Applied Categorical Structures (2024) 32:12
https://doi.org/10.1007/s10485-024-09769-w

A DG-Enhancement of $\mathrm{D}_{qc}(X)$ with Applications in Deformation Theory

Francesco Meazzini[1,2]

Received: 11 August 2022 / Accepted: 15 April 2024

Abstract

It is well-known that DG-enhancements of the unbounded derived category $\mathrm{D}_{qc}(X)$ of quasi-coherent sheaves on a scheme X are all equivalent to each other. Here we present an explicit model which leads to applications in deformation theory. In particular, we shall describe three models for derived endomorphisms of a quasi-coherent sheaf $\mathcal{F}$ on a finite-dimensional Noetherian separated scheme (even if $\mathcal{F}$ does not admit a locally free resolution). Moreover, these complexes are endowed with DG-Lie algebra structures, which we prove to control infinitesimal deformations of $\mathcal{F}$.

Keywords Deformation theory · Derived category of quasi-coherent sheaves · Model categories

Mathematics Subject Classification 14F05 · 14D15 · 16W50 · 18G55

Contents

Communicated by Vladimir Hinich.

✉ Francesco Meazzini
francesco.meazzini@uniroma1.it

1 Dipartimento di Matematica, Alma Mater Studiorum Università di Bologna, Piazza di Porta San Donato 5, 40126 Bologna, Italy

2 Dipartimento di Matematica Guido Castelnuovo, Sapienza, Università di Roma, Piazzale Aldo Moro 5, 00185 Rome, Italy

Published online: 08 May 2024

1 Introduction

A classical problem in deformation theory concerns the study of infinitesimal deformations of a quasi-coherent sheaf $\mathcal{F}$ on a scheme X over a field $\mathbb{K}$. Deformations up to isomorphisms define a functor $\mathrm{Def}_{\mathcal{F}}\colon \mathrm{Art}_{\mathbb{K}} \to \mathrm{Set}$, where $\mathrm{Art}_{\mathbb{K}}$ denotes the category of local Artin $\mathbb{K}$-algebras with residue field $\mathbb{K}$. The classical approach is based on a finite locally free resolution $\mathcal{E} \to \mathcal{F}$, which for instance exists provided that X is smooth projective. In fact, a deformation of $\mathcal{F}$ can be understood as the data of local deformations of $\mathcal{E}$ together with suitable gluing conditions. It is proven in [11] that $\mathrm{Def}_{\mathcal{F}}$ is controlled by the DG-Lie algebra of global sections of an acyclic resolution of the sheaf $\mathcal{E}nd^*(\mathcal{E})$ in the sense of [16, 27]. In particular, it is well-known that $T^1\,\mathrm{Def}_{\mathcal{F}} \cong \mathrm{Ext}^1(\mathcal{F},\mathcal{F})$ and obstructions are contained in $\mathrm{Ext}^2(\mathcal{F},\mathcal{F})$. This highlights the considerable role of derived endomorphisms $\mathrm{REnd}(\mathcal{F})$, and the importance of being able to compute its cohomology $\mathrm{Ext}^*(\mathcal{F},\mathcal{F})$. Classically $\mathrm{REnd}(\mathcal{F})$ is defined (up to quasi-isomorphisms) as the complex $\mathrm{Hom}^*_{\mathcal{O}_X}(\mathcal{F},\mathcal{I})$ for any injective resolution $\mathcal{F} \to \mathcal{I}$. Unfortunately, despite the outstanding fact that injective resolutions always exist, it is often very hard to describe them. Here comes the aim of this paper to present another approach to compute $\mathrm{REnd}(\mathcal{F})$ when dealing with concrete geometric situations, always trying to keep the exposition as clear as possible with the attempt to reduce the use of simplicial and model category techniques at minimum.

The main tool is the introduction of the category $\mathrm{Mod}(A.)$ of modules over the diagram $A.$ representing a separated $\mathbb{K}$-scheme X. Fix an open affine covering $\mathcal{U} = \{U_h\}$ for X, then the associated diagram $A.$ with respect to $\mathcal{U}$ is defined as

$$A.\colon \mathcal{N} \to \mathrm{Alg}_{\mathbb{K}}\ , \qquad \alpha \mapsto A_\alpha = \Gamma(U_\alpha, \mathcal{O}_X)$$

where $\mathcal{N} = \{\alpha = \{h_0,\ldots,h_k\} \mid U_\alpha = U_{h_0} \cap \ldots U_{h_k} \neq \emptyset\}$ is the nerve of $\mathcal{U}$. Recently, this way of thinking of a $\mathbb{K}$-scheme X has been used in [31] in order to study infinitesimal deformations of X by virtue of the general theory developed in [30].

An $A.$-module $\mathcal{G}$ can be understood as the following data

(1) a DG-module $\mathcal{G}_\alpha$ over A_α for every α in the nerve $\mathcal{N}$ of $\mathcal{U}$,
(2) a morphism $g_{\alpha\beta}\colon \mathcal{G}_\alpha \otimes_{A_\alpha} A_\beta \to \mathcal{G}_\beta$ of A_β-modules, for every $\alpha \subseteq \beta$ in $\mathcal{N}$,

satisfying the *cocycle condition*, see Definition 3.1. Similar notions were considered in [10, 13, 15, 37]. Taking advantage of the standard projective model structure on DG-modules, the category $\mathrm{Mod}(A.)$ can be endowed with a model structure, see Theorem 3.9, where weak equivalences are pointwise quasi-isomorphisms. The above model structure can be seen as a geometric example of an abstract recent result obtained in [2]. In order to work with quasi-coherent sheaves, we need a homotopical version of quasi-coherence for $A.$-modules: $\mathcal{G}$ is called quasi-coherent if all the maps $g_{\alpha\beta}$ introduced above are quasi-isomorphisms, see Definition 3.12. To the author knowledge the last definition does not appear in the existing literature, a part for the case of non-graded modules for which the theory is carried out in [10, 37]. Now, denote by $\mathrm{Ho}(\mathrm{QCoh}(A.))$ the category of quasi-coherent $A.$-modules localized with respect to the weak equivalences: Theorem 5.7 states that there exists an equivalence of triangulated categories

$$\mathrm{Ho}(\mathrm{QCoh}(A.)) \simeq \mathrm{D}_{qc}(X)$$

with the unbounded derived category of quasi-coherent sheaves on X, hence leading to an explicit description of a DG-enhancement of $D_{qc}(X)$, see Corollary 5.8. It is worth to notice that some of the functors involved in Sect. 5 have been somehow already considered in the literature, see [19, 21]. Moreover a result similar to the equivalence of Theorem 5.7 was partially proven in [7, Proposition 2.28].

In [25] it was shown the uniqueness of DG-enhancements for the derived category of a suitable Grothendieck category up to equivalence. In particular, this applies to $D_{qc}(X)$ under some mild hypothesis on X (e.g. if X is a quasi-projective $\mathbb{K}$-scheme). On the other hand, our construction turns out to be very useful when dealing with derived endomorphisms of a quasi-coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules. In fact, the category of $A_\cdot$-modules allows to easily describe $\mathrm{REnd}(\mathcal{F})$ in terms of a cofibrant replacement of $\mathcal{F}$, see Theorem 6.4. Moreover, Example 3.7 shows the feasibility of the computation of such cofibrant replacement in interesting cases. In Sect. 6 we propose two more models for $\mathrm{REnd}(\mathcal{F})$: the first is again in terms of a cofibrant replacement in the model category of $A_\cdot$-modules and involves the Thom–Whitney totalization, while the other assumes the existence of a locally free resolution for $\mathcal{F}$.

The last section is devoted to our main application in deformation theory; in particular, we deal with the functor $\mathrm{Def}_{\mathcal{F}}\colon \mathrm{Art}_{\mathbb{K}} \to \mathrm{Set}$ of classical infinitesimal deformations of $\mathcal{F}$. Recall that since the eighties the leading principle in deformation theory (due to Quillen, Deligne, Drinfeld, Kontsevich...) states that any deformation problem is controlled by a DG-Lie algebra via Maurer–Cartan solutions modulo gauge equivalence, see [16, 27, 32]. Around 2010 this was formally proven independently by Lurie [26, Theorem 5.3] and Pridham [34, Theorem 4.55]; it is dutiful to mention that partial results in this direction where previously obtained by Hinich and Manetti, see [18, 28, 34] and references therein. In Sect. 7 we adopt this point of view proving that the three complexes representing $\mathrm{REnd}(\mathcal{F})$ described in Sect. 6 are all equipped with a DG-Lie algebra structure, and each of them controls $\mathrm{Def}_{\mathcal{F}}$ via Maurer–Cartan elements modulo gauge equivalence. In particular, we give two proofs of this fact: the first (Sect. 7.1) involves the semicosimplicial machinery together with standard arguments of descent of Deligne groupoid, while the second (Sect. 7.2) relies on a direct computation in $\mathrm{Mod}(A_\cdot)$.

A remarkable fact is that our descriptions of $\mathrm{REnd}(\mathcal{F})$ in terms of $A_\cdot$-modules does not require the existence of a locally free resolution for $\mathcal{F}$, since cofibrant replacements always exist. Hence we recover that $T^1\,\mathrm{Def}_{\mathcal{F}} \cong \mathrm{Ext}^1(\mathcal{F}, \mathcal{F})$ and that obstructions are contained in $\mathrm{Ext}^2(\mathcal{F}, \mathcal{F})$ only assuming X to be a finite-dimensional Noetherian separated $\mathbb{K}$-scheme.

2 Preliminaries and Notation

This brief introductory section aims to fix the geometric framework where we shall work throughout all the paper, and to briefly recall some basic constructions.

We work on a fixed finite-dimensional Noetherian separated scheme X over a field $\mathbb{K}$ of characteristic 0. Actually, the assumption on the characteristic of $\mathbb{K}$ will be necessary only in Sects. 6 and 7, where applications to algebraic geometry will be discussed. For simplicity of exposition we shall work over $\mathbb{K}$ throughout all the paper, although the results of the first sections hold for schemes over $\mathbb{Z}$. Moreover, we fix an open affine covering $\mathcal{U} = \{U_h\}_{h\in H}$ together with its *nerve*

$$\mathcal{N} = \left\{\{h_0, \dots, h_k\} \mid U_{h_0} \cap \cdots \cap U_{h_k} \neq \emptyset\right\}$$

which carries a *degree function* $\deg\colon \mathcal{N} \to \mathbb{N}$ defined by $\deg(\{h_0,\dots,h_k\}) = k$. Moreover, for every $\alpha = \{h_0,\dots,h_k\} \in \mathcal{N}$ we denote by U_α the intersection $U_{h_0} \cap \cdots \cap U_{h_k}$, and define $A_\alpha = \Gamma(U_\alpha, \mathcal{O}_X)$. Each U_α is affine since X is assumed to be separated. The nerve $\mathcal{N}$ is a partially ordered set where $\alpha \le \beta$ if and only if $\alpha \subseteq \beta$; notice that if $\alpha \le \beta$ then $U_\beta \subseteq U_\alpha$ so that there exists a flat map of $\mathbb{K}$-algebras $A_\alpha \to A_\beta$ satisfying $A_\beta \cong A_\beta \otimes_{A_\alpha} A_\beta$. Hence, once we have fixed $\mathcal{U}$, the scheme X can be represented by the diagram

$$A.\colon \mathcal{N} \to \mathrm{Alg}_{\mathbb{K}}\ , \qquad \alpha \mapsto A_\alpha$$

where $A_\alpha \to A_\beta$ is the opposite map of $\mathrm{Spec}(A_\beta) \to \mathrm{Spec}(A_\alpha)$ for every $\alpha \le \beta$ in $\mathcal{N}$.

For any open subset $U \subseteq X$ let $\mathrm{DGMod}(\mathcal{O}_U)$ be the category of unbounded complexes of $\mathcal{O}_U$-modules, and by $\mathrm{QCoh}(U)$ the full subcategory of complexes of quasi-coherent sheaves.

For every inclusion $i\colon U \to V$ between open subsets there are three associated functors:

$$i_!, i_*\colon\ \mathrm{DGMod}(\mathcal{O}_U) \to \mathrm{DGMod}(\mathcal{O}_V), \qquad i^*\colon\ \mathrm{DGMod}(\mathcal{O}_V) \to \mathrm{DGMod}(\mathcal{O}_U).$$

Recall that $i^*\mathcal{G} = \mathcal{G}|_U$ because $\mathcal{O}_V|_U = \mathcal{O}_U$, and $i_!\mathcal{F}$ is the sheaf associated to the presheaf $i(\mathcal{F})$ defined by

$$\begin{cases} i(\mathcal{F})(W) = \mathcal{F}(W) & \text{if } W \subseteq U \\ i(\mathcal{F})(W) = 0 & \text{otherwise.} \end{cases}$$

The obvious retraction $i(\mathcal{F}) \to i_*(\mathcal{F}) \to i(\mathcal{F})$ of presheaves gives a retraction of sheaves $i_!\mathcal{F} \to i_*\mathcal{F} \to i_!\mathcal{F}$ and then a retraction of functors $i_! \to i_* \xrightarrow{r} i_!$. Notice also that for every $\mathcal{G} \in \mathrm{DGMod}(\mathcal{O}_V)$ there exists an injective morphism

$$i_!i^*\mathcal{G} \to \mathcal{G}$$

and therefore a natural morphism given by composition with the retraction r

$$i_*i^*\mathcal{G} \to \mathcal{G}\ ,$$

which is an isomorphism on stalks over every $x \in U$, and 0 over $x \notin U$.

If $\mathcal{F}$ and $\mathcal{G}$ are complexes of quasi-coherent sheaves, then also $i_*\mathcal{F}$ and $i^*\mathcal{G}$ are so, see e.g. [17, Proposition 5.8]. This is not true in general for $i_!\mathcal{F}$, see e.g. [17, Example 5.2.3].

In the above notation, if U is affine then the functor $i_*\colon\ \mathrm{QCoh}(U) \to \mathrm{QCoh}(V)$ is exact.

3 The Model Category of *A*.-Modules

In the following, for every ring R we denote by $\mathrm{DGMod}(R)$ the category of DG-modules over R. As explained in Sect. 2 we denote by $\mathcal{N}$ the nerve of the affine open covering $\mathcal{U}$ of X.

Definition 3.1 An A**.-module** $\mathcal{F}$ over the scheme X (with respect to the fixed covering $\mathcal{U}$) consists of the following data:

(1) an object $\mathcal{F}_\alpha \in \mathrm{DGMod}(A_\alpha)$, for every $\alpha \in \mathcal{N}$,
(2) a morphism $f_{\alpha\beta}\colon \mathcal{F}_\alpha \otimes_{A_\alpha} A_\beta \to \mathcal{F}_\beta$ in $\mathrm{DGMod}(A_\beta)$, for every $\alpha \le \beta$ in $\mathcal{N}$,

satisfying the *cocycle condition* $f_{\beta\gamma} \circ \left(f_{\alpha\beta} \otimes_{A_\beta} \mathrm{Id}_{A_\gamma}\right) = f_{\alpha\gamma}$, for every $\alpha \le \beta \le \gamma$ in $\mathcal{N}$.

In the setting of Definition 3.1, the data of the map $f_{\alpha\beta}\colon \mathcal{F}_\alpha \otimes_{A_\alpha} A_\beta \to \mathcal{F}_\beta$ in $\mathrm{DGMod}(A_\beta)$ is equivalent to its adjoint morphism $\mathcal{F}_\alpha \to \mathcal{F}_\beta$ in $\mathrm{DGMod}(A_\alpha)$, where the A_α-module structure on $\mathcal{F}_\beta$ is induced via $A_\alpha \to A_\beta$.

For instance, to any sheaf $\mathcal{G}$ of $\mathcal{O}_X$-modules it is associated the $A.$-module $\Upsilon^*\mathcal{G}$ defined as

$$\left(\Upsilon^*\mathcal{G}\right)_\alpha = \mathcal{G}(U_\alpha) \in \mathrm{DGMod}(A_\alpha) \qquad \text{and} \qquad g_{\alpha\beta}\colon \mathcal{G}(U_\alpha)\otimes_{A_\alpha} A_\beta \to \mathcal{G}(U_\beta)$$

for every $\alpha \leq \beta$ in $\mathcal{N}$, where the map $g_{\alpha\beta}$ is induced by the restriction map of the sheaf $\mathcal{G}$.

Definition 3.2 A **morphism of $A.$-modules** $\varphi\colon \mathcal{F} \to \mathcal{G}$ over X consists of the following data:

(1) a morphism $\varphi_\alpha\colon \mathcal{F}_\alpha \to \mathcal{G}_\alpha$ in $\mathrm{DGMod}(A_\alpha)$, for every $\alpha \in \mathcal{N}$,
(2) for every $\alpha \leq \beta$ in $\mathcal{N}$, the diagram

$$\begin{array}{ccc} \mathcal{F}_\alpha \otimes_{A_\alpha} A_\beta & \xrightarrow{\varphi_\alpha} & \mathcal{G}_\alpha \otimes_{A_\alpha} A_\beta \\ \downarrow{\scriptstyle f_{\alpha\beta}} & & \downarrow{\scriptstyle g_{\alpha\beta}} \\ \mathcal{F}_\beta & \xrightarrow[\varphi_\beta]{} & \mathcal{G}_\beta \end{array}$$

commutes in $\mathrm{DGMod}(A_\beta)$.

The set of morphisms between $\mathcal{F}$ and $\mathcal{G}$ is denoted by $\mathrm{Hom}_{A.}(\mathcal{F}, \mathcal{G})$.

Recall that for any ring R and any pair of DG-modules $M, N \in \mathrm{DGMod}(R)$ it is defined total-Hom complex $\mathrm{Hom}^*_R(M, N)$ as follows:

$$\mathrm{Hom}^p_R(M, N) = \prod_{n\in\mathbb{Z}} \mathrm{Hom}_R(M^n, N^{n+p})\ ,$$
$$\partial^p_{\mathrm{Hom}}\colon (f^n)_{n\in\mathbb{Z}} \mapsto (f^{n+1}d^n_N - (-1)^p d^{n+p}_N f^n)_{n\in\mathbb{Z}}\ .$$

Definition 3.3 The set of $*$**-morphisms** between $A.$-modules $\mathcal{F}$ and $\mathcal{G}$ over X is defined by:

$$\mathrm{Hom}^*_{A.}(\mathcal{F}, \mathcal{G}) \subseteq \prod_{\alpha\in\mathcal{N}} \mathrm{Hom}^*_{A_\alpha}(\mathcal{F}_\alpha, \mathcal{G}_\alpha)$$

where $\{\varphi_\alpha\}_{\alpha\in\mathcal{N}}$ belongs to $\mathrm{Hom}^*_{A.}(\mathcal{F}, \mathcal{G})$ if the diagram

$$\begin{array}{ccc} \mathcal{F}_\alpha \otimes_{A_\alpha} A_\beta & \xrightarrow{\varphi_\alpha} & \mathcal{G}_\alpha \otimes_{A_\alpha} A_\beta \\ \downarrow{\scriptstyle f_{\alpha\beta}} & & \downarrow{\scriptstyle g_{\alpha\beta}} \\ \mathcal{F}_\beta & \xrightarrow[\varphi_\beta]{} & \mathcal{G}_\beta \end{array}$$

commutes for every $\alpha \leq \beta \in \mathcal{N}$.

Notice that $\mathrm{Hom}_{A.}(\mathcal{F}, \mathcal{G})$ are precisely the 0-cocycles of the complex $\mathrm{Hom}^*_{A.}(\mathcal{F}, \mathcal{G})$, whose differential is the inherited (graded) commutator. We shall denote by $\mathrm{Mod}(A.)$ the category of $A.$-modules, with morphisms of $A.$-modules as in Definition 3.2, and by $\mathrm{Mod}^*(A.)$ the DG-category of $A.$-modules, with $*$-morphisms as in Definition 3.3. Since the covering $\mathcal{U}$ is assumed to be fixed at the beginning, we do not emphasise the dependence on it.

Recall that by [9, 22, 35] for every $\alpha \in \mathcal{N}$ the category $\mathrm{DGMod}(A_\alpha)$ is endowed with a model structure where

- weak equivalences are quasi-isomorphisms,

- fibrations are degreewise surjective morphisms,
- every object is fibrant
- $C \in \mathrm{DGMod}(A_\alpha)$ is cofibrant if and only if for every cospan $C \xrightarrow{f} D \xleftarrow{g} E$ with g a surjective quasi-isomorphism there exists a lifting $h: C \to E$.
- cofibrations are degreewise split injective morphisms with cofibrant cokernel.

Moreover if a complex in $\mathrm{DGMod}(A_\alpha)$ is bounded above then it is cofibrant if and only if it is degreewise projective, see [22, Lemma 2.3.6].

Our next goal is to endow the category $\mathrm{Mod}(A.)$ with a model structure. Fix $\mathcal{F} \in \mathrm{Mod}(A.)$ and $\alpha \in \mathcal{N}$; define the **latching module** of $\mathcal{F}$ at α to be

$$L_\alpha \mathcal{F} = \operatorname*{colim}_{\gamma < \alpha} \left(\mathcal{F}_\gamma \otimes_{A_\gamma} A_\alpha \right) \in \mathrm{DGMod}(A_\alpha)$$

and notice that there exists a natural map $L_\alpha \mathcal{F} \to \mathcal{F}_\alpha$. We call an $A.$-module $\mathcal{F} \in \mathrm{Mod}(A.)$ **cofibrant** if for every $\alpha \in \mathcal{N}$ the latching map $L_\alpha \mathcal{F} \to \mathcal{F}_\alpha$ is a cofibration in $\mathrm{DGMod}(A_\alpha)$. Cofibrant $A.$-modules define full subcategories $\mathrm{Mod}(A.)^c \subseteq \mathrm{Mod}(A.)$ and $\mathrm{Mod}^*(A.)^c \subseteq \mathrm{Mod}^*(A.)$.

Remark 3.4 Let $\{U_h\}_{h \in H}$ be an open cover of X and let $\mathcal{N}$ be its nerve. Choose a total order on H; then to every $\alpha \in \mathcal{N}$ it is associated the abstract oriented simplicial complex $\mathcal{P}(\alpha)$, whose faces are the subsets of α, and denote by C_α the corresponding chain complex. Recall that C_α in degree r is the free abelian group of rank $\binom{\deg(\alpha)+1}{r+1}$, and its homology is given by: $H^0(C_\alpha) = \mathbb{Z}$ and $H^j(C_\alpha) = 0$ for every $j \neq 0$. Now consider the category $\mathrm{Ch}(\mathbb{Z})$ of chain complexes of abelian groups; we define the diagram

$$C: \mathcal{N} \to \mathrm{Ch}(\mathbb{Z}); \qquad \alpha \mapsto C_\alpha$$

where for every $\alpha \leq \beta$ in $\mathcal{N}$ the map $C_\alpha \to C_\beta$ is the natural inclusion. We have a a short exact sequence

$$0 \to \operatorname*{colim}_{\gamma < \alpha} C_\gamma \xrightarrow{\iota_\alpha} C_\alpha \to \operatorname{coker}(\iota_\alpha) \to 0$$

where $\operatorname{coker}(\iota_\alpha)$ is $\mathbb{Z}$ concentrated in degree $\deg(\alpha)$.

Example 3.5 (Cofibrant $A.$-module associated to $\mathcal{O}_X$) To the scheme X it is associated a cofibrant $A.$-module $\mathcal{Q}_X \in \mathrm{Mod}(A.)$ as follows. Define

$$\mathcal{Q}^r_{X,\alpha} = C_\alpha^{-r} \otimes_{\mathbb{Z}} A_\alpha \qquad \text{and} \qquad d^r_{\mathcal{Q}_X} = d^{-r}_{C_\alpha} \otimes \mathrm{Id}_{A_\alpha}$$

for every $r \in \mathbb{Z}$ and every $\alpha \in \mathcal{N}$. For every $\alpha \leq \beta$ the map $\mathcal{Q}_{X,\alpha} \otimes_{A_\alpha} A_\beta \to \mathcal{Q}_{X,\beta}$ is induced by the natural inclusion $C_\alpha \to C_\beta$. Now denote by $\hat{C}_\alpha$ the cochain complex defined by $\hat{C}^r_\alpha = C^{-r}_\alpha$ and $d^r_{\hat{C}_\alpha} = d^{-r}_{C_\alpha}$, $r \in \mathbb{Z}$; hence $\mathcal{Q}_{X,\alpha} = \hat{C}_\alpha \otimes_{\mathbb{Z}} A_\alpha$ for every $\alpha \in \mathcal{N}$. Notice that by Remark 3.4 for every $\alpha \in \mathcal{N}$ we have a short exact sequence

$$0 \to L_\alpha \mathcal{Q}_X \xrightarrow{\iota_\alpha \otimes \mathrm{Id}_{A_\alpha}} \mathcal{Q}_{X,\alpha} \to \operatorname{coker}(\iota_\alpha) \otimes_{\mathbb{Z}} A_\alpha \to 0$$

so that the latching map $\iota_\alpha \otimes \mathrm{Id}_{A_\alpha}$ is degreewise injective and its cokernel is zero except for degree $\deg(\alpha)$. Finally, since $\operatorname{coker}(\iota_\alpha \otimes \mathrm{Id}_{A_\alpha})^{\deg(\alpha)} = A_\alpha$ is a free A_α-module, then the latching map is in fact a cofibration in $\mathrm{DGMod}(A_\alpha)$ by [22, Lemma 2.3.6].

A **cofibrant replacement** for a given $A.$-module $\mathcal{F} \in \mathrm{Mod}(A.)$ is a morphism $\mathcal{Q} \to \mathcal{F}$ in $\mathrm{Mod}(A.)$ such that

1: $\mathcal{Q}$ is a cofibrant $A.$-module,
2: the map $\mathcal{Q}_\alpha \to \mathcal{F}_\alpha$ is a surjective quasi-isomorphism for every $\alpha \in \mathcal{N}$.

Cofibrant replacements are not unique.

Example 3.6 (Cofibrant replacement for the structure sheaf $\mathcal{O}_X$) As already noticed, to any sheaf $\mathcal{G}$ of $\mathcal{O}_X$-modules it is associated an $A.$-module $\Upsilon^*\mathcal{G} \in \mathrm{Mod}(A.)$. In particular, $\Upsilon^*\mathcal{O}_X \in \mathrm{Mod}(A.)$ is defined as $(\Upsilon^*\mathcal{O}_X)_\alpha = A_\alpha$ on every $\alpha \in \mathcal{N}$, and the map $(\Upsilon^*\mathcal{O}_X)_\alpha \otimes_{A_\alpha} A_\beta \to (\Upsilon^*\mathcal{O}_X)_\beta$ is the identity for every $\alpha \leq \beta$.

Let $\mathcal{Q}_X \in \mathrm{Mod}(A.)$ be as in Example 3.5, then by Remark 3.4 the set of maps $\{C_\alpha \to H^0(C_\alpha) = \mathbb{Z}\}_{\alpha\in\mathcal{N}}$ induce a morphism $\mathcal{Q}_X \to \Upsilon^*\mathcal{O}_X$ in $\mathrm{Mod}(A.)$ which is a cofibrant replacement. In fact, by the flatness of the map $A_\alpha \to A_\beta$ it follows that

$$\pi_\alpha \colon \mathcal{Q}_{X,\alpha} = \hat{C}_\alpha \otimes_{\mathbb{Z}} A_\alpha \to A_\alpha = \left(\Upsilon^*\mathcal{O}_X\right)_\alpha$$

is a surjective quasi-isomorphism for every $\alpha \in \mathcal{N}$.

Example 3.7 (Cofibrant replacement for a locally free sheaf) Consider a locally free sheaf $\mathcal{E}$ on X, and take a cover $\{U_h\}_{h\in H}$ such that $\mathcal{E}|_{U_\alpha}$ is a free A_α-module for every $\alpha \in \mathcal{N}$. Since for every $\alpha \in \mathcal{N}$ the (DG-)module $\Upsilon^*\mathcal{E}_\alpha = \mathcal{E}(U_\alpha)$ is concentrated in degree 0, it is cofibrant in $\mathrm{DGMod}(A_\alpha)$ by [22, Lemma 2.3.6]. Nevertheless, the latching maps need not to be cofibrations in general; hence $\Upsilon^*\mathcal{E}$ provides an example of an $A.$-module which is pointwise cofibrant but not globally cofibrant. Following Example 3.6 we can explicitly construct a cofibrant replacement $\mathcal{Q}_\mathcal{E} \to \Upsilon^*\mathcal{E}$:

- $\mathcal{Q}_{\mathcal{E},\alpha} = \mathcal{Q}_{X,\alpha} \otimes_{A_\alpha} \mathcal{E}(U_\alpha)$ for every $\alpha \in \mathcal{N}$,
- for every $\alpha \leq \beta$ in $\mathcal{N}$ the morphism $\mathcal{Q}_{\mathcal{E},\alpha} \otimes_{A_\alpha} A_\beta \to \mathcal{Q}_{\mathcal{E},\beta}$ is induced by the corresponding restriction map of $\mathcal{E}$,
- the morphism $\mathcal{Q}_{\mathcal{E},\alpha} \to \mathcal{E}(U_\alpha) = (\Upsilon^*\mathcal{E})_\alpha$ is defined as $\pi_\alpha \otimes \mathrm{Id}_{\mathcal{E}(U_\alpha)}$ for every $\alpha \in \mathcal{N}$.

By Example 3.6 $\pi \colon \mathcal{Q}_X \to \Upsilon^*\mathcal{O}_X$ is a cofibrant replacement; therefore the map $\pi \otimes \mathrm{Id} \colon \mathcal{Q}_\mathcal{E} \to \Upsilon^*\mathcal{E}$ is a cofibrant replacement for $\Upsilon^*\mathcal{E}$.

Now fix $\alpha \in \mathcal{N}$; define $\mathcal{R}_\alpha = \{\gamma \in \mathcal{N} \mid \gamma < \alpha\}$ and recall that the category of diagrams $\mathrm{DGMod}(A_\alpha)^{\mathcal{R}_\alpha}$ is endowed with the Reedy model structure where a natural transformation $f \colon Y \to Z$ is a Reedy weak equivalence (respectively: Reedy fibration) if and only if for every $\gamma < \alpha$ the map $f_\gamma \colon Y_\gamma \to Z_\gamma$ is a quasi-isomorphism (respectively: degreewise surjective). Moreover, f is a Reedy cofibration if and only if the map

$$\operatorname*{colim}_{\beta<\gamma} Z_\beta \coprod_{\operatorname*{colim}_{\beta<\gamma} Y_\beta} Y_\gamma \to Z_\gamma$$

is a cofibration in $\mathrm{DGMod}(A_\gamma)$ for every $\gamma \in \mathcal{R}_\alpha$, see [20, Theorem 16.3.4].

We have a restriction functor $\mathrm{res}_\alpha \colon \mathrm{Mod}(A.) \to \mathrm{DGMod}(A_\alpha)^{\mathcal{R}_\alpha}$ defined by

$$(\mathrm{res}_\alpha \mathcal{F})_\gamma = \mathcal{F}_\gamma \otimes_{A_\gamma} A_\alpha\,, \qquad \gamma < \alpha$$

for every $\mathcal{F} \in \mathrm{Mod}(A.)$.

Lemma 3.8 *For every morphism $\varphi \colon \mathcal{F} \to \mathcal{G}$ in* $\mathrm{Mod}(A.)$ *the following conditions are equivalent.*

(1) *For every* $\alpha \in \mathcal{N}$*, the morphism* $\varphi_\alpha \colon \mathcal{F}_\alpha \to \mathcal{G}_\alpha$ *is a quasi-isomorphism in* $\mathrm{DGMod}(A_\alpha)$*, and the natural morphism*

$$L_\alpha \mathcal{G} \amalg_{(L_\alpha \mathcal{F})} \mathcal{F}_\alpha \longrightarrow \mathcal{G}_\alpha$$

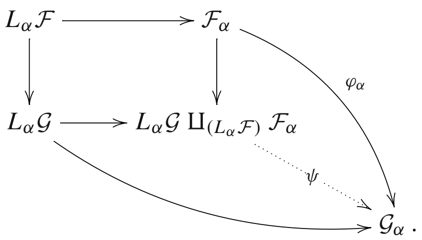
is a cofibration in $\mathrm{DGMod}(A_\alpha)$.

(2) *For every* $\alpha \in \mathcal{N}$*, the natural morphism*

$$L_\alpha \mathcal{G} \amalg_{(L_\alpha \mathcal{F})} \mathcal{F}_\alpha \longrightarrow \mathcal{G}_\alpha$$

is a trivial cofibration in $\mathrm{DGMod}(A_\alpha)$.

Proof Fix $\alpha \in \mathcal{N}$ and consider the following diagram



Now define two diagrams in $\mathrm{DGMod}(A_\alpha)^{\mathcal{R}_\alpha}$ as $Y = \mathrm{res}_\alpha \mathcal{F}$ and $Z = \mathrm{res}_\alpha \mathcal{G}$, and notice that if either (1) or (2) holds the morphism $Z \to Y$ induced by φ is a Reedy cofibration, since colimits commute with coproducts. Moreover, by [20, Theorem 15.3.15] it follows that $Y \to Z$ is a Reedy weak equivalence if either (1) or (2) holds, so that the vertical morphisms in the diagram above are trivial cofibrations in $\mathrm{DGMod}(A_\alpha)$; in fact $\mathrm{colim}\colon \mathrm{DGMod}(A_\alpha)^{\mathcal{R}_\alpha} \to \mathrm{DGMod}(A_\alpha)$ is a left Quillen functor and trivial cofibrations are closed under pushouts. Therefore, φ_α is a weak equivalence if and only if ψ is so. □

Theorem 3.9 (Model structure on $A.$-modules) *The category of* $A.$*-modules over* X *is endowed with a model structure, where a morphism* $\mathcal{F} \to \mathcal{G}$ *in* $\mathrm{Mod}(A.)$ *is*

(1) *a weak equivalence if and only if the morphism* $\mathcal{F}_\alpha \to \mathcal{G}_\alpha$ *is a quasi-isomorphism in* $\mathrm{DGMod}(A_\alpha)$ *for every* $\alpha \in \mathcal{N}$*,*
(2) *a fibration if and only if the morphism* $\mathcal{F}_\alpha \to \mathcal{G}_\alpha$ *is surjective in* $\mathrm{DGMod}(A_\alpha)$ *for every* $\alpha \in \mathcal{N}$*,*
(3) *a cofibration if and only if the natural morphism*

$$L_\alpha \mathcal{G} \amalg_{(L_\alpha \mathcal{F})} \mathcal{F}_\alpha \longrightarrow \mathcal{G}_\alpha$$

is a cofibration in $\mathrm{DGMod}(A_\alpha)$ *for every* $\alpha \in \mathcal{N}$.

Proof It is sufficient to prove that $\mathrm{Mod}(A.)$ with the classes defined in the statement satisfies the axioms of a model category. First notice that the category $\mathrm{Mod}(A.)$ is complete and cocomplete since limits and colimits are taken pointwise. Moreover, the class of weak equivalences satisfies the *2 out of 3* axiom by definition.

The closure with respect to retracts holds since if $\mathcal{F} \to \mathcal{G}$ is a retract of $\mathcal{F}' \to \mathcal{G}'$ in the category of maps of $\mathrm{Mod}(A.)$, then the natural morphism $L_\alpha \mathcal{G} \amalg_{(L_\alpha \mathcal{F})} \mathcal{F}_\alpha \longrightarrow \mathcal{G}_\alpha$ is a retract of the natural morphism $L_\alpha \mathcal{G}' \amalg_{(L_\alpha \mathcal{F}')} \mathcal{F}'_\alpha \longrightarrow \mathcal{G}'_\alpha$ in the category of maps of $\mathrm{DGMod}(A_\alpha)$, for every $\alpha \in \mathcal{N}$.

In order to show that the *lifting* axiom holds, observe that a morphism $\mathcal{F} \to \mathcal{G}$ is a trivial cofibration in $\mathrm{Mod}(A.)$ if and only if for every $\alpha \in \mathcal{N}$ the natural morphism $L_\alpha \mathcal{G} \amalg_{(L_\alpha \mathcal{F})}$

$\mathcal{F}_\alpha \longrightarrow \mathcal{G}_\alpha$ is a trivial cofibration in DGMod(A_α), see Lemma 3.8. Therefore the required lifting can be constructed inductively on the degree of α.

The *factorization* axiom can be proved inductively as follows. Take a morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$, we need to define (functorial) factorizations $\mathcal{F} \to \mathcal{Q} \to \mathcal{G}$ in Mod$(A.)$ as a cofibration (respectively, trivial cofibration) followed by a trivial fibration (respectively, fibration). Now, fix $\alpha \in \mathcal{N}$ of degree d and suppose φ_γ has been factored for all $\gamma \in \mathcal{N}$ of degree less that d. Consider a (functorial) factorization of the natural morphism

$$L_\alpha \mathcal{G} \amalg_{(L_\alpha \mathcal{F})} \mathcal{F}_\alpha \longrightarrow \mathcal{Q}_\alpha \longrightarrow \mathcal{G}_\alpha$$

in DGMod(A_α) as a cofibration (respectively, trivial cofibration) followed by a trivial fibration (respectively, fibration). Lemma 3.8 implies that $\mathcal{Q}$ satisfies the required properties by construction. □

Cofibrant $A.$-modules previously defined coincides with cofibrant objects in Mod$(A.)$ with respect to the above model structure.

Remark 3.10 A morphism $f\colon \mathcal{F} \to \mathcal{G}$ in Mod$(A.)$ is a weak equivalence (respectively: fibration, cofibration) with respect to the model structure of Theorem 3.9 if and only if for every $\alpha \in \mathcal{N}$ the induced morphism $\mathrm{res}_\alpha(f)$ is a Reedy weak equivalence (respectively: Reedy fibration, Reedy cofibration) in DGMod$(A_\alpha)^{\mathcal{R}_\alpha}$. This follows immediately by the flatness of the map $A_\beta \to A_\gamma$ for every $\beta \leq \gamma$.

The idea of Theorem 3.9 is not far from the one recently used in [33], where a similar argument provided a model structure on the category of certain quiver representations. On the other hand, in [33] such model structure has been applied in order to characterize Gorenstein projective modules over certain rings, while in the present paper we shall use it to provide results in a geometric deformation problem.

Remark 3.11 For any $\alpha \in \mathcal{N}$, consider the full subcategory DGMod$^{\leq 0}(A_\alpha) \subseteq$ DGMod(A_α) whose objects are complexes concentrated in non-positive degrees. This is endowed with a model structure where

- weak equivalences are quasi-isomorphisms,
- fibrations are surjections in negative degrees,
- cofibrations are degreewise injective morphisms with degreewise projective cokernel.

We may define the full subcategory of non-positively graded $A.$-modules Mod$^{\leq 0}(A.) \subseteq$ Mod$(A.)$ simply replacing DGMod(A_α) by DGMod$^{\leq 0}(A_\alpha)$. Notice that the same argument of Theorem 3.9 provides a model structure for Mod$^{\leq 0}(A.)$, where a morphism $\mathcal{F} \to \mathcal{G}$ in Mod$^{\leq 0}(A.)$ is a weak equivalence (respectively: cofibration, trivial fibration) if and only if it is a weak equivalence (respectively: cofibration, trivial fibration) in Mod$(A.)$. The same does not hold for fibrations. In particular, the natural inclusion functor Mod$^{\leq 0}(A.) \to$ Mod$(A.)$ is a left Quillen functor.

Definition 3.12 An $A.$-module $\mathcal{F}$ over X is called **quasi-coherent** if the morphism

$$f_{\alpha\beta}\colon \mathcal{F}_\alpha \otimes_{A_\alpha} A_\beta \to \mathcal{F}_\beta$$

is a weak equivalence in DGMod(A_β) for every $\alpha \leq \beta$ in $\mathcal{N}$.

We shall denote by QCoh$(A.) \subseteq$ Mod$(A.)$, and respectively by QCoh$^*(A.) \subseteq$ Mod$^*(A.)$, the full subcategories whose objects are quasi-coherent $A.$-modules. Every quasi-coherent sheaf over X induces a quasi-coherent $A.$-module in the obvious way.

Remark 3.13 Quasi-coherent $A.$-modules are closed under weak equivalences, i.e. given a weak equivalence $\varphi\colon \mathcal{F} \to \mathcal{G}$ in $\mathrm{Mod}(A.)$ then $\mathcal{F}$ is quasi-coherent if and only if $\mathcal{G}$ is so. To prove the claim it is sufficient to consider the commutative diagram

$$\begin{array}{ccc} F_\alpha \otimes_{A_\alpha} A_\beta & \xrightarrow{f_{\alpha\beta}} & F_\beta \\ \downarrow{\scriptstyle \varphi_\alpha \otimes \mathrm{Id}} & & \downarrow{\scriptstyle \varphi_\beta} \\ G_\alpha \otimes_{A_\alpha} A_\beta & \xrightarrow[g_{\alpha\beta}]{} & G_\beta \end{array}$$

for every $\alpha \leq \beta$ in $\mathcal{N}$. The statement follows by the flatness of the map $A_\alpha \to A_\beta$ and by the *2 out of 3* property. This implies in particular that the subcategory $\mathrm{QCoh}(A.) \subseteq \mathrm{Mod}(A.)$ is closed under both factorizations given by Theorem 3.9.

Lemma 3.14 *Let $\mathcal{Q} \in \mathrm{Mod}(A.)$ be a cofibrant $A.$-module. Given a cospan $\mathcal{Q} \xrightarrow{f} \mathcal{R} \xleftarrow{\pi} \mathcal{P}$ in $\mathrm{Mod}^*(A.)$, if π is degreewise surjective then there exists $h \in \mathrm{Hom}^*_{A.}(\mathcal{Q}, \mathcal{P})$ such that $\pi h = f$.*

Proof For simplicity we assume that $f \in \mathrm{Hom}^0_{A.}(\mathcal{Q}, \mathcal{R})$; the general case can be obtain by a shift. Fix $i \in \mathbb{Z}$; the map $\pi^i : \mathcal{R}^i \to \mathcal{P}^i$ induces the map of $A.$-modules

$$\begin{array}{cccccccccccc} \hat{\mathcal{R}}\colon & & \cdots & \longrightarrow & 0 & \longrightarrow & \mathcal{R}^i & \xrightarrow{\mathrm{Id}} & \mathcal{R}^i & \longrightarrow & 0 & \longrightarrow \cdots \\ \downarrow{\scriptstyle \hat{\pi}} & & & & \downarrow & & \downarrow{\scriptstyle \pi^i} & & \downarrow{\scriptstyle \pi^i} & & \downarrow & \\ \hat{\mathcal{P}}\colon & & \cdots & \longrightarrow & 0 & \longrightarrow & \mathcal{P}^i & \xrightarrow{\mathrm{Id}} & \mathcal{P}^i & \longrightarrow & 0 & \longrightarrow \cdots \end{array}$$

which is a trivial fibration. Moreover, $f^i : \mathcal{Q}^i \to \mathcal{R}^i$ induces the map of $A.$-modules

$$\begin{array}{cccccccccccc} \mathcal{Q}\colon & & \cdots & \longrightarrow & \mathcal{Q}^{i-2} & \xrightarrow{d_{\mathcal{Q}}^{i-2}} & \mathcal{Q}^{i-1} & \xrightarrow{d_{\mathcal{Q}}^{i-1}} & \mathcal{Q}^i & \longrightarrow & \mathcal{Q}^{i+1} & \longrightarrow \cdots \\ \downarrow{\scriptstyle \hat{f}} & & & & \downarrow & & \downarrow{\scriptstyle f^i d_{\mathcal{Q}}^{i-1}} & & \downarrow{\scriptstyle f^i} & & \downarrow & \\ \hat{\mathcal{P}}\colon & & \cdots & \longrightarrow & 0 & \longrightarrow & \mathcal{P}^i & \xrightarrow{\mathrm{Id}} & \mathcal{P}^i & \longrightarrow & 0 & \longrightarrow \cdots \end{array}$$

which can be lifted to $\hat{\mathcal{R}}$ because $\mathcal{Q}$ is cofibrant by assumption; i.e. there exists a map of $A.$-modules $\hat{h}\colon \mathcal{Q} \to \hat{\mathcal{R}}$ such that $\hat{\pi}\hat{h} = \hat{f}$. Now define $h^i = \hat{h}^i : \mathcal{Q}^i \to \mathcal{R}^i$; reproducing the same argument for every $i \in \mathbb{Z}$ we obtain the required map $h \in \mathrm{Hom}^0_{A.}(\mathcal{Q}, \mathcal{P})$. □

Notice that if X is an affine scheme then we can choose $\mathcal{N} = \{*\}$. Therefore $A.$-modules reduce to the category of DG-modules over $\Gamma(X, \mathcal{O}_X)$, and Lemma 3.14 states that cofibrant DG-modules are degreewise projective. In the general case, the liftings $\{h^i_\gamma : \mathcal{Q}^i_\gamma \to \mathcal{P}^i_\gamma\}_{\gamma \in \mathcal{N}}$ satisfy the commutativity relations induced by the nerve for any fixed $i \in \mathbb{Z}$.

3.1 *A.*-Modules as Sheaves Over the Nerve

Our next goal is to give a "sheaf theoretic" description of $A.$-modules. To this aim, we define a topology $\tau_{\mathcal{N}}$ on the nerve $\mathcal{N}$ as follows: $V \in \tau_{\mathcal{N}}$ if and only if the condition

$$\alpha \in V, \alpha \leq \beta \Rightarrow \beta \in V$$

is satisfied. This is called the Alexandroff topology, since $(\mathcal{N}, \tau_{\mathcal{N}})$ becomes an Alexandroff topological space, see [1]. For every fixed $\alpha \in \mathcal{N}$ the set $V_\alpha = \{\gamma \in \mathcal{N} \,|\, \alpha \leq \gamma\} \subseteq \mathcal{N}$ is open, and the collection $\{V_\alpha\}_{\alpha\in\mathcal{N}} \subseteq \tau_{\mathcal{N}}$ is a basis for the topology. Then consider the category $\mathrm{Sh}(\mathcal{N})$ of sheaves of complexes over $\mathcal{N}$, where moreover on every V_α it is given a structure of DG-module over A_α compatible with the restriction maps. Now, there is a pair of functors

$$\mathcal{S}\colon \mathrm{Mod}(A.) \to \mathrm{Sh}(\mathcal{N}) \qquad \Gamma\colon \mathrm{Sh}(\mathcal{N}) \to \mathrm{Mod}(A.)$$

defined by

$$\mathcal{S}(\mathcal{F})(V) = \lim_{\gamma\in V} \mathcal{F}_\gamma \qquad \text{and} \qquad \Gamma(\mathcal{G})_\alpha = \mathcal{G}(V_\alpha)$$

for every $\mathcal{F} \in \mathrm{Mod}(A.)$, every $\mathcal{G} \in \mathrm{Sh}(\mathcal{N})$, every $\alpha \in \mathcal{N}$ and every $V \in \tau_{\mathcal{N}}$. Notice that

$$\mathcal{S}(\mathcal{F})(V) = \left\{ \{s_\gamma\} \in \prod_{\gamma\in V} \mathcal{F}_\gamma \;\middle|\; f_{\gamma_1\gamma_2}(s_{\gamma_1} \otimes 1) = s_{\gamma_2} \text{ for every } \gamma_1 \leq \gamma_2 \right\}$$

and that $\mathcal{S}(\mathcal{F})(V_\alpha) = \mathcal{F}_\alpha$. for every $\alpha \in \mathcal{N}$. In particular, $\Gamma \circ \mathcal{S} = \mathrm{Id}_{\mathrm{Mod}(A.)}$. Given $\mathcal{G} \in \mathrm{Sh}_X(\mathcal{N})$ we have a natural map

$$\mathcal{G}(V) \xrightarrow{\cong} \lim_{\gamma\in V} \mathcal{G}(V_\gamma) = \mathcal{S}(\Gamma(\mathcal{G}))(V)$$

for every $V \in \tau_{\mathcal{N}}$, which is an isomorphism because $\mathcal{G}$ is a sheaf and $\bigcup_{\gamma\in V} V_\gamma = V$. Therefore the functors $\mathcal{S}\colon \mathrm{Mod}(A.) \leftrightarrows \mathrm{Sh}(\mathcal{N})\colon \Gamma$ are equivalences of categories. A similar result can be found in [6, Proposition 6.6].

Recall that a sheaf $\mathcal{G}$ of $\mathcal{O}_X$-modules is flasque if the restriction map $\mathcal{G}(U) \to \mathcal{G}(V)$ is surjective for every inclusion $V \to U$ between open subsets of X.

Definition 3.15 An $A.$-module $\mathcal{F} \in \mathrm{Mod}(A.)$ is called **flasque** if the associated sheaf $\mathcal{S}(\mathcal{F})$ is so.

3.2 Inverse and Direct Image for *A*.-Modules: $j_V^* \dashv j_{V,*}$

For any fixed open $V \in \tau_{\mathcal{N}}$, denote by $j_V\colon V \hookrightarrow \mathcal{N}$ the natural inclusion; the aim of this subsection is to introduce two functors j_V^* and $j_{V,*}$, which we defined the "inverse image" and "direct image" functors because of the equivalence described in Sect. 3.1.

First define $U_V = \bigcup_{\gamma\in V} U_\gamma \subseteq X$; recall that for every $\alpha \in \mathcal{N}$ we denoted $V_\alpha = \{\gamma \in \mathcal{N} \,|\, \gamma \geq \alpha\}$, so that in particular $U_{V_\alpha} = U_\alpha \subseteq X$. Then the "inverse image" and "direct image" functors are defined by

$$\begin{aligned} j_V^*\colon \mathrm{Mod}(A.) &\to \mathrm{Mod}(U_V) \\ \{\mathcal{F}_\gamma\}_{\gamma\in\mathcal{N}} &\mapsto \{\mathcal{F}_\gamma\}_{\gamma\in V} \end{aligned} \qquad \text{and} \qquad \begin{aligned} j_{V,*}\colon \mathrm{Mod}(U_V) &\to \mathrm{Mod}(A.) \\ \{\mathcal{G}_\alpha\}_{\alpha\in V} &\mapsto \left\{ \lim_{V\cap V_\alpha} \mathcal{G} \right\}_{\alpha\in\mathcal{N}} \end{aligned}$$

respectively. More explicitly:

$$(j_{V,*}\mathcal{G})_\alpha = \begin{cases} \lim\limits_{\gamma\in V\cap V_\alpha} \mathcal{G}_\gamma & \text{if } U_\alpha \cap U_V \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

where the limit is taken in $\mathrm{DGMod}(A_\alpha)$, and the A_α-module structure is induced via $A_\alpha \to A_\gamma$ on each $\mathcal{G}_\gamma$. Given $\alpha \leq \beta$ in $\mathcal{N}$ such that $U_\beta \cap U_V \neq \emptyset$, the limit induces a natural map

$$(j_{V,*}\mathcal{G})_\alpha = \lim_{\gamma \in V \cap V_\alpha} \mathcal{G}_\gamma \longrightarrow \lim_{\gamma \in V \cap V_\beta} \mathcal{G}_\gamma = (j_{V,*}\mathcal{G})_\beta$$

between DG-modules over A_α. Since the A_α structure on $\lim_{V\cap V_\beta} \mathcal{G}_\gamma$ is given by $A_\alpha \to A_\beta$, by adjunction the above map corresponds to a morphism

$$(j_{V,*}\mathcal{G})_\alpha \otimes_{A_\alpha} A_\beta \to (j_{V,*}\mathcal{G})_\beta$$

between DG-modules over A_β. Notice that in particular if $\alpha \in V$ then $(j_{V,*}\mathcal{G})_\alpha = \mathcal{G}_\alpha$.

Remark 3.16 For every open subset $j_V : V \hookrightarrow \mathcal{N}$, there is an adjunction $j_V^* \dashv j_{V,*}$. In fact $j_V^* j_{V,*}$ is the identity on $\mathrm{Mod}(U_V)$, and given $\mathcal{F} \in \mathrm{Mod}(A.)$ the unit $\eta\colon \mathrm{Id}_{\mathrm{Mod}(A.)} \to j_{V,*} j_V^*$ is defined by

$$\eta_{\mathcal{F}} = \begin{cases} \mathcal{F}_\alpha \to \lim\limits_{\gamma \in V\cap V_\alpha} \mathcal{F}_\gamma = (j_{V,*} j_V^* \mathcal{F})_\alpha & \text{if } U_V \cap U_\alpha \neq \emptyset \\ \mathcal{F}_\alpha \to 0 & \text{otherwise} \end{cases}$$

so that the *unit-counit equations* reduces to $\eta_{j_{V,*}\mathcal{G}} = \mathrm{Id}_{j_{V,*}\mathcal{G}}$ for every $\mathcal{G} \in \mathrm{Mod}(U_V)$.

Remark 3.17 The adjoint pair of Remark 3.16 is not necessarily a Quillen pair; in particular, the restriction $j_V^*\mathcal{F}$ of a cofibrant $A.$-module $\mathcal{F} \in \mathrm{Mod}(A.)$ may not be cofibrant. The crucial point is that the functor

$$\lim_{V\cap V_\alpha} : \mathrm{DGMod}(A_\alpha)^{V\cap V_\alpha} \to \mathrm{DGMod}(A_\alpha)$$

is right adjoint to the constant diagram, which does not preserve cofibrations in general. Nevertheless, if we choose $V = V_{\overline{\alpha}} = \{\gamma \in \mathcal{N} \mid \overline{\alpha} \leq \gamma\}$ for some $\overline{\alpha}$ then the adjunction $j_{V_{\overline{\alpha}}}^* \dashv j_{V_{\overline{\alpha}},*}$ is in fact a Quillen pair. To prove the claim, notice that for every $\alpha \in \mathcal{N}$ such that $U_{V_{\overline{\alpha}}} \cap U_\alpha \neq \emptyset$ we have $V_{\overline{\alpha}} \cap V_\alpha = V_{\overline{\alpha}\cup\alpha}$. Hence the constant functor

$$\mathrm{DGMod}(A_\alpha) \to \mathrm{DGMod}(A_\alpha)^{V_{\overline{\alpha}\cup\alpha}}$$

preserves cofibrations and trivial cofibrations; in fact for every $\beta \in \mathcal{N}$ the set $\{\gamma \in \mathcal{N} \mid \overline{\alpha}\cup\alpha \leq \gamma < \beta\}$ is connected. It follows that the functor $\lim_{V_{\overline{\alpha}\cup\alpha}}$ preserves fibrations and trivial fibrations, so that $j_{V_{\overline{\alpha}},*}\colon \mathrm{Mod}(U_{V_{\overline{\alpha}}}) \to \mathrm{Mod}(A.)$ is a right Quillen functor as required. In particular, given a cofibrant $A.$-module $\mathcal{F} \in \mathrm{Mod}(A.)$, its restriction $j_{V_{\overline{\alpha}}}^*\mathcal{F}$ to $V_{\overline{\alpha}}$ is cofibrant in $\mathrm{Mod}(U_{\overline{\alpha}})$.

Remark 3.18 Notice that in Remark 3.16 the differentials do not play any role, so that we have binatural isomorphisms

$$\mathrm{Hom}_{A.,V}\left(j_V^*\mathcal{Q}, \mathcal{G}\right) \cong \mathrm{Hom}_{A.}\left(\mathcal{Q}, j_{V,*}\mathcal{G}\right)$$
$$\mathrm{Hom}^*_{A.,V}\left(j_V^*\mathcal{Q}, \mathcal{G}\right) \cong \mathrm{Hom}^*_{A.}\left(\mathcal{Q}, j_{V,*}\mathcal{G}\right)$$

for every $\mathcal{Q} \in \mathrm{Mod}(A.)$ and every $\mathcal{G} \in \mathrm{Mod}(U_\alpha)$. To avoid possible confusion we denoted morphisms in $\mathrm{Mod}(U_V)$ by $\mathrm{Hom}_{A.,V}(-,-)$, and $*$-morphisms in $\mathrm{Mod}^*(U_V)$ by $\mathrm{Hom}^*_{A.,V}(-,-)$.

Lemma 3.19 *Fix an open subset* $j_V : V \hookrightarrow \mathcal{N}$. *Let* $\mathcal{Q}, \mathcal{G} \in \mathrm{Mod}(A.)$ *and assume* $\mathcal{Q}$ *to be cofibrant. Denote by* $\eta_{\mathcal{G}} : \mathcal{G} \to j_{V,*}\, j_V^* \mathcal{G}$ *the unit map of the adjunction given by Remark 3.16. If* $\eta_{\mathcal{G}}$ *is degreewise surjective, then the induced morphism*

$$\mathrm{Hom}^*_{A.}(\mathcal{Q}, \mathcal{G}) \xrightarrow{\eta_{\mathcal{G}}} \mathrm{Hom}^*_{A.}\left(\mathcal{Q}, j_{V,*}\, j_V^* \mathcal{G}\right) = \mathrm{Hom}^*_{A.,V}(j_V^* \mathcal{Q}, j_V^* \mathcal{G})$$

is degreewise surjective.

Proof We prove that the map $\mathrm{Hom}^0_{A.}(\mathcal{Q}, \mathcal{G}) \xrightarrow{\eta_{\mathcal{G}}} \mathrm{Hom}^0_{A.,V}\left(\mathcal{Q}, \Upsilon^V_* \Upsilon^*_V \mathcal{G}\right)$ is surjective, the same argument works for other degrees. We need to show that every $\{\varphi_\gamma\}_{\gamma \in \mathcal{N}} \in \mathrm{Hom}^0_{A.}\left(\mathcal{Q}, j_{V,*}\, j_V^* \mathcal{G}\right)$ factors through the unit map $\eta_{\mathcal{G}}$. Recall that since $\mathcal{Q}$ is cofibrant then $\mathcal{Q}^p$ is projective (see Lemma 3.14) for every $p \in \mathbb{Z}$, so that there exists the dotted morphism

$$\begin{array}{ccc} & & \mathcal{G}^p \\ & \nearrow & \big\downarrow \eta_{\mathcal{G}} \\ \mathcal{Q}^p & \longrightarrow & (j_{V,*}\, j_V^* \mathcal{G})^p \end{array}$$

whence the statement. □

Lemma 3.19 can be restated in terms of flasque $A.$-modules, see Definition 3.15. For every pair of $A.$-modules $\mathcal{Q}, \mathcal{G} \in \mathrm{Mod}(A.)$ it is defined an $A.$-module $\mathrm{Hom}^*_{A.}(\mathcal{Q}, \mathcal{G}) \in \mathrm{Mod}(A.)$ as follows

(1) $\mathrm{Hom}^*_{A.}(\mathcal{Q}, \mathcal{G})_\alpha = \mathrm{Hom}^*_{A.,V_\alpha}\left(j^*_{V_\alpha} \mathcal{Q}, j^*_{V_\alpha} \mathcal{G}\right) = \mathrm{Hom}^*_{A.}\left(\mathcal{Q}, j_{V_\alpha,*}\, j^*_{V_\alpha} \mathcal{G}\right)$ for every $\alpha \in \mathcal{N}$,
(2)

$$\mathrm{Hom}^*_{A.}(\mathcal{Q}, \mathcal{G})_\alpha \otimes_{A_\alpha} A_\beta \to \mathrm{Hom}^*_{A.}(\mathcal{Q}, \mathcal{G})_\beta$$
$$\{\varphi_\gamma\}_{\gamma \geq \alpha} \otimes x \mapsto \{x \cdot \varphi_\gamma\}_{\gamma \geq \beta}$$

for every $\alpha \leq \beta$ in $\mathcal{N}$.

Proposition 3.20 *Let* $\mathcal{Q}, \mathcal{G} \in \mathrm{Mod}(A.)$ *with* $\mathcal{Q}$ *cofibrant and* $\mathcal{G}$ *flasque. Then the* $A.$*-module* $\mathrm{Hom}^*_{A.}(\mathcal{Q}, \mathcal{G}) \in \mathrm{Mod}(A.)$ *is flasque.*

Proof If $\mathcal{G}$ is flasque then for every open subset $j_V : V \hookrightarrow \mathcal{N}$ the unit map $\eta_{\mathcal{G}} : \mathcal{G} \to j_{V,*}\, j_V^* \mathcal{G}$ described in Remark 3.16 is surjective. The statement follows by Lemma 3.19. □

4 Extended Lower-Shriek Functor

This section is devoted to the well posedness of a certain functor that we shall call the *extended lower-shriek*.

Definition 4.1 Define the poset $\mathbf{L}_{\mathcal{N}}$ as

(1) $\mathbf{L}_{\mathcal{N}} = \{(\beta, \gamma) \in \mathcal{N} \times \mathcal{N} \mid \beta \leq \gamma\}$,
(2) $(\beta, \gamma) \leq (\delta, \eta)$ if and only if $\beta \leq \delta$ and $\eta \leq \gamma$ in $\mathcal{N}$.

In particular, condition (2) of Definition 4.1 implies that for every $\beta \leq \delta \leq \eta \leq \gamma$ the diagram

$$\begin{array}{ccc} (\beta,\gamma) & \longrightarrow & (\delta,\gamma) \\ \downarrow & \searrow & \downarrow \\ (\beta,\eta) & \longrightarrow & (\delta,\eta) \end{array}$$

commutes in $\mathbf{L}_{\mathcal{N}}$. We shall call a morphism $(\beta,\gamma) \to (\delta,\gamma)$ an *horizontal* morphism, and similarly we call morphisms of the form $(\beta,\gamma) \to (\beta,\eta)$ *vertical* morphisms.

Remark 4.2 More generally, for every small category C we can consider the category $\mathbf{L}_C$ whose objects are maps in C and whose morphisms are commutative diagrams:

$$\begin{array}{ccc} \beta & \longrightarrow & \gamma \\ \downarrow & & \uparrow \\ \delta & \longrightarrow & \eta \end{array} \qquad \Longleftrightarrow \qquad \begin{array}{c} (\beta \to \gamma) \\ \big\downarrow {\scriptstyle \in \mathrm{Mor}_{\mathbf{L}_C}} \\ (\delta \to \eta) \end{array}$$

If C is a direct Reedy category, then $\mathbf{L}_C$ is an inverse Reedy category with degree function

$$\deg(\beta \to \gamma) = \deg(\gamma) - \deg(\beta) \geq 0.$$

For every $\alpha \leq \beta$ in $\mathcal{N}$ denote by

$$i_\beta \colon U_\beta \xrightarrow{i^\alpha_\beta} U_\alpha \xrightarrow{i_\alpha} X$$

the natural inclusions. Since the scheme is separated, then U_α is affine for every $\alpha \in \mathcal{N}$. Hence the datum of an $A.$-module $\mathcal{F} \in \mathrm{Mod}(A.)$ is equivalent to $\mathcal{F}_\alpha \in \mathrm{DGMod}(\mathcal{O}_{U_\alpha})$ for every $\alpha \in \mathcal{N}$ and morphisms

$$f_{\alpha\beta} \colon i^{\alpha *}_\beta \mathcal{F}_\alpha = \mathcal{F}_\alpha|_{U_\beta} \to \mathcal{F}_\beta, \qquad \alpha \leq \beta.$$

Now, we fix the $A.$-module $\mathcal{F}$ and define the following functors

$$\begin{array}{ll} \mathcal{F}_* \colon \mathbf{L}_{\mathcal{N}} \to \mathrm{DGMod}(\mathcal{O}_X) & \mathcal{F}_! \colon \mathbf{L}_{\mathcal{N}} \to \mathrm{DGMod}(\mathcal{O}_X) \\ (\beta,\gamma) \mapsto i_{\gamma *} i^{\beta *}_\gamma \mathcal{F}_\beta = i_{\gamma *} \mathcal{F}_\beta|_{U_\gamma} & (\beta,\gamma) \mapsto i_{\gamma !} i^{\beta *}_\gamma \mathcal{F}_\beta = i_{\gamma !}(\mathcal{F}_\beta|_{U_\gamma}) \; . \end{array}$$

If $(\beta,\gamma) \to (\delta,\eta)$ then $U_\gamma \subset U_\eta \subset U_\delta \subset U_\beta$, so that it is given the map $f_{\beta\delta} \colon \mathcal{F}_\beta|_{U_\delta} \to \mathcal{F}_\delta$ which in turn induces the morphism $\mathcal{F}_!(\beta,\gamma) \to \mathcal{F}_!(\delta,\eta)$ defined by the composition

$$i_{\gamma !}(\mathcal{F}_\beta|_{U_\gamma}) \xrightarrow{i_{\gamma !}(f_{\beta\delta}|_{U_\gamma})} i_{\gamma !}\mathcal{F}_\delta|_{U_\gamma} \to i_{\eta !}\mathcal{F}_\delta|_{U_\eta}.$$

Similarly, the morphisms $\mathcal{F}_*(\beta,\gamma) \to \mathcal{F}_*(\delta,\eta)$ is given by the composition

$$i_{\gamma *}(\mathcal{F}_\beta|_{U_\gamma}) \xrightarrow{i_{\gamma *}(f_{\beta\delta}|_{U_\gamma})} i_{\gamma *}\mathcal{F}_\delta|_{U_\gamma} \to i_{\eta *}\mathcal{F}_{\delta|U_\eta}.$$

Definition 4.3 In the above notation, the **extended lower-shriek** functor $\Upsilon_!$ is defined as

$$\begin{aligned} \Upsilon_! \colon \; \mathrm{Mod}(A.) &\to \mathrm{DGMod}(\mathcal{O}_X) \\ \mathcal{F} &\mapsto \operatorname*{colim}_{\mathbf{L}_{\mathcal{N}}} \mathcal{F}_! \; . \end{aligned}$$

Proposition 4.4 *The functors* $\Upsilon_! \colon \mathrm{Mod}(A.) \leftrightarrows \mathrm{DGMod}(\mathcal{O}_X) \colon \Upsilon^*$ *form an adjoint pair.*

Proof We need to show that there exists a bi-natural bijection of sets

$$\mathrm{Hom}_{\mathrm{DGMod}(\mathcal{O}_X)}(\Upsilon_!\mathcal{F},\mathcal{G}) \cong \mathrm{Hom}_{A.}(\mathcal{F},\Upsilon^*\mathcal{G})$$

for every $\mathcal{F} \in \mathrm{Mod}(A.)$ and every $\mathcal{G} \in \mathrm{DGMod}(\mathcal{O}_X)$. By the universal property of the colimit, the data of a morphism $\varphi \in \mathrm{Hom}_{\mathrm{DGMod}(\mathcal{O}_X)}(\Upsilon_!\mathcal{F},\mathcal{G})$ is equivalent to the following chain of one-to-one correspondences

$$\varphi \longleftrightarrow \left\{i_{\gamma!}\left(\mathcal{F}_\beta|_{U_\gamma}\right) \to \mathcal{G}\right\}_{(\beta,\gamma)\in\mathbf{L}_\mathcal{N}} \longleftrightarrow \left\{\left(\mathcal{F}_\beta|_{U_\gamma}\right) \to \mathcal{G}|_{U_\gamma}\right\}_{(\beta,\gamma)\in\mathbf{L}_\mathcal{N}} \overset{(*)}{\longleftrightarrow}$$
$$\overset{(*)}{\longleftrightarrow} \left\{\mathcal{F}_\beta(U_\beta)\otimes_{A_\beta} A_\gamma \to \mathcal{G}(U_\gamma)\right\}_{(\beta,\gamma)\in\mathbf{L}_\mathcal{N}} \overset{(**)}{\longleftrightarrow} \{\mathcal{F}_\gamma(U_\gamma)\to\mathcal{G}(U_\gamma)\}_{\gamma\in\mathcal{N}} \in$$
$$\mathrm{Hom}_{A.}(\mathcal{F},\Upsilon^*\mathcal{G})$$

where:

- $(*)$ is a bijection since the morphisms of sheaves are all determined by localizations of the module $\mathcal{F}_\beta \otimes_{A_\beta} A_\gamma$,
- $(**)$ is a bijection since for every $(\beta,\gamma) \in \mathbf{L}_\mathcal{N}$ we have a commutative diagram

$$\begin{array}{ccc} \mathcal{F}_\beta(U_\beta)\otimes_{A_\beta} A_\gamma & \xrightarrow{f_{\beta\gamma}} & \mathcal{F}_\gamma(U_\gamma) \\ & \searrow & \downarrow \\ & & \mathcal{G}(U_\gamma) \end{array}$$

  where the morphisms $f_{\beta\gamma}$ are given by the $A.$-module $\mathcal{F}$.

□

Recall that an object $\mathcal{F} \in \mathrm{DGMod}(\mathcal{O}_X)$ is called a **flasque complex** if it is degreewise flasque, see [23].

Theorem 4.5 [23, Theorem 5.2] *Let X be a separated finite-dimensional Noetherian scheme. Then the category* $\mathrm{DGMod}(\mathcal{O}_X)$ *is endowed with the* ***flat model structure****, where the weak equivalences are the quasi-isomorphisms, and fibrations are epimorphisms with flasque kernel.*

Remark 4.6 [17, Exercise II.1.6] Let $\varphi\colon \mathcal{F} \to \mathcal{G}$ be an epimorphism of sheaves of $\mathcal{O}_X$-modules with flasque kernel over a separated Noetherian scheme X. Then $\varphi_V : \mathcal{F}(V) \to \mathcal{G}(V)$ is surjective for every open subset $V \subseteq X$.

Theorem 4.7 *The adjoint functors*

$$\Upsilon_!\colon\ \mathrm{Mod}(A.) \rightleftarrows \mathrm{DGMod}(\mathcal{O}_X)\colon\ \Upsilon^*$$

form a Quillen pair with respect to the model structure of Theorem 3.9 on $\mathrm{Mod}(A.)$*, and the flat model structure on* $\mathrm{DGMod}(\mathcal{O}_X)$.

Proof The adjointness follows from Proposition 4.4, and the right adjoint Υ^* preserves fibrations by Remark 4.6. In order to prove that the functor Υ^* preserves trivial fibrations it is sufficient to observe that the complex of sections $\Gamma(V, \ker(f))$ is acyclic for every open $V \subseteq X$ and for any epimorphism with flasque kernel $f : \mathcal{F} \to \mathcal{G}$ in $\mathrm{DGMod}(\mathcal{O}_X)$; this immediately follows from [23, Lemma 4.1]. □

Notice that the proof of Theorem 4.7 relies on [23, Lemma 4.1], which applies because we assumed the scheme X to be Noetherian and finite-dimensional. As a consequence of Theorem 4.7, we obtain the existence of the total derived functors

$$\mathbb{L}\Upsilon_! \colon \operatorname{Ho}(\operatorname{Mod}(A.)) \leftrightarrows \operatorname{Ho}(\operatorname{DGMod}(\mathcal{O}_X)) \colon \mathbb{R}\Upsilon^* .$$

5 From *A.*-Modules to Derived Categories

The first goal of this section is to show that the total left derived functor of the extended lower-shriek introduced in the Sect. 4 maps (classes of) quasi-coherent $A.$-modules in (classes of) complexes of quasi-coherent sheaves, see Theorem 5.4. Hence there will be induced functors

$$\overline{\mathbb{L}\Upsilon_!} \colon \operatorname{Ho}(\operatorname{QCoh}(A.)) \leftrightarrows \operatorname{D}_{qc}(X) \colon \overline{\mathbb{R}\Upsilon}^* .$$

Our main result shows that the above functors are in fact equivalences of triangulated categories, see Theorem 5.7. To this aim, we shall first prove that

$$\overline{\mathbb{L}\Upsilon_!}[\mathcal{F}] = [\Upsilon_!\mathcal{F}] \qquad \text{for every } [\mathcal{F}] \in \operatorname{Ho}(\operatorname{QCoh}(A.))$$
$$\overline{\mathbb{R}\Upsilon}^*[\mathcal{G}] = [\Upsilon^*\mathcal{G}] \qquad \text{for every } [\mathcal{G}] \in \operatorname{D}_{qc}(X) .$$

As usual, X is a fixed separated finite-dimensional Noetherian scheme over $\mathbb{K}$; moreover $\mathcal{N}$ denotes the nerve of a fixed affine open covering $\{U_h\}_{h\in H}$. Recall that by Definition 3.12, an $A.$-module $\mathcal{F} \in \operatorname{Mod}(A.)$ is called *quasi-coherent* if the morphism

$$f_{\alpha\beta} \colon \mathcal{F}_\alpha \otimes_{A_\alpha} A_\beta \to \mathcal{F}_\beta$$

is a weak equivalence (i.e. a quasi-isomorphism) in $\operatorname{DGMod}(A_\beta)$ for every $\alpha \leq \beta$ in $\mathcal{N}$.

We need an easy preliminary result.

Lemma 5.1 *Let N be a small direct category and let R be a ring. Consider the category* $\operatorname{DGMod}(R)$ *of complexes of R-modules. Given a functor $F \colon N \to \operatorname{DGMod}(R)$ there exists a natural isomorphism of R-modules $H^j\left(\operatorname{colim}_{\beta\in N} F_\beta\right) \cong \operatorname{colim}_{\beta\in N}(H^j(F_\beta))$ for every $j \in \mathbb{Z}$.*

Proof Consider the exact sequence $0 \to Z^j F_\beta \to F^j_\beta \xrightarrow{d^j_\beta} Z^{j+1} F_\beta \to H^{j+1} F_\beta \to 0$, for every $\beta \in N$ and every $j \in \mathbb{Z}$. Now observe that the functor colim_N is exact, being direct on a category of modules. In particular,

$$\operatorname*{colim}_{\beta\in N} Z^j F_\beta \cong \ker\left\{\operatorname*{colim}_{\beta\in N} d^j_\beta\right\} = Z^j\left(\operatorname*{colim}_{\beta\in N} F_\beta\right),$$

and the thesis easily follows. □

Proposition 5.2 *Let $\mathcal{F} \in \operatorname{QCoh}(A.)$ be a quasi-coherent $A.$-module. Then for every $\alpha \in \mathcal{N}$ there exists a quasi-isomorphism $\widetilde{\mathcal{F}_\alpha} \to (\Upsilon_!\mathcal{F})|_{U_\alpha}$ in $\operatorname{DGMod}(\mathcal{O}_{U_\alpha})$.*

Proof We show that the natural morphism

$$\varphi \colon \widetilde{\mathcal{F}}_\alpha \to \left(\operatorname*{colim}_{(\beta,\gamma)\in\mathbf{L}_\mathcal{N}} i_{\gamma!}(\widetilde{\mathcal{F}}_\beta|_{U_\gamma})\right)\Big|_{U_\alpha} = (\Upsilon_!\mathcal{F})|_{U_\alpha}$$

is a quasi-isomorphism by showing that the induced morphism φ_x is so at each stalk, $x \in U_\alpha$. Consider the following chain of equalities

$$\left((\Upsilon_!\mathcal{F})|_{U_\alpha}\right)_x = \operatorname*{colim}_{(\beta,\gamma)\in \mathbf{L}_{\mathcal{N}}} \left(i_{\gamma !}(\widetilde{\mathcal{F}}_\beta|_{U_\gamma})\right)_x = \operatorname*{colim}_{\{(\beta,\gamma)\in \mathbf{L}_{\mathcal{N}}\,|\,x\in U_\gamma\}} \left(\widetilde{\mathcal{F}}_\beta|_{U_\gamma}\right)_x = \operatorname*{colim}_{\beta\in\mathcal{N}} \left(\widetilde{F}_\beta\right)_x$$

where the last equality holds since for every $\beta \leq \gamma_1 \leq \gamma_2$ the vertical morphism induced on the stalk $\left(\widetilde{\mathcal{F}}_\beta|_{U_{\gamma_1}}\right)_x \to \left(\widetilde{\mathcal{F}}_\beta|_{U_{\gamma_2}}\right)_x$ is an isomorphism, being $x \in U_{\gamma_2} \subseteq U_{\gamma_1}$. Now take $j \in \mathbb{Z}$ and notice that $\mathcal{N}$ is connected, whenever $\beta_1 \leq \beta_2$ the natural morphism $H^j(\widetilde{F}_{\beta_1})_x \to H^j(\widetilde{F}_{\beta_2})_x$ is an isomorphism by hypothesis; hence

$$H^j(\varphi_x)\colon H^j(\widetilde{F}_\alpha)_x \xrightarrow{\cong} \operatorname*{colim}_{\beta\in\mathcal{N}} H^j(\widetilde{F}_\beta)_x \cong [\text{Lemma 5.1}] \cong H^j\left(\operatorname*{colim}_{\beta\in\mathcal{N}}(\widetilde{F}_\beta)\right)_x$$

and the statement follows. □

Notice that there are inclusion functors

$$\mathrm{Ho}(\mathrm{QCoh}(A.)) \to \mathrm{Ho}(\mathrm{Mod}(A.)) \text{ and } \mathrm{D}_{qc}(X) \to \mathrm{Ho}(\mathrm{DGMod}(\mathcal{O}_X))\,.$$

Our goal is now to show that the total left derived functor $\mathbb{L}\Upsilon_!\colon \mathrm{Ho}(\mathrm{Mod}(A.)) \to \mathrm{Ho}(\mathrm{DGMod}(\mathcal{O}_X))$ maps $\mathrm{Ho}(\mathrm{QCoh}(A.))$ to $\mathrm{D}_{qc}(X)$.

Remark 5.3 Let $\mathrm{D}_{qc}(\mathcal{O}_X)$ be the derived category of cochain complexes of arbitrary $\mathcal{O}_X$-modules over X, with quasi-coherent cohomology. Then the natural functor $\mathrm{D}_{qc}(X) \to \mathrm{D}_{qc}(\mathcal{O}_X)$ is an equivalence of categories, see [5].

Theorem 5.4 *The functor* $\mathbb{L}\Upsilon_!\colon \mathrm{Ho}(\mathrm{Mod}(A.)) \to \mathrm{Ho}(\mathrm{DGMod}(\mathcal{O}_X))$ *maps (classes of) quasi-coherent* $A.$*-modules to (classes of) complexes of quasi-coherent sheaves.*

Proof The statement immediately follows by Proposition 5.2 and Remark 5.3. □

The functor Υ^* obviously maps quasi-coherent sheaves to quasi-coherent $A.$-modules. Therefore by Theorem 5.4 the restricted functors

$$\overline{\mathbb{L}\Upsilon_!}\colon \mathrm{Ho}(\mathrm{QCoh}(A.)) \leftrightarrows \mathrm{D}_{qc}(X)\colon \overline{\mathbb{R}\Upsilon^*}$$

are well-defined.

5.1 The Equivalence $\mathrm{Ho}(\mathrm{QCoh}(A.)) \simeq \mathrm{D}_{qc}(X)$

The aim of this subsection is to show that the adjoint pair

$$\overline{\mathbb{L}\Upsilon_!}\colon \mathrm{Ho}(\mathrm{QCoh}(A.)) \leftrightarrows \mathrm{D}_{qc}(X)\colon \overline{\mathbb{R}\Upsilon}^*$$

introduced in the section above is in fact an equivalence of triangulated categories.

Explicit models for the (unique) DG-enhancement of $\mathrm{D}_{qc}(X)$ already exist, e.g. the category of complexes of injectives. For a survey concerning this topic we refer to [8, 25]. As we shall see, cofibrant $A.$-modules provide another explicit DG-enhancement for $\mathrm{D}_{qc}(X)$, see Corollary 5.8.

Remark 5.5 The functor $\Upsilon^*\colon \mathrm{DGMod}(\mathcal{O}_X) \to \mathrm{Mod}(A.)$ maps quasi-isomorphisms between (complexes of) quasi-coherent sheaves to weak equivalences between quasi-coherent $A.$-modules. This easily follows recalling that cohomology commutes with direct colimits (hence with stalks), see Lemma 5.1. In particular, $\overline{\mathbb{R}\Upsilon}^*[\mathcal{F}] = [\Upsilon^*(\mathcal{F})]$ for every $[\mathcal{F}] \in \mathrm{D}_{qc}(X)$.

Lemma 5.6 *Let $\varphi\colon \mathcal{F} \to \mathcal{G}$ be a morphism in* $\mathrm{QCoh}(A.)$*. Then φ is a weak equivalence if and only if $\Upsilon_!(\varphi)$ is a weak equivalence in* $\mathrm{DGMod}(\mathcal{O}_X)$.

Proof For any $\alpha \in \mathcal{N}$ consider the commutative diagram

$$\begin{array}{ccc} \widetilde{\mathcal{F}_\alpha} & \longrightarrow & \Upsilon_!(\mathcal{F})|_{U_\alpha} \\ \downarrow & & \downarrow \\ \widetilde{\mathcal{G}_\alpha} & \longrightarrow & \Upsilon_!(\mathcal{G})|_{U_\alpha} \end{array}$$

where the horizontal arrows are quasi-isomorphisms in $\mathrm{DGMod}(\mathcal{O}_{U_\alpha})$ by Proposition 5.2. Observe that $\mathcal{F}_\alpha \to \mathcal{G}_\alpha$ is a quasi-isomorphism in $\mathrm{DGMod}(A_\alpha)$ if and only if $\widetilde{\mathcal{F}_\alpha} \to \widetilde{\mathcal{G}_\alpha}$ is so on each stalk in U_α. Then the statement follows by the *2 out of 3* property. □

Notice that Lemma 5.6 implies that $\overline{\mathbb{L}\Upsilon}_![\mathcal{G}] = [\Upsilon_!\mathcal{G}]$ for every $[\mathcal{G}] \in \mathrm{Ho}(\mathrm{QCoh}(A.))$. Hence it is convenient to simply denote by

$$\Upsilon_!\colon\ \mathrm{Ho}(\mathrm{QCoh}(A.)) \leftrightarrows \mathrm{D}_{qc}(X)\colon\ \Upsilon^*$$

the functors $\overline{\mathbb{L}\Upsilon}_!$ and $\overline{\mathbb{R}\Upsilon}^*$.

Theorem 5.7 *The functors $\Upsilon_!\colon\ \mathrm{Ho}(\mathrm{QCoh}(A.)) \leftrightarrows \mathrm{D}_{qc}(X)\colon\ \Upsilon^*$ are equivalences of triangulated categories.*

Proof In order to avoid possible confusion, throughout all the proof we shall keep the notation $\overline{\mathbb{L}\Upsilon}_!$ and $\overline{\mathbb{R}\Upsilon}^*$ to denote the functors in the statement.

First recall that the triangulated structure is preserved because the functors come from a Quillen adjunction. Hence we only need to prove that the natural morphisms

$$\overline{\mathbb{L}\Upsilon}_! \circ \overline{\mathbb{R}\Upsilon}^*[\mathcal{F}] \to [\mathcal{F}] \qquad \text{and} \qquad [\mathcal{G}] \to \overline{\mathbb{R}\Upsilon}^* \circ \overline{\mathbb{L}\Upsilon}_![\mathcal{G}]$$

are isomorphisms for every $[\mathcal{F}] \in \mathrm{D}_{qc}(X)$ and every $[\mathcal{G}] \in \mathrm{Ho}(\mathrm{Mod}(A.))$.

(1) First observe that $\overline{\mathbb{L}\Upsilon}_! \circ \overline{\mathbb{R}\Upsilon}^*[\mathcal{F}] = [\Upsilon_!\Upsilon^*(\mathcal{F})]$ by Remark 5.5 and Lemma 5.6. Moreover, since

$$\begin{aligned} \left(\Upsilon_!\Upsilon^*(\mathcal{F})\right)_x &= \operatorname*{colim}_{(\beta,\gamma)\in \mathbf{L}_{\mathcal{N}}} (i_{\gamma !}(\mathcal{F}|_{U_\gamma}))_x = \operatorname*{colim}_{\{(\beta,\gamma)\in \mathbf{L}_{\mathcal{N}}\,|\, x\in U_\gamma\}} (i_{\gamma !}(\mathcal{F}|_{U_\gamma}))_x \\ &= \operatorname*{colim}_{\beta\in I} (\mathcal{F}|_{U_\beta})_x = \mathcal{F}_x \end{aligned}$$

for every $x \in X$, then the natural map $\Upsilon_!\Upsilon^*(\mathcal{F}) \to \mathcal{F}$ is an isomorphism.

(2) The second natural isomorphism follows by Lemma 5.6 and Proposition 5.2.

□

Theorem 5.7 partially appears in [7, Proposition 2.28], where it is proven that Υ^* is an equivalence on its image.

Define the DG-category $\mathrm{QCoh}^*(A.)^c$ whose objects are cofibrant quasi-coherent $A.$-modules, and whose morphisms are $*$-morphisms, see Definition 3.3. Notice that

$$Z^0\left(\mathrm{QCoh}^*(A.)^c\right) = \mathrm{QCoh}(A.)^c\ .$$

Moreover, every weak equivalence $\mathcal{F} \to \mathcal{G}$ in $\mathrm{Mod}(A.)$ between cofibrant $A.$-modules is in fact an isomorphism up to homotopy; i.e. $H^0\left(\mathrm{QCoh}^*(A.)^c\right) \simeq \mathrm{Ho}\left(\mathrm{QCoh}(A.)^c\right)$.

Corollary 5.8 *The DG-category* $\mathrm{QCoh}^*(A.)^c$ *is a DG-enhancement for the unbounded derived category* $\mathrm{D}_{qc}(X)$.

Proof There are equivalences of triangulated categories

$$H^0\left(\mathrm{QCoh}^*(A.)^c\right) \simeq \mathrm{Ho}\left(\mathrm{QCoh}(A.)^c\right) \simeq \mathrm{Ho}\left(\mathrm{QCoh}(A.)\right) \simeq \mathrm{D}_{qc}(X),$$

where the last one follows by Theorem 5.7. □

6 Derived Endomorphisms of Quasi-coherent Sheaves

Throughout this section we shall consider a fixed finite-dimensional Noetherian separated scheme X over a field $\mathbb{K}$, together with a quasi-coherent sheaf $\mathcal{F}$ on it. Also, we fix an open affine covering $\{U_h\}_{h\in H}$, denoting by $\mathcal{N}$ its nerve.

The first main goal of this section is to give different constructions of the derived endomorphisms $\mathrm{REnd}(\mathcal{F})$. The interest in this object arises in several areas of Algebraic Geometry; for instance it carries a DG-Lie structure controlling infinitesimal deformations of $\mathcal{F}$ as we shall see in Sect. 7.

Recall that $\mathrm{REnd}(\mathcal{F})$ is represented (up to quasi-isomorphisms) by the complex $\mathrm{Hom}^*_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I})$, for any injective resolution $\mathcal{F} \to \mathcal{I}$. Notice that $\mathrm{Hom}^*_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I}) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I})$, up to a sign on the differential.

6.1 REnd($\mathcal{F}$) via $A.$-Modules

The aim of this subsection is to prove that given a cofibrant replacement $\varepsilon\colon \mathcal{Q} \to \Upsilon^*\mathcal{F}$ in $\mathrm{Mod}(A.)$, then the derived endomorphisms of $\mathcal{F}$ are represented by $\mathrm{End}^*_{A.}(\mathcal{Q})$.

For notational convenience we shall also denote by ε the induced map $\Upsilon_!\mathcal{Q} \to \Upsilon_!\Upsilon^*\mathcal{F} = \mathcal{F}$.

Proposition 6.1 *Let* $\mathcal{F}$ *be a quasi-coherent sheaf on* X, *and consider a cofibrant replacement* $\varepsilon\colon \mathcal{Q} \to \Upsilon^*\mathcal{F}$ *in* $\mathrm{Mod}(A.)$. *Then the induced map*

$$\mathrm{Hom}^*_{\mathcal{O}_X}(\Upsilon_!\mathcal{Q}, \mathcal{J}) \xleftarrow{-\circ\varepsilon} \mathrm{Hom}^*_{\mathcal{O}_X}(\mathcal{F}, \mathcal{J})$$

is a quasi-isomorphism for any bounded below complex of injectives $\mathcal{J}$.

Proof Since $\mathcal{J}$ is degreewise injective we have a short exact sequence

$$0 \to \mathrm{Hom}^*_{\mathcal{O}_X}(\mathcal{F}, \mathcal{J}) \to \mathrm{Hom}^*_{\mathcal{O}_X}(\Upsilon_!\mathcal{Q}, \mathcal{J}) \to \mathrm{Hom}^*_{\mathcal{O}_X}(\mathcal{H}, \mathcal{J}) \to 0$$

where $\mathcal{H} = \ker(\varepsilon)$ is acyclic. By standard arguments it is easy to show that any map from an acyclic complex to a bounded below complex of injectives is homotopic to zero, see e.g. [14, III.5.24]. Hence the complex $\mathrm{Hom}^*_{\mathcal{O}_X}(\mathcal{H}, \mathcal{J})$ is acyclic and the statement follows. □

Proposition 6.2 *Let* $\mathcal{F}$ *be a quasi-coherent sheaf on* X, *let* $\varphi\colon \mathcal{F} \to \mathcal{I}$ *be an injective resolution, and consider a cofibrant replacement* $\varepsilon\colon \mathcal{Q} \to \Upsilon^*\mathcal{F}$ *in* $\mathrm{Mod}(A.)$. *Then the maps*

$$\mathrm{Hom}^*_{A.}(\mathcal{Q}, \mathcal{Q}) \xrightarrow{\varepsilon\circ-} \mathrm{Hom}^*_{A.}(\mathcal{Q}, \Upsilon^*\mathcal{F}) \xrightarrow{\varphi\circ-} \mathrm{Hom}^*_{A.}(\mathcal{Q}, \Upsilon^*\mathcal{I})$$

are quasi-isomorphisms.

Proof We shall prove that the functor $\mathrm{Hom}^*_{A.}(\mathcal{Q}, -)\colon \mathrm{Mod}(A.) \to \mathrm{DGMod}(\mathbb{Z})$ maps weak equivalences to quasi-isomorphisms, being $\mathcal{Q}$ cofibrant. Since every object in $\mathrm{Mod}(A.)$ is fibrant, by Ken Brown's Lemma it is sufficient to show that $\mathrm{Hom}^*_{A.}(\mathcal{Q}, -)$ maps trivial fibrations to quasi-isomorphisms. To this aim, take a trivial fibration $f : \mathcal{G} \to \mathcal{H}$ in $\mathrm{Mod}(A.)$. Then we have a short exact sequence

$$0 \to \mathrm{Hom}^*_{A.}(\mathcal{Q}, \ker(f)) \to \mathrm{Hom}^*_{A.}(\mathcal{Q}, \mathcal{G}) \xrightarrow{f\circ -} \mathrm{Hom}^*_{A.}(\mathcal{Q}, \mathcal{H}) \to 0\ ;$$

where the surjectivity comes from Lemma 3.14.

To conclude we need to show that $\mathrm{Hom}^*_{A.}(\mathcal{Q}, \ker(f))$ is acyclic. Notice that every cocycle $[h] \in Z^n\left(\mathrm{Hom}^*_{A.}(\mathcal{Q}, \ker(f))\right)$ is given by a map $h\colon \mathcal{Q} \to \ker(f)[n]$ of $A.$-modules. Now, factor the weak equivalence $0 \to \ker(f)$ as

$$0 \xrightarrow{\iota} \mathrm{cocone}\left(\mathrm{Id}_{\ker(f)[n]}\right) \xrightarrow{\pi} \ker(f)[n]$$

and observe that ι is a weak equivalence and π is a trivial fibration. Hence the square of solid arrows

$$\begin{array}{ccc} 0 & \xrightarrow{\ \iota\ } & \mathrm{cocone}\left(\mathrm{Id}_{\ker(f)[n]}\right) \\ \big\downarrow & \overset{\overline{h}}{\nearrow} & \big\downarrow \pi \\ \mathcal{Q} & \xrightarrow[\ h\ ]{} & \ker(f)[n] \end{array}$$

admits the dotted lifting $\overline{h}\colon \mathcal{Q} \to \mathrm{cocone}\left(\mathrm{Id}_{\ker(f)[n]}\right)$, which in turn implies that h is homotopic to zero, i.e. $[h] = [0] \in H^n\left(\mathrm{Hom}^*_{A.}(\mathcal{Q}, \ker(f))\right)$. □

Remark 6.3 The same argument given in the proof of Proposition 6.2 leads to quasi-isomorphisms

$$\mathrm{Hom}^*_{A.}(\mathcal{Q}, \mathcal{Q})_\alpha \xrightarrow{\varepsilon\circ -} \mathrm{Hom}^*_{A.}(\mathcal{Q}, \Upsilon^*\mathcal{F})_\alpha \xrightarrow{\varphi\circ -} \mathrm{Hom}^*_{A.}(\mathcal{Q}, \Upsilon^*\mathcal{I})_\alpha$$

for every $\alpha \in \mathcal{N}$.

Theorem 6.4 *Let $\mathcal{F}$ be a quasi-coherent sheaf on X, and let $\varepsilon\colon \mathcal{Q} \to \Upsilon^*\mathcal{F}$ be a cofibrant replacement in* $\mathrm{Mod}(A.)$. *Then* $\mathrm{REnd}(\mathcal{F})$ *is represented by* $\mathrm{End}^*_{A.}(\mathcal{Q})$.

Proof First notice that $\mathrm{Hom}^*_{A.}(\mathcal{Q}, \Upsilon^*\mathcal{I}) \cong \mathrm{Hom}^*_{\mathcal{O}_X}(\Upsilon_!\mathcal{Q}, \mathcal{I})$, the proof being the same as Proposition 4.4. Now the statement follows by Propositions 6.2 and 6.1. □

Remark 6.5 Notice that the above proofs easily extend to the general case of a complex of sheaves $\mathcal{F}^* \in \mathrm{DGMod}(\mathcal{O}_X)$, so that the issue is to construct a cofibrant replacement $\varepsilon\colon \mathcal{Q} \to \Upsilon^*\mathcal{F}^*$.

6.1.1 Concrete Computations via *A.*-Modules

The approach via $A.$-modules seems to be fruitful in some geometric situations, see e.g. [3, Section 3] and [4, Section 5]. We shall now construct the cofibrant pseudo-module providing a description of the DG-Lie representative of derived endomorphisms of a complex of locally free sheaves.

Consider any bounded above complex of locally free sheaves $\mathcal{F}^*$ on X. For each $\alpha \in \mathcal{N}$, consider the abstract oriented simplicial complex Δ_α: the faces are given by non-empty subsets of α. The homology of its associated chain complex $C_*(\Delta_\alpha)$ is non-trivial only in degree 0: $H_0(C_*(\Delta_\alpha)) = \mathbb{Z}$. Let us now describe the cofibrant $A_\cdot$-module $\mathcal{Q}$. We begin with the dual cochain complex

$$C^k(\Delta_\alpha) = C_{-k}(\Delta_\alpha) \qquad \partial^k = \partial_{-k}.$$

Then define

$$\mathcal{Q}_\alpha = C^*(\Delta_\alpha) \otimes_{\mathbb{Z}} \mathcal{F}^*(U_\alpha)\ ,$$

whose cohomology gives back the desired complex: $H^*(\mathcal{Q}_\alpha) \cong \mathcal{F}^*(U_\alpha)$. Notice that the projection $C^*(\Delta_\alpha) \to H^0(C^*(\Delta_\alpha))$ induces a map $\mathcal{Q}_\alpha \to \mathcal{F}^*(U_\alpha)$. These data commute with each other (for any $\alpha \le \beta \in \mathcal{N}$); therefore we have constructed a morphism $\varepsilon\colon \mathcal{Q} \to \Upsilon^*\mathcal{F}$ in the category of $A_\cdot$-modules. It is not difficult to check that $\mathcal{Q}$ is cofibrant, see [3, Section 3.2] for details. Now from Theorem 6.4 and Remark 6.5 we obtain that the DG-Lie algebra $\mathrm{Hom}^*_{A_\cdot}(\mathcal{Q}, \mathcal{Q})$ represents the derived endomorphisms of the complex $\mathcal{F}^*$.

Notice that in order to compute cohomology, i.e. $\mathrm{Ext}^*(\mathcal{F}^*, \mathcal{F}^*)$, it can be useful to deal with the complex $\mathrm{Hom}^*_{A_\cdot}(\mathcal{Q}, \Upsilon^*\mathcal{F}^*)$ instead of $\mathrm{Hom}^*_{A_\cdot}(\mathcal{Q}, \mathcal{Q})$.

6.2 REnd($\mathcal{F}$) via Thom–Whitney Totalization

The aim of this subsection is to prove that given a cofibrant replacement $\mathcal{Q} \to \Upsilon^*\mathcal{F}$ in $\mathrm{Mod}(A_\cdot)$, then the derived endomorphisms of $\mathcal{F}$ are represented by the Thom–Whitney totalization of a certain semicosimplicial DG-Lie algebra described in terms of $\mathcal{Q}$, see Definition 6.6.

We begin by recalling the following construction. Let $\{U_j\}_{j\in J}$ be an affine open covering for a finite-dimensional Noetherian separated scheme X. Define

$$\overline{\mathcal{N}}_n = \{(j_0, \dots, j_n) \in J^n | U_{j_0} \cap \dots \cap U_{j_n} \neq \emptyset\}$$

for any $n \in \mathbb{N}$. The **ordered nerve** of $\{U_j\}$ is the disjoint union $\overline{\mathcal{N}} = \coprod_{n\ge 0} \overline{I}_n$. Notice that there exists a map

$$\overline{\mathcal{N}} \to \mathcal{N}\,, \qquad \overline{\alpha} = (j_0, \dots, j_n) \mapsto \alpha = \{j_0, \dots, j_n\}$$

where $\mathcal{N}$ is the nerve of $\{U_j\}$.

Consider $\mathcal{Q} \in \mathrm{Mod}(A_\cdot)$, and for every $n \in \mathbb{N}$ define

$$\mathfrak{L}_n = \prod_{\overline{\alpha} \in \overline{\mathcal{N}} n} \mathrm{Hom}^*_{A_\cdot}(\mathcal{Q}, \mathcal{Q})_\alpha$$

where the product is taken in the category of DG-vector spaces. Notice that $\mathfrak{L}^n$ is a DG-Lie algebra since every $\mathrm{Hom}^*_{A_\cdot}(\mathcal{Q}, \mathcal{Q})_\alpha \subseteq \prod_{\gamma \ge \alpha} \mathrm{Hom}^*_{A_\gamma}(\mathcal{Q}_\gamma, \mathcal{Q}_\gamma)$ inherits a DG-Lie structure, where the bracket is the (graded) commutator. Moreover, for every monotone map $f : [n] \to [m]$ it is induced a map

$$h_f : \overline{\mathcal{N}}_m \to \overline{\mathcal{N}}_n\,, \qquad \overline{\alpha} = (a_0, \dots, a_m) \mapsto h_f(\overline{\alpha}) = (a_{f(0)}, \dots, a_{f(n)})$$

satisfying $h_f(\overline{\alpha}) \le \overline{\alpha}$ for every $\overline{\alpha} \in \overline{\mathcal{N}}$. This in turn gives a map

$$f_* = \left\{f_{\overline{\beta}}\right\}_{\overline{\beta} \in \overline{\mathcal{N}}_m} : \mathfrak{L}_n \to \mathfrak{L}_m \quad \text{defined by} \quad f_{\overline{\beta}}\left(\{\varphi_{\overline{\alpha}}\}_{\overline{\alpha} \in \overline{\mathcal{N}}_n}\right) = \pi_{h_f(\overline{\beta})\overline{\beta}}\left(\varphi_{h_f(\overline{\beta})}\right) \in \mathfrak{L}_n\,,$$

where $\pi_{h_f(\overline{\beta})\overline{\beta}}\colon \operatorname{Hom}^*_{A.}(\mathcal{Q},\mathcal{Q})_{h_f(\overline{\beta})} \to \operatorname{Hom}^*_{A.}(\mathcal{Q},\mathcal{Q})_\beta$ is the natural projection.

Definition 6.6 For every $n \in \mathbb{N}$ and every $0 \le k \le n+1$, define $\delta^k\colon [n] \to [n+1]$ as

$$\delta^k(p) = \begin{cases} p & \text{if } p < k \\ p+1 & \text{if } p \ge k \end{cases}$$

Then the maps δ^k_* induce the semicosimplicial DG-Lie algebra

$$\mathfrak{L}\colon \qquad \mathfrak{L}_0 \rightrightarrows \mathfrak{L}_1 \substack{\textstyle\rightarrow\\[-1ex]\textstyle\rightarrow\\[-1ex]\textstyle\rightarrow} \mathfrak{L}_1 \substack{\textstyle\rightarrow\\[-1ex]\textstyle\rightarrow\\[-1ex]\textstyle\rightarrow\\[-1ex]\textstyle\rightarrow} \cdots$$

Similarly we now introduce three semicosimplicial complexes. Let $\mathcal{Q} \to \Upsilon^*\mathcal{F}$ be a cofibrant replacement for $\Upsilon^*\mathcal{F}$ in $\operatorname{Mod}(A.)$ and consider an injective resolution $\mathcal{F} \to \mathcal{I}$, then define

$$\mathfrak{B}^{\mathcal{QF}}\colon \qquad \mathfrak{B}^{\mathcal{QF}}_0 = \prod_{\overline{\alpha}\in\overline{\mathcal{N}}_0} \operatorname{Hom}^*_{A.}(\mathcal{Q},\Upsilon^*\mathcal{F})_\alpha \rightrightarrows \mathfrak{B}^{\mathcal{QF}}_1 = \prod_{\overline{\alpha}\in\overline{\mathcal{N}}_1} \operatorname{Hom}^*_{A.}(\mathcal{Q},\Upsilon^*\mathcal{F})_\alpha \substack{\textstyle\rightarrow\\[-1ex]\textstyle\rightarrow\\[-1ex]\textstyle\rightarrow} \cdots$$

$$\mathfrak{B}^{\mathcal{QI}}\colon \qquad \mathfrak{B}^{\mathcal{QI}}_0 = \prod_{\overline{\alpha}\in\overline{\mathcal{N}}_0} \operatorname{Hom}^*_{A.}(\mathcal{Q},\Upsilon^*\mathcal{I})_\alpha \rightrightarrows \mathfrak{B}^{\mathcal{QI}}_1 = \prod_{\overline{\alpha}\in\overline{\mathcal{N}}_1} \operatorname{Hom}^*_{A.}(\mathcal{Q},\Upsilon^*\mathcal{I})_\alpha \substack{\textstyle\rightarrow\\[-1ex]\textstyle\rightarrow\\[-1ex]\textstyle\rightarrow} \cdots$$

$$\mathfrak{B}^{\mathcal{FI}}\colon \qquad \mathfrak{B}^{\mathcal{FI}}_0 = \prod_{\overline{\alpha}\in\overline{\mathcal{N}}_0} \operatorname{Hom}^*_{\mathcal{O}_X}(i_{\alpha!}(\mathcal{F}|_{U_\alpha}),\mathcal{I}) \rightrightarrows \mathfrak{B}^{\mathcal{FI}}_1 = \prod_{\overline{\alpha}\in\overline{\mathcal{N}}_1} \operatorname{Hom}^*_{\mathcal{O}_X}(i_{\alpha!}(\mathcal{F}|_{U_\alpha}),\mathcal{I}) \substack{\textstyle\rightarrow\\[-1ex]\textstyle\rightarrow\\[-1ex]\textstyle\rightarrow} \cdots$$

where we denoted by $i_\alpha\colon U_\alpha \to X$ the natural inclusion. Notice that the maps defined in Propositions 6.1 and in Proposition 6.2 induce semicosimplicial morphisms

$$\mathfrak{L} \to \mathfrak{B}^{\mathcal{QF}} \to \mathfrak{B}^{\mathcal{QI}} \leftarrow \mathfrak{B}^{\mathcal{FI}}.$$

Recall that for a semicosimplicial DG-vector space V the Thom–Whitney–Sullivan totalization is the DG-vector space defined by

$$\operatorname{Tot}(V) = \left\{ (x_n) \in \prod_{n\ge 0} \Omega_n \otimes V_n \;\middle|\; (\delta^*_k \otimes \operatorname{Id})x_n = (\operatorname{Id}\otimes\delta_k)x_{n-1} \text{ for every } 0 \le k \le n \right\}$$

where $\Omega_n = \frac{\mathbb{K}[t_0,\ldots,t_n,dt_0,\ldots,dt_n]}{(\sum t_i - 1, \sum dt_i)}$ is the graded algebra of polynomial differential forms on the n-simplex. Moreover, to every semicosimplicial DG-vector space V is associated the complex

$$C(V)=\bigoplus_{p\in\mathbb{N}}\prod_{n\in\mathbb{N}}V_n[-n]^p=\bigoplus_{p\in\mathbb{N}}\prod_{n\in\mathbb{N}}V_n^{p-n}$$

which is quasi-isomorphic to the totalization via the Whitney integration map $\int\colon \operatorname{Tot}(V)\to C(V)$, see [38]. Given a map of DG-vector spaces $g\colon W\to V_0$ satisfying $\delta_0 g=\delta_1 g$, it is induced a morphism $\hat{g}\colon W\to\operatorname{Tot}(V)$ defined by $\hat{g}(w)=(1\otimes g(w), 1\otimes\delta_0 g(w), 1\otimes\delta_0^2 g(w),\dots)$. Using the semicosimplicial identities it is straightforward to prove that the composition $\int\circ g$ is in fact the composition of g with the natural inclusion $V_0\to C(V)$. In this way it is induced a natural map

$$\operatorname{Hom}^*_{A.}(\mathcal{Q},\mathcal{Q})\to\operatorname{Tot}(\mathfrak{L})$$

which respects the DG-Lie structure.

The aim of this subsection is to prove that $\operatorname{Hom}^*_{A.}(\mathcal{Q},\mathcal{Q})\to\operatorname{Tot}(\mathfrak{L})$ is a quasi-isomorphism of DG-associative algebras. Actually we shall prove much more: there exists a commutative diagram

$$\begin{array}{ccccccc}
\operatorname{Hom}^*_{A.}(\mathcal{Q},\mathcal{Q}) & \longrightarrow & \operatorname{Hom}^*_{A.}(\mathcal{Q},\Upsilon^*\mathcal{F}) & \longrightarrow & \operatorname{Hom}^*_{A.}(\mathcal{Q},\Upsilon^*\mathcal{I}) & \longleftarrow & \operatorname{Hom}^*_{\mathcal{O}_X}(\mathcal{F},\mathcal{I}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\xi} \\
\operatorname{Tot}(\mathfrak{L}) & \longrightarrow & \operatorname{Tot}(\mathfrak{B}^{\mathcal{QF}}) & \longrightarrow & \operatorname{Tot}(\mathfrak{B}^{\mathcal{QI}}) & \longleftarrow & \operatorname{Tot}(\mathfrak{B}^{\mathcal{FI}}) \\
\downarrow{\scriptstyle\int} & & \downarrow{\scriptstyle\int} & & \downarrow{\scriptstyle\int} & & \downarrow{\scriptstyle\int} \\
C(\mathfrak{L}) & \longrightarrow & C(\mathfrak{B}^{\mathcal{QF}}) & \longrightarrow & C(\mathfrak{B}^{\mathcal{QI}}) & \longleftarrow & C(\mathfrak{B}^{\mathcal{FI}})\,.
\end{array} \tag{6.1}$$

where all maps are quasi-isomorphisms.

Lemma 6.7 *The vertical map* $\operatorname{Hom}^*_{\mathcal{O}_X}(\mathcal{F},\mathcal{I})\xrightarrow{\xi}\operatorname{Tot}(\mathfrak{B}^{\mathcal{FI}})$ *appearing in diagram (6.1) is a quasi-isomorphism.*

Proof As already noticed above the Whitney integration map $\int\colon\operatorname{Tot}(\mathfrak{B}^{\mathcal{FI}})\to C(\mathfrak{B}^{\mathcal{FI}})$ is a quasi-isomorphism. Therefore, in order to prove the statement it is sufficient to show that the composition $\int\circ\xi$ is an isomorphism in cohomology. To this aim we introduce two double complexes

$$A_{ij}=\begin{cases}\operatorname{Hom}^i_{\mathcal{O}_X}(\mathcal{F},\mathcal{I}) & \text{if } j=0\\ 0 & \text{otherwise}\end{cases}\qquad B_{ij}=\prod_{\overline{\alpha}\in\overline{\mathcal{N}}_j}\operatorname{Hom}^i_{\mathcal{O}_{U_\alpha}}\left(\mathcal{F}|_{U_\alpha},\mathcal{I}|_{U_\alpha}\right)$$

defined for $i,j\geq 0$. Restrictions give a map of double complexes $\{A_{ij}\to B_{ij}\}_{i,j\geq 0}$, which in turn corresponds to a morphism between the associated complexes

$$f\colon A^{\cdot}=\bigoplus_{n\in\mathbb{N}}\operatorname{Hom}^n_{\mathcal{O}_X}(\mathcal{F},\mathcal{I})\to B^{\cdot}=\bigoplus_{n\in\mathbb{N}}\bigoplus_{i=0}^{n}B_{n-i,i}\,.$$

Now, consider the following complete and exhaustive filtrations

$$F^pA^{\cdot}=\bigoplus_{i\geq p}\operatorname{Hom}^i_{\mathcal{O}_X}(\mathcal{F},\mathcal{I})\,,\qquad F^pB^{\cdot}=\bigoplus_{i\geq p}\bigoplus_{j\geq 0}B_{i,j}\,,\qquad p\in\mathbb{N}$$

together with the induced morphism

$$\hat{f}^p\colon \frac{F^pA^\cdot}{F^{p+1}A^\cdot} = \mathrm{Hom}^p_{\mathcal{O}_X}(\mathcal{F},\mathcal{I}) \longrightarrow \bigoplus_{j\geq 0}\prod_{\overline{\alpha}\in\overline{\mathcal{N}}_j}\mathrm{Hom}^p_{\mathcal{O}_{U_\alpha}}\left(\mathcal{F}|_{U_\alpha},\mathcal{I}|_{U_\alpha}\right) = \frac{F^pB^\cdot}{F^{p+1}B^\cdot}\ .$$

Observe that for every $p\in\mathbb{N}$ the map $\hat{f}^p$ is a quasi-isomorphism; in fact by the degreewise injectivity of $\mathcal{I}$ it follows that the restriction map

$$\mathrm{Hom}^p_{\mathcal{O}_X}(\mathcal{F},\mathcal{I}) \to \mathrm{Hom}^p_{\mathcal{O}_X}(i_!\mathcal{F},\mathcal{I}) = \mathrm{Hom}^p_{\mathcal{O}_{X|U}}(\mathcal{F}|_V,\mathcal{I}|_V)$$

is surjective for every open subset $i\colon V\to X$, therefore the sequence

$$0\to \mathrm{Hom}^p_{\mathcal{O}_X}(\mathcal{F},\mathcal{I}) \to \prod_{\overline{\alpha}\in\overline{\mathcal{N}}_0}\mathrm{Hom}^p_{\mathcal{O}_{U_\alpha}}\left(\mathcal{F}|_{U_\alpha},\mathcal{I}|_{U_\alpha}\right) \to \prod_{\overline{\beta}\in\overline{\mathcal{N}}_1}\mathrm{Hom}^p_{\mathcal{O}_{U_\beta}}\left(\mathcal{F}|_{U_\beta},\mathcal{I}|_{U_\beta}\right) \to \cdots$$

is exact because flasque sheaves are acyclic. It follows that the map $f\colon A^\cdot\to B^\cdot$ is a quasi-isomorphism.

To conclude the proof it is sufficient to observe that f is indeed the composition $\int\circ\xi$. Clearly $A^\cdot = \mathrm{Hom}^*_{\mathcal{O}_X}(\mathcal{F},\mathcal{I})$; moreover

$$\begin{aligned} B^\cdot &= \bigoplus_{n\in\mathbb{N}}\bigoplus_{i=0}^{n} B_{n-i,i} = \bigoplus_{n\in\mathbb{N}}\bigoplus_{i=0}^{n}\prod_{\overline{\alpha}\in\overline{\mathcal{N}}_i}\mathrm{Hom}^{n-i}_{\mathcal{O}_{U_\alpha}}\left(\mathcal{F}|_{U_\alpha},\mathcal{I}|_{U_\alpha}\right) \\ &= \bigoplus_{n\in\mathbb{N}}\prod_{i=0}^{n}\mathrm{Hom}^{n-i}_{\mathcal{O}_{U_\alpha}}\left(\bigoplus_{\overline{\alpha}\in\overline{\mathcal{N}}_i} i_{\alpha !}(\mathcal{F}|_{U_\alpha}),\mathcal{I}\right) \end{aligned}$$

so that $B^\cdot = C(\mathfrak{B}^{\mathcal{F}\mathcal{I}})$. Now, the map $\int\circ\xi$ is the same as the composition

$$\mathrm{Hom}^*_{\mathcal{O}_X}(\mathcal{F},\mathcal{I}) \to \prod_{\alpha\in\mathcal{N}_0}\mathrm{Hom}^*_{\mathcal{O}_{U_\alpha}}\left(\mathcal{F}|_{U_\alpha},\mathcal{I}|_{U_\alpha}\right) \to C(\mathfrak{B}^{\mathcal{F}\mathcal{I}})$$

which is precisely f as claimed. □

Theorem 6.8 *All the maps appearing in diagram* (6.1) *are quasi-isomorphisms.*

Proof The maps in the first row have been discussed in Propositions 6.1 and 6.3. Now, recall that to prove that the map between complexes associated to semicosimplicial DG-vector spaces is a quasi-isomorphisms, it is sufficient to prove that it is induced by a semicosimplicial quasi-isomorphism between them. By Remark 6.3 and by Proposition 6.1 there are quasi-isomorphisms

$$\begin{array}{ccccc} \mathrm{Hom}^*_{A.}(\mathcal{Q},\mathcal{Q})_\alpha & \xrightarrow{\varepsilon\circ -} & \mathrm{Hom}^*_{A.}(\mathcal{Q},\Upsilon^*\mathcal{F})_\alpha & \xrightarrow{\varphi\circ -} & \mathrm{Hom}^*_{A.}(\mathcal{Q},\Upsilon^*\mathcal{I})_\alpha \\ & & & & \downarrow\cong \\ & & & & \mathrm{Hom}^*_{\mathcal{O}_{U_\alpha}}(\Upsilon_!\mathcal{Q},\mathcal{I}) \xleftarrow{-\circ\varepsilon} \mathrm{Hom}^*_{\mathcal{O}_{U_\alpha}}\left(\mathcal{F}|_{U_\alpha},\mathcal{I}|_{U_\alpha}\right) \end{array}$$

for every $\alpha\in\mathcal{N}$, which in turn induce semicosimplicial quasi-isomorphisms

$$\mathfrak{L}\to\mathfrak{B}^{\mathcal{Q}\mathcal{F}}\to\mathfrak{B}^{\mathcal{Q}\mathcal{I}}\leftarrow\mathfrak{B}^{\mathcal{F}\mathcal{I}}.$$

Therefore the maps in the bottom row are all quasi-isomorphisms. Moreover, since for every DG-vector space V the map $\int\colon \operatorname{Tot}(V) \to C(V)$ is a quasi-isomorphism, by the *2 out of 3* property also the maps in the middle row are quasi-isomorphisms.

To conclude the proof recall that the map $\xi\colon \operatorname{Hom}^*_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I}) \to \operatorname{Tot}(\mathfrak{B}^{\mathcal{F}\mathcal{I}})$ is a quasi-isomorphism by Lemma 6.7. Hence the statement follows again by the *2 out of 3* property. □

Corollary 6.9 *Let $\mathcal{F}$ be a quasi-coherent sheaf on X, and let $\varepsilon\colon \mathcal{Q} \to \Upsilon^*\mathcal{F}$ be a cofibrant replacement in* $\operatorname{Mod}(A.)$. *Denote by $\mathfrak{L}$ the semicosimplicial DG-Lie algebra introduced in Definition* 6.6. *Then* $\operatorname{REnd}(\mathcal{F})$ *is represented by* $\operatorname{Tot}(\mathfrak{L})$.

Proof Immediate consequence of Theorems 6.4 and 6.8 □

Remark 6.10 Another consequence of Theorem 6.8 is the existence of a quasi-isomorphism of differential graded Lie algebras $\operatorname{Hom}^*_{A.}(\mathcal{Q}, \mathcal{Q}) \to \operatorname{Tot}(\mathfrak{L})$. This implies that the associated deformations functors defined through Maurer–Cartan elements modulo gauge equivalence are isomorphic:

$$\operatorname{Def}_{\operatorname{Hom}^*_{A.}(\mathcal{Q},\mathcal{Q})} \cong \operatorname{Def}_{\operatorname{Tot}(\mathfrak{L})}$$

see [29, Corollary 5.52].

6.3 REnd($\mathcal{F}$) in Presence of a Locally Free Resolution

Let $\mathcal{E} \to \mathcal{F}$ be a locally free resolution for a quasi-coherent sheaf $\mathcal{F}$ over X. Recall that if X is smooth projective such a resolution always exists, but we keep working in full generality only assuming X to be a finite-dimensional separated Noetherian scheme over $\mathbb{K}$. Moreover we choose an affine open cover $\{U_h\}_{h\in H}$ for X such that the restriction $\mathcal{E}|_{U_\alpha}$ is a complex of free sheaves for every $\alpha \in \mathcal{N}$. Notice that:

(1) $\Upsilon^*\mathcal{E} \in \operatorname{Mod}(A.)$ is quasi-coherent,
(2) $(\Upsilon^*\mathcal{E})_\alpha$ is cofibrant in $\operatorname{DGMod}(A_\alpha)$ for every $\alpha \in \mathcal{N}$,
(3) $\Upsilon^*\mathcal{E}$ is not necessarily cofibrant in $\operatorname{Mod}(A.)$.

Lemma 6.11 *Let $\mathcal{E} \to \mathcal{F}$ be a locally free resolution, and consider a cofibrant replacement $\mathcal{Q} \xrightarrow{\pi} \Upsilon^*\mathcal{E}$ in* $\operatorname{Mod}(A.)$. *Fix $\alpha \in \mathcal{N}$; then all the maps in the commutative square*

$$\begin{array}{ccc}
\operatorname{Hom}^*_{A.}(\mathcal{Q}, \Upsilon^*\mathcal{F})_\alpha & \longleftarrow & \operatorname{Hom}^*_{A.}(\Upsilon^*\mathcal{E}, \Upsilon^*\mathcal{F})_\alpha \\
\downarrow & & \downarrow \\
\operatorname{Hom}^*_{A_\alpha}(\mathcal{Q}_\alpha, (\Upsilon^*\mathcal{F})_\alpha) & \longleftarrow & \operatorname{Hom}^*_{A_\alpha}((\Upsilon^*\mathcal{E})_\alpha, (\Upsilon^*\mathcal{F})_\alpha)
\end{array}$$

are quasi-isomorphisms, where the vertical arrows are the natural projections.

Proof First notice that the vertical arrow on the right is clearly an isomorphism. Moreover, the bottom arrow is a quasi-isomorphism because it is induced by the map $\mathcal{Q}_\alpha \to (\Upsilon^*\mathcal{E})_\alpha$, which is a weak equivalence between cofibrant objects in $\operatorname{DGMod}(A_\alpha)$. By the *2 out of 3* axiom it is then sufficient to prove that the projection

$$\pi\colon \operatorname{Hom}^*_{A.}(\mathcal{Q}, \Upsilon^*\mathcal{F})_\alpha \to \operatorname{Hom}^*_{A_\alpha}(\mathcal{Q}_\alpha, (\Upsilon^*\mathcal{F})_\alpha)$$

is a quasi-isomorphism. We begin by showing the surjectivity in cohomology. To this aim, take $\varphi_\alpha \in Z^0\left(\mathrm{Hom}^*_{A_\alpha}(\mathcal{Q}_\alpha, (\Upsilon^*\mathcal{F})_\alpha)\right) = \mathrm{Hom}_{A_\alpha}(\mathcal{Q}_\alpha, (\Upsilon^*\mathcal{F})_\alpha)$. By induction, fix $\beta \in \mathcal{N}$ such that $\alpha < \beta$ and suppose we have already constructed maps $\varphi_\gamma \in \mathrm{Hom}_{A_\gamma}(\mathcal{Q}_\gamma, (\Upsilon^*\mathcal{F})_\gamma)$ for every

$$\gamma \in \mathcal{R}_{\alpha\beta} = \{\gamma \in \mathcal{N} \mid \alpha \le \gamma < \beta\}$$

satisfying the necessary commutativity relations. In order to define $\varphi_\beta \in \mathrm{Hom}_{A_\beta}(\mathcal{Q}_\beta, (\Upsilon^*\mathcal{F})_\beta)$ first notice that the map

$$\operatorname*{colim}_{\gamma\in\mathcal{R}_{\alpha\beta}}(\mathcal{Q}_\gamma \otimes_{A_\gamma} A_\beta) \to \mathcal{Q}_\beta$$

is a cofibration in $\mathrm{DGMod}(A_\beta)$ by Remark 3.17. Notice that $\mathcal{Q}$ is a quasi-coherent $A_\cdot$-module by Remark 3.13, so that the map

$$\left\{\mathcal{Q}_\gamma \otimes_{A_\gamma} A_\beta \to \mathcal{Q}_\beta\right\}_{\gamma\in\mathcal{R}_{\alpha\beta}}$$

is a Reedy weak equivalence. Moreover, the diagram $\left\{\mathcal{Q}_\gamma \otimes_{A_\gamma} A_\beta\right\}_{\gamma\in\mathcal{R}_{\alpha\beta}}$ is Reedy cofibrant by Remark 3.17, and $\left\{\mathcal{Q}_\beta\right\}_{\gamma\in\mathcal{R}_{\alpha\beta}}$ is Reedy cofibrant since $\mathcal{R}_{\alpha\beta}$ is connected. It follows that the map

$$\operatorname*{colim}_{\gamma\in\mathcal{R}_{\alpha\beta}}(\mathcal{Q}_\gamma \otimes_{A_\gamma} A_\beta) \to \operatorname*{colim}_{\gamma\in\mathcal{R}_{\alpha\beta}} \mathcal{Q}_\beta \cong \mathcal{Q}_\beta$$

is a weak equivalence since the left Quillen functor $\mathrm{colim}\colon\ \mathrm{DGMod}(A_\beta)^{\mathcal{R}_{\alpha\beta}} \to \mathrm{DGMod}(A_\beta)$ preserves weak equivalences between Reedy cofibrant objects by Ken Brown's Lemma. Hence the diagram

$$\begin{array}{ccc} \operatorname*{colim}_{\gamma\in\mathcal{R}_{\alpha\beta}}(\mathcal{Q}_\gamma \otimes_{A_\gamma} A_\beta) & \longrightarrow & (\Upsilon^*\mathcal{F})_\beta \\ \mathcal{CW}\big\downarrow & \overset{\varphi_\beta}{\nearrow} & \\ \mathcal{Q}_\beta & & \end{array}$$

admits the required dotted lifting. This proves that π is surjective in cohomology in degree 0. For the general case it is sufficient to observe that

$$Z^n\left(\mathrm{Hom}^*_{A_\alpha}(\mathcal{Q}_\alpha, (\Upsilon^*\mathcal{F})_\alpha)\right) \cong Z^0\left(\mathrm{Hom}^*_{A_\alpha}(\mathcal{Q}_\alpha, (\Upsilon^*\mathcal{F})_\alpha[n])\right) .$$

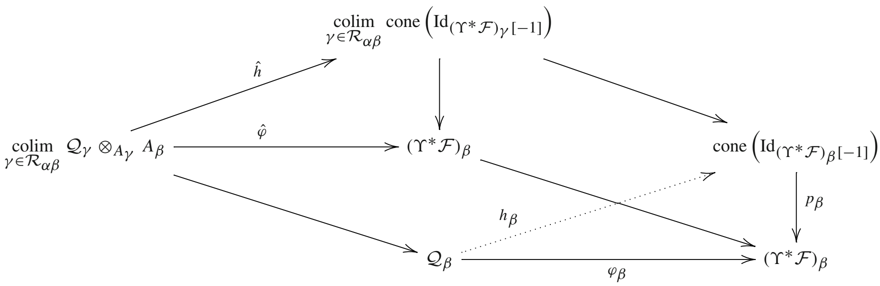
We are left with the proof of the injectivity of π in cohomology. To this aim, take $\{\varphi_\gamma\}_{\gamma\ge\alpha}$ in $\mathrm{Hom}_{A_\cdot}(\mathcal{Q}, \Upsilon^*\mathcal{F})_\alpha$ and suppose that $\varphi_\alpha\colon \mathcal{Q}_\alpha \to (\Upsilon^*\mathcal{F})_\alpha$ is homotopic to the zero map; i.e. $\pi(\{\varphi_\gamma\}) = 0$ in $H^0\left(\mathrm{Hom}^*_{A_\alpha}(\mathcal{Q}_\alpha, (\Upsilon^*\mathcal{F})_\alpha)\right)$. This is equivalent to say that the diagram of solid arrows

$$\begin{array}{ccc} & & \mathrm{cone}\left(\mathrm{Id}_{(\Upsilon^*\mathcal{F})_\alpha[-1]}\right) \\ & \overset{h_\alpha}{\nearrow} & \big\downarrow p_\alpha \\ \mathcal{Q}_\alpha & \xrightarrow[\varphi_\alpha]{} & (\Upsilon^*\mathcal{F})_\alpha \end{array}$$

admits the dotted lifting h_α. Recall that

$$\mathrm{cone}\left(\mathrm{Id}_{(\Upsilon^*\mathcal{F})_\alpha[-1]}\right) = (\Upsilon^*\mathcal{F})_\alpha \oplus (\Upsilon^*\mathcal{F})_\alpha[-1]$$

as graded A_α-modules, and p_α is the projection on the first summand. In order to prove that $\{\varphi_\gamma\}$ is exact we proceed by induction: fix $\beta \in \mathcal{N}$ such that $\alpha < \beta$ and suppose that the homotopy h_α has been lifted to $h_\gamma\colon \mathcal{Q}_\gamma \to \operatorname{cone}\left(\operatorname{Id}_{(\Upsilon^*\mathcal{F})_\gamma[-1]}\right)$ for every $\gamma \in \mathcal{R}_{\alpha\beta} = \{\gamma \in \mathcal{N} \mid \alpha \le \gamma < \beta\}$. We need to prove the existence of the dotted lifting in the diagram below



where $\hat{h}$ is induced by $\{h_\gamma\}_{\gamma\in\mathcal{R}_{\alpha\beta}}$ and $\hat{\varphi}$ is induced by $\{\varphi_\gamma\}_{\gamma\in\mathcal{R}_{\alpha\beta}}$. Notice that p_β is surjective (hence a fibration), and $\operatorname{colim}_{\gamma\in\mathcal{R}_{\alpha\beta}} \mathcal{Q}_\gamma \otimes_{A_\gamma} A_\beta \to \mathcal{Q}_\beta$ is a trivial cofibration as proved above; therefore the statement follows by the lifting property. □

Remark 6.12 Even if $\mathcal{F}$ does not admit a locally free resolution, we can consider a cofibrant replacement $\mathcal{Q} \to \Upsilon^*\mathcal{F}$ in $\operatorname{Mod}(A.)$: the same argument of Lemma 6.11 shows that the projection

$$\operatorname{Hom}^*_{A.}(\mathcal{Q}, \Upsilon^*\mathcal{F})_\alpha \to \operatorname{Hom}^*_{A_\alpha}(\mathcal{Q}_\alpha, (\Upsilon^*\mathcal{F})_\alpha)$$

is a quasi-isomorphism.

Remark 6.13 In the proof of Lemma 6.11, the fact that $\Upsilon^*\mathcal{F}$ is concentrated in degree 0 does not play any role. Therefore for every $\alpha \in \mathcal{N}$ the same argument leads to a quasi-isomorphism

$$-\circ\pi\colon \operatorname{Hom}^*_{A.}(\Upsilon^*\mathcal{E}, \Upsilon^*\mathcal{E})_\alpha \to \operatorname{Hom}^*_{A.}(\mathcal{Q}, \Upsilon^*\mathcal{E})_\alpha$$

where $\pi\colon \mathcal{Q} \to \Upsilon^*\mathcal{E}$ is a cofibrant replacement in $\operatorname{Mod}(A.)$.

Given a locally free resolution $\mathcal{E} \to \mathcal{F}$ on X, we consider the associated Čech semicosimplicial DG-Lie algebra

$$\mathfrak{h}\colon \qquad \prod_{\overline{\alpha}\in\mathcal{N}_0} \operatorname{Hom}^*_{\mathcal{O}_{U_\alpha}}(\mathcal{E}|_{U_\alpha}, \mathcal{E}|_{U_\alpha}) \rightrightarrows \prod_{\overline{\beta}\in\mathcal{N}_1} \operatorname{Hom}^*_{\mathcal{O}_{U_\beta}}(\mathcal{E}|_{U_\beta}, \mathcal{E}|_{U_\beta}) \rightrightarrows\!\!\!\!\rightarrow \cdots$$

which will give us another model for derived endomorphisms of $\mathcal{F}$.

Theorem 6.14 *Let $\mathcal{F}$ be a quasi-coherent sheaf on X, and let $\mathcal{E} \to \mathcal{F}$ be a locally free resolution. Denote by $\mathfrak{h}$ the Čech semicosimplicial DG-Lie algebra as above. Then* $\operatorname{REnd}(\mathcal{F})$ *is represented by* $\operatorname{Tot}(\mathfrak{h})$.

Proof Take a cofibrant replacement $\mathcal{Q} \to \Upsilon^*\mathcal{E}$ in $\operatorname{Mod}(A.)$ and fix $\alpha \in \mathcal{N}$. By Lemma 6.11 there exists a quasi-isomorphism

$$\begin{aligned}\operatorname{Hom}^*_{A.}(\mathcal{Q}, \Upsilon^*\mathcal{F})_\alpha \leftarrow \operatorname{Hom}^*_{A.}(\Upsilon^*\mathcal{E}, \Upsilon^*\mathcal{F})_\alpha &\cong \operatorname{Hom}^*_{A_\alpha}\left((\Upsilon^*\mathcal{E})_\alpha, (\Upsilon^*\mathcal{F})_\alpha\right)\\ &\cong \operatorname{Hom}^*_{\mathcal{O}_{U_\alpha}}(\mathcal{E}|_{U_\alpha}, \mathcal{F}|_{U_\alpha})\,.\end{aligned}$$

Moreover the map $\mathrm{Hom}^*_{\mathcal{O}_{U_\alpha}}(\mathcal{E}|_{U_\alpha}, \mathcal{E}|_{U_\alpha}) \to \mathrm{Hom}^*_{\mathcal{O}_{U_\alpha}}(\mathcal{E}|_{U_\alpha}, \mathcal{F}|_{U_\alpha})$ is a quasi-isomorphism, being $\mathcal{E}|_{U_\alpha}$ a complex of free sheaves. Therefore we obtain a quasi-isomorphism

$$\mathrm{Hom}^*_{\mathcal{O}_{U_\alpha}}(\mathcal{E}|_{U_\alpha}, \mathcal{E}|_{U_\alpha}) \to \mathrm{Hom}^*_{A.}(\mathcal{Q}, \Upsilon^*\mathcal{F})_\alpha$$

which extends to a semicosimplicial quasi-isomorphism $\mathfrak{h} \to \mathfrak{B}^{\mathcal{QF}}$, so that the induced map $\mathrm{Tot}(\mathfrak{h}) \to \mathrm{Tot}(\mathfrak{B}^{\mathcal{QF}})$ is a quasi-isomorphism. The statement follows by Theorem 6.8 and Corollary 6.9. $\square$

Theorem 6.13 essentially states that $H^k(\mathrm{Tot}(\mathfrak{h})) = \mathrm{Ext}^k_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$ for every $k \in \mathbb{N}$. For future purposes, we are now interested in a stronger result, namely that $\mathrm{Tot}(\mathfrak{h})$, $\mathrm{Tot}(\mathfrak{L})$ and $\mathrm{End}^*_{A.}(\mathcal{Q})$ are quasi-isomorphic as DG-Lie algebras, so that in particular the associated deformation functors $\mathrm{Def}_{\mathrm{Tot}(\mathfrak{h})}$, $\mathrm{Def}_{\mathrm{Tot}(\mathfrak{L})}$ and $\mathrm{Def}_{\mathrm{Hom}^*_{A.}(\mathcal{Q})}$ will be isomorphic to each other. Recall that it has been already proven in Sect. 6.2 that $\mathrm{Def}_{\mathrm{Tot}(\mathfrak{L})} \cong \mathrm{Def}_{\mathrm{Hom}^*_{A.}(\mathcal{Q})}$.

Lemma 6.15 *Let $\mathcal{E} \to \mathcal{F}$ be a locally free resolution, and consider a cofibrant replacement $\mathcal{Q} \xrightarrow{\pi} \Upsilon^*\mathcal{E}$ in $\mathrm{Mod}(A.)$. Fix $\alpha \in \mathcal{N}$ and define the DG-Lie algebra*

$$M_\alpha = \left\{(f, g) \in \mathrm{Hom}^*_{A.}(\mathcal{Q}, \mathcal{Q})_\alpha \times \mathrm{Hom}^*_{A.}(\Upsilon^*\mathcal{E}, \Upsilon^*\mathcal{E})_\alpha \mid \pi \circ f = g \circ \pi\right\} .$$

Then there exists a commutative square

$$\begin{array}{ccc} M_\alpha & \xrightarrow{p_2} & \mathrm{Hom}^*_{A.}(\Upsilon^*\mathcal{E}, \Upsilon^*\mathcal{E})_\alpha \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle -\circ\pi} \\ \mathrm{Hom}^*_{A.}(\mathcal{Q}, \mathcal{Q})_\alpha & \xrightarrow[\pi\circ -]{} & \mathrm{Hom}^*_{A.}(\mathcal{Q}, \Upsilon^*\mathcal{E})_\alpha \end{array}$$

where every map is a quasi-isomorphism.

Proof First notice that the map

$$\pi \circ - \colon \mathrm{Hom}^*_{A.}(\mathcal{Q}, \mathcal{Q})_\alpha \to \mathrm{Hom}^*_{A.}(\mathcal{Q}, \Upsilon^*\mathcal{E})_\alpha$$

is a quasi-isomorphism being $\mathcal{Q}$ cofibrant in $\mathrm{Mod}(A.)$, see Remark 3.17. Moreover, the map

$$- \circ \pi \colon \mathrm{Hom}^*_{A.}(\Upsilon^*\mathcal{E}, \Upsilon^*\mathcal{E})_\alpha \to \mathrm{Hom}^*_{A.}(\mathcal{Q}, \Upsilon^*\mathcal{E})_\alpha$$

is a quasi-isomorphism by Remark 6.13. By the functoriality of cohomology, to prove the statement it is sufficient to show that the projection p_1 is a quasi-isomorphism. To this aim, first observe that $\mathcal{Q}$ is cofibrant and π is surjective, so that the map p_1 is surjective by Lemma 3.14. Moreover, the complex $\ker(p_1) = \mathrm{Hom}^*_{A.}(\mathcal{Q}, \ker(\pi))_\alpha$ is acyclic, being $\mathcal{Q}$ cofibrant and $\ker(\pi)$ acyclic. The statement follows. $\square$

Theorem 6.16 *Let $\mathcal{E} \to \mathcal{F}$ be a locally free resolution, and consider a cofibrant replacement $\mathcal{Q} \xrightarrow{\pi} \Upsilon^*\mathcal{E}$ in $\mathrm{Mod}(A.)$. Let $\mathfrak{L}$ be the semicosimplicial DG-Lie algebra associated to $\mathcal{Q}$ as in Definition 6.6. Then $\mathrm{Tot}(\mathfrak{L})$ and $\mathrm{Tot}(\mathfrak{h})$ are quasi-isomorphic as DG-Lie algebras. In particular, the associated deformation functors $\mathrm{Def}_{\mathrm{Tot}(\mathfrak{L})}$ and $\mathrm{Def}_{\mathrm{Tot}(\mathfrak{h})}$ are naturally isomorphic.*

Proof It is sufficient to observe that by Lemma 6.15 there exists quasi-isomorphisms

$$\mathrm{Hom}^*_{A.}(\mathcal{Q}, \mathcal{Q})_\alpha \leftarrow M_\alpha \to \mathrm{Hom}^*_{A.}(\Upsilon^*\mathcal{E}, \Upsilon^*\mathcal{E})_\alpha$$

of DG-Lie algebras inducing quasi-isomorphisms of semicosimplicial DG-Lie algebras. To conclude the proof recall that the Whitney integration maps lift quasi-isomorphisms between complexes associated to semicosimplicial DG-Lie algebras to quasi-isomorphisms between their totalizations. □

7 Infinitesimal Deformations of Quasi-coherent Sheaves

It is well known that infinitesimal deformations of a coherent sheaf on a smooth projective variety are related to $\mathrm{Ext}^*(\mathcal{F}, \mathcal{F})$, see e.g. [11]. Using results of Sect. 6, our aim is now to prove that the DG-Lie algebras $\mathrm{End}^*_{A.}(\mathcal{Q}) = \mathrm{Hom}^*_{A.}(\mathcal{Q}, \mathcal{Q})$ and $\mathrm{Tot}(\mathfrak{L})$ control infinitesimal deformations of a quasi-coherent sheaf $\mathcal{F}$ over a finite-dimensional Noetherian separated scheme X. Here $\mathcal{Q} \to \Upsilon^*\mathcal{F}$ is any cofibrant replacement in $\mathrm{Mod}(A.)$.

For the reader convenience, we briefly recall the definition of the deformation functor associated to infinitesimal deformations of $\mathcal{F}$. A deformation of $\mathcal{F}$ over $A \in \mathrm{Art}_{\mathbb{K}}$ is a morphism $\pi : \mathcal{F}_A \to \mathcal{F}$ of sheaves of $\mathcal{O}_X \otimes A$-modules over $X \times \mathrm{Spec}(A)$, with $\mathcal{F}_A$ flat over A, such that the reduced map $\mathcal{F}_A \otimes_A \mathbb{K} \to \mathcal{F}$ is an isomorphism. We say that two deformations $\mathcal{F}_A$ and $\mathcal{F}'_A$ are isomorphic if there exists an isomorphism of sheaves $\varphi\colon \mathcal{F}_A \to \mathcal{F}'_A$ such that $\pi' \circ \varphi = \pi$. The functor of infinitesimal deformations of $\mathcal{F}$ up to isomorphism is denoted by $\mathrm{Def}_{\mathcal{F}}\colon \mathrm{Art}_{\mathbb{K}} \to \mathrm{Set}$.

The main result of this section will be the existence of natural isomorphisms

$$\mathrm{Def}_{\mathcal{F}} \cong \mathrm{Def}_{\mathrm{Tot}(\mathfrak{L})} \cong \mathrm{Def}_{\mathrm{End}^*_{A.}(\mathcal{Q})} .$$

We shall give different proofs. First recall that by Remark 6.10 there exists a natural isomorphism $\mathrm{Def}_{\mathrm{End}^*_{A.}(\mathcal{Q})} \to \mathrm{Def}_{\mathrm{Tot}(\mathfrak{L})}$. In Sect. 7.1 we will use a powerful result of [11], which will lead us to a natural isomorphism $\mathrm{Def}_{\mathcal{F}} \cong \mathrm{Def}_{\mathrm{Tot}(\mathfrak{L})}$. In Sect. 7.2 we will give an explicit natural isomorphism $\mathrm{Def}_{\mathrm{End}^*_{A.}(\mathcal{Q})} \to \mathrm{Def}_{\mathcal{F}}$.

7.1 Deformations via Descent of Deligne Groupoid

We begin by recalling the construction of the functors $Z^1_{\mathfrak{g}}, H^1_{\mathfrak{g}}\colon \mathrm{Art}_{\mathbb{K}} \to \mathrm{Set}$ for any given semicosimplicial DG-Lie algebra

$$\mathfrak{g}\colon \qquad \mathfrak{g}_0 \overset{\partial_{0,1}}{\underset{\partial_{1,1}}{\rightrightarrows}} \mathfrak{g}_1 \overset{\partial_{0,2}}{\underset{\partial_{2,2}}{\xrightarrow[\partial_{1,2}]{}}} \cdots .$$

For every $A \in \mathrm{Art}_{\mathbb{K}}$ define $Z^1_{\mathfrak{g}}(A) \subseteq (\mathfrak{g}^1_0 \oplus \mathfrak{g}^0_1) \otimes \mathfrak{m}_A$ to be the subset of elements $(l, m) \in (\mathfrak{g}^1_0 \oplus \mathfrak{g}^0_1) \otimes \mathfrak{m}_A$ satisfying

$$\begin{cases} dl + \frac{1}{2}[l, l] = 0 \\ \partial_{1,1} l = e^m * \partial_{0,1} l \\ \partial_{0,2} m \bullet (-\partial_{1,2} m) \bullet \partial_{2,2} m = dn + [\partial_{2,2}\partial_{0,1} l, n] \quad \text{for some } n \in \mathfrak{g}^{-1}_2 \otimes \mathfrak{m}_A \end{cases}$$

where $*$ denotes the gauge action and $\bullet$ denotes the Baker–Campbell–Hausdorff product; i.e. $x \bullet y = \log(e^x e^y)$. There is an equivalence relation on $Z^1_{\mathfrak{g}}(A)$: two elements $(l_0, m_0), (l_1, m_1) \in Z^1_{\mathfrak{g}}(A)$ are equivalent if and only if there exist $a \in \mathfrak{g}^0_0 \otimes \mathfrak{m}_A$ and $b \in \mathfrak{g}^{-1}_1 \otimes \mathfrak{m}_A$ such that

$$\begin{cases} e^a * l_0 = l_1 \\ -m_0 \bullet (-\partial_{1,1} a) \bullet \partial_{0,1} a = db + [\partial_{0,1} l_0, b] \ . \end{cases}$$

We shall denote by $\sim$ the equivalent relation defined above; the functor of Artin rings $H^1_{\mathfrak{g}} \colon \mathrm{Art}_{\mathbb{K}} \to \mathrm{Set}$ is defined as $H^1_{\mathfrak{g}}(A) = {Z^1_{\mathfrak{g}}(A)}/{\sim}$ for every $A \in \mathrm{Art}_{\mathbb{K}}$. This functor extends the one defined in [12] for semicosimplicial Lie algebras. It was proven in [11] that there exists a commutative diagram of functors

$$\begin{array}{ccc} \mathbf{DGLA}^{\Delta}_{H \geq 0} & \xrightarrow{\mathrm{Tot}(\cdot)} & \mathbf{DGLA} \\ & {\scriptstyle H^1_{\cdot}} \searrow \quad \swarrow {\scriptstyle \mathrm{Def}_{\cdot}} & \\ & \mathrm{Set}^{\mathrm{Art}_{\mathbb{K}}} & \end{array}$$

where $\mathbf{DGLA}^{\Delta}_{H \geq 0}$ is the category of semicosimplicial DG-Lie $\mathbb{K}$-algebras whose cohomology is concentrated non-negative degrees, $\mathbf{DGLA}$ is the category of DG-Lie $\mathbb{K}$-algebras, and $\mathrm{Set}^{\mathrm{Art}_{\mathbb{K}}}$ is the category of functors $\mathrm{Art}_{\mathbb{K}} \to \mathrm{Set}$. Moreover, the functor $\mathrm{Def}_{\cdot} \colon \mathbf{DGLA} \to \mathrm{Set}^{\mathrm{Art}_{\mathbb{K}}}$ is defined by Maurer–Cartan solution modulo gauge equivalence.

Our strategy is now clear: we first need to show that the semicosimplicial DG-Lie algebra $\mathfrak{L}$ defined in Definition 6.6 has cohomology concentrated in positive degrees, i.e. $\mathfrak{L} \in \mathbf{DGLA}^{\Delta}_{H \geq 0}$, then we conclude by showing that $\mathrm{Def}_{\mathcal{F}} \cong H^1_{\mathfrak{L}}$.

Lemma 7.1 *Let $\mathcal{F}$ be a quasi-coherent sheaf on X, and take a cofibrant replacement $\mathcal{Q} \to \Upsilon^* \mathcal{F}$ in* $\mathrm{Mod}(A_\cdot)$. *Then the associated semicosimplicial DG-Lie algebra $\mathfrak{L}$ defined in Definition 6.6 belongs to $\mathbf{DGLA}^{\Delta}_{H \geq 0}$.*

Proof Fix $\alpha \in \mathcal{N}$; we need to show that $\mathrm{Hom}^*_{A_\cdot}(\mathcal{Q}, \mathcal{Q})_\alpha$ is acyclic in negative degrees. Consider the composition

$$\mathrm{Hom}^*_{A_\cdot}(\mathcal{Q}, \mathcal{Q})_\alpha \to \mathrm{Hom}^*_{A_\cdot}(\mathcal{Q}, \Upsilon^* \mathcal{F})_\alpha \to \mathrm{Hom}^*_{A_\alpha}(\mathcal{Q}_\alpha, \mathcal{F}(U_\alpha))$$

where the first map is a quasi-isomorphism by Proposition 6.2, and the second map is a quasi-isomorphism by Remark 6.12. Now consider a projective resolution $P^\cdot \to \mathcal{F}(U_\alpha)$, which in particular is a cofibrant replacement in $\mathrm{DGMod}(A_\alpha)$, see e.g. [22, Lemma 2.3.6]. Therefore there exists a quasi-isomorphism $q \colon \mathcal{Q}_\alpha \to P^\cdot$ lifting $\mathcal{Q}_\alpha \to \mathcal{F}(U_\alpha)$. By Ken Brown's Lemma, the functor $\mathrm{Hom}^*_{A_\alpha}(-, \mathcal{F}(U_\alpha))$ maps weak equivalences between cofibrant objects to quasi-isomorphisms, so that the induced map

$$\mathrm{Hom}^*_{A_\alpha}\left(P^\cdot, \mathcal{F}(U_\alpha)\right) \xrightarrow{-\circ q} \mathrm{Hom}^*_{A_\alpha}(\mathcal{Q}_\alpha, \mathcal{F}(U_\alpha))$$

is a quasi-isomorphism. Now the statement follows since the complex $\mathrm{Hom}^*_{A_\alpha}(P^\cdot, \mathcal{F}(U_\alpha))$ does not have non-zero n-cocycles for $n < 0$. □

Fix $\alpha \in \mathcal{N}$ and $A \in \mathrm{Art}_{\mathbb{K}}$; a Maurer–Cartan element $\{l_\beta\}_{\beta \geq \alpha} \in \mathrm{Hom}^1_{A_\cdot}(\mathcal{Q}, \mathcal{Q})_\alpha \otimes \mathfrak{m}_A$ defines complexes $(\mathcal{Q}_\beta \otimes A, d_{\mathcal{Q}_\beta} + l_\beta)$ for every $\beta \geq \alpha$, hence deformations of the sheaf $\mathcal{F}|_{U_\beta}$ by taking the sheaf associated to the 0-th cohomology. In fact, the condition $(d_{\mathcal{Q}_\beta} + l_\beta)^2 = 0$ is equivalent to require $d_{\mathfrak{L}_0} l_\beta + \frac{1}{2}[l_\beta, l_\beta] = 0$, while the flatness follows from [36, Theorem A.31] since every cofibrant complex is degreewise projective, see e.g. [22, Lemma 2.3.6]. Notice that for every $\alpha \leq \beta \leq \gamma$ we have a quasi-isomorphism

$$(\mathcal{Q}_\beta \otimes A, d_{\mathcal{Q}_\beta} + l_\beta) \otimes_{(A_\beta \otimes A)} (A_\gamma \otimes A) \to (\mathcal{Q}_\gamma \otimes A, d_{\mathcal{Q}_\gamma} + l_\gamma)$$

so that the induced map between deformations

$$\begin{array}{ccc} H^0(\mathcal{Q}_\beta \otimes A, d_{\mathcal{Q}_\beta} + l_\beta) \otimes_{(A_\beta \otimes A)} (A_\gamma \otimes A) & \xrightarrow{\cong} & H^0(\mathcal{Q}_\gamma \otimes A, d_{\mathcal{Q}_\gamma} + l_\gamma) \\ & \searrow \quad \swarrow & \\ & \mathcal{F}(U_\beta) & \end{array}$$

is an isomorphism. This means that a Maurer–Cartan element $l^\alpha = \{l_\beta\}_{\beta \geq \alpha} \in \mathrm{Hom}^1_{A.}(\mathcal{Q}, \mathcal{Q})_\alpha \otimes \mathfrak{m}_A$ is essentially a deformation of the sheaf $\mathcal{F}|_{U_\alpha}$.

Now consider a Maurer–Cartan element $l = \{l^\alpha\}_{\alpha \in \mathcal{N}_0} \in \prod_{\alpha \in \mathcal{N}_0} \mathrm{Hom}^1_{A.}(\mathcal{Q}, \mathcal{Q})_\alpha \otimes \mathfrak{m}_A$, so that each l^α is a Maurer–Cartan element in $\mathrm{Hom}^1_{A.}(\mathcal{Q}, \mathcal{Q})_\alpha \otimes \mathfrak{m}_A$. In order to glue the deformations associated to each l^α, we need to require the existence of an isomorphism

$$(\mathcal{Q}_\beta \otimes A, d_{\mathcal{Q}_\beta} + l^\alpha_\beta) \otimes_{(A_\beta \otimes A)} (A_\gamma \otimes A) \xrightarrow{f} (\mathcal{Q}_\beta \otimes A, d_{\mathcal{Q}_\beta} + l^{\alpha'}_\beta) \otimes_{(A_\beta \otimes A)} (A_\gamma \otimes A)$$

lifting the identity for every $\alpha, \alpha' \in \mathcal{N}_0$ and every $\beta \in \overline{\mathcal{N}}$ such that $\alpha, \alpha' \leq \beta$. Since f lifts the identity on $\mathcal{Q}_\beta$, then $f = e^{m^{(\alpha,\alpha')}_\beta}$ for some $m^{(\alpha,\alpha')}_\beta \in \mathrm{Hom}^0_{A.}(\mathcal{Q}_\beta, \mathcal{Q}_\beta) \otimes \mathfrak{m}_A$. The commutativity with the differential is equivalent to the relation $d_{\mathcal{Q}_\beta} + l^\alpha_\beta = e^{m^{(\alpha,\alpha')}_\beta}(d_{\mathcal{Q}_\beta} + l^{\alpha'}_\beta)e^{-m^{(\alpha,\alpha')}_\beta}$, i.e. $l^{\alpha'}_\beta = e^{m^{(\alpha,\alpha')}_\beta} * l^\alpha_\beta$. Therefore for every $(\alpha, \alpha') \in \overline{\mathcal{N}}_1$ all these isomorphisms are collected by the element $(\alpha, \alpha') \in \mathrm{Hom}^0_{A.}(\mathcal{Q}, \mathcal{Q})_{\alpha \cup \alpha'} \otimes \mathfrak{m}_A$.

Observe that in order to satisfy the cocycle condition on the 0-th cohomology, we need to require that for every $(\alpha, \alpha', \alpha'') \in \overline{\mathcal{N}}_2$ there exists an element $n^{(\alpha,\alpha',\alpha'')} \in \mathrm{Hom}^{-1}_{A.}(\mathcal{Q}, \mathcal{Q})_{\alpha \cup \alpha' \cup \alpha''}$ such that

$$m^{(\alpha',\alpha'')}_\gamma \bullet (-m^{(\alpha,\alpha'')}_\gamma) \bullet m^{(\alpha,\alpha')}_\gamma = \left[d + l^{\alpha'}_\gamma, n^{(\alpha,\alpha',\alpha'')}_\gamma\right]$$

for every $\gamma \geq (\alpha, \alpha', \alpha'')$.

Summing up all the above discussion, we have a natural transformation defined for every $A \in \mathrm{Art}_{\mathbb{K}}$ by

$$\varphi_A \colon H^1_{\mathfrak{L}}(A) \longrightarrow \mathrm{Def}_{\mathcal{F}}(A)\,, \qquad \left(\{l^\alpha\}_{\alpha \in \mathcal{N}_0}, \{m^{(\alpha,\alpha')}\}_{(\alpha,\alpha') \in \overline{\mathcal{N}}_1}\right) \mapsto (\mathcal{F}_A \to \mathcal{F})$$

where $\mathcal{F}_A$ is the sheaf obtained gluing together the deformations associated to each l^α through the isomorphisms $e^{m^{(\alpha,\alpha')}}$.

Proposition 7.2 *The natural transformation $\varphi\colon \mathrm{Def}_{\mathcal{F}} \to H^1_{\mathfrak{L}}$ defined above is a natural isomorphism.*

Proof For simplicity we assume the replacement $\mathcal{Q}$ to belong to $\mathrm{Mod}^{\leq 0}(A.)$, i.e. $\mathcal{Q}_\alpha$ is concentrated in non-positive degrees for every $\alpha \in \mathcal{N}$. Notice that by Remark 3.11 such a replacement always exists, and our assumption is not restrictive since for every pair of cofibrant replacements $\mathcal{Q} \to \Upsilon^*\mathcal{F} \leftarrow \mathcal{Q}'$ the DG-Lie algebras $\mathrm{End}^*_{A.}(\mathcal{Q})$ and $\mathrm{End}^*_{A.}(\mathcal{Q}')$ are quasi-isomorphic.

In order to prove the claim, fix $A \in \mathrm{Art}_{\mathbb{K}}$ and take an isomorphism between deformations $f : \mathcal{F}_A$ and $\mathcal{F}'_A$, associated to $\left(\{l^\alpha\}_{\alpha \in \mathcal{N}_0}, \{m^{(\alpha,\alpha')}\}_{(\alpha,\alpha') \in \overline{\mathcal{N}}_1}\right)$ and $\left(\{\lambda^\alpha\}_{\alpha \in \mathcal{N}_0}, \{\mu^{(\alpha,\alpha')}\}_{(\alpha,\alpha') \in \overline{\mathcal{N}}_1}\right)$

respectively. For every $\alpha \in \mathcal{N}_0$ and every $\beta \geq \alpha$, the restriction of f to each U_α lifts to isomorphisms

$$(\mathcal{Q}_\beta \otimes A, d_{\mathcal{Q}_\beta} + l^\alpha_\beta) \to (\mathcal{Q}_\beta \otimes A, d_{\mathcal{Q}_\beta} + \lambda^\alpha_\beta)$$

that reduce to the identity modulo the maximal ideal $\mathfrak{m}_A$. Therefore all these isomorphisms are of the form $e^{a^\alpha_\beta}$ for some $\{a^\alpha\} \in \prod_{\alpha\in\mathcal{N}_0} \mathrm{Hom}^0_{A.}(\mathcal{Q}, \mathcal{Q})_\alpha \otimes \mathfrak{m}_A$. Again, the commutativity with the differentials is equivalent to the relations

$$e^{a^\alpha_\beta} * l^\alpha_\beta = \lambda^\alpha_\beta\,, \qquad \text{for every } \beta \geq \alpha\ .$$

We are only left with the proof that φ_A is surjective for every $A \in \mathrm{Art}_{\mathbb{K}}$. To this aim, take a deformation $\mathcal{F}_A \to \mathcal{F}$ in $\mathrm{Def}_{\mathcal{F}}$ and fix $\alpha \in \mathcal{N}_0$. Notice that for every $\beta \geq \alpha$ in $\mathcal{N}$ the map $\mathcal{Q}_\beta \to \mathcal{F}(U_\beta)$ lifts to surjective quasi-isomorphisms $(\mathcal{Q}_\beta \otimes A, d + l^\alpha_\beta) \to \mathcal{F}_A(U_\beta)$ of DG-modules over $A_\beta \otimes A$, for some $l^\alpha \in \mathrm{Hom}^1_{A.}(\mathcal{Q}, \mathcal{Q})_\alpha \otimes \mathfrak{m}_A$. The gluing data correspond to elements $m^{(\alpha,\alpha')} \in \mathrm{Hom}^0_{A.}(\mathcal{Q}, \mathcal{Q})_{\alpha\cup\alpha'} \otimes \mathfrak{m}_A$ for every $(\alpha, \alpha') \in \overline{\mathcal{N}}_1$; moreover, for every $\beta \geq \alpha \cup \alpha'$ each isomorphism $e^{m^{(\alpha,\alpha')}_\beta}$ lifts the identity in the 0-th cohomology, and liftings are unique up to homotopy. □

The argument used in Proposition 7.2 is similar to the Kodaira-Spencer approach to deformations of a locally free sheaf $\mathcal{E}$ of $\mathcal{O}_X$-modules on a complex manifold, [24], and in fact closely follows the one given in [11] to show that deformations of a quasi-coherent sheaf $\mathcal{F}$ are controlled by the sheaf of DG-Lie algebras $\mathcal{E}nd^*(\mathcal{E})$ for any given locally free resolution $\mathcal{E} \to \mathcal{F}$. The main advantage of our approach relies on the fact that we do not assume the existence of such a resolution.

Theorem 7.3 *Let X be a finite dimensional Noetherian separated scheme over $\mathbb{K}$, and let $\mathcal{F}$ be a quasi-coherent sheaf on it. Fix a cofibrant replacement $\mathcal{Q} \to \Upsilon^*\mathcal{F}$. Then there exists a natural isomorphism $\mathrm{Def}_{\mathrm{Tot}(\mathfrak{L})} \longrightarrow \mathrm{Def}_{\mathcal{F}}$, where $\mathfrak{L}$ is the semicosimplicial DG-Lie algebra associated to $\mathcal{Q}$, see Definition 6.6.*

Hence by Remark 6.10 we have natural isomorphisms $\mathrm{Def}_{\mathrm{End}^*_{A.}(\mathcal{Q})} \cong \mathrm{Def}_{\mathrm{Tot}(\mathfrak{L})} \cong \mathrm{Def}_{\mathcal{F}}$.

Proof It has been already observed in Remark 6.10 that $\mathrm{Def}_{\mathrm{End}^*_{A.}(\mathcal{Q})} \cong \mathrm{Def}_{\mathrm{Tot}(\mathfrak{L})}$. Therefore, by Lemma 7.1 and [11, Theorem 7.6], it is sufficient to prove that $\mathrm{Def}_{\mathcal{F}} = H^1_{\mathfrak{L}}$. The statement now follows by Proposition 7.2. □

In particular, by Corollary 6.9 we recover the well-known fact that $T^1\,\mathrm{Def}_{\mathcal{F}} = \mathrm{Ext}^1(\mathcal{F}, \mathcal{F})$ and obstructions are contained in $\mathrm{Ext}^2(\mathcal{F}, \mathcal{F})$.

7.2 Deformations via *A*.-Modules

In this subsection we present another proof of Theorem 7.3 without using semicosimplicial techniques.

Theorem 7.4 *Let X be a finite dimensional Noetherian separated scheme over $\mathbb{K}$, and let $\mathcal{F}$ be a quasi-coherent sheaf on it. Fix a cofibrant replacement $\mathcal{Q} \to \Upsilon^*\mathcal{F}$. Then there exists a natural isomorphism $\mathrm{Def}_{\mathrm{End}^*_{A.}(\mathcal{Q})} \longrightarrow \mathrm{Def}_{\mathcal{F}}$.*

Hence by Remark 6.10 we have natural isomorphisms $\mathrm{Def}_{\mathrm{Tot}(\mathfrak{L})} \cong \mathrm{Def}_{\mathrm{End}^*_{A.}(\mathcal{Q})} \cong \mathrm{Def}_{\mathcal{F}}$.

Proof For simplicity we assume the replacement $\mathcal{Q}$ to belong to $\mathrm{Mod}^{\leq 0}(A.)$, i.e. $\mathcal{Q}_\alpha$ is concentrated in non-positive degrees for every $\alpha \in \mathcal{N}$. Notice that by Remark 3.11 such a replacement always exists, and our assumption is not restrictive since for every pair of cofibrant replacements $\mathcal{Q} \to \Upsilon^*\mathcal{F} \leftarrow \mathcal{Q}'$ the DG-Lie algebras $\mathrm{End}^*_{A.}(\mathcal{Q})$ and $\mathrm{End}^*_{A.}(\mathcal{Q}')$ are quasi-isomorphic, hence inducing isomorphic deformation functors $\mathrm{Def}_{\mathrm{End}^*_{A.}(\mathcal{Q})} \cong \mathrm{Def}_{\mathrm{End}^*_{A.}(\mathcal{Q}')}$.

Our first goal is to explicitly define a natural transformation $\varphi\colon \mathrm{Def}_{\mathrm{End}^*_{A.}(\mathcal{Q})} \longrightarrow \mathrm{Def}_{\mathcal{F}}$. To every object $\eta = \{\eta_\alpha\}_{\alpha\in\mathcal{N}} \in \mathrm{MC}\left(\mathrm{Hom}^*_{A.}(\mathcal{Q},\mathcal{Q}) \otimes A\right)$ there are associated (local) deformations

$$H^0(\mathcal{Q}_\alpha \otimes A, d_{\mathcal{Q}_\alpha} + \eta_\alpha) \to \mathcal{F}(U_\alpha)\,, \qquad \alpha \in \mathcal{N}$$

where each $H^0(\mathcal{Q}_\alpha \otimes A, d_{\mathcal{Q}_\alpha} + \eta_\alpha)$ is A-flat by [36, Theorem A.31]. Here the Maurer–Cartan equation is equivalent to the condition $(d_{\mathcal{Q}_\alpha} + \eta_\alpha)^2 = 0$. Moreover, for every $\alpha \leq \beta$ the map

$$H^0(\mathcal{Q}_\alpha \otimes A, d_{\mathcal{Q}_\alpha} + \eta_\alpha) \otimes_{(A_\alpha \otimes A)} (A_\beta \otimes A) \to H^0(\mathcal{Q}_\beta \otimes A, d_{\mathcal{Q}_\beta} + \eta_\beta)$$

is an isomorphism because $\mathcal{Q}$ is quasi-coherent in $\mathrm{Mod}(A.)$ by Remark 3.13. Now, for every $\alpha \leq \beta \leq \gamma$ there is a commutative diagram

$$\mathcal{Q}_\alpha \otimes_{A_\alpha} A_\gamma \xrightarrow{q_{\alpha\beta}\otimes \mathrm{Id}_{A_\gamma}} \mathcal{Q}_\beta \otimes_{A_\beta} A_\gamma \xrightarrow{q_{\beta\gamma}} \mathcal{Q}_\gamma, \qquad q_{\alpha\gamma}\colon \mathcal{Q}_\alpha \otimes_{A_\alpha} A_\gamma \to \mathcal{Q}_\gamma$$

inducing the cocycle conditions on the deformations $\{H^0(\mathcal{Q}_\alpha \otimes A, d_{\mathcal{Q}_\alpha} + \eta_\alpha) \to \mathcal{F}(U_\alpha)\}_{\alpha\in\mathcal{N}}$. Hence they glue together in a global deformation $\mathcal{F}^\eta_A \to \mathcal{F}$, with $\mathcal{F}_A$ flat over $\mathrm{Spec}(A)$. Define the natural transformation $\varphi\colon \mathrm{Def}_{\mathrm{End}^*_{A.}(\mathcal{Q})} \longrightarrow \mathrm{Def}_{\mathcal{F}}$ as $\varphi_A\colon \eta \mapsto (\mathcal{F}^\eta_A \to \mathcal{F})$ on every $A \in \mathrm{Art}_{\mathbb{K}}$. In order to show that φ is well-defined, take two Maurer–Cartan elements $\eta, \xi \in \mathrm{Hom}^1_{A.}(\mathcal{Q},\mathcal{Q}) \otimes \mathfrak{m}_A$ and suppose that there exists an element $a = \{a_\alpha\}_{\alpha\in\mathcal{N}} \in \mathrm{Hom}^0_{A.}(\mathcal{Q},\mathcal{Q}) \otimes \mathfrak{m}_A$ such that $e^a * \eta = \xi$. The last condition is equivalent to require that the maps in the square

$$\begin{array}{ccc}
(\mathcal{Q}_\alpha \otimes A, d_{\mathcal{Q}_\alpha} + \eta_\alpha) \otimes_{(A_\alpha\otimes A)} (A_\beta \otimes A) & \xrightarrow{e^{a_\alpha}\otimes \mathrm{Id}_{(A_\beta\otimes A)}} & (\mathcal{Q}_\alpha \otimes A, d_{\mathcal{Q}_\alpha} + \xi_\alpha) \otimes_{(A_\alpha\otimes A)} (A_\beta \otimes A) \\
\downarrow & & \downarrow \\
(\mathcal{Q}_\beta \otimes A, d_{\mathcal{Q}_\beta} + \eta_\beta) & \xrightarrow{e_\beta} & (\mathcal{Q}_\beta \otimes A, d_{\mathcal{Q}_\beta} + \xi_\beta)
\end{array}$$

commute with differentials for every $\alpha \leq \beta$ in $\mathcal{N}$. Therefore the associated deformations $\mathcal{F}^\eta_A \to \mathcal{F}$ and $\mathcal{F}^\xi_A \to \mathcal{F}$ are isomorphic.

We are left with the proof that φ is a natural isomorphism. Fix $A \in \mathrm{Art}_{\mathbb{K}}$ and take an isomorphism between deformations $f\colon \mathcal{F}^\eta_A$ and $\mathcal{F}^\xi_A$, associated to $\eta = \{\eta_\alpha\}_{\alpha\in\mathcal{N}}$ and $\xi = \{\xi_\alpha\}_{\alpha\in\mathcal{N}}$ respectively. For every $\alpha \leq \beta$, the restriction of f to each U_α lifts to isomorphisms

$$(\mathcal{Q}_\alpha \otimes A, d_{\mathcal{Q}_\alpha} + \eta_\alpha) \to (\mathcal{Q}_\alpha \otimes A, d_{\mathcal{Q}_\alpha} + \xi_\alpha)$$

that reduce to the identity modulo the maximal ideal $\mathfrak{m}_A$. Therefore all these isomorphisms are of the form e^{a_α} for some $a = \{a_\alpha\}_{\alpha\in\mathcal{N}} \in \mathrm{Hom}^0_{A.}(\mathcal{Q},\mathcal{Q}) \otimes \mathfrak{m}_A$. As above, the commutativity with the differentials is equivalent to the relations $e^a * \eta = \xi$, so that φ_A is injective.

In order to show that φ is surjective, fix $A \in \mathrm{Art}_{\mathbb{K}}$ and take a deformation $\mathcal{F}_A \to \mathcal{F}$ in $\mathrm{Def}_{\mathcal{F}}$. Notice that for every α in $\mathcal{N}$ the map $\mathcal{Q}_\alpha \to \mathcal{F}(U_\alpha)$ lifts to surjective quasi-isomorphisms $(\mathcal{Q}_\alpha \otimes A, d + \eta_\alpha) \to \mathcal{F}_A(U_\alpha)$ of DG-modules over $A_\alpha \otimes A$, for some $\eta_\alpha \in \mathrm{Hom}^1_{A.}(\mathcal{Q}, \mathcal{Q}) \otimes \mathfrak{m}_A$. □

In particular, by Theorem 6.4 we recover the well-known fact that $T^1 \mathrm{Def}_{\mathcal{F}} = \mathrm{Ext}^1(\mathcal{F}, \mathcal{F})$ and obstructions are contained in $\mathrm{Ext}^2(\mathcal{F}, \mathcal{F})$.

If the sheaf $\mathcal{F}$ admits a locally free resolution $\mathcal{E} \to \mathcal{F}$ then there exists a natural isomorphism of deformation functors $\mathrm{Def}_{\mathrm{Tot}(\mathfrak{h})} \cong \mathrm{Def}_{\mathcal{F}}$ by Theorems 6.16 and 7.4.

Acknowledgements I am deeply in debt with Marco Manetti, who encouraged me to draw up this paper. Part of the material of the present paper has been carried out during my Ph.D., so I am grateful to the external referees Vladimir Hinich and Donatella Iacono for carefully reading my Ph.D. thesis and for their helpful suggestions.

Author Contributions F. Meazzini is the only author of the manuscript.

Funding Open access funding provided by Università degli Studi di Roma La Sapienza within the CRUI-CARE Agreement. The author did not receive any special fundings during the preparation of this paper.

Availability of Data and Materials All of the material is owned by the author and/or no permissions are required.

Declarations

Conflict of interest The author has no conflict of interest as defined by Springer, or other interests that might be perceived to influence the results and/or discussion reported in this paper.

References

1. Alexandroff, P.S.: Diskrete Räume. MatematiceskiiSbornik (N.S.) **2**, 501–518 (1937)
2. Balzin, E.: Reedy model structures in families. arXiv:1803.00681 (2018)
3. Bandiera, R., Manetti, M., Meazzini, F.: Deformations of polystable sheaves on surfaces: quadraticity implies formality. Moscow Math. J. **22**, 239–263 (2022)
4. Bandiera, R., Manetti, M., Meazzini, F.: Formality conjecture for minimal surfaces of Kodaira dimension 0. Compos. Math. **157**(2), 215–235 (2021)
5. Bökstedt, M., Neeman, A.: Homotopy limits in triangulated categories. Compos. Math. **86**(2), 209–234 (1993)
6. Brun, M., Bruns, W., Römer, T.: Cohomology of partially ordered sets and local cohomology of section rings. Adv. Math. **208**, 210–235 (2007)
7. Buchweitz, R.-O., Flenner, H.: A semiregularity map for modules and applications to deformations. Compos. Math. **137**, 135–210 (2003)
8. Canonaco, A., Stellari, P.: A tour about existence and uniqueness of dg enhancements and lifts. J. Geom. Phys. **122**, 28–52 (2017)
9. Dwyer, W.G., Spalinski, J.: Homotopy Theories and Model Categories. Handbook of Algebraic Topology (Amsterdam), pp. 73–126. North-Holland, Amsterdam (1995)

10. Enochs, E., Estrada, S.: Relative homological algebra in the category of quasi-coherent sheaves. Adv. Math. **194**, 284–295 (2005)
11. Fiorenza, D., Iacono, D., Martinengo, E.: Differential graded Lie algebra controlling infinitesimal deformations of coherent sheaves. J. Eur. Math. Soc. (JEMS) **14**(2), 521–540 (2012)
12. Fiorenza, D., Manetti, M., Martinengo, E.: Cosimplicial DGLAs in deformation theory. Commun. Algebra **40**, 2243–2260 (2012)
13. Forster, O., Knorr, K.: Ein beweis des grauertscwrn bildgarbensatzes nach ideen von B. Malgrange. Manuscripta Math., vol. 5, pp. 19–44. Springer (1971)
14. Gelfand, S.I., Manin, Y.I.: Methods of Homological Algebra. Springer Monographs in Mathematics, Springer, Berlin (2003)
15. Gerstenhaber, M., Schack, S.D.: On the deformation of algebra morphism and diagrams. Trans. Am. Math. Soc. **279**(1), 1–50 (1983)
16. Goldman, W.M., Millson, J.J.: The deformation theory of representations of fundamental groups of compact Kähler manifolds. Publ. Mathématiques de l'I.H.É.S. **67**(2), 43–69 (1988)
17. Hartshorne, R.: Algebraic Geometry. Graduate Texts in Mathematics, vol. 52 (1977)
18. Hinich, V.: DG coalgebras as formal stacks. J. Pure Appl. Algebra **162**(2–3), 209–250 (2001)
19. Hinich, V.: Deformations of homotopy algebras. Commun. Algebra **32**(2), 473–494 (2004)
20. Hirschhorn, P.S.: Model Categories and Their Localizations. Mathematical Surveys and Monographs, vol. 99. American Mathematical Society, Providence (2003)
21. Hirschowitz, A., Simpson, C.: Descente de n-champs. arXiv:math.AG/9807049 (1998)
22. Hovey, M.: Model Categories. Mathematical Surveys and Monographs, vol. 63. American Mathematical Society, Providence (1999)
23. Hovey, M.: Model category structures on chain complexes of sheaves. Trans. Am. Math. Soc. **353**(6), 2441–2457 (2001)
24. Kodaira, K., Spencer, D.C.: On deformations of complex analytic structures. II. Ann. Math. **67**(2), 403–466 (1958)
25. Lunts, V., Orlov, D.: Uniqueness for DG enhancements for triangulated categories. J. Am. Math. Soc. **23**(3), 853–908 (2010)
26. Lurie, J.: Moduli problems for ring spectra. In: Proceedings of the International Congress of Mathematicians, pp 1099–1125. Volume II, Hindustan Book Agency, New Delhi (2010)
27. Manetti, M.: Differential graded Lie algebras and formal deformation theory. In: Algebraic Geometry: Seattle 2005. Proc. Sympos. Pure Math. vol. 80, pp. 785–810 (2009)
28. Manetti, M.: Extended deformation functors. Int. Math. Res. Not. **14**, 719–756 (2002)
29. Manetti, M.: Lectures on deformations of complex manifolds. Rend. Mat. Serie VII **24**, 1–183 (2004)
30. Manetti, M., Meazzini, F.: Formal deformation theory in left-proper model categories. N. Y. J. Math. **25**, 1259–1311 (2019)
31. Manetti, M., Meazzini, F.: Deformations of algebraic schemes via Reedy–Palamodov cofibrant resolutions. Indag. Math. **31**, 7–32 (2020)
32. Manin, Y.: Mirrors, Functoriality, and Derived Geometry. arXiv:1708.02849 (2017)
33. Meazzini, F.: Quiver representations over a quasi-Frobenius ring and Gorenstein-projective modules. Rend. Mat. Appl. **7**(42), 1–33 (2021)
34. Pridham, J.: Unifying derived deformation theories. Adv. Math. **224**(3), 772–826 (2010)
35. Quillen, D.: Homotopical Algebra. Lecture Notes in Mathematics, vol. 43. Springer, New York (1967)
36. Sernesi, E.: Deformations of Algebraic Schemes. Grundlehren der mathematischen Wissenschaften, vol. 334. Springer, New York Berlin (2006)
37. Šťovíček, J.: Exact model categories, approximation theory, and cohomology of quasi-coherent sheaves. In: Advances in Representation Theory of Algebras (ICRA Bielefeld, Germany, 8–17 August, 2012), EMS Series of Congress Reports, European Mathematical Society Publishing House (2012)
38. Whitney, H.: Geometric Integration Theory. Princeton University Press, Princeton (1957)