Measurement of solvency capital requirement for the interest rate risk using the Standard Formula: a stochastic model to evaluate a participating life insurance contract

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Abstract: The interest rate risk is relevant in the creation of a life insurance company's solvency capital requirement. In the article we address the problem of its measurement when the company has with-profit insurance contracts with a minimum guaranteed rate in its portfolio and uses the Standard Formula. A stochastic model and the Monte Carlo simulation is needed to calculate the technical provision. We propose a Cox-Ingersoll-Ross model with an exogenous barrier extended with a deterministic function which allows to estimate negative rates and the perfect-fit of the term structure of interest rates, measured using the Smith-Wilson method. We also introduce an alternative method to define the upward and downward scenarios which is consistent with the regulatory framework.

Keywords: Interest rate risk, Scenario-based, With-profit insurance contract, Stochastic model, Monte Carlo simulation

1 Introduction

The market risk contributes towards the making of a life insurance company's capital requirement in a predominant way. Among its sub-risks, an important emphasis is connected to the interest rate risk ([EIOPA (2011)]), that is the risk derived from the variability of term structure of risk-free interest rates. In the Standard Formula, used by most Italian companies ([ANIA (2016)]), the Solvency Capital Requirement (SCR) against this risk is calculated by a scenario-based approach, that being by the maximum variation in Basic Own Funds (BOF) if positive, and is obtained by applying two stressed structures that define two types of markets, one is upward and the other is downward.
The technical provision's calculation is preliminary to the implementation of the Standard Formula. In the paper a valuation model of with-profit insurance contract, widely spread in the Italian market, is shown. The model allows for the perfect-fit of the reference structure of risk-free interest rates. Moreover, a different approach to the scenario measurement is proposed,

that helps to overcome some theoretical contradictions in the methodology applied by European Authority (EIOPA). Besides, it provides a more coherent estimation with the current European market, characterized by a strong fall in interest rates during the last few years.
The new approach proposed, using the statistical technique of Principal Component Analysis (PCA), is applied to determine the interest rate swaps that characterize both upward and downward markets, in agreement with the normative that defines them as benchmarks for the measurement of the term structure.
PCA is widely used as it is particularly suitable in reducing the problem size characterized by a high number of variables, such as the one of the analysis of interest rates dynamics. [Loretan (1997)] applies PCA to produce scenarios of the interest rate curve and the equity market; [Frey (1997)] calculates the Value-at-Risk of a portfolio made up of coupon bonds and interest rate options through a limited number of scenarios of the interest rate curve, obtained from a linear combination of correct one-way shocks of the first four principal components. Even [Jamshidian and Zhu (1997)] measure the interest rate risk exposure using a model based on scenarios obtained by PCA. They point out the convenience in terms of computational efficiency compared with the Monte Carlo simulation. [Novosyolov and Satchkov (2008)] apply it to describe the behaviour of rates in a global context analising three possible versions.
In [Kreinin et al. (1998)] and in [Fiori and Iannotti (2006)] the Value-at-Risk for the interest rate risk is estimated by using the combination of the Monte Carlo simulation and PCA. The articles are different regarding the distribution of risk factors, the first authors assume it to be normal and the second obtain it by the empirical distribution of principal components using a non-parametric technique. Differing from these, the proposed approach to fix the upward and downward interest rate swap does not use any distributional hypothesis and it does not need any technique for the estimation. Following that, the stressed structures are evaluated applying the Smith-Wilson method. The Regulator recommends this method as the technique for the measurement of the market structure and it chooses the prices of interest rate swaps as data for the calibration, concluding that it is coherent to obtain the scenarios for the interest rate structure by shocked swaps.
The theory for the valuation of the technical provision is found in [De Felice and Moriconi (2005)]. They apply the Cox-Ingersoll-Ross model ([Cox et al. (1985)]) to describe the uncertainty connected with interest rates and they explain that the model is economically sound as it does not produce non negative rates and is mathematically tractable. In the current market, characterized by negative rates, the model does not seem suitable. In addition, it allows the estimation of yield curves that only have a monotonic trend, and is not able of reproducing, for instance, a humped curve which is often observed. In the paper the Cox-Ingersoll-Ross model with a barrier, that can be also negative, is introduced and then it is extended with a deterministic shift as in [Brigo and Mercurio (2001)]. This allows the exact fit of an observed term structure

of interest rates.
The Italian with-profit insurance contracts have a minimum guaranteed rate, contractually fixed, that is cliquet. They are part of participating life insurance. There are various articles about this topic. [Zemp (2011)] analyses five different bonus distribution models, widespread in the European market. He implements the Black and Scholes model for the valuation as [Bacinello (2001)], [Bauer et al. (2006)] and [Kling et al. (2007)]. [Bernard et al. (2005)] and [Graf et al. (2011)] introduce the uncertainty of interest rate applying the Vasicek model. Other reference papers are [Briys and de Varenne (1997)], [Grosen and Jørgensen (2000)], [Grosen and Jørgensen (2002)]. [Eckert et al. (2016)] study a participating life insurance considering an asset portfolio with defaultable bonds and equities. They examine the impact of credit risk on the fair value of the contract and take the shortfall probability as risk measurement.
These papers are concentrated on the fair (or market-consistent) valuation. We thus contribute to the existing literature by examining the pricing problem to calculate the regulatory capital in the Standard Model framework.
[Floryszczak et al. (2016)] calculate the solvency capital of a company portfolio which has equities, risk free bonds and cash as assets and with-profit contracts with a guaranteed rate as liabilities. They use the standard Cox-Ingersoll-Ross model for the interest rate uncertainty and apply the least-squares Monte Carlo method to calculate the capital requirement, thus developing the assessment in an internal model context.
In the balance sheet of an Italian insurance company these contracts are financially significant as its portfolios have guaranteed minimum higher than the current rates, so they are unlikely to be realizable in the present market. Therefore, to measure the interest rate risk exposure is essential for the company solvency.
We emphasize that the valuation of assets (the reference portfolio) and liabilities (technical provision) is market-consistent as it uses the information provided by the financial market, represented by the term structure of interest rates measured according to technical rules. The pricing of with-profit policy requires the Monte Carlo simulation ([Boyle (1997)]) as the minimum guarantee determines a non-standard option (cliquet option) in the contract. [Bauer et al. (2006)] and [Eckert et al. (2016)] also implement the Monte Carlo simulation.
The paper is organized as follows. In Section 2 the calculation of solvency capital for the interest rate risk, as defined in the Standard Formula, is presented. The method to estimate the stressed structures of interest rates is illustrated and the insurance contract is introduced. It shows the formula of technical provision too. In Section 3 the Smith-Wilson method and the Cox-Ingersoll-Ross model with exogenous barrier (reference model) are presented and its extension too. Section 4 introduces the implementation technique for the Monte Carlo simulation and Section 5 shows some numerical results obtained by using different reference portfolios. The analysis is developed with regard to an individual policy and a policies portfolio. We also

calculate the solvency capital in relation to different values of the minimum guaranteed interest rate and a comparison is made with the current regulatory framework. The conclusions are drawn in the last section.

2 Standard Formula for the interest rate risk

2.1 Solvency Capital Requirement

In the Standard Formula, the amount of solvency capital for the interest rate risk is calculated by using the scenario-based methodology: two different structures of interest rates are calculated, one defines the upward market, the other the downward market. Then, the higher variation of the value of Basic Own Funds (BOF), obtained by recalculating the values of assets and liabilities, is the regulatory capital.

Indicated by $A(t)$ and $V(t)$ the market value of assets and liabilities at time t, respectively, the difference is called BOF at time t:

$$\mathrm{BOF}(t) = A(t) - V(t)\,,$$

given the structure of risk-free interest rates at time t, $\{i(t,s)\}$, with $s \geq t$.

If $\{i^1(t,s)\}$ and $\{i^2(t,s)\}$ are the structures that define the upward and downward market, respectively, the variation of BOF is calculated by:

$$\Delta\mathrm{BOF}^k = \mathrm{BOF}(t) - \mathrm{BOF}^k\,, \tag{1}$$

with $k = 1, 2$ and where

$$\mathrm{BOF}^k = A^k - V^k\,, \tag{2}$$

is the value of BOF obtained by recalculating the values of assets and liabilities portfolio A^k and V^k, according to the two structures. Thus, SCR is given by:

$$\mathrm{SCR} = \max\{\Delta\mathrm{BOF}^1, \Delta\mathrm{BOF}^2, 0\}\,. \tag{3}$$

2.2 Measurement of scenarios

To determine the upward and downward yield curves, EIOPA applies the Principal Component Analysis to four datasets of rates: euro and pound swap rates, euro and pound government zero coupon term structures. It takes the annual relative changes of rates over an historical 12 year interval. The shock applied to the interest rate, for a fixed maturity between 1 and 20 years, is calculated through the average of 4 shocks obtained by each dataset. Every shock is the quantile at confidence level 99.5% (upward) and 0.5% (downward). The shocks, for non-quoted maturity, are calculated by linear interpolation and for the maturity between 21 and 89 years too. If the maturity is equal or greater than 90 years, the shock is fixed at 20% in both directions. The upper maximum change is 1% and the downward shock is zero if the rate is negative[1].

We propose a different approach (M1) that allows getting over some shortcomings. It is the same used in Abdymomunov and Gerlach (2014). PCA is applied to determine the stressed swap rates at every quoted maturity by calibration at confidence level 99.5% and 0.5%. They define the upward and downward markets and represent data for the calibration of the Smith-Wilson method with which the stressed risk-free term structures are obtained.

Let $\boldsymbol{J}$ denote the $N \times T$ matrix of observed swap rates and $\boldsymbol{\Sigma}$ the covariance matrix. $\boldsymbol{\Sigma} = \boldsymbol{P\Lambda P}^T$, where $\boldsymbol{\Lambda}$ is the diagonal matrix of eigenvalues ranked in decreasing order. New data given by $\boldsymbol{Y} = \boldsymbol{P}^T(\boldsymbol{J} - \boldsymbol{\mu})$, where $\boldsymbol{\mu}$ is the vector of averages, is called principal components (PC). They are uncorrelated with each other and have variance equal to eigenvalues. The columns of matrix $\boldsymbol{P}$ are the coefficients used in the linear combination to calculate every single principal component and are called factor loadings.

The contribution of the principal component i to the total data variation is given by:

$$\frac{\lambda_i}{\sum_{i=1}^{N} \lambda_i} .$$

Choosing a number $n \ll T$ of principal components that explains almost all of the total data variance, the absolute annual changes of N factor loadings, related to the first n principal components, are calculated by $\boldsymbol{\Delta}_i^n = \boldsymbol{\Delta}_i(\cdot, 1:n)$. Then the changes $\boldsymbol{\Delta}_i^n$ be added to current factor loadings $\boldsymbol{P}_0^n = \boldsymbol{P}(N, 1:n)$, thus $N - 250$ swap rate curves are generated:

$$\widetilde{j}_m^i = (\boldsymbol{P}_0^n + \boldsymbol{\Delta}_i^n)\boldsymbol{Y}(i, 1:n) + \mu_i .$$

Upward and downward shocks, s_m and d_m, for the maturity m, are calculated by:

$$s_m : \boldsymbol{P}\big[(\widetilde{j}_m^i - j_m^0) \leq s_m\big] = 0.995 ,$$

$$d_m : \boldsymbol{P}\big[(\widetilde{j}_m^i - j_m^0) \leq d_m\big] = 0.005 ,$$

where j_m^0 is the interest rate swap at current time t. Upward and downward swaps, j_m^1 and j_m^2, are obtained by:

$$j_m^1 = \max\{j_m^0 + \frac{s_m}{\widehat{v}}|j_m^0|, j_m^0 + v^+\} , \tag{4}$$

$$j_m^2 = \max\{j_m^0 + \frac{d_m}{\widehat{v}}|j_m^0|, j_m^0 + v^-\} , \tag{5}$$

[1] The information about the calculation of shocked rates can be found in [EIOPA (2010)] and in the excel file, published every month, that contains term structures of risk-free interest rates for each currency and shocked term structures.

where $\widehat{v}$ is the average of absolute observed changes, whereas v^+ and v^- are the averages of positive and negative changes, respectively.
Starting from 2006-09-01 to 2018-04-30, $N = 3037$ observations of interest rate swaps are used and $n = 4$ principal components are considered. The first component identifies the level, the second one the slope, the third one the curvature and the latter the twist. Table 1 shows the contribution of each one to the total variance and the cumulated variance. Their values are reported in Table 2.

Tabella 1: Contribution of each principal component to the total variance and cumulated variance.

PC	Contribution (%)	Cumulated Var. (%)
1	97.824	97.824
2	2.071	99.895
3	0.081	99.975
4	0.021	99.996

Tabella 2: Principal components: eigenvectors of the covariance matrix.

m	PC1	PC2	PC3	PC4
1	-0.303	-0.590	0.614	0.297
2	-0.299	-0.413	-0.023	-0.331
3	-0.298	-0.257	-0.288	-0.333
4	-0.295	-0.130	-0.348	-0.139
5	-0.290	-0.026	-0.313	0.060
6	-0.285	-0.056	-0.238	0.193
7	-0.279	0.118	-0.151	0.255
8	-0.273	0.162	-0.065	0.260
9	-0.268	0.196	0.016	0.223
10	-0.263	0.223	0.090	0.178
12	-0.256	0.266	0.199	0.065
15	-0.248	0.306	0.290	-0.155
20	-0.240	0.319	0.323	-0.624

The curve of current swaps, j_m^0, and the estimated upward and downward ones, j_m^1 e j_m^2, are shown in Fig. 1[2]. The sharp drop in rates, that has characterized the market in last years, causes heavily declined levels of interest rate swaps.

2.3 Insurance contract

The benefit of the insurance contract is determined by the annual rate of return earned by the asset portfolio (reference portfolio) made up of coupon bonds, free from default risk. The

[2] We plot swap rates up to 20 years as the Smith-Wilson method is calibrated on quoted swaps rates up to that maturity, according to regulatory technical specifications (see [EIOPA (2018)]).

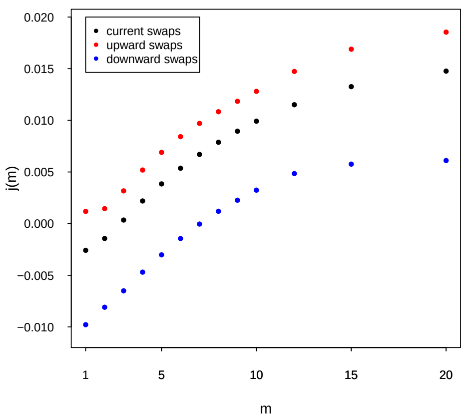


Figura 1: Swap rates at date t=2018-04-30 and estimated upward and downward swap rates.

potential revaluation of the benefit, compared to the previous year's one, is an indexation rule and the presence of the minimum guaranteed interest rate induces an optional non-standard component on the contract.

The contract is a single premium policy that pays the benefit in case of death and survival. Let x be the age of the policyholder at time t and i be set as the technical rate and the probability distribution of random variable T_x, that is the residual life maturity of the policyholder[3].

The benefit $\widetilde{Y}_n$, possibly paid at the end of the year $t_n = t + n$, with $n = 1, 2, \dots, m-1$, is:

$$\widetilde{Y}_n = Y_{n-1} \begin{cases} 1 + \rho_n & P_t(t_{n-1} < T_x \leq t_n) \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

At the maturity $T = t + m$, if the policyholder is alive at the age $x + m - 1$, is:

$$\widetilde{Y}_m = Y_{m-1}(1 + \rho_m). \tag{7}$$

Given i_{min} the minimum guaranteed interest rate, not inferior to i, and β the participation coefficient, with $0 \leq \beta \leq 1$, both of them contractually specified, the revaluation rate is:

$$\rho_n = \frac{\max\{\beta I_n, i_{min}\} - i}{1 + i}, \tag{8}$$

where $I_n = \frac{P_n}{P_{n-1}} - 1$ is the reference portfolio return over the year $[t_{n-1}, t_n]$.

[3]The probability distribution of T_x is obtained by the mortality table for males, for the year 2015, from Italian National Institute of Statistics.

It is easy to verify that the revaluable benefit can be written as:

$$Y_n = Y_0(1+i)^{-n}\prod_{k=1}^{n}(1+\max\{\beta I_k, i_{min}\})\,, \tag{9}$$

as well as:

$$\begin{aligned} Y_n &= Y_{n-1}(1+i)^{-1}(1+\beta I_n) + \\ & \quad Y_{n-1}(1+i)^{-1}\frac{\beta}{P_{n-1}}\max\left\{\left(1+\frac{i_{min}}{\beta}\right)P_{n-1}-P_n, 0\right\}\,. \end{aligned} \tag{10}$$

Eq. (10) defines the benefit as the sum of the base component (without any guarantee) and the put component having the reference portfolio as the underlying asset. This put is in-the-money if I_n is not greater than i_{min}/β. The insurance contract, then, generates a series of forward-starting at-the-money put options whose strike price is updated every year to the initial value. Thus, we can individuate an embedded cliquet option.

The cash-flow of the benefits, $\widetilde{\boldsymbol{Y}} = \{\widetilde{Y}_n, n = 1, 2, \ldots, m\}$, is exposed both to financial risks, that are in the reference portfolio, and to actuarial risks, as the result of the uncertainty of the residual life maturity T_x.

According to Solvency II Directive, the technical provision is calculated by[4]:

$$\begin{aligned} V(t) &= \sum_{n=1}^{m} V(t;\widetilde{Y}_n) \\ &= Y_0\sum_{n=1}^{m}(1+i)^{-n}V\left(t;\prod_{k=1}^{n}[1+\max\{\beta I_k, i_{min}\}]\right)\boldsymbol{P}_t^n\,, \end{aligned} \tag{11}$$

where $\boldsymbol{P}_t^n$ is the appropriate probability (of death or of survival)[5].

It can be expressed as:

$$V(t) = Y_0\sum_{n=1}^{m}(1+i)^{-n}\left[V\left(t;\prod_{k=1}^{n}\left[(1+\beta I_k)+\frac{\beta}{P_{k-1}}X(t_k;P_k)\right]\right)\right]\boldsymbol{P}_t^n\,, \tag{12}$$

where $X(t_k;P_k)$ is the pay-off of the put option at time t_k, as in Eq. (10).

[4]In [EIOPA (2014)] it is written that the risk margin is excluded within the calculation of SCR for elementary risks, thus the technical provision is equal to the best estimate.

[5]As we assume that the policyholder can get out of the policy before maturity T only because of death, we can multiple two expectations in Eq. (11). See [De Felice and Moriconi (2005)].

3 Models

This section introduces the models of term structure used to calculate the best estimate of the with-profit insurance contract and SCR for the interest rate risk, according to Standard Formula. The Smith-Wilson method is specified by illustrating technical details for the estimation, using quoted swap rates. Then, we present the Cox-Ingersoll-Ross model with barrier which is the reference model for the extension and we give a closed formula for the price of a unit zero coupon bond. At the end of the section we apply the extension proposed by [Brigo and Mercurio (2001)] to get the perfect-fit of observed rates.

3.1 Smith-Wilson method

The Smith-Wilson method is used to measure the term structure of risk free interest rates that is the input for the stochastic model, as suggested by [de Kort and Vellekoop (2016)]. We take the cue for this treatment from their article to which we refer for further insights regarding the topic.

Let $v_w(0,s)$ be the price of a unit zero coupon bond at time 0 with maturity s. In the Smith-Wilson method it has the following functional form:

$$v_w(0,s) = (1 + g(s))\, e^{-f_\infty s}\,, \tag{13}$$

where

$$g(s) = \sum_{i=1}^{n} \xi_i \sum_{j=1}^{m} x_{ij} e^{-f_\infty t_j} W(s,t_j)\,, \tag{14}$$

where f_∞ is a macroeconomic exogenous parameter, called ultimate forward rate, that is the asymptotic long term rate (mean-reverting property), t_j is the paying time of the amount x_{ij}, $\boldsymbol{\xi} = (\xi_i)$ is the vector of n parameters to be estimated and W is the Wilson function, well-known as exponential spline in tension:

$$W(s,t_j) = \alpha \min(s,t_j) - \frac{e^{-\alpha|s-t_j|} - e^{-\alpha(s+t_j)}}{2}\,,$$

where α is a positive constant.

The calibration of the method uses quoted swap rates, therefore the row i of the matrix $\boldsymbol{X} = (x_{ij})$ is the cash-flow generated by the quoted swap rate j_T^i with $T \in \{1, 2, \ldots, m\}$. It is, then, $\boldsymbol{x}_i = \{j_T^i, j_T^i, \ldots, 1 + j_T^i, 0, \ldots, 0\}$ and the number of zeros in $\boldsymbol{x}_i$ is $m - T$. As the swap rate is a par-yield, we can write:

$$\sum_{l=1}^{m} x_{il}(1 + g(l))e^{-f_\infty l} = 1\,, \tag{15}$$

that is a linear equation with n unknowns ξ_i. For $i = 1, 2, \ldots, n$, using Eqs. (13) and (14), we get:

$$\boldsymbol{X}\boldsymbol{\mu} + \boldsymbol{\xi}(\boldsymbol{X}\boldsymbol{W}_\mu\boldsymbol{X}^T) = \mathbf{1}\,, \tag{16}$$

where $\boldsymbol{\mu} = (e^{-f_\infty j})$ and $\boldsymbol{W}_\mu = \boldsymbol{\mu}^T\boldsymbol{W}\boldsymbol{\mu}$.
If $\det(\boldsymbol{X}\boldsymbol{W}_\mu\boldsymbol{X}^T) \neq 0$, the solution[6] is calculated by $\boldsymbol{\xi} = (\boldsymbol{X}\boldsymbol{W}_\mu\boldsymbol{X}^T)^{-1}(\mathbf{1} - \boldsymbol{X}\boldsymbol{\mu})$ and it completely defines the term structure of interest rates $\{i_w(0,s)\}$, where $i_w(0,s) = v_w(0,s)^{-1/s} - 1$.
To obtain the solution of Eq. (16), we need the values of the asymptotic long term rate f_∞ and the speed of convergence α. The first is decided by EIOPA and is calculated as the sum of the expected inflation and the long term average of short term real rates. The constant α is calculated in such a way that the difference between f_∞ and the forward rate $\delta_w(0,T)$ is less than 1 basis point. T is the convergence point and is equal to 60 years for the euro. The function $\delta_w(0,s) = -\frac{\partial}{\partial s}\ln v_w(0,s)$ is:

$$\delta_w(0,s) = f_\infty - \frac{1}{1+g(s)}\sum_{i=1}^{n}\xi_i\left(\sum_{j=1}^{m}x_{ij}e^{-f_\infty t_j}\frac{\partial}{\partial s}W(s,t_j)\right)\,. \tag{17}$$

We have:

$$\frac{\partial}{\partial s}W(s,t_j) = \begin{cases} \frac{\alpha}{2}e^{-\alpha s}(e^{\alpha t_j} - e^{-\alpha t_j}) & s \geq t_j \\ \alpha - \frac{\alpha}{2}e^{-\alpha t_j}(e^{\alpha s} - e^{-\alpha s}) & s < t_j \end{cases}$$

and, because of $T > t_j$ for every j, we get:

$$\delta_w(0,T) = f_\infty + \frac{\alpha e^{-\alpha T}}{2(1+g(T))}\sum_{i=1}^{n}\xi_i\left(\sum_{j=1}^{m}x_{ij}e^{-f_\infty t_j}(e^{\alpha t_j} - e^{-\alpha t_j})\right)\,. \tag{18}$$

It is easy to verify that $\lim_{s\to\infty}\delta_w(0,s) = f_\infty$ and

$$\lim_{s\to 0}\delta_w(0,s) = f_\infty - \sum_{i=1}^{n}\xi_i\sum_{j=1}^{m}e^{-f_\infty t_j}\,.$$

The value of the parameter α is determined in such a way that $|\delta_w(0,T) - f_\infty| \leq 0.0001$:

$$\left|\frac{\alpha e^{-\alpha T}}{2(1+g(T))}\sum_{i=1}^{n}\xi_i\left(\sum_{j=1}^{m}x_{ij}e^{-f_\infty t_j}(e^{\alpha t_j} - e^{-\alpha t_j})\right)\right| \leq 0.0001\,. \tag{19}$$

Note that, if in Eq. (18) the term multiplying α is equal to 1, the forward rate is constant, so there would be no chance for any convergence to the asymptotic value. Therefore, α must be

[6] No problems were encountered regarding the calculation of the inverse matrix over the time period from 2001-02-02 to 2018-04-30.

calculated in such a way that the following equation does not occur:

$$\sum_{i=1}^{n} \xi_i \sum_{j=1}^{m} x_{ij} e^{-f\infty t_j} (e^{-\alpha(T-t_j)} - e^{-\alpha(T+t_j)} - \alpha t_j) = 1\,.$$

The parameter f_∞ is set equal to ln(1.0405) and the parameter α is estimated by calculating the value that satisfies Eq. (19) using an iterative procedure with the lower bound equal to 0.05 (see [EIOPA (2018)]).

Fig. 2 plots the yield curve $\{i_w^0(0,s)\}$ obtained by swap rates j_m^0 at the date t =2018-04-30 and the yield curves $\{i_w^1(0,s)\}$ e $\{i_w^2(0,s)\}$ obtained by swap rates j_m^1 e j_m^2, calculated in section 2.2, from 1 up to 90 years. In the same figure we also draw the term structures obtained by applying shocks calculated by EIOPA (M2). Table 3 shows the estimated parameters of the Smith-Wilson method and the estimated values of the speed of convergence α.

We note that: 1. the upward structure by EIOPA is actually a parallel shift of the current one, 2. the downward structure by EIOPA is few far from the current one at short maturities because rates are near to zero, 3. both stressed structures by EIOPA do not converge to an asymptotic value and this is against the mean-reverting property usually required.

The proposed method produces a further downward curve and this reflects the decreasing trend of short term rates observed in the last few years. Furthermore, the stressed curves of the forward rate also converge to the parameter f_∞.

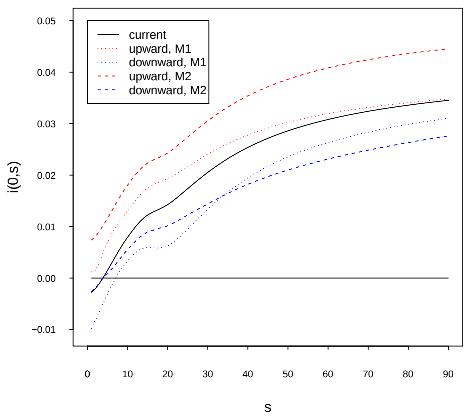


Figura 2: Term structure of risk free interest rates at time t=2018-04-30 and stressed structures.

3.2 Reference model

Let $\{x_t\}$ be a diffusion process whose dynamic in the real world is described by the following stochastic differential equation:

Tabella 3: Estimated parameters of the Smith-Wilson method and estimated value of the speed of convergence α.

$\boldsymbol{\xi}$	0	1	2
ξ_1	-0.176	-3.801	0.793
ξ_2	1.427	3.302	-1.286
ξ_3	0.317	0.826	1.783
ξ_4	-0.753	-1.557	-1.246
ξ_5	0.749	0.978	1.064
ξ_6	-0.449	-0.649	-0.705
ξ_7	-0.068	0.070	0.556
ξ_8	0.037	-0.136	-0.807
ξ_9	0.604	-0.683	0.356
ξ_{10}	-0.830	1.453	0.874
ξ_{11}	0.836	0.069	-0.711
ξ_{12}	-1.516	-1.461	-1.858
ξ_{13}	1.306	1.074	2.403
α	0.148	0.145	0.155

$$dx_t = \theta(\gamma - x_t)dt + \rho\sqrt{x_t - l}dZ_t\,, \tag{20}$$

where Z_t is a standard Brownian motion, θ, γ, ρ are positive constants and l is a fixed real number. The evolution of variable x_t is mean-reverting with asymptotic long term rate γ and θ is the speed for its return to the mean value. It is characterized by a barrier l not to be overcome. If $l = 0$, Eq. (20) is the well-known model proposed by [Cox et al. (1985)].
Let $y_t = x_t - l$ be set, Eq. (20) can be written as:

$$dy_t = \theta(\gamma_l - y_t)dt + \rho\sqrt{y_t}dZ_t\,,$$

with $\gamma_l = \gamma - l$. Using the no-arbitrage principle and the hedging argument, the price $V(t)$, depending on y_t at time t, of any contract, satisfies the following second order differential equation:

$$\frac{1}{2}g^2\frac{\partial^2 V(t)}{\partial y_t^2} + \widehat{f}\frac{\partial V(t)}{\partial y_t} + \frac{\partial V(t)}{\partial t} = y_t V(t)\,, \tag{21}$$

where $g = \rho\sqrt{y_t}$ is the diffusion function and $\widehat{f}$ is the risk-adjusted drift function given by:

$$\widehat{f} = f - qg\,. \tag{22}$$

$f = \theta(\gamma_l - y_t)$ is the natural drift and q is the market price of interest rate risk that we choose to be $q = -\frac{\pi\sqrt{y_t}}{\rho}$, with $\pi \in \mathbb{R}$. Eq. (22) is:

$$\widehat{f} = \widehat{\theta}(\widehat{\gamma} - y_t)\,, \tag{23}$$

where $\widehat{\theta} = \theta - \pi$ and $\widehat{\gamma} = \frac{\theta\gamma_l}{\theta-\pi}$.
Since $dx_t = dy_t$, the risk neutral dynamic of the variable x_t becomes:

$$dx_t = \widehat{\theta}[(\widehat{\gamma} + l) - x_t]dt + \rho\sqrt{x_t - l}dZ_t\,. \tag{24}$$

The parameter vector $\boldsymbol{b} = \{x_t, \widehat{\theta}, \widehat{\gamma}, \rho\}$ defines the risk neutral probability $\boldsymbol{Q}$ that is a non-central chi-squared distribution.
The price of the unit zero coupon bond is:

$$\begin{aligned} v_c(t,s) &= \mathrm{E}_t^{\boldsymbol{Q}}\left[e^{-\int_t^s x_u du}\right] \\ &= \mathrm{E}_t^{\boldsymbol{Q}}\left[e^{-\int_t^s (y_u+l) du}\right] \\ &= e^{-l(s-t)}\mathrm{E}_t^{\boldsymbol{Q}}\left[e^{-\int_t^s y_u du}\right]\,. \end{aligned} \tag{25}$$

Using the result in [Cox et al. (1985)], we obtain:

$$\begin{aligned} v_c(t,s) &= e^{-l(s-t)}A(t,s)e^{-y_t B(t,s)} \\ &= A(t,s)e^{-l(s-t)-(x_t-l)B(t,s)}\,, \end{aligned} \tag{26}$$

where

$$A(t,s) = \left[\frac{2de^{\frac{\widehat{\theta}+d}{2}(s-t)}}{(\widehat{\theta}+d)(e^{d(s-t)}-1)+2d}\right]^{\nu}, \tag{27}$$

and

$$B(t,s) = \frac{2(e^{d(s-t)}-1)}{(\widehat{\theta}+d)(e^{d(s-t)}-1)+2d}\,, \tag{28}$$

with $d = \sqrt{\widehat{\theta}^2 + 2\rho^2}$ and $\nu = 2\frac{\widehat{\theta}\widehat{\gamma}}{\rho^2}$. The price in Eq. (26) corresponds to the one in [Gorovoi and Linetsky (20
The forward rate is:

$$\begin{aligned} \delta_c(t,s) &= -\frac{\partial}{\partial s}\ln v(t,s) \\ &= -\frac{\partial}{\partial s}\ln A(t,s) + \frac{\partial}{\partial s}[l(s-t) + (x_t - l)B(t,s)] \\ &= -\frac{\partial}{\partial s}\ln A(t,s) + l + (x_t - l)\frac{\partial}{\partial s}B(t,s)\,, \end{aligned} \tag{29}$$

where

$$\frac{\partial}{\partial s}\ln A(t,s) = \frac{2\widehat{\theta}\widehat{\gamma}(e^{d(s-t)}-1)}{(\widehat{\theta}+d)(e^{d(s-t)}-1)+2d} \tag{30}$$

and

$$\frac{\partial}{\partial s}B(t,s) = \frac{4d^2 e^{d(s-t)}}{[(\widehat{\theta}+d)(e^{d(s-t)}-1)+2d]^2}\,. \tag{31}$$

To estimate the vector $\boldsymbol{b}$, we use a different specification of the price in Eq. (26) as it is more efficient. Setting $\sigma = \frac{\widehat{\theta}+d}{2}$ in Eqs. (27) and (28), we obtain[7]:

$$A(t,s) = \left[\frac{de^{\sigma s}}{\sigma(e^{d(s-t)}-1)+d}\right]^{\nu} \tag{32}$$

and

$$B(t,s) = \frac{e^{ds}-1}{\sigma(e^{d(s-t)}-1)+d}\,. \tag{33}$$

Given the term structures of interest rates $\{i_w^k(t,t+s)\}$, with $k=0,1,2$, we estimate three sets of parameters, $\boldsymbol{b}^0$, $\boldsymbol{b}^1$ e $\boldsymbol{b}^2$, by minimizing the sum of squared errors:

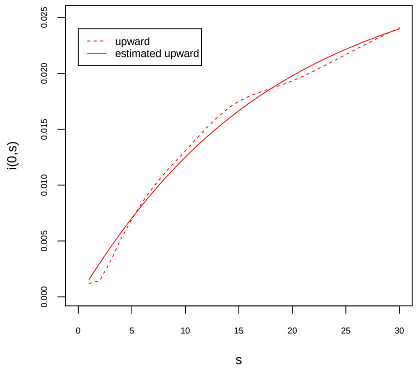
$$\min_{\boldsymbol{b}^k}\sum_{s=1}^{m}[i_w^k(t,t+s)-i_c(t,t+s)]^2\,, \tag{34}$$

where $i_c(t,t+s) = v_c(t,t+s)^{-\frac{1}{s}}-1$.
We solve the problem (34) under the following constraints: $x^k > l$, $\sigma > 0$, $d-\sigma > 0$ and $\nu > 1$. The first two are obvious, the third one is obtained by $\rho = \sqrt{2(d\sigma-\sigma^2)}$ and the last one is the so called Feller condition.
Table 4 shows the estimated parameters, Fig. 3 and 4 plot the term structures.

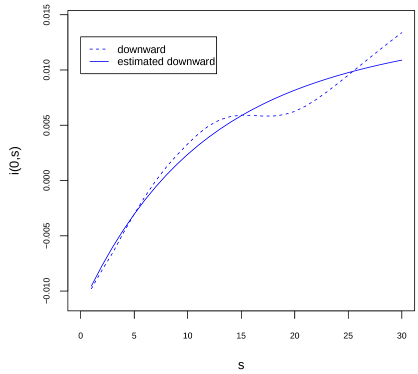
Tabella 4: Estimated parameters of the reference model.

	(a)	(b) M1		(c) M2	
	$\boldsymbol{b}^0$	$\boldsymbol{b}^1$	$\boldsymbol{b}^2$	$\boldsymbol{b}^1$	$\boldsymbol{b}^2$
l	-0.400%	0.000%	-1.200%	0.000%	-1.200%
x	-0.399%	0.000%	-1.162%	0.527%	-0.427%
d	0.133	0.116	0.167	0.149	0.153
σ	0.113	0.092	0.150	0.126	0.312
ν	1.563	1.394	1.738	1.611	1.436

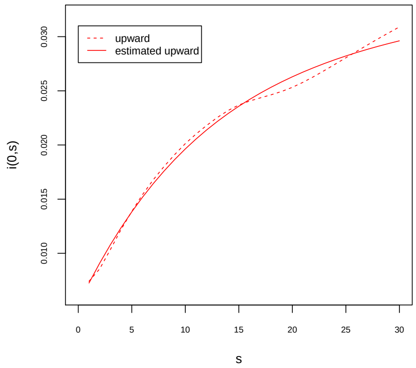
[7] After estimating the parameters d,σ,ν, for the implementation of Monte Carlo method we use: $\widehat{\theta} = 2\sigma - d$, $\rho = \sqrt{2(d\sigma-\sigma^2)}$ e $\widehat{\gamma} = \frac{\rho^2\nu}{2\widehat{\theta}}$.



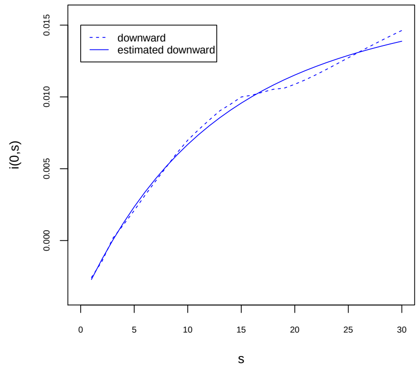
(a) Upward market, M1



(b) Downward market, M1



(c) Upward market, M2



(d) Downward market, M2

Figura 3: Estimated stressed term structures minimizing the squared errors.

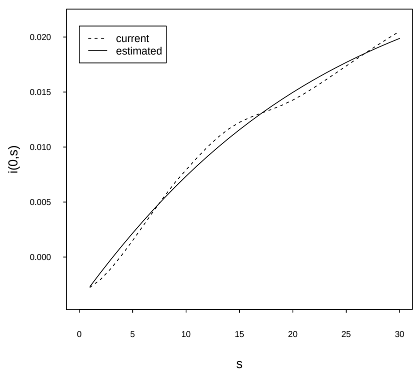


Figura 4: Estimated term structure at date t=2018-04-30 minimizing the squared errors.

3.3 Extended model

The reference portfolio is made up of risk free coupon bonds, thus the risk source is represented by the spot rate r_t, that is the interest rate of a zero coupon bond with infinitesimal maturity:

$$r_t = \lim_{s \to t} h(t,s)\,,$$

being $h(t,s) = \frac{1}{s-t}\int_t^s \delta(t,u)du$ the continuous-time yield. The gain by investing the amount x at time t over the interval $[t, t+dt]$ is $xr_t dt$.
The extension proposed defines the process of the spot rate, $\{r_t\}$, as:

$$r_t = x_t + \varphi(t)\,, \tag{35}$$

where $\{x_t\}$ is the stochastic process that evolves over time according to Eq. (20) and where φ is a deterministic function used to achieve the perfect-fit of the observed term structure. It is:

$$\varphi(t) = \delta_w(0,t) - \delta_c(0,t)\,, \tag{36}$$

where $\delta_w(0,t)$ and $\delta_c(0,t)$ are defined in Eqs. (17) and (29), respectively.
In fact, to get the perfect fit, it must be $v_w(0,t) = e^{-\int_0^t \varphi_u du} v_c(0,t)$, that is:

$$\ln v_w(0,t) = -\int_0^t \varphi_u du + \ln v_c(0,t)$$

and, differentiating both members, we obtain:

$$\varphi(t) = -\frac{\partial}{\partial t}\ln v_w(0,t) + \frac{\partial}{\partial t}\ln v_c(0,t)\,,$$

which is Eq. (36). See [Brigo and Mercurio (2001)].
In the extended model the price of the contract, that pays the amount $X(s)$ at future time s, is:

$$V(t) = \mathrm{E}_t^{\boldsymbol{Q}}\left[X(s)e^{-\int_t^s r_u du}\right]\,, \tag{37}$$

in particular, the price of the unit zero coupon bond is:

$$\begin{aligned} v(t,s) &= \mathrm{E}_t^{\boldsymbol{Q}}[e^{-\int_t^s r_u du}] \\ &= \mathrm{E}_t^{\boldsymbol{Q}}[e^{-\int_t^s (x_u+\varphi_u)du}] \\ &= e^{-\int_t^s [\delta_w(0,u)-\delta_c(0,u)]du} v_c(t,s)\,, \end{aligned} \tag{38}$$

from which we can easily obtain:

$$v(t,s) = \frac{v_w(0,s)v_c(0,t)}{v_w(0,t)v_c(0,s)} v_c(t,s)\,. \tag{39}$$

Substituting Eq. (26) in the previous one, we get:

$$v(t,s) = \overline{A}(t,s)e^{-l(s-t)-(r_t-l)B(t,s)}\,, \tag{40}$$

where

$$\overline{A}(t,s) = \frac{v_w(0,s)A(0,t)}{v_w(0,t)A(0,s)} e^{l(s-t)+(x_0-l)[B(0,s)-B(0,t)]} A(t,s)e^{\varphi(t)B(t,s)}\,.$$

4 Monte Carlo simulation

The calculation of the technical provision requires the implementation of a numerical method as a closed formula does not exist for the price in Eq. (11). Thus, the pricing problem of the insurance contract is solved by using the Feynman-Kac representation defined by Eq. (37). The market-consistent value of the provision, $V(t)$, is given by:

$$\begin{aligned} V(t) &= Y_t \sum_{n=1}^{m} (1+i)^{-n} \\ &E_t^{\boldsymbol{Q}}\left[\prod_{k=1}^{n}\left(1+\max\left\{\beta\left(\frac{P_k}{P_{k-1}}-1\right), i_{min}\right\}\right) e^{-\int_t^{t+n} r_u du}\right] \boldsymbol{P}_t^n\,. \end{aligned} \tag{41}$$

Indicated by $\{r_{\omega,t}\}$ the trajectories space, each possible future scenario is modeled through a time function:

$$\begin{aligned} r_\omega \quad : \quad & [0,+\infty) \longrightarrow S \\ & t \longmapsto r_{\omega,t} \end{aligned}$$

where S is the space of the states and $r_{\omega,t}$ is the state of the phenomenon at time t. The outcomes of the stochastic process are the trajectories, the sample space is the set of trajectories and the events are the set Ω of the trajectories:

$$\Omega = \{r_{w,t} : t \geq 0\}\,.$$

To generate a single trajectory, we make a partition of the time interval $[t,T]$ in N sub-intervals Δt long: $[t_{j-1}, t_j)$ with $t_j = t_{j-1} + \Delta t$. In each of them, using the discrete Eulero equivalent of

Eq. (24), the increase of the random variable x_t is given by:

$$\Delta x_j = \widehat{\theta}[(\widehat{\gamma} + l) - x_{j-1}]\Delta t + \rho \epsilon_j \sqrt{(x_{j-1} - l)\Delta t}\,, \tag{42}$$

where $\epsilon_j \sim N(0,1)$.

In the extended model $r_j = x_j + \varphi(t_j)$, where $\varphi(t_j)$ corresponds to Eq. (36). Thus, for $N \in \mathbb{N}$ we get the sequence $\{r_j\}_{j=1}^N$ and the simulated trajectory is $\{(t_j, r_j), j = 1, \ldots, N\}$. The approximation to the true trajectory is much better when N is a higher number.

Generated M trajectories, the estimated value of Eq. (41) is:

$$\begin{aligned} V(t) \;&=\; Y_t \sum_{n=1}^{m} (1+i)^{-n} \\ &\left(\frac{1}{M} \sum_{l=1}^{M} \left[\prod_{k=1}^{n} \left(1 + \max\left\{ \beta \left(\frac{P_k^l}{P_{k-1}^l} - 1 \right), i_{min} \right\} \right) \tilde{v}^l(t, t+n) \right] \boldsymbol{P}_t^n \right), \end{aligned} \tag{43}$$

where $\tilde{v}^l(t, t+n)$ is the simulated value of the discounting factor[8] from t to $t+n$ and P_k^l is the simulated value of the portfolio at time $t+k$, both values are related to the trajectory l:

$$P_k^l = \sum_{h=k}^{m} x_h v(t+k, t+h; r_k^l)\,. \tag{44}$$

$\boldsymbol{x} = \{x_1, x_2, \ldots, x_m\}$ is the cash-flow connected to the reference portfolio relating to $\{t+1, t+2, \ldots, t+m\}$. The amount x_k, earned at time $t+k$, is invested in the market by buying zero coupon bonds with one year maturity. Thus, we make continuous annual reinvestment up to the maturity $t+m$.

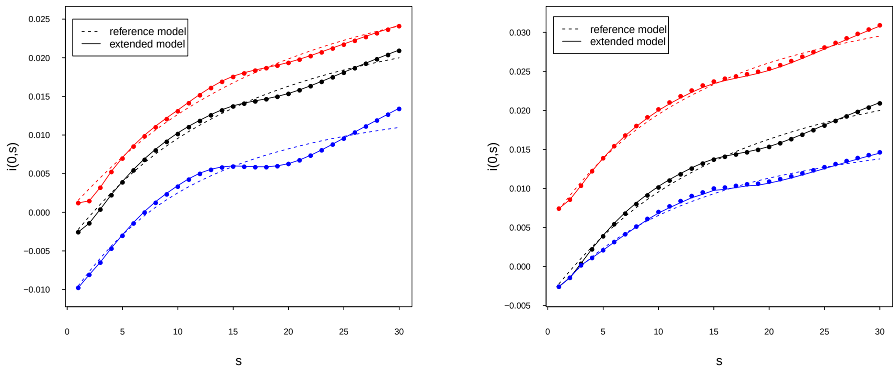
Fig. 5 shows the current term structures from 1 up to 30 years and the ones obtained by Monte Carlo simulation with the reference model and the extended model. We also display the ones obtained in the upward and downward markets. In the simulation we fix $\Delta t = 1/12$, $T = 30$ and $M = 10000$. The parameters are exhibited in Table 3 and 4. We underline the improvement given by the extended model[9].

5 Numerical results

The valuation (of technical provision and SCR) is carried out at the market state at date t =2018-04-30, taken as time origin ($t = 0$) and defined by the term structure shown in Fig. 2. The company liability is a single policy with a single premium earned before 0, with $\beta = 80\%$, maturity $m = 30$, where x is the policyholder's age of 35 years at time 0, and with initial insured

[8] To estimate the simulated discounting factor at every maturity, we sum the area of the single trapezoids.

[9] To simulate the stressed curves by EIOPA, the parameters of the Smith-Wilson method are estimated using the prices of zero coupon bonds obtained by the term structures. The function $g(s)$ is different from Eq. (14).



(a) M1 (b) M2

Figura 5: Estimated and simulated term structures with the reference model and the extended model.

capital $Y_0 = 100$. The company asset is a portfolio of risk free coupon bonds with $C = 100$ the nominal capital, ζ_i the annual yield and m_i the maturity (see Table 5).

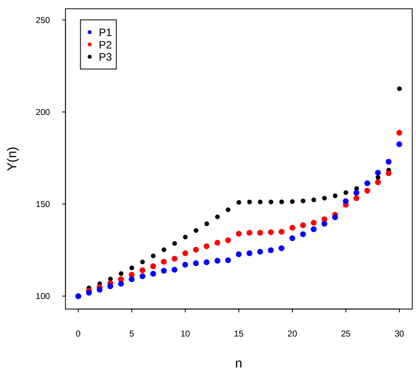
Tabella 5: Risk free bonds.

	$\zeta_i(\%)$	m_i	$V(0; \boldsymbol{x}_i)$
1	0.5	5	100.575
2	1.0	10	100.077
3	1.5	15	102.421
4	2.0	20	102.299
5	2.3	25	112.611
6	2.5	30	113.900

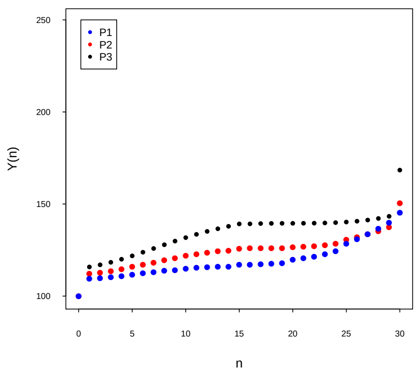
We create three portfolios $\boldsymbol{P}^j = \{\alpha_i \boldsymbol{x}_i\}_{i=1,\ldots,6}$ as follows: $\boldsymbol{t} = \{1, 2, \ldots, m\}$ is the time of payments and q_j is the percentage of bond $\boldsymbol{x}_6$:

$$\alpha_i = \frac{(1 - q_j)V(0)}{5V(0; \boldsymbol{x}_i)}, \ \alpha_6 = \frac{q_j V(0)}{V(0; \boldsymbol{x}_6)},$$

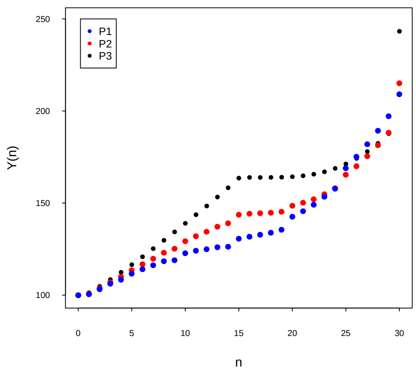
where $i = 1, 2, \ldots, 5$ and $V(0)$ is the value of the technical provision at time 0, calculated by Eq. (43). We consider the following values: $q_1 = 0.10$, $q_2 = 0.50$ e $q_3 = 1.00$.
By construction $V(0; \boldsymbol{P}^j) = V(0)$, that being $\mathrm{BOF}(0) = 0$. Thus the calculated SCR, if positive, is capital that the company must add.
Fig. 6 shows the mean value of annual benefits estimated for each portfolio, where the technical rate i and the minimum guaranteed rate i_{min} are both equal to 0.5%. The portfolio $\boldsymbol{P}^3$ produces higher revaluable benefits and lower over the period 25-29 years in the upward market. The revaluation is stronger applying shocks by EIOPA and this is much more evident in the upward market.



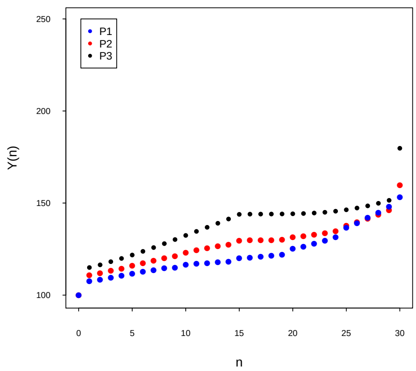
(a) Upward market, M1



(b) Downward market, M1



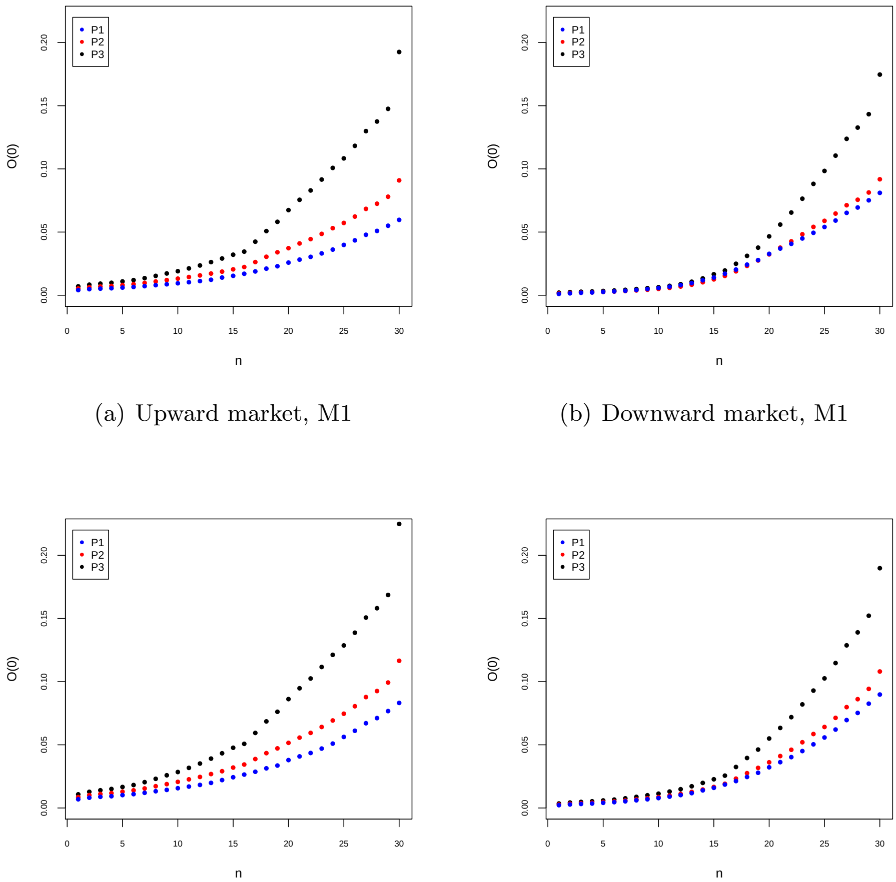
(c) Upward market, M2



(d) Downward market, M2

Figura 6: Average revaluable benefits.

Fig. 7 shows the trend of the optional component for each annual benefit. It increases for each portfolio. The value is higher for the portfolio $\boldsymbol{P}^3$, that being for the portfolio with stronger revaluation, as the optional component in one year is made up of the ones from previous years, then, if the revaluation rate is higher, the quantity of the in-the-money option increases (see Eqs. (9) and (10)).



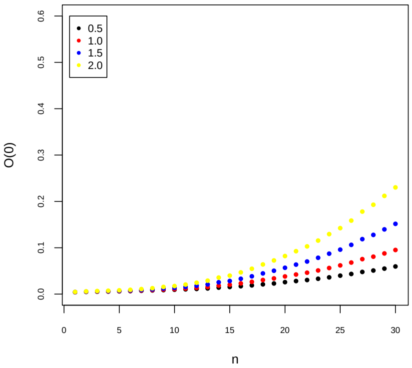
(a) Upward market, M1

(b) Downward market, M1

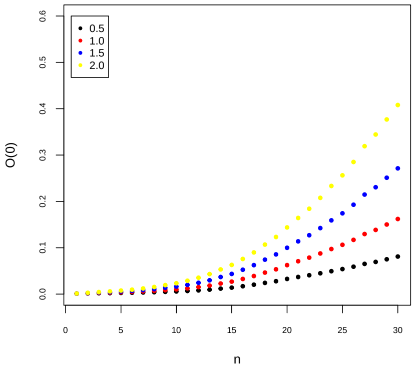
(c) Upward market, M2

(d) Downward market, M2

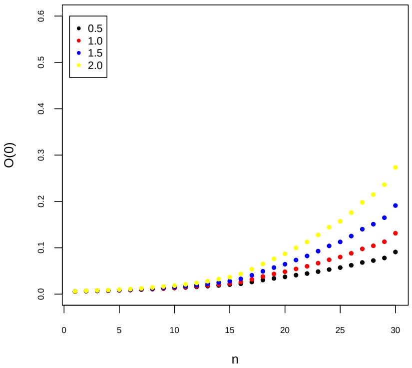
Figura 7: Put component.



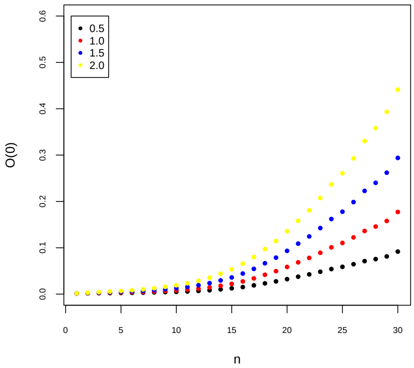
(a) Upward market, $\boldsymbol{P}^1$



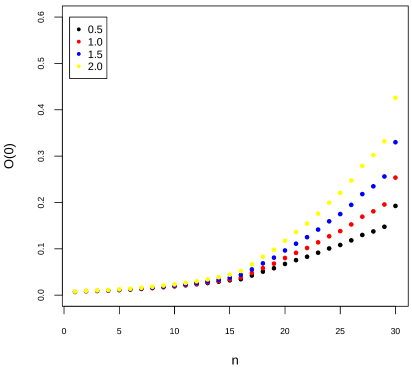
(b) Downward market, $\boldsymbol{P}^1$



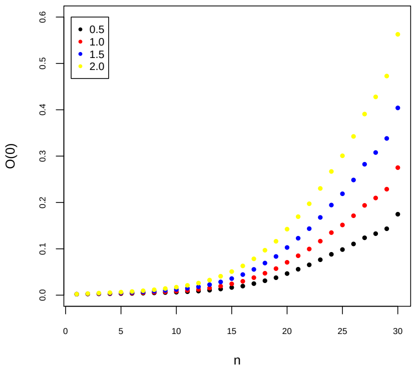
(c) Upward market, $\boldsymbol{P}^2$



(d) Downward market, $\boldsymbol{P}^2$



(e) Upward market, $\boldsymbol{P}^3$



(f) Downward market, $\boldsymbol{P}^3$

Figura 8: Put component, M1.

The value of SCR is strongly affected by the portfolio as the value of both components (of asset and liability) depends on it: it is the company asset and determines the benefits of the liability as they depend on the annual yield of the portfolio.
Table 6 shows the values calculated by Eq. (3). We also display the value of technical provision $V(0)$, the stressed values V^1 and V^2 and the ones of portfolio $\boldsymbol{P}^{j,1}$ and $\boldsymbol{P}^{j,2}$. For the first two portfolios the regulatory capital comes from the downward market. The conservative calibration of shocks by the European Authority is clear, the value in fact triples in the case of $\boldsymbol{P}^3$.
The value of the put option increases as the value of the minimum guaranteed rate increases and the gap increases as the maturity increases (see Fig. 8).

Tabella 6: SCR, $i = i_{min} = 0.5\%$.

q_j	$V(0)$	V^1	V^2	$\boldsymbol{P}^{j,1}$	$\boldsymbol{P}^{j,2}$	SCR
M1						
0.10	88.097	84.232	102.304	84.021	98.353	3.951
0.50	91.579	87.500	106.219	86.494	104.846	1.374
1.00	102.586	98.182	118.358	95.704	121.096	2.478
M2						
0.10	88.097	78.799	98.942	77.239	93.383	5.559
0.50	91.579	81.247	103.797	76.600	98.417	3.940
1.00	102.586	91.361	116.048	82.464	114.344	7.812

Table 7 shows the values of SCR for each portfolio that is again built by matching its value with the best estimate. Fixed q_j, SCR increases as the minimum guaranteed rate increases, except in two cases (shown in bold text). The comparison between the two methods again shows the conservative valuation by the European Regulator. In some cases the value is three times more.

Tabella 7: SCR to changing i_{min}.

i_{min}	0.5	1.0	1.5	2.0
M1				
$\boldsymbol{P}^1$	3.951	7.161	10.498	13.877
$\boldsymbol{P}^2$	1.374	4.371	7.853	11.511
$\boldsymbol{P}^3$	2.478	**2.031**	**1.722**	4.679
M2				
$\boldsymbol{P}^1$	5.559	8.671	11.874	15.127
$\boldsymbol{P}^2$	3.904	7.014	10.375	13.768
$\boldsymbol{P}^3$	7.812	**7.307**	**6.287**	8.566

Tables 8, 9 and 10 show the values of the base component B and the optional component O in each market. The stressed value of the portfolio is shown too. Obviously, the base value does not change as the minimum guaranteed rate increases, unlike the optional value which increases. Furthermore, the change of the optional component, compared to the current value, is higher in the market that produces SCR.

Tabella 8: Base and put components, $\boldsymbol{P}^1$.

i_{min}	0.5	1.0	1.5	2.0	0.5	1.0	1.5	2.0
	M1				M2			
$V(0)$	88.097	92.828	100.066	109.795	88.097	92.828	100.066	109.795
$B(0)$	82.629	82.629	82.629	82.629	82.629	82.629	82.629	82.629
$O(0)$	5.469	10.199	17.437	27.166	5.469	10.199	17.437	27.166
V^1	84.232	87.943	93.822	102.017	78.799	81.604	85.860	91.789
B^1	78.211	78.211	78.211	78.211	70.185	70.185	70.185	70.185
O^1	6.021	9.732	15.611	23.806	8.614	11.420	15.675	21.604
V^2	102.304	110.796	122.213	136.454	98.942	107.068	117.943	131.509
B^2	94.187	94.187	94.187	94.187	89.910	89.910	89.910	89.910
O^2	8.117	16.608	28.026	42.267	9.032	17.159	28.034	41.600
$P^{1,1}$	84.021	88.533	95.436	104.715	77.239	81.387	87.733	96.263
$P^{1,2}$	98.353	103.634	111.715	122.577	93.383	98.397	106.069	116.382

Tabella 9: Base and put components, $\boldsymbol{P}^2$.

i_{min}	0.5	1.0	1.5	2.0	0.5	1.0	1.5	2.0
	M1				M2			
$V(0)$	91.579	96.717	104.245	114.391	91.579	96.717	104.245	114.391
$B(0)$	83.300	83.300	83.300	83.300	83.300	83.300	83.300	83.300
$O(0)$	8.279	13.417	20.945	31.091	8.279	13.417	20.945	31.091
V^1	87.500	91.677	97.841	106.337	81.247	84.448	88.920	95.003
B^1	78.203	78.203	78.203	78.203	69.112	69.112	69.112	69.112
O^1	9.296	13.474	19.638	28.134	12.135	15.336	19.808	25.891
V^2	106.219	115.100	127.199	142.473	103.797	112.512	124.084	138.546
B^2	97.048	97.048	97.048	97.048	103.797	112.512	124.084	138.546
O^2	9.171	18.052	30.151	45.425	10.882	19.597	31.169	45.630
$P^{2,1}$	86.494	91.347	98.457	108.040	77.756	82.118	88.509	97.124
$P^{2,2}$	104.846	110.728	119.346	130.962	99.894	105.499	113.710	124.777

Tabella 10: Base and put components, $\boldsymbol{P}^3$.

i_{min}	0.5	1.0	1.5	2.0	0.5	1.0	1.5	2.0
	M1				M2			
$V(0)$	102.586	109.742	118.793	130.126	102.586	109.742	118.793	130.126
$B(0)$	84.338	84.338	84.338	84.338	84.338	84.338	84.338	84.338
$O(0)$	18.247	25.404	34.455	45.788	18.247	25.404	34.455	45.788
V^1	91.361	96.685	103.037	110.671	91.361	96.685	103.037	110.671
B^1	78.411	78.411	78.411	78.411	68.024	68.024	68.024	68.024
O^1	19.771	26.001	33.805	43.602	23.337	28.661	35.013	42.647
V^2	118.358	128.688	141.950	158.285	116.048	126.494	139.614	155.518
B^2	100.764	100.764	100.764	100.764	96.824	96.824	96.824	96.824
O^2	17.593	27.923	41.185	41.185	19.225	29.671	42.790	58.694
$P^{3,1}$	95.704	102.380	110.824	121.397	83.550	89.378	96.750	105.980
$P^{3,2}$	121.096	129.544	140.228	153.606	115.850	123.932	134.153	146.952

Now we consider a portfolio made up of 1000 policies whose characteristics are shown in Table (11). Each insurance contract has an initial capital equal to 100, ends at the age of 65 and the minimum guaranteed rate matches technical rate. The average age of the portfolio is 47 years and the policies with a minimum guaranteed rate equal to 4% accounts for 52.8% of the total.

Tabella 11: Portfolio with 1000 policies ($i = i_{min}$).

x	m	unit	i
35	30	20	0.0%
37	28	32	0.5%
39	26	52	1.0%
41	24	68	1.0%
43	22	68	1.5%
45	20	76	2.0%
47	18	84	2.0%
49	16	72	2.0%
51	14	72	4.0%
53	12	90	4.0%
55	10	102	4.0%
57	8	120	4.0%
60	5	144	4.0%

We consider ten different reference portfolios as illustrated in Table (12). Each one of them is again built by matching its value with the best estimate. We observe that at first the value of SCR decreases and then increases and that the gap rises with an increasing q_j. The amount of capital, by regulatory standards, is up to seven times higher. We also show the scenario from which the solvency capital comes in brackets. Using our stressed curves, the policies with a minimum guaranteed rate equal to 4% do not contribute to SCR if it comes from the downward market. On the other hand, if it comes from the upward market, the policies contribute most of SCR. For instance, if $q_j = 0.70$, their contribution is approximately equal to 90%. Using regulatory curves, these policies contribute to SCR regardless of the market which generates the solvency capital. The percentage of their contribution is between 34% and 65%.

Tabella 12: SCR of policies portfolio.

q_j	$V(0)$	SCR (M1)	SCR (M2)	Ratio
0.10	94148.281	3216.496(d)	6414.888(d)	1.994
0.20	94427.571	2870.731(d)	6329.559(d)	2.205
0.30	94836.639	2478.407(d)	6219.588(d)	2.510
0.40	95373.190	2041.830(d)	6083.633(d)	2.980
0.50	96032.905	1561.559(d)	6649.035(u)	4.258
0.60	96812.474	1033.083(d)	7457.530(u)	7.219
0.70	97711.181	1268.366(u)	8336.184(u)	6.572
0.80	98723.569	1557.587(u)	9286.516(u)	5.962
0.90	99842.478	1870.850(u)	10309.196(u)	5.510
1.00	101059.080	2210.109(u)	11403.817(u)	5.160

6 Conclusions

In the Solvency II context, even if the company uses the Standard Formula, the solvency capital quantification for the interest rate risk requires the use of a stochastic model for some types of insurance contracts.

In the paper we propose an extended Cox-Ingersoll-Ross model that allows negative interest rates and the perfect-fit of the term structure of risk free interest rates to be calculated using the Smith-Wilson method, as required by the technical regulations. The dynamic of the spot rate is characterised by an exogenous barrier and a deterministic component, as proposed by Brigo and Mercurio (2001). If the insurance contract has a minimum rate guaranteed each year, to apply the standard model, we need to implement the Monte Carlo simulation as the benefit is characterised by a non-standard option.

We also propose a method to measure the upward and downward term structures. It does not require any distributional assumption or a specific estimation technique. The two structures result from stressed swap rate markets. First we apply the technique of Principal Component Analysis to the set of the swap rates, then we calibrate the Smith-Wilson method. Doing this, we can overcome some theoretical criticalities.

In the current market situation, the analysis produces term structures significantly different from those used by legislation. Considering three asset portfolios, the amount of solvency capital requirement is considerably different, and applying the regulatory shocks it is higher. The valuation with different values of the minimum guaranteed rate allows the measurement of the impact of the embedded option on the company solvency. We also show that, if we consider a portfolio made up of policies with a high minimum guaranteed rate for more than half of it,

their contribution to the regulatory capital is different depending on the type of stressed market curve.
The parameters in the Standard Formula are calibrated to take into account the simplications and the approximations, but to hold capital, higher than those required to cover risks, has an impact on the company profitability. Above all, most small companies would use the simplified model as the internal one could be too expensive in terms of technology, personnel education and organisational structure.
We have two aims: to examine the topic of stressed scenarios measurement in depth, for instance using a generalized autoregressive model, as in Abdymomunov et al. (2014), and to extend the analysis including other risk factors. As in Eckert at al. (2016), we can consider a reference portfolio made up of equities and defaultable bonds. Undoubtedly, the complexity of the model and the computational process are significant.

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