Analytic structure of the high-energy gravitational amplitude: Multi-*H* diagrams and classical 5PM logarithms

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We investigate the high-energy, small-angle limit of two-body gravitational scattering. Using power counting arguments and dispersion relations in an effective field theory for the Regge regime, we derive the general loop expansion that determines how the leading Regge logarithms and their complex structure arise as a power series in t/s. Focusing on the tower of multi-*H* diagrams that govern the leading logarithmic behavior, we compute the leading double logarithm at four loops (or equivalently fifth post-Minkowskian order, 5PM) using both effective field theory methods and the multi-Regge expansion, finding complete agreement. Finally, using the aforementioned dispersion relations, we extract the single logarithmic contribution to the imaginary part of the eikonal phase at 5PM in the Regge limit.

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I. MOTIVATION AND INTRODUCTION

The Regge limit of QCD has a long and rich history.[1] In this limit $s \gg |t|$, radiative corrections to $2 \to 2$ amplitudes exhibit iterative evolution in the rapidity $y \simeq \log(s/|t|)$. This evolution can be formulated either in the t channel (Reggeized exchanges) or in the s channel (rapidity evolution of projectile/target degrees of freedom), and both viewpoints are needed for unitarity [4]. In practice, only a few iterative structures have been studied in detail. The first is the Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation [5–8], which describes one- and two Reggeized-gluon ladders in the t-channel and has been further generalized to multi-Reggeon states in the planar limit [9–12]. Interestingly, its dynamics can be formulated as a rapidity renormalization group equation (RRGE) [13,14], with the BFKL kernel playing the role of the anomalous dimension [15]. The second is the Balitsky-Jalilian-Marian–Iancu–McLerran–Weigert–Leonidov–Kovner hierarchy [16–18], which governs the rapidity evolution on the s-channel side and provides a complementary, unitarity description at high energies.

In this paper, we will be interested in the behavior of gravity in the Regge limit, for which much less is known. The leading high-energy contributions to the elastic amplitude are governed by the eikonal phase, which arises from multiple soft graviton exchanges in the t-channel; see Ref. [19] for a review. By contrast, the graviton Regge trajectory is suppressed by t/s [20–23] and thus represents a quantum correction. Similarly, soft exchanges between Glauber gravitons beyond the single-emission case arise only quantum mechanically: these correspond to unitarity cuts of $2 \to 2 + n$ amplitudes with $n \geq 2$ [24,25], which generate the gravitational BFKL evolution [20,23,26]. The single soft emission yields the well-known $\mathcal{O}(G_N^3 s^3 \log(s/|t|))$ contribution to the imaginary part of the two-loop amplitude—the *H diagram* [27]—obtained

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[1]See Refs. [1–3] for recent reviews on this topic.

from the three-particle unitarity cut in multi-Regge kinematics (MRK).

The leading Regge logarithms at higher loop orders arise by iterating the same structure that appears in the H diagram: each new rung corresponds to inserting an additional three-particle unitarity cut in MRK, with a single on-shell graviton connecting pair of different rungs. This construction defines a *multi-H generalization* of the H diagram [28–30], which contribute at $(2N+3)$ post-Minkowskian (PM) order and provides a correction of $\mathcal{O}((G_N^2 s \log(s/|t|))^N)$ to the H diagram appearing at 3PM order [27]. These classical logs are particularly relevant for the gravitational wave community, especially given the ongoing interest in targeting the 5PM order [31–34]. Here, we take this opportunity to compute the double logarithm appearing in the real part, as well as the single logarithm in the imaginary part, of the amplitude at 5PM order, using both effective field theory (EFT) and Regge techniques [23,35].

The S-matrix viewpoint suggests that the high-energy, fixed-t amplitude is constrained by analyticity, crossing, and unitarity, yielding subtracted dispersion relations that organize the tower of $\log(s/|t|)$ terms [36]. The behavior of the gravitational amplitude in the forward limit is well captured by the familiar eikonal formula (see Ref. [37] for a review), which reproduces the graviton pole [38] and saturates the relevant spin-2 dispersive sum rules [39–41], suggesting that a twice-subtracted fixed-t dispersion relation holds for the $2 \to 2$ amplitude [42,43]. This has indeed been applied at 3PM in the Regge limit [27,44], which motivates us to revisit and extend such arguments to higher loops in the ultrarelativistic regime.

This paper is organized as follows. We first establish the relevant power counting for the EFT of forward scattering in gravity in Sec. II, emphasizing the distinction between classical and quantum contributions. In Sec. III, we then review the application of this formalism to amplitudes and set up the relevant RRGE at leading-log accuracy, making contact with the gravitational BFKL kernel. In Sec. IV, we construct the multi-H series from both the EFT perspective and the Regge-theory approach in MRK, demonstrating the complete agreement between these approaches and computing explicitly the single and double-H diagrams in D dimensions. Finally, in Sec. V, we derive new dispersion relations for gravitational amplitudes as an expansion in powers of t/s obtaining, under the assumption of eikonalization, the leading logarithmic terms in the real and imaginary parts of the classical eikonal phase at 5PM order.

A. Conventions

We use the mostly minus metric signature $\eta = (+,-,-,-)$. Gravitational coupling and Newton's constant are related by $\kappa = \sqrt{8\pi G_N}$. Momentum-space integrals are defined with the measure $\hat{\mathrm{d}}^D p = \mathrm{d}^D p/(2\pi)^D$ in $D = 4-2\epsilon$ dimensionally regularized dimensions. We adopt the shorthand notation $\hat{\delta}^{(D)}(\cdot) = (2\pi)^D \delta^{(D)}(\cdot)$ and $\delta^+(k^2) = \Theta(k^0)\delta(k^2)$ and employ light-cone coordinates $(p^+, p^-, p_\perp)$, with $p_\perp$ a transverse vector in $d = D-2$ dimensions. Finally, we define the symbol $\int_{q_\perp} = \int \hat{\mathrm{d}}^d q_\perp$ for the integration in transverse space.

II. POWER COUNTING IN GRAVITY AND WHEN IT FAILS

The effective field theory describing forward scattering was developed for QCD in Ref. [15] and extended to gravity in Ref. [23]. These constructions generalize the framework of soft-collinear effective theory (SCET), originally formulated for hard scattering in QCD [45–47] and subsequently adapted to gravity [48–50]. The forward scattering theory differs from its hard scattering counterpart through the presence of a Glauber (off-shell) mode, which dominates the interaction in the Regge limit. Moreover, the power counting is especially distinct in the gravitational case due to the mass dimension of the coupling.

We consider Einstein's gravity with the hierarchy

$$s \gg \frac{1}{G_N} \gg |t|, \tag{1}$$

where s and t are the Mandelstam invariants of the two-body scattering amplitude $\mathcal{M}_{2\to 2}$. The corresponding EFT is organized in terms of the expansion parameters

$$\alpha_Q \equiv G_N t, \qquad \alpha_C \equiv G_N^2 s t, \qquad \lambda \equiv \frac{t}{s}, \tag{2}$$

and we work only to leading order in λ^2 but to all orders in α_C and α_Q, corresponding to the "classical" and "quantum" couplings, respectively. As shown in Appendix B, this is equivalent to working at leading logarithmic accuracy in the rapidity logarithms $\log(s/|t|)$; correspondingly, we neglect subleading-power corrections in λ as well as hard-region contributions, such as double logarithms, that lie outside the forward/Glauber EFT.

The difficulty with the gravitational perturbative series arises from the fact that the effective gravitational charge grows with energy, scaling as $G_N s = \alpha_C/\alpha_Q$. This means that at high energy the expansion cannot be uniformly controlled unless we restrict to observables that are insensitive to short-distance (contact) interactions. Such observables correspond to scattering between well-separated particles, described by localized wave packets with large angular momentum [51]. Even within this regime, however, local UV-sensitive effects can generate nonlocal contributions at higher loops. Calculational control is regained if the amplitude eikonalizes in impact-parameter space [19], where it takes the form

[2] While t/s is not independent of the other two parameters, it is convenient to treat it as a separate expansion parameter.

$$i\int_{q_\perp} e^{iq\cdot b}\frac{\mathcal{M}_{2\to2}(s,t=q^2)}{2s} = (1+\Delta_Q)e^{2i\delta_{\rm Cl}} - 1, \quad (3)$$

with $\delta_{\rm Cl}$ and Δ_Q representing the classical and quantum contributions, respectively, expanded as

$$\delta_{\rm Cl} = G_N s\sum_{j=0}^{\infty}\alpha_C^j\delta_{\rm Cl}^{(2j)}(b),$$
$$\Delta_Q = \sum_{n,k=0}^{\infty}\alpha_Q^n\alpha_C^{k+1}\Delta_Q^{(n,k)}(b). \quad (4)$$

For massive scattering, with $m^2 \gg |t|$, eikonalization is known to hold [52–54], whereas in the strictly massless case, evidence from $\mathcal{N}=8$ supergravity suggests that it can fail at subleading-logarithmic accuracy starting at 4PM [55]. Although in the EFT we will work in the massless case, our Regge theory approach originates from the massive regime and smoothly extends to the massless limit at leading order in λ. Consequently, at leading logarithmic accuracy, our target calculations will not be affected by any possible eikonalization breakdown.

III. EFT OF FORWARD SCATTERING FOR GRAVITY

Within the EFT framework, it can be shown [15,56,57] that the $2\to2$ amplitude factorizes, at leading power, into a convolution in transverse momentum space

$$\mathcal{M}_{2\to2} = i\sum_{M,N}J_{(M)}\otimes S_{(M,N)}\otimes\bar{J}_{(N)}, \quad (5)$$

where $J_{(M)}$ and $\bar{J}_{(N)}$ depend on the collinear degrees of freedom, while $S_{(N,M)}$ depends on the soft degrees of freedom. In general, the soft function can mix sectors with different numbers of Glaubers, but in gravity at the order of interest $S_{(M,N)}\propto\delta_{MN}S_{(M)}$ [23], so the sum reduces to a single index.

The dependence on the large ratio $s/|t|$ enters through logarithms of the rapidity renormalization scale ν (the analog of μ in dimensional regularization), which separate the collinear and soft modes. The resummation of these rapidity logarithms is governed by the RRGE, which can be implemented by evolving either the collinear functions down to t or, equivalently, the soft function up to s; here we adopt the latter viewpoint. Consequently, all leading logarithms can be obtained by studying the evolution of $S_{(N)}$ in isolation, while subleading logarithms may receive collinear or finite-mass corrections. In particular, the leading-log structure is universal and identical in the massive and massless theories.

Each matrix element satisfies a RRGE [13,14], with the soft function in particular satisfying

$$\nu\frac{d}{d\nu}S_{(M)} = -\gamma_{(M)}\otimes S_{(M)} - S_{(M)}\otimes\gamma_{(M)}, \quad (6)$$

where $\gamma_{(M)}$ is the anomalous dimension of the M-Glauber soft matrix element. At leading-logarithmic order (i.e., for the one-loop anomalous dimension), it takes the form

$$\gamma_{(M)} \sim \sum_i \omega_G(q_i)I_{\perp(M-1)} + \sum_{\text{pairs } i,j}\mathcal{K}^{\rm GR}(q_i,q_j;q)I_{\perp(M-2)}. \quad (7)$$

In the above, ω_G is the graviton Regge trajectory, and $\mathcal{K}^{\rm GR}$ is the gravitational BFKL kernel [20,26]

$$\mathcal{K}^{\rm GR}(q_1,q_2;q) \equiv \mathcal{K}^{\rm GR}(q_1,q_2;q-q_1,q-q_2), \quad (8)$$

where the four-argument form reads

$$\begin{aligned}\mathcal{K}^{\rm GR}(q_1,q_2;q_3,q_4) = \frac{4}{(q_1-q_2)_\perp^4}\Big[&q_{1\perp}^2q_{2\perp}^2q_{3\perp}^2q_{4\perp}^2 \\ &- q_{3\perp}^2q_{4\perp}^2(q_{1\perp}\cdot q_{2\perp})^2 \\ &- q_{1\perp}^2q_{2\perp}^2(q_{3\perp}\cdot q_{4\perp})^2\Big] \\ &+\Big[(q_{1\perp}+q_{3\perp})\cdot(q_{2\perp}+q_{4\perp}) \\ &- \frac{q_{2\perp}^2q_{3\perp}^2+q_{1\perp}^2q_{4\perp}^2}{(q_1-q_2)_\perp^2}\Big]^2. \end{aligned} \quad (9)$$

The first sum in (7) corresponds to the insertion of the Regge trajectory on any individual Glauber exchange, while the second term runs over all pairs of Glauber rungs connected by a BFKL kernel. The operator $I_{\perp(M)}$ acts as the identity on the convolution space for M Glauber exchanges. High-energy logarithms in the amplitude can then be predicted by evolving in ν from the soft rapidity scale $\sqrt{-t}$ up to the collinear rapidity scale $\sqrt{s}$.

At tree level, the soft and collinear matrix elements scale as $J_{(M)}\otimes S_{(M)}\otimes\bar{J}_{(M)}\sim(G_Ns)^M s/t$, while the one-loop anomalous dimension scales as $\gamma_{(M)}\sim G_N t$. Taken together, the RRGE (6) then predicts a classical contribution from $S_{(M)}$ to the amplitude at $(2M-1)$-PM order of the form $\log^{M-1}(s)$ [23]. In general, this term will contain both genuinely classical contributions as well as quantum iterations, the latter involving insertions of the graviton Regge trajectory as well as higher-order BFKL-type ladders with two or more graviton exchanges between the same pair of Glauber rungs. When solving Eq. (6), these nonclassical pieces can be directly discarded, leaving only the contributions in which each Glauber rung exchanges a graviton with at least one other rung. These configurations are exactly the classical s-channel multi-H diagrams [29,30].

IV. MULTI-H DIAGRAMS FROM AMPLITUDES AND EFT TOOLS

In this section, we will calculate the leading logarithmic contributions to the classical amplitude, recovering the single $\log(s/|t|)$ at 3PM and then computing the double $\log^2(s/|t|)$ at 5PM. To do so, we will use both the EFT approach and Regge-theory techniques, demonstrating the complete agreement between the two approaches.

A. EFT approach

In the EFT framework, the leading rapidity logarithms arise from the evolution of the soft function under the RRGE (6), where each iteration inserts an additional soft graviton exchange between distinct Glauber lines and generates one power of $\log s$.

The normalization of the soft function and its anomalous dimension is fixed by the factorization formula (5), which can be made explicit as

$$\mathcal{M}_{2\to 2} = i\sum_M \int_{\perp(M\times M)} J_{(M)}(\{l_{i\perp}\}) \times S_{(M)}(\{l_{i\perp}\};\{l'_{i\perp}\})\bar{J}_{(M)}(\{l'_{i\perp}\}), \qquad (10)$$

where the transverse convolution measure is

$$\int_{\perp(M\times M)} \equiv \int_{\perp(M)}\int_{\perp(M)},$$

$$\int_{\perp(M)} = \frac{(-i)^M}{M!}\left[\prod_{i=1}^{M}\int_{l_{i\perp}}\frac{1}{l_{i\perp}^2}\right]\hat{\delta}^{(d)}\left(\sum_i l_{i\perp} - q_\perp\right). \qquad (11)$$

While we do not know how to solve the RRGE equation (6) in general, the leading logarithm can be computed systematically by iterating such an equation. The classical (Regge-cut) part of $\gamma_{(M+1)}$ in (7) corresponds to the exchange of a single soft graviton connecting any pair of the $(M+1)$ Glauber lines and is given by [23]

$$\gamma^{\mathrm{Cl}}_{(M+1)} = -i^{M+1}(M+1)!\sum_{i<j}\frac{\kappa^2}{8\pi}\mathcal{K}^{\mathrm{GR}}(k_{i\perp},k_{j\perp};\ell_{i\perp},\ell_{j\perp}) \times \prod_{m\neq i,j}\ell^2_{m\perp}\hat{\delta}^{(d)}(\ell_{m\perp}-k_{m\perp}). \qquad (12)$$

The tree-level value of the soft function is

$$S^{(0)}_{(M)}(\{l_{i\perp}\};\{l'_{i\perp}\}) = 2i^M M!\left[\prod_{a=1}^{M}(l'_{a\perp})^2\right] \times \left[\prod_{n=1}^{M-1}\hat{\delta}^{(d)}(l_{n\perp}-l'_{n\perp})\right]. \qquad (13)$$

A single iteration of the RRGE (6) acting on the two-Glauber soft function $S^{(0)}_{(2)}$ reproduces the single H diagram [23], giving the $\mathcal{O}(G_N^3 s^3 \log s)$ contribution to the amplitude at 3PM order. The next iteration involves the three-Glauber soft function $S^{(0)}_{(3)}$, whose running yields the $\mathcal{O}(G_N^5 s^4 \log^2 s)$ double-H contribution at 5PM order. We will discuss those more in details in the next sections, using the complementary Regge-theory perspective.

B. Regge theory approach

The multi-H diagrams obtained from the rapidity renormalization group (RRG) can be equivalently constructed in the traditional amplitude approach in MRK [35]. One starts from the M-loop ladder diagram, which contains $M+1$ vertical graviton propagators and thus carries the usual eikonal factor $1/(M+1)!$ from the symmetrization of identical exchanges. The M horizontal soft gravitons are then inserted by connecting distinct pairs of vertical lines, with no repeated pairs in order to exclude vertical graviton loops, which are absent in the classical limit. The number of available pairs is $N_{\text{pairs}} = \binom{M+1}{2}$, so the number of topologically distinct multi-H topologies is

$$C(N_{\text{pairs}}, M) = \frac{N_{\text{pairs}}!}{(N_{\text{pairs}}-M)!}. \qquad (14)$$

In addition, the soft graviton phase space with strong rapidity ordering $y_1 \gg y_2 \gg \ldots \gg y_M$ gives

$$\frac{\log^M(s)}{M!} = \int \mathrm{d}y_1\cdots\mathrm{d}y_M\Theta(y_1>y_2>\cdots>y_M), \qquad (15)$$

where the factor $1/M!$ arises from cutting M identical soft gravitons, i.e., from restricting to the ordered rapidity domain. In (15) and throughout, the argument of the logarithm is understood to be s/μ^2, with μ^2 a scale of order $q_\perp^2$. In the Regge limit, these soft graviton exchanges are placed on shell and thus correspond to genuine multiparticle intermediate states. Their effect is to generate Regge cuts, which may be understood as arising from overlapping discontinuities in multiparticle kinematics. We now evaluate explicitly the contributions from the single and double-H diagrams.

Single H diagram: Here, we revisit the computation of the H diagram originally carried out in Ref. [27], where it was used—together with real analyticity and crossing symmetry—to extract the full 3PM eikonal in the ultrarelativistic limit. More recently, this analysis was extended to the massive case [44] and employed as a tool to obtain the 3PM eikonal in $\mathcal{N}=8$ supergravity [37] (see also Refs. [58–68] for related studies at 3PM order).

The s-channel discontinuity of the two-loop $2\to 2$ amplitude is proportional to the imaginary part of the amplitude, which is determined by the three-particle cut involving the square of two tree-level $2\to 3$ amplitudes,

$$2\,\mathrm{Im}\,\mathcal{M}^{(2)}_{2\to2}(s,q^2) = \int \mathrm{d}\mathcal{P}_3 |\mathcal{M}^{(0)}_{2\to3}|^2,$$
$$\mathrm{d}\mathcal{P}_3 = \mathrm{d}\Phi(k_0,k_1,k_2)\hat{\delta}^{(D)}\times\left(p_1+p_2+\sum_{i=0}^{2}k_i\right), \quad (16)$$

with the phase space measure defined as $\mathrm{d}\Phi(k_0,k_1,k_2) = \prod_{i=0}^{2}\hat{\mathrm{d}}^D k_i \delta^+(k_i^2)$ and where the on-shell momenta satisfy $p_1^2 = k_0^2 = p_4^2 = 0$, $p_2^2 = k_2^2 = p_3^2 = 0$ and $k_1^2 = 0$. Using the conventions in Fig. 1, the overall momentum conservation implies $p_4 = -p_1 - q$ and $p_3 = -p_2 + q$ and thus $q_3 = q - q_1$ and $q_4 = q - q_2$, with $t = q^2 \simeq -q_\perp^2$.

Combining MRK approximations for the tree-level $2\to3$ amplitudes and for the three-particle phase space [1],

$$\int \mathrm{d}\mathcal{P}_3 \simeq \frac{1}{4s}\left(\frac{\log(s)}{2\pi}\right)\int_{q_{1\perp},q_{2\perp}}, \quad (17)$$

one can integrate out the rapidities obtaining

$$2\,\mathrm{Im}\,\mathcal{M}^{(2)}_{2\to2}(s,q^2) \simeq \frac{(8\pi G_N)^3 s^3}{8\pi}\log(s) H_1(q_\perp^2), \quad (18a)$$

$$H_1(q_\perp^2) \equiv \varsigma^{2\epsilon}\int_{q_{1\perp},q_{2\perp}} \frac{\mathcal{K}^{\mathrm{GR}}(q_1,q_2;q)}{q_{1\perp}^2 q_{2\perp}^2 (q-q_1)_\perp^2 (q-q_2)_\perp^2}. \quad (18b)$$

The term $\varsigma = \mu^2\exp(\gamma_E)$ fixes the renormalization scheme. Notice that the appearance of $\log(s)$ is entirely due to the integration over the soft graviton's rapidity and that in MRK the computation of $2\,\mathrm{Im}\,\mathcal{M}^{(2)}_{2\to2}(s,q^2)$ drastically simplifies from a two-loop four-point integral in D dimensions to a two-loop massless two-point integral in $d = D-2$ dimensions, graphically represented in Fig. 1, which is significantly easier. By power counting, the result must be proportional to $(q_\perp^2)^{d-2}$.

Using integration-by-parts (IBPs) [69], one can reduce $H_1(q_\perp^2)$ to two simple master integrals which are product and iterations of bubbles in (A1a). The ϵ expansion of the discontinuity reads in momentum and impact-parameter space, respectively,

$$2\,\mathrm{Im}\,\mathcal{M}^{(2)}_{2\to2}(s,q^2) = 8G_N^3 s^3 \log(s)\left(\frac{4\pi\mu^2}{q^2}\right)^{2\epsilon}\times\left(-\frac{1}{\epsilon^2}+\frac{2}{\epsilon}+\zeta_2+\mathcal{O}(\epsilon^0)\right), \quad (19a)$$

$$2\,\mathrm{Im}\,\tilde{\mathcal{M}}^{(2)}_{2\to2}(s,b^2) = \frac{8G_N^3 s^2}{\pi b^2}\log(s)(\pi b^2 \varsigma e^{\gamma_E})^{3\epsilon}\times\left(-\frac{1}{\epsilon}+2+\mathcal{O}(\epsilon^0)\right). \quad (19b)$$

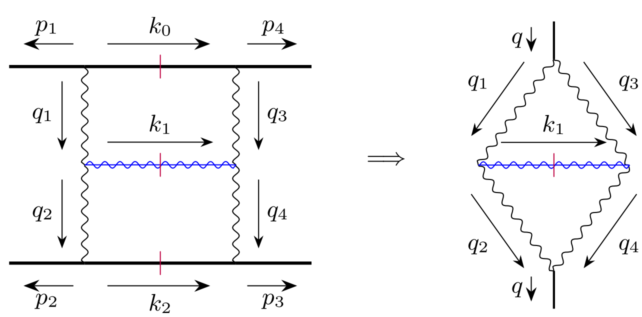

FIG. 1. Vertical and horizontal wavy lines denote Glauber and soft gravitons, respectively. Integrating over rapidities contracts the massive lines to effective vertices through which the momentum q flows.

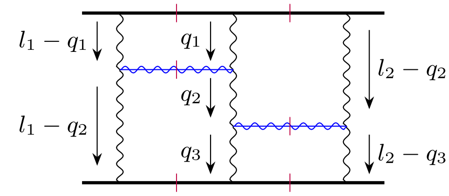

FIG. 2. Momenta parametrization for H_2.

Notice that this identically agrees with Eq. (3.7) of Ref. [44].

Double-H diagrams: In general, at four loops, there are six diagrams contributing to the 5PM amplitude at leading logarithmic accuracy, in agreement with the counting (14). However, it turns out they are all equal and equivalent to the one in Fig. 2, where two horizontal soft gravitons are simultaneously cut in the Regge limit. Such equivalence is a direct consequence of target-projectile symmetry and factorization properties of the high-energy scattering. Using again the MRK approximation and accounting for both the eikonal symmetry factor from the Glauber graviton exchanges and the rapidity-ordering factor from the two soft emissions, one finds

$$2\mathrm{Re}\mathcal{M}^{(4)}_{2\to2}(s,q^2) \simeq -\frac{(8\pi G_N)^5 s^4}{64\pi^2}\log^2(s) H_2(q_\perp^2), \quad (20a)$$

$$H_2(q_\perp^2) \equiv \varsigma^{4\epsilon}\int_{l_{1\perp},l_{2\perp},q_{2\perp}} J(q_2,l_1) q_2^2 J(q_2,l_2)\times\hat{\delta}^{(d)}(l_{1\perp}+l_{2\perp}+q_\perp-q_{2\perp}), \quad (20b)$$

where we used the momentum parametrization in Fig. 2 and we have defined

$$J(q,l) = \int_{k_\perp}\frac{\mathcal{K}^{\mathrm{GR}}(k,q;l)}{k_\perp^2(k-l)_\perp^2 q_\perp^2 (q-l)_\perp^2} = \sum_{i=1,2} J_i(q,l), \quad (21)$$

with $J_i(q,l)$ given in Appendix A. Therefore, one can decompose $H_2(q_\perp^2) = H_2^{(A)}(q_\perp^2) + H_2^{(B)}(q_\perp^2)$ where we define

$$H_2^{(A)}(q_\perp^2) = \varsigma^{4\epsilon}\int_{l_{1\perp},l_{2\perp},q_{2\perp}} \hat{\delta}^{(d)}(l_{1\perp}+l_{2\perp}+q_\perp-q_{2\perp}) \times q_2^2\big(J_2(q_2,l_1)J_2(q_2,l_2) + 2J_1(q_2,l_1)J_2(q_2,l_2)\big), \tag{22}$$

$$H_2^{(B)}(q_\perp^2) = \varsigma^{4\epsilon}\int_{l_{1\perp},l_{2\perp},q_{2\perp}} \hat{\delta}^{(d)}(l_{1\perp}+l_{2\perp}+q_\perp-q_{2\perp}) \times q_2^2 J_1(q_2,l_1)J_1(q_2,l_2). \tag{23}$$

The first integral $H_2^{(A)}(q_\perp^2)$ can be solved in terms of bubbles (A1a) and tensor bubbles (A2) integrals; see Appendix A for further details. Interestingly, the second integral $H_2^{(B)}(q_\perp^2)$ is proportional to

$$H_2^{(B)}(q_\perp^2) \propto \int_{l_{1\perp},l_{2\perp}} \frac{(l_{1\perp}^2)^{\frac{d}{2}}(l_{2\perp}^2)^{\frac{d}{2}}}{(q-l_1)_\perp^2(q-l_2)_\perp^2(q-l_1-l_2)_\perp^2}, \tag{24}$$

which is a scalar kite topology with noninteger exponents, similar to the one found at four loops in QCD [70]. As shown in Appendix A, it can be solved [71] in d dimensions using the Gegenbauer polynomial technique [72–74] in terms of the hypergeometric function ${}_3F_2$.

Summing together the six diagrams contributing to the 5PM order, we get the following ϵ-expanded results:

$$2\mathrm{Re}\mathcal{M}_{2\to2}^{(4)}(s,q^2) \simeq -4G_N^5 s^4\log^2(s)\frac{q^2}{\pi}\left(\frac{4\pi\mu^2}{q^2}\right)^{4\epsilon} \times\left[-\frac{1}{\epsilon^3}-\frac{1}{\epsilon^2}-\frac{9}{\epsilon}+\frac{2}{\epsilon}\zeta_2+\frac{2}{\epsilon}\zeta_3+O(1)\right], \tag{25}$$

$$2\mathrm{Re}\tilde{\mathcal{M}}_{2\to2}^{(4)}(s,b^2) \simeq -256G_N^5\frac{s^3\log^2(s)}{(\pi b^2)^2}(\pi b^2\varsigma e^{\gamma_E})^{5\epsilon} \times\left[\frac{1}{8\epsilon^2}-\frac{1}{\epsilon}+\frac{5}{2}+\frac{5}{16}\zeta_2-\frac{1}{4}\zeta_3+O(\epsilon)\right]. \tag{26}$$

V. PREDICTING SUB-LEADING LOGS FROM LEADING LOGS THROUGH ANALYTICITY AND CROSSING

The computation of the leading classical double logarithm at 5PM allows us to extract further information on the amplitude at the same loop order by using $s\leftrightarrow u$ crossing-symmetric dispersion relations. In particular, the result we obtained is related to the single classical logarithm in the imaginary part of $\mathcal{M}_{2\to2}^{(4)}(s,t)$, which will be our starting point to get information on the eikonal phase. Crossing-symmetric constraints of this type were first identified at 3PM in the Regge limit [27] and later extended to massive scattering in Ref. [44]. It was further shown in QCD [75] that crossing symmetry dictates how imaginary terms appear in the leading expansion in t/s. In particular, when logarithms are written in the combination $\log|s/t| - i\pi/2$, their coefficients are real (imaginary) for amplitudes with minus (plus) crossing signature. Here, we generalize the result of Ref. [75] to gravity, or equivalently extend the analysis of Ref. [27] to higher PM orders.

Since we consider scalar scattering, the amplitude has positive crossing signature. As discussed above, the scaling of the gravitational coupling as Gs implies that we must retain all orders in t/s. For this reason, we replace s/t by the crossing-covariant variable $z_t = 1+2s/t$, whose powers encode the correct transformation properties under $s\leftrightarrow u$ crossing. The same variable enters the natural crossing-symmetric definition of the logarithm, $L\equiv(\log(-z_t)+\log(z_t))/2 = \log|z_t| - i\pi/2$. In Appendix B, we use dispersion relations and crossing symmetry to derive the following form of the amplitude,

$$\mathcal{M}_{2\to2}^{(\ell)} = \frac{G_N^{\ell+1}(tz_t)^{\ell+2}}{t}\sum_{j=0}^{\infty}\sum_{k=0}^{\min(\ell,j)} i^{\ell+j} f_{(j,k)}^{(\ell)}(t) z_t^{-j} L^k, \tag{27}$$

where the real coefficients $f_{(j,k)}^{(\ell)}$ scale as $(\mu^2/t)^{\ell\epsilon}$. In this representation, all s-dependence is explicit, appearing only through L or the overall powers of z_t. Combining this structure with the assumption of eikonalization,[3] we obtain a prediction for the single radiative logarithm.

To do so, it is useful to expand δ_{Cl} and Δ_Q in powers of $\log(s/|t|)$. The same counting rules for the amplitude apply to these at each order in G_N. In particular, for the 3PM and 5PM classical phases, we write

$$\delta_{\mathrm{Cl}}^{(2j)} = \sum_{k=0}^{j}\delta_{\mathrm{Cl}}^{(2j),k}(b)\log^k(s/|t|), \tag{28}$$

with a similar expansion for Δ_Q. Equating Eqs. (27) and (3), we may solve for the unknown coefficients $f_{(j,k)}^{(\ell)}$ in terms of δ_{Cl} and Δ_Q. Doing so then gives the relations between the real and imaginary parts of the classical eikonal phase at two loops,

$$\mathrm{Re}\delta_{\mathrm{Cl}}^{(2),0} = \frac{\pi}{2}\mathrm{Im}\,\delta_{\mathrm{Cl}}^{(2),1} - 8\epsilon^2(\delta_{\mathrm{Cl}}^{(0)})^3, \tag{29}$$

and at four loops,

$$\mathrm{Im}\,\delta_{\mathrm{Cl}}^{(4),1} = 16\epsilon(1-4\epsilon)(\delta_{\mathrm{Cl}}^{(0)})^2\,\mathrm{Im}\,\delta_{\mathrm{Cl}}^{(2),1} - \pi\mathrm{Re}\delta_{\mathrm{Cl}}^{(4),2}. \tag{30}$$

Such a relation at 3PM has been previously derived in Refs. [27,44], and it is identical to Eq. (29) if one uses the

[3] As noted, for the massless case of interest, the known violations of strict eikonalization do not affect our result.

fact that $\delta_{\rm Cl}^{(2j)} \sim (b^2)^{(2j+1)\epsilon}$ in dimensional regularization. It should be mentioned that the formulas given above are only valid in dimensional regularization; however, expressions valid for other IR regulators may be obtained from the procedure described above. We also note that in this work we have only obtained

$$\mathrm{Re}\delta_{\rm Cl}^{(4),2} = \frac{1}{2}\mathrm{Re}\tilde{\mathcal{M}}_{2\to 2}^{(4)}(s,b^2) \tag{31}$$

from the above calculations at 5PM using (3). We then obtain $\mathrm{Im}\,\delta_{\rm Cl}^{(4),1}$ from the above relations (30). Using the known results for $\delta_{\rm Cl}^{(2)}$ to $O(\epsilon)$ in Refs. [23,55], we find

$$\mathrm{Im}\,\delta_{\rm Cl}^{(4),1} = \frac{16}{\pi}(\pi b^2 \varsigma e^{\gamma_E})^{5\epsilon}\left[-\frac{1}{\epsilon} - \zeta_3 + 6 + O(\epsilon)\right]. \tag{32}$$

It is curious that, similar to the case at 3PM, there is a cancellation between the leading $1/\epsilon^2$ divergences in the dispersion relation between the two terms. Although we derived Eq. (32) using only analyticity and crossing symmetry, it would be interesting to rederive it independently through an MRK discontinuity cut or the RRGE method described above. We leave this question for future investigations.

It is important to note that this result along with (26) must agree with the high-energy limit of the massive case. In the EFT, this is expected: the leading logarithms are fixed entirely by the soft sector, which is insensitive to the particle masses, while subleading contributions from collinear radiation do depend on the mass.

Finally, the EFT also implies additional consistency relations for rapidity anomalous dimensions [76]. In QCD, an infinite tower of relations between soft functions with different indices was derived in Ref. [57], and analogous structures are expected to hold in gravity.

VI. CONCLUSIONS

The behavior of gravitational amplitudes in the Regge limit encodes rich physics, both theoretically—as an effective field theory description—and phenomenologically, as it underlies resummations relevant to the two-body problem in the ultrarelativistic regime. Yet, compared to gauge theory where Regge methods have long been a powerful tool for strong interactions, the gravitational case remains largely unexplored at higher-loop order.

Building on Lipatov's seminal work, the eikonal formalism, and recent advances in soft-collinear effective theory, we have initiated a systematic study of classical leading-logarithmic contributions to the $2 \to 2$ amplitude of massive particles in gravity. The resulting series of multi-H diagrams is the main subject of this work.

We begin by revisiting the imaginary part of the single-H diagram at 3PM order, confirming previous results, and we then compute for the first time the leading-logarithmic contribution of the real part of the double-H diagram at 5PM order. This requires the evaluation of a nontrivial topology involving the massless two-point function at higher-loop order, which also appears in the calculation of four-loop anomalous dimensions in QCD [70]. For all diagrams, the SCET framework and the multi-Regge expansion produce the same integrand representation, showing complete agreement.

Finally, we develop a new set of dispersion relations for the gravitational scattering amplitude in the Regge limit, combining signature symmetry in the ultrarelativistic regime [75] with eikonal exponentiation [27,44], to extract some pieces of the ultrarelativistic limit of the classical eikonal phase. In this way, we recover both the real and imaginary parts of the 3PM eikonal phase in the ultrarelativistic limit found in Refs. [27,55] and determine the leading 5PM real contribution from the double-H diagrams together with the subleading logarithmic correction to the imaginary part. Despite the fact that we work in the massless limit, our results are valid in the high-energy limit of the massive case at 5PM order.

Our work opens several avenues for future investigation. An immediate direction is the study of the triple-H diagram and higher-loop contributions, with the long-term goal of achieving an all-order resummation of the $2 \to 2$ amplitude in the Regge limit. Moreover, our results provide useful input to the gravitational S-matrix bootstrap program [38,43], as the Regge-cut contributions directly inform the nonperturbative analytic structure of the elastic amplitude. We also plan to study the contribution of these effects to the corresponding classical observables, such as the scattering angle, and to extend the methods presented here beyond the leading-logarithmic approximation, paving the way for a systematic resummation in the ultrarelativistic regime.

Finally, the high-energy limit explored here may also find phenomenological applications. In particular, it could inform self-force and effective-one-body (EOB)-based resummations of scattering observables [77,78], where logarithmic growth at high energy [79,80] challenges current resummation schemes for the scattering angle when compared to numerical simulations [81,82]. Our EFT approach suggests a physically motivated resummation strategy that systematically organizes these logarithms, potentially shedding light on the high-energy puzzle [30,83–89]. We hope to come back to this problem in the near future.

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DATA AVAILABILITY

No data were created or analyzed in this study, all the results (the final equations, no numerical data or plots) are available in this paper.

APPENDIX A: DETAILS OF THE MASTER INTEGRAL EVALUATION AND DISPERSION RELATIONS

1. Relevant master integrals

The *generalised bubble* integrals are defined as

$$\Lambda_{a,b}(q_\perp) = \int_{l_\perp} \frac{1}{[(q-l)_\perp^2]^a [l_\perp^2]^b} = c_{a,b} (q_\perp^2)^{\frac{d}{2}-a-b},$$
$$c_{a,b} = \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(a+b-\frac{d}{2})}{\Gamma(a)\Gamma(b)} \frac{\Gamma(\frac{d}{2}-a)\Gamma(\frac{d}{2}-b)}{\Gamma(d-a-b)}, \qquad \text{(A1a)}$$

whereas the *vector bubble* integrals are

$$\Lambda^i_{a,b}(q_\perp) = \int_{l_\perp} \frac{l^i_\perp}{[(q-l)_\perp^2]^a [l_\perp^2]^b} = \frac{q^i_\perp}{2}\left[\Lambda_{a,b}(q_\perp) + \frac{1}{q_\perp^2}(\Lambda_{a,b-1}(q_\perp) - \Lambda_{a-1,b}(q_\perp))\right]. \qquad \text{(A2)}$$

Using Eq. (A1a), one gets for $J(q,l)$ in (21)

$$J(q,l) = J_1(q,l) + J_2(q,l), \qquad \text{(A3)}$$

$$J_1(q,l) = c_{1,1} \frac{(l_\perp^2)^{\frac{d}{2}}}{q_\perp^2 (q-l)_\perp^2}, \qquad \text{(A4)}$$

$$J_2(q,l) = \left[\frac{d}{2} - \frac{d}{2}\frac{l_\perp^2}{q_\perp^2} - \left(1-\frac{d}{2}\right)\frac{(l-q)_\perp^2}{q_\perp^2}\right]\Lambda_{1,1}(l_\perp - q_\perp) + (q_\perp \leftrightarrow l_\perp - q_\perp). \qquad \text{(A5)}$$

The result for $H_2^{(A)}(q_\perp^2)$ in Eq. (22) can be evaluated using the bubble integrals above, getting

$$H_2^{(A)}(q) = \frac{\varsigma^{4\epsilon}}{(q_\perp^2)^3}\left(\frac{q_\perp^2}{4\pi}\right)^{2d} g^{(d)}\left[f_1^{(d)} + f_2^{(d)}\left(h_a^{(d)} + h_b^{(d)}\right)\right],$$

with the coefficients

$$g^{(d)} = \frac{\pi 2^{5-4d}\Gamma(\frac{d}{2})^2}{2\Gamma(\frac{d-1}{2})^2\Gamma(\frac{5d}{2}-3)}, \qquad \text{(A6)}$$

$$f_1^{(d)} = 4^{d+1}\left(d(5d^2+d-9)-6\right) \times \Gamma(3-2d)\Gamma(d-2)^2\Gamma\left(\frac{d}{2}\right), \qquad \text{(A7)}$$

$$f_2^{(d)} = \Gamma\left(1-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}-1\right), \qquad \text{(A8)}$$

$$h_a^{(d)} = -2^{2d+1}\left(d(2d-1)-4\right) \times \frac{\Gamma(1-2d)\Gamma(1-d)\Gamma(\frac{d}{2}-1)\Gamma(d)\Gamma(2d)}{\Gamma(1-\frac{3d}{2})\Gamma(\frac{3d}{2})}, \qquad \text{(A9)}$$

$$h_b^{(d)} = -\frac{1}{\sqrt{\pi}}3\left((d-3)d+4\right)\left(d(3d-2)-4\right) \times \Gamma\left(\frac{3}{2}-d\right)\Gamma\left(1-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{3d}{2}-3\right). \qquad \text{(A10)}$$

Both $H_2^{(A)}(q)$ and $H_2^{(B)}(q)$ exhibit a leading $1/\epsilon^4$ behavior that cancels out in their sum in Eq. (25). We express the integral in (24) as $H_2^{(B)}(q_\perp^2) = \varsigma^{4\epsilon}(c_{1,1})^2 K(-d/2, 1, -d/2, 1, 1)$, making use of a generalized massless kite topology K defined as

$$K(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5) = \int_{l_{1\perp},l_{2\perp}} \frac{(l_{1\perp})^{-2\nu_1}(l_{2\perp})^{-2\nu_3}}{[(l_2-q)_\perp]^{2\nu_2}[(l_1-q)_\perp]^{2\nu_4}[(l_1+l_2-q)_\perp]^{2\nu_5}} \qquad \text{(A11)}$$

and schematically represented in Fig. 3. The integral K has been solved in Ref. [71] using the Gegenbauer polynomial technique [73,74] and in Ref. [90] through a Mellin-Barnes representation. In Ref. [90], the authors proved that the expansion for $\epsilon \to 0$ of such integral involves only rational numbers and multiple Zeta values. Using result in Appendix B of Ref. [91], we obtain

$$H_2^{(B)}(q) = \frac{\varsigma^{4\epsilon}}{(q_\perp^2)^3}\left(\frac{q_\perp^2}{4\pi}\right)^{2d}\left[\bar{f}_1^{(d)} + \bar{f}_2^{(d)}\left(\bar{h}_a^{(d)} + \bar{h}_b^{(d)}\right)\right], \qquad \text{(A12)}$$

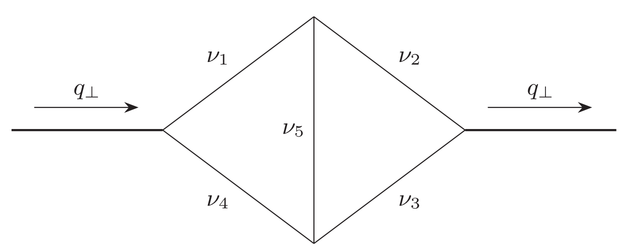

FIG. 3. The scalar massless kite integral topology K.

with the following coefficients:

$$\bar{f}_1^{(d)} = \frac{8\Gamma(1-\frac{3d}{2})\Gamma(1-d)\Gamma(1-\frac{d}{2})\Gamma(\frac{d}{2})^4\Gamma(d)\Gamma(\frac{3d}{2})}{\Gamma(d-1)\Gamma(\frac{5d}{2}-3)}, \qquad \bar{f}_2^{(d)} = \frac{(d-2)\Gamma(1-\frac{d}{2})^5\Gamma(\frac{d}{2})^5}{\Gamma(2-\frac{d}{2})^4\Gamma(-\frac{d}{2})},$$
$$\bar{h}_a^{(d)} = -\frac{(d-2)^2\Gamma(1-2d)\Gamma(2d)\,{}_3F_2(1,d-2,\frac{3d}{2}-2;1-\frac{d}{2},\frac{3d}{2}-1;1)}{(3d-4)\Gamma(2d-2)\Gamma(\frac{5d}{2}-3)},$$
$$\bar{h}_b^{(d)} = -\frac{\Gamma(1-d)^2\Gamma(1-\frac{d}{2})\Gamma(\frac{d}{2})\Gamma(d)^2\,{}_3\tilde{F}_2(d-2,\frac{3d}{2}-2,\frac{5d}{2}-3;\frac{3d}{2}-1,2d-3;1)}{\Gamma(1-\frac{3d}{2})\Gamma(d-2)\Gamma(\frac{3d}{2})}. \tag{A13}$$

APPENDIX B: DISPERSION RELATION CONSTRAINTS FOR GRAVITY AMPLITUDES IN THE REGGE LIMIT

The fixed-t dispersion relation in Mellin space for a crossing-symmetric amplitude is given by

$$\mathcal{M}_{2\to2}(s,t) = \frac{1}{\pi}\int_{\gamma-i\infty}^{\gamma+i\infty} d\omega \frac{a(t,\omega)}{\sin(\pi\omega)} \times\left[\left(\frac{-s-i\epsilon}{-t}\right)^{\omega} + \left(\frac{s+t-i\epsilon}{-t}\right)^{\omega}\right], \tag{B1}$$

where we have defined the ϵ'-regularized Mellin transform of the s-channel discontinuity,

$$a(t,\omega) = \int_{-t}^{\infty}\frac{ds'}{s'}\mathrm{Disc}_s\mathcal{M}_{2\to2}(s',t)\left(\frac{s'}{-t}\right)^{\omega}e^{-\epsilon' s'}, \tag{B2}$$

and we have taken the subtraction point to be $-t$. The poles in ϵ' correspond to local (real) counterterms, which play no role in our analysis since we make no claims regarding contact interactions.

Now, we change the variables from (s,t) to (z_t,t), where

$$z_t = \frac{2s}{t} + 1 \tag{B3}$$

is crossing odd and rewrite the dispersion relation in terms of the crossing even $L \equiv \log|z_t| - i\pi/2$,

$$\mathcal{M}_{2\to2}(z_t,t) = \frac{1}{\pi}\int_{\gamma-i\infty}^{\gamma+i\infty} d\omega \frac{a(t,\omega)}{\sin(\pi\omega)} e^{\omega L} \times\left\{\cos\left(\frac{\pi\omega}{2}\right)\left[\left(\frac{1+1/z_t}{2}\right)^{\omega} + \left(\frac{1-1/z_t}{2}\right)^{\omega}\right] + i\sin\left(\frac{\pi\omega}{2}\right)\left[\left(\frac{1-1/z_t}{2}\right)^{\omega} - \left(\frac{1+1/z_t}{2}\right)^{\omega}\right]\right\}, \tag{B4}$$

where $a(t,\omega)$ is now written as a polynomial in z_t. We can then rewrite the dispersion relation as

$$\mathcal{M}_{2\to2}(z_t,t) = \frac{1}{\pi}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{d\omega}{2^{\omega}}\frac{a(t,\omega)}{\sin(\pi\omega)}e^{\omega L} \times\left[\cos\left(\frac{\pi\omega}{2}\right)\sum_n\binom{\omega}{2n}|z_t|^{-2n} + i\sin\left(\frac{\pi\omega}{2}\right)\sum_n\binom{\omega}{2n+1}|z_t|^{-2n-1}\right]. \tag{B5}$$

This manifestly crossing-invariant representation ties the powers of i in each term to the corresponding power of z_t. To obtain our expression (27), we must establish both the reality properties of the coefficients $f_{(j,k)}^{(\ell)}$ and the structure of the powers of L. The former follows from identifying the locations of the poles in ω, which are fixed by the powers of z_t appearing in the amplitude. A pole at $\omega_0 \in \mathbb{Z}$ contributes a factor $(i|z_t|)^{\omega_0}$ through the exponential in (B5), yielding a real or imaginary term depending on whether ω_0 is even or odd. Since $a(t,\omega)$ is real for $\omega \in \mathbb{R}$, we conclude that even (odd) powers of z_t in the amplitude come with real (imaginary) coefficients.

We now turn to the powers of L. Any power of L appearing in the discontinuity $\mathrm{Disc}_s\mathcal{M}_{2\to2}$ produces a higher-order pole in $a(t,\omega)$ from (B2): a term proportional to L^r in the discontinuity leads to a pole of order $r+1$ in ω. In the dispersion integral (B5), such a pole contributes a factor of L^r to the full amplitude through derivatives acting on $e^{\omega L}$. The EFT power counting implies that the discontinuity can contain at most one logarithm per loop [14,92]. Furthermore, logarithms in $\mathrm{Disc}_s\mathcal{M}_{2\to2}$ arise only from collinear or soft corrections; inserting an additional Glauber exchange does not generate any new logs. So, at ℓ loops, the series will be

$$\mathcal{M}_{2\to2}^{(\ell)} \sim c_0 z_t^{\ell+2} + z_t^{\ell+1}(c_{1,1}L + c_{1,0}) + z_t^{\ell}(c_{2,2}L^2 + c_{2,1}L + c_{2,0}) + \cdots z_t^2(c_{\ell,\ell}L^{\ell} + c_{\ell,\ell-1}L^{\ell-1} + \cdots c_{\ell,0}). \tag{B6}$$

Thus, the maximal power of logarithms at ℓ loops is ℓ. Furthermore, each power z_t^j is accompanied by a polynomial in L whose highest power is $\min(\ell+2-j,\ell)$. Combining this structure with the correlation between powers of z_t and phases of i obtained from (B5) leads to our final expression for the ℓ-loop amplitude, Eq. (27).

Finally, we note that our logarithmic power counting excludes hard-region corrections. As shown in Ref. [21], hard contributions can generate double logarithms, which we have not incorporated here. These double logs are purely quantum and are parametrically suppressed, although at sufficiently high orders in G_N, they may be enhanced to a (super)classical scaling. Their appearance represents yet another manifestation of the breaking of eikonalization.

As a cross-check, we can use this result to reproduce the dispersion-relation formula quoted in Appendix A of Ref. [44], which states that if $\mathrm{Im}\,\mathcal{M}_{2\to2}(s,t)\sim s^n\log^p(s/|t|)$ at leading logarithmic accuracy then

$$\frac{\mathrm{Re}\mathcal{M}_{2\to2}}{\mathrm{Im}\,\mathcal{M}_{2\to2}}=-\frac{2\log(s/|t|)}{(1+p)\pi}+O(\log^0(s/|t|)),\qquad n\text{ even},$$
$$\frac{\mathrm{Re}\mathcal{M}_{2\to2}}{\mathrm{Im}\,\mathcal{M}_{2\to2}}=\frac{\pi p}{2\log(s/|t|)}+O(\log^{-2}(s/|t|)),\qquad n\text{ odd}. \tag{B7}$$

To see that Eq. (27) reproduces this behavior, consider the component of the amplitude scaling as $z_t^nL^p$. Using $z_t^n\propto s^n+O(t/s)$ and $\log(z_t)=\log(s/|t|)+\log(2)+O(t/s)$, we obtain, for even n,

$$\mathcal{M}_{2\to2}\overset{n\text{ even}}{\sim}s^nL^{p+1}=s^n\left(\log^{p+1}(s/|t|)-(p+1)\frac{i\pi}{2}\log^p(s/|t|)+\cdots\right), \tag{B8}$$

while for odd n, we find

$$\mathcal{M}_{2\to2}\overset{n\text{ odd}}{\sim}is^nL^p=s^n\left(i\log^p(s/|t|)+p\frac{\pi}{2}\log^{p-1}(s/|t|)+\cdots\right), \tag{B9}$$

where ellipses denote terms subleading in powers of $\log(s/|t|)$. Taking ratios of the real and imaginary parts in the two cases reproduces Eq. (B7). As a further cross-check on Eq. (27), we have compared it with the existing amplitudes in the literature, specifically the high-energy limit of amplitude for GR and two massive scalars [67,93] and the $\mathcal{N}=8$ supergravity amplitude through two loops [37], and found agreement in all cases.

[1] V. Del Duca, An introduction to the perturbative QCD pomeron and to jet physics at large rapidities, arXiv:hep-ph/9503226.

[2] V. Del Duca and L. J. Dixon, The SAGEX review on scattering amplitudes Chapter 15: The multi-Regge limit, J. Phys. A **55**, 443016 (2022).

[3] H. Raj and R. Venugopalan, QCD-Gravity double copy in Regge asymptotics: From $2\to n$ amplitudes to radiation in shockwave collisions, Acta Phys. Pol. B **56**, 11 (2025).

[4] L. N. Lipatov, Pomeron in quantum chromodynamics, Adv. Ser. Dir. High Energy Phys. **5**, 411 (1989).

[5] L. N. Lipatov, Reggeization of the vector meson and the vacuum singularity in non-Abelian gauge theories, Sov. J. Nucl. Phys. **23**, 338 (1976).

[6] E. A. Kuraev, L. N. Lipatov, and V. S. Fadin, Multi—Reggeon processes in the Yang-Mills theory, Sov. Phys. JETP **44**, 443 (1976).

[7] E. A. Kuraev, L. N. Lipatov, and V. S. Fadin, The Pomeranchuk singularity in non-Abelian gauge theories, Sov. Phys. JETP **45**, 199 (1977).

[8] I. I. Balitsky and L. N. Lipatov, The Pomeranchuk singularity in quantum chromodynamics, Sov. J. Nucl. Phys. **28**, 822 (1978).

[9] L. N. Lipatov, High-energy asymptotics of multicolor QCD and two-dimensional conformal field theories, Phys. Lett. B **309**, 394 (1993).

[10] L. N. Lipatov, Integrability of scattering amplitudes in $N=4$ SUSY, J. Phys. A **42**, 304020 (2009).

[11] L. N. Lipatov, Asymptotic behavior of multicolor QCD at high energies in connection with exactly solvable spin models, JETP Lett. **59**, 596 (1994).

[12] L. D. Faddeev and G. P. Korchemsky, High-energy QCD as a completely integrable model, Phys. Lett. B **342**, 311 (1995).

[13] J.-y. Chiu, A. Jain, D. Neill, and I. Z. Rothstein, The rapidity renormalization group, Phys. Rev. Lett. **108**, 151601 (2012).

[14] J.-Y. Chiu, A. Jain, D. Neill, and I. Z. Rothstein, A formalism for the systematic treatment of rapidity logarithms in quantum field theory, J. High Energy Phys. 05 (2012) 084.

[15] I. Z. Rothstein and I. W. Stewart, An effective field theory for forward scattering and factorization violation, J. High Energy Phys. 08 (2016) 025.

[16] I. Balitsky, Operator expansion for high-energy scattering, Nucl. Phys. **B463**, 99 (1996).

[17] Y. V. Kovchegov, Small-x f_2 structure function of a nucleus including multiple pomeron exchanges, Phys. Rev. D **60**, 034008 (1999).

[18] E. Iancu, A. Leonidov, and L. McLerran, The renormalization group equation for the color glass condensate, Phys. Lett. B **510**, 133 (2001).

[19] P. Di Vecchia, C. Heissenberg, R. Russo, and G. Veneziano, The gravitational eikonal: From particle, string and brane collisions to black-hole encounters, Phys. Rep. **1083**, 1 (2024).

[20] L. N. Lipatov, Graviton reggeization, Phys. Lett. **116B**, 411 (1982).

[21] J. Bartels, L. N. Lipatov, and A. Sabio Vera, Double-logarithms in Einstein-Hilbert gravity and supergravity, J. High Energy Phys. 07 (2012) 056.

[22] S. Melville, S. G. Naculich, H. J. Schnitzer, and C. D. White, Wilson line approach to gravity in the high energy limit, Phys. Rev. D **89**, 025009 (2014).

[23] I. Z. Rothstein and M. Saavedra, A systematic Lagrangian formulation for quantum and classical gravity at high energies, arXiv:2412.04428.

[24] R. Britto, R. Gonzo, and G. R. Jehu, Graviton particle statistics and coherent states from classical scattering amplitudes, J. High Energy Phys. 03 (2021) 214.

[25] A. Cristofoli, R. Gonzo, N. Moynihan, D. O'Connell, A. Ross, M. Sergola, and C. D. White, The uncertainty principle and classical amplitudes, J. High Energy Phys. 06 (2021) 181.

[26] L. N. Lipatov, Multi—Regge processes in gravitation, Sov. Phys. JETP **55**, 582 (1982).

[27] D. Amati, M. Ciafaloni, and G. Veneziano, Higher order gravitational deflection and soft bremsstrahlung in Planckian energy superstring collisions, Nucl. Phys. **B347**, 550 (1990).

[28] D. Amati, M. Ciafaloni, and G. Veneziano, Effective action and all order gravitational eikonal at Planckian energies, Nucl. Phys. **B403**, 707 (1993).

[29] D. Amati, M. Ciafaloni, and G. Veneziano, Towards an S-matrix description of gravitational collapse, J. High Energy Phys. 02 (2008) 049.

[30] M. Ciafaloni, D. Colferai, F. Coradeschi, and G. Veneziano, Unified limiting form of graviton radiation at extreme energies, Phys. Rev. D **93**, 044052 (2016).

[31] M. Driesse, G. U. Jakobsen, G. Mogull, J. Plefka, B. Sauer, and J. Usovitsch, Conservative black hole scattering at fifth post-Minkowskian and first self-force order, Phys. Rev. Lett. **132**, 241402 (2024).

[32] M. Driesse, G. U. Jakobsen, A. Klemm, G. Mogull, C. Nega, J. Plefka, B. Sauer, and J. Usovitsch, Emergence of Calabi–Yau manifolds in high-precision black-hole scattering, Nature (London) **641**, 603 (2025).

[33] C. Dlapa, G. Kälin, Z. Liu, and R. A. Porto, Local-in-time conservative binary dynamics at fifth post-Minkowskian and first self-force orders, Phys. Rev. Lett. **135**, 251401 (2025).

[34] Z. Bern, E. Herrmann, R. Roiban, M. S. Ruf, A. V. Smirnov, V. A. Smirnov, and M. Zeng, Second-order self-force potential-region binary dynamics at $O(G^5)$ in supergravity, Phys. Rev. Lett. **136**, 081401 (2026).

[35] F. Alessio, V. Del Duca, R. Gonzo, and E. Rosi, Gravitational amplitudes in the Regge limit: Waveforms, shock waves and unitarity cuts, arXiv:2601.2168.

[36] P. D. B. Collins, *An Introduction to Regge Theory and High Energy Physics* (Cambridge University Press, Cambridge, England, 1977).

[37] P. Di Vecchia, C. Heissenberg, R. Russo, and G. Veneziano, The eikonal approach to gravitational scattering and radiation at $\mathcal{O}(G^3)$, J. High Energy Phys. 07 (2021) 169.

[38] K. Häring and A. Zhiboedov, What is the graviton pole made of?, arXiv:2410.21499.

[39] S. Caron-Huot, D. Mazáč, L. Rastelli, and D. Simmons-Duffin, Sharp boundaries for the Swampland, J. High Energy Phys. 07 (2021) 110.

[40] S. Caron-Huot, D. Mazáč, L. Rastelli, and D. Simmons-Duffin, AdS bulk locality from sharp CFT bounds, J. High Energy Phys. 11 (2021) 164.

[41] C.-H. Chang and J. Parra-Martinez, Graviton loops and negativity, J. High Energy Phys. 08 (2025) 175.

[42] S. Caron-Huot, Y.-Z. Li, J. Parra-Martinez, and D. Simmons-Duffin, Causality constraints on corrections to Einstein gravity, J. High Energy Phys. 05 (2022) 122.

[43] K. Häring and A. Zhiboedov, Gravitational Regge bounds, SciPost Phys. **16**, 034 (2024).

[44] P. Di Vecchia, C. Heissenberg, R. Russo, and G. Veneziano, Universality of ultra-relativistic gravitational scattering, Phys. Lett. B **811**, 135924 (2020).

[45] C. W. Bauer, S. Fleming, and M. E. Luke, Summing Sudakov logarithms in $B \to X_s \gamma$ in effective field theory., Phys. Rev. D **63**, 014006 (2000).

[46] C. W. Bauer, D. Pirjol, and I. W. Stewart, Soft collinear factorization in effective field theory, Phys. Rev. D **65**, 054022 (2002).

[47] C. W. Bauer, S. Fleming, D. Pirjol, and I. W. Stewart, An effective field theory for collinear and soft gluons: Heavy to light decays, Phys. Rev. D **63**, 114020 (2001).

[48] M. Beneke and G. Kirilin, Soft-collinear gravity, J. High Energy Phys. 09 (2012) 066,

[49] T. Okui and A. Yunesi, Soft collinear effective theory for gravity, Phys. Rev. D **97**, 066011 (2018).

[50] M. Beneke, P. Hager, and R. Szafron, Soft-collinear gravity beyond the leading power, J. High Energy Phys. 03 (2022) 080; 04 (2024) 141(E).

[51] D. A. Kosower, B. Maybee, and D. O'Connell, Amplitudes, observables, and classical scattering, J. High Energy Phys. 02 (2018) 137.

[52] M. Levy and J. Sucher, Eikonal approximation in quantum field theory, Phys. Rev. **186**, 1656 (1969).

[53] R. Akhoury, R. Saotome, and G. Sterman, High energy scattering in perturbative quantum gravity at next to leading power, Phys. Rev. D **103**, 064036 (2021).

[54] Y. Du, S. Ajith, R. Rajagopal, and D. Vaman, Worldline proof of eikonal exponentiation, J. High Energy Phys. 09 (2024) 161.

[55] P. Di Vecchia, S. G. Naculich, R. Russo, G. Veneziano, and C. D. White, A tale of two exponentiations in $\mathcal{N} = 8$ supergravity at subleading level, J. High Energy Phys. 03 (2019) 173.

[56] A. Gao, I. Moult, S. Raman, G. Ridgway, and I. W. Stewart, A collinear perspective on the Regge limit, J. High Energy Phys. 05 (2024) 328.

[57] I. Z. Rothstein and M. Saavedra, Relations between anomalous dimensions in the Regge limit, J. High Energy Phys. 07 (2024) 073.

[58] Z. Bern, C. Cheung, R. Roiban, C.-H. Shen, M. P. Solon, and M. Zeng, Scattering amplitudes and the conservative Hamiltonian for binary systems at third post-Minkowskian order, Phys. Rev. Lett. **122**, 201603 (2019).

[59] T. Damour, Classical and quantum scattering in post-Minkowskian gravity, Phys. Rev. D **102**, 024060 (2020).

[60] J. Parra-Martinez, M. S. Ruf, and M. Zeng, Extremal black hole scattering at $\mathcal{O}(G^3)$: Graviton dominance, eikonal exponentiation, and differential equations, J. High Energy Phys. 11 (2020) 023.

[61] G. Kälin, Z. Liu, and R. A. Porto, Conservative dynamics of binary systems to third post-Minkowskian order from the effective field theory approach, Phys. Rev. Lett. **125**, 261103 (2020).

[62] T. Damour, Radiative contribution to classical gravitational scattering at the third order in G, Phys. Rev. D **102**, 124008 (2020).

[63] E. Herrmann, J. Parra-Martinez, M. S. Ruf, and M. Zeng, Radiative classical gravitational observables at $\mathcal{O}(\mathrm{G}^3)$ from scattering amplitudes, J. High Energy Phys. 10 (2021) 148,

[64] A. Brandhuber, G. Chen, G. Travaglini, and C. Wen, Classical gravitational scattering from a gauge-invariant double copy, J. High Energy Phys. 10 (2021) 118.

[65] Z. Bern, H. Ita, J. Parra-Martinez, and M. S. Ruf, Universality in the classical limit of massless gravitational scattering, Phys. Rev. Lett. **125**, 031601 (2020).

[66] C. Cheung and M. P. Solon, Classical gravitational scattering at $\mathcal{O}(\mathrm{G}^3)$ from Feynman diagrams, J. High Energy Phys. 06 (2020) 144,

[67] N. E. J. Bjerrum-Bohr, P. H. Damgaard, L. Planté, and P. Vanhove, The amplitude for classical gravitational scattering at third post-Minkowskian order, J. High Energy Phys. 08 (2021) 172.

[68] N. E. J. Bjerrum-Bohr, P. H. Damgaard, L. Planté, and P. Vanhove, Classical gravity from loop amplitudes, Phys. Rev. D **104**, 026009 (2021).

[69] R. N. Lee, Presenting LiteRed: A tool for the loop InTEgrals REDuction, arXiv:1212.2685.

[70] S. Caron-Huot, When does the gluon reggeize?, J. High Energy Phys. 05 (2013) 093,

[71] A. V. Kotikov and S. Teber, Two-loop fermion self-energy in reduced quantum electrodynamics and application to the ultrarelativistic limit of graphene, Phys. Rev. D **89**, 065038 (2014).

[72] K. G. Chetyrkin, A. L. Kataev, and F. V. Tkachov, New approach to evaluation of multiloop Feynman integrals: The Gegenbauer polynomial x space technique, Nucl. Phys. **B174**, 345 (1980).

[73] A. V. Kotikov, The Gegenbauer polynomial technique: The evaluation of a class of Feynman diagrams, Phys. Lett. B **375**, 240 (1996).

[74] A. V. Kotikov and S. Teber, Multi-loop techniques for massless Feynman diagram calculations, Phys. Part. Nucl. **50**, 1 (2019).

[75] S. Caron-Huot, E. Gardi, and L. Vernazza, Two-parton scattering in the high-energy limit, J. High Energy Phys. 06 (2017) 016.

[76] I. Z. Rothstein and M. Saavedra, Extracting the asymptotic behavior of S-matrix elements from their phases, J. High Energy Phys. 11 (2023) 155.

[77] T. Damour and P. Rettegno, Strong-field scattering of two black holes: Numerical relativity meets post-Minkowskian gravity, Phys. Rev. D **107**, 064051 (2023).

[78] A. Buonanno, G. Mogull, R. Patil, and L. Pompili, Post-Minkowskian theory meets the spinning effective-one-body approach for bound-orbit waveforms, Phys. Rev. Lett. **133**, 211402 (2024).

[79] C. Dlapa, G. Kälin, Z. Liu, J. Neef, and R. A. Porto, Radiation reaction and gravitational waves at fourth post-Minkowskian order, Phys. Rev. Lett. **130**, 101401 (2023).

[80] D. Bini, T. Damour, and A. Geralico, Radiated momentum and radiation reaction in gravitational two-body scattering including time-asymmetric effects, Phys. Rev. D **107**, 024012 (2023).

[81] S. Swain, G. Pratten, and P. Schmidt, Strong field scattering of black holes: Assessing resummation strategies, Phys. Rev. D **111**, 064048 (2025).

[82] O. Long, H. P. Pfeiffer, A. Buonanno, G. U. Jakobsen, G. Mogull, A. Ramos-Buades, H. R. Rüter, L. E. Kidder, and M. A. Scheel, Highly accurate simulations of asymmetric black-hole scattering and cross validation of effective-one-body models, Phys. Rev. D **112**, 124039 (2025).

[83] P. D. D'Eath, High speed black hole encounters and gravitational radiation, Phys. Rev. D **18**, 990 (1978).

[84] S. J. Kovacs and K. S. Thorne, The generation of gravitational waves. 3. Derivation of bremsstrahlung formulas, Astrophys. J. **217**, 252 (1977).

[85] S. J. Kovacs and K. S. Thorne, The generation of gravitational waves. 4. Bremsstrahlung, Astrophys. J. **224**, 62 (1978).

[86] A. Gruzinov and G. Veneziano, Gravitational radiation from massless particle collisions, Classical Quantum Gravity **33**, 125012 (2016).

[87] M. Ciafaloni, D. Colferai, and G. Veneziano, Infrared features of gravitational scattering and radiation in the eikonal approach, Phys. Rev. D **99**, 066008 (2019).

[88] P. Di Vecchia, C. Heissenberg, R. Russo, and G. Veneziano, The eikonal operator at arbitrary velocities I: The soft-radiation limit, J. High Energy Phys. 07 (2022) 039.

[89] R. Gonzo and A. Ilderton, Wave scattering event shapes at high energies, J. High Energy Phys. 10 (2023) 108.

[90] I. Bierenbaum and S. Weinzierl, The massless two loop two point function, Eur. Phys. J. C **32**, 67 (2003).

[91] A. V. Kotikov and S. Teber, New results for a two-loop massless propagator-type Feynman diagram, Teor. Mat. Fiz. **194**, 331 (2018).

[92] C. Duhr, Mathematical aspects of scattering amplitudes, in *Journeys Through the Precision Frontier: Amplitudes for Colliders* (World Scientific, Singapore, 2015), pp. 419–476.

[93] A. Koemans Collado, P. Di Vecchia, and R. Russo, Revisiting the second post-Minkowskian eikonal and the dynamics of binary black holes, Phys. Rev. D **100**, 066028 (2019).