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Structural estimation of counterparty credit risk under recovery risk

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ABSTRACT

Counterparty Credit Risk (CCR) represents one of the major sources of uncertainty in many financial contracts. The role of credit value adjustment (CVA) is, in fact, that of rewarding the parties for the exposure to such risk. A key driver of CVA is the recovery risk, generated by the variability of recovery rates. In this paper, we develop a framework to assess the CCR accounting for the recovery risk that arises from the introduction of stochastic recovery rates. Adopting the structural model for the time to default that exploits a time-changed Lévy process for the risk driver of the equity value, we provide a complete picture to monitor the CCR and gauge the effects of the stochastic recovery rates. The model extracts information on the creditworthiness of the parties in the OTC contract combining Fourier Cosine Expansion and Monte Carlo simulations methods to price CDS spreads, the related underlying, and to retrieve the default barrier. We apply the model proposed to a business case analyzing the CCR of two parties involved in the OTC contract with underlying energy commodities. Low average recovery rates reveal to be associated with high implied volatility and depart from the fixed value of 40%, especially during periods of market distress.

1. Introduction

The global financial crisis has unveiled several weaknesses of the financial system within the derivatives market. This prompted financial regulators to strengthen the assessment of the Counterparty Credit Risk (CCR) which has contributed to the disruption of the stability of the financial system during the crisis (Basel Committee on Banking Supervision, 2018). CCR affects the creditworthiness of the counterparties and exacerbates fluctuations of the underlying risk factors, addressing both the credit and the market risk. Moreover, the incorporation of CCR metrics such as Credit Value Adjustments (CVA) and Debt Value Adjustment (DVA) into the determination of capital requirements has encouraged academics and practitioners to investigate the drivers of CCR. One of the major challenges for risk managers and regulators becomes the development of models that unravel the links of the CCR with other sources of uncertainty and promote risk mitigation. Our paper tackles this issue.

The increasing attention to the CCR hinges on the exponential growth of the Over the Counter (OTC) market over the last fifteen years which has highlighted the difference between the traditional credit risk and the CCR arising from the stochastic nature of derivative payoffs. For this purpose, regulators have introduced Central Clearing Counterparts to foster market transparency and mitigate the risk of occurrence of large insolvencies. Chief among the risks that contribute to the CCR is the recovery risk, defined as the risk that the contracts of the defaulting institution cannot be fully honored. In general, market participants tend to assume constant recovery rates of around 40% within the pricing models (Das and Hanouna, 2009). However, in real market conditions, recovery rates reveal to be stochastic (Schläfer and Uhrig-Homburg, 2014). The impact of the recovery risk on the CCR is mostly captured by the weight that recovery rates assign to the CVA, therefore relevant deviations of the predicted recovery rate from the observed counterpart lead to highly biased models that consequently affect the capital requirements (Altman et al., 2005) and the solvency of financial institutions.

In this paper, we propose a novel framework to assess the CCR allowing recovery rates to be stochastic. We model the risk driver of the asset price underlying the OTC contract with a *time-changed* Lévy process (Ballotta et al., 2019) obtained subordinating the Brownian motion to a Normal Inverse Gaussian (NIG) process. This approach yields a superior capability in replicating non-null short-

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term default probabilities compared to Geometric Brownian motion. We adopt the structural approach of Black and Cox (1976) to identify the condition of default of either one of the two parties in the contract. That is, we study the first time the equity value hits the default barrier. This set-up is in line with the regulatory recommendations of the Fundamental Review of the Trading Book (Basel Committee on Banking Supervision, 2013) which strongly encourages the adoption of the structural approach for purposes of credit risk management along with the consideration of stochastic recovery rates. We link CCR to stochastic recovery rates building a ratio that quantifies the distance of the equity value to the default barrier. In particular, we conveniently model stochastic recovery rates adopting the Beta distribution which ranges in [0,1] and captures the typical skewness and kurtosis of the recovery rates (Chen et al., 2013). In this way, we account for the variability of the recovery rates empirically detected for instance in Acharya et al. (2003). The methodology allows us to price counterparty risk claims under two perspectives. The first is strictly dependent on deterministic recovery rates, while the second includes stochastic recovery rates sampled from the Beta distribution that directly act upon the estimated CVA.

We apply our modeling framework to study the role of the CCR combined with stochastic recovery rates for two financial institutions committed in the OTC contract with underlying energy commodities. We select energy derivatives since they represent the largest asset class in terms of market participants, according to the European Securities and Markets Authorities (2017). The chosen financial institutions are BNP Paribas, dealer of the contract, and Enel, the counterpart. Concerning the underlying commodities, we rely on Vanilla options on Brent Crude Oil and Natural Gas. The analysis is carried out through a *hybrid* approach that combines Monte Carlo simulations with the standard Fourier-Cosine Expansion (COS) method (Fang and Oosterlee, 2008) and the recursive COS method (Fang et al., 2010) to calibrate the parameters of the NIG process and the default barriers. More explicitly, we price Vanilla options on the two energy commodities using the COS method to retrieve the parameters, kurtosis, skewness, and implied volatility for the stochastic representation of the risk driver through the NIG process. Then, we price the CDS spread for BNP Paribas and Enel with the recursive COS method to estimate the related default barriers. The use of such a *hybrid* model is justified by the proposal of the Basel Committee of hybrid applications of the Internal Ratings-Based (IRB) approach for the CCR (Basel Committee on Banking Supervision, 2016)[2].

The following findings can be gathered from our analysis. From the theoretical point of view, we provide a semi-analytical expression for the no-arbitrage price of the counterparty risk derivative claim. Overall, we find that Enel shows higher stochastic recovery rates. This suggests that on average the corporate firm contributes less to the CCR with respect to the dealer, BNP Paribas. The discrimination between the two distributions of stochastic recovery rates is probably related to the difference in the magnitude of the implied volatility. The higher degree of uncertainty in the equity value of the dealer implies that the proportion of the asset value compared to the default losses that are likely to be recovered is lower. Hence, volatility exacerbates the CCR. Moreover, analyzing the effect of the negative shock on the volatility of BNP Paribas, we observe that the distribution of the stochastic recovery rates is closer to the theoretical distribution parametrized on the standard fixed recovery rate equal to 40% (Das and Hanouna, 2009). Thus, stochastic recovery rates reveal to be more appropriate during periods of high volatility.

The major element of novelty that differentiates our paper from the existing literature in this field is the theoretical assessment of the counterparty credit risk within a framework that considers stochastic recovery rates[3]. As pointed out in Szegö (2002), current market conditions call for the need for a complex and thorough modeling set-ups to provide proper support to the regulators for the design of quantitative policies. However, most of the previous works have separated the study of the CCR (Brigo and Tarenghi, 2005 ; Brigo and Bakkar, 2009; Lipton and Sepp, 2009; Bielecki et al., 2011; Arora et al., 2012; Albanese et al., 2013; Brigo et al., 2013; Ballotta and Fusai, 2015; Bo and Capponi, 2015; Kim and Leung, 2016; Cohen and Costanzino, 2017; Ballotta et al., 2019; Li and Zhang, 2019) to that of stochastic recovery rates (Chiang and Tsai, 2010; Amraoui et al., 2012; Schläfer and Uhrig-Homburg, 2014).

Concerning the instruments of the methodology, we accommodate market incompleteness including the *time-changed* pure jump Lévy process which, unlike pure diffusion models, succeeds to reproduce frequent spikes in the asset price and the non-null short-term CDS quotes observed in the market. Moreover, the NIG process reflects the non-zero skewness and kurtosis featured by energy commodities used in the application. We mainly contribute to the strands of the literature that exploit COS related methods for pricing exposure to default (Ballotta and Fusai, 2015; Lian et al., 2017; Alonso-García et al., 2018; Tour et al., 2018) and Monte Carlo simulations in the context of structural models for credit risks (Merton, 1974; Black and Cox, 1976; Brigo et al., 2011; Ballotta et al., 2019). Differently from the most commonly used *reduced-form* models of default risk (also known as *intensity-based models*), we derive the distribution of the random default time through Monte Carlo simulations. According to this approach, the occurrence of the credit event can be explained by the economic process that leads the firm to the condition of not honoring its obligations. Moreover, the information set available to the policy maker is the same as that of the firm's manager (Jarrow and Protter, 2004), thus making the default time predictable on the basis of the asset-liability profile of the firm. In line with this view, we believe that it is a suitable choice to model stochastic recovery rates as a function of the severity of the default of the firm or, equivalently, in terms of the distance of the equity from the barrier.

The paper is organized as follows. Section 2 delineates the theoretical model for the assessment of the CCR combined with stochastic recovery rates. Section 3 discusses the simulation analysis through the calibration of the CCR metrics and the role of stochastic recovery rates. Section 4 provides conclusive remarks.

2. A model for the pricing of the counterparty credit risk

In the following, we outline the framework to model the CCR. Consider the filtered probability space $(\Omega, \mathbb{Q}, \mathcal{G}_t)$ with sample space Ω, *risk-neutral* martingale measure $\mathbb{Q}$, and enlarged filtration $\{\mathcal{G}_t\}_{0\le t\le T}$ defined as:

$$\mathcal{G}_t \equiv \mathcal{F}_t \vee \mathcal{H}_t.$$

Such synthetic filtration jointly collects the information on the market available at time t, given in $\{\mathcal{F}_t\}_{0\le t\le T}$, and the information on the default history of the two counterparts involved in the contract, contained in $\{\mathcal{H}_t\}_{0\le t\le T}$[4]. In particular, the latter is generated by the $\mathcal{G}$-stopping time τ which represents the first time that the creditworthiness of either one of the two parties is compromised. We denote by B and C the dealer of the OTC derivative contract and its counterpart, respectively. Following the structural approach

[2] Possibly, with the introduction of an exogenous floor for the estimation of the risk weighted assets, in order to balance their excessive variability.

[3] This is also suggested in Brigo and Vrins (2018) and Ballotta et al. (2019).

[4] We assume that the filtrations $\{\mathcal{F}_t\}_{0\le t\le T}$, $\{\mathcal{H}_t\}_{0\le t\le T}$, and $\{\mathcal{G}_t\}_{0\le t\le T}$ satisfy the usual conditions of completeness and right-continuity.

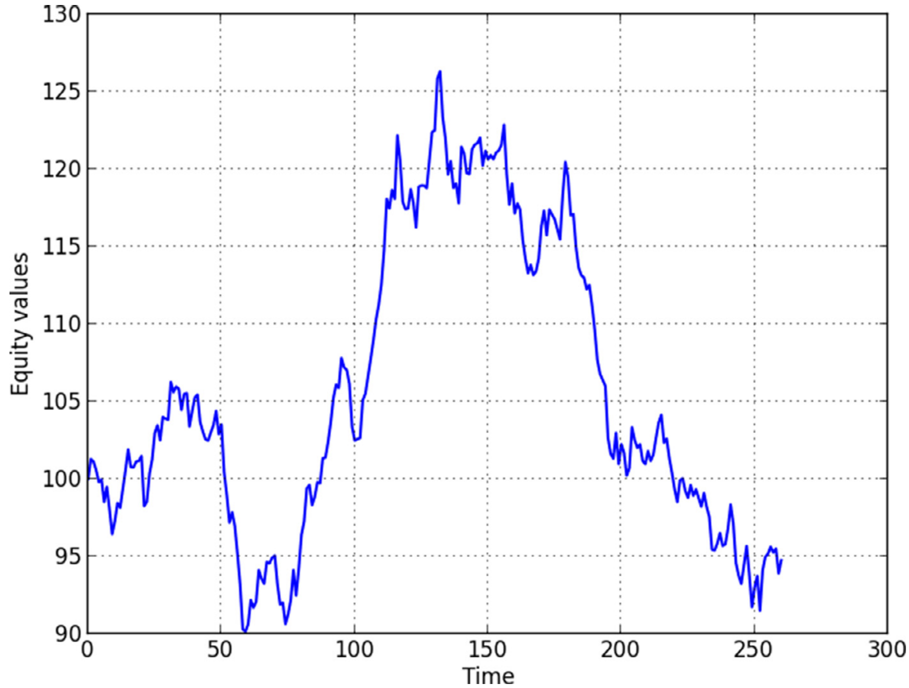

Fig. 1. Simulation of one possible path of the asset prices described by equation 1, setting $S_0 = 100$. The simulation is run so to generate weekly observations of S_t over five years.

in Black and Cox (1976), the equity value, S_t, and the related risk driver, X_t, are described by:

$$S_t = S_0 e^{[r-q-\varphi_X(-\imath)]t+X(t)}, \tag{1}$$

where $X(t)$ is a Lévy process with characteristic exponent $\varphi_X(\cdot)$, $r > 0$ is the proxy of risk-free rate, $q > 0$ is the continuous *convenience yield* paid for holding the underlying inventories prior to the maturity of the contract, $\bar{\mu} = r - q - \varphi_X(-\imath)$ is the *mean-correcting drift* needed to allow S_t to be an exponential martingale. The risk driver X_t of the price of the underlying is described with the Normal Inverse Gaussian (NIG) process introduced in Barndorff-Nielsen (1997). This process results in pure jump Lévy process obtained subordinating the Brownian motion, W_t, with drift μ and volatility σ to an independent Inverse Gaussian process, z_t. In other words, the NIG process results in the *time-changed* Lévy process indexed to a *stochastic clock*:

$$X_t = \mu z_t + \sigma W_{z_t}. \tag{2}$$

The underlying distribution of the process is in general featured by skewness, kurtosis, and it is infinitely divisible. The characteristic exponent $\varphi_X(u)$ is defined for any $u \in \mathbb{R}$:

$$\varphi_X(u) = \frac{1 - \sqrt{1 - 2\imath\theta\kappa + u^2\sigma^2\kappa}}{\kappa}, \tag{3}$$

where $\imath$ is the imaginary unit, $\theta \in \mathbb{R}$ **describes the sign of the skewness,** and $\kappa > 0$ controls the excess kurtosis of the distribution.

The rationale for the use of time-changed processes lies in the economic application that concerns the switch from calendar time to *business time*. This implies that the asset price is mainly driven by relevant news with both random arrival time and impact on the market[5]. Moreover, the successful adaptability of Lévy processes to model asset prices that exhibit systematic spikes and high volatility (see Figure 1 for an example) endorses our choice for the employment of this class of processes.

Among all the possible financial events likely to occur during the history of the price process, we are interested in the first time that the firm's equity value crosses the fixed default barrier M:

$$\tau^i = \inf\{t \in (0, T] : S_t^i \leq M^i\} \quad \text{with} \quad i \in \{B, C\}. \tag{4}$$

According to the structural approach used for CCR, we model the *first-passage time* τ^i as in Black and Cox (1976). Moreover, through the unpleasant event $\{S_t^i \leq M^i\}$, we assume that we can also extract the information on the state of creditworthiness of the financial institutions B and C[6].

2.1. Bilateral counterparty credit risk pricing

The arising of the CCR calls for the need to quantify the economic value of the derivative contract with two different measures. The *default-free* economic value of the contract, $V(S_t)$, and the correspondent value of the counterparty risk claim, $\hat{V}(S_t, \mathcal{D}_t^B, \mathcal{D}_t^C)$, estimated when the probability of default for either B or C is considered.

The state processes $\{\mathcal{D}_t^i\}_{0 \leq t \leq T}$ indicate the occurrence of the credit events:

$$\mathcal{D}_t^i \equiv \mathbb{1}_{\{\tau_i \leq t\}} \quad \text{with} \quad i \in \{B, C\}.$$

In the case of no default before the maturity T, the buyer will be paid back the derivative payoff $\Phi(S_{[t,T]})$, while the dealer will earn the opposite cash flows $-\Phi(S_{[t,T]})$. According to International Swap Dealers Association (2002), if premature default verifies when the surviving party is *out of the money*, it has to settle all the debt. Conversely, it can claim just a recovery fraction of the credit if *in the money*.

[5] Interestingly, gaussianity seems to be recovered under such trading time. Lévy processes are able to replicate implied volatility surfaces without over stressing model parameters and can accommodate for jumps (Cont and Tankov, 2004).

[6] The convenience yield in the dynamics of the assets is assumed to be zero.

Proposition 1. *Let ε_t^+ denote the Positive Exposure (PE) at time t, ε_t^- the Negative Exposure (NE) at time t, $V(S_t)$ the default-free value of the contract, $\delta(t,\tau)$ the risk-free discount factor, R_B and R_C the client and the bank's recovery functions upon default. Suppose that the following border conditions hold at the stopping times τ^B and τ^C:*

$$\hat{\Phi}(\tau_B) = \varepsilon_{\tau_B}^+ + R_B(\varepsilon_{\tau_B}^+) \quad \text{and} \quad \hat{\Phi}(\tau_C) = R_C(\varepsilon_{\tau_C}^+) - \varepsilon_{\tau_C}^+, \tag{5}$$

then the no arbitrage price $\hat{V}(S_t)$ of the counterparty risk derivative claim is:

$$\hat{V}(S_t, \mathcal{D}_t^B, \mathcal{D}_t^C) = V(S_t) - \underbrace{\mathbb{E}_t^{\mathbb{Q}}[\mathbb{1}_{\{\tau=\tau_C\}}\delta(t,\tau_C)(\varepsilon_{\tau_C}^+ - R_C(\varepsilon_{\tau_C}^+))]}_{CVA} + \underbrace{\mathbb{E}_t^{\mathbb{Q}}[\mathbb{1}_{\{\tau=\tau_B\}}\delta(t,\tau_B)(\varepsilon_{\tau_B}^- - R_B(\varepsilon_{\tau_B}^-))]}_{DVA}. \tag{6}$$

Setting **the** border conditions in compliance with the close-out agreements in International Swap Dealers Association (2002) and assuming that both the recovery functions take values between 0 and 1, the CVA can be seen as a call option on the uncollateralized exposure with zero strike and random maturity which represents the expected loss on the credit due to counterparty default risk. Contrariwise, the DVA can be regarded as a put option issued on the uncollateralized exposure with zero strike and random maturity, thus representing the expected debt saving due to the own default risk.

2.2. State-dependent stochastic recovery rates

Recovery risk plays a crucial role in the assessment of the CCR. It mainly concerns the risk that, after the credit default event, the contracts of the defaulting party cannot be fully honored, thereby exacerbating the financial losses of the counterparty. In real market conditions, stochastic recovery rates are differentiated according to the claims issued by the defaulted company. More precisely, the holders of senior bonds or other secured instruments have priority in collecting their credits during the liquidation process. As a consequence, unlike the junior creditor, the senior creditor is likely to collect a higher fraction of the distressed assets of the defaulted company. Besides, the funds' availability to manage likely adverse financial events changes with the degree of distress in the market which affects the likelihood of valuable amounts of recovered assets after default. Hence, neglecting the volatility of the recovery risk can be harmful even for the most sophisticated models since it would directly impact the stability of the company through incorrect calculation of the required capital buffers.

One possibility to provide a quantitative modeling framework for stochastic recovery rates is to assume an explicit statistical distribution of the recovery rates basing on the bankruptcy of the company. We choose the Beta distribution, $\mathcal{B}(\alpha,\beta)$, a continuous distribution taking values in [0,1], a suitable domain for recovery rates, which captures both skewness and kurtosis with respect to e.g, the Uniform[0,1] distribution.

Our goal is to relate stochastic recovery rates to the severity of credit events. We incorporate the volatility of stochastic recovery rates into the monitoring of CCR building a random variable which depends on the relative Gaussian distance of the equity value upon simulated defaults with respect to the default barrier M. Let $\eta_\tau = f(S_\tau)$ be such random variable defined as:

$$\eta_\tau = \frac{\phi_{M,1}(S_\tau)}{\phi_{M,1}(M)}, \tag{7}$$

where $\phi_{M,1}(\cdot)$ is the cumulative distribution function (*cdf*) of a normal distribution with mean M and unit variance. The stochastic recovery rate is therefore defined as the inverse *cdf* of a Beta distribution calculated in η_τ:

$$\mathcal{R}(\eta_\tau) = \mathcal{B}^{-1}(\alpha^*, \beta^*)(\eta_\tau), \tag{8}$$

where $\mathcal{B}^{-1}(\alpha,\beta)$ is the Beta quantile function and α^* and β^* are the distribution parameters. According to the magnitude of η_τ which proxies the probability of default in terms of distance to the barrier, M, we retrieve the related quantile corresponding to the stochastic recovery rate. Stochastic recovery rates are thus vested by the *state-dependent* nature captured in the distance ratio from the barrier.

Accounting for the recovery risk within the history of the contract between the two counterparties, the formula for bilateral CCR adjustments extended to stochastic recovery rates modifies Proposition 1 as follows:

$$\hat{V}(S_t, \mathcal{D}_t^B, \mathcal{D}_t^C) = V(S_t) + - \underbrace{\mathbb{E}^{\mathbb{Q}}[\mathbb{1}_{\{\tau=\tau^C\}}\delta(t,\tau^C)[1-R(\eta_{\tau^C})]V_{\tau^C}^+ \mid \mathcal{G}_t]}_{CVA} + \underbrace{\mathbb{E}^{\mathbb{Q}}[\mathbb{1}_{\{\tau=\tau^B\}}\delta(t,\tau^B)[1-R(\eta_{\tau^B})]V_{\tau^B}^- \mid \mathcal{G}_t]}_{DVA}, \tag{9}$$

where $\mathcal{R}(\eta_{\tau_C})$ and $\mathcal{R}(\eta_{\tau_B})$ are the stochastic recovery rates of the firm and the bank upon simulated default, respectively. The effect of the variability of stochastic recovery rates on the CVA is thus formally captured in (9) where stochastic recovery rates directly act upon the CVA.

2.3. The fourier cosine expansion method for plain vanilla options

For the purpose of the application, we briefly recall the Fourier Cosine Expansion (COS) method introduced in Fang and Oosterlee (2008) that we employ in this paper to compute the value of the options when default occurs and to calibrate the parameters of the underlying. The approach guarantees higher efficiency than the Fast Fourier Transform (FFT) proposed in Carr and Madan (1999) or the Convolution method (CONV) discussed in Lord et al. (2008). In addition, the COS method handles more general dynamics for the underlying compared to other approaches.

Through inverse Fourier integrals, the procedure succeeds to recover the unknown conditional *pdf* of the price of the derivative using the expansion of the cosine series. This method is proved to adapt well to smooth densities, a property which holds for Lévy processes defined on a finite support. The COS method yields the following general pricing formula for the derivative claim:

$$v(x,t) = e^{-r(T-t)} {\sum_{k=0}^{N-1}}' \Re\left\{ e^{-\frac{ik\pi a}{b-a}} \phi_{y|x}\left(\frac{k\pi}{b-a}; \mathbf{x}\right)\right\} V_k, \tag{10}$$

where r is the risk-free rate, $\Re\{\cdot\}$ is the real part function, $\phi_{y|x}$ is the characteristic function of the model[7], V_k is the payoff series coefficients, and a and b are constants chosen such that the truncated integral well approximates the infinite counterpart. For Plain Vanilla Call and Put options the *k-th* coefficient V_k of the derivative payoff is given by:

$$\begin{aligned} V_k^{Call} &= \frac{2}{b-a}\int_0^b K(e^y-1)\cos\left(k\pi\frac{y-a}{b-a}\right)dy = \frac{2}{b-a}K(\chi_k(0,b) - \psi_k(0,b)) \\ V_k^{Put} &= \frac{2}{b-a}\int_a^0 K(1-e^y)\cos\left(k\pi\frac{y-a}{b-a}\right)dy = \frac{2}{b-a}K(-\chi_k(a,0) + \psi_k(a,0)), \end{aligned} \tag{11}$$

where K is the strike price and y represents the log-asset price at maturity time T[8].

[7] As previously discussed, we adopt the NIG model.

2.4. The recursive COS method for CDS

Under the structural modelling perspective, the information set available to the policy-maker is the same as the firm's manager which includes complete knowledge on the asset-liability management (Jarrow and Protter, 2004). To recover information provided by the market on the creditworthiness of the firms, we apply the recursive COS method (Fang et al., 2010) to CDS. Indeed, these contracts hedge the buyer against the default of the financial institution, therefore parameters calibrated to CDS spreads reflect the necessary information on default probabilities. For the application of the COS method for CDS, we explicit the risk-neutral survival probability at time t as:

$$\begin{aligned}\mathbb{Q}\{\tau > t\} &= \mathbb{Q}\left\{Y_s > \log\left(\frac{K}{S_0}\right), \forall\, 0 \le s \le t\right\} \\ &= \mathbb{Q}\left\{\min_{0\le s\le t} Y_s > \log\left(\frac{K}{S_0}\right)\right\} \\ &= \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}\left\{\min_{0\le s\le t} Y_s > \log\left(\frac{K}{S_0}\right)\right\}\right],\end{aligned} \tag{13}$$

where $Y_t = \log\left(\frac{S_t}{S_0}\right)$. Define $\Upsilon = \log\left(\frac{K}{S_0}\right)$ the reference value for the bankruptcy and suppose that the time interval $(0, T]$ can be split into L time-frames, $\mathcal{T} \equiv \{T_0, T_1, \dots, T_L\}$, such that $\Delta T = \frac{T}{L}$ and $T_l = l\Delta T$. The generic time T_l is the default monitoring date which, considered along with the others, allows us to write the survival probability as:

$$\mathbb{Q}\{\tau > T\} = \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{Y_{T_1}\in[\Upsilon,\infty]\}}\mathbb{1}_{\{Y_{T_2}\in[\Upsilon,\infty]\}}\cdots\mathbb{1}_{\{Y_{T_L}\in[\Upsilon,\infty]\}}\right]. \tag{14}$$

The probability in (14) corresponds to the pricing formula for discrete digital options without discounting. The integral form develops into:

$$\begin{aligned}&\mathbb{Q}\{\tau > T\} = \int_\Upsilon^\infty \dots \int_\Upsilon^\infty \dots \int_\Upsilon^\infty f_{Y_{T_L}|Y_{T_{L-1}}}(y_{T_L} \mid y_{T_{L-1}})\times \\ &\times f_{Y_{T_l}|Y_{T_{l-1}}}(y_{T_l} \mid y_{T_{l-1}})dy_{T_l}\dots f_{Y_{T_1}|Y_{T_0}}(y_{T_1} \mid y_{T_0})dy_{T_1}.\end{aligned} \tag{15}$$

The computation of the survival probabilities can therefore be performed via a backward loop of the COS scheme applied to the conditional *pdf* of the equity value $f_{Y_{T_l}|Y_{T_{l-1}}}(y \mid x)$:

$$f_{Y_{T_l}|Y_{T_{l-1}}}(y \mid x) = \frac{2}{b-a}\sum_{k=0}^{N}{}' \Re\left\{e^{-ik\pi\frac{x-a}{b-a}}\phi_{y|x}\left(\frac{k\pi}{b-a}, \Delta T\right)\right\}\cos\left(k\pi\frac{y-a}{b-a}\right). \tag{16}$$

The theoretical *fair CDS spread* at the start date, T_0, for a running CDS[9] with maturity T, given the recovery rate, R, is the one which makes equal the premium leg and the protection leg. Applying the composite trapezoidal rule, a good approximation of the CDS spread, s, is obtained:

$$s = (1-R)\left(\frac{1-e^{-rT}\mathbb{Q}\{\tau > T\}}{\sum_{l=0}^{L} w_l e^{-rT_l}\mathbb{Q}\{\tau > T_l\}\Delta T} - r\right) \quad \text{with} \quad w_0 = \frac{1}{2} \quad \text{and} \quad w_L = 1. \tag{17}$$

The calibration in the numerical analysis is generated by (17). The theoretical CDS spread depends on a stream of survival probabilities, which have been computed within **the** structural approach in (15).

[8] The functions χ_k and ψ_k are defined for the general arguments c and d with $[c, d] \subset [a, b]$ as:

$$\begin{aligned}\chi_k(c,d) &:= \int_c^d e^y \cos\left(k\pi\frac{y-a}{b-a}\right)dy \\ \psi_k(c,d) &:= \int_c^d \cos\left(k\pi\frac{y-a}{b-a}\right)dy.\end{aligned} \tag{12}$$

[9] Running CDS are characterized by no upfront payment at inception.

Table 1
Calibration results based on the CDS quotes and on Plain Vanilla options data for the underlying energy commodities. RMSEs are measured in basis points.

Parameters Calibration					
	M^i	κ^*	θ^*	σ^*	RMSE
ENEL	0.65033	0.23639	-0.20818	0.18073	1.5445
BNP	0.63839	0.24519	-0.20520	0.23570	1.1972
BR	-	0.86128	-0.14939	0.19686	0.5373
NG	-	0.87215	-0.13590	0.21107	0.4565

3. Numerical analysis

We implement the theoretical framework so far described to assess the creditworthiness of two firms within the OTC derivative contract as measured by the relationship between the stochastic recovery rates and the CCR. The chosen representative bank is BNP Paribas (BNP) while Enel (ENEL) is the corporate firm. The OTC derivatives are issued on the two most liquid energy commodities, Brent Crude Oil (BR) and Natural Gas (NA), traded on the NYMEX. Since BNP Paribas and Enel belong to different market sectors, we assume a minor degree of connection between the two equities which translates into a null correlation between S_t^{BNP} and S_t^{ENEL}.

Following Black and Cox (1976), the first passage time of the equity value under the barrier is obtained via Monte Carlo simulations. At the first default of either one of the two parties, the simulation stops, the default is registered, and the algorithm moves on to the next iteration. Both S_t^{BNP} and S_t^{ENEL} evolve according to (1) with the parameters of the NIG process calibrated using the recursive COS for CDS, which is also exploited to estimate the default barriers M^i. The exposure at default is priced using (10) for Plain Vanilla options, discounted at t_0 and then multiplied by the stochastic recovery rate.

3.1. Calibration results - CDS

We rely on CDS spreads data available from the iTraxx series for the European CDS market, and perform the calibration minimizing the root mean square error (RMSE) of the theoretical fair spread produced by the difference between market quotes and the model counterparts. Define $\xi = (M, \kappa, \theta, \sigma)$ as the vector of the parameters required to describe the dynamics of the bank, of the firm, and of the underlyings considered, then the optimal calibration is:

$$\xi^* = \arg\min_{\xi}\sqrt{\sum_{\text{CDS}}\frac{(\text{market CDS spread - model CDS spread})^2}{\text{Number of benchmark CDS}}}. \tag{18}$$

The energy commodities selected are assumed to be non-defaultable, thus the default barrier is not estimated. The other parameters are calibrated using available Plain Vanilla option prices listed in the Chicago Mercantile Exchange (CME). Regarding BNP Paribas and Enel, we have collected up to 5 years of CDS quotes to calibrate ξ. In the application of the recursive COS method for the pricing of CDS, the Fourier summation has been truncated to $N = 2^{10}$ terms in order to reach a satisfying level of accuracy. For Brent Crude Oil and Natural Gas Plain Vanilla options, we set $N = 2^8$.

Table 1 reports the results of the calibration of ξ for BNP Paribas, Enel, Brent Crude Oil, and Natural Gas[10].

[10] Regarding the computational time, relying on 210 terms in the iterative COS for the pricing of CDS, the algorithm takes on average 67.5 seconds. Moreover, 28 terms are applied in the calibration of the standard COS which in this work is used for the

Table 2
CVAs and DVAs for ATM Plain Vanilla Call on Brent Crude Oil and Natural Gas incorporating *state-dependent* stochastic recovery rates. Relative valuation adjustments are expressed in basis points.

Counterparty Credit Risk Valuation Adjustments						
Contract type	V_0	CVA	CVA/V_0	DVA	DVA/V_0	$\hat{V}_0$
ATM Call Oil	5.82103	0.93075	1598.940	0	0	4.89028
ATM Call Gas	0.41155	0.06477	1573.876	0	0	0.34677

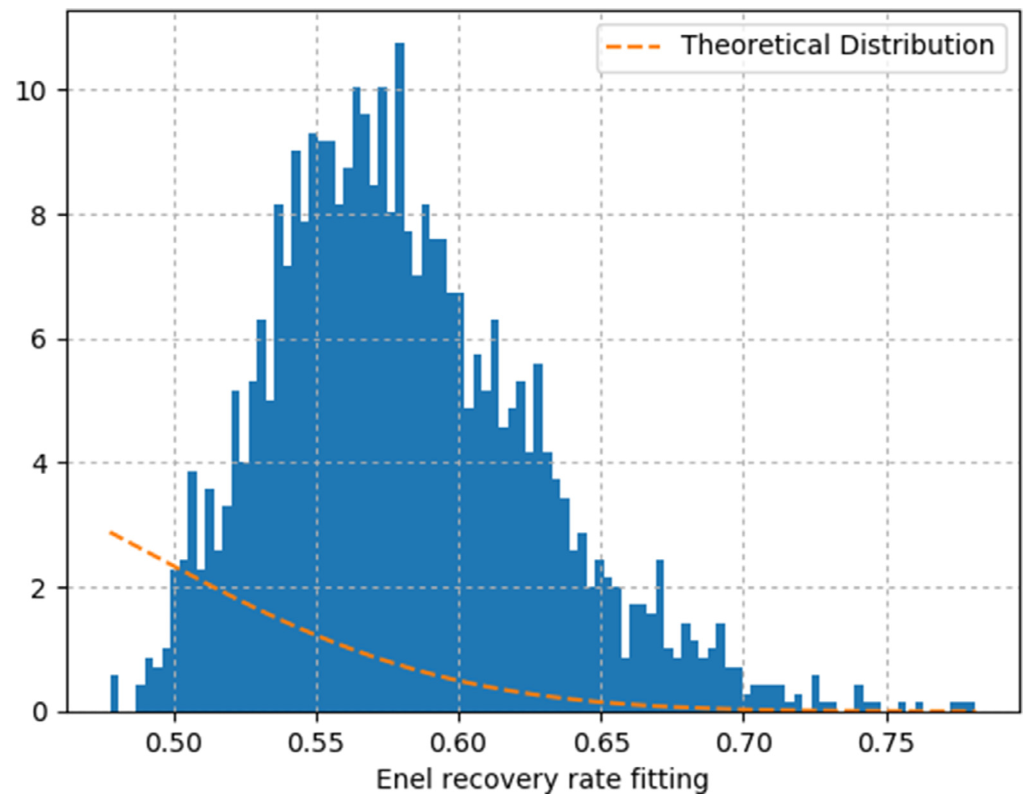

Fig. 2. Simulated distribution of Enel recovery rates under the baseline calibration.

The similarity between the CDS spreads of Enel and BNP Paribas is reflected in the estimation of the default barrier and distribution parameters. The most pronounced difference concerns implied volatility which is remarkably higher for BNP Paribas suggesting an overall increased risk exposure. The lower default barrier is probably related to the propensity to reach highly variable magnitudes. The calibrated distributions are negatively skewed and slightly fat-tailed, especially for the two energy commodities.

3.2. Calibration results - stochastic recovery rates

We estimate the distribution of the stochastic recovery rates parameterizing α and β so that the Beta distribution possesses small variance and it is centered around 40%, the historical average of corporate recovery rates.

The rationale for this procedure lies in two major issues: i) there are no historical series of recovery rates upon default and ii) there are not sufficiently liquid markets that allow the calibration for risk-neutral stochastic recovery rates (O'Kane and Turnbull, 2003). Following this criteria, we obtain the optimal parameters: $\alpha^* = 10.464$ and $\beta^* = 15.696$.

computation of EAD for the plain vanilla claims taken into account. For the latter, the computational time is of the order of a few seconds.

Table 2 shows the results for bilateral CCR adjustments computed through the *hybrid* COS-MC model which is extended in order to allow for state-dependent recovery rates upon simulated defaults. The parameters are calibrated using the COS method and MC is exploited for the joint simulation of defaults. The parameter η is calculated only when the default occurs and it yields the stochastic recovery rate through the Beta quantile function. Figures 2 and 3 show the goodness of fit of the sampling distribution of the recovery rates to the theoretical Beta (dashed red line), $\mathcal{B}(\alpha^*, \beta^*)$, for BNP Paribas and Enel[11]. Flat recovery rates reveal an acceptable assumption for the distribution of the recovery rate of BNP Paribas where the mode is close to 40% and the distribution is featured by light tails that suggest weak exposure to extreme values. On the contrary, the sample distribution of the stochastic recovery rate of Enel remarkably departs from the theoretical due to the right-shifting of the distribution. This suggests that overall the default of Enel leads to a higher propensity of recovery of the resulting losses. Such evidence is probably anchored to the fact that equity values of BNP Paribas are remarkably more volatile than Enel. As a consequence, the relative Gaussian distance of the equity

[11] Focusing on the Monte Carlo simulations for the proposed hybrid approach, the computational time with 100,000 iterations is modified, on average, from 2,477 to 2,484 seconds when activating the sampling from the inverse cdf of the Beta distribution for the computation of stochastic recoveries.

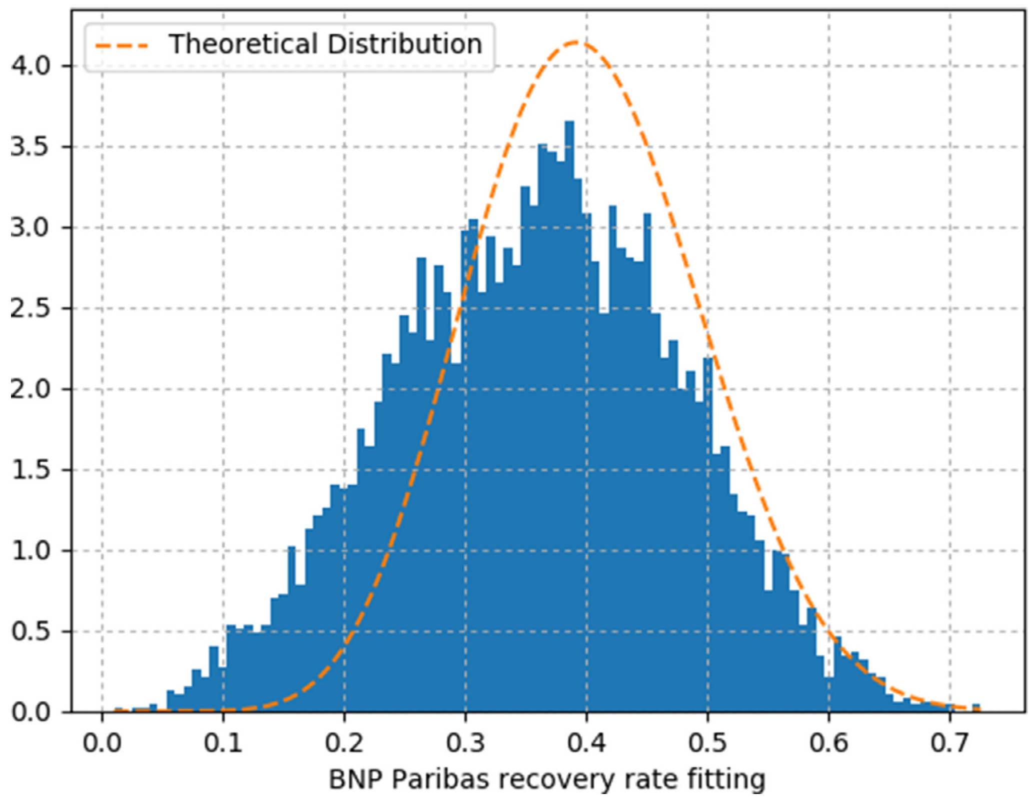

Fig. 3. Simulated distribution of BNP Paribas recovery rates under the baseline calibration.

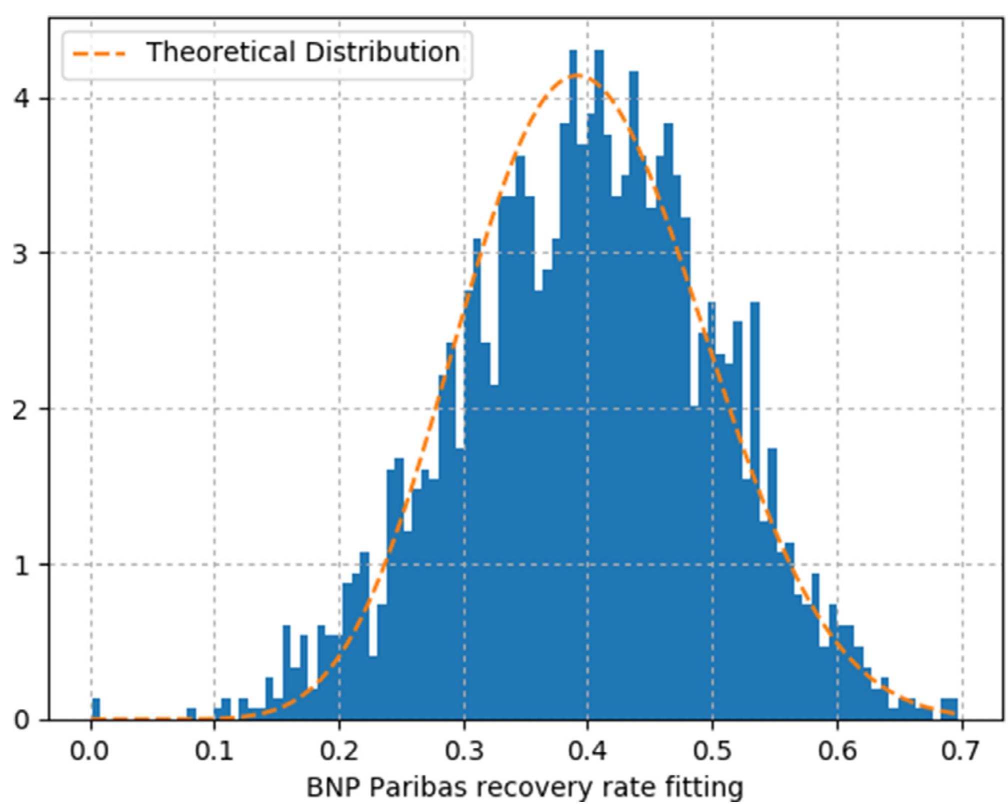

Fig. 4. Simulated distribution of BNP Paribas recovery rates in case of a negative 25% shock on volatility.

Table 3
Descriptive statistics of the recovery rates relative to the simulation of the CCR measures of the Plain Vanilla Call option on Natural Gas.

Stochastic Recoveries Descriptive Statistics						
Firm	Min	Max	**Mean**	Variance	Skewness	Kurtosis
ENEL	0.47762	0.78154	**0.58083**	0.00218	0.69119	0.59477
BNP Paribas	0.01008	0.72569	**0.36009**	0.01390	0.02283	-0.38029

Table 4
Average stochastic recovery rates and CVAs for the Plain Vanilla Call option on Oil as a consequence of α-shifts (for fixed β).

α-shift	Recovery Rate	CVA
-50%	0.21589	1.19436
-25%	0.29601	1.02486
-10%	0.33219	0.96775
+10%	0.37934	0.90627
+25%	0.41147	0.87570
+50%	0.45876	0.80633

Table 5
Average stochastic recovery rates and CVAs for the Plain Vanilla Call option on Oil as a consequence of β-shifts (for fixed α).

β-shift	Recovery Rate	CVA
-50%	0.51696	0.67555
-25%	0.42850	0.85702
-10%	0.38527	0.91130
+10%	0.33735	0.97825
+25%	0.31115	0.99610
+50%	0.27611	1.02569

value from the default barrier increases as the simulated stochastic recovery rates decrease. Thus, the adoption of constant recoveries requires careful attention by the regulators because it may lead to either over or underestimation of the recovery risk, and hence the expected severity of the default.

Figure 4 shows the effect of a multiplicative negative shock of 25% on the volatility of BNP Paribas. The simulated distribution of stochastic recovery rates results closer to the theoretical Beta indicating that the 40% recovery rate is a suitable choice during periods of markedly low volatility for BNP Paribas. Moreover, the CVA reduces the default-free value of the Natural Gas Plain Vanilla option by 7.07%, highlighting the possible implicit leverage effect triggered by volatility easing. Therefore, the lack of consideration of the stochastic nature of recovery rates can severely affect the estimation of CVA and the ensuing determination of the capital requirements.

We perform the sensitivity analysis on the CVA/DVA obtained using stochastic recovery rates and on the parameters, α and β, to gauge the degree of influence of the value of the CVA on the distribution of the stochastic recovery rates. For this purpose, we observe the recovery rates and related CVA for the following values of α: $\pm 10\%, \pm 25\%, \pm 50\%$. Results show that with increasing vales of α, the recovery rate increases and the CVA decreases. Conversely, increasing values of β amplify the CCR leading to the reduction of recovery rates and to the growth of the CVA. Furthermore, CVA shows slightly more sensitivity to the parameter β. Results are reported in Tables 4 and 5.

4. Conclusions

In this paper, we develop a quantitative framework for the assessment of Counterparty Credit Risk (CCR) accounting for the introduction of stochastic recovery rates. Recovery risk affects CCR, calling for the need for considering models that reflect the stochastic nature of recovery rates (Acharya et al., 2003; Das and Hanouna, 2009). The lack of a framework that fully incorporates this feature is harmful to the promotion of the stability of the financial system, since recovery rates directly enter into the computation of the Credit Value Adjustment (CVA) used in the determination of capital requirements.

The framework devised exploits the Beta distribution to model stochastic recovery rates of the two parties involved in the Over the Counter (OTC) derivative contract. In particular, we compute the stochastic recovery rate building a ratio that gauges CCR in terms of the distance of the equity value to default. We proxy the creditworthiness of the two parties of the contract with the related equity value and model the risk driver by a *time-changed* Lévy process obtained subordinating the Brownian motion to an independent Inverse Gaussian process. Then, pricing Plain Vanilla options on energy commodities and CDS spreads of the parties in the contract through Fourier-Cosine Expansion (COS) methods (Fang and Oosterlee, 2008; Fang et al., 2010), we calibrate the parameters that allow us to describe the time-varying evolution of the asset price. We empirically identify the first time to default adopting the structural approach of Black and Cox (1976) and exploiting Monte Carlo simulations.

Our main finding concerns the relationship between the implied volatility of the equity value of the parties in the contract and the expected value of stochastic recovery rates. We observe that deterministic recovery rates are not appropriate to capture the correct exposure to the CCR during high volatility periods.

The theoretical model delineated in this paper is especially valuable to risk managers and investors for daily monitoring activities and trading operations. For instance, given the spreading of the Covid-19 pandemic, many corporate defaults have occurred. Relevant bias in the assessment of the CCR through improper calculation of capital requirements may be due to the use of flat recovery rates which we have proved to be not appropriate during periods of financial distress.

Inspired by the lesson in Szegö (2002), this framework corroborates the need for complex analytical and numerical techniques to depict objective macro-economic landscapes of the stability of the financial system and promote the safeguard of market participants.

CRediT authorship contribution statement

Rosella Castellano: Conceptualization, Supervision, Writing – review & editing. **Vincenzo Corallo:** Methodology, Software, Writing – original draft. **Giacomo Morelli:** Writing – original draft, Writing – review & editing.

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