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# An $\alpha$ -regret analysis of Adversarial Bilateral Trade

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## Abstract

We study sequential bilateral trade where sellers and buyers valuations are completely arbitrary (*i.e.*, determined by an adversary). Sellers and buyers are strategic agents with private valuations for the good and the goal is to design a mechanism that maximizes efficiency (or gain from trade) while being incentive compatible, individually rational and budget balanced. In this paper we consider gain from trade which is harder to approximate than social welfare.

We consider a variety of feedback scenarios and distinguish the cases where the mechanism posts one price and when it can post different prices for buyer and seller. We show several surprising results about the separation between the different scenarios. In particular we show that (a) it is impossible to achieve sublinear  $\alpha$ -regret for any  $\alpha < 2$ , (b) but with full feedback sublinear 2-regret is achievable (c) with a single price and partial feedback one cannot get sublinear  $\alpha$  regret for any constant  $\alpha$  (d) nevertheless, posting two prices even with one-bit feedback achieves sublinear 2-regret, and (e) there is a provable separation in the 2-regret bounds between full and partial feedback.

## 1 Introduction

The bilateral trade problem arises when two rational agents, a seller and a buyer, wish to trade a good; they both hold a private valuation for it, and their goal is to maximize their utility. The solution of the problem consists in designing a mechanism that intermediates between the two parties to make the trade happen. Ideally, the mechanism should maximize social welfare even though the agents act strategically (*incentive compatibility*) and should guarantee non-negative utility to the agents (*individual rationality*). Furthermore, we are interested in mechanisms for bilateral trade that do not subsidize the agents (*budget balance*). Obvious mechanisms that satisfy incentive compatibility, individual rationality, and budget balanced, are posted price mechanisms. Two common metrics are used to measure the efficiency of a mechanism: social welfare subsequent to trade and gain from trade (*i.e.*, the increase in social welfare). Consider a mechanism that posts prices  $p$  (price for the seller) and  $q$  (price for the buyer) to agents with valuations  $s$  and  $b$ , formally we have:

- **Social Welfare:**  $\text{SW}(p, q, s, b) = s + (b - s) \cdot \mathbb{I}\{s \leq p \leq q \leq b\}$ <sup>1</sup>
- **Gain from trade:**  $\text{GFT}(p, q, s, b) = (b - s) \cdot \mathbb{I}\{s \leq p \leq q \leq b\}$

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<sup>1</sup>We use  $\mathbb{I}\{Q\}$  for the indicator variable that takes the value 1 if the predicated  $Q$  is true and zero otherwise.

It is clear from these expressions that if we are interested in exact optimality then maximizing gain from trade is equivalent to maximizing social welfare. However, Myerson and Satterthwaite [1983] showed that there are *no mechanisms* for bilateral trade that are simultaneously social welfare maximizing (alternately, gain from trade maximizing), incentive compatible, individually rational, and budget balanced<sup>2</sup>. It follows that the best one can hope for is an incentive compatible, individually rational, and budget balanced mechanism that approximates the optimal social welfare. This creates an asymmetry between the two metrics: a multiplicative  $c$  approximation to the maximal gain from trade implies an approximation at least as good ( $\geq c$ ) to the maximal social welfare but not vice versa. Ergo, it is harder to approximate the gain from trade than to approximate social welfare. For example, consider an instance where the seller has valuation 0.99 and the buyer is willing to pay up to 1: irrespective of if a trade occurs or not, 99% of the optimal social welfare is guaranteed. In particular, a mechanism that posts a price of zero and generates no trade gets a good approximation to the social welfare. Contrariwise, the gain from trade is non-zero only if the mechanism manages to post prices in the narrow  $[0.99, 1]$  interval.

The vast body of work subsequent to Myerson and Satterthwaite [1983] primarily considers the Bayesian version of the problem, where agents' valuations are drawn from some distribution and the efficiency is evaluated in expectation with respect to the valuations' randomness. There are many incentive compatible mechanisms that give a constant approximation to the social welfare (in the Bayesian setting), e.g., see Blumrosen and Dobzinski [2014]. On the other hand, finding a constant approximation to the gain from trade has been a long standing problem and only a very recent paper of Deng et al. [2022] has given a Bayesian incentive compatible mechanism for this problem. In this paper we deal with the harder scenario where an adversary determines seller and buyer valuations (*i.e.*, valuations are not drawn from some distribution). Ergo, positive results in the Bayesian setting are inapplicable in our setting.

Following Cesa-Bianchi et al. [2021a], we consider the sequential adversarial bilateral trade problem, where at each time step  $t$ , a new seller-buyer pair arrives. The seller has some private valuation  $s_t \in [0, 1]$  representing the smallest price she is willing to accept; conversely, the buyer holds as private information  $b_t \in [0, 1]$ , *i.e.*, the largest price she is willing to pay to get the good. Concurrently, the mechanism posts price  $p_t$  to the seller and  $q_t$  to the buyer. If they both accept ( $s_t \leq p_t$  and  $q_t \leq b_t$ ), then the trade happens at those prices, otherwise the agents leave forever. By the requirement that the mechanism be budget balanced, the prices posted by the mechanism are such that  $p_t \leq q_t$ . At the end of each time step, the mechanism receives some feedback that depends on the outcome of the trade. Ideally, we would like to have a strategy for the sequential bilateral trade problem whose average gain from trade converges to that of the best fixed posted price mechanism in hindsight. However, as Cesa-Bianchi et al. [2021a] showed, this is a hopeless task.

Our goal in this work is then to achieve mechanisms whose average performance converges to a constant factor of the best fixed posted price mechanism in hindsight. We would like to find the smallest  $\alpha \geq 1$  such that the  $\alpha$ -regret [Kakade et al., 2009] is sublinear in the time horizon  $T$ :

$$\max_{p, q} \sum_{t=1}^T \text{GFT}(p, q, s_t, b_t) - \alpha \cdot \mathbb{E} \left[ \sum_{t=1}^T \text{GFT}(p_t, q_t, s_t, b_t) \right].$$

If the goal is only to maximize gain from trade, there is never any sense in offering two different prices (to the seller and buyer). However, critically, offering two prices is provably helpful in the context of a learning algorithm.

To conclude the description of our learning framework, we specify the type of feedback received by the mechanism. We focus on the two extremes of the feedback spectrum. On the one hand we study the full feedback model, where, after prices are posted, the mechanism learns both seller and buyer valuations ( $s_t, b_t$ ). On the other hand, we investigate a more realistic *partial feedback* model, the *one-bit feedback*, where the learner only discovers if a trade took place or not. We also consider an intermediate (partial feedback) model, called the *two-bit feedback model*. In this model, the learner posts (one or two) prices, and learns if the buyer is willing to trade and if the seller is willing to trade, at these prices. Clearly, a trade actually occurs only if both are willing to trade. Note that these two models enforce the desirable property that buyers and sellers only communicate to the mechanism a minimal amount of information useful for the trade, without disclosing their actual valuations.

<sup>2</sup>This impossibility result holds even when the (private) agents valuations are assumed to be drawn from some (public) random distributions and the incentive compatibility is only enforced in expectation.

	Full Feedback	Two-bit feedback	one-bit feedback
Single price	$O(\sqrt{T})$ - Theorem 2	$\Omega(T)$ - Theorem 4	
Two prices	$\Omega(\sqrt{T})$ - Theorem 3	$\Omega(T^{2/3})$ - Theorem 6	$O(T^{3/4})$ - Theorem 5

Table 1: Summary of 2-regret results in various settings.

## 1.1 Overview of our Results

We present our results for the adversarial sequential bilateral trade problem (see also Table 1).

- We show that no learning algorithm can achieve sublinear  $\alpha$ -regret for any  $\alpha < 2$  (Theorem 1). This holds in the full feedback model (and thus for both partial feedback models).
- We give a learning algorithm with full feedback that achieves  $\tilde{O}(\sqrt{T})$  2-regret<sup>3</sup> (Theorem 2) and show that no algorithm can improve upon this (Theorem 3).
- We show that if limited to a single price, no learning algorithm achieves sublinear  $\alpha$ -regret for any constant  $\alpha$  in either partial feedback models, *i.e.*, one or two-bit feedback (Theorem 4).
- Given the negative results above, we show that allowing the learning algorithm to post two prices gives sublinear 2-regret even for one-bit feedback (Theorem 5). This means that our learning algorithm achieves, on average, at least half of the gain from trade of the best fixed price in hindsight, using only one-bit of feedback at each step! We show a separation between partial versus full feedback by giving a  $\Omega(T^{2/3})$  lower bound in the former model on the 2-regret for any learning algorithm (Theorem 6).

The gaps in Table 1 may appear misleading because upper bounds in weaker models apply in stronger models and lower bounds in stronger models apply in weaker models. The only remaining open gap in our results is between the  $\Omega(T^{2/3})$  lower bound and the  $O(T^{3/4})$  upper bound that hold for two prices and partial feedback (either one or two-bit feedback). It is also worth noting that in our worst case model two prices are required but one-bit suffices for sublinear 2-regret. This is a different qualitative behaviour that the one observed in the stochastic case [Cesa-Bianchi et al., 2021a], where it is enough to use one single price but the two-bit feedback is required to achieve sublinear (1-)regret. One may wonder why two prices are helpful at all in our adversarial setting, given their suboptimality in maximizing the gain from trade. It turns out that randomizing over two prices it is possible to estimate the (non-stochastic) valuations of the agents.

## 1.2 Technical challenges

**From experts to prices.** As already observed in Cesa-Bianchi et al. [2021a], the full feedback model nicely fits into the prediction with experts framework [Cesa-Bianchi and Lugosi, 2006]: there is a clear mapping between expert and prices and the mechanism can easily reconstruct the gain that each price/expert experiences using the feedback received. The main challenge here is given by the continuous nature of the possible prices, as the usual experts framework assumes a finite number of experts. There are workarounds that exploit some regularity of the gain function such as the Lipschitz property or convexity/concavity [see, e.g., Cesa-Bianchi and Lugosi, 2006, Hazan, 2016, Slivkins, 2019]. Unfortunately, gain from trade is not such a function. Moreover, in our adversarial setting we cannot adopt the smoothing trick used in Cesa-Bianchi et al. [2021a], where they assume some regularity on the agents distribution to argue that  $\mathbb{E}[\text{GFT}(\cdot)]$  becomes Lipschitz. Our main technical tool to address this issue is a discretization claim that allows us to compare the performance of the best fixed price in  $[0, 1]$  with that of the best on a finite grid.

**A magic estimator.** Consider any of the two partial feedback models; there at each time step  $t$  the learner only receives a minimal information about what happened at time  $t$ : namely, one or two-bit versus the full knowledge of  $\text{GFT}_t(\cdot)$ . Note that this type of feedback is strictly more difficult than the classic bandit feedback [Cesa-Bianchi and Lugosi, 2006], where the learner always observes at least the gain its action incurred. Our main technical tool to circumvent this issue is given by the design of a procedure that, posting two randomized prices, is able to estimate the  $\text{GFT}_t$  in a given price. This unbiased estimator is then used in a carefully designed block decomposition of the time horizon to achieve sublinear 2-regret in presence of this very poor feedback.

<sup>3</sup>The  $\tilde{O}$  hides poly-logarithmic terms

**Lower bounds.** For our lower bounds we adopt two different strategies. In Theorems 1 and 4 we construct randomized instances where no algorithm can learn anything: the only prices the learner could use to discriminate between different instances are cautiously hidden, while all the other prices do not reveal any useful information, given the type of feedback considered. The randomized instances used in Theorems 3 and 6 involve instead a more structured approach; this is due to the challenge posed by the contemporary handling of the multiplicative and additive part of the 2-regret. To this end, we carefully hide the optimal ex-post prices and make hard to the learner to achieve small (1-)regret with respect to some “second best” prices. A crucial task we often face is to “hide” some small finite sets of critical prices from the learning algorithm. We employ two techniques to do so: random shifts (as in the proof of Theorem 4) and repeatedly dividing overlaps (Theorems 1 and 3).

### 1.3 Related work

The work that is most closely related to ours is Cesa-Bianchi et al. [2021a]. There, the authors study the same sequential bilateral trade problem as we do, with the objective of minimizing the (1-)regret with respect to the best fixed price. They focus on the (easier) stochastic model, where the adversary chooses a distribution over valuations and not a deterministic sequence like in our model. A full characterization of the minimax regret regimes is offered, for the same type of feedback we consider (not that the one-bit feedback is only addressed in their extended version [Cesa-Bianchi et al., 2021b]) and with various regularity assumptions on the underlying random distributions. Cesa-Bianchi et al. [2021a] also give the first result for the adversarial setting we consider, showing that no learning algorithm can achieve sublinear 1-regret.

Regret minimization in the context of economics has been studied in many papers [e.g., Morgenstern and Roughgarden, 2015, Cesa-Bianchi et al., 2015, Ho et al., 2016, Daskalakis and Syrgkanis, 2016, Lykouris et al., 2016]. In particular, Kleinberg and Leighton [2003] studied the one-sided pricing problem, offering a  $\tilde{O}(T^{2/3})$  upper bound on the regret in the adversarial setting and opening a fruitful line of research [Blum et al., 2004, Blum and Hartline, 2005, Bubeck et al., 2019]. The notion of  $\alpha$ -regret has been formally introduced by Kakade et al. [2009], but was already present in Kalai and Vempala [2005]. It has then found applications in linear [Garber, 2021] and submodular optimization [Roughgarden and Wang, 2018], learning with sleeping actions [Emamjomeh-Zadeh et al., 2021], combinatorial auctions [Roughgarden and Wang, 2019] and market design [Niazadeh et al., 2021]. We mention that our work fits in the line of research that studies online learning with feedback models different from full information and the bandit ones; our one and two-bit feedback models share similarities with the feedback graphs model (see e.g., Alon et al. [2017], van der Hoeven et al. [2021], Esposito et al. [2022]) and the partial monitoring framework (see e.g., Bartók et al. [2014], Lattimore and Szepesvári [2019]).

While Myerson and Satterthwaite [1983] were the first to thoroughly investigate the bilateral trade problem in the Bayesian setting with their famous impossibility result, it was the seminal paper of Vickrey [Vickrey, 1961] that introduced the problem, proving that any mechanism that is welfare maximizing, individually rational, and incentive compatible may not be budget balanced. In the Bayesian setting, it was only very recently that Deng et al. [2022] gave the first (Bayesian) incentive compatible, individually rational and budget balanced mechanism achieving a constant factor approximation of the optimal gain from trade. Prior to this paper, a posted price  $O(\log \frac{1}{r})$  approximation bound was achieved by [Colini-Baldeschi et al., 2017], with  $r$  being the probability that a trade happens (i.e., the value of the buyer is higher than the value of the seller). The literature also includes many individually rational, incentive compatible and budget balanced mechanisms achieving a constant factor approximation of the optimal social welfare. Blumrosen and Dobzinski [2014] proposed a simple posted price mechanism, the median mechanism, yielding a 2-approximation of the optimal social welfare; the same authors then implemented a randomized fixed price mechanism improving the approximation to  $e/(e-1)$  [Blumrosen and Dobzinski, 2021]. Recently, Dütting et al. [2021] showed that even posting one single sample from the seller distribution as price is enough to achieve a 2 approximation to the optimal social welfare. The class of fixed price mechanism is of particular interest as it has been showed that all (dominant strategy) incentive compatible and individually rational mechanisms that enforce a stricter notion of budget balance, i.e., the so-called strong budget balance (where the mechanism is not allowed to subsidize or extract revenue from the agents) are indeed fixed price [Hagerty and Rogerson, 1987, Colini-Baldeschi et al., 2016].

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## Learning Protocol of Sequential Bilateral Trade

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**for** time  $t = 1, 2, \dots$  **do**  
 a new seller/buyer pair arrives with (hidden) valuations  $(s_t, b_t) \in [0, 1]^2$   
 the learner posts prices  $p_t, q_t \in [0, 1]$   
 the learner receives a (hidden) reward  $\text{GFT}_t(p_t, q_t) := \text{GFT}(p_t, q_t, s_t, b_t) \in [0, 1]$   
 a feedback  $z_t$  is revealed

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## 2 Preliminaries

The formal protocol for the sequential bilateral trade follows Cesa-Bianchi et al. [2021a]. At each time step  $t$ , a new pair of seller and buyer arrives, each with private valuations  $s_t$  and  $b_t$  in  $[0, 1]$ ; the learner posts two prices:  $p_t \in [0, 1]$  to the seller and  $q_t \in [0, 1]$  to the buyer. A trade happens if and only if both agents agree to trade, i.e., when  $s_t \leq p_t$  and  $q_t \leq b_t$ . Since we want our mechanism to enforce budget balance, we require that  $p_t \leq q_t$  for all  $t$ . When a trade occurs, the learner is awarded with the resulting increase in social welfare, i.e.,  $b_t - s_t$ . The learner then observes some feedback  $z_t$ . The gain from trade at time  $t$  depends on the valuations  $s_t$  and  $b_t$  and on the price posted. To simplify the notation we introduce the following:

$$\text{GFT}_t(p, q) := \text{GFT}(p, q, s_t, b_t) = \mathbb{I}\{s_t \leq p \leq q \leq b_t\} \cdot (b_t - s_t)$$

When the two prices are equal, we omit one of the arguments to simplify the notation.

Given any constant  $\alpha \geq 1$ , the  $\alpha$ -regret of a learning algorithm  $\mathcal{A}$  against a sequence of valuations  $\mathcal{S}$  on time horizon  $T$  is defined as follows

$$R_T^\alpha(\mathcal{A}, \mathcal{S}) := \max_{p, q \in [0, 1]^2} \sum_{t=1}^T \text{GFT}_t(p, q) - \alpha \cdot \sum_{t=1}^T \mathbb{E}[\text{GFT}_t(p_t, q_t)].$$

In the right side of the equation the dependence on  $\mathcal{S}$  is contained in the  $\text{GFT}_t(\cdot)$ . Note that the expectation in the previous formula is with respect to the internal randomization of the learning algorithm:  $p_t$  and  $q_t$  are the (possibly random) prices posted by  $\mathcal{A}$ .

The  $\alpha$ -regret of a learning algorithm  $\mathcal{A}$ , without specifying the dependence of the sequence, is defined as its  $\alpha$ -regret against the “worst” sequence of valuations:  $R_T^\alpha(\mathcal{A}) := \sup_{\mathcal{S}} R_T^\alpha(\mathcal{A}, \mathcal{S})$ . Stated differently, the performance of an algorithm is measured against an oblivious adversary that generates the sequence of valuations ahead of time: the learner has to perform well on *all* possible sequences. In this paper we study the minimax  $\alpha$ -regret,  $R_T^{\alpha, \star}$ , that measures the performance of the best (learning) algorithm versus the optimal fixed price in hindsight, on the worst possible instance:  $R_T^{\alpha, \star} := \inf_{\mathcal{A}} R_T^\alpha(\mathcal{A})$ . The set of learning algorithms we consider depends on which of the various settings we are dealing with. In this paper we consider a variety of such settings (i.e., how many prices are posted, what feedback is available, see below Sections 2.1 and 2.2).

### 2.1 Single Price vs. Two Prices — Seller price and Buyer price

We consider two families of learning algorithms, differing in the nature of the probe they perform, corresponding to two notions of what it means to be budget balanced:

**Single price mechanisms.** If we want to enforce a stricter notion of budget balance, namely strong budget balance, the mechanism is neither allowed to subsidize nor extract revenue from the system. This is modeled by imposing  $p_t = q_t$ , for all  $t$ . If  $p_t = q_t$  we use the notation  $\text{GFT}_t(p_t)$  to represent the gain from trade at time  $t$ .

**Two price mechanisms.** If we require that the mechanism enforces (weak) budget balance, it can post two different prices,  $p_t$  to the seller and  $q_t$  to the buyer, as long as  $p_t \leq q_t$ . I.e., we only require that the mechanism never subsidize a trade, we do not require that the mechanism not make a profit. In this setting we use the notation  $\text{GFT}_t(p_t, q_t)$  to represent the gain from trade at time  $t$ .

**Observation 1.** *Note that the only reason to post two prices is to obtain information. For any pair of prices  $(p, q)$  with  $p < q$  posting any single price  $\pi \in [p, q]$  guarantees no less gain from trade.*

In particular, any budget balanced algorithm that knows the future and seeks to maximize gain from trade while repeatedly posting the same prices will never choose two different prices.

## 2.2 Feedback models

We consider three types of feedback, presented here in increasing order of difficulty for the learner. (Note that full feedback “implies” two-bit feedback which in turn implies one-bit feedback):

**Full feedback.** In the full feedback model, the learner receives both seller and buyer valuations, immediately after posting prices the feedback to the learner at time  $t$ : formally,  $z_t = (s_t, b_t)$ . E.g., both seller and buyer send sealed bids that are opened immediately after the [one or two] price[s] are revealed. It follows from Observation 1 that in the full feedback model there is never any reason to post two prices, as all the relevant information is revealed anyway.

**Two-bit feedback.** In two-bit feedback the algorithm observes separately if the two agents agree on the given price, *i.e.*, the feedback at time  $t$  is  $z_t = (\mathbb{I}\{s_t \leq p_t\}, \mathbb{I}\{q_t \leq b_t\})$ .

**One-bit feedback** The one-bit feedback is arguably the minimal feedback the learner could get: the only information revealed is whether the trade occurred or not, *i.e.*,  $z_t = \mathbb{I}\{s_t \leq p_t \leq q_t \leq b_t\}$ .

## 2.3 Lower bounds via Yao’s Minimax Theorem

An important technical tool we use to prove our lower bounds is the well known Yao’s Minimax Theorem [Yao, 1977]. In particular, we apply the easy direction of the theorem, which reads (using our terminology) as follows: the  $\alpha$ -regret of a randomized learner against the worst-case valuations sequence is at least the minimax regret of the optimal deterministic learner against a stochastic sequence of valuations. Formally,

$$R_T^{\alpha, \star} \geq \sup_{\mathcal{A}} \mathbb{E} \left[ \max_{p, q \in [0, 1]^2} \sum_{t=1}^T \text{GFT}_t(p, q) - \alpha \cdot \sum_{t=1}^T \text{GFT}_t(p_t, q_t) \right],$$

where the expectation is with respect to the stochastic valuation sequence  $\mathcal{S}$ , while  $\mathcal{A}$  denotes a deterministic learner. We remark that — for the minimax theorem to be applicable — the random instance  $\mathcal{S}$  has to be oblivious of the learner.

## 2.4 Regret due to discretization

Our first theoretical result concerns the study of how discretization impacts the regret. In particular, we compare the performance of the best fixed price taken from the continuous set  $[0, 1]$  to that of the best fixed price chosen from some discrete grid  $Q \subset [0, 1]$ . Optimizing over a continuous set may seemingly be a problem because our object, gain from trade, is discontinuous (thus non-Lipschitz), non-convex and non-concave; one cannot use the “standard approach” that makes use of such regularity conditions. What we show in the following Claim is that it is possible to compare the performance of the best continuous fixed price with *twice* that of the best fixed price on the grid.

**Claim 1** (Discretization error). *Let  $Q = \{q_0 = 0 \leq q_1 \leq q_2 \leq \dots \leq q_n = 1\}$  be any finite grid of prices in  $[0, 1]$  and let  $\delta(Q)$  be the largest difference between two contiguous prices, *i.e.*,  $\max_{i=1, \dots, n} |q_i - q_{i-1}|$ , then for any sequence  $\mathcal{S} = (s_1, b_1), \dots, (s_T, b_T)$  and any price  $p$  we have*

$$\sum_{t=1}^T \text{GFT}_t(p) \leq 2 \cdot \max_{q \in Q} \sum_{t=1}^T \text{GFT}_t(q) + \delta(Q) \cdot T.$$

Let  $\mathcal{A}$  be any learning algorithm that posts prices  $(p_t, q_t)$ , then the following inequality holds:

$$R_T^2(\mathcal{A}) \leq 2 \sup_{\mathcal{S}} \left\{ \max_{q \in Q} \sum_{t=1}^T \text{GFT}_t(q) - \sum_{t=1}^T \mathbb{E} [\text{GFT}_t(p_t, q_t)] \right\} + \delta(Q) \cdot T. \quad (1)$$

*Proof.* Fix any sequence of valuations  $\mathcal{S}$  and let  $p^*$  be the corresponding best fixed price:  $p^* \in \arg \max_{p \in [0, 1]} \sum_{t=1}^T \text{GFT}_t(p)$ . If  $p^* \in Q$ , then there is nothing to prove; alternatively let  $q^+$  and  $q^-$  be the consecutive prices on the grid such that  $p^* \in [q^-, q^+]$ . For any time  $t$  where  $\text{GFT}_t(p^*) > 0$ , either  $p^* \in [s_t, b_t] \subseteq [q^-, q^+]$ , in which case

$$\text{GFT}_t(p^*) \leq (b_t - s_t) \leq (q^+ - q^-) \leq \delta(Q),$$

or  $[s_t, b_t] \cap \{q^+, q^-\} \neq \emptyset$ , and therefore  $GFT_t(p^*) = \max\{GFT_t(q^+), GFT_t(q^-)\}$ . All in all, we have that, for each time  $t$ , the following inequality holds:

$$GFT_t(p^*) \leq GFT_t(q^+) + GFT_t(q^-) + \delta(Q)$$

Summing up over all times  $t$  we get:

$$\sum_{t=1}^T GFT_t(p^*) \leq \sum_{t=1}^T GFT_t(q^+) + \sum_{t=1}^T GFT_t(q^-) + \delta(Q)T \leq 2 \cdot \max_{q \in Q} \sum_{t=1}^T GFT_t(q) + \delta(Q)T. \quad (2)$$

Focus now on the second part of the claim and fix any learning algorithm  $\mathcal{A}$ , we have:

$$\begin{aligned} R_T^2(\mathcal{A}) &= \sup_S \left\{ \sum_{t=1}^T GFT_t(p^*) - 2 \cdot \sum_{t=1}^T \mathbb{E}[GFT_t(p_t, q_t)] \right\} \\ &\leq \sup_S \left\{ \sum_{t=1}^T GFT_t(p^*) - 2 \cdot \max_{q \in Q} \sum_{t=1}^T GFT_t(q) \right\} \\ &\quad + \sup_S \left\{ 2 \max_{q \in Q} \sum_{t=1}^T GFT_t(q) - 2 \sum_{t=1}^T \mathbb{E}[GFT_t(p_t, q_t)] \right\} \\ &\leq T \cdot \delta(Q) + 2 \sup_S \left\{ \max_{q \in Q} \sum_{t=1}^T GFT_t(q) - \sum_{t=1}^T \mathbb{E}[GFT_t(p_t, q_t)] \right\}, \end{aligned}$$

where the last inequality follows from Equation (2) that holds for all sequences.  $\square$

Before moving to the next section, we spend some words to compare our discretization result with the one in Cesa-Bianchi et al. [2021a] (Second decomposition Lemma). There the authors exploit the stochastic nature of the valuations to argue that  $\mathbb{E}[GFT_t(\cdot)]$  is Lipschitz, under some regularity assumptions on the random distributions. We study the adversarial model, thus we cannot use this “smoothing” procedure; this is why we lose an extra multiplicative factor of 2.

### 3 Full Feedback

In this section we study the full feedback model, where the learner receives as feedback both seller and buyer valuations after posting a single price (see Observation 1). The learner can thus evaluate  $GFT_t(p)$  for all  $p \in [0, 1]$ , independently by the price posted. Even with this very rich feedback we show that the impossibility result from Cesa-Bianchi et al. [2021a], *i.e.*, no learning algorithm achieves sublinear regret (1-regret) in the sequential bilateral trade problem, can be extended to hold for  $\alpha$ -regret for all  $\alpha \in [1, 2)$ .

To prove this result, formalized in Theorem 1, we use Yao’s Minimax Theorem: a randomized family of valuations sequences is constructed, with the property that any deterministic learner would suffer, on average, linear  $2 - \varepsilon$  regret against it. The detailed proof is provided below, but we sketch here the main ideas. Specifically, any valuations sequence from the randomized family consists of (sell, buy) prices that have the form  $(0, b_i)$  or  $(s_i, 1)$ , for some carefully designed  $\{s_i\}_i$  and  $\{b_i\}_i$ . These sequences are generated iteratively in a way such that all realized [sell, buy] segments overlap and the next segment is chosen at random among two disjoint options  $(0, b_i)$  or  $(s_i, 1)$ . Since all realized [sell, buy] segments overlap, there is at least one price in the intersection of all intervals: this is the optimal fixed price in hindsight. Conversely, at each time step no learner can post a price that guarantees a trade with probability greater than  $1/2$ , thus yielding the lower bound.

**Theorem 1** (Lower bound on  $(2 - \varepsilon)$ -regret). *In the full-feedback model, for all  $\varepsilon \in (0, 1]$  and horizons  $T$ , the minimax  $(2 - \varepsilon)$ -regret satisfies  $R_T^{2-\varepsilon, \star} \geq \frac{1}{8}\varepsilon T$ .*

*Proof.* We prove this lower bound via Yao’s Theorem. Fix any  $\varepsilon \in (0, 1]$ , we argue that there exist some constant  $c_\varepsilon$  and a distribution over sequences such that the  $(2 - \varepsilon)$ -regret of any deterministic learning algorithm  $\mathcal{A}$  against it is, on average, at least  $c_\varepsilon \cdot T$ . Our construction is reminiscent—and to some extent simplifies—the one given in Theorem 4.6 of Cesa-Bianchi et al. [2021a], but presents

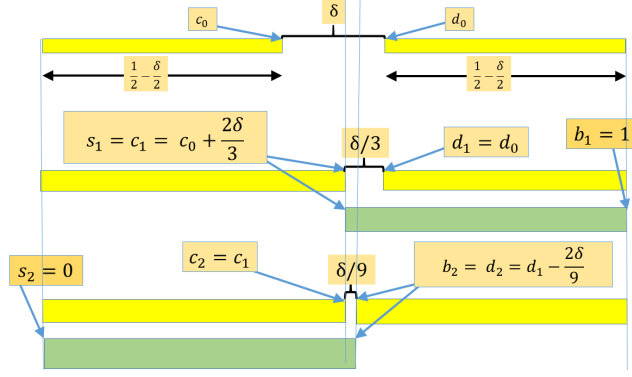


Figure 1: Lower bound construction that “hides” the optimal price.

one main difference: here we construct a family of instances that is oblivious to the learner, whereas in Cesa-Bianchi et al. [2021a] they construct a single instance tailored to the learning algorithm  $\mathcal{A}$ . It is critical for the application of Yao’s Theorem that the sequence distribution be independent of the actual algorithm.

Let  $\delta < \varepsilon/8$ , the adversary initiates two auxiliary sequences of points  $c_0 = \frac{1}{2} - \frac{1}{2}\delta$  and  $d_0 = \frac{1}{2} + \frac{1}{2}\delta$  then, inductively constructs the auxiliary sequences and draws  $s_{t+1}$  and  $b_{t+1}$  as follows:

$$\begin{cases} c_{t+1} := c_t, d_{t+1} := d_t - \frac{2\delta}{3^t}, s_{t+1} := 0, b_{t+1} := d_{t+1}, & \text{with probability } 1/2 \\ c_{t+1} := c_t + \frac{2\delta}{3^t}, d_{t+1} := d_t, s_{t+1} := c_{t+1}, b_{t+1} := 1, & \text{with probability } 1/2. \end{cases}$$

A quick description of the procedure: at the beginning of each time step  $t + 1$  the adversary has two points,  $c_t$  and  $d_t$ , with  $d_t - c_t = \delta/3^t$ . Then, it chooses uniformly at random between *left* or *right*. If *left* is chosen (first line of the construction), then  $c_{t+1} = c_t$ , while  $d_{t+1}$  is moved to the first third of the  $[c_t, d_t]$  interval:  $d_{t+1} = d_t - \frac{2\delta}{3^t} = c_t + \frac{\delta}{3^t}$ , and the adversary posts prices  $s_{t+1} = 0$  and  $b_{t+1} = d_{t+1}$ . If *right* is chosen, then the symmetric event happens:  $d_{t+1} = d_t$ ,  $c_{t+1}$  moves to the second third of  $[c_t, d_t]$  and the adversary posts  $s_{t+1} = c_{t+1}$  and  $b_{t+1} = 1$ . A pictorial representation of a sample run of this procedure is given in Figure 1.

At each time step the two possible realizations (for a fixed past) of the  $[s_t, b_t]$  intervals are disjoint: it implies that *any price* the learner posts results in a trade with probability (over the randomness of the adversary) of at most  $1/2$ . As we are in a full feedback scenario, there is no point for the learner to post two prices, so we assume that  $\mathcal{A}$  posts a single price.

For any realization of the randomness used in the construction of the sequence,  $[s_t, b_t]$  intervals have a non-empty intersection; let  $p^*$  be some price in this intersection. Moreover, at all time steps  $t$  it holds that  $(b_t - s_t) \geq \frac{1}{2} - \frac{\delta}{2}$ . All in all, this gives a simple bound on the total gain from trade of the best price in hindsight that holds for any realization of the valuations sequence:

$$\max_{p \in [0,1]} \sum_{t=1}^T \text{GFT}_t(p) = \sum_{t=1}^T \text{GFT}_t(p^*) \geq \frac{T}{2} (1 - \delta).$$

Consider now what happens to the learner. We already argued that at each time step the learner obtains a trade with probability at most  $1/2$ . Furthermore,  $(b_t - s_t) \leq \frac{1}{2} + \frac{\delta}{2}$  for all realizations. Thus:

$$\mathbb{E} \left[ \sum_{t=1}^T \text{GFT}_t(p_t) \right] \leq \sum_{t=1}^T \frac{1}{2} \left( \frac{1}{2} + \frac{\delta}{2} \right) = \frac{T}{2} \left( \frac{1}{2} + \frac{\delta}{2} \right)$$

At this point we have the desired explicit bound on the  $2 - \varepsilon$  regret via Yao’s Theorem:

$$R_T^{2-\varepsilon}(\mathcal{A}) \geq \frac{T}{2} (1 - \delta) - (2 - \varepsilon) \frac{T}{2} \left( \frac{1}{2} + \frac{\delta}{2} \right) = \frac{T}{4} (\varepsilon + \varepsilon\delta - 4\delta) \geq \frac{1}{8} \varepsilon T.$$

□



If we look for positive results, we note that there is a clear connection of our problem in the full feedback and the prediction with experts framework [Cesa-Bianchi and Lugosi, 2006]. In particular, if we simplify the task of the learner and ask it to be competitive against *the best price in a finite grid*, we can use classical results on prediction with experts as a black box. Combining this fact with our discretization result (Claim 1), we can show an  $\tilde{O}(\sqrt{T})$  upper bound on the 2-regret.

**Theorem 2** (Upper bound on 2-regret given full feedback). *In the full-feedback setting, there exists a learning algorithm  $\mathcal{A}$  whose 2-regret, for  $T$  large enough, respects  $R_T^2(\mathcal{A}) \leq 5 \cdot \sqrt{T \cdot \log T}$ .*

*Proof.* Consider a grid of prices  $Q$  composed by  $T + 1$  equally spaced points:  $q_i = i/T$  for  $i = 0, 1, \dots, T$  and choose your favourite prediction with experts learning algorithm, e.g., Multiplicative Weights [Arora et al., 2012]. Given the full feedback regime, and the fact that the grid is finite, we can run expert algorithm using as experts the points on the grid. Typically, the best experts learning algorithm exhibit a bound on the regret  $O(\sqrt{T \log K})$ , that becomes  $O(\sqrt{T \log T})$  in our case since we have  $T + 1$  experts. If we use the Multiplicative Weights algorithm against the best fixed price on the grid  $Q$  with  $\eta = \sqrt{\frac{\log T}{T}}$ , we get by Theorem 2.5 of Arora et al. [2012]:

$$\sup_S \left\{ \max_{q \in Q} \sum_{t=1}^T \text{GFT}_t(q) - \sum_{t=1}^T \mathbb{E}[\text{GFT}_t(p_t)] \right\} \leq 2\sqrt{T \log(T+1)}$$

Plugging this bound in Claim 1, we get the desired order of regret.

$$\begin{aligned} R_T^2(\mathcal{A}) &\leq 2 \sup_S \left\{ \max_{q \in Q} \sum_{t=1}^T \text{GFT}_t(q) - \sum_{t=1}^T \mathbb{E}[\text{GFT}_t(p_t)] \right\} + \delta(Q) \cdot T \\ &\leq 4\sqrt{T \log(T+1)} + 1 \leq 5\sqrt{T \log(T)}. \end{aligned}$$

The first inequality is just a restatement of Equation (2) from Claim 1. The second inequality follows by combining the bound on the regret of multiplicative weight and the fact that the grid is equally spaced, thus  $\delta(Q) = 1/T$ .  $\square$

We conclude the analysis of repeated bilateral trade in the full feedback model with a lower bound that shows that the previous result is tight up to a logarithmic factor: the minimax 2-regret of the full feedback problem is  $\tilde{\Theta}(\sqrt{T})$ . The proof uses once again Yao's Theorem and consists in constructing a randomized family of sequences such that any deterministic learning algorithm suffers, in expectation, a  $\Omega(\sqrt{T})$  2-regret. The detailed construction is described below and it involves the careful combination of two scaled copies of the hard sequences used in the proof of Theorem 1. As a technical ingredient, we need the following property of Random walks.

**Lemma 1** (Property of Random Walks). *Let  $S_T$  be a symmetric random walk on the line after  $T$  steps, starting from 0. Then, for  $T$  large enough, it holds that  $\mathbb{E}[|S_T|] \geq \frac{2}{3}\sqrt{T}$ .*

*Proof.* It is well known that the expected distance of a random walk from the origin grows like  $\Theta(\sqrt{T})$ . Formally, the following asymptotic result holds [e.g., Palacios, 2008]

$$\lim_{T \rightarrow \infty} \frac{\mathbb{E}[|S_T|]}{\sqrt{T}} = \sqrt{\frac{2}{\pi}}.$$

Observe that  $\sqrt{\frac{2}{\pi}} > 2/3$ , thus there exists a finite  $T_0$  such that  $\mathbb{E}[|S_T|] \geq \frac{2}{3}\sqrt{T}$  for all  $T \geq T_0$ .  $\square$

**Theorem 3** (Lower bound on 2-regret given full feedback). *In the full-feedback model, for all horizons  $T$  large enough, the minimax 2-regret satisfies  $R_T^{2,*} \geq \frac{1}{13}\sqrt{T}$ .*

*Proof.* We show that there exists a distribution over valuations sequences such that any deterministic learning algorithm  $\mathcal{A}$  achieves, on average, at least a 2-regret of  $\frac{1}{13} \cdot \sqrt{T}$ . This is enough to conclude the proof via Yao's Theorem. It may be helpful to consider Figure 2 to visualize this construction. Fix some small  $\delta$  to be set later and consider two scaled copies of the lower bound construction from Theorem 1, one in  $[0, \frac{1}{2} - \delta]$  and the other is  $[\frac{1}{2} + \delta, 1]$ . Starting from  $c_0^L = \frac{1}{4} - \delta$ ,  $d_0^L = \frac{1}{4}$ ,  $c_0^R = \frac{3}{4}$ ,  $d_0^R = \frac{3}{4} + \delta$ , the left  $L$  and right  $R$  pair of sequences evolve over time and generate

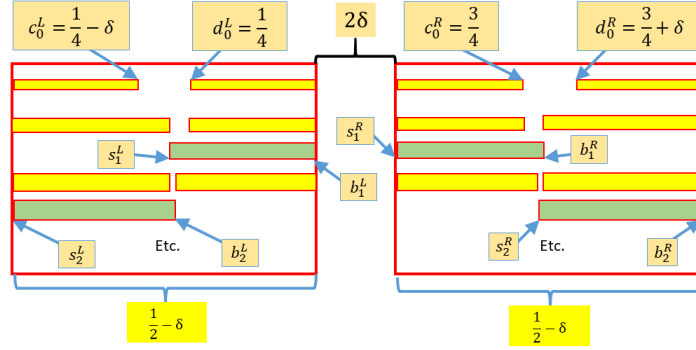


Figure 2: The proof of Theorem 3 makes use of two (appropriately scaled and shifted) copies of the lower bound from Theorem 1 (See Figure 1). In this example the left hand copy choose right and then left, while the right hand copy happened to choose left and then right. The (seller,buyer) bids at time  $t$  are then chosen independently at random from  $(s_t^L, b_t^L)$  and  $(s_t^R, b_t^R)$ .

two distinct sequences of valuations:  $(s_t^L, b_t^L) \subseteq [0, \frac{1}{2} - \delta]$  and  $(s_t^R, b_t^R) \subseteq [\frac{1}{2} + \delta, 1]$ . The actual sequence of valuations presented to the learner is based on these two sequences as follows: at each time step, the adversary tosses a fair coin, if it is a head, then it selects  $(s_t, b_t) = (s_t^L, b_t^L)$ , otherwise  $(s_t, b_t) = (s_t^R, b_t^R)$ . Observe that there are two independent sources of randomness in the adversary construction: the one responsible of the generation of the auxiliary sequences and the one toss of the left-right coin. We give now an upper bound on the expected performance of the learner at each time step  $t$ . Reasoning similarly to what we did in Theorem 1, there are four disjoint intervals of the  $[0, 1]$  interval where a price could cause a trade, and each one of them is the one chosen by the adversary with probability  $1/4$  ( $1/2$  given by the left-right coin and another  $1/2$ , independently, by the evolution of the sequences  $(s_t^L, b_t^L)$  and  $(s_t^R, b_t^R)$ ). All in all, this implies that for any price the algorithm posts, it results in a trade with probability at most  $1/4$ . Moreover, we have the property that  $(b_t - s_t) \leq (1/4 + \delta)$  at all times and for all realizations, therefore:  $\mathbb{E}[\text{GFT}_t(p_t)] \leq \frac{1}{4} (\frac{1}{4} + \delta)$ . We move now our attention to lower bounding the gain from trade of the best price in hindsight. Consider any realization of the sequence of coin tosses, we know that there exist two prices  $p_L^*$  and  $p_R^*$  such that  $p_L^*$  guarantees a trade in every time step where the result of the left-right coin gives left, and  $p_R^*$  does the same when the coin gives right. In addition, we know that  $(b_t - s_t) \geq \frac{1}{4} - \delta$ . All in all we have that, for all realizations of the randomness,

$$\sum_{t=1}^T \text{GFT}_t(p_L^*) + \sum_{t=1}^T \text{GFT}_t(p_R^*) \geq \frac{1}{4} - \delta.$$

At this point, fix the randomness of the auxiliary sequences and focus on the the one given by the coin tosses, and call  $X_t$  the indicator random variable of observing left from the coin at time  $t$ . We have:

$$\begin{aligned} \mathbb{E} \left[ \max_{p \in [0,1]} \sum_{t=1}^T \text{GFT}_t(p) \right] &= \mathbb{E} \left[ \max_{p \in \{p_L^*, p_R^*\}} \sum_{t=1}^T \text{GFT}_t(p) \right] \\ &\geq \left( \frac{1}{4} - \delta \right) \mathbb{E} \left[ \max \left\{ \sum_{t=1}^T \mathbb{I}\{p_L^* \in [s_t, b_t]\}, \sum_{t=1}^T \mathbb{I}\{p_R^* \in [s_t, b_t]\} \right\} \right] \\ &= \left( \frac{1}{4} - \delta \right) \mathbb{E} \left[ \max \left\{ \sum_{t=1}^T X_t, T - \sum_{t=1}^T X_t \right\} \right] \\ &= \left( \frac{1}{4} - \delta \right) \mathbb{E} \left[ \frac{T}{2} + \frac{1}{2} \max \left\{ 2 \sum_{t=1}^T X_t - T, T - 2 \sum_{t=1}^T X_t \right\} \right] \\ &= \left( \frac{1}{4} - \delta \right) \left( \frac{T}{2} + \frac{1}{2} \mathbb{E}[\|S_T\|] \right) \geq \left( \frac{1}{4} - \delta \right) \left( \frac{T}{2} + \frac{\sqrt{T}}{3} \right), \end{aligned}$$

where in the last inequality we used Lemma 1. Since the previous bound holds for any realization of the auxiliary sequences, it holds also in expectation over all the randomness. We can finally combine

the two results and conclude by Yao's Theorem that

$$\begin{aligned}
R_T^{2,*} &\geq \mathbb{E} \left[ \max_{p \in [0,1]} \sum_{t=1}^T \text{GFT}_t(p) - 2 \sum_{t=1}^T \text{GFT}_t(p_t) \right] \\
&\geq \left( \frac{1}{4} - \delta \right) \left( \frac{T}{2} + \frac{\sqrt{T}}{3} \right) - \frac{T}{2} \left( \frac{1}{4} + \delta \right) \\
&\geq \frac{1}{12} \sqrt{T} - \delta T - \delta \frac{\sqrt{T}}{3} \geq \frac{1}{13} \sqrt{T}
\end{aligned}$$

where in the last inequality we took  $\delta$  small enough, e.g.,  $\delta = 1/T$ .  $\square$

## 4 Partial Feedback

In this section, we study the partial feedback models where the learner receives very limited information on the realizations of the gain from trade. Specifically, one or two bits that describe the relative positions of the prices proposed to the agents and their valuations.

### 4.1 Lower bound on $\alpha$ -regret posting single price given two-bit feedback

Consider a learner that is constrained to post one single price at every iteration; the same to both seller and buyer. For this class of algorithms we show a very strong impossibility result, namely that for any constant  $\alpha$ , there exists no algorithm achieving sublinear  $\alpha$ -regret. We prove this in the two-bit feedback model and thus it trivially holds also if given one-bit feedback. The core of the lower bound construction resides in the possibility for the adversary to hide a *large* interval between many shorter ones; a learner posting only one price will not be able to locate it using partial feedback (which consists in just *counting* the number of intervals on the left and on the right).

**Theorem 4** (Lower bound on  $\alpha$ -regret posting single price, two-bit feedback). *In the two-bit feedback model where the learner is allowed to post one single price, for all horizons  $T \in \mathbb{N}$  and any constant  $\alpha > 1$ , the minimax  $\alpha$ -regret satisfies  $R_T^{\alpha,*} \geq \frac{1}{128\alpha^3} T$ .*

*Proof.* In this proof, we construct a randomized family of sequences that are impossible to distinguish using a single price and given two bid feedback. Furthermore, no deterministic algorithm is capable of achieving good regret against them in expectation. It may be useful to refer to Figure 3 for visualization. We first prove the claim under a “grid hiding” assumption that the learning algorithm is disallowed from posting prices in some fixed finite grid (to be defined below). We later justify the grid hiding assumption by introducing some minor perturbation to the grid.

Let  $\delta$  and  $\Delta$  two positive constants, with  $1 \geq \Delta > \delta$  to set later such that  $1/\Delta$ ,  $1/\delta$  and  $\Delta/\delta$  are integers. The grid used in the grid hiding assumption is composed by all integral multiples of  $\delta$ . For each  $i$  from 0 to  $1/\Delta - 1$ , consider the following sets of valuations:

$$\begin{aligned}
S_i &= \left\{ (i \cdot \Delta, (i+1)\Delta) \right\} \\
&\bigcup_{j \neq i} \bigcup_{k=0}^{\Delta/\delta-1} \left\{ (j \cdot \Delta + k \cdot \delta, j \cdot \Delta + (k+1)\delta) \right\} \bigcup_{k=1}^{\Delta/\delta-1} \left\{ (i \cdot \Delta + k \cdot \delta, i \cdot \Delta + k \cdot \delta) \right\} \quad (3)
\end{aligned}$$

The adversary constructs the first family of sequences as follows: to start, it selects uniformly at random  $i$  from 0 to  $1/\Delta - 1$ , then generates the sequence by repeatedly drawing independently and uniformly at random  $(s_t, b_t)$  from  $S_i$ . Note that the cardinality of  $S_i$  is  $N_\delta := 1/\delta$  for all  $i$ . As a first step, we give a lower bound on the expected gain from trade of the best fixed price in hindsight: fix any realization of the random draws from  $S_i$  and any price  $p_i^*$  in  $(i \cdot \Delta, (i+1) \cdot \Delta)$ . We have then that

$$\mathbb{E} \left[ \max_{p \in [0,1]} \sum_{t=1}^T \text{GFT}_t(p) \right] \geq \mathbb{E} \left[ \sum_{t=1}^T \text{GFT}_t(p_i^*) \right] = \frac{\Delta}{N_\delta} T = T\delta\Delta. \quad (4)$$

Since Equation (4) holds for any realization of the initial choice of  $S_i$ , it also holds in expectation over all the randomness of the adversary, i.e. also over the random choice of  $S_i$ .

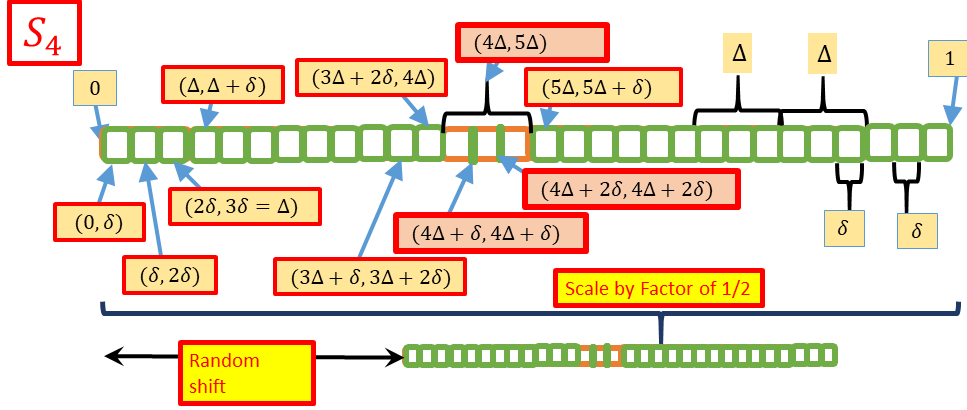


Figure 3: Example of sets used in the lower bound of Theorem 4 and how the grid is hidden. This example has  $\Delta = 1/10$ ,  $\delta = 1/30$ , so each section of size  $\Delta$  is partitioned into 3 sections of size  $\delta$ . The  $(sell, buy)$  pairs in  $S_4$  are as described in Equation 3 (not all such pairs are shown, there are  $1/\delta = 30$  such pairs in  $S_4$ ). Note that the gain from trade is  $\Delta$  if the bids are  $(4\Delta, 5\Delta)$  and if a price in between is posted. Also note that that seller and buyer valuations are equal for  $(4\Delta + \delta, 4\Delta + \delta)$  and for  $(4\Delta + 2\delta, 4\Delta + 2\delta)$ .

Let us focus now on the expected performance of any deterministic learner  $\mathcal{A}$ . The crux of this proof is that any price that is not a multiple of  $\delta$  is not able to discriminate between  $S_i$  and  $S_j$ , for any  $j \neq i$ . To see this, let  $p$  be any price that is not a multiple of  $\delta$ , there exist unique  $j(p) \in \{0, 1, \dots, 1/\Delta - 1\}$  and  $k(p) \in \{0, 1, \dots, \Delta/\delta - 1\}$  such that

$$p \in \left( j(p) \cdot \Delta + k(p) \cdot \delta, j(p) \cdot \Delta + (k(p) + 1) \cdot \delta \right).$$

The crucial observation is now that *regardless of the  $S_i$  selected by the adversary*, the random variable  $(\mathbb{P}(s_t \leq p), \mathbb{P}(p \leq b_t))$  follows the same distribution, in particular, we get:

$$(\mathbb{P}(S_t \leq p), \mathbb{P}(p \leq B_t)) = \begin{cases} (1, 1) & \text{with probability } \frac{1}{N_\delta} \\ (1, 0) & \text{with probability } \frac{1}{N_\delta} (j(p) \frac{\Delta}{\delta} + k(p) - 1) \\ (0, 1) & \text{with the remaining probability} \end{cases}$$

Stated differently, the learner observes a trade with a fixed probability  $1/N_\delta$ , while the probability masses on the left and on the right are determined by the position of  $p$ , and are constant across all choices of  $S_i$  by the adversary. Any trade that the learner observes corresponds to a  $\Delta$  gain with probability  $\Delta$  and to a  $\delta$  gain with the remaining probability. All in all, the gain from trade for any price posted by the learner (in expectation over both the random choice of  $S_i$  and the randomness at that specific round) is:

$$\mathbb{E}[\text{GFT}_t(p_t)] = \frac{1}{N_\delta} \left[ \frac{\Delta}{N_\Delta} + \delta \left( 1 - \frac{1}{N_\Delta} \right) \right] = \delta(\Delta^2 + \delta - \Delta\delta) \leq \delta(\Delta^2 + \delta),$$

where  $N_\Delta := 1/\Delta$  and represents the number of possible  $S_i$  that the adversary randomly select at the beginning. Summing up the inequality over all times  $t$ , we get (for all learners that do not post multiples of  $\delta$ )

$$\mathbb{E} \left[ \sum_{t=1}^T \text{GFT}_t(p_t) \right] \leq \delta(\Delta^2 + \delta)T \quad (5)$$

Via Yao's Theorem and combining Equation (4) and Equation (5), we get:

$$R_T^{\alpha, \star} \geq \mathbb{E} \left[ \max_{p \in [0, 1]} \sum_{t=1}^T \text{GFT}_t(p) - \alpha \sum_{t=1}^T \text{GFT}_t(p_t) \right] \geq (\Delta - \alpha(\Delta^2 + \delta))\delta \cdot T$$

At this point we set<sup>4</sup>  $\Delta = 1/(2\alpha)$  and  $\delta = 1/(8\alpha^2)$ :

$$R_T^{\alpha, \star} \geq \mathbb{E} \left[ \max_{p \in [0, 1]} \sum_{t=1}^T \text{GFT}_t(p) - \alpha \sum_{t=1}^T \text{GFT}_t(p_t) \right] \geq \frac{1}{64\alpha^3} T.$$

<sup>4</sup>At the beginning of the proof we assumed  $1/\Delta$ ,  $1/\delta$  and  $\Delta/\delta$  to be integer. It is easy to see that this is without loss of generality, given these choices of  $\delta$  and  $\Delta$

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**Estimation procedure of GFT using two prices and one-bit feedback**


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**Input:** price  $p$   
Toss a biased coin with head probability  $p$   
**if** head **then** Draw  $U$  u.a.r. in  $[0, p]$  and set  $\hat{p} \leftarrow U, \hat{q} \leftarrow p$   
**else** Draw  $V$  u.a.r. in  $[p, 1]$  and set  $\hat{p} \leftarrow p, \hat{q} \leftarrow V$   
Post price  $\hat{p}$  to the seller and  $\hat{q}$  to the buyer and observe the one-bit feedback  $\mathbb{I}\{s \leq \hat{p} \leq \hat{q} \leq b\}$   
**Return:**  $\widehat{\text{GFT}}(p) \leftarrow \mathbb{I}\{s \leq \hat{p} \leq \hat{q} \leq b\}$  ▷ Unbiased estimator of  $\text{GFT}(p)$

---

The proof above requires the “grid hiding” assumption that the learning algorithm cannot post prices that are on the grid (multiples of  $\delta$ ).

One way to proceed is to scale down the instance by a constant factor, say  $1/2$ , so that all prices and valuations will be in the range  $0$  to  $1/2$ . Then the adversary adds a random uniform number (called a shift) between  $0$  up to  $1/2$  (see Figure 3). It is clear that any algorithm has zero probability to pinpoint the exact value of the shift, thus the learning algorithm can post a multiple of  $\delta/2$  plus the required shift with probability  $0$ . In this scaled down instance the total gain from trade derived from the optimal fixed price goes down by a factor of  $1/2$ , while the the gain of the learning algorithm is scaled down by further factor of at least  $2$  (since the learner has to deal with the extra uncertainty due to the random shift). Ergo, the  $\alpha$ -regret will be at least  $\frac{1}{128\alpha^3}T$  which completes the proof.  $\square$

## 4.2 Upper bound on the 2-regret, posting two prices and given one-bit feedback

The main result in this section is presented in Theorem 5: it is possible to achieve sublinear 2-regret with one-bit feedback (and by posting two prices). We find this to be the most surprising result in this paper. The crucial ingredient of our approach is an unbiased estimator,  $\widehat{\text{GFT}}$ , of the gain from trade that uses two prices and *one single bit* of feedback. This seems quite remarkable. The gain from trade is a discontinuous function composed by two different objects: the difference  $(b - s)$  and the indicator variable  $\mathbb{I}\{p \in [s, b]\}$ . Both these two objects are easy to estimate *independently*, but for the gain from trade we need an estimator of their product. To estimate  $\text{GFT}(p)$  for any fixed price  $p$ , we construct an estimation procedure that considers both features at the same time: it tosses a biased coin with head probability  $p$ ; if head, it posts price  $p$  to the buyer and a price drawn u.a.r. in  $[0, p]$  to the seller; if tails, it posts price  $p$  to the seller and a price drawn u.a.r. in  $[p, 1]$  to the buyer. The formal procedure is described in the pseudocode, while the following lemma proves that this procedure yields an unbiased estimator of the gain from trade.

**Lemma 2.** *Fix any agents’ valuations  $s, b \in [0, 1]$ . For any price  $p \in [0, 1]$ , it holds that  $\widehat{\text{GFT}}(p)$  is an unbiased estimator of  $\text{GFT}(p)$ :  $\mathbb{E}[\widehat{\text{GFT}}(p)] = \text{GFT}(p)$ , where the expectation is with respect to the randomness of the estimation procedure.*

*Proof.* Note that  $p$  is fixed and known to the learner,  $s$  and  $b$  are fixed but unknown and the learner has to estimate the fixed but unknown quantity  $\text{GFT}(p) = \mathbb{I}\{s \leq p \leq q \leq b\} \cdot (b_t - s_t)$  using only the two-bit feedback. To analyze the expected value of  $\widehat{\text{GFT}}(p)$  we define two random variables:

$$X_s(p) = I_{\{s \leq U \leq p \leq b\}}, X_b(p) = I_{\{s \leq p \leq V \leq b\}}, \text{ where } U \sim \text{Unif}(0, p) \text{ and } V \sim \text{Unif}(p, 1).$$

Clearly, if  $p \notin [s, b]$ , the two random variables attains value  $0$  with probability  $1$  (and therefore are both unbiased estimators of  $\text{GFT}(p)$  in that case). Consider now the case in which  $p \in [s, b]$  and compute their expected value:

$$\begin{aligned} \mathbb{E}[X_s(p)] &= \mathbb{P}(s \leq U \leq p \leq b) = \mathbb{P}(s \leq U) = \frac{p - s}{p}, \\ \mathbb{E}[X_b(p)] &= \mathbb{P}(s \leq p \leq V \leq b) = \mathbb{P}(V \leq b) = \frac{b - p}{1 - p}. \end{aligned}$$

The estimator  $\widehat{\text{GFT}}(p)$  works as follows: with probability  $p$  it posts prices  $(U, p)$ , otherwise  $(p, V)$ , then receives the one-bit feedback from the agents and returns it. Conditioning on the result of the toss of the biased coin it is then easy to compute the expected value of  $\widehat{\text{GFT}}(p)$ :

$$\mathbb{E}[\widehat{\text{GFT}}(p)] = p \mathbb{E}[X_s(p)] + (1 - p) \mathbb{E}[X_b(p)] = \mathbb{I}\{s \leq p \leq q \leq b\} (b - s) = \text{GFT}(p). \quad \square$$

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**BLOCK-DECOMPOSITION (BD)**


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- 1: **Input:** time horizon  $T$ , number of blocks  $S$ , grid  $Q$  and expert algorithm  $\mathcal{E}$
  - 2:  $\Delta \leftarrow T/S, K \leftarrow |Q|$
  - 3:  $B_\tau \leftarrow \{(\tau - 1) \cdot \Delta + 1, \dots, \tau \cdot \Delta\}$ , for all  $\tau = 1, 2, \dots, S$
  - 4: Initialize  $\mathcal{E}$  with time horizon  $S$  and  $K$  actions, one for each  $p_i \in Q$
  - 5: **for** each round  $\tau = 1, 2, \dots, S$  **do**
  - 6: Receive from  $\mathcal{E}$  the price  $p_\tau$
  - 7: Select uniformly at random an injection  $h_\tau : Q \rightarrow B_\tau$   $\triangleright$  We need  $\Delta \gg |Q|$
  - 8: **for** each round  $t \in B_\tau$  **do**
  - 9: **if**  $h_\tau(p_i) = t$  for some price  $p_i$  **then**
  - 10: Use the estimator  $\widehat{\text{GFT}}(p_i)$  at time  $t$  and call its output  $\widehat{\text{GFT}}_\tau(p_i)$
  - 11: **else:** Post price  $p_\tau$  and ignore feedback
  - 12: Feed to  $\mathcal{E}$  the estimated gains  $\{\widehat{\text{GFT}}_\tau(p_i)\}_{i=1, \dots, K}$
- 

This estimation procedure becomes a powerful tool to estimate the gain from trade that the learner would have extracted at time  $t$  posting price  $p$  using randomization and *one single bit* of feedback. Note here that the possibility of posting two different prices is crucial: as we have argued in the previous section, one single price is not able to do that, even for two-bit feedback. Given the estimator  $\widehat{\text{GFT}}$  (actually it consists of a family of estimators: one for each price  $p$ ) we present our learning algorithm **BLOCK-DECOMPOSITION**. Similarly to what is done in Chapter 4 of Nisan et al. [2007], the learner divides the time horizon in  $S$  time blocks  $B_\tau$  of equal length<sup>5</sup> and uses as subroutine some expert algorithm  $\mathcal{E}$  on a meta-instance that considers each time block as a time step and each price in a suitable grid  $Q$  as an action. In each block the learner posts the same price  $p_\tau$  in all but  $|Q|$  time steps, where it uses  $\widehat{\text{GFT}}$  to estimate the total gain from trade obtained in  $B_\tau$  by all prices in  $Q$ . The details of **BLOCK-DECOMPOSITION** are presented in the pseudocode.

Consider any instantiation of the algorithm **BLOCK-DECOMPOSITION**, fix any block  $B_\tau$  and price  $p$ . With a slight abuse of notation we denote the average gain from trade posting price  $p$  in  $B_\tau$  as  $\text{GFT}_\tau$ ; formally,

$$\text{GFT}_\tau(p) = \frac{1}{\Delta} \sum_{t \in B_\tau} \text{GFT}_t(p).$$

We show that  $\widehat{\text{GFT}}_\tau(p)$  as defined in the pseudocode is an unbiased estimator of  $\text{GFT}_\tau(p)$ , where the randomization is due to the random choice of the injective function  $h_\tau$  and the inherent randomness in the estimator  $\widehat{\text{GFT}}$ .

**Lemma 3.** *Fix any sequence of valuations, then the random variable  $\widehat{\text{GFT}}_\tau(p_i)$  is an unbiased estimator of  $\text{GFT}_\tau(p_i)$  for any  $\tau \in \{1, 2, \dots, S\}$  and price  $p_i$  on the grid  $Q$ .*

*Proof.* For any fixed price  $p_i$  it is clear that  $h_\tau(p_i)$  is distributed uniformly at random in the time steps contained in the block  $B_\tau$ . Moreover, given  $h_\tau$ , the  $\widehat{\text{GFT}}$  are still unbiased estimators of the corresponding time steps. Thus, we have the following:

$$\begin{aligned} \mathbb{E} \left[ \widehat{\text{GFT}}_\tau(p_i) \right] &= \sum_{t \in B_\tau} \mathbb{P}(h_\tau(p_i) = t) \mathbb{E} \left[ \widehat{\text{GFT}}_t(p_i) \mid h_\tau(p_i) = t \right] \\ &= \sum_{t=1}^T \frac{1}{\Delta} \mathbb{E} \left[ \widehat{\text{GFT}}_t(p_i) \mid h_\tau(p_i) = t \right] \\ &= \sum_{t=1}^T \frac{1}{\Delta} \text{GFT}_t(p_i) = \text{GFT}_\tau(p_i). \end{aligned}$$

A notational observation: with the random variable  $\widehat{\text{GFT}}_t(p)$  we refer to the result of the estimation procedure in  $p$  run at time  $t$ , which is an unbiased estimator of the gain from trade of price  $p$  achievable at time  $t$ .  $\square$

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<sup>5</sup>For ease of exposition we assume that  $S$  divides  $T$ . This is without loss of generality in our case, as one can always add some dummy time steps for an additive regret of at most  $T/S$ .

**Theorem 5** (Upper bound on 2-regret posting two prices, one-bit feedback). *In one-bit feedback model where the learner is allowed to post two prices, the 2-regret of BLOCK-DECOMPOSITION (BD) is such that  $R_T^2(\text{BD}) \leq 5T^{3/4}\sqrt{\log(T)}$ , for appropriate choices of the expert algorithm  $\mathcal{E}$ , grid  $Q$  and number of blocks  $S$ .*

*Proof.* We consider a grid  $Q$  of equally spaced prices (we set the step later), and denote with  $\Delta = T/S$  the length of every time block. The learner keeps playing the same price in each block, apart from the explorations steps, that are drawn uniformly at random. The learner decides which action to play according to some routine  $\mathcal{E}$  that is run on  $S$  time steps and  $|Q|$  actions: this is the reason we talk interchangeably of actions and prices.

From Lemma 3 we know that the estimators in a block, i.e.,  $\widehat{\text{GFT}}_\tau(p_i)$  are indeed unbiased estimators of  $\text{GFT}_\tau(p_i)$ . Since this holds for any price  $p_i \in Q$ , it also holds for any random price  $\hat{p}$  whose randomness is independent on the choice of the injection  $h_\tau$  and the internal randomization of the estimators. Thus, the same holds even if instead of a fixed price  $p_i$  we consider price  $p_\tau$  posted by the algorithm because it depends *only* on what happened in past blocks. Let now  $\mathcal{E}$  be the Multiplicative Weights algorithm. If we fix the randomness in the exploration and in the estimation upfront and consider only the inherent randomness in  $\mathcal{E}$  we inherit the bound on the regret of  $\mathcal{E}$  on the realized estimated gain from trades (note that they are all bounded in  $[0, 1]$ )

$$\max_{p \in Q} \sum_{\tau=1}^S \widehat{\text{GFT}}_\tau(p) - \mathbb{E} \left[ \sum_{\tau=1}^S \widehat{\text{GFT}}_\tau(p_\tau) \right] \leq 2\sqrt{S \log(|Q|)} \quad (6)$$

The randomness of  $\mathcal{E}$  depends somehow on the realizations of the random injections and estimators, but if we look at any block  $B_\tau$ , we see that the random price output by the routine is independent from  $h_\tau$  and the estimators in that block. Therefore, we can safely take the expected value (on the randomness of the  $h_\tau$  and the estimators) on both sides of Equation (6), apply Lemma 3, and get

$$\max_{p \in Q} \sum_{\tau=1}^S \text{GFT}_\tau(p) - \mathbb{E} \left[ \sum_{\tau=1}^S \text{GFT}_\tau(p_\tau) \right] \leq 2\sqrt{S \log(|Q|)}. \quad (7)$$

Note that we have derived the first inequality using that the max of the expectation is smaller than the expectation of the max. Now, we move from the blocks time scale to the normal one and multiply everything by a factor  $\Delta$ . Our algorithm does not always play  $p_\tau$ , but for each one of the block, it spends  $|Q|$  steps exploring. Therefore, we need to consider an extra  $|Q|S$  additive term. At this point, we have all the ingredients to bound the 2-regret of our algorithm. Plugging Equation (7) and the observation about the extra losses incurred by the exploration into the discretization inequality (Claim 1) we get:

$$R_T^2(\text{BD}) \leq 2\Delta\sqrt{S \log |Q|} + |Q|S + \delta(Q)T.$$

The theorem then follows by optimizing the free parameters: we set  $\Delta = \sqrt{T}$  and we choose  $Q$  to be the uniform grid of multiples of  $T^{-1/4}$  (thus  $S = \sqrt{T}$ ,  $\delta(Q) = T^{-1/4}$  and  $|Q| = T^{1/4} + 1$ ).  $\square$

### 4.3 Lower bound on 2-regret, posting two prices and two-bit feedback

In this section, we complement the positive results for the single price and two-bit feedback setting with a lower bound on the 2-regret achievable in the (easier) two price and two-bit feedback setting. This lower bound strongly depends on a powerful characterization result from the partial monitoring literature [Bartók et al., 2014] and consists in constructing a class of instances with the following structure that mimic an “hard” partial monitoring game. The  $[0, 1]$  interval is divided into 4 disjoint regions, the first one is composed by a single optimal price  $p^*$ , then two intervals that are candidate to be the second best after  $p^*$ . The only way for the learner to actually discriminate between the two candidates and assess which is the actual second best is to post prices in the last, suboptimal region of the  $[0, 1]$  interval. The construction is such that there is a multiplicative factor 2 between the gain from trade of  $p^*$  and that of the second best. For the learner it is impossible to locate the single point  $p^*$  (given the structure of the feedback), and its regret with respect to the second best prices is at least  $\Omega(T^{2/3})$ . The reader familiar with the learning literature would recognize the similarity of this structure to the classical revealing action problem [Cesa-Bianchi et al., 2006].

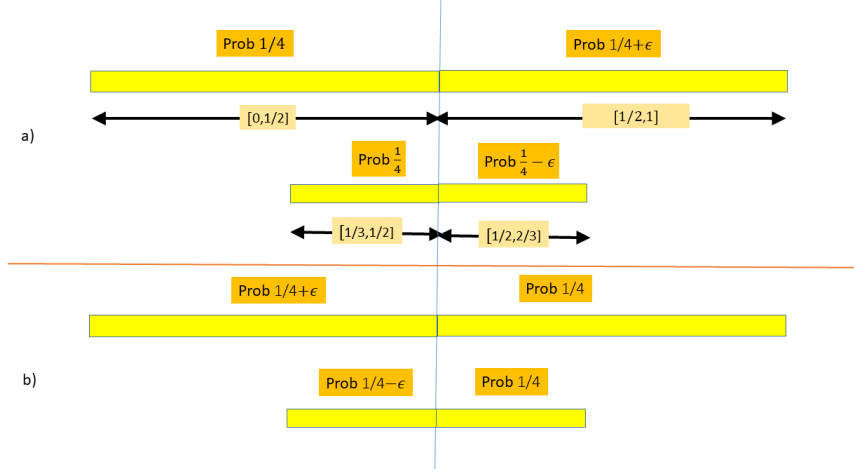


Figure 4: Construction used in the lower bounds of Theorem 6. The adversary chooses either option a) or option b). To distinguish between the two cases the algorithm must set prices in the (suboptimal) ranges  $[0, 1/3]$  or  $(2/3, 1]$ . The expected gain from trade for a price in these segments is about  $1/8$ , smaller by an additive term, approximately equal to  $1/6$ , from the expected gain from trade achieved by placing a price somewhere in the range  $[1/3, 2/3] \setminus \{1/2\}$ . The optimal price is  $1/2$  for the examples as described above in this figure — as done previously —  $1/2$  can be slightly perturbed and thus cannot be found by the online learner. By choosing the [perturbed] price (about  $1/2$ ) the optimal gain from trade is about twice the expected gain from trade achieved by any other price.

The randomized family of instances that are hard to learn for any deterministic learner is easy to describe, (see Figure 4 for a pictorial representation): at the beginning the adversary randomly and uniformly select one of the two following distributions over valuations  $(s, b)$  and then draws  $T$  i.i.d. samples from it:

$$\left\{ \begin{array}{ll} (0, \frac{1}{2}) & \text{with probability } \frac{1}{4} + \varepsilon \\ (\frac{1}{3}, \frac{1}{2}) & \text{with probability } \frac{1}{4} - \varepsilon \\ (\frac{1}{2}, \frac{2}{3}) & \text{with probability } \frac{1}{4} \\ (\frac{1}{2}, 1) & \text{with probability } \frac{1}{4} \end{array} \right. \quad \left\{ \begin{array}{ll} (0, \frac{1}{2}) & \text{with probability } \frac{1}{4} \\ (\frac{1}{3}, \frac{1}{2}) & \text{with probability } \frac{1}{4} \\ (\frac{1}{2}, \frac{2}{3}) & \text{with probability } \frac{1}{4} - \varepsilon \\ (\frac{1}{2}, 1) & \text{with probability } \frac{1}{4} + \varepsilon \end{array} \right.$$

We can compute the expected performance  $\mathbb{E}[GFT(p)]$  of any price  $p$  against them (the first, respectively second, column corresponds to the first, respectively second, distribution)

$$\left\{ \begin{array}{ll} \frac{1}{8} + \frac{\varepsilon}{2} & \text{if } p \in [0, \frac{1}{3}) \\ \frac{1}{6} + \frac{\varepsilon}{3} & \text{if } p \in [\frac{1}{3}, \frac{1}{2}) \\ \frac{1}{3} + \frac{\varepsilon}{3} & \text{if } p = \frac{1}{2} \\ \frac{1}{6} & \text{if } p \in (\frac{1}{2}, \frac{2}{3}] \\ \frac{1}{8} & \text{if } p \in (\frac{2}{3}, 1] \end{array} \right. \quad \left\{ \begin{array}{ll} \frac{1}{8} & \text{if } p \in [0, \frac{1}{3}) \\ \frac{1}{6} & \text{if } p \in [\frac{1}{3}, \frac{1}{2}) \\ \frac{1}{3} + \frac{\varepsilon}{3} & \text{if } p = \frac{1}{2} \\ \frac{1}{6} + \frac{\varepsilon}{3} & \text{if } p \in (\frac{1}{2}, \frac{2}{3}] \\ \frac{1}{8} + \frac{\varepsilon}{2} & \text{if } p \in (\frac{2}{3}, 1] \end{array} \right.$$

It is clear that the best price is  $\frac{1}{2}$ , that yields an expected gain from trade that is approximately a multiplicative factor 2 larger than the one induced by the second best price, i.e.  $p \in [\frac{1}{3}, \frac{1}{2})$  or  $p \in (\frac{1}{2}, \frac{2}{3}]$  depending on the instance in question. The two candidates to be the second best price are an additive  $\Theta(\varepsilon)$  factor away, while posting prices in  $[0, \frac{1}{3}) \cup (\frac{2}{3}, 1]$  gives a constant loss. The crucial property is that the only way the learner can discriminate between the two instances is to post prices in the low gain region  $[0, \frac{1}{3}) \cup (\frac{2}{3}, 1]$ . For example, posting a price of  $\frac{1}{3}$  the learner observes a trade with probability exactly  $\frac{1}{2}$  in both the distributions, while posting 0 yields a trade with probability  $\frac{1}{4} + \varepsilon$  in the first case while exactly  $\frac{1}{4}$  in the second (thus allowing some learning to happen). Moreover, the learner cannot take advantage of the possibility of posting more than one price: the only useful thing to do it to learn is where the extra  $\varepsilon$  probability is, and there is no way of doing it without suffering a constant instantaneous regret; not even with two-bits of feedback. We formalize these considerations in the following lemma, whose proof is deferred to the Appendix.

**Lemma 4.** Consider the class of learning algorithms that can post two prices (both different from  $1/2$ ) and receive two-bit feedback. For any  $\mathcal{A}$  in this class, there exists a sequence from the family we



described such that the following bound on the regret holds, for some constant  $c > 0$ :

$$\max_{p \neq \frac{1}{2}} \sum_{t=1}^T \text{GFT}(p) - \mathbb{E} \left[ \sum_{t=1}^T \text{GFT}_t(p_t, q_t) \right] \geq cT^{2/3}.$$

To conclude our lower bound, we need to show how to *hide* to the learner the price  $1/2$  that is clearly optimal. It is sufficient to add a small, random, perturbation of the instance.

**Theorem 6** (Lower bound on regret for two prices and two-bit feedback). *In the two-bit feedback model where the learner is allowed to post two prices, for all horizons  $T \in \mathbb{N}$ , the minimax 2-regret satisfies  $R_T^{2,*} \geq \tilde{c}T^{2/3}$  for some constant  $\tilde{c}$ .*

*Proof.* Let  $\delta > 0$  be an arbitrarily small constant. We perturb each instance of the family we have constructed earlier in the following way: the adversary draws uniformly at random a shift  $x \in [0, \delta]$ , then adds it to all valuations and finally it divides them all by  $1 + \delta$ . The valuations are still in  $[0, 1]$  and the optimal price  $p^*$  has now become  $\frac{1}{2(1+\delta)} + \frac{x}{1+\delta}$ . The learner has now no way of pinpointing the exact location of  $p^*$ , since it is impossible to locate a specific point in  $[0, \delta]$  using two-bit feedback. Finally, the addition of new, independent, random noise does not make the learning of the second best price easier, i.e., the bound of Lemma 4 holds. All in all, for any learning algorithm  $\mathcal{A}$ , we have:

$$\begin{aligned} \mathbb{E} [R_T^2(\mathcal{A})] &\geq \mathbb{E} \left[ \max_{p \in [0,1]} \sum_{t=1}^T \text{GFT}_t(p) - 2 \max_{p \neq p^*} \sum_{t=1}^T \text{GFT}_t(p) \right] \\ &\quad + 2 \cdot \mathbb{E} \left[ \max_{p \neq p^*} \sum_{t=1}^T \text{GFT}_t(p) - \sum_{t=1}^T \text{GFT}_t(p_t, q_t) \right] \\ &\geq \delta \Theta(T) + cT^{2/3} \geq \tilde{c} \cdot T^{2/3}. \end{aligned}$$

□

## 5 Discussion, Extensions, and Open Problems

In this paper, we investigate the sequential bilateral trade problem with adversarial valuations. We study various feedback scenarios and consider the possibility for the mechanism to post one price vs. when it can post different prices for buyer and seller. We identify the exact threshold of  $\alpha$  that allows sublinear  $\alpha$ -regret. We show that with a partial feedback it is impossible to achieve sublinear  $\alpha$ -regret for any constant  $\alpha$  with a single price while 2-regret is achievable with 2 prices. Finally, we show a separation in the minimax 2-regret between full and partial feedback. Although in this paper we only consider the gain from trade, our positive results trivially also hold with respect to social welfare. Furthermore, modifying our lower bound from Theorem 1 it is possible to show that sublinear  $\alpha$ -regret is not achievable for  $\alpha < 2$  with respect to social welfare. An obvious open problem, with respect to both gain from trade and to social welfare, consists in determining the exact regret term as a function of  $T$ . Clearly there is a gap in our Table of results, and the exact term is yet unclear also for social welfare. We focus on the sequential problem where at each step one buyer and one seller appears. It would be interesting to study the model where multiple buyers and multiple sellers arrive at each time step and sellers have values for their goods, buyers have values for the different goods.

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## References

- Noga Alon, Nicolò Cesa-Bianchi, Claudio Gentile, Shie Mannor, Yishay Mansour, and Ohad Shamir. Nonstochastic multi-armed bandits with graph-structured feedback. *SIAM J. Comput.*, 46(6):1785–1826, 2017.
- Sanjeev Arora, Elad Hazan, and Satyen Kale. The multiplicative weights update method: a meta-algorithm and applications. *Theory Comput.*, 8(1):121–164, 2012.
- Gábor Bartók, Dean P. Foster, Dávid Pál, Alexander Rakhlin, and Csaba Szepesvári. Partial monitoring - classification, regret bounds, and algorithms. *Math. Oper. Res.*, 39(4):967–997, 2014.
- Avrim Blum and Jason D. Hartline. Near-optimal online auctions. In *SODA*, pages 1156–1163. SIAM, 2005.
- Avrim Blum, Vijay Kumar, Atri Rudra, and Felix Wu. Online learning in online auctions. *Theoretical Computer Science*, 324(2-3):137–146, 2004.
- Liad Blumrosen and Shahar Dobzinski. Reallocation mechanisms. In *EC*, page 617. ACM, 2014.
- Liad Blumrosen and Shahar Dobzinski. (almost) efficient mechanisms for bilateral trading. *Games Econ. Behav.*, 130:369–383, 2021.
- Sébastien Bubeck, Nikhil R. Devanur, Zhiyi Huang, and Rad Niazadeh. Multi-scale online learning: Theory and applications to online auctions and pricing. *J. Mach. Learn. Res.*, 20:62:1–62:37, 2019.
- Nicolò Cesa-Bianchi and Gábor Lugosi. *Prediction, learning, and games*. Cambridge University Press, UK, 2006.
- Nicolò Cesa-Bianchi, Gábor Lugosi, and Gilles Stoltz. Regret minimization under partial monitoring. *Math. Oper. Res.*, 31(3):562–580, 2006.
- Nicolò Cesa-Bianchi, Claudio Gentile, and Yishay Mansour. Regret minimization for reserve prices in second-price auctions. *IEEE Trans. Inf. Theory*, 61(1):549–564, 2015.
- Nicolò Cesa-Bianchi, Tommaso R. Cesari, Roberto Colomboni, Federico Fusco, and Stefano Leonardi. A regret analysis of bilateral trade. In *EC*, pages 289–309. ACM, 2021a.
- Nicolò Cesa-Bianchi, Tommaso R. Cesari, Roberto Colomboni, Federico Fusco, and Stefano Leonardi. Bilateral trade: A regret minimization perspective. *CoRR*, abs/2109.12974, 2021b.
- Riccardo Colini-Baldeschi, Bart de Keijzer, Stefano Leonardi, and Stefano Turchetta. Approximately efficient double auctions with strong budget balance. In *SODA*, pages 1424–1443. SIAM, 2016.
- Riccardo Colini-Baldeschi, Paul W. Goldberg, Bart de Keijzer, Stefano Leonardi, and Stefano Turchetta. Fixed price approximability of the optimal gain from trade. In *WINE*, volume 10660 of *Lecture Notes in Computer Science*, pages 146–160. Springer, 2017.
- Constantinos Daskalakis and Vasilis Syrgkanis. Learning in auctions: Regret is hard, envy is easy. In *FOCS*, pages 219–228. IEEE Computer Society, 2016.
- Yuan Deng, Jieming Mao, Balasubramanian Sivan, and Kangning Wang. Approximately efficient bilateral trade. In *STOC*, pages 718–721. ACM, 2022.
- Paul Dütting, Federico Fusco, Philip Lazos, Stefano Leonardi, and Rebecca Reiffenhäuser. Efficient two-sided markets with limited information. In *STOC*, pages 1452–1465. ACM, 2021.
- Ehsan Emamjomeh-Zadeh, Chen-Yu Wei, Haipeng Luo, and David Kempe. Adversarial online learning with changing action sets: Efficient algorithms with approximate regret bounds. In *ALT*, volume 132 of *Proceedings of Machine Learning Research*, pages 599–618. PMLR, 2021.
- Emmanuel Esposito, Federico Fusco, Dirk van der Hoeven, and Nicolò Cesa-Bianchi. Learning on the edge: Online learning with stochastic feedback graphs. *To appear in NeurIPS*, Preprint on the arXiv abs/2210.04229, 2022.

- Dan Garber. Efficient online linear optimization with approximation algorithms. *Math. Oper. Res.*, 46(1):204–220, 2021.
- Kathleen M Hagerty and William P Rogerson. Robust trading mechanisms. *Journal of Economic Theory*, 42(1):94–107, 1987.
- Elad Hazan. Introduction to online convex optimization. *Found. Trends Optim.*, 2(3-4):157–325, 2016.
- Chien-Ju Ho, Aleksandrs Slivkins, and Jennifer Wortman Vaughan. Adaptive contract design for crowdsourcing markets: Bandit algorithms for repeated principal-agent problems. *J. Artif. Intell. Res.*, 55:317–359, 2016.
- Sham M. Kakade, Adam Tauman Kalai, and Katrina Ligett. Playing games with approximation algorithms. *SIAM J. Comput.*, 39(3):1088–1106, 2009.
- Adam Tauman Kalai and Santosh S. Vempala. Efficient algorithms for online decision problems. *J. Comput. Syst. Sci.*, 71(3):291–307, 2005.
- Robert D. Kleinberg and Frank Thomson Leighton. The value of knowing a demand curve: Bounds on regret for online posted-price auctions. In *FOCS*, pages 594–605. IEEE Computer Society, 2003.
- Tor Lattimore and Csaba Szepesvári. Cleaning up the neighborhood: A full classification for adversarial partial monitoring. In *ALT*, volume 98 of *Proceedings of Machine Learning Research*, pages 529–556. PMLR, 2019.
- Thodoris Lykouris, Vasilis Syrgkanis, and Éva Tardos. Learning and efficiency in games with dynamic population. In *SODA*, pages 120–129. SIAM, 2016.
- Jamie Morgenstern and Tim Roughgarden. On the pseudo-dimension of nearly optimal auctions. In *NIPS*, pages 136–144, 2015.
- Roger B Myerson and Mark A Satterthwaite. Efficient mechanisms for bilateral trading. *Journal of economic theory*, 29(2):265–281, 1983.
- Rad Niazadeh, Negin Golrezaei, Joshua R. Wang, Fransisca Susan, and Ashwinkumar Badanidiyuru. Online learning via offline greedy algorithms: Applications in market design and optimization. In *EC*, pages 737–738. ACM, 2021.
- Noam Nisan, Tim Roughgarden, Éva Tardos, and Vijay V. Vazirani, editors. *Algorithmic Game Theory*. Cambridge University Press, 2007.
- José Luis Palacios. On the simple symmetric random walk and its maximal function. *The American Statistician*, 62(2):138–140, 2008.
- Tim Roughgarden and Joshua R. Wang. An optimal learning algorithm for online unconstrained submodular maximization. In *COLT*, volume 75 of *Proceedings of Machine Learning Research*, pages 1307–1325. PMLR, 2018.
- Tim Roughgarden and Joshua R. Wang. Minimizing regret with multiple reserves. *ACM Trans. Economics and Comput.*, 7(3):17:1–17:18, 2019.
- Aleksandrs Slivkins. Introduction to multi-armed bandits. *Found. Trends Mach. Learn.*, 12(1-2): 1–286, 2019.
- Dirk van der Hoeven, Federico Fusco, and Nicolò Cesa-Bianchi. Beyond bandit feedback in online multiclass classification. In *NeurIPS*, pages 13280–13291, 2021.
- William Vickrey. Counterspeculation, auctions, and competitive sealed tenders. *The Journal of finance*, 16(1):8–37, 1961.
- Andrew Chi-Chih Yao. Probabilistic computations: Toward a unified measure of complexity (extended abstract). In *FOCS*, pages 222–227. IEEE Computer Society, 1977.

## Appendix

### Proof of Lemma 4

Recall the construction of the hard randomized instance for this problem: the adversary selects uniformly at random one of these two distributions at the beginning and draws  $T$  i.i.d. samples from it. We report here the distributions for completeness.

$$\left\{ \begin{array}{l} (0, \frac{1}{2}) \\ (\frac{1}{3}, \frac{1}{2}) \\ (\frac{1}{2}, \frac{2}{3}) \\ (\frac{1}{2}, 1) \end{array} \right. \begin{array}{l} \text{with probability } \frac{1}{4} + \varepsilon \\ \text{with probability } \frac{1}{4} - \varepsilon \\ \text{with probability } \frac{1}{4} \\ \text{with probability } \frac{1}{4} \end{array} \quad \left\{ \begin{array}{l} (0, \frac{1}{2}) \\ (\frac{1}{3}, \frac{1}{2}) \\ (\frac{1}{2}, \frac{2}{3}) \\ (\frac{1}{2}, 1) \end{array} \right. \begin{array}{l} \text{with probability } \frac{1}{4} \\ \text{with probability } \frac{1}{4} \\ \text{with probability } \frac{1}{4} - \varepsilon \\ \text{with probability } \frac{1}{4} + \varepsilon \end{array}$$

As we mentioned in the main text, the expected gain from trade achievable by the learner against these two distributions are:

$$\left\{ \begin{array}{l} \frac{1}{8} + \frac{\varepsilon}{2} \\ \frac{1}{6} + \frac{\varepsilon}{3} \\ \frac{1}{3} + \frac{\varepsilon}{3} \\ \frac{1}{6} \\ \frac{1}{8} \end{array} \right. \begin{array}{l} \text{if } p \in [0, \frac{1}{3}) \\ \text{if } p \in [\frac{1}{3}, \frac{1}{2}) \\ \text{if } p = \frac{1}{2} \\ \text{if } p \in (\frac{1}{2}, \frac{2}{3}] \\ \text{if } p \in (\frac{2}{3}, 1] \end{array} \quad \left\{ \begin{array}{l} \frac{1}{8} \\ \frac{1}{6} \\ \frac{1}{3} + \frac{\varepsilon}{3} \\ \frac{1}{6} + \frac{\varepsilon}{3} \\ \frac{1}{8} + \frac{\varepsilon}{2} \end{array} \right. \begin{array}{l} \text{if } p \in [0, \frac{1}{3}) \\ \text{if } p \in [\frac{1}{3}, \frac{1}{2}) \\ \text{if } p = \frac{1}{2} \\ \text{if } p \in (\frac{1}{2}, \frac{2}{3}] \\ \text{if } p \in (\frac{2}{3}, 1] \end{array}$$

It is clear that the best price is  $\frac{1}{2}$ , that yields an expected gain from trade that is approximately a multiplicative factor 2 larger than the one induced by the second best price, i.e.  $p \in [\frac{1}{3}, \frac{1}{2})$  or  $p \in (\frac{1}{2}, \frac{2}{3}]$  depending on the instance in question. The two candidates to be the second best price are an additive  $\Theta(\varepsilon)$  factor away, while posting prices in  $[0, \frac{1}{3}) \cup (\frac{2}{3}, 1]$  gives a constant loss. The crucial property is that the only way the learner can discriminate between the two instances is to post prices in the low gain region  $[0, \frac{1}{3}) \cup (\frac{2}{3}, 1]$ . For example, posting a price of  $\frac{1}{3}$  the learner observes a trade with probability exactly  $\frac{1}{2}$  in both the distributions, while posting 0 yields a trade with probability  $\frac{1}{4} + \varepsilon$  in the first case while exactly  $\frac{1}{4}$  in the second (thus allowing some learning to happen). Moreover, the learner cannot take advantage of the possibility of posting more than one price: the only useful thing to do it to learn is where the extra  $\varepsilon$  probability is, and there is no way of doing it without suffering a constant instantaneous regret; not even with two-bits of feedback. We formalize these considerations in the following lemma.

**Lemma 4.** *Consider the class of learning algorithms that can post two prices (both different from  $1/2$ ) and receive two-bit feedback. For any  $\mathcal{A}$  in this class, there exists a sequence from the family we described such that the following bound on the regret holds, for some constant  $c > 0$ :*

$$\max_{p \neq \frac{1}{2}} \sum_{t=1}^T \text{GFT}(p) - \mathbb{E} \left[ \sum_{t=1}^T \text{GFT}_t(p_t, q_t) \right] \geq cT^{2/3}.$$

To prove the Lemma, we show that the family of sequences presented above fits into the proof scheme of Theorem 4 of Bartók et al. [2014]. To formally show this, we need to introduce the theoretical framework of partial monitoring and argue that our instance is a special case of the one used in the  $\Omega(T^{2/3})$  lower bound of Theorem 4 of Bartók et al. [2014].

We recall from Bartók et al. [2014] that an  $N$  actions,  $M$  outcomes partial monitoring game is characterized by two matrices, the loss (or gain, as in our case) matrix  $L$  and the signal matrix  $H$ . In each round  $t$ , the learner chooses an action  $I_t \in [N]$  and, simultaneously, the adversary chooses an outcome  $J_t \in [M]$ . The learner experiences a gain  $L_{I_t, J_t}$  and receives as feedback  $H_{I_t, J_t}$ . The notion of regret is defined as in the classical online learning framework as the difference between the total gain of the best fixed action in hindsight and the expected gain of the learning algorithm.

In our family of instances, we have  $M = 4$  possible outcomes, according to the valuations:  $(s, b) = (0, \frac{1}{2}), (\frac{1}{3}, \frac{1}{2}), (\frac{1}{2}, \frac{2}{3})$  or  $(\frac{2}{3}, 1)$ . The possible actions corresponds to all the possible (continuous) prices posted by the learner; however, given the structure of the problem, it is enough to consider only a finite representative set of  $N = 10$  of them:  $(q, p) \in \{0, \frac{1}{3}, \frac{2}{3}, 1\}^2$  such that  $q \leq p$ . Note that this is without loss of generality since prices in the same interval gets the same gain and feedback. To get

	$(0, \frac{1}{2})$	$(\frac{1}{3}, \frac{1}{2})$	$(\frac{1}{2}, \frac{2}{3})$	$(\frac{1}{2}, 1)$
$(0, 0)$	$\frac{1}{2}$	0	0	0
$(0, \frac{1}{3})$	$\frac{1}{2}$	0	0	0
$(0, \frac{2}{3})$	0	0	0	0
$(0, 1)$	0	0	0	0
$(\frac{1}{3}, \frac{1}{3})$	$\frac{1}{2}$	$\frac{1}{6}$	0	0
$(\frac{1}{3}, \frac{2}{3})$	0	0	0	0
$(\frac{1}{3}, 1)$	0	0	0	0
$(\frac{2}{3}, \frac{2}{3})$	0	0	$\frac{1}{6}$	$\frac{1}{2}$
$(\frac{2}{3}, 1)$	0	0	0	$\frac{1}{2}$
$(1, 1)$	0	0	0	$\frac{1}{2}$

Table 2: Gain Matrix  $L$ 

	$(0, \frac{1}{2})$	$(\frac{1}{3}, \frac{1}{2})$	$(\frac{1}{2}, \frac{2}{3})$	$(\frac{1}{2}, 1)$
$(0, 0)$	(1, 1)	(0, 1)	(0, 1)	(0, 1)
$(0, \frac{1}{3})$	(1, 1)	(0, 1)	(0, 1)	(0, 1)
$(0, \frac{2}{3})$	(1, 0)	(0, 0)	(0, 1)	(0, 1)
$(0, 1)$	(1, 0)	(0, 0)	(0, 0)	(0, 1)
$(\frac{1}{3}, \frac{1}{3})$	(1, 1)	(1, 1)	(0, 1)	(0, 1)
$(\frac{1}{3}, \frac{2}{3})$	(1, 0)	(1, 0)	(0, 1)	(0, 1)
$(\frac{1}{3}, 1)$	(1, 0)	(1, 0)	(0, 0)	(0, 1)
$(\frac{2}{3}, \frac{2}{3})$	(1, 0)	(1, 0)	(1, 1)	(1, 1)
$(\frac{2}{3}, 1)$	(1, 0)	(1, 0)	(1, 0)	(1, 1)
$(1, 1)$	(1, 0)	(1, 0)	(1, 0)	(1, 1)

Table 3: Feedback Matrix  $H$ 

The two tables represent the gain and feedback matrices of the family of sequences we introduced in the main body. Note that the rows refer to the actions, i.e., the prices posted to seller and buyers, while the columns to the outcomes, i.e., the valuations of the agents. The row-colors reflect the properties of the corresponding actions: green for Pareto-Optimal, yellow for degenerate and white for dominated.

more intuition: the action *posting prices*  $(0, 1/3)$  represents all the actions where the price to the seller is in the  $[0, \frac{1}{3})$  interval and the price to the buyer is in the  $[\frac{1}{3}, \frac{1}{2})$  interval. The gain and feedback matrices are reported in Tables 2 and 3.

Let  $\Delta_M$  denote the  $M$ -dimensional probability simplex, and let  $\ell_i$  denote the  $i^{\text{th}}$  row of  $L$  as a column array. Given a vector  $\pi \in \Delta_M$ , this induces a probability distribution over the outcomes; an action  $i$  is optimal under  $\pi$  if it is the best response of the learner to  $\pi$ :  $\langle \ell_i, \pi \rangle \geq \langle \ell_{i'}, \pi \rangle$  for all  $i' \neq i$ . The notion of optimal action induces a cell decomposition of  $\Delta_M$ .

**Definition 1** (Cell Decomposition). *For every action  $i \in [N]$ , let*

$$C_i = \{\pi \in \Delta_M : \text{action } i \text{ is optimal under } \pi\}.$$

*The sets  $C_1, \dots, C_N$  constitute the cell decomposition of  $\Delta_M$ .*

As an example, consider the first action in Table 2, corresponding to posting price 0 to both agents. Its cell  $C_1$  in  $\Delta_4$  is composed by all  $\pi = (\pi_1, \pi_2, \pi_3, \pi_4) \in \Delta_4$  such that action 1 is the best response to it. Clearly,  $\pi_2 = 0$ , because otherwise the fifth action  $(\frac{1}{3}, \frac{1}{3})$  would guarantee strictly larger gain from trade. The only other constraint is given by the last two actions:

$$\frac{1}{2}\pi_1 \geq \frac{1}{6}\pi_3 + \frac{1}{2}\pi_4.$$

Therefore  $C_1 = \{\pi \in \Delta_4 : \pi_2 = 0, 3\pi_1 - \pi_3 - 3\pi_4 \geq 0\}$ . Using the cell decomposition, we can characterize the actions.

- Action  $i$  is called dominated if  $C_i = \emptyset$ , otherwise it is called non-dominated. In our instance, actions 3, 4, 6 and 7 are dominated, the others are non-dominated.
- Action  $i$  is called degenerate if it is non-dominated and there exists an action  $i'$  such that  $C_i \subsetneq C_{i'}$ . Our first two actions and the last two are degenerate.
- If an action is neither dominated nor degenerate, then it is called Pareto-optimal. In our instance actions 5 and 8 are Pareto-optimal.
- Two Pareto-optimal actions  $i$  and  $j$  are neighbors if  $C_i \cap C_j$  is an  $(M - 2)$ -dimensional polytope. The neighborhood action set of two neighboring actions  $i$  and  $j$  is defined as  $N_{i,j}^+ = \{k \in [N] : C_i \cap C_j \subseteq C_k\}$ . In our example  $C_5 \cap C_8 = \{\pi \in \Delta_4 : 3\pi_1 + \pi_2 = \pi_3 + 3\pi_4\}$ , therefore the two Pareto-optimal actions are neighbours and their neighborhood contains only the two of them.

	$(0, \frac{1}{2})$	$(\frac{1}{3}, \frac{1}{2})$	$(\frac{1}{2}, \frac{2}{3})$	$(\frac{1}{2}, 1)$
(1, 1)	1	0	0	0
(1, 0)	0	0	0	0
(0, 1)	0	1	1	1
(0, 0)	0	0	0	0

Table 4: Signal matrices of actions 1 and 2

	$(0, \frac{1}{2})$	$(\frac{1}{3}, \frac{1}{2})$	$(\frac{1}{2}, \frac{2}{3})$	$(\frac{1}{2}, 1)$
(1, 1)	1	1	0	0
(1, 0)	0	0	0	0
(0, 1)	0	0	1	1
(0, 0)	0	0	0	0

Table 5: Signal matrix of action 5

	$(0, \frac{1}{2})$	$(\frac{1}{3}, \frac{1}{2})$	$(\frac{1}{2}, \frac{2}{3})$	$(\frac{1}{2}, 1)$
(1, 1)	0	0	0	0
(1, 0)	1	1	0	0
(0, 1)	0	0	0	0
(0, 0)	0	0	1	1

Table 6: Signal matrix of action 8

	$(0, \frac{1}{2})$	$(\frac{1}{3}, \frac{1}{2})$	$(\frac{1}{2}, \frac{2}{3})$	$(\frac{1}{2}, 1)$
(1, 1)	0	0	0	0
(1, 0)	1	1	1	0
(0, 1)	0	0	0	0
(0, 0)	0	0	0	1

Table 7: Signal matrices of actions 9 and 10

We now move our attention to the feedback matrix.

**Definition 2.** Let  $s_i$  be the number of symbols in the  $i^{\text{th}}$  row of  $H$  and let  $\sigma_1, \dots, \sigma_{s_i}$  be an enumeration of those symbols. Then the signal matrix  $S_i \in \{0, 1\}^{s_i \times M}$  of action  $i$  is defined as  $(S_i)_{k, \ell} = \delta_{H_{i, \ell} = \sigma_k}$ .

To get a better understanding of the definition, note that in our example the symbols are 4 and correspond to the possible two-bit feedback: (1, 1), (1, 0), (0, 1) and (0, 0). The signals matrices of the non-dominated actions are reported in Tables 4, 5, 6 and 7 (for the sake of uniformity we reported all the symbols in each signal matrix, this does not affect the results in any way).

We are now ready to introduce the key definitions of observability we need to invoke the characterization theorem.

**Definition 3.** A partial monitoring game admits the global observability condition, if for all pairs  $i$  and  $j$  of actions, the vector  $\ell_i - \ell_j$  belongs to the span generated by all the rows of the signal matrices:

$$\ell_i - \ell_j \in \bigoplus_{k \in [N]} \text{Im}(S_k^T).$$

This definition seems fairly abstract, but in our case is extremely easy to verify: the first row of  $S_1$  (Table 4), the first and third rows of  $S_5$  (Table 5) and the last row of  $S_9$  (Table 7) generate all  $\mathbb{R}^4$ , so our game respects the global observability condition.

**Definition 4.** A pair of neighboring actions  $i, j$  is said to be locally observable if

$$\ell_i - \ell_j \in \bigoplus_{k \in N_{i,j}^+} \text{Im}(S_k^T).$$

A game satisfies the local observability condition if every pair of neighboring actions is locally observable.

Our instance does not respect the local observability condition. We already argued that  $N_{5,8}^+$  contains only the two actions 5 and 8. If we look at the span generated by the row vectors of their signal matrices, we observe that it consists of all the vectors  $v = (v_1, v_2, v_3, v_4) \in \mathbb{R}^4$  that can be written as  $(\lambda, \lambda, \mu, \mu)$  for some parameters  $\lambda$  and  $\mu$ , while  $\ell_5 - \ell_8 = (\frac{1}{2}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{2})$ . What we have shown so far is enough to claim that the instance we have built is an “hard” partial monitoring game [Bartók et al., 2014].

*Proof of Lemma 4.* Theorem 4 of Bartók et al. [2014] states that the minimax regret of a partial monitoring instance that does not respect the local observability condition is at least  $c \cdot T^{2/3}$ , for some instance specific constant  $c$ . To conclude the proof of the Lemma, we note that the probability

distributions over the instances that we stated in the previous section are indeed a special case of the ones used in the main body of Theorem 4 of Bartók et al. [2014]. More in the specific, actions 1 and 2 in that proof correspond to our actions 5 and 8, probability vector  $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$  corresponds to what is there denoted with  $p_0$  (note that that this choice of  $p_0$  does belong in ours  $C_5$  and  $C_8$ ) and our choice of vector  $v$  is  $v = (1, -1, 0, 0)$  (up to a rescaling of the  $\varepsilon$  small enough).  $\square$