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Zero-temperature stochastic Ising model on quasi-transitive graphs

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Abstract

In this thesis, we examine the question of fixation for zero-temperature stochastic Ising model on some connected quasi-transitive graphs. The initial spin configuration is distributed according to a Bernoulli product measure with parameter $p \in (0, 1)$. Each vertex, at rate 1, changes its spin value if it disagrees with the majority of its neighbours and determines its spin value by a fair coin toss in case of a tie between the spins of its neighbours.

Depending on the graph where the process evolves and the initial density, the behavior of the model can be of three distinct types: if no vertex fixates the model is of type $\mathcal{I}$; if all vertices fixate the model is of type $\mathcal{F}$, and if there are vertices that fixate and vertices that do not, the model is called of type $\mathcal{M}$. We prove that the shrink property for the underlying graph is a necessary condition in order for the zero-temperature Ising model to be of type $\mathcal{I}$. This property requires that each finite set of vertices has at least one vertex whose neighborhood falls mostly outside of this set.

Our main result shows that if $p = 1/2$ and the graph is connected, quasi-transitive, invariant under rotations and translations, then a strenghening of the shrink property, called the planar shrink property, implies that the model is of type $\mathcal{I}$. Finally we prove that for one-dimensional translation invariant graphs, the shrink property is a necessary and sufficient condition for the model to be of type $\mathcal{I}$.

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Chapter 1

Introduction

In this thesis, we study the zero-temperature stochastic Ising model $(\sigma_t)_{t\geq 0}$ on a graph G with homogeneous ferromagnetic interactions (see e.g. [19, 29]).

The initial spin configuration is distributed according to a Bernoulli product measure with parameter $p \in (0,1)$, see e.g. [18, 28, 29]. The dynamic evolves in the following way: each vertex, at rate 1, changes its spin value if it disagrees with the majority of its neighbours and determines its spin value by a fair coin toss in case of a tie between the spins of its neighbours. This process is often referred to as *domain coarsening* or *majority dynamics* and it is sometimes used as an opinion model.

A question of particular relevance is whether for each vertex v its spin flips only finitely many times almost surely, i.e. in other words whether σ_t has an almost sure limit. We say that a vertex v *fixates* if the spin at v flips only finitely many times. According to the classification given in [19], a model is of type $\mathcal{I}$ if no site fixates almost surely, i.e all sites flip infinitely often a.s.; a model is of type $\mathcal{F}$ if all sites fixate almost surely, i.e. all sites flip only finitely many times a.s. and it is said of type $\mathcal{M}$ if there are both vertices that fixate and vertices that do not fixate almost surely. Whether a model is type $\mathcal{I}$, $\mathcal{F}$, or $\mathcal{M}$ depends on the initial configuration and on the structure of the underlying graph G.

The literature, in order to investigate this question, in the early years focused on the case $G = \mathbb{Z}^d$ and mainly with $d = 2$. It is known that the zero-temperature stochastic Ising model on $\mathbb{Z}$ with homogeneous ferromagnetic interactions is of type $\mathcal{I}$ for any initial density $p \in (0,1)$ (see [1, 29]).

The disordered model on $\mathbb{Z}^d$, if the interactions $J_{x,y}$ are independent random variables with continuous distribution, is of type $\mathcal{F}$ (see [15, 29]). Moreover, in $d = 2$ and in the case of the homogeneous ferromagnet, the model is of type $\mathcal{I}$ (see [29]). An important consequence of the methods used in [29] is that σ_t has an almost sure limit (i.e. the model is of type $\mathcal{F}$) if G

has all vertices with odd degree, such as for example the hexagonal lattice and the homogeneous tree of degree K with K odd. In [19], an analysis of the zero-temperature stochastic Ising model on $\mathbb{Z}^d$ with nearest-neighbour interactions distributed according to a measure μ_J (disordered model) is performed. In particular, it is proved that if the interactions are i.i.d. taking only the values $\pm J$ then in $d = 2$ the model is of type $\mathcal{M}$. An analogous result for $d > 2$ with a temperature fast decreasing to zero is obtained in [7]. On the cubic lattice $\mathbb{Z}^d$, if the initial configuration is distributed according to a Bernoulli product measure with parameter p sufficiently close to 1 (i.e. if $p > p_d^\star$), then the model is of type $\mathcal{F}$, in particular each vertex fixates at the value $+1$ (see [18]). Moreover in [28] it is shown that $p_d^\star \to 1/2$ as $d \to \infty$. For homogeneous trees of degree at least 3 and p sufficiently close to 1, it has been shown that the model is of type $\mathcal{F}$ (see [6, 17]).

In [10, 12], the case in which one or infinitely many vertices are frozen, i.e. their spins are not allowed to flip, is studied. The main result of the first paper is that for $d = 2$ the model, with infinitely many frozen vertices, is of type $\mathcal{F}$. On the contrary, in the second paper the authors show that the model in $d = 2$ is of type $\mathcal{I}$ when only one spin is frozen.

For articles on the stochastic Ising model on graphs other than $\mathbb{Z}^d$ see for example [5, 7, 8, 11, 14, 20, 21]; in particular in [5] it is shown that the zero-temperature Ising model on the hexagonal lattice is of type $\mathcal{F}$ and in [7] it is proved that it is not of type $\mathcal{F}$ if simultaneous spin flips are allowed. Inspired by the fact that on $\mathbb{Z}^d$ for $d = 2$ the model is of type $\mathcal{I}$ and that the discussion for $d = 3$ is an open problem, in [11] the authors studied zero-temperature Ising Glauber dynamics on $2D$ slabs of thickness $k \geq 2$, i.e. on $S_k = \mathbb{Z}^2 \times \{0, \ldots, k-1\}$, with free or periodic boundary conditions in the third coordinate. They proved that the model is of type $\mathcal{F}$ if $k = 2$ under free boundary conditions and for $k = 2$ or $k = 3$ under periodic boundary conditions; for thicker slabs they proved that the model is of type $\mathcal{M}$. In [20] the authors studied the Dilute Curie–Weiss Model, i.e. the Ising Model on a dense Erdős Rényi random graph, and proved that depending on the distribution of interactions there are different behaviors.

The graphs G we consider are mainly connected planar quasi-transitive graphs. These graphs will be defined in Chapter 2. The quasi-transivity of the graph will be given by the invariance under translations and rotations. We will show that, under mild assumptions, the only rotations to consider are those of an angle $\theta \in \left\{\frac{\pi}{3}, \frac{\pi}{2}, \frac{2}{3}\pi, \pi\right\}$ (see Lemma 2.1 and Proposition 2.1). For reasons that will become clear in the following, we do not deal with $\theta = \pi$. Such a class of graphs includes, for instance, the square, the triangular and the hexagonal lattice.

Our first result (Theorem 3.1) shows that a necessary condition for the model to be of type $\mathcal{I}$ is that underlying graph has the *shrink property.* This property, as we will see, requires that each finite set of vertices has at least one vertex whose neighborhood falls mostly outside of this set. Indeed, for example, the hexagonal lattice does not have the shrink property and the model on the hexagonal lattice is not of type $\mathcal{I}$ (it is of type $\mathcal{F}$) (see [29]). Thus, we will focus on a class of graphs having also the shrink property. Actually, for technical reasons, we will use a potentially stronger definition of the shrink property that is the *planar shrink property.*

Our main result (Theorem 3.2) shows that if $p = 1/2$ and the graph is invariant under rotations, translations and has the planar shrink property, then the model is of type $\mathcal{I}$.

Here we briefly present the general strategy to prove this achievement. First we show two preliminary results on general attractive spin systems with initial density $p \in (0, 1]$ (see Lemmas 1.1-1.2 in Section 1.1). More precisely we show that, for an attractive system, if a spin fixates to $+1$ with positive probability then the probability that it is constantly equal to $+1$ for all times $t \in [0, \infty)$ is positive. After this general analysis, we specifically study the zero-temperature stochastic Ising model. First we show that, under the shrink property and the translation-ergodicity, the cardinality of any cluster grows to infinity almost surely (Proposition 4.1). By this preliminary result, we are able to show that the cluster at the origin will intersect the boundary of any finite region infinitely often almost surely. As already mentioned, we consider a planar graph that is invariant under translations and a rotation of $\theta \in \left\{\frac{\pi}{3}, \frac{\pi}{2}, \frac{2}{3}\pi\right\}$. Then, we construct a planar regular region centered at the origin that has the same rotation invariance of the graph. By the FKG inequality and the rotation invariance of the region, the cluster in the origin will intersect all sides of the regular region with a positive probability larger or equal to a quantity, denoted by p_{cross}. We stress that, for t growing to infinity, p_{cross} does not depend on the size of the region. By these properties and by the previous results, we show that any ball centered in the origin has its spins equal to $+1$ infinitely often with a probability larger of p_{cross} (see Lemmas 4.3-4.7). Thus, with probability at least p_{cross} no site will be able to fixate at the value -1. Finally, by considering the initial density $p = 1/2$ and by Lemma 1.2 and Lemmas 4.2-4.7, we show that all sites flip infinitely often almost surely (see Theorem 3.2).

We emphasize again that, by Theorem 3.1, a necessary condition for the model to be of type $\mathcal{I}$ is that underlying graph has the shrink property. If the underlying graph G is planar and invariant under rotations and translations, in order to have a sufficient condition we have to introduce a potentially stronger property, the planar shrink property. Our last result (Theorem 3.3) shows that for one-dimensional translation invariant graphs, the shrink property is a necessary and sufficient condition for the model to be of type $\mathcal{I}$.

The set of results, presented in this thesis, was done in collaboration with my advisor E. De Santis (see [13]). The plan of the thesis is the following. In Section 1.1, we define the Markov process by the infinitesimal generator and by the Harris' graphical representation. Moreover, we state two general lemmas (Lemma 1.1 and Lemma 1.2) for Glauber attractive dynamics. In Section 1.2, we describe in detail the zero-temperature stochastic Ising model $I(G, p)$, where G is the underlying graph and p is the initial density. In Chapter 2, we introduce some notation on graphs, define the collection of graphs in which we are interested in and present an infinite class of graphs having the planar shrink property (see Proposition 2.3). We also provide examples of graphs that have and do not have the shrink property, cases where the Ising model is of type $\mathcal{I}$ in the first case, and of type either $\mathcal{M}$ or $\mathcal{F}$, in the latter. In Chapter 3, we present Theorem 3.1, which, as already said, provides a necessary condition for the model to be of type $\mathcal{I}$. Moreover, we state the two main results of the thesis, Theorem 3.2 and Theorem 3.3. In Chapter 4, Theorem 3.2 is proved through some lemmas. In Chapter 5, we provide the ideas for the proof of Theorem 3.3 that, as we will see, is similar to that of the Theorem 3.2.

1.1 Attractive spin systems

We now introduce the spin systems referring mainly to [26, Chapter 3]. We consider a spin system $(\sigma_t)_{t\geq 0}$, which describes ± 1 spin flips dynamics on a countable set of vertices V. The state space is $\Sigma = \{+1, -1\}^V$. The value of the spin at vertex $v \in V$ at time t will be denoted by $\sigma_t(v)$. We introduce the usual order relation $\leq$ on Σ: given two configurations $\sigma, \sigma' \in \Sigma$, we say that $\sigma \leq \sigma'$ if for each $v \in V$, $\sigma(v) \leq \sigma'(v)$. The system evolves as a Markov process on the state space Σ with infinitesimal generator L_t, which acts on local functions f, and defined as

$$(L_t f)(\sigma) = \sum_{v \in V} c_t(v, \sigma)\big(f(\sigma^v) - f(\sigma)\big), \tag{1.1}$$

where $t \geq 0$, $c_t(v, \sigma)$ is the flip rate of the spin at vertex v, and σ^v is defined in the following way:

$$\sigma^v(u) = \begin{cases} \sigma(u) & \text{if } u \neq v \\ -\sigma(u) & \text{if } u = v. \end{cases}$$

We assume that $c_t(v,\sigma)$ is a uniformly bounded non-negative function, which is continuous on σ and satisfies the condition

$$\sup_{v\in V}\sum_{w\in V}\sup_{\sigma\in\Sigma}|c_t(v,\sigma)-c_t(v,\sigma^w)|<\infty, \tag{1.2}$$

where the behavior on t of the rates is arbitrary. The condition in (1.2) guarantees the existence of the Markov process with infinitesimal generator L_t and determined by the flip rates $c_t(v,\sigma)$ (see in detail [26, p.122-123]). Moreover the condition in (1.2) implies that this spin system is a Feller process (see in detail [26, p.122-123] and the theorems referred to therein). We take the process $(\sigma_t)_{t\geq 0}$ right continuous.

We say that a spin system is *attractive* if $c_t(v,\sigma)$ is increasing in σ when $\sigma(v)=-1$ and decreasing in σ when $\sigma(v)=+1$. We are in particular interested to study Glauber dynamics, for which the relation

$$c_t(v,\sigma)=1-c_t(v,\sigma^v) \tag{1.3}$$

holds for each $v\in V$, $\sigma\in\Sigma$ and $t\geq 0$. If the relation (1.3) holds, then $0\leq c_t(v,\sigma)\leq 1$. We write the flip rates in the form

$$c_t(v,\sigma)=\tilde{c}_t(v,(\sigma(u))_{u\in A_v}), \tag{1.4}$$

where A_v is a subset of V. Under assumptions $\sup_{v\in V}|A_v|<\infty$ and (1.3), the process defined in (1.1) can be constructed by the Harris' graphical representation, which we now describe (see e.g. [22, 24, 25, 26]). We consider a collection $(\mathcal{P}_v)_{v\in V}$ of independent Poisson processes with rate 1 interpreted as counting processes. For each $v\in V$, let $\mathcal{T}_v=(\tau_{v,n}:n\in\mathbb{N})$ be the ordered sequence of arrivals of the Poisson process $\mathcal{P}_v$, associated with the vertex v. The probability that there is a flip at vertex v at time t (conditioning on the event $\{t\in\mathcal{T}_v\}$) is equal to $c_t(v,\sigma_{t^-})$, where $\sigma_{t^-}:=\lim_{s\to t^-}\sigma_s$. For convenience, to describe these events in more detail, we can use a family of i.i.d. random variables $(U_{v,n}:v\in V,n\in\mathbb{N})$ distributed according to a uniform random variable in $[0,1]$ and such that if $U_{v,n}<c_{\tau_{v,n}}(v,\sigma_{\tau_{v,n}^-})$, then the spin at v flips at time $\tau_{v,n}$ (see [24] and [26]).

Lemma 1.1 is well known, but we present a proof in order to construct the coupling that will be used in the proof of Lemma 1.2.

Lemma 1.1. *Given two Glauber attractive dynamics $(\sigma_t)_{t\geq 0}$ and $(\sigma'_t)_{t\geq 0}$ having the same generator and such that $\sigma_0\leq\sigma'_0$, there exists a coupling such that $\sigma_t\leq\sigma'_t$ for each $t\geq 0$.*

Proof. By hypothesis the order relation is satisfied at the initial time. Hence, it is sufficient to consider a single arrival of the Poisson process, i.e. only a

possible spin flip, in order to show that the order relation is maintained.
Let us explicitly construct the desired coupling. We use the same Poisson processes for the two systems, but different families of i.i.d. uniform random variables $(U_{v,n} : v \in V, n \in \mathbb{N})$ and $(U'_{v,n} : v \in V, n \in \mathbb{N})$ for $(\sigma_t)_{t\geq 0}$ and $(\sigma'_t)_{t\geq 0}$ respectively. For each $v, v' \in V$ and $n \in \mathbb{N}$, let us consider the stopping time $\tau_{v,n}$. Moreover, for $v' \in V \setminus \{v\}$ let us define

$$N(n, v, v') := \sup\{\ell \in \mathbb{N} : \tau_{v',\ell} \leq \tau_{v,n}\}.$$

We notice that $N(n, v, v) = n$, moreover for $v' \neq v$ one has $\tau_{v',N(n,v,v')} < \tau_{v,n}$ almost surely.

If $\sigma_{\tau^-_{v,n}}(v) = \sigma'_{\tau^-_{v,n}}(v)$ then we define $U'_{v,n} = U_{v,n}$, if $\sigma_{\tau^-_{v,n}}(v) = -1 < \sigma'_{\tau^-_{v,n}}(v) = +1$ then we define $U'_{v,n} = 1 - U_{v,n}$. Hence, the random variable $U'_{v,n}$ is a function of

$$(U_{v,1}, \ldots, U_{v,n-1}) \tag{1.5}$$

and of the independent sequences $(U_{v',\ell} : v' \in V \setminus \{v\}, \ell \leq N(n, v, v'))$ and $(\mathcal{P}_{v'}(t) : v' \in V, t \leq \tau_{v,n})$.

If $U'_{v,n} = U_{v,n}$, by construction, $U'_{v,n}$ is independent from $U'_{v,1}, \ldots, U'_{v,n-1}$, which altogether are functions of $U_{v,1}, \ldots, U_{v,n-2}$, of $\{U_{v',\ell} : v' \in V \setminus \{v\}, \ell \leq N(n-1, v, v')\}$ and $(\mathcal{P}_{v'}(t) : v' \in V, t \leq \tau_{v,n-1})$. Otherwise, if $U'_{v,n} = 1 - U_{v,n}$ independence follows by:

$$\begin{aligned}
&\mathbb{P}(U'_{v,n} \in [a, b] | U_{v,1} = u_1, \ldots, U_{v,n-1} = u_{n-1}) = \\
&\qquad = \mathbb{P}(U_{v,n} \in [1 - b, 1 - a] | U_{v,1} = u_1, \ldots, U_{v,n-1} = u_{n-1}) = \\
&\qquad\qquad = \mathbb{P}(U_{v,n} \in [1 - b, 1 - a]) = b - a = \mathbb{P}(U'_{v,n} \in [a, b]),
\end{aligned}$$

where $0 < a < b < 1$ and $u_1, \ldots u_{n-1} \in (0, 1)$. This implies that the distribution of $U'_{v,n}$ and the conditional distribution of $U'_{v,n}$ given $U'_{v,1}, \ldots, U'_{v,n-1}$ coincide, hence $U'_{v,n}$ is independent from $U'_{v,1}, \ldots, U'_{v,n-1}$ and $U'_{v,n}$ is a uniform random variable on $[0, 1]$. The independence of different sequences of uniform random variables can be proved in a similar way.

Whenever there is an arrival of a Poisson process, for example at time t for a vertex v (i.e. $t \in \mathcal{T}_v$), the following situations can arise:

Case $\sigma_{\tau^-_{v,n}}(v) = \sigma'_{\tau^-_{v,n}}(v) = -1$.
Since $U_{v,n} = U'_{v,n}$ and $c_{\tau_{v,n}}(v, \sigma_{\tau^-_{v,n}})$ is increasing in σ, we have the following three situations: if $U_{v,n} < c_{\tau_{v,n}}(v, \sigma_{\tau^-_{v,n}}) \leq c_{\tau_{v,n}}(v, \sigma'_{\tau^-_{v,n}})$ then both systems change the spin value at v; if $c_{\tau_{v,n}}(v, \sigma_{\tau^-_{v,n}}) \leq U_{v,n} < c_{\tau_{v,n}}(v, \sigma'_{\tau^-_{v,n}})$ then in the system $(\sigma_t)_{t\geq 0}$ the spin at v does not change its value, while in $(\sigma'_t)_{t\geq 0}$ the spin flip occurs at v; if $c_{\tau_{v,n}}(v, \sigma_{\tau^-_{v,n}}) \leq c_{\tau_{v,n}}(v, \sigma'_{\tau^-_{v,n}}) \leq U_{v,n}$ then both

systems do not have the spin flip at v. In all these three situations the order relation is maintained.

Case $\sigma_{\tau_{v,n}^-}(v) = \sigma'_{\tau_{v,n}^-}(v) = +1$.
Since $U_{v,n} = U'_{v,n}$ and $c_{\tau_{v,n}}(v, \sigma_{\tau_{v,n}^-})$ is decreasing in σ, we have the following three situations: if $U_{v,n} < c_{\tau_{v,n}}(v, \sigma'_{\tau_{v,n}^-}) \le c_{\tau_{v,n}}(v, \sigma_{\tau_{v,n}^-})$ then both systems change the spin value at v; if $c_{\tau_{v,n}}(v, \sigma'_{\tau_{v,n}^-}) \le U_{v,n} < c_{\tau_{v,n}}(v, \sigma_{\tau_{v,n}^-})$ then in the system $(\sigma_t)_{t\ge 0}$ the spin at v change its value, while in $(\sigma'_t)_{t\ge 0}$ the spin flip does not occur at v; if $c_{\tau_{v,n}}(v, \sigma'_{\tau_{v,n}^-}) \le c_{\tau_{v,n}}(v, \sigma_{\tau_{v,n}^-}) \le U_{v,n}$ then both systems do not have the spin flip at v. In all these three situations the order relation is maintained.

Case $\sigma_{\tau_{v,n}^-}(v) = -1 < \sigma'_{\tau_{v,n}^-}(v) = +1$.
If $U_{v,n} < c_{\tau_{v,n}}(v, \sigma_{\tau_{v,n}^-})$ then, since in this case $U'_{v,n} = 1 - U_{v,n}$, by using the relation (1.3) for Glauber dynamics and by attractivity, one has that

$$U'_{v,n} = 1 - U_{v,n} > 1 - c_{\tau_{v,n}}(v, \sigma_{\tau_{v,n}^-}) = c_{\tau_{v,n}}(v, \sigma^v_{\tau_{v,n}^-}) \ge c_{\tau_{v,n}}(v, \sigma'_{\tau_{v,n}^-}).$$

Thus, in the system $(\sigma_t)_{t\ge 0}$ the spin at v changes its value, while in $(\sigma'_t)_{t\ge 0}$ the spin flip does not occur at v, maintaining the order relation. If $U_{v,n} \ge c_{\tau_{v,n}}(v, \sigma_{\tau_{v,n}^-})$, the spin at v in $(\sigma_t)_{t\ge 0}$ does not change and therefore the order relation is maintained.

By previous cases and since $\sigma_0 \le \sigma'_0$, one deduces that the order relation is maintained at any time. Hence $\sigma_{\tau_{v,n}^-}(v) = +1$ and $\sigma'_{\tau_{v,n}^-}(v) = -1$ does not occur. $\square$

Now, we give the following definition.

Definition 1.1. *We say that a vertex v* fixates *if the spin at v flips only finitely many times and we say that a vertex* fixates from time zero *if its spin never flips.*

In the following Lemma 1.2, for Glauber attractive dynamics, we compare the probability that a spin fixates or fixates from time zero.

Lemma 1.2. *Consider a Glauber attractive dynamics $(\sigma_t)_{t\ge 0}$ where σ_0 has density $p \in (0, 1]$. If a vertex w fixates at the value $+1$ with positive probability, then the vertex w fixates at the value $+1$ from time zero with positive probability.*

Proof. We define $T_w := \inf\{s \ge 0 : \sigma_t(w) = +1 \ \ \forall t \ge s\}$, where $\inf \emptyset = +\infty$. We assume that $\mathbb{P}(T_w < \infty) = \rho > 0$ and choose $\bar{t}$ such that $\mathbb{P}(T_w < \bar{t}) \ge \rho/2$.

We consider a spin system $(\sigma_t)_{t\geq 0}$, described through the Harris' graphical representation with the independent Poisson processes of rate 1 $(\mathcal{P}_v : v \in V)$ and the i.i.d. uniform random variables $(U_{v,n} : v \in V, n \in \mathbb{N})$, with initial configuration σ_0 distributed according to a Bernoulli product measure with parameter p. Now, we construct another system $(\sigma'_t)_{t\geq 0}$ with the same distribution. We make a resampling (indipendently by all other random variables already introduced) of the spin at vertex w in the initial configuration, such that

$$\sigma'_0(w) = \begin{cases} +1 & \text{with probability } p \\ -1 & \text{with probability } 1-p, \end{cases}$$

and $\sigma'_0(u) = \sigma_0(u)$, for each $u \neq w$.

We define a new Poisson process $\mathcal{P}'_w$ that after time $\bar{t}$ has the same arrivals of $\mathcal{P}_w$. In the interval $[0, \bar{t}]$, $\mathcal{P}'_w$ is a Poisson process of rate 1 indipendent by $\mathcal{P}_w$. This new process, by independent increments property, is still a Poisson process of rate 1. With positive probability one has $\mathcal{P}'_w(\bar{t}) = 0$. Thus, by independence, with probability at least $\frac{\varrho}{2} p e^{-\bar{t}}$, the following three events occur:

$$\{T_w < \bar{t}\}, \quad \{\sigma'_0(w) = +1\}, \quad \{\mathcal{P}'_w(\bar{t}) = 0\}. \tag{1.6}$$

Whenever these three independent events occur, we define $(U'_{v,n} : v \in V, n \in \mathbb{N})$ as follows:

- for $v \neq w$, $U'_{v,n} = U_{v,n}$ when $\sigma_{\tau^-_{v,n}}(v) = \sigma'_{\tau^-_{v,n}}(v)$, otherwise $U'_{v,n} = 1 - U_{v,n}$;
- for $v = w$, $U'_{w,n} = U_{w,n+\mathcal{P}_w(\bar{t})}$ when $\sigma_{\tau^-_{w,n+\mathcal{P}_w(\bar{t})}}(w) = \sigma'_{\tau^-_{w,n}}(w)$, otherwise $U'_{w,n} = 1 - U_{w,n+\mathcal{P}_w(\bar{t})}$.

If one of the three events in (1.6) does not occur, the uniform random variables $(U'_{v,n} : v \in V, n \in \mathbb{N})$ will be defined as $U'_{v,n} = U_{v,n}$ for each $v \in V$ and $n \in \mathbb{N}$.

Now, we suppose that the three events in (1.6) occur. By construction, we have that $\sigma_0 \leq \sigma'_0$. We show that for $t \in [0, \bar{t}]$, one has $\sigma_t \leq \sigma'_t$. Since $\mathcal{P}'_w(\bar{t}) = 0$, then in the process $(\sigma'_t)_{t\geq 0}$ the spin at w remains equal to $+1$ until time $\bar{t}$; hence $\sigma_t(w) \leq \sigma'_t(w)$ for all $t \leq \bar{t}$. When there is an arrival of a Poisson process $\mathcal{P}_v$ with $v \neq w$, by using the same coupling of Lemma 1.1, it follows that the desired order relation is maintained until time $\bar{t}$. In particular $\sigma_{\bar{t}} \leq \sigma'_{\bar{t}}$. Now, applying the result of Lemma 1.1 by considering $\bar{t}$ as initial time, it follows that $\sigma_t \leq \sigma'_t$ for each $t \geq 0$. Hence, the vertex w fixates from time zero with probability at least $\frac{\varrho}{2} p e^{-\bar{t}}$, concluding the proof. $\square$

1.2 The $\mathbf{I}(\mathbf{G}, \mathbf{p})$-model

We consider the stochastic process $(\sigma_t)_{t\geq 0}$, which describes ± 1 spin flips dynamics on an infinite graph $G = (V, E)$ with $\Delta(G) < \infty$, where $\Delta(G)$ is the maximum degree of G (which will be defined in detail in Section 2.1). The state space is $\Sigma = \{+1, -1\}^V$ and the initial state is distributed according to a Bernoulli product measure (denoted by μ_p) with density $p \in [0, 1]$ of spins $+1$ and $1 - p$ of spins -1. The process corresponds to the zero-temperature limit of Glauber dynamics for an Ising model with formal Hamiltonian

$$\mathcal{H}(\sigma) = -\sum_{\substack{u,v\in V:\\ \{u,v\}\in E}} \sigma(u)\sigma(v), \tag{1.7}$$

where $\sigma \in \Sigma$. The definition (1.7) is not well posed for infinite graphs. For this reason, we introduce the changes in energy at vertex $v \in V$ as

$$\Delta\mathcal{H}_v(\sigma) = 2\sum_{\substack{u\in V:\\ \{u,v\}\in E}} \sigma(u)\sigma(v).$$

The process $(\sigma_t)_{t\geq 0}$ is a Markov process on Σ with infinitesimal generator having as flip rates

$$c_t(v, \sigma) = \begin{cases} 0 & \text{if } \Delta\mathcal{H}_v(\sigma) > 0 \\ \frac{1}{2} & \text{if } \Delta\mathcal{H}_v(\sigma) = 0 \\ 1 & \text{if } \Delta\mathcal{H}_v(\sigma) < 0. \end{cases} \tag{1.8}$$

It is immediate to notice that this stochastic process is well defined, indeed the supremum in (1.2) is bounded by $\Delta(G)$ (see [26]). We note that the process is a Glauber attractive dynamics. Furthermore, since $\Delta(G) < \infty$, the flip rates in (1.8) satisfy the condition in (1.4) with $A_v = N_V(v)$, where $N_V(v)$ is the set of neighbours of v in G. Therefore, this process can be constructed by the Harris' graphical representation (see [22, 24, 25, 26]). In the following, we will refer to this model as $I(G, p)$-model where G is the underlying graph and p is the density of the Bernoulli product measure. Let $\mathbb{P}_{dyn}$ be the probability measure for the realization of clock rings and tie-breaking coin tosses. Moreover, we denote by $\mathbb{P}_p = \mu_p \times \mathbb{P}_{dyn}$ the joint probability measure on the space Ω of the initial configurations and realizations of the dynamics. An element of the sample space Ω will be denoted by ω.

Chapter 2

Underlying graphs

2.1 Some notation on graphs

We begin this section by presenting Lemma 2.1 that holds in general for subsets of $\mathbb{R}^2$ that are invariant by translations and rotations with the purpose of applying it to connected planar infinite quasi-transitive graphs. Later in the subsection, we introduce some definitions and notation on the graphs (see e.g [16] and [21]) and we present the graphs on which the stochastic Ising model will be constructed.

Let us denote by $\|\cdot\|$ the Euclidean norm and by $B(x, r)$ the ball of radius $r > 0$ centered in x. For any $S \subset \mathbb{R}^2$ and $\bar{x} \in \mathbb{R}^2$, we define the translation of a set as

$$S + \bar{x} := \{x + \bar{x} : x \in S\}.$$

Given $\theta \in (0, \pi]$, we say that S is invariant under rotation of θ if there exists a point $O \in \mathbb{R}^2$, which we assume to be the origin, such that $R_\theta(S) \subset S$, where R_θ is the rotation in the plane with center O and angle θ.

Lemma 2.1. *Given a non-zero vector $\bar{x} \in \mathbb{R}^2$ and a rotation in the plane R_θ with center O and angle $\theta \in (0, \pi]$. Let $S \subset \mathbb{R}^2$ be a non-empty set such that*

- *S has a finite number of points in any ball;*
- *$S + \bar{x} \subset S$;*
- *$R_\theta(S) \subset S$.*

Then $S + \bar{x} = S$, $R_\theta(S) = S$ and $\theta \in \left\{\frac{\pi}{3}, \frac{\pi}{2}, \frac{2}{3}\pi, \pi\right\}$.

Proof. By hypothesis S is non-empty. Thus, by $S + \bar{x} \subset S$, there exists a point $v \in S$, with $v \neq O$. Regarding the rotation, we write $\theta = 2\pi\alpha$ where

$\alpha \in \mathbb{R}$. Now, if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ the set $\{R_\theta^n(v) : n \in \mathbb{N}\} \subset S$ is dense in $\|v\|S^1$ that contradicts the property that each ball in $\mathbb{R}^2$ contains a finite number of points of S. Hence, α necessarily belongs to $\mathbb{Q}$.

Let $\alpha \in \mathbb{Q}$, we write $\alpha = \frac{m}{n}$ with $m, n \in \mathbb{N}$ coprime. By Bézout's lemma, there exist $a, b \in \mathbb{Z}$ such that $am+bn = 1$. Let us select an integer $k \in \mathbb{N}$ such that $a + kn \in \mathbb{N}$. One has $R_\theta^{a+kn} = R_{\frac{2\pi}{n}-2\pi(b-km)} = R_{\frac{2\pi}{n}}$. Thus $R_{\frac{2\pi}{n}}(S) \subset S$. Hence, we can consider only the angles of the form $\theta = 2\pi\frac{1}{n}$, for $n \in \mathbb{N}$.

By applying $(n-1)$ times the rotation $R_{\frac{2\pi}{n}}$ one obtains $R_{-\frac{2\pi}{n}}(S) \subset S$, therefore the rotations with rational α are surjective onto S and consequently also invertible on S. In particular, $R_{2\pi\alpha}(S) = S$ for any $\alpha \in \mathbb{Q}$.

Now we define

$$r := \min\{||w|| : w \in \mathbb{R}^2, \quad S + w \subset S\}.$$

that is well-posed because, by hypothesis, there exists $\bar{x} \in \mathbb{R}^2$ such that $S + \bar{x} \subset S$ and S has a finite number of points in any ball.

Therefore there exists $\bar{v}_0 \in \mathbb{R}^2$ such that $S+\bar{v}_0 \subset S$ with $||\bar{v}_0|| = r$. Notice that, without loss of generality, one can assume that $r = 1$ and $\bar{v}_0 = (1, 0)$. Let us observe that, by $R_{2\pi\frac{1}{n}}(S) = S$, it follows

$$S + \bar{v}_k \subset S,$$

where $\bar{v}_k = \big(\cos(\frac{2\pi k}{n}), \sin(\frac{2\pi k}{n})\big)$ for $k = 0, \ldots, n-1$. From the fact that

$$-\bar{v}_0 = \sum_{k=1}^{n-1} \bar{v}_k,$$

one has that $S - \bar{v}_0 \subset S$. Therefore the translation with respect to $\bar{v}_0$ is surjective onto S and being also injective it is invertible on S. Therefore $S \pm \bar{v}_k = S$ for $k = 0, \ldots, n-1$. Hence, one has

$$S + \bar{v}_1 - \bar{v}_0 = S.$$

The norm of $\bar{v}_1 - \bar{v}_0$ is $\sqrt{2 - 2\cos(\frac{2\pi}{n})}$. Since $\bar{v}_0$ is a minimal norm vector such that $S + \bar{v}_0 = S$, one has

$$\|\bar{v}_1 - \bar{v}_0\| = \sqrt{2 - 2\cos\left(\frac{2\pi}{n}\right)} \geq 1. \tag{2.1}$$

By (2.1) one obtains that $\cos\big(\frac{2\pi}{n}\big) \leq \frac{1}{2}$, which gives $n \in \{2, 3, 4, 5, 6\}$.

In order to get the result, we need to show that $n = 5$ contradicts $r = 1$. In this regard, we consider

$$S + \bar{v}_0 + \bar{v}_2 = S$$

where $\bar{v}_2 = (\cos(4\pi/5), \sin(4\pi/5))$. Notice that $\|\bar{v}_0 + \bar{v}_2\| = \sqrt{2 + 2\cos\left(\frac{4}{5}\pi\right)} < 1$, which contradicts $r = 1$. Therefore $\theta \in \left\{\frac{\pi}{3}, \frac{\pi}{2}, \frac{2}{3}\pi, \pi\right\}$ and this concludes the proof. □

Remark 2.1. *We note that a set S satisfying the hypotheses of Lemma 2.1 must be countable. Notice that for $\bar{x} = (0, 1)$ and $\theta \in \left\{\frac{\pi}{3}, \frac{\pi}{2}, \frac{2}{3}\pi, \pi\right\}$ there exist examples of sets S such that $S + \bar{x} = S$ and $R_\theta(S) = S$. Moreover, there are examples where $R_{2\pi/3}(S) = S$ but $R_{\pi/3}(S) \neq S$ and examples such that $R_\pi(S) = S$ but $R_{\pi/2}(S) \neq S$ (see e.g. the graphs on the right in Figures 2.1 and 2.4, the graph in Figure 2.6 and related comments in Section 2.3).*

Let us now recall some definitions and notation on graph theory (see e.g. [16, 21]).

Let $G = (V, E)$ be a graph, where V is the set of its vertices (or sites) and E, the set of its edges, is a set of unordered pairs $\{u, v\} \subset V$. The *degree* of a vertex $v \in V$, denoted by $deg(v)$, is the number of neighbours of v, i.e. $deg(v) := \left|\left\{u \in V : \{u, v\} \in E\right\}\right|$. The *maximum degree* of G is $\Delta(G) := \sup\{deg(v) : v \in V\}$. Given $v \in V$ and $S \subset V$, we denote by $N_S(v)$ the set of neighbours of v in S, i.e. $N_S(v) := \left\{u \in S : \{u, v\} \in E\right\}$ and we define the degree of v in S as $deg_S(v) := |N_S(v)|$. Given $U \subset V$, the induced subgraph $G[U]$ is the graph whose vertex set is U and whose edge set consists precisely of the edges $\{u, v\} \in E$ with $u, v \in U$. A *path* connecting a vertex v to a vertex u is a non-empty graph $P = (V(P), E(P))$, where $V(P) = \{v_0 = v, v_1, \ldots, v_{m-1}, v_m = u\}$, the vertices v_i are all distinct and $E(P) = \{\{v_i, v_{i+1}\} \in E : i = 0, \ldots, m-1\}$; m is the length of the path P. If $P = (V(P), E(P))$ is a path connecting v to u, then the graph $C := (V(P), E(P) \cup \{\{u, v\}\})$ is called a *cycle*. A graph $G = (V, E)$ is said to be *connected* if for any two vertices $u, v \in V$ there exists a path connecting them. We say $U \subset V$ is connected if the induced subgraph $G[U]$ is connected. We denote by $d_G(u, v)$ the distance in G of two vertices u and v defined as the length of a shortest path connecting u to v. Given a subset $U \subset V$, we define the *external boundary* of U as the set $\partial_{ext} U := \{v \in V \setminus U : \exists u \in U \text{ s.t. } \{v, u\} \in E\}$. Now, we provide the following definitions (see e.g. [21]).

Definition 2.1 (Graph automorphism)**.** *Let $G = (V, E)$ be a graph. A bijective map $\phi \colon V \to V$ is said to be a* graph automorphism *if $\{u, v\} \in E \iff \{\phi(u), \phi(v)\} \in E$.*

Definition 2.2 (Transitive graph). *A graph $G = (V, E)$ is called* transitive *if for any $u, v \in V$ there is a graph automorphism mapping u on v.*

Definition 2.3 (Quasi-transitive graph). *A graph $G = (V, E)$ is said to be* quasi-transitive *if V can be partitioned into a finite number of vertex sets $V_1, \dots, V_N$ such that for any $i = 1, \dots, N$ and any $u, v \in V_i$, there exists a graph automorphism mapping u on v.*

Now we introduce planar graphs, which play a central role in our thesis. An *arc* is a subset of $\mathbb{R}^2$ that is the union of finitely many segments and is homeomorphic to the closed interval $[0, 1]$. The images of 0 and 1 under such a homeomorphism are the *endpoints* of this arc. If A is an arc with endpoints x and y, the *interior* of A is $A \setminus \{x, y\}$ (see [16]).

A *plane graph* is a pair $G = (V, E)$ that satisfies the following properties:

1. $V \subset \mathbb{R}^2$ is at most countable;
2. every edge is an arc between two vertices;
3. different edges have different sets of endpoints;
4. the interior of an edge contains no vertex and no point of any other edge.

A graph $G = (V, E)$ is said to be *planar* if it can be embedded in the plane, i.e. it is isomorphic to a plane graph $\tilde{G}$. The plane graph $\tilde{G}$ is called a *drawing* of G or *embedding* of G in the plane $\mathbb{R}^2$. We can identify a planar graph with its embedding in $\mathbb{R}^2$. Similarly, we say that $G = (V, E)$ is embedded in $\mathbb{R}^d$ if $V \subset \mathbb{R}^d$ is at most countable and (2)-(4) hold.

Given a plane graph $G = (V, E)$ and a set $S \subset V$, let $Conv_G(S) := Conv(S) \cap V$, where $Conv(S)$ denotes the convex hull of S. We now define the shrink and planar shrink property, which are important for our discussion.

Definition 2.4 (Shrink property). *Given a graph $G = (V, E)$, we say that G has the* shrink property *if for each subset $S \subset V$ with finite cardinality, there exists $u \in S$ such that $deg_{V \setminus S}(u) \geq deg_S(u)$.*

Given a line $\ell \in \mathbb{R}^2$, we denote by $H_1^\ell \subset \mathbb{R}^2$ and $H_2^\ell \subset \mathbb{R}^2$ the closed half-planes having ℓ as boundary. Given a non-empty subset $S \subset V$ and a line ℓ, we define $S_1^\ell = S \cap H_1^\ell$ and $S_2^\ell = S \cap H_2^\ell$; clearly $S = S_1^\ell \cup S_2^\ell$.

Definition 2.5 (Planar shrink property). *For a plane graph $G = (V, E)$, we say that G has the* planar shrink property *when, for any non-empty set $S \subset V$ and for any line ℓ, one has:*

for $i = 1, 2$, if S_i^ℓ is non-empty and has finite cardinality, then there exists $u \in S_i^\ell$ such that $deg_{V \setminus S}(u) \geq deg_S(u)$.

We say that a planar graph G has the planar shrink property *if there exists an embedding of G in the plane for which such a property holds.*

For a planar graph, it is immediate to note that the planar shrink property implies the shrink property.

2.2 Definition of the graph classes of interest

We are interested in a connected planar infinite graph $G = (V, E)$ with a specified embedding in $\mathbb{R}^2$ such that the following conditions hold:

(C1) There exists a non-zero vector $\bar{x}$ such that $V + \bar{x} \subset V$ and for any $u, v \in V$,

$$\{u, v\} \in E \iff \{u + \bar{x}, v + \bar{x}\} \in E.$$

Then we say that G is translation invariant with respect to the vector $\bar{x}$.

(C2) There exists a point $O \in \mathbb{R}^2$ and $\theta \in (0, \pi]$ such that $R_\theta(V) \subset V$ and for any $u, v \in V$,

$$\{u, v\} \in E \iff \{R_\theta(u), R_\theta(v)\} \in E.$$

Then we say that G is rotation invariant with respect to R_θ.

(C3) Each ball in $\mathbb{R}^2$ contains a finite number of vertices of G.

By Lemma 2.1, it follows that a graph satisfying conditions (C1), (C2) and (C3) has $\theta \in \left\{\frac{\pi}{3}, \frac{\pi}{2}, \frac{2}{3}\pi, \pi\right\}$ and the translations and rotations in (C1), (C2) are graph automorphisms. For reasons that will become clear in the following, we do not deal with $\theta = \pi$. The translations and rotations provide a partition of V in classes, in any case this partition can be finer than the partition given in Definitions 2.2 and 2.3. For $\theta = \pi$, it is straightforward to exhibit an example where the number of classes is infinite. For example, we can consider $G = (V, E)$ where $V = \mathbb{Z}^2$ and the edge set is

$$E = \big\{\{(i, j), (i, j + 1)\} : i, j \in \mathbb{Z}\big\} \cup \big\{\{(i, 0), (i + 1, 0)\} : i \in \mathbb{Z}\big\}.$$

It is immediate to note that G is invariant under translation with respect to the vector $(1, 0)$ and is invariant under rotation of π but not of $\pi/2$. Moreover, the classes of G are $C_n = \{(i, \pm n) : i \in \mathbb{Z}\}$ for $n \in \mathbb{N}_0$ (see Figure 2.6 in Section 2.3). In any case we have the following result.

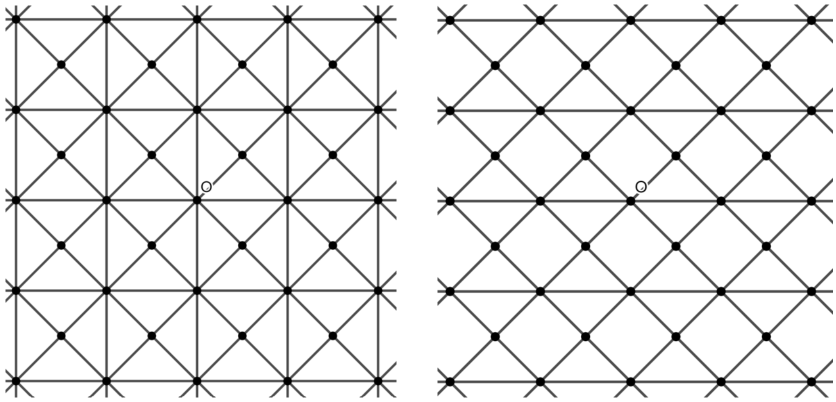

Figure 2.1: The graph on the left belongs to $\mathcal{G}(4)$, hence it is invariant under rotation of $\pi/2$. The graph on the right is only invariant under rotation of π.

Proposition 2.1. *If G is a plane graph satisfying the conditions (C1), (C2) and (C3), then $\theta \in \left\{\frac{\pi}{3}, \frac{\pi}{2}, \frac{2}{3}\pi, \pi\right\}$. Moreover if $\theta \in \left\{\frac{\pi}{3}, \frac{\pi}{2}, \frac{2}{3}\pi\right\}$ then the plane graph G is either transitive or quasi-transive.*

Proof. The first part of the statement has already been discussed above. It is sufficient to prove that for $\theta \in \left\{\frac{\pi}{3}, \frac{\pi}{2}, \frac{2}{3}\pi\right\}$ the number of classes is finite. Let $\theta \in \left\{\frac{\pi}{3}, \frac{\pi}{2}, \frac{2}{3}\pi\right\}$, the plane graph G is translation invariant with respect to the linear independent vectors $\bar{x}$ and $\bar{y} := R_\theta \bar{x}$. The number of classes is at most the number of vertices belonging to the closed parallelogram spanned by vectors $\bar{x}$ and $\bar{y}$. By (C3) follows that the number of vertices in this parallelogram is finite. $\square$

Now, we introduce the class of plane graphs $\mathcal{G}(a)$ with $a \in \{3, 4, 6\}$ that is the collection of connected infinite graphs with finite maximal degree satisfying conditions (C1)-(C3) with $\theta = \theta(a) = 2\pi/a$. It is immediate to notice that $\mathcal{G}(6) \subset \mathcal{G}(3)$. Let, furthermore, $\mathcal{G} := \mathcal{G}(3) \cup \mathcal{G}(4)$.

Now, we introduce a class of one-dimensional graphs which are the graphs underlying the model considered in our second main result, Theorem 3.3.

Let $G = (\mathbb{Z}, E)$ be a connected graph satisfying the following properties:

(A1) There exists $L \in \mathbb{Z}$ such that for any $x, y \in \mathbb{Z}$

$$\{x, y\} \in E \iff \{x + L, y + L\} \in E.$$

Then we say that G is translation invariant with respect to L.

(A2) $\Delta(G) < \infty$.

Proposition 2.2. *If $G = (\mathbb{Z}, E)$ is a connected graph satisfying the properties (A1) and (A2) then*

(a) *G is quasi-transitive;*

(b) *There exists $K < \infty$ such that $|x - y| < K$ for each $\{x, y\} \in E$.*

Proof. The translation invariance with respect to L provides a partition of $\mathbb{Z}$ in classes and the number of classes is at most L. This implies (a).
Finally, since G is translation invariant and $\Delta(G) < \infty$, to show (b) it suffices to consider $K = \max_{x,y\in\mathbb{Z}:\{x,y\}\in E} |x - y|$. □

2.3 Construction of a class of graphs having the planar shrink property and examples

We begin by introducing a class of graphs that have the planar shrink property, as we show in Proposition 2.3. Let $\mathcal{H}$ be the collection of infinite plane graphs $G = (V, E)$ satisfying the following properties:

(P1) every edge is a closed line segment, i.e. a line segment which includes its two end-points;

(P2) for each $e \in E$, let us consider the unique straight line ℓ which contains e. Then for any $x \in \ell$ there exists $f \in E$ such that $x \in f$.

(P3) $\Delta(G) < \infty$.

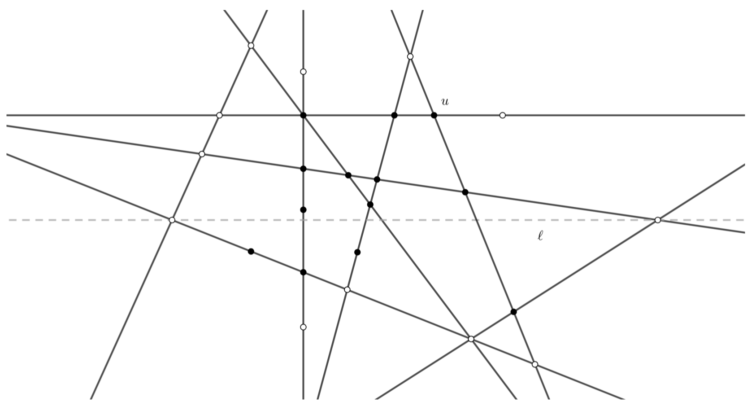

Figure 2.2: In black the vertices in S and in white those in $V \setminus S$. This figure illustrates a graph $G \in \mathcal{H}$ and the vertex u used in the proof of Proposition 2.3.

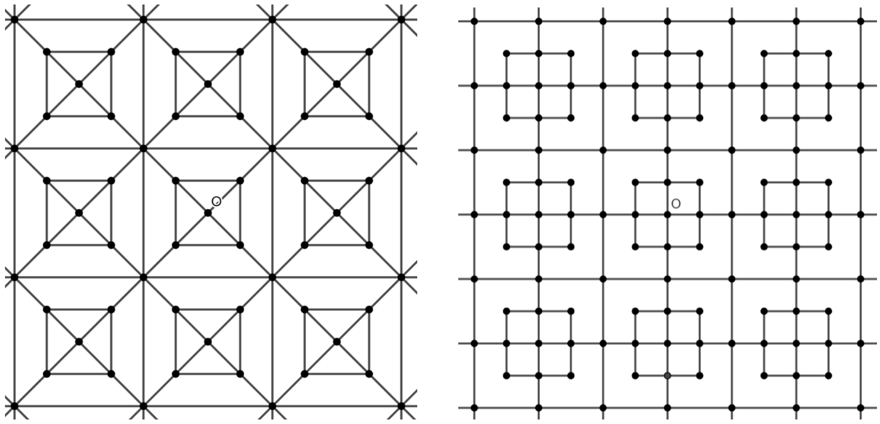

Figure 2.3: Examples of graphs in $\mathcal{G}(4)$ that do not have the shrink property.

Proposition 2.3. *If $G \in \mathcal{H}$, then G has the planar shrink property.*

Proof. Given a non-empty subset $S \subset V$ and a straight line ℓ, suppose without loss of generality that $0 < |S_1^\ell| < \infty$ (see Definition 2.5). We need to prove that there exists $u \in S_1^\ell$ such that $deg_{V \setminus S}(u) \geq deg_S(u)$. First, we note that by property (P2) every vertex has a degree that is even. Without loss of generality, we can consider the line ℓ coincident with the axis x (by applying a translation and a rotation) such that the vertices in S_1^ℓ have a non-negative ordinate. We write every vertex $v \in V$ as $v = (v_x, v_y) \in \mathbb{R}^2$ and let $r_y := \max_{v \in S_1^\ell} v_y$ and $r_x := \max\{v_x \in \mathbb{R} : (v_x, r_y) \in S_1^\ell\}$.

We consider the vertex $u = (r_x, r_y) \in V$. Given $w = (w_x, w_y) \in N_S(u)$, there are two cases to consider (see Figure 2.2). If $w_y = r_y$ then, by property (P2), there exists a vertex $w' = (w'_x, w'_y) \in N_{V \setminus S}(u)$ with $w'_y = r_y$ and $w'_x > r_x$. If instead $w_y < r_y$ then, by property (P2), there exists a vertex $w' \in N_{V \setminus S}(u)$ with $w'_y > r_y$. In both cases, w' belongs to the linear extension of edge $\{u, w\}$ out of S. In other words, it is possible to define an injective function

$$f_u \colon w \in N_S(u) \mapsto w' \in N_{V \setminus S}(u),$$

where the vertices u, w and w' are aligned. This implies that $deg_{V \setminus S}(u) \geq deg_S(u)$. $\square$

We note that the shrink property holds even if we replace $\mathcal{H}$ with a class of graphs embedded in $\mathbb{R}^d$ having the properties (P1), (P2) and (P3), i.e. they are obtained by intersection of lines. The proof of this fact is analogous to the proof of Proposition 2.3.

We want to provide some explicit examples here.

The square lattice $\mathbb{Z}^2$, the triangular lattice (see Figure 2.4) and the graphs in Figures 2.1, 2.4 and 2.5 belong to class $\mathcal{H}$ and therefore, by Proposition 2.3, have the planar shrink property. In particular, we note that the graph on the right in Figure 2.4 belongs to $\mathcal{G}(3) \setminus \mathcal{G}(6)$, i.e. it is invariant under rotation of an angle of $2\pi/3$, but not of $\pi/3$. Indeed, if we consider O a vertex of the blue triangular lattice or the barycenter of suitable triangles of the black triangular lattice, then the graph is invariant under rotation of $2\pi/3$ with center O. However, it is not possible to identify points O on the plane for which the graph is invariant under rotation of $\pi/3$ with center O.

In Figure 2.3 we give two examples of graphs that do not have the shrink property

In Figure 2.6 we show a graph G with infinite classes that is invariant by a rotation of π but not invariant under a rotation of $\pi/2$.

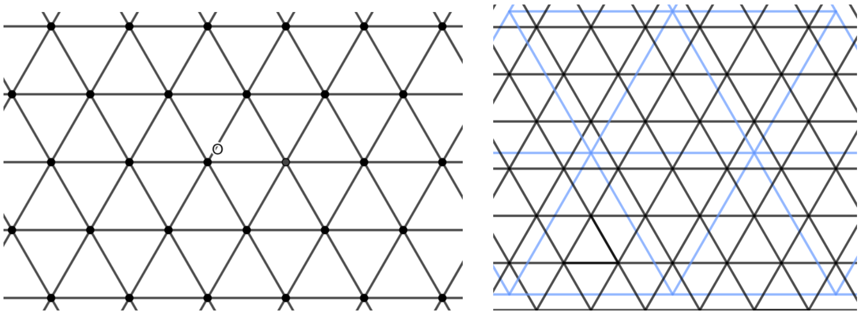

Figure 2.4: On the left the triangular lattice, example of a graph $G \in \mathcal{G}(6) \cap \mathcal{H}$. On the right a double triangular lattice, example of a graph $G \in (\mathcal{G}(3) \setminus \mathcal{G}(6)) \cap \mathcal{H}$.

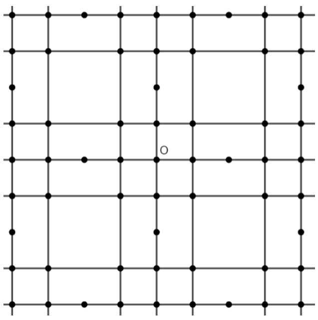

Figure 2.5: Modified double lattice $\mathbb{Z}^2$: example of a graph $G \in \mathcal{G}(4) \cap \mathcal{H}$.

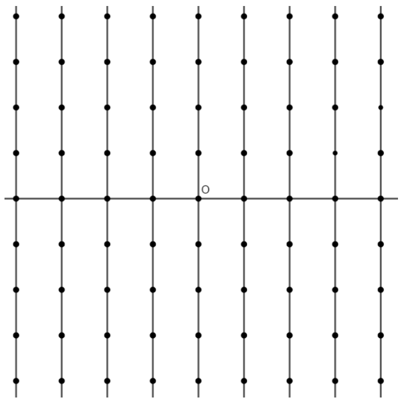

Figure 2.6: Example of a graph $G \in \mathcal{H}$ that is invariant under translation and rotation of π, but not of $\pi/2$. The number of classes of G is infinite.

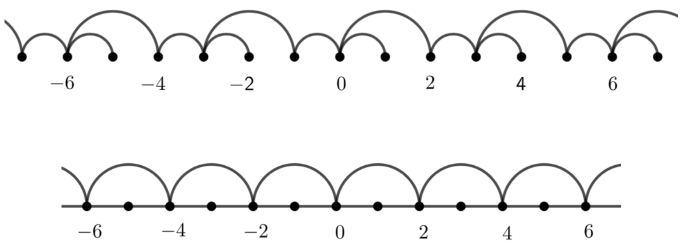

Figure 2.7: Examples of connected graphs $G = (\mathbb{Z}, E)$ satisfying the properties (A1) and (A2). The top graph does not have the shrink property, but the bottom graph does.

2.4 Construction of graphs in $d = 1$

In this section, we show a simple way to construct one-dimensional graphs $G = (\mathbb{Z}, E)$ satisfying the properties (A1) and (A2) (see [9]). Given $G^\star = (V^\star, E^\star, f)$ a directed graph with integer weights associated with the edges, we define the graph $\tilde{G} = (\tilde{V}, \tilde{E})$ in the following way:

$$\tilde{V} = \big\{(x, v) : x \in \mathbb{Z}, v \in V^\star\big\},$$

$$\tilde{E} = \big\{\{(x, u), (x + f(\{u, v\}), v)\} : \{u, v\} \in E^\star\big\}.$$

It is immediate to notice that G is isomorphic to a graph having $G = (\mathbb{Z}, E)$ satisfying the properties (A1) and (A2). This way of constructing graphs can be very useful for visually verifying some properties of the graph.

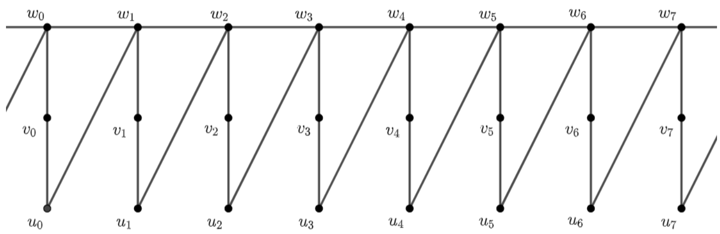

Figure 2.8: Example of a graph $\tilde{G} = (\tilde{V}, \tilde{E})$ constructed as above starting from a weighted directed graph $G^\star = (V^\star, E^\star, f)$, where $V^\star = \{u, v, w\}$, $E^\star = \{\{w, v\}, \{w, u\}, \{v, u\}, \{w, w\}\}$, $f(\{w, v\}) = f(\{v, u\}) = 0$, $f(\{w, u\}) = -1$ and $f(\{w, w\}) = +1$.

Chapter 3

Statements of main results

In this chapter, we state the two main results. We recall that we deal with the zero-temperature stochastic Ising model, where the initial spin configuration is distributed according to a Bernoulli product measure μ_p with parameter $p \in (0, 1)$. It is possible to notice that all our results remain valid if μ_p is replaced by any measure satisfying the following conditions: invariance and mixing under translations for which the graph is invariant; invariance under rotation for which the graph is invariant; the FKG property; symmetry under global spin flips.

First, we present Theorem 3.1, which show that the shrink property is a necessary condition to obtain that the $I(G, p)$-model is of type $\mathcal{I}$.

Theorem 3.1. *Let $G = (V, E)$ be a graph with $\Delta(G) < \infty$. If G does not have the shrink property then for any $p \in [0, 1]$ the $I(G, p)$-model is not of type $\mathcal{I}$.*

Proof. First let us consider the case $p \in (0, 1]$. Since G does not have the shrink property then there exists a finite subset $S \subset V$ such that $deg_{V \setminus S}(u) < deg_S(u)$ for any $u \in S$. Moreover, one has

$$\mathbb{P}_p\left(\bigcap_{u \in S} \{\sigma_0(u) = +1\}\right) = p^{|S|} > 0. \tag{3.1}$$

Since $deg_{V \setminus S}(u) < deg_S(u)$ for every $u \in S$, no site in S can change the value of its spin if $\sigma_0(u) = +1$ for each $u \in S$.

This fact implies that

$$\mathbb{P}\big(\text{each vertex in } S \text{ fixates at the value } +1 \text{ from time zero}\big) \geq p^{|S|} > 0.$$

Thus, for any $p \in (0, 1]$ the $I(G, p)$-model is not of type $\mathcal{I}$.

Let us now consider $p = 0$. In this case, all sites fixate from time 0 at the value -1 almost surely and the $I(G, 0)$-model is of type $\mathcal{F}$. $\square$

Remark 3.1. *Note that Theorem 3.1 does not imply that the $I(G,p)$-model is of type $\mathcal{F}$ or $\mathcal{M}$.*

If G does not have the shrink property, it is possible to show examples in which the $I(G,p)$-model is respectively of type $\mathcal{F}$ or $\mathcal{M}$. It is known (see [5, 29]) that if G is the hexagonal lattice then the $I(G,p)$-model is of type $\mathcal{F}$. Now, we show in Example 3.1 that if we consider $G \in \mathcal{G}(4)$ as in Figure 3.1 then for any $p \in (0,1)$ the $I(G,p)$-model is of type $\mathcal{M}$.

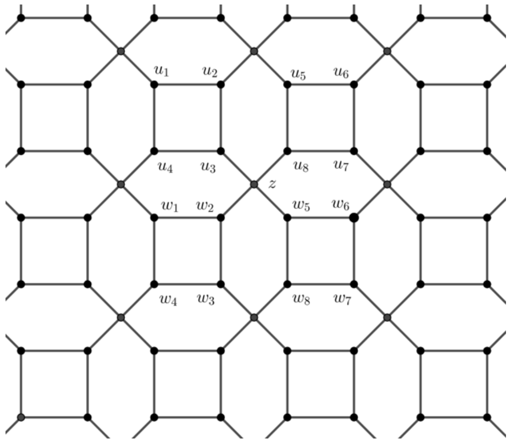

Figure 3.1: Example of a graph $G \in \mathcal{G}(4)$ that does not have the shrink property and, for $p \in (0,1)$, the $I(G,p)$-model is of type $\mathcal{M}$.

Example 3.1. *Let $G \in \mathcal{G}$ be the graph in Figure 3.1. For any $p \in (0,1)$ the $I(G,p)$-model is of type $\mathcal{M}$.*
Now, we show this statement. Let $u_i \in V$ with $deg(u_i) = 3$ for $i = 1,\dots,8$ as in Figure 3.1. Let $S = \{u_1,\dots,u_4\}$. Since $deg_{V\setminus S}(u_1) = 1 < 2 = deg_S(u_1)$, it is immediate to notice that

$$\mathbb{P}_p\big(u_1 \text{ fixates at the value } +1\big) \geq \mathbb{P}_p\big(\forall i = 1,\dots,4\ \sigma_0(u_i) = +1\big) = p^4 > 0. \tag{3.2}$$

Hence, by ergodicity (see [27, 29]), there exist vertices that fixate at the value $+1$ almost surely. Now, let $z, w_i \in V$ be the vertices such that $deg(z) = 4$ and $deg(w_i) = 3$ for $i = 1,\dots,8$ as in Figure 3.1. Let E_z be the event that the vertices u_i and w_i fixate respectively at the value $+1$ and -1 for each $i = 1,\dots,8$. With the same argument used in (3.2), we deduce that $\mathbb{P}_p(E_z) \geq p^8(1-p)^8 > 0$. Moreover, conditioning on the event E_z, whenever there is an arrival of the Poisson process $\mathcal{P}_z$ the spin flip at z occurs with probability $1/2$. Thus, by Lévy's extension of the Borel-Cantelli Lemmas, z flips infinitely often with positive probability. By ergodicity (see [27, 29]) *again, it follows that there exist vertices that flip infinitely often almost surely. Therefore, the $I(G, p)$-model is of type $\mathcal{M}$.*

In light of Theorem 3.1, for the two main results we focus on a class of graphs having also the shrink property. Recall that $\mathcal{G}$ is the class of graphs introduced in Section 2.2, which consists of planar graphs invariant under translations and rotations of $\theta \in \left\{\frac{\pi}{3}, \frac{\pi}{2}, \frac{2}{3}\pi\right\}$. Now, we are ready to state our main result, Theorem 3.2, whose proof is given in Chapter 4.

Theorem 3.2. *If $G = (V, E) \in \mathcal{G}$ has the planar shrink property, then the $I(G, 1/2)$-model is of type $\mathcal{I}$, i.e., all sites flip infinitely often almost surely.*

Now, we state Theorem 3.3, whose proof is given in Chapter 5.

Theorem 3.3. *If $G = (\mathbb{Z}, E)$ is a connected graph satisfying the properties (A1) and (A2), then the $I(G, 1/2)$-model is of type $\mathcal{I}$ if and only if G has the shrink property.*

We note that, unlike the graphs belonging to the class $\mathcal{G}$, for a connected graph $G = (\mathbb{Z}, E)$ satisfying the properties (A1) and (A2) the shrink property is a necessary and sufficient condition for the model to be of type $\mathcal{I}$. This depends on the one-dimensionality of the graphs considered. Indeed in proving Theorem 3.2, we use the planar shrink property to make all sites belonging to a suitable two-dimensional convex hull have spin equal to $+1$. It is immediate to understand that this planar shrink property is no longer useful for one-dimensional graphs since trivially we have no two-dimensional convex hull to consider. For the graphs considered in Theorem 3.3, the following simple idea can be used. Given $I_r(0)$ a symmetric interval centered at 0, for each graph it is possible to identify two disjoint subsets of fixed cardinality $\{r+1, \dots, r+K\}$ and $\{-r-K, \dots, -r-1\}$ so that if all the vertices in these two sets have spin equal to $+1$ at time t then the shrink property implies that all the vertices in $I_r(0)$ have spin equal to $+1$ in some time $s \in (t, t+1)$ with positive probability. The probability that this event occurs infinitely often is positive and does not depend on the size r of the interval since the two disjoint sets considered have fixed cardinality. Once a lower bound on this probability has been obtained, the continuation of the proof is analogous to the proof of Theorem 3.2. For details on the strategy of this proof, as already mentioned, refer to Chapter 5. This simple idea, however, would not work to prove Theorem 3.2, because if all the vertices of the boundary of a finite region have spin equal to $+1$ at time t then we would obtain a lower bound on the probability that all the sites of the region have spin equal to $+1$ infinitely often that depends on the size of the region. This size-dependent lower bound could not be used in the proof of Theorem

3.2, as instead we will do using the lower bound that comes from Lemmas 4.6-4.7 (see Chapter 4).

Chapter 4

Proof of Theorem 3.2

In the following, given $\sigma \in \Sigma$ and $v \in V$, we denote by $C_v(\sigma)$ the cluster at site v for σ, defined as the maximal connected subset of V such that $v \in C_v(\sigma)$ and for any $u \in C_v(\sigma)$ one has $\sigma(u) = \sigma(v)$.

First, we state the following known result (see [7, 29]), which will be useful to prove Proposition 4.1.

Lemma 4.1 ([7, 29])**.** *For $d \in \mathbb{N}$, take a $I(G,p)$-model, where $p \in [0,1]$ and G is a graph embedded in $\mathbb{R}^d$ with $\Delta(G) < \infty$. If in addition G is translation invariant with respect to d linearly independent vectors, then for each vertex there are finitely many energy-lowering spin flips almost surely.*

We do not provide the proof of Lemma 4.1 here, but refer to the proof of Theorem 3 in [29] (as emphasized in the related remark) or to Lemma 5 in [7].

Now, we are ready to present the following proposition, which is an extension of Proposition 3.1 in [4].

Proposition 4.1. *For $d \in \mathbb{N}$, take a $I(G,p)$-model, where $p \in [0,1]$ and G is a graph embedded in $\mathbb{R}^d$ that is translation invariant with respect to d linearly independent vectors. Moreover, suppose that G has the shrink property and $\Delta(G) < \infty$. Then, the size of the cluster at a vertex $v \in V$ diverges almost surely as $t \to \infty$, i.e.*

$$\forall v \in V, \quad \lim_{t\to\infty} |C_v(\sigma_t)| = \infty \quad \textit{almost surely.}$$

Proof. We prove the proposition by contradiction. Hence, for a vertex $v \in V$, let us define the event

$$\mathcal{A} := \{\omega \in \Omega : \liminf_{t\to\infty} |C_v(\sigma_t)| < \infty\}.$$

By contradiction assumption we suppose $\mathbb{P}(\mathcal{A}) > 0$. By continuity of the measure there exists $M > 0$ such that $\mathbb{P}(\mathcal{A}_M) > 0$, where

$$\mathcal{A}_M := \{\omega \in \Omega : \liminf_{t\to\infty} |C_v(\sigma_t)| < M\}.$$

Then, for any $\omega \in \mathcal{A}_M$, one can define $(T_k(\omega))_{k\in\mathbb{N}}$ such that

$$T_1(\omega) = \inf\{t \geq 0 : |C_v(\sigma_t)| < M\},$$

and, for $k \in \mathbb{N}$, one recursively defines

$$T_{k+1}(\omega) = \inf\{t \geq T_k(\omega) + 1 : |C_v(\sigma_t)| < M\}.$$

Let $\mathcal{F}$ be the σ-algebra generated by the process $(\sigma_t)_{t\geq 0}$ and $\mathcal{F}_t$ be the σ-algebra generated by the process up to time t. It is immediate to note that T_k is a stopping time with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ for any $k \in \mathbb{N}$. We consider $\mathcal{F}_{T_k}$ for any $k \in \mathbb{N}$. We notice that for $\omega \in \mathcal{A}_M$, since $\Delta(G) < \infty$ and $|C_v(\sigma_{T_k})| < M$, the cluster $C_v(\sigma_{T_k})$ can be equal only to a finite number of sets of vertices. For each of these sets of vertices, by the shrink property there is an ordered finite sequence of clock rings and outcomes of tie-breaking coin tosses inside a fixed finite ball that would cause the cluster to shrink to a single site $w \in V$ with $d_G(w, v) < M$ (w could, in principle, depend on $C_v(\sigma_{T_k})$). Then, since the vertex w would have all neighbours with opposite spins, it could have an energy-lowering spin flip (with change in energy equal to $-2deg(w)$) and the cluster would vanish with positive probability. We define

$$\mathcal{B}_{M,k} := \bigcup_{\substack{w\in V:\\ d_G(w,v)<M}} \big\{\text{spin at } w \text{ flips at time } t \in (T_k, T_k + 1) \text{ with } \Delta\mathcal{H}_w(\sigma_t) \leq -1\big\}.$$

From the previous statements in this proof, one has that there exists $\delta > 0$ such that

$$\mathbb{P}_p\big(\mathcal{B}_{M,k}|\sigma_{T_k}\big) \geq \delta.$$

Now, by the Strong Markov property of the process, for any $k \in \mathbb{N}$ we have the following lower bound

$$\xi_k(\omega) := \mathbb{P}_p\big(\mathcal{B}_{M,k}|\mathcal{F}_{T_k}\big)(\omega) = \mathbb{P}_p\big(\mathcal{B}_{M,k}|\sigma_{T_k}\big) \geq \delta,$$

for almost every $\omega \in \mathcal{A}_M$. Thus $\sum_{k=1}^{\infty} \xi_k(\omega) = \infty$, for almost every $\omega \in \mathcal{A}_M$. Now, by using the Lévy's extension of Borel-Cantelli Lemmas (see e.g. [30]) with the sequence of events $(\mathcal{B}_{M,k})_{k\in\mathbb{N}}$ and the filtration $(\mathcal{F}_{T_k})_{k\in\mathbb{N}}$, we have that

$$\left\{\omega \in \Omega : \sum_{k=1}^{\infty} \xi_k(\omega) = \infty\right\} \subset \left\{\omega \in \Omega : \sum_{k=1}^{\infty} \mathbf{1}_{\mathcal{B}_{M,k}}(\omega) = \infty\right\} \cup C$$

where $C \in \mathcal{F}$ is a set of zero measure. Then

$$\mathbb{P}_p\big(\limsup_{k\to\infty} \mathcal{B}_{M,k}\big) \geq \mathbb{P}_p\big(\mathcal{A}_M\big) > 0.$$

Thus, there exists a vertex $\tilde{w}$ with $d_G(\tilde{w}, v) < M$ such that energy-lowering spin flips occur at $\tilde{w}$ infinitely many times with positive probability. By translation invariance with respect to d linearly independent vectors and ergodicity, there exists a positive density of vertices for which energy-lowering spin flips occur infinitely often almost surely. This fact contradicts Lemma 4.1. This concludes the proof. $\square$

Remark 4.1. *Proposition 4.1 is a generalization of Proposition 3.2 in [4] for graphs G having the shrink property. In [4], the result was given only for the cubic lattice Z^d that in particular has the shrink property.*

In the following, we consider the $I(G, p)$-model having $G \in \mathcal{G}(a)$ for $a \in \{3, 4\}$ and it is invariant under translation with respect to $\bar{x}$. Without loss of generality we take $\bar{x} = (1, 0)$. Let us consider a vertex $\tilde{v}$ having minimal Euclidean distance from the origin O. Clearly, $\tilde{v} = O$ when O belongs to V. In the case $\tilde{v} \neq O$ we consider the two distinct vertices $\tilde{v}, R_\theta(\tilde{v})$; since G is a connected graph, we can select a connected finite set $S \subset V$ such that $\tilde{v}, R_\theta(\tilde{v}) \in S$. Finally we define the set of vertices

$$U = \begin{cases} \{O\} & \text{if } O \in V \\ \bigcup_{k=0}^{a-1} R_{k\theta(a)}(S) & \text{otherwise,} \end{cases} \tag{4.1}$$

where $\theta(a) = 2\pi/a$. By construction U is connected and $R_{\theta(a)}(U) = U$.

For $a \in \{3, 4\}$ we construct a region of size $L \in \mathbb{R}_+$ centered in O as follows. Let us consider the point $P_1(L, a) = (L \tan(\theta(a)/2), L) \in \mathbb{R}^2$ and let

$$P_{i+1}(L, a) = R_{i\theta(a)}(P_1),$$

for $i = 1, \ldots, a-1$. We define the region of size $L \in \mathbb{R}_+$ centered in O as follows

$$T_L(a) := Conv(\{P_1(L, a), \ldots, P_a(L, a)\}).$$

For $a = 3, 4$ one respectively obtains that $T_L(a)$ is an equilateral triangle or a square.

Now, let us consider the class $V_1 \subseteq V$ (see Proposition 2.1 and Definition 2.3). If the graph $G = (V, E)$ is transitive then $V_1 = V$. We write every vertex $v \in V$ as $v = (v_x, v_y) \in \mathbb{R}^2$ and let $v_{0,y} := \max_{v \in V_1 \cap T_L(a)} v_y$ and $v_{0,x} := \min\{v_x \in \mathbb{R} : (v_x, v_y) \in V_1 \cap T_L(a) \text{ and } v_y = v_{0,y}\}$. We define $v_0 = (v_{0,x}, v_{0,y}) \in V_1 \cap T_L(a)$. By translation invariance of G with respect to

$\bar{x} = (1, 0)$, one has $v_0 + \bar{x} \in V_1$. Now we can select a connected set of vertices U_0 such that $V_1 \cap B(v_0, 2) \subset U_0$. Finally we choose $r_1 \geq 2$ such that $U_0 \subset B(v_0, r_1)$. We are ready to present the following lemma.

Lemma 4.2. *For any $G \in \mathcal{G}(a)$ with $a \in \{3, 4\}$ and for any $L \in \mathbb{R}_+$ there exists a connected set of vertices $W_L \subset (T_{L+2r_1}(a) \setminus T_{L-2r_1}(a))$ such that $R_{\theta(a)}(W_L) = W_L$.*

Proof. For $k \in \mathbb{N}$, let

$$v_k := v_0 + k\bar{x}.$$

We define $k_{max} = \max\{k \in \mathbb{N} : v_k \in T_L(a)\}$. The set of vertices

$$\mathcal{S} = \bigcup_{k=0}^{k_{max}} (U_0 + k\bar{x})$$

is connected because for any $k = 0, \ldots, k_{max} - 1$ it turns out that $G[U_0 + k\bar{x}]$ is connected and $v_k, v_{k+1} \in U_0 + k\bar{x}$.

We define the set of vertices $W_L = \bigcup_{i=0}^{a-1} R_{i\theta(a)}(\mathcal{S})$. Now, we show that W_L is connected. Since

$$||v_0 - P_2(L, a)|| \leq 1, \quad ||P_1(L, a) - v_{k_{max}}|| \leq 1$$

and by using the triangle inequality, one obtains

$$\begin{aligned} ||v_0 - R_{\theta(a)}(v_{k_{max}})|| &\leq ||v_0 - P_2(L, a)|| + ||P_2(L, a) - R_{\theta(a)}(v_{k_{max}})|| = \\ &= ||v_0 - P_2(L, a)|| + ||P_1(L, a) - v_{k_{max}}|| \leq 2. \end{aligned}$$

The previous inequality and $V_1 \cap B(v_0, 2) \subset U_0$ imply that W_L is connected. Clearly $W_L \subset (T_{L+2r_1}(a) \setminus T_{L-2r_1}(a))$ and $R_{\theta(a)}(W_L) = W_L$. □

Let W_L as in Lemma 4.2. One can select a cycle $U_L \subset W_L$; we call $f_{L,\infty}$ its outer face and $f_{L,0}$ its inner face.

Remark 4.2. *We notice that, by translational invariance with respect to the vectors $\bar{x} = (1, 0)$ and $\bar{y} = (\cos\theta(a), \sin\theta(a))$, one has*

$$|W_L| \asymp L \quad \textit{and} \quad |V \cap f_{L,0}| \asymp L^2,$$

where we write $a_n \asymp b_n$ to mean that there exist two positive constants c_1 and c_2 such that $c_1 \leq \frac{a_n}{b_n} \leq c_2$ for all $n \in \mathbb{N}$. This implies that $G \in \mathcal{G}(a)$ is amenable for any $a \in \{3, 4, 6\}$. Under the assumptions of amenability of the graph, the translation invariance of the measure μ and finite-energy of the measure μ, it is known that the infinite cluster is at most one almost *surely (see [2, 3, 21]). For the zero-temperature stochastic Ising model, it is not known whether the measure induced at time t has finite-energy property. Therefore, we are not able to prove the uniqueness of the infinite cluster at time t. Instead, if the temperature is positive and decreases to zero, one has the property of finite-energy (see [7]). In this last case one obtains the uniqueness of the infinite cluster.*

Now, given an integer $q \in \mathbb{N}$ and $\delta < \frac{1}{2q}$, we consider $T_{1+\delta}(a) \setminus T_{1-\delta}(a)$ and we show that there exists a collection of balls $(B(c_i, 4/q) : i = 1, \dots, aq)$ such that:

a. $T_{1+\delta}(a) \setminus T_{1-\delta}(a) \subset \bigcup_{i=1}^{aq} B(c_i, 4/q)$;

b. for any $i = 1, \dots, aq$, the center c_i belongs to $\partial T_1(a)$;

c. for any $i = 1, \dots, q$ and $m = 0, \dots, a-1$ one has $c_{i+mq} = R_{m\theta(a)}(c_i)$. In particular, $R_{\theta(a)}(\bigcup_{i=1}^{aq} B(c_i, 4/q)) = \bigcup_{i=1}^{aq} B(c_i, 4/q)$.

It is clear that such a construction exists, for example by taking the centers of the balls almost equally spaced. The chosen balls in this construction will be maintained also in the sequel.

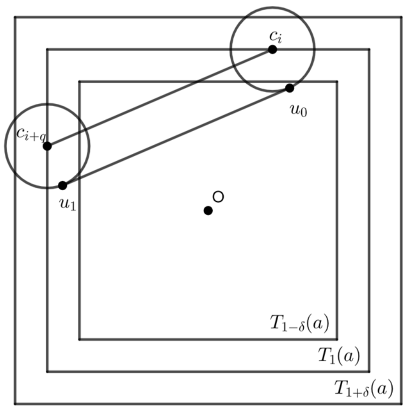

Figure 4.1: The distance between the segment having $u_0 \in B(c_i, 4/q)$ and $u_1 \in B(c_{i+q}, 4/q)$ as its endpoints and O can decrease at most of $4/q$ (the length of the radius) with respect to the distance between the segment having endpoins c_i and c_{i+q} and O.

Lemma 4.3 (Geometric Lemma). *Let $a \in \{3, 4\}$, $q \geq 10$ and $\delta < \frac{1}{2q}$ and consider the cover of $T_{1+\delta}(a) \setminus T_{1-\delta}(a)$ introduced in items a, b, and c. Then, for any $(u_k)_{k=0,\dots,a-1}$ such that $u_k \in B(c_{i+kq}, 4/q)$ for $k = 0, \dots, a-1$, one has $Conv(\{u_0, \dots, u_{a-1}\}) \supset B(O, \frac{1}{2} - \frac{4}{q})$.*

Proof. For a fixed $i = 1, \ldots, q$, let us consider $(c_{i+kq})_{k=0,\ldots,a-1}$. For $a = 3, 4$, we note that $Conv(\{c_{i+kq} : k = 0, \ldots, a-1\})$ is an equilateral triangle or a square. Therefore, since $c_{i+kq} \in \partial T_1(a)$ one has $Conv(\{c_{i+kq} : k = 0, \ldots, a-1\}) \supset B(O, \frac{1}{2})$. Let us now consider the segment having $u_0 \in B(c_i, 4/q)$ and $u_1 \in B(c_{i+q}, 4/q)$ as its endpoints. The distance between this segment and the origin O can decrease at most of $4/q$ with respect to the distance between O and the segment having endpoins c_i, c_{i+q} (see Figure 4.1). Then one obtains that $Conv(\{u_0, \ldots, u_{a-1}\}) \supset B(O, \frac{1}{2} - \frac{4}{q})$. □

As already announced, we do not deal with $\theta = \pi$. Indeed if we consider a=2 which corresponds to $\theta(a) = \pi$, this statement is false because $Conv(\{u_0, u_1\})$ would be a segment and there is no ball contained in it. From now on we take $q \geq 24$ and hence $\frac{1}{2} - \frac{4}{q} \geq \frac{1}{3}$.

Now, we present the following definition.

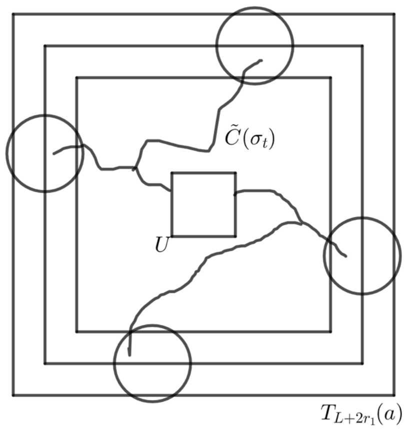

Figure 4.2: Example of a realization of an L-cross.

Definition 4.1 (L-Cross)**.** *Given $G \in \mathcal{G}(a)$ with $a \in \{3, 4\}$, we say that an* L-cross of $+1$ *occurs at time t if there exists a cluster $\tilde{C}(\sigma_t)$ of $G[V \cap T_{L+2r_1}(a)]$ such that*

- *$\sigma_t(v) = +1$ for each $v \in \tilde{C}(\sigma_t)$;*
- *$\tilde{C}(\sigma_t) \supset U$, where U is defined in* (4.1)*;*
- *there exists $i \in \{1, \ldots, q\}$ such that $\tilde{C}(\sigma_t) \cap B(Lc_{i+kq}, 4L/q) \neq \emptyset$ for each $k = 0, \ldots, a-1$.*

We denote by E_L^t with $t \in \mathbb{R}_0^+$ the event that an L-cross of $+1$ *occurs at time t. Moreover, we define*

$$A_L := \limsup_{t \to \infty} E_L^t.$$

We define the set of vertices $S_L(t) := \tilde{C}(\sigma_t) \cap W_L$, where the properties of W_L are given in Lemma 4.2. The previous Lemma 4.3 shows that for each time $t \in \mathbb{R}_0^+$ in which an L-cross of $+1$ occurs (see Figure 4.2), one has

$$Conv(S_L(t)) \supset B\left(O, r_L\right) \quad \text{where } r_L = \frac{1}{3}L. \tag{4.2}$$

In other words, Lemma 4.3 says that $|Conv_G(S_L(t))| \asymp L^2$.

Lemma 4.4. *Consider the $I(G,p)$-model, where $p \in [\frac{1}{2}, 1)$ and $G \in \mathcal{G}$. If G has the shrink property, then*

$$\mathbb{P}_p(A_L) \geq p_{cross} := \frac{1}{(aq)^a}\left(\frac{1}{2}\right)^{a|U|}.$$

We now explain the strategy for proving Lemma 4.4. Let $U \subset V$ as defined in (4.1). If the initial density $p \geq 1/2$, then U is contained in a cluster of $+1$ with probability at least $(1/2)^{|U|}$, at any time t. By Proposition 4.1, the size of this cluster diverges almost surely as $t \to \infty$. Now, by FKG inequality and by rotation invariance, one obtains that $\liminf_{t\to\infty} \mathbb{P}_p(E_L^t) > 0$, i.e. the cluster satisfies the properties in Definition 4.1 with positive probability. Note that this lower bound does not depend on L. By Reverse Fatou Lemma, we obtain the same lower bound on $\mathbb{P}_p(A_L)$. We are now ready to prove the lemma.

Proof of Lemma 4.4. Let $\mathcal{U}_t$ be the event that all vertices in U have spin equal to $+1$ at time t. By Lemma 1.1, FKG inequality and Harris' inequality (see [23, 26]), it follows that

$$\mathbb{P}_p(\mathcal{U}_t) \geq \left(\frac{1}{2}\right)^{|U|}. \tag{4.3}$$

If $\mathcal{U}_t$ occurs, then, since U is connected, at time t all vertices in U belong to a same cluster, we call it $C_U(\sigma_t)$. Moreover, let $\tilde{C}(\sigma_t)$ be the cluster of $G[V \cap T_{L+2r_1}(a)]$ that contains U. We denote by $\mathcal{C}_{W_L}(t)$ the event that the cluster $\tilde{C}(\sigma_t)$ intersects W_L, i.e. $\mathcal{C}_{W_L}(t) := \{\tilde{C}(\sigma_t) \cap W_L \neq \emptyset\}$. By Proposition 4.1, we have that $\lim_{t\to\infty} |C_U(\sigma_t)| = \infty$ almost surely. Thus, by planarity of G and $V \cap f_{L,0}$ has finite cardinality (see Remark 4.2), we get

$$\lim_{t\to\infty} \mathbb{P}_p\big(\mathcal{C}_{W_L}(t)\big) = 1. \tag{4.4}$$

By (4.3) and (4.4), it follows that

$$\liminf_{t\to\infty} \mathbb{P}_p\big(\mathcal{C}_{W_L}(t) \cap \mathcal{U}_t\big) = \liminf_{t\to\infty} \mathbb{P}_p(\mathcal{U}_t) \geq \left(\frac{1}{2}\right)^{|U|}. \tag{4.5}$$

Now, we write $W_L = \cup_{i=1}^{q} \cup_{k=0}^{a-1} P_{L,k}^i$ where $P_{L,k}^i := W_L \cap B(Lc_{i+kq}, 4L/q)$ for $i = 1, \ldots, q$ and $k = 0, \ldots, a-1$. We define the event $\mathcal{C}_{L,i,k}(t) := \{\tilde{C}(\sigma_t) \cap P_{L,k}^i \neq \emptyset\}$ for $i = 1, \ldots, q$ and $k = 0, \ldots, a-1$. Hence, we have

$$\mathcal{C}_{W_L}(t) \cap \mathcal{U}_t = \bigcup_{i=1}^{q} \bigcup_{k=0}^{a-1} \mathcal{C}_{L,i,k}(t) \cap \mathcal{U}_t.$$

Thus, by rotation invariance and by the union bound, we have

$$\mathbb{P}_p\big(\mathcal{C}_{W_L}(t) \cap \mathcal{U}_t\big) = \mathbb{P}_p\left(\bigcup_{i=1}^{q} \bigcup_{k=0}^{a-1} \big(\mathcal{C}_{L,i,k}(t) \cap \mathcal{U}_t\big)\right) \leq \sum_{i=1}^{q} \sum_{k=0}^{a-1} \mathbb{P}_p\big(\mathcal{C}_{L,i,k}(t) \cap \mathcal{U}_t\big) \leq$$
$$\leq aq \mathbb{P}_p\big(\mathcal{C}_{L,\bar{i},0}(t) \cap \mathcal{U}_t\big), \quad (4.6)$$

where $\bar{i} \in \{1, \ldots, q\}$ is such that $\mathbb{P}_p\big(\mathcal{C}_{L,\bar{i},0}(t) \cap \mathcal{U}_t\big) = \max_{i=1,\ldots,q} \mathbb{P}_p\big(\mathcal{C}_{L,i,0}(t) \cap \mathcal{U}_t\big)$. We note that $\mathcal{C}_{L,\bar{i},k}(t) \cap \mathcal{U}_t$ is an increasing event for $k = 0, \ldots, a-1$; therefore

$$\mathbb{P}_p\left(\bigcap_{k=0}^{a-1} \big(\mathcal{C}_{L,\bar{i},k}(t) \cap \mathcal{U}_t\big)\right) \geq \left(\mathbb{P}_p\big(\mathcal{C}_{L,\bar{i},0}(t) \cap \mathcal{U}_t\big)\right)^a \geq \left(\frac{1}{aq} \mathbb{P}_p\big(\mathcal{C}_{W_L}(t) \cap \mathcal{U}_t\big)\right)^a, \quad (4.7)$$

where the first inequality follows by FKG inequality and by rotation invariance, and the last inequality follows by (4.6). We also notice that, by definition of $P_{L,k}^i$, one has

$$\mathcal{C}_{L,i,k}(t) = \{\tilde{C}(\sigma_t) \cap P_{L,k}^i \neq \emptyset\} \subset \{\tilde{C}(\sigma_t) \cap B(Lc_{i+kq}, 4L/q) \neq \emptyset\}.$$

Thus, by Definition 4.1, we have

$$E_L^t = \bigcup_{i=1}^{q} \bigcap_{k=0}^{a-1} \{\tilde{C}(\sigma_t) \cap B(Lc_{i+kq}, 4L/q) \neq \emptyset\} \cap \mathcal{U}_t \supseteq$$
$$\supseteq \bigcup_{i=1}^{q} \bigcap_{k=0}^{a-1} \mathcal{C}_{L,i,k}(t) \cap \mathcal{U}_t \supseteq \bigcap_{k=0}^{a-1} \mathcal{C}_{L,\bar{i},k}(t) \cap \mathcal{U}_t,$$

and hence

$$\mathbb{P}_p(E_L^t) \geq \mathbb{P}_p\left(\bigcap_{k=0}^{a-1} \mathcal{C}_{L,\bar{i},k}(t) \cap \mathcal{U}_t\right). \quad (4.8)$$

Now, by (4.5), (4.7) and (4.8), we obtain the following lower bound

$$\liminf_{t\to\infty} \mathbb{P}_p(E_L^t) \geq \liminf_{t\to\infty} \left(\frac{1}{aq} \mathbb{P}_p\big(\mathcal{C}_{W_L}(t) \cap \mathcal{U}_t\big)\right)^a \geq \frac{1}{(aq)^a} \left(\frac{1}{2}\right)^{a|U|}.$$

Finally, by Reverse Fatou Lemma we get

$$\mathbb{P}_p(A_L) \geq \limsup_{t\to\infty} \mathbb{P}_p(E_L^t) \geq \liminf_{t\to\infty} \mathbb{P}_p(E_L^t) \geq \frac{1}{(aq)^a}\left(\frac{1}{2}\right)^{a|U|} > 0$$

that concludes the proof. □

Now we give a simple definition that will be useful when related to E_L^t through Lemma 4.3. For $t \in \mathbb{R}_0^+$, let F_L^t be the event that all sites belonging to $B(O, \frac{L}{3})$ have spin equal to $+1$ at some time $s \in (t, t+1)$.

Lemma 4.5. *Consider the $I(G,p)$-model, where $p \in [\frac{1}{2}, 1)$ and $G \in \mathcal{G}$. If G has the planar shrink property, then there exists $\epsilon_L > 0$ such that*

$$\mathbb{P}_p(F_L^t | \sigma_t = \sigma) \geq \epsilon_L$$

for any $\sigma \in \Sigma$ such that $\{\sigma_t = \sigma\} \subset E_L^t$.

Proof. Let $\sigma \in \Sigma$ and $(\sigma_s)_{s\geq 0}$ be the $I(G,p)$-model such that $\{\sigma_t = \sigma\} \subset E_L^t$. We define another zero-temperature stochastic Ising model $(\sigma'_s)_{s\geq t}$ with infinitesimal generator having the flip rates as in (1.8) and such that

$$\sigma'_t(v) = \begin{cases} +1 & \text{for each } v \in \tilde{C}(\sigma_t) \\ -1 & \text{otherwise,} \end{cases}$$

where $\tilde{C}(\sigma_t)$ is the cluster of $G[V \cap T_{L+2r_1}(a)]$ in the configuration σ, as in Definition 4.1. By definition of σ'_t and E_L^t, $\tilde{C}(\sigma_t) \supset U$ is the unique cluster of $+1$ sites in the configuration σ'_t. In configuration σ'_t, we have that $\big(V \cap T_{L+2r_1}(a)\big) \setminus \tilde{C}(\sigma_t) = D_1(\sigma'_t) \sqcup \cdots \sqcup D_k(\sigma'_t)$, where $D_i(\sigma'_t)$ for $i = 1, \ldots, k$ are clusters of -1 sites (we stress that $k < \infty$ because $V \cap T_{L+2r_1}(a)$ has finite cardinality).

We notice that, by planarity of G, for each $i = 1, \ldots, k$ we have that $\partial_{ext} D_i(\sigma'_t) \subset \tilde{C}(\sigma_t)$. By planar shrink property, for each $i = 1, \ldots, k$ there exists an ordered finite sequence of updates (i.e. of clock rings and outcomes of tie-breaking coin tosses inside $V \cap T_{L+2r_1}(a)$) that would cause all sites of $D_i(\sigma'_t) \cap Conv_G(S_L(t))$ (see definition above formula (4.2)) to have spin equal to $+1$ in some time $s \in (t, t+1)$ with positive probability. Therefore, we get an ordered finite sequence of updates inside $V \cap T_{L+2r_1}(a)$ that would cause all sites of $Conv_G(S_L(t))$ to have spin equal to $+1$ in σ'_s (with $s \in (t, t+1)$), but since $\sigma'_t \leq \sigma_t$ this sequence of updates works, by the coupling in Lemma 1.1, in the same way for the original process $(\sigma_s : s \in (t, t+1))$. Moreover, by Lemma 4.3, one has $V \cap B(O, \frac{L}{3}) \subset Conv_G(S_L(t))$.

Thus, there exists $\epsilon_L > 0$ such that

$$\mathbb{P}_p(F_L^t | \sigma_t = \sigma) \geq \epsilon_L$$

for any $\sigma \in \Sigma$ having $\{\sigma_t = \sigma\} \subset E_L^t$. □

We recall that $A_L := \limsup_{t\to\infty} E_L^t$. Now, we define $B_L := \limsup_{t\to\infty} F_L^t$. We are ready to present the following lemma.

Lemma 4.6. *Consider the $I(G,p)$-model, where $p \in [\frac{1}{2}, 1)$ and $G \in \mathcal{G}$. If G has the planar shrink property, then $\mathbb{P}_p(B_L) \geq p_{cross} > 0$.*

Proof. First, let $\omega \in A_L$, i.e. an L-cross of $+1$ occurs infinitely often. Then one can define $(T_k(\omega))_{k\in\mathbb{N}}$ such that

$$T_1(\omega) = \inf\{t \geq 0 : E_L^t \text{ occurs}\},$$

and, for $k \in \mathbb{N}$, we recursively define

$$T_{k+1}(\omega) = \inf\{t \geq T_k + 1 : E_L^t \text{ occurs}\}.$$

Let $\mathcal{F}$ be the σ-algebra generated by the process $(\sigma_t)_{t\geq 0}$ and $\mathcal{F}_t$ be the σ-algebra generated by the process up to time t. It is immediate to note that T_k is a stopping time with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ for any $k \in \mathbb{N}$. We consider $\mathcal{F}_{T_k}$ for any $k \in \mathbb{N}$. By the Strong Markov property of the process and by Lemma 4.5, for any $k \in \mathbb{N}$ we have the following lower bound

$$\xi_k(\omega) := \mathbb{P}_p\big(F_L^{T_k} | \mathcal{F}_{T_k}\big)(\omega) = \mathbb{P}_p\big(F_L^{T_k} | \sigma_{T_k}\big) \geq \epsilon_L > 0,$$

for almost every $\omega \in A_L$. Thus $\sum_{k=1}^{\infty} \xi_k(\omega) = \infty$, for almost every $\omega \in A_L$. Now, by using the Lévy's extension of Borel-Cantelli Lemmas (see e.g. [30]) with the sequence of events $(F_L^{T_k})_{k\in\mathbb{N}}$ and the filtration $(\mathcal{F}_{T_k})_{k\in\mathbb{N}}$, we have that

$$\left\{\omega \in \Omega : \sum_{k=1}^{\infty} \xi_k(\omega) = \infty\right\} \subset \left\{\omega \in \Omega : \sum_{k=1}^{\infty} \mathbf{1}_{F_L^{T_k}}(\omega) = \infty\right\} \cup C,$$

where $C \in \mathcal{F}$ is a set of zero measure. Then, by Lemma 4.4, we get

$$\mathbb{P}_p\big(B_L\big) \geq \mathbb{P}_p\big(A_L\big) \geq \frac{1}{(aq)^a}\left(\frac{1}{2}\right)^{a|U|} = p_{cross} > 0.$$

This concludes the proof. □

Now, for any time $t_2 > t_1 + 1$ we define

$$D(L; t_1, t_2) := \bigcup_{s \in [t_1, t_2 - 1]} F_L^s \quad \text{and} \quad D(L; t_1, \infty) := \bigcup_{s \geq t_1} F_L^s.$$

Lemma 4.7. *For any $L \in \mathbb{R}_+$ and $t_1 \geq 0$, one has*

$$\mathbb{P}_p\big(D(L; t_1, \infty)\big) \geq p_{cross}.$$

Moreover, for any $\epsilon > 0$ there exists a time $s > t_1 + 1$ such that

$$\mathbb{P}_p\big(D(L; t_1, s)\big) \geq (1 - \epsilon) p_{cross}.$$

Proof. We observe that $D(L; t_1, \infty) \supset B_L$. In particular, by Lemma 4.6,

$$\mathbb{P}_p\big(D(L; t_1, \infty)\big) \geq p_{cross}.$$

Now, we note that for $t_2 \leq t_2'$ one has $D(L; t_1, t_2) \subset D(L; t_1, t_2')$. Thus, by continuity of measure

$$\lim_{t_2 \to \infty} \mathbb{P}_p\big(D(L; t_1, t_2)\big) = \mathbb{P}_p\left(\bigcup_{s \geq t_1} D(L; t_1, s)\right) = \mathbb{P}_p\big(D(L; t_1, \infty)\big) \geq p_{cross}.$$

Hence for all $\epsilon > 0$ there exists a time $s > t_1 + 1$ such that

$$\mathbb{P}_p\big(D(L; t_1, s)\big) \geq (1 - \epsilon) p_{cross}.$$

$\square$

Let $\mathcal{F}$ be the σ-algebra generated by the process $(\sigma_t)_{t \geq 0}$. All the events introduced belong to $\mathcal{F}$. Given a configuration $\sigma \in \Sigma = \{+1, -1\}^V$ and a non-zero vector $\bar{v}$ such that G is translation invariant with respect to $\bar{v}$, we define the configuration translated $\tau_{\bar{v}}\sigma$ with respect to $\bar{v}$ as

$$\tau_{\bar{v}}\sigma(v) := \sigma(v + \bar{v}) \quad \text{for any } v \in V.$$

Let X be a $\mathcal{F}$-measurable random variable. Then $X = f((\sigma_t)_{t \geq 0})$ where f is a measurable function. We define

$$X + \bar{v} := f((\tau_{-\bar{v}}\sigma_t)_{t \geq 0}). \tag{4.9}$$

If X is an indicator function then f takes only the values 0 or 1. Let $A \in \mathcal{F}$, one can define

$$\mathbf{1}_A + \bar{v} = f((\tau_{-\bar{v}}\sigma_t)_{t \geq 0}) =: \mathbf{1}_{A + \bar{v}},$$

that defines $A + \bar{v}$ for any $A \in \mathcal{F}$.

In the following result we will apply the ergodic theorem. We note that these processes are ergodic with respect to the translation if the initial conditions are given for instance by a Bernoulli product measure, see e.g. [22, 26, 27] and references therein.

Now, we introduce some notation, which we will use in the proof of Theorem 3.2. For $v \in V$ and $t \in \mathbb{R}_+$, let

$$A_v^+(t) := \{\sigma_s(v) = +1, \ \ \forall s \in [0,t]\}, \quad A_v^-(t) := \{\sigma_s(v) = -1, \ \ \forall s \in [0,t]\}.$$

We denote by $A_v^+(\infty)$ (resp. $A_v^-(\infty)$) the event that the vertex v fixates at the value $+1$ (resp. -1) from time zero. Clearly, $A_v^\pm(t) \subset A_v^\pm(t')$, for any $t' \leq t$. We recall that $\{V_1, \dots, V_N\}$ is the partition of the vertex set V that comes from the quasi-transitivity of $G \in \mathcal{G}$, see Proposition 2.1. We note that, since G is quasi-transitive, $\mathbb{P}(A_v^\pm(t))$ depends only on the class to which the vertex v belongs and does not depend explicitly on the vertex itself. Thus, for $p = 1/2$, for each $i = 1, \dots, N$, $v \in V_i$, and $t \in \mathbb{R}_+ \cup \{\infty\}$, we set

$$\rho_i(t) := \mathbb{P}_{1/2}(A_v^+(t)) = \mathbb{P}_{1/2}(A_v^-(t)).$$

The last equality follows by symmetry under the global spin flip for $p = 1/2$. Now, we are ready to prove our main result.

Proof of Theorem 3.2. We will prove the theorem by contradiction. Suppose that there exists $j \in \{1, \dots, N\}$ such that $\rho_j(\infty) > 0$ that by Lemma 1.2 is equivalent to have a site that fixates with positive probability. We fix the following constants: $\epsilon = \frac{1}{3} p_{cross} \rho_j(\infty)$, $\epsilon_1 = \frac{\rho_j(\infty)}{5}$, $\epsilon_2 = \frac{1}{4} p_{cross}$ and $\tilde{\epsilon} = \frac{1}{8}$.

We notice that, by continuity of the measure, the limit of $\rho_i(t)$ as $t \to \infty$ exists and is equal to

$$\lim_{t \to \infty} \rho_i(t) = \lim_{t \to \infty} \mathbb{P}_{1/2}(A_v^+(t)) = \mathbb{P}_{1/2}\left(\bigcap_{m=1}^{\infty} A_v^+(m) \right) = \mathbb{P}_{1/2}(A_v^+(\infty)) = \rho_i(\infty),$$

for each $v \in V_i$. This implies that there exists a time $t_\epsilon > 0$ such that

$$0 \leq \rho_j(t_\epsilon) - \rho_j(\infty) < \epsilon. \tag{4.10}$$

Since $G \in \mathcal{G}$, there exist two linearly independent vectors $\bar{x}_1$ and $\bar{x}_2$ such that G is translation invariant with respect to them. We want to construct on the graph G disjoint regions of a suitable size L_0 centered in $n_1 \bar{x}_1 + n_2 \bar{x}_2$ with $n_1, n_2 \in \mathbb{Z}$. By ergodicity (see [22, 27, 29]), one has

$$\lim_{r \to \infty} \frac{1}{n(r,j)} \sum_{v \in B(O,r) \cap V_j} \mathbf{1}_{A_v^-(t_\epsilon)} = \rho_j(t_\epsilon) > 0 \quad \text{almost surely,} \tag{4.11}$$

where $n(r,j) := |B(O,r) \cap V_j|$. Thus, (4.11) implies that there exists $\tilde{r} \in \mathbb{R}_+$ such that

$$\mathbb{P}_{1/2}\bigg(\frac{1}{n(\tilde{r},j)} \sum_{v \in B(O,\tilde{r}) \cap V_j} \mathbf{1}_{A_v^-(t_\epsilon)} \notin [\rho_j(t_\epsilon) - \epsilon_1, \rho_j(t_\epsilon) + \epsilon_1]\bigg) \leq \epsilon_2.$$

Then, in particular

$$\mathbb{P}_{1/2}\bigg(\sum_{v \in B(O,\tilde{r}) \cap V_j} \mathbf{1}_{A_v^-(t_\epsilon)} < n(\tilde{r},j)\big(\rho_j(t_\epsilon) - \epsilon_1\big)\bigg) \leq \epsilon_2. \tag{4.12}$$

Now, we construct disjoint regions of size L_0 on the graph G in the following way. Let $L_0 = 3\tilde{r}$, where L_0 and $\tilde{r}$ play the same role of L and r_L in (4.2). We define the event

$$G(L;t,\eta) := \bigg\{ \sum_{v \in B(O,L/3) \cap V_j} \mathbf{1}_{A_v^-(t)} \geq n(L/3,j)\big(\rho_j(t) - \eta\big) \bigg\},$$

where $L, t, \eta > 0$. By (4.12), one has

$$\mathbb{P}_{1/2}\big(G(L_0; t_\epsilon, \epsilon_1)\big) \geq 1 - \epsilon_2. \tag{4.13}$$

Now, let

$$Y_{L_0}(t) := \sum_{v \in B(O,L_0/3) \cap V_j} \mathbf{1}_{A_v^-(t)}.$$

Let $n_0 \in \mathbb{N}$ such that $T_{L_0+2r_1}(a) \cap (T_{L_0+2r_1}(a) + n_0\bar{x}_i) = \emptyset$ for $i = 1,2$. We define $Y_{L_0,m_1,m_2}(t) := Y_{L_0}(t) + m_1 n_0 \bar{x}_1 + m_2 n_0 \bar{x}_2$ with $m_1, m_2 \in \mathbb{Z}$ (see (4.9)). By ergodicity, it follows that for any $t \in \mathbb{R}_+$

$$\lim_{M \to \infty} \frac{1}{(2M+1)^2} \sum_{\substack{m_1,m_2 \in \mathbb{Z}: \\ |m_1|,|m_2| \leq M}} \frac{1}{n(L_0/3,j)} Y_{L_0,m_1,m_2}(t) = \rho_j(t), \quad \text{a. s.} \tag{4.14}$$

We define the translated events

$$D(L; m_1, m_2; t_1, t_2) := D(L; t_1, t_2) + m_1 n_0 \bar{x}_1 + m_2 n_0 \bar{x}_2$$

and

$$G(L; m_1, m_2; t, \eta) := G(L; t, \eta) + m_1 n_0 \bar{x}_1 + m_2 n_0 \bar{x}_2.$$

Now, by Lemma 4.7 and by translation invariance, there exists a time $t_{\tilde{\epsilon}} > t_\epsilon + 1$ such that

$$\mathbb{P}_{1/2}\big(D(L_0; m_1, m_2; t_\epsilon, t_{\tilde{\epsilon}})\big) \geq (1 - \tilde{\epsilon}) p_{cross}. \tag{4.15}$$

By ergodicity, (4.13) and (4.15), it follows that

$$\lim_{M\to\infty}\frac{1}{(2M+1)^2}\sum_{\substack{m_1,m_2\in\mathbb{Z}:\\|m_1|,|m_2|\le M}}\mathbf{1}_{G(L_0;m_1,m_2;t_\epsilon,\epsilon_1)\cap D(L_0;m_1,m_2;t_\epsilon,t_{\tilde\epsilon})}=$$
$$=\mathbb{P}_{1/2}\big(G(L_0;m_1,m_2;t_\epsilon,\epsilon_1)\cap D(L_0;m_1,m_2;t_\epsilon,t_{\tilde\epsilon})\big)\ge$$
$$\ge\mathbb{P}_{1/2}\big(D(L_0;m_1,m_2;t_\epsilon,t_{\tilde\epsilon})\big)-\mathbb{P}_{1/2}\big(G(L_0;m_1,m_2;t_\epsilon,\epsilon_1)^c\big)\ge(1-\tilde\epsilon)p_{cross}-\epsilon_2\quad\text{a.s.}\tag{4.16}$$

Over the event $G(L_0;m_1,m_2;t_\epsilon,\epsilon_1)$ one has

$$Y_{L_0,m_1,m_2}(t_\epsilon)\ge n(L_0/3,j)\big(\rho_j(t_\epsilon)-\epsilon_1\big).\tag{4.17}$$

By (4.14), we get

$$\rho_j(t_\epsilon)-\rho_j(t_{\tilde\epsilon})=\lim_{M\to\infty}\frac{1}{(2M+1)^2}\sum_{\substack{m_1,m_2\in\mathbb{Z}:\\|m_1|,|m_2|\le M}}\frac{1}{n(L_0/3,j)}\Big[Y_{L_0,m_1,m_2}(t_\epsilon)-Y_{L_0,m_1,m_2}(t_{\tilde\epsilon})\Big]\ge$$
$$\ge\lim_{M\to\infty}\frac{1}{(2M+1)^2}\sum_{\substack{m_1,m_2\in\mathbb{Z}:\\|m_1|,|m_2|\le M}}\frac{1}{n(L_0/3,j)}Y_{L_0,m_1,m_2}(t_\epsilon)\mathbf{1}_{G(L_0;m_1,m_2;t_\epsilon,\epsilon_1)\cap D(L_0;m_1,m_2;t_\epsilon,t_{\tilde\epsilon})}.\tag{4.18}$$

The inequality in (4.18) follows by

$$Y_{L_0,m_1,m_2}(t_\epsilon)-Y_{L_0,m_1,m_2}(t_{\tilde\epsilon})\ge Y_{L_0,m_1,m_2}(t_\epsilon)\mathbf{1}_{D(L_0;m_1,m_2;t_\epsilon,t_{\tilde\epsilon})}\ge$$
$$\ge Y_{L_0,m_1,m_2}(t_\epsilon)\mathbf{1}_{G(L_0;m_1,m_2;t_\epsilon,\epsilon_1)\cap D(L_0;m_1,m_2;t_\epsilon,t_{\tilde\epsilon})}.$$

Indeed if $D(L_0;m_1,m_2;t_\epsilon,t_{\tilde\epsilon})$ occurs then $Y_{L_0,m_1,m_2}(t_{\tilde\epsilon})=0$ and one has an equality. Otherwise, if $D(L_0;m_1,m_2;t_\epsilon,t_{\tilde\epsilon})$ does not occur then $Y_{L_0,m_1,m_2}(t_\epsilon)-Y_{L_0,m_1,m_2}(t_{\tilde\epsilon})\ge 0$ since $Y_{L_0,m_1,m_2}(t)$ is a decreasing function in t. Now, by (4.16) and (4.17), the last term in (4.18) is lower bounded by

$$\big(\rho_j(t_\epsilon)-\epsilon_1\big)\lim_{M\to\infty}\frac{1}{(2M+1)^2}\sum_{\substack{m_1,m_2\in\mathbb{Z}:\\|m_1|,|m_2|\le M}}\mathbf{1}_{G(L_0;m_1,m_2;t_\epsilon,\epsilon_1)\cap D(L_0;m_1,m_2;t_\epsilon,t_{\tilde\epsilon})}\ge$$
$$\ge\big(\rho_j(t_\epsilon)-\epsilon_1\big)\big((1-\tilde\epsilon)p_{cross}-\epsilon_2\big)\quad\text{a.s.}\tag{4.19}$$

Combining (4.10) with (4.18) and (4.19) and recalling the value of the constants ϵ, ϵ_1, ϵ_2 and $\tilde\epsilon$, we obtain

$$\frac{1}{3}p_{cross}\rho_j(\infty)=\epsilon>\rho_j(t_\epsilon)-\rho_j(\infty)\ge\rho_j(t_\epsilon)-\rho_j(t_{\tilde\epsilon})\ge$$
$$\ge\big(\rho_j(t_\epsilon)-\epsilon_1\big)\big((1-\tilde\epsilon)p_{cross}-\epsilon_2\big)\ge\big(\rho_j(\infty)-\epsilon_1\big)\big((1-\tilde\epsilon)p_{cross}-\epsilon_2\big)=$$
$$=\frac{1}{2}p_{cross}\rho_j(\infty),$$

which is obviously false when $\rho_j(\infty) > 0$. Thus, for each $i \in \{1, \ldots, N\}$ we have that $\rho_i(\infty) = 0$, i.e., each site fixates at the value $+1$ (or -1) from time 0 with zero probability. This implies, by Lemma 1.2, that each site fixates at the value $+1$ (or -1) with zero probability. Hence, all sites flip infinitely often almost surely, i.e., the model is of type $\mathcal{I}$. $\square$

Chapter 5

Proof of Theorem 3.3

By Theorem 3.1, if the $I(G, 1/2)$-model is of type $\mathcal{I}$ then G has the shrink property. Therefore, it remains to prove the converse implication, i.e. that the shrink property of G is also a sufficient condition to obtain that the $I(G, 1/2)$-model is of type $\mathcal{I}$. In this chapter, we provide details of the strategy to prove the converse implication. In particular, we show how, starting from the idea already mentioned in Chapter 3, the proof of Theorem 3.3 consists in a simple and immediate revisiting of some of the proofs of Chapter 4.

From now on, as in the proof of Proposition 2.2, we set

$$K = \max_{x,y \in \mathbb{Z}: \{x,y\} \in E} |x - y|.$$

Now, we define the following subsets of $\mathbb{Z}$:

$$I_r(0) := \big\{x \in \mathbb{Z} : |x| \le r\big\}$$

and

$$U_{r,K}(0) := I_{r+K}(0) \setminus I_r(0) = \big\{-r-1, \ldots, -r-K\big\} \sqcup \big\{r+1, \ldots, r+K\big\}.$$

We define the event

$$\mathcal{U}_{t,K} := \bigcap_{x \in U_{r,K}(0)} \big\{\sigma_t(x) = +1\big\}.$$

By Lemma 1.1 and FKG inequality, it follows that

$$\mathbb{P}_{1/2}(\mathcal{U}_{t,K}) \ge \left(\frac{1}{2}\right)^{2K} > 0. \tag{5.1}$$

Moreover, by Reverse Fatou Lemma and (5.1) we get

$$\mathbb{P}_{1/2}(\limsup_{t\to\infty} \mathcal{U}_{t,K}) \geq \limsup_{t\to\infty} \mathbb{P}_{1/2}(\mathcal{U}_{t,K}) \geq \left(\frac{1}{2}\right)^{2K} > 0. \tag{5.2}$$

For $t \geq 0$, we define the event

$$F^t_{r,K} := \bigcup_{s\in(t,t+1)} \bigcap_{x\in I_{r+K}(0)} \{\sigma_s(x) = +1\}.$$

Now, we are ready to prove Lemma 5.1, which is nothing more than a simple revisitation of the proof of Lemma 4.5 for these one-dimensional graphs that we are considering in this chapter.

Lemma 5.1. *Consider the $I(G, 1/2)$-model, where $G = (\mathbb{Z}, E)$ is a connected graph satisfying the properties (A1) and (A2). If G has the shrink property, then there exists $\delta_r > 0$ such that*

$$\mathbb{P}_{1/2}(F^t_{r,K}|\sigma_t = \sigma) \geq \delta_r \tag{5.3}$$

for any $\sigma \in \Sigma$ such that $\{\sigma_t = \sigma\} \subset \mathcal{U}_{t,K}$.

Proof. Let $\sigma \in \Sigma$ and $(\sigma_s)_{s\geq 0}$ be the $I(G, 1/2)$-model such that $\{\sigma_t = \sigma\} \subset \mathcal{U}_{t,K}$. We define another zero-temperature stochastic Ising model $(\sigma'_s)_{s\geq t}$ with infinitesimal generator having the same flip rates and such that

$$\sigma'_t(x) = \begin{cases} +1 & \text{for each } x \in U_{r,K}(0) \\ -1 & \text{otherwise.} \end{cases}$$

By definition of σ'_t and K, one has that $I_r(0)$ is a cluster of -1 sites in configuration σ'_t and $\partial_{ext}\big(I_r(0)\big) \subset U_{r,K}(0)$. Since $\sigma'_t(x) = +1$ for each $x \in U_{r,K}(0)$, by shrink property there exists an ordered finite sequence of updates (i.e. of clock rings and outcomes of tie-breaking coin tosses inside $I_{r+K}(0)$) that would cause all sites of $I_r(0)$ to have spin equal to $+1$ in σ'_s for some $s \in (t, t + 1)$, but since $\sigma'_t \leq \sigma_t$ this sequence of updates works in the same way for the original process $(\sigma_s : s \in (t, t + 1))$. Thus, there exists $\delta_r > 0$ such that

$$\mathbb{P}_{1/2}(F^t_{r,K}|\sigma_t = \sigma) \geq \delta_r$$

for any $\sigma \in \Sigma$ having $\{\sigma_t = \sigma\} \subset \mathcal{U}_{t,K}$. □

Now, we define $B_{r,K} := \limsup_{t\to\infty} F^t_{r,K}$. We are ready to present the following lemma (analogous to Lemma 4.6).

Lemma 5.2. *We consider the $I(G, 1/2)$-model, where $G = (\mathbb{Z}, E)$ is a connected graph satisfying the properties (A1) and (A2). If G has the shrink property, then*

$$\mathbb{P}_{1/2}(B_{r,K}) \geq \left(\frac{1}{2}\right)^{2K} =: \tilde{p} > 0.$$

Proof. See proof of Lemma 4.6. □

Finally, repeating in an almost identical way the proofs of Lemma 4.7 and Theorem 3.2, Theorem 3.3 is proved.

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