$.

Remark 3.3: The integral form (10) rewrites as (see [18])

$$\begin{aligned}\eta_e(k+1) =& \eta_e^+(0,0)\\ &+ \int_0^{u_1(k)} G_{e1}(\eta_e^+(u_1,u_2(k)),u_1,u_2(k))\mathrm{d}u_1\\ &+ \int_0^{u_2(k)} G_{e2}(\eta_e^+(0,u_2),0,u_2)\mathrm{d}u_2\end{aligned}$$

because by definition the vector fields $G_{e1}(\eta_e,u_1,u_2)$ and $G_{e2}(\eta_e,u_1,u_2)$ verify the so called compatibility conditions

$$\begin{aligned}&\nabla_{u_1}G_{e2}(\cdot,u_1,u_2) - \nabla_{u_2}G_{e1}(\cdot,u_1,u_2)\\ &= \left[G_{e1}(\cdot,u_1,u_2), G_{e2}(\cdot,u_1,u_2)\right]\end{aligned}$$

with $\left[G_{e1},G_{e2}\right] = (\nabla_{\eta_e}G_{e2})G_{e1} - (\nabla_{\eta_e}G_{e1})G_{e2}$.

IV. Stabilization of the extended reduced dynamics-an I&I approach

Hereinafter, we discuss the design of a stabilizing controller for the reduced dynamics (7) by assuming the existence of a stabilizing feedback when there is no mismatch in the delays acting over the input channels of (5) (i.e., when $N_2 - N_1 = 0$).

Assumption 4.1 (Uniform delay): When $N = N_2 - N_1 = 0$, there exists a feedback $u_1 = \gamma_1(\eta)$, $u_2 = \gamma_2(\eta)$ which makes the origin a GAS equilibrium for the "ideal" reduced-dynamics

$$\eta(k+1) = F_r(\eta(k), u_1(k), u_2(k)) \tag{11}$$

computed over the uniformly delayed system (1).

Remark 4.1: Assumption 4.1 can be inferred from the stabilizability of the delay-free dynamics associated to (5). For further details the reader is referred to [12].

In the following, we denote $\gamma(\cdot) = (\gamma_1(\cdot),\gamma_2(\cdot))$. Under Assumption 4.1, the existence of a stabilizing feedback over the multi-delayed dynamics (5) can be proved by defining an I&I feedback over the extended reduced model (7). For this purpose, the I&I design over the dynamics (7) proceeds along the steps sketched below.

Target dynamics - From Assumption 4.1, one deduces the target dynamics over $\mathbb{R}^n$ as

$$\zeta(k+1) = F_r(\zeta(k), \gamma_1(\zeta(k)), \gamma_2(\zeta(k))) \tag{12}$$

possessing a GAS equilibrium at the origin.

[1] Because $F_0(\cdot)$ admits an inverse, then $F_r(\cdot,u_1,u_2)$ is smooth enough and admits an inverse $F_r^{-1}(\cdot,u_1,u_2)$ for $(u_1,u_2)\in\mathbb{R}^2$ sufficiently small.

Immersion mapping - The immersion mapping $\pi : \mathbb{R}^n \to \mathbb{R}^{n+p_2N}$ is defined as

$$\pi(\zeta) = \begin{pmatrix} \zeta(k) \\ \pi_2(\zeta(k)) \end{pmatrix} = \begin{pmatrix} \zeta(k) \\ \gamma_2(\zeta(k)) \\ \gamma_2(\zeta(k+1)) \\ \dots \\ \gamma_2(\zeta(k+N-1)) \end{pmatrix}$$

where, for $i = 1, \dots, N$

$$\zeta(k+i) = \underbrace{F_r(\cdot, \gamma(\cdot)) \circ \dots \circ F_r(\zeta(k), \gamma(\zeta(k)))}_{i \text{ times}}.$$

The on-the set feedback is thus given by

$$c(\zeta(k)) = (c_1(\zeta(k)), c_2(\zeta(k))) = (\gamma_1(\zeta(k)), \gamma_2(\zeta(k+N)))$$

so that the following invariance condition is verified

$$\begin{pmatrix} F_r(\zeta(k), c_1(\zeta(k)), \gamma_2(\zeta(k)) \\ A\pi_2(\zeta(k)) + Bc_2(\zeta(k)) \end{pmatrix} = \pi(F_r(\zeta(k), \gamma(\zeta(k)))).$$

Invariant set - The invariant set is described as the null set of the mapping $\phi(\eta, \xi) : \mathbb{R}^{n+p_2N} \to \mathbb{R}^{p_2N}$ with $\phi(\eta, \xi) = \text{col}\{\phi_1(\eta, \xi), \dots, \phi_N(\eta, \xi)\}$ and for any $i = 1, \dots, N$

$$\phi_i(\eta(k), \xi(k)) = \xi_i(k) - \gamma_2(\eta(k+i-1))$$

with for $i = 1, \dots, N$ and $u_1 = \gamma_1(\eta)$

$$\eta(k+i) = F_r(\cdot, \gamma_1(\cdot), \xi_i(k)) \circ \dots \circ F_r(\eta(k), \gamma_1(\eta(k)), \xi_1(k))$$

i.e., one sets

$$\mathscr{M} = \{(\eta, \xi) \in \mathbb{R}^n \times \mathbb{R}^{p_2N} \text{ s.t. } \phi_i(\eta, \xi) = 0 \text{ for } i = 1, \dots, N\}.$$

Accordingly, the off-the-set component is defined over $\mathbb{R}^{p_2N}$ as $z = \text{col}(z_1, \dots, z_N)$ with $z_i = \phi_i(\eta, \xi)$ for $i = 1, \dots, N$.

The following result can now be enhanced by showing that Assumption 4.1 is sufficient to infer I&I stabilizability of the extended reduced dynamics (7). The proof is omitted as it follows the lines of [8].

Proposition 4.1: Under Assumption 4.1, any feedback $\psi(\eta_e, z) : \mathbb{R}^{n+p_2N} \times \mathbb{R}^{p_2N} \to \mathbb{R}^p$ making the trajectories of the closed-loop system

$$\begin{aligned} z(k+1) =& Az(k) + B\psi_2(\eta_e(k), z(k)) \\ \eta(k+1) =& F_r(\eta(k), \psi_1(\eta_e(k), z(k)), \xi_1(k))) \\ \xi(k+1) =& A\xi(k) + B\psi_2(\eta_e(k), z(k)) \end{aligned}$$

bounded for all $k \geq 0$ with $\lim_{k\to\infty} z(k) = 0$ and $\psi(\pi(\zeta), 0) = c(\zeta)$ ensures I&I stabilizability of the reduced dynamics (7). Accordingly, the origin is a GAS equilibrium for

$$\eta(k+1) = F_r(\eta(k), \psi_1(\eta_e(k), \phi(\eta_e(k))), \xi_1(k)) \quad (13a)$$
$$\xi(k+1) = A\xi(k) + B\psi_2(\eta_e(k), \phi(\eta_e(k))). \quad (13b)$$

The I&I stabilizing feedback is given in the theorem below.

Theorem 4.1: Let the system (5) verify Assumption 4.1. Then, the reduced model (7) is I&I stabilizable with target dynamics (12) under the I&I feedback $u = \psi(\eta_e, z)$

$$u_1(k) = \psi_1(\eta_e(k), z(k)) = \gamma_1(\eta(k)) \quad (14a)$$
$$u_2(k) = \psi_2(\eta_e(k), z(k)) = \ell z(k) + \gamma_2(\eta(k+N)) \quad (14b)$$

verifying $\psi(\pi(\zeta), 0) = c(\zeta)$ and with ℓ making $A + B\ell$ Schur.

Proof. The proof follows the lines of the main result in [9]. It is a matter of computations to verify that by construction of the immersion mapping, invariance of the closed-loop dynamics is ensured by the choice $c_1(\zeta(k)) = \gamma_1(\zeta(k))$, $c_2(\zeta(k)) = \gamma_2(\zeta(k+N))$. Thus, the associated set is feedback invariant and the overall design aims at making it attractive while ensuring boundedness of the extended dynamics

$$\begin{aligned} z(k+1) =& Az(k) + B(u_2(k) - \gamma_2(\eta(k+N))) \\ \eta(k+1) =& F_r(\eta(k), u_1(\eta(k)), \gamma_2(\eta(k)) + z_1(k)) \\ =& F_r(\eta(k), u_1(k), \gamma_2(\eta(k))) + \mathscr{F}(\eta(k), u_1(k), z_1(k)) \\ \xi(k+1) =& A\xi(k) + Bu_2(k) \end{aligned}$$

with

$$\mathscr{F}(\eta, u_1, z_1) := \sum_{i=1}^{p_2} \int_0^{z_1^i} \nabla_{v_i} F_r(\eta, u_1, \mathbf{c}^i \gamma_2(\eta) + \mathbf{v}^i) \mathrm{d}v_i$$

with $z_1 = \text{col}(z_1^1, \dots, z_1^{p_2})$, $\mathbf{c}^i = (\mathbf{1}^\top_{p_2-i+1} \ \ \mathbf{0}_{1\times i-1})$ and $\mathbf{v}^i = (z_1^1, \dots, z_1^{i-1}, v_i, \mathbf{0}_{1\times i-1})$. As a result, I&I stability is ensured by any feedback of the form (14) making $A + B\ell$ Schur. $\triangleleft$

Remark 4.2: Contrarily to classical prediction methods, the feedback (14) requires the computation of the trajectories of the reduced dynamics (7) over N steps ahead by also minimizing the prediction horizon.

V. A Chained Dynamics as an Example

As an example consider the chained dynamics [19]

$$\begin{aligned} &\dot{x}_1(t) = x_2(t), \ \dot{x}_2(t) = u_1(t-\tau_1), \ \dot{x}_3(t) = x_5(t) \\ &\dot{x}_4(t) = x_6(t), \ \dot{x}_5(t) = u_2(t-\tau_2) \\ &\dot{x}_6(t) = -x_3(t)(1 + u_1(t-\tau_1)) \end{aligned}$$

and let the control be piecewise constant over time intervals of length δ (the sampling period) with respective delays $\tau_i = N_i\delta, N_i \in \mathbb{N}$ for $i = 1, 2$.

Remark 5.1: The above system might represent the dynamics provided after suitable coordinates change and feedback as described in [19].

Setting $x = \text{col}(x_1, \dots, x_6)$ and $x(k) := x(k\delta)$, one exactly computes the sampled-data equivalent model as

$$\begin{aligned} x(k+1) =& A^\delta x(k) + B_0^\delta(u_1(k-N_1), u_2(k-N_2)) \\ &+ B_1^\delta(u_1(k-N_1))x(k) \end{aligned} \quad (15)$$

with

$$A^\delta = \begin{pmatrix} 1 & \delta & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \delta & 0 \\ 0 & 0 & -\frac{\delta^2}{2} & 1 & -\frac{\delta^3}{6} & \delta \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -\delta & 0 & -\frac{\delta^2}{2} & 1 \end{pmatrix}$$

$$B_0^\delta(u_1, u_2) = \begin{pmatrix} \frac{\delta^2 u_1}{2} & \delta u_1 & \frac{\delta^2 u_2}{2} & \frac{-\delta^4(1+u_1)u_2}{24} & \delta u_2 & \frac{-\delta^3(1+u_1)u_2}{6} \end{pmatrix}^\top$$

$$B_1^\delta(u_1) = \begin{pmatrix} & & \mathbf{0}_{3\times 6} & & & \\ 0 & 0 & -\frac{\delta^2}{2}u_1 & 0 & -\frac{\delta^3}{6}u_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\delta u_1 & 0 & -\frac{\delta^2}{2}u_1 & 0 \end{pmatrix}.$$

A. A reduced feedback for the uniformly delayed case

Assuming $N_1 = N_2 = 1$, we want to solve a steering problem for the chained dynamics. Basically, we aim at defining a feedback law so that the full state $x(k)$ reaches a desired value $x_d = (x_d^1,\ 0,\ 0,\ x_d^2,\ 0,\ 0)^\top$ in exactly one step over δ (deadbeat). For this purpose, we first rewrite the error dynamics as $\varepsilon(k) = x(k) - x_d$ and compute the corresponding dynamics

$$\begin{aligned}\varepsilon(k+1) =& A^\delta[\varepsilon(k)+x_d] + B_0^\delta(u_1(k-1),u_2(k-1)) \\ &+ B_1^\delta(u_1(k-1))[\varepsilon(k)+x_d] - x_d\end{aligned} \tag{16}$$

possessing an equilibrium at the origin to be stabilized. Accordingly, we apply our procedure to stabilize (16). By noticing that $B_1^\delta(u_1)x_d = 0$, one defines the reduction as

$$\begin{aligned}\eta(k) :=& \varepsilon(k) + A^{-\delta}B_0^\delta(u_1(k-1),u_2(k-1)) \\ &+ A^{-\delta}B_1^\delta(u_1(k-1))\varepsilon(k)\end{aligned}$$

and the corresponding reduced dynamics as

$$\begin{aligned}\eta(k+1) =& A^\delta\eta(k) + A^{-\delta}B_0^\delta(u_1(k),u_2(k)) \\ &+ A^{-\delta}B_1^\delta(u_1(k))A^\delta\eta(k).\end{aligned} \tag{17}$$

As far as control design is concerned, we first build the feedback stabilizing (17) in the uniformly delayed case (i.e., when $N_1 = N_2$) so to guarantee the requirements in Assumption 4.1. For this purpose, we set up a multi-rate strategy of orders $m_1 = 2$ and $m_2 = 4$ over, respectively, u_1 and u_2 by setting

$$\begin{aligned}u_1(t) &= u_1^j(k),\ t \in [(k+\frac{j-1}{2})\delta,(k+\frac{j}{2})\delta[,\ \ j = 1,2 \\ u_2(t) &= u_2^j(k),\ t \in [(k+\frac{j-1}{4})\delta,(k+\frac{j}{4})\delta[,\ \ j = 1,\dots,4.\end{aligned}$$

At any sampling instant $t = k\delta$ by denoting $\bar\delta = \frac{\delta}{4}$ and by dropping the k-argument in the right hand side, the multi-rate reduced model gets the form

$$\begin{aligned}\eta(k+1) =& (A^{\bar\delta} + A^{-\bar\delta}B_1^{\bar\delta}(u_1^2)A^{\bar\delta})^2(A^{\bar\delta} + A^{-\bar\delta}B_1^{\bar\delta}(u_1^1)A^{\bar\delta})^2\eta(k) \\ &+ (A^{\bar\delta} + A^{-\bar\delta}B_1^{\bar\delta}(u_1^2)A^{\bar\delta})^2(I + A^{-\bar\delta}B_1^{\bar\delta}(u_1^1))B_0^{\bar\delta}(u_1^1,u_2^1) \\ &+ (A^{\bar\delta} + A^{-\bar\delta}B_1^{\bar\delta}(u_1^2)A^{\bar\delta})^2B_0^{\bar\delta}(u_1^1,u_2^2) \\ &+ (I + A^{-\bar\delta}B_1^{\bar\delta}(u_1^1))B_0^{\bar\delta}(u_1^2,u_2^3) + A^{-\bar\delta}B_0^{\bar\delta}(u_1^2,u_2^4)\end{aligned}$$

with six control inputs. Accordingly, one computes the feedback $u_1^i(k) = \gamma_1^i(\eta(k))$ and $u_2^j(k) = \gamma_2^j(\eta(k))$ ($i = 1,2$ and $j = 1,\dots,4$) as the unique solution to $\eta(k+1) \equiv 0$ also ensuring global exponential stability of (16) when $N_1 = N_2 = 1$ and, thus, Assumption 4.1.

B. The double-channel case

Assuming now $N_1 = 1$ and $N_2 = 2$, one computes the extended reduced model of the error dynamics under multi-rate sampling as

$$\begin{aligned}\eta(k+1) =& (A^{\bar\delta} + A^{-\bar\delta}B_1^{\bar\delta}(u_1^2)A^{\bar\delta})^2(A^{\bar\delta} + A^{-\bar\delta}B_1^{\bar\delta}(u_1^1)A^{\bar\delta})^2\eta \\ &+ (A^{\bar\delta} + A^{-\bar\delta}B_1^{\bar\delta}(u_1^2)A^{\bar\delta})^2(I + A^{-\bar\delta}B_1^{\bar\delta}(u_1^1))B_0^{\bar\delta}(u_1^1,\xi^1) \\ &+ (A^{\bar\delta} + A^{-\bar\delta}B_1^{\bar\delta}(u_1^2)A^{\bar\delta})^2B_0^{\bar\delta}(u_1^1,\xi^2) \\ &+ (I + A^{-\bar\delta}B_1^{\bar\delta}(u_1^1))B_0^{\bar\delta}(u_1^2,\xi^3) + A^{-\bar\delta}B_0^{\bar\delta}(u_1^2,\xi^4) \\ \xi^j(k+1) =& u_2^j(k), \quad i = 1,2,3,4.\end{aligned}$$

By applying the I&I procedure in Section IV, one computes the off-the-set component as $z^j(k) = \xi^j(k) - \gamma_2^j(\eta(k))$. Accordingly, for $i = 1,2$ and $j = 1,\dots,4$, the final multi-rate feedback gets the form

$$u_1^i(k) = \gamma_1^i(\eta(k)),\ u_2^j(k) = \gamma_2^j(\eta(k+1)) + \ell_j z^j(k), \quad |\ell_j| < 1.$$

C. Simulations

Simulations of the proposed deadbeat maneuver are reported when applying the I&I reduced feedback and setting $\ell_j = 0$ for $j = 1,2,3,4$ and desired final configuration $x_d^\top = (10,0,0,10,0,0)$ when starting from the origin. The red solid lines represent the evolution of the target and the controls when a uniform delay affects all of the input channels, while the blue solid lines represent the actual behavior in the multichannel (MC) case with $N_1 = 1$ and $N_2 = 2$.

The proposed strategy ensures convergence of the dynamical system toward the desired final position in the desired number of steps while ensuring $\eta(k) \equiv 0$ in exactly one step (simulations of this last scenario are omitted for the sake of space). Furthermore, we note that the proposed feedback still ensures stability for larger values of the sampling period while still guaranteeing small control effort.

VI. Conclusions

In this paper, we show how to extend the reduction approach to handle time-delay systems affected by two distinct input-delays. Moreover, we exhibit one among the possible controllers by combining reduction and Immersion and Invariance arguments for achieving stabilization in closed loop. The proposed methodology applies to sampled-data delayed dynamics under entire delays. Future works are toward different directions: sampled-data systems under non-entire delays [11]; a comparison with the continuous-time prediction framework with special emphasis on the cascade like representations provided by transport PDEs [14], [16] with respect to the discrete-time one [9]; the specialization of this methodology to different scenarios where time delays are unavoidable as in networked systems [20].

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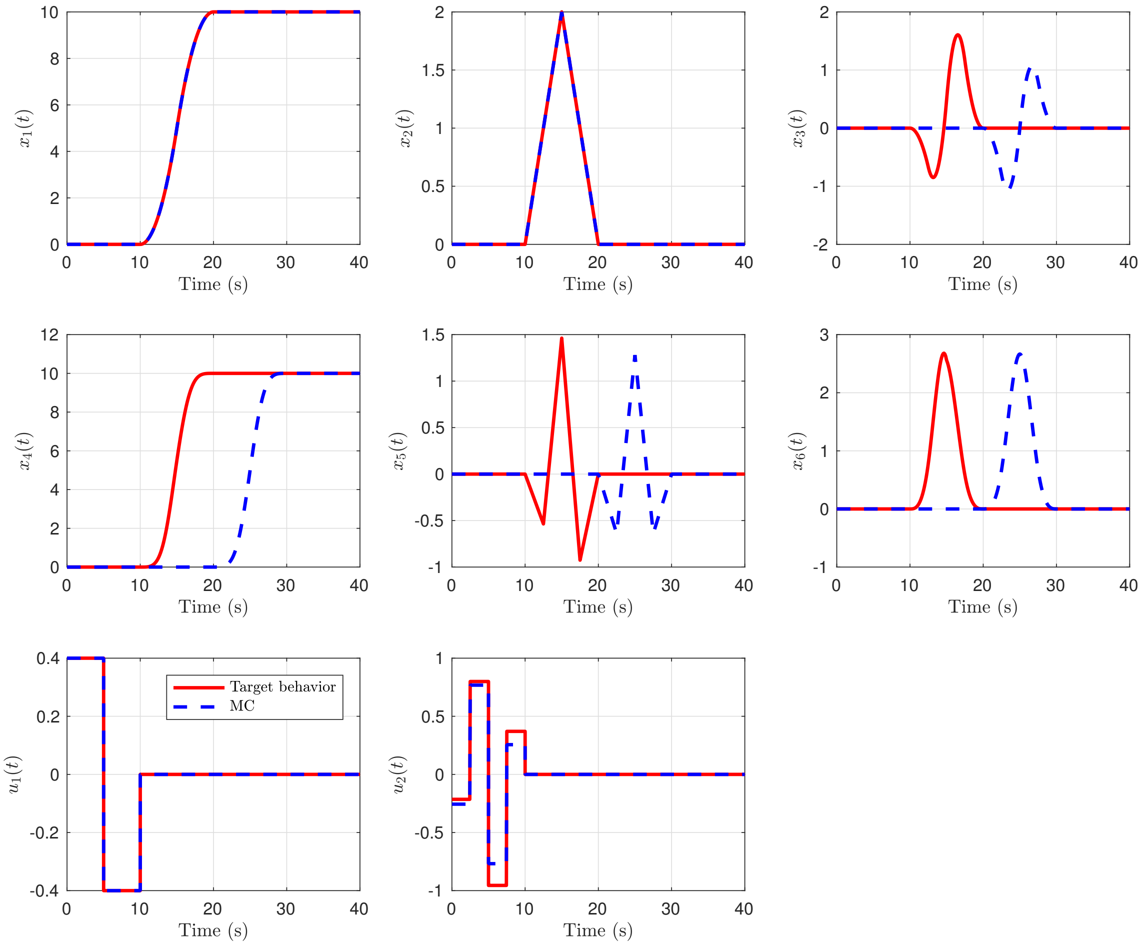

Fig. 1. $\delta = 10$ seconds and $x_d = (10, 0, 0, 10, 0, 0)$

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