RESEARCH ARTICLE

WILEY

How vulnerable is an undirected planar graph with respect to max flow

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Abstract

We study the problem of computing the vitality of edges and vertices with respect to the st-max flow in undirected planar graphs, where the vitality of an edge/vertex is the st-max flow decrease when the edge/vertex is removed from the graph. This allows us to establish the vulnerability of the graph with respect to the st-max flow. We give efficient algorithms to compute an additive guaranteed approximation of the vitality of edges and vertices in planar undirected graphs. We show that in the general case high vitality values are well approximated in time close to the time currently required to compute st-max flow $O(n \log \log n)$. We also give improved, and sometimes optimal, results in the case of integer capacities. All our algorithms work in $O(n)$ space.

KEYWORDS

max flow, planar graphs, undirected graphs, vitality, vulnerability

1 | INTRODUCTION

Max flow problems have been intensively studied in the last 60 years, we refer to [1, 2] for a comprehensive bibliography. Currently, the best known algorithms for general graphs [31, 42] compute the max flow between two vertices in $O(mn)$ time, where m is the number of edges and n is the number of vertices.

Italiano et al. [29] presented an algorithm for max flow that solves the problem in $O(n \log \log n)$ time for undirected planar graphs. For directed st-planar graphs (i.e., graphs allowing a planar embedding with s and t on the same face) finding a max flow was reduced by Hassin [25] to the single source shortest path (SSSP) problem, that can be solved in $O(n)$ time by the algorithm in [26]. For the planar directed case, Borradaile and Klein [15] presented an $O(n \log n)$ time algorithm. In the special case of directed planar unweighted graphs, an $O(n)$ time algorithm was proposed by Eisenstat and Klein [20].

The effect of edges deletion on the max flow value has been studied since 1963, only a few years after the seminal paper by Ford and Fulkerson [21] in 1956. Wollmer [49] presented a method for determining the most vital edge (i.e., the edge whose deletion causes the largest decrease of the max flow value) in a railway network; in the same years, other studies about max flow interdiction were carried out on the planar Russian railway [24, 39]. A more general problem was studied in [44], where an enumerative approach is proposed for finding the k edges whose simultaneous removal causes the largest decrease in max flow. Wood [50] showed that this problem is NP-hard in the strong sense, while its approximability has been studied in [4, 43].

In this paper we deal with the computation of *vitality* of edges and vertices with respect to the value of an st-max flow in an undirected planar graph G, denoted by MF_G or MF if no confusion arises, where s and t are two fixed vertices. The vitality of

an edge e (resp., of a vertex v) measures the st-max flow decrease observed after the removal of edge e (resp., all edges incident on v) from the graph.

A reasonable measure of the overall vulnerability of a network can be the number of edges/vertices with high vitality. So, if all edges and vertices have small vitality, then the graph is robust. We stress that verifying the robustness/vulnerability of the graph by using previous algorithms requires to compute the exact vitality of each edge and/or vertex. We refer to [3, 38, 41] for surveys on several kind of robustness and vulnerability problems discussed by an algorithmic point of view.

A survey on vitality with respect to the max flow problems can be found in [6]. In the same paper, it is shown that for st-planar graphs (both directed or undirected) the vitality of each edge and each vertex can be found in optimal $O(n)$ time. Ausiello et al. [7] proposed a recursive algorithm that computes the vitality of each edge in an undirected unweighted planar graph in $O(n \log n)$ time. The max flow algorithms in [29, 35] have also a dynamic version that can be used to compute edge and vertex vitality (see Theorem 4 and corresponding paragraph for discussion).

Our contribution: We propose fast algorithms for computing an additive guaranteed approximation of the vitality of each edge and vertex whose capacity is less than an arbitrary threshold c. Later, we explain that these results can be used to obtain a useful approximation of vitality for general distribution of capacities and in the case of power-law distribution.

Formally, the st-max flow vitality of a set $X \subseteq (V(G) \setminus \{s, t\}) \cup E(G)$, denoted by $vit(X)$, is equal to $MF_G - MF_{G-X}$, where $G - X$ is the graph obtained from G by removing set X. Our main results are summarized in the following two theorems. For a graph G, we denote by $E(G)$ and $V(G)$ its set of edges and vertices, respectively. Let $c : E(G) \rightarrow \mathbb{R}^+$ be the *edge capacity function*, we define the capacity $c(v)$ of a vertex v as the sum of the capacities of all edges incident on v. We show that we can compute an approximated value of edge and vertex vitalities with an additive error lower than δ, where δ is an arbitrarily fixed positive vitality. All our algorithms work in $O(n)$ space.

> **Theorem 1.** *Let G be a planar graph with positive edge capacities. Then for any $c, \delta > 0$, we can compute a value $vit^\delta(e) \in (vit(e) - \delta, vit(e)]$ for each $e \in E(G)$ satisfying $c(e) \le c$, in $O\left(\frac{c}{\delta}n + n \log \log n\right)$ time.*

> **Theorem 2.** *Let G be a planar graph with positive edge capacities. Then for any $c, \delta > 0$, we can compute a value $vit^\delta(v) \in (vit(v) - \delta, vit(v)]$ for each $v \in V(G)$ satisfying $c(v) \le c$, in $O\left(\frac{c}{\delta}n + n \log n\right)$ time.*

By the following theorem we can compute the exact vitality of some vertices. It could be extended similarly to edge vitality, but the result would not improve the results in [29, 35] (see Paragraph "Real world applications" and Theorem 4 for further details).

> **Theorem 3.** *Let G be a planar graph with positive edge capacities. Then for any $S \subseteq V(G)$, we can compute $vit(v)$ for each $v \in S$ in $O(|S|n + n \log \log n)$ time.*

1.1 | Small integer case

In the case of integer capacity values that do not exceed a small constant, or in the more general case in which capacity values are integers with bounded sum we also prove the following corollaries by using the results in [8, 9, 34, 51].

> **Corollary 1.** *Let G be a planar graph with integer edge capacity and let L be the sum of all the edge capacities. Then*
>
> - *for any $H \subseteq E(G) \cup V(G)$, we can compute $vit(x)$ for each $x \in H$, in $O(|H|n + L)$ time,*
> - *for any $c \in \mathbb{N}$, we can compute $vit(e)$ for each $e \in E(G)$ satisfying $c(e) \le c$, in $O(cn + L)$ time.*

> **Corollary 2.** *Let G be a planar graph with unit edge capacity. Let $n_{>d}$ be the number of vertices whose degree is greater than d. We can compute the vitality of each edge in $O(n)$ time and the vitality of each vertex in $O(\min\{n^{3/2}, n(n_{>d} + d + \log n)\})$ time.*

> **Corollary 3.** *Let G be a planar graph with unit edge capacity where only a constant number of vertices have degree greater than a fixed constant d. Then we can compute the vitality of each vertex in $O(n)$ time.*

Real world applications: Planar graphs naturally arise in various fields of science: they are studied in graph theory, in combinatorics [16, 47], in quantum gravity [5] and VLSI layout [22]. In biology, planar graphs play a central role: their function lies in describing veination patterns of leaves or insect wings [30, 40] and the plasmodium of the slime mould Physarum polycephalum [13].

Our results are linked to the max flow problem, and planar graphs are extensively utilized in city science to depict, either directly or with high approximation, various infrastructure networks [11], as water distribution networks [19, 27] and lots of streets patterns [36, 46]. The cities structure are the subject of many studies [12, 17, 18, 37] based on their planar aspects; see [14, 48] for a complete bibliography.

Before arguing with our results, we report in one unique theorem the state of the art about edge and vertex vitality in general planar graphs. Even if the vitality has not been studied directly in [29, 35], their dynamic algorithm leads to the following result.

Theorem 4 ([29, 35])**.** *Let G be a planar graph with positive edge capacities. Then it is possible to compute the vitality of h single edges or the vitality of a set of h edges in* $O\left(\min\left\{\frac{hn}{\log n} + n\log\log n,\ hn^{2/3}\log^{8/3} n + n\log n\right\}\right)$ *time.*

We stress that in the above described real world applications we are usually interested in finding edges and/or vertices with high vitality, that is, edges or vertices whose removal involves relevant decrease on the max flow value. This fact is motivated by understanding the robustness/vulnerability of the network and/or where maintenance activity is more critical.

Now we explain better how to apply the results in Theorems 1 and 2 to the real world. We note that in the general case the capacities are not bounded by any function of n. Despite this in many cases we can assume c/δ constant, implying that the time complexity of Theorem 1 is equal to the best current time bound for computing the st-max flow. The following remark is crucial, where $c_{\max} = \max_{e \in E(G)} c(e)$.

Remark 1 (bounding capacities)**.** We can bound all edge capacities higher than MF to MF, obtaining a new bounded edge capacity function. This change has no impact on the st-max flow value or the vitality of any edge/vertex. Thus w.l.o.g., we can assume that $c_{\max} \leq MF$.

By using Remark 1 we can explain why c/δ can be assumed constant. We study separately the case of general distribution of capacities and the case of power-law distribution. Note that the power-law distribution of capacities is frequent in some real world networks, especially in distribution networks.

- *General distribution* (after bounding capacities as in Remark 1). If we set $c = c_{\max}$ and $\delta = c/k$, for some constant k, then we obtain the capacities with an additive error less than MF/k, because of Remark 1. In the previous real world applications, in order to detect edges with high vitality, it is reasonable to consider this error acceptable even for small values of k, for example, $k = 10, 50, 100$. In this way we obtain small percentage error of vitality for edges with high vitality—edges whose vitality is comparable with MF—while edges with small vitality—edges whose vitality is smaller than MF/k—are badly approximated. We recall that we are usually interested in high capacity edges, and that with these choices the time complexity is $O(n\log\log n)$, that is, the time currently required for the computation of the st-max flow.
- *Power-law distribution* (after bounding capacities as in Remark 1). The previous method cannot be applied to power-law distribution because most of the edges have capacity lower than MF/k, even for high value of k. Thus we have to separate edges with high capacity and edges with low capacity. Let $c = \frac{c_{\max}}{\ell}$ for some constant ℓ and let $H_c = \{e \in E(G) \mid c(e) > c\}$. By power-law distribution, $|H_c|$ is small even for high values of ℓ, and thus we compute the exact vitality of edges in H_c by Theorem 4. For edges with capacity less than c, we set $\delta = c/k$, for some constant k. By Remark 1 we compute the vitality of these edges with an additive error less than $\frac{MF}{k\ell}$. Again, the overall time complexity is equal or close to the time currently required for the computation of the st-max flow.

The result in Theorem 1 is useful even in the case in which $c = MF$ and $vit^\delta(e) = 0$ for each $e \in E(G)$. This implies that all edges have vitality in $[0, \delta]$, where δ is the acceptable error in Theorem 1. Thus we certify that all edges in the network have low vitality, so the network is robust.

To apply the same arguments to vertex vitality, and so w.r.t. Theorem 2, we need some observations. If G's vertices have maximum degree d, then, after bounding capacities as in Remark 1, we have $\max_{v \in V(G)} c(v) \leq dMF$. Otherwise, we note that a real-world planar graph is expected to have few vertices with high degree (it is also implied by Euler's formula for planar graphs). The exact vitality of these vertices can be computed by Theorem 4 from [29, 35] or by Theorem 3 of this paper.

Now we compare the results stated in Theorem 4 with our results in Theorems 1 and 2. It is clear that Theorem 4 computes the exact vitalities, thus a direct comparison is impossible. But, because of the previous arguments, under a general distribution of edge capacities, we can compute a useful approximation of all edge vitalities in $O(n\log\log n)$ time and a useful approximation of all vertex vitalities in $O(n\log n)$ time by using the results in Theorems 1 and 2, respectively; while computing the exact vitality of each edge and/or each vertex by using Theorem 4 costs $O(n^{5/3}\log^{8/2} n)$ time. Thus the time complexity is significantly decreased.

With respect to Theorem 3, if we denote by $E_S = \sum_{v \in S} deg(v)$, where $deg(v)$ is the degree of vertex v, then Theorem 3 improves on Theorem 4 if either $|S| < \log n$ and $E_S > |S| \log n$ or $|S| \geq \log n$ and $E_S > \frac{|S| n^{1/3}}{\log^{8/3}}$.

Our approach: We adopt Itai and Shiloach's approach [28], that first computes a modified version D of a dual graph of G, then reduces the computation of the max flow to the computation of noncrossing shortest paths between pairs of vertices of the infinite face of D. We first study the effect on D of an edge or a vertex removal in G, showing that computing the vitality of an edge or a vertex can be reduced to computing some distances in D (see Propositions 2 and 3).

Then we determine required distances by solving SSSP instances. To decrease the cost we use a divide-and-conquer strategy: we slice D in regions delimited by some of the noncrossing shortest paths computed above. We choose noncrossing shortest paths with similar lengths, so that we compute an additive guaranteed approximation of each distance by looking into a single region instead of examining the whole graph D (see Lemma 3).

Finally we have all the machinery to compute an approximation of required distances of Propositions 2 and 3 and obtain edge and vertex vitalities.

Structure of the article: In Section 2, we report main results about how to compute max flow in planar graphs; we focus on the approach in [28] on which our algorithms are based. In Section 3, we show some preliminary results that allow us to compute edge and vertex vitality. In Section 4, we explain our divide-and-conquer strategy. In Section 5, we state our main result about edge vitality. Vertex vitality is described in Section 6 and in Section 7, we obtain some corollaries about planar graphs with small integer capacities. Finally, in Section 8 conclusions and open problems are given.

2 | MAX FLOW IN PLANAR GRAPHS

In this section we report some well-known results concerning max flow, focusing on planar graphs.

Given a connected undirected graph $G = (V(G), E(G))$ with n vertices, we denote an edge $e = \{i, j\} \in E(G)$ by the shorthand notation ij, and we define $\text{dist}_G(u, v)$ as the length of a shortest path in G joining vertices u and v. Moreover, for two sets of vertices $S, T \subseteq V(G)$, we define $\text{dist}_G(S, T) = \min_{u \in S, v \in T} \text{dist}_G(u, v)$. We write for short $v \in G$ and $e \in G$ in place of $v \in V(G)$ and $e \in E(G)$, respectively. We say that a path p is an ab path if its extremal vertices are a and b. Basic notations used throughout the paper are summarized in Figure 1.

Let $s, t \in G$, $s \neq t$, be two fixed vertices. We denote by $c(e)$ the capacity of an edge e. A *feasible flow* in G assigns to each edge $e = ij \in G$ two real values $x_{ij} \in [0, c(e)]$ and $x_{ji} \in [0, c(e)]$ such that: $\sum_{j:ij \in E(G)} x_{ij} = \sum_{j:ij \in E(G)} x_{ji}$, for each $i \in V(G) \setminus \{s, t\}$. The *flow from s to t* under a feasible flow assignment x is defined as $F(x) = \sum_{j:sj \in E(G)} x_{sj} - \sum_{j:sj \in E(G)} x_{js}$. The maximum flow from s to t, denoted by MF, is the maximum value of $F(x)$ over all feasible flow assignments x.

An *st-cut* is a partition of $V(G)$ into two subsets S and T such that $s \in S$ and $t \in T$. The capacity of an *st*-cut is the sum of the capacities of the edges $ij \in E(G)$ such that $|S \cap \{i, j\}| = 1$ and $|T \cap \{i, j\}| = 1$. The well known Min-Cut Max-Flow theorem [21] states that the maximum flow from s to t is equal to the capacity of a minimum *st*-cut for any weighted graph G.

We denote by $G - e$ the graph G after the removal of edge e. Similarly, we denote by $G - v$ the graph G after the removal of vertex v and all edges adjacent to v. The following definition about edge and vertex vitality is written according to the general concept of vitality in [33].

Definition 1. The vitality $vit(e)$ (resp., $vit(v)$) of an edge e (resp., vertex v) with respect to the maximum flow from s to t is defined as the difference between the maximum flow in G and the maximum flow in $G - e$ (resp., $G - v$).

We deal with planar undirected graphs. A *plane graph* is a planar graph with a fixed embedding. The dual of a plane undirected graph G is an undirected planar multigraph G^* whose vertices correspond to faces of G and such that for each edge e in G there is an edge $e^* = \{u^*, v^*\}$ in G^*, where u^* and v^* are the vertices in G^* that correspond to faces f and g adjacent to e in G. Length of e^* equals the capacity of e; for this reason we denote the length of e^* by $c(e^*)$; clearly $c(e^*) = c(e)$. Moreover, for a subgraph H of G^* we define $c(H) = \sum_{e \in H} c(e)$.

G	primal graph
G^*	dual graph
$vit(x)$	vitality of (edge or vertex) x
δ	a fixed positive vitality error
$c(e)$	capacity of edge e in G
$c(e^*)$	length of edge e^* in G^*
π	shortest path in G^* from a face containing s to a face containing t
D	graph obtained from G^* by doubling vertices in π
$v_1^*, \ldots, v_k^*$	vertices of π in G^*
x_i, y_i	vertices in copies of v_i^* in D
d_i	distance in D from x_i to y_i
$d_i(X)$	distance in D from x_i to y_i if all vertices of X are contracted into one vertex
π_x, π_y	path in D formed by $x_1, \ldots, x_k$ and path in D formed by $y_1, \ldots, y_k$
$q_{f_v^*}^x, q_{f_v^*}^y$	projection in D of $f_v^* \cap \pi_y$ on π_x and projection in D of $f_v^* \cap \pi_x$ on π_y

FIGURE 1 Summary of notation used throughout the article.

We fix a planar embedding of the graph, and we work on the dual graph G^* defined by this embedding. A vertex v in G generates a face in G^* denoted by f_v^*. We choose in G^* a vertex v_s^* in f_s^* and a vertex v_t^* in f_t^*. A cycle in the dual graph G^* that separates vertex v_s^* from vertex v_t^* is called an *st-separating cycle*. Moreover, we choose a shortest path π in G^* from v_s^* to v_t^*.

Proposition 1 ([28, 45]). *A (minimum) st-cut in G corresponds to a (shortest) cycle in G^* that separates vertex v_s^* from vertex v_t^*.*

2.1 | Itai and Shiloach's approach/decomposition

According to the approach by Itai and Shiloach in [28] used to find a min-cut by searching for minimum *st*-separating cycles, graph G^* is "cut" along the fixed shortest path π from v_s^* to v_t^*, obtaining graph D_G, in which each vertex v_i^* in π is split into two vertices x_i and y_i; when no confusion arises we omit the subscript G. In Figure 2 there is a plane graph G in black continuous lines and in Figure 3 on the right graph D. Now we explain the construction of the latter.

Let us assume that $\pi = \{v_1^*, v_2^*, \ldots, v_k^*\}$, with $v_1^* = v_s^*$ and $v_k^* = v_t^*$. For convenience, let π_x be the duplicate of π in D whose vertices are $\{x_1, \ldots, x_k\}$ and let π_y be the duplicate of π in D whose vertices are $\{y_1, \ldots, y_k\}$. We assume that π splits the plane into two parts A and B in the following way: we add two dummy edges, the first joining v_s^* to a dummy vertex α inside face f_s^* and the second joining v_t^* to a dummy vertex β inside face f_t^*, and then we extend these two edges to the infinity. For convenience, we state that part A is "above" π and part B is "below" π. These two parts can be chosen arbitrarily.

For any $i \in [k]$, where $[k] = \{1, \ldots, k\}$, edges in G^* incident on each v_i^* from below π are moved to y_i and edges incident on v_i^* from above π are moved to x_i. In Figure 2 there is a graph G in black continuous line, G^* in red dashed lines and shortest path π from v_1^* to v_k^*. In Figure 3, on the left there are the graph G and G^* of Figure 2 where path π is doubled.

For each $e^* \in \pi$ in G^*, in D we denote by e_x^* the copy of e^* in π_x and e_y^* the copy of e^* in π_y. Note that each $v \in V(G) \setminus \{s, t\}$ generates a face f_v^* in G^* that still remains a face in D, still denoted by f_v^*. There are not faces f_s^* and f_t^* in D because the dummy vertices α and β are inside faces f_s^* and f_t^* in G^*, respectively. Both faces f_s^* and f_t^* in G^* "correspond" in D to the leftmost $x_1 y_1$ path and to the rightmost $x_k y_k$ path, respectively. Since we are not interested in removing vertices s and t, then faces f_s^* and f_t^* are not needed in D. In Figure 3, on the right there is graph D built on G in Figure 2.
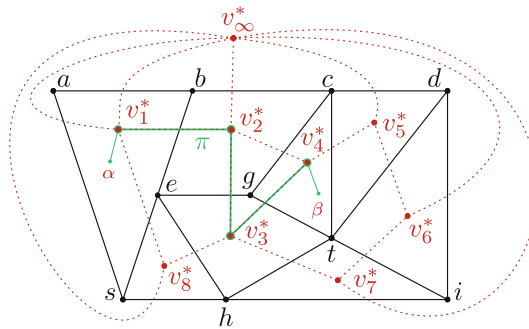

FIGURE 2 Graph G in black continuous line, G^* in red dashed lines, shortest path π from v_s^* (v_1^*) to v_t^* (v_k^*) in green, α and β are dummy vertices.
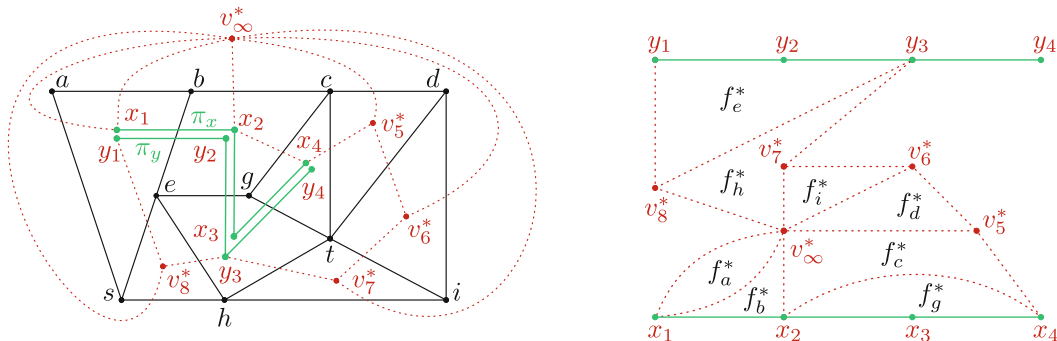

FIGURE 3 On the left green path π is doubled into paths π_x and π_y, and edges incident on x_1, y_1, x_4, y_4 in G^* are moved according to the dummy vertices α and β in Figure 2. On the right graph D.

TABLE 1 Notations of edges, vertices and faces between graphs G, G^*, and D.

G	G^*	D
edge e	edge e^*	edge e^*, if $e^* \notin \pi$ in G^*
		edge $e_x^* \in \pi_x$ and edge $e_y^* \in \pi_y$, if $e^* \in \pi$ in G^*
vertex v	face f_v^*	face f_v^*, if $v \notin \{s, t\}$
		nothing, if $v \in \{s, t\}$
face f_i	vertex v_i^*	vertex v_i^*, if $v_i^* \notin \pi$ in G^* (iff $i > k$)
		vertex x_i and vertex y_i, if $v_i^* \in \pi$ in G^* (iff $i \le k$)
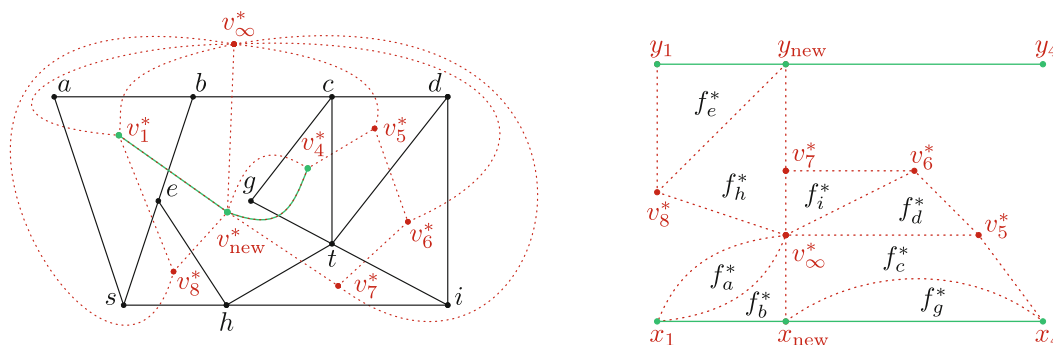
FIGURE 4 Starting from graph G in Figure 2, we show on the left graph $G - eg$ and $(G - eg)^*$, and graph D_{G-eg} on the right.

If $e^* \notin \pi$ in G^*, then we still denote the corresponding edge in D by e^*. Similarly, if $v_i^* \notin \pi$ in G^* (that is, $i > k$), then we still denote the corresponding vertex in D by v_i^*. In Table 1 all notations for graph G, G^* and D are summarized.

3 | PRELIMINARY RESULTS

In this section we show preliminary but crucial results (Propositions 2 and 3) that allow us to compute edge and vertex vitality. In Section 3.1, we show the effects in G^* and D of removing an edge or a vertex from G. In Section 3.2, we prove that we can focus only on st-separating cycles that cross π exactly once, and in Section 3.3, we state the two main propositions about edge and vertex vitality.

3.1 | Effects on G^* and D of deleting an edge or a vertex of G

We observe that removing an edge e from G corresponds to contracting endpoints of e^* into one vertex in G^*. With respect to D, if $e^* \notin \pi$ in G^*, then the removal of e corresponds to the contraction into one vertex of endpoints of e^* in D. If $e^* \in \pi$ in G^*, then both copies of e^* (e_x^* and e_y^*) have to be contracted. In Figure 4 we show the effects of removing edge eg from graph G in Figure 2.

Let v be a vertex of $V(G)$. Removing v corresponds to contracting vertices of face f_v^* in G^* into a single vertex. If f_v^* and π have no common vertices in G^*, then in D all vertices of f_v^* are contracted into one. Otherwise, in G^* face f_v^* intersects π on vertices $\bigcup_{i \in I_v} \{v_i^*\}$ for some non empty set $I_v \subseteq [k]$. Then, in D, all vertices of f_v^* are contracted into one vertex, all vertices of $\bigcup_{i \in I_v} \{x_i\}$ not belonging to f_v^* are contracted into another vertex and all vertices of $\bigcup_{i \in I_v} \{y_i\}$ not belonging to f_v^* are contracted into a third vertex. For convenience, we define $q_{f_v^*}^x = (\bigcup_{i \in I_v} \{x_i\}) \setminus V(f_v^*)$ and $q_{f_v^*}^y = (\bigcup_{i \in I_v} \{y_i\}) \setminus V(f_v^*)$. To better understand these definitions, see Figure 5. In Figure 6 it is shown what happens when we remove vertex g of graph G in Figure 2.

3.2 | Single-crossing st-separating cycles

Itai and Shiloach [28] consider only shortest st-separating cycles that cross π exactly once, that correspond in D to paths from x_i to y_i, for some $i \in [k]$. Formally, given two paths p_1, p_2 in a plane graph, a *crossing* between p_1 and p_2 is a minimal subpath of p_1 defined by vertices $v_1, v_2, \ldots, v_k$, with $k \ge 3$, such that vertices $v_2, \ldots, v_{k-1}$ are contained in p_2, and, fixing an orientation of p_2, edge $v_1 v_2$ lies to the left of p_2 and edge $v_{k-1} v_k$ lies to the right of p_2, or vice-versa. We say p_1 *crosses* p_2 t *times* if there are t different crossings between p_1 and p_2.
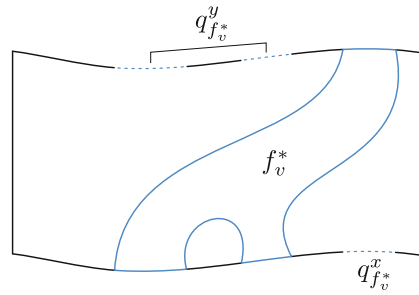
FIGURE 5 A face f_v^*, for some $v \in V(G)$, and sets $q_{f_v^*}^x$ and $q_{f_v^*}^y$. Removing v from G corresponds in D to contracting vertices of f_v^*, $q_{f_v^*}^x$ and $q_{f_v^*}^y$ in three distinct vertices.
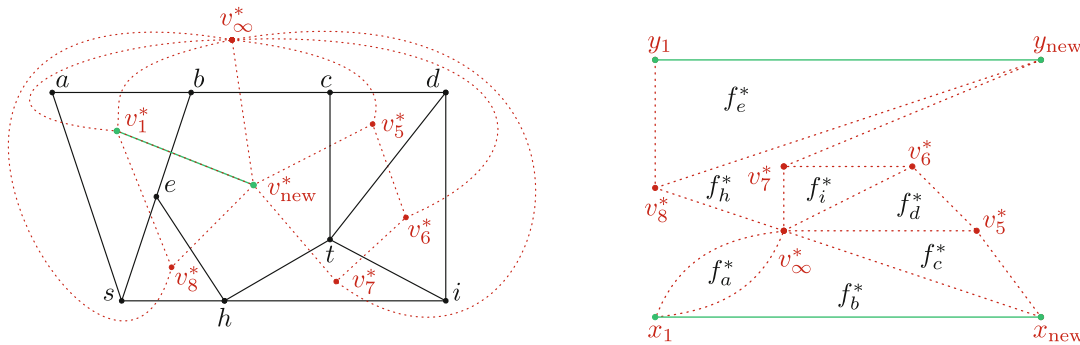

FIGURE 6 Starting from graph G in Figure 2, we show on the left graph $G - g$ and $(G - g)^*$, and graph D_{G-g} on the right.

In our approach, we contract vertices of an edge or a face of G^*. Despite this we can still consider only st-separating cycles that cross π exactly once. The proof of this is the goal of this subsection.

Lemma 1. *Let γ be a simple st-separating cycle and let S be either an edge or a face of G^*. Let $r = |V(\gamma) \cap V(S)|$. After contracting vertices of S into one vertex, then γ becomes the union of r simple cycles and exactly one of them is an st-separating cycle.*

Proof. Since an edge can be seen as a degenerate face, we prove the statement only in the case in which S is a face f. Let $v^* \in V(\gamma)$ and let $u_1^*, u_2^*, \ldots, u_r^*$ be the vertices of $V(\gamma) \cap V(f)$ ordered in clockwise order starting from v^*. For convenience, let $u_{r+1}^* = u_1^*$. For $i \in [r]$, let q_i be the clockwise $u_i^* u_{i+1}^*$ path on γ. After contracting the vertices of f into one, q_i becomes a cycle. Every q_i's joined with the counterclockwise $u_i^* u_{i+1}^*$ path on the border cycle of f defines a region R_i of G^*. We remark that if q_i is composed by a single edge e^*, then q_i becomes a self-loop and region R_i is a composed only by e^*.

Cycle γ splits graph G^* into two regions: a region internal to γ called R_{in} and an external region called R_{out}. W.l.o.g., we assume that $s \in R_{in}$ and $t \in R_{out}$. Now we split the proof into two cases: $f \subseteq R_{in}$ and $f \subseteq R_{out}$.

• Case $f \subseteq R_{in}$. By above, it holds that $R_1, \ldots, R_r \subseteq R_{in}$ (see Figure 7 on the left). Being γ an st-separating cycle, then there exists a unique $j \in [r]$ such that $s \in R_j$. Thus, after contracting vertices of f into one, p_j becomes the unique st-separating cycle, while all others R_i's become cycles that split G^* into two regions, and each region contains neither s nor t (see Figure 7 on the right).

• Case $f \subseteq R_{out}$. By above there exists a unique $j \in [r]$ such that $R_i \subseteq R_{out}$ for all $i \neq j$ and $R_{in} \subseteq R_j$ (see Figure 8 on the left). W.l.o.g., we assume that $j = r$. After contracting the vertices of f into one, all regions $R_1, \ldots, R_{r-1}$ become regions inside R_r because of the embedding (see Figure 8 on the right). We recall that $s \in R_{in}$, thus there are two cases: if $t \in R_i$ for some $i \in [r - 1]$, then p_i becomes the unique st-separating cycle; otherwise, $t \in R_{out}$, and thus p_r becomes the unique st-separating cycle. ∎

Let Γ be the set of all st-separating cycles in G^*, and let Γ_1 be the set of all st-separating cycles in G^* that cross π exactly once. Given $\gamma \in \Gamma$ and either an edge or a face S of G^*, thanks to Lemma 1 we can define $\Delta_S(\gamma)$ as "the length of the unique st-separating cycle contained in γ after contracting vertices of S into one". Being MF equal to the length of a minimum
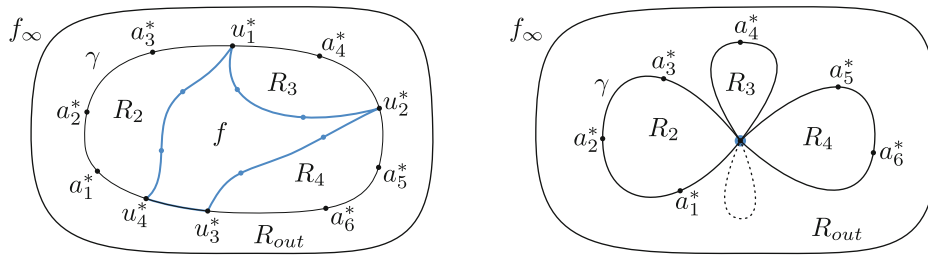
FIGURE 7 On the left, cycle γ and face f belonging to R_{in}. On the right, cycle γ after contracting all vertices of f into one, the dashed edge represents a self-loop.
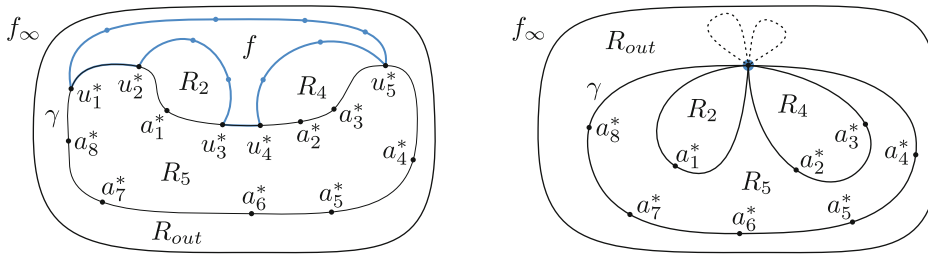

FIGURE 8 On the left, cycle γ and face f belonging to R_{out}. On the right, cycle γ after contracting all vertices of f into one, dashed edges represent self-loops.

st-separating cycle, the following relations hold:

$$\text{for any } e \in E(G), \quad vit(e) = MF - \min_{\gamma \in \Gamma} \Delta_{e^*}(\gamma),$$

$$\text{for any } v \in V(G) \setminus \{s, t\}, \quad vit(v) = MF - \min_{\gamma \in \Gamma} \Delta_{f_v^*}(\gamma).$$

Now we show that in the above equations we can replace set Γ with set Γ_1.

Lemma 2. *Let $e \in E(G)$ and $v \in V(G) \setminus \{s, t\}$. It holds that $vit(e) = MF - \min_{\gamma \in \Gamma_1} \Delta_{e^*}(\gamma)$ and $vit(v) = MF - \min_{\gamma \in \Gamma_1} \Delta_{f_v^*}(\gamma)$.*

Proof. We recall that removing an edge e from G corresponds to contracting endpoints of e^* into one vertex, while removing a vertex v from G corresponds to contracting all the vertices in face f_v^* into one vertex. So we prove the thesis only in the more general case of vertex removal. For convenience, we denote f_v^* by f. Let $\gamma \in \Gamma$ be such that $vit(f) = MF - \Delta_f(\gamma)$ and assume that $\gamma \notin \Gamma_1$. If $V(\gamma) \cap V(f) = \emptyset$, then $vit(f) = 0$, hence it suffices to remove crossings between γ and π, see [28]. Thus let us assume that $V(\gamma) \cap V(f) \neq \emptyset$.

By Lemma 1, there exist unique $a^*, b^* \in V(f) \cap V(\gamma)$ such that the clockwise a^*b^* path p on γ becomes an st-separating cycle after the contraction of vertices of f into one. Then we remove crossing between p and π in order to obtain a path p' not longer than p as above. Finally, let $\gamma' = p \circ q$, where q is the clockwise a^*b^* path on f. It holds that $\gamma' \in \Gamma_1$ and $\Delta_f(\gamma') \leq \Delta_f(\gamma)$, the thesis follows. ∎

3.3 | Vitality versus distances in D

The main results of this subsection are Propositions 2 and 3. The first proposition shows which distances in D are needed to obtain edge vitality and in the latter proposition we do the same for vertex vitality. In Section 3.1 we have proved that removing an edge or a vertex from G corresponds to contracting in single vertices some sets of vertices of D. The main result of Propositions 2 and 3 is that we can consider these vertices individually.

We observe that the capacities of edges in G become lengths in G^* and D. We recall that, given an edge $e \in G$, we denote by $c(e^*)$ the length of e^* in G^* and $c(e^*) = c(e)$. The same happens passing from G^* to D: given an edge $e^* \in G^*$ satisfying $e^* \notin \pi$, we denote the length of edge e^* in D by $c(e^*)$; given an edge $e^* \in G^*$ satisfying $e^* \in \pi$ in G^*, we denote the lengths of edges e_x^* and e_y^* by $c(e_x^*)$ and $c(e_y^*)$, respectively.

Let e be an edge of G. The removal of e from G corresponds to the contraction of endpoints of e^* into one vertex in G^*. Thus if an st-separating cycle γ of G^* contains e^*, then the removal of e from G reduces the length of γ by $c(e^*)$. Thus e has strictly positive vitality if and only if there exists an st-separating cycle γ in G^* whose length is strictly less than $MF + c(e^*)$ and $e^* \in \gamma$. This is the main idea to compute the vitality of each edge. Now we have to translate it to D.

For $i \in [k]$, we define $d_i = \text{dist}_D(x_i, y_i)$. We observe that $MF = \min_{i \in [k]} d_i$. For a subset S of $V(D)$ and any $i \in [k]$ we define $d_i(S) = \min\{d_i, \text{dist}_D(x_i, S) + \text{dist}_D(y_i, S)\}$. We observe that $d_i(S)$ represents the distance in D from x_i to y_i if all vertices of S are contracted into one.

For every $x \in V(G) \cup E(G)$ we define MF_x as the max flow in graph $G - x$. By definition, $vit(x) = MF - MF_x$ and, trivially, x has strictly positive vitality if and only if $MF_x < MF$.

Proposition 2. *For every edge e of G, if $e^* \notin \pi$ in G^*, then $MF_e = \min_{i \in [k]}\{d_i(e^*)\}$. If $e^* \in \pi$ in G^*, then $MF_e = \min_{i \in [k]} \left\{\min\{d_i(e_x^*), d_i(e_y^*)\}\right\}$.*

Proof. Let e be an edge of G. If $vit(e) = 0$, then $MF_e = MF$ and the thesis trivially holds. Hence let us assume $vit(e) > 0$, then Lemma 2 there exists an st-separating cycle in G^* that crosses π exactly once satisfying $c(\gamma) < MF + c(e^*)$ and $e^* \in \gamma$. If $e^* \notin \pi$, then e corresponds in D to edge e^*, thus the thesis holds. If $e^* \in \pi$ in G^*, then we note that every path in D containing both e_x^* and e_y^* corresponds in G^* to an st-separating cycle that passes through e^* twice, thus its length is equal or greater than $MF + 2c(e)$. Thus we consider only paths that contain e_x^* or e_y^* but not both. The thesis follows. ∎

Note that if f_v^* and π have some common vertices in G^*, then one among $q_{f_v^*}^x$ and $q_{f_v^*}^*$ could be empty. For this reason, we set $d_i(\emptyset) = +\infty$, for all $i \in [k]$.

Proposition 3. *For every vertex v of G, if f_v^* and π have no common vertices in G^*, then $MF_v = \min_{i \in [k]} \{d_i(f)\}$, where $f = f_v^*$ in D, otherwise*

$$MF_v = \min \begin{cases} \min_{i \in [k]}\{d_i(f)\} \\ \min_{i \in [k]}\{d_i(q_f^x)\} \\ \min_{i \in [k]}\{d_i(q_f^y)\} \\ \text{dist}_D(f, q_f^x) \\ \text{dist}_D(f, q_f^y) \end{cases}. \tag{1}$$

Proof. If f_v^* and π have no common vertices in G^*, then the proof is analogous to the edge case. Thus let us assume that f_v^* and π have common vertices in G^*. Let D' be the graph obtained from D by adding vertices u, v, z connected with all vertices of q_f^x, of q_f^y, of f, respectively, with zero weight edges; for convenience we assume that q_f^x and q_f^y are both not empty. By Lemma 2 and discussion in Section 3.1, $MF_v = c(p)$, where p is a shortest $x_i y_i$ path in D', varying $i \in [k]$.

Note that after contracting vertices of f into one vertex there exists an $x_i y_i$ path whose length is $dist_D(f, x_i)$, for each $x_i \in q_f^x$. In particular, there exists an $x_i y_i$ path whose length is $dist_D(f, q_f^x)$, for some i satisfying $x_i \in q_f^x$. The same argument applies for q_f^y. This implies that if $vit(v) = 0$, then Equation (1) is correct. Hence we assume that $vit(v) > 0$, so at least one among u, v and z belongs to p.

If $u \in p$ and $v, z \notin p$ (resp., $v \in p$ and $u, z \notin p$), then $c(p) = \min_{i \in [k]}\{d_i(q_f^x)\}$ (resp., $c(p) = \min_{i \in [k]}\{d_i(q_f^y)\}$). If $z \in p$ and $u, v \notin p$ then $c(p) = \min_{i \in [k]}\{d_i(f)\}$. We have analyzed all cases in which p contains exactly one vertex among u, v and z. To complete the proof, we prove that, for any $i \in [k]$, every $x_i y_i$ path that contains at least two vertices among u, v and z also contains a subpath whose length is at least $\min\{\text{dist}_D(f, q_f^x), \text{dist}_D(f, q_f^y)\}$.

Let ℓ be an $x_i y_i$ path, for some $i \in [k]$. If $u, z \in \ell$, then there exists a subpath ℓ' of ℓ from a vertex x_j of q_f^x to a vertex r of f. If we add to ℓ' the two zero weigthed edges rz and zy_j we obtain a $x_j y_j$ path whose length is at least $\text{dist}_D(f, q_f^x)$. We can use a symmetric strategy if $v, z \in \ell$.

It remains only the case in which $u, v \in \ell$. If q_f^x and q_f^y are both nonempty, then f splits D and D' into two or more parts and no part contains vertices of both q_f^x and q_f^y (see Figure 5). Thus if $u, v \in \ell$, then ℓ passes through at least one vertex of f, implying that ℓ has a subpath from a vertex of f to a vertex of q_f^x, or q_f^y. As above, this path can be transformed in a $x_j y_j$ path shorter than ℓ whose length is at least $\min\{\text{dist}_D(f, q_f^x), \text{dist}_D(f, q_f^y)\}$, for some $j \in [k]$. ∎

4 | SLICING GRAPH D PRESERVING APPROXIMATED DISTANCES

In this section we explain our divide-and-conquer strategy. We slice graph D along shortest $x_i y_i$'s paths. If these paths have lengths that differ at most δ, then we have a δ additive approximation of distances required in Propositions 2 and 3 by looking

into a single slice instead of the whole graph D. This result is stated in Lemma 3. These slices can share boundary vertices and edges, implying that their dimension might be $O(n^2)$. In Lemma 4 we compute an implicit representation of these slices in total linear time.

From now on, we mainly work on graph D, thus we omit the superscript D unless we refer to G or G^*. To work in D we need a shortest $x_i y_i$ path and its length, for each $i \in [k]$. In the following theorem we show time complexities for obtaining elements in D. We say that two paths are *single-touch* if their intersection is still a path.

Given two graphs $A = (V(A), E(A))$ and $B = (V(B), E(B))$ we define $A \cup B = (V(A) \cup V(B), E(A) \cup E(B))$ and $A \cap B = (V(A) \cap V(B), E(A) \cap E(B))$.

Theorem 5 ([10, 23, 29]). *If G is a positive edge-weighted planar graph,*

- *we compute $U = \bigcup_{i \in [k]} p_i$ and $c(p_i)$ for each $i \in [k]$, where p_i is a shortest $x_i y_i$ path in D and $\{p_i\}_{i \in [k]}$ is a set of pairwise noncrossing single-touch paths, in $O(n \log \log n)$ time—see [29] for computing U and [10] for computing $c(p_i)$'s,*
- *for every $I \subseteq [k]$, we compute $\bigcup_{i \in I} p_i$ in $O(n)$ time—see [23] by noting that U is a forest and the paths can be found by using nearest common ancestor queries.*

From now on, for each $i \in [k]$ we fix a shortest $x_i y_i$ path p_i, and we assume that $\{p_i\}_{i \in [k]}$ is a set of pairwise single-touch noncrossing shortest paths. Let $U = \bigcup_{i \in [k]} p_i$, see Figure 9A.

Given an ab path p and a bc path q, we define $p \circ q$ as the (possibly not simple) ac path obtained by the union of p and q. Each p_i's splits D into two parts as shown in the following definition and in Figure 9B.

Definition 2. For every $i \in [k]$, we define Left_i as the subgraph of D bounded by the cycle $\pi_y[y_1, y_i] \circ p_i \circ \pi_x[x_i, x_1] \circ l$, where l is the leftmost $x_1 y_1$ path in D. Similarly, we define Right_i as the subgraph of D bounded by the cycle $\pi_y[y_i, y_k] \circ r \circ \pi_x[x_k, x_i] \circ p_i$, where r is the rightmost $x_k y_k$ path in D.

Based on Definition 2, for every $i, j \in [k]$, with $i < j$, we define $\Omega_{i,j} = \text{Right}_i \cap \text{Left}_j$, see Figure 9C. We classify (x_i, y_i)'s pairs according to the difference between d_i and MF. Each class contains pairs for which this difference is between δr and $\delta(r + 1)$; we recall that $\delta > 0$ is an arbitrarily fixed value.

For every $r \in \mathbb{N}$, we define $L_r = (\ell_1^r, \dots, \ell_{z_r}^r)$ as the ordered list of indices in $[k]$ such that $d_j \in [MF + \delta r, MF + \delta(r + 1))$, for all $j \in L_r$, and $\ell_j^r < \ell_{j+1}^r$ for all $j \in [z_r - 1]$. It is possible that $L_r = \emptyset$ for some $r > 0$, while $L_0 \neq \emptyset$ because $MF = \min_{i \in [k]} d_i$ and there is at least one couple x_i, y_i whose distance in D is exactly MF. If no confusion arises, we omit the superscript r; thus we write ℓ_i in place of ℓ_i^r.

The following lemma is the key of our slicing strategy. In particular, Lemma 3 can be applied for computing distances required in Propositions 2 and 3, since the vertex set of a face or an edge of D is always contained in a slice. An application is in Figure 10.

Lemma 3. *Let $r > 0$ and let $L_r = (\ell_1, \ell_2, \dots, \ell_z)$. Let S be a set of vertices of D with $S \subseteq \Omega_{\ell_i, \ell_{i+1}}$ for some $i \in [z - 1]$. Then*

$$\min_{\ell \in L_r} d_\ell(S) > \min\{d_{\ell_i}(S), d_{\ell_{i+1}}(S)\} - \delta.$$

Moreover, if $S \subseteq \text{Left}_{\ell_1}$ (resp., $S \subseteq \text{Right}_{\ell_z}$) then $\min_{\ell \in L_r} d_\ell(S) > d_{\ell_1}(S) - \delta$ (resp., $\min_{\ell \in L_r} d_\ell(S) > d_{\ell_z}(S) - \delta$).

Proof. We need the following crucial claim.

a) Let $i < j \in L_r$. Let L be a set of vertices in Left_i and let R be set of vertices in Right_j. Then $d_i(L) < d_j(L) + \delta$ and $d_j(R) < d_i(R) + \delta$.
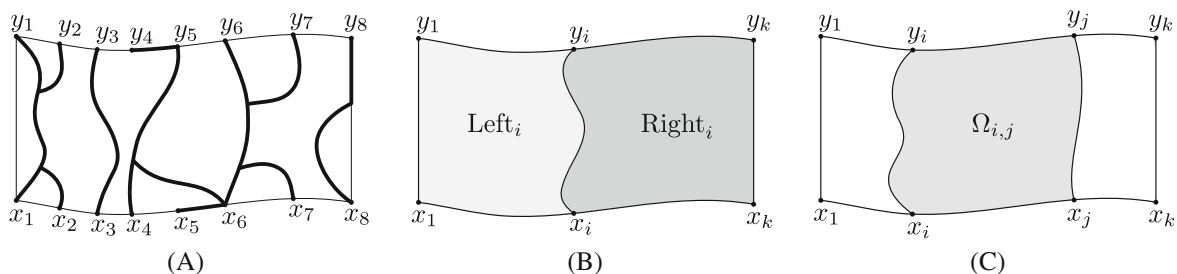

FIGURE 9 (A) The graph U in bold and (B) subgraphs Left_i and Right_i are highlighted. (C) Subgraph $\Omega_{i,j}$, for some $i < j$.
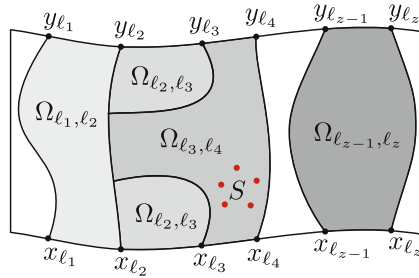
FIGURE 10 By Lemma 3, it holds that $\min_{\ell \in L_r} d_\ell(S) \geq \min\{d_{\ell_3}(S), d_{\ell_4}(S)\} - \delta$.
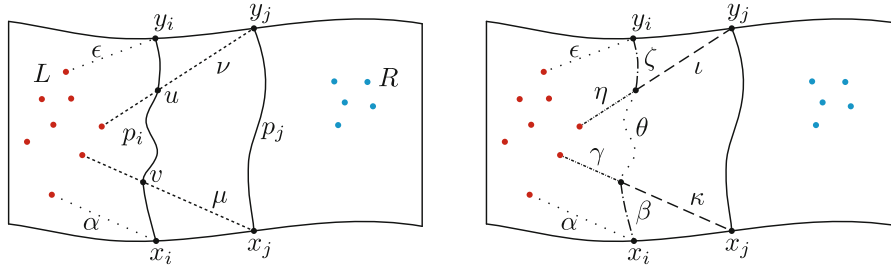

FIGURE 11 Example of paths and subpaths used in the proof of (a).

Proof of (a): we prove that $d_i(L) < d_j(L) + \delta$. By symmetry, it also proves that $d_j(R) < d_i(R) + \delta$. Let us assume by contradiction that $d_i(L) \geq d_j(L) + \delta$.

Let α (resp., ϵ, μ, ν) be a path from x_i (resp., y_i, x_j, y_j) to z_α (resp., z_ϵ, z_μ, z_ν) whose length is $d(x_i, L)$ (resp. $d(y_i, L), d(x_j, L), d(y_j, L)$), see Figure 11 on the left. Being $x_j, y_j \in \text{Right}_i$ and $L \subseteq \text{Left}_i$, then μ and ν cross p_i. Let v be the vertex that appears first in $p_i \cap \mu$ starting from x_j on μ and let u be the vertex that appears first in $p_i \cap \nu$ starting from y_j on ν. An example of these paths is in Figure 11 on the left. Let $\zeta = p_i[y_i, u]$, $\theta = p_i[u, v]$, $\beta = p_i[x_i, v]$, $\kappa = \mu[x_j, v]$, $\iota = \nu[y_j, u]$, $\eta = \nu[u, z_\nu]$ and $\gamma = \mu[v, z_\mu]$, see Figure 11 on the right.

Now $c(\beta) + c(\gamma) \geq c(\alpha)$, otherwise α would not be a shortest path from x_i to L. Similarly $c(\zeta) + c(\eta) \geq c(\epsilon)$. Moreover, being $c(\zeta) + c(\theta) + c(\beta) = d_i$, then $c(\theta) \leq d_i - c(\alpha) + c(\gamma) - c(\epsilon) + c(\eta)$. Being $d_i(L) \geq d_j(L) + \delta$, then $c(\alpha) + c(\epsilon) \geq c(\mu) + c(\nu) + \delta$, this implies $c(\alpha) + c(\epsilon) \geq c(\kappa) + c(\gamma) + c(\iota) + c(\eta) + \delta$.

It holds that $c(\theta) + c(\kappa) + c(\iota) \leq d_i - c(\alpha) + c(\gamma) - c(\epsilon) + c(\eta) + c(\alpha) + c(\epsilon) - c(\gamma) - c(\eta) - \delta = d_i - \delta < d_j$ because $i, j \in L_r$ imply $|d_i - d_j| < \delta$. Thus $\kappa \circ \theta \circ \iota$ is a path from x_j to y_j strictly shorter than d_j, absurdum. End proof of (a).

Being $S \subseteq \text{Right}_{\ell_j}$ for all $j < i$ and $S \subseteq \text{Left}_{\ell_{j'}}$ for all $j' > i + 1$, then the first part of the thesis follows from (a). The second part follows also from (a) by observing that if $S \subseteq \text{Left}_{\ell_1}$, then $S \subseteq \text{Left}_{\ell_i}$ for all $i \in L_r$. ∎

To compute distances in D we have to solve some SSSP instances in some $\Omega_{i,j}$'s subsets. These subsets can share boundary edges, thus the sum of their edges might be $O(n^2)$. We note that, by the single-touch property, if an edge e belongs to $\Omega_{i,j}$ and $\Omega_{j,\ell}$ for some $i < j < \ell \in [k]$, then $e \in p_j$.

To overcome this problem we introduce subsets $\tilde{\Omega}_{i,j}$ in the following way: for any $i < j \in [k]$, if $p_i \cap p_j$ is a nonempty path q, then we define $\tilde{\Omega}_{i,j}$ as $\Omega_{i,j}$ in which we replace path q by an edge with the same length; note that the single-touch property implies that all vertices in q but its extremal have degree two. Otherwise, we define $\tilde{\Omega}_{i,j} = \Omega_{i,j}$. Note that distances between vertices in $\tilde{\Omega}_{i,j}$ are the same as in $\Omega_{i,j}$. It the following lemma we show how to compute some $\tilde{\Omega}_{i,j}$'s in $O(n)$ time.

Lemma 4. *Let $A = (a_1, a_2, \ldots, a_z)$ be any increasing sequence of indices of $[k]$. It holds that $\sum_{i \in [z-1]} |E(\tilde{\Omega}_{a_i, a_{i+1}})| = O(n)$. Moreover, given U, we compute $\tilde{\Omega}_{a_i, a_{i+1}}$, for all $i \in [z - 1]$, in $O(n)$ total time.*

Proof. For convenience, we denote by Ω_i the set $\Omega_{a_i, a_{i+1}}$, for all $i \in [z - 1]$. We note that if $e \in \Omega_i \cap \Omega_{i+1}$, then $e \in p_{i+1}$. Thus, if e belongs to more than two Ω_i's, then e belongs to exactly two $\tilde{\Omega}$'s because it is contracted in all other Ω_i's by definition of the $\tilde{\Omega}_i$'s. Thus $\sum_{i \in [z-1]} |E(\tilde{\Omega}_i)| = O(n) + O(z) = O(n)$ because $z \leq k \leq n$.

To obtain all $\tilde{\Omega}_i$'s, we compute $U_z = \bigcup_{a \in A} p_a$ in $O(n)$ time by Theorem 5. Then we preprocess all trees in U_z in $O(n)$ time by using Gabow and Tarjan's result [23] in order to obtain the intersection path $p_{a_i} \cap p_{a_{i+1}}$ via lowest common ancestor queries, and its length in $O(1)$ time with a similar approach. Finally, we build $\tilde{\Omega}_i$ in $O(|E(\tilde{\Omega}_i)|)$, for all $i \in [z - 1]$, with a BFS visit of Ω_i that excludes vertices of $p_{a_i} \cap p_{a_{i+1}}$. ∎

5 | COMPUTING EDGE VITALITY

Now we can give our main result about edge vitality stated in Theorem 1. We need the following preliminary lemma that is an easy consequence of Lemmas 3 and 4.

Lemma 5. *Let* $r \in \mathbb{N}$, *given* U, *we compute a value* $\alpha_r(e) \in [\min_{i \in L_r}\{d_i(e)\}, \min_{i \in L_r}\{d_i(e)\} + \delta)$ *for each* $e \in E(D)$ *in* $O(n)$ *total time.*

Proof. We compute $U_r = \bigcup_{i \in L_r} p_i$ in $O(n)$ time by Theorem 5. Let $e \in E(D)$. If $e \in U_r$, we set $\alpha_r(e) = MF + \delta(r + 1) - c(e)$. If $e \notin U_r$, then either $e \in \tilde{\Omega}_{\ell_i, \ell_{i+1}}$ for some $i \in [z_r - 1]$, or $e \in \text{Left}_{\ell_1}$, or $e \in \text{Right}_{\ell_z}$.

For each $e \subseteq \text{Left}_{\ell_1}$, we set $\alpha_r(e) = d_{\ell_1}(e)$, similarly, for each $e \subseteq \text{Right}_{\ell_z}$, we set $\alpha_r(e) = d_{\ell_z}(e)$. Finally, if $e \in \tilde{\Omega}_{\ell_i, \ell_{i+1}}$, then we set $\alpha_r(e) = \min\{d_{\ell_i}(e), d_{\ell_{i+1}}(e)\}$. All these choices satisfy the required estimation by Lemma 3.

To compute required distances, it suffices to solve two SSSP instances with sources x_i and y_i to vertices of $\tilde{\Omega}_i \cup \tilde{\Omega}_{i+1}$, for every $i \in L_r$. In total we spend $O(n)$ time by Lemma 4 by using algorithm in [26] for SSSP instances. ∎

Theorem 1. *Let* G *be a planar graph with positive edge capacities. Then for any* $c, \delta > 0$, *we can compute a value* $vit^{\delta}(e) \in (vit(e) - \delta, vit(e)]$ *for each* $e \in E(G)$ *satisfying* $c(e) \leq c$, *in* $O(\frac{c}{\delta}n + n \log \log n)$ *time.*

Proof. We compute U in $O(n \log \log n)$ time by Theorem 5. If $d_i > MF + c(e)$, then $d_i(e^*) > MF$, so we are only interested in computing (approximate) values of $d_i(e^*)$ for each $i \in [k]$ satisfying $d_i < MF + c$. By Lemma 5, for each $r \in \{0, 1, \ldots, \lceil \frac{c}{\delta} \rceil\}$, we compute $\alpha_r(e^*) \in [\min_{i \in L_r} d_i(e^*), \min_{i \in L_r} d_i(e^*) + \delta)$, for each $e^* \in E(D)$, in $O(n)$ total time. Then, for each $e^* \in E(D)$, we compute $\alpha(e^*) = \min_{r \in \{0, 1, \ldots, \frac{c}{\delta}\}} \alpha_r(e^*)$; it holds that $\alpha(e^*) \in [\min_{i \in [k]}\{d_i(e^*)\}, \min_{i \in [k]}\{d_i(e^*)\} + \delta)$. Then, by Proposition 2, for each $e \in E(G)$ satisfying $c(e) \leq c$, we compute a value $vit^{\delta}(e) \in (vit(e) - \delta, vit(e)]$ in $O(1)$ time. ∎

6 | COMPUTING VERTEX VITALITY

In this section, we show how to compute vertex vitality by computing an additive guaranteed approximation of distances required in Proposition 3.

Let us denote by F the set of faces of D. By Proposition 3, for every face $f \in F$ we need $\min_{i \in [k]}\{d_i(f)\}$, this is discussed in Lemma 6. For faces $f \in F^x = \{f \in F \mid f \text{ and } \pi_x \text{ have common vertices}\}$ we need also $\min_{i \in [k]}\{d_i(q_f^y)\}$ and $\text{dist}_D(f_v, q_f^y)$. Similarly, for faces $f \in F^y = \{f \in F \mid f \text{ and } \pi_y \text{ have common vertices}\}$ we need also $\min_{i \in [k]}\{d_i(q_f^x)\}$ and $\text{dist}_D(f_v, q_f^x)$.

We observe that there is symmetry between q_f^x and q_f^y. Thus we restrict some definitions and results to the "y case" and then we use the same results for the "x case". In this way, we have to show only how to compute $\text{dist}_D(f, q_f^y)$ (it is done in Section 6.1) and $\min_{i \in [k]}\{d_i(q_f^y)\}$ (see Section 6.2) for every face $f \in F$ that intersects π_y on vertices.

By using the same procedure of Lemma 5, we can also computing $d_i(f)$ for $f \in F$. Thus we can state the following lemma.

Lemma 6. *Let* $r \in \mathbb{N}$, *given* U, *we compute a value* $\alpha_r(f) \in [\min_{i \in L_r}\{d_i(f)\}, \min_{i \in L_r}\{d_i(f)\} + \delta)$ *for each* $f \in F$ *in* $O(n)$ *total time.*

6.1 | Computing $\text{dist}_D(f, q_f^y)$

The unique our result of this subsection is stated in Lemma 7. To obtain it, we use the following result that easily derives from Klein's algorithm about the multiple source shortest path problem [32].

Theorem 6 ([32]). *Given an* n *vertices undirected planar graph* G *with nonnegative edge-lengths, given* r *pairs* $\{(a_i, b_i)\}_{i \in [r]}$ *where the* b_i's *are on the boundary of the infinite face and the* a_i's *are anywhere, it is possible to compute* $\text{dist}_G(a_i, b_i)$, *for each* $i \in [r]$, *in* $O(r \log n + n \log n)$ *total time and* $O(n)$ *space.*

Lemma 7. *We compute* $\text{dist}_D(f, q_f^y)$, *for each* $f \in F^y$, *in* $O(n \log n)$ *total time.*

Proof. For every $i \in [k]$ let $F_i \subseteq F^y$ be the set of faces such that $x_i \in f$, for each $f \in F_i$. We observe that if $|F_i| = m$, then $deg_D(x_i) \geq m + 1$, where $deg_D(x_i)$ is the degree of x_i in D.

Let D' be the graph obtained by adding a new vertex u_f for each face $f \in F^y$ and connecting u_f to all vertices of f by an edge of length L, where $L = \sum_{e \in E(D)} c(e)$ (see Figure 12 for an example of construction of graph D'). Thus $\text{dist}_D(y_i, f) = \text{dist}_{D'}(y_i, u_f) - L$.
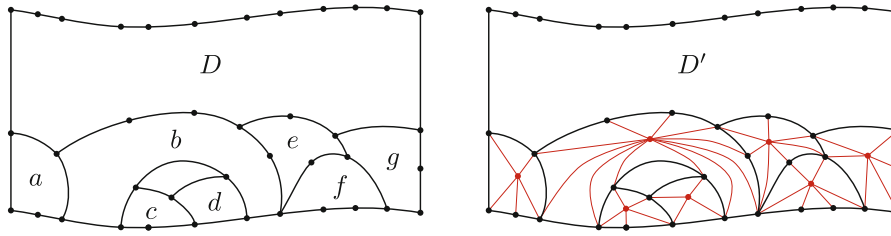
FIGURE 12 Graph D, faces in F^y and graph D' used in the proof of Lemma 7.
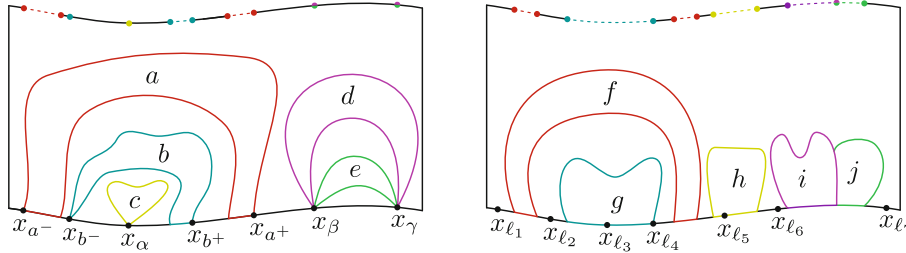

FIGURE 13 On the left $c \prec b \prec a$ and $e \prec d$; a and d are the only maximal faces; it holds that $c^- = c^+ = \alpha$, $[d^-, d^+] = [e^-, e^+] = [\beta, \gamma]$. On the right let $L_r = (\ell_1, \ell_2, \ldots, \ell_7)$, it holds that: $[f_r^-, f_r^+] = [\ell_1, \ell_5]$, $[g_r^-, g_r^+] = [\ell_2, \ell_5]$, $[h_r^-, h_r^+] = [\ell_4, \ell_6]$, $[i_r^-, i_r^+] = [j_r^-, j_r^+] = [\ell_6, \ell_7]$.

We compute $d_{D'}(y_i, u_f)$, for $i \in [k]$ and $f \in F_i$, by using the result stated in Theorem 6. Being $|V(D')| = O(n)$, we spend $O\left(\log n \sum_{i \in [k]} |F_i| + n \log n\right) \leq O\left(\log n \sum_{i \in [k]} (deg_D(x_i) - 1) + O(n \log n)\right) = O(n \log n + n \log n) = O(n \log n)$ time. Finally, for each $f \in F^y$

$$\text{dist}_D(f, q_f^y) = \min_{\{i \in [k] \mid x_i \in f\}} \text{dist}_D(y_i, f) = \min_{\{i \in [k] \mid x_i \in f\}} \text{dist}_{D'}(y_i, u_f) - L.$$

Thus we obtain what we need adding no more time than $\sum_{\{i \in [k] \mid x_i \in f\}} O(1) \leq O(\sum_{f \in F^y} |V(f)|) \leq O(\sum_{f \in F} |V(f)|) = O(n)$. ∎

6.2 | Computing $d_i(q_f^y)$

We note that for computing the $d_i(q_f^y)$'s we can not directly use Lemma 3 as we have done for the $d_i(e)$'s and the $d_i(f)$'s. Indeed, it is possible that vertices in q_f^y are not contained in any slice $\Omega_{i,j}$, with i,j consecutive indices in L_r. To overcome this, we have to introduce a partial order on faces of D.

For each $f \in F^y$, we define f^- and f^+ as the minimum and maximum indices in $[k]$, respectively, such that $x_{f^-}, x_{f^+} \in V(f)$. Now we introduce the concept of *maximal face*. Let $f \in F^y$ and let p_f and q_f be the two subpaths of the border cycle of f from x_{f^-} to x_{f^+}. We say that $g \prec f$ if g is contained in the region R bounded by $\pi_x[x_{f^-}, x_{f^+}] \circ p_f$, this implies that g is also contained in the region R' bounded by $\pi_x[x_{f^-}, x_{f^+}] \circ q_f$, thus the definition does not depend on the choice of p_f and q_f. Finally, we say that f is *maximal* if it does not exist any face $g \in F^y$ satisfying $f \prec g$, and we define $F_{\max} = \{f \in F^y \mid f \text{ is maximal}\}$, see the left part of Figure 13. We find $F_{\max}$ in $O(n)$ time.

Given $r \in \mathbb{N}$ and $f \in F^y$, we define f_r^+ as the smallest index in L_r such that $f^+ < f_r^+$ (if $f^+ > \ell_{z_r}^r$, then we define $f_r^+ = \ell_{z_r}^r$). Similarly, we define f_r^- as the largest index in L_r such that $f_r^- > f^-$ (if $f^- < \ell_1^r$, then we define $f_r^- = \ell_1^r$), see the right part of Figure 13.

Now we deal with computing $d_i(q_f^y)$, for each $f \in F^y$. By following Equation (1), we can restrict only to the easier case in which f satisfies $d_i(q_f^y) < \text{dist}_D(f, q_f^y)$; indeed, if f does not satisfy it, then we are not interested in the value of $d_i(q_f^y)$.

Lemma 8. *Let $r \in \mathbb{N}$. Given $\text{dist}_D(f, q_f^y)$ and given U, for each $f \in F^y$ satisfying $\min_{i \in L_r} d_i(q_f^y) < \text{dist}_D(f, q_f^y)$ we compute a value $\beta_r(f) \in [\min_{i \in L_r} d_i(q_f^y), \min_{i \in L_r} d_i(q_f^y) + \delta)$ in $O(n)$ total time.*

Proof. Let $f \in F^y$. We observe that if $i \in [f^-, f^+]$, then every path from x_i to q_f^y passes through either x_{f^-} or x_{f^+}. Thus, for every $i \in [f^-, f^+]$, it holds that $d_i(q_f^y) \geq \text{dist}_D(f, q_f^y)$. Hence for any $f \in F^y$ satisfying $\min_{i \in L_r} d_i(q_f^y) < \text{dist}_D(f, q_f^y)$ it holds that

$$\min_{i \in L_r} d_i(q_f^y) = \min_{i \in L_r, i \notin [f^-, f^+]} \{d_i(q_f^y)\}, \tag{2}$$

being $q_f^y \subseteq \text{Right}_{f^-}$ and $q_f^y \subseteq \text{Left}_{f^+}$, then Lemma 3 and Equation (2) imply

$$\min_{i \in L_r} d_i(q_f^y) = \min_{i \in L_r, i \notin [f^-, f^+]} \{d_i(q_f^y)\} \geq \min\{d_{f_r^-}(q_f^y), d_{f_r^+}(q_f^y)\} - \delta. \qquad (3)$$

To complete the proof, we need to show how to compute $d_{f_r^-}(q_f^y)$ and $d_{f_r^+}(q_f^y)$, for each $f \in F^y$ satisfying $\min_{i \in L_r} d_i(q_f^y) < \text{dist}_D(f, q_f^y)$ in $O(n)$ total time. In the following claim we prove it by removing the request that every face $f \in F^y$ has to satisfy $\min_{i \in L_r} d_i(q_f^y) < \text{dist}_D(f, q_f^y)$.

1. We compute $d_{f_r^-}(q_f^y)$ and $d_{f_r^+}(q_f^y)$, for each $f \in F^y$, in $O(n)$ total time.

Proof of 1: we recall that $d_i(q_f^y) = \text{dist}_D(x_i, q_f^y) + \text{dist}_D(y_i, q_f^y)$, for all $i \in [k]$ and $f \in F^y$. Being $q_f^y \subseteq V(\pi_y)$ we compute $\text{dist}_D(y_i, q_f^y)$ in $O(|V(q_f^y)|)$ time. Thus we have to compute only $\text{dist}_D(x_i, q_f^y)$, for required $i \in L_r$ and $f \in F^y$.

For every $f \in F^y$, let $R_f = \Omega_{f_r^-, f_r^+}$, and let $\mathcal{R} = \bigcup_{f \in F_{\max}} R_f$. We observe that, given two maximal faces f and g, it is possible that $R_f = R_g$. This happens if and only if $f_r^- = g_r^-$ and $f_r^+ = g_r^+$ (see face i and face j in Figure 14). We overcome this abundance by introducing $\tilde{F}$ as a minimal set of faces such that $\mathcal{R} = \bigcup_{f \in \tilde{F}} R_f$ and $R_f \neq R_g$, for all distinct $f, g \in \tilde{F}$ (see Figure 14 for an example of $\tilde{F}$).

For every $f \in \tilde{F}$, it holds that $\pi_y[f_r^-, f_r^+] \subseteq R_f$. Thus, by the above argument, if $g \in F^y$ and $R_g \subseteq R_f$, then $q_g^y \subseteq R_f$. We solve 4 SSSP instances in R_f with sources x_j, for each $j \in \{f_r^-, f_r^+, f^-, f^+\}$ (possibly, $f_r^- = f^-$ and/or $f_r^+ = f^+$ and/or $f^- = f^+$). Now we have to prove that this suffices to compute $d_{f_r^-}(q_f^y)$ and $d_{f_r^+}(q_f^y)$, for each $f \in F^y$. In particular we show that, after solving the SSSP instances, we compute $d_{g_r^-}(q_g^y)$ and $d_{g_r^+}(q_g^y)$ in $O(|V(g)|)$ total time, for each $g \in F^y$.

Let $g \in F^y$, and let $f \in \tilde{F}$ be such that $g \subseteq R_f$. There are two cases: either $g_r^- = f_r^-$ and $g_r^+ = f_r^+$, or $g_r^- \neq f_r^-$ and/or $g_r^+ \neq f_r^+$.

If the first case occurs, then we compute $\text{dist}_D(x_{g_r^-}, q_g^y) = \text{dist}_D(x_{f_r^-}, q_g^y)$ and $\text{dist}_D(x_{g_r^+}, q_g^y) = \text{dist}_D(x_{f_r^+}, q_g^y)$ in $O(|V(g)|)$ time, because $q_g^y \subseteq R_f$ and $|V(q_g^y)| < |V(g)|$. Otherwise, w.l.o.g., we assume that $g_r^- \neq f_r^-$ (if $g_r^+ \neq f_r^+$, then the proof is similar). By definitions of $\tilde{F}$, Ω_g, and Ω_f, it holds that $g \prec f$. Thus $g_r^- \in [f^-, f^+]$, therefore every path from $x_{g_r^-}$ to q_g^y passes through either f^- or f^+ (see g_3 and f_5 in Figure 14). By this discussion, it follows that

$$\text{dist}_D(x_{g_r^-}, q_g^y) = \min \begin{Bmatrix} \text{dist}_D(x_{g_r^-}, x_{f^-}) + \text{dist}_D(x_{f^-}, q_f^y) \\ \text{dist}_D(x_{g_r^-}, x_{f^+}) + \text{dist}_D(x_{f^+}, q_f^y) \end{Bmatrix}$$

$$= \min \begin{Bmatrix} |\pi[x_{g_r^-}, x_{f^-}]| + \text{dist}_D(x_{f^-}, q_f^y) \\ |\pi[x_{g_r^-}, x_{f^+}]| + \text{dist}_D(x_{f^+}, q_f^y) \end{Bmatrix}.$$

We compute all these distances by the solutions of previous SSSP instances in $O(|V(q_g^y)|)$ time, and thus we compute $\text{dist}_D(x_{g_r^-}, q_g^y)$ in $O(|V(q_g^y)|)$ time. By symmetry, the same cost is required to compute $\text{dist}_D(x_{g_r^+}, q_g^y)$.

We have proved that, after solving the described SSSP instances, we compute $d_{f_r^-}(q_f^y)$ and $d_{f_r^+}(q_f^y)$, for all $f \in F^y$, in $O(|V(f)|)$ time for each $f \in F^y$. Being $\sum_{f \in F^y} |V(f)| = O(n)$, it remains to show that we can solve all the previous SSSP instances in $O(n)$ total time. We want to use Lemma 4 (we recall that, for our purposes, distances in $\Omega_{f_r^-, f_r^+}$ are the same in $\tilde{\Omega}_{f_r^-, f_r^+}$).

Let us fix $i \in [h]$ and let $a = f_i$, $b = f_{i+1}$, $c = f_{i+2}$ and $d = f_{i+3}$. We can not use directly Lemma 4 because it is possible that $a_r^- < b_r^+$ (see in Figure 14 $a = f_3$ and $b = f_4$, thus $b_r^- = \ell_4 < \ell_5 = a_r^+$) and thus we might have not an increasing set of indices. But, by definition of $\tilde{F}$, it holds that $a_r^- \leq d_r^-$, indeed $a_r^- \in [b_r^-, c_r^+]$ otherwise $R_b = R_c$; these relations do not depend on i. Similarly, $d_r^- \geq a_r^+$. Thus we solve first the SSSP instances in R_{f_i}, for each $i \in [h]$ such that $i \equiv 0 \mod 3$; then for $i \equiv 1 \mod 3$ and finally for $i \equiv 2 \mod 3$. By Lemma 4 it costs $O(n)$ time. End proof of 1.
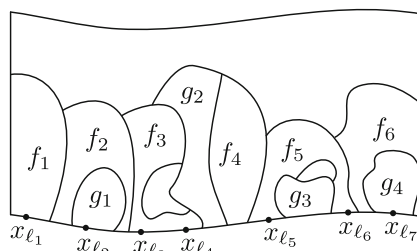

FIGURE 14 Assume that $L_r = (\ell_1, \dots, \ell_7)$. A possible $\tilde{F}$ is $\tilde{F} = \{f_1, \dots, f_6\}$. Moreover, g_1, g_3, g_4 are not in $F_{\max}$, $g_2 \in F_{\max}$ and $R_{g_2} = R_{f_4}$ thus $g_2 \notin \tilde{F}$.

Finally, by Equation (3), we set $\beta_r(f) = \min\{d_{f_r^-}(q_f^y), d_{f_r^+}(q_f^y)\}$ for each $f \in F^y$ satisfying $\beta_r(f) < \text{dist}_D(f, q_f^y)$ and we ignore faces in F^y that do not satisfy it. ∎

6.3 | Computational complexity of vertex vitality

Now we give our theorems about vertex vitality. To prove Theorem 2 we follow the same approach used in Theorem 1, by referring to Proposition 3 in place of Proposition 2.

We recall that the result stated in Theorem 3 is more efficient than the result in Theorem 4 if either $|S| < \log n$ and $E_S > |S| \log n$ or $|S| \geq \log n$ and $E_S > \frac{|S|n^{1/3}}{\log^{8/3}}$, where $E_S = \sum_{v \in S} deg(v)$.

Theorem 2. *Let G be a planar graph with positive edge capacities. Then for any $c, \delta > 0$, we can compute a value $vit^\delta(v) \in (vit(v) - \delta, vit(v)]$ for each $v \in V(G)$ satisfying $c(v) \leq c$, in $O\left(\frac{c}{\delta}n + n \log n\right)$ time.*

Proof. We compute D and U in $O(n \log \log n)$ time by Theorem 5. If $c(v) < c$, then $c(f_v^*) < c$. For convenience, in D, we denote f_v^* by f. By Lemma 7, we compute $\text{dist}_D(f, q_f^y)$ (resp., $\text{dist}_D(f, q_f^x)$) in $O(n \log n)$ time, for each $f \in F^y$ (resp., for each $f \in F^x$). Now we have to show how to obtain $\min_{i \in [k]}\{d_i(f)\}$, $\min_{i \in [k]}\{d_i(q_f^y)\}$ and $\min_{i \in [k]}\{d_i(q_f^x)\}$ that we may compute with an error depending on δ.

We note that $\min_{i \in [k]} d_i(f) = \min_{i \in [k], d_i < MF+c(f)} d_i(f)$. Indeed, if $d_i(f) = MF - z$, for some $z > 0$, then d_i is at most $MF - z + c(f)$. For the same reason, $\min_{i \in [k]} d_i(q_f^x) = \min_{i \in [k], d_i < MF+c(f)} d_i(q_f^x)$ and, similarly, $\min_{i \in [k]} d_i(q_f^y) = \min_{i \in [k], d_i < MF+c(f)} d_i(q_f^y)$.

By using Lemma 5, for each $r \in \{0, 1, \ldots, \lceil \frac{c}{\delta} \rceil\}$, we compute a value $\alpha_r(f) \in [\min_{i \in L_r} d_i(f), \min_{i \in L_r} d_i(f) + \delta)$, for each $f \in F$, in $O(n)$ total time. Then, for each $f \in F$, we compute $\alpha(f) = \min_{r \in \{0, 1, \ldots, \frac{c}{\delta}\}} \alpha_r(f)$. By above, for any $f \in F$ satisfying $c(f) < c$, it holds that $\alpha(f)$ satisfies $\alpha(f) \in [\min_{i \in [k]}\{d_i(f)\}, \min_{i \in [k]}\{d_i(f)\} + \delta)$.

With a similar strategy, by replacing Lemma 5 with Lemma 8, for each $f \in F^y$ satisfying $c(f) < c$ and $\min_{i \in L_r} d_i(q_f^y) < \text{dist}_D(f, q_f^y)$, we compute a value $\beta(f) \in [\min_{i \in [k]}\{d_i(q_f^y)\}, \min_{i \in [k]}\{d_i(q_f^y)\} + \delta)$. The same results hold for the "x case": for each $f \in F^x$ satisfying $c(f) < c$ and $\min_{i \in L_r} d_i(q_f^x) < \text{dist}_D(f, q_f^y)$, we compute a value $\gamma(f) \in [\min_{i \in [k]}\{d_i(q_f^x)\}, \min_{i \in [k]}\{d_i(q_f^x)\} + \delta)$.

Then, by Proposition 3, for each $v \in V(G)$ satisfying $c(v) \leq c$ ($c(f) < c$) we compute a value $vit^\delta(v)$ satisfying $vit^\delta(v) \in (vit(v) - \delta, vit(v)]$ in $O(1)$ time by using $\text{dist}_D(f, q_f^x)$, $\text{dist}_D(f, q_f^y)$, $\alpha(f)$, $\beta(f)$ and $\gamma(f)$. ∎

Theorem 3. *Let G be a planar graph with positive edge capacities. Then for any $S \subseteq V(G)$, we can compute $vit(v)$ for each $v \in S$ in $O(|S|n + n \log \log n)$ time.*

Proof. We compute D and U in $O(n \log \log n)$ time by Theorem 5. For convenience, we denote f_v^* by f. To compute $\min_{i \in [k]}\{d_i(f)\}$, $\text{dist}_D(f, q_f^x)$ and $\text{dist}_D(f, q_f^y)$ we put a vertex u_f in the face f and we connect it to all vertices of f with zero weighted edges. Then we solve an SSSP instance with source u_f and we compute $\min_{i \in [k]}\{d_i(f)\}$, $\text{dist}_D(f, q_f^x)$ and $\text{dist}_D(f, q_f^y)$ in $O(n)$. With a similar strategy, we compute $\min_{i \in [k]}\{d_i(q_f^y)\}$ and $\min_{i \in [k]}\{d_i(q_f^x)\}$ in $O(n)$ time. Finally, by Proposition 3, for each $v \in S$, we compute $vit(v)$ in $O(1)$ time. ∎

7 | SMALL INTEGER CAPACITIES AND UNIT CAPACITIES

If the edges capacities are integer, then we can compute the max flow in $O(n + L)$ time [20] and also U in $O(n + L)$ time [8, 51], where L is the sum of all the edges capacities.

Corollary 1. *Let G be a planar graph with integer edge capacity and let L be the sum of all the edges capacities. Then*

- *for any $H \subseteq E(G) \cup V(G)$, we can compute $vit(x)$ for each $x \in H$, in $O(|H|n + L)$ time,*
- *for any $c \in \mathbb{N}$, we can compute $vit(e)$ for each $e \in E(G)$ satisfying $c(e) \leq c$, in $O(cn + L)$ time.*

Proof. Note that, being all the edge capacities integer, then every edge or vertex vitality is an integer. Thus, by taking $\delta = 1$ in Theorems 1 and 2, we obtain all the vitalities without error. The two statements follow from the proof of Theorems 1 and 2 by taking $\delta = 1$ and by computing U in $O(n + L)$ time instead of $O(n \log \log n)$ time by using algorithm in [8]. ∎

Corollary 2. *Let G be a planar graph with unit edge capacity. Let $n_{>d}$ be the number of vertices whose degree is greater than d. We can compute the vitality of each edge in $O(n)$ time and the vitality of each vertex in $O(\min\{n^{3/2}, n(n_{>d} + d + \log n)\})$ time.*

Proof. The complexity of edge vitality is implied by Corollary 1 by taking $c = 1$ and because $L = O(n)$. Being the vitality integers, then we compute the vitality of each vertex in $O((n_{>d} + d)n + n \log n)$ time by Theorem 2.

To compute the vitality of each vertex in $O(n^{3/2})$ total time we note that in a planar graph, by Euler formula, there are at most $6\sqrt{n}$ vertices whose degree is greater than $\sqrt{n}$. Thus it suffices to take $d = \sqrt{n}$, that implies $n_{>d} \leq 6\sqrt{n}$ and $O((n_{>d} + d)n + n \log n) = O(n^{3/2})$. ∎

Kowalik and Kurowski [34] described an algorithm that, given an unweighted planar graph G and a constant d, with a $O(n)$ time preprocessing can establish in $O(1)$ time if the distance between two vertices in G is at most d and, if so, computes it in $O(1)$ time.

Corollary 3. *Let G be a planar graph with unit edge capacity where only a constant number of vertices have degree greater than a fixed constant d. Then we can compute the vitality of each vertex in $O(n)$ time.*

Proof. By above discussion, Corollary 2 and the proof of Theorem 2, it suffices to show that we can compute $\text{dist}_D(f_v, q_f^y)$, for each $f \in F^y$, in $O(n)$ total time. For every $i \in [k]$ let $F_i \subseteq F^y$ be the set of faces such that $x_i \in f$, for each $f \in F_i$. Note that if $|F_i| = m$, then $deg_D(x_i) = m + 1$.

Let d be the maximum degree of G. We need $d_D(y_i, z)$, for each $i \in [k]$, and $z \in V(f)$, for each $f \in F_i$. If we use the algorithm in [34], then we spend $\sum_{i \in [k]} \left(\sum_{f \in F_i} |V(f)| \right) \leq \sum_{i \in [k]} |F_i| d = \sum_{i \in [k]} (deg_D(x_i) + 1)d = O(n)$ time. ∎

8 | CONCLUSIONS AND OPEN PROBLEMS

We proposed algorithms for computing an additive guaranteed approximation of the vitality of each edge or vertex with bounded capacity with respect to the max flow from s to t in undirected planar graphs. These results are relevant for determining the vulnerability of real world networks, under various capacity distributions.

It is still open the problem of computing the exact vitality of each edge of an undirected planar graph within the same time bound as computing the max flow value, as is already known for the *st*-planar case.

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