Conditional decisions under objective and subjective ambiguity in Dempster-Shafer theory

Davide Petturiti[a,*], Barbara Vantaggi[b]

[a]*Dip. Economia, Università di Perugia, 06100 Perugia, Italy*
[b]*Dip. MEMOTEF, "La Sapienza" Università di Roma, 00185 Roma, Italy*

Abstract

This paper deals with conditional decisions on generalized Anscombe-Aumann acts mapping states of the world to finitely additive probabilities on the set of menus of consequences, the latter conveying a form of "objective" ambiguity. If the decision maker has a systematic pessimistic/optimistic attitude towards "objective" ambiguity, acts reduce to functions mapping states of the world to belief/plausibility functions on consequences. We provide a system of axioms assuring the representability of a family of conditional preference relations on such acts by a conditional functional in which "subjective" uncertainty is modeled through a conditional belief/plausibility function on the states of the world, obeying to a suitable axiomatic definition.

Keywords: Belief functions, Finitely additive probabilities, Ambiguity, Conditional preferences

1. Introduction

Choices based on subjective probabilities find their roots in the seminal works by Savage [41] and Anscombe-Aumann [2]: such models, though relying on different decision settings, assume additivity in the quantification of decision maker's uncertainty. In particular, in the classical formulation proposed by Anscombe-Aumann we distinguish between "objective" uncertainty related to consequences and "subjective" uncertainty related to states

*Corresponding author.
Email addresses: davide.petturiti@unipg.it (Davide Petturiti), barbara.vantaggi@uniroma1.it (Barbara Vantaggi)

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of the world: the first is exogenously quantified, while the second is encoded in the decision maker's preferences.

In many circumstances, due to partial knowledge, uncertainty cannot be expressed by a single probability measure, but we need to manage a class of probability measures: this configures a situation of *ambiguity*.

We consider generalized Anscombe-Aumann acts mapping states of the world to finitely additive probabilities on the set of menus of consequences [33]. Each of such finitely additive probabilities encodes a form of "objective" ambiguity as it induces a closed convex set of finitely additive probabilities on the set of consequences whose lower/upper envelope is a belief/plausibility function in the Dempster-Shafer theory [9, 44]. Hence, we interpret such finitely additive probabilities as generalized lotteries since they allow to model partially known randomizing devices (like an urn or a roulette wheel), that result in a class of finitely additive probabilities whose lower/upper envelope is a belief/plausibility function, like in the well-known Ellsberg's urn paradox [12]. The term "objective" ambiguity is borrowed from [48] (see also [1, 37]) where the author considers acts mapping states of the world to non-empty compact convex polyhedral sets of probabilities on consequences.

As acknowledged by the survey papers [13] and [21], most of the efforts in decision theory have been devoted to model "subjective" ambiguity, that is to ambiguity in "subjective" uncertainty evaluations. This research stream starts with the seminal papers by Gilboa and Schmeidler, dealing with a capacity [43] or a set of probability measures [22], respectively. For this, we assume that also "subjective" uncertainty can be ambiguous, still referring to the Demspter-Shafer framework. Further, in order to deal with dynamic decision problems, a suitable notion of conditioning for belief/plausibility functions is necessary.

Starting from Dempster's original work [9], many researches have tried to propose a suitable notion of conditioning for belief/plausibility functions and, more generally, for non-additive uncertainty measures. Conditioning has been approached essentially proposing *conditiong rules* either working directly with a non-additive measure or working with the set of probability measures dominating/dominated by it. In particular, for rules of the first group we refer to [5, 7, 23, 27] and for those of the second group to [31, 49]. For a discussion on both approaches to conditioning we cite, for instance, [28, 32]. Here, we consider axiomatic definitions of conditional belief and plausibility functions that allow to define the conditional measure of every conditional event $E|H$ with $H \neq \emptyset$ (see [6]). Such axiomatic definitions generalize the so-called *Dempster rule* [9] and *product rule* [45, 46] and have conditional probability in the sense of Dubins [11] as a particular case. These axiomatic definitions make sense even if H is an event of null measure, as a pandemic or a terror attack. We stress that, as shown in [34], "null" events play a crucial role in the analysis of a game (see also [36]), thus this axiomatic approach seems to be the most suitable to model "subjective" uncertainty.

We consider a conditional decision model involving the above generalization of Anscombe-Aumann acts, assuming that the decision maker is able to provide a family of preference relations on acts indexed by the set of non-impossible events. Every preference relation can be interpreted as comparing acts under a particular hypothesis (or "scenario").

We provide a system of axioms assuring the representability of the family of conditional preference relations through a conditional functional parametrized by a bounded utility function on consequences and a full conditional belief function on the states. Such functional turns out to be a "subjective" conditional Choquet expectation of a state-contingent "objective" mixture of convex combinations of infima and suprema of utility on menus. Thus, the conditional functional involves a double integration, where the inner integral is inspired to the functional proposed by [17] in a Savage's setting, through acts that map states of the world to non-empty sets of consequences. The obtained conditional functional is then specialized to a "subjective" conditional Choquet expectation of a state-contingent "objective" lower/upper expected utility, the latter reducing to a state-contingent Choquet expected utility. The last two conditional functionals generalize to arbitrary sets of consequences and states of the world (besides introducing "subjective" ambiguity), those introduced in [40] that, in turn, generalize the conditional version of the Anscombe-Aumann model given in [36] by introducing "objective" ambiguity. Then, we consider analogous models in which the external full conditional belief function is replaced by a full conditional plausibility function. In the end, we consider the special case of a Bayesian decision maker, whose "subjective" uncertainty is unambiguous and expressed by a conditional probability in the sense of Dubins [11]. In this case, the preference relations are shown to satisfy a strong form of *dynamic consistency* (for a discussion on dynamic consistency we refer, e.g., to [47]) that takes into account the possibility of conditioning on "null" events as in [36].

The paper is structured as follows. Section 2 presents a motivating decision problem for the models discussed in this paper. Section 3 deals with conditioning in Dempster-Shafer theory, while Section 4 introduces the decision-theoretic setting and the representation theorems. Section 5 shows that the preference pattern singled out in Section 2 is not consistent with a

Bayesian decision maker and that conditioning to a "null" event is crucial. Finally, Section 6 collects conclusions. Proofs of results are reported in Appendix A to improve readability.

2. A motivating decision problem

An agent in an Italian insurance company needs to decide between four investment strategies where the adopted financial instrument is contingent on the state of the world.

Consider the set of states of the world $S = \{s_1, s_2, s_3, s_4\}$, where:

- s_1 = "At least one Covid-19 lockdown in Italy next year and Italian GDP increases next year",
- s_2 = "At least one Covid-19 lockdown in Italy next year and Italian GDP does not increase next year",
- s_3 = "No Covid-19 lockdown in Italy next year and Italian GDP increases next year",
- s_4 = "No Covid-19 lockdown in Italy next year and Italian GDP does not increase next year".

Such states of the world give rise to the events:

- $C = \{s_1, s_2\}$ = "At least one Covid-19 lockdown in Italy next year",
- $C^c = \{s_3, s_4\}$ = "No Covid-19 lockdown in Italy next year".

The agent has at his/her disposal three different financial instruments maturing after one year. The result in one year of each financial instrument can be a loss of €1000, a null gain, or a gain of €1000, i.e., instruments take values in $X = \{-€1000, €0, €1000\}$.

The usual decision-theoretic approach would be to identify each instrument with a probability distribution on X. Unfortunately, due to partial knowledge on previous performances of instruments, the agent has only the following information:

Instrument 1: it is only known that it results in a null gain in 50% of cases.

Instrument 2: it is only known that it results in a gain of €1000 in 30% of cases.

Instrument 3: it is only known that it results in a loss of €1000 in 30% of cases.

We have that instrument i determines a class of probability measures $\mathbf{P}^i$ on $\mathcal{P}(X)$ compatible with the available information:

$$\begin{aligned}
\mathbf{P}^1 &= \{P : P \text{ is a probability measure on } \mathcal{P}(X), \gamma \in [0, 0.5],\\
&\qquad P(\{-1000\}) = \gamma, P(\{0\}) = 0.5, P(\{1000\}) = 0.5 - \gamma\},\\
\mathbf{P}^2 &= \{P : P \text{ is a probability measure on } \mathcal{P}(X), \gamma \in [0, 0.7],\\
&\qquad P(\{-1000\}) = \gamma, P(\{0\}) = 0.7 - \gamma, P(\{1000\}) = 0.3\},\\
\mathbf{P}^3 &= \{P : P \text{ is a probability measure on } \mathcal{P}(X), \gamma \in [0, 0.7],\\
&\qquad P(\{-1000\}) = 0.3, P(\{0\}) = \gamma, P(\{1000\}) = 0.7 - \gamma\}.
\end{aligned}$$

For every $i = 1, 2, 3$, if we consider the lower envelope on the elements of $\mathcal{P}(X)$ defined as $\varphi_i = \min \mathbf{P}^i$, we get a non-additive uncertainty measure which reveals to be a belief function (whose definition and properties are recalled in Section 3). Denoting $x_1 = -1000$, $x_2 = 0$, $x_3 = 1000$, $A_i = \{x_i\}$ and $A_{ij} = \{x_i, x_j\}$, we have that

$\mathcal{P}(X)$	$\emptyset$	A_1	A_2	A_3	A_{12}	A_{13}	A_{23}	X
φ_1	0	0	0.5	0	0.5	0.5	0.5	1
φ_2	0	0	0	0.3	0.7	0.3	0.3	1
φ_3	0	0.3	0	0	0.3	0.3	0.7	1

Now, consider the following four investment strategies in which the adopted financial instrument is contingent on the state of the world:

S	s_1	s_2	s_3	s_4
f	**Instr. 2**	**Instr. 2**	**Instr. 2**	**Instr. 2**
g	**Instr. 1**	**Instr. 3**	**Instr. 3**	**Instr. 3**
h	**Instr. 3**	**Instr. 3**	**Instr. 2**	**Instr. 2**
l	**Instr. 3**	**Instr. 3**	**Instr. 3**	**Instr. 3**

We assume that the agent needs to decide among the above investment strategies, immediately after the end of 2021 Covid-19 lockdown. This makes of particular importance to take into account the scenarios C and C^c.

Suppose the agent is *ex ante* indifferent between f, g and f, h, while strictly prefers l to f. Moreover, he/she strictly prefers g to f conditionally on both the scenarios C and C^c. In symbol, we write

$$f \sim_S g, \quad f \sim_S h, \quad f \prec_S l, \quad f \prec_C g, \quad \text{and} \quad f \prec_{C^c} g.$$

Further, suppose, as natural, that such an agent is a profit maximizer. We also assume that the agent is systematically pessimistic in coping with "objective" uncertainty, meaning that he/she always identifies each menu of consequences with the less preferred consequence in it.

Representing instrument i with the corresponding belief function φ_i on $\mathcal{P}(X)$, for $i = 1, 2, 3$, each investment strategy can be seen as a function mapping states of the world to belief functions on consequences. This configures a two-stage decision problem in which first the state is chosen by nature and then the consequence is chosen according to a partially known randomizing device acting in agreement with one of the belief functions above.

In [40], we proposed a model able to deal with a decision problem like the one presented so far. Such model assumes the agent is Bayesian, that is the agent uses conditional probability to encode his/her "subjective" beliefs on the states of the world.

As will be shown in Section 4, the preference pattern $f \prec_C g$, $f \prec_{C^c} g$ and $f \sim_S g$ violates the *strong dynamic consistency* axiom, that would require $f \prec_S g$, instead. Such a property must be satisfied by a family of conditional preference relations to be consistent with a Bayesian agent. Further, the preference statements $f \sim_S h$ and $f \prec_S l$ imply that event C is a "null" event, that is the agent finds C as unexpected. This evaluation of C is motivated by the current status of the vaccination campaign in several countries.

The decision problem above shows two different kinds of ambiguity: "objective" ambiguity on the consequences and "subjective" ambiguity on the states. This requires a departure from probability theory, in modeling both "objective" and "subjective" uncertainty. Moreover, the example shows a situation in which conditioning on a "null" event is crucial.

We maintain that Dempster-Shafer theory is the theory of uncertainty closest to probability and yet, under a suitable notion of conditioning, able to treat situations like those described above.

3. Conditioning in Dempster-Shafer theory

In this section we collect some preliminary material working on an abstract non-empty set Ω. Denote by $\mathcal{P}(\Omega)$ the power set of Ω and by $\mathcal{U} = \mathcal{P}(\Omega)^0 = \mathcal{P}(\Omega) \setminus \{\emptyset\}$.

A *belief function* (or *completely monotone capacity*) [9, 44] on $\mathcal{P}(\Omega)$ is a function $\varphi : \mathcal{P}(\Omega) \to [0, 1]$ satisfying the following properties:

(i) $\varphi(\emptyset) = 0$ and $\varphi(\Omega) = 1$;

(ii) for every $k \geq 2$ and every $A_1, \ldots, A_k \in \mathcal{P}(\Omega)$,

$$\varphi\left(\bigcup_{i=1}^{k} A_i\right) \geq \sum_{\emptyset \neq I \subseteq \{1,\ldots,k\}} (-1)^{|I|+1} \varphi\left(\bigcap_{i \in I} A_i\right).$$

In particular, φ is a *finitely additive probability (measure)* if it satisfies the above inequality as an equality. The *dual* function ψ defined, for every $A \in \mathcal{P}(\Omega)$, as $\psi(A) = 1 - \varphi(A^c)$, is said *plausibility function* (or *completely alternating capacity*) and satisfies the above inequality in the opposite direction, switching intersections and unions.

Every belief function φ on $\mathcal{P}(\Omega)$ with dual ψ induces a non-empty closed convex set of finitely additive probabilities (see, e.g., [25, 43]) on $\mathcal{P}(\Omega)$:

$$\mathbf{core}(\varphi) = \{P \,:\, P \text{ is a finitely additive probability on } \mathcal{P}(\Omega), \varphi \leq P\},$$

such that $\varphi = \min \mathbf{core}(\varphi)$ and $\psi = \max \mathbf{core}(\varphi)$, where the minimum and maximum are pointwise on $\mathcal{P}(\Omega)$. As shown in [42], the Choquet integral of every bounded function $f : \Omega \to \mathbb{R}$ with respect to φ and ψ has a lower/upper expectation interpretation

$$\begin{aligned} \oint_\Omega f(\omega)\varphi(\mathrm{d}\omega) &= \min_{P \in \mathbf{core}(\varphi)} \int_\Omega f(\omega) P(\mathrm{d}\omega), \\ \oint_\Omega f(\omega)\psi(\mathrm{d}\omega) &= \max_{P \in \mathbf{core}(\varphi)} \int_\Omega f(\omega) P(\mathrm{d}\omega). \end{aligned}$$

In all the paper, integrals with respect to a finitely additive probability are of Stieltjes type [3].

Theorem A in [24] shows that every belief function φ on $\mathcal{P}(\Omega)$ corresponds to a unique finitely additive probability measure μ_φ defined on an algebra $\mathfrak{A}$ (possibly strictly contained in $\mathcal{P}(\mathcal{U})$) such that, for every $A \in \mathcal{P}(\Omega)$

$$\varphi(A) = \int_{\mathcal{U}} \nu_B(A) \mu_\varphi(\mathrm{d}B), \tag{1}$$

where $\nu_B : \mathcal{P}(\Omega) \to [0,1]$ is a *vacuous belief function* (or *unanimity game*) on B such that $\nu_B(A) = 1$ if $B \subseteq A$ and 0 otherwise. In analogy with the case of a finite Ω [4, 25], we refer to the finitely additive probability μ_φ on $\mathfrak{A}$ as the *Möbius inverse* of φ and $\mathfrak{A}$ is the algebra on $\mathcal{U}$ generated by $\{\tilde{A} \,:\, A \in \mathcal{U}\}$ where, for every $A \in \mathcal{P}(\Omega)$,

$$\tilde{A} = \{B \in \mathcal{U} \,:\, B \subseteq A\},$$

In case of a finite Ω we have that $\mathfrak{A} = \mathcal{P}(\mathcal{U})$.

Notice, that μ_φ can be extended, generally not in a unique way, to a finitely additive probability defined on the whole $\mathcal{P}(\mathcal{U})$.

Conversely, every finitely additive probability measure μ on $\mathcal{P}(\mathcal{U})$ determines a belief function φ_μ on $\mathcal{P}(\Omega)$ but, in general, there can be more finitely additive probabilities giving rise to the same belief function. The main feature of finitely additive probabilities on $\mathcal{P}(\mathcal{U})$ is that only their restriction to $\mathfrak{A}$ is meaningful in order to get a representation of the corresponding belief function.

With a dual argument we have that every finitely additive probability measure μ on $\mathcal{P}(\mathcal{U})$ determines also a plausibility function ψ_μ on $\mathcal{P}(\Omega)$.

A finitely additive probability μ defined on $\mathcal{P}(\mathcal{U})$ allows to have an additive expression of the Choquet integral of a bounded function $f : \Omega \to \mathbb{R}$ with respect to φ_μ and ψ_μ. Indeed, by Corollary in [35] (see also Proposition 6 in [17]) we have

$$\begin{aligned}
\oint_\Omega f(\omega)\varphi_\mu(\mathrm{d}\omega) &= \int_{\mathcal{U}} \left(\inf_{\omega\in B} f(\omega)\right) \mu(\mathrm{d}B),\\
\oint_\Omega f(\omega)\psi_\mu(\mathrm{d}\omega) &= \int_{\mathcal{U}} \left(\sup_{\omega\in B} f(\omega)\right) \mu(\mathrm{d}B).
\end{aligned}$$

Starting from the seminal work by Dempster [9] (see also [14, 30, 50]), conditioning has been introduced either by providing a conditioning rule "directly" for non-additive uncertainty measures, or applying the conditioning to every probability in the core and then defining the conditional non-additive measure as their envelope.

In the decision theory literature both alternatives have been considered, in particular for the first group we refer to [5, 8, 23, 27] and for the second group to [31, 49]. For a discussion on both approaches to conditioning we cite, for instance, [28, 32].

Given a belief function $\varphi : \mathcal{P}(\Omega) \to [0,1]$ and its dual plausibility function ψ, the most popular conditioning rules are, for every $E|H \in \mathcal{P}(\Omega) \times \mathcal{P}(\Omega)^0$:

(Dempster rule [9]) provided $\psi(H) > 0$,

$$\varphi_D(E|H) = 1 - \psi_D(E^c|H) \text{ and } \psi_D(E|H) = \frac{\psi(E\cap H)}{\psi(H)};$$

(Product rule [45, 46]) provided $\varphi(H) > 0$,

$$\varphi_P(E|H) = \frac{\varphi(E\cap H)}{\varphi(H)} \text{ and } \psi_P(E|H) = 1 - \varphi_P(E^c|H);$$

(Bayesian rule [31, 50]) provided $\varphi(E \cap H) + \psi(E^c \cap H) > 0$,

$$\varphi_B(E|H) = \frac{\varphi(E \cap H)}{\varphi(E \cap H) + \psi(E^c \cap H)} \text{ and } \psi_B(E|H) = 1 - \varphi_B(E^c|H).$$

This paper adopts the conditioning rule expressed by the axiomatic definition of conditional belief and plausibility functions generalizing the product and Dempster rule, respectively. For a discussion on the different axiomatic definitions see [6] (see also [39]).

Definition 1. *A function $\varphi : \mathcal{P}(\Omega) \times \mathcal{P}(\Omega)^0 \to [0, 1]$ is a* **full conditional belief function on** $\mathcal{P}(\Omega)$ *if it satisfies the following conditions:*

(i) $\varphi(E|H) = \varphi(E \cap H|H)$*, for every* $E \in \mathcal{P}(\Omega)$ *and* $H \in \mathcal{P}(\Omega)^0$*;*

(ii) $\varphi(\cdot|H)$ *is a belief function on* $\mathcal{P}(\Omega)$*, for every* $H \in \mathcal{P}(\Omega)^0$*;*

(iii) $\varphi(E \cap F|H) = \varphi(E|H) \cdot \varphi(F|E \cap H)$*, for every* $H, E \cap H \in \mathcal{P}(\Omega)^0$ *and* $E, F \in \mathcal{P}(\Omega)$*.*

Definition 2. *A function $\psi : \mathcal{P}(\Omega) \times \mathcal{P}(\Omega)^0 \to [0, 1]$ is a* **full conditional plausibility function on** $\mathcal{P}(\Omega)$ *if it satisfies the following conditions:*

(i) $\psi(E|H) = \psi(E \cap H|H)$*, for every* $E \in \mathcal{P}(\Omega)$ *and* $H \in \mathcal{P}(\Omega)^0$*;*

(ii) $\psi(\cdot|H)$ *is a plausibility function on* $\mathcal{P}(\Omega)$*, for every* $H \in \mathcal{P}(\Omega)^0$*;*

(iii) $\psi(E \cap F|H) = \psi(E|H) \cdot \psi(F|E \cap H)$*, for every* $H, E \cap H \in \mathcal{P}(\Omega)^0$ *and* $E, F \in \mathcal{P}(\Omega)$*.*

Notice that condition *(iii)* in Definition 1 can be replaced by the chain rule condition *(iii')*

$$\varphi(A|C) = \varphi(A|B) \cdot \varphi(B|C),$$

for every $A \in \mathcal{P}(\Omega)$ and $B, C \in \mathcal{P}(\Omega)^0$, with $A \subseteq B \subseteq C$. An analogous chain rule condition can replace condition *(iii)* in Definition 2. Also, notice that condition *(iii)* in Definition 1 implies, in particular, that, for every $E|H \in \mathcal{P}(\Omega) \times \mathcal{P}(\Omega)^0$ with $\varphi(H|\Omega) > 0$

$$\varphi(E|H) = \frac{\varphi(E \cap H|\Omega)}{\varphi(H|\Omega)}.$$

An analogous property holds for $\psi(\cdot|\cdot)$. As usual, we can identify $\varphi(\cdot|\Omega)$ with the unconditional belief function $\varphi(\cdot)$ and $\psi(\cdot|\Omega)$ with the unconditional plausibility function $\psi(\cdot)$.

Let us stress that the conditional measures $\varphi(\cdot|\cdot)$ and $\psi(\cdot|\cdot)$ defined above are generally not *dual* (see [39]), as it can be $\varphi(E|H) \neq 1 - \psi(E^c|H)$, for some $E \in \mathcal{P}(\Omega)$ and $H \in \mathcal{P}(\Omega)^0$ with $H \neq \Omega$. The lack of preservation of duality is already known for the classical Dempster and product rules: as shown in the following example, in this context, even more degrees of freedom are introduced in case of conditioning on events of null measure.

In particular, a full conditional probability $P(\cdot|\cdot)$ on $\mathcal{P}(\Omega)$ in the sense of Dubins [11] is both a full conditional belief function and a full conditional plausibility function on $\mathcal{P}(\Omega)$ according to Definitions 1 and 2.

Example 1. *Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and consider the full conditional belief function on $\mathcal{P}(\Omega)$ given below*

$\mathcal{P}(\Omega)$	$\emptyset$	$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_3\}$	$\{\omega_1,\omega_2\}$	$\{\omega_1,\omega_3\}$	$\{\omega_2,\omega_3\}$	Ω
$\varphi(\cdot\|\Omega)$	0	0	0	0	0	0.5	0	1
$\varphi(\cdot\|\{\omega_1,\omega_2\})$	0	0.3	0.7	0	1	0.3	0.7	1
$\varphi(\cdot\|\{\omega_1,\omega_3\})$	0	0	0	0	0	1	0	1
$\varphi(\cdot\|\{\omega_2,\omega_3\})$	0	0	0.7	0.3	0.7	0.3	1	1
$\varphi(\cdot\|\{\omega_1\})$	0	1	0	0	1	1	0	1
$\varphi(\cdot\|\{\omega_2\})$	0	0	1	0	1	0	1	1
$\varphi(\cdot\|\{\omega_3\})$	0	0	0	1	0	1	1	1

The function $\varphi(\cdot|\cdot)$ is easily verified to agree with Definition 1. In particular, $\varphi(\cdot|\{\omega_1,\omega_2\})$ is well-defined, even though $\varphi(\{\omega_1,\omega_2\}|\Omega) = 0$.

Now, setting $\psi(E|\Omega) = 1 - \varphi(E^c|\Omega)$ for every $E \in \mathcal{P}(\Omega)$, we get a plausibility function $\psi(\cdot|\Omega)$ on $\mathcal{P}(\Omega)$ which is strictly positive on $\mathcal{P}(\Omega)^0$. Therefore, defining, for every $E|H \in \mathcal{P}(\Omega) \times \mathcal{P}(\Omega)^0$

$$\psi(E|H) = \frac{\psi(E \cap H|\Omega)}{\psi(H|\Omega)},$$

we get a full conditional plausibility on $\mathcal{P}(\Omega)$ agreeing with Definition 2. Note that $\varphi(\cdot|\Omega)$ and $\psi(\cdot|\Omega)$ are dual, while we have that

$$\varphi(\{\omega_1\}|\{\omega_1,\omega_2\}) = 0.3 \neq 0.5 = 1 - \psi(\{\omega_1\}^c|\{\omega_1,\omega_2\}).$$

This shows that $\varphi(\cdot|\cdot)$ and $\psi(\cdot|\cdot)$ are not dual in general. ♦

4. Main model description

Consider the following decision-theoretic setting:

- X, an arbitrary non-empty set of consequences;
- $\mathcal{U} = \mathcal{P}(X) \setminus \{\emptyset\}$, the set menus, i.e., non-empty sets of consequences;
- $\mathbf{M}(\mathcal{U}) = \{\mu : \mathcal{P}(\mathcal{U}) \to [0,1]\}$, the set of all finitely additive probability measures defined on $\mathcal{P}(\mathcal{U})$;
- S, an arbitrary non-empty set of states of the world;
- $\mathcal{P}(S)$, the set of all events;
- $\mathcal{P}(S)^0 = \mathcal{P}(S) \setminus \{\emptyset\}$, the set of non-impossible events (i.e., scenarios or hypotheses);
- $\mathcal{F}_{\text{const}} = \{\overline{\mu} \in \mathbf{M}(\mathcal{U})^S \;:\; \mu \in \mathbf{M}(\mathcal{U})\}$, the set of constant acts, where $\overline{\mu} \in \mathcal{F}_{\text{const}}$ is such that $\overline{\mu}(s) = \mu$ for all $s \in S$;
- $\mathcal{F}_{\text{simple}}$, the set of simple acts (functions in $\mathbf{M}(\mathcal{U})^S$ with finite image);
- $\mathcal{F}$, a set of acts closed under pointwise convex combination, with $\mathcal{F}_{\text{simple}} \subseteq \mathcal{F} \subseteq \mathbf{M}(\mathcal{U})^S$: this implies $\mathcal{F}_{\text{const}} \subseteq \mathcal{F}$.

This decision-theoretic setting uses power sets in order to have a lighter presentation. The entire formulation of the problem can be restated using algebras of sets, provided to take care of measurability restrictions and some structural assumptions.

Consider a family $\{\precsim_H\}_{H \in \mathcal{P}(S)^0}$ of preference relations on $\mathcal{F}$, indexed by the set of non-impossible events $\mathcal{P}(S)^0$. As usual $\precsim_H$ is said to be a *weak order* if it is complete and transitive.

For every $H \in \mathcal{P}(S)^0$, we denote by $\prec_H$ and $\sim_H$ the asymmetric and symmetric parts of $\precsim_H$. Moreover, for every $f, g \in \mathcal{F}$, $f \precsim_H g$ means "f is not preferred to g under the hypothesis H", $f \prec_H g$ means "g is preferred to f under the hypothesis H", and $f \sim_H g$ means "f is indifferent to g under the hypothesis H".

Notice that the set $\mathbf{M}(\mathcal{U})$ contains the set

$$\mathbf{M}_0(\mathcal{U}) = \{\delta_B \in \mathbf{M}(\mathcal{U}) \;:\; B \in \mathcal{U}\},$$

of *degenerate finitely additive probabilities*, where δ_B is the finitely additive probability such that $\delta_B(\{B\}) = 1$. Let us notice that $\mathbf{M}(\mathcal{U})$ is closed under the following operations:

Countable convex combination: For every $\mu_1, \mu_2, \ldots \in \mathbf{M}(\mathcal{U})$ and $\alpha_1, \alpha_2, \ldots \in [0,1]$ with $\sum_{n=1}^{\infty} \alpha_n = 1$, the finitely additive probability $\sum_{n=1}^{\infty} \alpha_n \mu_n$ defined, for every $A \in \mathcal{P}(\mathcal{U})$, as $\sum_{n=1}^{\infty} \alpha_n \mu_n(A)$, belongs to $\mathbf{M}(\mathcal{U})$.

Conditioning: For every $\mu \in \mathbf{M}(\mathcal{U})$ and $H \in \mathcal{P}(\mathcal{U})$ with $\mu(H) > 0$, the finitely additive probability μ_H defined, for every $A \in \mathcal{P}(\mathcal{U})$, as $\mu_H(A) = \frac{\mu(A \cap H)}{\mu(H)}$, belongs to $\mathbf{M}(\mathcal{U})$.

For every $H \in \mathcal{P}(S)^0$, the relation $\precsim_H$ determines a relation $\trianglelefteq_H$ on $\mathbf{M}(\mathcal{U})$ through constant acts defined, for every $\mu_1, \mu_2 \in \mathbf{M}(\mathcal{U})$, as

$$\mu_1 \trianglelefteq_H \mu_2 \Longleftrightarrow \overline{\mu_1} \precsim_H \overline{\mu_2}.$$

In turn, the relation $\trianglelefteq_H$ determines a relation $\leq_H^\bullet$ on $\mathcal{U}$ defined, for every $A, B \in \mathcal{U}$, as

$$A \leq_H^\bullet B \Longleftrightarrow \delta_A \trianglelefteq_H \delta_B.$$

Finally, the relation $\leq_H^\bullet$ induces a relation $\leq_H^*$ on X defined, for every $x, y \in X$, as

$$x \leq_H^* y \Longleftrightarrow \{x\} \leq_H^\bullet \{y\}.$$

For $H \in \mathcal{P}(S)^0$, if there are $x_*, x^* \in X$ such that $x_* <_H^* x^*$ then, for every $E \in \mathcal{P}(S)$, define the act

$$\mathbf{1}_E(s) = \begin{cases} \delta_{\{x^*\}} & \text{if } s \in E, \\ \delta_{\{x_*\}} & \text{if } s \notin E. \end{cases} \tag{2}$$

For every fixed $H \in \mathcal{P}(S)^0$, two acts $f, g \in \mathcal{F}$ are *comonotonic* if there are no $s, t \in S$ such that $f(s) \triangleleft_H f(t)$ and $g(t) \triangleleft_H g(s)$, where $\triangleleft_H$ is the asymmetric part of the relation $\trianglelefteq_H$ induced by $\precsim_H$ on $\mathbf{M}(\mathcal{U})$ through constant acts.

Here, we assume that "subjective" uncertainty is modeled through a full conditional belief function according to Definition 1.

We look for a representation in terms of a conditional functional $\underline{\Lambda}$ defined, for every $f \in \mathcal{F}$ and every $H \in \mathcal{P}(S)^0$, as

$$\underline{\Lambda}(f|H) = \oint_S \left[\int_{\mathcal{U}} \left(\alpha(B) \cdot \inf_{x \in B} u(x) + (1 - \alpha(B)) \cdot \sup_{x \in B} u(x) \right) f(s)(\mathrm{d}B) \right] \varphi(\mathrm{d}s|H), \tag{3}$$

where $\varphi : \mathcal{P}(S) \times \mathcal{P}(S)^0 \to [0,1]$ is a full conditional belief function according to Definition 1, $u : X \to \mathbb{R}$ is a bounded utility function and $\alpha : \mathcal{U} \to [0,1]$ is a function that can be interpreted as a pessimism index function [17], which depends on the menu B. To simplify computations, denoting

$$V(f(s)) = \int_{\mathcal{U}} \left(\alpha(B) \cdot \inf_{x \in B} u(x) + (1-\alpha(B)) \cdot \sup_{x \in B} u(x) \right) f(s)(\mathrm{d}B), \quad (4)$$

we can write

$$\underline{\Lambda}(f|H) = \oint_S V(f(s))\varphi(\mathrm{d}s|H). \quad (5)$$

For every $H \in \mathcal{P}(S)^0$, the (conditional) functional $\underline{\Lambda}(\cdot|H)$ turns out to be a "subjective" conditional Choquet expectation of a state-contingent "objective" mixture of convex combinations of infima and suprema of utility on menus. We point out the resemblance of the internal menu-contingent convex combination with the Hurwicz's criterion [29].

More in detail, for every menu $B \in \mathcal{U}$, the corresponding utility is obtained as a convex combination of the "worst" and "best" utility values on the related consequences, where the weight of the convex combination is $\alpha(B)$. This gives rise to a utility function on menus that captures the attitude of the decision maker towards "objective" ambiguity, through the function α. Then, for every state $s \in S$, we take an "objective" expectation with respect to the finitely additive probability measure $f(s)$ defined on menus that, in turn, gives rise to a state-contingent "objective" expected utility. Finally, we take a "subjective" conditional Choquet expectation with respect to the conditional belief function $\varphi(\cdot|H)$, defined on states. Notice that, due to Definition 1, $\underline{\Lambda}(f|H)$ is well-defined even though the *ex ante* belief function $\varphi(H|S)$ is equal to 0. Further, the properties of the Choquet integral imply that the external conditional Choquet expectation is actually a lower conditional expectation computed with respect to $\mathbf{core}(\varphi(\cdot|H))$. In other terms, a decision maker deciding according to the conditional functional $\underline{\Lambda}(\cdot|\cdot)$ has a systematically pessimistic attitude towards "subjective" uncertainty.

The conditional functional $\underline{\Lambda}$ is said to *represent* the family of preference relations $\{\precsim_H\}_{H \in \mathcal{P}(S)^0}$ if, for every $H \in \mathcal{P}(S)^0$ and every $f, g \in \mathcal{F}$, it holds that

$$f \precsim_H g \Longleftrightarrow \underline{\Lambda}(f|H) \leq \underline{\Lambda}(g|H). \quad (6)$$

Our aim is to study the representability of $\{\precsim_H\}_{H \in \mathcal{P}(S)^0}$ through a conditional functional $\underline{\Lambda}(\cdot|\cdot)$. For this, consider the following axioms.

(A1) Weak order: For every $H \in \mathcal{P}(S)^0$, $\precsim_H$ is a weak order on $\mathcal{F}$.

(A2) Continuity: For every $H \in \mathcal{P}(S)^0$, for every $f, g, h \in \mathcal{F}$, if $f \prec_H g \prec_H h$, there exist $\alpha, \beta \in (0,1)$ such that

$$\alpha f + (1-\alpha)h \prec_H g \prec_H \beta f + (1-\beta)h.$$

(A3) Comonotonic independence: For every $H \in \mathcal{P}(S)^0$, for every pairwise comonotonic $f, g, h \in \mathcal{F}$ and for every $\alpha \in (0,1)$

$$f \precsim_H g \Longleftrightarrow \alpha f + (1-\alpha)h \precsim_H \alpha g + (1-\alpha)h.$$

(A4) Monotonicity: For every $H \in \mathcal{P}(S)^0$, for every $f, g \in \mathcal{F}$, if $f(s) \trianglelefteq_H g(s)$, for all $s \in S$, then $f \precsim_H g$.

(A5) Non-triviality: For every $H \in \mathcal{P}(S)^0$, there exist $f, g \in \mathcal{F}$ such that $f \prec_H g$.

(A6) Complete monotonicity: For every $H \in \mathcal{P}(S)^0$, for every $k \geq 2$ and every $E_1, \ldots, E_k \in \mathcal{P}(S)$, if $x_*, x^* \in X$ are such that $x_* <_H x^*$ and for all $\emptyset \neq I \subseteq \{1, \ldots, k\}$

$$\mathbf{1}_{\bigcup_{i=1}^k E_i} \sim_H \overline{\mu^{\cup}_{\{1,\ldots,k\}}} \text{ and } \mathbf{1}_{\bigcap_{i \in I} E_i} \sim_H \overline{\mu^{\cap}_I}$$

with $\mu^{\cup}_{\{1,\ldots,k\}}, \mu^{\cap}_I \in \mathbf{M}(\mathcal{U})$, then the acts

$$g = \frac{1}{2^{k-1}} \left(\overline{\mu^{\cup}_{\{1,\ldots,k\}}} + \sum_{\substack{\emptyset \neq I \subseteq \{1,\ldots,k\} \\ |I| \text{ even}}} \overline{\mu^{\cap}_I} \right) \text{ and } h = \frac{1}{2^{k-1}} \left(\sum_{\substack{\emptyset \neq I \subseteq \{1,\ldots,k\} \\ |I| \text{ odd}}} \overline{\mu^{\cap}_I} \right)$$

are such that

$$h \precsim_H g.$$

(A7) Relevance: For every $H \in \mathcal{P}(S)^0$, for every $f, g \in \mathcal{F}$ with $f(s) = g(s)$, for all $s \in H$, then $f \sim_H g$.

(A8) Chain rule: For every $A \in \mathcal{P}(S)$ and $B, C \in \mathcal{P}(S)^0$ with $A \subseteq B \subseteq C$, for every $\mu_1, \mu_2, \mu_3 \in \mathbf{M}(\mathcal{U})$ with $\mu_1 \trianglelefteq_B \mu_2 \trianglelefteq_B \mu_3$ and $\mu_1 \trianglelefteq_C \mu_2 \trianglelefteq_C \mu_3$, if $f, g \in \mathcal{F}$ are such that $f_{|A} = \overline{\mu_3}_{|A}$, $f_{|A^c} = \overline{\mu_1}_{|A^c}$, $g_{|B} = \overline{\mu_2}_{|B}$ and $g_{|B^c} = \overline{\mu_1}_{|B^c}$, then

$$f \sim_B g \Longrightarrow f \sim_C g.$$

(A9) Scenario neutrality: For every $H, K \in \mathcal{P}(S)^0$ and for every $\overline{\mu_1}, \overline{\mu_2} \in \mathcal{F}_{\mathrm{const}}$,

$$\overline{\mu_1} \precsim_H \overline{\mu_2} \Longleftrightarrow \overline{\mu_1} \precsim_K \overline{\mu_2}.$$

(A10) Sure-thing: For every $H \in \mathcal{P}(S)^0$, for every $\mathcal{B} \in \mathcal{P}(\mathcal{U})$, for every $\mu_1, \mu_2 \in \mathbf{M}(\mathcal{U})$ it holds that

$$\begin{cases} \mu_1(\mathcal{B}) = 1, \overline{\mu_2} \prec_H \overline{\delta_B} \text{ for all } B \in \mathcal{B} \Longrightarrow \overline{\mu_2} \precsim_H \overline{\mu_1}; \\ \mu_1(\mathcal{B}) = 1, \overline{\delta_B} \prec_H \overline{\mu_2} \text{ for all } B \in \mathcal{B} \Longrightarrow \overline{\mu_1} \precsim_H \overline{\mu_2}. \end{cases}$$

(A11) Contingencywise dominance: For every $H \in \mathcal{P}(S)^0$, for every $B \in \mathcal{U}$, for every $\mu \in \mathbf{M}(\mathcal{U})$ whose corresponding belief function φ_μ on $\mathcal{P}(X)$ is a finitely additive probability, it holds that

$$\begin{cases} \overline{\delta_{\{x\}}} \prec_H \overline{\mu} \text{ for all } x \in B \Longrightarrow \overline{\delta_B} \precsim_H \overline{\mu}; \\ \overline{\mu} \prec_H \overline{\delta_{\{x\}}} \text{ for all } x \in B \Longrightarrow \overline{\mu} \precsim_H \overline{\delta_B}. \end{cases}$$

(A12) Outcome boundedness: For every $H \in \mathcal{P}(S)^0$, for every $B \in \mathcal{U}$, there exist $x_1, x_2 \in X$ such that

$$\overline{\delta_{\{x_1\}}} \precsim_H \overline{\delta_B} \precsim_H \overline{\delta_{\{x_2\}}}.$$

Axioms **(A1)**–**(A5)** are the usual Schmeidler's adaptations of Anscombe-Aumann axioms [43] in the formulation of [20], stated for generalized Anscombe-Aumann acts and every preference relation in $\{\precsim_H\}_{H \in \mathcal{P}(S)^0}$. Such axioms essentially deal with the external Choquet integral.

Axiom **(A6)** assures the complete monotonicity of the "subjective" conditional capacity used in the external Choquet integral. In particular, due to the properties of the Choquet integral [10, 25, 42], this consists in a conditional lower expectation computed with respect to $\mathbf{core}(\varphi(\cdot|H))$. We point out that requiring the property in **(A6)** to hold for every $2 \leq k \leq n$ with a fixed $n \geq 2$ we get a conditional n-monotone capacity. In particular, for $n = 2$ the corresponding axiom turns out to be equivalent, in presence of other axioms, to the *uncertainty aversion* axiom of Schmeidler [43], for every $H \in \mathcal{P}(S)^0$. We further notice that if **(A6)** is strengthened by requiring $f \sim_H g$, then $\varphi(\cdot|\cdot)$ reduces to a full conditional probability on $\mathcal{P}(S)$. Therefore, the strengthened version of **(A6)** plus axiom **(A3)**, in presence of other axioms, imply the *independence axiom* of Schmeidler [43], for every $H \in \mathcal{P}(S)^0$.

Axioms **(A7)**–**(A8)** cope with conditioning. In particular, axiom **(A7)** expresses a focusing conditioning rule, i.e., it states that in conditioning on H, only the part of acts inside of H counts. Such axiom is also referred to as *consequentialism* in the literature (see, e.g., [18]) but we maintain the name *relevance* as in [36].

Axiom **(A8)** copes with relating different conditioning events through a qualitative chain rule. In particular, axiom **(A8)** connects the *ex ante* preference relation $\precsim_S$ with any other preference relation $\precsim_H$ relating indifferences of $\precsim_H$ on suitable acts to indifferences of $\precsim_S$. We point out that the implication in axiom **(A8)** cannot be reverted and is the key feature of the axiomatic notion of conditioning captured by Definition 1 that allows conditioning also on events H such that $\varphi(H|S) = 0$.

Example 2. *Let $S = \{s_1, s_2, s_3\}$ and $X = \{x_1, x_2, x_3\}$, and take the full conditional probability $P(\cdot|\cdot)$ on $\mathcal{P}(S)$ such that*

$\mathcal{P}(S)$	$\emptyset$	$\{s_1\}$	$\{s_2\}$	$\{s_3\}$	$\{s_1, s_2\}$	$\{s_1, s_3\}$	$\{s_2, s_3\}$	S
$P(\cdot|S)$	0	0	0	1	0	1	1	1
$P(\cdot|\{s_1, s_2\})$	0	0.5	0.5	0	1	0.5	0.5	1

while all other values can be recovered through ratios. We have that $P(\cdot|\cdot)$ is also a full conditional belief function on $\mathcal{P}(S)$ according to Definition 1. Let α be an arbitrary function $\alpha : \mathcal{U} \to [0, 1]$ and u be such that $u(x_1) = 0$, $u(x_2) = 2$ and $u(x_3) = 3$.

Considering the acts

S	s_1	s_2	s_3
f	$\delta_{\{x_3\}}$	$\delta_{\{x_1\}}$	$\delta_{\{x_1\}}$
g	$\delta_{\{x_2\}}$	$\delta_{\{x_2\}}$	$\delta_{\{x_1\}}$

since, for $i = 1, 2, 3$, it holds that

$$\int_{\mathcal{U}} \left(\alpha(B) \cdot \inf_{x \in B} u(x) + (1 - \alpha(B)) \cdot \sup_{x \in B} u(x) \right) \delta_{\{x_i\}}(\mathrm{d}B) = u(x_i),$$

by equation (4) *we have that*

S	s_1	s_2	s_3
$V(f(s))$	3	0	0
$V(g(s))$	2	2	0

Let $A = \{s_1\}$, $B = \{s_1, s_2\}$ and $C = S$. By equation (5) and the additivity of $P(\cdot|C)$ and $P(\cdot|B)$ we get

$$
\begin{aligned}
\underline{\Lambda}(f|C) &= \oint_S V(f(s))P(\mathrm{d}s|C) = \sum_{s\in S} V(f(s))P(\{s\}|C) = 0,\\
\underline{\Lambda}(g|C) &= \oint_S V(g(s))P(\mathrm{d}s|C) = \sum_{s\in S} V(g(s))P(\{s\}|C) = 0,\\
\underline{\Lambda}(f|B) &= \oint_S V(f(s))P(\mathrm{d}s|B) = \sum_{s\in S} V(f(s))P(\{s\}|B) = 1.5,\\
\underline{\Lambda}(g|B) &= \oint_S V(g(s))P(\mathrm{d}s|B) = \sum_{s\in S} V(g(s))P(\{s\}|B) = 2.
\end{aligned}
$$

Hence, it holds that

$$
\begin{aligned}
\underline{\Lambda}(f|C) &= \underline{\Lambda}(g|C),\\
\underline{\Lambda}(f|B) &< \underline{\Lambda}(g|B).
\end{aligned}
$$

Referring to the family of preference relations $\{\precsim_H\}_{H\in\mathcal{P}(S)^0}$ on $\mathcal{F} = \mathbf{M}(\mathcal{U})^S$ induced by $\underline{\Lambda}(\cdot|\cdot)$, we have that acts f and g satisfy the conditions in **(A8)** *with A, B and C as above, but $f \sim_C g$ does not imply $f \sim_B g$.* ♦

Axiom **(A9)** assures the independence of the utility function on the conditioning event. Axiom **(A10)** is a restating of condition **A4a** in [15] assuring the boundedness of the utility and the inner integral expression. Finally, axioms **(A11)**–**(A12)** are inspired to **Axiom 8** and **Axiom 9** of [17], respectively, and guarantee the inner convex combination expression.

The following theorem shows that axioms **(A1)**–**(A12)** are sufficient to get a representation by a conditional functional $\underline{\Lambda}(\cdot|\cdot)$.

Theorem 1. *If the family of relations $\{\precsim_H\}_{H\in\mathcal{P}(S)^0}$ on $\mathcal{F}$ satisfies* **(A1)**–**(A12)** *then there exist a full conditional belief function $\varphi : \mathcal{P}(S)\times\mathcal{P}(S)^0 \to [0,1]$, a non-constant bounded utility function $u : X \to \mathbb{R}$, and a pessimism index function $\alpha : \mathcal{U} \to [0,1]$ such that, for every $H \in \mathcal{P}(S)^0$, defining for every $f \in \mathcal{F}$,*

$$
\underline{\Lambda}(f|H) = \oint_S \left[\int_{\mathcal{U}} \left(\alpha(B)\cdot \inf_{x\in B} u(x) + (1-\alpha(B))\cdot \sup_{x\in B} u(x)\right) f(s)(\mathrm{d}B)\right] \varphi(\mathrm{d}s|H),
$$

for every $f, g \in \mathcal{F}$ it holds that

$$
f \precsim_H g \Longleftrightarrow \underline{\Lambda}(f|H) \le \underline{\Lambda}(g|H).
$$

Moreover, φ is unique, u is unique up to positive linear transformations, and α is uniquely defined whenever $\inf_{x\in B} u(x) < \sup_{x\in B} u(x)$.

The detailed proof of the above result is reported in Appendix A. In such proof, axioms **(A1)**–**(A5)** are used, for every $H \in \mathcal{P}(S)^0$, to get a non-constant *affine* function $V_H : \mathbf{M}(\mathcal{U}) \to \mathbb{R}$ and a capacity $\varphi_H : \mathcal{P}(S) \to [0,1]$ such that the functional defined, for all $f \in \mathcal{F}$, as

$$\underline{\Lambda}_H(f) = \oint_S V_H(f(s))\varphi_H(\mathrm{d}s),$$

represents the preference $\precsim_H$. Affinity of V_H means that

$$V_H(\alpha\mu_1 + (1-\alpha)\mu_2) = \alpha V_H(\mu_1) + (1-\alpha)V_H(\mu_2),$$

for every $\mu_1, \mu_2 \in \mathbf{M}(\mathcal{U})$ and $\alpha \in [0,1]$. In particular, V_H is determined by the relation $\precsim_H$ on constant acts, thus it does not depend on "subjective" uncertainty.

In turn, axiom **(A10)** is used to get an integral expression of V_H through a non-constant bounded utility function $v_H : \mathcal{U} \to \mathbb{R}$ on menus. Then, axioms **(A11)** and **(A12)** are used to build a non-constant bounded utility function $u_H : X \to \mathbb{R}$ on consequences such that v_H satisfies internality with respect to u_H, i.e., for every $B \in \mathcal{U}$,

$$\inf_{x\in B} u_H(x) \leq v_H(B) \leq \sup_{x\in B} u_H(x).$$

Note that $\alpha_H(B) \in [0,1]$ is a pessimism index allowing to write $v_H(B)$ as the convex combination of the "worst" and "best" utility values in the menu B, that is

$$v_H(B) = \alpha_H(B) \cdot \inf_{x\in B} u_H(x) + (1-\alpha_H(B)) \cdot \sup_{x\in B} u_H(x).$$

Now, axiom **(A9)** is used to get the independence of V_H, v_H,u_H and α_H of the conditioning event $H \in \mathcal{P}(S)^0$. Thus, for all conditioning events we can take the same functions V, v, u and α, respectively.

Finally, axioms **(A6)**–**(A8)** are used to show that the function $\varphi : \mathcal{P}(S) \times \mathcal{P}(S)^0 \to [0,1]$ obtained setting, for every $E|H \in \mathcal{P}(S) \times \mathcal{P}(S)^0$,

$$\varphi(E|H) = \varphi_H(E),$$

is a full conditional belief function according to Definition 1.

The sketched construction shows that u and α (and, so, v and V) only depend on "objective" uncertainty, since they are determined by the relations on constant acts. In particular, u and α do not depend on "subjective" uncertainty which is encoded in the full conditional belief function $\varphi(\cdot|\cdot)$

on $\mathcal{P}(S)$. Hence, the three parameters of the model u, α and φ clearly separate the treatment of "objective" and "subjective" ambiguity. More precisely, "objective" ambiguity is dealt with a precise utility function u on consequences in X endowed with a pessimism index function α. By contrast, (conditional) "subjective" ambiguity is dealt with the belief function $\varphi(\cdot|H)$ on $\mathcal{P}(S)$, i.e., with the set of finitely additive probability measure $\mathbf{core}(\varphi(\cdot|H))$ on $\mathcal{P}(S)$.

The following proposition states that all axioms **(A1)**–**(A11)** are necessary to get a representation of $\{\precsim_H\}_{H\in\mathcal{P}(S)^0}$, through a conditional functional $\underline{\Lambda}(\cdot|\cdot)$ of the form discussed above.

Proposition 1. *If there exist a full conditional belief function $\varphi : \mathcal{P}(S) \times \mathcal{P}(S)^0 \to [0,1]$, a non-constant bounded utility function $u : X \to \mathbb{R}$, and a pessimism index function $\alpha : \mathcal{U} \to [0,1]$ such that the conditional functional $\underline{\Lambda}(\cdot|\cdot)$ defined as in* (3) *represents the family of relations $\{\precsim_H\}_{H\in\mathcal{P}(S)^0}$, then the family $\{\precsim_H\}_{H\in\mathcal{P}(S)^0}$ satisfies* **(A1)**–**(A11)**.

We stress that, axiom **(A12)** is not necessary as shown by the following example. On the other hand, axiom **(A12)** is vacuously satisfied if X is finite, thus in that case all axioms are necessary and sufficient. Thus, it follows that **(A12)** has a "regularity" role when the set of state is not finite.

Example 3. *Take $S = X = \mathbb{N} = \{1, 2, \ldots\}$, $u(x) = 1 - \frac{1}{x}$, α constantly equal to* 0 *on $\mathcal{U}$, and φ any full conditional belief function on $\mathcal{P}(S)$, we have that*

$$\underline{\Lambda}(\overline{\delta_X}|S) = \sup_{x\in X} u(x) = 1,$$

but there is no $x_2 \in X$ such that $\underline{\Lambda}(\overline{\delta_{\{x_2\}}}|S) = u(x_2) \geq 1$. Thus, we have that there is no $x_2 \in X$ such that $\overline{\delta_X} \precsim_S \overline{\delta_{\{x_2\}}}$ and axiom **(A12)** *is not satisfied.* ♦

The conditional functional $\underline{\Lambda}(\cdot|\cdot)$ can be specialized by incorporating a systematically pessimistic or optimistic attitude towards "objective" ambiguity. This consists in selecting an α which is constantly equal to 1 or 0, respectively, and this is possible adding a further axiom.

In case of a systematically pessimistic attitude we search for a conditional functional $\underline{\Phi}$ defined, for every $f \in \mathcal{F}$ and every $H \in \mathcal{P}(S)^0$, as

$$
\begin{aligned}
\underline{\Phi}(f|H) &= \oint_S \left[\int_{\mathcal{U}} \left(\inf_{x\in B} u(x)\right) f(s)(\mathrm{d}B)\right] \varphi(\mathrm{d}s|H) \\
&= \oint_S \left[\oint_X u(x)\varphi_{f(s)}(\mathrm{d}x)\right] \varphi(\mathrm{d}s|H) \\
&= \oint_S \left[\min_{\nu\in\mathbf{core}(\varphi_{f(s)})} \int_X u(x)\nu(\mathrm{d}x)\right] \varphi(\mathrm{d}s|H), \qquad (7)
\end{aligned}
$$

where $\varphi_{f(s)}$ is the belief function on $\mathcal{P}(X)$ induced by the finitely additive probability $f(s)$ on $\mathcal{P}(\mathcal{U})$. Thus, the internal integral reduces to a state-contingent "objective" lower expected utility. The following axiom is inspired to **Axiom 12** in [17].

(A13) Ignorance pessimism: For every $H \in \mathcal{P}(S)^0$, for every $A, B \in \mathcal{U}$, if for all $x \in A$, there exists $y \in B$ such that $\overline{\delta_{\{y\}}} \precsim_H \overline{\delta_{\{x\}}}$ then $\overline{\delta_B} \precsim_H \overline{\delta_A}$.

The above axiom requires that, for every pair of menus of consequences A, B and scenario H, if for every consequence x in A we can find a consequence y in B such that x selected with certainty is weakly preferred to y selected with certainty, then menu A selected with certainty must be weakly preferred to menu B selected with certainty.

Theorem 2. *If the family of relations $\{\precsim_H\}_{H\in\mathcal{P}(S)^0}$ on $\mathcal{F}$ satisfies* **(A1)**–**(A12)** *and* **(A13)** *then there exist a full conditional belief function $\varphi : \mathcal{P}(S) \times \mathcal{P}(S)^0 \to [0,1]$ and a non-constant bounded utility function $u : X \to \mathbb{R}$ such that, for every $H \in \mathcal{P}(S)^0$, defining for every $f \in \mathcal{F}$,*

$$
\underline{\Phi}(f|H) = \oint_S \left[\int_{\mathcal{U}} \left(\inf_{x\in B} u(x)\right) f(s)(\mathrm{d}B)\right] \varphi(\mathrm{d}s|H),
$$

for every $f, g \in \mathcal{F}$ it holds that

$$
f \precsim_H g \Longleftrightarrow \underline{\Phi}(f|H) \leq \underline{\Phi}(g|H).
$$

Moreover, φ is unique and u is unique up to positive linear transformations.

In case of a systematically optimistic attitude we search for a conditional functional $\underline{\Psi}$ defined, for every $f \in \mathcal{F}$ and every $H \in \mathcal{P}(S)^0$, as

$$
\begin{aligned}
\underline{\Psi}(f|H) &= \oint_S \left[\int_{\mathcal{U}} \left(\sup_{x\in B} u(x)\right) f(s)(\mathrm{d}B)\right] \varphi(\mathrm{d}s|H) \\
&= \oint_S \left[\oint_X u(x)\psi_{f(s)}(\mathrm{d}x)\right] \varphi(\mathrm{d}s|H) \\
&= \oint_S \left[\max_{\nu\in\mathbf{core}(\varphi_{f(s)})} \int_X u(x)\nu(\mathrm{d}x)\right] \varphi(\mathrm{d}s|H), \qquad (8)
\end{aligned}
$$

where $\psi_{f(s)}$ is the dual of the belief function $\varphi_{f(s)}$. Thus, the internal integral reduces to a state-contingent "objective" upper expected utility. The following axiom is inspired to **Axiom 12** in [17].

(A13') Ignorance optimism: For every $H \in \mathcal{P}(S)^0$, for every $A, B \in \mathcal{U}$, if for all $y \in B$, there exists $x \in A$ such that $\overline{\delta_{\{y\}}} \precsim_H \overline{\delta_{\{x\}}}$ then $\overline{\delta_B} \precsim_H \overline{\delta_A}$.

The above axiom requires that, for every pair of menus of consequences A, B and scenario H, if for every consequence y in B we can find a consequence x in A such that x selected with certainty is weakly preferred to y selected with certainty, then menu A selected with certainty must be weakly preferred to menu B selected with certainty.

Theorem 3. *If the family of relations $\{\precsim_H\}_{H\in\mathcal{P}(S)^0}$ on $\mathcal{F}$ satisfies* **(A1)**–**(A12)** *and* **(A13')** *then there exist a full conditional belief function $\varphi : \mathcal{P}(S) \times \mathcal{P}(S)^0 \to [0,1]$ and a non-constant bounded utility function $u : X \to \mathbb{R}$ such that, for every $H \in \mathcal{P}(S)^0$, defining for every $f \in \mathcal{F}$,*

$$
\underline{\Psi}(f|H) = \oint_S \left[\int_{\mathcal{U}} \left(\sup_{x\in B} u(x)\right) f(s)(\mathrm{d}B)\right] \varphi(\mathrm{d}s|H),
$$

for every $f, g \in \mathcal{F}$ it holds that

$$
f \precsim_H g \Longleftrightarrow \underline{\Psi}(f|H) \leq \underline{\Psi}(g|H).
$$

Moreover, φ is unique and u is unique up to positive linear transformations.

The following proposition states that axioms **(A1)**–**(A11)** and **(A13)** [**(A13')**] are necessary to get a representation of $\{\precsim_H\}_{H\in\mathcal{P}(S^0)}$ through a conditional functional $\underline{\Phi}(\cdot|\cdot)$ [$\underline{\Psi}(\cdot|\cdot)$]. Also in this case, axiom **(A12)** is only sufficient, in general, but becomes necessary when X is finite.

Proposition 2. *If there exist a full conditional belief function $\varphi : \mathcal{P}(S) \times \mathcal{P}(S)^0 \to [0,1]$ and a non-constant bounded utility function $u : X \to \mathbb{R}$ such that the conditional functional $\underline{\Phi}(\cdot|\cdot)$ [$\underline{\Psi}(\cdot|\cdot)$] defined as in* (7) *[*(8)*] represents the family of relations $\{\precsim_H\}_{H\in\mathcal{P}(S)^0}$, then the family $\{\precsim_H\}_{H\in\mathcal{P}(S)^0}$ satisfies* **(A1)**–**(A11)** *and* **(A13)** *[***(A13')***].*

We point out that a family of preference relations cannot satisfy simultaneously **(A1)–(A12)**, **(A13)** and **(A13')**. Indeed, this would lead to a constant utility function $u : X \to \mathbb{R}$ but this is in contrast with axiom **(A5)**.

4.1. A systematically optimistic attitude towards subjective uncertainty

The three conditional functionals $\underline{\Lambda}(\cdot|\cdot)$, $\underline{\Phi}(\cdot|\cdot)$ and $\underline{\Psi}(\cdot|\cdot)$ introduced so far assume a systematically pessimistic attitude in "subjective" uncertainty evaluations. A completely opposite behavior is encoded in the conditional functionals defined, for every $f \in \mathcal{F}$ and $H \in \mathcal{P}(S)^0$, as

$$\overline{\Lambda}(f|H) = \oint_S \left[\int_{\mathcal{U}} \left(\alpha(B)\cdot \inf_{x\in B} u(x) + (1-\alpha(B))\cdot \sup_{x\in B} u(x)\right) f(s)(\mathrm{d}B)\right] \psi(\mathrm{d}s|H), \tag{9}$$

$$\overline{\Phi}(f|H) = \oint_S \left[\int_{\mathcal{U}} \left(\inf_{x\in B} u(x)\right) f(s)(\mathrm{d}B)\right] \psi(\mathrm{d}s|H), \tag{10}$$

$$\overline{\Psi}(f|H) = \oint_S \left[\int_{\mathcal{U}} \left(\sup_{x\in B} u(x)\right) f(s)(\mathrm{d}B)\right] \psi(\mathrm{d}s|H), \tag{11}$$

where $\psi : \mathcal{P}(S) \times \mathcal{P}(S)^0 \to [0,1]$ is a full conditional plausibility function according to Definition 2.

We focus on the conditional functional $\overline{\Lambda}(\cdot|\cdot)$, of which $\overline{\Phi}(\cdot|\cdot)$ and $\overline{\Psi}(\cdot|\cdot)$ are particular cases. Also in this case, the internal integral gives rise to a utility function on menus that captures the attitude of the decision maker towards "objective" ambiguity, relying on the function α. Then, we take a "subjective" conditional Choquet expectation with respect to the conditional plausibility function $\psi(\cdot|H)$, defined on states. Again, due to Definition 2, $\overline{\Lambda}(f|H)$ is well-defined even though the *ex ante* plausibility function $\psi(H|S)$ is equal to 0. Further, the external conditional Choquet expectation is actually an upper conditional expectation computed with respect to the core of the dual belief function of $\psi(\cdot|H)$. In other terms, a decision maker deciding according to the conditional functional $\overline{\Lambda}(\cdot|\cdot)$ has a systematically optimistic attitude towards "subjective" uncertainty.

Further, recalling (4), we can simplify computations by writing

$$\overline{\Lambda}(f|H) = \oint_S V(f(s))\psi(\mathrm{d}s|H). \tag{12}$$

To get representation theorems for conditional functionals $\overline{\Lambda}(\cdot|\cdot)$, $\overline{\Phi}(\cdot|\cdot)$ and $\overline{\Psi}(\cdot|\cdot)$ analogous to Theorems 1, 2 and 3 it is sufficient to replace axiom **(A6)** with

(A6') Complete alternance: For every $H \in \mathcal{P}(S)^0$, for every $k \geq 2$ and every $E_1, \ldots, E_k \in \mathcal{P}(S)$, if $x_*, x^* \in X$ are such that $x_* <_H x^*$ and for all $\emptyset \neq I \subseteq \{1, \ldots, k\}$

$$\mathbf{1}_{\bigcap_{i=1}^k E_i} \sim_H \overline{\mu^{\cap}_{\{1,\ldots,k\}}} \text{ and } \mathbf{1}_{\bigcup_{i \in I} E_i} \sim_H \overline{\mu^{\cup}_I}$$

with $\mu^{\cap}_{\{1,\ldots,k\}}, \mu^{\cup}_I \in \mathbf{M}(\mathcal{U})$, then the acts

$$g = \frac{1}{2^{k-1}} \left(\overline{\mu^{\cap}_{\{1,\ldots,k\}}} + \sum_{\substack{\emptyset \neq I \subseteq \{1,\ldots,k\} \\ |I| \text{ even}}} \overline{\mu^{\cup}_I} \right) \text{ and } h = \frac{1}{2^{k-1}} \left(\sum_{\substack{\emptyset \neq I \subseteq \{1,\ldots,k\} \\ |I| \text{ odd}}} \overline{\mu^{\cup}_I} \right)$$

are such that

$$g \precsim_H h.$$

Also for axiom **(A6')** we have that requiring the property to hold for every $2 \leq k \leq n$ with a fixed $n \geq 2$ we get a conditional n-alternating capacity. In particular, for $n = 2$ the corresponding axiom turns out to be equivalent, in presence of other axioms, to the *uncertainty appeal* axiom of Schmeidler [43], for every $H \in \mathcal{P}(S)^0$.

4.2. A Bayesian decision maker

So far we have assumed that "subjective" uncertainty is not necessarily additive. Now, we aim to model a Bayesian decision maker whose "subjective" uncertainty is unambiguous and can be expressed by a full conditional probability. We then consider the following strengthening of axiom **(A3)**

(A3') Independence: For every $H \in \mathcal{P}(S)^0$, for every $f, g, h \in \mathcal{F}$ and for every $\alpha \in (0, 1)$

$$f \precsim_H g \Longleftrightarrow \alpha f + (1-\alpha)h \precsim_H \alpha g + (1-\alpha)h.$$

If in the previous representation theorems (see [43]) we substitute axiom **(A3)** with **(A3')**, then $\varphi(\cdot|\cdot) = \psi(\cdot|\cdot) = P(\cdot|\cdot)$ where $P(\cdot|\cdot)$ is a full conditional probability on $\mathcal{P}(S)$, that is we have a decision maker with unambiguous "subjective" beliefs. We stress that, in presence of other axioms, axiom **(A3')** alone implies the three axioms **(A3)**, **(A6)** and **(A6')**. Further, still in presence of other axioms, **(A3')** also implies the following (see [47] for an analysis of dynamic consistency)

(A14) Strong dynamic consistency: For every $f, g \in \mathcal{F}$ and every $H, K \in \mathcal{P}(S)^0$, if $f \precsim_H g$, $f \precsim_K g$, and $H \cap K = \emptyset$ then $f \precsim_{H \cup K} g$, where the last preference is strict if both the first two are.

In turn, property **(A14)** implies the weaker

(A14') Weak dynamic consistency: For every $f, g \in \mathcal{F}$ and every $H, K \in \mathcal{P}(S)^0$, if $f \precsim_H g$, $f \precsim_K g$, and $H \cap K = \emptyset$ then $f \precsim_{H \cup K} g$.

We stress that axiom **(A14')** is generally not satisfied (and so neither is **(A14)**) by conditional preferences representable by a conditional functional $\underline{\Lambda}(\cdot|\cdot)$ or $\overline{\Lambda}(\cdot|\cdot)$ as the following example shows. In other terms, in presence of other axioms, the combination of **(A3)** and **(A6)** or **(A3)** and **(A6')** does not imply **(A14')** (and so neither **(A14)**).

Example 4. *Let $S = \{s_1, s_2, s_3\}$ and $X = \{x_1, x_2, x_3\}$ and consider the full conditional belief function on $\mathcal{P}(S)$ such that*

$\mathcal{P}(S)$	$\emptyset$	$\{s_1\}$	$\{s_2\}$	$\{s_3\}$	$\{s_1, s_2\}$	$\{s_1, s_3\}$	$\{s_2, s_3\}$	S
$\varphi(\cdot\|S)$	0	0.2	0.2	0.1	0.4	0.6	0.3	1

and, for every $E|H \in \mathcal{P}(S) \times \mathcal{P}(S)^0$, $\varphi(E|H) = \frac{\varphi(E \cap H|S)}{\varphi(H|S)}$.

Let α be an arbitrary function $\alpha : \mathcal{U} \to [0, 1]$ and u be such that $u(x_1) = 0$, $u(x_2) = 1$, $u(x_3) = 2$. Consider the acts f and g defined as

S	s_1	s_2	s_3
f	$\delta_{\{x_2\}}$	$\delta_{\{x_2\}}$	$\delta_{\{x_2\}}$
g	$\delta_{\{x_3\}}$	$\delta_{\{x_3\}}$	$\delta_{\{x_1\}}$

Since, for $i = 1, 2, 3$, it holds that

$$\int_{\mathcal{U}} \left(\alpha(B) \cdot \inf_{x \in B} u(x) + (1 - \alpha(B)) \cdot \sup_{x \in B} u(x) \right) \delta_{\{x_i\}}(\mathrm{d}B) = u(x_i),$$

for $H = \{s_2, s_3\}$ and $H^c = \{s_1\}$, simple computations show that

$$\begin{aligned} \underline{\Lambda}(f|H) = 1 &\quad < \quad 1.\overline{3} = \underline{\Lambda}(g|H), \\ \underline{\Lambda}(f|H^c) = 1 &\quad < \quad 2 = \underline{\Lambda}(g|H^c), \\ \underline{\Lambda}(f|S) = 1 &\quad > \quad 0.8 = \underline{\Lambda}(g|S), \end{aligned}$$

hence the family of preference relations induced by $\underline{\Lambda}(\cdot|\cdot)$ does not satisfy **(A14')**.

Now, for every $E \in \mathcal{P}(S)$, define $\psi(E|S) = 1 - \varphi(E^c|S)$ that is

$\mathcal{P}(S)$	$\emptyset$	$\{s_1\}$	$\{s_2\}$	$\{s_3\}$	$\{s_1, s_2\}$	$\{s_1, s_3\}$	$\{s_2, s_3\}$	S
$\psi(\cdot\|S)$	0	0.7	0.4	0.6	0.9	0.8	0.8	1

and, for every $E|H \in \mathcal{P}(S) \times \mathcal{P}(S)^0$, $\psi(E|H) = \frac{\psi(E\cap H|S)}{\psi(H|S)}$. The function $\psi(\cdot|\cdot)$ is a conditional plausibility function according to Definition 2. Take α, u as before, and let $\mu = 0.8 \cdot \delta_{\{x_2\}} + 0.2 \cdot \delta_{\{x_1\}}$. Consider the act h defined as

S	s_1	s_2	s_3
h	$\delta_{\{x_1\}}$	$\delta_{\{x_3\}}$	μ

Since it holds that

$$\int_{\mathcal{U}} \left(\alpha(B) \cdot \inf_{x\in B} u(x) + (1-\alpha(B)) \cdot \sup_{x\in B} u(x)\right) \mu(\mathrm{d}B) = 0.8,$$

referring to $\overline{\Lambda}(\cdot|\cdot)$ and taking $K = \{s_1, s_2\}$ and $K^c = \{s_3\}$, we have that

$$\begin{aligned} \overline{\Lambda}(h|K) = 0.\overline{8} \quad &< \quad 1 = \overline{\Lambda}(f|K), \\ \overline{\Lambda}(h|K^c) = 0.8 \quad &< \quad 1 = \overline{\Lambda}(f|K^c), \\ \overline{\Lambda}(h|S) = 1.12 \quad &> \quad 1 = \overline{\Lambda}(f|S), \end{aligned}$$

hence the family of preference relations induced by $\overline{\Lambda}(\cdot|\cdot)$ does not satisfy **(A14')**.

This example further shows that the violation of **(A14')** *both for $\underline{\Lambda}(\cdot|\cdot)$ and $\overline{\Lambda}(\cdot|\cdot)$ is not necessarily due to events with null unconditional measure.* ♦

We point out that, even though axiom **(A14')** (and so **(A14)**) may fail to hold for the entire family of relations $\{\precsim_H\}_{H\in\mathcal{P}(S)^0}$, it can happen that **(A14')** (or even **(A14)**) holds for all the relations in a subfamily $\{\precsim_H\}_{H\in\mathcal{H}}$, where $\mathcal{H} \subseteq \mathcal{P}(S)^0$ is a non-empty set closed with respect to finite unions. For instance, this is the case if $\varphi(\cdot|H)$ is a finitely additive probability and $\mathcal{H} = \{K \in \mathcal{P}(S)^0 : K \subseteq H, \varphi(K|H) > 0\}$. In particular, if $\varphi(\cdot|\cdot)$ is a full conditional probability on $\mathcal{P}(S)$ then the entire family of relations $\{\precsim_H\}_{H\in\mathcal{P}(S)^0}$ satisfies **(A14)**.

It is important to notice that, contrary to [47], axiom **(A14)** asks that both preferences $f \prec_H g$ and $f \prec_K g$ are strict for implying that $f \prec_{H\cup K} g$ is strict. Indeed, even restricting to conditional probability for expressing "subjective" uncertainty, it could happen that $f \sim_H g$, $f \prec_K g$ and $f \sim_{H\cup K} g$, as the following example shows. This is essentially due to the fact that we allow situations in which $P(H|H \cup K) = 0$ or $P(K|H \cup K) = 0$.

Example 5. *Let $S = \{s_1, s_2, s_3, s_4\}$ and $X = \{x_1, x_2, x_3\}$ and consider the full conditional probability $P(\cdot|\cdot)$ on $\mathcal{P}(S)$, where $H = \{s_1, s_2\}$ and $H^c = \{s_3, s_4\}$, such that*

S	s_1	s_2	s_3	s_4
$P(\cdot\|S)$	0.5	0.5	0	0
$P(\cdot\|H^c)$	0	0	0.5	0.5

while all other values $P(A|B)$ can be determined through ratios taking, respectively, $P(\cdot|S)$ if $P(B|S) > 0$ and $P(\cdot|H^c)$ otherwise.

Let α be an arbitrary function $\alpha : \mathcal{U} \to [0,1]$ and u be such that $u(x_1) = 0$, $u(x_2) = 1$, $u(x_3) = 2$. Consider the acts f and g defined as

S	s_1	s_2	s_3	s_4
f	$\delta_{\{x_2\}}$	$\delta_{\{x_2\}}$	$\delta_{\{x_1\}}$	$\delta_{\{x_2\}}$
g	$\delta_{\{x_2\}}$	$\delta_{\{x_2\}}$	$\delta_{\{x_3\}}$	$\delta_{\{x_1\}}$

Since, for $i = 1, 2, 3$, it holds that

$$\int_{\mathcal{U}} \left(\alpha(B) \cdot \inf_{x \in B} u(x) + (1 - \alpha(B)) \cdot \sup_{x \in B} u(x) \right) \delta_{\{x_i\}}(\mathrm{d}B) = u(x_i),$$

simple computations show that

$$\begin{aligned} \underline{\Lambda}(f|H) = 1 &\;=\; 1 = \underline{\Lambda}(g|H), \\ \underline{\Lambda}(f|H^c) = 0.5 &\;<\; 1 = \underline{\Lambda}(g|H^c), \\ \underline{\Lambda}(f|S) = 1 &\;=\; 1 = \underline{\Lambda}(g|S), \end{aligned}$$

hence the family of preference relations induced by $\underline{\Lambda}(\cdot|\cdot)$ is such that $f \sim_H g$, $f \prec_{H^c} g$ and $f \sim_S g$. ♦

5. A motivating decision problem (continuation)

We get back to the decision problem introduced in Section 2. We have that, for $i = 1, 2, 3$, the class of probabilities $\mathbf{P}^i$ on $\mathcal{P}(X)$ gives rise to a probability measure μ_i on $\mathcal{P}(\mathcal{U})$, where $\mathcal{U} = \mathcal{P}(X) \setminus \{\emptyset\}$. In particular, the distribution m_i of μ_i on $\mathcal{U}$ coincides with the Möbius inverse of a belief function φ_i on $\mathcal{P}(X)$ (see, e.g., [25]). Denoting $x_1 = -1000$, $x_2 = 0$, $x_3 = 1000$, $A_i = \{x_i\}$ and $A_{ij} = \{x_i, x_j\}$, we have

$\mathcal{U}$	A_1	A_2	A_3	A_{12}	A_{13}	A_{23}	X
m_1	0	0.5	0	0	0.5	0	0
m_2	0	0	0.3	0.7	0	0	0
m_3	0.3	0	0	0	0	0.7	0

Notice that each μ_i gives also rise to a plausibility function ψ_i on $\mathcal{P}(X)$ that is dual to φ_i.

Hence, identifying instrument i with the probability measure μ_i on $\mathcal{P}(\mathcal{U})$, the investment strategies f, g, h, l introduced in Section 2 reduce to the generalized Anscombe-Aumann acts

S	s_1	s_2	s_3	s_4
f	μ_2	μ_2	μ_2	μ_2
g	μ_1	μ_3	μ_3	μ_3
h	μ_3	μ_3	μ_2	μ_2
l	μ_3	μ_3	μ_3	μ_3

where, in particular, $f = \overline{\mu_2}$ and $l = \overline{\mu_3}$.

We stress that, assuming that the agent is a profit maximizer consists in selecting a strictly increasing utility function $u : X \to [0,1]$, that is $u(x_1) < u(x_2) < u(x_3)$. Further, assuming that the agent is systematically pessimistic in coping with "objective" ambiguity consists in taking the pessimism index function $\alpha : \mathcal{U} \to [0,1]$ constantly equal to 1.

For $C = \{s_1, s_2\}$ and $C^c = \{s_3, s_4\}$, given the preference pattern introduced in Section 2

$$f \sim_S g, \quad f \sim_S h, \quad f \prec_S l, \quad f \prec_C g, \quad \text{and} \quad f \prec_{C^c} g,$$

our aim is to show its representability with a conditional functional $\underline{\Lambda}(\cdot|\cdot)$, that, due to the assumption on α, reduces to $\underline{\Phi}(\cdot|\cdot)$.

Notice that statements $f \prec_S l$ and $f \sim_S h$ identify the event C as a "null" event for the agent. Indeed, if $\varphi(\cdot|\cdot)$ is a full conditional belief function on $\mathcal{P}(S)$ according to Definition 1, we have

$$\begin{aligned}
\underline{\Phi}(f|S) = \underline{\Phi}(\overline{\mu_2}|S) = v_2 \quad &< \quad v_3 = \underline{\Phi}(\overline{\mu_3}|S) = \underline{\Phi}(l|S), \\
\underline{\Phi}(f|S) = \underline{\Phi}(\overline{\mu_2}|S) = v_2 \quad &= \quad (v_3 - v_2) \cdot \varphi(C|S) + v_2 = \underline{\Phi}(h|S),
\end{aligned}$$

which implies that $\varphi(C|S) = 0$. Moreover, the statements $f \sim_S g$, $f \prec_C g$, $f \prec_{C^c} g$ violate axiom **(A14)**, thus $\varphi(\cdot|\cdot)$ cannot be a full conditional probability on $\mathcal{P}(S)$.

Denote $B_i = \{s_i\}$, $B_{ij} = \{s_i, s_j\}$, $B_{ijk} = \{s_i, s_j, s_k\}$. The above preferences can be represented taking $u(x_1) = 0$, $u(x_2) = 1$, $u(x_3) = 2$ together with the full conditional belief function $\varphi(\cdot|\cdot)$ on $\mathcal{P}(S)$ such that

$\mathcal{P}(S)$	$\emptyset$	B_1	B_2	B_3	B_4	B_{12}	B_{13}	B_{14}	B_{23}	B_{24}	B_{34}	B_{123}	B_{124}	B_{134}	B_{234}	S
$\varphi(\cdot\|S)$	0	0	0	0.25	0.25	0	0.25	0.25	0.25	0.25	0.5	0.25	0.25	0.5	0.5	1
$\varphi(\cdot\|C)$	0	0.25	0.75	0	0	1	0.25	0.25	0.75	0.75	0	1	1	0.25	0.75	1
$\varphi(\cdot\|C^c)$	0	0	0	0.5	0.5	0	0.5	0.5	0.5	0.5	1	0.5	0.5	1	1	1

while for every $E|H \in \mathcal{P}(S) \times \mathcal{P}(S)^0$ with $H \neq C$ and $H \neq C^c$, we set

$$\varphi(E|H) = \begin{cases} \frac{\varphi(E\cap H|C)}{\varphi(H|C)}, & \text{if } H \subseteq C, \\ \frac{\varphi(E\cap H|S)}{\varphi(H|S)}, & \text{otherwise.} \end{cases}$$

Notice that both $\varphi(\cdot|C)$ and $\varphi(\cdot|C^c)$ are probability measures on $\mathcal{P}(S)$, while $\varphi(\cdot|S)$ is not. The choice of the u above implies that

$$\int_{\mathcal{U}} \left(\inf_{x \in B} u(x) \right) \mu_i(\mathrm{d}B) = \begin{cases} 0.5 & \text{if } i = 1, \\ 0.6 & \text{if } i = 2, \\ 0.7 & \text{if } i = 3. \end{cases}$$

so, by (4), we get

S	s_1	s_2	s_3	s_4
$V(f(s))$	0.6	0.6	0.6	0.6
$V(g(s))$	0.5	0.7	0.7	0.7
$V(h(s))$	0.7	0.7	0.6	0.6
$V(l(s))$	0.7	0.7	0.7	0.7

Finally, simple computations show that

$$\begin{aligned} \underline{\Phi}(l|S) = \underline{\Phi}(g|C^c) &= 0.7, \\ \underline{\Phi}(g|C) &= 0.65, \\ \underline{\Phi}(f|S) = \underline{\Phi}(g|S) = \underline{\Phi}(h|S) = \underline{\Phi}(f|C) = \underline{\Phi}(f|C^c) &= 0.6, \end{aligned}$$

thus all the stated preferences are represented.

6. Conclusions

In this paper we assume that a decision maker is able to provide a family of conditional preference relations on acts mapping states of the world to finitely additive probabilities on menus of consequences. In particular, the latter can be seen as generalized lotteries conveying a form of "objective" ambiguity that reduces to belief/plausibility functions.

We first assume "subjective" uncertainty is ambiguous and modeled by a full conditional belief function. In this case, we propose a system of axioms that assure the representability of preferences by a "subjective" conditional Choquet expectation of a state-contingent "objective" mixture of convex combinations of infima and suprema of utility on menus. Then, we axiomatise a systematically pessimistic/optimistic attitude towards "objective" ambiguity. The last two models generalize to arbitrary sets of consequences and states of the world (besides incorporating "subjective" ambiguity), those introduced in [40] that, in turn, generalize the conditional version of the Anscombe-Aumann model given in [36] by introducing "objective" ambiguity.

This leads to consider a systematically optimistic approach to "subjective" ambiguity by referring to a conditional plausibility function. We stress that all the models we introduce allow conditioning on "null" (i.e., "unexpected") events, like a pandemic, characterized by a "null" *ex ante* measure.

Here we consider comparisons of acts only conditionally on the same event H, i.e., $f \precsim_H g$: a possible extension of our models could be to consider comparisons under different conditioning events $(f, H) \precsim (g, K)$ as done by Fishburn in [16] without taking care of ambiguity.

The main similarity of our models is with the *Choquet expected utility (CEU)* model by Schmeidler [42] and the *maximin expected utility (MEU)* model by Gilboa and Schmeidler [22]. Nevertheless, our models differ from them since they do not cope with conditioning and with "objective" ambiguity in the form addressed in the present paper. In particular, the form of "objective" ambiguity dealt with in the paper is of the same type of *generalized lotteries* due to Jaffray [30]. Still referring to the classical finite Anscombe-Aumann formulation without conditioning and "objective" ambiguity, in [19] Giang proposes a "subjective" foundation of possibility measures, that is maxitive plausibility functions.

In [48], limiting to a finite S and X, Vierø considers acts mapping states of the world to non-empty compact convex polyhedral sets of probability measures on X. If we restrict to the finite setting as well and, for every act f, we refer to $\mathbf{core}(f(s))$ for $s \in S$, our decision objects turn out to be similar to those in [48]. Nevertheless, in [48] the author takes into account unconditional preferences only and refers to a representation functional in which "subjective" uncertainty is modeled with a probability measure. Further, she considers a state-contigent convex combination of expected utilities computed with respect to the "best" and "worst" lotteries in each set of probabilities given by an act on a state. Hence, the numerical reference model in [48] is quite different from ours.

In the paper [26] Gul and Pesendorfer consider Savage acts mapping the states of the world to a closed interval X of real numbers, expressing monetary payoffs. Limiting to an unconditional decision problem, the authors provide a characterization of a preference relation $\precsim$ on the set of all such acts. Such characterization refers to a numeric representation of $\precsim$, namely *expected uncertainty utility (EUU)*, parametrized by an interval utility function u defined on the set of all closed sub-intervals of X and a countably additive complete and non-atomic prior probability μ on a suitable σ-algebra on the states. In detail, the proposed representation maps each act f to the expectation with respect to μ of u evaluated on the envelope of f. The main result in [26] requires an infinite state space. For this, the authors restrict to a finite state space S (seen as a partition of the original state space) and consider discrete acts defined on S. They show that the original prior μ induces a probability π on $\mathcal{P}(S)^0$. In this finite case, they get a representation that consists in an expectation with respect to π of the interval utility u evaluated on the infimum and supremum of f, varying $A \in \mathcal{P}(S)^0$. The probability π turns out to be the Möbius inverse of a belief function on $\mathcal{P}(S)$, thus the model in [26] provides a subjective foundation for the Dempster-Shafer theory of evidence. Our models work in a different setting with respect to [26] and consider conditional belief/plausibility functions according to Definitions 1 and 2 as primitive concepts for modeling "subjective" uncertainty.

The issue of modeling updating for ambiguous beliefs in terms of preferences has been considered by Pacheco Pires in [38], where the *full Bayesian conditioning rule* is taken as reference. Still referring to a generalized Bayesian conditioning rule, Ghirardato, Maccheroni and Marinacci in [18] deal with the issue of dynamic consistency advocating that such condition should hold only on acts that are not subject to "subjective" ambiguity.

We point out that, as already discussed in Section 4, the models proposed in this paper can be generalized by weakening axiom **(A6)** [**(A6')**] so as to obtain a "subjective" full conditional uncertainty measure $\varphi(\cdot|\cdot)$ on $\mathcal{P}(S)$ that is only 2-monotone [2-alternating]. Further, replacing axioms **(A3)** and **(A6)** with a straightforward adaptation of *C-independence* and *uncertainty aversion [appeal]* axioms of [22], the "subjective" full conditional uncertainty measure $\varphi(\cdot|\cdot)$ on $\mathcal{P}(S)$ is substituted by a family $\{\mathcal{C}_H\}_{H\in\mathcal{P}(S)^0}$ of non-empty closed convex sets of finitely additive probability measures on $\mathcal{P}(S)$. The family $\{\mathcal{C}_H\}_{H\in\mathcal{P}(S)^0}$ is used to compute a lower [upper] expectation, locally on every $H \in \mathcal{P}(S)^0$. Moreover, in presence of other axioms, $\{\mathcal{C}_H\}_{H\in\mathcal{P}(S)^0}$ is such that the corresponding family of lower [upper] envelopes obeys to a form of focusing and chain rule.

An interesting line for future research is the modeling of "objective" generalized lotteries in the framework of 2-monotone [2-alternating] capacities. Referring to results in [24, 35], in this case we should substitute the set $\mathbf{M}(\mathcal{U})$ with a (suitable) set of signed finitely additive measures.

Appendix A. Proofs of results

Proof of Theorem 1. For every $H \in \mathcal{P}(S)^0$, axioms **(A1)**–**(A5)** imply the existence of a non-constant affine function $V_H : \mathbf{M}(\mathcal{U}) \to \mathbb{R}$ and a capacity $\varphi_H : \mathcal{P}(S) \to [0,1]$ such that, for every $f, g \in \mathcal{F}_{\text{simple}}$, it holds that

$$f \precsim_H g \Longleftrightarrow \oint_S V_H(f(s))\varphi_H(\mathrm{d}s) \leq \oint_S V_H(g(s))\varphi_H(\mathrm{d}s), \tag{A.1}$$

where V_H is unique up to a positive linear transformation and φ_H is unique. The proof of this claim follows by the proof of the main Theorem in [43]. Notice that in [43] acts are functions mapping S to the set Y of finite support probability measures defined on $\mathcal{P}(X)$, nevertheless, only the fact that Y is a mixture set is used in such proof. Hence, our claim simply follows by taking $Y = \mathbf{M}(\mathcal{U})$.

Let $\trianglelefteq_H$ be the relation induced by $\precsim_H$ on $\mathbf{M}(\mathcal{U})$ through constant acts with asymmetric part $\triangleleft_H$. The function V_H is such that

- $\mu_1 \trianglelefteq_H \mu_2 \Longleftrightarrow V_H(\mu_1) \leq V_H(\mu_2)$, for every $\mu_1, \mu_2 \in \mathbf{M}(\mathcal{U})$;
- $V_H(\alpha\mu_1 + (1-\alpha)\mu_2) = \alpha V_H(\mu_1) + (1-\alpha)V_H(\mu_2)$, for every $\mu_1, \mu_2 \in \mathbf{M}(\mathcal{U})$ and $\alpha \in [0,1]$;

moreover, V_H is unique up to a positive linear transformation.

Axiom **(A10)** implies that, defining for every $B \in \mathcal{U}$

$$v_H(B) = V_H(\delta_B),$$

the function $v_H : \mathcal{U} \to \mathbb{R}$ is bounded by virtue of Lemma 10.5 in [15], and furthermore $V_H(\mu) = \int_{\mathcal{U}} v_H(B)\mu(\mathrm{d}B)$, for every $\mu \in \mathbf{M}(\mathcal{U})$, by virtue of Theorem 10.1 in [15]. Moreover, since $\mathbf{M}(\mathcal{U})$ is the set of all finitely additive probabilities on $\mathcal{P}(\mathcal{U})$, there exist $\mu_*, \mu^* \in \mathcal{P}(\mathcal{U})$ such that

$$V_H(\mu_*) = \inf_{B \in \mathcal{U}} v_H(B) \quad \text{and} \quad V_H(\mu^*) = \sup_{B \in \mathcal{U}} v_H(B).$$

In turn, since for every $\mu \in \mathbf{M}(\mathcal{U})$ we have $V_H(\mu_*) \leq V_H(\mu) \leq V_H(\mu^*)$ and V_H represents $\trianglelefteq_H$, for every $f \in \mathcal{F}$ it follows that $\mu_* \trianglelefteq_H f(s) \trianglelefteq_H \mu^*$ for all $s \in S$. Hence, by the Corollary in [43] we have that (A.1) holds for all $f, g \in \mathcal{F}$.

We show that axioms **(A11)** and **(A12)** imply that, for every $B \in \mathcal{U}$, it holds that

$$\inf_{x \in B} v_H(\{x\}) \leq v_H(B) \leq \sup_{x \in B} v_H(\{x\}).$$

At this aim, suppose $v_H(B) > \sup_{x\in B} v_H(\{x\}) = \alpha$. By axiom **(A12)** there exists $x_2 \in X$ such that $\delta_B \trianglelefteq_H \delta_{\{x_2\}}$ implying that, since V_H represents $\trianglelefteq_H$, $v_H(\{x_2\}) \geq v_H(B) > \alpha$. Let $y \in B$ be such that $v_H(\{y\}) \leq \alpha < v_H(B) \leq v_H(\{x_2\})$. Since V_H is affine, there exists $\beta \in [0,1]$ such that

$$\begin{aligned}\beta v_H(\{x_2\}) + (1-\beta) v_H(\{y\}) &= \beta V_H(\delta_{\{x_2\}}) + (1-\beta) V_H(\delta_{\{y\}}) \\ &= V_H(\beta\delta_{\{x_2\}} + (1-\beta)\delta_{\{y\}}) \in (\alpha, v_H(B)).\end{aligned}$$

Moreover, the belief functions $\varphi_{\delta_{\{x_2\}}}, \varphi_{\delta_{\{y\}}}$ induced by $\delta_{\{x_2\}}, \delta_{\{y\}}$ on $\mathcal{P}(X)$ are finitely additive probabilities and the same holds for the belief function φ_μ induced on $\mathcal{P}(X)$ by $\mu = \beta\delta_{\{x_2\}} + (1-\beta)\delta_{\{y\}}$. Since $V_H(\mu) > \alpha = \sup_{x\in B} v_H(\{x\})$ it follows that $V_H(\mu) > V_H(\delta_{\{x\}})$ for every $x \in B$ and axiom **(A11)** implies

$$\delta_B \trianglelefteq_H \mu,$$

so, it cannot be $\mu \triangleleft_H \delta_B$ reaching a contradiction since

$$V_H(\mu) < V_H(\delta_B) \Longrightarrow \mu \triangleleft_H \delta_B.$$

The other inequality is proved analogously.

Hence, for every $B \in \mathcal{U}$, there exists $\alpha_H(B) \in [0,1]$ such that

$$v_H(B) = \alpha_H(B) \cdot \inf_{x\in B} v_H(\{x\}) + (1-\alpha_H(B)) \cdot \sup_{x\in B} v_H(\{x\}),$$

where $\alpha_H(B)$ is unique whenever $\inf_{x\in B} v_H(\{x\}) < \sup_{x\in B} v_H(\{x\})$, and is arbitrary otherwise. Making B vary we get a function $\alpha_H : \mathcal{U} \to [0,1]$.

Now, defining $u_H : X \to \mathbb{R}$ setting, for every $x \in X$,

$$u_H(x) = v_H(\{x\}),$$

it follows that

$$V_H(\mu) = \int_{\mathcal{U}} \left(\alpha_H(B) \cdot \inf_{x\in B} u_H(x) + (1-\alpha_H(B)) \cdot \sup_{x\in B} u_H(x)\right) \mu(\mathrm{d}B),$$

for every $\mu \in \mathbf{M}(\mathcal{U})$. Moreover, u_H is non-constant, bounded and unique up to a positive linear transformation since V_H is.

Axiom **(A9)** implies that, for every $H, K \in \mathcal{P}(S)^0$, the relations $\trianglelefteq_H$ and $\trianglelefteq_K$ on $\mathbf{M}(\mathcal{U})$ coincide, so, up to a positive linear transformation, we can assume $V_H = V_K = V$, which implies $v_H = v_K = v$ and $u_H = u_K = u$, so we can take $\alpha_H = \alpha_K = \alpha$.

For every $f \in \mathcal{F}$ and $H \in \mathcal{P}(S)^0$, define the functional

$$\underline{\Lambda}_H(f) = \oint_S \left[\int_{\mathcal{U}} \left(\alpha(B) \cdot \inf_{x \in B} u(x) + (1 - \alpha(B)) \cdot \sup_{x \in B} u(x) \right) f(s)(\mathrm{d}B) \right] \varphi_H(\mathrm{d}s).$$

By the previous discussion we have that, for every $f, g \in \mathcal{F}$ and every $H \in \mathcal{P}(S)^0$, it holds that

$$f \precsim_H g \Longleftrightarrow \underline{\Lambda}_H(f) \leq \underline{\Lambda}_H(g).$$

Axioms **(A4)** and **(A5)** imply that there exist $x_*, x^* \in X$ such that $x_* <_H^* x^*$. For every $E \in \mathcal{P}(S)$ define the act $\mathbf{1}_E$ as in (2) and, since u is unique up to a positive linear transformation assume $u(x_*) = 0$ and $u(x^*) = 1$. For every $H \in \mathcal{P}(S)^0$ we have that

$$\begin{aligned} \underline{\Lambda}_H(\mathbf{1}_E) &= \oint_S \left[\int_{\mathcal{U}} \left(\alpha(B) \cdot \inf_{x \in B} u(x) + (1 - \alpha(B)) \cdot \sup_{x \in B} u(x) \right) \mathbf{1}_E(s)(\mathrm{d}B) \right] \varphi_H(\mathrm{d}s) \\ &= \oint_S \chi_E(s) \varphi_H(\mathrm{d}s) = \varphi_H(E), \end{aligned}$$

where χ_E denotes the indicator of E. Furthermore, define the acts $\mathbf{1} = \mathbf{1}_S$ and $\mathbf{0} = \mathbf{1}_\emptyset$.

We have that

$$\underline{\Lambda}_H(\mathbf{1}_E) = \varphi_H(E) = \underline{\Lambda}_H(\varphi_H(E)\mathbf{1} + (1 - \varphi_H(E))\mathbf{0})$$

and since $\underline{\Lambda}_H$ represents $\precsim_H$ it follows that

$$\mathbf{1}_E \sim_H \varphi_H(E)\mathbf{1} + (1 - \varphi_H(E))\mathbf{0}. \tag{A.2}$$

We show that φ_H is a belief function on $\mathcal{P}(S)$. For every $k \geq 2$ and every $E_1, \ldots, E_k \in \mathcal{P}(S)$, for all $\emptyset \neq I \subseteq \{1, \ldots, k\}$ define the acts

$$\begin{aligned} \overline{\mu_{\{1,\ldots,k\}}^{\cup}} &= \varphi_H\left(\bigcup_{i=1}^k E_i \right) \mathbf{1} + \left(1 - \varphi_H\left(\bigcup_{i=1}^k E_i \right) \right) \mathbf{0}, \\ \overline{\mu_I^{\cap}} &= \varphi_H\left(\bigcap_{i \in I} E_i \right) \mathbf{1} + \left(1 - \varphi_H\left(\bigcap_{i \in I} E_i \right) \right) \mathbf{0}, \end{aligned}$$

and let g, h be defined as in axiom **(A6)**. By (A.2) and axiom **(A6)** we have that, since $\underline{\Lambda}_H$ represents $\precsim_H$, $\underline{\Lambda}_H(h) \leq \underline{\Lambda}_H(g)$ which is equivalent to

$$\frac{1}{2^{k-1}} \left(\sum_{\substack{\emptyset \neq I \subseteq \{1,\ldots,k\} \\ |I| \text{ odd}}} \underline{\Lambda}_H(\overline{\mu_I^{\cap}}) \right) \leq \frac{1}{2^{k-1}} \left(\underline{\Lambda}_H(\overline{\mu_{\{1,\ldots,k\}}^{\cup}}) + \sum_{\substack{\emptyset \neq I \subseteq \{1,\ldots,k\} \\ |I| \text{ even}}} \underline{\Lambda}_H(\overline{\mu_I^{\cap}}) \right),$$

that is finally equivalent to

$$\varphi_H\left(\bigcup_{i=1}^{k} E_i\right) \geq \sum_{\emptyset \neq I \subseteq \{1,\ldots,k\}} (-1)^{|I|+1} \varphi_H\left(\bigcap_{i \in I} E_i\right),$$

and so φ_H is a belief function.

Define the function $\varphi : \mathcal{P}(S) \times \mathcal{P}(S)^0 \to [0,1]$ setting, for every $E|H \in \mathcal{P}(S) \times \mathcal{P}(S)^0$,

$$\varphi(E|H) = \varphi_H(E).$$

We show that $\varphi(\cdot|\cdot)$ is a full conditional belief function, i.e., it satisfies conditions *(i)–(iii')* of Definition 1.

Condition ***(i)*****.** For every $H \in \mathcal{P}(S)^0$, since $\mathbf{1}_E(s) = \mathbf{1}_{E\cap H}(s)$, for all $s \in H$, axiom **(A7)** implies, as $\underline{\Lambda}_H$ represents $\precsim_H$, that

$$\varphi(E|H) = \varphi_H(E) = \varphi_H(E \cap H) = \varphi(E \cap H|H).$$

Condition ***(ii)*****.** For every $H \in \mathcal{P}(S)^0$, $\varphi(\cdot|H)$ is a belief function on $\mathcal{P}(S)$ since $\varphi_H(\cdot)$ is.

Condition ***(iii')*****.** For every $A \in \mathcal{P}(S)$ and $B, C \in \mathcal{P}(S)^0$ with $A \subseteq B \subseteq C$, we need to show that

$$\varphi(A|C) = \varphi_C(A) = \varphi_B(A) \cdot \varphi_C(B) = \varphi(A|B) \cdot \varphi(B|C).$$

Since $\mathbf{1}_B(s) = \mathbf{1}(s)$, for all $s \in B$, axiom **(A7)** implies $\mathbf{1}_B \sim_B \mathbf{1}$, moreover, from (A.2) it holds

$$\mathbf{1}_A \sim_B \varphi_B(A)\mathbf{1} + (1 - \varphi_B(A))\mathbf{0}.$$

Applying axiom **(A3)** we get

$$\varphi_B(A)\mathbf{1}_B + (1 - \varphi_B(A))\mathbf{0} \sim_B \varphi_B(A)\mathbf{1} + (1 - \varphi_B(A))\mathbf{0},$$

and axiom **(A1)** implies

$$\mathbf{1}_A \sim_B \varphi_B(A)\mathbf{1}_B + (1 - \varphi_B(A))\mathbf{0}.$$

Since we have

$$\mathbf{1}_A = \begin{cases} \delta_{\{x^*\}} & \text{if } s \in A, \\ \delta_{\{x_*\}} & \text{if } s \in A^c, \end{cases} \quad \text{and}$$

$$\varphi_B(A)\mathbf{1}_B + (1 - \varphi_B(A))\mathbf{0} = \begin{cases} \varphi_B(A)\delta_{\{x^*\}} + (1 - \varphi_B(A))\delta_{\{x_*\}} & \text{if } s \in B, \\ \delta_{\{x_*\}} & \text{if } s \in B^c, \end{cases}$$

with

$$\begin{array}{lcl} \delta_{\{x_*\}} & \trianglelefteq_B & \varphi_B(A)\delta_{\{x^*\}} + (1-\varphi_B(A))\delta_{\{x_*\}} \quad \trianglelefteq_B \quad \delta_{\{x^*\}}, \\ \delta_{\{x_*\}} & \trianglelefteq_C & \varphi_B(A)\delta_{\{x^*\}} + (1-\varphi_B(A))\delta_{\{x_*\}} \quad \trianglelefteq_C \quad \delta_{\{x^*\}}, \end{array}$$

axiom **(A8)** implies

$$\mathbf{1}_A \sim_C \varphi_B(A)\mathbf{1}_B + (1-\varphi_B(A))\mathbf{0}.$$

Moreover, by (A.2) it follows

$$\mathbf{1}_B \sim_C \varphi_C(B)\mathbf{1} + (1-\varphi_C(B))\mathbf{0},$$

and applying axiom **(A3)** we have

$$\begin{array}{rcl} \varphi_B(A)\mathbf{1}_B + (1-\varphi_B(A))\mathbf{0} & \sim_C & \varphi_B(A)[\varphi_C(B)\mathbf{1} + (1-\varphi_C(B))\mathbf{0}] + (1-\varphi_B(A))\mathbf{0} \\ & = & \varphi_B(A)\varphi_C(B)\mathbf{1} + (1-\varphi_B(A)\varphi_C(B))\mathbf{0}, \end{array}$$

and by axiom **(A1)** we get

$$\mathbf{1}_A \sim_C \varphi_B(A)\varphi_C(B)\mathbf{1} + (1-\varphi_B(A)\varphi_C(B))\mathbf{0}.$$

Since $\underline{\Lambda}_C$ represents $\precsim_C$, and $\underline{\Lambda}_C(\mathbf{1}_A) = \varphi_C(A)$ and $\underline{\Lambda}_C(\varphi_B(A)\varphi_C(B)\mathbf{1} + (1-\varphi_B(A)\varphi_C(B))\mathbf{0}) = \varphi_B(A)\varphi_C(B)$ it follows

$$\varphi(A|C) = \varphi_C(A) = \varphi_B(A)\cdot\varphi_C(B) = \varphi(A|B)\cdot\varphi(B|C).$$

Moreover, $\varphi(\cdot|\cdot)$ is unique since every φ_H is.

Finally, define the conditional functional $\underline{\Lambda}(\cdot|\cdot)$ setting, for every $f \in \mathcal{F}$ and $H \in \mathcal{P}(S)^0$,

$$\begin{array}{rcl} \underline{\Lambda}(f|H) & = & \underline{\Lambda}_H(f) \\ & = & \displaystyle\oint_S \left[\int_{\mathcal{U}} \left(\alpha(B)\cdot\inf_{x\in B} u(x) + (1-\alpha(B))\cdot\sup_{x\in B} u(x)\right) f(s)(\mathrm{d}B)\right] \varphi(\mathrm{d}s|H). \end{array}$$

$\square$

Proof of Proposition 1. For every $H \in \mathcal{P}(S)^0$, by equations (4) and (5), for all $f \in \mathcal{F}$, we have that

$$\underline{\Lambda}(f|H) = \oint_S V(f(s))\varphi(\mathrm{d}s|H),$$

where $V : \mathbf{M}(\mathcal{U}) \to \mathbb{R}$ is defined, for every $\mu \in \mathbf{M}(\mathcal{U})$, as

$$V(\mu) = \int_{\mathcal{U}} \left(\alpha(B)\cdot\inf_{x\in B} u(x) + (1-\alpha(B))\cdot\sup_{x\in B} u(x)\right)\mu(\mathrm{d}B).$$

Notice that the boundedness of u implies the boundedness of V, further, for every $H \in \mathcal{P}(S)^0$, it holds that

$$
\begin{aligned}
\mu_1 \trianglelefteq_H \mu_2 &\iff V(\mu_1) \leq V(\mu_2), \\
x_1 \leq_H^* x_2 &\iff u(x_1) \leq u(x_3),
\end{aligned}
$$

for all $\mu_1, \mu_2 \in \mathbf{M}(\mathcal{U})$ and all $x_1, x_2 \in X$.

Since $\mathbf{M}(\mathcal{U})$ is a mixture set and the function V is affine, i.e., it is such that, for every $\mu_1, \mu_2 \in \mathbf{M}(\mathcal{U})$ and $\alpha \in [0,1]$,

$$
V(\alpha\mu_1 + (1-\alpha)\mu_2) = \alpha V(\mu_1) + (1-\alpha)V(\mu_2),
$$

then necessity of axioms **(A1)**–**(A5)** follows by the main Theorem in [43]. As already noticed in the proof of our Theorem 1, in the proof of the quoted theorem we can take $Y = \mathbf{M}(\mathcal{U})$.

We prove necessity of **(A6)**. For every $H \in \mathcal{P}(S)^0$, for every $k \geq 2$ and every $E_1, \ldots, E_k \in \mathcal{P}(S)$, if $x_*, x^* \in X$ are such that $x_* <_H x^*$ and for all $\emptyset \neq I \subseteq \{1, \ldots, k\}$

$$
\mathbf{1}_{\bigcup_{i=1}^k E_i} \sim_H \overline{\mu_{\{1,\ldots,k\}}^{\cup}} \text{ and } \mathbf{1}_{\bigcap_{i\in I} E_i} \sim_H \overline{\mu_I^{\cap}}
$$

with $\mu_{\{1,\ldots,k\}}^{\cup}, \mu_I^{\cap} \in \mathbf{M}(\mathcal{U})$, consider the acts

$$
g = \frac{1}{2^{k-1}} \left(\overline{\mu_{\{1,\ldots,k\}}^{\cup}} + \sum_{\substack{\emptyset \neq I \subseteq \{1,\ldots,k\} \\ |I| \text{ even}}} \overline{\mu_I^{\cap}} \right) \text{ and } h = \frac{1}{2^{k-1}} \left(\sum_{\substack{\emptyset \neq I \subseteq \{1,\ldots,k\} \\ |I| \text{ odd}}} \overline{\mu_I^{\cap}} \right).
$$

Since $\underline{\Lambda}(\cdot|H)$ represents $\precsim_H$, we have that

$$
\begin{aligned}
\underline{\Lambda}\left(\mathbf{1}_{\bigcup_{i=1}^k E_i} \middle| H\right) &= (u(x^*) - u(x_*))\varphi\left(\bigcup_{i=1}^k E_i \middle| H\right) + u(x_*) = \beta_{\{1,\ldots,k\}}^{\cup}, \\
\underline{\Lambda}\left(\overline{\mu_{\{1,\ldots,k\}}^{\cup}} \middle| H\right) &= V\left(\mu_{\{1,\ldots,k\}}^{\cup}\right) = \beta_{\{1,\ldots,k\}}^{\cup}, \\
\underline{\Lambda}\left(\mathbf{1}_{\bigcap_{i\in I} E_i} \middle| H\right) &= (u(x^*) - u(x_*))\varphi\left(\bigcap_{i\in I} E_i \middle| H\right) + u(x_*) = \beta_I^{\cap}, \\
\underline{\Lambda}\left(\overline{\mu_I^{\cap}} \middle| H\right) &= V\left(\mu_I^{\cap}\right) = \beta_I^{\cap},
\end{aligned}
$$

where $\beta^{\cup}_{\{1,\ldots,k\}}, \beta^{\cap}_{I} \in \mathbb{R}$. Further, since f and g are constant acts, by the affinity of V, we have that

$$\begin{aligned}
\underline{\Lambda}(g|H) &= \frac{1}{2^{k-1}}\left(\beta^{\cup}_{\{1,\ldots,k\}} + \sum_{\substack{\emptyset\neq I\subseteq\{1,\ldots,k\}\\ |I| \text{ even}}} \beta^{\cap}_{I}\right),\\
\underline{\Lambda}(h|H) &= \frac{1}{2^{k-1}}\left(\sum_{\substack{\emptyset\neq I\subseteq\{1,\ldots,k\}\\ |I| \text{ odd}}} \beta^{\cap}_{I}\right).
\end{aligned}$$

Finally, the complete monotonicity of $\varphi(\cdot|H)$ implies that

$$\frac{1}{2^{k-1}}\left(\beta^{\cup}_{\{1,\ldots,k\}} + \sum_{\substack{\emptyset\neq I\subseteq\{1,\ldots,k\}\\ |I| \text{ even}}} \beta^{\cap}_{I}\right) \geq \frac{1}{2^{k-1}}\left(\sum_{\substack{\emptyset\neq I\subseteq\{1,\ldots,k\}\\ |I| \text{ odd}}} \beta^{\cap}_{I}\right),$$

and since $\underline{\Lambda}(\cdot|H)$ represents $\precsim_H$, this is equivalent to $h \precsim_H g$.

We prove necessity of **(A7)**. For every $H \in \mathcal{P}(S)^0$, let $f, g \in \mathcal{F}$ be such that $f(s) = g(s)$, for all $s \in H$. Denoting by χ_H the indicator of H, for every $t \in \mathbb{R}$ we have that

$$\begin{aligned}
\varphi(\{(V\circ f) \geq t\}|H) &= \varphi(\{(V\circ f) \geq t\}\cap H|H)\\
&= \varphi(\{(V\circ f)\chi_H \geq t\}\cap H|H)\\
&= \varphi(\{(V\circ f)\chi_H \geq t\}|H)\\
&= \varphi(\{(V\circ g)\chi_H \geq t\}|H)\\
&= \varphi(\{(V\circ g)\chi_H \geq t\}\cap H|H)\\
&= \varphi(\{(V\circ g) \geq t\}\cap H|H) = \varphi(\{(V\circ g) \geq t\}|H).
\end{aligned}$$

Hence, by the definition of the Choquet integral (see, e.g., [10, 25, 42]) we have that $\underline{\Lambda}(f|H) = \underline{\Lambda}(g|H)$, and since $\underline{\Lambda}(\cdot|H)$ represents $\precsim_H$, this is equivalent to $f \precsim_H g$.

We prove necessity of **(A8)**. For every $A \in \mathcal{P}(S)$ and $B, C \in \mathcal{P}(S)^0$ with $A \subseteq B \subseteq C$, for every $\mu_1, \mu_2, \mu_3 \in \mathbf{M}(\mathcal{U})$ with $\mu_1 \trianglelefteq_B \mu_2 \trianglelefteq_B \mu_3$ and $\mu_1 \trianglelefteq_C \mu_2 \trianglelefteq_C \mu_3$, let $f, g \in \mathcal{F}$ be such that $f_{|A} = \overline{\mu_3}_{|A}$, $f_{|A^c} = \overline{\mu_1}_{|A^c}$, $g_{|B} = \overline{\mu_2}_{|B}$

and $g_{|B^c} = \overline{\mu_1}_{|B^c}$. Let $V(\mu_i) = \beta_i \in \mathbb{R}$, for $i = 1, 2, 3$. We have that

$$\begin{aligned}
\underline{\Lambda}(f|B) &= (\beta_3 - \beta_1)\varphi(A|B) + \beta_1, \\
\underline{\Lambda}(g|B) &= (\beta_2 - \beta_1)\varphi(B|B) + \beta_1 = \beta_2, \\
\underline{\Lambda}(f|C) &= (\beta_3 - \beta_1)\varphi(A|C) + \beta_1, \\
\underline{\Lambda}(g|C) &= (\beta_2 - \beta_1)\varphi(B|C) + \beta_1.
\end{aligned}$$

Since $\underline{\Lambda}(\cdot|B)$ represents $\precsim_B$, the statement $f \sim_B g$ is equivalent to $\underline{\Lambda}(f|B) = \underline{\Lambda}(g|B)$, that is

$$(\beta_3 - \beta_1)\varphi(A|B) + \beta_1 = \beta_2.$$

If the above equation holds then

$$\begin{aligned}
\underline{\Lambda}(g|C) &= [(\beta_3 - \beta_1)\varphi(A|B) + \beta_1 - \beta_1]\varphi(B|C) + \beta_1 \\
&= (\beta_3 - \beta_1)\varphi(A|B)\varphi(B|C) + \beta_1 \\
&= (\beta_3 - \beta_1)\varphi(A|C) + \beta_1 = \underline{\Lambda}(f|C),
\end{aligned}$$

and since $\underline{\Lambda}(\cdot|C)$ represents $\precsim_C$, this is equivalent to $f \sim_C g$.

We prove necessity of **(A9)**. For every $H, K \in \mathcal{P}(S)^0$ and for every $\overline{\mu_1}, \overline{\mu_2} \in \mathcal{F}_{\mathrm{const}}$, since $\underline{\Lambda}(\overline{\mu_i}|H) = \underline{\Lambda}(\overline{\mu_i}|K) = V(\mu_i)$, for $i = 1, 2$, and $\underline{\Lambda}(\cdot|H)$ and $\underline{\Lambda}(\cdot|K)$ represent $\precsim_H$ and $\precsim_K$, respectively, we immediately get that

$$\overline{\mu_1} \precsim_H \overline{\mu_2} \Longleftrightarrow \overline{\mu_1} \precsim_K \overline{\mu_2}.$$

We prove necessity of **(A10)**. For $H \in \mathcal{P}(S)^0$, $\mathcal{B} \in \mathcal{P}(\mathcal{U})$, and $\mu_1, \mu_2 \in \mathbf{M}(\mathcal{U})$, we have that

$$\underline{\Lambda}(\overline{\mu_1}|H) = V(\mu_1), \quad \underline{\Lambda}(\overline{\mu_2}|H) = V(\mu_2), \quad \text{and} \quad \underline{\Lambda}(\overline{\delta_B}|H) = V(\delta_B).$$

Suppose $\mu_1(\mathcal{B}) = 1$ and $\overline{\mu_2} \prec_H \overline{\delta_B}$ for all $B \in \mathcal{B}$. Since $\underline{\Lambda}(\cdot|H)$ represents $\precsim_H$, the statement $\overline{\mu_2} \prec_H \overline{\delta_B}$ is equivalent to $V(\mu_2) < V(\delta_B)$ for all $B \in \mathcal{B}$. Hence, we get

$$V(\mu_2) \leq \inf_{B \in \mathcal{B}} V(\delta_B) \leq V(\mu_1),$$

and this is equivalent to $\overline{\mu_2} \precsim_H \overline{\mu_1}$. Analogously, suppose $\mu_1(\mathcal{B}) = 1$ and $\overline{\delta_B} \prec_H \overline{\mu_2}$ for all $B \in \mathcal{B}$. Since $\underline{\Lambda}(\cdot|H)$ represents $\precsim_H$, the statement $\overline{\delta_B} \prec_H \overline{\mu_2}$ is equivalent to $V(\delta_B) < V(\mu_2)$ for all $B \in \mathcal{B}$. Hence, we get

$$V(\mu_1) \leq \sup_{B \in \mathcal{B}} V(\delta_B) \leq V(\mu_2),$$

and this is equivalent to $\overline{\mu_1} \precsim_H \overline{\mu_2}$.

We prove necessity of **(A11)**. For $H \in \mathcal{P}(S)^0$, $B \in \mathcal{U}$, and $\mu \in \mathbf{M}(\mathcal{U})$ whose corresponding belief function φ_μ on $\mathcal{P}(X)$ is a finitely additive probability, we have that

$$\begin{aligned} \underline{\Lambda}(\delta_{\{x\}}|H) &= u(x), \\ \underline{\Lambda}(\delta_B|H) &= \alpha(B) \cdot \inf_{x \in B} u(x) + (1 - \alpha(B)) \cdot \sup_{x \in B} u(x), \\ \underline{\Lambda}(\overline{\mu}|H) &= V(\mu). \end{aligned}$$

Suppose $\overline{\delta_{\{x\}}} \prec_H \overline{\mu}$ for all $x \in B$. Since $\underline{\Lambda}(\cdot|H)$ represents $\precsim_H$, the statement $\overline{\delta_{\{x\}}} \prec_H \overline{\mu}$ is equivalent to $u(x) < V(\mu)$ for all $x \in B$, which implies

$$\underline{\Lambda}(\delta_B|H) \leq \sup_{x \in B} u(x) \leq V(\mu) = \underline{\Lambda}(\overline{\mu}|H),$$

that is equivalent to $\overline{\delta_B} \precsim_H \overline{\mu}$. Analogously, suppose $\overline{\mu} \prec_H \overline{\delta_{\{x\}}}$ for all $x \in B$. Since $\underline{\Lambda}(\cdot|H)$ represents $\precsim_H$, the statement $\overline{\mu} \prec_H \overline{\delta_{\{x\}}}$ is equivalent to $V(\mu) < u(x)$ for all $x \in B$, which implies

$$\underline{\Lambda}(\overline{\mu}|H) = V(\mu) \leq \inf_{x \in B} u(x) \leq \underline{\Lambda}(\delta_B|H),$$

that is equivalent to $\overline{\mu} \precsim_H \overline{\delta_B}$. □

Proof of Theorem 2. The proof follows the same steps of the proof of Theorem 1. We show that, for every $B \in \mathcal{U}$, for the function $v_H : \mathcal{U} \to \mathbb{R}$ derived in the proof of Theorem 1 it holds

$$v_H(B) = \inf_{x \in B} v_H(\{x\}).$$

We already know that $v_H(B) \geq \inf_{x \in B} v_H(\{x\})$, so, it is sufficient to prove only the other inequality. At this aim, consider axiom **(A13)** with $x \in B$ fixed arbitrarily. For $A = \{x\}$ and B there exists $y \in B$ such that $\delta_{\{y\}} \trianglelefteq_H \delta_{\{x\}}$, possibly $y = x$. This implies $\delta_B \trianglelefteq_H \delta_{\{x\}}$ and from this it follows $v_H(B) \leq v_H(\{x\})$. Since this holds for every $x \in B$ we have that

$$v_H(B) \leq \inf_{x \in B} v_H(\{x\}).$$

Now, defining $u_H : X \to \mathbb{R}$ setting, for every $x \in X$,

$$u_H(x) = v_H(\{x\}),$$

it follows that, for every $\mu \in \mathbf{M}(\mathcal{U})$,

$$V_H(\mu) = \int_{\mathcal{U}} \left(\inf_{x \in B} u_H(x) \right) \mu(\mathrm{d}B),$$

that is the function $\alpha_H : \mathcal{U} \to [0, 1]$ is constantly equal to 1. □

Proof of Theorem 3. The proof is analogous to the proof of Theorem 2, showing that axiom **(A13')** implies that the function $v_H : \mathcal{U} \to \mathbb{R}$ derived in the proof of Theorem 1 is such that

$$v_H(A) = \sup_{x\in A} v_H(\{x\}).$$

We already know that $v_H(A) \leq \sup_{x\in A} v_H(\{x\})$, so, it is sufficient to prove only the other inequality. At this aim, consider axiom **(A13')** with $y \in A$ fixed arbitrarily. For $B = \{y\}$ and A there exists $x \in A$ such that $\delta_{\{y\}} \trianglelefteq_H \delta_{\{x\}}$, possibly $x = y$. This implies $\delta_{\{y\}} \trianglelefteq_H \delta_A$ and from this it follows $v_H(\{y\}) \leq v_H(A)$. Since this holds for every $y \in A$ we have that

$$v_H(A) \geq \sup_{y\in A} v_H(\{y\}).$$

Now, defining $u_H : X \to \mathbb{R}$ setting, for every $x \in X$,

$$u_H(x) = v_H(\{x\}),$$

it follows that, for every $\mu \in \mathbf{M}(\mathcal{U})$,

$$V_H(\mu) = \int_{\mathcal{U}} \left(\sup_{x\in B} u_H(x)\right) \mu(\mathrm{d}B),$$

that is the function $\alpha_H : \mathcal{U} \to [0,1]$ is constantly equal to 0. □

Proof of Proposition 2. Necessity of axioms **(A1)**–**(A11)**, both for a representation through $\underline{\Phi}(\cdot|\cdot)$ and $\underline{\Psi}(\cdot|\cdot)$, comes from Proposition 1. To prove necessity of **(A13)** for the representation through $\underline{\Phi}(\cdot|\cdot)$, let $A, B \in \mathcal{U}$ be such that for all $x \in A$, there exists $y \in B$ such that $\overline{\delta_{\{y\}}} \precsim_H \overline{\delta_{\{x\}}}$. We have that

$$\underline{\Phi}(\overline{\delta_{\{y\}}}|H) = u(y), \quad \underline{\Phi}(\overline{\delta_{\{x\}}}|H) = u(x),$$
$$\underline{\Phi}(\overline{\delta_A}|H) = \inf_{x\in A} u(x), \quad \underline{\Phi}(\overline{\delta_B}|H) = \inf_{y\in B} u(y),$$

and since $\underline{\Phi}(\cdot|H)$ represents $\precsim_H$, we get that for all $x \in A$, there exists $y \in B$ such the $u(y) \leq u(x)$, therefore

$$\inf_{y\in B} u(y) \leq \inf_{x\in A} u(x),$$

which is equivalent to $\overline{\delta_B} \precsim_H \overline{\delta_A}$. Analogously, to prove necessity of **(A13')** for the representation through $\underline{\Psi}(\cdot|\cdot)$, let $A, B \in \mathcal{U}$ be such that for all $y \in B$, there exists $x \in A$ such that $\overline{\delta_{\{y\}}} \precsim_H \overline{\delta_{\{x\}}}$. We have that

$$\underline{\Psi}(\overline{\delta_{\{y\}}}|H) = u(y), \quad \underline{\Psi}(\overline{\delta_{\{x\}}}|H) = u(x),$$

$$\underline{\Psi}(\overline{\delta_A}|H) = \sup_{x\in A} u(x), \quad \underline{\Psi}(\overline{\delta_B}|H) = \sup_{y\in B} u(y),$$

and since $\underline{\Psi}(\cdot|H)$ represents $\precsim_H$, we get that for all $y \in B$, there exists $x \in A$ such the $u(y) \leq u(x)$, therefore

$$\sup_{y\in B} u(y) \leq \sup_{x\in A} u(x),$$

which is equivalent to $\overline{\delta_B} \precsim_H \overline{\delta_A}$. □

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