Improved Adams-type inequalities and their extremals in dimension $2m$

Azahara DelaTorre*
Albert-Ludwigs-Universität Freiburg
Mathematisches Institut. Abteilung für Reine Mathematik
Ernst-Zermelo-Str. 1, D-79104 Freiburg im Breisgau
azahara.de.la.torre@math.uni-freiburg.de

Gabriele Mancini*
Sapienza Università di Roma
Dipartimento di Scienze di Base e Applicate per l'Ingegneria
Via Antonio Scarpa 16, 00161 Roma
gabriele.mancini@uniroma1.it

April 1, 2019

Abstract

In this paper we prove the existence of extremal functions for the Adams-Moser-Trudinger inequality on the Sobolev space $H_0^m(\Omega)$, where Ω is any bounded, smooth, open subset of $\mathbb{R}^{2m}$, $m \geq 1$. Moreover, we extend this result to improved versions of Adams' inequality of Adimurthi-Druet type. Our strategy is based on blow-up analysis for sequences of subcritical extremals and introduces several new techniques and constructions. The most important one is a new procedure for obtaining capacity-type estimates on annular regions.

1 Introduction

Given $m \in \mathbb{N}$, $m \geq 1$, let $\Omega \subseteq \mathbb{R}^{2m}$ be a bounded open set with smooth boundary. For any $\beta > 0$, we consider the Moser-Trudinger functional

$$F_\beta(u) := \int_\Omega e^{\beta u^2} dx$$

and the set

$$M_0 := \left\{ u \in H_0^m(\Omega) \,:\, \|u\|_{H_0^m(\Omega)} \leq 1 \right\},$$

where

$$\|u\|_{H_0^m(\Omega)} = \|\Delta^{\frac{m}{2}} u\|_{L^2(\Omega)} \qquad \text{and} \qquad \Delta^{\frac{m}{2}} u := \begin{cases} \Delta^n u & \text{if } m = 2n,\ n \in \mathbb{N}, \\ \nabla \Delta^n u & \text{if } m = 2n+1,\ n \in \mathbb{N}. \end{cases}$$

*The authors have been supported by the Swiss National Science Foundation, projects nr. PP00P2-144669 and PP00P2-170588/1. The fist author is also partially supported by the Spanish government grants MTM2017-85757-P and MTM2014-52402-C3-1-P.

The Adams-Moser-Trudinger inequality (see [1]) implies that

$$\sup_{M_0} F_\beta < +\infty \qquad \Longleftrightarrow \qquad \beta \le \beta^*, \tag{1.1}$$

where $\beta^* := m(2m-1)!Vol(\mathbb{S}^{2m})$. This result is an extension to dimension $2m$ of the work done by Moser [26] and Trudinger [33] in the case $m = 1$, and can be considered as a critical version of the Sobolev inequality for the space $H_0^m(\Omega)$. A classical problem related to Moser-Trudinger and Sobolev-type embeddings consists in investigating the existence of extremal functions. While it is rather simple to prove that the supremum in (1.1) is attained for any $\beta < \beta^*$, lack of compactness due to concentration phenomena makes the critical case $\beta = \beta^*$ challenging. The first proof of existence of extremals for (1.1) was given by Carleson and Chang [5] in the special setting $m = 1$ and $\Omega = B_1(0)$. The case of arbitrary domains $\Omega \subseteq \mathbb{R}^2$ was treated by Flucher in [8]. These results are based on sharp estimates on the values that F_β can attain on concentrating sequences of functions. Recently, a different approach was proposed in [20] and [7]. Concerning the higher order case, as far as we know, the existence of extremals was proved only for $m = 2$ by Lu and Yang in [18] (see also [13]). In this work, we are able to study the problem for any arbitrary $m \ge 1$ and any arbitrary domain in $\mathbb{R}^{2m}$. Indeed, we prove here the following result.

Theorem 1.1. *Let $\Omega \subseteq \mathbb{R}^{2m}$ be a smooth bounded domain, then for any $m \ge 1$ and $\beta \le \beta^*$ the supremum in* (1.1) *is attained, i.e. there exists a function $u^* \in M_0$ such that $F_\beta(u^*) = \sup_{M_0} F_\beta$.*

More generally, we are interested in studying extremal functions for a larger family of inequalities. Let us denote

$$\lambda_1(\Omega) := \inf_{u \in H_0^m(\Omega), u \neq 0} \frac{\|u\|^2_{H_0^m(\Omega)}}{\|u\|^2_{L^2(\Omega)}}.$$

For the 2-dimensional case, in [2] it was proved that if $\Omega \subseteq \mathbb{R}^2$ and $0 \le \alpha < \lambda_1(\Omega)$, then

$$\sup_{u \in M_0} \int_\Omega e^{\beta^* u^2 (1 + \alpha \|u\|^2_{L^2(\Omega)})} dx < +\infty. \tag{1.2}$$

Moreover the bound on α is sharp, i.e. the supremum is infinite for any $\alpha \ge \lambda_1(\Omega)$. A stronger form of this inequality can be deduced from the results in [32]:

$$\sup_{u \in H_0^1(\Omega),\ \|u\|^2_{H_0^1(\Omega)} - \alpha \|u\|^2_{L^2(\Omega)} \le 1} F_{\beta^*} < +\infty. \tag{1.3}$$

Surprisingly, the study of extremals for the stronger inequality (1.3) is easier than for (1.2). In fact, it was proved in [35] that the supremum in (1.3) is attained for any $0 \le \alpha < \lambda_1(\Omega)$, while existence of extremal functions for (1.2) is known only for small values of α (see [17]). Such results have been extended to dimension 4 in [18] and [27]. In this paper, we consider the case of an arbitrary $m \ge 1$. For any $0 \le \alpha < \lambda_1(\Omega)$ we denote

$$\|u\|^2_\alpha := \|u\|^2_{H_0^m(\Omega)} - \alpha \|u\|^2_{L^2(\Omega)},$$

and we consider the set

$$M_\alpha := \{u \in H_0^m(\Omega) \ : \ \|u\|_\alpha \le 1\}$$

and the quantity

$$S_{\alpha,\beta} := \sup_{M_\alpha} F_\beta. \tag{1.4}$$

Observe that Poincare's inequality implies that for any $0 \le \alpha < \lambda_1(\Omega)$, $\|\cdot\|_\alpha$ is a norm on H_0^m which is equivalent to $\|\cdot\|_{H_0^m}$. Our main result is the following:

Theorem 1.2. *Let $\Omega \subseteq \mathbb{R}^{2m}$ be a smooth bounded domain, then for any $m \ge 1$ the following holds:*

1. *For any $0 \le \beta \le \beta^*$ and $0 \le \alpha < \lambda_1(\Omega)$ we have $S_{\alpha,\beta} < +\infty$, and there exists a function $u^* \in M_\alpha$ such that $F_\beta(u^*) = S_{\alpha,\beta}$.*

2. If $\alpha \geq \lambda_1(\Omega)$, or $\beta > \beta^*$, we have $S_{\alpha,\beta} = +\infty$.

The proof of the first part of Theorem 1.2 for $\beta = \beta^*$ is the most difficult one and it is based on blow-up analysis for sequences of sub-critical extremals. We will take a sequence $\beta_n \nearrow \beta^*$ and find $u_n \in M_\alpha$, such that $F_{\beta_n}(u_n) = S_{\alpha,\beta_n}$. If u_n is bounded in $L^\infty(\Omega)$, then standard elliptic regularity proves that u_n converges in $H^m(\Omega)$ to a function $u_0 \in M_\alpha$ such that $F_{\beta^*}(u_0) = S_{\alpha,\beta^*}$. Hence, one has to exclude that u_n blows-up, i.e. that $\mu_n := \max_{\overline{\Omega}} |u_n| \to +\infty$. This is done through a contradiction argument. On the one hand, if $\mu_n \to +\infty$, one can show that u_n admits a unique blow-up point x_0 and give a precise description of the behavior of u_n around x_0. Specifically, we will prove (see Proposition 4.2) that blow-up implies

$$S_{\alpha,\beta^*} = \lim_{n\to+\infty} F_{\beta_n}(u_n) \leq |\Omega| + \frac{Vol(\mathbb{S}^{2m})}{2^{2m}} e^{\beta^*\left(C_{\alpha,x_0} - I_m\right)},$$

where C_{α,x_0} is the value at x_0 of the trace of the regular part of the Green's function for the operator $(-\Delta)^m - \alpha$, and I_m is a dimensional constant. On the other hand, by exhibiting a suitable test function, we will prove (see Proposition 5.3) that such upper bound cannot hold, concluding the proof.

Considering that the rather standard strategy in the study of this kind of problems (see e.g. [2], [8], [10], [11], [12], [17], [18], [27] and [35]) is not easy to generalize, we need to introduce several elements of novelty.

First, our description of the behaviour of u_n near its blow-up point x_0 is sharper than the one given for $m = 2$ in [18] and [27]. There, in order to compensate the lack of sufficiently sharp standard elliptic estimates on a small scale, the authors needed to modify the standard scaling for the Euler-Lagrange equation satisfied by u_n. Instead, following the approach first introduced in [21], we are able to use the standard scaling replacing classical elliptic estimates with Lorentz-Zygmund type regularity estimates.

Secondly, in order to describe the behaviour of u_n far from x_0, we extend to higher dimension the approach of Adimurthi and Druet [2], which is based on the properties of truncations of u_n. To preserve the high-order regularity required in the high-dimensional setting, we introduce polyharmonic truncations. This step, requires precise pointwise estimates on the derivatives of u_n, which are a generalisation of the ones in [25], where the authors study sequences of positive critical points of F_β constrained to spheres in H^m_0. We stress that the results of [25] cannot be directly applied to our case, since here subcritical maximizers are not necessarily positive in Ω if $m \geq 2$. In addition, the presence of the parameter α modifies the Euler-Lagrange equation. While the differences in the nonlinearity do not create significant issues, the argument in [25] relies strongly on the positivity assumption. Therefore, here we propose a different proof.

The most important feature of our proof of Theorem 1.2 is that it does not rely on explicit capacity estimates. A crucial step in our blow-up analysis consists in finding sharp lower bounds for the integral of $|\Delta^{\frac{m}{2}} u_n|^2$ on annular regions. In all the earlier works, this is achieved by comparing the energy of u_n with the quantity

$$i(a, b, R_1, R_2) := \min_{u\in E_{a,b}} \int_{\{R_1\leq|x|\leq R_2\}} |\Delta^{\frac{m}{2}} u|^2 dy$$

for suitable choices of $a = (a_0, \ldots, a_{m-1})$, $b = (b_0, \ldots, b_{m-1})$, and where $E_{a,b}$ denotes the set of all the H^m functions on $\{R_1 \leq |x| \leq R_2\}$ satisfying $\partial^i_\nu u_n = a_i$ on $\partial B_{R_1}(0)$ and $\partial^i_\nu u_n = b_i$ on $\partial B_{R_2}(0)$ for $i = 0, \ldots, m-1$. While for $m = 1$ or $m = 2$, $i(a, b, R_1, R_2)$ can be explicitly computed, finding its expression for an arbitrary m appears to be very hard. In our work we show that these capacity estimates are unnecessary, since equivalent lower bounds can be obtained by directly comparing the Dirichlet energy of u_n with the energy of a suitable polyharmonic function. This results in a considerable simplification of the proof, even for $m = 1, 2$.

Finally, working with arbitrary values of m makes much harder the construction of good test functions and the study of blow-up near $\partial\Omega$, since standard moving planes techniques are not available for $m \geq 2$. To address the last issue, we will apply the Pohozaev-type identity introduced in [30] and applied in [24] to Liouville-type equations.

It would be interesting to extend our result to different Adams-type inequalities. Let $\Omega \subset \mathbb{R}^N$, if we consider the space $W_0^{m,\frac{N}{m}}(\Omega)$, without the dimension restriction $N = 2m$, the existence of extremals for the inequality (see [1]) was proved only for $m = 1$ by K-C Lin in [15]. More generally, one could consider the non-local Moser-Trudinger inequality for fractional-order Sobolev spaces proved in [23], for which the existence of extremals is still completely open. In this fractional setting, the behavior of blowing-up subcritical extremals was studied in [19] (at least for nonnegative functions). However, obtaining capacity-type estimates becomes much more challenging, and our argument to avoid them relies strongly on the local nature of the operator $(-\Delta)^m$.

This paper is organized as follows. In Section 2, we will introduce some notation and state some preliminary results. In Section 3, we will focus on the subcritical case $\beta < \beta^*$. In Section 4, we will analyze the blow up behavior of subcritical extremals. Since this part of the paper will discuss the most important elements of our work, it will be divided into several subsections. Finally, in Section 5, we will introduce new test functions and we will complete the proof of Theorem 1.2. For the reader convenience, we will recall in Appendix some known results concerning elliptic estimates for the operator $(-\Delta)^m$.

Acknowledgments

We are grateful to Professor Luca Martinazzi for introducing us to the problem and for supporting us in the preparation of this work with his encouragement and with many invaluable suggestions.

A consistent part of this work was carried out while we were employed by the University of Basel and later by the University of Padua. We would like to thank the Department of Mathematics and Computer Science in Basel and the Department of Mathematics Tullio Levi-Civita in Padua for their hospitality and support.

2 Preliminaries

Throughout the paper we will denote by ω_l the $l-$dimensional Hausdorff measure of the unit sphere $\mathbb{S}^l \subseteq \mathbb{R}^{l+1}$. We recall that, for any $m \geq 1$,

$$\omega_{2m-1} = \frac{2\pi^m}{(m-1)!} \qquad \text{and} \qquad \omega_{2m} = \frac{2^{m+1}\pi^m}{(2m-1)!!}. \tag{2.1}$$

It is known that the fundamental solution of $(-\Delta)^m$ in $\mathbb{R}^{2m}$ is given by $-\frac{1}{\gamma_m}\log|x|$, where

$$\gamma_m := \omega_{2m-1}2^{2m-2}[(m-1)!]^2 = \frac{\beta^*}{2m},$$

with β^* defined as in (1.1). In other words, one has

$$(-\Delta)^m\left(-\frac{2m}{\beta^*}\log|x|\right) = \delta_0 \qquad \text{in } \mathbb{R}^{2m}.$$

More generally, for any $1 \leq l \leq m-1$, we have

$$\Delta^l(\log|x|) = \tilde{K}_{m,l}\frac{1}{|x|^{2l}},$$

where

$$\tilde{K}_{m,l} = (-1)^{l+1}2^{2l-1}\frac{(l-1)!(m-1)!}{(m-l-1)!}. \tag{2.2}$$

This also yields

$$\Delta^{l+\frac{1}{2}}(\log|x|) = -2l\tilde{K}_{m,l}\frac{x}{|x|^{2l+2}}.$$

For any $1 \le j \le 2m-1$, we define

$$K_{m,\frac{j}{2}} := \begin{cases} \tilde{K}_{m,\frac{j}{2}} & \text{for } j \text{ even} \\ -(j-1)\tilde{K}_{m,\frac{j-1}{2}} & \text{for } j \text{ odd}, j \ge 3, \\ 1 & \text{for } j = 1. \end{cases} \tag{2.3}$$

Then, we obtain

$$\Delta^{\frac{j}{2}}(\log|x|) = \frac{K_{m,\frac{j}{2}}}{|x|^j} e_j(x), \qquad \text{where} \qquad e_j(y) := \begin{cases} 1 & j \text{ even}, \\ \frac{y}{|y|} & j \text{ odd}. \end{cases} \tag{2.4}$$

In order to use the same notation for all the values of m, we will use the symbol $\cdot$ to denote both the scalar product between vectors in $\mathbb{R}^{2m}$ and the standard Euclidean product between reals numbers. This turns out to be very useful to have compact integration by parts formulas. For instance, we will use several times the following Proposition:

Proposition 2.1. *Let $\Omega \subseteq \mathbb{R}^{2m}$ be a bounded open domain with Lipschitz boundary. Then, for any $u \in H^m(\Omega)$, $v \in H^{2m}(\Omega)$, we have*

$$\int_\Omega \Delta^{\frac{m}{2}} u \cdot \Delta^{\frac{m}{2}} v \, dx = \int_\Omega u(-\Delta)^m v \, dx - \sum_{j=0}^{m-1} \int_{\partial\Omega} (-1)^{m+j} \nu \cdot \Delta^{\frac{j}{2}} u \, \Delta^{\frac{2m-j-1}{2}} v \, d\sigma,$$

where ν denotes the outer normal to $\partial\Omega$.

A crucial role in our proof will be played by Green's functions for operators of the form $(-\Delta)^m - \alpha$. We recall here that for any $x_0 \in \Omega$, and $0 \le \alpha < \lambda_1(\Omega)$, there exists a unique distributional solution G_{α,x_0} of

$$\begin{cases} (-\Delta)^m G_{\alpha,x_0} = \alpha G_{\alpha,x_0} + \delta_{x_0} & \text{in } \Omega, \\ G_{\alpha,x_0} = \partial_\nu G_{\alpha,x_0} = \ldots = \partial_\nu^{m-1} G_{\alpha,x_0} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

Some of the main properties of the function G_{α,x_0} are listed in the following Proposition. We refer to [4] and [6] for the proof of the case $\alpha = 0$, while the general case can be obtained with minor modifications.

Proposition 2.2. *Let Ω be a bounded open set with smooth boundary. Then, for any $x_0 \in \Omega$ and $0 \le \alpha < \lambda_1(\Omega)$, we have:*

1. *There exist $C_{\alpha,x_0} \in \mathbb{R}$ and $\psi_{\alpha,x_0} \in C^{2m-1}(\overline{\Omega})$ such that $\psi_{\alpha,x_0}(x_0) = 0$ and*

$$G_{\alpha,x_0}(x) = -\frac{2m}{\beta^*} \log|x - x_0| + C_{\alpha,x_0} + \psi_{\alpha,x_0}(x), \qquad \text{for any } x \in \Omega \setminus \{x_0\}.$$

2. *There exists a constant $C = C(m, \alpha, \Omega)$ independent of x_0, such that*

$$|G_{\alpha,x_0}(x)| \le C|\log|x - x_0||,$$

and

$$|\nabla^l G_{\alpha,x_0}(x)| \le \frac{C}{|x - x_0|^l},$$

for any $1 \le l \le 2m-1, x \in \Omega \setminus \{x_0\}$.

3. *$G_{\alpha,x_0}(x) = G_{\alpha,x}(x_0)$, for any $x \in \Omega \setminus \{x_0\}$.*

In addition, using integration by parts and Proposition 2.2, we can establish the following new property.

Lemma 2.3. *For any $x_0 \in \Omega$ and $0 \le \alpha < \lambda_1(\Omega)$, we have*

$$\int_{\Omega\setminus B_\delta(x_0)} |\Delta^{\frac{m}{2}} G_{\alpha,x_0}|^2 dx = \alpha \|G_{\alpha,x_0}\|^2_{L^2(\Omega)} - \frac{2m}{\beta^*}\log\delta + C_{\alpha,x_0} + H_m + O(\delta|\log\delta|),$$

as $\delta \to 0$, where C_{α,x_0} is as in Proposition 2.2 and

$$H_m := \begin{cases} \left(\frac{2m}{\beta^*}\right)^2 \omega_{2m-1} \sum_{j=1}^{m-1} (-1)^{j+m} K_{m,\frac{j}{2}} K_{m,\frac{2m-j-1}{2}} & \text{if } m \ge 2, \\ 0 & \text{if } m = 1. \end{cases} \tag{2.6}$$

Proof. From Proposition 2.1 applied in $\Omega \setminus B_\delta(x_0)$ and (2.5), we find

$$\int_{\Omega\setminus B_\delta(x_0)} |\Delta^{\frac{m}{2}} G_{\alpha,x_0}|^2 dx = \alpha \int_{\Omega\setminus B_\delta(x_0)} G^2_{\alpha,x_0} dx + \sum_{j=0}^{m-1} \int_{\partial B_\delta(x_0)} (-1)^{m+j} \nu \cdot \Delta^{\frac{j}{2}} G_{\alpha,x_0} \Delta^{\frac{2m-j-1}{2}} G_{\alpha,x_0}\, d\sigma.$$

On $\partial B_\delta(x_0)$, Proposition 2.2, (2.4), and the identity $\frac{2m}{\beta^*} K_{m,\frac{m-1}{2}} = \frac{(-1)^{m-1}}{\omega_{2m-1}}$ yield

$$\begin{aligned} \nu \cdot G_{\alpha,x_0} \Delta^{\frac{2m-1}{2}} G_{\alpha,x_0} &= \left(-\frac{2m}{\beta^*}\log\delta + C_{\alpha,x_0} + O(\delta)\right)\left(\frac{-2m}{\beta^*} K_{m,\frac{2m-1}{2}} \delta^{1-2m} + O(1)\right) \\ &= \frac{(-1)^m}{\omega_{2m-1}} \delta^{1-2m}\left(-\frac{2m}{\beta^*}\log\delta + C_{\alpha,x_0} + O(\delta) + O(\delta^{2m-1}|\log\delta|)\right), \end{aligned}$$

and, for $m \ge 2$ and $1 \le j \le m-1$, that

$$\begin{aligned} \nu \cdot \Delta^{\frac{j}{2}} G_{\alpha,x_0} \Delta^{\frac{2m-j-1}{2}} G_{\alpha,x_0} &= \left(-\frac{2m}{\beta^*} K_{m,\frac{j}{2}} \delta^{-j} + O(1)\right)\left(-\frac{2m}{\beta^*} K_{m,\frac{2m-j-1}{2}} \delta^{1+j-2m} + O(1)\right) \\ &= \left(\frac{2m}{\beta^*}\right)^2 K_{m,\frac{j}{2}} K_{m,\frac{2m-j-1}{2}} \delta^{1-2m}(1 + O(\delta^j)). \end{aligned}$$

Then, we get

$$\sum_{j=0}^{m-1} \int_{\partial B_\delta(x_0)} (-1)^{m+j} \nu \cdot \Delta^{\frac{j}{2}} G_{\alpha,x_0} \Delta^{\frac{2m-j-1}{2}} G_{\alpha,x_0}\, d\sigma = -\frac{2m}{\beta^*}\log\delta + C_{\alpha,x_0} + H_m + O(\delta|\log\delta|), \tag{2.7}$$

with H_m as in (2.6). Finally, applying again Proposition 2.2, we find

$$\int_{\Omega\setminus B_\delta(x_0)} G^2_{\alpha,x_0} dx = \|G_{\alpha,x_0}\|^2_{L^2(\Omega)} + O(\delta^{2m}\log^2\delta). \tag{2.8}$$

The conclusion follows by (2.7) and (2.8). □

Remark 2.4. *One can further observe that*

$$H_m = \frac{m}{\beta^*} \sum_{j=1}^{m-1} \frac{(-1)^{[\frac{2j}{m}]}}{j}.$$

Indeed, we have the identity

$$(-1)^m \omega_{2m-1} \frac{2m}{\beta^*} K_{m,\frac{j}{2}} K_{m,m-\frac{j}{2}-\frac{1}{2}} = \begin{cases} \frac{1}{j} & j \text{ even}, \\ \frac{1}{2m-j-1} & j \text{ odd}. \end{cases}$$

Hence,

$$\omega_{2m-1}\frac{2m}{\beta^*}\sum_{j=1}^{m-1}(-1)^{j+m}K_{m,\frac{j}{2}}K_{m,m-\frac{j}{2}-\frac{1}{2}} = \sum_{j=1,\,j\ even}^{m-1}\frac{1}{j}-\sum_{j=1,\,j\ odd}^{m-1}\frac{1}{2m-j-1}$$
$$= \sum_{j=1,\,j\ even}^{m-1}\frac{1}{j}-\sum_{j=m,\,j\ even}^{2m-2}\frac{1}{j}$$
$$= \frac{1}{2}\sum_{j=1}^{m-1}\frac{(-1)^{[\frac{2j}{m}]}}{j}.$$

We conclude this section, by recalling the following standard consequence of Adams' inequality and the density of $C_c^\infty(\Omega)$ in $H_0^m(\Omega)$.

Lemma 2.5. *For any $u\in H_0^m(\Omega)$ and $\beta\in\mathbb{R}^+$, we have $e^{\beta u^2}\in L^1(\Omega)$.*

Proof. For any $\varepsilon>0$ we can find a function $v_\varepsilon\in C_0^\infty(\Omega)$ such that $\|v_\varepsilon-u\|^2_{H_0^m(\Omega)}\le\varepsilon$. Since

$$u^2=v_\varepsilon^2+(u-v_\varepsilon)^2+2v_\varepsilon(u-v_\varepsilon)\le 2v_\varepsilon^2+2(u-v_\varepsilon)^2,$$

we have

$$e^{\beta u^2}\le\|e^{2\beta v_\varepsilon^2}\|_{L^\infty(\Omega)}e^{2\beta(u-v_\varepsilon)^2}\le\|e^{2\beta v_\varepsilon^2}\|_{L^\infty(\Omega)}e^{2\beta\varepsilon\left(\frac{u-v_\varepsilon}{\|u-v_\varepsilon\|_{H_0^m(\Omega)}}\right)^2}.$$

If we choose $\varepsilon>0$ small enough, we get $2\varepsilon\beta\le\beta^*$ and, applying Adam's inequality (1.1), we find

$$\int_\Omega e^{\beta u^2}dx\le\|e^{2\beta v_\varepsilon^2}\|_{L^\infty(\Omega)}F_{\beta^*}\left(\frac{u-v_\varepsilon}{\|u-v_\varepsilon\|_{H_0^m(\Omega)}}\right)<+\infty.$$

□

3 Subcritical inequalities and their extremals

In this section, we prove the existence of extremal functions for F_β on M_α in the subcritical case $\beta<\beta^*$, $0\le\alpha<\lambda_1(\Omega)$. As in the case $m=1$, this is a consequence of Vitali's convergence theorem and of the following improved Adams-type inequality, which is a generalization of Theorem 1.6 in [16].

Proposition 3.1. *Let $u_n\in H_0^m(\Omega)$ be a sequence of functions such that $\|u_n\|_{H_0^m(\Omega)}\le 1$ and $u_n\rightharpoonup u_0$ in $H_0^m(\Omega)$. Then, for any $0<p<\frac{1}{1-\|u_0\|^2_{H_0^m}}$, we have*

$$\limsup_{n\to+\infty}F_{p\beta^*}(u_n)<+\infty.$$

Proof. First, we observe that

$$\|u_n-u_0\|^2_{H_0^m(\Omega)}=\|u_n\|^2_{H_0^m(\Omega)}+\|u_0\|^2_{H_0^m(\Omega)}-2(u_n,u_0)_{H_0^m(\Omega)}\le 1-\|u_0\|^2_{H_0^m(\Omega)}+o(1).$$

Hence, there exists $\sigma>0$ such that

$$p\|u_n-u_0\|^2_{H_0^m(\Omega)}\le\sigma<1,$$

for sufficiently large n. For any $\gamma>0$, we have

$$u_n^2\le(1+\gamma^2)u_0^2+(1+\frac{1}{\gamma^2})(u_n-u_0)^2.$$

Since $0<\sigma<1$, we can choose γ sufficiently large so that $\sigma\left(1+\frac{1}{\gamma^2}\right)<1$. Applying Hölder's inequality with exponents $q=\frac{1}{\sigma\left(1+\frac{1}{\gamma^2}\right)}$ and $q'=\frac{q}{q-1}$, we get

$$F_{p\beta^*}(u_n)\le\int_\Omega e^{p\beta^*(1+\gamma^2)u_0^2}e^{p\beta^*(1+\frac{1}{\gamma^2})(u_n-u_0)^2}dx\le\|e^{p\beta^*(1+\gamma^2)u_0^2}\|_{L^{q'}(\Omega)}\|e^{p\beta^*(1+\frac{1}{\gamma^2})(u_n-u_0)^2}\|_{L^q(\Omega)}.$$

Lemma 2.5 guarantees that $\|e^{p\beta^*(1+\gamma^2)u_0^2}\|_{L^{q'}(\Omega)}<+\infty$. Moreover, since

$$pq(1+\frac{1}{\gamma^2})\|u_n-u_0\|^2_{H_0^m(\Omega)}=\frac{p}{\sigma}\|u_n-u_0\|^2_{H_0^m(\Omega)}\le 1,$$

for large n, Adams' inequality (1.1) yields

$$\|e^{p\beta^*(1+\frac{1}{\gamma^2})(u_n-u_0)^2}\|_{L^q(\Omega)}=F_{\beta^*}\left(\sqrt{pq(1+\frac{1}{\gamma^2})}(u_n-u_0)\right)^{\frac{1}{q}}\le S_{0,\beta^*}^{\frac{1}{q}}<+\infty.$$

Hence, $\limsup_{n\to+\infty}F_{p\beta^*}(u_n)<+\infty$. □

Next we recall the following consequence of Vitali's convergence theorem (see e.g. [31]).

Theorem 3.2. *Let $\Omega\subseteq\mathbb{R}^{2m}$ be a bounded open set and take a sequence $\{f_n\}_{n\in\mathbb{N}}\subseteq L^1(\Omega)$. Assume that:*

1. *For a.e. $x\in\Omega$ the pointwise limit $f(x):=\lim_{n\to+\infty}f_n(x)$ exists.*
2. *There exists $p>1$ such that $\|f_n\|_{L^p(\Omega)}\le C$.*

Then, $f\in L^1(\Omega)$ and $f_n\to f$ in $L^1(\Omega)$.

We can now prove the existence of subcritical extremals.

Proposition 3.3. *For any $\beta<\beta^*$ and $0\le\alpha<\lambda_1(\Omega)$, we have $S_{\alpha,\beta}<+\infty$. Moreover $S_{\alpha,\beta}$ is attained, i.e., there exists $u_{\alpha,\beta}\in M_\alpha$ such that $S_{\alpha,\beta}=F_\beta(u_{\alpha,\beta})$.*

Proof. Let $u_n\in M_\alpha$ be a maximizing sequence for F_β, i.e. such that $F_\beta(u_n)\to S_{\alpha,\beta}$ as $n\to+\infty$. Since $F_\beta(u_n)\le F_\beta(\frac{u_n}{\|u_n\|_\alpha})$, w.l.o.g we can assume $\|u_n\|_\alpha=1$, for any $n\in\mathbb{N}$. Since $\alpha<\lambda_1(\Omega)$, u_n is uniformly bounded in $H_0^m(\Omega)$. In particular, extracting a subsequence, we can find $u_0\in H_0^m(\Omega)$ such that $u_n\rightharpoonup u_0$ in $H_0^m(\Omega)$, $u_n\to u_0$ in $L^2(\Omega)$ and $u_n\to u_0$ a.e. in Ω. Observe that

$$\|u_0\|_\alpha^2=\|u_0\|^2_{H_0^m(\Omega)}-\alpha\|u_0\|^2_{L^2(\Omega)}\le\liminf_{n\to+\infty}\|u_n\|^2_{H_0^m(\Omega)}-\alpha\|u_n\|^2_{L^2(\Omega)}=\liminf_{n\to+\infty}\|u_n\|_\alpha^2=1,$$

hence $u_0\in M_\alpha$. If we prove that there exists $p>1$ such that

$$\|e^{\beta u_n^2}\|_{L^p(\Omega)}\le C, \tag{3.1}$$

then we can apply Theorem 3.2 to $f_n:=e^{\beta u_n^2}$ and we obtain $F_\beta(u_0)=S_{\alpha,\beta}$ and $S_{\alpha,\beta}<+\infty$, which concludes the proof. To prove (3.1) we shall treat two differnt cases.

Assume first that $u_0=0$. Then we have

$$\beta\|u_n\|^2_{H_0^m(\Omega)}=\beta(1+\alpha\|u_n\|^2_{L^2(\Omega)})=\beta+o(1)<\beta^*,$$

and we can find $p>1$ such that

$$p\beta\|u_n\|^2_{H_0^m(\Omega)}\le\beta^*,$$

for n large enough. In particular, using (1.1), we obtain

$$\|e^{\beta u_n^2}\|^p_{L^p(\Omega)}=\int_\Omega e^{p\beta u_n^2}dx\le F_{\beta^*}\left(\frac{u_n}{\|u_n\|_{H_0^m(\Omega)}}\right)\le S_{0,\beta^*}<+\infty.$$

Assume instead $u_0 \neq 0$. Consider the sequence $v_n := \frac{u_n}{\|u_n\|_{H_0^m(\Omega)}}$, and observe that $v_n \rightharpoonup v_0$ in $H_0^m(\Omega)$ where $v_0 = \frac{u_0}{\sqrt{1+\alpha\|u_0\|_{L^2}^2}}$. Since

$$\begin{aligned}\|u_n\|_{H_0^m}^2(1-\|v_0\|_{H_0^m}^2) &= \left(1+\alpha\|u_n\|_{L^2}^2\right)\left(1-\frac{\|u_0\|_{H_0^m}^2}{1+\alpha\|u_0\|_{L^2}^2}\right)\\ &= 1+\alpha\|u_0\|_{L^2}^2 - \|u_0\|_{H_0^m}^2 + o(1)\\ &= 1-\|u_0\|_\alpha^2 + o(1),\end{aligned}$$

and $u_0 \neq 0$, we get

$$\limsup_{n\to+\infty} \|u_n\|_{H_0^m}^2 < \frac{1}{1-\|v_0\|_{H_0^m}^2}.$$

In particular, there exist $p, q > 1$ such that

$$p\|u_n\|_{H_0^m}^2 \le q < \frac{1}{1-\|v_0\|_{H_0^m}^2},$$

for n large enough. Then, we get

$$\|e^{\beta u_n^2}\|_{L^p}^p \le \|e^{\beta^* u_n^2}\|_{L^p}^p = \|e^{\beta^*\|u_n\|_{H_0^m}^2 v_n^2}\|_{L^p}^p \le \|e^{\beta^* q v_n^2}\|_{L^1} = F_{q\beta^*}(v_n) \le C,$$

where the last inequality follows from Proposition 3.1. Therefore, the proof of (3.1) is complete. □

Finally, we stress that, as $\beta \to \beta^*$, the family $u_{\alpha,\beta}$ is a maximizing family for the critical functional F_{β^*}.

Lemma 3.4. *For any* $0 \le \alpha < \lambda_1(\Omega)$, *we have*

$$\lim_{\beta\nearrow\beta^*} S_{\alpha,\beta} = S_{\alpha,\beta^*}.$$

Proof. Clearly, $S_{\alpha,\beta}$ is monotone increasing with respect to β. In particular, we must have

$$\lim_{\beta\nearrow\beta^*} S_{\alpha,\beta} \le S_{\alpha,\beta^*}.$$

To prove the opposite inequality, we observe that, for any function $u \in M_\alpha$, the monotone convergence theorem implies

$$F_{\beta^*}(u) = \lim_{\beta\nearrow\beta^*} F_\beta(u) \le \lim_{\beta\nearrow\beta^*} S_{\alpha,\beta}.$$

Since u is an arbitrary function in M_α, we get

$$S_{\alpha,\beta^*} \le \lim_{\beta\nearrow\beta^*} S_{\alpha,\beta_n}.$$

□

4 Blow-up analysis at the critical exponent

This section is the most important one in this work. Here, we will provide the main ingredient for the proof of Theorem 1.2. We will study the behaviour of subcritical extremals as β approaches the critical exponent β^* from below. We will show that either they converge to an extremal of the critical problem, or they blow-up at one point. In the last case, we will obtain an upper bound for S_{α,β^*}. This is the result stated in Proposition 4.2, and together with Proposition 5.3 prove Theorem 1.2. More details are given at the end of Section 5.

In the following, we will take a sequence $(\beta_n)_{n\in\mathbb{N}}$ such that

$$0 < \beta_n < \beta^* \quad \text{and} \quad \beta_n \to \beta^*, \text{ as } n \to +\infty. \tag{4.1}$$

Due to Proposition 3.3, for any $n \in \mathbb{N}$, we can find a function $u_n \in M_\alpha$ such that

$$F_{\beta_n}(u_n) = S_{\alpha,\beta_n}. \tag{4.2}$$

Lemma 4.1. *If $u_n \in M_\alpha$ satisfies (4.2), then u_n has the following properties*

1. $\|u_n\|_\alpha = 1$.
2. u_n *is a solution to*
$$\begin{cases} (-\Delta)^m u_n = \lambda_n u_n e^{\beta_n u_n^2} + \alpha u_n & \text{in } \Omega, \\ u_n = \partial_\nu u_n = \cdots = \partial_\nu^{m-1} u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.3}$$
where
$$\lambda_n = \left(\int_\Omega u_n^2 e^{\beta_n u_n^2} dx \right)^{-1}. \tag{4.4}$$
3. $u_n \in C^\infty(\overline{\Omega})$.
4. $F_{\beta_n}(u_n) \to S_{\alpha,\beta^*}$ *as* $n \to +\infty$.
5. *If λ_n is as in (4.4), then* $\limsup_{n\to+\infty} \lambda_n < +\infty$.

Proof. *1.* Since $u_n \in M_\alpha$, we have $\|u_n\|_\alpha \le 1$, $\forall\, n \in \mathbb{N}$. Moreover, the maximality of u_n implies $u_n \neq 0$. If $\|u_n\|_\alpha < 1$, then we would have

$$S_{\alpha,\beta_n} = F_{\beta_n}(u_n) < F_{\beta_n}\left(\frac{u_n}{\|u_n\|_\alpha}\right),$$

which is a contradiction.

2. Since u_n is a critical point for F_{β_n} constrained to M_α, there exists $\gamma_n \in \mathbb{R}$ such that

$$\gamma_n \left((u_n, \varphi)_{H_0^m} - \alpha (u_n, \varphi)_{L^2} \right) = \beta_n \int_\Omega u_n e^{\beta_n u_n^2} \varphi dx, \tag{4.5}$$

for any function $\varphi \in H_0^m(\Omega)$. Taking u_n as test function and using *1.*, we find

$$\gamma_n = \beta_n \int_\Omega u_n^2 e^{\beta_n u_n^2} dx. \tag{4.6}$$

In particular, $\gamma_n \neq 0$ and (4.5) implies that u_n is a weak solution of (4.3) with $\lambda_n := \frac{\beta_n}{\gamma_n}$. Finally, (4.6) is equivalent to (4.4).

3. By Lemma 2.5, we know that u_n and $e^{\beta_n u_n^2}$ belong to every L^p space, $p > 1$. Then, applying standard elliptic regularity results (see e.g. Proposition A.4) and Sobolev embedding theorem, we find $u_n \in W^{2m,p}(\Omega) \subseteq C^{2m-1,\gamma}(\Omega)$, for any $\gamma \in (0,1)$. Then, we also have $(-\Delta)^m u_n \in C^{2m-1,\gamma}(\Omega)$ and, applying recursively Schauder estimates (Proposition A.3), we conclude that $u_n \in C^\infty(\overline{\Omega})$.

4. This is a direct consequence of Lemma 3.4.

5. Assume by contradiction that there exists a subsequence for which $\lambda_n \to +\infty$, as $n \to +\infty$. Then, by (4.4), we have

$$\int_\Omega u_n^2 e^{\beta_n u_n^2} dx \to 0,$$

as $n \to +\infty$. Exploiting the basic inequality $e^t \le 1 + te^t$ for $t \ge 0$, we obtain

$$F_{\beta_n}(u_n) \le |\Omega| + \beta_n \int_\Omega u_n^2 e^{\beta_n u_n^2} dx \to |\Omega|.$$

Since, by *4.*, $F_{\beta_n}(u_n) = S_{\alpha,\beta_n} \to S_{\alpha,\beta^*} > |\Omega|$, we get a contradiction.

□

In order to prove that S_{α,β^*} is finite and attained, we need to show that u_n does not blow-up as $n \to +\infty$. Let us take a point $x_n \in \Omega$ such that

$$\mu_n := \max_{\overline{\Omega}} |u_n| = |u_n(x_n)|. \tag{4.7}$$

Extracting a subsequence and changing the sign of u_n we can always assume that

$$u_n(x_n) = \mu_n \quad \text{and} \quad x_n \to x_0 \in \overline{\Omega}, \text{ as } n \to +\infty. \tag{4.8}$$

The main purpose of this section consists in proving the following Proposition.

Proposition 4.2. *Let β_n, u_n, μ_n, x_n, and x_0 be as in (4.1), (4.2), (4.7), and (4.8). If $\mu_n \to +\infty$, then $x_0 \in \Omega$ and we have*

$$S_{\alpha,\beta^*} = \lim_{n\to+\infty} F_{\beta_n}(u_n) \le |\Omega| + \frac{\omega_{2m}}{2^{2m}} e^{\beta^*\left(C_{\alpha,x_0} - I_m\right)},$$

where C_{α,x_0} is as in Proposition 2.2 and

$$I_m := -\frac{m4^{2m}}{\beta^*\omega_{2m}} \int_{\mathbb{R}^{2m}} \frac{\log\left(1 + \frac{|y|^2}{4}\right)}{(4+|y|^2)^{2m}} dy. \tag{4.9}$$

The proof of Proposition 4.2 is quite long and it will be divided into several subsections. Some standard properties of u_n will be established in section 4.1. Then, in section 4.2, as a consequence of Lorentz-Zygmund elliptic estimates, we will prove uniform bounds for Δu_n^2. Such bounds will be crucial in the analysis given in section 4.3, where we will study the behaviour of u_n on a small scale. Sections 4.4, 4.5 and 4.6 contain respectively estimates on the derivatives of u_n, the definition of suitable polyharmonic truncations of u_n, and the description of the behaviour of u_n far from x_0. In section 4.7 we will deal with blow-up at the boundary, which will be excluded using Pohozaev-type identities. Finally, we conclude the proof in section 4.8 by introducing a new technique to obtain lower bounds on the Dirichlet energy for u_n on suitable annular regions.

In the rest of this section β_n, u_n, μ_n, x_n, and x_0 will always be as in Proposition 4.2 and we will always assume that $\mu_n \to +\infty$.

4.1 Concentration near the blow-up point

In this subsection we will prove that, if $\mu_n \to +\infty$, u_n must concentrate around the blow-up point x_0. We start by proving that its weak limit in $H_0^m(\Omega)$ is 0.

Lemma 4.3. *If $\mu_n \to +\infty$, then $u_n \rightharpoonup 0$ in $H_0^m(\Omega)$ and $u_n \to 0$ in $L^p(\Omega)$ for any $p \ge 1$.*

Proof. Since u_n is bounded in $H_0^m(\Omega)$, we can assume that $u_n \rightharpoonup u_0$ in $H_0^m(\Omega)$ with $u_0 \in H_0^m(\Omega)$. The compactness of the embedding of $H_0^m(\Omega)$ into $L^p(\Omega)$ implies $u_n \to u_0$ in $L^p(\Omega)$, for any $p \ge 1$.

If $u_0 \ne 0$, then, by Proposition 3.1, $e^{\beta_n u_n^2}$ is bounded in $L^{p_0}(\Omega)$ for some $p_0 > 1$. By Lemma 4.1, we know that λ_n is bounded. Hence $(-\Delta)^m u_n$ is bounded in $L^s(\Omega)$ for any $1 < s < p_0$. Then, by elliptic estimates (see Proposition A.4), we find that u_n is bounded in $W^{2m,s}(\Omega)$ and, by Sobolev embeddings, in $L^\infty(\Omega)$. This contradicts $\mu_n \to +\infty$. Hence, we have $u_0 = 0$.

□

In fact, u_n converges to 0 in a much stronger sense if we stay far from the blow-up point x_0, while $|\Delta^{\frac{m}{2}} u_n|^2$ concentrates around x_0.

Lemma 4.4. *If $\mu_n \to +\infty$, then we have:*

1. $|\Delta^{\frac{m}{2}} u_n|^2 \rightharpoonup \delta_{x_0}$ *in the sense of measures.*
2. $e^{\beta_n u_n^2}$ *is bounded in* $L^s(\Omega \setminus B_\delta(x_0))$*, for any* $s \ge 1$, $\delta > 0$.

3. $u_n \to 0$ *in* $C^{2m-1,\gamma}(\Omega \setminus B_\delta(x_0))$*, for any* $\gamma \in (0,1)$*,* $\delta > 0$*.*

Proof. First of all, for any function $\xi \in C^{2m}(\overline{\Omega})$, we observe that

$$\Delta^{\frac{m}{2}}(u_n\xi) = \xi\Delta^{\frac{m}{2}}u_n + f_n,$$

with

$$|f_n| \le C_1 \sum_{l=0}^{m-1} |\nabla^l u_n||\nabla^{m-l}\xi| \le C_2 \sum_{l=0}^{m-1} |\nabla^l u_n|,$$

for some constants $C_1, C_2 > 0$, depending only on m, l, and ξ. Since $u_n \rightharpoonup 0$ in $H_0^m(\Omega)$, and $H_0^m(\Omega)$ is compactly embedded in $H^{m-1}(\Omega)$, we get that $f_n \to 0$ in $L^2(\Omega)$. In particular, we have

$$\begin{aligned} \|\Delta^{\frac{m}{2}}(u_n\xi)\|_{L^2(\Omega)}^2 &= \int_\Omega \xi^2|\Delta^{\frac{m}{2}}u_n|^2dx + 2\int_\Omega \Delta^{\frac{m}{2}}u_n \cdot f_n dx + \int_\Omega |f_n|^2dx \\ &= \int_\Omega \xi^2|\Delta^{\frac{m}{2}}u_n|^2dx + o(1). \end{aligned} \tag{4.10}$$

We can now prove the first statement of this lemma. Assume by contradiction that there exists $r > 0$ such that

$$\limsup_{n\to+\infty} \|\Delta^{\frac{m}{2}}u_n\|_{L^2(B_r(x_0)\cap\Omega)}^2 < 1. \tag{4.11}$$

Take a function $\xi \in C_c^\infty(\mathbb{R}^{2m})$ such that $\xi \equiv 1$ on $B_{\frac{r}{2}}(x_0)$, $\xi \equiv 0$ on $\mathbb{R}^{2m} \setminus B_r(x_0)$ and $0 \le \xi \le 1$. By (4.10) and (4.11), we have that $\limsup_{n\to+\infty} \|\Delta^{\frac{m}{2}}(u_n\xi)\|_{L^2(\Omega)}^2 < 1$. Adams' inequality implies that we can find $s > 1$ such that $e^{\beta_n(u_n\xi)^2}$ is bounded in $L^s(\Omega)$. In particular, $e^{\beta_n u_n^2}$ is bounded in $L^s(B_{\frac{r}{2}}(x_0))$. By Lemma 4.3, $u_n \to 0$ in $L^p(\Omega)$ for any $p \ge 1$. Therefore, we get that $(-\Delta)^m u_n \to 0$ in $L^q(\Omega)$ for any $1 < q < s$. Then, Proposition A.4 yields $u_n \to 0$ in $W^{2m,q}(\Omega)$ and, since $q > 1$, in $L^\infty(\Omega)$. This contradicts $\mu_n \to +\infty$.

To prove *2.*, we fix a cut-off function $\xi_2 \in C_c^\infty(\mathbb{R}^{2m})$ such that $\xi_2 \equiv 1$ in $\mathbb{R}^{2m} \setminus B_\delta(x_0)$, $\xi_2 \equiv 0$ in $B_{\frac{\delta}{2}}(\Omega)$, and $\xi \le 1$. Since $|\Delta^{\frac{m}{2}}u_n| \rightharpoonup \delta_{x_0}$, from (4.10) we get $\|\Delta^{\frac{m}{2}}(u_n\xi_2)\|_{L^2(\Omega)} \to 0$. Then, Adams' inequality implies that $e^{\beta_n(u_n\xi_2)^2}$ is bounded in $L^s(\Omega)$, for any $s > 1$. Because of the definition of ξ_2, we get the conclusion.

To prove *3.*, we apply standard elliptic estimates. By part *2.*, we know that u_n and $e^{\beta_n u_n^2}$ are bounded in $L^s(\Omega \setminus B_\delta(x_0))$ for any $s \ge 1$. Since λ_n is bounded, the same bound holds for $(-\Delta)^m u_n$. Then, elliptic estimates (Propostion A.6) imply that u_n is bounded in $W^{2m,s}(\Omega \setminus B_{2\delta}(x_0))$. By Sobolev embedding theorem, it is also bounded in $C^{2m-1,\gamma}(\Omega \setminus B_{2\delta}(x_0))$, for any $\gamma \in (0,1)$. Then, up to a subsequence, we can find a function $u_0 \in C^{2m-1,\gamma}(\Omega \setminus B_{2\delta}(x_0))$ such that $u_n \to u_0$ in $C^{2m-1,\gamma}(\Omega \setminus B_{2\delta}(x_0))$. Since $u_n \rightharpoonup 0$ in $H_0^m(\Omega)$, we must have $u_0 \equiv 0$ in $\Omega \setminus B_{2\delta}(x_0)$ and $u_n \to 0$ in $C^{2m-1,\gamma}(\Omega \setminus B_{2\delta}(x_0))$. □

4.2 Lorentz-Sobolev elliptic estimates

In this subsection, we prove uniform integral estimates on the derivatives of u_n. Notice that Sobolev's inequality implies $\|\nabla^l u_n\|_{L^{\frac{2m}{l}}(\Omega)} \le C$ for any $1 \le l \le m-1$. In addition, standard elliptic estimates (Proposition A.11) yield $\|\nabla^l u_n\|_{L^p(\Omega)} \le C$, for any $p < \frac{2m}{l}$ and $m \le l \le 2m-1$. Arguing as in [21], we will prove that sharper estimates can be obtained thanks to Lorentz-Zygmund elliptic regularity theory (see Proposition A.10 in Appendix). In the following, for any $\alpha \ge 0$, $1 < p < +\infty$, and $1 \le q \le +\infty$, $(L(\log L)^\alpha, \|\cdot\|_{L(\log L)^\alpha})$ and $(L^{(p,q)}(\Omega), \|\cdot\|_{(p,q)})$, will denote respectively the Zygmund and Lorentz spaces on Ω. We refer to the Appendix for the precise definitions (see (A.2)-(A.8)).

Lemma 4.5. *For any* $1 \le l \le 2m-1$*, we have*

$$\|\nabla^l u_n\|_{(\frac{2m}{l},2)} \le C.$$

Proof. Set $f_n := (-\Delta)^m u_n$. By Proposition A.10, there exists a constant $C > 0$ such that

$$\|\nabla^l u_n\|_{(\frac{2m}{l},2)} \le C\|f_n\|_{L(LogL)^{\frac{1}{2}}},$$

for any $1 \le l \le 2m-1$, $n \in \mathbb{N}$. Therefore, it is sufficient to prove that f_n is bounded in $L(\log L)^{\frac{1}{2}}$.For any $x \in \mathbb{R}^+$, let $\log^+ x := \max\{0, \log x\}$ be the positive part of $\log x$. Since β_n and λ_n are bounded, using the simple inequalities

$$\log(x+y) \le x + \log^+ y \qquad \text{and} \qquad \log^+(xy) \le \log^+ x + \log^+ y, \qquad x, y \in \mathbb{R}^+,$$

we find

$$\begin{aligned}\log(2+|f_n|) &\le 2 + \log^+|u_n| + \log^+\left(\lambda_n e^{\beta_n u_n^2} + \alpha\right)\\ &\le C + \log^+|u_n| + \beta_n u_n^2\\ &\le C(|u_n|+1)^2.\end{aligned}$$

Then,

$$|f_n|\log^{\frac{1}{2}}(2+|f_n|) \le C|f_n|(1+|u_n|) \le C\left(\lambda_n|u_n|e^{\beta_n u_n^2} + \lambda_n u_n^2 e^{\beta_n u_n^2} + \alpha|u_n| + \alpha u_n^2\right),$$

and, by Lemma 4.3 and (4.4), as $n \to +\infty$ we get

$$\begin{aligned}\int_\Omega |f_n|\log^{\frac{1}{2}}(2+|f_n|)dx &\le C\left(\lambda_n \int_\Omega |u_n|e^{\beta_n u_n^2}dx + 1 + o(1)\right)\\ &\le C\left(\lambda_n \int_{\{|u_n|<1\}} |u_n|e^{\beta_n u_n^2}dx + \lambda_n \int_{\{|u_n|\ge 1\}} |u_n|^2 e^{\beta_n u_n^2}dx + 1 + o(1)\right)\\ &\le C\left(\lambda_n e^{\beta_n}|\Omega| + 2 + o(1)\right) = O(1).\end{aligned}$$

Hence, f_n is bounded in $L(LogL)^{\frac{1}{2}}$. □

As a consequence of Lemma 4.5, we obtain an integral estimate on the derivatives of u_n^2, which will play an important role in Sections 4.3 and 4.4. The idea behind this estimate is based on the following remark: up to terms involving only lower order derivatives, which can be controlled using Lemma 4.5, $(-\Delta)^m u_n^2$ coincides with $u_n(-\Delta)^m u_n$, which is bounded in $L^1(\Omega)$. Then, estimates on u_n^2 can be obtained via Green's representation formula.

Lemma 4.6. *There exists a constant $C > 0$ such that for any $1 \le l \le 2m-1$, $x \in \Omega$, and $\rho > 0$ with $B_\rho(x) \subseteq \Omega$, we have*

$$\int_{B_\rho(x)} |\nabla^l u_n^2|dy \le C\rho^{2m-l}.$$

Proof. We start by observing that $(-\Delta)^m u_n^2$ is bounded in $L^1(\Omega)$. Clearly

$$|(-\Delta)^m u_n^2| \le 2|u_n(-\Delta)^m u_n| + C\sum_{j=1}^{2m-1} |\nabla^j u_n||\nabla^{2m-j}u_n|.$$

Equation (4.4) and Lemma 4.3 imply that $u_n(-\Delta)^m u_n$ is bounded in $L^1(\Omega)$. As a consequence of Hölder's inequality for Lorentz spaces (Proposition (A.9)) and Lemma 4.5, we find

$$\int_\Omega |\nabla^{2m-j}u_n||\nabla^j u_n|dx \le \|\nabla^{2m-j}u_n\|_{(\frac{2m}{2m-j},2)}\|\nabla^j u_n\|_{(\frac{2m}{j},2)} \le C.$$

Thus, $(-\Delta)^m u_n^2$ is bounded in $L^1(\Omega)$.

Now, we apply Green's representation formula to u_n^2 to get

$$u_n^2(y) = \int_\Omega G_y(z)(-\Delta)^m u_n^2(z)dz,$$

for any $y \in \Omega$ where $G_y := G_{0,y}$ is defined as in (2.5). By the properties of G_y (see Proposition 2.2), we have

$$|\nabla_y^l G_y(z)| \le \frac{C}{|y-z|^l},$$

for any $y, z \in \Omega$ with $z \neq y$. Hence

$$|\nabla^l u_n^2(y)| \le \int_\Omega \frac{C|(-\Delta)^m u_n^2(z)|}{|y-z|^l} dz.$$

Let $x \in \Omega$ and $\rho > 0$ be as in the statement. Then, we find

$$\begin{aligned}\int_{B_\rho(x)} |\nabla^l u_n^2| dy &\le \int_{B_\rho(x)} \int_\Omega \frac{C|(-\Delta)^m u_n^2(z)|}{|y-z|^l} dzdy \\ &= C\int_\Omega |(-\Delta)^m u_n^2(z)| \int_{B_\rho(x)} \frac{1}{|y-z|^l} dydz.\end{aligned}$$

Since

$$\int_{B_\rho(x)} \frac{1}{|y-z|^l} dy \le \int_{B_\rho(x)} \frac{1}{|y-x|^l} dy = C\rho^{2m-l},$$

and $(-\Delta)^m u_n^2$ is bounded in $L^1(\Omega)$, we get the conclusion. □

4.3 The behavior on a small scale

Let u_n, μ_n and x_n be as in (4.2), (4.7), (4.8). In this subsection, we will study the behavior of u_n on small balls centered at the maximum point x_n. Define $r_n > 0$ so that

$$\omega_{2m} r_n^{2m} \lambda_n \mu_n^2 e^{\beta_n \mu_n^2} = 1, \tag{4.12}$$

with ω_{2m} as in (2.1).

Remark 4.7. *Note that, as $n \to +\infty$, we have $r_n^{2m} = o(\mu_n^{-2})$ and, in particular, $r_n \to 0$.*

Proof. Indeed, by (4.4), we have

$$\frac{1}{\lambda_n e^{\beta_n \mu_n^2}} = \frac{1}{e^{\beta_n \mu_n^2}} \int_\Omega u_n^2 e^{\beta_n u_n^2} dx \le \|u_n\|_{L^2(\Omega)}^2.$$

Since $u_n \to 0$ in $L^2(\Omega)$, the definition of r_n^{2m} yields $r_n^{2m}\mu_n^2 \to 0$ as $n \to +\infty$. □

Let us now consider the scaled function

$$\eta_n(y) := \mu_n(u_n(x_n + r_n y) - \mu_n), \tag{4.13}$$

which is defined on the set

$$\Omega_n := \{y \in \mathbb{R}^{2m} \ : \ x_n + r_n y \in \Omega\}.$$

The main purpose of this subsection consists in proving the following convergence result.

Proposition 4.8. *We have* $\frac{d(x_n,\partial\Omega)}{r_n} \to +\infty$ *and, in particular,* Ω_n *approaches* $\mathbb{R}^{2m}$ *as* $n \to +\infty$. *Moreover,* η_n *converges to the limit function*

$$\eta_0(y) = -\frac{m}{\beta^*}\log\left(1+\frac{|y|^2}{4}\right) \tag{4.14}$$

in $C^{2m-1,\gamma}_{loc}(\mathbb{R}^{2m})$*, for any* $\gamma \in (0,1)$.

In order to avoid repetitions, it is convenient to see Proposition 4.8 as a special case of the following more general result, which will be useful also in the proof of Proposition 4.15.

Proposition 4.9. *Given two sequences* $\tilde{x}_n \in \Omega$ *and* $s_n \in \mathbb{R}^+$*, consider the scaled set* $\tilde{\Omega}_n := \{y \in \mathbb{R}^{2m} : \tilde{x}_n + s_n y \in \Omega\}$ *and the functions* $v_n(y) := u_n(\tilde{x}_n + s_n y)$ *and* $\tilde{\eta}_n(y) := \tilde{\mu}_n(v_n(y) - \tilde{\mu}_n)$*, where* $\tilde{\mu}_n := u_n(\tilde{x}_n)$. *Assume that*

1. $\omega_{2m} s_n^{2m}\lambda_n\tilde{\mu}_n^2 e^{\beta_n\tilde{\mu}_n^2} = 1$ *and* $|\tilde{\mu}_n| \to +\infty$, $s_n^{2m} \to 0$, *as* $n \to +\infty$.
2. *For any* $R > 0$ *there exists a constant* $C(R) > 0$ *such that*

$$\left|\frac{v_n}{\tilde{\mu}_n}\right| \le C(R) \quad \text{and} \quad v_n^2 - \tilde{\mu}_n^2 \le C(R) \quad \text{in } \tilde{\Omega}_n \cap B_R(0). \tag{4.15}$$

Then, we have $\frac{d(\tilde{x}_n,\partial\Omega)}{s_n} \to +\infty$ *and* $\frac{v_n}{\mu_n} \to 1$ *in* $C^{2m-1,\gamma}_{loc}(\mathbb{R}^{2m})$*, for any* $\gamma \in (0,1)$. *Moreover* $\tilde{\eta}_n \to \eta_0$ *in* $C^{2m-1,\gamma}_{loc}(\mathbb{R}^{2m})$*, where* η_0 *is defined as in* (4.14).

Note that the assumptions of Proposition 4.9 are satisfied when $\tilde{x}_n = x_n$ and $s_n = r_n$. Hence, Proposition 4.8 follows from Proposition 4.9. We split the proof of Proposition 4.9 into four steps. The first two steps (Lemma 4.10 and Lemma 4.11) are stated under more general assumptions, since they will be reused in the proof of Proposition 4.16.

Lemma 4.10. *Given two sequences* $\tilde{x}_n \in \Omega$ *and* $s_n \in \mathbb{R}^+$*, let* $\tilde{\Omega}_n$ *and* v_n *be defined as in Proposition 4.9. Let also* Σ *be a finite (possibly empty) subset of* $\mathbb{R}^{2m} \setminus \{0\}$. *Assume that*

1. $s_n \to 0$ *and* $D_n := \max\limits_{0 \le i \le 2m-1} |\nabla^i v_n(0)| \to +\infty$ *as* $n \to +\infty$.
2. *For any* $R > 0$*, there exist* $C(R) > 0$ *and* $N(R) \in \mathbb{N}$ *such that*

$$|v_n(y)| \le C(R)D_n \qquad \text{and} \qquad |(-\Delta)^m v_n(y)| \le C(R)D_n,$$

for any $y \in \tilde{\Omega}_{n,R} := \tilde{\Omega}_n \cap B_R(0) \setminus \bigcup_{\xi\in\Sigma} B_{\frac{1}{R}}(\xi)$ *and any* $n \ge N(R)$.

Then, we have

$$\lim_{n\to+\infty} \frac{d(\tilde{x}_n,\partial\Omega)}{s_n} = +\infty.$$

Proof. Let us consider the functions $w_n(y) := \frac{v_n(y)}{D_n}$. First, we observe that the assumptions on $\tilde{x}_n$ and s_n imply

$$w_n = O(1), \tag{4.16}$$

and

$$|(-\Delta)^m w_n| = O(1), \tag{4.17}$$

uniformly in $\tilde{\Omega}_{n,R}$, for any $R > 0$. Moreover, by Sobolev's inequality, for any $1 \le j \le m$ we have that

$$\|\nabla^j w_n\|_{L^{\frac{2m}{j}}(\tilde{\Omega}_n)} = D_n^{-1}\|\nabla^j u_n\|_{L^{\frac{2m}{j}}(\Omega)} \le C D_n^{-1}\|\Delta^{\frac{m}{2}} u_n\|_{L^2(\Omega)} = O(D_n^{-1}). \tag{4.18}$$

Then, using Hölder's inequality, (4.16) and (4.18) give

$$\|w_n\|_{W^{m,1}(\tilde{\Omega}_{n,R})} = O(1). \tag{4.19}$$

Now, we assume by contradiction that for a subsequence

$$\frac{d(\tilde{x}_n, \partial\Omega)}{s_n} \to R_0 \in [0, +\infty).$$

Then, the sets $\tilde{\Omega}_n$ converge in C^∞_{loc} to a hyperplane $\mathcal{P}$ such that $d(0, \partial\mathcal{P}) = R_0$. For any sufficiently large $R > 0$ and any $p > 1$, using (4.17), (4.19), Proposition A.6, and Remark A.7, we find a constant $C = C(R)$ such that $\|w_n\|_{W^{2m,p}(\tilde{\Omega}_{n,\frac{R}{2}})} \le C$. Then, Sobolev's embeddings imply that $\|w_n\|_{C^{2m-1,\gamma}(\tilde{\Omega}_{n,\frac{R}{2}})} \le C$, for any $\gamma \in (0,1)$. Reproducing the standard proof of the Ascoli-Arzelà theorem, we find a function $w_0 \in C^{2m-1,\gamma}_{loc}(\overline{\mathcal{P}} \setminus \Sigma)$ such that, up to a subsequence, we have

$$w_n \to w_0 \qquad \text{in } C^{2m-1}_{loc}(\mathcal{P} \setminus \Sigma) \tag{4.20}$$

and

$$\nabla^j w_n(\xi_n) \to \nabla^j w_0(\xi), \quad 0 \le j \le 2m-1, \tag{4.21}$$

for any $\xi \in \overline{\mathcal{P}} \setminus \Sigma$ and any sequence $\{\xi_n\}_{n\in\mathbb{N}}$ such that $\xi_n \to \xi$. Since $w_n = 0$ on $\partial\tilde{\Omega}_n$ and $\tilde{\Omega}_n$ converges to $\mathcal{P}$, (4.21) yields $w_0 \equiv 0$ in $\partial\mathcal{P} \setminus \Sigma$. Furthermore, (4.18) and (4.20) imply that $\nabla w_0 \equiv 0$ in $\mathcal{P} \setminus \Sigma$. Therefore, $w_0 \equiv 0$ on $\overline{\mathcal{P}} \setminus \Sigma$. But, by definition of D_n and w_n, we have

$$\max_{0 \le i \le 2m-1} |\nabla^i w_n(0)| = 1,$$

which contradicts either (4.20) (if $R_0 > 0$) or (4.21) (if $R_0 = 0$). □

Lemma 4.11. *Let s_n, $\tilde{x}_n$, v_n, $\tilde{\Omega}_n$, D_n and Σ be as in Lemma 4.10. Then, $|v_n(0)| \to +\infty$ and*

$$\frac{v_n}{v_n(0)} \to 1 \qquad \text{in } C^{2m-1,\gamma}_{loc}(\mathbb{R}^{2m} \setminus \Sigma),$$

for any $\gamma \in (0,1)$.

Proof. Consider the function $w_n(y) := \dfrac{v_n(y)}{D_n}$, $y \in \tilde{\Omega}_n$. As in (4.16), (4.17) and (4.18), we have

$$w_n = O(1) \qquad \text{and} \qquad (-\Delta)^m w_n = O(1), \tag{4.22}$$

uniformly in $B_R(0) \setminus \bigcup_{\xi\in\Sigma} B_{\frac{1}{R}}(\xi)$, for any $R > 0$, and

$$\|\nabla w_n\|_{L^{2m}(\tilde{\Omega}_n)} \to 0. \tag{4.23}$$

By (4.22), Proposition A.5, Sobolev's embeddings, and (4.23), a subsequence of w_n must converge to a constant function w_0 in $C^{2m-1,\gamma}_{loc}(\mathbb{R}^{2m} \setminus \Sigma)$, for any $\gamma \in (0,1)$. In particular, we have $|\nabla^j w_n(0)| \to 0$ for any $1 \le j \le 2m-1$. Then, the definitions of D_n and w_n give

$$1 = \max_{0 \le j \le 2m-1} |\nabla^j w_n(0)| = |w_n(0)|,$$

which implies that $|v_n(0)| = D_n \to +\infty$ and that $|w_0| \equiv 1$ in $\mathbb{R}^{2m} \setminus \Sigma$. Hence,

$$\frac{v_n}{v_n(0)} = \frac{w_n}{w_n(0)} \to 1 \qquad \text{in } C^{2m-1,\gamma}_{loc}(\mathbb{R}^{2m} \setminus \Sigma).$$

□

Next, we let $\tilde{x}_n, s_n$, $\tilde{\mu}_n$ and $\tilde{\eta}_n$ be as in Proposition 4.9 and we apply Lemma 4.6 to prove bounds for $\Delta\tilde{\eta}_n$ in $L^1_{loc}(\mathbb{R}^{2m})$.

Lemma 4.12. *Under the assumptions of Proposition 4.9, there exists a constant $C > 0$ such that*

$$\|\Delta\tilde{\eta}_n\|_{L^1(B_R(0))} \le CR^{2m-2},$$

for any $R > 1$ and for sufficiently large n.

Proof. First, we observe that $\tilde{x}_n$ and s_n satisfy the assumptions of Lemma 4.10 and Lemma 4.11. Indeed, equation (4.3), the definition of v_n, and the assumptions on $\tilde{x}_n$ and s_n yield $v_n = O(|\tilde{\mu}_n|)$ and

$$\begin{aligned}(-\Delta)^m v_n &= s_n^{2m}\lambda_n v_n e^{\beta_n v_n^2} + s_n^{2m}\alpha v_n \\ &= \omega_{2m}^{-1}\frac{v_n}{\tilde{\mu}_n^2}e^{\beta_n(v_n^2-\tilde{\mu}_n^2)} + s_n^{2m}\alpha v_n \\ &= O(|\tilde{\mu}_n^{-1}|) + O(s_n^{2m}|\tilde{\mu}_n|),\end{aligned} \tag{4.24}$$

uniformly in $\tilde{\Omega}_n \cap B_R(0)$, for any $R > 0$. Then, Lemma 4.10 and Lemma 4.11 imply that $\tilde{\Omega}_n$ approaches $\mathbb{R}^{2m}$ and

$$\frac{v_n}{\tilde{\mu}_n} \to 1 \qquad \text{in } C^{2m-1,\gamma}_{loc}(\mathbb{R}^{2m}), \text{ for any } \gamma \in (0,1). \tag{4.25}$$

Next, we rewrite the estimates of Lemma 4.6 in terms of $\tilde{\eta}_n$. On the one hand, by Lemma 4.6, there exists $C > 0$, such that

$$\|\Delta u_n^2\|_{L^1(B_{Rs_n}(\tilde{x}_n))} \le C(s_nR)^{2m-2},$$

for any $R > 0$ and $n \in \mathbb{N}$. On the other hand, we have

$$\begin{aligned}\|\Delta u_n^2\|_{L^1(B_{Rs_n}(\tilde{x}_n))} &\ge 2\|u_n\Delta u_n\|_{L^1(B_{Rs_n}(\tilde{x}_n))} - 2\|\nabla u_n\|^2_{L^2(B_{Rs_n}(\tilde{x}_n))} \\ &= 2s_n^{2m-2}\left(\|v_n\Delta v_n\|_{L^1(B_R(0))} - \|\nabla v_n\|^2_{L^2(B_R(0))}\right).\end{aligned}$$

Then, we obtain

$$\|v_n\Delta v_n\|_{L^1(B_R(0))} \le CR^{2m-2} + \|\nabla v_n\|^2_{L^2(B_R(0))}. \tag{4.26}$$

By (4.25) and the definition of $\tilde{\eta}_n$, we infer

$$\begin{aligned}\|v_n\Delta v_n\|_{L^1(B_R(0))} = |\tilde{\mu}_n|\|\Delta v_n\|_{L^1(B_R(0))}(1+o(1)) &= \|\Delta\tilde{\eta}_n\|_{L^1(B_R(0))}(1+o(1)) \\ &\ge \frac{1}{2}\|\Delta\tilde{\eta}_n\|_{L^1(B_R(0))},\end{aligned} \tag{4.27}$$

for sufficiently large n. Finally, applying Hölder's inequality,

$$\|\nabla v_n\|^2_{L^2(B_R(0))} \le \|\nabla v_n\|^2_{L^{2m}(B_R(0))}|B_R|^{1-\frac{1}{m}} \le \|\nabla u_n\|^2_{L^{2m}(\Omega)}|B_R|^{1-\frac{1}{m}} \le CR^{1-\frac{1}{m}}. \tag{4.28}$$

Since $1 - \frac{1}{m} \le 2m-2$, the conclusion follows from (4.26), (4.27), and (4.28). □

We can now complete the proof of Proposition 4.9.

Proof of Proposition 4.9. Arguing as in the previous Lemma, we have that $\frac{d(\tilde{x}_n,\partial\Omega)}{s_n} \to +\infty$ and that (4.25) holds. Observe that (4.25) implies

$$(1+o(1))s_n^{2m}\tilde{\mu}_n^2 = \frac{s_n^{2m}}{\omega_{2m}}\int_{B_1(0)} v_n^2(y)dy = \frac{1}{\omega_{2m}}\int_{B_{s_n}(\tilde{x}_n)} u_n^2(x)dx = O(\|u_n\|^2_{L^2(\Omega)}) = o(1). \tag{4.29}$$

Moreover, as in (4.24), by the definitions of $\tilde{\eta}_n$ and v_n, and the assumptions on $\tilde{\mu}_n$, s_n and $\tilde{x}_n$, we get

$$(-\Delta)^m\tilde{\eta}_n = O(1) + O(s_n^{2m}\tilde{\mu}_n^2) = O(1), \tag{4.30}$$

uniformly in $B_R(0)$, for any $R > 0$. In addition, Lemma 4.12 implies that $\Delta\tilde{\eta}_n$ is bounded in $L^1_{loc}(\mathbb{R}^{2m})$. By Proposition A.5 and Sobolev's embedding theorem, $\Delta\tilde{\eta}_n$ is bounded in $L^\infty_{loc}(\mathbb{R}^{2m})$. As a consequence of (4.15) and (4.25), we have

$$C(R) \geq v_n^2 - \tilde{\mu}_n^2 = (v_n - \tilde{\mu}_n)(v_n + \tilde{\mu}_n) = \tilde{\eta}_n(2 + o(1))$$

in $B_R(0)$. Since $\tilde{\eta}_n(0) = 0$, Proposition A.8 shows that $\tilde{\eta}_n$ is bounded in $L^\infty_{loc}(\mathbb{R}^{2m})$. Together with (4.30), Proposition A.5, and Sobolev's embeddings, this implies that η_n it is bounded in $C^{2m-1,\gamma}_{loc}(\mathbb{R}^{2m})$, for any $\gamma \in (0,1)$. Then, we can extract a subsequence such that $\tilde{\eta}_n$ converges in $C^{2m-1,\gamma}_{loc}(\mathbb{R}^{2m})$ to a limit function $\eta_0 \in C^{2m-1,\gamma}_{loc}(\mathbb{R}^{2m})$. Observe that, as $n \to +\infty$,

$$(-\Delta)^m \tilde{\eta}_n = \left(1 + \frac{\tilde{\eta}_n}{\tilde{\mu}_n^2}\right)\left(\omega_{2m}^{-1} e^{2\beta_n \tilde{\eta}_n + \beta_n \frac{\tilde{\eta}_n^2}{\tilde{\mu}_n^2}} + \alpha s_n^{2m} \tilde{\mu}_n^2\right) \to \omega_{2m}^{-1} e^{2\beta^* \eta_0},$$

locally uniformly in $\mathbb{R}^{2m}$. This implies that η_0 must be a weak solution of

$$\begin{cases} (-\Delta)^m \eta_0 = \omega_{2m}^{-1} e^{2\beta^* \eta_0}, \\ e^{2\beta^* \eta_0} \in L^1(\mathbb{R}^{2m}), \\ \eta_0 \leq 0, \eta_0(0) = 0. \end{cases} \tag{4.31}$$

Solutions of problem (4.31) have been classified in [22] (see also [14] and [34]). In particular, Theorems 1 and 2 in [22] imply that there exists a real number $a \leq 0$, such that $\lim_{|y| \to +\infty} \Delta\eta_0(y) = a$. Moreover, either $a \neq 0$, or $\eta_0(y) = -\frac{m}{\beta^*} \log\left(1 + \frac{|y|^2}{4}\right)$, for any $y \in \mathbb{R}^{2m}$. To exclude the first possibility we observe that, if $a \neq 0$, then we can find $R_0 > 0$ such that $|\Delta\eta_0| \geq \frac{|a|}{2}$ for $|y| \geq R_0$. This yields

$$\int_{B_R(0)} |\Delta\eta_0| dy \geq \int_{B_{R_0}(0)} |\Delta\eta_0| dy + \frac{|a|}{2} \omega_{2m}(R^{2m} - R_0^{2m}), \tag{4.32}$$

for any $R > R_0$. But Lemma 4.12 implies

$$\int_{B_R(0)} |\Delta\eta_0| dy \leq C R^{2m-2}, \tag{4.33}$$

for any $R > 1$. For large values of R, (4.33) contradicts (4.32). □

This completes the proof of Proposition 4.9. Now, we state some properties of the function η_0 that will play a crucial role in the next sections.

Lemma 4.13. *Let η_0 be as in* (4.14). *Then, as $R \to +\infty$, we have*

$$\omega_{2m}^{-1} \int_{B_R(0)} e^{2\beta^* \eta_0} dy = 1 + O(R^{-2m}) \tag{4.34}$$

and

$$\int_{B_R(0)} |\Delta^{\frac{m}{2}} \eta_0|^2 dy = \frac{2m}{\beta^*} \log \frac{R}{2} + I_m - H_m + O(R^{-2} \log R), \tag{4.35}$$

where H_m is defined as in (2.6) *and*

$$I_m = \int_{\mathbb{R}^{2m}} \eta_0 (-\Delta)^m \eta_0 \, dy = -\frac{m 4^{2m}}{\beta^* \omega_{2m}} \int_{\mathbb{R}^{2m}} \frac{\log\left(1 + \frac{|y|^2}{4}\right)}{(4 + |y|^2)^{2m}} dy \tag{4.36}$$

is as in (4.9).

Proof. First, using a straightforward change of variable and the representation of $\mathbb{S}^{2m}$ through the standard stereographic projection, we observe that

$$\int_{\mathbb{R}^{2m}} e^{2\beta^*\eta_0}dy = \int_{\mathbb{R}^{2m}} \frac{4^m}{(1+|y|^2)^{2m}}dy = \omega_{2m}.$$

Since $e^{2\beta^*\eta_0} = O(\frac{1}{|y|^{4m}})$ as $|y| \to +\infty$, we get (4.34).

The proof of (4.35) relies on the integration by parts formula of Proposition 2.1. For any $1 \le l \le m-1$, we have

$$\Delta^l \eta_0(y) = \frac{m}{\beta^*}\sum_{k=0}^{l} a_{k,l}\frac{|y|^{2k}}{(4+|y|^2)^{2l}}, \quad a_{k,l} = (-1)^l(l-1)!\binom{l}{k}\frac{(m+l-1)!(m-l+k-1)!}{(m+k-1)!(m-l-1)!}2^{4l-2k},$$

and

$$\Delta^{l+\frac{1}{2}}\eta_0(y) = \frac{m}{\beta^*}\sum_{k=0}^{l} b_{k,l}\frac{|y|^{2k}y}{(4+|y|^2)^{2l+1}}, \quad b_{k,l} = \begin{cases} 8(k+1)a_{k+1,l} + (2k-4l)a_{k,l} & 0 \le k \le l-1, \\ -2la_{ll} & k = l. \end{cases}$$

Note that $a_{ll} = -2\tilde{K}_{m,l}$, where $\tilde{K}_{m,l}$ is as in (2.2). In any case, for $1 \le j \le 2m-1$, we find

$$\Delta^{\frac{j}{2}}\eta_0 = -\frac{2m}{\beta^*}K_{m,\frac{j}{2}}\frac{e_j(y)}{|y|^j} + O(|y|^{-2-j}), \tag{4.37}$$

as $|y| \to +\infty$, where $K_{m,\frac{j}{2}}$ and e_j are defined as in (2.3) and (2.4). Integrating by parts, we find

$$\int_{B_R(0)} |\Delta^{\frac{m}{2}}\eta_0|^2 dy = \int_{B_R(0)} \eta_0(-\Delta)^m\eta_0\, dy - \sum_{j=0}^{m-1}\int_{\partial B_R(0)} (-1)^{j+m}\nu\cdot\Delta^{\frac{j}{2}}\eta_0\Delta^{\frac{2m-j-1}{2}}\eta_0\, d\sigma.$$

On $\partial B_R(0)$, (4.37) and the identity $\frac{2m}{\beta^*}K_{m,\frac{2m-1}{2}} = \frac{(-1)^{m-1}}{\omega_{2m-1}}$ imply

$$\begin{aligned}\nu\cdot\eta_0\Delta^{\frac{2m-1}{2}}\eta_0 &= \left(-\frac{2m}{\beta^*}\log\frac{R}{2} + O(R^{-2})\right)\left(\frac{-2m}{\beta^*}K_{m,\frac{2m-1}{2}}R^{1-2m} + O(R^{-2m-1})\right) \\ &= \frac{(-1)^m}{\omega_{2m-1}}R^{1-2m}\left(-\frac{2m}{\beta^*}\log\frac{R}{2} + O(R^{-2}\log R)\right),\end{aligned}$$

and, for $1 \le j \le m-1$, that

$$\begin{aligned}\nu\cdot\Delta^{\frac{j}{2}}\eta_0\Delta^{\frac{2m-j-1}{2}}\eta_0 &= \left(-\frac{2m}{\beta^*}K_{m,\frac{j}{2}}R^{-j} + O(R^{-j-2})\right)\left(-\frac{2m}{\beta^*}K_{m,\frac{2m-j-1}{2}}R^{1+j-2m} + O(R^{j-2m-1})\right) \\ &= \left(\frac{2m}{\beta^*}\right)^2 K_{m,\frac{j}{2}}K_{m,\frac{2m-j-1}{2}}R^{1-2m} + O(R^{-2m-1}).\end{aligned}$$

Hence, we have

$$\int_{B_R(0)} |\Delta^{\frac{m}{2}}\eta_0|^2 dy = \int_{B_R(0)} \eta_0(-\Delta)^m\eta_0\, dy + \frac{2m}{\beta^*}\log\frac{R}{2} - H_m + O(R^{-2}\log R). \tag{4.38}$$

Finally, since $\eta_0(-\Delta)^m\eta_0$ decays like $|y|^{-4m}\log|y|$ as $|y| \to +\infty$, we get

$$\int_{B_R(0)} \eta_0(-\Delta)^m\eta_0\, dy = I_m + O(R^{-2m}\log R),$$

which, together with (4.38), gives the conclusion. □

Remark 4.14. *Proposition 4.8 and Lemma 4.13 imply*

1. $\lim_{n\to+\infty}\int_{B_{Rr_n(x_n)}} \lambda_n u_n^2 e^{\beta_n u_n^2}dx = 1 + O(R^{-2m}).$

2. $\lim_{n\to+\infty}\int_{B_{Rr_n(x_n)}} \lambda_n \mu_n u_n e^{\beta_n u_n^2}dx = 1 + O(R^{-2m}).$

3. $\lim_{n\to+\infty}\int_{B_{Rr_n(x_n)}} \lambda_n \mu_n |u_n| e^{\beta_n u_n^2}dx = 1 + O(R^{-2m}).$

4. $\lim_{n\to+\infty}\int_{B_{Rr_n(x_n)}} \lambda_n \mu_n^2 e^{\beta_n u_n^2}dx = 1 + O(R^{-2m}).$

Indeed, all the integrals converge to $\omega_{2m}^{-1}\int_{B_R(0)} e^{2\beta^*\eta_0}dy.$

4.4 Estimates on the derivatives of u_n

In this subsection, we prove some pointwise estimates on u_n and its derivatives that are inspired from the ones in Theorem 1 of [21] and Proposition 11 of [25] (where the authors assume $\alpha = 0$ and $u_n \geq 0$).

Proposition 4.15. *There exists a constant $C > 0$, such that*

$$|x - x_n|^{2m}\lambda_n u_n^2 e^{\beta_n u_n^2} \leq C,$$

for any $x \in \Omega$.

Proof. Let us denote

$$L_n := \sup_{x\in\overline{\Omega}} |x - x_n|^{2m}\lambda_n u_n^2(x) e^{\beta_n u_n^2(x)}. \tag{4.39}$$

Assume by contradiction that $L_n \to +\infty$ as $n \to +\infty$. Take a point $\tilde{x}_n \in \Omega$ such that

$$L_n = |\tilde{x}_n - x_n|^{2m}\lambda_n u_n^2(\tilde{x}_n) e^{\beta_n u_n^2(\tilde{x}_n)}, \tag{4.40}$$

and define $\tilde{\mu}_n := u_n(\tilde{x}_n)$ and $s_n \in \mathbb{R}^+$ such that

$$\omega_{2m} s_n^{2m}\lambda_n \tilde{\mu}_n^2 e^{\beta_n \tilde{\mu}_n^2} = 1. \tag{4.41}$$

We will show that $\tilde{x}_n$ and s_n satisfy the assumptions of Proposition 4.9. Clearly, since $L_n \to +\infty$, (4.40) and (4.41) imply that

$$|\tilde{\mu}_n| \to +\infty \qquad \text{and} \qquad \frac{|x_n - \tilde{x}_n|}{s_n} \to +\infty. \tag{4.42}$$

In particular, $s_n \to 0$. Let v_n and $\tilde{\Omega}_n$ be as in Proposition 4.9. Using (4.39) and (4.40), we obtain

$$\frac{v_n^2}{\tilde{\mu}_n^2} e^{v_n^2 - \tilde{\mu}_n^2} \leq \frac{|y_n|^{2m}}{|y - y_n|^{2m}}, \tag{4.43}$$

for any $y \in \tilde{\Omega}_n$, where $y_n := \frac{x_n - \tilde{x}_n}{s_n}$. Since $|y_n| \to +\infty$, (4.43) yields

$$\frac{v_n^2}{\tilde{\mu}_n^2} e^{v_n^2 - \tilde{\mu}_n^2} \leq C(R) \qquad \text{in } \tilde{\Omega}_n \cap B_R(0), \tag{4.44}$$

for sufficiently large n. Thanks to (4.44), we infer that

$$\left|\frac{v_n}{\tilde{\mu}_n}\right| \leq C(R) \quad \text{and} \quad v_n^2 - \tilde{\mu}_n^2 \leq C(R)$$

on the set $\{|v_n| \geq |\tilde{\mu}_n|\} \cap B_R(0)$, and therefore on $\tilde{\Omega}_n \cap B_R(0)$. Then, all the assumptions of Proposition 4.9 are satisfied. In particular, as in Remark 4.14, by Proposition 4.9 and Lemma 4.13, we get

$$\lim_{n\to+\infty} \int_{B_{Rs_n}(\tilde{x}_n)} \lambda_n u_n^2 e^{\beta_n u_n^2} dx = \omega_{2m}^{-1} \int_{B_R(0)} e^{2\beta^* \eta_0} dy = 1 + O(R^{-2m}). \tag{4.45}$$

Besides, if r_n is as in (4.12), we have $r_n \leq s_n$ and, by (4.42), $B_{Rs_n}(\tilde{x}_n) \cap B_{Rr_n}(x_n) = \emptyset$, for any $R > 0$. Then, (4.4), Remark 4.14, and (4.45) imply

$$1 = \lim_{n\to+\infty} \int_\Omega \lambda_n u_n^2 e^{\beta_n u_n^2} dx \geq \lim_{n\to+\infty} \int_{B_{Rr_n}(x_n)\cup B_{Rs_n}(\tilde{x}_n)} \lambda_n u_n^2 e^{\beta_n u_n^2} dx = 2 + O(R^{-2m}),$$

which is a contradiction for large values of R. □

Next, we prove pointwise estimates on $|\nabla^l u_n|$ for any $1 \leq l \leq 2m-1$.

Proposition 4.16. *There exists a constant $C > 0$ such that*

$$|x - x_n|^l |u_n \nabla^l u_n| \leq C,$$

for any $x \in \Omega$ and $1 \leq l \leq 2m-1$.

The proof of Proposition 4.16 follows the same steps of the ones of Propositions 4.9. However, in this case it will be more difficult to obtain uniform bounds on u_n on a small scale. For any $1 \leq l \leq 2m-1$, we denote

$$L_{n,l} := \sup_{x\in\overline{\Omega}} |x - x_n|^l |u_n||\nabla^l u_n|. \tag{4.46}$$

Let $x_{n,l} \in \Omega$ be such that

$$|x_{n,l} - x_n|^l |u_n(x_{n,l}) \nabla^l u_n(x_{n,l})| = L_{n,l}. \tag{4.47}$$

We define $s_{n,l} := |x_{n,l} - x_n|$, $\mu_{n,l} := u_n(x_{n,l})$, and $y_{n,l} := \frac{x_n - x_{n,l}}{s_{n,l}}$. Up to subsequences, we can assume $y_{n,l} \to \overline{y}_l \in \mathbb{S}^{2m-1}$ as $n \to +\infty$. Consider now the scaled functions

$$v_{n,l}(y) = u_n(x_{n,l} + s_{n,l} y),$$

which are defined on the sets $\Omega_{n,l} := \{y \in \mathbb{R}^{2m} : x_{n,l} + s_{n,l} y \in \Omega\}$. Observe that $v_{n,l}$ satisfies

$$\begin{cases} (-\Delta)^m v_{n,l} = s_{n,l}^{2m} \lambda_n v_{n,l} e^{\beta_n v_{n,l}^2} + s_{n,l}^{2m} \alpha v_{n,l} & \text{in } \Omega_{n,l}, \\ v_{n,l} = \partial_\nu v_{n,l} = \ldots = \partial_\nu^{m-1} v_{n,l} = 0, & \text{on } \partial\Omega_{n,l}. \end{cases} \tag{4.48}$$

Moreover, Proposition 4.15 yields

$$s_{n,l}^{2m} \lambda_n v_{n,l}^2 e^{\beta_n v_{n,l}^2} \leq \frac{C}{|y - y_{n,l}|^{2m}}, \tag{4.49}$$

for any $y \in \Omega_{n,l}$, and (4.47) can be rewritten as

$$L_{n,l} = |v_{n,l}(0)||\nabla^l v_{n,l}(0)| = |\mu_{n,l}||\nabla^l v_{n,l}(0)|. \tag{4.50}$$

Remark 4.17. *If $L_{n,l} \to +\infty$ as $n \to +\infty$, then Lemma 4.4 implies that $s_{n,l} \to 0$. In particular, (4.49) gives*

$$s_{n,l}^{2m} \lambda_n v_{n,l} e^{\beta_n v_{n,l}^2} \to 0$$

as $n \to +\infty$, uniformly in $\Omega_{n,l} \setminus B_{\frac{1}{R}}(\overline{y}_l)$, for any $R > 0$. Indeed, if we choose a sequence $\{a_n\}_{n\in\mathbb{N}}$ such that $a_n \to +\infty$ and $s_{n,l}^{2m} \lambda_n a_n e^{\beta_n a_n^2} \to 0$ as $n \to +\infty$, then we have

$$\left| s_{n,l}^{2m} \lambda_n v_{n,l} e^{\beta_n v_{n,l}^2} \right| \leq s_{n,l}^{2m} \lambda_n a_n e^{\beta_n a_n^2},$$

on the set $\{|v_{n,l}| \le a_n\}$*, while* (4.49) *gives*

$$\left|s_{n,l}^{2m}\lambda_n v_{n,l} e^{\beta_n v_{n,l}^2}\right| \le \frac{s_{n,l}^{2m}\lambda_n v_{n,l}^2 e^{\beta_n v_{n,l}^2}}{a_n} \le \frac{C}{a_n|y-y_{n,l}|},$$

on the set $\{|v_{n,l}| \ge a_n\}$*.*

In the following, we will treat separately the cases $l = 1$ and $2 \le l \le 2m-1$.

Lemma 4.18. *If* $L_{n,1} \to +\infty$ *as* $n \to +\infty$*, then we have* $\frac{d(x_{n,1},\partial\Omega)}{s_{n,1}} \to +\infty$*. Moreover,* $\frac{v_{n,1}}{\mu_{n,1}} \to 1$ *in* $C^{2m-1,\gamma}_{loc}(\mathbb{R}^{2m} \setminus \{\overline{y}_1\})$*, for any* $\gamma \in (0,1)$*.*

Proof. It is sufficient to prove that $x_{n,1}$, $s_{n,1}$ and $v_{n,1}$ satisfy the assumptions of Lemma 4.10 and Lemma 4.11, with $\Sigma = \{\overline{y}_1\}$. First of all, we observe that, for any $R > 0$, the definition of $L_{n,1}$ implies $|\nabla v_{n,1}^2| \le C(R)L_{n,1}$ in $\Omega_{n,1} \setminus B_{\frac{1}{R}}(\overline{y}_1)$. Then, a Taylor expansion and (4.50) yield

$$v_{n,1}^2 \le \mu_{n,1}^2 + C(R)L_{n,1} \le C(R)D_{n,1}^2 \tag{4.51}$$

in $\Omega_{n,1} \cap B_R(0) \setminus B_{\frac{1}{R}}(\overline{y}_1)$, where $D_{n,1} := \max_{0\le i\le 2m-1} |\nabla^i v_{n,1}(0)|$. Moreover, by equation (4.48), Remark 4.17, and (4.51), we get

$$|(-\Delta)^m v_{n,1}| = o(1) + s_{n,1}^{2m}\alpha v_{n,1} = o(1) + O(s_{n,1}^{2m}D_{n,1}),$$

uniformly in $\Omega_{n,1} \cap B_R(0) \setminus B_{\frac{1}{R}}(\overline{y}_1)$. Finally, Remark 4.17 gives $s_{n,1} \to 0$, while (4.50) and the condition $L_{n,1} \to +\infty$ imply $D_{n,1} \to +\infty$. □

We can now prove Proposition 4.16 for $l = 1$.

Proof of Proposition 4.16 for $l = 1$. Assume by contradiction that $L_{n,1} \to +\infty$, as $n \to +\infty$. Consider the function $z_n(y) := \dfrac{v_{n,1}(y) - \mu_{n,1}}{|\nabla v_{n,1}(0)|}$. On the one hand, by the definitions of $L_{n,1}$ and $x_{n,1}$ in (4.46) and (4.47), and by Lemma 4.18, we have

$$|\nabla v_{n,1}(y)| \le \frac{|\nabla v_{n,1}(0)|(1+o(1))}{|y-y_{n,1}|} \le C(R)|\nabla v_{n,1}(0)|,$$

uniformly in $B_R(0) \setminus B_{\frac{1}{R}}(\overline{y}_1)$, for any $R > 0$. In particular,

$$|\nabla z_n(y)| \le C(R) \qquad \text{in } B_R(0) \setminus B_{\frac{1}{R}}(\overline{y}_1).$$

Since $z_n(0) = 0$, z_n is bounded in $L^\infty_{loc}(\mathbb{R}^{2m} \setminus \{\overline{y}_1\})$. On the other hand, arguing as in (4.29), Lemma 4.18 implies that

$$s_{n,1}^{2m}\mu_{n,1}^2 = o(1),$$

and, using also (4.49), that

$$(-\Delta)^m z_n = \frac{\lambda_n s_{n,1}^{2m} v_{n,1} e^{\beta_n v_{n,1}^2} + \alpha s_{n,1}^{2m} v_{n,1}}{|\nabla v_{n,1}(0)|} = O\left(\frac{1}{\mu_{n.1}|\nabla v_{n,1}(0)|}\right) = o(1), \qquad \text{in } B_R(0) \setminus B_{\frac{1}{R}}(\overline{y}_1).$$

By Proposition A.5, we find a function z_0, harmonic in $\mathbb{R}^{2m}\setminus\{\overline{y}_1\}$, such that, up to subsequences, $z_n \to z_0$ in $C^{2m-1,\gamma}_{loc}(\mathbb{R}^{2m} \setminus \{\overline{y}_1\})$, for any $\gamma \in (0,1)$. We claim now that z_0 must be constant on $\mathbb{R}^{2m} \setminus \{\overline{y}_1\}$. To prove this, we observe that, by Lemma 4.6, for any $R > 0$ there exists a constant $C(R) > 0$ such that

$$\|\nabla v_{n,1}^2\|_{L^1(B_R(0))} \le C(R).$$

Applying Lemma 4.18 and (4.50), we obtain

$$\begin{aligned}\|\nabla v_{n,1}^2\|_{L^1(B_R(0))} &\geq 2\int_{B_R(0)\setminus B_{\frac{1}{R}}(\overline{y}_1)} |v_{n,1}||\nabla v_{n,1}|dy \\ &= 2|\mu_{n,1}|(1+o(1))\|\nabla v_{n,1}\|_{L^1(B_R(0)\setminus B_{\frac{1}{R}}(\overline{y}_1))} \\ &= 2L_{n,1}(1+o(1))\|\nabla z_n\|_{L^1(B_R(0)\setminus B_{\frac{1}{R}}(\overline{y}_1))}.\end{aligned}$$

Thus, as $n\to+\infty$, we have

$$\|\nabla z_n\|_{L^1(B_R(0)\setminus B_{\frac{1}{R}}(\overline{y}_1))} \leq \frac{C(R)}{L_{n,1}} \to 0.$$

Hence, z_0 must be constant, which contradicts

$$|\nabla z_0(0)| = \lim_{n\to+\infty} |\nabla z_n(0)| = 1.$$

□

We shall now deal with the case $2\leq l\leq 2m-1$. Since Proposition 4.16 has been proved for $l=1$, we know that $L_{n,1}$ is bounded, i.e.

$$|x-x_n||u_n(x)||\nabla u_n(x)|\leq C,$$

for any $x\in\Omega$. Equivalently, given any $1\leq l\leq 2m-1$, we have

$$|v_{n,l}(y)||\nabla v_{n,l}(y)| \leq \frac{C}{|y-y_{n,l}|}, \tag{4.52}$$

for any $y\in\Omega_{n,l}$. In particular, (4.52) yields

$$\|\nabla v_{n,l}^2\|_{L^\infty(\Omega_{n,l}\setminus B_{\frac{1}{R}}(\overline{y}_l))} \leq C(R), \tag{4.53}$$

for any $R>0$.

Lemma 4.19. *Fix any $2\leq l\leq 2m-1$. If $L_{n,l}\to+\infty$ as $n\to+\infty$, then we have $\frac{d(x_{n,l},\partial\Omega)}{s_{n,l}}\to+\infty$. Moreover, $\frac{v_{n,l}}{\mu_{n,l}}\to 1$ in $C^{2m-1,\gamma}_{loc}(\mathbb{R}^{2m}\setminus\{\overline{y}_l\})$, for any $\gamma\in(0,1)$.*

Proof. As in Lemma 4.18, we show that $x_{n,l}$, $s_{n,l}$ and $v_{n,l}$ satisfy the assumptions of Lemma 4.10 and Lemma 4.11, with $\Sigma=\{\overline{y}_l\}$. Let us denote $D_{n,l} := \max_{0\leq i\leq 2m-1}|\nabla^i v_{n,l}(0)|$. Note that (4.50) and the condition $L_{n,l}\to+\infty$ imply $D_{n,l}\to+\infty$. Then, for any $R>0$, a Taylor expansion and (4.53) yield

$$v_{n,l}^2 \leq \mu_{n,l}^2 + C(R) \leq C(R)D_{n,l}^2 \tag{4.54}$$

in $\Omega_{n,l}\cap B_R(0)\setminus B_{\frac{1}{R}}(\overline{y}_l)$. Moreover, by equation (4.48), Remark 4.17, and (4.54), we get

$$|(-\Delta)^m v_{n,l}| = o(1) + s_{n,l}^{2m}\alpha v_{n,l} = o(1) + O(s_{n,l}^{2m}D_{n,l}),$$

uniformly in $\Omega_{n,l}\cap B_R(0)\setminus B_{\frac{1}{R}}(\overline{y}_l)$. □

Proof of Proposition 4.16 for $2\leq l\leq 2m-1$. Assume by contradiction that $L_{n,l}\to+\infty$ as $n\to+\infty$. Consider the function $z_n := \frac{v_{n,l}-\mu_{n,l}}{|\nabla^l v_n(0)|}$. Observe that (4.50), (4.52), and Lemma 4.19, yield

$$|\nabla z_n(y)| \leq \frac{C(R)}{L_{n,l}} \to 0, \tag{4.55}$$

uniformly in $B_R(0) \setminus B_{\frac{1}{R}}(\overline{y}_l)$, for any $R > 0$. Since $z_n(0) = 0$, (4.55) implies that

$$|z_n| \le \frac{C(R)}{L_{n,l}} \to 0,$$

uniformly in $B_R(0) \setminus B_{\frac{1}{R}}(\overline{y}_l)$. Similarly, as a consequence of equation (4.48), (4.49), and Lemma 4.19, one has

$$|(-\Delta)^m z_n| \le \frac{C(R)}{L_{n,l}},$$

in $B_R(0) \setminus B_{\frac{1}{R}}(\overline{y}_l)$. Therefore, up to subsequnces, $z_n \to 0$ in $C^{2m-1,\gamma}_{loc}(\mathbb{R}^{2m} \setminus \{\overline{y}_l\})$, for any $\gamma \in (0,1)$. Since $|\nabla^l z_n(0)| = 1$ for any n, we get a contradiction. □

4.5 Polyharmonic truncations

In this subsection, we will generalize the truncation argument introduced in [2] and [11]. For any $A > 1$ and $n \in \mathbb{N}$, we will introduce a new function u_n^A whose values are close to $\frac{\mu_n}{A}$ in a small ball centered at x_n, and which coincides with u_n outside the same ball.

Lemma 4.20. *For any $A > 1$ and $n \in \mathbb{N}$, there exists a radius $0 < \rho_n^A < d(x_n, \partial\Omega)$ and a constant $C = C(A)$ such that*

1. *$u_n \ge \frac{\mu_n}{A}$ in $B_{\rho_n^A}(x_n)$.*
2. *$|u_n - \frac{\mu_n}{A}| \le C\mu_n^{-1}$ on $\partial B_{\rho_n^A}(x_n)$.*
3. *$|\nabla^l u_n| \le \dfrac{C}{\mu_n(\rho_n^A)^l}$ on $\partial B_{\rho_n^A}(x_n)$, for any $1 \le l \le 2m-1$.*
4. *If r_n is defined as in (4.12), then $\frac{\rho_n^A}{r_n} \to +\infty$ as $n \to +\infty$.*

Proof. For any $\sigma \in \mathbb{S}^{2m-1}$, the function $t \mapsto u_n(x_n + t\sigma)$ ranges from μ_n to 0 in the interval $[0, t_n^*(\sigma)]$, where $t_n^*(\sigma) := \sup\{t > 0 \,:\, x_n + s\sigma \in \Omega \text{ for any } s \in [0,t]\}$. Since $u_n \in C(\overline{\Omega})$, one can define

$$t_n^A(\sigma) := \inf\{t \in [0, t_n^*(\sigma)) \,:\, u_n(x_n + t\sigma) = \frac{\mu_n}{A}\}.$$

Clearly, one has $0 < t_n^A(\sigma) < t_n^*(\sigma)$ and $u_n(x_n + t_n^A(\sigma)\sigma) = \frac{\mu_n}{A}$, for any $\sigma \in \mathbb{S}^{2m-1}$. Moreover, the function $\sigma \longmapsto t_n^A(\sigma)$ is lower semi-continuous on $\mathbb{S}^{2m-1}$. In particular, we can find $\overline{\sigma}_n^A$ such that $t_n^A(\overline{\sigma}_n^A) = \min_{\sigma \in \mathbb{S}^{2m-1}} t_n^A(\sigma)$. We define $\rho_n^A := t_n^A(\overline{\sigma}_n)$, and $y_n^A := x_n + \rho_n^A \overline{\sigma}_n^A \in \partial B_{\rho_n^A}(x_n)$. By construction we have, $0 < \rho_n^A < d(x_n, \partial\Omega)$, $u_n \ge \frac{\mu_n}{A}$ on $B_{\rho_n^A}(x_n)$, and $u_n(y_n^A) = \frac{\mu_n}{A}$. Thus, applying Proposition 4.16, we get

$$|\nabla^l u_n| \le \frac{CA}{\mu_n(\rho_n^A)^l},$$

on $\partial B_{\rho_n^A}(x_n)$, for any $1 \le l \le 2m-1$. Furthermore, for any $x \in \partial B_{\rho_n^A}(x_n)$, one has

$$|u_n(x) - \frac{\mu_n}{A}| = |u_n(x) - u_n(y_n^A)| \le \pi \rho_n^A \sup_{\partial B_{\rho_n^A}(x_n)} |\nabla u_n| \le \frac{C}{\mu_n}.$$

Finally, if r_n is as in (4.12), Proposition 4.8 and (4.13) imply that $u_n = \mu_n + O(\mu_n^{-1})$ uniformly in $B_{r_n R}(x_n)$, for any $R > 0$. Therefore, for sufficiently large n, we have $r_n R < \rho_n^A$. Since R is arbitrary, we get the conclusion. □

Let ρ_n^A be as in the previous lemma and let $v_n^A \in C^{2m}(\overline{B_{\rho_n^A}(x_n)})$ be the unique solution of

$$\begin{cases} (-\Delta)^m v_n^A = 0 & \text{in } B_{\rho_n^A}(x_n), \\ \partial_\nu^i v_n^A = \partial_\nu^i u_n & \text{on } \partial B_{\rho_n^A}(x_n), 0 \le i \le m-1. \end{cases}$$

We consider the function

$$u_n^A(x) := \begin{cases} v_n^A & \text{in } B_{\rho_n^A}(x_n), \\ u_n & \text{in } \Omega \setminus B_{\rho_n^A}(x_n). \end{cases} \tag{4.56}$$

By definition, we have $u_n^A \in H_0^m(\Omega)$. The main purpose of this section is to study the properties of u_n^A.

Lemma 4.21. *For any $A > 1$, we have*

$$u_n^A = \frac{\mu_n}{A} + O(\mu_n^{-1}),$$

uniformly on $\overline{B_{\rho_n^A}(x_n)}$.

Proof. Define $\tilde{v}_n(y) := v_n^A(x_n + \rho_n^A y) - \frac{\mu_n}{A}$ for $y \in B_1(0)$. Then, by elliptic estimates (Proposition A.2), we have

$$\begin{aligned} \|v_n^A - \frac{\mu_n}{A}\|_{L^\infty(B_{\rho_n^A}(x_n))} = \|\tilde{v}_n\|_{L^\infty(B_1(0))} &\le C \sum_{l=0}^{m-1} \|\nabla^l \tilde{v}_n\|_{L^\infty(\partial B_1(0))} \\ &= C \sum_{l=0}^{m-1} (\rho_n^A)^l \|\nabla^l v_n^A\|_{L^\infty(\partial B_{\rho_n^A}(x_n))} \\ &= C \sum_{l=0}^{m-1} (\rho_n^A)^l \|\nabla^l u_n\|_{L^\infty(\partial B_{\rho_n^A}(x_n))} \end{aligned}$$

By Lemma 4.20, we know that $(\rho_n^A)^l \|\nabla^l u_n\|_{L^\infty(\partial B_{\rho_n^A}(x_n))} \le \frac{C}{\mu_n}$ and the proof is complete. □

Proposition 4.22. *For any $A > 1$, we have*

$$\limsup_{n\to+\infty} \int_\Omega |\Delta^{\frac{m}{2}} u_n^A|^2 dx \le \frac{1}{A}.$$

Proof. Since $u_n^A \equiv u_n$ in $\Omega \setminus B_{\rho_n^A}(x_n)$, u_n^A is $m-$harmonic in $B_{\rho_n^A}(x_n)$, and $\partial_\nu^j u_n^A = \partial_\nu^j u_n$ on $\partial B_{\rho_n^A}(x_n)$ for $0 \le j \le m-1$, we have

$$\begin{aligned} \int_\Omega |\Delta^{\frac{m}{2}}(u_n - u_n^A)|^2 dx &= \int_{B_{\rho_n^A}(x_n)} \Delta^{\frac{m}{2}}(u_n - u_n^A) \Delta^{\frac{m}{2}} u_n \, dx \\ &= \int_{B_{\rho_n^A}(x_n)} (u_n - u_n^A)(-\Delta)^m u_n \, dx. \end{aligned} \tag{4.57}$$

As a consequence of Lemma 4.20, we get $(-\Delta)^m u_n \ge 0$ in $B_{\rho_n^A}(x_n)$. Therefore, the maximum principle guarantees $u_n \ge u_n^A$ in $B_{\rho_n^A}(x_n)$. Hence, if r_n is as in (4.12), we have

$$\begin{aligned} \int_{B_{\rho_n^A}(x_n)} (u_n - u_n^A)(-\Delta)^m u_n dx &\ge \int_{B_{Rr_n}(x_n)} (u_n - u_n^A)(-\Delta)^m u_n dx \\ &\ge \int_{B_{Rr_n}(x_n)} (u_n - u_n^A)\lambda_n u_n e^{\beta_n u_n^2} dx, \end{aligned} \tag{4.58}$$

for any $R > 0$. By Lemma 4.21, (4.12), and Proposition 4.8, we find

$$\begin{aligned}
&\int_{B_{Rr_n}(x_n)} (u_n - u_n^A)\lambda_n u_n e^{\beta_n u_n^2} dx \\
&\quad = r_n^{2m}\lambda_n \int_{B_R(0)} \left(\mu_n + \frac{\eta_n}{\mu_n} - \frac{\mu_n}{A} + O(\mu_n^{-1})\right)\left(\mu_n + \frac{\eta_n}{\mu_n}\right) e^{\beta_n\left(\mu_n^2 + 2\eta_n + \frac{\eta_n^2}{\mu_n^2}\right)} dy \qquad (4.59)\\
&\quad = \omega_{2m}^{-1}\left(1 - \frac{1}{A}\right)\int_{B_R(0)} e^{2\beta^*\eta_0} dy + o(1),
\end{aligned}$$

where η_n and η_0 are as in (4.13) and (4.14). Using (4.57), (4.58), (4.59), and Lemma 4.13, as $n \to +\infty$ and $R \to +\infty$ we find

$$\liminf_{n\to+\infty} \int_\Omega |\Delta^{\frac{m}{2}}(u_n - u_n^A)|^2 dx \geq 1 - \frac{1}{A}. \qquad (4.60)$$

Finally, since u_n^A is $m-$harmonic in $B_{\rho_n^A}(x_n)$, we have

$$\begin{aligned}
1 + o(1) &= \int_\Omega |\Delta^{\frac{m}{2}} u_n|^2 dx \\
&= \int_\Omega |\Delta^{\frac{m}{2}} u_n^A|^2 dx + \int_\Omega |\Delta^{\frac{m}{2}}\left(u_n - u_n^A\right)|^2 dx + 2\int_\Omega \Delta^{\frac{m}{2}} u_n^A \cdot \Delta^{\frac{m}{2}}(u_n - u_n^A) dx \qquad (4.61)\\
&= \int_\Omega |\Delta^{\frac{m}{2}} u_n^A|^2 dx + \int_\Omega |\Delta^{\frac{m}{2}}\left(u_n - u_n^A\right)|^2 dx.
\end{aligned}$$

Thus, (4.60) and (4.61) yield the conclusion. □

As a consequence of Proposition 4.22, we get some simple but crucial estimates.

Lemma 4.23. *Let $0 \leq \alpha < \lambda_1(\Omega)$ and let S_{α,β^*} be as in (1.4). Then, we have*

$$S_{\alpha,\beta^*} = |\Omega| + \lim_{n\to+\infty} \frac{1}{\lambda_n \mu_n^2}.$$

In particular, $\lambda_n\mu_n \to 0$ as $n \to +\infty$.

Proof. Fix $A > 1$ and let u_n^A be as in (4.56). By Adams' inequality (1.1) and Proposition 4.22, we know that $e^{\beta_n (u_n^A)^2}$ is bounded in $L^p(\Omega)$, for any $1 < p < A$. Since $u_n^A \to 0$ a.e. in Ω, Theorem 3.2 gives

$$\lim_{n\to+\infty} \int_{\Omega\setminus B_{\rho_n^A}(x_n)} e^{\beta_n u_n^2} dx = \lim_{n\to+\infty} \int_{\Omega\setminus B_{\rho_n^A}(x_n)} e^{\beta_n (u_n^A)^2} dx = |\Omega|. \qquad (4.62)$$

By Lemma 4.20, $u_n \geq \frac{\mu_n}{A}$ on $B_{\rho_n^A}(x_n)$. Hence,

$$\int_{B_{\rho_n^A}(x_n)} e^{\beta_n u_n^2} dx \leq \frac{A^2}{\mu_n^2} \int_{B_{\rho_n^A}(x_n)} u_n^2 e^{\beta_n u_n^2} dx \leq \frac{A^2}{\lambda_n \mu_n^2}. \qquad (4.63)$$

Moreover, for $R > 0$ large enough, Lemma 4.20 (part *4*) and Remark 4.14 imply

$$\limsup_{n\to+\infty} \int_{B_{\rho_n^A}(x_n)} e^{\beta_n u_n^2} dx \geq \limsup_{n\to+\infty} \int_{B_{r_n R}(x_n)} e^{\beta_n u_n^2} dx = (1 + O(R^{-2m})) \limsup_{n\to+\infty} \frac{1}{\lambda_n \mu_n^2}. \qquad (4.64)$$

From (4.2), (4.62), (4.63), (4.64), and Lemma 3.4, we get

$$|\Omega| + \limsup_{n\to+\infty} \frac{1}{\lambda_n \mu_n^2} \leq S_{\alpha,\beta^*} \leq |\Omega| + A^2 \liminf_{n\to+\infty} \frac{1}{\lambda_n \mu_n^2}.$$

Since A is an arbitrary number greater than 1, we get the conclusion. □

We conclude this section with the following lemma, which gives L^1 bounds on $(-\Delta)^m(\mu_n u_n)$. This will be important in the analysis of the behaviour of u_n far from x_0, which is given in the next section.

Lemma 4.24. *The sequence $\lambda_n\mu_n u_n e^{\beta_n u_n^2}$ is bounded in $L^1(\Omega)$. Moreover, $\lambda_n\mu_n u_n e^{\beta_n u_n^2} \rightharpoonup \delta_0$ in the sense of measures.*

Proof. By Remark 4.14, it is sufficient to show that

$$\lim_{R\to 0}\limsup_{n\to+\infty}\lambda_n\int_{\Omega\setminus B_{r_nR}(x_n)}\mu_n|u_n|e^{\beta_n u_n^2}dx = 0.$$

Let us denote $f_n = \lambda_n\mu_n u_n e^{\beta_n u_n^2}$. Fix $A > 1$ and let ρ_n^A and u_n^A be as in Lemma 4.20 and (4.56). Then, for any $R > 0$ and n sufficiently large, we have

$$\int_{\Omega\setminus B_{r_nR}(x_n)}|f_n(x)|dx = \int_{B_{\rho_n^A}(x_n)\setminus B_{r_nR}(x_n)}|f_n(x)|dx + \int_{\Omega\setminus B_{\rho_n^A}(x_n)}|f_n(x)|dx =: I_n^1 + I_n^2.$$

By Lemma 4.20, (4.4), and Remark 4.14, we obtain

$$\begin{aligned}I_n^1 \le A\int_{B_{\rho_n^A}(x_n)\setminus B_{r_nR}(x_n)}\lambda_n u_n^2 e^{\beta_n u_n^2}dx &\le A\int_{\Omega\setminus B_{r_nR}(x_n)}\lambda_n u_n^2 e^{\beta_n u_n^2}dx\\ &= A\left(1-\int_{B_{r_nR}(x_n)}\lambda_n u_n^2 e^{\beta_n u_n^2}dx\right)\\ &= A\,O(R^{-2m}).\end{aligned}$$

Therefore,

$$\limsup_{n\to+\infty} I_n^1 \le A\,O(R^{-2m}). \tag{4.65}$$

For the second integral, we observe that Proposition 4.22 and Adams' inequality imply that $e^{\beta_n(u_n^A)^2}$ is bounded in $L^p(\Omega)$, for any $1 < p < A$. In particular, applying Hölder's inequality and Lemma 4.23, we get

$$\begin{aligned}I_n^2 \le \int_{\Omega\setminus B_{\rho_n^A}(x_n)}|f_n(x)|dx &\le \lambda_n\mu_n\|e^{\beta_n(u_n^A)^2}\|_{L^p(\Omega)}\|u_n\|_{L^{\frac{p}{p-1}}(\Omega)}\\ &\le C\lambda_n\mu_n\|u_n\|_{L^{\frac{p}{p-1}}(\Omega)} \to 0,\end{aligned} \tag{4.66}$$

as $n\to+\infty$. Since R is arbitrary, the conclusion follows from (4.65) and (4.66). □

4.6 Convergence to Green's fuction

In this subsection, we will study the behavior of the sequence $\mu_n u_n$ according to the position of the blow-up point x_0. First, we will show that, if $x_0\in\Omega$, we have $\mu_n u_n \to G_{\alpha,x_0}$ locally uniformly in $\overline{\Omega}\setminus\{x_0\}$, where G_{α,x_0} is the Green's function for $(-\Delta)^m - \alpha$, defined as in (2.5).

Lemma 4.25. *The sequence $\mu_n u_n$ is bounded in $W_0^{m,p}(\Omega)$, for any $p\in[1,2)$.*

Proof. Let v_n be the unique solution to

$$\begin{cases}(-\Delta)^m v_n = \lambda_n\mu_n u_n e^{\beta_n u_n^2} =: f_n & \text{in } \Omega,\\ v_n = \partial_\nu v_n = \ldots = \partial_\nu^{m-1}v_n = 0 & \text{on } \partial\Omega.\end{cases}$$

By Lemma 4.24, we know that f_n is bounded in $L^1(\Omega)$. By Proposition A.11, we can conclude that v_n is bounded in $W_0^{m,p}(\Omega)$ for any $1\le p<2$. Define now $w_n = \mu_n u_n - v_n$. Then w_n solves

$$\begin{cases}(-\Delta)^m w_n = \alpha w_n + \alpha v_n & \text{in } \Omega,\\ w_n = \partial_\nu w_n = \ldots = \partial^{m-1}w_n = 0 & \text{on } \partial\Omega.\end{cases}$$

If we test the equation against w_n, using Poincare's and Sobolev's inequalities, we find that

$$\begin{aligned}\|w_n\|^2_{H^m_0(\Omega)} = \alpha\|w_n\|^2_{L^2(\Omega)} + \alpha\int_\Omega w_n v_n dx &\le \alpha\|w_n\|^2_{L^2(\Omega)} + \alpha\|w_n\|_{L^2(\Omega)}\|v_n\|_{L^2(\Omega)} \\ &\le \frac{\alpha}{\lambda_1(\Omega)}\|w_n\|^2_{H^m_0(\Omega)} + \frac{\alpha}{\sqrt{\lambda_1(\Omega)}}\|w_n\|_{H^m_0(\Omega)}\|v_n\|_{L^2(\Omega)} \\ &\le \frac{\alpha}{\lambda_1(\Omega)}\|w_n\|^2_{H^m_0(\Omega)} + C\|w_n\|_{H^m_0(\Omega)}.\end{aligned}$$

Then,

$$\|w_n\|_{H^m_0(\Omega)}\left(1 - \frac{\alpha}{\lambda_1(\Omega)}\right) \le C,$$

which implies that w_n is bounded $H^m_0(\Omega)$. This yields the conclusion. □

Lemma 4.26. *Let x_0 be as in* (4.8)*. If $x_0 \in \Omega$, then we have:*

1. $\mu_n u_n \rightharpoonup G_{\alpha,x_0}$ *in* $W^{m,p}_0(\Omega)$ *for any* $1 < p < 2$;
2. $\mu_n u_n \to G_{\alpha,x_0}$ *in* $C^{2m-1,\gamma}_{loc}(\overline{\Omega}\setminus\{x_0\})$.

Proof. Fix $1 < p < 2$. By Lemma 4.25, we can find $\tilde{u} \in W^{m,p}_0(\Omega)$ such that $\mu_n u_n \rightharpoonup \tilde{u}$ in $W^{m,p}_0(\Omega)$. Let φ be any test function in $C^\infty_c(\Omega)$. Applying Lemma 4.24 and the compactness of the embedding of $W^{m,p}_0(\Omega)$ into $L^1(\Omega)$, we obtain

$$\int_\Omega (\mu_n\lambda_n u_n e^{\beta_n u_n^2} + \alpha\mu_n u_n)\varphi dx = \varphi(x_0) + \alpha\int_\Omega \tilde{u}\varphi\, dx + o(1).$$

Hence necessarily $\tilde{u} = G_{\alpha,x_0}$. To conclude the proof, it remains to show that $\mu_n u_n \to G_{\alpha,x_0}$ in $C^{2m-1,\gamma}_{loc}(\overline{\Omega}\setminus\{x_0\})$. By elliptic estimates (Proposition A.6), it is sufficient to show that $(-\Delta)^m(\mu_n u_n)$ is bounded in $L^s(\Omega\setminus B_\delta(x_0))$, for any $s > 1$, $\delta > 0$. This follows from Lemma 4.4 and Lemma 4.23. □

Lemma 4.26 describes the behaviour of $\mu_n u_n$ when $x_0 \in \Omega$. The following Lemma deals with the case $x_0 \in \partial\Omega$. In fact, we will prove in the next subsection that blow-up at the boundary is not possible.

Lemma 4.27. *If $x_0 \in \partial\Omega$, we have:*

1. $\mu_n u_n \rightharpoonup 0$ *in* $W^{m,p}_0(\Omega)$ *for any* $1 < p < 2$.
2. $\mu_n u_n \to 0$ *in* $C^{2m-1,\gamma}_{loc}(\overline{\Omega}\setminus\{x_0\})$, *for any* $\gamma \in (0,1)$.

Proof. As before, using Lemma 4.25 and Lemma 4.24, we can find $\tilde{u} \in W^{m,p}_0(\Omega)$, $p \in (1,2)$, such that $\mu_n u_n \rightharpoonup \tilde{u}$ in $W^{m,p}_0(\Omega)$ for any $p \in (1,2)$ and $\mu_n u_n \to \tilde{u}$ in $C^{2m-1,\gamma}_{loc}(\overline{\Omega}\setminus\{x_0\})$, for any $\gamma \in (0,1)$. Moreover, as $n \to +\infty$, we have

$$\int_\Omega (\mu_n\lambda_n u_n e^{\beta_n u_n^2} + \alpha\mu_n u_n)\varphi dx = \alpha\int_\Omega \tilde{u}\varphi\, dx + o(1),$$

Then, $\tilde{u}$ is a weak solution of $(-\Delta)^m\tilde{u} = \alpha\tilde{u}$ in Ω. Since $\tilde{u} \in W^{m,p}_0(\Omega)$, elliptic regularity (Proposition A.4) implies $\tilde{u} \in W^{3m,p}(\Omega)$, for any $p \in (1,2)$. In particular, we have $\tilde{u} \in H^m_0(\Omega)$, and

$$\|\tilde{u}\|^2_{H^m_0(\Omega)} = \alpha\|\tilde{u}\|^2_{L^2(\Omega)}.$$

Since $0 \le \alpha < \lambda_1(\Omega)$, we must have $\tilde{u} \equiv 0$. □

4.7 The Pohozaev identity and blow-up at the boundary

In this subsection, we prove that the blow-up point x_0 cannot lie on $\partial\Omega$. The proof is based on the following Pohozaev-type identity.

Lemma 4.28. *Let $\Omega \subseteq \mathbb{R}^{2m}$ be a bounded open set with Lipschitz boundary. If $u \in C^{2m}(\overline{\Omega})$ is a solution of*

$$(-\Delta)^m u = h(u), \tag{4.67}$$

with $h : \mathbb{R} \longrightarrow \mathbb{R}$ continuous, then for any $y \in \mathbb{R}^{2m}$ the following identity holds:

$$\frac{1}{2}\int_{\partial\Omega} |\Delta^{\frac{m}{2}} u|^2 (x-y)\cdot\nu\, d\sigma(x) + \int_{\partial\Omega} f(x) d\sigma(x) = \int_{\partial\Omega} H(u(x))(x-y)\cdot\nu d\sigma(x) - 2m\int_{\Omega} H(u(x))dx,$$

where $H(t) := \int_0^t h(s)ds$ and

$$f(x) := \sum_{j=0}^{m-1} (-1)^{m+j}\nu\cdot\left(\Delta^{\frac{j}{2}}((x-y)\cdot\nabla u)\Delta^{\frac{2m-j-1}{2}} u\right).$$

Proof. We multiply equation (4.67) for $(x-y)\cdot\nabla u$ and integrate on Ω to obtain

$$\int_{\Omega} (x-y)\cdot\nabla u\, (-\Delta)^m u\, dx = \int_{\Omega} (x-y)\cdot\nabla u\, h(u) dx. \tag{4.68}$$

On the one hand, using the divergence Theorem, we can rewrite the RHS of (4.68) as

$$\begin{aligned}\int_{\Omega} (x-y)\cdot\nabla u\, h(u)dx &= \int_{\Omega} (x-y)\cdot\nabla H(u) dx \\ &= \int_{\Omega} \operatorname{div}\left((x-y)H(u)\right) dx - 2m\int_{\Omega} H(u)dx \\ &= \int_{\partial\Omega} H(u)(x-y)\cdot\nu\, d\sigma(x) - 2m\int_{\Omega} H(u)dx.\end{aligned}$$

On the other hand, we can integrate by parts the LHS of (4.68) to find

$$\int_{\Omega} (x-y)\cdot\nabla u\, (-\Delta)^m u\, dx = \int_{\Omega} \Delta^{\frac{m}{2}}\left((x-y)\cdot\nabla u\right)\Delta^{\frac{m}{2}} u\, dx + \int_{\partial\Omega} f d\sigma.$$

As proved in Lemma 14 of [24], we have the identity

$$\Delta^{\frac{m}{2}}\left((x-y)\cdot\nabla u\right)\cdot\Delta^{\frac{m}{2}} u = \frac{1}{2}\operatorname{div}\left((x-y)|\Delta^{\frac{m}{2}} u|^2\right).$$

Hence, the divergence theorem yields

$$\int_{\Omega} (x-y)\cdot\nabla u\, (-\Delta)^m u\, dx = \frac{1}{2}\int_{\partial\Omega} (x-y)\cdot\nu\, |\Delta^{\frac{m}{2}} u|^2 d\sigma(x) + \int_{\partial\Omega} f d\sigma.$$

□

We now apply Lemma 4.28 to u_n in a neighborhood of x_0, and we use Lemma 4.27 to prove that x_0 must be in Ω. A smart choice of the point y is crucial to control the boundary terms in the identity. This strategy was first introduced in [30] and was applied in [24] to Liouville equations in dimension $2m$.

Lemma 4.29. *Let x_0 be as in (4.8). Then $x_0 \in \Omega$.*

Proof. We assume by contradiction that $x_0 \in \partial\Omega$. If we fix a sufficiently small $\delta > 0$, we have that $\frac{1}{2} \le \nu \cdot \nu(x_0) \le 1$ on $\partial\Omega \cap B_\delta(x_0)$. Then we can define

$$\rho_n := \frac{\int_{\partial\Omega\cap B_\delta(x_0)} |\Delta^{\frac{m}{2}} u_n|^2 (x-x_0)\cdot \nu d\sigma(x)}{\int_{\partial\Omega\cap B_\delta(x_0)} |\Delta^{\frac{m}{2}} u_n|^2 \nu\cdot\nu(x_0) d\sigma(x)} \quad \text{and} \quad y_n := x_0 + \rho_n \nu(x_0).$$

Observe that $|y_n - x_0| \le 2\delta$. Applying the Pohozaev identity of Lemma 4.28 on $\Omega_\delta = \Omega \cap B_\delta(x_0)$, we obtain

$$\begin{aligned} \frac{1}{2}\int_{\partial\Omega_\delta} |\Delta^{\frac{m}{2}} u_n|^2 (x-y_n)\cdot\nu \, d\sigma(x) + \int_{\partial\Omega_\delta} f_n(x) d\sigma(x) \\ = \int_{\partial\Omega_\delta} H_n(u_n(x))(x-y_n)\cdot\nu d\sigma(x) - 2m\int_{\Omega_\delta} H_n(u_n(x))dx, \end{aligned} \tag{4.69}$$

where $H_n(t) = \frac{\lambda_n}{2\beta_n} e^{\beta_n t^2} + \frac{\alpha}{2} t^2$, and

$$f_n := \sum_{j=0}^{m-1} (-1)^{m+j} \nu \cdot \left(\Delta^{\frac{j}{2}}((x-y_n)\cdot\nabla u_n)\Delta^{\frac{2m-j-1}{2}} u_n\right).$$

Observe that the definitions of y_n and ρ_n imply

$$\int_{\partial\Omega\cap B_\delta(x_0)} |\Delta^{\frac{m}{2}} u_n|^2 (x-y_n)\cdot\nu \, d\sigma(x) = 0, \tag{4.70}$$

and thus, by Lemma 4.27, we have

$$\int_{\partial\Omega_\delta} |\Delta^{\frac{m}{2}} u_n|^2 (x-y_n)\cdot\nu \, d\sigma(x) = \int_{\Omega\cap\partial B_\delta(x_0)} |\Delta^{\frac{m}{2}} u_n|^2 (x-y_n)\cdot\nu \, d\sigma(x) = o(\mu_n^{-2}). \tag{4.71}$$

Similarly, since $f_n = -|\Delta^{\frac{m}{2}} u_n|^2 (x-y_n)\cdot\nu$ on $\partial\Omega\cap B_\delta(x_0)$, applying (4.70) and Lemma 4.27, we get

$$\int_{\partial\Omega_\delta} f_n(x)d\sigma(x) = \int_{\Omega\cap\partial B_\delta(x_0)} f_n(x)d\sigma(x) = o(\mu_n^{-2}). \tag{4.72}$$

Furthermore, we have

$$\begin{aligned} \int_{\partial\Omega_\delta} e^{\beta_n u_n^2}(x-y_n)\cdot\nu d\sigma(x) &= \int_{\Omega\cap\partial B_\delta(x_0)} e^{\beta_n u_n^2}(x-y_n)\cdot\nu d\sigma(x) + \int_{\partial\Omega\cap B_\delta(x_0)} (x-y_n)\cdot\nu d\sigma(x) \\ &= I_{\delta,n} + o(\mu_n^{-2}), \end{aligned}$$

where $I_{\delta,n} = \int_{\partial\Omega_\delta} (x-y_n)\cdot\nu d\sigma(x) = O(\delta)$ uniformly with respect to n. In particular,

$$\begin{aligned} \int_{\partial\Omega_\delta} H_n(u_n(x))(x-y_n)\cdot\nu d\sigma(x) &= \frac{\lambda_n}{2\beta_n}\int_{\partial\Omega_\delta} e^{\beta_n u_n^2}(x-y_n)\cdot\nu d\sigma(x) + \frac{\alpha}{2}\int_{\Omega\cap\partial B_\delta(x_0)} u_n^2 (x-y_n)\cdot\nu d\sigma(x) \\ &= \frac{\lambda_n}{2\beta_n} I_{\delta,n} + o(\mu_n^{-2}). \end{aligned} \tag{4.73}$$

Finally, we have

$$\begin{aligned} \int_{\Omega_\delta} H_n(u_n(x))dx &= \frac{\lambda_n}{2\beta_n}\int_{\Omega_\delta} e^{\beta_n u_n^2}dx + \frac{\alpha}{2}\int_{\Omega_\delta} u_n^2 dx \\ &= \frac{\lambda_n}{2\beta_n}\int_{\Omega_\delta} e^{\beta_n u_n^2}dx + o(\mu_n^{-2}). \end{aligned} \tag{4.74}$$

Therefore, (4.71), (4.72), (4.73), (4.74) allow to rewrite the identity in (4.69) as

$$\lambda_n\mu_n^2\left(2m\int_{\Omega_\delta} e^{\beta_n u_n^2}dx - I_{\delta,n}\right) = o(1). \tag{4.75}$$

Lemma 4.27, (4.2) and Lemma 3.4, assure

$$\int_{\Omega_\delta} e^{\beta_n u_n^2}dx = F_{\beta_n}(u_n) - \int_{\Omega\setminus B_\delta(x_0)} e^{\beta_n u_n^2}dx \to S_{\alpha,\beta^*} - |\Omega\setminus B_\delta(x_0)| \geq S_{\alpha,\beta^*} - |\Omega| > 0,$$

as $n\to+\infty$. Then, for δ sufficiently small, the quantity $\int_{\Omega_\delta} e^{\beta_n u_n^2}dx - I_{n,\delta}$ is bounded away from 0. Hence, the identity (4.75) implies $\lambda_n\mu_n^2\to 0$ and, since $I_{n,\delta} = O(\delta)$,

$$\lambda_n\mu_n^2\int_{\Omega_\delta} e^{\beta_n u_n^2}dx = o(1). \tag{4.76}$$

But (4.76) contradicts Remark 4.14, since for any large $R>0$ one has

$$\lambda_n\mu_n^2\int_{\Omega_\delta} e^{\beta_n u_n^2}dx \geq \lambda_n\mu_n^2\int_{B_{Rr_n}(x_n)} e^{\beta_n u_n^2}dx = 1 + O(R^{-2m}).$$

□

4.8 Neck analysis

In this subsection, we complete the proof of Proposition 4.2 by giving a sharp upper bound on $\dfrac{1}{\lambda_n\mu_n^2}$. Let us fix a large $R>0$ and a small $\delta>0$ and let us consider the annular region

$$A_n(R,\delta) := \{x\in\Omega : r_nR \leq |x-x_n| \leq \delta\},$$

where r_n is given by (4.12). Note that, by Lemma 4.29, we have $A_n(R,\delta)\subseteq\Omega$, for any $0<\delta<d(x_0,\partial\Omega)$ and any sufficiently large $n\in\mathbb{N}$. Our main idea is to compare the Dirichlet energy of u_n on $A_n(R,\delta)$ with the energy of the $m-$harmonic function

$$\mathcal{W}_n(x) := -\frac{2m}{\beta^*\mu_n}\log|x-x_n|.$$

As a consequence of Proposition 4.8 and (4.13), on $\partial B_{Rr_n}(x_n)$, we have

$$u_n(x) = \mu_n + \frac{\eta_0(\frac{x-x_n}{r_n})}{\mu_n} + o(\mu_n^{-1}) = \mu_n - \frac{2m}{\beta^*\mu_n}\log\frac{R}{2} + \frac{O(R^{-2})}{\mu_n} + o(\mu_n^{-1}),$$

as $n\to+\infty$. Similarly, using also (4.37), we find

$$\Delta^{\frac{j}{2}}u_n(x) = \frac{\Delta^{\frac{j}{2}}\eta_0(\frac{x-x_n}{r_n})}{r_n^j\mu_n} + o(r_n^{-j}\mu_n^{-1}) = -\frac{2mK_{m,\frac{j}{2}}}{\beta^* r_n^j\mu_n R^j}e_{n,j} + \frac{O(R^{-j-2})}{r_n^j\mu_n} + o(r_n^{-j}\mu_n^{-1}),$$

for any $1\leq j\leq 2m-1$, where $e_{n,j} := e_j(x-x_n)$ with e_j is as in (2.4). The function $\mathcal{W}_n$ has an analog behaviour. Indeed, remembering the definition of r_n in (4.12), we get

$$\mathcal{W}_n(x) = \frac{\beta_n}{\beta^*}\mu_n - \frac{2m}{\beta^*\mu_n}\log R + \frac{1}{\beta^*\mu_n}\log\left(\omega_{2m}\lambda_n\mu_n^2\right), \tag{4.77}$$

and, by (2.4),

$$\Delta^{\frac{j}{2}}\mathcal{W}_n = -\frac{2mK_{m,\frac{j}{2}}}{\beta^*\mu_n r_n^j R^j}e_{n,j}, \qquad \text{for any } 1\leq j\leq 2m-1, \tag{4.78}$$

on $\partial B_{Rr_n}(x_n)$. We can so conclude that, as $n \to +\infty$, on $\partial B_{Rr_n}(x_n)$, we have the expansions

$$u_n - \mathcal{W}_n = \left(1 - \frac{\beta_n}{\beta^*}\right)\mu_n + \frac{1}{\beta^*\mu_n}\log\left(\frac{2^{2m}}{\omega_{2m}\lambda_n\mu_n^2}\right) + \frac{O(R^{-2})}{\mu_n} + o(\mu_n^{-1}), \tag{4.79}$$

and

$$\Delta^{\frac{j}{2}}(u_n - \mathcal{W}_n) = \frac{O(R^{-j-2})}{r_n^j\mu_n} + o(r_n^{-j}\mu_n^{-1}), \qquad \text{for any } 1 \le j \le 2m-1. \tag{4.80}$$

Similarly, on $\partial B_\delta(x_n)$, we can use Lemma 4.26 and Propositon 2.2 to get

$$u_n - \mathcal{W}_n = \frac{C_{\alpha,x_0}}{\mu_n} + \frac{O(\delta)}{\mu_n} + o(\mu_n^{-1}), \tag{4.81}$$

and

$$\Delta^{\frac{j}{2}}(u_n - \mathcal{W}_n) = \frac{O(1)}{\mu_n} + o(\mu_n^{-1}), \qquad \text{for any } 1 \le j \le 2m-1. \tag{4.82}$$

Here we have also used that $\frac{|x-x_n|}{|x-x_0|} \to 1$, uniformly on $\partial B_\delta(x_n)$. The asymptotic formulas in (4.77)-(4.82) allow to compare $\|\Delta^{\frac{m}{2}}u_n\|_{L^2(A_n(R,\delta))}$ and $\|\Delta^{\frac{m}{2}}\mathcal{W}_n\|_{L^2(A_n(R,\delta))}$. Since the quantity $\lambda_n\mu_n^2$ appears in (4.79), this will result in the desired upper bound.

Lemma 4.30. *Under the assumptions of Proposition 4.2, we have*

$$\lim_{n\to+\infty}\frac{1}{\lambda_n\mu_n^2} \le \frac{\omega_{2m}}{2^{2m}}e^{\beta^*\left(C_{\alpha,x_0}-I_m\right)}.$$

Proof. First, Young's inequality yields

$$\|\Delta^{\frac{m}{2}}u_n\|^2_{L^2(A_n(R,\delta))} - \|\Delta^{\frac{m}{2}}\mathcal{W}_n\|^2_{L^2(A_n(R,\delta))} \ge 2\int_{A_n(R,\delta)}\Delta^{\frac{m}{2}}(u_n - \mathcal{W}_n)\cdot\Delta^{\frac{m}{2}}\mathcal{W}_n dx. \tag{4.83}$$

Integrating by parts, the integral in the RHS equals to

$$\int_{A_n(R,\delta)}\Delta^{\frac{m}{2}}(u_n - \mathcal{W}_n)\cdot\Delta^{\frac{m}{2}}\mathcal{W}_n dx = -\int_{\partial A_n(R,\delta)}\sum_{j=0}^{m-1}(-1)^{m+j}\nu\cdot\left(\Delta^{\frac{j}{2}}(u_n - \mathcal{W}_n)\Delta^{\frac{2m-j-1}{2}}\mathcal{W}_n\right)d\sigma. \tag{4.84}$$

Let us denote, $\Lambda_n := \frac{2^{2m}}{\omega_{2m}\lambda_n\mu_n^2}$. On $\partial B_{Rr_n}(x_n)$, by (4.78), (4.79), (4.80), and the explicit expression of $K_{\frac{2m-1}{2}}$ (see (2.3)), we find

$$\begin{aligned}(u_n - \mathcal{W}_n)\Delta^{\frac{2m-1}{2}}\mathcal{W}_n\cdot\nu &= -\frac{2m}{\beta^*}\left(1 - \frac{\beta_n}{\beta^*} + \frac{1}{\beta^*\mu_n^2}\log(\Lambda_n) + \frac{O(R^{-2})}{\mu_n^2} + o(\mu_n^{-2})\right)\frac{K_{m,\frac{2m-1}{2}}}{(r_nR)^{2m-1}} \\ &= \frac{(-1)^m}{\omega_{2m-1}(r_nR)^{2m-1}}\left(1 - \frac{\beta_n}{\beta^*} + \frac{1}{\beta^*\mu_n^2}\log(\Lambda_n) + \frac{O(R^{-2})}{\mu_n^2} + o(\mu_n^{-2})\right),\end{aligned} \tag{4.85}$$

and, for $1 \le j \le m-1$,

$$\Delta^{\frac{j}{2}}(u_n - \mathcal{W}_n)\Delta^{\frac{2m-j-1}{2}}\mathcal{W}_n\cdot\nu = \left(\frac{O(R^{-2})}{\mu_n^2} + o(\mu_n^{-2})\right)O(r_nR)^{1-2m}. \tag{4.86}$$

Similarly, on $\partial B_\delta(x_0)$, (2.4), (4.81) and (4.82) yield

$$(u_n - \mathcal{W}_n)\Delta^{\frac{2m-1}{2}}\mathcal{W}_n\cdot\nu = \frac{(-1)^m}{\omega_{2m-1}\delta^{2m-1}}\left(\frac{C_{\alpha,x_0}}{\mu_n^2} + \frac{O(\delta)}{\mu_n^2} + o(\mu_n^{-2})\right), \tag{4.87}$$

and

$$\Delta^{\frac{j}{2}}(u_n - \mathcal{W}_n)\Delta^{\frac{2m-j-1}{2}}\mathcal{W}_n \cdot \nu = \left(\frac{O(1)}{\mu_n^2} + o(\mu_n^{-2})\right) O(\delta^{1+j-2m}), \tag{4.88}$$

for any $1 \le j \le m-1$. Using (4.85), (4.86), (4.87), (4.88), we can rewrite (4.84) as

$$\int_{A_n(R,\delta)} \Delta^{\frac{m}{2}}(u_n - \mathcal{W}_n) \cdot \Delta^{\frac{m}{2}}\mathcal{W}_n dx = \Gamma_n + \frac{O(R^{-2})}{\mu_n^2} + \frac{O(\delta)}{\mu_n^2} + o(\mu_n^{-2}),$$

with

$$\Gamma_n := 1 - \frac{\beta_n}{\beta^*} + \frac{1}{\beta^* \mu_n^2} \log(\Lambda_n) - \frac{C_{\alpha,x_0}}{\mu_n^2}. \tag{4.89}$$

Therefore, (4.83) reads as

$$\|\Delta^{\frac{m}{2}} u_n\|^2_{L^2(A_n(R,\delta))} - \|\Delta^{\frac{m}{2}}\mathcal{W}_n\|^2_{L^2(A_n(R,\delta))} \ge 2\Gamma_n + \frac{O(R^{-2})}{\mu_n^2} + \frac{O(\delta)}{\mu_n^2} + o(\mu_n^{-2}). \tag{4.90}$$

We shall now compute the difference in the LHS of (4.90) in a precise way. Since $\|u_n\|_\alpha = 1$, we have

$$\|\Delta^{\frac{m}{2}} u_n\|^2_{L^2(A_n(R,\delta))} = 1 + \alpha\|u_n\|^2_{L^2(\Omega)} - \int_{\Omega\setminus B_\delta(x_0)} |\Delta^{\frac{m}{2}} u_n|^2 dx - \int_{B_{r_n R}(x_n)} |\Delta^{\frac{m}{2}} u_n|^2 dx.$$

By Lemma 4.26 and Lemma 2.3, we infer

$$\|u_n\|^2_{L^2(\Omega)} = \frac{\|G_{\alpha,x_0}\|^2_{L^2(\Omega)}}{\mu_n^2} + o(\mu_n^{-2}),$$

and

$$\int_{\Omega\setminus B_\delta(x_0)} |\Delta^{\frac{m}{2}} u_n|^2 dx = \mu_n^{-2}\left(\alpha\|G_{\alpha,x_0}\|^2_{L^2(\Omega)} - \frac{2m}{\beta^*}\log\delta + C_{\alpha,x_0} + H_m + O(\delta|\log\delta|) + o(1)\right).$$

Moreover, Proposition 4.8 and Lemma 4.13 imply

$$\int_{B_{r_n R}(x_n)} |\Delta^{\frac{m}{2}} u_n|^2 dx = \mu_n^{-2}\left(\frac{2m}{\beta^*}\log\frac{R}{2} + I_m - H_m + O(R^{-2}\log R) + o(1)\right).$$

Therefore,

$$\|\Delta^{\frac{m}{2}} u_n\|^2_{L^2(A_n(R,\delta))} = 1 + \frac{2m}{\beta^*\mu_n^2}\log\frac{2\delta}{R} - \frac{C_{\alpha,x_0} + I_m}{\mu_n^2} + \frac{O(R^{-2}\log R)}{\mu_n^2} + \frac{O(\delta|\log\delta|)}{\mu_n^2} + o(\mu_n^{-2}).$$

The identity $\omega_{2m-1}\frac{2m}{\beta^*}K^2_{m,\frac{m}{2}} = 1$ and a direct computation show that

$$\begin{aligned}\|\Delta^{\frac{m}{2}}\mathcal{W}_n\|^2_{L^2(A_n(R,\delta))} &= \omega_{2m-1}\left(\frac{2mK_{m,\frac{m}{2}}}{\beta^*\mu_n}\right)^2 \log\frac{\delta}{Rr_n} \\ &= \frac{2m}{\beta^*\mu_n^2}\log\frac{\delta}{R} + \frac{\beta_n}{\beta^*} + \frac{1}{\beta^*\mu_n^2}\log\left(\omega_{2m}\lambda_n\mu_n^2\right).\end{aligned}$$

Hence,

$$\|\Delta^{\frac{m}{2}} u_n\|^2_{L^2(A_n(R,\delta))} - \|\Delta^{\frac{m}{2}}\mathcal{W}_n\|^2_{L^2(A_n(R,\delta))} = \Gamma_n - \frac{I_m}{\mu_n^2} + \frac{O(R^{-2}\log R)}{\mu_n^2} + \frac{O(\delta|\log\delta|)}{\mu_n^2} + o(\mu_n^{-2}), \tag{4.91}$$

with Γ_n as in (4.89). Comparing (4.90) and (4.91), we find the upper bound

$$\Gamma_n \le -\frac{I_m}{\mu_n^2} + \frac{O(R^{-2}\log R)}{\mu_n^2} + \frac{O(\delta|\log\delta|)}{\mu_n^2} + o(\mu_n^{-2}). \tag{4.92}$$

Since $\beta_n < \beta^*$, the definition of Γ_n in (4.89) implies

$$\Gamma_n \ge \frac{1}{\beta^*\mu_n^2}\log(\Lambda_n) - \frac{C_{\alpha,x_0}}{\mu_n^2},$$

Then, (4.92) yields

$$\log(\Lambda_n) \le \beta^*(C_{\alpha,x_0} - I_m) + O(R^{-2}\log R) + O(\delta|\log\delta|) + o(1).$$

Passing to the limit as $n \to +\infty$, $R \to +\infty$ and $\delta \to 0$, we can conclude

$$\lim_{n\to+\infty} \Lambda_n \le e^{\beta^*\left(C_{\alpha,x_0} - I_m\right)}.$$

□

We have so concluded the proof of Proposition 4.2, which follows directly from Lemma 4.23, Lemma 4.29, and Lemma 4.30.

5 Test functions and the proof of Theorem 1.2

In this section, we complete the proof of Theorem 1.2 by showing that the upper bound on S_{α,β^*}, given in Proposition 4.2, cannot hold. Consequently, any sequence $u_n \in M_\alpha$ satisfying (4.2) must be uniformly bounded in Ω.

Lemma 5.1. *For any $x_0 \in \mathbb{R}^{2m}$, and ε,R,$\mu > 0$, there exists a unique radially symmetric polynomial $p_{\varepsilon,R,\mu,x_0}$ such that*

$$\partial_\nu^i p_{\varepsilon,R,\mu,x_0}(x) = -\partial_\nu^i\left(\mu^2 + \eta_0\left(\frac{x-x_0}{\varepsilon}\right) + \frac{2m}{\beta^*}\log|x-x_0|\right) \quad \text{on } \partial B_{\varepsilon R}(x_0), \tag{5.1}$$

for any $0 \le i \le m-1$, where η_0 is as in (4.14). Moreover, $p_{\varepsilon,R,\mu,x_0}$ has the form

$$p_{\varepsilon,R,\mu,x_0}(x) = -\mu^2 + \sum_{j=0}^{m-1} c_j(\varepsilon,R)|x-x_0|^{2j}, \tag{5.2}$$

with

$$c_0(\varepsilon,R) = -\frac{2m}{\beta^*}\log(2\varepsilon) + d_0(R) \quad \text{and} \quad c_j(\varepsilon,R) = \varepsilon^{-2j}R^{-2j}d_j(R), \quad 1 \le j \le m-1,$$

where $d_j(R) = O(R^{-2})$ as $R \to +\infty$, for $0 \le j \le m-1$.

Proof. We can construct $p_{\varepsilon,R,\mu,x_0}$ in the following way. Let $d_1(R)$,...,$d_{m-1}(R)$ be the unique solution of the non-degenerate linear system

$$\sum_{j=\left[\frac{i+1}{2}\right]}^{m-1} \frac{(2j)!}{(2j-i)!}d_j(R) = \frac{2m}{\beta^*}(-1)^i(i-1)! - R^i\eta_0^{(i)}(R), \quad i = 1,\dots,m-1. \tag{5.3}$$

Set also

$$\tilde{d}_0(\varepsilon,R,\mu) := -\left(\mu^2 + \eta_0(R) + \frac{2m}{\beta^*}\log(\varepsilon R)\right) - \sum_{j=1}^{m-1} d_j(R), \tag{5.4}$$

and

$$q(x) := \tilde{d}_0(\varepsilon, R, \mu) + \sum_{j=1}^{m-1} d_j(R)|x|^{2j}.$$

If we define $p_{\varepsilon,R,\mu,x_0}(x) := q\left(\frac{x-x_0}{\varepsilon R}\right)$, then $p_{\varepsilon,R,\mu,x_0}(x)$ satisfies (5.1) for any $0 \le i \le m-1$. Since, as $R \to +\infty$,

$$\eta_0^{(i)}(R) = \frac{2m}{\beta^*}(-1)^i (i-1)! R^{-i} + O(R^{-i-2}), \qquad \text{for } 1 \le i \le m-1,$$

and the system in (5.3) is nondegenerate, we find $d_j = O(R^{-2})$ as $R \to +\infty$ for $1 \le j \le m-1$. Similarly, we have

$$\tilde{d}_0(\varepsilon, R, \mu) = -\mu^2 - \frac{2m}{\beta^*}\log(2\varepsilon) + d_0(R),$$

where

$$d_0(R) := -\eta_0(R) - \frac{2m}{\beta^*}\log\frac{R}{2} - \sum_{j=1}^{m-1} d_j(R),$$

and, by (5.4) and the asymptotic behavior at infinity of η_0, $d_0(R) = O(R^{-2})$ as $R \to +\infty$. Then $p_{\varepsilon,R,\mu,x_0}$ has the form (5.2) with $c_0(\varepsilon, R) := \tilde{d}_0(\varepsilon, R, \mu) + \mu^2$ and $c_j(\varepsilon, R) := (\varepsilon R)^{-2j} d_j(R)$. ☐

Remark 5.2. *Observe that Lemma 5.1 gives*

$$\left| p_{\varepsilon,R,\mu,x_0} + \mu^2 + \frac{2m}{\beta^*}\log(2\varepsilon) \right| \le CR^{-2} \quad \text{and} \quad |\Delta^{\frac{m}{2}} p_{\varepsilon,R,\mu,x_0}| \le C\varepsilon^{-m} R^{-m-2},$$

in $B_{\varepsilon R}(x_0)$, where C depends only on m.

Proposition 5.3. *For any $x_0 \in \Omega$, and $0 \le \alpha < \lambda_1(\Omega)$, we have*

$$S_{\alpha,\beta^*} > |\Omega| + \frac{\omega_{2m}}{2^{2m}} e^{\beta^*\left(C_{\alpha,x_0} - I_m\right)},$$

where C_{α,x_0} and I_m are respectively as in Proposition 2.2 and (4.36).

Proof. We consider the function

$$u_{\varepsilon,\alpha,x_0}(x) := \begin{cases} \mu_\varepsilon + \dfrac{\eta_0\left(\frac{x-x_0}{\varepsilon}\right)}{\mu_\varepsilon} + \dfrac{C_{\alpha,x_0} + \psi_{\alpha,x_0}(x) + p_\varepsilon(x)}{\mu_\varepsilon} & \text{for } |x-x_0| < \varepsilon R_\varepsilon, \\ \dfrac{G_{\alpha,x_0}(x)}{\mu_\varepsilon} & \text{for } |x-x_0| \ge \varepsilon R_\varepsilon, \end{cases}$$

where ψ_{α,x_0} is as in the expansion of G_{α,x_0} given in Proposition 2.2, $R_\varepsilon = |\log\varepsilon|$, μ_ε is a constant that will be fixed later, and $p_\varepsilon := p_{\varepsilon,R_\varepsilon,\mu_\varepsilon,x_0}$ is the polynomial defined in Lemma 5.1. To simplify the notation, in this proof we will write u_ε in place of $u_{\varepsilon,\alpha,x_0}$ without specifying the dependence on α and x_0.

Note that the choice of p_ε (specifically (5.1)) implies that, for sufficiently small ε, $u_\varepsilon \in H_0^m(\Omega)$. Moreover, we can write $u_\varepsilon = \frac{\tilde{u}_\varepsilon}{\mu_\varepsilon}$, where

$$\tilde{u}_\varepsilon(x) = \begin{cases} \eta_0\left(\frac{x-x_0}{\varepsilon}\right) + C_{\alpha,x_0} + \psi_{\alpha,x_0}(x) + p_\varepsilon + \mu_\varepsilon^2 & \text{if } |x-x_0| < \varepsilon R_\varepsilon, \\ G_{\alpha,x_0} & \text{if } |x-x_0| \ge \varepsilon R_\varepsilon, \end{cases} \tag{5.5}$$

is a function that does not depend on the choice of μ_ε, because of Lemma 5.1. In particular, if we fix $\mu_\varepsilon := \|\tilde{u}_\varepsilon\|_\alpha$, we get $\|u_\varepsilon\|_\alpha = 1$, and so $u_\varepsilon \in M_\alpha$. In order to compute $F_{\beta^*}(u_\varepsilon)$, we need a precise expansion of μ_ε. Observe that, by Lemma 4.13, the function $\eta_\varepsilon(x) := \eta_0\left(\frac{x-x_0}{\varepsilon}\right)$, satisfies

$$\begin{aligned} \int_{B_{\varepsilon R_\varepsilon}(x_0)} |\Delta^{\frac{m}{2}}\eta_\varepsilon|^2 dx &= \int_{B_{R_\varepsilon}(0)} |\Delta^{\frac{m}{2}}\eta_0|^2 dx \\ &= \frac{2m}{\beta^*}\log\frac{R_\varepsilon}{2} + I_m - H_m + O(R_\varepsilon^{-2}\log R_\varepsilon). \end{aligned} \tag{5.6}$$

Since $\psi_{\alpha,x_0} \in C^{2m-1}(\overline{\Omega})$, we have

$$\int_{B_{\varepsilon R_\varepsilon}(x_0)} |\Delta^{\frac{m}{2}} \psi_{\alpha,x_0}|^2 dx = O(\varepsilon^{2m} R_\varepsilon^{2m}), \tag{5.7}$$

Remark 5.2 gives $|\Delta^{\frac{m}{2}} p_\varepsilon| = O(\varepsilon^{-m} R_\varepsilon^{-m-2})$ in $B_{\varepsilon R_\varepsilon}(x_0)$. Therefore,

$$\int_{B_{\varepsilon R_\varepsilon}(x_0)} |\Delta^{\frac{m}{2}} p_\varepsilon|^2 dx = O(R_\varepsilon^{-4}). \tag{5.8}$$

Using Hölder's inequality, (5.6) and (5.7), we find

$$\begin{aligned} \int_{B_{\varepsilon R}(x_0)} \Delta^{\frac{m}{2}} \eta_\varepsilon \cdot \Delta^{\frac{m}{2}} \psi_{\alpha,x_0} dx &\le \|\Delta^{\frac{m}{2}} \eta_\varepsilon\|_{L^2(B_{\varepsilon R_\varepsilon}(x_0))} \|\Delta^{\frac{m}{2}} \psi_{\alpha,x_0}\|_{L^2(B_{\varepsilon R_\varepsilon}(x_0))} \\ &= O(\varepsilon^m R_\varepsilon^m \log^{\frac{1}{2}} R_\varepsilon). \end{aligned} \tag{5.9}$$

Similarly, by (5.6), (5.7) and (5.8), we get

$$\int_{B_{\varepsilon R_\varepsilon}(x_0)} \Delta^{\frac{m}{2}} \eta_\varepsilon \cdot \Delta^{\frac{m}{2}} p_\varepsilon dx = O(R_\varepsilon^{-2} \log^{\frac{1}{2}} R_\varepsilon), \tag{5.10}$$

and

$$\int_{B_{\varepsilon R_\varepsilon}(x_0)} \Delta^{\frac{m}{2}} p_\varepsilon \cdot \Delta^{\frac{m}{2}} \psi_{\alpha,x_0} dx = O(\varepsilon^m R_\varepsilon^{m-2}). \tag{5.11}$$

By (5.6), (5.7), (5.8), (5.9), (5.10) and (5.11), we infer

$$\int_{B_{\varepsilon R_\varepsilon}(x_0)} |\Delta^{\frac{m}{2}} \tilde{u}_\varepsilon|^2 dx = \frac{2m}{\beta^*} \log \frac{R_\varepsilon}{2} + I_m - H_m + O(R_\varepsilon^{-2} \log R_\varepsilon).$$

Furthermore, applying Lemma 2.3, we have

$$\begin{aligned} \int_{\Omega \setminus B_{\varepsilon R_\varepsilon}(x_0)} |\Delta^{\frac{m}{2}} \tilde{u}_\varepsilon|^2 dx &= \int_{\Omega \setminus B_{\varepsilon R_\varepsilon}(x_0)} |\Delta^{\frac{m}{2}} G_{\alpha,x_0}|^2 dx \\ &= -\frac{2m}{\beta^*} \log(\varepsilon R_\varepsilon) + C_{\alpha,x_0} + H_m + \alpha \|G_{\alpha,x_0}\|_{L^2(\Omega)}^2 + O(\varepsilon R_\varepsilon |\log(\varepsilon R_\varepsilon)|). \end{aligned}$$

Hence,

$$\int_\Omega |\Delta^{\frac{m}{2}} \tilde{u}_\varepsilon|^2 dx = -\frac{2m}{\beta^*} \log(2\varepsilon) + C_{\alpha,x_0} + I_m + \alpha \|G_{\alpha,x_0}\|_{L^2(\Omega)}^2 + O(R_\varepsilon^{-2} \log R_\varepsilon). \tag{5.12}$$

Finally, since (5.5) and Remark 5.2, imply $\tilde{u}_\varepsilon = O(|\log \varepsilon|)$ on $B_{\varepsilon R_\varepsilon}(x_0)$, and since $G_{\alpha,x_0} = O(|\log |x - x_0||)$ near x_0, we find

$$\begin{aligned} \|\tilde{u}_\varepsilon\|_{L^2(\Omega)}^2 &= \|G_{\alpha,x_0}\|_{L^2(\Omega \setminus B_{\varepsilon R_\varepsilon})}^2 + O(\varepsilon^{2m} R_\varepsilon^{2m} \log^2 \varepsilon) \\ &= \|G_{\alpha,x_0}\|_{L^2(\Omega)}^2 + O(\varepsilon^{2m} R_\varepsilon^{2m} \log^2 \varepsilon). \end{aligned} \tag{5.13}$$

Therefore, using (5.12) and (5.13), we obtain

$$\mu_\varepsilon^2 = \|\tilde{u}_\varepsilon\|_\alpha^2 = -\frac{2m}{\beta^*} \log(2\varepsilon) + C_{\alpha,x_0} + I_m + O(R_\varepsilon^{-2} \log R_\varepsilon). \tag{5.14}$$

We can now estimate $F_{\beta^*}(u_\varepsilon)$. On $B_{\varepsilon R_\varepsilon}(x_0)$, by definition of u_ε, we get

$$u_\varepsilon^2 \ge \mu_\varepsilon^2 + 2 \left(\eta_0 \left(\frac{x - x_0}{\varepsilon} \right) + C_{\alpha,x_0} + \psi_{\alpha,x_0}(x) + p_\varepsilon(x) \right).$$

Then, Lemma 5.1, Remark 5.2, and (5.14), give

$$u_\varepsilon^2 \geq -\frac{2m}{\beta^*}\log(2\varepsilon) + 2\eta_0\left(\frac{x-x_0}{\varepsilon}\right) + C_{\alpha,x_0} - I_m + O(R_\varepsilon^{-2}\log R_\varepsilon).$$

Hence, using a change of variables and Lemma 4.13,

$$\begin{aligned}\int_{B_{\varepsilon R_\varepsilon}(x_0)} e^{\beta^* u_\varepsilon^2}dx &\geq \frac{1}{2^{2m}}e^{\beta^*\left(C_{\alpha,x_0}-I_m\right)}(1+O(R_\varepsilon^{-2}\log R_\varepsilon))\int_{B_{R_\varepsilon}(0)} e^{2\beta^*\eta_0}dy \\ &= \frac{\omega_{2m}}{2^{2m}}e^{\beta^*\left(C_{\alpha,x_0}-I_m\right)} + O(R_\varepsilon^{-2}\log R_\varepsilon)).\end{aligned} \tag{5.15}$$

Outside $B_{\varepsilon R_\varepsilon}(x_0)$, the basic inequality $e^{t^2} \geq 1+t^2$ gives

$$\begin{aligned}\int_{\Omega\setminus B_{\varepsilon R_\varepsilon}(x_0)} e^{\beta^* u_\varepsilon^2}dx &= \int_{\Omega\setminus B_{\varepsilon R_\varepsilon}(x_0)} e^{\frac{\beta^*}{\mu_\varepsilon^2}G_{\alpha,x_0}^2}dx \\ &\geq |\Omega| + \frac{\beta^*}{\mu_\varepsilon^2}\|G_{\alpha,x_0}\|_{L^2(\Omega)}^2 + o(\mu_\varepsilon^{-2}) + O(\varepsilon^{2m}R_\varepsilon^{2m}).\end{aligned} \tag{5.16}$$

Since $R_\varepsilon = O(\mu_\varepsilon^2)$, by (5.15) and (5.16), we conclude that

$$F_{\beta^*}(u_\varepsilon) \geq |\Omega| + \frac{\omega_{2m}}{2^{2m}}e^{\beta^*\left(C_{\alpha,x_0}-I_m\right)} + \frac{\beta^*}{\mu_\varepsilon^2}\|G_{\alpha,x_0}\|_{L^2(\Omega)}^2 + o(\mu_\varepsilon^{-2}).$$

In particular, for sufficiently small ε, we find

$$S_{\alpha,\beta^*} \geq F_{\beta^*}(u_\varepsilon) > |\Omega| + \frac{\omega_{2m}}{2^{2m}}e^{\beta^*\left(C_{\alpha,x_0}-I_m\right)}.$$

□

We can now prove Theorem 1.2 using Proposition 4.2 and Proposition 5.3.

Proof of Theorem 1.2.

1. Let β_n, u_n and μ_n be as in (4.1), (4.2), (4.7) and (4.8). Since $\|u_n\|_\alpha = 1$ and $0 \leq \alpha < \lambda_1(\Omega)$, u_n is bounded in $H_0^m(\Omega)$. In particular, we can find a function $u_0 \in H_0^m(\Omega)$ such that, up to subsequences, $u_n \rightharpoonup u_0$ in $H_0^m(\Omega)$ and $u_n \to u_0$ a.e. in Ω. The weak lower semicontinuity of $\|\cdot\|_\alpha$ implies that $u_0 \in M_\alpha$. By Propositions 4.2 and 5.3, we must have $\limsup_{n\to+\infty}\mu_n \leq C$. Then, Fatou's Lemma and the dominated convergence theorem imply respectively $F_{\beta^*}(u_0) < +\infty$ and $F_{\beta_n}(u_n) \to F_{\beta^*}(u_0)$. Since, by Lemma 3.4, u_n is maximizing sequence for S_{α,β^*}, we conclude that $S_{\alpha,\beta^*} = F_{\beta^*}(u_0)$. Then, S_{α,β^*} is finite and attained.

2. Clearly, if $\beta > \beta^*$, using (1.1), we get

$$S_{\alpha,\beta} \geq S_{0,\beta} = +\infty, \qquad \text{for any } \alpha \geq 0.$$

Assume now $\alpha \geq \lambda_1(\Omega)$ and $0 \leq \beta \leq \beta^*$. Let φ_1 be an eigenfuntion for $(-\Delta)^m$ on Ω corresponding to $\lambda_1(\Omega)$, i.e. a nontrivial solution of

$$\begin{cases} (-\Delta)^m\varphi_1 = \lambda_1(\Omega)\varphi_1 & \text{in } \Omega, \\ \varphi_1 = \partial_\nu\varphi_1 = \ldots = \partial_\nu^{m-1}\varphi_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

Observe that, for any $t \in \mathbb{R}$,

$$\|t\varphi_1\|_\alpha^2 = t^2(\lambda_1(\Omega)-\alpha)\|\varphi_1\|_{L^2}^2 \leq 0.$$

In particular, $t\varphi_1 \in M_\alpha$. Then we have

$$S_{\alpha,\beta} \geq F_{\alpha,\beta}(t\varphi_1) \to +\infty,$$

as $t \to +\infty$. □

Appendix: Some elliptic estimates

In this appendix, we recall some useful elliptic estimates which have been used several times throughout the paper. We start by recalling that $m-$harmonic functions are of class C^∞ and that bounds on their L^1-norm give local uniform estimates on all their derivatives.

Proposition A.1. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set. Then, for any $m \geq 1$, $l \in \mathbb{N}$, $\gamma \in (0,1)$, and any open set $V \subset\subset \Omega$, there exists a constant $C = C(m, l, \gamma, V, \Omega)$ such that every m-harmonic function u in Ω satisfies*

$$\|u\|_{C^{l,\gamma}(V)} \leq C\|u\|_{L^1(\Omega)}.$$

Proposition A.1 can be deduced e.g. from Proposition 12 in [21], and its proof is based on Pizzetti's formula [29], which is a generalization of the standard mean value property for harmonic functions.

If $m \geq 2$, in general $m-$harmonic functions on a bounded open set Ω do not satisfy the maximum principle, unless Ω is one of the so called positivity preserving domains (balls are the simplest example). However, it is always true that the C^{m-1} norm of a $m-$harmonic function can be controlled in terms of the L^∞ norm of its derivatives on $\partial\Omega$.

Proposition A.2. *Let $\Omega \subseteq \mathbb{R}^N$ be a smooth bounded open set. Then, there exists a constant $C = C(\Omega) > 0$ such that*

$$\|u\|_{C^{m-1}(\Omega)} \leq C \sum_{l=0}^{m-1} \|\nabla^l u\|_{L^\infty(\partial\Omega)},$$

for any $m-$harmonic function $u \in C^{m-1}(\overline{\Omega})$.

We recall now the main results concerning Schauder and L^p elliptic estimates for $(-\Delta)^m$.

Proposition A.3 (see Theorem 2.18 of [9])**.** *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set with smooth boundary, and take $k, m \in \mathbb{N}$, $k \geq 2m$, and $\gamma \in (0,1)$. If $u \in H^m(\Omega)$ is a weak solution of the problem*

$$\begin{cases} (-\Delta)^m u = f & \text{in } \Omega, \\ \partial_\nu^j u = h_j & \text{on } \partial\Omega,\ 0 \leq j \leq m-1, \end{cases} \tag{A.1}$$

with $f \in C^{k-2m,\gamma}(\Omega)$ and $h_j \in C^{k-j,\gamma}(\partial\Omega)$, $0 \leq j \leq m-1$, then $u \in C^{k,\gamma}(\Omega)$ and there exists a constant $C = C(\Omega, k, \gamma)$ such that

$$\|u\|_{C^{k,\gamma}(\Omega)} \leq C \left(\|f\|_{C^{k-2m,\gamma}(\Omega)} + \sum_{j=0}^{m-1} \|h_j\|_{C^{k-j,\gamma}(\partial\Omega)} \right).$$

Proposition A.4 (see Theorem 2.20 of [9])**.** *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set with smooth boundary, and take $m, k \in \mathbb{N}$, $k \geq 2m$, and $p > 1$. If $u \in H^m(\Omega)$ is a weak solution of (A.1) with $f \in W^{k-2m,p}(\Omega)$ and $h_j \in W^{k-j-\frac{1}{p},p}(\partial\Omega)$, $0 \leq j \leq m-1$, then $u \in W^{k,p}(\Omega)$ and there exists a constant $C = C(\Omega, k, p)$ such that*

$$\|u\|_{W^{k,p}(\Omega)} \leq C \left(\|f\|_{W^{k-2m,p}(\Omega)} + \sum_{j=0}^{m-1} \|h_j\|_{W^{k-j-\frac{1}{p},\gamma}(\partial\Omega)} \right).$$

In the absence of boundary conditions one can obtain local estimates combining Propositions A.3 and A.4 with Proposition A.1.

Proposition A.5. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set with smooth boundary and take $m, k \in \mathbb{N}$, $k \geq 2m$, $p > 1$. If $f \in W^{k-2m,p}(\Omega)$ and u is a weak solution of $(-\Delta)^m u = f$ in Ω, then $u \in W^{k,p}_{loc}(\Omega)$ and, for any open set $V \subset\subset \Omega$, there exists a constant $C = C(k, p, V, \Omega)$ such that*

$$\|u\|_{W^{k,p}(V)} \leq C \left(\|f\|_{W^{k-2m,p}(\Omega)} + \|u\|_{L^1(\Omega)} \right).$$

Similarly, if $f \in C^{k-2m,\gamma}(\Omega)$ and u is a weak solution of $(-\Delta)^m u = f$ in Ω, then $u \in C^{k,\gamma}_{loc}(\Omega)$ and, for any open set $V \subset\subset \Omega$, there exists a constant $C = C(k, \gamma, V, \Omega)$ such that

$$\|u\|_{C^{k,\gamma}(V)} \leq C \left(\|f\|_{C^{k-2m,\gamma}(\Omega)} + \|u\|_{L^1(\Omega)} \right).$$

In many cases, one has to deal with solutions of $(-\Delta)^m u = f$ in Ω, with boundary conditions satisfied only on a subset of $\partial\Omega$. For instance, as a consequence of Proposition A.4, Green's representation formula, and the continuity of trace operators on $W^{m,1}(\Omega)$, one obtains the following Proposition.

Proposition A.6. *Let $\Omega \subseteq \mathbb{R}^N$ be an open set with smooth boundary, and fix $x_0, x_1 \in \mathbb{R}^{2m}$ and $p > 1$. For any $\delta, R > 0$ such that $\Omega \cap B_R(x_1) \setminus B_{2\delta}(x_0) \neq \emptyset$, there exists a constant $C = C(\Omega, x_0, x_1, \delta, R)$ such that every weak solution u of problem* (A.1)*, with $f \in L^p(\Omega)$ and $h_j = 0$, $0 \leq j \leq m-1$, satisfies*

$$\|u\|_{W^{2m,p}(\Omega \cap B_R(x_1) \setminus B_{2\delta}(x_0))} \leq C(\|f\|_{L^p(\Omega \cap B_{2R}(x_1) \setminus B_\delta(x_0))} + \|u\|_{W^{m,1}(\Omega \setminus B_{2R}(x_1) \cap B_\delta(x_0))}).$$

Remark A.7. *The constant C appearing in Proposition A.6 depends on Ω only through the C^{2m} norms of the local maps that define $B_{2R}(x_1) \cap \partial\Omega$. In particular, Proposition A.6 can be applied uniformly to sequences $\{\Omega_n\}_{n \in \mathbb{N}}$, which converge in the C^{2m}_{loc} sense to a limit domain Ω.*

The following Proposition holds only in the special case $m = 1$. It gives a Harnack-type inequality which is useful to control the local behavior of a sequence of solutions of $-\Delta u = f$, when the behavior at one point is known.

Proposition A.8. *Let $u_n \in H^1(B_R(0))$ be a sequence of weak solutions of $-\Delta u_n = f_n$ in $B_R(0) \subseteq \mathbb{R}^N$, $R > 0$. Assume that f_n is bounded in $L^\infty(B_R(0))$, and there exists $C > 0$ such that $u_n \leq C$ and $u_n(0) \geq -C$. Then, u_n is bounded in $L^\infty(B_{\frac{R}{2}}(0))$.*

Proof. We write $u_n = v_n + h_n$, with h_n harmonic in $B_R(0)$, and v_n solving

$$\begin{cases} \Delta v_n = f_n & \text{in } B_R(0), \\ v_n = 0 & \text{on } \partial B_R(0). \end{cases}$$

By Proposition A.4, v_n is bounded in $W^{2,p}(B_R(0))$, for any $p > 1$. In particular, it is bounded in $L^\infty(B_R(0))$. Then, we have

$$h_n = u_n - v_n \leq C + \|v_n\|_{L^\infty(B_R(0))} \leq \tilde{C},$$

and

$$h_n(0) = u_n(0) - v_n(0) \geq -C - \|v_n\|_{L^\infty(B_R(0))} \geq -\tilde{C}.$$

By the mean value property, for any $x \in B_{\frac{R}{2}}(0)$, we get

$$\begin{aligned} h_n(x) - \tilde{C} &= \frac{2^N}{\omega_N R^N} \int_{B_{\frac{R}{2}}(x)} (h_n - \tilde{C}) dy \\ &\geq \frac{2^N}{\omega_N R^N} \int_{B_R(0)} (h_n - \tilde{C}) dy \\ &= 2^N (h_n(0) - \tilde{C}) \\ &\geq -2^{N+1} \tilde{C}. \end{aligned}$$

Hence, h_n is bounded in $L^\infty(B_{\frac{R}{2}}(0))$. □

Finally, we recall some Lorentz-Zygmund type elliptic estimates. For any $\alpha \geq 0$, let $L(\log L)^\alpha$ be defined as the space

$$L(\log L)^\alpha = \left\{ f : \Omega \longrightarrow \mathbb{R} \text{ s.t. } f \text{ is measurable and } \int_\Omega |f| \log^\alpha(2 + |f|) dx < +\infty \right\}, \tag{A.2}$$

and endowed with the norm

$$\|f\|_{L(LogL)^\alpha} := \int_\Omega |f| \log^\alpha(2+|f|)dx. \tag{A.3}$$

Given $1 < p < +\infty$, and $1 \le q \le +\infty$, let $L^{(p,q)}(\Omega)$ be the Lorentz space

$$L^{(p,q)}(\Omega) := \{u : \Omega \longrightarrow \mathbb{R} \,:\, u \text{ is measurable and } \|u\|_{(p,q)} < +\infty\}, \tag{A.4}$$

where

$$\|u\|_{(p,q)} := \left(\int_0^{|\Omega|} t^{\frac{q}{p}-1} u^{**}(t)^q dt\right)^{\frac{1}{q}}, \quad \text{for } 1 \le q < +\infty, \tag{A.5}$$

and

$$\|u\|_{(p,\infty)} = \sup_{t\in(0,|\Omega|)} t^{\frac{1}{p}} u^{**}(t), \tag{A.6}$$

with

$$u^{**}(t) := t^{-1} \int_0^t u^*(s)ds, \tag{A.7}$$

and

$$u^*(t) := \inf\{\lambda > 0 \,:\, |\{|u| > \lambda\}| \le t\}. \tag{A.8}$$

Among the many properties of Lorentz spaces we recall the following Hölder-type inequality (see [28]).

Proposition A.9. *Let $1 < p, p' < +\infty$, $1 \le q, q' \le +\infty$, be such that $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$. Then, for any $u \in L^{(p,q)}(\Omega)$, $v \in L^{(p',q')}(\Omega)$, we have*

$$\|uv\|_{L^1(\Omega)} \le \|u\|_{(p,q)} \|v\|_{(p',q')}.$$

As proved in Corollary 6.16 of [3] (see also Theorem 10 in [21]) one has the following:

Proposition A.10. *Let $\Omega \subseteq \mathbb{R}^N$, $N \ge 2m$, be a bounded smooth domain and take $0 \le \alpha \le 1$. If $f \in L(\log L)^\alpha$, and u is a weak solution of (A.1), then $\nabla^{2m-l}u \in L^{(\frac{N}{N-l},\frac{1}{\alpha})}(\Omega)$, for any $1 \le l \le 2m-1$. Moreover, there exists a constant $C = C(\Omega, l) > 0$ such that*

$$\|\nabla^{2m-l}u\|_{(\frac{N}{N-l},\frac{1}{\alpha})} \le C\|f\|_{L(LogL)^\alpha}.$$

Note that, if $\alpha = 0$, we have $L(\log^\alpha L) = L^1(\Omega)$. Moreover, $L^{(\frac{N}{N-l},\frac{1}{\alpha})}(\Omega) = L^{(\frac{N}{N-l},\infty)}(\Omega)$ coincides with the weak $L^{\frac{N}{N-l}}$ space on Ω. In particular, $L^{(\frac{N}{N-l},\infty)}(\Omega) \subseteq L^p(\Omega)$ for any $1 \le p < \frac{N}{N-l}$. Therefore, as a consequence of Proposition A.10, we recover the following well known result, whose classical proof relies on Green's representation formula.

Proposition A.11. *Let $\Omega \subseteq \mathbb{R}^N$, $N \ge 2m$, be a bounded smooth domain. Then, for any $1 \le l \le 2m-1$ and $1 \le p < \frac{N}{N-l}$, there exists a constant $C = C(p, l, \Omega)$ such that every weak solution of (A.1) with $f \in L^1(\Omega)$ satisfies*

$$\|\nabla^{2m-l}u\|_{L^p(\Omega)} \le C\|f\|_{L^1(\Omega)}.$$

References

[1] D. Adams, *A sharp inequality of J. Moser for higher order derivatives*, Ann. of Math. **128** (1988), 385-398.

[2] Adimurthi, O. Druet, *Blow-up analysis in dimension 2 and a sharp form of Trudinger-Moser inequality*, Comm. Partial Differential Equations **29** (2004), 295-322.

[3] C. Bennett, R. Sharpley, *Interpolation of operators*, Pure and Applied Mathematics vol. **129**, Academic Press (1988).

[4] T. Boggio, *Sulle funzioni di green d'ordine m*, Rendiconti del Circolo Matematico di Palermo **20** (1905), 97–135.

[5] L. Carleson, S.-Y. A. Chang, *On the existence of an extremal function for an inequality of J. Moser*, Bull. Sci. Math. **110** (1986), 113-127.

[6] A. Dall'Acqua, G. Sweers, *Estimates for Green function and Poisson kernels of higher-order Dirichlet boundary value problems*, J. Differential Equations **205** (2004), 466–487.

[7] O. Druet, P-D. Thizy, *Multi-bumps analysis for Trudinger-Moser nonlinearities I - Quantification and location of concentration points*, Preprint 2017, `https://arxiv.org/abs/1710.08811`.

[8] M. Flucher, *Extremal functions for the Trudinger-Moser inequality in 2 dimensions*, Comment. Math. Helv. **67** (1992) 471-497.

[9] F. Gazzola, H-C. Grunau, G. Sweers, *Polyharmonic boundary value problems*, Lecture Notes in Mathematics, Springer-Verlag, Berlin (2010), ISBN: 978-3-642-12244-6.

[10] S. Iula, G. Mancini, *Extremal functions for singular Moser-Trudinger embeddings*, Nonlinear Anal. **156**, (2017), 215–248.

[11] Y. Li, *Moser-Trudinger inequality on compact Riemannian manifolds of dimension two*, J. Partial Differential Equations **14** (2001), 163–192.

[12] Y. Li, *Extremal functions for the Moser-Trudinger inequalities on compact Riemannian manifolds*, Sci. China Ser. A **48** (2005), 618–648.

[13] Y. Li, C.B. Ndiaye, *Extremal functions for Moser-Trudinger type inequality on compact closed 4-manifolds*, J. Geom. Anal. **17** (2007), 669–699.

[14] C. S. Lin, *A classification of solutions of conformally invariant fourth order equations in* $\mathbb{R}^n$, Comm. Math. Helv **73** (1998), 206-231.

[15] K.-C. Lin, *Extremal functions for Moser's inequality.*, Trans.Amer. Math. Soc. **348** (1996), no. 7 2663-2671.

[16] P.-L. Lions, *The concentration-compactness principle in the calculus of variations. The limit case. I*, Rev. Mat. Iberoamericana **1** (1985), 145-201.

[17] G. Lu, Y. Yang, *The sharp constant and extremal functions for Moser-Trudinger inequalities involving* L^p *norms in two dimension*, Discrete Contin. Dyn. Syst. **25** (2009), 963-979.

[18] G. Lu, Y. Yang, *Adams' inequalities for bi-Laplacian and extremal functions in dimension four*, Adv. Math. **220** (2009), 1135-1170.

[19] A. Maalaoui, L. Martinazzi, A. Schikorra, *Blow-up behavior of a fractional Adams-Moser-Trudinger-type inequality in odd dimension*, Comm. Partial Differential Equations, **41** (2016), 1593–1618.

[20] G. Mancini, L. Martinazzi, *The Moser-Trudinger inequality and its extremals on a disk via energy estimates*, Calc. Var. Partial Differential Equations, **56** (2017).

[21] L. Martinazzi, *A threshold phenomenon for embeddings of* H_0^m *into Orlicz spaces*, Calc. Var. Partial Differential Equations **36** (2009), 493-506.

[22] L. Martinazzi, *Classification of solutions to the higher order Liouville's equation on* $\mathbb{R}^{2m}$, Math. Z. **263** (2009), 307–329.

[23] L. Martinazzi, *Fractional Adams-Moser-Trudinger type inequalities*, Nonlinear Anal., **127** (2015), 263–278.

[24] L. Martinazzi, M. Petrache, *Asymptotics and Quantization for a Mean-Field Equation of Higher Order*, Comm. Partial Differential Equations **35** (2010), 443-464.

[25] L. Martinazzi, M. Struwe, *Quantization for an elliptic equation of order 2m with critical exponential non-linearity*, Math. Z. **270** (2012), 453-486.

[26] J. Moser, *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J. **20** (1971), 1077-1092.

[27] V. H. Nguyen, *A sharp Adams inequality in dimension four and its extremal functions*, Preprint 2017, https://arxiv.org/abs/1701.08249.

[28] R. O'Neil, *Convolution operators and $L(p,q)$ spaces*, Duke Math. J. **30** (1963), 129-142.

[29] P. Pizzetti, *Sulla media dei valori che una funzione dei punti dello spazio assume alla superficie di una sfera*, Rend. Lincei **18** (1909), 182-185.

[30] F. Robert, J. Wei, *Asymptotic behavior of a fourth order mean field equation with Dirichlet boundary condition*, Indiana Univ. Math. J. **57** (2008), 2039–2060.

[31] W. Rudin, *Real and complex analysis. Third edition*, McGraw-Hill Book Co., New York (1987), ISBN: 0-07-054234-1.

[32] C. Tintarev, *Trudinger-Moser inequality with remainder terms*, J. Funct. Anal. **266** (2014), 55-66.

[33] N. S. Trudinger, *On embedding into Orlicz spaces and some applications*, J. Math. Mech. **17** (1967), 473-483.

[34] J. Wei, X-W. Xu, *Classification of solutions of higher order conformally invariant equations*, Math. Ann. **313** (1999), 207-228.

[35] Y. Yang, *Extremal functions for Trudinger-Moser inequalities of Adimurthi-Druet type in dimension two*, J. Differential Equations **258** (2015), 3161–3193.