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Effects and Phenomenology of Cut-Off Physics on the Primordial Universe

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Abstract

The main focus of this thesis is the implementation of alternative quantization procedures, developed to easily introduce quantum gravitational corrections, on cosmological models. The aim is to study the fate of cosmological singularities and to derive possible signatures of more fundamental Quantum Gravity theories.

Polymer Quantum Mechanics (PQM) is a procedure of quantization on a lattice, inspired by Loop Quantum Gravity but derived independently. The discretization of a position-like variable implies a cut-off on the corresponding momentum, and its implementation on the cosmological minisuperspaces, called Polymer Cosmology, is usually able to remove the Big Bang and Big Crunch singularities, replacing them with a Big Bounce similarly to Loop Quantum Cosmology. However the properties of the Bounce, i.e. its universality or its dependence on initial conditions, depend on the geometrical nature of the variable chosen to describe the model. After a comparison of different sets of variables in the isotropic Friedmann-Lemaître-Robertson-Walker model, the privileged variable to obtain a universal Bounce is identified in the cubed scale factor corresponding to the comoving volume. The recovery of the equivalence between different sets of variables after the discretization has been implemented is still an open question, and it can have implications for Loop Quantum Cosmology.

The Generalized Uncertainty Principle representation (GUP), inspired by String Theories, introduces higher-order corrections on the canonical commutation relations and implies an absolute minimal uncertainty on position. Its cosmological implementation however is not usually able to remove the singularities; furthermore, the recreation of Brane Cosmology (a cosmological sector of String Theories) happens only for a different variation of this representation. Indeed, it is possible to extend the GUP formulation to different functions of the momentum, obtaining deformed commutation relations known as Modified Algebras. Not all of them imply minimal uncertainties, but some can introduce energy cut-offs. In particular, there are specific formulations able to reproduce Polymer Cosmology and Brane Cosmology. Another interesting form is able to remove singularities not through a Bounce but through an asymptotic behaviour; this way it is possible to reproduce the so-called Emergent Universe model without the fine tuning needed on a classical level.

Modified Algebras can be used to easily derive corrections to the primordial Power Spectrum of scalar perturbations, thus yielding a possibly observable signature of quantum gravitational corrections. When applied to the gravitational collapse of a dust cloud, they are able to halt the collapse and prevent the formation of Black Holes, thus possibly explaining the recently observed violations of star mass limits.

This thesis represents a starting point for the development of alternative quantization procedures as a somewhat phenomenological approach to implement corrections from more fundamental Quantum Gravity theories. There is still the need to define and develop canonical transformations between different sets of variables, since at the moment they seem to yield inequivalent dynamics; furthermore, the algebras can have different operatorial representations whose equivalence must still be formally proven. Finally, their implementation on more complex systems such as the anisotropic and chaotic Bianchi IX model or inhomogeneous solutions could be useful towards the further development of a complete theory of Quantum Gravity.

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Chapter 1

Introduction

One of the most relevant open questions in Relativistic Cosmology concerns the existence of the initial singularity [133, 142, 167, 220]. Indeed, as proved in the well-known Singularity Theorems [114, 115], the existence of a singular instant in the past of the Universe where the curvature invariant diverges and the Einstein equations are no longer predictive is a general feature of the cosmological problem, which has nothing to do with the highly symmetric nature of the Robertson-Walker (RW) geometry describing the isotropic Universe. For this reason, any physics possibly able to overcome the singularity of the primordial Universe acquires a particular relevance; given the high-energy and high-curvature regime expected close to the singularity, the solution has to be searched in the quantum realm.

Modern Quantum Gravity approaches are mainly based on the use of Ashtekar-Barbero-Immirzi (first order) variables [16, 17], which constitute the starting point for the construction of both Canonical Quantum Gravity [78, 88] and the Loop Quantum Gravity (LQG) theory [18, 23, 28] (LQG is still considered a canonical approach, but the quantization procedure is not the standard one).

The canonical quantization in the Wheeler-DeWitt (WDW) formulation has rarely been able to provide a non-singular quantum cosmology [45, 49]; on the other hand, LQG constitutes a significant change in the point of view on how to approach the quantization of the gravitational degrees of freedom, especially because this formulation was able to construct a kinematical Hilbert space and to justify spontaneously the emergence of discrete area and volume spectra [195]. LQG relies on the possibility to reduce the gravitational phase space to that of a $SU(2)$ non-Abelian gauge theory [194, 196], and then the quantization scheme is performed by using “smeared” (non-local) variables, as suggested by the original Wilson loop formulation and by non-Abelian gauge theories on a lattice. Indeed, when adopting the Ashtekar variables, the invariance of the gravitational action under the local rotation of the triad adapted to the spacetime foliation is expressed in the form of a Gauss constraint.

The implementation of this new approach to the cosmological setting, i.e. the reformulation of the minisuperspace dynamics in terms of Ashtekar variables, called Loop Quantum Cosmology (LQC) [19, 24, 25, 26, 29, 50, 51, 52, 54], has determined the existence of a Universe with a non-zero minimal volume where the collapsing and the expanding branches of the dynamics are connected by the so-called Big Bounce and the singularity is avoided; this construction offers a non-singular framework to implement the cosmological history of the Universe (actually, after the Planckian time the Universe thermal history remains faithful to the original formulation [99, 133, 167, 220]). Indeed, despite the minisuperspace model associated to homogeneous cosmological Universes prevents a full implementation of the $SU(2)$ symmetry of the full theory, it is possible to extract a notion of cut-off on the Universe volume and then a maximum critical density for the Planck era with a suitable procedure, recovering the general theory prescription and formalisms. However, the fact that the basic $SU(2)$ symmetry of the full LQG theory is essentially lost in LQC is a non-trivial limitation and the discretization of the area operator spectrum is somewhat introduced ad hoc, in contrast with LQG where it takes place naturally on a kinematical level. The difficulties of LQC in reproducing the fundamental character of the general quantum theory have been discussed in [79], and a more thorough criticism of the whole cosmological setting of LQG has been given in [55].

Despite these limitations, LQC remains an interesting attempt to regularize the cosmological singularity, opening a new perspective on the origin and evolution of the Universe. Furthermore, the effective formulation of LQC [205, 211] is isomorphic to the implementation of Polymer Quantum Mechanics (PQM) [81, 82], a procedure of quantization on a lattice, to the minisuperspace variables, typically the Universe scale factor or functions of it. This correspondence allows to investigate some features of the LQC formulation by applying simplified formalisms to more complicated models, thus making them viable.

On the same footing, the low energy phenomenology of String Theories, which constitute another highly developed proposal to move classical General Relativity towards a quantum formulation [42, 48, 185, 186, 187], can be interpreted as a modification of the Heisenberg Uncertainty Principle, as applied to the generalized coordinates and momenta of a given dynamical system; this is known as the Generalized Uncertainty Principle representation (GUP) [62, 95, 126, 153, 184, 199], and when applied to the minisuperspace it reproduces dynamics similar to that of Brane Cosmology [65, 138, 149, 152], a cosmological sector of String Theories.

The similarities between Polymer Cosmology (the implementation of PQM on cosmological minisuperspaces) and LQC on one hand, and between the GUP representation and Brane Cosmology on the other, beg the question of whether it is possible to develop other formalisms able to implement Quantum Gravity features through simple, independent frameworks. The answer has been found in Modified Algebras, a generalization of the GUP representation to other forms; these are usually functions of the momentum operator, since quantum gravitational corrections are expected to be relevant at high energies. Modified Algebras can be found to reproduce more fundamental quantum cosmological theories through the introduction of fixed structures, such as a minimal length in the form of an absolute minimal uncertainty on position [126] or a maximum in the allowed eigenvalues for the momentum operator [202]. Such limits are exactly what it is expected from a theory of quantum gravity and quantum cosmology, in that they usually correspond to energy cut-offs that could therefore mitigate the problem of singularities. Furthermore, Modified Algebras constitute a powerful tool in regards to the derivation of possible phenomenological consequences of the resulting cosmological models. Their versatility can for example allow for easier computations of corrections to the behaviour under dissipative phenomenons or to the primordial power spectrum of perturbations. The simplicity of their semiclassical limit can also be used to easily and readily obtain semiclassical effective dynamics, which besides being useful by itself, can give an idea of what to expect from the implementation of fundamental quantum gravitational theories to more complex settings such as anisotropic models.

This thesis focuses on Polymer Cosmology and Modified Algebra Cosmology, with the aim of better characterizing the removal of cosmological singularities and to give some possible phenomenological consequences of more fundamental Quantum Gravity theories.

After a brief review of standard Hamiltonian cosmology on the classical and quantum levels, the latter both in the Canonical and in the Loop frameworks, I will introduce the alternative representations of quantum mechanics used in my research: Polymer Quantum Mechanics, the Generalized Uncertainty Principle representation, and Modified Algebras. I will highlight the procedures of quantization and the consequences that they imply, as well as how to implement the corresponding semiclassical limits. These are the main tools that will be used throughout the thesis.

Then I will show my original research. I will first focus on Isotropic Polymer Cosmology, that is the implementation of PQM on the isotropic Friedmann-Lemaître-Robertson-Walker model (FLRW). I will study the model in different sets of variables, both in a semiclassical effective representation and on a quantum level, and highlight how the replacement of the cosmological singularities of the Big Bang and the Big Crunch with a non-singular Big Bounce is not always universal: the nature of the Bounce is related to the geometrical nature of the variable chosen to describe the model, and this feature has implications for LQC.

Secondly I will show how Modified Algebras can be used to introduce quantum gravitational corrections by implementing some different forms on the anisotropic Bianchi I model, including the original GUP formulation and a Polymer Algebra able to reproduce the same effective dynamics of Polymer Cosmology. I will then focus on a particular algebra that is able to remove the cosmological singularities not through a Bounce but through the introduction of an asymptotic behaviour; this is a way to naturally obtain the so-called Emergent Universe model without the fine-tuning needed on a classical level.

Finally, I will study the collapse of a spherical dust cloud in the framework of Modified Algebras. The motivation for this study comes from the fact that the internal metric is identical to that of a closed FLRW model, and therefore it is expected that the gravitational singularity is removed and the formation of a Black Hole is avoided. This represents an example of Modified Algebras introducing quantum corrections in other gravitational systems beyond the cosmological setting.

To conclude, I will mention a few possibilities in which the research explained within this thesis can be continued and expanded. This is a starting point for the exploration of Polymer Quantum Mechanics, of Modified Algebras, and of alternative quantization procedures in general, with the hope of furthering the search for a complete theory of Quantum Gravity.

The thesis is based on the following original papers.

- Reference [168]: Giovanni Montani, Claudia Mantero, Flavio Bombacigno, Francesco Cianfrani and **Gabriele Barca**, "Semiclassical and quantum analysis of the isotropic Universe in the polymer paradigm", *Physical Review D* 99, 063534 (2019).
- Reference [31]: **Gabriele Barca**, Paolo Di Antonio, Giovanni Montani and Alberto Patti, "Semiclassical and quantum polymer effects in the flat isotropic universe", *Physical Review D* 99, 123509 (2019).
- Reference [33]: **Gabriele Barca**, Eleonora Giovannetti and Giovanni Montani, "An Overview on the Nature of the Bounce in LQC and PQM", *Universe* 7, 327 (2021).
- Reference [34]: **Gabriele Barca**, Eleonora Giovannetti and Giovanni Montani, "Comparison of the semiclassical and quantum dynamics of the Bianchi I cosmology in the polymer and GUP extended paradigms", *International Journal of Geometric Methods in Modern Physics* 19, 2250097 (2022).
- Reference [102]: Eleonora Giovannetti, **Gabriele Barca**, Federico Mandini, Giovanni Montani, "Polymer Dynamics of Isotropic Universe in Ashtekar and in Volume Variables", *Universe* 8, 302 (2022).
- Reference [37]: **Gabriele Barca**, Giovanni Montani and Alessandro Melchiorri, "Emergent Universe model from modified Heisenberg algebra", *Physical Review D* 108, 063505 (2023).
- Reference [36]: **Gabriele Barca** and Giovanni Montani, "Non-Singular Gravitational Collapse through Modified Heisenberg Algebra", *European Physical Journal C* 84, 261 (2024).

Some results are also included in the following conference Proceedings.

- Reference [32]: **Gabriele Barca**, Eleonora Giovannetti, Federico Mandini and Giovanni Montani, "Polymer Quantization of the Isotropic Universe: Comparison with the Bounce of Loop Quantum Cosmology", in *The 16th Marcel Grossmann Meeting on Recent Developments in Theoretical and Experimental General Relativity, Astrophysics and Relativistic Field Theories* (2021).
- Reference [35]: **Gabriele Barca**, Eleonora Giovannetti and Giovanni, Montani, "PQM and the GUP: Implications of Lattice Dynamics and Minimal Uncertainties in Quantum Mechanics and Cosmology", in *56th Rencontres de Moriond on Gravitation* (2022).

Chapter 2

Standard Cosmology

In this chapter I will give an overview of standard cosmology, both on the classical and on the quantum level. However, I will not derive it from the Einstein equations, but directly from the Hamiltonian formulation of General Relativity (GR).

I will start from the $SU(2)$ reformulation of GR in terms of the Ashtekar-Barbero-Immirzi variables [16, 17], and then focus on the isotropic Friedmann-Lemaître-Robertson-Walker model (FLRW). I will derive the solutions for the scale factor $a(t)$ in the presence of different kinds of matter-energy, and then show how, due to the Hamiltonian being constrained to zero, this formulation allows the freedom to choose an internal time variable; this is usually identified in a free scalar field. Then I will give a brief introduction of the anisotropic Bianchi models, with focus on Bianchi I which will be used later in the thesis.

Regarding Quantum Cosmology, I will first implement the canonical Wheeler-DeWitt formulation (WDW) [88], based on the reinterpretation of the fundamental equation $\hat{\mathcal{H}} \left|\psi\right\rangle = 0$ as a Klein-Gordon-like (KG) wave equation. I will show how to construct wavepackets and how to compute expectation values of operators with the KG conserved probability current. Then I will focus on Loop Quantum Gravity (LQG) [18, 23, 28], a more recent (although still canonical) quantization procedure where the geometrical operators of area and volume result to have a discrete spectrum [195]. Its cosmological implementation, called Loop Quantum Cosmology (LQC) [29, 54, 53], is able to overcome a few problems of the WDW formulation; in particular, it is possible to obtain a cut-off on the energy density operator that replaces the classical cosmological singularities with a Big Bounce.

The models and solutions presented here will be used for comparison when in later chapters I will implement alternative quantization procedures.

2.1 Classical Hamiltonian Cosmology

A cosmological spacetime is defined by a line element

$$ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta, \tag{2.1}$$

where $g_{\alpha\beta}$ is the metric tensor. From the metric tensor it is possible to construct connections i.e. Christoffel symbols $\Gamma^\gamma_{\alpha\beta}$, and then the curvature of spacetime is defined by the Riemann tensor $R^\alpha_{\ \beta\gamma\delta}$, constructed from the derivatives of the connections. The two other quantities that encode curvature are the Ricci tensor $R_{\alpha\beta}$, obtained from the contraction of the first and third indices of the Riemann, and its trace $R = R^{\alpha}_{\alpha}$ also called Ricci curvature scalar. They are of primary importance in gravity and cosmology because they appear in the Einstein field equations:

$$R_{\alpha\beta} - \frac{1}{2}\, g_{\alpha\beta}\, R = \chi\, T_{\alpha\beta}, \tag{2.2}$$

where χ is the Einstein constant (that usually will be set to 1) and $T_{\alpha\beta}$ is the energy-momentum tensor parametrizing the matter distribution. However, as mentioned earlier, I will introduce General Relativity and cosmological models in their Hamiltonian formulations, where the Ricci scalar R represents the gravitational Lagrangian, and will make little reference to the Einstein field equations.

2.1.1 Hamiltonian Formulation of the Isotropic Universe

The Hamiltonian formulation of General Relativity (GR) is based on a $3+1$ foliation of spacetime and a Gauge reformulation of the three spatial dimensions as a $SU(2)$ theory; through this process and with the help of Lagrange multipliers, three fundamental constraints are derived: the Gauss constraint which encodes the $SU(2)$ symmetry, the SuperMomentum or spatial constraint that generates diffeomorphisms, and the SuperHamiltonian or scalar constraint responsible of time evolution.

The action for GR in its standard form is given by

$$S_{\mathrm{GR}} = \frac{1}{2}\int d^4x\, \sqrt{-g}\, R, \tag{2.3}$$

where g is the determinant of the metric tensor and R is the Ricci curvature scalar; after foliation, it can be rewritten in the Holt form:

$$S_{\mathrm{GR}} = \int dt\, d^3x \left(\mathcal{A}^a_j\, \mathcal{E}^j_a - \left(\mathcal{A}^a_0\, \mathcal{G}_a + \mathcal{N}^j\, \mathcal{H}_j + \mathcal{N}\, \mathcal{H}\right)\right), \tag{2.4}$$

where $\mathcal{A}^a_\alpha$ are gauge connections describing curvature (and their time components act here as Lagrange multipliers), $\mathcal{E}^\alpha_a$ are densitized triads containing information about the spatial geometry, $\mathcal{G}_a$, $\mathcal{H}_j$ and $\mathcal{H}$ are the three constraints in the order mentioned above, and the lapse function $\mathcal{N}$ and the shift vector $\mathcal{N}^j$ are Lagrange multipliers. It is quite straightforward to see that varying the action with respect to the three Lagrange multipliers yields the three constraints $\mathcal{G}_a = 0$, $\mathcal{H}_j = 0$, and $\mathcal{H} = 0$. This reformulation of GR as a $SU(2)$ theory was performed by Ashtekar [16, 17], and the quantities $\mathcal{A}$ are called Ashtekar-Barbero-Immirzi connections.

In homogeneous cosmology the relevant quantity is the SuperHamiltonian, since the Gauss and the SuperMomentum constraints are automatically satisfied by the symmetry requirements and contain no dynamical information for the model.

In homogeneous cosmological models, given the symmetry requirements, the shift vector can be ignored, so only the lapse function will appear; the isotropic Robertson-Walker (RW) metric then is

$$ds^2 = \mathcal{N}^2 dt^2 - a^2 d\ell^2_{\mathrm{RW}}, \qquad d\ell^2_{\mathrm{RW}} = \frac{dr^2}{1 - K\, r^2} + r^2 d\theta^2 + r^2 \sin^2(\theta)\, d\varphi^2, \tag{2.5}$$

where $d\ell^2_{\rm RW}$ is the metric of isotropic three-dimensional space in the polar coordinates (r,θ,φ), K is the spatial curvature with dimensions of an inverse squared length, and $a(t)$ is the dimensionless cosmic scale factor, which is the only real degree of freedom available to the dynamical problem and whose time evolution encodes the whole dynamics. Note that in cosmology when $K \neq 0$ it is always possible to rescale the scale factor a and the coordinate r so that $|K| = 1$. Therefore the only different possibilities for the geometry of spacetime are $K = 0$ (a flat hyper-plane), $K = -1$ (an open hyper-saddle), and $K = 1$ (a closed hyper-sphere). Matter is then introduced in the form of a generic energy density ρ. Plugging everything together into the action, after a spatial integration and a Legendre transform, the FLRW action and Hamiltonian result to be

$$S_{\rm FLRW} = \int dt\, \mathcal{V} \left(\frac{3}{\mathcal{N}}\,(a^2\,\ddot{a} + \dot{a}^2\,a) + 3\,\mathcal{N}\,K\,a - \mathcal{N}\rho\,a^3\right), \tag{2.6}$$

$$\mathcal{N}\mathcal{H}_{\rm FLRW}(a,p_a) = -\frac{\mathcal{N}}{12}\frac{p_a^2}{a} - 3\,\mathcal{N}\,K\,a + \mathcal{N}\rho\,a^3 = 0, \qquad p_a = -\frac{6}{\mathcal{N}}\,a\,\dot{a}, \tag{2.7}$$

where $\mathcal{V}$ is a volume scale that will be set to 1 (unless otherwise specified in later chapters) and I defined the momentum p_a conjugate to the scale factor through the standard procedure of the Legendre transform.

The role of the Lapse Function $\mathcal{N}$ here is to parameterize the freedom of choosing a time variable, as I will show later. For the moment, since I am working in synchronous time t, it will be set to 1. The equations of motion are

$$\dot{a} = \frac{\partial\mathcal{H}}{\partial p_a} = -\frac{p_a}{6a}, \qquad \dot{p}_a = -\frac{\partial\mathcal{H}}{\partial a} = -\frac{1}{12}\frac{p_a^2}{a^2} + 3\,K - \frac{\partial(\rho\,a^3)}{\partial a}. \tag{2.8}$$

From the first one, squaring and substituting the Hamiltonian constraint itself yields the Friedmann equation, which is the fundamental equation of cosmology linking the expansion rate to the matter content:

$$H^2 = \frac{\dot{a}^2}{a^2} = \frac{p_a^2}{36a^4} = \frac{\rho}{3} - \frac{K}{a^2}, \tag{2.9}$$

where $H = \dot{a}/a$ is the Hubble parameter encoding the expansion rate. Then, defining the pressure as $P = -\partial(\rho v)/\partial v$, from the equation of motion for $\dot{p}_a$ it is possible to derive the acceleration equation:

$$\frac{\ddot{a}}{a} = -\frac{\rho + 3P}{6}. \tag{2.10}$$

In order to solve these equations and derive the dynamics of the model, i.e. the evolution of $a(t)$, an expression for $\rho = \rho(a)$ is needed. This can be derived by combining the two equations above:

$$\dot{\rho} + 3\,H\,(\rho + P) = 0. \tag{2.11}$$

In GR, this is usually derived from the energy-momentum tensor being divergence-free, which is a representation of energy conservation.

The Friedmann equation (2.9), the acceleration equation (2.10) and the continuity equation (2.11) completely describe the dynamics of the FLRW model. Actually, since any one of the three can be derived from the other two, only two of them are strictly necessary, supplemented by an equation of state (EoS) linking pressure and energy density. In general it is customary to choose the Friedmann and the continuity equations, but any pair is equivalent.

Regarding matter, the EoS is needed in order to solve the continuity equation for $\rho(a)$; various kinds of EoS exist, but the one most used in cosmology is the relation

$$P = w\,\rho, \tag{2.12}$$

where w is the equation of state parameter, which is constant and must obey $-1 \leq w \leq 1$ in order not to break causality and not to have a superluminal speed of sound. Then, when $w = -1$, $\rho =$ const. which corresponds to a Cosmological Constant, while for any other allowed value of w the energy density is

$$\rho(a) = \overline{\rho}\, a^{-3(1+w)}, \tag{2.13}$$

where $\overline{\rho}$ is an integration constant. Different values of w describe different kinds of matter: $w = 0$ corresponds to ordinary pressureless matter (often called dust), $w = 1/3$ corresponds to radiation i.e. relativistic matter such as photons and neutrinos, and $w = 1$ corresponds to the so-called stiff matter. In general, the energy density ρ represents a sum of all types of matter, but usually, due to their different dependence on a, only one dominates for specific ranges of the scale factor and the others can be neglected.

A very useful tool for matter is a scalar field ϕ: depending on the assigned potential, it is able to mimic any type of perfect fluid. Even better, if the potential evolves with time it can be used to represent transitions from one domination era to the next. The expression for the equation of state parameter of a scalar field ϕ is

$$w_\phi = \frac{P_\phi}{\rho_\phi} = \frac{\dot{\phi}^2 - 2\,U(\phi)}{\dot{\phi}^2 + 2\,U(\phi)}\,, \tag{2.14}$$

where $\dot{\phi}^2/2$ is the kinetic term and U is the potential. It is immediate to see that, when the former dominates over the latter, $w_\phi \approx 1$, while in the opposite case if $U \gg \dot{\phi}^2$ then $w_\phi \approx -1$; for other cases in between, it is possible to obtain every allowed value of w.

Once an expression for $\rho(a)$ is provided, the Friedmann equation becomes a differential equation for $a(t)$, and the assumption of the existence of a singularity implies the initial condition $a(0) = 0$, making the differential equation solvable for any $w \neq -1$. The solutions for the flat case are relatively easy; on the other hand, when $K \neq 0$, the solutions are not always solvable analytically, and even then they are usually in implicit form $t = t(a)$; however, it is still possible to deduce the behaviour from the form of the Friedmann equation, and then solve them numerically. The case $w = -1/3$ that implies $\ddot{a} = 0$ and the case $w = -1$ corresponding to a Cosmological Constant deserve particular attention, since the evolution will be radically different.

The solutions in the flat case with $K = 0$ are given by

$$a(t) \propto (t^2)^{\frac{1}{3(1+w)}}; \tag{2.15}$$

Clearly, depending on the choices $t > 0$ and $\dot{a} > 0$ or $t < 0$ and $\dot{a} < 0$, the dynamics consists either of an expansion from a past singularity, called Big Bang, or a collapse towards a future singularity, called Big Crunch (with these expressions, both happen at $t = 0$). The different values of w only slightly change the incline i.e. the speed of approach to the singularity.

In the open case with $K < 0$, H^2 is greater than in the previous case because the curvature term $-K/a^2$ will contribute positively; therefore it is expected that the approach to the singularity will be slightly faster. Furthermore, since all known forms of matter (except the Cosmological Constant) decrease faster than $1/a^2$, the bigger the scale factor gets the more the curvature term becomes important and the matter term negligible; therefore the behaviour far from the singularity will be quite different. However this thesis will only be concerned with the dynamics close to the singularity where quantum effects become relevant.

The closed models with $K > 0$ are different, because in this case there will exist a value of a such that $H = 0$, corresponding to a critical point $\dot{a} = 0$. In correspondence of this value then H will change its sign and the model will start to contract again with $\dot{a} < 0$. This is why this is called a closed model: it will reach a maximum value for the scale factor and then recollapse to a singularity in a finite time, presenting both the Big Bang and the Big Crunch in the same branch.

When $w = -1/3$, the acceleration equation (2.10) implies $\ddot{a} = 0$ and it is immediately obvious that the expansion will be linear. From that same equation it is also quite easy to see that for any acceptable $w < -1/3$ the expansion will be accelerated since $\ddot{a} > 0$; this implies that for these values the recollapse as seen in the cases with $K > 0$ will never happen.

The three cases with different curvature and with $w \neq -1$ are shown in Figure 2.1 for the three relevant cases of matter, together with the case $w = -1/3$ of linear evolution.

The case with $w = -1$ is peculiar: in the Friedmann equation (2.9), if the right-hand side contains only a Cosmological Constant ρ_Λ, the solution is simply an exponential of the form

$$a(t) = \exp\left(\pm\sqrt{\frac{\rho_\Lambda}{3}}\,t\right). \tag{2.16}$$

The fact that $H =$ const. can be quite useful in specific situations, and it was used to develop the theory of Inflation [133, 146, 147, 150, 151, 167, 169].

Now, starting from the Hamiltonian formulation of FLRW cosmology presented above, I will restate the whole system in terms of the new volume variable $v = a^3$. As I will show later in this thesis, this variable will play a very important role in the semiclassical and quantum implementation of alternative quantum mechanics.

From the canonical relation, the momentum p_v conjugate to the volume is found and thus the Hamiltonian can be rewritten as follows:

$$p_a\, da = p_v\, dv, \qquad p_v = \frac{p_a}{3a^2}, \tag{2.17}$$

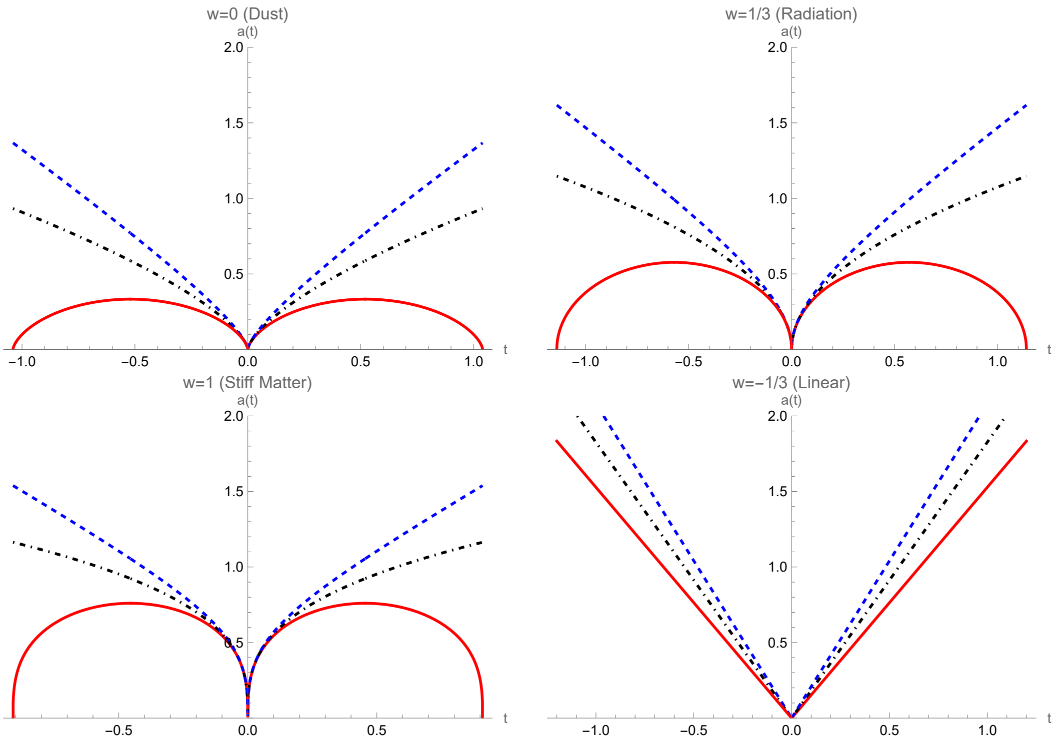

Figure 2.1. The solutions $a(t)$ for different values of w. Each panel contains all the different cases of curvature: the flat case with $K = 0$ (black dot-dashed lines), the open case with $K < 0$ (blue dashed lines) and the closed case with $K > 0$ (red continuous lines). All of them present singularities in that $a \to 0$ and consequently $\rho \to \infty$.

$$\mathcal{H}_{\text{FLRW}}(v, p_v) = -\frac{3}{4}\, p_v^2\, v + 3\, K\, v^{\frac{1}{3}} + \rho\, v = 0. \tag{2.18}$$

Then the equations of motion and the Friedmann equation are

$$\dot{v} = -\frac{3}{2}\, p_v\, v, \qquad \dot{p}_v = \frac{3}{4}\, p_v^2 + \frac{K}{v^{\frac{2}{3}}} - \frac{\partial(\rho\, v)}{\partial v}\,, \tag{2.19}$$

$$H^2 = \left(\frac{\dot{v}}{3v}\right)^2 = \frac{p_v^2}{4} = \frac{\rho}{3} - \frac{K}{v^{\frac{2}{3}}}. \tag{2.20}$$

The continuity equation and its solution are exactly the same, but rewritten in terms of v:

$$\dot{\rho} + \frac{\dot{v}}{v}\,(\rho + P) = 0, \qquad \rho(v) = \overline{\rho}\, v^{-(1+w)}. \tag{2.21}$$

The solutions $v(t)$ for different kinds of matter will then be the same obtained previously for $a(t)$, but cubed.

2.1.2 About Time

A very delicate issue is presented by the time-gauge freedom. As mentioned above, the cosmological (but also more in general the gravitational) Hamiltonian function is constrained to zero. This can be problematic on a quantum level: if the Hamiltonian operator responsible for time evolution only has the zero eigenvalue, the states will never evolve; this is known as the Problem of Time, and will be analyzed more in depth later. Fortunately, the presence of the Lapse Function $\mathcal{N}$ comes in handy: it parameterizes the fact that it is possible to choose a different time variable other than synchronous time t.

In the previous paragraphs the FLRW Lagrangian only depended on the scale factor a and its time derivatives. By assuming now that it depends also on another variable, new degrees of freedom appear. Then it is possible to choose the Lapse function $\mathcal{N}$ in such a way that the new variable can play the role of time. This way a new Hamiltonian is defined, corresponding to the opposite of the momentum conjugate to that variable. This procedure is a time-gauge fixing through a reduction of the dynamics, and is quite useful in gravity and cosmology [14, 15].

I will now show a more concrete example. In cosmology, a good and often used candidate for time variable is a free scalar field i.e. without a potential, given its often monotonic relation with synchronous time; given that it corresponds to stiff matter with $w = 1$ and $\rho \propto 1/a^6$, it is also expected to be the dominant form of matter-energy close to the singularity.

The energy density of a free scalar field is given by $\rho_\phi = \dot{\phi}^2/2$, so its conjugate momentum reads as

$$p_\phi = \frac{\dot{\phi}\, v}{\mathcal{N}} \,. \tag{2.22}$$

The Hamiltonian and the equations of motion therefore become

$$\mathcal{H}_{\text{FLRW}}(v, p_v, \phi, p_\phi) = -\frac{3\mathcal{N}}{4}\, p_v^2\, v + 3\mathcal{N}\, K\, v^{\frac{1}{3}} + \mathcal{N}\,\frac{p_\phi^2}{2v^2}\, v, \tag{2.23}$$

$$\dot{v} = -\frac{3\mathcal{N}}{2}\, p_v\, v, \qquad \dot{\phi} = \mathcal{N}\,\frac{p_\phi}{v}, \tag{2.24}$$

where the momentum conjugate to the field results to be a constant due to the absence of a potential and the momentum conjugate to the volume is not relevant at the moment. By temporarily setting $\mathcal{N} = 1$ and $K = 0$, the Friedmann equation in synchronous time can be derived and solved:

$$\frac{\dot{v}^2}{9v^2} = \frac{p_\phi^2}{6v^2}, \qquad v(t) = \pm\frac{3}{2}\, p_\phi\, t, \qquad \phi(t) = \pm\ln\!\left(\frac{2}{3}\,|p_\phi\, t|\right), \tag{2.25}$$

where I used the boundary conditions $v(0) = 0$, $\phi(0) \to \pm\infty$. Therefore, regardless of the chosen branch, the field ϕ will have a monotonic relation to t and is thus a good candidate to be used as a time variable.

Now, going back to equation (2.24) and setting $\dot{\phi} = 1$, a value of $\mathcal{N}$ is found that changes the equation of motion for v:

$$\mathcal{N} = \frac{v}{p_\phi}, \qquad \frac{\mathrm{d}v}{\mathrm{d}\phi} = -\frac{3}{2}\,\frac{p_v}{p_\phi}\, v^2. \tag{2.26}$$

Then the Friedmann equation in terms of the scalar field yields a differential equation for $v(\phi)$ that is easily solvable:

$$\left(\frac{1}{3v}\,\frac{\mathrm{d}v}{\mathrm{d}\phi}\right)^2 = \frac{1}{4}\,\frac{p_v^2\, v^2}{p_\phi^2} = \frac{1}{6}, \qquad v(\phi) = v_1 \exp\!\left(\pm\sqrt{\frac{3}{2}}\,\phi\right), \tag{2.27}$$

where v_1 is a generic integration constant; again there are two branches with an exponential behaviour, and the singularities are at $\phi \to \pm\infty$.

Alternatively, a reduced Hamiltonian can be obtained by solving the constraint $\mathcal{H}_{\text{FLRW}} = 0$ for p_ϕ; then the equation of motion for $dv/d\phi$ is simply the Hamilton equation of this new Hamiltonian:

$$\mathcal{H}_\phi = -p_\phi = \pm\sqrt{\frac{3}{2}\, p_v^2\, v^2 - 6\, K\, v^{\frac{4}{3}}}\ , \tag{2.28}$$

$$\mathcal{H}_\phi(K=0) = \pm\sqrt{\frac{3}{2}}\, p_v\, v, \qquad \frac{\mathrm{d}v}{\mathrm{d}\phi} = \frac{\mathrm{d}\mathcal{H}_\phi}{\mathrm{d}p_v} = \pm\sqrt{\frac{3}{2}}\, v, \tag{2.29}$$

which yields the same exact solution $v(\phi)$ found above.

The evolution in terms of the scalar field shown here will be used for comparison later, when I will implement modified semiclassical or quantum dynamics.

2.1.3 The Bianchi Models

In 1897, Luigi Bianchi listed all the possible three-dimensional Lie algebras such that each of them determines the local properties of a three-dimensional group. Given a homogeneous spacetime, its symmetry group can be described by one of nine different (classes of) Lie algebras, with uniquely determined structure constants [46]. The Bianchi models are the homogeneous spaces associated with these nine algebras; in cosmology, even if the three cases (flat, open or closed) of the isotropic FLRW model are technically included in the Bianchi classification, usually the Bianchi universes are considered anisotropic models.

The metric tensor $g_{\alpha\beta}$ can be immediately written by considering a basis of dual vector fields z^α preserved under spatial isometries. The four-dimensional line element is then expressed as

$$ds_{\text{B}}^2 = \mathcal{N}^2\, dt^2 - A_{ij}\, z^i\, z^j, \tag{2.30}$$

where B stands for Bianchi and, thanks to the homogeneity constraint, A_{ij} is a symmetric matrix encoding the whole time dependence of the model similarly to the scale factor a of the isotropic FLRW model.

The simplest solution of the Einstein field equations in the framework of the Bianchi classification is the so-called Kasner solution, i.e. the type I model where all the structure constants are zero [124, 137]. In the case of vacuum where $\rho = 0$ and $T_{\alpha\beta} = 0$, it is possible to diagonalize the matrix A_{ij} as

$$A_{ij} = \text{Diag}(a_1, a_2, a_3), \tag{2.31}$$

where a_i are three different scale factors for the three spatial directions; then the metric can be recast in a form similar to the FLRW one in cartesian coordinates, but with the three different scale factors:

$$ds_{\text{BI}}^2 = dt^2 - a_1^2(dx^1)^2 - a_2^2(dx^2)^2 - a_3^2(dx^3)^2, \tag{2.32}$$

where BI stands for Bianchi I. This way the model is easily solvable:

$$a_i(t) \propto t^{2c_i}, \qquad \sum_i c_i = 1, \qquad \sum_i c_i^2 = 1, \tag{2.33}$$

where the c_i are constants called Kasner indices that obey the given relations. One Kasner solution is then a set of c_i obeying these relations. Note that in order for those relations to be valid one of the indices will always be negative

The line element (2.32) describes an anisotropic space where volumes linearly grow with time, while linear distances grow along two directions and decrease along the third one, differently from the Friedmann solution where all distances contract towards the singularity with the same behavior. This metric has only one singularity in $t = 0$.

The Bianchi models are easily seen to be anisotropic from the metric (2.32). However, it is often more useful to study them in a different set of variables, especially for the Hamiltonian formulation: the Misner variables $(\alpha, \beta_\pm)$ defined as [162, 163]

$$\alpha = \frac{1}{3}\ln(a_1 a_2 a_3), \qquad \beta_+ = \frac{1}{6}\ln\left(\frac{a_1 a_2}{a_3^2}\right), \qquad \beta_- = \frac{1}{2\sqrt{3}}\ln\left(\frac{a_1}{a_2}\right). \tag{2.34}$$

From these definition, it is evident that the variable α is linked to the total volume and will contain the isotropic information, while the variables $\beta_\pm$ parametrize the anisotropies.

Inserting the solutions $a_i(t)$, the evolution of the Misner variables is found to be

$$\begin{cases} \alpha(t) = \frac{1}{3}\ln\left(\frac{t}{t_0}\right), \\ \beta_+(t) = \frac{c_1+c_2-2c_3}{6}\ln\left(\frac{t}{t_0}\right) = \frac{1-3c_3}{2}\alpha, \\ \beta_-(t) = \frac{c_1-c_2}{2\sqrt{3}}\ln\left(\frac{t}{t_0}\right) = \frac{\sqrt{3}}{2}(c_1-c_2)\alpha, \end{cases} \tag{2.35}$$

where t_0 is an integration constant. It is clear that this system has a singularity $\alpha \to -\infty$ at $t \to 0$. Given that α has a monotonic relation with synchronous time t, it can be chosen as time variable and a very useful quantity is then identified in the anisotropy velocity vector $\boldsymbol{\beta'} = (d\beta_+/d\alpha, d\beta_-/d\alpha)$; then the conditions on the Kasner indices simply become $|\boldsymbol{\beta'}|^2 = 1$. In the Misner variables, when using α as time, the Bianchi I model is simply a free particle moving in the (β_+, β_-) plane with constant, unitary velocity.

At this point, after the $3+1$ foliation of spacetime, the Bianchi metric (2.30) in terms of the Misner variables rewrites in the following form:

$$ds_{\mathrm{B}}^2 = \mathcal{N}^2\, dt^2 - e^{2\alpha}\left(e^{2\beta}\right)_{ij} z^i\, z^j, \tag{2.36}$$

where $\mathcal{N}$ is again the Lapse Function, z^α are the same one-forms as before specifying the symmetry group, and β_{ij} is a matrix containing combinations of the anisotropies $\beta_\pm$. Then the action and the Hamiltonian are

$$S_{\mathrm{B}} = \int dt\left(p_\alpha\,\dot{\alpha} + p_+\,\dot{\beta}_+ + p_-\,\dot{\beta}_- - \mathcal{N}\,\mathcal{H}_{\mathrm{B}}\right), \tag{2.37}$$

$$\mathcal{H}_{\rm B} = \frac{1}{12e^{3\alpha}}\Big(-p_\alpha^2 + p_+^2 + p_-^2\Big) + \frac{e^\alpha}{4}\, U_{\rm B}(\beta_\pm) = 0, \tag{2.38}$$

$$p_\alpha = -\frac{6}{\mathcal{N}}\, e^{3\alpha}\, \dot\alpha, \qquad\qquad p_\pm = \frac{6}{\mathcal{N}}\, e^{3\alpha}\, \dot\beta_\pm. \tag{2.39}$$

where $U_{\rm B}$ is a potential that depends on the chosen model i.e. on the symmetry group that the one-forms z^α obey, and p_α, $p_\pm$ are the momenta conjugate to the corresponding variables. Note that, with this definition, an expanding universe with $\dot\alpha > 0$ implies $p_\alpha < 0$.

In the simple case of the Bianchi I model, the potential is zero: $U_{\rm BI} = 0$, which implies that $p_\pm$ are constants of motion since the anisotropies do not appear; furthermore, given that α appears only as a total prefactor and that the Hamiltonian is constrained to zero, p_α is also a constant of motion. Then the equations of motion and the solutions for α and $\beta_\pm$ in synchronous time, i.e. with $\mathcal{N} = 1$, are given by

$$\dot\alpha = -\frac{\mathcal{N}}{6}\,\frac{p_\alpha}{e^{3\alpha}}, \qquad\qquad \dot\beta_\pm = \frac{\mathcal{N}}{6}\,\frac{p_\pm}{e^{3\alpha}}, \tag{2.40}$$

$$\alpha(t) = \frac{1}{3}\ln\!\Big(-\frac{p_\alpha}{2}\,t\Big), \qquad\qquad \beta_\pm(t) = \beta_{0\pm} - \frac{p_\pm}{p_\alpha}\ln\!\Big(-\frac{p_\alpha}{2}\,t\Big), \tag{2.41}$$

where $\beta_{0\pm}$ are integration constants; for $p_\alpha = -2/t_0$ these are the same solutions as before, except the Kasner indices are now encoded in the constant momenta.

The other Bianchi models will not be analyzed in this thesis, but I will give a very brief conceptual introduction. I have shown above how in the Misner variables, when using α as an internal time variable, the dynamics of the Bianchi I model coincides with that of a free particle moving in a straight line on the (β_+, β_-) plane. The other Bianchi models will have a non-zero potential in the form of infinite, exponentially steep walls. Away from them the dynamics will still be well approximated by a Kasner solution and the potential is only relevant when the particle is close to the walls; when a wall is approached, the particle-universe will rebound off of it and change the values of the Kasner indices. Then, depending on the Bianchi model and the number of walls, the dynamics consists in a series of Kasner solutions linked by these rebounds. In particular, the Bianchi IX model has three walls in an equilateral triangle configuration, so that the particle universe will keep rebounding off the walls. Considering that the walls are moving with α, going backwards in time they will keep closing in on each other, and the Kasner solutions be shorter and shorter until the singularity is reached. The dynamics then acquires a chaotic character, and indeed the chaotic properties of the Bianchi IX model, earning it the name Mixmaster Universe, have been and still are the subject of many studies with many different approaches [38, 39, 43, 44, 96, 103, 116, 144, 145, 163, 164, 166, 167, 192, 210].

2.2 Standard Quantum Cosmology

In this section I will briefly introduce the most relevant and studied approaches to quantum cosmology, namely the canonical quantization and Loop Quantum Gravity (LQG), highlighting the features that are relevant for later comparison.

The canonical approach consists in the quantization of the Hamiltonian constraints according to Standard Quantum Mechanics (SQM). Since they are classically vanishing, this leads to the infamous Wheeler-DeWitt equation (WDW) [78, 88] that is affected by the Problem of Time [58, 123, 131, 135, 155, 193, 217].

LQG is a background-independent quantization scheme based on the $SU(2)$ reformulation of GR, where the fundamental variables are taken to be holonomies of the connections $\mathcal{A}_j^a$ and fluxes of the densitized triads $\mathcal{E}_a^j$ [18, 23, 28, 194, 196]. It is able to overcome some of the problems of the previous approach, and its main results are the discreteness of the geometrical operators [195] and the avoidance of cosmological singularities on a quantum level [24, 26].

2.2.1 Canonical Quantum Cosmology and the Problem of Time

The starting point for Canonical Quantum Cosmology is the action (2.4). Since they are constrained to zero, promoting the SuperHamiltonian and the SuperMomentum to operators results in the wavefunctions admitting only the zero eigenvalue:

$$\hat{\mathcal{H}}\,|\psi\rangle = 0, \qquad \hat{\mathcal{H}}_j\,|\psi\rangle = 0. \tag{2.42}$$

This is the WDW equation of Quantum Gravity. Therefore, when trying to implement the time-dependent Schrödinger equation in the standard way, the wavefunctions will not evolve. This frozen formalism constitutes the Problem of Time. However, there are a few approaches to work around it.

The first approach consists in the recovery of an internal time variable after quantization. It is based on a reinterpretation of the WDW equation as a Klein-Gordon-like wave equation. For example, consider the flat FLRW model expressed in terms of the Misner variable $\alpha = \ln(a)$, filled with a free scalar field ϕ; the Hamiltonian (2.7) with $\mathcal{N} = 1$, $K = 0$ and $\rho_\phi = p_\phi^2/2a^6$ rewrites as

$$\mathcal{H}_{\text{FLRW}}(\alpha, p_\alpha; \phi, p_\phi) = -\frac{1}{12}\frac{p_\alpha^2}{e^{3\alpha}} + \frac{1}{2}\frac{p_\phi^2}{e^{3\alpha}} = 0. \tag{2.43}$$

By promoting the fundamental variables to quantum operators in the standard way the momenta will act differentially, and given that $\mathcal{H} = 0$ the prefactor $e^{-3\alpha}$ disappears. Then the WDW equation reduces to

$$\hat{\mathcal{H}}\,\psi = 0 \qquad \rightarrow \qquad \left(\frac{1}{6}\frac{\mathrm{d}^2}{\mathrm{d}\alpha^2} - \frac{\mathrm{d}^2}{\mathrm{d}\phi^2}\right)\psi = 0 \tag{2.44}$$

(note that I set $\hbar = 1$); this is a Klein-Gordon wave equation where α is the space variable and ϕ plays the role of time variable. The generic solution will then be a superposition of plane waves i.e. a wavepacket of the form

$$\Psi(\alpha, \phi) = \int dk\; W(k)\; e^{ik(\frac{\alpha}{\sqrt{6}} \pm \phi)}, \qquad k = k_\alpha = \pm\sqrt{6}\,k_\phi, \tag{2.45}$$

where the wavenumbers k_i are the eigenvalues of the corresponding momenta $\hat{p}_i$, the second equality comes from the solution to the dispersion relation $k_\alpha^2 = 6k_\phi^2$, and

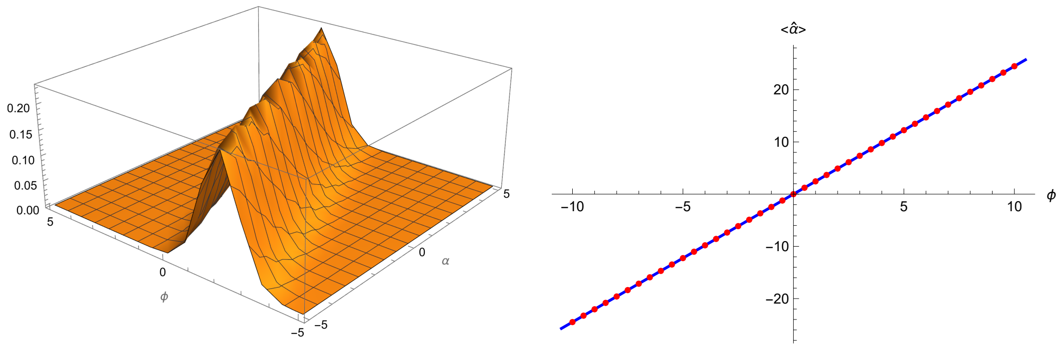

Figure 2.2. Left: absolute value of the wavepacket Ψ in the (ϕ, α) plane, calculated numerically. Right: expectation value $\langle \hat{\alpha} \rangle$ as function of time ϕ (red dots, computed numerically), compared with the classical solution $\alpha(\phi)$ (blue line, derived analytically).

$W(k)$ is a generic, usually Gaussian-like weighing profile; then expectation values of relevant operators are computed through a Klein-Gordon scalar product:

$$\left\langle \hat{O} \right\rangle = \langle \Psi | \, \hat{O} \, | \Psi \rangle = \int d\alpha \; i \left(\Psi^* \, \partial_\phi (\hat{O} \Psi) - (\hat{O} \Psi) \, \partial_\phi \Psi^* \right). \tag{2.46}$$

As an example, Figure 2.2 shows the norm $|\Psi|^2$ of the wavepacket in the (ϕ, α) plane and the expectation value $\langle \hat{\alpha} \rangle$ as function of the scalar field time; it is evident how both the peak of the wavepacket and the expectation value closely follow the classical expanding trajectory of α (and in the right panel it is actually superimposed), which from the Hamiltonian (2.43) can be easily calculated to be $\alpha(\phi) = \pm\sqrt{6}\,\phi$ where the plus corresponds to an expanding universe (and is the one shown in Figure 2.2) while the minus to a contracting one.

It is important to note how the width of the wavepacket remains constant; this is due to the solutions of the dispersion relation being linear and implies that the wavepacket can be peaked at will and the approach to the singularity at $\alpha \to \infty$ is well-described by the corresponding classical dynamics. I will show in later chapters that this not always the case, and with alternative quantization procedure the phenomenon of spreading can happen. This means that in that case the (semi)classical description can be trusted only up to a certain point, and the singularity must be treated in a purely quantum framework.

The Klein-Gordon approach is known as *time after quantization*, and is the one that I will mostly use during this thesis. However, there are other approaches to Quantum Cosmology and to the Problem of Time.

One other approach consists in finding a *time before quantization* through the gauge fixing and the reduction presented above in Section 2.1.2; this implies that the system to be quantized is the one with a reduced Hamiltonian similar to (2.28) and (2.29) where ϕ is a time variable already on the classical level. In simple cases, this yields more or less the same results as the first one, but the two approaches are in general not equivalent. Indeed there are a few conceptual differences: the after quantization approach is based on a full quantization of the system, while the second one relies on a quantization only of some degrees of freedom, and the others have been reduced through the time-gauge fixing and the solution of the classical constraints.

Finally, there are *timeless* approaches based on the idea that there is no need of time at a fundamental level. The quantum theory of gravity can be constructed without a notion of time and such concept may arise only in some special situations and only relationally. Even classical mechanics can be reformulated in a timeless framework, where any motion is only a relative evolution between observables.

To conclude, I must stress that in general the resolution of cosmological singularities in the canonical WDW formalism, although possible, is not a common result [49, 106]. Indeed, according to the Ehrenfest theorem [92], it is expected that the expectation values of physical observables follow the corresponding classical trajectories, and therefore the quantum wavepackets will fall into the singularity in the same way as the GR solutions.

2.2.2 Loop Quantum Cosmology

The name Loop Quantum Cosmology (LQC) refers to a specific quantum cosmological model, i.e. the quantization of the FLRW spacetime according to the methods of Loop Quantum Gravity (LQG) [19, 24, 25, 26, 27, 29, 50, 51, 52, 53, 54, 181, 209]. More in general, it is often also used to indicate cosmological models that are quantized through LQG procedures [20, 57, 68, 100, 221]. In this section, I quickly introduce the formalism of kinematical LQG and then show without too many details its implementation to the isotropic Universe in both the old and new prescriptions, that are the μ_0 scheme and the $\overline{\mu}$ scheme. In later chapters, when I will perform a comparison with Polymer Cosmology, I will reiterate some of the relevant features.

LQG is based on the reformulation of GR as a $SU(2)$ theory mentioned in Section 2.1.1, and then the quantization scheme is performed by using smeared i.e. non-local variables such as the holonomy and flux variables, as suggested by non-Abelian gauge theories on lattices [194, 196, 213, 214]. For recent reviews, see [18, 28].

The smearing of the Ashtekar variables $\mathcal{A}$ and $\mathcal{E}$ is achieved defining holonomies of the connections along edges ℓ and fluxes of the triads across surfaces S:

$$h_\ell[\mathcal{A}] = \exp\left(\int_\ell d\ell^a \; \tau_j \, \mathcal{A}_a^j\right), \qquad \Phi_S[\mathcal{E}] = \int_S dS_a \; \tau_j \, \mathcal{E}_a^j, \tag{2.47}$$

where τ^j are the $SU(2)$ generators. The trace of the holonomies for a closed edge results in the so-called Wilson loop that gives the theory its name.

Now, the quantum kinematics is obtained by promoting these objects to operators and defining their commutator; a very important consequence of the requirement of background independence, i.e. of diffeomorphism invariance, is that the holonomy-flux algebra results to have a unique representation and, therefore, a unique Hilbert space $\mathbb{H}^{\text{kin}}_{\text{LQG}}$. This is called a *spin network space*, defined as a graph γ_{LQG} made of edges identified by a half-integer spin number J and nodes with an intertwiner tensor I. The basis vectors of this Hilbert space are therefore *spin network states* denoted as $|\gamma_{\text{LQG}}, J, I\rangle$, which are orthonormal; they depend on the connections only through holonomies and are square-integrable with respect to the Haar measure.

A key result of the kinematical framework of LQG is the quantization of the geometrical operators of area and volume. For example, the area operator and its action on a functional can be defined through the flux operator and the eigenvalues result in being dependent on how many edges of $\gamma_{\rm LQC}$ intersect the considered surface. In particular, the smallest non-zero eigenvalue of the area operator is a constant quantity depending on fundamental constants and on the Immirzi parameter only; it is called the area gap $\Delta_{\rm LQG}$, and it is a key parameter of the theory. Note that this result is purely kinematical [21, 22, 195].

The dynamics is derived through the implementation of the operators corresponding to the three constraints mentioned in Section 2.1.1 (i.e. the Gauss, the SuperMomentum and the SuperHamiltonian constraints); in order to do this, they must first be expressed in terms of the fundamental holonomies and fluxes, and then quantized through the Dirac procedure [90, 91]. However I will not show the dynamics of full LQG, only the implementation of the Loop quantization procedure on cosmological models.

The implementation of the LQG approach to the cosmological setting leads to define in a rather rigorous mathematical way the concept of a primordial Big Bounce, thus removing the cosmological singularity. However LQC has the intrinsic limitation that the basic $SU(2)$ symmetry underlying the LQG formulation is unavoidably lost when the minisuperspace dynamics is addressed [55, 79]. This is due to the fact that the homogeneity constraint reduces the cosmological problem to a finite number of degrees of freedom; in particular, it becomes impossible to perform the local rotation and preserve the structure constants of the Lie algebra associated to the specific isometry group. In this respect, LQC requires some sort of gauge fixing of the full $SU(2)$ invariance [80]. In addition, the problem of translating the quantum constraints from the full to the reduced level remains still open. This is the reason that LQC is not the cosmological sector of LQG, i.e. it is not a symmetry reduction to cosmological minisuperspaces of the full LQG theory; rather it is the implementation of the Loop quantization procedure to cosmological spacetimes whose symmetry-reduction has been performed beforehand on a classical level. I will now briefly show how this implementation works on the isotropic cosmological model.

When the symmetries of the FLRW model are implemented, the Ashtekar variables simply reduce to powers of the scale factor and its derivatives: the triad $\mathcal{E}$ becomes the area variable $s = a^2$, while $\mathcal{A}$ becomes the connection $p_s \propto \dot{a}$. While the area has already the correct non-local features, the connection p_s must be smeared through the construction of the holonomy, which in the diagonal case reads as

$$h^j_\lambda(p_s) = \cos\left(\frac{\lambda\, p_s}{2}\right)\mathbb{I} + 2\sin\left(\frac{\lambda\, p_s}{2}\right)\tau^j, \tag{2.48}$$

where λ is a generic parameter and $\mathbb{I}$ is the identity matrix; the elementary configurational variables then are s and $\exp(i\lambda p_s/2)$.

The Hilbert space $\mathbb{H}^{\rm kin}_{\rm LQC}$ is the space $L^2(\mathbb{R}_{\rm B}, dm_H)$ of square-integrable function on the Bohr compactification of the real line $\mathbb{R}_{\rm B}$ endowed with the Haar measure dm_H. In the s-polarization then the fundamental states are eigenstates of $\hat{s}$ labeled by a real number μ on which the fundamental operators act as

$$\widehat{e^{i\frac{\lambda p_s}{2}}}\;|\mu\rangle = |\mu+\lambda\rangle\,, \qquad \hat{s}\;|\mu\rangle = \frac{\Gamma}{6}\,\mu\;|\mu\rangle\,, \tag{2.49}$$

where Γ is the Immirzi parameter [122, 197].

In the cosmological minisuperspace, the dynamics is defined by the introduction of an operator corresponding to the SuperHamiltonian constraint (the Gauss and SuperMomentum constraints are identically satisfied as mentioned in Section 2.1). This must be done by returning to the integral expression of the constraint and expressing it as function of the fundamental variables before quantization. After implementing the symmetry reductions and the Thiemann strategy [212], the gravitational constraint can be written as the limit of a $\lambda-$dependent Hamiltonian that can be easily promoted to operator:

$$\mathcal{H}_{\text{LQC}} = \lim_{\lambda\to 0}\mathcal{H}_\lambda, \qquad \hat{\mathcal{H}}_\lambda = \frac{24\,i\,s\,\sin^2(\lambda\,p_s)\,\hat{\mathcal{O}}_\lambda}{\Gamma^3\lambda^3}\,, \tag{2.50}$$

$$\hat{\mathcal{O}}_\lambda = \sin\!\left(\frac{\lambda\,p_s}{2}\right)\hat{s}^{\frac{3}{2}}\,\cos\!\left(\frac{\lambda\,p_s}{2}\right) - \cos\!\left(\frac{\lambda\,p_s}{2}\right)\hat{s}^{\frac{3}{2}}\,\sin\!\left(\frac{\lambda\,p_s}{2}\right). \tag{2.51}$$

However, in LQC the limit $\lambda\to 0$ does not exist by construction. This can be interpreted as a reminder of the underlying quantum geometry, where the area operator has a discrete spectrum with a smallest non-zero eigenvalue corresponding to the area gap Δ; as a consequence, λ must be set to a certain value that can be appropriately related to the area gap. This relation is where the two different schemes of LQC arise from: depending on whether the area gap is assigned a kinematical or dynamical character, the result will be the μ_0 scheme or the $\overline{\mu}$ scheme respectively.

The μ_0 scheme, introduced in [24], considers the area gap as a kinematical quantity and sets the parameter $\lambda = \mu_0 =$ const. by considering that holonomies are eigenstates of the kinematical area operator:

$$\hat{s}\,h_{\mu_0} \propto \mu_0^2\,h_{\mu_0} \propto \Delta_{\text{LQC}}\,h_{\mu_0}. \tag{2.52}$$

Then the quantum Hamiltonian constraint is simply the $\lambda-$dependent operator (2.50) with $\lambda = \mu_0$.

After the introduction of matter in the form of a free scalar field ϕ as $\hat{\mathcal{H}}_\phi = \hat{p}_\phi^2/\hat{s}^{\frac{3}{2}}$, where p_ϕ is the momentum conjugate to the field, the WDW equation can be rewritten as a difference operator (instead of differential):

$$\frac{\partial^2\Psi}{\partial\phi^2} = \frac{1}{B_\mu}\bigg(\mathcal{H}_+\,\Psi(\mu+4\mu_0,\phi) - \Big(\mathcal{H}_+ + \mathcal{H}_-\Big)\,\Psi(\mu,\phi) + \mathcal{H}_-\,\Psi(\mu-4\mu_0,\phi)\bigg), \tag{2.53}$$

$$\mathcal{H}_\pm \propto \frac{\Big||\mu\pm 3\mu_0|^{\frac{3}{2}} - |\mu\pm\mu_0|^{\frac{3}{2}}\Big|}{\mu_0^3}, \tag{2.54}$$

where B_μ is the eigenvalue of the inverse-volume operator $1/\hat{s}^{\frac{3}{2}}$ and is a function of the label μ. Wavepackets can be constructed, and then it is possible to evolve them and compute the expectation values of relevant observables numerically. I will briefly summarize the most important results.

An initially semiclassical state remains sharply peaked around the classical trajectories for most of the evolution, but when matter density approaches a critical value the state Bounces from the expanding branch to a contracting one with the same value of $\langle \hat{p}_\phi \rangle$. This universally solves the cosmological singularities by replacing the Big Bang and the Big Crunch with a Big Bounce.

However, the critical value of the matter density results in being inversely proportional to the expectation value $\langle \hat{p}_\phi \rangle$ and can, therefore, be made arbitrarily small by choosing a sufficiently large initial condition. This fact, besides being physically unreasonable because it could imply departures from the classical trajectories well away from the Planck regime, becomes even more problematic in the case of a closed model: the point of maximum expansion depends on $\langle \hat{p}_\phi \rangle$ as well. In order to have a Bounce density comparable with that of Planck, a very small value is needed, but in that case, the Universe would never become big enough to be considered classical; on the other hand, a closed Universe that grows to become classical needs a larger value but would have a very low Bounce density where it is well known that quantum effects are negligible. This is the reason for which a different scheme was needed.

The $\overline{\mu}$ scheme was developed in [25, 26] to solve the shortcomings of the the previous one, in particular regarding the Bounce density. The idea is that the quantization of the area operator should refer to physical geometries, and therefore the parameter λ is not a constant anymore but is a function $\overline{\mu}(\mu)$ linked to the area gap through the relation

$$\overline{\mu}^2 \, \mu \propto \Delta_{\text{LQC}}. \tag{2.55}$$

This way, the translational operator depends both on the connection and the geometry, and more care is needed in the definition of the exponential operator because now $\exp(i \, \overline{\mu} \, p_s/2)$ depends also on the eigenvalues μ of $\hat{s}$.

From geometric consideration and a comparison with the standard Schrödinger representation, it is possible to set

$$\widehat{e^{i \frac{\overline{\mu} p_s}{2}}} \, \Psi(\mu) = e^{\overline{\mu} \frac{\mathrm{d}}{\mathrm{d}\mu}} \, \Psi(\mu), \tag{2.56}$$

that is, the exponential operator translates the state by a unit affine-parameter distance along the curve of the vector field $\overline{\mu} \, d/d\mu$. This affine parameter is given by

$$\nu \propto \text{sgn}(\mu) \, |\mu|^{\frac{3}{2}}, \tag{2.57}$$

and since this expression is invertible it is useful to perform a change of variable from μ to ν, so that the exponential operator now acts as

$$\widehat{e^{i \frac{\overline{\mu} p_s}{2}}} \, \Psi(\nu) = \Psi(\nu + 1). \tag{2.58}$$

The kets ν are an orthonormal basis and they result to be eigenstates of the volume operator. Then the gravitational constraint can be constructed in the same way as before.

Repeating the same steps of the previous μ_0 scheme, the WDW equation is again a difference operator but this time in terms of ν:

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}\phi^2} = \frac{1}{B_\nu}\Bigg(\mathcal{H}_+\,\Psi(\nu+4,\phi) - \Big(\mathcal{H}_+ + \mathcal{H}_-\Big)\,\Psi(\nu,\phi) + \mathcal{H}_-\,\Psi(\nu-4,\phi)\Bigg), \tag{2.59}$$

$$\mathcal{H}_\pm \propto |\nu \pm 2|\,\Bigg|\,|\nu\pm 1| - |\nu \pm 3|\,\Bigg|. \tag{2.60}$$

The previous WDW equation (2.53) of the μ_0 scheme involves step that are constant in the eigenvalues of the area operator $\hat{s}$, while this new one involves steps that are constant in the eigenvalues of the volume operator $\hat{\nu} \propto \hat{s}^{\frac{3}{2}}$.

After the construction of wavepackets and numerical calculations, the state still remains sharply peaked and, when approaching a critical energy density, it Bounces to the other branch also in this case; so the singularity is still removed and replace by a Big Bounce. However, and most importantly, the critical value of the density at which the Bounce happens results to be a universal constant.

The semiclassical limit of LQC, i.e. the inclusion of quantum corrections in the classical dynamics, can be obtained through a geometric formulation of quantum mechanics where the Hilbert space is treated as an infinite-dimensional phase space [205]. In simpler cases with coherent states that are preserved by the full quantum dynamics, the resulting Hamiltonian coincides with the classical one; however, in more general systems it is possible to choose suitable semiclassical states that are preserved up to a desired accuracy (e.g. in a $\hbar$ expansion), and the corresponding effective Hamiltonian preserving this evolution is generally different from the classical one [211].

The leading order Loop corrections on the FLRW Hamiltonian result to be:

$$\mathcal{H}^{\mu_0}_{\mathrm{LQC}} = -\frac{3}{\Gamma^2\mu_0^2}\; s^{\frac{1}{2}}\,\sin^2(\mu_0\, p_s) + \frac{B_\mu}{2}\,p_\phi^2 = 0, \tag{2.61}$$

$$\mathcal{H}^{\overline{\mu}}_{\mathrm{LQC}} = -\frac{3}{\Gamma^2\overline{\mu}^2}\; s^{\frac{1}{2}}\,\sin^2(\overline{\mu}\, p_s) + \frac{B_\nu}{2}\,p_\phi^2 = 0. \tag{2.62}$$

From the Hamilton equations, it is possible to obtain modified Friedmann equations:

$$H^2_{\mu_0} = \left(\frac{\dot{s}}{2s}\right)^2 = \frac{\rho}{3}\left(1 - \frac{\rho}{\rho_{\mu_0}}\right), \qquad \rho_{\mu_0} = \left(\frac{3}{\Gamma^2\mu_0^2}\right)^{\frac{3}{2}}\frac{\sqrt{2}}{p_\phi}, \tag{2.63}$$

$$H^2_{\overline{\mu}} = \left(\frac{\dot{\nu}}{3\nu}\right)^2 = \frac{\rho}{3}\left(1 - \frac{\rho}{\rho_{\overline{\mu}}}\right), \qquad \rho_{\overline{\mu}} = \frac{4\sqrt{3}}{\Gamma^3}. \tag{2.64}$$

It is clear how in both cases, when the total energy density ρ is equal to the regularizing densities ρ_{μ_0} or $\rho_{\overline{\mu}}$, the Hubble parameter goes to zero and therefore a critical point $\dot{s} = 0$ or $\dot{\nu} = 0$ appears corresponding to a Bounce; therefore the singularity is removed also in the (corrected) classical dynamics. However, in the effective μ_0 dynamics this critical energy density depends on the constant of motion p_ϕ, and therefore on initial conditions, while in the $\overline{\mu}$ scheme it is a universal constant. This is the main reason for which the improved model is much more appealing than the standard one.

Over the years, many criticisms have been made on the LQC framework, mainly about the fact that the quantum dynamics is not derived by a symmetry reduction of the full LQG theory, but by quantizing cosmological models that are reduced before quantization. The spatial geometry of a cosmological spacetime is fixed, and it is not possible to perform local $SU(2)$ transformations in the minisuperspace. Furthermore, as it is well known, in LQG the implementation of the scalar constraint is not yet a viable task, and it is worth noting how this problem is somehow bypassed in LQC, where the dynamics for the cosmological models is constructed; however, this procedure is far from being completely clear. Another way to see the problem of the $SU(2)$ symmetry is that the resulting algebra on the reduced model is different from the holonomy-flux algebra of the full theory, and therefore, LQC is not equivalent to LQG [79]. Indeed, the improved $\overline{\mu}$ scheme of LQC does not use the symmetry-reduced counterpart of the original Ashtekar connection, which is the only viable $SU(2)$ variable of full LQG. Another problem that is often raised is that an external parameter (the area gap) fixing the discretization scale must be introduced from the full theory by hand, because LQC is derived independently from LQG and the area gap is not introduced naturally in the cosmological setting.

LQG and LQC attempt to provide a promising framework for a quantum mechanical description of general relativity and of cosmological models, but as outlined in this paragraph, both—the latter in particular—need to be substantially improved. In more recent years there have been a few new LQG-like approaches to the quantization of General Relativity and of cosmological models that attempt to solve or work around these issues, or even to derive LQG from a more fundamental theory. Some examples are Quantum Reduced Loop Gravity [1, 2, 3, 4, 5, 154], Causal Dynamical Triangulation [8, 9, 10, 104, 148], Modified LQC [141, 198, 223] and Group Field Theory [98, 101, 177, 178, 179, 180]; however they will not be analyzed here.

On the other hand, the new quantum mechanical framework of Polymer Quantum Mechanics (PQM) was developed in order to reproduce LQC effects independently from LQG. This makes it much more versatile and easily applicable to any Hamiltonian system, and it is helpful to explore Loop-like quantum effects in cosmological models without the need to address the problems of the more fundamental LQG theory. The quantum mechanical framework of PQM will be the subject of the next chapter, together with other alternative quantization procedures of similar scope.

Chapter 3

Alternative Quantum Mechanics

In this chapter I introduce a series of alternative quantization procedures that will later be implemented on the gravitational and cosmological actions.

The alternative procedures that I will present must of course violate one or more hypotheses of the Stone-von Neumann theorem [207, 208, 218, 219], otherwise they would just be equivalent representations of Standard Quantum Mechanics (SQM) and would give the same predictions.

The first alternative representation is Polymer Quantum Mechanics (PQM), a quantization procedure on a lattice where one of the fundamental variables (usually position or a similar quantity) is assigned a discrete character, and as a consequence the generator of translations is not weakly continuous anymore and the corresponding conjugate momentum cannot be constructed in the usual way.

The second alternative quantization is the Generalized Uncertainty Principle representation (GUP), where a modified uncertainty relation is obtained through a deformation of the standard Heisenberg commutators. This implies an absolute minimal uncertainty on position and is a simple and phenomenological way to introduce a fundamental length, a concept whose existence is indicated by most approaches to a theory of Quantum Gravity.

The GUP representation can be extended by generalising the modified commutation relations to other functions. Since quantum gravitational corrections are expected to be relevant at high energies, these deformations of the commutators will usually be in the form of functions of momenta. Similarly to the GUP, they represent a quick and simple procedure to introduce in Hamiltonian systems corrections coming from more fundamental Quantum Gravity theories, especially in view of their straightforward classical limit.

3.1 Polymer Quantum Mechanics

Polymer Quantum Mechanics (PQM) is a non-regular alternative representation of SQM, non-unitarily connected to the ordinary Schrödinger representation. It has been used to explore both mathematical and physical issues in background-independent theories. It was conceived in the framework of Loop Quantum Gravity (LQG) but through an independent procedure and, when applied to minisuperspace models, has given way to what I will call Polymer Cosmology with results similar to

Loop Quantum Cosmology (LQC).

The Polymer quantization is made of several steps. The first one is to build a representation of the Heisenberg-Weyl algebra on a background-independent Kinematic Hilbert space, which will be referred to as the Polymer Hilbert space $\mathbb{H}_{\text{PQM}}$. The second and most important part is the implementation of the dynamics and needs the definition of a Hamiltonian function or constraint on this space. The second step is more difficult than the previous because one of the main features of this representation is that one among the standard quantum operators $\hat{q}$ and $\hat{p}$ will always be not well-defined (nor will be their analogues in systems with more elaborate variables). This is because the fundamental feature of PQM is to assign a discrete character to one of the phase space variables. Thus any operator that is a function of the not-defined variable has to be regulated by a well-defined operator which usually involves the introduction of some extra structure on the configuration space, namely a lattice. The downside is that this extra structure can not be removed when working in $\mathbb{H}_{\text{PQM}}$, and one is left with the ambiguity of the regularization, an ambiguity that is usually associated with a length scale i.e. the lattice spacing. However, when applying the semiclassical Polymer deformation to various settings in later chapters, usually cosmological models but not only, it is possible to find some constraints on this scale by imposing reasonable conditions, such as for example on the reasonable order of magnitude of the density of the Universe near the singularity or on the validity of a non-relativistic approximation.

Here, I introduce PQM following Corichi et al. [82]. Given the similarities, here I will use the same labels of LQC; from here on, except where indicated, the symbol μ (and other similar ones) will refer to the Polymer representation and not to the Loop quantization.

3.1.1 Polymer Kinematics

In order to introduce the Polymer representation without any reference to the Schrödinger one, consider the abstract kets $|\mu\rangle$ labeled by a real number $\mu \in \mathbb{R}$ and taken from the Hilbert space $\mathbb{H}_{\text{PQM}}$.

A generic cylindrical state can be defined as a finite linear combination of the form

$$|\psi\rangle = \sum_{i=1}^{N} c_i \, |\mu_i\rangle \, , \tag{3.1}$$

where $\mu_i \in \mathbb{R}$, $i = 1, \ldots, N \in \mathbb{N}$. The inner product is chosen so that the fundamental kets are orthonormal:

$$\langle \mu_i | \mu_j \rangle = \delta_{ij} \, . \tag{3.2}$$

From this choice, it follows that the inner product between two cylindrical states $|\psi_1\rangle = \sum_i c_{1i} \, |\mu_i\rangle$ and $|\psi_2\rangle = \sum_j c_{2j} \, |\mu_j\rangle$ is

$$\langle \psi_2 | \psi_1 \rangle = \sum_{i,j} c^*_{2j} \, c_{1i} \, \delta_{ij} = \sum_i c^*_{2i} \, c_{1i}. \tag{3.3}$$

It can be demonstrated that the Hilbert space $\mathbb{H}_{\text{PQM}}$ is the Cauchy completion of the finite linear combination of the form (3.1) with respect to the inner product (3.2) and that it results to be non-separable.

Two fundamental operators can be defined on this Hilbert space: the symmetric label operator $\hat{\epsilon}_\mu$ and the shift operator $\hat{S}_\mu(\lambda)$ with $\lambda \in \mathbb{R}$. They act on the kets $|\mu\rangle$ as follows:

$$\hat{\epsilon}_\mu \, |\mu\rangle = \mu \; |\mu\rangle \; , \qquad \hat{S}_\mu(\lambda) \, |\mu\rangle = |\mu + \lambda\rangle \; . \tag{3.4}$$

The shift operator defines a one-parameter family of unitary operators on $\mathbb{H}_{\text{poly}}$. However, since the kets $|\mu\rangle$ and $|\mu + \lambda\rangle$ are orthogonal for any $\lambda \neq 0$, the shift operator $\hat{S}_\mu(\lambda)$ is discontinuous in λ and therefore there is no Hermitian operator that can generate it by exponentiation.

Now that the abstract structure of the Hilbert space is described, I can proceed to define the physical states and operators. In the following, I will consider a one-dimensional system identified by the phase-space coordinates (q, p), and I will separate the discussion into two cases referred to the two possible polarizations for the wave function. I will also assume that the configurational coordinate q has a discrete character, due to the relation that it often possess with geometrical quantities. This is a way to investigate the physical effects of discreteness at a given scale, for example when introducing quantum gravity effects on the cosmological dynamics.

In the momentum polarization the fundamental kets and a generic wave function can be written as:

$$\psi_\mu(p) = \langle p|\mu\rangle = e^{i\mu p}, \qquad \psi(p) = \langle p|\psi\rangle \; . \tag{3.5}$$

The shift operator $\hat{S}_\mu(\lambda)$ is identified with the multiplicative exponential operator $\hat{\varsigma}_\mu(\lambda)$:

$$\hat{\varsigma}(\lambda) \, \psi_\mu(p) = e^{i\lambda p} \, e^{i\mu p} = e^{i(\mu+\lambda)p} = \psi_{\mu+\lambda}(p); \tag{3.6}$$

$\hat{\varsigma}$ is discontinuous by definition and as a result, the momentum p cannot be promoted to a well-defined operator. On the other hand, $\hat{q}$ corresponds to the label operator $\hat{\epsilon}_\mu$ and in this polarization acts differentially:

$$\hat{q} \, \psi_\mu(p) = -i \, \frac{\partial \psi_\mu}{\partial p} = \mu \, \psi_\mu(p). \tag{3.7}$$

Additionally, it has to be considered as a discrete operator since the states $\psi_\mu(p)$ are orthonormal for all μ, even though μ belongs to a continuous set.

By means of C^*-algebra it can be seen that $\mathbb{H}_{\text{PQM}}$ is isomorphic to $L^2(\mathbb{R}_B, dm_H)$, where $\mathbb{R}_B$ is the Bohr compactification of the real line, i.e. the dual group of the real line equipped with discrete topology, and dm_H is the Haar measure. Note how this is the same Hilbert space of kinematic LQC introduced in 2.2.2.

In the position polarization, the wave functions depend on the configurational variable q and can be written as

$$\psi(q) = \langle q|\psi\rangle \; , \tag{3.8}$$

where the basis functions can be derived using a Fourier-like transform:

$$\begin{aligned}\psi_\mu(q) = \langle q|\mu\rangle = \langle q| \int_{\mathbb{R}_B} dm_H\, |p\rangle \langle p|\mu\rangle = \langle q| \int_{\mathbb{R}_B} dm_H\, |p\rangle\, \psi_\mu(p) = \\ = \int_{\mathbb{R}_B} dm_H\, e^{-iqp} e^{i\mu p} = \delta_{q\mu},\end{aligned} \tag{3.9}$$

through which it is evident that the $\hat{p}$ operator does not exist since the derivative of the Kronecker delta is not well defined. However, the shift operator $\hat{\varsigma}$ acts as

$$\hat{\varsigma}(\lambda)\, \psi(q) = \psi(q+\lambda). \tag{3.10}$$

As in the previous case, the $\hat{q}$ operator corresponds to $\hat{\epsilon}_\mu$, but in this polarization it acts multiplicatively:

$$\hat{q}\, \psi_\mu(q) = \mu\, \psi_\mu(q). \tag{3.11}$$

The Hilbert space has analogous features as before, resulting to be isomorphic to $L^2(\mathbb{R}_D, dm_C)$ where $\mathbb{R}_D$ is the real line equipped with the discrete topology and dm_C is the counting measure. The inner product then is

$$\left\langle \psi_{\mu_i}(q) \middle| \psi_{\mu_j}(q) \right\rangle = \delta_{ij}, \tag{3.12}$$

so it is clear how the $\hat{q}$ operator is discrete also in this polarization.

3.1.2 Polymer Dynamics

In the previous section, the Polymer kinematic Hilbert space was introduced. In particular, the discussion above has highlighted that it is not possible to well define the $\hat{q}$ and $\hat{p}$ operators simultaneously in the Polymer framework. So, it is necessary to understand how to implement the dynamics in order to apply the Polymer framework to a physical system.

Consider the p-polarization of a one-dimensional system described by a standard Hamiltonian:

$$\mathcal{H} = \frac{p^2}{2m} + U(q). \tag{3.13}$$

Assuming $\hat{q}$ to be a discrete operator, an approximate form for $\hat{p}$ must be constructed. For this reason, the required regularization procedure consists in the introduction of a lattice with constant spacing μ_0:

$$\gamma_\mu = \{q \in \mathbb{R} : q = n\mu_0\, \forall\, n \in \mathbb{Z}\}. \tag{3.14}$$

In order to remain on the lattice then only states with label $\mu_n = n\mu_0$ are permitted, and the Hilbert space becomes $\mathbb{H}_{\gamma_\mu} \subset \mathbb{H}_{\text{poly}}$ containing all the functions ψ such that $\sum_n |c_n|^2 < \infty$, where c_n are the coefficients as in equation (3.1).

Now, an approximate form for $\hat{p}$ must be constructed in order to have a well-defined Hamiltonian operator through which to implement the dynamics in both the polarizations. The operator $\widehat{e^{i\lambda p}}$ is well defined and acts as the shift operator on the kets $|\mu\rangle$; in particular, by restricting it to the lattice through the imposition $\lambda = \mu_0$, its action becomes

$$\hat{\varsigma}(\mu_0)\, |\mu_n\rangle = |\mu_n + \mu_0\rangle = |\mu_{n+1}\rangle\,. \tag{3.15}$$

Therefore, it is possible to use the shift operator to introduce the following approximation:

$$p \sim \frac{\sin(\mu_0\, p)}{\mu_0} = \frac{e^{i\mu_0 p} - e^{-i\mu_0 p}}{2\, i\, \mu_0}, \tag{3.16}$$

valid in the limit $\mu_0 p \ll 1$, so that the regularized $\hat{p}$ operator acts as

$$\hat{p}_\mu \, |\mu_n\rangle = \frac{1}{2i\mu_0}\Big(\hat{\varsigma}(\mu_0) - \hat{\varsigma}(-\mu_0)\Big)\, |\mu_n\rangle = \frac{1}{2i\mu_0}\big(|\mu_{n+1}\rangle - |\mu_{n-1}\rangle\big); \tag{3.17}$$

when moving onto the momentum polarization, its action on the wavefunction $\psi(p)$ will be multiplicative with a modified eigenvalue:

$$\hat{p}_\mu\, \psi(p) = \frac{\sin(\mu_0\, p)}{\mu_0}\, \psi(p). \tag{3.18}$$

Note that this implies that the spectrum of the momentum operator is limited both from above and from below, given that the sine is a limited function. I will show in later chapter how this corresponds to an energy cut-off.

There are various possible definitions for the squared momentum, but the simplest and most consistent one is the composition of the operator $\hat{p}_\mu$ defined above with itself, resulting in $\hat{p}^2_\mu\, \psi(p) = \frac{\sin^2(\mu_0\, p)}{\mu_0^2}\, \psi(p)$.

Since $\hat{q}$ is always a well-defined operator, the regularized version of the Hamiltonian is simply

$$\hat{\mathcal{H}}_\mu = \frac{\hat{p}^2_\mu}{2m} + \hat{U}(\hat{q}) \tag{3.19}$$

that represents a symmetric and well-defined operator on $\mathbb{H}_{\gamma_\mu}$.

Note that this Hamiltonian provides an effective description at the given scale μ_0. More specifically, the question about the consistency between the effective theories at different scales and the existence of the continuum limit is deeply investigated in [81]. In particular, it is shown that the continuum Hamiltonian can be represented in a Hilbert space unitarily equivalent to the ordinary L^2 space of the Schrödinger theory by means of a renormalization procedure that involves coarse graining as well as rescaling, following Wilson's renormalization group ideas.

When implementing PQM on the cosmological minisuperspaces, the geometrical variables will usually be discretized and therefore will play the same role as the position q; consequently, all the conjugate variables will play the same role as the momentum p and will therefore act multiplicatively with a modified eigenvalue in the form of the sine function as in (3.18).

It is also possible to implement Polymer corrections on a semiclassical level by modifying the Hamiltonian, substituting every instance of the momentum conjugate to the discretized variable with the corresponding sine function, i.e. using the substitution (3.16) where $p \to \sin(\mu_0\, p)/\mu_0$, and then deriving the equations of motion through the standard Hamilton equations. Most of the time throughout this thesis I will study the semiclassical system obtained in this way before moving on to quantization.

3.2 Generalized Uncertainty Principle Representation

The existence of a fundamental minimal scale is supposed and expected in a quantum theory of gravity. Therefore many possibilities have been studied involving minimal length uncertainty relations, and most of them imply corrections to the usual Heisenberg uncertainty principle such that it becomes what is known as Generalized Uncertainty Principle (GUP).

Interest in a minimal length or GUP has been motivated by studies on perturbative String Theory [7, 11, 107, 108, 222], and considerations regarding the properties of black holes [153] and the de Sitter space [206]. From the String Theory point of view, a minimal length is a consequence of the fact that strings cannot probe distances below the string scale.

There are different approaches to the Generalized Uncertainty Principle; I will present the first and most straightforward one, proposed by Kempf, Mangano and Mann in 1995 (KMM) [126]. Even though this formalism can be constructed independently from String Theories, it represents a way in which some of their features can be made manifest in Hamiltonian models.

3.2.1 Modified Commutation Relations

The KMM approach consists in a modified commutation relation between position and momentum, that for a one-dimensional system reads

$$[\hat{q},\hat{p}] = i\,(1+B_0\,\hat{p}^2), \qquad B_0>0, \tag{3.20}$$

where B_0 is a positive parameter related to the string length and to the scale at which quantum-gravitational effects are expected to become relevant. This modification leads to the modified uncertainty relation

$$\Delta q\,\Delta p \geq \frac{1}{2}\left(1+B_0\,\Delta p^2 + B_0\,\langle\hat{p}\rangle^2\right), \tag{3.21}$$

that appears in perturbative String Theory [7, 11, 107]. When considering $\langle\hat{p}\rangle = 0$, this implies an absolute minimal uncertainty in position $\Delta q_{\text{min}} = \sqrt{B_0}\,$.

The existence of a nonzero uncertainty in position implies that a position eigenstate cannot exists, because eigenstates have by definition zero uncertainty. (Actually it is possible to construct position eigenvectors, but they will be only formal eigenvectors and not physical states.) This consideration forces the use of the momentum polarization, where the states are functions of the momentum p and the basic operators' action is

$$\hat{p}\,\psi(p) = p\,\psi(p), \qquad \hat{q}\,\psi(p) = i\,(1+B_0\,p^2)\frac{\mathrm{d}\psi}{\mathrm{d}p}. \tag{3.22}$$

It is easy to verify that this way the commutation relations (3.20) are recovered.

3.2.2 Quasi-Position Wavefunction

In order to recover information on positions, it is possible to construct states that realize the maximum allowed localization, i.e. the minimal possible uncertainty.

Such states of maximal localization around a position q_0, denoted by $\left|\psi^{\mathrm{ml}}_{q_0}\right\rangle$, have therefore the properties

$$\left\langle\psi^{\mathrm{ml}}_{q_0}\middle|\hat{q}\middle|\psi^{\mathrm{ml}}_{q_0}\right\rangle = q_0, \qquad \Delta q\Big|_{\left|\psi^{\mathrm{ml}}_{q_0}\right\rangle} = \Delta q_{\mathrm{min}} = \sqrt{B_0}\,, \tag{3.23}$$

and they obey the equation

$$\left(\hat{q} - \langle\hat{q}\rangle + \frac{\langle[\hat{q},\hat{p}]\rangle}{2\,\Delta p^2}\,\hat{p}\right)\left|\psi^{\mathrm{ml}}_{q_0}\right\rangle = 0, \tag{3.24}$$

where the absolute maximal localization can be obtained only for $\langle\hat{p}\rangle = 0$. By solving this, the explicit form of these minimal uncertainty states is

$$\psi^{\mathrm{ml}}_{q_0}(p) \propto \frac{1}{1 + B_0\,p^2}\,\exp\!\left(-i\,q_0\,\frac{\arctan\!\left(\sqrt{B_0}\,p\right)}{\sqrt{B_0}}\right); \tag{3.25}$$

they reduce to the standard plane waves in the limit $B_0 \to 0$.

At this point, the probability amplitude for a particle to be maximally localized around a position q_0 with standard deviation Δq_{min} can be obtained by projecting an arbitrary state $|\psi\rangle$ on $\left|\psi^{\mathrm{ml}}_{q_0}\right\rangle$. This projection $\psi(q_0) = \left\langle\psi^{\mathrm{ml}}_{q_0}\middle|\psi\right\rangle$ is called quasi-position wave function and is given by

$$\psi(q_0) \propto \int_{-\infty}^{+\infty} \frac{dp}{(1 + B_0\,p^2)^{\frac{3}{2}}}\,\exp\!\left(iq_0\,\frac{\arctan\!\left(\sqrt{B_0}\,p\right)}{\sqrt{B_0}}\right)\psi(p). \tag{3.26}$$

This is nothing but a generalized Fourier transformation, and in the limit $B_0 \to 0$ the ordinary position wave function $\psi(q_0) = \langle q_0|\psi\rangle$ is recovered.

3.3 Modified Algebras

As mentioned earlier, it is possible to generalize the KMM construction presented above to different forms of deformed algebras. I will first introduce the general features of the kind of modified algebras that are relevant for Quantum Gravity and Quantum Cosmology, i.e. those of the form $[\hat{q},\hat{p}] = i\hbar f(\hat{p})$ (although I will set $\hbar = 1$ also in this section), and then extract specific properties of the algebras that I will use in this thesis.

A few constructions have already been analyzed in the literature [41, 95, 202], and they are usually referred to as variations of the GUP; however, I will use the term GUP to refer only to the KMM construction, and the other ones will have names more specific to the properties they exhibit.

Note that it is possible to implement modified algebras (including the standard GUP) in more than one dimensions; in that case the algebra would be $[\hat{q}_i, \hat{p}_j] = i\,f(|p|)\,\delta_{ij}$ or similar forms, but by requiring that the Jacobi identities be valid a non-commutativity between different space direction is automatically induced, as explained in [95, 203]. However, given the symmetries of cosmological models, in later chapters I will mostly deal with one-dimensional systems where the Jacobi identities are usually automatically satisfied; therefore this property will not be explored here, and the following introduction to modified algebras will only talk about one-dimensional systems. Furthermore, even in multi-dimensional systems it is possible to implement a modified algebra on a single variable and, as long as this is independent from the other ones, there will be no spatial non-commutativity; as an example, for the Bianchi models in Misner variables, it is possible to implement the algebra on the isotropic variable α and leave the anisotropic sector parametrized by the variables $\beta_\pm$ completely unaffected.

The results about the specific algebras used in this thesis (as well as a brief cosmological implementation) are published in the paper [34], while their implementation to gravitational and cosmological models, presented later in Chapter 5, are reported in the papers [36, 37].

3.3.1 Different Representations

First of all, there are a few different ways to implement modified algebras, corresponding to different representations i.e. different actions for the operators. Given a generic system with variables q and p, whose corresponding operators obey a modified algebra of the form

$$[\hat{q},\hat{p}] = i\, f(\hat{p}), \tag{3.27}$$

it is easy to see that it is much simpler to study this kind of systems in the momentum polarization, where wavefunctions $\psi = \psi(p)$ are functions of the momentum. Then there are many possible representations; here I will introduce those two that are relevant for this thesis.

The first is a KMM-like representation, where the momentum operator acts multiplicatively in the same way as SQM and the position operator is differential but modified:

$$\hat{q}\,\psi(p) = i\, f(p)\,\frac{\mathrm{d}\psi}{\mathrm{d}p}, \qquad \hat{p}\,\psi(p) = p\,\psi(p); \tag{3.28}$$

then the domain of the system $\mathbb{D}_q$ (where the subscript q indicates that it is the position operator to be modified with respect to SQM) and the corresponding scalar product in the Schrödinger picture will be

$$\mathbb{D}_q \subseteq L^2\Big(\mathbb{R}, \frac{dp}{f(p)}\Big), \qquad \langle\psi_1|\psi_2\rangle_q = \int \frac{dp}{f(p)}\,\psi_1^*\,\psi_2; \tag{3.29}$$

the presence of the measure $1/f(p)$ in the scalar product is needed to ensure the symmetry of the position operator.

The second representation is somewhat opposite: here the position operator will act differentially in the same way as SQM, and it will be the momentum operator to be modified (but still multiplicative) in a similar way to the operators in PQM. Through arguments introduced in [34, 202], the action of the operators on a wavefuction $\psi(p)$ in this representation results to be

$$\hat{q}\,\psi(p) = i\,\frac{\mathrm{d}\psi}{\mathrm{d}p}, \qquad \hat{p}\,\psi(p) = G(p)\,\psi(p), \qquad G^{-1} = \int \frac{dp}{f(p)}, \tag{3.30}$$

where G^{-1} indicates the inverse function. Correspondingly, the domain $\mathbb{D}_p$ and the scalar product (in the Schrödinger picture) will be

$$\mathbb{D}_p \subseteq L^2(\mathbb{R}, dp), \qquad \langle\psi_1|\psi_2\rangle_p = \int dp\;\psi_1^*\,\psi_2. \tag{3.31}$$

Note how there is no need for a modified measure here.

The equivalence between these two representations is not yet completely clear. In a few cases, constructing wavepackets with specific profiles, they yield the same expectation values and standard deviations for observables; even in some cases where they are not the same, the dynamics appears to be similar. Nevertheless, given that the domains are different, a deeper and more thorough study from the functional analysis point of view is required, and a complete and formal equivalence has not been proven yet.

As a final note, I specified that the scalar products presented above are in the Schrödinger picture, but it is not the only possibility. Indeed, as shown in Section 2.2.1, in most cosmological models the Wheeler-DeWitt equation can be reinterpreted as a wave equation in the Klein-Gordon picture; in those cases, the term $\psi_1^*\psi_2$ is replaced by the Klein-Gordon current $i(\psi_1^*\partial_t\psi_2 - \psi_2\partial_t\psi_1^*)$, where ∂_t represents a time derivative.

3.3.2 Minimum Lengths vs Maximum Momenta

Here I introduce the specific algebras that will be used in the thesis, and show their properties mainly regarding the introduction of fundamental structures such as minimal or maximal values for some quantities. In particular, the presence of a minimal uncertainty can be obtained from the uncertainty relations: given a generic algebra $[\hat{q},\hat{p}] = i\,f(\hat{p})$, the relations are

$$\Delta q\,\Delta p \geq \frac{|\langle[\hat{q},\hat{p}]\rangle|}{2}\,. \tag{3.32}$$

In order to better highlight this procedure, I will start again from the original KMM construction before moving on to the other algebras.

In the original KMM GUP formulation, the algebra is

$$f_{\text{KMM}}(p) = 1 + B_0\,p^2. \tag{3.33}$$

Using the relation $\Delta p^2 = \langle\hat{p}^2\rangle - \langle\hat{p}\rangle^2$ and solving the corresponding equality for Δp yields

$$(\Delta q\,\Delta p)_{\text{KMM}} \geq \frac{1}{2}\left(1 + B_0\,\Delta p^2 + B_0\,\langle\hat{p}\rangle^2\right), \tag{3.34}$$

$$\Delta p = \frac{1}{B}\left(\Delta q \pm \sqrt{\Delta q^2 - B_0\,(1 - B_0\,\langle\hat{p}\rangle^2)}\right). \tag{3.35}$$

Then, in order for the square root to be real, a lower limit on Δq is obtained whose absolute minimum value appears for $\langle\hat{p}\rangle = 0$:

$$\Delta q \geq \sqrt{B_0\,(1 + B_0\,\langle\hat{p}\rangle^2)}\,, \qquad \Delta q_{\text{min}} = \sqrt{B_0}\,. \tag{3.36}$$

This is exactly the absolute minimal uncertainty mentioned in the previous section.

Even though for this formulation only the representation (3.28) with the modified position operator is used, for thoroughness I report here also the action of the momentum in the other representation (3.30):

$$\hat{p}_{\text{KMM}}\,\psi(p) = \frac{\tan\left(\sqrt{B_0}\,p\right)}{\sqrt{B_0}}\,\psi(p). \tag{3.37}$$

The KMM construction, being one of the first and most successful GUP formulations, has been widely studied and implemented in various contexts, and is probably the most famous one [62, 153, 184, 199].

The second algebra that I will study comes from the observation that the representation (3.30) where the action of the momentum p is modified looks suspiciously similar to the PQM operators, as mentioned above. Therefore, it should be possible to find an algebra that yields the exact same operators of that quantization procedure. Indeed, by asking that the momentum operator acts in the same way as PQM, i.e.

$$\hat{p}_{\mathrm{PA}}\, \psi(p) = \frac{\sin(\mu_0\, p)}{\mu_0}\, \psi(p), \tag{3.38}$$

where the subscript PA stands for Polymer Algebra, the corresponding deformed algebra results to be

$$f_{\mathrm{PA}}(p) = \sqrt{1 - \mu_0^2\, p^2}\ . \tag{3.39}$$

Note that I used the same symbol μ_0 of PQM; however, once the algebra is implemented, it has nothing to do with the lattice spacing anymore and its physical meaning is just that of a deformation parameter.

Finding the uncertainty relations and solving again for Δp, the following equation is obtained:

$$\Delta p = \frac{\sqrt{1 - \mu_0^2\, \langle \hat{p} \rangle^2}}{\sqrt{4 \Delta q^2 + \mu_0^2}}\ . \tag{3.40}$$

it is evident how in this case there is no limit on Δq, and therefore this construction does not introduce an absolute minimal uncertainty on position like the KMM GUP. On the other hand, the numerator instead introduces an upper limit on the expectation value of the momentum operator:

$$|\langle \hat{p} \rangle| \leq \frac{1}{\mu_0}. \tag{3.41}$$

This same upper limit can be obtained from other relations, such as the form of the modified eigenvalue (3.18) and the form of the algebra (3.39), since the sine is a limited function, and the commutator needs to be purely imaginary in order to allow unitarity.

This algebra was only briefly studied in the past [41, 105], but given the similarity it was then mostly replaced by Polymer Quantum Mechanics.

The third algebra is a hybrid of the previous two, i.e. with a square root similar to the Polymer Algebra but with the same plus sign of the KMM algebra. This is probably the one the has been studied the most after the original KMM construction and is sometimes called Extended GUP [41, 95, 202], but for reasons that will become clear later I will refer to it as the BGUP algebra, where the B stands for Brane.

The algebra is

$$f_{\mathrm{BGUP}}(p) = \sqrt{1 + B_0\, p^2}, \tag{3.42}$$

and the corresponding uncertainty relations, solved for Δp, are

$$\Delta p = \frac{\sqrt{1 + B_0 \langle \hat{p} \rangle^2}}{\sqrt{4\Delta q^2 - B_0}}; \tag{3.43}$$

in this case there is clearly a minimum uncertainty: requiring the denominator to be real yields

$$\Delta q_{\min} = \frac{\sqrt{B_0}}{2} . \tag{3.44}$$

This is very similar to the KMM minimal uncertainty; indeed, if instead in this case I had defined $f = \sqrt{1 + 2B_0 \, p^2}$, I would have obtained the exact same minimal uncertainty as the KMM case; it is worth noting that the KMM algebra would then correspond to the first two terms of the Taylor expansion of the BGUP algebra with the 2.

An important subtlety must be mentioned here. While in the KMM formulation the minimal value of Δq is reached for a finite value of Δp, here the minimum is attained in the limit $\Delta p \to \infty$. On the other hand, the action of the operator-valued function $\sqrt{1 + B_0 \, \hat{p}^2}$ is complicated due to the square root, and in order to compute its expectation value and obtain the numerator in equation (3.43) a series expansion must be performed that is convergent if and only if the operator $\hat{p}$ has finite norm, in particular if and only if $|\langle \hat{p} \rangle| \leq 1/\sqrt{B_0}$; consequently, in that case, also Δp will be bounded and the absolute minimal uncertainty cannot be obtained for this construction. Then a truncation of momentum space must be imposed externally in order for this framework to yield physical results. (This same feature of the series expansion and the convergence radius is present also in the Polymer Algebra (3.39) above because of the square-root operator, but given that in that case there is no minimal uncertainty regardless and that the truncation of momentum is intrinsic, this problem is automatically solved.) Nevertheless, since I will implement this particular algebra only on a semiclassical level, I do not need to worry about this issue. For more information on the series expansion and the truncation procedure, see [95, 202].

For thoroughness, the action of the momentum operator in the modified-p representation for the BGUP algebra is

$$\hat{p}_{\mathrm{BGUP}} \, \psi(p) = \frac{\sinh\left(\sqrt{B_0} \, p\right)}{\sqrt{B_0}}. \tag{3.45}$$

Note the pattern where the difference of a sign under the square root with respect to the Polymer Algebra amounts to the operator being a hyperbolic sine instead of a trigonometric sine.

The final algebra that I will deal with is the other hybrid between KMM and the Polymer Algebra, i.e. with just a quadratic term but with a minus sign:

$$f_{\mathrm{PUP}}(p) = 1 - \mu_0^2 \, p^2, \tag{3.46}$$

where PUP stands for Polymer Uncertainty Principle. This loosely corresponds to the first two term of the Taylor expansion of the Polymer Algebra, in a similar way to the KMM algebra being the Taylor expansion of the BGUP. Sometimes, when studying the KMM construction, the literature will briefly considers the possibility for the KMM deformation parameter B_0 to be negative, which corresponds to this algebra; however this case has not been analyzed too in depth. I studied some of its quantum mechanical properties in [31], and its application to some cosmological and gravitational systems in [34, 36]

The corresponding uncertainty relations, solved for Δp, are

$$(\Delta q \, \Delta p)_{\mathrm{PUP}} \geq \frac{1 - \mu_0^2 \, \Delta p^2 - \mu_0^2 \, \langle \hat{p} \rangle^2}{2}, \tag{3.47}$$

$$\Delta p = \frac{1}{\mu_0^2} \left(-\Delta q + \sqrt{\Delta q^2 - \mu_0^2 \left(\mu_0^2 \, \langle \hat{p} \rangle^2 - 1 \right)} \right); \tag{3.48}$$

imposing that the square root be real then yields

$$\Delta q^2 \geq \mu_0^2 \left(\mu_0^2 \, \langle \hat{p} \rangle^2 - 1 \right), \tag{3.49}$$

and since the right-hand side can go to zero there is no minimal uncertainty on position in this case. On the other hand, even though values of $\langle \hat{p} \rangle$ greater than $1/\mu_0$ are not prohibited and therefore here there is not a physical cut-off on momentum, differently from the Polymer Algebra, the value $\langle \hat{p} \rangle = 1/\mu_0$ is still somewhat important given that in that case one would have $\langle [\hat{q}, \hat{p}] \rangle = 0$. However also this algebra will be implemented on a semiclassical level only, and the consequences of this particular value have not been explored yet from a purely quantum mechanical point of view.

To conclude, the representation where the momentum operator is multiplicative but modified for this algebra is

$$\hat{p}_{\mathrm{PUP}} \, \psi(p) = \frac{\tanh(\mu_0 \, p)}{\mu_0} \, \psi(p). \tag{3.50}$$

Again, the different sign with respect to the KMM formulation changes the trigonometric function to a hyperbolic one. Furthermore, the hyperbolic tangent is also limited function, and this will be relevant in later chapters.

In this section I have shown how various forms of deformed algebra can introduce some structures that are expected in quantum gravitational theories, such as minimal lengths or momentum (i.e. energy) cut-offs, in an independent and somewhat phenomenological way. The motivations to study this kind of alternative quantization procedures are therefore clear.

3.3.3 Semiclassical Implementation

The usefulness of modified algebras can be made even more evident through their semiclassical limit, which will be the subject of this paragraph. However also for this there are a few different possibilities.

The first and most straightforward one is to downgrade the modified commutation relations to modified rules for Poisson brackets instead:

$$[\hat{q}, \hat{p}] = i \, f(\hat{p}) \qquad \rightarrow \qquad \{q, p\} = f(p). \tag{3.51}$$

Then, given a unmodified i.e. classical Hamiltonian $\mathcal{H}$, it is possible to derive the (modified) equations of motion as Poisson brackets with the Hamiltonian:

$$\dot{q} = \{q, \mathcal{H}\} = \frac{\partial \mathcal{H}}{\partial p} f(p), \qquad \dot{p} = \{p, \mathcal{H}\} = -\frac{\partial \mathcal{H}}{\partial q} f(p). \tag{3.52}$$

Obviously it is expected that, in the limit $f \to 1$ (that in the cases presented above corresponds to $\mu_0 \to 0$ or $B_0 \to 0$), the classical unmodified equations of motion are recovered.

The second possibility comes from the semiclassical implementation of Polymer Quantum Mechanics, and effectively it is the semiclassical limit of the representation (3.30) where the action of the momentum is modified. It consists of a modification of the Hamiltonian where every instance of the momentum is replaced by the corresponding modified eigenvalue, i.e. the function G defined in (3.30):

$$\mathcal{H}_{\text{class}}(q, p) \qquad \to \qquad \mathcal{H}_{\text{mod}}(q, p) = \mathcal{H}_{\text{class}}\Big(q, G(p)\Big). \tag{3.53}$$

Then the equations of motion are derived with the standard Hamilton equations for the modified Hamiltonian:

$$\dot{q} = \frac{\partial \mathcal{H}_{\text{mod}}}{\partial p}, \qquad \dot{p} = -\frac{\partial \mathcal{H}_{\text{mod}}}{\partial q}. \tag{3.54}$$

I have shown that this procedure yields the same exact modified dynamics of the previous one where the Poisson brackets are modified, at least on cosmological models [34].

Sadly, a way to implement the semiclassical limit from the KMM-like representation (3.28), where the differential operator $\hat{q}$ is modified, is currently being researched but has not been found yet. The hope is that this third semiclassical implementation, once found, could give insight also on the possible equivalence between the two different quantum representations.

The semiclassical limit of these modified algebras is a powerful tool to explore quantum corrections to classical Hamiltonian systems. Besides, due to the Ehrenfest theorem [92], it gives an idea of what to expect from the full quantization of the system, as well as a term of comparison for the trajectories of expectation values of observables. Indeed, as mentioned earlier, I will often analyze the semiclassical system before moving on to quantization.

Chapter 4

Isotropic Polymer Cosmology

In this chapter I will implement the Polymer formulation presented in Section 3.1 to isotropic Friedmann-Lemaître-Robertson-Walker model (FLRW) introduced in Section 2.1.

I will start with the semiclassical implementation, meaning that I will modify the standard Hamiltonian constraint with the Polymer substitution (3.16), replacing the momentum conjugate to the geometrical variable with a sine function. The resulting dynamics will present a non-singular Bounce, but certain features will require to compare different sets of variables.

Then I will move on to the quantum setting, promoting the variables to operators according to the Polymer formulation where, in the momentum polarization, the corresponding operator acts multiplicatively with a modified eigenvalue. I will show how PQM is able to remove the singularity also on a quantum level, in accordance with the corresponding semiclassical dynamics.

Finally I will give some comments on the dependence of the Bounce features on the geometrical variables chosen to describe the model; an attempt to recover the equivalence after discretization will have some implications for Loop Quantum Cosmology.

4.1 Semiclassical Polymer Cosmology

Here I will implement PQM on the isotropic FLRW model on a semiclassical level. As mentioned, in order to highlight the similarities and differences with LQC, I will discretize different sets of variables.

I will first consider the scale factor a as the most natural variable, showing that it provides a representation of the Universe dynamics characterized by a Bounce scenario only for supra-radiation equation of state, i.e. $P > \rho/3$, where P and ρ are the Universe pressure and energy density as in Chapter 2. This unpleasant feature implies the need to search for a suitable configuration variable such that the Polymer quantization predicts a Bounce whose features are independent of the matter filling the space, so that it can be interpreted as an intrinsic geometrodynamical property of the considered quantum gravity approach. This variable is identified in the cubed scale factor $v = a^3$, which characterizes the geometrical volume of the Universe and therefore seems to have a privileged dynamical role. These results are reported in the paper [168].

I must stress that the choice of the cubed scale factor as dynamical variable allows a direct comparison of the obtained modified Friedman equation with the one proper of the improved $\overline{\mu}$ scheme of LQC presented in Section 2.2.2. In fact, on an effective level, the two equations retain the same form, which allows to find a precise link between the Immirzi parameter and the Polymer cut-off value. As mentioned in previous chapters, the reason for the development of this second scheme over the initial μ_0 scheme is that the latter is a straightforward minisuperspace implementation of LQG, and it has the non-trivial limitation that the basic $SU(2)$ symmetry is essentially lost and the discretization of the area operator spectrum is somewhat introduced ad hoc, in contrast with LQG where it takes place naturally on a kinematical level.

In this respect, I will then study and compare the Polymer semiclassical dynamics of the FLRW model, as constructed in LQC, in two sets of variables: the Ashtekar connection and its conjugate variable, or the volume and its new generalized conjugate coordinate. Moreover, in order to make some comparison also on a phenomenological level between the the two sets of variables in Polymer semiclassical dynamics, I will briefly introduce in the model an additional matter component that satisfies a continuity equation with a dissipative term, namely particle creation; this way on a semiclassical level a different behaviour of entropy emerges, regarding its dependence on initial conditions. These results are included in the paper [102].

The semiclassical analysis will be the starting point for the actual quantization, that will be performed later.

4.1.1 Discretization of the Scale Factor

The homogeneous and isotropic universe is described by the RW line element (2.5), which I report here for better reference:

$$ds^2_{\mathrm{RW}} = dt^2 - a^2(t)\, d\ell^2_{\mathrm{RW}}, \qquad d\ell^2_{\mathrm{RW}} = \frac{dr^2}{1 - K\, r^2} + r^2 d\theta^2 + r^2 \sin^2(\theta)\, d\varphi^2. \tag{4.1}$$

The model is filled with a generic energy density of the form (2.13). I also choose the synchronous time gauge $\mathcal{N} = 1$ and zero spatial curvature $K = 0$. The Hamiltonian (2.7) then becomes

$$\mathcal{H}_{\mathrm{FLRW}}(a, p_a) = -\frac{1}{12}\frac{p_a^2}{a} + \overline{\rho}\, a^{-3w} = 0. \tag{4.2}$$

Assuming the scale factor a to be discretized with lattice spacing μ_a, PQM is implemented on the semiclassical level by using the Polymer substitution (3.16) on the conjugate momentum p_a; thus the modified Hamiltonian is

$$\mathcal{H}^{\mathrm{PQM}}_{\mathrm{FLRW}}(a, p_a) = -\frac{1}{12}\frac{\sin^2(\mu_a\, p_a)}{\mu_a^2 a} + \overline{\rho}\, a^{-3w} = 0. \tag{4.3}$$

Then, substituting the Hamiltonian constraint in the equation of motion for a, a modified Friedmann equation is derived where some sort of critical density appears:

$$\dot{a} = -\frac{1}{6}\frac{\sin(\mu_a\, p_a)\cos(\mu_a\, p_a)}{\mu_a\, a}, \tag{4.4}$$

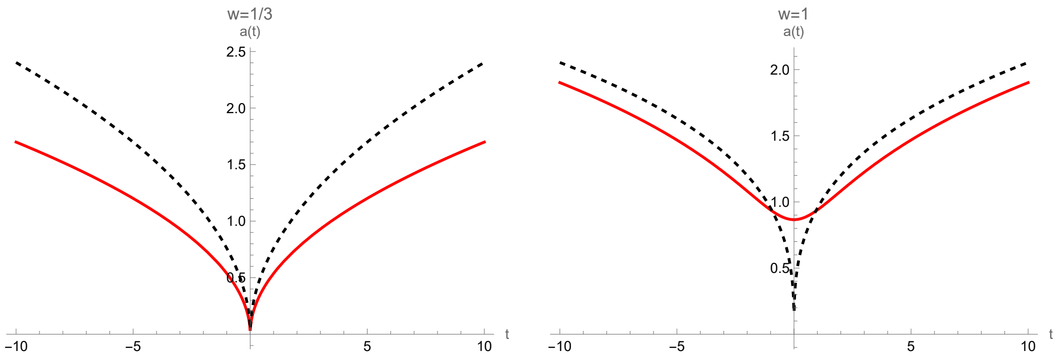

Figure 4.1. The Polymer-modified solutions $a(t)$ (red continuous lines) compared with the corresponding classical solutions (black dashed lines) for the two cases $w = 1/3$ (left) and $w = 1$ (right).

$$H^2 = \frac{\dot{a}^2}{a^2} = \frac{\rho}{3}\left(1 - \frac{\rho}{\rho_{a\mu}}\right), \qquad \rho_{a\mu} = \frac{1}{12\,\mu_a^2\,a^4}. \tag{4.5}$$

Note how both this modified Friedmann equation and the Hamiltonian (4.3) are similar to the equations of effective LQC presented in Section 2.2.2; however the dynamics here was derived independently from LQG. The differences and similarities between Polymer Cosmology and LQC will be analyzed more in depth later.

The correction factor between parentheses in equation (4.5) modifies the Friedmann equation (2.9); its main implication, similarly to LQC, is the existence of a Big Bounce and turning points for the scale factor evolution, occurring at $a = a_{\rm B}$ where

$$a_{\rm B} = (12\,\mu_a^2\,\overline{\rho})^{\frac{1}{3w-1}}. \tag{4.6}$$

However, this is true only for $w > 1/3$ i.e. for a supraradiation equation of state parameter; when $w = 1/3$, as long as $12\mu_a^2\overline{\rho} < 1$, the correction factor in the modified Friedmann equation is a positive constant and the resulting dynamics has the same features of the classical case (although with a different constant that slightly changes the approach to the singularity); for all other cases, there exist no real solution for the scale factor such that $\dot{a} = 0$, and therefore a Bounce cannot be derived.

The solutions $a(t)$ for the two cases $w = 1/3$ and $w = 1$ are shown in Figure 4.1. It is evident how, in the case $w = 1$, the Polymer-modified solution does not go the singularity but has a positive minimum value at $a = a_{\rm B}$ and the classical Big Bang and Big Crunch are replaced with a Big Bounce.

The presence and position of the Bounce are therefore determined by the matter filling the Universe. Thus, with the aim of obtaining a Bounce and turning points related to geometrical properties only, it is possible to implement a canonical transformation which allows to find a critical density independent from the scale factor. The same issue is solved in LQC by the $\overline{\mu}$ scheme, in which the holonomy correction is assumed to be metric dependent in order to derive a constant critical energy density, as explained in Section 2.2.2. The analysis below achieves the same result from the point of view of Polymer quantization, fixing a proper correspondence between PQM in the minisuperspace and the $\overline{\mu}$ scheme of LQC.

4.1.2 Specialization to the Volume Variable

I will now introduce a new configurational variable $f = f(a)$ defined as a generic function of the scale factor and whose conjugate momentum reads as

$$p_f = \frac{p_a}{f'(a)}. \tag{4.7}$$

This way, the classical Hamiltonian (2.7) becomes

$$\mathcal{H}_{\text{FLRW}}(f, p_f) = -\frac{1}{12}\,\frac{p_f^2 (f')^2}{a} + \overline{\rho} a^{-3w} = 0; \tag{4.8}$$

then I implement the Polymer substitution (3.16) on p_f, thus obtaining

$$\mathcal{H}_{\text{FLRW}}(f, p_f) = -\frac{1}{12}\,\frac{(f')^2}{\mu_f^2 a}\,\sin^2(\mu_f\, p_f) + \overline{\rho} a^{-3w} = 0. \tag{4.9}$$

Note that by design in PQM the relevant variable to be discretized is chosen before the implementation of the substitution (or before quantization in the quantum setting); the implications of performing a canonical transformation after discretization will be analyzed later.

Now the equation of motion for a and the corresponding modified Friedmann equation will contain f and its derivative (note that there is also a derivative dp_f/dp_a that simplifies one power of f'):

$$\dot{a} = -\frac{1}{6}\,\frac{f'}{\mu_f\, a}\,\sin(\mu_f\, p_f)\cos(\mu_f\, p_f), \tag{4.10}$$

$$H^2 = \frac{\rho}{3}\left(1 - \frac{\rho}{\rho_{f\mu}}\right), \qquad \rho_{f\mu} = \frac{(f')^2}{12\,\mu_f^2\, a^4}. \tag{4.11}$$

Therefore, in order to have a constant critical density $\rho_{f\mu}$, it must be $f \propto a^3$; I will choose the following volume variable:

$$f(a) = a^3 = v, \qquad p_v = \frac{p_a}{3a^2}. \tag{4.12}$$

The corresponding Polymer-modified Hamiltonian, modified Friedmann equation and solution then are

$$\mathcal{H}_{\text{FLRW}}^{\text{PQM}}(v, p_v) = \frac{3}{4\mu_v^2}\,\sin^2(\mu_v\, p_v)\, v + \overline{\rho}\, v^{-w} = 0, \tag{4.13}$$

$$H^2 = \left(\frac{\dot{v}}{3v}\right)^2 = \frac{\rho}{3}\left(1 - \frac{\rho}{\rho_{v\mu}}\right), \qquad \rho_{v\mu} = \frac{3}{4\,\mu_v^2}, \tag{4.14}$$

$$v(t) = \left(v_{\text{B}}^{1+w} + \frac{3}{4}\,(1+w)^2\,\overline{\rho}\, t^2\right)^{\frac{1}{1+w}}, \qquad v_{\text{B}}^{1+w} = \frac{\overline{\rho}}{\rho_{v\mu}} = \frac{4\,\mu_v^2\,\overline{\rho}}{3}, \tag{4.15}$$

where $v_{\text{B}} = v(0)$ is the value of the geometrical volume at the Bounce. As expected, the critical energy density $\rho_{v\mu}$ is now a constant, and a Bouncing solution can be found for all $w \neq -1$. Besides, it is clear how in the limit $\mu_v \to 0$, $v_{\text{B}} \to 0$ and the classical solutions presented previously in Section 2.1 are recovered.

4.1.3 Ashtekar Connection vs Volume

I will now apply the Polymer representation to the FLRW Universe filled with matter in the form of a free scalar field ϕ. I will use both the Ashtekar variables (s, p_s) and the volume variables (v, p_v). I will consider the variables $s = a^2$ and $v = a^3$ as discrete and therefore use the substitution (3.16) on the conjugate momenta $p_s \propto \dot{a}$ and $p_v \propto \dot{a}/a$. In order to make the comparison with LQC more evident, I start from a slightly different Hamiltonian that has some different constants with respect to the usual FLRW Hamiltonian (2.7), in particular the Immirzi parameter [122, 197]; therefore the two Polymer-modified Hamiltonian constraints in the area variable s and in the volume variable v are

$$\mathcal{H}^{\text{PQM}}_{\text{FLRW}}(s, p_s; \phi, p_\phi) = -\frac{3}{\Gamma^2 \mu_s^2}\sqrt{s}\, \sin^2(\mu_s\, p_s) + \rho_\phi\, s^{\frac{3}{2}} = 0, \qquad \rho_\phi = \frac{p_\phi^2}{2s^3}; \tag{4.16}$$

$$\mathcal{H}^{\text{PQM}}_{\text{FLRW}}(v, p_v; \phi, p_\phi) = -\frac{27}{4\Gamma^2 \mu_v^2}\, v\, \sin^2(\mu_v\, p_v) + \rho_\phi\, v = 0, \qquad \rho_\phi = \frac{p_\phi^2}{2v^2}; \tag{4.17}$$

where Γ is the aforementioned Immirzi parameter, p_ϕ is the momentum conjugate to the scalar field ϕ and is a constant of motion, and μ_v and μ_s are the Polymer lattice steps related to the corresponding variables and are constants.

Thanks to the equations of motion and the Hamiltonian constraints, the analytic expressions for the modified Friedmann equations are derived:

$$H_s^2 = \left(\frac{\dot{s}}{2s}\right)^2 = \frac{\rho_\phi}{3}\left(1 - \frac{\rho_\phi}{\rho_{s\mu}}\right), \qquad \rho_{s\mu} = \frac{3}{\Gamma^2\, \mu_s^2\, s}; \tag{4.18}$$

$$H_v^2 = \left(\frac{\dot{v}}{3v}\right)^2 = \frac{\rho_\phi}{3}\left(1 - \frac{\rho_\phi}{\rho_{v\mu}}\right), \qquad \rho_{v\mu} = \frac{27}{4\,\Gamma^2\, \mu_v^2}. \tag{4.19}$$

Two regularizing energy densities ρ_μ appear in the correction factors; both introduce a critical point in the evolution of H^2. On one hand, $\rho_{s\mu}$ depends on time ϕ and on the constant of motion p_ϕ through s, as I will show below, but its presence still makes it so that, when $\rho_\phi = \rho_{s\mu}$, the critical point is reached and a Big Bounce appears; on the other hand, $\rho_{v\mu}$ depends only on fundamental constants and is a proper critical energy density. The dynamics is again very similar to effective LQC, but the expression for the Bounce densities are different from those in equations (2.63) and (2.64) since the corrections come from different, although related, quantization procedures.

I will now consider the scalar field ϕ as the internal time for the dynamics by fixing the gauge $\dot{\phi} = 1$. The effective Friedmann equations can be solved analytically:

$$\left(\frac{1}{s}\frac{ds}{d\phi}\right)^2 = \frac{2}{3}\left(1 - \frac{\Gamma^2\, \mu_0^2}{6}\frac{p_\phi^2}{s^2}\right), \qquad s(\phi) = \frac{\Gamma\, \mu_0}{\sqrt{6}}\, p_\phi\, \cosh\!\left(\sqrt{\frac{2}{3}}\, \phi\right); \tag{4.20}$$

$$\left(\frac{1}{v}\frac{dv}{d\phi}\right)^2 = \frac{3}{2}\left(1 - \frac{4\,\Gamma^2\, \mu_0^2}{54}\frac{p_\phi^2}{v^2}\right), \qquad v(\phi) = \frac{2\,\Gamma\, \mu_0}{3\sqrt{6}}\, p_\phi\, \cosh\!\left(\sqrt{\frac{3}{2}}\, \phi\right). \tag{4.21}$$

As shown in figure 4.2, the Polymer trajectories of s and v decrease (as expected classically) until they reach the quantum era where the effects of quantum geometry

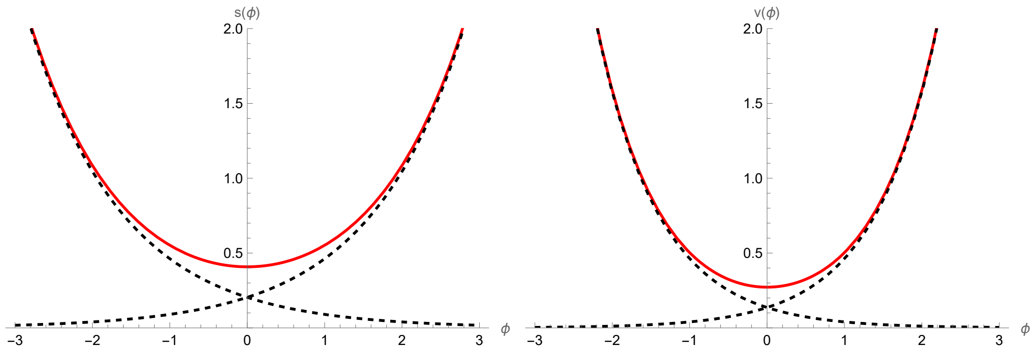

Figure 4.2. The Polymer-modified trajectories (red continuous lines) of s (left) and v (right) as functions of time ϕ, compared with the classical unmodified solutions (black dashed lines).

become dominant; they then reach a non-zero minimum and start to increase again. The resulting dynamics is that of a Bouncing Universe replacing the classical singularities.

4.1.4 Phenomenology with Particle Creation

Here I introduce particle creation with the aim of finding a phenomenological signature of the two semiclassical schemes presented above. Since this is a dissipative phenomenon, the assumption of constant entropy is replaced by that of constant entropy per particle.

By introducing a non-zero chemical potential, a new term parametrizing particle creation appears in the continuity equation for the energy density [71, 165, 167]:

$$\dot{\rho} + 3\,H\,\rho\,(1+w) = 0 \quad \rightarrow \quad \dot{\rho} + 3\,H\,\rho\,(1+w)\left(1 - \left|\frac{\mathrm{d}\ln N}{\mathrm{d}\ln v}\right|\right) = 0, \tag{4.22}$$

where N here is the number of particles in the comoving volume. Given that a constant entropy per particle $\frac{\mathrm{d}S}{\mathrm{d}N} = \text{const.}$ is assumed, the number of particles is directly proportional to the total entropy S produced. This request ensures that the entropy production is strictly due to the particle creation process, and it has no relation with the physics or the dynamics of each single particle. Furthermore, the resulting proportionality between the entropy and the particle number is immediately translated into the proportionality of the entropy density and the particle number density. This feature preserves a property valid in the standard cosmological picture [133], since, for instance, for the radiation component considered in the numerical analysis below (whose presence is naturally expected in the very early Universe), the following relation holds:

$$\frac{\mathrm{d}S_\gamma}{\mathrm{d}N_\gamma} = \frac{\rho_\gamma + p_\gamma}{T} = \frac{4}{3}\,\frac{\rho_\gamma}{T} \propto \frac{\mathrm{d}N_\gamma}{\mathrm{d}v}, \tag{4.23}$$

where the subscript γ is used here to indicate the radiation fluid. There is no physical reason that the request of a constant entropy per particle be an assumption phenomenologically inadequate to the quantum evolution of the Universe, especially in the present scenario, in which the matter creation phenomenon is considered on an expanding Polymer-modified background.

With the addition of this new component, the Friedmann equations (4.18) and (4.19) rewrite as

$$H_s^2 = \frac{\rho_\phi + \rho_\gamma}{3}\left(1 - \frac{\rho_\phi + \rho_\gamma}{\rho_{s\mu}}\right), \tag{4.24}$$

$$H_v^2 = \frac{\rho_\phi + \rho_\gamma}{3}\left(1 - \frac{\rho_\phi + \rho_\gamma}{\rho_{v\mu}}\right). \tag{4.25}$$

In order to solve these equations, I make the usual ansatz [167]

$$\left|\frac{\mathrm{d}\ln N}{\mathrm{d}\ln v}\right| \propto H^{2b}, \tag{4.26}$$

where b is a free parameter. Therefore the continuity equation for $\rho_\gamma(v)$ rewrites as

$$\frac{\mathrm{d}\rho_\gamma}{\mathrm{d}v} + \frac{\rho_\gamma}{v}(1+w)\left(1 - \left(\frac{H}{\overline{H}}\right)^{2b}\right) = 0. \tag{4.27}$$

The ansatz above has a phenomenological character and, by the direct proportionality between the particle creation rate and the Universe expansion rate H, the physical origin of particle creation is identified with the rapid time variation of the primordial gravitational field. In the context of a Bouncing cosmology, this proportionality has the significant implication that near the Bounce, where the expansion rate H vanishes, the matter creation is correspondingly suppressed. In other words, the process of matter creation has a major impact on the Universe dynamics in an intermediate region between the minimal volume and the late Universe. However, this maximum of matter creation concerns a very primordial phase, when the energy is still of comparable order to the critical value.

The ansatz (4.26) and the corresponding continuity equation (4.27) contain two phenomenological parameters b and $\overline{H}$. The first is taken in the numerical analysis below of order unity, since there are no reasonable indications for its deviation from the classical setting. For what concerns $\overline{H}$, the only important constraint comes from avoiding that matter creation affects the de Sitter phase of an inflationary scenario. Otherwise, in the opposite case, a spectrum of inhomogeneous perturbations that is different than the natural scale invariant one could be obtained [220]. This consideration suggests that the value of $\overline{H}$ must be such that the matter creation is strongly suppressed before inflation starts, for example for a Universe temperature of order $\lesssim 10^{15}\,GeV$. However, in this study the value of $\overline{H}$ is chosen to obtain the maximum of matter creation close the Planckian phase of the Universe (where the Polymer modifications are relevant), as stressed above. Finally, I consider the contribution of all relativistic species as a single effect, and therefore the value of $\overline{H}$ is actually meant to represent an average effect over all ultrarelativistic matter species.

Note that modifying the continuity equation as in (4.22) while keeping the same form for the modified Friedmann equations (except for the presence of the new

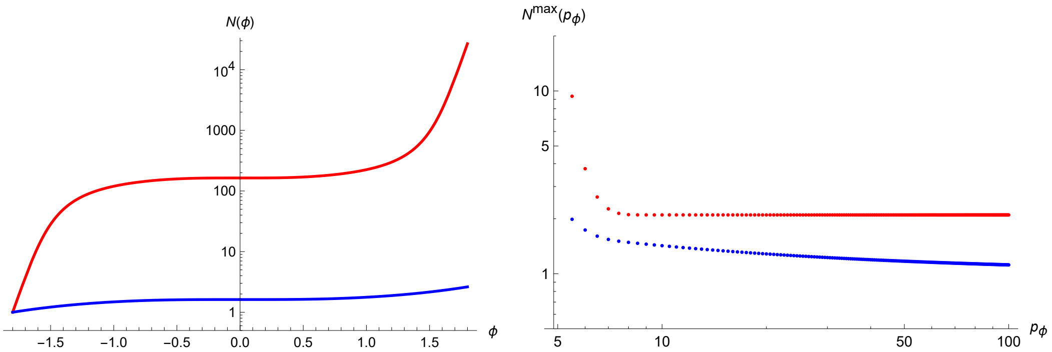

Figure 4.3. Left: number of particles as function of time ϕ. Right: number of final particles as function of the initial condition p_ϕ. Ashtekar variables in blue, volume variables in red.

radiation component), unavoidably leads to a modified form of the acceleration equations as follows:

$$\frac{\ddot{s}}{2s} = \frac{\rho}{2}\left(\frac{1}{3} - w\left(1 - 2\frac{\rho}{\rho_{s\mu}}\right) + \left|\frac{\mathrm{d}\ln N}{\mathrm{d}\ln v}\right|(1+w)\left(1 - 2\frac{\rho}{\rho_{s\mu}}\right)\right), \tag{4.28}$$

$$\frac{\ddot{v}}{3v} = \frac{\rho}{2}\left(1 - w\left(1 - 2\frac{\rho}{\rho_{v\mu}}\right) + \left|\frac{\mathrm{d}\ln N}{\mathrm{d}\ln v}\right|(1+w)\left(1 - 2\frac{\rho}{\rho_{v\mu}}\right)\right). \tag{4.29}$$

The new term $\left|\frac{\mathrm{d}\ln N}{\mathrm{d}\ln v}\right|$ acts as a positive energy density and could therefore drive the acceleration when $\rho < \frac{\rho_\mu}{2}$, that is away from the Planck regime; however, given the ansatz (4.26), in the late universe the Hubble parameter decays and particle creation is strongly suppressed, so the additional term cannot act as a suitable candidate to explain a de Sitter-like phase of expansion, as suggested for example in [30, 143, 189].

Now it is possible to numerically solve eq. (4.27) for $\rho_\gamma(v)$ and to find $v(\phi)$ and $N(\phi)$ from the Friedmann equations (4.24), (4.25) and from the ansatz (4.26) respectively. In the left panel of Figure 4.3 the evolution of the total number of particles as function of time ϕ is shown; it is clear how, in the case of the Ashtekar variables, the production of particles (and therefore of entropy) is negligible with respect to the volume representation. The right panel shows instead the final number of particles in the two cases as function of the initial condition p_ϕ; a greater value of p_ϕ (and therefore a more dominant scalar field density) suppresses the creation of particles in both cases, while on the other hand for small values of p_ϕ the creation grows appreciably. It is worth stressing that the different behaviour outlined here between the entropy creation in the polymerization of the Ashtekar connection or of the volume variable has a precise physical meaning in the Polymer paradigm only. However, since in the considered semiclassical dynamics when matter creation is absent the two Polymer pictures mimic the μ_0 and $\overline{\mu}$ schemes of LQC respectively, it is legit to suppose that these results might be valid also in the LQC theory and could represent an important phenomenological difference between the two schemes.

Another possible avenue to derive phenomenological predictions could be the computation of corrections to the primordial Power Spectrum of perturbations, which could be observable on the Cosmic Microwave Background (CMB). Some work in the context of LQC has already been done [59, 60, 70], and a comparison with the corrections induced by a PQM-modified background could be used to further differentiate between the two sets of variables presented in this section, clearly linked to the two different schemes of LQC.

I studied corrections to the primordial Power Spectrum in a different cosmological background, which does not present a Bounce but an asymptotically Einstein-static phase that is still able to remove the singularity; this was obtained through a variation of the GUP representation inspired by PQM, and will be presented later in Section 5.2.

4.2 Quantum Polymer Cosmology

In this section the main purpose is to promote the system to a quantum level, starting from the Hamiltonian constraint in its quantum counterpart and applying Dirac quantization [90, 91] directly to the fundamental variables in order to obtain the WDW equation. The variables are directly promoted to a quantum level, the Poisson brackets to commutators and the constraints to operators; the latter, when applied to the quantum states, will select physical states and yield the WDW equation (2.42). This procedure will lead to the dynamics whereby the system will fix ψ as an eigenstate for the Hamiltonian with vanishing eigenvalue. The results of this section are included in [102].

4.2.1 Quantum Analysis in the Ashtekar Variables

To implement the Dirac quantization method, the fundamental variables are promoted to quantum operators according to the Polymer prescription (3.18):

$$\hat{s} = -i\,\frac{\Gamma}{3}\,\frac{\mathrm{d}}{\mathrm{d}p_s}, \qquad \hat{p}_s = \frac{\sin(\mu_s\, p_s)}{\mu_s}, \qquad \hat{p}_\phi = -i\,\frac{\mathrm{d}}{\mathrm{d}\phi}. \tag{4.30}$$

Given the Hamiltonian in the Ashtekar variable s, the corresponding constraint operator in the momentum representation acts as

$$\hat{\mathcal{H}}^{\mathrm{PQM}}_{\mathrm{FLRW}}\,\psi(p_s,\phi) = \left(-\frac{2}{3\,\mu_s^2}\left(\sin(\mu_s\, p_s)\frac{\mathrm{d}}{\mathrm{d}p_s}\right)^2 + \frac{\mathrm{d}^2}{\mathrm{d}\phi^2}\right)\psi(p_s,\phi) = 0. \tag{4.31}$$

This mixed factor ordering for the term $\hat{s}^2\hat{p}_s^2$ allows for the interpretation of this differential equation as a Klein-Gordon-like equation; indeed it admits a conserved current and the expectation value of a generic operator $\hat{O}$ can be found through a Klein-Gordon-like scalar product similar to (2.46):

$$\langle\psi|\hat{O}|\psi\rangle = \int_{-\frac{\pi}{\mu_s}}^{+\frac{\pi}{\mu_s}} \frac{dp_s}{\frac{\sqrt{2}}{\mu_s\sqrt{3}}\sin(\mu_s\, p_s)}\; i\left(\psi^*\,\partial_\phi(\hat{O}\psi) - (\hat{O}\psi)\,\partial_\phi\psi^*\right). \tag{4.32}$$

Thanks to the substitution to the auxiliary variable $x_s = \sqrt{\frac{3}{2}}\,\ln\left|\tan\left(\frac{\mu_s\, p_s}{2}\right)\right| + \overline{x}_s$, where $\overline{x}_s$ is an irrelevant integration constant that will therefore be set to zero, equation (4.31) becomes a massless Klein-Gordon equation:

$$\frac{\mathrm{d}^2}{\mathrm{d}x_s^2}\,\psi(x_s,\phi) = \frac{\mathrm{d}^2}{\mathrm{d}\phi^2}\,\psi(x_s,\phi), \tag{4.33}$$

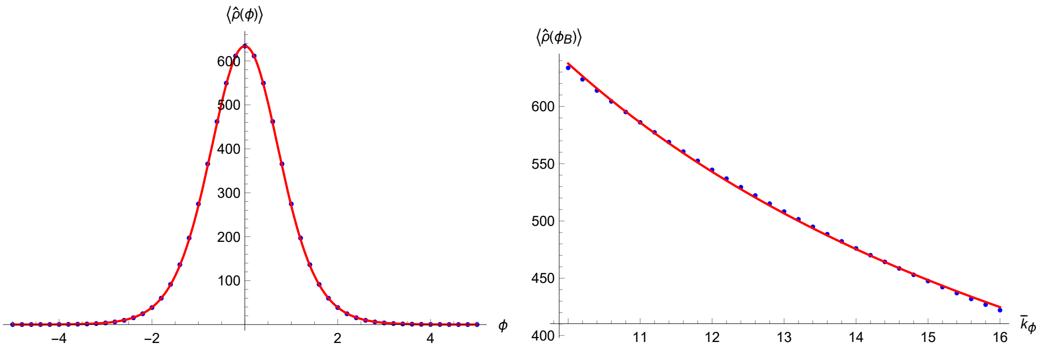

Figure 4.4. Left: the expectation value of the energy density as function of time for a fixed value of $\overline{k}_\phi$ (blue dots); right: the expectation value of the energy density at the time ϕ_B of the Bounce as function of $\overline{k}_\phi$ (blue dots). Both have been fitted with a function in accordance with the semiclassical evolution (continuous red line).

where ψ can be written as a planewave superposition: $\psi(x_s, \phi) = e^{-ik_s x_s} e^{-ik_\phi \phi}$, and the dispersion relation yields $k_s^2 = k_\phi^2$. The solution to this equation can be stated in the form of a Gaussian-like localized wavepacket:

$$\Psi(x_s, \phi) = \int_0^\infty dk_\phi \, \frac{e^{-\frac{(k_\phi - \overline{k}_\phi)^2}{2\sigma^2}}}{\sqrt{4\pi\sigma^2}} \, k_\phi \, e^{ik_\phi x_s} \, e^{-ik_\phi \phi}, \tag{4.34}$$

where I have chosen incoming planewaves with $k_x = -k_\phi$. Here σ and $\overline{k}_\phi$ are the variance and the position of the peak of the Gaussian profile, k_ϕ is the energy-like eigenvalue of the operator $\hat{p}_\phi$, and its positive or negative values select collapsing or expanding solutions respectively.

Now, in order to investigate the non-singular behaviour of the model, I will compute the expectation value of the energy density operator $\hat{\rho}_\phi = \frac{\hat{p}_\phi^2}{2\hat{s}^3}$. In what follows, all the mean values and variances of the relevant operators have been computed using the auxiliary variable x_s inside the scalar product (4.32), so that the results are given by

$$\langle \Psi | \hat{O} | \Psi \rangle = \int_{-\infty}^{\infty} dx_s \; i \left(\psi^* \, \partial_\phi (\hat{O}\psi) - (\hat{O}\psi) \, \partial_\phi \psi^* \right), \tag{4.35}$$

where the wavefunctions are assumed to be normalized. I remark that the transformation from p_s to x_s in the minisuperspace preserves the expectation values of the physical observables.

Figure 4.4 shows the time evolution of $\langle \hat{\rho}_\phi(\phi) \rangle$ for a fixed value of $\overline{k}_\phi$ and the maximum $\langle \hat{\rho}_\phi(\phi_B) \rangle$, i.e. the expectation value of the Bounce density, that results to be inversely proportional to $\overline{k}_\phi$, in accordance with the semiclassical critical energy density (4.18). The points representing the quantum expectation values were obtained through numerical integration and have been fitted with the full lines; they are in accordance with the semiclassical trajectories when taking into account numerical effects and quantum fluctuations. Note that the action of the energy density operator expressed as function of x_s has been simplified thanks to the hypothesis of a sufficiently localized wavepacket.

The non-diverging nature of the energy density of the primordial Universe clearly implies, in view of its scalar and physical nature, the existence of a minimum non-zero volume (although strongly dependent on the initial conditions), thus confirming the replacement of the singularity with a Bounce also in the quantum system. A more precise assessment of the nature of the Bounce would require also a careful analysis of the variance of the density and the moments of the quantum probability, as done for instance in [56], but already at this level the existence of a Bounce is a solid prediction of this model.

4.2.2 Quantum Analysis in the Volume Variable

I will now implement the quantization procedure on the system expressed in the volume variable v and its conjugate momentum p_v. This procedure is very similar to the previous one and to the equivalent Loop quantization performed in [25, 29]. Therefore I will report only the most relevant results.

The fundamental variables, when promoted to operators in the Polymer framework (3.18), act as

$$\hat{v} = -i\,\frac{\Gamma}{3}\frac{\mathrm{d}}{\mathrm{d}p_v}, \qquad \hat{p_v} = \frac{\sin(\mu_v\, p_v)}{\mu_v}, \qquad \hat{p}_\phi = -i\,\frac{\mathrm{d}}{\mathrm{d}\phi}, \tag{4.36}$$

and the action of the quantum Hamiltonian constraint becomes

$$\hat{\mathcal{H}}^{\mathrm{PQM}}_{\mathrm{FLRW}}\psi(p_v,\phi) = \left(-\frac{3}{2\mu_v^2}\Big(\sin(\mu_v\, p_v)\frac{\mathrm{d}}{\mathrm{d}p_v}\Big)^2 + \frac{\mathrm{d}^2}{\mathrm{d}\phi^2}\right)\psi(p_v,\phi) = 0. \tag{4.37}$$

In this case the appropriate substitution to use is simply $x_v = \sqrt{\frac{2}{3}}\;\ln\left|\tan\!\left(\frac{\mu_v\, p_v}{2}\right)\right| + \overline{x}_v$; this leads to the same massless Klein-Gordon equation of the previous case and allows to write its solution in the x_v-representation as a Gaussian-like wavepacket:

$$\Psi(x_v,\phi) = \int_0^\infty dk_\phi\;\frac{e^{-\frac{(k_\phi-\overline{k}_\phi)^2}{2\sigma^2}}}{\sqrt{4\pi\sigma^2}}\,k_\phi\, e^{ik_\phi x_v}\, e^{-ik_\phi\phi}. \tag{4.38}$$

Now the procedure is exactly the same as before; the operators of interest in this case are both the volume $\hat{v}$ and the energy density $\hat{\rho}_\phi = \frac{\hat{p}_\phi^2}{2\hat{v}^2}$. Again I will use the same Klein-Gordon scalar product (4.32) (except for a numerical constant) to calculate their expectation values; their action is derived from (4.36). In figures 4.5 the expectation values of the volume $\langle\hat{v}(\phi)\rangle$ and density $\langle\hat{\rho}_\phi(\phi)\rangle$ as functions of time are shown. In the left panel of figure 4.6 the value $\langle\hat{v}(\phi_B)\rangle$ of the volume at the Bounce is shown as function of the initial value $\overline{k}_\phi$; the minimum volume scales linearly with the energy-like eigenvalue, in accordance with the semiclassical expression for $v(\phi)$ given in (4.21). Then in the right panel of the same figure the Bounce density $\langle\hat{\rho}_\phi(\phi_B)\rangle$ is shown for different values of $\overline{k}_\phi$; in accordance with the semiclassical critical energy density, the density at the Bounce in the new variables does not depend on the initial conditions of the system (or of the wavepacket in the quantum analysis).

The fact that in this set of variables the critical energy density of the Universe is fixed and does not depend on initial conditions is related to the volume itself being

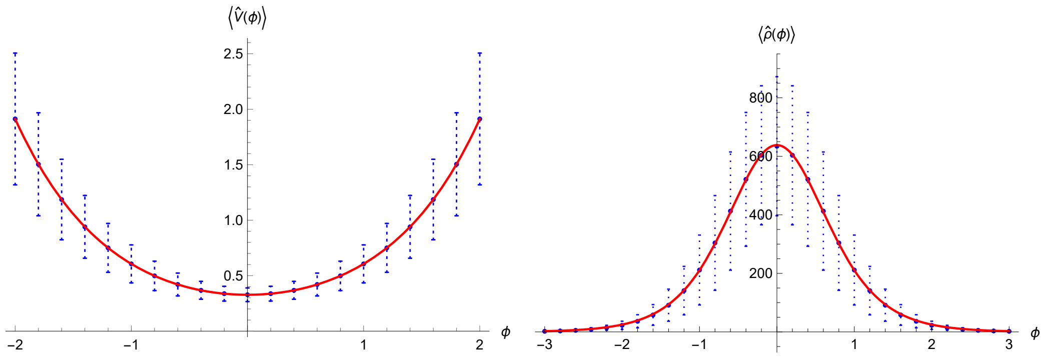

Figure 4.5. Expectation values of the volume (left) and of the energy density (right) as functions of time for a fixed value of $\bar{k}_\phi$ (blue dots), fitted with functions in accordance with the semiclassical evolution (continuous red lines). The error bars are calculated as the standard deviation of the corresponding operators.

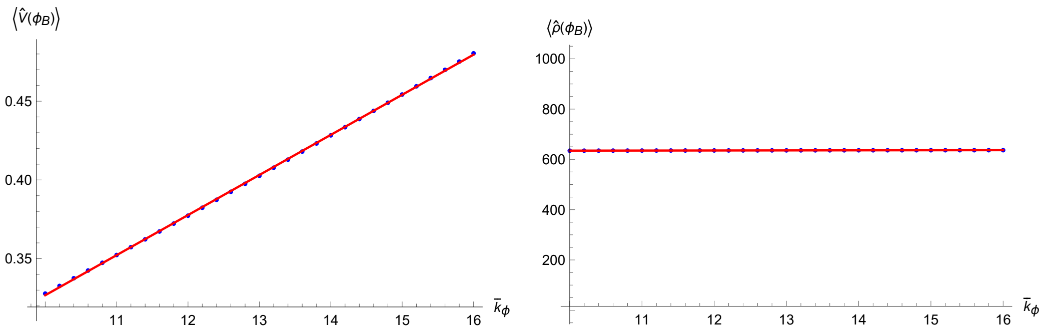

Figure 4.6. Expectation values of the volume (left) and of the density (right) at the time ϕ_B of the Bounce as functions of $\bar{k}_\phi$ (blue dots), fitted with functions in accordance with the semiclassical evolution (full red lines).

chosen as the configurational variable for the Polymer quantization of the system. However, even if this representation is preferable on the grounds that it yields an universal Bounce scenario that is physically more acceptable, this choice is clearly dynamically inequivalent to the Ashtekar variables (s, p_s) that are the only $SU(2)$ choice in LQC. Actually, in the next section I will show that, although it is possible on a semiclassical level to recover the physical equivalence between the two sets of variables, this leads in the Polymer representation to a translational operator whose implementation on the states constitutes a non-trivial issue.

4.3 The Problem of Equivalence

The focus of the this analysis is the physical interpretation of the two frameworks, both in the Polymer and in the LQC theories, and the link between the improved $\bar{\mu}$ scheme performed in [25, 26] and the original μ_0 scheme presented in [24]. The analysis performed via the improved LQC Hamiltonian in [25] seems to be affected by an ambiguous change of variables, which is required in order to restore a standard translational operator with constant step. In fact, the Universe volume (i.e. the cubed cosmic scale factor) has its own conjugate variable corresponding to a redefined generalized coordinate which does not implement LQG features into the symmetries of the minisuperspace in the same way as with the original $SU(2)$ Ashtekar connection.

In this respect, from the point of view of PQM I will show how, on a semiclassical level, restoring the natural Ashtekar gauge connection after the lattice has been implemented on the volume variable is formally equivalent to considering the basic lattice parameter as a function of the momentum variable; accounting for this redefinition of the Polymer parameter, the Universe volume obeys the same dynamical equations in the two sets of variables. Thus, the contribution of this analysis is twofold: on one hand, the Polymer quantization of the isotropic Universe implies that considering the connection induced by the full theory as the privileged variable leads to a Bouncing dynamics whose maximum density is not fixed a priori by fundamental constants; on the other hand, I will also provide a brief argument in favor of the viability of the $\overline{\mu}$ scheme of LQC in the volume variable, by showing in the Polymer framework its semiclassical equivalence to the analysis in the natural Ashtekar connection.

Above I stressed that in the Polymer framework the Universe always possesses a Bouncing point in the past both in a semiclassical and in a pure quantum description, with the difference that when the natural connection p_s is used the maximal density is fixed by the initial conditions on the system, while when using the redefined variable v the Bounce density depends on fundamental constants and the Immirzi parameter only. In this respect, the Polymer quantization introduces a minimal value to the geometrical operators area and volume with $s \propto a^2$ and $v \propto a^3$ respectively. In the first case, by discretizing the area element also the volume results to be regularized since a Bouncing cosmology emerges, but with different implications on the behaviour of the critical energy density. Consequently, these two representations clearly appear dynamically and physically not equivalent (see [13, 84, 103] for similar not equivalent behaviours in Polymer Cosmology). However, I must stress that both polymerization procedures, in terms of the Ashtekar connection and the volume variable, have some physical link to the background LQC kinematics. The first is justified by the direct interpretation in terms of the right $SU(2)$ connection adopted in LQC, while the latter refers to the kinematical result about its spectrum discretization [195]. In other words, polymerizing the Ashtekar connection means giving LQG features to the natural variable in which the Loop setup is formulated; on the same footing, polymerizing the volume corresponds to attributing a discrete structure to the quantum representation of this geometrical operator, as in the original LQG theory.

Let me now compare the semiclassical dynamics in both sets of variables in search of a physical link between the two representations. Starting from the Polymer-modified system in the (v, p_v) representation, the canonical transformation to the natural Ashtekar connection is

$$s = v^{\frac{2}{3}}, \qquad p_s = \frac{3}{2}\, p_v\, v^{\frac{1}{3}}. \tag{4.39}$$

To realize a transformation in the Polymer construction, the condition $\mu_v\, p_v = \mu_1 p_s$ must hold in order to map the Polymer Hamiltonian written in the variables (v, p_v) to the one in terms of the new variables (s, p_s) and make the Polymer-modified

Poisson brackets (3.39) formally invariant:

$$\{p_v, v\} = \frac{\Gamma}{3}\sqrt{1-\mu_v^2\,p_v^2} = \frac{\Gamma}{3}\sqrt{1-\mu_1^2\,p_s^2} = \{p_s, s\}. \tag{4.40}$$

A new Polymer parameter μ_1 appears that depends on the configurational variable:

$$\mu_1 = \frac{2}{3}\,\mu_v\,v^{-\frac{1}{3}} = \frac{2}{3}\,\frac{\mu_v}{\sqrt{s}}. \tag{4.41}$$

The last dependence of μ_1 on s is the same of $\overline{\mu}(\mu)$ in equation (2.55) for the $\overline{\mu}$ scheme of LQC .

After introducing a dependence of the Polymer parameter from the configurational variable under a transformation, it is commutative to write the transformed Hamiltonian and to introduce the Polymer substitution (3.16). Therefore it is expected that also the equations of motion for the two different sets of variables will be mapped using (4.39) and (4.41):

$$\dot{s} = \frac{2}{3}\,v^{-\frac{1}{3}}\,\dot{v} = \frac{\Gamma}{3}\,\frac{\partial\mathcal{H}}{\partial p_s} = -\frac{2}{\Gamma\mu_1}\sqrt{s}\,\sin(\mu_1\,p_s)\cos(\mu_1\,p_s), \tag{4.42}$$

$$\begin{aligned}\dot{p}_s =& \frac{3}{2}\,\dot{p_v}\,v^{\frac{1}{3}} + \frac{1}{2}\,p_v\,v^{-\frac{2}{3}}\,\dot{v} = -\frac{\Gamma}{3}\,\frac{\partial\mathcal{H}}{\partial s} = \\ =& \frac{1}{\sqrt{s}}\left(\frac{\sin^2(\mu_1\,p_s)}{2\,\Gamma\,\mu_1^2} - \frac{p_s}{\Gamma\,\mu_1}\sin(\mu_1\,p_s)\cos(\mu_1\,p_s) + \frac{\Gamma\,p_\phi^2}{4\,s^2}\right).\end{aligned} \tag{4.43}$$

For comparison, the equations of motion in the (s, p_s) representation are

$$\dot{s} = -\frac{2}{\Gamma\,\mu_s}\sqrt{s}\,\sin(\mu_s\,p_s)\cos(\mu_s\,p_s), \tag{4.44}$$

$$\dot{p}_s = \frac{1}{3}\left(\frac{3}{\Gamma\,\mu_0^2}\,\frac{1}{2\sqrt{s}}\,\sin^2(\mu_s\,p_s) + \frac{3\,\Gamma}{4}\,\frac{p_\phi^2}{s^{5/2}}\right). \tag{4.45}$$

Note that (4.42) is formally the same as (4.44), but the dependence of μ_1 on s changes the solution, while (4.43) for the connection p_s results to be different from (4.45) because of the dependence of the Polymer parameter μ_1 on s. Therefore, on a semiclassical level, there exists a physical equivalence in the evolution of s and v. Indeed, thanks to (4.41), the regularizing density $\rho_{s\mu}$ in (4.18) turns out to be the same critical energy density $\rho_{v\mu}$ of (4.19):

$$\rho_{s\mu} = \frac{3}{\Gamma^2\,\mu_1^2\,s} = \frac{27}{4\,\Gamma^2\,\mu_v^2} = \rho_{v\mu}\,. \tag{4.46}$$

Also the effective Friedmann equation in the time gauge $\dot{\phi} = 1$ reads as

$$\left(\frac{1}{s}\,\frac{ds}{d\phi}\right)^2 = \frac{2}{3}\left(1 - \frac{4\,\Gamma^2\,\mu_v^2}{54}\,\frac{p_\phi^2}{s^3}\right) \tag{4.47}$$

and it clearly reduces to (4.21) using (4.39). The Bounce of the Universe volume has the same properties in the two sets of variables only by considering the new Polymer parameter μ_1 to be dependent on s.

However, in its natural formulation PQM is associated to a lattice (that has a constant spacing by construction) only after the dynamics is assigned and after the change of variables in the classical Hamiltonian has been performed, and indeed the difference between the two schemes consists in choosing the variable for which the lattice parameter is constant: transforming into the natural Ashtekar connection from the volume-like momentum after the Polymer framework has been implemented yields a Polymer parameter depending on the configurational coordinate, and unfortunately this request prevents a full quantum analysis of the problem since it produces a translational operator that cannot be implemented. This analysis highlights the privileged nature of the variable for which the Polymer parameter is taken constant, since the physical results depend on it. In particular, if in Polymer semiclassical cosmology one starts from assigning a lattice in the Ashtekar variables and then performs a transformation to the volume ones, obtaining a non-constant spacing, the resulting cosmology would still be a Bouncing one with a cut-off energy density depending on initial conditions; on the other hand, starting from the volume variables, one would have a universal Bouncing cosmology in both sets, with the difference that the Polymer parameter would not be constant in the Ashtekar ones. Hence in Polymer Cosmology the use of the Ashtekar or volume variables leads to two different, inequivalent physical pictures. Far from defining an equivalence between the μ_0 and the $\overline{\mu}$ schemes of LQC, I simply stress that the latter would just correspond to dealing with a Polymer parameter depending on the configurational variable when seen in the Ashtekar connection instead of the volume one. This result encourages the thought that the conclusions gained in [26] after a change of basis has been performed can be obtained on the semiclassical level (and hopefully in the quantum too) also in the Ashtekar variables: the critical density of the Universe takes an absolute value, independent on the initial conditions on the semiclassical system or on the quantum wave packet.

The problem of equivalence addressed here has a deep physical meaning since it involves the real nature of the so-called Big Bounce: is it an intrinsic cut-off on the cosmological dynamics or is it a primordial turning point fixed by initial conditions on the quantum Universe? The present analysis suggests that the second case appears more natural in Polymer Cosmology if it is referred to LQG, since the quantum implementation of the Ashtekar connection produces results in accordance with the original analysis in [26].

As described in Section 2.2.2, in the original formulation of LQC the dynamics of the isotropic Universe is described via the canonical couple (s, p_s). The fundamental states of the theory, denoted by $|\mu\rangle$, are eigenstates of the momentum operator; while the operator $\hat{p}_s$ remains undefined, an exponential translational operator can be defined via equation (2.49). Now, the choice of the μ_0 or $\overline{\mu}$ scheme amounts to giving the minimum area eigenvalue a kinematical or dynamical character, i.e. in choosing (2.52) or (2.55) [25, 26, 29, 54]. In the $\overline{\mu}$ scheme of LQC the basic relation for the minimum area states the necessity to deal with physical values of the area spectrum properly scaled by the momentum (i.e. the squared scale factor). In this case, the implementation of the translational operator (2.56) becomes more natural through the change to the variable ν, i.e. to a volume-like coordinate (this is immediate observing that $\overline{\mu} \propto 1/\sqrt{\mu}$). This change of basis can be thought of as the passage from a translational operator with s-dependent step to a representation that defines a natural constant spacing in the μ space. This situation is similar to the case discussed in the Polymer formulation, when in equation (4.41) the lattice step was promoted to a function of the generalized coordinate to restore the invariance of the semiclassical dynamics.

Since on a semiclassical level the Polymer and LQC dynamics are comparable and I have shown that the physical properties of the Universe are dictated by the representation with a constant lattice step, it is possible to infer an interesting feature of LQC. At least on a semiclassical level, the use of the volume coordinate is legitimated on the same level as the Ashtekar connection (the privileged choice in LQG), because the universal value of the critical energy density (that is independent from $\overline{k}_\phi$), observed in the former case, would remain valid also for the evolution of the latter framework as long as the non-constant lattice parameter is taken into account. In this respect, the proof of the complete equivalence of the μ_0 and $\overline{\mu}$ schemes of LQC (clearly absent in the quantum Polymer picture, as shown by this analysis, where a constant lattice step is considered) would require the non-trivial technical question of addressing the action of a translational operator with a coordinate-dependent step.

Chapter 5

Cosmology with Modified Algebras

In this chapter I will implement the Modified Algebras introduced in Section 3.3 to different cosmological models, both on a semiclassical and on a quantum level.

I will start by the semiclassical implementation of the four algebras on the Bianchi I model presented in Section 2.1.3, with particular attention on whether they are able to remove the singularity and in their ability to reproduce other quantum cosmological theories such as Polymer Cosmology and Brane Cosmology. Then I will focus on two specific algebra to implement on a quantum level, showing that although the expectation values do follow the corresponding semiclassical trajectories, the fact that wavepackets spread means that the semiclassical formulation is predictive only to a certain point and the cosmological singularity must be treated in a full quantum picture. This cosmological model has been chosen as the first and most straightforward generalization of the isotropic Universe; indeed, in the limit of zero anisotropies $\beta_\pm \to 0$, the Bianchi I model reduces exactly to the flat FLRW model.

One particular algebra will then be analyzed more in depth, due to its ability to remove the singularity through an asymptotic phase and to reproduce the so-called Emergent Universe model. Classically, the latter can be obtained only through a specific fine-tuning of initial conditions, while this algebra implements the asymptotically static phase in a natural way. Then I will also study corrections to the primordial Power Spectrum of perturbations, giving an observational signature and a procedure to compute possible phenomenological consequences for these Modified Algebras.

5.1 Modified Algebras in the Bianchi I Model

The main focus of this section is the comparison of the implementation of the phenomenological approaches of PQM, the GUP representation and the other algebras of Section 3.3 on a Bianchi I Universe expressed in Misner-like variables, i.e. the ones presented in Section 2.1.3 but using the volume $v = a^3$ instead of the logarithmic α. As mentioned in Section 3.3, the reasons for choosing these four algebras are that the KMM formulation is one of the first and most studied ones, the Polymer Algebra was constructed to reproduce Polymer Quantum Mechanics, and the remaining two are the simplest modifications to those, namely through the change of a sign. Furthermore, the two algebras with the square root (the Polymer Algebra and the BGUP) are also the only ones allowed by the Jacobi identities when including spin [95], and therefore the results obtained with them could be straightforwardly extended to higher dimensions or to more complicated models.

The leading idea of this analysis is that the most important difference between the various algebras is the sign, that seems to affect deeply the cosmological dynamics since it is reflected on the Friedmann equation for the isotropic Universe, as analyzed also in [41]. I will show here that, regarding the presence of the singularity, the different sign affects the modified dynamics of a Bianchi I model in the same way as for the isotropic Universe. Note that I will implement the modified algebra on the volume variable only, using instead the standard Poisson brackets for the anisotropic degrees of freedom $\beta_\pm$. I will also show that, similarly to Polymer Cosmology (PC) being able to reproduce effective Loop Quantum Cosmology (LQC), the GUP representation will yield the effective equations of Brane Cosmology (a cosmological sector of String Theories [65, 138, 149, 152]) in the form of a Randall-Sundrum model [190, 191] but only in the case where the BGUP algebra with a square root is used; this is the reason why that algebra was named Brane GUP in Section 3.3.

On a quantum level, I will rewrite the Wheeler-DeWitt equation in an isomorphic form in both the PQM and the GUP formulations by suitably choosing a substitution for the momentum variable and, in the GUP case, the correct measure for the scalar product. The analysis confirms the existence of a Big Bounce in the PQM description and of a singular cosmology in the GUP formulation also for the quantum wavepackets. The analysis of the standard deviation shows that the localized packets inevitably spread, but this fact has different implications in the two frameworks: while for the polymer analysis the initial conditions can be set so that the wavepacket is peaked at the Bounce and then it is possible to have a quasi-classical description across it, in the GUP case the existence of a minimal uncertainty on position (in this setting the volume variable) forces the treatment of the singularity as a full quantum phenomenon.

The results of this section are included partly in [31], although in regards to the isotropic FLRW model; they have been expanded and improved later in [34] in regards to the anisotropic Bianchi I model.

5.1.1 Semiclassical Implementation

I will begin with the semiclassical implementation of the various algebras on the volume variable. The starting point is the unmodified Hamiltonian constraint (2.38) but with $U_{\mathrm{BI}} = 0$ and with v instead of α; the model is filled with matter in the form of a free scalar field ϕ, which will be chosen to play the role of time. I will then derive the dynamics through the different modified Poisson brackets of Section 3.3. Then the Hamiltonian constraint is

$$\mathcal{H} = \frac{1}{12v}(-9\,p_v^2\,v^2 + p_+^2 + p_-^2) + \rho_\phi\,v = 0. \tag{5.1}$$

The classical unmodified equations of motion and Friedmann equation are

$$\dot{v} = -\frac{3}{2}\,p_v\,v, \qquad \dot{\beta}_\pm = \frac{p_\pm}{6v}, \tag{5.2}$$

$$H^2 = \left(\frac{\dot{v}}{3v}\right)^2 = \frac{\rho_\phi + \rho_\beta}{3}, \qquad \rho_\beta = \frac{p_+^2 + p_-^2}{12v^2}. \tag{5.3}$$

Note how the anisotropies contribute to the total matter-energy through the density-like quantity ρ_β; indeed, as shown in Section 2.1.3, the Bianchi models are dynamical even in vacuum differently from the isotropic FLRW model.

Fixing the time gauge so that the scalar field ϕ acts as time, the solutions are

$$\left(\frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}\phi}\right)^2 = \frac{3}{2}\left(1 + \frac{\rho_\beta}{\rho_\phi}\right) = v_1^2, \qquad v(\phi) = v_0\, e^{\pm v_1 \phi}, \tag{5.4}$$

$$\frac{\mathrm{d}\beta_\pm}{\mathrm{d}\phi} = \sqrt{\frac{\rho_\pm}{6\rho_\phi}} = \beta_{1\pm}, \qquad \beta_\pm(\phi) = \beta_{1\pm}\,\phi + \beta_{0\pm}, \tag{5.5}$$

where I separated the anisotropy density ρ_β into its two contributions $\rho_\pm = p_\pm^2/12v^2$, v_1 and $\beta_{1\pm}$ are constants because all involved densities behave as $1/v^2$, and v_0 and $\beta_{0\pm}$ come from integration. The anisotropies behave linearly, and their evolution will not change in other settings, since the modified Poisson brackets will act only on the volume. Regarding the volume, the classical evolution is exponential and has singularities at $\phi \to \pm\infty$. I will show that with some deformed algebra this will change.

I will start with the standard KMM GUP algebra (3.33). The equation of motion for v and the modified Friedmann equation then are:

$$\{v, p_v\}_{\mathrm{KMM}} = 1 + B_0\, p_v^2, \qquad \dot{v} = -\frac{3}{2}\, p_v\, v\,(1 + B_0\, p_v^2), \tag{5.6}$$

$$H^2 = \frac{\rho_\phi + \rho_\beta}{3}\left(1 + \frac{\rho_\phi + \rho_\beta}{2\rho_{\mathrm{GUP}}}\right)^2, \qquad \rho_{\mathrm{GUP}} = \frac{3}{8\, B_0}. \tag{5.7}$$

I defined the regularizing GUP density ρ_{GUP} in that way in order to make the comparison with other algebras more straightforward.

The modified Friedmann equation (5.7), although similar to the PQM and LQC ones (4.19) and (2.64), has a different sign that will not introduce a critical point. Indeed, fixing the gauge to use ϕ as time, the evolution results to be

$$\left(\frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}\phi}\right)^2 = v_1^2\left(1 + \frac{\rho_\phi + \rho_\beta}{2\rho_{\mathrm{GUP}}}\right)^2, \qquad v_1 = \frac{3}{2}\left(1 + \frac{\rho_\beta}{\rho_\phi}\right), \tag{5.8}$$

$$v(\phi) = \sqrt{\frac{(\rho_\phi + \rho_\beta)\, v^2}{2\rho_{\mathrm{GUP}}}}\;\sqrt{e^{\pm 2 v_1 \phi} - 1}\,, \tag{5.9}$$

where the quantity $(\rho_\phi + \rho_\beta)\, v^2$ is a constant since the densities scale as $1/v^2$. The two solutions are shown later in Figure 5.1 compared with the classical (unmodified) evolution and with the BGUP solutions that will be derived shortly. Clearly the singularity is still present since $v(0) = 0$, but the way it is approached is drastically different since it happens at a finite value of time ϕ.

Now I will implement the Polymer Algebra (3.39). Since it was derived from the PQM action (3.18) of the operators, it is expected that the solution will be the same as Polymer Cosmology.

The algebra, the equations of motion and the modified Friedmann equation are

$$\{v, p_v\}_{\text{PA}} = \sqrt{1 - \mu_0^2\, p_v^2}\,, \qquad \dot{v} = -\frac{3}{2}\, p_v\, v\, \sqrt{1 - \mu_0^2\, p_v^2}\,, \tag{5.10}$$

$$H^2 = \frac{\rho_\phi + \rho_\beta}{3}\left(1 - \frac{\rho_\phi + \rho_\beta}{\rho_\mu}\right), \qquad \rho_\mu = \frac{3}{4\,\mu_0^2}. \tag{5.11}$$

These are exactly the same modified Friedmann equation and critical energy density ρ_μ appearing in equation (4.14) of isotropic Polymer Cosmology. Therefore it is already expected that the singularities will be removed and replaced by a Big Bounce.

In the scalar field time gauge, the solution is

$$\left(\frac{1}{v}\,\frac{\mathrm{d}v}{\mathrm{d}\phi}\right)^2 = v_1^2\left(1 - \frac{\rho_\phi + \rho_\beta}{\rho_\mu}\right), \qquad v(\phi) = \sqrt{\frac{(\rho_\phi + \rho_\beta)\, v^2}{\rho_\mu}}\,\cosh(v_1\,\phi). \tag{5.12}$$

It is clear how a Big Bounce appears since the hyperbolic cosine has a positive minimum at $\phi = 0$. The evolution is shown later in Figure 5.2, compared with the classical solution and with the PUP solution that I will derive later.

The third modification is the BGUP algebra (3.42). However, in order to make the comparison more consistent, I will define it with the factor 2 mentioned in Section 3.3 so that the standard KMM formulation is the Taylor expansion of this one.

The equation of motion and the modified Friedmann equation then are

$$\{v, p_v\}_{\text{BGUP}} = \sqrt{1 + 2\,B_0\, p_v^2}\,, \qquad \dot{v} = -\frac{3}{2}\, p_v\, v\, \sqrt{1 + 2\,B_0\, p_v^2}\,, \tag{5.13}$$

$$H^2 = \frac{\rho_\phi + \rho_\beta}{3}\left(1 + \frac{\rho_\phi + \rho_\beta}{\rho_{\text{GUP}}}\right). \tag{5.14}$$

Here the regularizing density ρ_{GUP} has the same definition (5.7) of the KMM case. Now the reason for the factor 2 is clear: even though the Friedmann equations of the two GUP cases are slightly different, expending the square in the KMM one yields at first order the same one of the BGUP case, and the squared term is a higher order correction. Indeed, as shown in Figure 5.1, at low energies i.e. large volumes the two solutions are the same, while the approach to the singularity $v = 0$ is slightly different, although in both cases it happens at $\phi = 0$. Furthermore, the BGUP Friedmann equation (5.14) is the same Friedmann equation of Brane Cosmology [152, 190, 191], where the role of the regularizing density ρ_{GUP} is played by the Brane tension (and of course here there are no other terms such as curvature or a cosmological constant).

The solution in the scalar field time gauge is

$$\left(\frac{1}{v}\,\frac{\mathrm{d}v}{\mathrm{d}\phi}\right)^2 = v_1^2\left(1 + \frac{\rho_\phi + \rho_\beta}{\rho_{\text{GUP}}}\right), \qquad v(\phi) = \pm\sqrt{\frac{(\rho_\phi + \rho_\beta)\, v^2}{2\rho_{\text{GUP}}}}\,\sinh(v_1\,\phi). \tag{5.15}$$

The evolution is shown in Figure 5.1, compared with the classical evolution and with the standard KMM GUP solution.

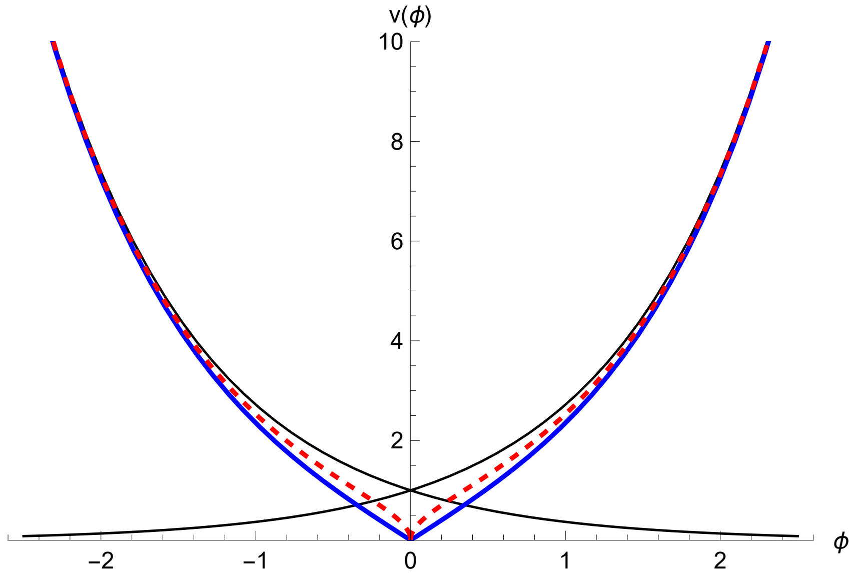

Figure 5.1. The solutions $v(\phi)$ obtained with the KMM algebra (red dashed lines) and with the BGUP formulation (continuous blue lines), compared to the classical unmodified solutions (thin black lines). The singularity is still present at $\phi = 0$.

The last modification is the PUP algebra (3.46). However, also here I will insert a factor of 2 to make the comparison with the Polymer algebra more evident.

The algebra, the equations of motion and the modified Friedmann equations are

$$\{v, p_v\}_{\text{PUP}} = 1 - \frac{\mu_0^2}{2}\, p_v, \qquad \dot{v} = -\frac{3}{2}\, p_v\, v \left(1 - \frac{\mu_0^2}{2}\, p_v\right), \tag{5.16}$$

$$H^2 = \frac{\rho_\phi + \rho_\beta}{3} \left(1 - \frac{\rho_\phi + \rho_\beta}{2\rho_\mu}\right)^2, \tag{5.17}$$

where again the regularizing density ρ_μ is the same defined before in equation (5.11). Also here, comparing the modified Friedmann equation to the Polymer Algebra one, the factor of 2 and the square make it so that the first terms, when the square is expanded, coincide. However, the higher order term here changes the solution drastically. Indeed in the scalar field time gauge the result is

$$\left(\frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}\phi}\right)^2 = v_1^2 \left(1 - \frac{\rho_\phi + \rho_\beta}{2\rho_\mu}\right)^2, \qquad v(\phi) = \sqrt{\frac{(\rho_\phi + \rho_\beta)\, v^2}{2\rho_\mu}}\, \sqrt{e^{\pm 2v_1\phi} + 1}\,. \tag{5.18}$$

The $+1$ under the square root makes it so that the solution does not go to zero in a finite a time, and is instead asymptotic to a positive value. The evolution is shown in Figure 5.2, compared with the classical solution and with the Bouncing Polymer Algebra solution. It is clear how with the PUP algebra the singularity is still removed, but with an asymptotic behaviour instead of with a Bounce. This is called an Emergent Universe model [93, 94] and will be analyzed more in detail in Section 5.2 that constitutes the second part of this chapter.

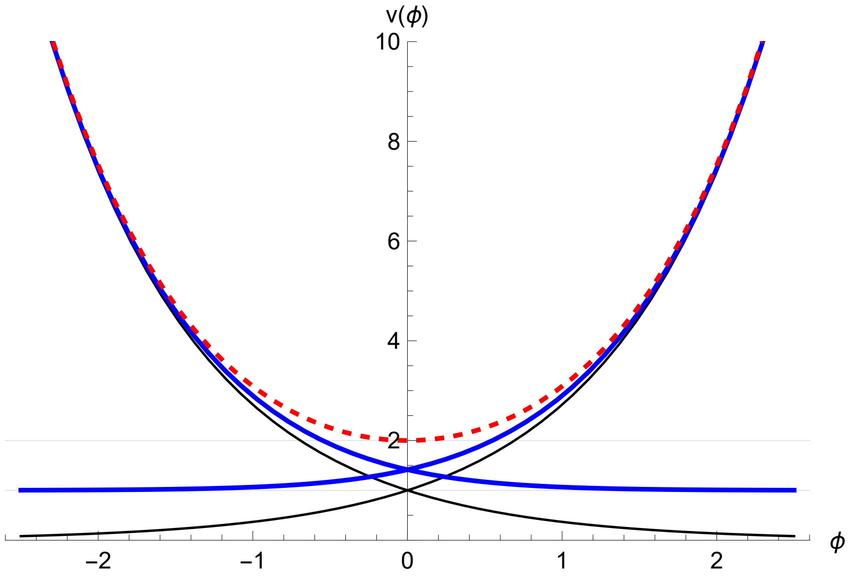

Figure 5.2. Comparison between the solutions $v(\phi)$ in the classical model (thin black lines), with the Polymer Algebra (red dashed lines) and with the PUP formulation (continuous blue line). The singularity is removed, replaced with a Big Bounce or with an asymptotic behaviour respectively. The values of v at the Bounce and at the asymptotes are highlighted with thinner grey lines.

Finally, as a further semiclassical consequence of modified algebras, I will show that the implementation performed as explained in equation (3.53), i.e. by modifying the Hamiltonian substituting the momentum with the modified eigenvalue $G(p)$ of the corresponding momentum operator, yields the exact same semiclassical dynamics. In this case, since the Hamiltonian is already modified, the Poisson brackets will always be the standard ones $\{v, p_v\} = 1$.

The four Hamiltonians then are

$$\mathcal{H}_{\mathrm{KMM}} = \frac{1}{12v}\left(-9\,\frac{\tan^2(\sqrt{B_0}\,p_v)}{B_0}\,v^2 + p_+^2 + p_-^2\right) + \rho_\phi\, v = 0, \tag{5.19}$$

$$\mathcal{H}_{\mathrm{PA}} = \frac{1}{12v}\left(-9\,\frac{\sin^2(\mu_0\,p_v)}{\mu_0^2}\,v^2 + p_+^2 + p_-^2\right) + \rho_\phi\, v = 0, \tag{5.20}$$

$$\mathcal{H}_{\mathrm{BGUP}} = \frac{1}{12v}\left(-9\,\frac{\sinh^2(\sqrt{2B_0}\,p_v)}{2B_0}\,v^2 + p_+^2 + p_-^2\right) + \rho_\phi\, v = 0, \tag{5.21}$$

$$\mathcal{H}_{\mathrm{PUP}} = \frac{1}{12v}\left(-9\,\frac{2\tanh^2\left(\frac{\mu_0\,p_v}{\sqrt{2}}\right)}{\mu_0^2}\,v^2 + p_+^2 + p_-^2\right) + \rho_\phi\, v = 0. \tag{5.22}$$

Given the fact that the Hamiltonians are vanishing, they yield the following constraints (in the same order as the four Hamiltonians above):

$$\frac{\tan^2(\sqrt{B_0}\,p_v)}{B_0} = \frac{4}{3}\,(\rho_\phi+\rho_\beta), \tag{5.23}$$

$$\frac{\sin^2(\mu_0\,p_v)}{\mu_0^2} = \frac{4}{3}\,(\rho_\phi+\rho_\beta), \tag{5.24}$$

$$\frac{\sinh^2(\sqrt{2B_0}\,p_v)}{2B_0} = \frac{4}{3}\,(\rho_\phi+\rho_\beta), \tag{5.25}$$

$$\frac{2\tanh^2\left(\frac{\mu_0\,p_v}{\sqrt{2}}\right)}{\mu_0^2} = \frac{4}{3}\,(\rho_\phi+\rho_\beta). \tag{5.26}$$

It is quite easy to see that, thanks to trigonometric and hyperbolic relations and to the four constraints above linking the modified eigenvalues to the densities, the resulting modified Friedmann equations are exactly the same obtained above for the corresponding algebras.

$$\begin{cases} \dot{v}_{\text{KMM}} = -\frac{3}{2}\,v\,\frac{\tan(\sqrt{B_0}\,p_v)}{\sqrt{B_0}}\left(1+\tan^2(\sqrt{B_0}\,p_v)\right), \\ H^2_{\text{KMM}} = \frac{\rho_\phi+\rho_\beta}{3}\left(1+\frac{\rho_\phi+\rho_\beta}{2\rho_{\text{GUP}}}\right)^2, \end{cases} \tag{5.27}$$

$$\begin{cases} \dot{v}_{\text{PQM}} = -\frac{3}{2}\,v\,\frac{\sin(\mu_0\,p_v)\cos(\mu_0\,p_v)}{\mu_0} = -\frac{3}{2}\,v\,\frac{\sin(\mu_0\,p_v)}{\mu_0}\sqrt{1-\sin^2(\mu_0\,p_v)}\,, \\ H^2_{\text{PQM}} = \frac{\rho_\phi+\rho_\beta}{3}\left(1-\frac{\rho_\phi+\rho_\beta}{\rho_\mu}\right), \end{cases} \tag{5.28}$$

$$\begin{cases} \dot{v}_{\text{BGUP}} = -\,\dfrac{3}{2}\,v\,\dfrac{\sinh(\sqrt{2B_0}\,p_v)\cosh(\sqrt{2B_0}\,p_v)}{\sqrt{2B_0}} = \\ \qquad = -\,\dfrac{3}{2}\,v\,\dfrac{\sinh(\sqrt{2B_0}\,p_v)}{\sqrt{2B_0}}\sqrt{1+\sinh^2\left(\sqrt{2B_0}\,p_v\right)}\,, \\ H^2_{\text{BGUP}} = \frac{\rho_\phi+\rho_\beta}{3}\left(1+\frac{\rho_\phi+\rho_\beta}{\rho_{\text{GUP}}}\right), \end{cases} \tag{5.29}$$

$$\begin{cases} \dot{v}_{\text{PUP}} = -3\,v\,\frac{\tanh(\mu_0\,p_v)}{\mu_0}\left(1-\tanh^2(\mu_0\,p_v)\right), \\ H^2_{\text{PUP}} = \frac{\rho_\phi+\rho_\beta}{3}\left(1-\frac{\rho_\phi+\rho_\beta}{2\rho_\mu}\right)^2, \end{cases} \tag{5.30}$$

It is therefore clear how the two ways to implement the semiclassical limit of modified algebras presented at the end of Section 3.3 yield exactly the same semiclassical dynamics, at least for cosmological models where the Hamiltonians are vanishing.

Given that on a quantum level the commutation relations are more fundamental than the corresponding operatorial representations, it is reasonable to assume that the correct way to implement the (semi)classical limit of deformed algebras is to modify Poisson brackets; however it is very interesting that two kinds of corrections that are apparently very different (although obviously linked) can yield the exact same dynamics. It might be of high value to find a semiclassical implementation corresponding to the representation (3.28), since it could help to better understand the relation between different representations on a quantum level, that still lacks a formal proof.

5.1.2 Modified Quantization

In this section I will promote the fundamental variables to operators. The PUP formalism will be analyzed more in depth later, and in any case it is not able to reproduce the full polymer dynamics like the Polymer Algebra; on the other hand, the BGUP corrections do not differ too much from the standard KMM ones. Therefore I will restrict the quantum analysis to just the Polymer Algebra and the original KMM GUP. I will work in a hybrid polarization, i.e. momentum polarization for v and position polarization for the other variables $\beta_\pm$ and ϕ; this is due to the fact that in the two representations the position polarization is not viable: in PQM it would be non-trivial as showed in Section 3.1 [82], while in the GUP it is not available and the quasi-position representation should be used instead, as explained in Section 3.2 [126].

The standard procedure is to perform some substitutions such that the WdW equations become Klein-Gordon-like wave equations where ϕ plays the role of time, similarly to what was done in Sections 2.2.1 and 4.2. The resulting solutions will be therefore expressed in a new variable x_i with $i =$ KMM,PA and, although the substitutions will be different, the wavepackets as functions of x will have a similar form. When studying the physics, the spreading of the wavepacket will be the same in all pictures; however, the physical operators will of course have different expressions with respect to x and their expectation values will be different, yielding inequivalent physical predictions in accordance with the semiclassical dynamics. This reflects the different nature of the coordinates in the two representations.

The Hamiltonian constraints, when promoted to a quantum level, yield a WDW equation that is the same in all cases:

$$(3\,\hat{p}_v\,\hat{v})^2\psi = (\hat{p}_\phi^2 + \hat{p}_+^2 + \hat{p}_-^2)\psi, \tag{5.31}$$

where I absorbed some constants in the scalar field momentum p_ϕ to lighten the notation (this is equivalent to a rescaling of the scalar field). The action of the operators not linked to the volume will always be the standard one (i.e. the variables $\beta_\pm$ and ϕ will act multiplicatively and their conjugate momenta differentially); the differential volume operator and its conjugate momentum will have different expressions depending on the formalism used:

$$\text{KMM}: \quad [\hat{v},\hat{p_v}] = i(1 + B_0\,\hat{p_v}^2), \quad \hat{v}\,\psi = i(1 + B_0\,p_v^2)\frac{\mathrm{d}\psi}{\mathrm{d}p_v}, \quad \hat{p}_v\,\psi = p_v\,\psi; \tag{5.32}$$

$$\text{PA}: \quad [\hat{v},\hat{p_v}] = i\sqrt{1-\mu_0^2\,\hat{p}_v^2}\,, \quad \hat{v}\,\psi = i\frac{\mathrm{d}\psi}{\mathrm{d}p_v}, \quad \hat{p}_v\psi = \frac{\sin(\mu_0\,p_v)}{\mu_0}\,\psi. \tag{5.33}$$

The different actions will yield different terms on the left-hand side of the WDW equation (5.31); by choosing a mixed factor ordering, it is possible to use different transformations so that the two equations become KG-like, as done previously in Section 4.2:

$$\text{KMM}: \qquad (3\,\hat{p}_v\,\hat{v})^2\psi = -\left(3\,p_v\,(1 + B_0\,p_v^2)\frac{\mathrm{d}}{\mathrm{d}p_v}\right)^2\psi; \tag{5.34}$$

$$\text{PA}: \qquad (3\,\hat{p}_v\,\hat{v})^2\psi = -\left(3\,\frac{\sin(\mu_0 p_v)}{\mu_0}\,\frac{\mathrm{d}}{\mathrm{d}p_v}\right)^2\psi; \tag{5.35}$$

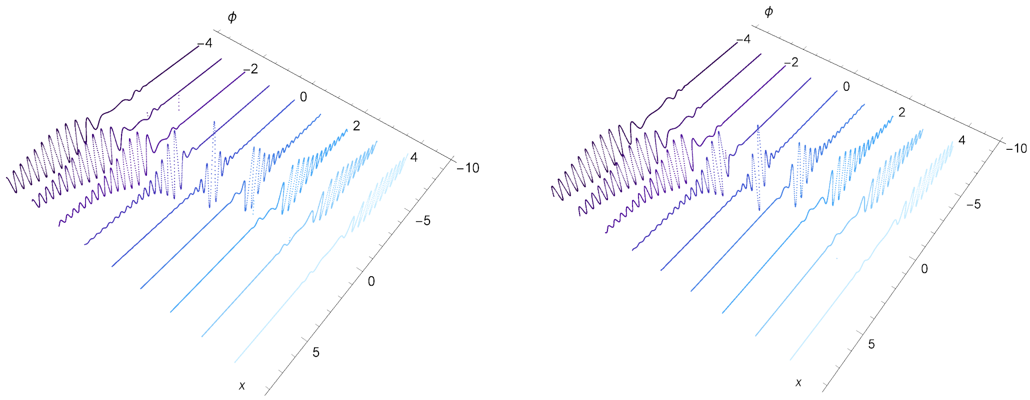

Figure 5.3. The wavepacket Ψ (real part on the left, imaginary part on the right) as function of the variable x for different values of ϕ. It is evident how it delocalizes at greater times.

$$x_{\text{KMM}} = \frac{1}{3}\ln\left|\frac{p_v}{\sqrt{1+B_0\,p_v^2}}\right|, \qquad x_{\text{PA}} = \frac{1}{3}\ln\left|\tan\left(\frac{\mu_0\,p_v}{2}\right)\right|; \tag{5.36}$$

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x_i^2} = \left(\frac{\mathrm{d}^2}{\mathrm{d}\phi^2} + \frac{\mathrm{d}^2}{\mathrm{d}\beta_+^2} + \frac{\mathrm{d}^2}{\mathrm{d}\beta_-^2}\right)\psi, \qquad i = \text{KMM,PA}. \tag{5.37}$$

Thanks to these substitutions, the WdW equations (5.31) are rewritten in the KG form (5.37) and their general solution can be expressed as a wavepacket Ψ with Gaussian weights W peaked around specific values $\overline{k}_\pm$, $\overline{k}_\phi$ of the momenta eigenvalues that obey a non-linear dispersion relation:

$$\Psi(x_i, \beta_\pm, \phi) = \int_{-\infty}^{+\infty} dk_+\, dk_-\, dk_\phi\, W(k_+)\, W(k_-)\, W(k_\phi)\, \psi, \tag{5.38}$$

$$\psi = \psi(x_i, \beta_\pm, \phi, k_\pm, k_\phi) = \exp\Big(i\,(k_i\,x_i + k_+\,\beta_+ + k_-\,\beta_- + k_\phi\,\phi)\Big), \tag{5.39}$$

$$W(k) = \frac{\exp\left(-\frac{(k-\overline{k})^2}{2\sigma^2}\right)}{(\pi\sigma^2)^{\frac{1}{4}}}, \tag{5.40}$$

$$k_x = \sqrt{k_+^2 + k_-^2 + k_\phi^2}\,. \tag{5.41}$$

Figure 5.3 highlights the spreading of the wavepacket Ψ with the passing of time ϕ, as expected for a wavepacket propagating in more than one "spatial" dimension (indeed, as shown in Section 2.2, in the flat isotropic FLRW model where there is only one dimension represented by the volume v, the wavepackets do not spread). Note that this phenomenon restricts the validity of the semiclassical limit of the quantum theory: if the wavepackets spread and delocalize, the quantum expectation values will follow the classical trajectory only for the brief interval when the packet is still localized enough, but the standard deviation will grow appreciably.

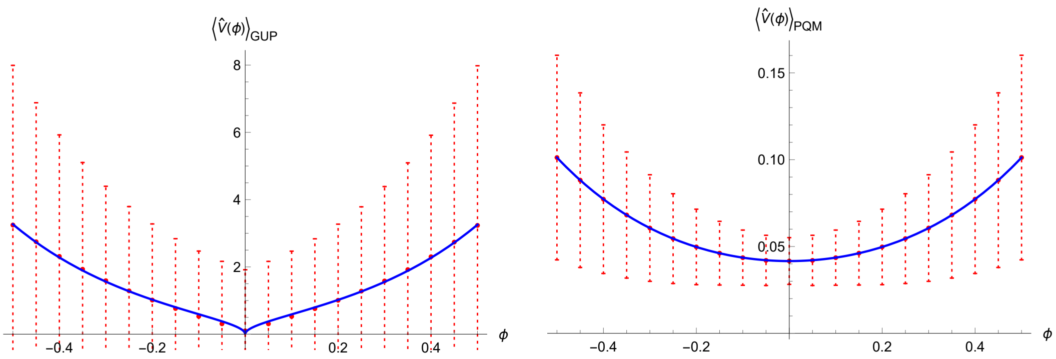

Figure 5.4. Expectation value $\langle \hat{v}(\phi) \rangle$ of the volume as function of time ϕ for the KMM formulation (left) and the Polymer Algebra (right). The red dots have been computed numerically and fitted with the blue lines in accordance with the semiclassical trajectories; the error bars correspond to the standard deviation $\sqrt{\langle \hat{v}^2 \rangle - \langle \hat{v} \rangle^2}$.

It is possible compute the expectation value of the volume operator $\hat{v}$ in the two different cases. In order to do this, I will use a KG-like probability density, and the integral for the expectation value will be the same for the two cases except that the KMM one will have a measure, which is required to ensure the symmetry of the operators as explained in Section (3.2).

$$\langle \hat{v} \rangle_{\text{KMM}} = \int_{-\infty}^{+\infty} \frac{dx\, d\beta_+\, d\beta_-}{\sqrt{1 + B_0\, p_v^2(x)}}\; i \left(\Psi^* \,\partial_\phi (\hat{v} \Psi) - (\hat{v} \Psi)\, \partial_\phi \Psi^* \right), \tag{5.42}$$

$$\langle \hat{v} \rangle_{\text{PA}} = \int_{-\infty}^{+\infty} dx\, d\beta_+\, d\beta_- \;\; i \left(\Psi^* \,\partial_\phi (\hat{v} \Psi) - (\hat{v} \Psi)\, \partial_\phi \Psi^* \right), \tag{5.43}$$

where $p_v(x)$ means that the substitution to x has been performed. On this note, also the action of the volume operator must be expressed in terms of the variables x:

$$\hat{v}_{\text{KMM}} = i\, (1 + B_0\, p_v^2)\, \frac{\mathrm{d}}{\mathrm{d}p_v} = -\frac{i}{3}\, \frac{(1 + B_0\, e^{6x})^{\frac{1}{2}}}{e^{3x}}\, \frac{\mathrm{d}}{\mathrm{d}x}, \tag{5.44}$$

$$\hat{v}_{\text{PA}} = i\, \frac{\mathrm{d}}{\mathrm{d}p_v} = -\frac{i\,\mu_0}{3}\, \cosh(x)\, \frac{\mathrm{d}}{\mathrm{d}x}, \tag{5.45}$$

where I omitted the subscripts on x to lighten notation. The integrals are then computed numerically.

Figure 5.4 shows the expectation value of the volume operator for the two different formalisms: the semiclassical trajectories are reproduced closely enough at small times, but the standard deviations start to grow quite soon; therefore these expectation values are reliable only in a small interval of time.

To conclude, a pure quantum description of the Planckian regime is more meaningful in the case of an anisotropic model, especially in the KMM formulation where a minimal uncertainty on position prevents the localization of the wavepackets at will; on the other hand, with the Polymer Algebra this possibility is always available and the dynamics can present a quasi-classical morphology. As a consequence, the inability of the GUP representation to remove the singularity in the semiclassical description, contrary to PQM, is not conclusive, since a predictive quantum picture of the primordial Universe is still lacking.

The main goal of this analysis was to better characterize the physical meaning of PQM, of the GUP representation and of Modified Algebras in general, especially in view of their cosmological implementation. The obtained equivalence of the effective Bianchi I dynamics with the corresponding LQC and Brane Cosmology predictions constitutes a significant starting point to extend the present picture and infer general features about the removal of the singularity in the so-called generic cosmological solution. Actually, the direct implementation of the two fundamental theories of LQC and Brane Cosmology would be forbidden by the complexity of the Superspace model, while the PQM and GUP versions of the so-called Belinskii-Khalatnikov-Lifshitz conjecture (BKL) are viable [13, 44]. However, in String Theories the quantized gravitational modes arise in a perturbative scheme and the relic notion of a classical background still survives, differently from LQG. Moreover, the original KMM formulation of the GUP introduces cut-off physics effects on a non-perturbative level in the low energy limit of String Theories, so this feature could suggest that the cosmological GUP implementation cannot fulfill all the features of a Planckian Universe in Brane Cosmology.

5.2 The Emergent Universe Scenario

Here, I will consider a non-singular cosmology that comes from assigning specific initial conditions on the closed RW model dynamics, known as the "Emergent Universe" (EU) [93, 94]. The interest for such an EU model was recently renewed by the analyses of the Planck data sets [173, 174], which seem to allow for a present-day positive curvature of the Universe [89, 111].

The possibility to classically solve the singularity is an interesting subject, but for the Emergent Universe this result is valid only for a specific fine-tuning of the initial conditions on the cosmological dynamics. Here I overcome this shortcoming of the original idea by considering suitable modified Poisson brackets inspired by cut-off physics such as PQM and the GUP presented in previous chapters, specifically the Polymer Uncertainty Principle algebra (PUP) (3.46), which induces an Emergent Universe scenario still on the (semi)classical level, valid for any assignment of the Cauchy problem. After reviewing the original literature on the subject of the classical EU model, I will show how it can be obtained thanks to the PUP algebra; the possibility to use it and introduce a non-singular, asymptotic state for the isotropic model was mentioned already in the previous Section 5.1 and in the paper [34]. The relevance of this formulation of the EU picture relies on the generality of its non-singular behavior, without the need for a constraint on the initial conditions to be required ab initio. In other words, including a quasi-classical modification of the symplectic algebra similar in its phenomenology to a modified gravity approach, this algebra is able to yield an EU with an asymptotic non-singular beginning for the synchronous time approaching negative infinity. I also properly characterize the different phases of the Universe evolution, starting with a radiation-dominated era close to the classical singularity, passing through an inflationary de Sitter period obtained including a constant energy density term, and ending again with a radiation-dominated Universe (the study of a late-time dark energy-dominated era, possible for an EU as mentioned in [93], is beyond the scope of this thesis).

An important part of this analysis is dedicated to the calculation of corrections to the primordial Spectrum when the inflaton field obeys the same symplectic algebra at the ground of the obtained semiclassical dynamics, but implemented on the pure quantum sector. I will treat the additional term emerging in the Fourier-decomposed Hamiltonian for the Mukhanov-Sasaki variable, which parameterizes the scalar perturbations [170], as a small perturbation and will determine the corrections it induces on the standard states (associated to a time-dependent harmonic oscillator). As a result, I will determine the Spectrum corrections due to the new physics at the ground of this study, and I will show under which constraints on the model parameters and initial conditions the modification is a reliably small and potentially observable feature.

The present analysis offers an interesting new perspective on the origin of a non-singular isotropic Universe, whose underlying cut-off physics can leave a precise fingerprint on the profile of the microwave background temperature distribution.

In order to better characterize the deformation parameter of the modified algebra, here I will keep fundamental constants such as the reduced Planck constant $\hbar$, the Einstein constant χ and the speed of light c explicit. The results of this sections are reported in paper [37].

5.2.1 Classical Emergent Universe Model

Here I compactly present the standard Emergent Universe model [93, 94] starting from the Hamiltonian formulation of the FLRW homogeneous and isotropic model. I derive a non-singular, ever-expanding solution and then show the potential used to end the inflationary expansion.

The configurational variables that I will use for the gravitational sector are the volume $v = a^3$, and its conjugate momentum $p_v \propto \dot{v}/v$ since they have been shown to be the suitable variables to yield an universal critical energy density in Polymer Cosmology, as previously mentioned in Chapter 4 [31, 168].

The Hamiltonian for a FLRW model with curvature filled with matter in the form of perfect fluids is the same as equation 2.18:

$$\mathcal{H}_{\text{FLRW}}(v, p_v) = -\frac{3\,\chi}{4\,\mathcal{V}}\,v\,p_v^2 - \frac{3}{\chi}\,K\,c^2\,v^{\frac{1}{3}}\,\mathcal{V} + \rho\,v\,\mathcal{V} = 0, \tag{5.46}$$

where here I explicitly kept some constants: $\mathcal{V}$ is the volume scale appearing when preforming the spatial integral in the action (2.6), $K > 0$ is the positive spatial curvature, and $\rho = \rho(v)$ is the energy density containing all necessary components, each separately obeying the standard continuity equation (2.11); a simple EU model contains a radiation fluid ρ_γ with $w_\gamma = 1/3$ and a Cosmological Constant $\rho_\Lambda = \overline{\rho_\Lambda}$ corresponding to $w_\Lambda = -1$.

From the equations of motion and the Hamiltonian constraint, the Friedmann equation is

$$H^2 = \left(\frac{\dot{v}}{3v}\right)^2 = \frac{\chi}{3}\,(\rho_\gamma + \rho_\Lambda) - \frac{K\,c^2}{v^{2/3}}. \tag{5.47}$$

By requiring the existence of a unique positive minimum v_i for the volume, the

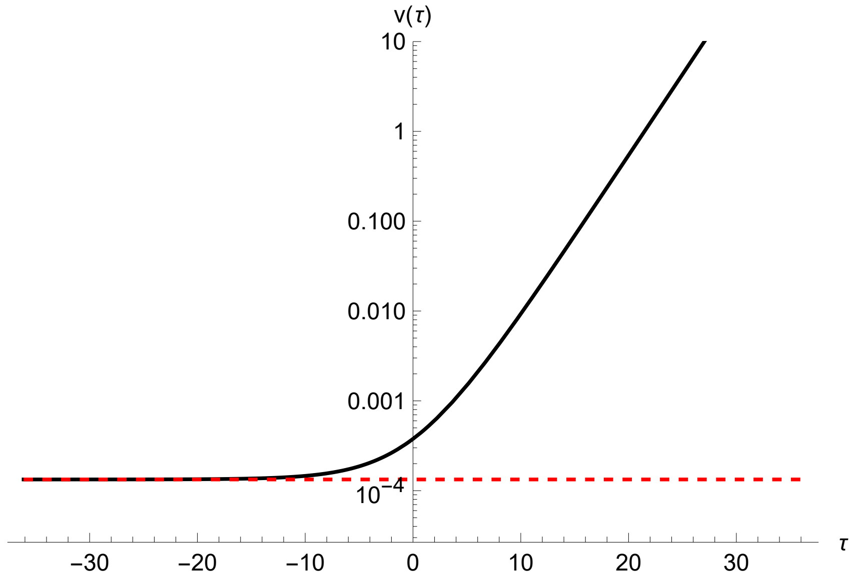

Figure 5.5. Evolution of the volume $v(\tau)$ in the standard EU model. The time variables are rescaled by the time t_s of the beginning of standard inflation: $\tau = \frac{t}{t_s}$. The minimum value is highlighted with a red dashed line.

following constraint on the free parameters of the densities is obtained:

$$v_i = \left(\frac{3}{2}\frac{K\,c^2}{\chi\,\rho_\Lambda}\right)^{\frac{3}{2}}, \qquad \overline{\rho_\gamma}\,\rho_\Lambda = \left(\frac{3}{2}\frac{K\,c^2}{\chi}\right)^2, \tag{5.48}$$

so that the Friedmann equation can be rewritten in terms of the minimum volume and easily solved:

$$\dot{v} = \pm 3c\sqrt{\frac{K}{2}}\,\left(\frac{v}{v_i}\right)^{\frac{1}{3}}\left(v^{\frac{2}{3}} - v_i^{\frac{2}{3}}\right), \qquad v(t) = v_i\left(1 + \exp\!\Big(\pm\frac{\sqrt{2\,K}\;c}{v_i^{1/3}}\,t\Big)\right)^{\frac{3}{2}}. \tag{5.49}$$

There are two solutions, one expending to infinity and one contracting from infinity, depending on the sign of the exponential; the expanding one with the + sign is the one of interest. The solution is shown in Figure 5.5: as expected it is asymptotically Einstein static, since $v \to v_i > 0$ when $t \to -\infty$, and exponentially expanding. Note that in the picture I rescaled the time variable as $\tau = t/t_s$, where $t_s \sim 10^{-36}s$ is the start of inflation in the standard cosmological model; this will be useful in later sections.

Even though inflation occurs for an infinite time in the past, at any given time $t_f \gg v_i^{1/3}/\sqrt{Kc^2}$ there is a finite amount of $e-$folds given by

$$N_e = \frac{1}{3}\,\ln\!\left(\frac{v(t_f)}{v_i}\right) \approx \frac{\sqrt{K}\;c\,t_f}{v_i^{1/3}}. \tag{5.50}$$

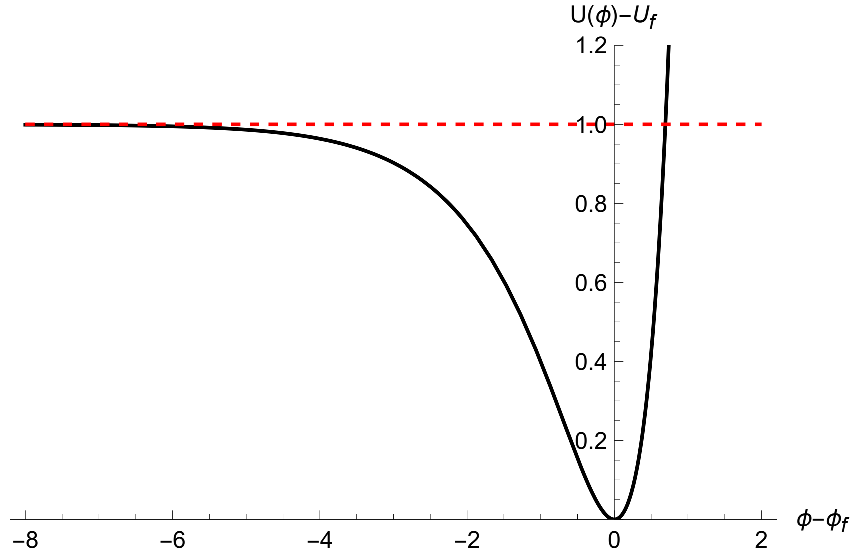

Figure 5.6. The potential $U(\phi) - U_f$ as function of $\phi - \phi_f$ for $U_i - U_f = 1$, $\mathcal{E} = 1$. The asymptote is highlighted with a red dashed line.

A simple way to create a past-infinite exponential expansion and end it at a finite time t_f is to use a scalar field. Therefore the matter term in the Hamiltonian (5.46) will contain the scalar field energy density

$$\rho_\phi(v, \phi, p_\phi) = \frac{p_\phi^2}{2v^2} - U(\phi), \tag{5.51}$$

where $U(\phi)$ is a potential and $p_\phi = \dot{\phi}\, v/c$ is the momentum conjugate to the scalar field. From the equations of motion, the scalar field obeys a Klein-Gordon-like equation:

$$\ddot{\phi} + \frac{\dot{v}}{v}\,\dot{\phi} + c^2\,\frac{\partial U}{\partial \phi} = 0. \tag{5.52}$$

The ideal potential for an EU model has a plateau (i.e. an asymptote) at $\phi \to -\infty$ and a well with an absolute minimum at $\phi = \phi_f$; it takes the form

$$U(\phi) = U_f + (U_i - U_f)\left(\exp\Big(\frac{\phi - \phi_f}{\mathcal{E}}\Big) - 1\right)^2, \tag{5.53}$$

where U_i is the asymptotic value in the infinite past, U_f is the minimum value and $\mathcal{E}$ is a constant scale parametrizing the width of the well. The form of the potential is shown in Figure 5.6 for generic values of the parameters.

For $t \to -\infty$, $\phi \to -\infty$ and the field is in the plateau of the potential; this implies $\dot{\phi}^2 \ll U$ and therefore the energy density $\rho_\phi \approx U_i$ is nearly constant, playing the role of the Cosmological Constant ρ_Λ of the previous subsection. When $t = t_f$ is approached, the field falls into the well until it reaches the minimum $U_f \ll U_i$, and the exponential expansion ends. Here it could possible to set $U_f \neq 0$ to represent the late-time cosmological constant [93], but again, this is beyond the scope of this study.

As mentioned before, the infinite time of inflation produces a finite amount of expansion; provided that v_i and t_f are respectively chosen small and large enough, a very large amount of $e-$folds can be produced as follows from equation (5.50), thus solving all the paradoxes of Friedmann evolution in a similar manner to the standard inflationary theory. However, analogously to the latter, this model is also subject to some form of fine-tuning and criticisms.

Indeed, some fine-tuning is needed in the classical EU to reproduce observational parameters, such as density perturbations of the order $\mathcal{O}(10^{-5})$ and a late-time Cosmological Constant $\Omega_\Lambda \approx 0.7$; however, all inflationary universe models need some amount of fine-tuning. The specific geometrical fine-tuning problem in the EU models is the requirement of a particular choice of the initial volume v_i and of the primordial cosmological constant ρ_Λ or U_i. This choice must then be supplemented by a further fine-tuning: a choice of initial kinetic energy such that the inequality $\dot{\phi}^2 \ll U_i$ holds. Both conditions are required to attain an asymptotically Einstein-static state.

The authors of [93, 94] acknowledge the necessity of fine-tuning in this model, but claim that the situation is not too different from any other inflationary model. Besides, they argue that the advantages of having a non-singular, highly symmetric initial state overcome the troubles of fine-tuning. However, the scope of this work is to provide a mechanism to generate an EU model with the minimum fine-tuning necessary.

5.2.2 Non-Singular Emergent Universe from Modified Algebra

I will now introduce the modified Heisenberg algebra, that in the classical limit translates to modified Poisson brackets as explained in Section 3.3, which is able to yield an EU-like solution without the need of much fine-tuning.

The modified algebra, that is the PUP algebra inspired by PQM [41, 82] and the GUP representation [34, 62, 95, 126, 153, 202] and presented in previous chapters, here takes the form

$$[\hat{q}, \hat{p}] = i\hbar \left(1 - \frac{\mu_0^2\, \ell_P^2\, \hat{p}^2}{\hbar^2}\right); \tag{5.54}$$

$\mu_0 > 0$ is a free real parameter that is reminiscent of the lattice spacing in PQM (hence the use of the same symbol), but here takes the role of just a deformation parameter similarly to the GUP representation. As mentioned earlier, differently from Section 3.3 where I introduced the algebra, in this case I have kept the appropriate fundamental constants in order to always have μ_0 as a dimensionless parameter, as is sometimes done in GUP literature [95, 184]; for example, in the commutator (5.54) I assumed q and p to be the standard position and momentum respectively, so I inserted the Planck length ℓ_P and the reduced Planck constant $\hbar$ to keep both the term in parentheses and the deformation parameter μ_0 dimensionless.

In the classical limit, the commutator (5.54) becomes a rule for Poisson brackets, as explained in previous chapters. In this first example, I will not consider curvature and will not assume any specific kind of matter but leave a generic energy density $\rho(v) = \overline{\rho}\, v^{-(1+w)}$.

The Hamiltonian constraint is the same as (5.46), but with the modified Poisson

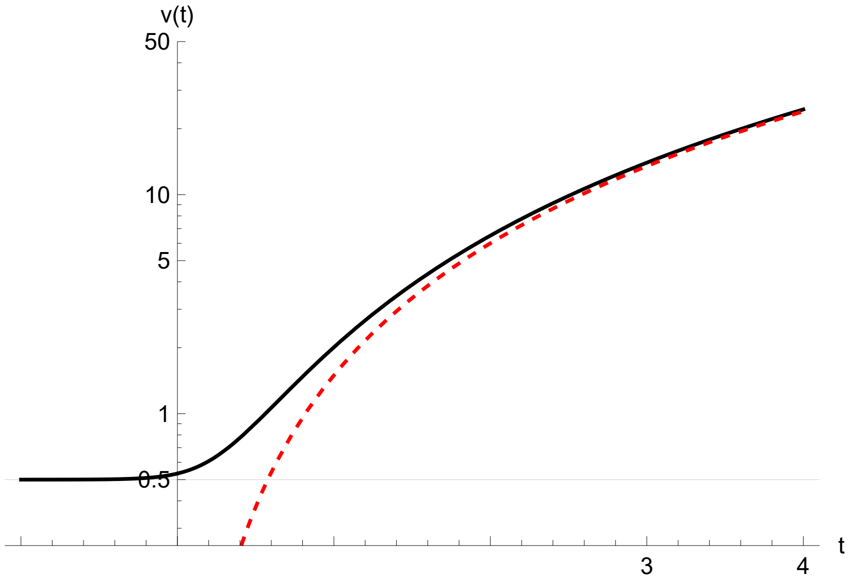

Figure 5.7. The asymptotic solution $v(t)$ for the simple model with a generic energy density (black continuous line), compared with the standard evolution (red dashed line) which falls into the singularity. The asymptotic volume v_i is highlighted with the grey faded line.

brackets and without curvature:

$$\mathcal{H}_{\text{FLRW}}(v, p_v) = -\frac{3\,\chi}{4\,\mathcal{V}}\, v\, p_v^2 + \rho(v)\, v\, \mathcal{V} = 0, \qquad \{v, p_v\} = 1 - \frac{\mu_0^2\, p_v^2}{\hbar^2}; \tag{5.55}$$

note that, since the volume v is dimensionless, p_v has the dimensions of an action and therefore I divided the correction term by $\hbar^2$ to keep μ_0 dimensionless. Then from the equations of motion and the constraint I derive a modified Friedmann equation:

$$H^2 = \frac{\chi}{3}\,\rho\left(1 - \frac{\rho}{\rho_\mu}\right)^2, \qquad \rho_\mu = \frac{3\,\chi\,\hbar^2}{4\,\mu_0^2\,\mathcal{V}^2}, \tag{5.56}$$

this is very similar to the modified Friedmann equations obtained in other sections and chapters with PQM, the GUP or other algebras, but with a slightly different expression for ρ_μ that however still introduces a critical point on the dynamics; the critical point is calculated as the value v_μ such that $\dot{v} = 0$, which, as long as $w \neq -1$, implies

$$1 - \frac{\rho(v_\mu)}{\rho_\mu} = 0, \quad v_\mu = \left(\frac{\overline{\rho}}{\rho_\mu}\right)^{\frac{1}{1+w}}. \tag{5.57}$$

The solution $v(t)$ then has the following implicit form:

$$\left(\frac{v(t)}{v_\mu}\right)^{\frac{1+w}{2}} - \text{atanh}\left(\left(\frac{v(t)}{v_\mu}\right)^{-\frac{1+w}{2}}\right) = \pm\frac{1+w}{2}\, t\,\sqrt{3\,\rho_\mu\,\chi}\,; \tag{5.58}$$

again there are two solutions, one contracting and one expanding, depending on the sign. The solution of interest (the expanding one with the + sign) is shown in

Figure 5.7 for generic values of the parameters. Of course this does not present an exponential behaviour, since at this stage I did not include a Cosmological Constant; however this is just a simplified model (but still more detailed than in the previous Section 5.1) to show the ability of the modified PUP algebra (5.54) to naturally implement an asymptotic minimum value.

The main result of this simple construction is that I did not have to impose any fine-tuning such as the constraint (5.48) in order to obtain a positive minimum for the volume; it naturally follows from the form of the correction factor $(1-\frac{\rho}{\rho_\mu})^2$ in the modified Friedmann equation (5.56). I obtained a non-singular, asymptotically Einstein-static model that in the future yields the standard Friedmann evolution; indeed, note that for $v \gg v_\mu$, $\rho(v) \ll \rho_\mu$ and the modified Friedmann equation reduces to the standard one $H^2 = \chi\,\rho/3$; this can be also seen from equation (5.58): in the limit $v \gg v_\mu$ corresponding to $t \to +\infty$, the argument of the hyperbolic arctangent goes to zero and, given the relation (5.57) between ρ_μ, $\overline{\rho}$ and v_i, the standard Friedmann evolution is recovered.

I will now consider the full model. I will consider different phases: the first, near the classical singularity, where the matter-energy content is dominated by a relativistic component; the second where a scalar field potential grows, yielding an inflationary phase dominated by a Cosmological Constant; a final one where the scalar field has again decayed into photons and the late-time evolution becomes Friedmann-like. In all phases I will consider positive curvature, even though in the modified algebra scheme it is not needed to obtain an asymptotic behaviour, in order to make the comparison with the standard EU model more immediate.

The Hamiltonian of the full model is

$$\mathcal{H}_{\text{FLRW}}(v,p_v,\phi,p_\phi) = -\frac{3\chi}{4\mathcal{V}}\,v\,p_v^2 - \frac{3}{\chi}\,K\,c^2\,v^{\frac{1}{3}}\,\mathcal{V} + \rho\,v\,\mathcal{V} = 0, \tag{5.59}$$

$$\text{phase 1)}\quad \rho = \rho_\gamma = \overline{\rho_\gamma^{\text{pre}}}\;v^{-\frac{4}{3}}, \tag{5.60}$$

$$\text{phase 2)}\quad \rho = \rho_\phi(U \gg \dot{\phi}^2) = U_i = \rho_\Lambda, \tag{5.61}$$

$$\text{phase 3)}\quad \rho = \rho_\gamma = \overline{\rho_\gamma^{\text{post}}}\;v^{-\frac{4}{3}}, \tag{5.62}$$

where the constants $\overline{\rho_\gamma^{\text{pre/post}}}$ and ρ_Λ have been chosen to maintain continuity for v and $\dot{v}$.

Given the complexity of the corresponding Friedmann equations, the resolution has been performed numerically. Again, I rescaled all quantities by their corresponding value at the beginning of inflation, that is, I used as time variable $\tau = t/t_s$ and all densities have been rescaled accordingly. The result is shown in Figure 5.8 for the whole evolution and compared with the classical case (i.e. the one obtained with standard Poisson brackets $\{v, p_v\} = 1$); it also shows a zoom near the singularity, to better highlight the asymptotic behaviour.

I must stress that I did not have to impose any condition such as (5.48) in order to obtain an asymptotic behaviour, it is implemented naturally by the modified algebra (5.54). Besides, the standard dynamics is recovered pretty soon and already shortly before the inflationary epoch the evolution is practically indistinguishable;

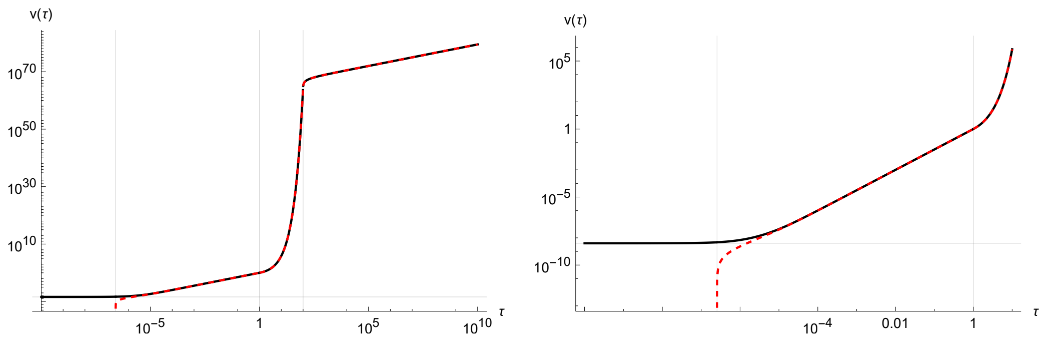

Figure 5.8. The evolution of $v(\tau)$ for the full model (black continuous line) compared with the standard dynamics (dashed red line). The minimum volume v_i is highlighted with a grey faded horizontal line, while the grey faded vertical lines separate the different phases (from left to right they indicate the classical Big Bang, the start of inflation and its end). The right panel is the same figure, but zoomed-in to give a better view of the behaviour near the classical singularity.

this will allow the use of the classical Friedmann equation for Inflation when in the next section I will compute corrections to the primordial Power Spectrum.

5.2.3 Modified Power Spectrum of Scalar Perturbations

In this section, as a phenomenological consequence of this model and in particular of the modified algebra (5.54), I will derive the modified Power Spectrum of primordial scalar perturbations during the inflationary epoch. I will partially follow [66] but compute the Spectrum through a different method. For other approaches to the computation of quantum gravity corrections on the inflationary Spectrum, see [156, 157].

The general action for a scalar field takes the form

$$S_\phi = \int \frac{dt\, d^3x}{2}\, \sqrt{-g}\, \Big(g^{\alpha\beta}\, \partial_\alpha\phi\, \partial_\beta\phi - 2\, U(\phi)\Big); \tag{5.63}$$

introducing conformal time η, the zero-order homogeneous action for the scalar field can be rewritten as

$$d\eta = \frac{dt}{a}, \tag{5.64}$$

$$S_\phi = \int \frac{dt}{2}\, a^3\, \mathcal{V} \left(\frac{\dot\phi^2}{c^2} - 2U(\phi)\right) = \int \frac{d\eta}{2}\, \mathcal{V}\Big(a^2\, \frac{(\phi')^2}{c^2} - 2\, a^4\, U(\phi)\Big), \tag{5.65}$$

where a prime denotes a derivative with respect to η and $\mathcal{V}$ is the usual volume scale that appears when performing the volume integral.

At this point it is useful to introduce the so-called Mukhanov-Sasaki variable ξ, a master gauge-invariant variable which is sufficient to fully describe the scalar sector of perturbations [170]:

$$\xi(x, \eta) = a \left(\delta\phi_{\rm GI} + \frac{\phi'\, \Phi_{\rm B}}{a\, H}\right), \tag{5.66}$$

where $\delta\phi_{\rm GI}$ is the gauge-invariant form of the scalar field perturbations and $\Phi_{\rm B}$ is a Bardeen potential depending on the perturbative scalar functions in the perturbed metric [66]. The action for the variable ξ is obtained as the scalar part of the second variation of the total action, that is, of both the gravitational sector and the matter action (5.63):

$$\delta^2 S = \int \frac{d\eta\, d^3x}{c^2}\left((\xi')^2 - \delta^{ij}\, c^2\, \partial_i\xi\, \partial_j\xi + \xi^2\, \frac{z''}{z}\right), \qquad z = a\sqrt{\epsilon}\,, \tag{5.67}$$

where $\epsilon = -\dot{H}/H^2$ is the first slow-roll parameter.

Now, since I am working with linear perturbations where each mode evolves independently, a Fourier decomposition can be performed so that the action greatly simplifies:

$$\xi(x,\eta) = \frac{1}{\sqrt{\mathcal{V}}}\sum_k \xi_k(\eta)\, e^{ikx}\,, \tag{5.68}$$

$$\delta^2 S = \int \frac{d\eta}{c^2}\sum_k\left(\xi_k^{*\,\prime}\, \xi_k' - \omega_k^2\, \xi_k^*\, \xi_k\right), \qquad \omega_k^2 = k^2\, c^2 - \frac{z''}{z}, \tag{5.69}$$

where $\omega_k = \omega_k(\eta)$ is a time-dependent frequency. Note that if I assume ξ to be real, then $\xi_k^* = \xi_{-k}$. The momentum conjugate to ξ_k is defined as $\pi_k = \xi_k'/c^2$ and finally the Hamiltonian for the scalar perturbations is

$$\mathcal{H} = \sum_k \mathcal{H}_k, \qquad \mathcal{H}_k = \frac{c^2}{2}\, \pi_k^*\, \pi_k + \frac{\omega_k^2(\eta)}{2\, c^2}\, \xi_k^*\, \xi_k. \tag{5.70}$$

In order to calculate the Power Spectrum, I will make the assumption that during the inflationary era the evolution is dominated by the Cosmological Constant and therefore all other components are negligible; besides, if inflation starts late enough, $\rho \ll \rho_\mu$ and, as mentioned in the last section, the correction factor in (5.56) is negligible:

$$H^2 = \frac{\chi}{3}\, \rho_\Lambda = H_s^2, \tag{5.71}$$

where H_s is the constant Hubble parameter of inflation. Note that, to be precise, in a pure de Sitter universe the background matter field is set to a constant value and thus, in principle, it does not make sense to speak about its perturbations. This is shown explicitly in the appearance of the slow-roll parameter ϵ, which in this limit should be vanishing; indeed in this case a Power Spectrum cannot be obtained since inflation never stops. Nonetheless, the computations can be performed by keeping the slow-roll parameter as a non-vanishing constant, and this particular case represents a very good and easy-to-compute example to derive a Power Spectrum. In this regime, conformal time acquires a precise dependence on the scale factor and the frequency ω_k greatly simplifies:

$$\eta = -\frac{1}{a\, H_s}, \qquad \omega_k^2(\eta) = k^2\, c^2 - \frac{2}{\eta^2}. \tag{5.72}$$

Before implementing quantization, in principle real counterparts of ξ_k and π_k should be defined, otherwise the procedure is not entirely consistent [159]. However this will make no difference in later calculations, and therefore I will not define such new variables to avoid cluttering the notation, as done in [66]. The only modification that I will implement is a rescaling of both variables by the speed of light c to simplify the constraint; in particular I will substitute $\pi_k^{\rm new} = \pi_k^{\rm old} c$ and $\xi_k^{\rm new} = \xi_k^{\rm old}/c$. Therefore the Hamiltonian operator that I will use is

$$\hat{\mathcal{H}} = \sum_k \hat{\mathcal{H}}_k, \qquad \hat{\mathcal{H}}_k = \frac{\hat{\pi}_k^2}{2} + \frac{\omega_k^2(\eta)}{2}\,\hat{\xi}_k^2; \tag{5.73}$$

the right units will be recovered later in the definition of the Power Spectrum.

Now the quantization of the system can be performed and the Power Spectrum can be computed. I will first briefly present the standard Spectrum derived through the canonical quantization, and then find the modified Spectrum coming from the algebra (5.54).

In Standard Quantum Mechanics, the two operators corresponding to the Fourier modes will obey the standard commutation relations and will have the standard action:

$$\left[\hat{\xi}_k, \hat{\pi}_k\right] = i\,\hbar, \qquad \hat{\xi}_k\,\psi(\xi_k) = \xi_k\,\psi(\xi_k), \qquad \hat{\pi}_k\,\psi(\xi_k) = -i\,\hbar\,\frac{\partial}{\partial\xi_k}\psi(\xi_k). \tag{5.74}$$

A single Fourier mode has Hamiltonian $\mathcal{H}_k$ with a time-dependent frequency $\omega_k(\eta)$; therefore the wavefunctions $\psi(\eta,\xi_k)$ will obey a time-dependent Schrödinger equation of the form

$$i\,\hbar\,\frac{\partial}{\partial\eta}\psi(\eta,\xi_k) = \frac{1}{2}\left(-\hbar^2\frac{\partial^2}{\partial\xi_k^2} + \omega_k^2(\eta)\,\xi_k^2\right)\psi(\eta,\xi_k). \tag{5.75}$$

This is the Schrödinger equation of a harmonic oscillator with time-dependent frequency. The solution to such a system can be found through the method of invariants [139, 140, 182] and is a superposition of the following normalized wavefunctions:

$$\psi_n(\eta,\xi_k) = \frac{h_n\!\left(\frac{\xi_k}{\sqrt{\hbar}\,F}\right)}{\sqrt{2^n\,n!}}\,\frac{e^{-\frac{\xi_k^2}{2\,\hbar\,F^2}}}{(\pi\,\hbar\,F^2)^{\frac{1}{4}}}\;e^{i\,\frac{F'}{2\,\hbar\,F}\,\xi_k^2}\;e^{i\,\sigma_n}, \tag{5.76}$$

where $\sigma_n = \sigma_n(\eta) = -(n+\frac{1}{2})\int F^{-2}d\eta$ is a time-dependent phase, h_n are Hermite polynomials and $F = F(\eta)$ is an auxiliary function with units of the square root of time that is solution to the following differential equation:

$$F'' + \omega_k^2\,F - F^{-3} = 0; \tag{5.77}$$

the solution to the time-independent harmonic oscillator can be easily recovered by making the substitution $F \to 1/\sqrt{\omega_k}$ and making it constant.

The Spectrum for ξ_k can then be calculated by linking its perturbations to the curvature perturbations [66], yielding

$$\mathcal{P}^{\rm std}(k) = \frac{c^2\,k^3}{4\,\pi^2}\,\frac{\langle 0|\hat{\xi}_k^2|0\rangle}{a^2\,\epsilon}\Bigg|_{-ck\eta\ll 1} \tag{5.78}$$

where $\eta \to 0^-$ corresponds to $t \to +\infty$ so that $-c\,k\,\eta \to 0$ is the large scale limit (the factor c^2 appears due to the rescaling of ξ_k performed earlier). The expectation value of $\hat{\xi}_k^2$ is computed on the vacuum state i.e. the ground state of the time-dependent oscillator; therefore I must find the expression for $\hat{\xi}_k\,\psi_n$. This can be done by constructing ladder operators for the time-dependent system: they take the form [112]

$$\hat{A}^\dagger = \frac{\frac{\hat{\xi}_k}{F} - i(F\,\hat{\pi}_k - F'\,\hat{\xi}_k)}{\sqrt{2\,\hbar}}, \qquad \hat{A}^\dagger\,\psi_n = \sqrt{n+1}\,e^{i\varphi}\,\psi_{n+1}; \tag{5.79}$$

$$\hat{A} = \frac{\frac{\hat{\xi}_k}{F} + i(F\,\hat{\pi}_k - F'\,\hat{\xi}_k)}{\sqrt{2\,\hbar}}, \qquad \hat{A}\,\psi_n = \sqrt{n}\,e^{-i\varphi}\,\psi_{n-1}; \tag{5.80}$$

from these I can derive the expressions of $\hat{\xi}_k$ and $\hat{\pi}_k$ as functions of the ladder operators, and their actions on an eigenstate ψ_n:

$$\hat{\xi}_k = \sqrt{\hbar}\,F\,\frac{\hat{A}^\dagger + \hat{A}}{\sqrt{2}}, \qquad \hat{\pi}_k = i\,\frac{\sqrt{\hbar}}{F}\,\frac{\hat{A}^\dagger - \hat{A}}{\sqrt{2}} + \sqrt{\hbar}\,F'\,\frac{\hat{A}^\dagger + \hat{A}}{\sqrt{2}}, \tag{5.81}$$

$$\hat{\xi}_k\,\psi_n = \sqrt{\hbar}\,F\left(\sqrt{\frac{n+1}{2}}\,e^{i\varphi}\,\psi_{n+1} + \sqrt{\frac{n}{2}}\,e^{-i\varphi}\,\psi_{n-1}\right), \tag{5.82}$$

$$\hat{\pi}_k\,\psi_n = i\,\frac{\sqrt{\hbar}}{F}\left(\mathcal{R}\sqrt{\frac{n+1}{2}}\,e^{i\varphi}\,\psi_{n+1} - \mathcal{R}^*\sqrt{\frac{n}{2}}\,e^{-i\varphi}\,\psi_{n-1}\right), \tag{5.83}$$

where I defined

$$\mathcal{R} = 1 - i\,F\,F', \qquad \varphi = \int \frac{d\eta}{F^2}\,. \tag{5.84}$$

Finally, I can write the single-mode Hamiltonian operator as function of the ladder operators, and find its action on a state ψ_n:

$$\begin{aligned}\hat{\mathcal{H}}_k =& \frac{\hbar}{4\,F^2}\left(\omega_k^2\,F^4\,(\hat{A}^\dagger + \hat{A})^2 - (\hat{A}^\dagger - \hat{A})^2\right) + \\ &+ \frac{\hbar}{4}\,\frac{F'}{F}\,(\hat{A}^\dagger + \hat{A})\Big(F\,F'\,(\hat{A}^\dagger + \hat{A}) + 2\,i\,(\hat{A}^\dagger - \hat{A})\Big)\,,\end{aligned} \tag{5.85}$$

$$\begin{aligned}\hat{\mathcal{H}}_k\,\psi_n =& \frac{\hbar}{4\,F^2}\,(2n+1)\Big(F^4\,\omega_k^2 + F^2\,F'^2 + 1\Big)\,\psi_n + \\ &+ \frac{\hbar}{4\,F^2}\,\sqrt{(n+1)(n+2)}\,e^{2i\varphi}\Big(F^4\,\omega_k^2 - \mathcal{R}^2\Big)\,\psi_{n+2} + \\ &+ \frac{\hbar}{4\,F^2}\,\sqrt{n(n-1)}\,e^{-2i\varphi}\Big(F^4\,\omega_k^2 - (\mathcal{R}^*)^2\Big)\,\psi_{n-2}\,.\end{aligned} \tag{5.86}$$

Obviously this expression implies that the states ψ_n are not eigenstates of the Hamiltonian operator, which was to be expected since it is explicitly time-dependent; however, by making the substitution $F \to 1/\sqrt{\omega_k}$ and making it a constant, so that $F' = 0$, $\mathcal{R} = \mathcal{R}^* = 1$, and $F^4\,\omega_k^2 = 1$, all these relations reduce to the standard formulas of the time-independent harmonic oscillator, including $\hat{\mathcal{H}}_k = \hbar\,\omega_k\,\hat{A}^\dagger\,\hat{A}$. Nevertheless, in the time-dependent system it is still possible to construct the operator

$$\hat{I} = \hbar\,\hat{A}^\dagger\,\hat{A} = \frac{\frac{\hat{\xi}_k^2}{F^2} + (F\,\hat{\pi}_k - F'\,\hat{\xi}_k)^2}{2}, \qquad \hat{I}\,\psi_n = \hbar\left(n + \frac{1}{2}\right)\psi_n, \tag{5.87}$$

which is actually the quantum version of the original invariant defined by Lewis and Riesenfeld [139, 140], and it can be used to find coherent states for the time-dependent harmonic oscillator; they reduce to the standard coherent states of the time-independent harmonic oscillator under the substitution $F = 1/\sqrt{\omega_k} =$ const. [112]. As a final comment, note that all these relations in term of the ladder operators are the same regardless if the states ψ_n are expressed in the ξ_k or the π_k polarization.

Now it is possible to compute the expectation value of $\hat{\xi}_k^2$ on the ground state. Since $\hat{\xi}_k\,\psi_0 = e^{i\varphi}\,\sqrt{\hbar}\,F\,\psi_1/\sqrt{2}$ and the wavefunctions are already normalized as $\langle\psi_n|\psi_n\rangle = \int d\xi_k\,|\psi_n|^2 = 1$, then

$$\begin{aligned}\langle 0|\hat{\xi}_k^2|0\rangle &= \int_{-\infty}^{+\infty} d\xi_k\;\psi_0^*\,\hat{\xi}_k^2\,\psi_0 = \int_{-\infty}^{+\infty} d\xi_k\;\left|\hat{\xi}_k\,\psi_0\right|^2 = \\ &= \int_{-\infty}^{+\infty} d\xi_k\;\left|\frac{\sqrt{\hbar}\,F\,\psi_1\,e^{i\varphi}}{\sqrt{2}}\right|^2 = \frac{\hbar}{2}\,F^2.\end{aligned} \tag{5.88}$$

To calculate the Spectrum only the expression of $F(\eta)$ is needed.

The solution to the auxiliary equation (5.77) can be constructed from the solutions F_1 and F_2 of the corresponding homogeneous equation:

$$F'' + \omega_k^2\,F = 0, \tag{5.89}$$

$$F_1(\eta) = \frac{1}{\sqrt{c\,k}}\left(\cos(c\,k\,\eta) - \frac{\sin(c\,k\,\eta)}{c\,k\,\eta}\right), \tag{5.90}$$

$$F_2(\eta) = \frac{1}{\sqrt{c\,k}}\left(\frac{\cos(c\,k\,\eta)}{c\,k\,\eta} + \sin(c\,k\,\eta)\right). \tag{5.91}$$

Then the function F takes the form

$$F(\eta) = \frac{1}{\mathcal{W}}\left(C_1^2\,F_1^2 + C_2^2\,F_2^2 + 2\,F_1\,F_2\sqrt{C_1^2\,C_2^2 - \mathcal{W}^2}\,\right)^{\frac{1}{2}}, \tag{5.92}$$

where C_1, C_2 are η-independent constants and $\mathcal{W}$ is the Wronskian:

$$\mathcal{W} = F_1\,F_2' - F_1'\,F_2 = 1. \tag{5.93}$$

The two constants must be set through initial conditions: the standard requirement is that at the beginning of inflation, when all the modes of astrophysical interest today have a physical wavelength smaller than the Hubble radius $c\,k/a\,H \gg 1$, the expansion of the Universe does not affect perturbations and therefore each mode behaves as a harmonic oscillator with constant frequency. Hence I impose that modes asymptotically approach Minkowskian quantum harmonic oscillators with frequency $c\,k$:

$$\lim_{-ck\eta\to\infty} F(\eta) = \frac{1}{\sqrt{c\,k}}\,; \tag{5.94}$$

this is satisfied by setting $C_1^2 = C_2^2 = 1$, so that the expression for F is

$$F(\eta) = \sqrt{\frac{1 + c^2\,k^2\,\eta^2}{c^3\,k^3\,\eta^2}}\,. \tag{5.95}$$

Then, inserting this expression into the Spectrum, taking the large scale limit $-c\,k\,\eta \ll 1$ and remembering the dependence of η on the scale factor (5.72), the final expression for the Spectrum is

$$\mathcal{P}^{\rm std}(k) = \frac{c^2\,k^3}{4\,\pi^2}\,\frac{\hbar\,F^2(\eta)}{2\,a^2\,\epsilon}\bigg|_{-ck\eta\ll 1} = \frac{\hbar}{c}\,\frac{H_s^2}{8\,\pi^2\,\epsilon}(1+c^2\,k^2\,\eta^2)\bigg|_{-ck\eta\ll 1} = \frac{\hbar}{c}\,\frac{H_s^2}{8\,\pi^2\,\epsilon}. \quad (5.96)$$

The result is the usual flat, scale-invariant Spectrum [220].

Now I will instead derive the Power Spectrum that arises from the Fourier-transformed Mukhanov-Sasaki variable ξ_k obeying the modified algebra (3.46):

$$\left[\hat{\xi}_k, \hat{\pi}_k\right] = i\,\hbar\left(1 - \frac{\mu_0^2\,t_P\,\hat{\pi}_k^2}{\hbar}\right), \quad (5.97)$$

where π_k^2 has the dimensions of an energy so I introduced the Planck constant and Planck time t_P to keep the deformation parameter μ_0 still dimensionless. Due to the modified commutator depending on π_k, it will be easier to work in the momentum polarization, i.e. with wavefunctions $\psi = \psi(\eta, \pi_k)$, as explained in Section 3.3.

By using arguments similar to those in [34, 202] and explained in (3.30), imposing that in the momentum polarization the scalar field operator acts simply differentially, the action of the multiplicative momentum operator $\hat{\pi}_k\,\psi(\pi_k) = G(\pi_k)\,\psi(\pi_k)$ can be found as

$$\frac{\mathrm{d}G}{\mathrm{d}\pi_k} = 1 - \frac{t_P}{\hbar}\,\mu_0^2\,G^2, \qquad \sqrt{\frac{\hbar}{t_P}}\,\frac{\operatorname{atanh}\left(\sqrt{\frac{t_P}{\hbar}}\,\mu_0\,G\right)}{\mu_0} = \pi_k\,; \quad (5.98)$$

therefore the action of the fundamental operators is

$$\hat{\pi}_k\,\psi = \sqrt{\frac{\hbar}{t_P}}\,\frac{\tanh\left(\sqrt{\frac{t_P}{\hbar}}\,\mu_0\,\pi_k\right)}{\mu_0}\,\psi\,, \qquad \hat{\xi}_k\,\psi = i\hbar\,\frac{\partial}{\partial\pi_k}\psi\,, \quad (5.99)$$

as expected from (3.50).

Given the action (5.99) for the modified operator $\hat{\pi}_k$, the Hamiltonian $\mathcal{H}_k$ for a single Fourier mode yields a time-dependent Schrödinger equation with a modified kinetic term:

$$i\,\hbar\,\frac{\partial\psi}{\partial\eta} = \frac{1}{2}\left(\frac{\hbar}{t_P}\,\frac{\tanh^2\left(\sqrt{\frac{t_P}{\hbar}}\,\mu_0\,\pi_k\right)}{\mu_0^2} - \hbar^2\,\omega_k^2(\eta)\,\frac{\partial^2}{\partial\pi_k^2}\right)\psi\,. \quad (5.100)$$

This partial differential equation (PDE) is quite difficult to solve, so I need to use perturbation theory and perform an expansion in powers of μ_0^2:

$$\frac{\hbar}{t_P}\,\frac{\tanh^2\left(\sqrt{\frac{t_P}{\hbar}}\,\mu_0\,\pi_k\right)}{\mu_0^2} = \pi_k^2 - \mu_0^2\,\frac{t_P}{\hbar}\,\frac{2\pi_k^4}{3} + \mathcal{O}\left(\mu_0^4\right), \quad (5.101)$$

$$\psi(\eta,\pi_k) = \psi^0(\eta,\pi_k) + \mu_0^2\,\psi^1(\eta,\pi_k) + \mathcal{O}\left(\mu_0^4\right). \quad (5.102)$$

Plugging these expansions back into the Schrödinger equation (5.100) and separating the different powers of μ_0^2, the two new PDEs for the two components ψ^0 and ψ^1 are:

$$i\,\hbar\,\frac{\partial\psi^0}{\partial\eta} = \frac{1}{2}\left(\pi_k^2 - \hbar^2\,\omega_k^2(\eta)\,\frac{\partial^2}{\partial\pi_k^2}\right)\psi^0, \tag{5.103}$$

$$i\,\hbar\,\frac{\partial\psi^1}{\partial\eta} = \frac{1}{2}\left(\pi_k^2 - \hbar^2\,\omega_k^2(\eta)\,\frac{\partial^2}{\partial\pi_k^2}\right)\psi^1 + \Sigma, \tag{5.104}$$

$$\Sigma = \Sigma(\pi_k) = -\frac{t_P}{\hbar}\,\frac{\pi_k^4}{3}\,\psi^0, \tag{5.105}$$

where Σ indicates a source term for the μ_0^2-order equation that results to be dependent on the zero-order solution.

Now, the zero-order PDE (5.103) is the Schrödinger equation of a time-dependent harmonic oscillator with standard operators, but in the momentum polarization; therefore the solution $\psi^0(\eta,\pi_k)$ is just the Fourier transform of $\psi_n(\eta,\xi_k)$. In order to derive it, I first rewrite the ξ_k solution (5.76) as

$$\psi_n(\eta,\xi_k) = \frac{h_n\!\left(\frac{\xi_k}{\sqrt{\hbar}\,F}\right)}{\sqrt{2^n\,n!}}\,\frac{e^{-\frac{\mathcal{R}\,\xi_k^2}{2\,\hbar\,F^2}}}{(\pi\,\hbar\,F^2)^{\frac{1}{4}}}\,e^{i\sigma_n}, \tag{5.106}$$

where $\mathcal{R}$ has been defined in (5.84) and depends on η. Even though this depends on time through F, $\mathcal{R}$ and σ_n, this dependence doesn't affect the implementation of a Fourier transform. Indeed, it is possible to write

$$\psi_n(\eta,\pi_k) = \int_{-\infty}^{+\infty}\frac{d\xi_k}{\sqrt{2\,\pi\,\hbar}}\;\psi_n(\eta,\xi_k)\;e^{-i\,\frac{\xi_k\,\pi_k}{\hbar}}, \tag{5.107}$$

$$\psi_n(\eta,\xi_k) = \int_{-\infty}^{+\infty}\frac{d\pi_k}{\sqrt{2\,\pi\,\hbar}}\;\psi_n(\eta,\pi_k)\;e^{i\,\frac{\xi_k\,\pi_k}{\hbar}}, \tag{5.108}$$

and then to insert the last expression inside equation (5.75). It is not too hard to see that the Fourier transform of the solution of equation (5.75) satisfies equation (5.103). Now, the expression (5.106) for $\psi_n(\eta,\xi_k)$ is just a slightly more complicated version of a Gaussian times a Hermite polynomial, so it is expected that its Fourier transform will have a similar form; indeed, by computing the integral (5.107), I obtain

$$\psi_n^0 = (-i)^n\left(\frac{\mathcal{R}^*}{\mathcal{R}}\right)^{\frac{n}{2}} h_n\!\left(\frac{\pi_k\,F}{\sqrt{\hbar}\,|\mathcal{R}|}\right)\sqrt{\frac{F}{2^n\,n!\,\mathcal{R}\,\sqrt{\pi\,\hbar}}}\;e^{-\frac{F^2\,\pi_k^2}{2\,\mathcal{R}\,\hbar}}\,e^{i\sigma_n}, \tag{5.109}$$

which is again a Hermite polynomial times a Gaussian with inverted variance; the phase term containing σ_n depends only on time and is thus unaffected, the term $(\mathcal{R}^*/\mathcal{R})^{n/2}$ normalizes the Hermite polynomials, and the factor $(-i)^n$ is needed to make the action of the ladder operators consistent. This expression is a solution of the momentum-space Schrödinger equation (5.103), is normalized and satisfies all the needed relations; besides, it again reduces to the standard momentum-space solution of the time-independent harmonic oscillator under the substitution $F \to 1/\sqrt{\omega_k}$ =const.

Looking at the first order PDE (5.104), it is the same of the zero order one but with the addition of the source term $\Sigma(\eta,\pi_k)$. In order to solve it, consider that the eigenfunctions $\psi_n^0(\eta,\pi_k)$ form a complete orthonormal basis such that $\langle\psi_{n_1}|\psi_{n_2}\rangle=\delta_{n_1,n_2}$ and any function can be expressed as a linear combination of them. Therefore I can write ψ^1 and ψ^0 as

$$\psi^0=\sum_n b_n(\eta)\,\psi_n(\eta,\pi_k)\,,\qquad \psi^1=\sum_n d_n(\eta)\,\psi_n(\eta,\pi_k)\,,\tag{5.110}$$

where $b_n(\eta)$, $d_n(\eta)$ are time-dependent coefficients; by plugging these expansions back into the first order Schrödinger equation (5.104), a recurrence relation for the coefficients is obtained, since all the eigenfunctions ψ_n^0 satisfy the zero-order equation (5.103) that corresponds to the homogeneous part of the first order one:

$$i\,\hbar\sum_n\frac{\mathrm{d}\,d_n}{\mathrm{d}\eta}\,\psi_n^0(\eta,\pi_k)=-\frac{t_P}{\hbar}\,\frac{\pi_k^4}{3}\,\sum_n b_n(\eta)\,\psi_n^0(\eta,\pi_k).\tag{5.111}$$

Considering just the ground state, the result (5.83) for π_k yields

$$\pi_k^4\,\psi_0=\frac{3\,\hbar^2}{4}\,\frac{\mathcal{R}^2\,(\mathcal{R}^*)^2}{F^4}\,\psi_0-\frac{3\,\hbar^2}{\sqrt{2}}\,\frac{\mathcal{R}^3\,\mathcal{R}^*}{F^4}\,e^{2i\varphi}\,\psi_2+\sqrt{\frac{3}{2}}\,\frac{\hbar^2\,\mathcal{R}^4}{F^4}\,e^{4i\varphi}\,\psi_4;\tag{5.112}$$

it is thus clear that, when $b_n=\delta_{0,n}$, the only non-zero coefficients on the left hand side of equation (5.111) are d_0, d_2 and d_4. Therefore the relations for these coefficients are:

$$i\,\frac{\mathrm{d}\,d_0}{\mathrm{d}\eta}=-\frac{t_P}{4}\,\frac{(1+F^2\,F'^2)^2}{F^4},\tag{5.113}$$

$$i\,\frac{\mathrm{d}\,d_2}{\mathrm{d}\eta}=+\frac{t_P}{\sqrt{2}}\,\frac{(1+F^2\,F'^2)(1-i\,F\,F')^2}{F^4}\,e^{2i\varphi},\tag{5.114}$$

$$i\,\frac{\mathrm{d}\,d_4}{\mathrm{d}\eta}=-\frac{t_P}{\sqrt{6}}\,\frac{(1-i\,F\,F')^4}{F^4}\,e^{4i\varphi}.\tag{5.115}$$

Finally, the ground state of this system in the π_k representation is

$$\psi_0^{\mathrm{tot}}(\eta,\pi_k)=\psi_0^0(\eta,\pi_k)+\mu_0^2\,\sum_{n=0}^{2}d_{2n}(\eta)\,\psi_{2n}^0(\eta,\pi_k).\tag{5.116}$$

From here on I will omit the superscript indicating the order, since I expressed ψ^1 and ψ^0 as linear combinations of $\psi_n(\eta,\pi_k)$.

In order to find the final Spectrum of perturbations I have to evaluate the expectation value $\left\langle\hat{\xi}_k^2\right\rangle$ on the ground state; the expression (5.82) allows to write

$$\begin{aligned}\left\langle\psi_0^{\mathrm{tot}}\middle|\hat{\xi}_k^2\middle|\psi_0^{\mathrm{tot}}\right\rangle&=\int d\pi_k\;\psi_0^{\mathrm{tot}*}\,\hat{\xi}_k^2\,\psi_0^{\mathrm{tot}}=\int d\pi_k\,\left|\hat{\xi}_k\,\psi_0^{\mathrm{tot}}\right|^2=\\&=\int d\pi_k\;\hbar\,F^2\left|\frac{1+\mu_0^2\,d_0}{\sqrt{2}}\,e^{i\varphi}\,\psi_1+\mu_0^2\,d_2\,e^{-i\varphi}\,\psi_1+\ldots\right|^2,\end{aligned}\tag{5.117}$$

where the dots indicate terms proportional to ψ_3 and ψ_5, whose square modulus would contribute with terms of order μ_0^4 which would be neglected. The norm of

$\psi_0^{\rm tot}$ is easily calculated to be $|\psi_0^{\rm tot}|^2 = 1 + 2\,\mu_0^2\,{\rm Re}(d_0)$, since $\int d\pi_k |\psi_n|^2 = 1$, and thus the normalized expectation value of $\hat{\xi}_k^2$ results to be

$$\begin{aligned}\frac{\left\langle \hat{\xi}_k^2 \right\rangle}{|\psi_0^{\rm tot}|^2} &= \frac{\hbar\,F^2}{2|\psi_0^{\rm tot}|^2}\left(1 + 2\,\mu_0^2\,{\rm Re}(d_0) + 2\,\sqrt{2}\,\mu_0^2\,{\rm Re}(d_2\,e^{-2i\varphi})\right) = \\ &= \frac{\hbar\,F^2}{2}\left(1 + \frac{2\,\sqrt{2}\,\mu_0^2\,{\rm Re}(d_2\,e^{-2i\varphi})}{1 + 2\,\mu_0^2\,{\rm Re}(d_0)}\right).\end{aligned} \tag{5.118}$$

As expected, the zero-order term is the same as for the standard Spectrum (5.88); on the other hand, for the μ_0^2-order correction only d_0 and d_2 are needed among the coefficients of the expansion.

Looking at equation (5.113), the right hand side is real; therefore d_0 has a purely imaginary time derivative, and its real time-independent part must be set through initial conditions. I will adopt the same prescription as in [158, 201] and assume that the wavefunction is in the instantaneous ground state at the beginning of inflation: I can therefore write $d_0(\eta_s) = 0$ and, since its real part is independent of time, it will remain zero throughout the evolution. Then I solve the integral (5.114) for $d_2(\eta)$, insert it into the Spectrum and find the asymptotic behaviour:

$$\begin{aligned}\mathcal{P}^{\rm mod}(k) &= \frac{c^2\,k^3}{4\,\pi^2}\,\frac{\hbar\,F^2}{2\,a^2\,\epsilon}\left(1 - 2\,\sqrt{2}\,\mu_0^2\,{\rm Re}(d_2\,e^{-2i\varphi})\right)\Bigg|_{-ck\eta\ll 1} = \\ &= \frac{\hbar}{c}\,\frac{H_s^2}{8\,\pi^2\,\epsilon}\left(1 - \frac{4\,t_P\,\mu_0^2}{7\,c^5\,k^5\,\eta^6}\right)\Bigg|_{-ck\eta\ll 1}.\end{aligned} \tag{5.119}$$

Now, by performing the limit $-ck\eta \to 0$ this correction term would diverge; however, inflation doesn't actually go on forever but ends at some finite instant; therefore it is possible to choose to compute the Spectrum at the value $\eta = \eta_f$ that is the end of inflation. Then I can set $\eta_f = 2\,\pi/c\,\overline{k}$ where $\overline{k}$ is a pivot scale and, by choosing the standard pivotal scale $\overline{k} = 0.002 Mpc^{-1}$ used in the analyses of CMB Spectra, the Spectrum can then be rewritten as

$$\begin{aligned}\mathcal{P}^{\rm mod}(k) &= \frac{\hbar}{c}\,\frac{H_s^2}{8\,\pi^2\,\epsilon}\left(1 - \frac{4}{7}\,\frac{c\,t_P\,\overline{k}}{(2\,\pi)^6}\,\mu_0^2\left(\frac{\overline{k}}{k}\right)^5\right) = \\ &\approx \mathcal{P}^{\rm std}\left(1 - 10^{-65}\,\mu_0^2\left(\frac{\overline{k}}{k}\right)^5\right).\end{aligned} \tag{5.120}$$

Finally, by asking that at the pivot scale $k = \overline{k}$ corrections be of order lower than 10^{-3}, I obtain a constraint on the deformation parameter:

$$\mu_0 < 10^{31}. \tag{5.121}$$

In Figure 5.9 the modified Power Spectrum (rescaled to the standard one) is shown for different values of the deformation parameter μ_0; the result is a suppression of the Spectrum for small values of k (corresponding to large scales), and the magnitude of the suppression depends on the deformation parameter μ_0.

I conclude by noting that this construction is similar to the implementation of modified dispersion relations, with the exception that those would be implemented

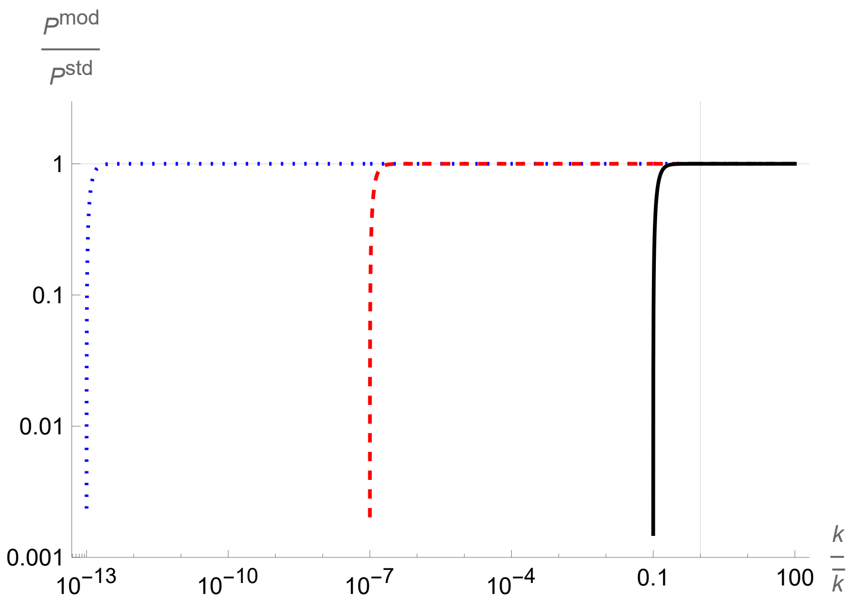

Figure 5.9. The modified Power Spectrum $\mathcal{P}^{\text{mod}}$ rescaled to the standard one $\mathcal{P}^{\text{std}}$ for $\mu_0 = 10^{30}$ (black continuous line), $\mu_0 = 10^{15}$ (red dashed line) and $\mu_0 = 1$ (blue dotted line). The pivotal scale $k = \bar{k}$ and the standard flat Spectrum are indicated by faded grey lines.

as having a different frequency ω_k, i.e. they would affect the form of the auxiliary equation (5.77), while these modifications are implemented at a more fundamental level on the commutation relations. Different forms of modified dispersion relations have been analyzed in the past, but they usually predict a red tilt of the Spectrum (either in the form of a suppression at high energies or of an infrared divergence), exotic behaviours such as oscillations in certain ranges, or no correction at all (see for example [63, 64, 69, 161]). On the other hand, computations of the Primordial Power Spectrum in Emergent Universe models obtained with different mechanisms have been shown to sometimes yield a suppression at large scale similar to the one found here, although with different magnitude and features [117, 118, 160]. Therefore this might perhaps be a general prediction of the kind of constructions that allow for an Einstein-static beginning of the Universe.

To conclude, there have been other attempts to generate an Emergent Universe scenario, but they usually have specific requirements, such as specific shapes of the potential, the presence of exotic matter or a modified continuity equation for matter (for example, see [128, 171, 172] and references therein). The relevance of the construction presented here consists in the possibility to have a finite volume limit in the distant past of the Universe without any fine-tuning of the initial conditions or any need for strange forms of matter, but just as a natural and general feature of the modified PUP algebra (3.46).

The present model has the merit to make the non-singular EU model a general feature of the isotropic Universe when a specific sector of cut-off physics is addressed. Furthermore, such a non purely classical feature of the Universe dynamics is expected to leave a specific trace on the primordial Spectrum, which could in principle be identified as a fingerprint on the temperature distribution of the microwave background.

It remains as a future research objective to determine how general the proposed scenario is, for example by considering more general cosmological models. Clearly, these cosmological frameworks can be reconciled to the isotropic late Universe by the inflationary de Sitter phase [132], whose associated Spectrum should have a corrected morphology which is expected to be similar to the one presented here.

Chapter 6

Non-Singular Gravitational Collapse

The problem of understanding the final fate of the gravitational collapse of an astrophysical object is a long standing question in literature [73, 183, 188]. In particular, the existence of the upper limits for the mass of compact stars [61, 77, 176, 215], above which the gravitational collapse is no longer contrasted by the matter pressure with the consequent formation of a black hole, constitutes one of the most outstanding and still debated results [76, 86, 113]. Indeed the observation of neutron stars with mass potentially greater than two Solar masses [12, 40, 83] opened the way to a series of conjectures concerning the possible physical explanation for this unexpected evidence, including scenarios with new physics for the gravitational field (see for instance the scalarization phenomenon in modified gravity [85, 127, 204]).

In this section I consider the collapse of a spherical dust cloud, infalling under the effect of its self-gravity, both in the Newtonian and in the fully relativistic limit. In the Newtonian limit, I adopt the representation of the spherical collapse proposed by Hunter in [119], which consists of a Lagrangian description for the dynamics of the background configuration and of an Euler formulation of the behavior characterizing small perturbations. While the background dynamics is characterized by a pressureless free fall, when studying the perturbations I will adopt a polytropic equation of state and the pressure contribution is relevant for the system stability.

In the general relativistic case, I adopt the Oppenheimer-Snyder model [175] in which the region external to the cloud is, according to the Birkhoff theorem [47, 164], a Schwarzschild spacetime, while the interior of the collapsing object is associated to a Robertson-Walker geometry with positive curvature. The two spacetime regions are then suitably matched on the boundary of the collapsing cloud. The stability of this collapse is then studied by considering the dynamics of the interior as the background, in agreement with the Lifshitz formulation of the cosmological perturbations [136, 142]. The equation of state for these perturbations has been taken in the isothermal form and the constant sound velocity is a free parameter, replacing the polytropic index of the Newtonian formulation.

The motivation of this study comes exactly from the fact that the interior metric of the cloud in the general relativistic formulation is equivalent to a closed RW geometry. Therefore, by introducing cut-off physics effect, it is expected that the singularity be removed and the collapse be stopped, similarly to a cosmological model. In particular, I will use the PUP algebra (3.46) introduced in 3.3, whose ability to implement an asymptotically static phase has been explored in [34, 37] as shown in Section 5.2. I stress that the assumption of a free falling background configuration, made both in the non-relativistic and relativistic cases, has been chosen in order to emphasize the effect of the repulsive gravity induced by cut-off physics, simply because they are not hidden here by the presence of a matter pressure contribution.

The present analysis is characterized by two main relevant results. First, it is always possible to obtain an asymptotically static configuration of the background collapse in correspondence to a radius greater than the Schwarzschild value; second, for a suitable range of the free parameters of the perturbation dynamics, the background configuration results to be stable to small perturbations. Furthermore, it is worth stressing that these two results remain valid in the limit of a very small (even sub-Planckian) value for the cut-off parameter μ_0 that characterizes the modification of the Poisson brackets. This fact suggests that the presence of a cut-off physics in the gravitational collapse constitutes an intrinsic modification of the gravitational force with respect to the standard Newtonian or Einsteinian gravity and that the singular collapse is never recovered in the modified dynamics.

In other words, by including quantum corrections in the description of a spherical dust collapse, as expected in an effective quantum gravity scenario, the resulting dynamics is always associated to the existence of a physical (super-Schwarzschild) static and, for a given range of the free parameters of the model, stable configuration. This results, and in particular the capability of cut-off physics effects to determine a macroscopic modification i.e. the stabilization of the dust collapse above the event horizon, open a new perspective in understanding the basic ingredients to fix the morphology and the final fate of astrophysical bodies. More specifically, once a real equation of state is considered and the star radial inhomogeneity properly accounted for, it could be possible to give constraints on the value of the cut-off parameter that could accommodate the observed violation of the Chandrasekhar or Tolman-Oppenheimer-Volkoff limits.

The results of this chapter are included in the paper [36].

6.1 Semiclassical Collapse in Different Formulations

In this section I first introduce the Newtonian description for the collapse of a dust cloud, developed by Hunter [119], in its Hamiltonian formulation. The Hunter model consists in a homogeneous and isotropic sphere of dust, initially at rest, collapsing under the action of its own gravity; therefore the density ρ is a function of time only and the pressure gradients are identically zero (this won't be valid anymore when studying perturbations later). Then I implement the modified algebra (3.46) on a semiclassical level to show how the singularity is removed and also derive some bounds on the deformation parameter μ_0 by requiring that the non-relativistic assumption hold.

Then I will also study the collapse from a general relativistic point of view. Therefore the starting point will be the Oppenheimer-Snyder collapse model [87, 175], for which I will present the Hamiltonian formulation and then implement on it the same PUP algebra.

Note that, similarly to Section 5.2, I will keep the necessary constants explicit in order to have the deformation parameter μ_0 dimensionless.

6.1.1 The Newtonian Hunter Model

Given spherical symmetry, it is enough to study the evolution of the radius r of the sphere; using the Newtonian gravitational potential, the Hamiltonian (actually the Hamiltonian per unit mass) results to be

$$\mathcal{H} = \frac{p_r^2}{2} - \frac{GM}{r}, \tag{6.1}$$

where p_r is the momentum conjugate to r, G is Newton's gravitational constant, and M is the total mass of the cloud. The Hamilton equations are

$$\dot{r} = \frac{\partial \mathcal{H}}{\partial p_r} = p_r, \qquad \dot{p}_r = -\frac{\partial \mathcal{H}}{\partial r} = -\frac{GM}{r^2}; \tag{6.2}$$

dividing the second equation by the first, a differential equation for $p_r(r)$ is obtained that can be integrated with the initial conditions $r = r_0$ and $\dot{r} = p_r = 0$ at $t = 0$, where r_0 is the initial radius of the cloud:

$$\frac{\partial p_r}{\partial r} = \frac{\dot{p}_r}{\dot{r}} = -\frac{GM}{r^2\, p_r}, \quad p_r(r) = \pm\sqrt{2GM\left(\frac{1}{r} - \frac{1}{r_0}\right)}. \tag{6.3}$$

Then, substituting the solution with the minus sign (since in a collapse $\dot{r} < 0$) in the equation for $\dot{r}$ and defining $a = r/r_0$ yields a solution for $a(t)$ in implicit form:

$$\dot{a} = -\sqrt{\frac{2GM}{r_0^3}\,\frac{1-a}{a}}, \tag{6.4}$$

$$\sqrt{\frac{2GM}{r_0^3}}\; t = \sqrt{a(1-a)} + \operatorname{acos}\sqrt{a}. \tag{6.5}$$

By setting $a(t_0) = 0$, the time of collapse t_0 is found to be

$$t_0 = \frac{\pi}{2}\sqrt{\frac{r_0^3}{2GM}}. \tag{6.6}$$

The solution is shown in Figure 6.1 compared with the modified non-singular solution that I will now derive.

To obtain the modified evolution, I start from the same Hamiltonian (6.1) but derive the equations of motion through the modified Poisson brackets (3.46) implemented on the variables (r, p_r):

$$\{r, p_r\} = 1 - \frac{\mu_0^2\, p_r^2}{c^2}, \qquad \dot{r} = p_r\left(1 - \frac{\mu_0^2\, p_r^2}{c^2}\right), \qquad \dot{p}_r = -\frac{GM}{r^2}\left(1 - \frac{\mu_0^2\, p_r^2}{c^2}\right), \tag{6.7}$$

where p_r has the dimensions of a velocity and therefore I inserted the speed of light c to keep μ_0 dimensionless. Now, dividing the second equation by the first yields the same relation (6.3), and substituting back into the deformed equation for $\dot{a}$ a differential equation for $a(t)$ is obtained:

$$\dot{a} = -\sqrt{\frac{2GM}{r_0^3}\,\frac{1-a}{a}}\left(1 - \frac{2GM\,\mu_0^2}{r_0\,c^2}\,\frac{1-a}{a}\right). \tag{6.8}$$

Similarly to the situations in Chapter 5, the modified algebra has introduced a critical point: the value $a_\infty < 1$ such that $\dot{a} = 0$ is found as

$$1 - c_\mu\,\frac{1-a_\infty}{a_\infty} = 0, \qquad a_\infty = \frac{c_\mu}{1+c_\mu}, \tag{6.9}$$

where I defined $c_\mu = 2GM\,\mu_0^2/c^2\,r_0$ to shorten the notation. The solution for $a(t)$ can again be found only in implicit form:

$$\begin{aligned}\sqrt{\frac{2GM}{r_0^3}}\;t = {} & \frac{\sqrt{a(1-a)}}{1+c_\mu} + \frac{1+3c_\mu}{(1+c_\mu)^2}\,\mathrm{acos}\,\sqrt{a} + \\ & + \frac{2c_\mu^{\frac{3}{2}}}{(1+c_\mu)^{\frac{5}{2}}}\sqrt{1+2b_-}\,(1+b_+)\,\mathrm{atanh}\left(\frac{\sqrt{a}-1}{\sqrt{(1-a)(1+2b_-)}}\right) + \\ & - \frac{2c_\mu^{\frac{3}{2}}}{(1+c_\mu)^{\frac{5}{2}}}\sqrt{1+2b_+}\,(1+b_-)\,\mathrm{atanh}\left(\frac{\sqrt{a}-1}{\sqrt{(1-a)(1+2b_+)}}\right),\end{aligned} \tag{6.10}$$

where $b_\pm = c_\mu \pm \sqrt{c_\mu(1+c_\mu)}$. It is trivial to see that, in the limit $\mu_0 \to 0$, also $c_\mu, b_\pm \to 0$ and the standard solution (6.5) is recovered; it is also easy to verify that, when $a = a_\infty$, the arguments of both inverse hyperbolic tangents become 1 and the right-hand-side diverges, meaning that the inverse function $a(t)$ has an horizontal asymptote such that $a \to a_\infty$ when $t \to \infty$. In Figure 6.1 the classical and the modified solutions are compared.

At this point it is possible to find some constraints on the deformation parameter μ_0 by requiring that the Newtonian description be valid. In particular, I impose that the minimum radius be much greater than the Scharzschild radius $r_S = 2GM/c^2$: the condition $a_\infty \gg a_S$ implies

$$\frac{c_\mu}{1+c_\mu} \gg a_S = \frac{r_S}{r_0}, \qquad \mu_0 \gg \sqrt{\frac{1}{1-a_S}} \tag{6.11}$$

(note that $c_\mu = a_S\,\mu_0^2$). For most objects of astrophysical interest, the Schwarzschild radius is much smaller than the physical radius; even for more compact objects such as neutron stars, the Schwarzschild radius is $a_S \lesssim 1/2$ and the square root would be smaller than $\sqrt{2} \approx 1.4$, which is still of order unity. Therefore the condition

$$\mu_0 \gg 1, \tag{6.12}$$

is enough to ensure the validity of the Newtonian description.

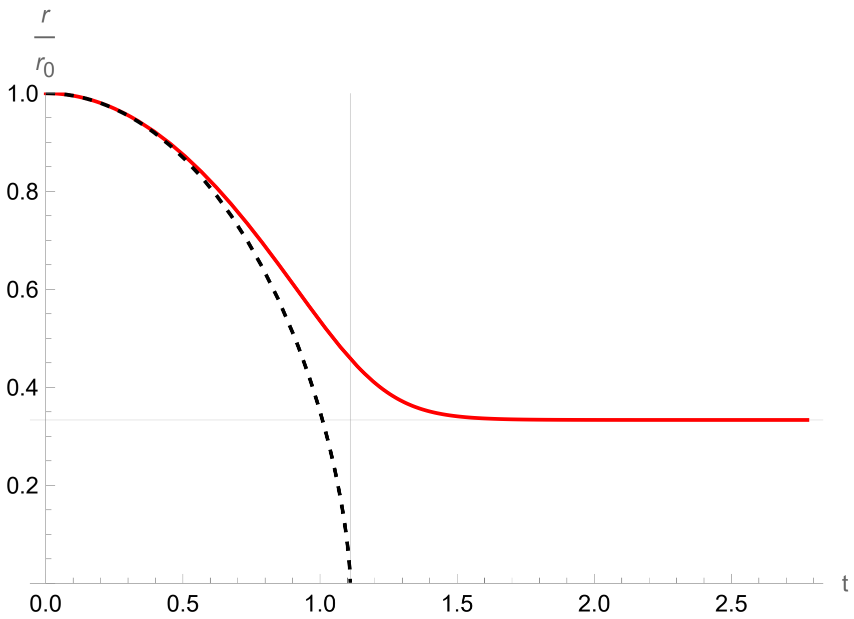

Figure 6.1. Comparison between the classical collapse (dashed black line) and the modified non-singular evolution (red continuous line) for generic values of the parameters; the time t_0 of the classical collapse and the minimum value a_∞ for the modified dynamics are highlighted by faded grey lines.

As a secondary check, I require that the maximum speed reached during the collapse be non-relativistic. The maximum speed is found by setting $\ddot{r} = 0$ and substituting in $\dot{r}$; then the requirement $\dot{r} \ll c$ yields

$$\mu_0 \gg \frac{2}{3\sqrt{3}} \approx 0.4, \tag{6.13}$$

which is slightly smaller but still of order unity. Note how this constraint, differently from the previous one, does not depend on any parameter. Therefore I conclude that, by taking the deformation parameter just a couple of orders of magnitude greater than unity, the Newtonian dynamics is still a good description for this model and the collapse stops before the formation of a horizon.

6.1.2 The Relativistic Oppenheimer-Snyder Model

The Oppenheimer-Snyder (OS) model is the simplest and most widely known model of gravitational collapse. Its importance lies in highlighting the need to consider two different observers, one stationary outside the collapsing matter and one comoving with it. The original paper [175] starts from the outside Schwarzschild metric in the standard form

$$ds^2 = \left(1 - \frac{r_S}{R}\right) c^2\, dT^2 - \frac{dR^2}{1 - \frac{r_S}{R}} - R^2 d\theta^2 - R^2 \sin^2(\theta)\, d\varphi^2, \tag{6.14}$$

where $T = T(t, r)$ and $R = R(t, r)$ are the external variables as functions of the internal t, r; then, by requiring spherical symmetry and homogeneity and implementing matching conditions on the surface r_0, the equations that the internal metric must satisfy are found and the collapse time as seen from a comoving observer can be computed. As mentioned, the internal metric results to be the same as a closed FLRW model, i.e. equation (2.5) with $K > 0$. While in the actual FLRW model the latter can always be set to ± 1 by rescaling the variables, here it can be linked to the initial parameters of the cloud both through physical arguments [110] and through a comparison of the solutions, as I will show shortly.

To obtain the Hamiltonian formulation for the OS model one starts with the ADM-reduced action S for spherically symmetric spacetimes [130, 134, 216] filled with Brown-Kuchař dust [67]; then, by implementing matching conditions between the Schwarzschild (6.14) and the FLRW (2.5) metrics and performing a partial symmetry reduction, the Hamiltonian gets split in three different contributions:

$$\begin{aligned} S = &\int dt\ (p_a\,\dot{a} + P_\tau\,\dot{\tau} - \mathcal{N}\,\mathcal{H} - M_+\,\dot{T}_+) + \\ &+ \int dt \int_{r_0}^{\infty} dr\ (P_R\,\dot{R} + P_L\,\dot{L} - \mathcal{N}^0\,\mathcal{H}_0 - \mathcal{N}^j\,\mathcal{H}_j)\,, \end{aligned} \tag{6.15}$$

$$\mathcal{H}_0 = \frac{P_L^2\,L}{2\,R^2} - \frac{P_R\,P_L}{R} + \frac{R'^2 + 2\,R\,R''}{2\,L} - \frac{L'\,R\,R'}{L^2} - \frac{L}{2}, \tag{6.16}$$

$$\mathcal{H}_j = P_R\,R' - P_L'\,L, \tag{6.17}$$

$$\mathcal{H} = -\frac{\chi}{6\,V_S}\,\frac{p_a^2}{a} - \frac{3\,V_S}{2\,\chi}\,K\,c^2\,a + P_\tau, \tag{6.18}$$

where τ is dust proper time, L and R are the functions appearing in the spherically symmetric external metric and can be obtained by comparison with (6.14), $\mathcal{N}$, $\mathcal{N}^0$ and $\mathcal{N}^j$ are Lagrange multipliers analogous to the Lapse Function and the Shift Vector introduced in Section 2.1, P_R, P_L and P_τ are the momenta conjugate to their respective variables, $M_+\,\dot{T}_+$ is a boundary term containing the ADM mass and the Schwarzschild-Killing time at asymptotic infinity, $\mathcal{H}_0$ and $\mathcal{H}_j$ are the super-Hamiltonian and super-momentum for the exterior of the dust cloud, and $\mathcal{H}$ is the Hamiltonian for the interior.

The internal Hamiltonian (6.18) is of course the one of interest: besides the already known quantities, it contains the momentum conjugate to dust proper time P_τ related to the energy density of the cloud, Einstein's constant χ, and the internal volume V_S of the sphere given by

$$V_S = \int_0^{r_0} dr\,\frac{4\,\pi\,r^2}{\sqrt{1 - K\,r^2}} = \frac{4\,\pi}{2\,K}\left(\frac{\operatorname{asin}\!\left(r_0\sqrt{K}\right)}{\sqrt{K}} - r_0\,\sqrt{1 - K\,r_0^2}\right). \tag{6.19}$$

The matching conditions imply the following identifications:

$$L = \frac{a}{\sqrt{1 - K\,r_0^2}}, \qquad R = a\,r_0; \tag{6.20}$$

then, by studying the dynamics of a in the interval $0 < a \leq 1$, the FLRW metric describes the interior of the dust cloud from the point of view of a comoving observer. After deriving the solution it will be sufficient to multiply the comoving scale factor

a by the initial radius r_0 to obtain the dynamics of the physical radius of the cloud; this is similar to the usual FLRW description where, even though the scale factor is defined only up to a constant and it is possible to rescale it in order to have $|K| = 1$, its value today is taken to be 1 in order to be able to find physical distances. In this description I will keep the scale factor as physical in order to keep the notation more compact, and only reintroduce the initial radius r_0 if needed for numerical purposes. For more information on the derivation of the action and the Hamiltonians see [74, 75, 129, 200].

Focusing on the interior Hamiltonian, the equations of motion are

$$\dot{a} = -\frac{\chi}{3\,V_S}\,\frac{p_a}{a}\,, \qquad \dot{p}_a = -\frac{\chi}{6\,V_S}\,\frac{p_a^2}{a^2} + \frac{3\,K\,c^2\,V_S}{2\,\chi}; \tag{6.21}$$

similarly to the non-relativistic case, it is useful to compute $\dot{p}_a/\dot{a}$ in order to have an easily solvable differential equation for $p_a(a)$:

$$\frac{\partial p_a}{\partial a} = \frac{p_a}{2\,a} - \frac{9}{2}\,\frac{K\,c^2\,V_S^2}{\chi^2}\,\frac{a}{p_a}\,, \qquad p_a(a) = \frac{3\,c\,V_S}{\chi}\sqrt{a\,(1-a)\,K}\,, \tag{6.22}$$

where I used the standard initial conditions $a = 1$ and $p_a = 0$ at $t = 0$. Substituting this in equation (6.21) for $\dot{a}$, the same differential equation (6.4) of the standard case is obtained, but with a different constant:

$$\dot{a} = -\sqrt{K\,c^2\,\frac{1-a}{a}}\;; \tag{6.23}$$

therefore the solution is already known and furthermore the curvature can be related to the initial parameters of the cloud:

$$\sqrt{a\,(1-a)} + \text{acos}\,\sqrt{a} = \sqrt{K}\,c\,t\,, \tag{6.24}$$

$$K = \frac{2\,GM}{c^2\,r_0^3} = \frac{r_S}{r_0^3}; \tag{6.25}$$

this is the same identification found from physical arguments in [110], where the authors find a link between the Schwarzschild and the FLRW metrics. With this identification the expression of V_S can be rewritten as

$$V_S = \frac{2\,\pi\,r_0^3}{\sqrt{a_S^3}}\left(\text{asin}\,\sqrt{a_S} - \sqrt{a_S\,(1-a_S)}\right), \tag{6.26}$$

where I have again used $a_S = r_S/r_0$.

The solution is shown later in Figure 6.2, compared with the modified solution which I will now derive.

To find the modified dynamics, I again start from the same Hamiltonian (6.18) but use the modified algebra (3.46) on a and p_a. The new equations of motion then are

$$\{a, p_a\} = 1 - \frac{\mu_0^2\,p_a^2}{\hbar^2} \tag{6.27}$$

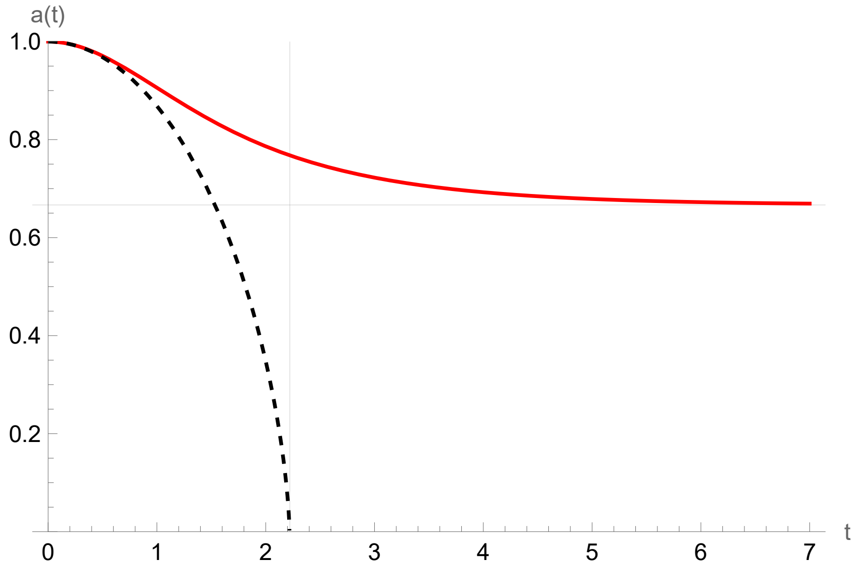

Figure 6.2. Comparison between the classical relativistic collapse (dashed black line) and the modified non-singular evolution (red continuous line) for generic values of the parameters. The collapse time t_0 and the minimum value a_∞ are highlighted by faded grey lines; it is evident how $a_\infty > 1/2$.

$$\dot{a} = -\frac{\chi}{3\,V_S}\,\frac{p_a}{a}\left(1-\frac{\mu_0^2\,p_a^2}{\hbar^2}\right), \qquad \dot{p}_a = \left(-\frac{\chi}{6\,V_S}\,\frac{p_a^2}{a^2}+\frac{3\,K\,c^2\,V_S}{2\chi}\right)\left(1-\frac{\mu_0^2\,p_a^2}{\hbar^2}\right), \tag{6.28}$$

where p_a has the dimensions of an action so I introduced a Planck constant to still have μ_0 dimensionless; dividing the second equation by the first, the same relation (6.22) is obtained, so the final differential equation for $a(t)$ becomes

$$\dot{a} = -\sqrt{K\,c^2\,\frac{1-a}{a}}\,\Big(1-g_\mu\,(1-a)\,a\Big)\,, \tag{6.29}$$

where I defined $g_\mu = K(3\,\mu_0\,c\,V_S/\hbar\,\chi)^2$. Already from here it is clear that there is still a critical point, but its expression is different from the Newtonian case:

$$1-g_\mu\,(1-a_\infty)\,a_\infty = 0, \qquad a_\infty = \frac{1}{2}\pm\frac{1}{2}\sqrt{1-\frac{4}{g_\mu}}\,. \tag{6.30}$$

First of all, in order for a_∞ to be real, the condition $g_\mu \geq 4$ must hold, otherwise there will still be the collapse $a \to 0$; this condition will imply a lower limit on the deformation parameter μ_0 as I will show later. Secondly, when that condition is satisfied, the asymptotic scale factor will obey $a_\infty \geq 1/2$ regardless of the values of the parameters (note that the solution with the minus sign will never be reached in this model, but only by starting below it with a positive derivative). Now, imposing the condition $a_\infty \gg a_S$, the following lower limits for μ_0 appear:

$$\mu_0 \gg \frac{8\,\pi\,\hbar\,\sqrt{r_S\,r_0^3}}{3\,c\,M\,V_S} = \frac{2\,\hbar}{c\,\rho_0\,V_S\,\sqrt{K}} \qquad \text{for} \qquad r_S < \frac{r_0}{2}, \tag{6.31}$$

$$\mu_0 \gg \frac{4\,\pi\,\hbar\,r_0^{\frac{5}{2}}}{3\,c\,M\,V_S\,\sqrt{r_0 - r_S}} = \frac{\hbar\,\sqrt{K}}{c\,\rho_0\,V_S\,\sqrt{a_S\,(1-a_S)}} \qquad \text{for} \qquad r_S > \frac{r_0}{2}; \tag{6.32}$$

note that when $r_S < r_0/2$, the condition $a_\infty > a_S$ is already satisfied by construction; indeed, the constraint (6.31) actually corresponds to the reality condition $g_\mu > 4$. By inserting reasonable values for the parameters, very small numbers of many orders of magnitude below unity are obtained for these lower limits (depending on the initial mass and radius, they can go from about 10^{-40} to 10^{-90}); this is due to g_μ being a very large number. It is safe to assume that the asymptotic radius of the cloud is always greater than its Schwarzschild radius as long as $\mu_0 \neq 0$.

Now, the non-singular solution can again be expressed only in implicit form:

$$\frac{\sqrt{8}\,\text{atan}\Big(\sqrt{\frac{2}{d_-}\,\frac{1-a}{a}}\Big)}{\sqrt{g_\mu\,(g_\mu-4)\,d_-}} - \frac{\sqrt{8}\,\text{atan}\Big(\sqrt{\frac{2}{d_+}\,\frac{1-a}{a}}\Big)}{\sqrt{g_\mu\,(g_\mu-4)\,d_+}} = \sqrt{K}\;c\,t, \tag{6.33}$$

where I defined $d_\pm = 2 - g_\mu \pm \sqrt{g_\mu(g_\mu - 4)}$ to shorten the notation. The solution is presented in Figure 6.2, compared with the unmodified relativistic evolution (6.24).

The existence of a stable and asymptotically static configuration of the collapse, established at a radius greater than the Schwarzschild one in both the two considered regimes, constitutes a very remarkable result. Furthermore, this feature takes place in correspondence to a sufficiently small value of the parameter accounting for the new cut-off physics. In other words, it is always possible to accommodate the stabilization of the gravitational collapse at super-Schwarzschild scales even when the deformation parameter is defined as a Planckian quantity, i.e. as regularizing physics only for very high energy scales.

There are some other recent attempts to analyze the gravitational collapse of a dust cloud through effective Loop Quantum Gravity, to which the modified PUP algebra (3.46) used in this chapter is linked through its derivation from Polymer Quantum Mechanics, and they mostly agree with the resolution of the singularity [6, 125].

6.2 Perturbations and Stability

In this section I will introduce perturbations on the backgrounds derived above, and will study the stability of the resulting object for the singular and non-singular evolution both in the Newtonian and in the relativistic frameworks.

As mentioned at the beginning of this chapter, while in the background formulation the contribution of pressure has been ignored, in the perturbed framework I will assume small pressure perturbations, that will be linked to the density perturbations through appropriate equations of state.

Note that, while the background configurations are determined by including cut-off physics effects, the evolution of the perturbations follows standard dynamics; this choice is justified by the observation that, while the background evolution is non-perturbatively sensitive to cut-off physics, the smallness of the perturbations ensures that their dynamics can be satisfactorily described via standard gravity effects.

6.2.1 Eulerian Perturbations on the Hunter Model

Still following Hunter [119], for the description of perturbations in the Newtonian model it is better to use an Eulerian representation. The system is then described by the following quantities (bold symbols represent three-dimensional vectors):

$$\mathbf{v} = (r_0\, \dot{a}\ ,\ 0\ ,\ 0), \qquad \rho = \rho_0\, a^{-3}, \qquad \Phi = -2\, G\, \pi\, \rho\, r_0^2 \left(1 - \frac{a^2}{3}\right), \tag{6.34}$$

where $\mathbf{v}$ is the velocity vector, ρ and ρ_0 are the mass density of the cloud and its initial value, and Φ is the gravitational potential. These quantities are linked by the continuity, Euler and Poisson equations [136]:

$$\dot{\rho} + \boldsymbol{\nabla} \cdot (\rho\, \mathbf{v}) = 0, \qquad \dot{\mathbf{v}} + (\mathbf{v} \cdot \boldsymbol{\nabla})\, \mathbf{v} = -\boldsymbol{\nabla}\, \Phi - \frac{\boldsymbol{\nabla} P}{\rho}, \qquad \nabla^2\, \Phi = 4\, \pi\, G\, \rho, \tag{6.35}$$

where $P = P(\rho)$ is the pressure that depends only on the density due to the barotropic assumption. Now I will perturb the quantities (6.34) to first order (higher-order corrections were investigated by Hunter later in [120, 121]) as $\mathbf{v} = \overline{\mathbf{v}} + \delta\mathbf{v}$, $\rho = \overline{\rho} + \delta\rho$, $\Phi = \overline{\Phi} + \delta\Phi$, where the unperturbed quantities (those with the overline) already satisfy equations (6.35). By substituting into the Euler equation and taking the rotor, an equation for the vorticity $\delta\mathbf{w} = \boldsymbol{\nabla} \times \delta\mathbf{v}$ is obtained:

$$\dot{\delta\mathbf{w}} = -\boldsymbol{\nabla} \times (\delta\mathbf{w} \times \mathbf{v}), \tag{6.36}$$

with solution

$$\delta\mathbf{w} = \left(\frac{w_r}{a^2} + W\ ,\ \frac{w_\theta}{a^2}\ ,\ \frac{w_\varphi}{a^2}\right), \tag{6.37}$$

where w_r, w_θ, w_ϕ and W are arbitrary functions in spherical coordinates which must satisfy $\boldsymbol{\nabla} \cdot \delta\mathbf{w} = 0$ (the divergence of a curl is identically zero in any system of coordinates); note that W can be ignored since it represents a static distribution. Substituting this result back in equations (6.35), eliminating $\delta\Phi$ and using the polytropic relation $P = \kappa\rho^\gamma$, a term involving the Laplacian of $\delta\rho$ is obtained (for more details see [72, 119]); in order to get rid of it, I will separate the variables as

$$\delta\rho(t, r_0\, a, \theta, \varphi) = \delta\varrho(t)\, F(r_0\, a, \theta, \varphi), \tag{6.38}$$

and then exploit the spherical symmetry of the problem by choosing F to be an eigenfunction of the Laplacian operator whose eigenvalues are the wavenumbers k:

$$F_{klm}(r_0\, a, \theta, \varphi) = \Big(A_{lm}\, j_l(k\, r_0\, a) + B_{lm}\, y_l(k\, r_0\, a)\Big)\mathcal{Y}_{lm}(\theta, \varphi), \tag{6.39}$$

where j_l and y_l are spherical Bessel functions of the first and second kind, A_{lm} and B_{lm} are constant coefficients, and $\mathcal{Y}_{lm}$ are spherical harmonics; this way it is possible to write $\nabla^2\delta\rho = -k^2\delta\rho$, simplify the spatial part F and obtain a differential equation just for the time dependent part of the density perturbation $\delta\varrho(t)$:

$$a^3\, \ddot{\delta\varrho} + 8\, a^2\, \dot{a}\, \dot{\delta\varrho} + \left(-4\, \pi\, \rho_0\, G + k^2\, v_s^2\, a + 12\, a\, \dot{a}^2 + 3\, a^2\, \ddot{a}\right)\, \delta\varrho = 0, \tag{6.40}$$

where $v_s^2 = \partial P/\partial\rho = \kappa\gamma\rho^{\gamma-1}$.

Until now, no reference to any solution was made. At this point I can insert the different expressions for $a(t)$ and its derivatives to obtain the solution $\delta\varrho(t)$ for the two cases.

In the classical Hunter model the expressions for the derivatives of a are

$$\dot{a} = -\sqrt{\frac{2GM}{r_0^3}\frac{1-a}{a}}\,, \qquad \ddot{a} = -\frac{GM}{r_0^3\,a^2}, \tag{6.41}$$

so the perturbation equation (6.40) becomes

$$a^3\,\ddot{\delta\varrho} + 8\,a^2\,\dot{a}\,\dot{\delta\varrho} + \left(\frac{6GM}{r_0^3}(3-4a) + k^2\,v_0^2\,a^{4-3\gamma}\right)\delta\varrho = 0, \tag{6.42}$$

where $v_0^2 = \kappa\gamma\rho_0^{\gamma-1}$. Now, since I am interested in the asymptotic behaviour near the singularity, I can take the solution (6.5) and perform an asymptotic expansion for $t \to t_0$, $a \to 0$, where t_0 is the collapse time given by equation (6.6), obtaining the explicit expression

$$a^{\text{asymp}}(t) = \left(\frac{3}{4}\pi\right)^{\frac{2}{3}}\left(1-\frac{t}{t_0}\right)^{\frac{2}{3}}. \tag{6.43}$$

After some manipulation, substituting this in equation (6.42) yields

$$y^2\,\frac{\mathrm{d}^2\delta\varrho}{\mathrm{d}y^2} - \frac{16}{3}\,y\,\frac{\mathrm{d}\delta\varrho}{\mathrm{d}y} + \left(4 + \frac{3^{\frac{2}{3}-2\gamma}\,\pi^{\frac{8}{3}-2\gamma}}{2^{\frac{13}{3}-4\gamma}}\,\frac{k^2\,v_0^2\,r_0^3}{GM}\,y^{\frac{8}{3}-2\gamma}\right)\delta\varrho = 0, \tag{6.44}$$

where $y = 1 - t/t_0$ so that the limit $t \to t_0$ corresponds to $y \to 0$; the general solution is

$$\delta\varrho(y) = A_+\,f_+(y) + A_-\,f_-(y), \tag{6.45}$$

$$f_\pm(y) = \frac{J_{\frac{\pm 5}{8-6\gamma}}\left((\alpha\,y)^{\frac{4}{3}-\gamma}\right)}{y^{\frac{13}{6}}}, \tag{6.46}$$

where $A_\pm$ are integration constants, α is a dimensionless constant containing the parameters v_0, k, r_0 and M, and J_n is the Bessel function of the first kind. The asymptotic behaviour for $1 \le \gamma < 4/3$ is

$$f_+ \sim y^{-3}, \quad f_- \sim y^{-\frac{4}{3}}, \tag{6.47}$$

while for $4/3 < \gamma \le 5/3$ it is

$$f_\pm \sim \frac{\cos\left((\alpha\,y)^{\frac{4}{3}-\gamma}\right)}{y^{\frac{17}{6}-\frac{\gamma}{2}}}; \tag{6.48}$$

remembering that in the asymptotic regime the background density scales as $\overline{\rho} \propto a^{-3} \propto y^{-2}$, the density contrast $\delta\varrho/\overline{\rho}$ will behave in the following ways:

$$\frac{\delta\varrho}{\overline{\rho}} \propto \frac{1}{y} \qquad \text{for} \qquad 1 \le \gamma < \frac{4}{3}, \tag{6.49}$$

$$\frac{\delta\varrho}{\overline{\rho}} \propto \frac{1}{y^{\frac{13}{6}}} \qquad \text{for} \qquad \gamma = \frac{4}{3}, \tag{6.50}$$

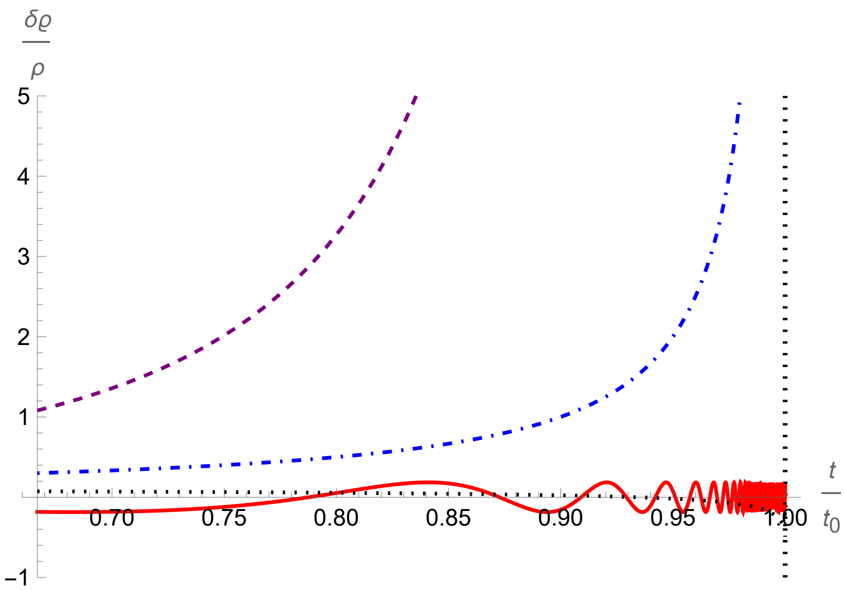

Figure 6.3. The asymptotic behaviour of the density contrast $\delta\varrho/\overline{\rho}$ for different values of the polytropic index in the classical Hunter model: $1 \leq \gamma < \frac{4}{3}$ (blue dot-dashed line), $\gamma = \frac{4}{3}$ (purple dashed line), $\frac{4}{3} < \gamma < \frac{5}{3}$ (black dotted line), $\gamma = \frac{5}{3}$ (red continuous line). The parameters have been chosen to have as initial value $\delta\varrho/\overline{\rho} \sim 10^{-1}$.

$$\frac{\delta\varrho}{\overline{\rho}} \propto \frac{\cos\left((\alpha\, y)^{\frac{4}{3}-\gamma}\right)}{y^{\frac{5}{6}-\frac{\gamma}{2}}} \qquad \text{for} \qquad \frac{4}{3} < \gamma < \frac{5}{3}, \tag{6.51}$$

$$\frac{\delta\varrho}{\overline{\rho}} \propto \cos\left(\frac{1}{\alpha\, y}\right) \qquad \text{for} \qquad \gamma = \frac{5}{3}. \tag{6.52}$$

Therefore I conclude that, except for the last case where the frequency of oscillations increases but the amplitude remains constant, the perturbations collapse faster than the background and a fragmentation process is favoured [109]. The behaviour of $\delta\varrho/\overline{\rho}$ for different values of γ is depicted in figure 6.3.

In the modified non-singular model, the expressions for the derivatives of a are different:

$$\begin{aligned} \dot{a} &= -\sqrt{\frac{2GM}{r_0^3}\,\frac{1-a}{a}}\,\left(1 - c_\mu\,\frac{1-a}{a}\right), \\ \ddot{a} &= -\frac{GM}{r_0^3\, a^2}\left(1 - c_\mu\,\frac{1-a}{a}\right)\left(1 - 3\,c_\mu\,\frac{1-a}{a}\right), \end{aligned} \tag{6.53}$$

so the differential equation for the amplitude $\delta\varrho$ of the perturbations is more complicated; however, the fact that the asymptotic behaviour at lowest order is just $a(t) \to a_\infty$ and $\dot{a}, \ddot{a} \to 0$ greatly simplifies the equation. Therefore the perturbation equation (6.40) becomes simply

$$a_\infty^3\, \ddot{\delta\varrho} + \left(k^2\, v_0^2\, a_\infty^{4-3\gamma} - \frac{3\, c^2\, a_\infty}{2\,(1-a_\infty)\, r_0^2\, \mu_0^2}\right) \delta\varrho = 0, \tag{6.54}$$

and the solution is a simple sum of two exponential functions:

$$\delta\varrho(t) = B_+\, e^{+\lambda t} + B_-\, e^{-\lambda t}, \qquad \lambda = \sqrt{\frac{3\,c^2}{2\,(1-a_\infty)\,a_\infty^2\,r_0^2\,\mu_0^2} - k^2\,v_0^2\,a_\infty^{1-3\gamma}}\,; \quad (6.55)$$

therefore the behaviour of perturbations depends entirely on the sign of the quantity in the square root. First of all, the value of v_0 corresponding to the speed of sound at the start of the collapse can be computed using a quasi-static approximation:

$$-\frac{G\,M^2}{r_0^2} + 4\,\pi\,r_0^2\,P_0 = 0, \qquad v_0^2 = \kappa\,\gamma\,\rho_0^{\gamma-1} = \gamma\,\frac{P_0}{\rho_0} = \frac{\gamma}{3}\,\frac{GM}{r_0}; \qquad (6.56)$$

then, from the expression of a_∞ the value of λ rewrites as

$$\lambda = \sqrt{\frac{3\,GM}{r_0^3\,a_\infty^3}\left(1 - \frac{\gamma}{9}\,k^2\,r_0^2\,a_\infty^{4-3\gamma}\right)}\,. \qquad (6.57)$$

Now, when $\lambda = 0$, a pivot scale k_0 is obtained that allows to rewrite λ:

$$k_0 = \frac{3}{r_0}\sqrt{\frac{a_\infty^{3\gamma-4}}{\gamma}}\,, \qquad \lambda = \sqrt{\frac{3\,GM}{r_0^3\,a_\infty^3}\left(1 - \frac{k^2}{k_0^2}\right)}\,. \qquad (6.58)$$

Therefore, for $k < k_0$, $\lambda^2 > 0$ so $\delta\varrho$ and $\delta\varrho/\overline{\rho}$ diverge while, for $k > k_0$, $\lambda^2 < 0$ so $\delta\varrho$ oscillates with constant amplitude and the density contrast $\delta\varrho/\overline{\rho}$ is ultimately damped to zero. This translates to a Jeans-like length scale $\ell_0 = 2\pi/k_0$ of the form

$$\ell_0 = \frac{2\,\pi\,r_0}{3}\sqrt{\frac{\gamma}{a_\infty^{3\gamma-4}}}\,, \qquad (6.59)$$

above which perturbation diverge and the fragmentation process is initiated while below it the perturbations are damped and erased. Note that, since $0 < a_\infty < 1$ and it is constant, for each value of the polytropic parameter γ both behaviours are possible depending only on the initial scale of the perturbation. However, for some values of γ it turns out that the scale ℓ_0 is bigger than the initial radius of the cloud, so all perturbations will disappear. Figure 6.4 shows the length scale ℓ_0 as function of the deformation parameter μ_0 for different values of γ: in order to have $\ell_0 < r_0$ and allow the fragmentation process, first of all it must be that $\gamma < 4/3$ since, for $\gamma = 4/3$, ℓ_0 does not depend on a_∞ (and therefore on μ_0) and is a constant already greater than r_0; this upper limit is further reduced by the condition $r_\infty \gg r_S$, and therefore it must be that $1 \leq \gamma < \gamma_1 < 4/3$, where γ_1 is such that $\ell_0 = r_0$ at a the value of μ_0 for which $r_\infty = r_S$.

6.2.2 General Relativistic Perturbations on the OS Model

I will now study the behaviour of density perturbations in the relativistic setting. I will mainly follow [142], meaning that I will study linear perturbations of the Einstein equations.

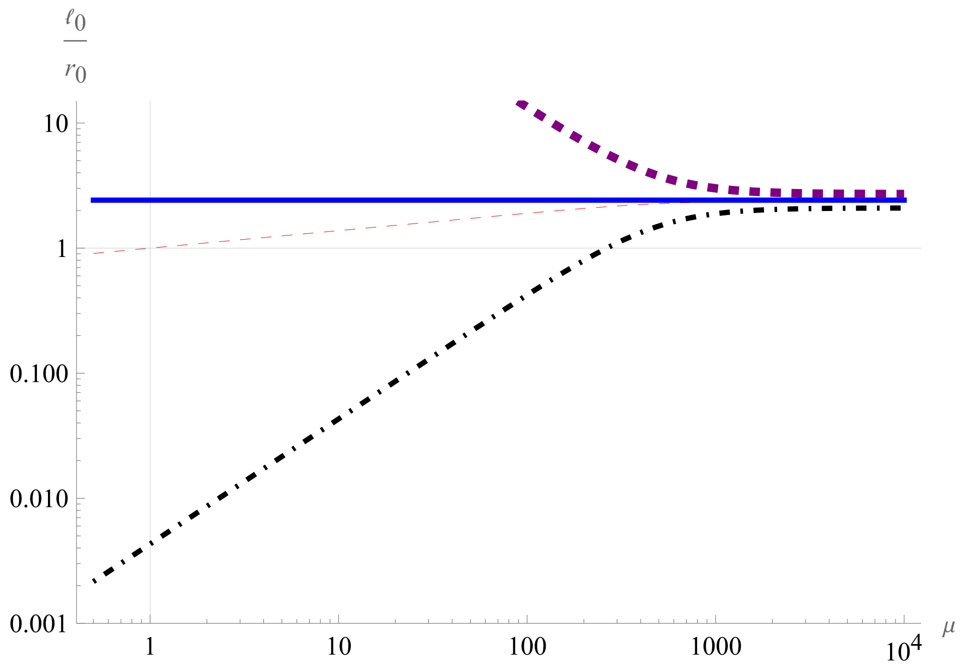

Figure 6.4. The Jeans-like length scale ℓ_0 as function of the deformation parameter μ_0 for different values of γ in the modified non-singular Hunter model. From top to bottom: $\gamma = \frac{5}{3}$ (thick purple dashed line), $\gamma = \frac{4}{3}$ (continuous blue line) for which ℓ_0 is constant, $\gamma = \gamma_1$ (thin red dashed line) that crosses the point $(1, 1)$, and $\gamma = 1$ (black dot-dashed line). The faded gray lines correspond to $\ell_0 = r_0$ and $\mu_0 = 1$.

First of all it is convenient to rewrite the FLRW metric in an easier form, introducing conformal time η and the new variable X defined as

$$d\eta = \frac{dt}{a}; \qquad dX^2 = \frac{dr^2}{1 - K\,r^2}, \quad X = \frac{\operatorname{asin}\left(\sqrt{K}\,r\right)}{\sqrt{K}}, \tag{6.60}$$

where the expression of X as function of r is valid for $K > 0$; this way, the FLRW metric inside of the cloud rewrites as

$$ds^2 = a^2(\eta)\left(c^2\,d\eta^2 - dX^2 - r^2\,d\theta^2 - r^2\,\sin^2(\theta)\,d\varphi^2\right), \tag{6.61}$$

$$r = r(X) = \frac{\sin\left(\sqrt{K}\,X\right)}{\sqrt{K}}. \tag{6.62}$$

Small perturbations are described by changes in the metric tensor, in the four-velocity and in the scalar density, parametrized as $g_{jk} = \overline{g}_{jk} + \delta g_{jk}$, $u^j = \overline{u}^j + \delta u^j$ and $\rho = \overline{\rho} + \delta\rho$. Without loss of generality, the synchronous gauge can be imposed setting $\delta g_{00} = \delta g_{0\alpha} = 0$. Note that in this case, differently from previous chapters, I will align with the notation of [142], meaning that latin indices go from 0 to 3, while greek indices refer to the spatial part and therefore go from 1 to 3. If the unperturbed system is comoving, it is possible to set $u^\alpha = 0$ and $u^0 = 1/a$, and then the unitarity of the four-velocity implies $\delta u^0 = 0$. Perturbations of the metric tensor then become perturbations of the Ricci tensor R_j^k and of the Ricci scalar R of the form

$$\delta R_\alpha^{\ \beta} = \frac{1}{2\,a^2}\left(\delta g_\alpha^{\ \gamma;\beta}{}_{;\gamma} + \delta g^\beta_{\ \gamma;\alpha}{}^{;\gamma} - \delta g_\alpha^{\ \beta;\gamma}{}_{;\gamma} - \delta g_{;\alpha}^{\ ;\beta}\right) + \\ + \frac{1}{c^2\,a^2}\left(\frac{1}{2}\,\delta g_\alpha^{\ \beta\,\prime\prime} + \frac{a'}{a}\,\delta g_\alpha^{\ \beta\,\prime} + \frac{a'}{2\,a}\delta g'\,\delta_\alpha^{\ \beta} - 2\,K\,c^2\,\delta g_\alpha^{\ \beta}\right), \tag{6.63}$$

$$\delta R_0^0 = \frac{1}{2\,a^2}\left(\delta g'' + \frac{a'}{a}\,\delta g'\right), \qquad \delta R_\alpha^0 = \frac{1}{2\,a^2}\left(\delta g'_{\ ;\alpha} - \delta g^\beta_{\ \alpha}{}'_{;\beta}\right), \tag{6.64}$$

$$\delta R = \frac{1}{a^2}\left(\delta g_\alpha^{\ \gamma;\alpha}{}_{;\gamma} - \delta g_{;\alpha}^{\ ;\alpha}\right) + \frac{1}{c^2\,a^2}\left(\delta g'' + 3\,\frac{a'}{a}\,\delta g' - \frac{2\,K\,c^2}{a}\,\delta g\right), \tag{6.65}$$

where δg is the trace of the metric perturbations, δ_α^β is Kronecker's delta, a semicolon indicates a covariant derivative and a prime corresponds to a derivative after η. On the other hand, the perturbed components of the Energy-Momentum tensor $T_j^{\ k}$ can be expressed as

$$\delta T_j^{\ k} = (P + c^2\rho)(u_j\,\delta u^k + u^k\,\delta u_j) + (\delta P + c^2\,\delta\rho)\,u_j\,u^k + \delta_j^{\ k}\,\delta P, \tag{6.66}$$

$$\delta T_\alpha^{\ \beta} = -\delta_\alpha^{\ \beta}\,\frac{\mathrm{d}P}{\mathrm{d}\rho}\,\frac{\delta T_0^{\ 0}}{c^2}, \qquad \delta T_0^{\ \alpha} = -a\,(P + c^2\,\rho)\,\delta u^\alpha, \qquad \delta T_0^{\ 0} = -c^2\,\delta\rho, \tag{6.67}$$

where I have made use of the relation $\delta P = (dP/d\rho)\delta\rho$. In the linear approximation, the perturbations satisfy the equation

$$\delta R_j^{\ k} - \frac{1}{2}\,\delta_j^{\ k}\,\delta R = \frac{\chi}{c^2}\,\delta T_j^{\ k}, \tag{6.68}$$

which yields the following equations for the perturbations of the metric:

$$\left(\delta g_\alpha^{\ \gamma;\beta}{}_{;\gamma} + \delta g^\beta_{\ \gamma;\alpha}{}^{;\gamma} - \delta g_\alpha^{\ \beta;\gamma}{}_{;\gamma} - \delta g_{;\alpha}^{\ ;\beta}\right) + \\ + \frac{1}{c^2}\left(\delta g_\alpha^{\ \beta\,\prime\prime} + 2\,\frac{a'}{a}\,\delta g_\alpha^{\ \beta\,\prime} - K\,c^2\,\delta g_\alpha^{\ \beta}\right) = 0, \tag{6.69}$$

$$\frac{1}{2}\,(\delta g_{;\gamma}^{\ ;\gamma} - \delta g_\gamma^{\ \delta;\gamma}{}_{;\delta}) - \frac{1}{c^2}\left(\delta g'' + 2\frac{a'}{a}\delta g' - K\,c^2\,\delta g\right) = \\ = \frac{3}{c^4}\,\frac{\mathrm{d}P}{\mathrm{d}\rho}\left(\frac{1}{2}\,(\delta g_\gamma^{\ \delta;\gamma}{}_{;\delta} - \delta g_{;\gamma}^{\ ;\gamma}) + \frac{a'}{a}\,\delta g' - K\,c^2\,\delta g\right). \tag{6.70}$$

Putting everything together, the final equation for the density perturbations turns out to be

$$\chi\,c^2\,\delta\rho = \frac{1}{2\,a^2}\left(\delta g_\alpha^{\ \beta;\alpha}{}_{;\beta} - \delta g_{;\alpha}^{\ ;\alpha} + \frac{2\,a'}{c^2\,a}\,\delta g' - 2K\,\delta g\right). \tag{6.71}$$

For more details on the derivation of these expressions, see [142].

Now, any perturbation in a hyperspherical geometry such as the positively-curved FLRW model can be expanded in four-dimensional spherical harmonics (similarly to the expansion in three-dimensional spherical harmonics performed in the non relativistic case in Section 6.2.1). The scalar hyperspherical harmonics Q^n can be expressed as [97]:

$$Q^n = \sum_{l=0}^{n-1}\sum_{m=-l}^{l} A^n_{lm}\,\mathcal{Y}_{lm}(\theta,\varphi)\,\Pi_{nl}(X), \tag{6.72}$$

$$\Pi_{nl} = \sin^l\left(\sqrt{K}\,X\right) \frac{\mathrm{d}^{l+1} \cos\left(n\sqrt{K}\,X\right)}{\mathrm{d}\cos\left(\sqrt{K}\,X\right)^{l+1}}, \tag{6.73}$$

where l can only go from 0 to $n-1$, A^n_{lm} are constant coefficients and $\mathcal{Y}_{lm}$ are the standard three-dimensional spherical harmonics. As an example, the most symmetric hyperspherical harmonics with $l = 0$ take the form

$$Q^n = \frac{\sin\left(n\sqrt{K}\,X\right)}{\sin\left(\sqrt{K}\,X\right)}. \tag{6.74}$$

From here on I will drop the superscript n to avoid cluttering the notation. All hyperspherical harmonics are scalar eigenfunctions of the Laplacian operator on the surface of a hypersphere with unit radius, and therefore they satisfy the relation

$$Q_{;\alpha}^{\;;\alpha} = -(n^2-1)\,Q. \tag{6.75}$$

Here the order n of the harmonics is an integer, and will play a similar role to the wave number k of the non-relativistic perturbations; it can be roughly interpreted as "the number of wavelengths that fit inside the radius of the sphere" i.e. as the ratio of the radius of the sphere to the length scale of a given perturbation. Note that there exist also vector and tensor hyperspherical harmonics, but they are not needed to study density perturbations.

Now, from the scalar harmonics Q it is possible to construct the following tensors and vectors with the following symmetries:

$$Q_\alpha^{\;\beta} = \frac{\delta_\alpha^{\;\beta}}{3}\,Q, \qquad Q_\alpha^{\;\alpha} = Q, \tag{6.76}$$

$$Z_\alpha = \frac{Q_{;\alpha}}{n^2-1}, \qquad Z_\alpha^{\;;\alpha} = -Q, \tag{6.77}$$

$$Z_\alpha^{\;\beta} = \frac{Q_{;\alpha}^{\;;\beta}}{n^2-1} + Q_\alpha^{\;\beta}, \qquad Z_\alpha^{\;\alpha} = 0. \tag{6.78}$$

Then it is possible to define

$$\delta g_\alpha^{\;\beta} = \Lambda(\eta)\,Z_\alpha^{\;\beta} + \Omega(\eta)\,Q_\alpha^{\;\beta}, \qquad \delta g = \Omega\,Q, \tag{6.79}$$

so that the whole spatial evolution is contained within the two tensors $Q_\alpha^{\;\beta}$ and $Z_\alpha^{\;\beta}$ while the time evolution i.e. the amplitude is just given by the two functions Λ and Ω. Now the equation for the density perturbations becomes

$$\chi\,c^2\,\delta\rho = \frac{Q}{3a^2}\left(K\,c^2\,(n^2-4)\,(\Lambda+\Omega) + 3\,\frac{a'}{a}\,\Omega'\right). \tag{6.80}$$

Inserting expression (6.79) into equations (6.69) and (6.70), two differential equations for the two functions Λ and Ω are obtained:

$$\Lambda'' + 2\,\frac{a'}{a}\,\Lambda' - \frac{K\,c^2}{3}\,(n^2-1)\,(\Lambda+\Omega) = 0, \tag{6.81}$$

$$\Omega'' + \left(2 + \frac{3}{c^2}\frac{\mathrm{d}P}{\mathrm{d}\rho}\right)\frac{a'}{a}\,\Omega' + \frac{K\,c^2}{3}\,(n^2-4)\,(\Lambda+\Omega)\left(1+\frac{3}{c^2}\frac{\mathrm{d}P}{\mathrm{d}\rho}\right) = 0. \tag{6.82}$$

Note that in the relativistic context it is not possible to use the polytropic relation because it is not a solution of the relativistic continuity equation; in this case I will make an isothermal assumption and leave the speed of sound $v_s^2 = dP/d\rho$ as a free constant parameter.

It is important to consider that only harmonics with $n > 2$ correspond to physical perturbations. For $n = 1, 2$ the tensor Z_α^β cannot be constructed, and therefore it is necessary to put $\Lambda = 0$; then just a second-order equation for Ω remains. When $n = 2$ both solutions for Ω can be ruled out by a transformation of the coordinates. When $n = 1$ only one of the two solutions can be ruled out by such a transformation; the second solution corresponds to a perturbation in the entire mass of the cloud, but space remains fully uniform and isotropic. Thus only $n > 2$ correspond to real physical perturbations of the metric. For more details, see [142].

Now the evolution of the perturbations is found by inserting the solutions for the scale factor a in the two cases of the singular and non-singular OS model; however they must first be reformulated in terms of the new time variable η.

I will start from the classical non-singular OS model. In order to find the expression for $a(\eta)$, it is necessary to go back to equation (6.23) and substitute $dt = a\,d\eta$, thus obtaining a differential equation in η that is easily solved:

$$a' = -\sqrt{K\,c^2\,(1-a)\,a}\,, \qquad a(\eta) = \cos^2\!\left(\frac{\sqrt{K}\,c\,\eta}{2}\right). \tag{6.83}$$

Now, equations (6.81) and (6.82) have two particular integrals that correspond to those fictitious changes in the metric that can be ruled out by a transformation of the reference system; these integrals are useful to lower the order of the two equations. They are

$$\Lambda_1 = -\Omega_1 = \text{const.}, \tag{6.84}$$

$$\Lambda_2 = -\sqrt{K}\,c\,(n^2-1)\int\frac{d\eta}{a}, \qquad \Omega_2 = \sqrt{K}\,c\,(n^2-1)\int\frac{d\eta}{a} - \frac{3\,a'}{\sqrt{K}\,c\,a^2}. \tag{6.85}$$

At this point the following change of variables from Λ, Ω to ξ, ζ can be performed:

$$\Lambda + \Omega = (\Lambda_2+\Omega_2)\,\sqrt{K}\,c\int\xi\,d\eta = -\frac{3\,a'}{a^2}\int\xi\,d\eta, \tag{6.86}$$

$$\begin{aligned}\Lambda' - \Omega' &= \sqrt{K}\,c\,(\Lambda_2' - \Omega_2')\int\xi\,d\eta + \sqrt{K}\,c\,\frac{\zeta}{a} = \\ &= \left(3\left(\frac{a''}{a^2} - 2\,\frac{a'^{\,2}}{a^3}\right) - \frac{2\,K\,c^2(n^2-1)}{a}\right)\int\xi\,d\eta + \sqrt{K}\,c\,\frac{\zeta}{a};\end{aligned} \tag{6.87}$$

this leads to two coupled first-order equations for the new functions ξ and ζ:

$$\xi' + \xi\left(\frac{2\,a''}{a'} + \frac{a'}{a}\left(\frac{3}{2\,c^2}\frac{\mathrm{d}P}{\mathrm{d}\rho} - 2\right)\right) + \frac{\sqrt{K}}{2\,c}\frac{\mathrm{d}P}{\mathrm{d}\rho}\,\zeta = 0, \tag{6.88}$$

$$\zeta' + \left(1 + \frac{3}{2\,c^2}\frac{\mathrm{d}P}{\mathrm{d}\rho}\right)\frac{a'}{a}\,\zeta + \xi\Bigg(-2\sqrt{K}\,c\,(n^2-1) + \\ + \frac{3}{\sqrt{K}\,c}\left(\frac{a''}{a} - \frac{2\,a'^{\,2}}{a^2} + \frac{3}{2\,c^2}\frac{a'^{\,2}}{a^2}\frac{\mathrm{d}P}{\mathrm{d}\rho}\right)\Bigg) = 0. \tag{6.89}$$

Now asymptotic expansions must be performed; in particular, close to the singularity, the scale factor (6.83) behaves as

$$a^{\mathrm{asymp}}(\eta) = \left(\frac{\pi}{2}\right)^2\left(1 - \frac{\eta}{\eta_0}\right)^2, \qquad \eta_0 = \frac{\pi}{\sqrt{K}\,c}, \tag{6.90}$$

where I have defined the time of singularity η_0. Then, inserting everything in equations (6.88) and (6.89) and introducing the velocity parameter $\beta = v_s/c < 1$, I obtain the following differential equations for ξ and ζ:

$$\frac{\mathrm{d}\xi}{\mathrm{d}x} + \frac{2+3\,\beta^2}{x}\,\xi - \sqrt{K}\,c\,\beta^2\,\eta_0\,\zeta = 0, \qquad \frac{\mathrm{d}\zeta}{\mathrm{d}x} + \frac{2+3\,\beta^2}{x}\,\zeta + \frac{18\,(1-\beta^2)}{\sqrt{K}\,c\,\eta_0^2\,x^2}\,\xi = 0, \tag{6.91}$$

where I defined $x = 1 - \eta/\eta_0$; the solutions are

$$\xi(x) = \frac{D_-\,x^{-\frac{\sigma}{2}} + D_+\,x^{+\frac{\sigma}{2}}}{x^{\frac{5}{2}+3\beta^2}}, \qquad \sigma = \sqrt{72\;\beta^4 - 72\;\beta^2 + 1}\,, \tag{6.92}$$

$$\zeta(x) = \frac{\sqrt{K}\,c\,\eta_0\Big(D_-\,(\sigma+1)\,x^{-\frac{\sigma}{2}} - D_+\,(\sigma-1)\,x^{+\frac{\sigma}{2}}\Big)}{36\,(1-\beta^2)\,x^{\frac{3}{2}+3\beta^2}}, \tag{6.93}$$

where $D_\pm$ are integration constants. From these, I can obtain the expressions for Λ and Ω and therefore for $\delta\rho$; remembering that $\overline{\rho} \propto a^{-3} \propto x^{-6}$, in the asymptotic limit $\eta \to \eta_0$ corresponding to $x \to 0$ the leading-order behaviour of the perturbations results to be

$$\frac{\delta\rho}{\overline{\rho}} \propto x^{-(\frac{5}{2}+3\beta^2+\frac{\sigma}{2})}. \tag{6.94}$$

Now, when $\sigma^2 > 0$ this quantity diverges in the limit $x \to 0$; on the other hand, when $\sigma^2 < 0$ the exponent is complex, but it still has a negative real part so that, even if there are some oscillations, the amplitude is still divergent close to the singularity.

I conclude that in the classical unmodified relativistic case all perturbations can diverge and initiate the fragmentation process, differently from the Newtonian case where for $\gamma = 5/3$ the amplitude remained constant.

Moving on to the modified non-singular OS model, in that case finding the expression for $a(\eta)$ is not necessary (although possible) because asymptotically the leading term is simply $a = a_\infty$ as in the Newtonian case (although obviously the expression of a_∞ is different). Therefore the density perturbations (6.80) will depend only on the sum $\Lambda + \Omega$, which can be found by summing (6.81) and (6.82) and solving the resulting equation for $\Lambda + \Omega$:

$$(\Lambda + \Omega)'' - K\,c^2\left(1 - (n^2-4)\,\beta^2\right)(\Lambda + \Omega) = 0, \tag{6.95}$$

$$\Lambda(\eta) + \Omega(\eta) = E_+ \, e^{+\nu \sqrt{K} \, c\eta} + E_- \, e^{-\nu \sqrt{K} \, c\eta}, \qquad \nu = \sqrt{1 - (n^2 - 4)\, \beta^2}\,, \quad (6.96)$$

where $E_\pm$ are constants of integration. Similarly to the Newtonian case, given that when $a' = 0$ the perturbations depend directly on the sum $\Lambda + \Omega$, the fate of the perturbations in the asymptotic limit $\eta \to \infty$ depends entirely on the nature of this parameter ν i.e. on the sign of the term inside the square root. Therefore, since only the values $n > 2$ represent physical perturbations, for each value of n there exist a critical value of β such that above it all perturbations oscillate and are ultimately damped, while below it all perturbations diverge. Conversely, for each value of β, there exist a value of n large enough such that above it all perturbations oscillate and are damped, while below it they all diverge. Given that higher values of n correspond to perturbations with shorter scales, this is again a Jeans-like length. The requirement that the collapse be stable to all perturbations must then be found for the lowest value of n; setting $n = 3$, a lower limit on β is found:

$$\beta > \beta_0 = \frac{1}{\sqrt{5}} \approx 0.45. \quad (6.97)$$

This suggests that, even if the repulsive character of the modified gravitational dynamics creates a static macroscopic configuration also when matter pressure is negligible, the request that this configuration be also stable under small perturbations still requires that the elementary constituents of the collapsing gas have a significant free-streaming effect.

To conclude, with both the Newtonian and the relativistic formulations I obtained specific ranges for the free equation of state parameters by requiring that the asymptotic configurations be stable to small perturbations. These ranges result to be wider in the asymptotic non-singular cases than in the original unmodified singular collapses, where perturbations would diverge for almost all physically acceptable values of the equation of state parameters.

The present analysis must be regarded as the starting point for subsequent investigations in which the gravitational collapse is modelled in a realistic astrophysical context, in order to better understand the implications that the repulsive gravitational dynamics can have on the formation of compact objects. In particular, the impact of the repulsive effects on the equilibrium of a real relativistic star [176] is of interest in order to determine possible corrections to the mass limits in the proposed scenarios.

Chapter 7

Summary and Outlook

In this thesis I explored the effects of alternative quantization procedures on cosmological and gravitational models, with particular focus on the quantization of different sets of variables and on the fate of singularities.

In Chapter 2 I gave a review of the Hamiltonian formulation of cosmology. First I briefly showed the basics of the reformulation of General Relativity as a $SU(2)$ gauge theory; then I derived the Hamiltonian for the isotropic Friedmann-Lemaître-Robertson-Walker model, and presented its evolution in the presence of different kinds of matter-energy. In particular, I showed how it is possible to use a scalar field to mimic various forms of matter, and that a free scalar field i.e. without a potential can be used as an internal time variable. This is particularly useful for quantization, since the fact that the gravitational Hamiltonians are constrained to zero would imply that in the standard quantum framework no evolution happens; this is known as the Problem of Time of Quantum Gravity. On a cosmological level the recovery of a time variable after quantization is made possible by the reinterpretation of the Wheeler-DeWitt equation $\hat{\mathcal{H}}\,|\psi\rangle = 0$ as a Klein-Gordon-like wave equation where the free scalar field plays the role of time. The chapter ends with an introduction to Loop Quantum Gravity and Loop Quantum Cosmology, with a detailed derivation of the different Loop quantization schemes and of how it is possible to remove the cosmological singularity both on a quantum level and with effective corrections to the classical dynamics. However, LQG and LQC are far from being complete theories of Quantum Gravity and Quantum Cosmology.

In Chapter 3 I introduced the alternative quantization procedures that I have used in my research. They were developed to introduce high-energy effects coming from more fundamental quantum gravitational theories with simple, independent frameworks and to give a somewhat phenomenological approach to derive quantum gravity corrections. The first is Polymer Quantum Mechanics, which consists in a procedure of quantization on a lattice inspired by LQG; usually a position-like variable is assigned a discrete character, and as a consequence the conjugate momentum cannot be implemented as an operator and must be regularized, resulting in an energy cut-off. The second is the Generalized Uncertainty Principle representation, inspired by String Theory; through a deformation of the canonical commutation relations, it is possible to obtain a higher-order correction to the standard Heisenberg Uncertainty Principle that implies an absolute minimal uncertainty on position.

Then I showed how it is possible to extend the GUP formulation to different kinds of deformed commutation relations; the most relevant ones for gravity and cosmology are usually functions of the momentum operator $\hat{p}$ which can have (at least) two different representations whose equivalence is still not completely clear. Then I showed the functions that I used in my research, highlighting how not all of them yield an absolute minimal uncertainty but some can introduce a momentum cut-off similarly to PQM. The usefulness of these modified algebras lies mainly in their ability to reproduce corrections from other more fundamental quantum gravitational theories, as well as in the straightforwardness of their semiclassical limit.

In Chapter 4 I implemented PQM on the isotropic FLRW model. I studied different sets of variables both from the semiclassical effective point of view and on the quantum level. In particular, I showed how PQM effective corrections are able to introduce a critical point on the evolution of the scale factor a, but the replacement of the singularity with a LQC-like Big Bounce can happen only in the presence of matter whose equation of state parameter is $w > 1/3$; besides, even in that case, the energy density at the Bounce results to be dependent on initial conditions. The requirement of the Bounce to be universal and to have a fixed critical energy density then selects the volume variable $v = a^3$ as the favoured variable in Polymer Cosmology. This makes the similarity with LQC more evident: while its original μ_0 scheme quantizes the symmetry reduced version of the Ashtekar variables corresponding to the area $s = a^2$, the improved $\overline{\mu}$ scheme performs a transformation to a volume-like variable which also in this case seems to be privileged in order to have a universal Bounce. Indeed, after a comparison between Isotropic Polymer Quantum Cosmology in different sets of variables, the last part of the chapter is dedicated to the recovery of equivalence between them and to the implication that this formulation can have for LQC.

Chapter 5 was dedicated to the cosmological implementation of Modified Algebras, starting from the anisotropic Bianchi I model expressed in Misner-like variables. I showed how on a semiclassical level both the sign and the "power" of the algebras (meaning the presence or absence of a square root) are reflected on the corresponding correction factor appearing in the modified Friedmann equations. In particular, the GUP-like algebras with a plus sign do not introduce critical points and yield modified dynamics where the singularity is still present; on the other hand, the PQM-inspired algebras with a minus sign are instead able to remove the singularity, but in different ways: while the Polymer Algebra with the square root reproduces exactly the same Big Bounce of Polymer Cosmology, the PUP algebra with a simple quadratic term introduces an asymptotic behaviour. This asymptotic behaviour was then studied more in depth: it represents a way to construct the so-called Emergent Universe model, which links the standard late-universe evolution to a non-singular, asymptotically Einstein-static beginning. Classically this behaviour can be obtained only through a specific fine tuning of the initial conditions, while with the PUP algebra it is implemented in a natural way. I also computed what corrections the PUP algebra introduces on the Primordial Power Spectrum of scalar perturbations, thus obtaining a possible phenomenological implication of this framework.

Finally, in Chapter 6 I implemented the PUP algebra on the gravitational collapse of a spherical dust cloud; this study was motivated by the fact that the internal metric of such objects is isomorphic to the closed isotropic FLRW metric with positive curvature. I studied both the Newtonian Hunter model and the relativistic Oppenheimer-Snyder model, obtaining in both cases that the collapse, which classically ends in a singularity, is halted and ends in a configuration that is asymptotic to a finite radius. Furthermore, this asymptotic radius can be above the horizon even for Planckian values of the deformation parameter in the algebra. Then I also studied the behaviour of the resulting configurations to small perturbations, finding that the non-singular backgrounds are stable for a wider range of the parameters with respect to the classical singular configurations. This could help explain the observed violations of the mass limits in astrophysical objects.

There are a few interesting remarks to make, which lead to various possibilities in which the research presented in this thesis can be continued and expanded.

To start, it is interesting how alternative quantization procedures that all introduce fundamental lengths, although in different ways, can have such different impacts on the dynamics and on the cosmological singularities. In particular, PQM introduces a fundamental length in the form of a lattice spacing, and is able to replace the Big Bang and Big Crunch singularities with a Big Bounce, while the GUP representation that introduces an absolute minimal uncertainty modifies the approach to the singularity but is not able to remove it; this might be due to a few different reasons. First of all, both procedures are implemented on the scale factor or its powers, which are comoving quantities and not physical observables; the actual reason that makes PQM able to remove the singularity is the momentum cut-off, that thanks to the Hamiltonian being constrained to zero implies a cut-off on the energy density which is an actual physical observable. In this respect, it might be interesting to implement the KMM GUP formulation on other systems, where the deformed variable could acquire a physical status and the minimal uncertainty could actually play a role in the expectation values of observables. However, on a quantum level, a minimal uncertainty does not necessarily correspond to a physical energy cut-off, and therefore it is not given that the KMM GUP could eliminate a singularity (except perhaps in a probabilistic sense, meaning that a wavepacket cannot be peaked arbitrarily and be made semiclassical) even when applied to variables representing physical observables. Finally, the existence and the nature of the Bounce in Polymer Cosmology, or anyway the nature of the corrections induced more in general by these frameworks, are clearly linked to the geometrical nature of the variable chosen to describe the model. These points must be explored further, and a way to link the different representations is needed, perhaps in the form of canonical transformations between different variables that are able to preserve the nature of the resulting cosmological dynamics.

Regarding Modified Algebras, there are still a lot of possible avenues to explore. The four examples that I studied in this thesis were motivated by the similarities between them and with other more fundamental quantum gravitational theories, such as LQG and String Theories; however some of them were explored only on a semiclassical level, and a full quantum analysis is still needed. Furthermore, as mentioned during their introduction, the equivalence between different representations for the same algebra has not been fully proven yet, but only verified for specific models on a semiclassical effective level; in order to obtain a definitive answer, it is necessary to find a way to rigorously map the different operators and their different domains. In this respect I am confident that a procedure to implement the semiclassical equivalent of the representation where the differential position operator is modified would be of great help.

Another possibility is to explore different forms of deformed algebras, corresponding to different functions. As mentioned earlier, in multidimensional spaces the Jacobi identities constrain the available forms for the algebras and can result in additional effects such as non-commutativity; however in one-dimensional systems such as most cosmological models the Jacobi identities are automatically satisfied, and therefore many functions of the momentum are available. It would be interesting to develop some kind of reconstruction method that, given effective corrections such as those found in this thesis on the Friedmann equation, is able to yield the corresponding deformed Poisson brackets; this way it could be possible to find a modified algebra for other quantum gravitational and quantum cosmological theories, and help give an idea of what to expect from those theories when applied in more complex systems.

Finally, there are many observational evidences that the observable Universe is homogeneous and isotropic, and it is well described by the FLRW model. However there is no reason to assume that it started that way, and more complicated systems such as the anisotropic Bianchi models or an inhomogeneous solution can become relevant for the Planckian era of the Universe. It would be of great interest to better characterize the influence that alternative quantization procedures such as modified algebras have for example on the chaotic behaviour of the Bianchi IX model. In this respect, an important achievement would be to compute the quantum corrections to the chaotic Belinskii-Khalatnikov-Lifshitz map due to different kinds of modified commutation relations, since this would give some insights on the fate of chaos in the Bianchi models in different quantum cosmological theories such as for example LQC and Brane Cosmology. The general character of the Bianchi models close to the classical singularity makes them the ideal arena for the analysis of quantum gravitational effects and Planck-scale physics, and their study constitutes the natural next step of implementation for alternative quantization procedures and for Quantum Gravity theories.

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