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Fractional truncated Laplacians: representation formula, fundamental solutions and applications

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Abstract. We introduce some nonlinear extremal nonlocal operators that approximate the, so called, truncated Laplacians. For these operators we construct representation formulas that lead to the construction of what, with an abuse of notation, could be called "fundamental solutions". This, in turn, leads to Liouville type results. The interest is double: on one hand we wish to "understand" what is the right way to define the nonlocal version of the truncated Laplacians, on the other, we introduce nonlocal operators whose nonlocality is on one dimensional lines, and this dramatically changes the prospective, as is quite clear from the results obtained that often differ significantly with the local case or with the case where the nonlocality is diffused. Surprisingly this is true also for operators that approximate the Laplacian.

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1. Introduction

In the last decades there has been an increasing interest in the comprehension of second order degenerate elliptic equations. The general idea being that new phenomena may occur when the uniform ellipticity condition is replaced by weaker form of ellipticity, while other fundamental properties like e.g. the comparison principle may still hold. It would be impossible and far too long to enumerate all the works and the "kind "of degeneracies that have been considered: degeneracy may depend of the point of application of the operator, on the value of the gradient of the solution, or it may be the case that the operator is simply "monotone" i.e. for any couple of symmetric matrices X and Y

$$X \leq Y \Rightarrow F(X) \leq F(Y).$$

In the realm of nonlocal equations, these very degenerate operators have only just begun to be considered, but they seem to open very interesting and surprising results as will be evident later on, for example in the strong maximum principle of Proposition 2.7 or the Liouville Theorem 4.1. In order to start a theory on nonlocal degenerate elliptic fully nonlinear operators, one needs to define general operators that are "extremal "among that class. So that sub or supersolutions of these extremal operators are sub or supersolutions for any degenerate operator. We will now define the two classes of nonlocal extremal operators we will consider in this paper. In both cases, the fractional order of the operator is given by $s \in (0,1)$.

We start with the first model, the description is somehow long for an introduction, so we ask for some patience from the reader: let $N \in \mathbb{N}$, $k \in \{1, 2, ..., N\}$, given $\xi \in \mathbb{S}^{N-1}$, $x \in \mathbb{R}^N$ and $u : \mathbb{R}^N \to \mathbb{R}$, we denote by

$$\mathcal{I}_\xi u(x) = C_s \int_0^{+\infty} [u(x + \tau\xi) + u(x - \tau\xi) - 2u(x)]\tau^{-(1+2s)} d\tau\,, \qquad (1.1)$$

where $C_s = C_{1,s} > 0$ is a normalizing constant related to the well-known fractional Laplacian $-(-\Delta)^s$, see (1.5) below. Roughly speaking, $\mathcal{I}_\xi$ acts as the one dimensional fractional $2s$-derivative in the direction of ξ. More precisely, for each $\xi \in \mathbb{S}^{N-1}$ and with an appropriate modification in the choice of the normalizing constant C_s in the definition, $\mathcal{I}_\xi$ is identified with the pseudo-differential operator in the Schwartz space defined through the symbol

$$v \mapsto -|\langle \xi, v\rangle|^{2s}.$$

Throughout the paper, when convenient, we shall use instead the formula

$$\mathcal{I}_\xi u(x) = C_s \text{P.V.} \int_{-\infty}^{+\infty} [u(x + \tau\xi) - u(x)]|\tau|^{-(1+2s)} d\tau$$

where P.V. stands for the Cauchy Principal Value.

An important fact related to the choice of the normalizing constant C_s and to the understanding of the definition of $\mathcal{I}_\xi u(x)$ is the asymptotic

$$\mathcal{I}_\xi u(x) \to \langle D^2 u(x)\xi, \xi\rangle, \quad \text{as } s \to 1^-,$$

under suitable regularity assumptions on u. We can now define the extremal operators

$$\mathcal{I}_k^+ u(x) = \sup\left\{ \sum_{i=1}^k \mathcal{I}_{\xi_i} u(x) \ : \ \{\xi_i\}_{i=1}^k \in \mathcal{V}_k \right\}, \qquad (1.2)$$

and similarly for $\mathcal{I}_k^-$ taking instead the infimum, where $\mathcal{V}_k$ is the family of k-dimensional orthonormal sets in $\mathbb{R}^N$. Notice that $\mathcal{I}_k^- u = -\mathcal{I}_k^+(-u)$. Let us emphasize that these operators are nonlocal, but the nonlocality is in some sense one dimensional. As far as the case $k = 1$ is concerned, let us mention that $\mathcal{I}_1^-$ has been recently considered by Del Pezzo–Quaas–Rossi [17] in order to introduce the notion of fractional convexity.

The second class of operators are instead k-dimensionally nonlocal. For $V \in \mathcal{V}_k$, we denote $\langle V\rangle$ the k-dimensional subspace generated by V. Then, for

$x \in \mathbb{R}^N$ and $u : \mathbb{R}^N \to \mathbb{R}$ and $V = \{\xi_1, ..., \xi_k\} \in \mathcal{V}_k$ we denote

$$\mathcal{J}_V u(x) = \frac{C_{k,s}}{2} \int_{\mathbb{R}^k} \left[u\left(x + \sum_{i=1}^k \tau_i \xi_i\right) + u\left(x - \sum_{i=1}^k \tau_i \xi_i\right) - 2u(x)\right] \left(\sum_{i=1}^k \tau_i^2\right)^{-\frac{k+2s}{2}} d\tau_1 ... d\tau_k$$

where $C_{k,s} > 0$ is the normalizing constant of the fractional Laplacian in the k-Euclidean space [c.f. (6.1)]. Using the change of variables formula (see [19]), we have the equivalent formulation

$$\mathcal{J}_V u(x) = \tfrac{C_{k,s}}{2} \int_{\langle V \rangle} [u(x+z) + u(x-z) - 2u(x)] |z|^{-(k+2s)} d\mathcal{H}^k(z),$$

where $\mathcal{H}^k$ is the k-dimensional Hausdorff measure in $\mathbb{R}^N$.

Then, the extremal operator we consider here is

$$\mathcal{J}_k^+ u(x) = \sup_{V \in \mathcal{V}_k} \mathcal{J}_V u(x), \tag{1.3}$$

and analogously for $\mathcal{J}_k^-$ replacing sup by inf in the above definition. Notice that $\mathcal{J}_k^- u = -\mathcal{J}_k^+(-u)$. Moreover $\mathcal{J}_1^\pm = \mathcal{I}_1^\pm$ and $\mathcal{J}_N^\pm = -(-\Delta)^s$. For this reason, concerning $\mathcal{J}_k^\pm$, we only concentrate on the cases $1 < k < N$.

Clearly for both classes of operators, in a suitable functional framework, say for bounded smooth functions u, $\mathcal{I}_k^\pm u(x)$ and $\mathcal{J}_k^\pm u(x)$ converge to the so called truncated Laplacian $\mathcal{P}_k^\pm u(x)$ as $s \to 1$, where

$$\mathcal{P}_k^+ u(x) := \sum_{i=N-k+1}^N \lambda_i(D^2u(x)) = \max\left\{ \sum_{i=1}^k \langle D^2u(x)\xi_i, \xi_i\rangle \ : \ \{\xi_i\}_{i=1}^k \in \mathcal{V}_k \right\}, \tag{1.4}$$

$\lambda_i(D^2u) \le \lambda_{i+1}(D^2u)$ being the eigenvalues of D^2u arranged in nondecreasing order, and, mutatis mutandis, similarly for $\mathcal{P}_k^- u(x)$ which is the sum of the smallest k-eigenvalues, we replace max by min in the above formula. The truncated Laplacians have received a certain interest, both in geometry and PDE. We wish to remember the works of: Harvey–Lawson [21,22], Caffarelli–Li–Nirenberg [11], Capuzzo Dolcetta–Leoni–Vitolo [14], Blanc–Rossi [9] and of two of the authors of this note with Ishii and Leoni [5–7]. One of the scopes of this paper is to shed some light on different ways of defining generalizations of these extremal degenerate elliptic operators.

The above definitions seem to be natural extensions of the nonlinear second-order operator to the nonlocal setting, in view of the definition of the fractional Laplacian $(-\Delta)^s$. Evaluated on a measurable function u satisfying regularity and growth condition at infinity, its precise definition reads as

$$(-\Delta)^s u(x) = -\tfrac{C_{N,s}}{2} \int_{\mathbb{R}^N} [u(x+z) + u(x-z) - 2u(x)] |z|^{-(N+2s)} dz, \tag{1.5}$$

where $C_{N,s} > 0$ is a normalizing constant making $-(-\Delta)^s \to \Delta$ as $s \to 1^-$. See (6.1) in the Appendix for details on this constant.

The classes of summable functions used in the evaluations of the integral operators considered in this paper are $L^1_{k,\sigma}$, which are spaces of measurable functions whose definition is given in Sect. 2. It will be clear from the definition that a function $u \in L^1(\mathbb{R}^N)$ may not belong to $L^1_{k,\sigma}$.

The first necessity has been to find representation formulas, at least say for radial functions with completely monotonic profile, for example, for power type functions. If we focus on the evaluation of $\mathcal{I}_k^+$ at a function $u(x) = |x|^{-\gamma}$ for $\gamma > 0$, the heuristic makes it reasonable to think that the operator preferably picks a frame $\{\xi_i\}_i$ which includes the direction $\hat{x} = x/|x|$ (we assume $x \neq 0$), since along this radial direction the one dimensional profile of u shows a sharper convexity. Then, the integral associated to the component $\mathcal{I}_{\hat{x}} u(x)$ at (1.2) involves the singularity of u at the origin, which immediately restricts the exponent $\gamma < 1$. The mentioned representation formulas is depicted in Theorem 3.4 below. Concerning the maximal operator $\mathcal{I}_k^+$, the idea discussed above about the preference of the radial direction is confirmed.

Concerning $\mathcal{I}_k^-$, the representation formula shows that in the case $k < N$ the operator picks a frame which is orthogonal to $\hat{x}$. More intriguing is the case of $\mathcal{I}_N^-$. We start noticing that it does not matches $-(-\Delta)^s$, and in fact $\mathcal{I}_N^- \neq \mathcal{I}_N^+$, while the equality occurs in the limit $s \to 1^-$ with the asymptotics $\mathcal{I}_N^\pm \to \Delta$ as $s \to 1^-$. We prove that for radial functions u with convex one dimensional profile, the operator chooses a frame in which all its elements form the same angle with respect to $\hat{x}$, and therefore we have the beautiful geometric symmetry result

$$\mathcal{I}_N^- u(x) = N \mathcal{I}_{\xi^*} u(x) \quad \text{for } x \neq 0, \tag{1.6}$$

where ξ^* is a unit vector such that $\langle \xi^*, \hat{x} \rangle = 1/\sqrt{N}$.

We would like to mention that our representation formulas are obtained under rather strong convexity assumptions on the one dimensional profile of u. Such conditions allow to provide a representation formula for every $x \neq 0$, and therefore we believe they can be relaxed if we look for instance, for the evaluation on bounded domains.

The representation formulas will be used in order to prove Liouville type results, i.e. existence or nonexistence of entire solutions (or supersolutions) bounded from below. First we will consider "superharmonic "functions i.e. supersolutions of

$$\mathcal{I}_k^+ u = 0 \quad \text{in } \mathbb{R}^N.$$

When $k = 1$ and $s \in [\frac{1}{2}, 1)$ there are no nonconstant supersolutions bounded from below while, in the other cases, such supersolutions do exist. Interestingly this result is in contrast to the local second order counter-part, and it is really due to the fractional nature of the operator. This is explained by the existence of a "fundamental solution" of logarithmic profile in the later case. Roughly speaking, since $s < 1$, there is a "gap" between the order of the operator and the dimensionality, and it is this gap that allows us to construct power-type fundamental solutions. We refer to the Appendix for a discussion about the asymptotic behaviour of the exponent of this fundamental solution when we approach the local regime (that is, when $s \to 1^-$), see Lemma 6.2.

We also consider semilinear Liouville theorems for the equations

$$\mathcal{I}_k^{\pm} u + u^p = 0, \quad u \geq 0 \text{ in } \mathbb{R}^N.$$

These semilinear Liouville theorems usually determine a critical value of the exponent p above which there exists supersolutions and below which such nontrivial supersolutions don't exist.

In the case $\mathcal{I}_k^+$, as it can be seen in Theorem 4.2 , and in view of the above discussion, the critical exponent p leading to existence/nonexistence of nontrivial supersolutions for this equation is determined by the exponent of the power-type fundamental solution, which, by the nonlocal nature of the problem, is restricted to be less than 1. As a consequence, we see that the Liouville result does not meet its local counterpart (1.9) as $s \to 1^-$, in the sense that the critical exponent of the nonlocal equation diverges to infinity (equivalently the exponent of the fundamental solution vanishes, see the Appendix). This is a remarkably nonlocal phenomena that is influenced by the tails of the kernel of the operator more than by its singularity.

Concerning $\mathcal{I}_k^-$, the representation formula shows that in the case $k < N$ the operator picks a frame which is orthogonal to $\hat{x}$. This allows us to conclude the existence of nontrivial supersolutions to

$$\mathcal{I}_k^- u + u^p = 0, \quad u \geq 0 \text{ in } \mathbb{R}^N, \tag{1.7}$$

for every $p > 0$, see Theorem 4.7. This phenomena is closely related with its local counterpart presented in [7].

In the case of the equation

$$\mathcal{I}_N^- u + u^p = 0, \quad u \geq 0 \text{ in } \mathbb{R}^N \tag{1.8}$$

let us emphasize that the representation formula (1.6) shows that for $x \neq 0$ the evaluation of the integral operator $\mathcal{I}_N^-$ does not observe possible singularities of u at the origin. Thus, we are able to construct adequate fundamental solutions for $\mathcal{I}_N^-$ (at the expense of a technical redefinition of a power-type function) leading to a Liouville result for Eq. (1.8) which is more in the direction of classical results, and more interesting, with a critical exponent that passes to the limit as $s \to 1^-$.

The local counterpart of these Liouville theorems concerns the equations

$$\mathcal{P}_k^{\pm} u + u^p = 0, \quad u \geq 0 \text{ in } \mathbb{R}^N. \tag{1.9}$$

This problem was studied by two of the authors and F. Leoni in [7]. The construction of fundamental solutions for $\mathcal{P}_k^+$ follows a careful analysis of the eigenvalues of the Hessian of radial functions and the use of the formula (1.4). Once fundamental solutions are at disposal, Liouville-type results associated to the so-called Serrin exponent in space dimension k, i.e. $\frac{k}{k-2}$, follow the directions of [16]. Results concerning $\mathcal{P}_k^-$ are also provided there.

Concerning the other possible extremal operator $\mathcal{J}_k^{\pm}$, we also obtain representation formulas for its evaluation on radial, convex functions, leading to power-type fundamental solutions for these operators. Here we would like to mention that $\mathcal{J}_1^{\pm} = \mathcal{I}_1^{\pm}$, meanwhile $\mathcal{J}_N^{\pm} = -(-\Delta)^s$, from which we restrict ourselves to the case in which k is neither 1 nor N.

In view of the definition (1.3), the higher dimensionality of the integrand allows to prove, in the case of $\mathcal{J}_k^+$, that the fundamental solutions meet the ones of the k-th dimensional fractional Laplacian $-(-\Delta_{\mathbb{R}^k})^s$. This makes the analysis simpler and closer to the local context in the sense that the critical exponent associated to the problem

$$\mathcal{J}_k^+ u + u^p = 0, \quad u \geq 0 \text{ in } \mathbb{R}^N \tag{1.10}$$

meets the critical exponent of (1.9) as $s \to 1$. In particular, this shows that operators $\mathcal{J}_k^+$ and $\mathcal{I}_k^+$ are not equivalent, raising an interesting question related to which of them is more adequate for applications.

The paper is organized as follows: in Sect. 2 we introduce the notion of viscosity solution and discuss comparison/maximum principles. In Sects. 3 and 4 we concentrate on $\mathcal{I}_k^{\pm}$: in Sect. 3 we provide the representation formulas for radial, convex functions, and in Section 4 we present the Liouville-type results for semilinear problems. In Sect. 5 we discuss the results for $\mathcal{J}_k^{\pm}$. Finally, in the Appendix we discuss the asymptotics as $s \to 1^-$.

2. Preliminaries and maximum principles

We start with the definition of viscosity solution. We require certain structural assumptions. We will say that a function u is admissible for $\mathcal{I}_k^{\pm}$, resp. for $\mathcal{J}_k^{\pm}$, if $u \in L^1_{1,2s}$, resp. $u \in L^1_{k,2s}$, where for $k > 1$ and $\sigma \in (0,2)$ we denote the set

$$L^1_{k,\sigma} = \Big\{ u \;\Big|\; u \in L^1_{loc}(V),\ \int_V \frac{|u(y)| d\mathcal{H}^k(y)}{1 + |y|^{k+\sigma}} < +\infty \quad \forall V \ \text{ affine subspace of } \mathbb{R}^N,\, \dim(V) = k \Big\}$$

where $\mathcal{H}^k$ is the k-dimensional Hausdorff measure in $\mathbb{R}^N$.

For viscosity evaluation, we make precise some notation. Given $\xi \in \mathbb{S}^{N-1}$ we denote

$$\mathcal{I}_{\xi,\delta}\phi(x) = C_s \int_0^\delta [\phi(x+\tau\xi) + \phi(x-\tau\xi) - 2\phi(x)]\tau^{-(1+2s)} d\tau$$

$$\mathcal{I}_\xi^\delta \phi(x) = C_s \int_\delta^{+\infty} [\phi(x+\tau\xi) + \phi(x-\tau\xi) - 2\phi(x)]\tau^{-(1+2s)} d\tau$$

and, for each $k = 1, \ldots, N$, $\xi = \{\xi_i\}_{i=1}^k \in \mathcal{V}_k$, we denote

$$\mathcal{J}_{\xi,\delta}\phi(x) = \frac{C_{k,s}}{2} \int_{B_\delta} \left[\phi\left(x + \sum_{i=1}^k z_i\xi_i\right) + \phi\left(x - \sum_{i=1}^k z_i\xi_i\right) - 2\phi(x) \right] |z|^{-(k+2s)} dz$$

$$\mathcal{J}_\xi^\delta\phi(x) = \frac{C_{k,s}}{2} \int_{B_\delta^c} \left[\phi\left(x + \sum_{i=1}^k z_i\xi_i\right) + \phi\left(x - \sum_{i=1}^k z_i\xi_i\right) - 2\phi(x) \right] |z|^{-(k+2s)} dz,$$

where $z = (z_1, \ldots, z_k) \in \mathbb{R}^k$.

Notice that in the case $k = 1$ then $\mathcal{I}_{\xi,\delta} = \mathcal{J}_{\xi,\delta}$ and $\mathcal{I}_\xi^\delta = \mathcal{J}_\xi^\delta$ for each $\xi \in \mathbb{S}^{N-1}$.

Definition 2.1. Let $f \in C(\mathbb{R}^N)$. An upper (lower) semicontinuous function $u : \mathbb{R}^N \to \mathbb{R}$ ($u \in USC(\mathbb{R}^N)$ ($u \in LSC(\mathbb{R}^N)$) for short), admissible with respect to $\mathcal{I}_k^+$, is a viscosity subsolution (supersolution) to

$$\mathcal{I}_k^+ u = f(x) \tag{2.1}$$

at a point $x_0 \in \mathbb{R}^N$ if for every function $\varphi \in C^2(B_\delta(x_0))$, $\delta > 0$, such that x_0 is a global maximum (minimum) point of $u-\varphi$, then there exists $\xi = \{\xi_i\}_{i=1}^k \in \mathcal{V}_k$ such that (for any $\xi = \{\xi_i\}_{i=1}^k \in \mathcal{V}_k$)

$$\begin{aligned} &\sum_{i=1}^k \left(\mathcal{I}_{\xi_i,\varepsilon}\varphi(x_0) + \mathcal{I}_{\xi_i}^\varepsilon u(x_0)\right) \geq f(x_0) \qquad \forall \varepsilon \in (0,\delta) \\ &\left(\sum_{i=1}^k \left(\mathcal{I}_{\xi_i,\varepsilon}\varphi(x_0) + \mathcal{I}_{\xi_i}^\varepsilon u(x_0)\right) \leq f(x_0) \qquad \forall \varepsilon \in (0,\delta)\right). \end{aligned} \tag{2.2}$$

Similarly, $u \in USC(\mathbb{R}^N)$ ($u \in LSC(\mathbb{R}^N)$), admissible with respect to $\mathcal{J}_k^+$, is a viscosity subsolution (supersolution) to

$$\mathcal{J}_k^+ u = f(x) \tag{2.3}$$

at a point $x_0 \in \mathbb{R}^N$ if for every function $\varphi \in C^2(B_\delta(x_0))$, $\delta > 0$, such that x_0 is a global maximum (minimum) point of $u - \varphi$, then there exists $\xi = \{\xi_i\}_{i=1^k} \in \mathcal{V}_k$ such that (for any $\xi = \{\xi_i\}_{i=1}^k \in \mathcal{V}_k$)

$$\begin{aligned} &\mathcal{J}_{\xi,\varepsilon}\varphi(x_0) + \mathcal{J}_\xi^\varepsilon u(x_0) \geq f(x_0) \qquad \forall \varepsilon \in (0,\delta) \\ &\left(\mathcal{J}_{\xi,\varepsilon}\varphi(x_0) + \mathcal{J}_\xi^\varepsilon u(x_0) \leq f(x_0) \qquad \forall \varepsilon \in (0,\delta)\right). \end{aligned}$$

A continuous function is a viscosity solutions of (2.1), or of (2.3), if it is both sub and supersolution.

Lastly we define viscosity sub/supersolutions for $\mathcal{I}_k^-$ and $\mathcal{J}_k^-$ in the same fashion.

The Definition 2.1 admits unbounded or singular sub and/or supersolutions as soon the nonlocal operator is well-defined.

Some comments about the consistence of the above definition with the notion of classical sub and supersolutions are in order. In the following we shall focus on the operator $\mathcal{I}_k^+$. Similar comments can be easily adapted for $\mathcal{I}_k^-$ and $\mathcal{J}_k^\pm$.

The first remark we want to make is the following: whenever a viscosity subsolution u of (2.1) can be touched from above at x_0 by a test function $\varphi \in C^2(B_\delta(x_0))$, then there exists $\xi = \{\xi_i\}_{i=1}^k \in \mathcal{V}_k$ such that

$$\sum_{i=1}^k C_s \int_0^{+\infty} [u(x_0 + \tau\xi_i) + u(x_0 - \tau\xi_i) - 2u(x_0)]\tau^{-(1+2s)}\, d\tau \geq f(x_0). \tag{2.4}$$

That is the operator $\sum_{i=1}^{k}\mathcal{I}_{\xi_i}u(x)$ can be evaluated classically at $x = x_0$ and then

$$\mathcal{I}_k^+ u(x_0) = \sup_{\xi\in\mathcal{V}_k}\sum_{i=1}^{k} C_s \int_0^{+\infty} \frac{u(x_0+\tau\xi_i)+u(x_0-\tau\xi_i)-2u(x_0)}{\tau^{1+2s}} d\tau \geq f(x_0). \quad (2.5)$$

Indeed the very definition of viscosity subsolution implies that there exists $\xi = \{\xi_i\}_{i=1}^k \in \mathcal{V}_k$ such that (2.2) holds. On the other hand,

$$\begin{aligned} u(x_0+\tau\xi_i)+u(x_0-\tau\xi_i)-2u(x_0) &\leq \varphi(x_0+\tau\xi_i)+\varphi(x_0-\tau\xi_i)-2\varphi(x_0) \\ &\leq C\tau^2 \qquad \text{for all } \tau \in \left(-\frac{\delta}{2},\frac{\delta}{2}\right), \end{aligned} \quad (2.6)$$

where $C = \max_{x\in\overline{B}_{\frac{\delta}{2}}(x_0)} \left\|D^2\varphi(x)\right\|$. Then the integrals

$$\int_0^{\frac{\delta}{2}} [u(x_0+\tau\xi_i)+u(x_0-\tau\xi_i)-2u(x_0)]\tau^{-(1+2s)}\, d\tau$$

are bounded from above for any $i = 1,\ldots,k$. Similarly, we have

$$\lim_{\varepsilon\to 0}\mathcal{I}_{\xi_i,\varepsilon}\varphi(x_0) = 0. \quad (2.7)$$

Denoting by χ_E is the characteristic function of the set E and using (2.6), it also holds that for any $\tau > 0$ and $\varepsilon < \frac{\delta}{2}$

$$\begin{aligned} [u(x_0+\tau\xi_i)+u(x_0-\tau\xi_i)-2u(x_0)]\tau^{-(1+2s)}\chi_{[\varepsilon,+\infty)} &\leq C\tau^{1-2s}\chi_{\left(0,\frac{\delta}{2}\right)} \\ &+[u(x_0+\tau\xi_i)+u(x_0-\tau\xi_i)-2u(x_0)]\tau^{-(1+2s)}\chi_{\left[\frac{\delta}{2},+\infty\right)}. \end{aligned}$$

Since the right hand side of the above inequality is an integrable function in $(0,\infty)$, for any $i = 1,\ldots,k$, by the assumption $u \in L^1_{1,2s}$, we are in position to use Fatou's lemma in (2.2) to infer that (2.4) holds.

From the above, we immediately obtain the following proposition.

Proposition 2.2. *If $u \in C^2(B_\rho(x_0)) \cap USC(\mathbb{R}^N) \cap L^1_{1,2s}$, $\rho > 0$, is viscosity subsolutions of (2.1) at x_0, then u is a subsolutions in the classical sense at x_0.*

On the other hand, under the assumption (2.4), classical subsolutions are in turn viscosity subsolutions as showed in the next proposition.

Proposition 2.3. *Suppose that $u \in C^2(B_\rho(x_0)) \cap USC(\mathbb{R}^N) \cap L^1_{1,2s}$, $\rho > 0$, satisfies pointwise the inequality*

$$\mathcal{I}_k^+ u(x_0) \geq f(x_0)$$

and suppose that there exists an orthonormal frame $\xi \in \mathcal{V}_k$ such that (2.4) holds. Then u is a viscosity subsolution of (2.1) at x_0.

Proof. Let $\varphi \in C^2(B_\delta(x_0))$, $\delta < \rho$, be such that

$$u(x) - u(x_0) \le \varphi(x) - \varphi(x_0) \qquad \text{for all } x \in B_\delta(x_0).$$

Then for any $\varepsilon \in (0, \delta)$ we have

$$\begin{aligned} &\sum_{i=1}^{k} C_s \int_0^{\varepsilon} [u(x_0 + \tau\xi_i) + u(x_0 - \tau\xi_i) - 2u(x_0)]\tau^{-(1+2s)}\, d\tau \\ &\quad \le \sum_{i=1}^{k} C_s \int_0^{\varepsilon} [\varphi(x_0 + \tau\xi_i) + \varphi(x_0 - \tau\xi_i) - 2\varphi(x_0)]\tau^{-(1+2s)}\, d\tau \end{aligned} \tag{2.8}$$

and so

$$\sum_{i=1}^{k} \left(\mathcal{I}_{\xi_i,\varepsilon}\varphi(x_0) + \mathcal{I}_{\xi_i}^{\varepsilon} u(x_0)\right) \ge f(x_0) \qquad \forall \varepsilon \in (0, \delta).$$

□

Remark 2.4. A sufficient condition to guarantee the validity of (2.4) from (2.5) is the following: $u \in C^2(B_\rho(x_0)) \cap USC(\mathbb{R}^N) \cap L^1_{1,2s}$ and there exist $C > 0$ and $0 < \alpha < 2s$ such that

$$u(x) \le C(1 + |x|^\alpha) \qquad \forall x \in \mathbb{R}^N.$$

Indeed, in this case, and for any unit vector ξ_i

$$\begin{aligned} &[u(x_0 + \tau\xi_i) + u(x_0 - \tau\xi_i) - 2u(x_0)]\tau^{-(1+2s)} \\ &\quad \le \max_{x \in \overline{B}_{\frac{\rho}{2}}(x_0)} \left\|D^2 u(x)\right\| \tau^{1-2s} \chi_{\left(0,\frac{\rho}{2}\right)} \\ &\qquad + \left(2|u(x_0)| + 2C\left(1 + (|x_0| + \tau)^\alpha\right)\right)\tau^{-(1+2s)}\chi_{\left[\frac{\rho}{2},+\infty\right)} \in L^1(0,+\infty)\,. \end{aligned}$$

Since the above estimate is independent of ξ_i, by Fatou's lemma we infer that the maps $\xi_i \mapsto \mathcal{I}_{\xi_i} u(x_0)$ are upper semicontinuous and the supremum in (2.5) is in fact a maximum.

For supersolutions the situation is similar but easier. If u is a viscosity supersolution of (2.1) which can be touched from below at $x_0 \in \mathbb{R}^N$ by $\varphi \in C^2(B_\delta(x_0))$, then one can prove, as above, that for any choice of $\xi = \{\xi_i\}_{i=1}^k \in \mathcal{V}_k$ the operators $\sum_{i=1}^{k} \mathcal{I}_{\xi_i} u(x_0)$ are defined in classical sense and that

$$\begin{aligned} \mathcal{I}_k^+ u(x_0) = \sup_{\xi \in \mathcal{V}_k} \sum_{i=1}^{k} C_s \int_0^{+\infty} &[u(x_0 + \tau\xi_i) + u(x_0 - \tau\xi_i) \\ &- 2u(x_0)]\tau^{-(1+2s)}\, d\tau \le f(x_0). \end{aligned} \tag{2.9}$$

In particular every viscosity supersolution $u \in C^2(B_\rho(x_0)) \cap LSC(\mathbb{R}^N) \cap L^1_{1,2s}$, $\rho > 0$, of (2.1) at a point x_0 is also a classical supersolution at x_0.

Conversely if $u \in C^2(B_\rho(x_0)) \cap LSC(\mathbb{R}^N) \cap L^1_{1,2s}$, $\rho > 0$, satisfies (2.9), then for any $\xi = \{\xi_i\}_{i=1}^k \in \mathcal{V}_k$ it turns out that

$$\sum_{i=1}^{k} C_s \int_0^{+\infty} [u(x_0 + \tau\xi_i) + u(x_0 - \tau\xi_i) - 2u(x_0)]\tau^{-(1+2s)}\, d\tau \le f(x_0). \quad (2.10)$$

Arguing as in (2.8), but with reversed inequalities, we also infer that (2.9) is fulfilled in viscosity sense.

It is worth to point out that the situation here is a little bit different from the subsolution case, where we had to assume (2.4) in order to prove the equivalence between classical and viscosity subsolutions. For supersolutions, instead, the validity of (2.10) is just a consequence of the definition of $\mathcal{I}_k^+$ which is a sup-type operator.

The inequalities (2.4) and (2.10) are convenient to prove comparison theorems.

Theorem 2.5. (Comparison principle) *Let $\Omega \subset \mathbb{R}^N$ be a bounded open subset of $\mathbb{R}^N$ and let $f \in C(\Omega)$. If $u \in C^2(\Omega) \cap USC(\overline{\Omega}) \cap L^1_{1,2s}$ and $v \in LSC(\overline{\Omega}) \cap L^1_{1,2s}$ are respectively classical subsolution and viscosity supersolution of*

$$\mathcal{I}_k^+ u = f(x) \qquad in\ \Omega$$

such that $u \le v$ in $\mathbb{R}^N \backslash \Omega$, then $u \le v$ in Ω.

Remark 2.6. The above result is still true if $\mathcal{I}_k^+$ is replaced by the operators $\mathcal{I}_k^-$, for $k = 1, \dots, N$. Moreover the hypothesis $u \in C^2(\Omega) \cap USC(\overline{\Omega}) \cap L^1_{1,2s}$ and $v \in LSC(\overline{\Omega}) \cap L^1_{1,2s}$ can be replaced by $u \in USC(\overline{\Omega}) \cap L^1_{1,2s}$ and $v \in C^2(\Omega) \cap LSC(\overline{\Omega}) \cap L^1_{1,2s}$.

We emphasize that the Theorem 2.5 continues to hold if u and v are merely semicontinuous and bounded functions (see [8]). The boundedness assumption of the solutions seems to be quite natural in the nonlocal viscosity framework, e.g. [4, Theorem 3]–[13, Theorem 5.2], see also [1, 3, 23]. On the other hand in this paper we are interested in fundamental solutions which in fact play a key role for Liouville theorems. Hence we allow sub/supersolutions to be unbounded. Nevertheless the validity of the comparison principle between classical subsolutions and lower semicontinuous supersolutions is sufficent for our purposes. For this reason we require $u \in C^2(\Omega) \cap USC(\overline{\Omega}) \cap L^1_{1,2s}$ in Theorem 2.5.

We end this remark by stating that Theorem 2.5 and the above comments also apply for $\mathcal{J}_k^\pm$ provided u, v are admissible for these operators, namely $u, v \in L^1_{k,2s}$.

Proof of Theorem 2.5. We suppose by contradiction that $\sup_\Omega (u - v) > 0$. Hence by semicontinuity there exists $x_0 \in \Omega$ such that

$$u(x_0) - v(x_0) = \max_{x \in \mathbb{R}^N} u(x) - v(x) > 0. \quad (2.11)$$

Since u is a test function for v, then the operator $\mathcal{I}_k^+ v(x_0)$ can be evaluated in classical sense and for any $\xi = \{\xi_i\}_{i=1}^k \in \mathcal{V}_k$ we have

$$\sum_{i=1}^k C_s \int_0^{+\infty} [v(x_0+\tau\xi_i) + v(x_0-\tau\xi_i) - 2v(x_0)]\tau^{-(1+2s)}\, d\tau \leq f(x_0). \quad (2.12)$$

Moreover for any $\varepsilon > 0$ there exists $\xi(\varepsilon) \in \mathcal{V}_k$ such that

$$\sum_{i=1}^k C_s \int_0^{+\infty} [u(x_0+\tau\xi_i(\varepsilon)) + v(x_0-\tau\xi_i(\varepsilon)) - 2v(x_0)]\tau^{-(1+2s)}\, d\tau \geq f(x_0) - \varepsilon. \quad (2.13)$$

Using (2.11) and (2.12)–(2.13) we have

$$\begin{aligned} -\varepsilon &\leq \sum_{i=1}^k C_s \int_0^{+\infty} \frac{(u-v)(x_0+\tau\xi_i(\varepsilon)) + (u-v)(x_0-\tau\xi_i(\varepsilon)) - 2(u-v)(x_0)}{\tau^{1+2s}} d\tau \\ &\leq \sum_{i=1}^k C_s \int_{\mathrm{diam}(\Omega)}^{+\infty} \frac{(u-v)(x_0+\tau\xi_i(\varepsilon)) + (u-v)(x_0-\tau\xi_i(\varepsilon)) - 2(u-v)(x_0)}{\tau^{1+2s}} d\tau \\ &\leq \sum_{i=1}^k C_s \int_{\mathrm{diam}(\Omega)}^{+\infty} \frac{-2(u-v)(x_0)}{\tau^{1+2s}} d\tau = -(u-v)(x_0) k \frac{C_s}{s} (\mathrm{diam}(\Omega))^{-2s}. \end{aligned} \quad (2.14)$$

In the last inequality we used that $\Omega \subseteq B_{\mathrm{diam}(\Omega)}(x_0)$ and that $x_0 \pm \tau\xi(\varepsilon) \notin B_{\mathrm{diam}(\Omega)}(x_0)$ for any $\tau \geq \mathrm{diam}(\Omega)$, which yields $(u-v)(x_0 \pm \tau\xi_i(\varepsilon)) \leq 0$. From (2.14) we obtain a contradiction for ε small enough. □

We now state the basic statement regarding the failure and the validity of the strong maximum/minimum principles for the operators $\mathcal{I}_k^\pm$.

Proposition 2.7. *For any $1 \leq k < N$ there exist nonconstant smooth solutions of*

$$\mathcal{I}_k^- u \leq 0 \quad \text{in } \mathbb{R}^N \quad (2.15)$$

which attain their minimum at some point in $\mathbb{R}^N$.

If u satisfies

$$\mathcal{I}_N^- u \leq 0 \quad \text{in } \mathbb{R}^N, \quad (2.16)$$

in the viscosity sense, and it attains its minimum at some $x_0 \in \mathbb{R}^N$, then u is constant.

Remark 2.8. In a dual fashion, for any $1 \leq k < N$ there exist nonconstant smooth solutions of

$$\mathcal{I}_k^+ u \geq 0 \quad \text{in } \mathbb{R}^N \quad (2.17)$$

which attain their maximum at some point in $\mathbb{R}^N$.

If u satisfies

$$\mathcal{I}_N^+ u \geq 0 \quad \text{in } \mathbb{R}^N, \quad (2.18)$$

in the viscosity sense, and it attains its maximum at some $x_0 \in \mathbb{R}^N$, then u is constant.

Remark 2.9. By the general fact

$$\mathcal{I}_k^+ u \le 0 \quad \Rightarrow \quad \mathcal{I}_N^- u \le 0\,, \tag{2.19}$$

we immediately obtain, via Proposition 2.7, the validity of the strong minimum principle for supersolutions of $\mathcal{I}_k^+ u = 0$.

To see (2.19), let $x_0 \in \mathbb{R}^N$ and let $\varphi \in C^2(B_\delta(x_0))$ be a test function touching u from below at x_0 (if there are no such φ then there is nothing to check). Then, from the viscosity inequality $\mathcal{I}_k^+ u(x_0) \le 0$, we infer that for every $\xi = \{\xi_i\}_{i=1}^k \in \mathcal{V}_k$ the operators $\mathcal{I}_{\xi_i} u(x_0)$ are well defined for any $i = 1, \dots, k$ and

$$\sum_{i=1}^k \mathcal{I}_{\xi_i} u(x_0) \le 0. \tag{2.20}$$

Let $\left\{\bar\xi_1, \dots, \bar\xi_N\right\}$ be an orthonormal basis of $\mathbb{R}^N$. Without loss of generality we may further assume that $\mathcal{I}_{\bar\xi_i} u(x_0) \le \mathcal{I}_{\bar\xi_{i+1}} u(x_0)$ for $i = 1, \dots, N-1$. Using (2.20) we have

$$\sum_{i=1}^N \mathcal{I}_{\bar\xi_i} u(x_0) \le \frac{N}{k} \sum_{i=N-k+1}^N \mathcal{I}_{\bar\xi_i} u(x_0) \le 0.$$

From this we deduce that

$$\sum_{i=1}^N \left(\mathcal{I}_{\bar\xi_i,\varepsilon} \varphi(x_0) + \mathcal{I}_{\bar\xi_i}^{\varepsilon} u(x_0)\right) \le 0 \qquad \forall \varepsilon \in (0, \delta),$$

hence u is a viscosity supersolution of $\mathcal{I}_N^- u \le 0$ at x_0.

Note that, since $\left\{\bar\xi_1, \dots, \bar\xi_N\right\}$ is arbitrary, then we may conclude in fact that $\mathcal{I}_N^+ u(x_0) \le 0$.

In a similar way and using Remark 2.8, we infer that the strong maximum principle for subsolution of $\mathcal{I}_k^- u = 0$ holds.

Proof of Proposition 2.7. Let φ be a nonconstant smooth and bounded function of one variable which attains the minimum at some point in $\mathbb{R}$. Consider φ as a function of N variables just by setting $u(x) := \varphi(x_N)$. It is clear that u is a nontrivial function attaining its minimum at some point in $\mathbb{R}^N$. If $\{e_i\}_{i=1}^N$ denote the canonical basis in $\mathbb{R}^N$, then for any $x \in \mathbb{R}^N$ and any $\tau \in \mathbb{R}$ we have

$$u(x + \tau e_i) = u(x) \quad \text{for } i = 1, \dots, N-1.$$

Hence $\mathcal{I}_{e_i} u(x) = 0$ for any $i = 1, \dots, N-1$ and

$$\mathcal{I}_k^- u(x) \le \sum_{i=1}^k \mathcal{I}_{e_i} u(x) = 0 \quad \text{in } \mathbb{R}^N.$$

This concludes the first part of the proof.

For the second part, we use the argument of propagation of maxima through the support of the kernel of the nonlocal operator, see [15].

Let $y \in \mathbb{R}^N$ and denote $d_0 = |y - x_0|$. Since x_0 is a minimum point for u we can use the constant function $\varphi(x) = u(x_0)$ in $B_\delta(x_0)$, $\delta > 0$, in the Definition 2.1 of viscosity supersolution of $\mathcal{I}_N^- u \le 0$ at x_0. Then there is $\xi = \{\xi_i\}_{i=1}^N \in \mathcal{V}_N$ such that

$$\sum_{i=1}^{N} \left(\mathcal{I}_{\xi_i,\varepsilon}\varphi(x_0) + \mathcal{I}^{\varepsilon}_{\xi_i}u(x_0)\right) \le 0 \qquad \forall \varepsilon \in (0,\delta).$$

Since $\mathcal{I}_{\xi_i,\varepsilon}\varphi(x_0) = 0$ and x_0 is a global minimum point of u, the above inequality yields $u(x_0 + \tau\xi_i) = u(x_0)$ for any $\tau \in \mathbb{R}$ and $i = 1,\dots,N$. Now, since ξ is a basis of $\mathbb{R}^N$, there exists at least one $\xi_i \in \xi$ such that

$$\left|\langle \widehat{y - x_0}, \xi_i\rangle\right| \ge 1/\sqrt{N},$$

where $\widehat{y - x_0} = \frac{y-x_0}{|y-x_0|}$. From this there exists τ such that $x_1 := x_0 + \tau\xi_i$ simultaneously satisfies $u(x_1) = u(x_0)$ and $d_1 := |x_1 - y| \le d_0\sqrt{1 - N^{-1}}$. Using the same argument above but with x_1 and d_1 replacing x_0 and d_0, it is possible to find $x_2 \in \mathbb{R}^N$ such that $u(x_2) = u(x_0)$ and $|x_2 - y| =: d_2 \le d_1\sqrt{1 - N^{-1}}$. Then, repeating this argument, we find a sequence $(x_k)_{k\in\mathbb{N}}$ such that $u(x_k) = u(x_0)$ and $x_k \to y$. By lower semicontinuity, we conclude that $u(y) \le u(x_0)$ and then $u(y) = u(x_0)$, x_0 being the global minimum point of u. Since y is arbitrary we get the result. □

3. Representation formula for $\mathcal{I}_k^{\pm}$

Here and in what follows we use the following notation: for $x \ne 0$, denote $\hat{x} = x/|x|$ and denote $V_x = \langle\{\hat{x}\}\rangle^{\perp}$ the orthogonal subspace to $\hat{x}$. Given a subspace V, we denote π_V the projection onto V. Then

Lemma 3.1. *Let $\xi \in \mathbb{S}^{N-1}$, $x \in \mathbb{R}^N$ and $u \in C^2(\mathbb{R}^N) \cap L^1_{1,2s}$. Then*

(a) $\mathcal{I}_\xi u(x) = \mathcal{I}_{-\xi}u(x)$.

(b) *If R is any rotation matrix in $\mathbb{R}^N$ and if we denote $\tilde{u}(x) = u(Rx)$, then*

$$\mathcal{I}_\xi \tilde{u}(x) = \mathcal{I}_{R\xi}u(Rx).$$

(c) *If u is radial, that is $u(x) = g(|x|)$ for some real valued function g, then*

$$\mathcal{I}_\xi u(x) = \mathcal{I}_{\xi_x}u(x),$$

where $\xi_x = \pi_{V_x}(\xi) - \langle\xi,\hat{x}\rangle\hat{x}$ is the unit vector, symmetric to ξ with respect to the hyperplane V_x.

(d) *If u is radial and $R : \mathbb{R}^N \to \mathbb{R}^N$ is a rotation matrix leaving invariant V_x, then*

$$\mathcal{I}_\xi u(x) = \mathcal{I}_{R\xi}u(x).$$

Proof. The proof of (a) and (b) are immediate, and do not require u to be radial. For (c), we see that

$$|x + \tau\xi_x|^2 = ||x| - \tau\langle\xi,\hat{x}\rangle|^2 + |\tau\pi_{V_x}(\xi)|^2,$$

and using the symmetry of the kernel, we make the change of variables $\tau = -\tau$, and noticing that $\xi = \pi_{V_x}(\xi) + \langle\xi,\hat{x}\rangle\hat{x}$ we conclude the result.

For (d), we notice that

$$|x+\tau R\xi|^2 = |x+\tau\langle \xi,\hat{x}\rangle\hat{x}|^2 + |R\pi_{V_x}(\xi)\tau|^2,$$

and using that a rotation matrix is an isometry, we conclude the result. □

Remark 3.2. By the previous lemma, for every radial function u and every orthonormal frame $\{\xi_i\}_{i=1}^k$ the definition of the operator $\mathcal{I}_k^{\pm}u(x)$ can be taken in such a way that the angle between x and each ξ_i is in $[0,\pi/2]$.

Now we present the main technical result of this section.

Lemma 3.3. *Assume $u(x)=\tilde{g}(|x|^2)$ is such that $u \in C^2(\mathbb{R}^N\setminus\{0\})\cap L^1_{1,2s}$. For $x\neq 0$ and $\theta\in[0,1]$ we denote*

$$I(|x|,\theta) := C_s|x|^{-2s} \times \int_0^{+\infty} \frac{\tilde{g}(|x|^2(1+\tau^2+2\tau\theta))+\tilde{g}(|x|^2(1+\tau^2-2\tau\theta))-2\tilde{g}(|x|^2)}{\tau^{1+2s}}d\tau.$$

- *If $\tilde{g}$ is convex and $k=1,...,N-1$, then*
$$\mathcal{I}_k^- u(x) = kI(|x|,0). \tag{3.1}$$
- *If $\tilde{g}''$ is convex, then*
$$\mathcal{I}_N^- u(x) = NI(|x|,\frac{1}{\sqrt{N}}). \tag{3.2}$$
- *If $\tilde{g},\tilde{g}''$ are convex, for all $k=1,...,N$ we have*
$$\mathcal{I}_k^+ u(x) = I(|x|,1)+(k-1)I(|x|,0). \tag{3.3}$$

Proof. For $a\geq b\geq 0$, let $h:[-1,1]\to\mathbb{R}$ be the function

$$h(t)=\tilde{g}(a+bt)+\tilde{g}(a-bt),$$

and $p:[0,1]\to\mathbb{R}$ defined as $p(t)=h(\sqrt{t})$.

Note that h is even. If $\tilde{g}$ is convex, so is h. From this, 0 is a minimum point for h and h is nondecreasing in $[0,1]$. The same analysis in the case $\tilde{g}''$ is convex implies h'' is even, convex and nondecreasing in $[0,1]$.

We start with (3.1). Using the monotonicity of h, in particular $h(0)\leq h(1)$, for each $a\geq b\geq 0$ we get

$$2\tilde{g}(a)\leq \tilde{g}(a+b)+\tilde{g}(a-b). \tag{3.4}$$

Take an orthonormal set $\{\xi_i\}_{i=1}^k$ and $\tau>0$. Using the last inequality with $a=|x|^2+\tau^2$ and $b=2\tau|\langle x,\xi_i\rangle|$ we have

$$2\tilde{g}(|x|^2+\tau^2)\leq \tilde{g}(|x|^2+\tau^2+2\tau|\langle x,\xi_i\rangle|)+\tilde{g}(|x|^2+\tau^2-2\tau|\langle x,\xi_i\rangle|).$$

Substracting $2\tilde{g}(|x|^2)$ in both sides, multiplying by the factor $\tau^{-(1+2s)}$, integrating from 0 to $+\infty$, and summing-up in $i=1,\dots,k$, we conclude that

$$kI(|x|,0)\leq \sum_{i=1}^k \mathcal{I}_{\xi_i}u(x).$$

Since $k<N$, we can select an orthonormal set such that $\langle x,\xi_i\rangle=0$ for all i and the lower bound is attained, from which we arrive at (3.1).

Now we continue with (3.2). For this, we use that $\tilde{g}''$ is convex. Observe that $h'(0) = 0$. Now, $p'(t) = \frac{1}{2\sqrt{t}} h'(\sqrt{t})$ and then

$$\begin{aligned} p''(t) &= \frac{1}{4t} h''(\sqrt{t}) - \frac{1}{4t^{3/2}} (h'(\sqrt{t}) - h'(0)) \\ &= \frac{1}{4t} h''(\sqrt{t}) - \frac{1}{4t^{3/2}} \int_0^{\sqrt{t}} h''(\theta) d\theta \\ &= \frac{1}{4t} \int_0^1 [h''(\sqrt{t}) - h''(\sqrt{t}\theta)] d\theta \,. \end{aligned}$$

Since h'' is nondecreasing in $[0, 1]$ we obtain $p'' \geq 0$, which shows that p is convex in $[0, 1]$.

Consider the simplex $\Lambda = \{\lambda = (\lambda_1, ..., \lambda_k) : \lambda_i \geq 0$ for $i = 1, \ldots, k$, $\sum_{i=1}^k \lambda_i = 1\}$ and let $P : \Lambda \to \mathbb{R}$ given by

$$P(\lambda) = \sum_{i=1}^k p(\lambda_i).$$

Let $\lambda = (\lambda_1, ..., \lambda_k) \in \Lambda$. Using the convexity of p, we can write

$$p(\lambda_i) \geq p\left(\frac{1}{k}\right) + p'\left(\frac{1}{k}\right)\left(\lambda_i - \frac{1}{k}\right)$$

for each i. Then, we conclude that

$$P(\lambda) \geq kp\left(\frac{1}{k}\right).$$

In particular, since $(\frac{1}{k}, ..., \frac{1}{k}) \in \Lambda$, we get

$$\min_{\lambda \in \Lambda} P(\lambda) = kp\left(\frac{1}{k}\right) = k\left(\tilde{g}\left(a + b\frac{1}{\sqrt{k}}\right) + \tilde{g}\left(a - b\frac{1}{\sqrt{k}}\right)\right).$$

When $k = N$, for each orthonormal set $\{\xi_i\}_{i=1}^N$ we have $\sum_{i=1}^N \langle \hat{x}, \xi_i \rangle^2 = 1$, where $\hat{x} = \frac{x}{|x|}$. Using the last equality with $a = |x|^2 + \tau^2$, $b = 2|x|\tau$ and $\lambda_i = |\langle \hat{x}, \xi_i \rangle|^2$ we conclude that

$$\begin{aligned} &\sum_{i=1}^N \tilde{g}(|x|^2 + \tau^2 + 2\tau|x||\langle \hat{x}, \xi_i \rangle|) + \tilde{g}(|x|^2 + \tau^2 - 2\tau|x||\langle \hat{x}, \xi_i \rangle|) \\ &\quad \geq N\left(\tilde{g}\left(|x|^2 + \tau^2 + 2\tau|x|\frac{1}{\sqrt{N}}\right) + \tilde{g}\left(|x|^2 + \tau^2 - 2\tau|x|\frac{1}{\sqrt{N}}\right)\right). \end{aligned}$$

Again, substracting $2N\tilde{g}(|x|^2)$ in both sides, multiplying by $\tau^{-(1+2s)}$ and integrating, we see that

$$NI\left(|x|, \frac{1}{\sqrt{N}}\right) \leq \sum_{i=1}^N \mathcal{I}_{\xi_i} u(x).$$

The infimum is attained. For this, let consider $O : \mathbb{R}^N \to \mathbb{R}^N$ the orthonormal map so that $O\hat{x} = \frac{1}{\sqrt{N}} \sum_{i=1}^N e_i$, where $\{e_i\}_{i=1}^N$ the standard basis in $\mathbb{R}^N$. Set $\xi_i = O^{-1} e_i$.

Then, using the rotation invariance of the operator, together with the radiality of the function we conclude that

$$\sum_{i=1}^{N} \mathcal{I}_{\xi_i} u(x) = \sum_{i=1}^{N} \mathcal{I}_{O^{-1}e_i} u(x) = \sum_{i=1}^{N} \mathcal{I}_{e_i} u(Ox) = NI\left(|x|, \frac{1}{\sqrt{N}}\right).$$

Now we deal with (3.3). Let $\{e_i\}_{i=1}^k$ the standard basis of $\mathbb{R}^k$. Since P is convex in Λ, then we have

$$\max_{\lambda\in\Lambda} P(\lambda) = \max_i P(e_i).$$

Observe that $P(e_i) = p(1) + (k-1)p(0)$ for each i, from which we conclude that

$$\max_{\lambda\in\Lambda} \sum_{i=1}^{k} [\tilde{g}(a + b\sqrt{\lambda_i}) + \tilde{g}(a - b\sqrt{\lambda_i})] = \tilde{g}(a+b) + \tilde{g}(a-b) + 2(k-1)\tilde{g}(a). \tag{3.5}$$

Let $x \neq 0$ and $\{\xi_i\}_{i=1}^k$ and orthonormal set in $\mathbb{R}^N$. Let $\rho = \sum_{i=1}^k \langle \hat{x}, \xi_i\rangle^2 \leq 1$ and assume that $\rho > 0$. Denote $\lambda_i = \frac{\langle \hat{x}, \xi_i\rangle^2}{\rho}$. Then, for each $\tau > 0$, by (3.5) with $a = |x|^2 + \tau^2$ and $b = 2\sqrt{\rho}|x|\tau$, and using the monotonicity of h, we have

$$\begin{aligned}
&\sum_{i=1}^{k} [\tilde{g}(|x|^2 + \tau^2 + 2|x|\tau|\langle \hat{x}, \xi_i\rangle|) + \tilde{g}(|x|^2 + \tau^2 - 2|x|\tau|\langle \hat{x}, \xi_i\rangle|)] \\
&\quad = \sum_{i=1}^{k} [\tilde{g}(|x|^2 + \tau^2 + 2|x|\tau\sqrt{\rho\lambda_i}) + \tilde{g}(|x|^2 + \tau^2 - 2|x|\tau\sqrt{\rho\lambda_i})] \\
&\quad \leq \tilde{g}(|x|^2 + \tau^2 + 2\sqrt{\rho}|x|\tau) + \tilde{g}(|x|^2 + \tau^2 - 2\sqrt{\rho}|x|\tau) + 2(k-1)\tilde{g}(|x|^2 + \tau^2) \\
&\quad \leq \tilde{g}(|x|^2 + \tau^2 + 2|x|\tau) + \tilde{g}(|x|^2 + \tau^2 - 2|x|\tau) + 2(k-1)\tilde{g}(|x|^2 + \tau^2).
\end{aligned}$$

Thus, we arrive at

$$\begin{aligned}
&\sum_{i=1}^{k} [\tilde{g}(|x|^2 + \tau^2 + 2|x|\tau|\langle \hat{x}, \xi_i\rangle|) + \tilde{g}(|x|^2 + \tau^2 - 2|x|\tau|\langle \hat{x}, \xi_i\rangle|) - 2\tilde{g}(|x|^2)] \\
&\quad \leq \Big(\tilde{g}(|x|^2 + \tau^2 + 2|x|\tau) + \tilde{g}(|x|^2 + \tau^2 - 2|x|\tau) - 2\tilde{g}(|x|^2)\Big) \\
&\qquad + 2(k-1)\Big(\tilde{g}(|x|^2 + \tau^2) - \tilde{g}(|x|^2)\Big),
\end{aligned}$$

from which, after multiplying by the kernel $\tau^{-(1+2s)}$ and integration, we get

$$\sum_{i=1}^{k} \mathcal{I}_{\xi_i} u(x) \leq I(|x|, 1) + (k-1)I(|x|, 0). \tag{3.6}$$

When $\rho = 0$, we use again the monotonicity of h (with $a = |x|^2 + \tau^2$ and $b = 2\tau|x|$) to conclude

$$\begin{aligned}\sum_{i=1}^{k}&[\tilde g(|x|^2+\tau^2+2|x|\tau|\langle \hat x,\xi_i\rangle|)+\tilde g(|x|^2+\tau^2\\ &\quad -2|x|\tau|\langle \hat x,\xi_i\rangle|)-2\tilde g(|x|^2)-2\tilde g(|x|^2)]\\ &=2k[\tilde g(|x|^2+\tau^2)-\tilde g(|x|^2)]\\ &\le \Big(\tilde g(|x|^2+\tau^2+2|x|\tau)+\tilde g(|x|^2+\tau^2-2|x|\tau)\\ &\quad -2\tilde g(|x|^2)\Big)+2(k-1)\Big(\tilde g(|x|^2+\tau^2)-\tilde g(|x|^2)\Big),\end{aligned}$$

which leads us to (3.6) as well. Noting that if we pick an orthonormal set $\{\bar\xi_i\}_{i=1}^k$ such that $\hat x = \bar\xi_1$, we see that

$$\sum_{i=1}^{k}\mathcal{I}_{\bar\xi_i}u(x) = I(|x|,1)+(k-1)I(|x|,0)$$

and this concludes (3.3). □

As a consequence of the above result we have the following representation formulas.

Theorem 3.4. *Assume $u(x) = \tilde g(|x|^2) \in C^2(\mathbb{R}^N\setminus\{0\}) \cap L^1_{1,2s}$. Let $x \neq 0$ and denote $\hat x = x/|x|$.*

(i) For all $N, k \in \mathbb{N}$ with $1 \le k \le N$, if $\tilde g$ and $\tilde g''$ are convex

$$\mathcal{I}_k^+u(x) = \mathcal{I}_{\hat x}u(x) + (k-1)\mathcal{I}_{x^\perp}u(x),$$

where $x^\perp \in V_x$ with $|x^\perp| = 1$.

(ii) If $1 \le k < N$ and $\tilde g$ is convex, we have

$$\mathcal{I}_k^-u(x) = k\mathcal{I}_{x^\perp}u(x),$$

where $x^\perp$ is as in the previous point.

(iii) If $\tilde g''$ is convex

$$\mathcal{I}_N^-u(x) = N\mathcal{I}_{\xi^*}u(x),$$

where $\xi^* \in \mathbb{R}^N$ is a unit vector such that $\langle \hat x, \xi^*\rangle = \frac{1}{\sqrt N}$.

Remark 3.5. It is easy to see that if $u(x) := g(|x|) := \tilde g(|x|^2)$ then the condition $\tilde g$ convex implies, for smooth g, that $g''(r) \ge \frac{g'(r)}{r}$. Also examples of functions $\tilde g$ satisfying the above assumptions include $\tilde g(t) = \sqrt{t}^{-\gamma}$ with $\gamma \in (0,1)$, $\tilde g(t) = \left(a+\sqrt t\right)^{-\gamma}$, $\tilde g(t) = (a+t)^{-\gamma}$, $\tilde g(t) = e^{-at}$ for $a > 0$ and $\gamma > 0$. Another example is the function $\tilde g(t) = -\sqrt{t}^{\gamma}$ for $\gamma \in (0, 2s)$.

3.1. Computation on power-type functions

We start with the following lemma that can be found in [2], but that we present here for the readers convenience.

Lemma 3.6. *For $\gamma > 0$, denote $v_\gamma(x) = |x|^\gamma$. Then for $s \in (0,1)$ and $\gamma \in (0, 2s)$, there exists a constant $\hat{c}(\gamma) \in \mathbb{R}$ such that*

$$\mathcal{I}_{\hat{x}} v_\gamma(x) = \hat{c}(\gamma)|x|^{\gamma-2s} \quad \textit{for all } x \neq 0$$

and

- $\hat{c}(\gamma) < 0$ *if* $\gamma \in (0, (2s-1)_+)$
- $\hat{c}(\gamma) = 0$ *if* $\gamma = (2s-1)_+$
- $\hat{c}(\gamma) > 0$ *of* $\gamma \in ((2s-1)_+, 2s)$.

Similarly, for $\gamma \in (0,1)$, denote $w_\gamma(x) = |x|^{-\gamma}$. Then for $s \in (0,1)$ there exists a constant $\hat{c}(\gamma) \in \mathbb{R}$ such that

$$\mathcal{I}_{\hat{x}} w_\gamma(x) = \hat{c}(\gamma)|x|^{-\gamma-2s} \quad \textit{for all } x \neq 0$$

and

- $\hat{c}(\gamma) < 0$ *if* $\gamma \in (0, (1-2s)_+)$
- $\hat{c}(\gamma) = 0$ *if* $\gamma = (1-2s)_+$
- $\hat{c}(\gamma) > 0$ *of* $\gamma \in ((1-2s)_+, 1)$.

Proof. We only consider the case $\mathcal{I}_{\hat{x}} v_\gamma(x)$, since the proof concerning $\mathcal{I}_{\hat{x}} w_\gamma(x)$ follows the same ideas. We notice that $\mathcal{I}_{\hat{x}} v_\gamma(x)$ is well-defined since $v_\gamma \in L^1_{1,2s}$. We have

$$\mathcal{I}_{\hat{x}} v_\gamma(x) = C_s \mathrm{P.V.} \int_{-\infty}^{+\infty} [|x + \tau\hat{x}|^\gamma - |x|^\gamma]|\tau|^{-(1+2s)} d\tau = C_s |x|^{\gamma-2s} I,$$

where

$$I := \mathrm{P.V.} \int_{-\infty}^{+\infty} [|1+\tau|^\gamma - 1]|\tau|^{-(1+2s)} d\tau.$$

We split the last integral as

$$I = \int_{-\infty}^{-1} [|1+\tau|^\gamma - 1]|\tau|^{-(1+2s)} d\tau + \mathrm{P.V.} \int_{-1}^{+\infty} [|1+\tau|^\gamma - 1]|\tau|^{-(1+2s)} d\tau,$$

and using the change of variables $1+\tau = -e^z$, $z \in \mathbb{R}$, for the first integral, and $1+\tau = e^z$, $z \in \mathbb{R}$, for the second, we obtain

$$\begin{aligned} I &= \int_{-\infty}^{+\infty} [e^{z\gamma} - 1](1+e^z)^{-(1+2s)} e^z dz \\ &\quad + \mathrm{P.V.} \int_{-\infty}^{+\infty} [e^{\gamma z} - 1]|e^z - 1|^{-(1+2s)} e^z dz \\ &= 2^{-2s} \int_{-\infty}^{+\infty} e^{z((\gamma+1)/2 - s)} \sinh(\gamma z/2)(\cosh(z/2))^{-(1+2s)} dz \\ &\quad + 2^{-2s} \mathrm{P.V.} \int_{-\infty}^{+\infty} e^{z((\gamma+1)/2 - s)} \sinh(\gamma z/2)|\sinh(z/2)|^{-(1+2s)} dz\,. \end{aligned}$$

Notice that in both integrals, the term $e^{z((\gamma+1)/2-s)}$ is integrated against an odd kernel. Thus, if $s > 1/2$ and $\frac{\gamma+1}{2} - s = 0$ (that is $\gamma = 2s - 1$), we have $I = 0$. Similarly, if $\gamma < 2s - 1$, then $I < 0$, and if $\gamma > 2s - 1$ then $I > 0$. This concludes the lemma. □

Using Theorem 3.4 we have the following identity

Proposition 3.7. *Let $\gamma \in (0, 1)$ and denote $w_\gamma(x) = |x|^{-\gamma}$ for $x \neq 0$. Then*

$$\mathcal{I}_k^+ w_\gamma(x) = c_k(\gamma)|x|^{-(\gamma+2s)}, \tag{3.7}$$

where $c_k(\gamma) = \hat{c}(\gamma) + (k-1)c^\perp(\gamma)$ with

$$\hat{c}(\gamma) := C_s \text{P.V.} \int_{-\infty}^{+\infty} [|1+\tau|^{-\gamma} - 1]|\tau|^{-(1+2s)} d\tau$$

$$c^\perp(\gamma) := 2C_s \int_0^{+\infty} [(1+\tau^2)^{-\gamma/2} - 1]\tau^{-(1+2s)} d\tau.$$

For $k \geq 1$, the function $c_k : (0, 1) \to \mathbb{R}$ satisfies $c_k(0^+) = 0$, $c_k(1^-) = +\infty$, it is strictly convex in $(0, 1)$ and there exists a unique $\bar{\gamma} = \bar{\gamma}(k, s) \in (0, 1)$ such that $c_k(\bar{\gamma}) = 0$ in the following cases:

(i) *$k = 1$ and $s \in (0, \frac{1}{2})$*
(ii) *$k \geq 2$ and $s \in (0, 1)$.*

Proof. Formula (3.7) follows directly by the characterization provided in Theorem 3.4 and the fact that for each $x \neq 0$ we have

$$\mathcal{I}_{\hat{x}} w_\gamma(x) = \hat{c}(\gamma)|x|^{-(\gamma+2s)}$$
$$\mathcal{I}_{\hat{x}^\perp} w_\gamma(x) = c^\perp(\gamma)|x|^{-(\gamma+2s)},$$

where we have used the homogeneity of the nonlocal operator and the function w.

Using Dominated Convergence Theorem, for each k we have

$$c_k(\gamma) \to 0 \quad \text{as } \gamma \to 0^+.$$

Hence, defining $c_k(0) = 0$, we have $c_k : [0, 1) \to \mathbb{R}$ is a continuous function.

On the other hand, noticing that $c^\perp(\gamma)$ is uniformly bounded for $\gamma \in (0, 1)$ and that $\hat{c}(\gamma) \to +\infty$ as $\gamma \to 1^-$, we have $c_k(\gamma) \to +\infty$ as $\gamma \to 1^-$.

In addition, $c_k \in C^2(0, 1)$ and for $\gamma \in (0, 1)$ we see that

$$c_k'(\gamma) = -C_s \Bigg(\int_0^{+\infty} \Big[|1+\tau|^{-\gamma} \ln|1+\tau| + |1-\tau|^{-\gamma} \ln|1-\tau| \Big] \tau^{-(1+2s)} d\tau$$
$$+ (k-1) \int_0^{+\infty} (1+\tau^2)^{-\gamma/2} \ln(1+\tau^2)\tau^{-(1+2s)} d\tau \Bigg)$$
$$c_k''(\gamma) = C_s \Bigg(\int_0^{+\infty} \Big[|1+\tau|^{-\gamma} \ln^2|1+\tau| + |1-\tau|^{-\gamma} \ln^2|1-\tau| \Big] \tau^{-(1+2s)} d\tau$$
$$+ \frac{k-1}{2} \int_0^{+\infty} (1+\tau^2)^{-\gamma/2} \ln^2(1+\tau^2)\tau^{-(1+2s)} d\tau \Bigg)$$

and from here we clearly have $c_k'' > 0$. Hence c_k is a strictly convex function in $[0, 1)$.

Now we prove the existence of a unique $\bar{\gamma} \in (0, 1)$ such that $c_k(\gamma) < 0$ for $\gamma \in (0, \bar{\gamma})$, $c_k(\bar{\gamma}) = 0$ and $c_k(\gamma) > 0$ for $\gamma \in (\bar{\gamma}, 1)$.

The case $k = 1$ and $s \in (0, \frac{1}{2})$ trivially follows from Lemma 3.6, since

$$c_1(1 - 2s) = \hat{c}(1 - 2s) = 0.$$

In this case $\bar{\gamma} = 1-2s$, $c_1(\gamma) < 0$ if $\gamma \in (0, 1-2s)$ and $c_1(\gamma) > 0$ if $\gamma \in (1-2s, 1)$.

In what follows we assume $k \geq 2$. If $s \in (0, \frac{1}{2})$ we have

$$c_k(1 - 2s) = \hat{c}(1 - 2s) + (k - 1)c^{\perp}(1 - 2s) = c^{\perp}(1 - 2s) < 0.$$

Then, by the convexity of c_k, there exists a unique $\bar{\gamma} \in (1 - 2s, 1)$ such that $c_k(\bar{\gamma}) = 0$. Moreover $c_k(\gamma) < 0$ for $\gamma \in (0, \bar{\gamma})$ and $c_k(\gamma) > 0$ for $\gamma \in (\bar{\gamma}, 1)$.

Now we consider $s \in [\frac{1}{2}, 1)$. It is easy to see that $c_k'(0^+)$ exists and we have the expression

$$\begin{aligned} c_k'(0^+) = -C_s \bigg(&\int_0^{+\infty} \ln|1 - \tau^4| \tau^{-(1+2s)} d\tau \\ &+ (k - 2) \int_0^{\infty} \ln(1 + \tau^2) \tau^{-(1+2s)} d\tau \bigg) \quad \text{for } k \geq 2. \end{aligned}$$

We claim that

$$c_k'(0^+) < 0 \quad \text{for all } k \geq 2. \tag{3.8}$$

From (3.8) we easily obtain the result, again by means of the convexity of c_k.

To complete the proof it remains to show (3.8). Since $c_{k+1}'(0^+) < c_k'(0^+)$ for any $k \geq 2$, it is then sufficient to prove the claim for $k = 2$.

Note that

$$c_2'(0^+) = -\frac{C_s}{2} F(s),$$

where

$$F(s) = \int_0^{+\infty} \ln|1 - \tau^2| \tau^{-(1+s)} d\tau. \tag{3.9}$$

The function $F : [1/2, 1] \to \mathbb{R}$ is well defined, and we shall prove that $F(s) > 0$ for any $s \in [1/2, 1)$.

A straightforward computation leads to

$$F(1) = 0. \tag{3.10}$$

Moreover for any $s \in \left[\frac{1}{2}, 1\right]$ and for a.e. $\tau \in (0, +\infty)$

$$\begin{aligned} \left| \frac{\ln|1 - \tau^2|}{\tau^{1+s}} \right| &\leq \left| \ln|1 - \tau^2| \right| \max\left\{ \frac{1}{\tau^2}, \frac{1}{\tau^{3/2}} \right\} \in L^1\left((0, +\infty)\right) \\ \left| \frac{\partial}{\partial s} \frac{\ln|1 - \tau^2|}{\tau^{1+s}} \right| &= \left| \frac{\ln|1 - \tau^2| \ln \tau}{\tau^{1+s}} \right| \\ &\leq \left| \ln|1 - \tau^2| \ln \tau \right| \max\left\{ \frac{1}{\tau^2}, \frac{1}{\tau^{3/2}} \right\} \in L^1\left((0, +\infty)\right) \end{aligned} \tag{3.11}$$

By (3.11), $F \in C^1\left(\left[\frac{1}{2},1\right]\right)$ and via integrations by parts we obtain

$$\begin{aligned} F'(s) &= -\int_0^{+\infty} \frac{\ln|1-\tau^2|\ln\tau}{\tau^{1+s}}d\tau \\ &= \frac{2}{s}\int_0^{+\infty} \frac{\tau^{1-s}\ln\tau}{1-\tau^2}d\tau - \frac{1}{s}F(s) \\ &\le -\frac{2}{s}I - \frac{1}{s}F(s), \end{aligned} \tag{3.12}$$

where

$$\int_0^{+\infty} \frac{\tau^{1-s}\ln\tau}{1-\tau^2}d\tau \le -I := \int_0^1 \frac{\sqrt{\tau}\ln\tau}{1-\tau^2}d\tau + \int_1^{+\infty} \frac{\ln\tau}{1-\tau^2}d\tau < 0.$$

From (3.12) we have

$$(sF(s))' \le -2I \quad \text{for } s \in \left[\frac{1}{2},1\right].$$

Integrating the above inequality between s and 1, and recalling (3.10), we obtain

$$F(s) \ge 2\,I\,\frac{1-s}{s}$$

which in particular implies that $F(s) > 0$ for any $s \in [1/2, 1)$. □

Remark 3.8. If $k = 1$ and $s \in \left[\frac{1}{2},1\right)$ the function $c_1(\gamma)$ is still strictly convex, as in the above proposition, but now it is positive in $(0,1)$. To show this note that $c_1(0) = 0$ and that

$$c_1'(0^+) = -C_s H(s), \tag{3.13}$$

where $H(s) = \displaystyle\int_0^{+\infty} [\ln|1-\tau^2|]\tau^{-(1+2s)}\,d\tau$. Integrating by parts yields

$$\begin{aligned} H'(s) &= -\frac{1}{s}H(s) + \frac{2}{s}\int_0^{+\infty} \frac{\tau^{1-2s}\ln\tau}{1-\tau^2}\,d\tau \\ &\le -\frac{1}{s}H(s) - \frac{2}{s}I \end{aligned}$$

where $I := -\displaystyle\int_0^1 \frac{\ln\tau}{1-\tau^2}\,d\tau > 0$.

Hence $(sH(s))' \le -2I$ for any $s \in \left[\frac{1}{2},1\right)$. Integrating between $\frac{1}{2}$ and s and using the fact that $H(\frac{1}{2}) = F(1) = 0$ (see (3.10)),

$$H(s) \le -2I\left(1 - \frac{1}{2s}\right) \le 0 \qquad \forall s \in \left[\frac{1}{2},1\right). \tag{3.14}$$

By (3.13)–(3.14) we conclude, in view of the strict convexity of $c_1(\gamma)$, that $c_1(\gamma) > 0$ for any $\gamma \in (0,1)$ and $s \in [1/2, 1)$.

Remark 3.9. In order to give an estimate of $\bar{\gamma}$, we mention that a tedious, but straightforward computation shows that if we compute $\hat{c}$ and $c^{\perp}$ at $\gamma = 2(1-s)$ for $s \in (\frac{1}{2}, 1)$, we get

$$\hat{c}(2(1-s)) = C_s \frac{1}{s(2s-1)}, \quad c^{\perp}(2(1-s)) = -C_s \frac{1}{s}.$$

Thus, when $k = 2$ we have $c_2\,(2(1-s)) = C_s \frac{2(1-s)}{s(2s-1)} > 0$, and therefore $\bar{\gamma} < 2(1-s)$.

4. Liouville-type results for $\mathcal{I}_k^{\pm}$

In this section we will prove a certain number of theorems of Liouville type i.e. of classifications of entire solutions or supersolutions that are bounded from below.

4.1. Liouville results for superharmonic functions

We state the results for $\mathcal{I}_k^+$. A dual result concerning $\mathcal{I}_k^-$ can be also given, but we omit the details. The computations in Proposition 3.7 play a crucial role.

Theorem 4.1. *Consider the equation*

$$\mathcal{I}_k^+ u = 0 \quad \textit{in } \mathbb{R}^N. \tag{4.1}$$

(i) *If $s \in [1/2, 1)$ and $k = 1$, every viscosity supersolution u to problem (4.1) which is bounded from below, is a constant.*
(ii) *If $s \in (0, 1/2)$ and $k = 1$, or $s \in (0,1)$ and $2 \le k \le N$, then there exist bounded from below nontrivial viscosity supersolutions of Eq. (4.1).*

Proof. (i). By adding a constant, we can assume that $u \ge 0$. Consider first the case $s > \frac{1}{2}$ and fix $\gamma \in (0, 2s-1]$. Let $w_\gamma(x) = w_\gamma(|x|) = -|x|^\gamma$. By Lemma 3.6, $\mathcal{I}_{\hat{x}} w_\gamma(x) \ge 0$ for any $x \ne 0$. In particular, we have that

$$\mathcal{I}_1^+ w_\gamma(x) \ge 0 \quad \text{for } |x| \ge 1.$$

Thus, for every $R > 1$ and denoting $m(1) = \min_{x \in \overline{B}_1} u(x)$, the function ϕ defined as

$$\phi(x) = m(1)\, \frac{w_\gamma(|x|) - w_\gamma(R)}{-w_\gamma(R)},$$

is a classical subsolution to (4.1) for $1 < |x| < R$ and moreover $\phi \le u$ for $|x| \le 1$ and for $|x| \ge R$. Then, by the comparison principle, see Theorem 2.5,we have $\phi \le u$ in $\mathbb{R}^N$. Thus, for each $|x| > 1$ fixed, we let $R \to +\infty$ and then

$$u(x) \ge m(1),$$

from which we infer that $u \geq m(1)$ in $\mathbb{R}^N$. By the strong minimum principle, see Proposition 2.7 and Remark 2.9, we conclude that u is constant. Now we consider the case $s = \frac{1}{2}$. Fix $\gamma \in (0,1)$ and let

$$w_\gamma(|x|) = \begin{cases} \varepsilon^{-\gamma} - \ln \varepsilon & \text{if } |x| \leq \varepsilon \\ |x|^{-\gamma} - \ln |x| & \text{if } |x| > \varepsilon. \end{cases}$$

We claim that, for ε sufficiently small, one has

$$\mathcal{I}_1^+ w_\gamma(x) \geq 0 \quad \text{for } |x| \geq 1. \tag{4.2}$$

Once this is proved we conclude as above by using the comparison function

$$\phi(x) = m(1)\, \frac{w_\gamma(|x|) - w_\gamma(R)}{w_\gamma(\varepsilon) - w_\gamma(R)}$$

and letting $R \to +\infty$.

It is convenient to write, for $x \neq 0$, the function w_γ as

$$w_\gamma(|x|) = |x|^{-\gamma} - \ln |x| + v(x),$$

where

$$v(x) = \begin{cases} \varepsilon^{-\gamma} - \ln \varepsilon - (|x|^{-\gamma} - \ln |x|) & \text{if } |x| \leq \varepsilon \\ 0 & \text{if } |x| > \varepsilon. \end{cases}$$

In this way, for $x \neq 0$,

$$\mathcal{I}_1^+ w_\gamma(x) \geq \mathcal{I}_{\hat{x}}(|x|^{-\gamma}) - \mathcal{I}_{\hat{x}}(\ln |x|) + \mathcal{I}_{\hat{x}} v(x). \tag{4.3}$$

Using Lemma 3.6 and (3.9)–(3.10) it turns out that

$$\mathcal{I}_{\hat{x}}(|x|^{-\gamma}) - \mathcal{I}_{\hat{x}}(\ln |x|) = \mathcal{I}_{\hat{x}}(|x|^{-\gamma}) = \hat{c}(\gamma)|x|^{-\gamma-1}, \tag{4.4}$$

where $\hat{c}(\gamma) > 0$. Moreover it is not difficult to see that for $|x| \geq 1$

$$\mathcal{I}_{\hat{x}} v(x) \geq -|x|^{-\gamma-1} g(x,\varepsilon), \tag{4.5}$$

where $g(x,\varepsilon)$ is a positive function such that

$$\lim_{\varepsilon \to 0^+} g(x,\varepsilon) = 0 \quad \text{uniformly for } |x| \geq 1. \tag{4.6}$$

Then (4.2) follows by (4.3)–(4.6) and the positivity of the quantity $\hat{c}(\gamma)$.

(ii). Let $\bar{\gamma}$ be the constant given in Proposition 3.7 and consider for $0 < \gamma < \bar{\gamma}$ the function $u(x) = \min\{1, |x|^{-\gamma}\}$. By the basic principle saying that minima of supersolutions are supersolutions, then u is a nontrivial viscosity supersolution to (4.1) which is bounded from below.

□

Recall that $\bar{\gamma} \to 0$ and that $\mathcal{I}_2^+ \to \mathcal{P}_2^+$ as $s \to 1^-$. So even if Liouville type Theorems are valid for $\mathcal{P}_2^+$ see Theorem 2.2 in [7], the result of Theorem 4.1 are not in contradiction, since in a certain sense the solution construct here converges to the trivial solution.

4.2. Liouville-type result for the maximal operator $\mathcal{I}_k^+$

In this subsection we assume $k = 1$ and $s \in (0, 1/2)$, or $k \geq 2$ and $s \in (0, 1)$. The aim is to prove the following

Theorem 4.2. *Let $\bar{\gamma} \in (0, 1)$ be as in Proposition 3.7. The equation*

$$\mathcal{I}_k^+ u(x) + u^p(x) = 0, \quad u \geq 0 \text{ in } \mathbb{R}^N \tag{4.7}$$

has nontrivial viscosity supersolutions if, and only if

$$p > 1 + \frac{2s}{\bar{\gamma}}.$$

We divide the proof in several partial results. We start with the sufficient condition in the previous theorem.

Proposition 4.3. *For any $p > 1 + \frac{2s}{\bar{\gamma}}$ there exist positive viscosity supersolutions of (4.7).*

Proof. For any $q \in \left[\frac{1}{p-1}, \frac{\bar{\gamma}}{2s}\right)$, let

$$u(x) = \frac{1}{(1 + |x|)^{2sq}}.$$

As a consequence of Theorem 3.4, we have for any $x \in \mathbb{R}^N \setminus \{0\}$ (note that there are no test functions touching u from below at $x = 0$)

$$\begin{aligned}
\mathcal{I}_k^+ u(x) &= C_s \, \text{P.V.} \int_{-\infty}^{+\infty} \left[(1 + ||x| + \tau|)^{-2sq} - (1 + |x|)^{-2sq} \right] |\tau|^{-(1+2s)} \, d\tau \\
&\quad + (k-1) C_s \int_{-\infty}^{+\infty} \left[\left(1 + \sqrt{|x|^2 + \tau^2}\right)^{-2sq} - (1 + |x|)^{-2sq} \right] |\tau|^{-(1+2s)} \, d\tau \\
&= \frac{1}{(1 + |x|)^{2sq}} \left(C_s \, \text{P.V.} \int_{-\infty}^{+\infty} \left[\left(\frac{1}{1 + |x|} + \left| \frac{|x|}{1 + |x|} + \frac{\tau}{1 + |x|} \right| \right)^{-2sq} \right.\right. \\
&\quad \left. - 1 \right] |\tau|^{-(1+2s)} \, d\tau \\
&\quad + (k-1) C_s \int_{-\infty}^{+\infty} \left[\left(\frac{1}{1 + |x|} + \sqrt{\frac{|x|^2}{(1 + |x|)^2} + \frac{\tau^2}{(1 + |x|)^2}} \right)^{-2sq} \right. \\
&\quad \left.\left. - 1 \right] |\tau|^{-(1+2s)} \, d\tau \right).
\end{aligned}$$

By the triangular inequality, for any $x \in \mathbb{R}^N$ and $\tau \in \mathbb{R}$,

$$\frac{1}{1 + |x|} + \left| \frac{|x|}{1 + |x|} + \frac{\tau}{1 + |x|} \right| \geq \left| 1 + \frac{\tau}{1 + |x|} \right|$$

and a straightforward computation yields

$$\left(\frac{1}{1 + |x|} + \sqrt{\frac{|x|^2}{(1 + |x|)^2} + \frac{\tau^2}{(1 + |x|)^2}} \right)^2 \geq 1 + \frac{\tau^2}{(1 + |x|)^2}.$$

Hence, using the change of variable $t = \frac{\tau}{1+|x|}$, we obtain

$$\mathcal{I}_k^+ u(x) \le \frac{1}{(1+|x|)^{2s(q+1)}} \left(C_s \, \text{P.V.} \int_{-\infty}^{+\infty} \left[|1+t|^{-2sq} - 1 \right] |t|^{-(1+2s)} \, dt \right.$$
$$\left. + (k-1) C_s \int_{-\infty}^{+\infty} \left[\left(1+t^2\right)^{-sq} - 1 \right] |t|^{-(1+2s)} \, dt \right)$$
$$= \frac{1}{(1+|x|)^{2s(q+1)}} \, c_k(2sq),$$

where $c_k(\cdot)$ is the function defined in Proposition 3.7. Since $2sq < \bar{\gamma}$, then $c_k(2sq) < 0$. For $\varepsilon \in (0, (-c_k(2sq))^{1/(p-1)})$ and $v(x) = \varepsilon u(x)$ we conclude

$$\mathcal{I}_k^+ v(x) + v^p(x) \le \frac{\varepsilon}{(1+|x|)^{2s(q+1)}} \left(c_k(2sq) + \frac{\varepsilon^{p-1}}{(1+|x|)^{2s(qp-q-1)}} \right)$$
$$\le \frac{\varepsilon}{(1+|x|)^{2s(q+1)}} \left(c_k(2sq) + \varepsilon^{p-1} \right) \le 0 \, .$$

□

For the necessary condition, we shall follow the ideas in [16] and we require some preliminary lemmas.

Lemma 4.4. *Let $\bar{\gamma}$ be as in Proposition 3.7. Given $r > 0$ and $u \in LSC(\mathbb{R}^N)$, we denote $m(r) = \min\limits_{\overline{B}_r} u$.*

(i) If u is a nonnegative viscosity supersolution of (4.1), for any $\gamma > \bar{\gamma}$ there exists a positive constant $c = c(\gamma)$ such that

$$m(r) \ge c \, m(1) r^{-\gamma} \quad \forall r \ge 1. \tag{4.8}$$

(ii) If u is a positive supersolution of (4.7) for some $p < \frac{1+2s}{\bar{\gamma}}$, then there exists a positive constant $\bar{c} = \bar{c}(\bar{\gamma}, p, s, m(1))$ such that

$$m(r) \ge \bar{c} \, r^{-\bar{\gamma}} \quad \forall r \ge 1. \tag{4.9}$$

Proof. (i) The statement (4.8) is trivial if $u \equiv 0$. By the strong minimum principle, see Proposition 2.7, we can then assume $u > 0$ in $\mathbb{R}^N$.

We claim that for ε small enough (depending on γ) the function

$$w(|x|) = \begin{cases} \varepsilon^{-\gamma} & \text{if } |x| \le \varepsilon \\ |x|^{-\gamma} & \text{if } |x| > \varepsilon \end{cases}$$

is a classical subsolution of $\mathcal{I}_k^+ u(x) = 0$ for $|x| \ge 1$. Then (4.8) follows from the claim, since the function

$$\phi(x) = m(1) \frac{w(|x|) - w(R)}{w(\varepsilon) - w(R)}$$

is, for any $R > 1$, classical subsolution of $\mathcal{I}_k^+ u(x) = 0$ for $1 < |x| < R$. Moreover $u(x) \ge m(1) \ge \phi(x)$ for $|x| \le 1$ and $u(x) \ge 0 \ge \phi(x)$ if $|x| \ge R$. The comparison principle, see Theorem 2.5, yields $u(x) \geq \phi(x)$ for $1 < |x| < R$ and letting $R \to +\infty$ we infer that

$$m(r) \geq m(1)\varepsilon^{\gamma} r^{-\gamma},$$

leading to (4.8) with $c = \varepsilon^{\gamma}$.

We proceed with the proof of the claim. For $|x| \geq 1$, we use that

$$\mathcal{I}_k^+ w(x) \geq \mathcal{I}_{\hat{x}} w(x) + (k-1)\mathcal{I}_{x^\perp} w_\gamma(x)$$

where $w_\gamma(x) = |x|^{-\gamma}$. Now we concentrate on $\mathcal{I}_{\hat{x}} w(x)$. For $|x| \geq 1$ we see that

$$\mathcal{I}_{\hat{x}} w(x) = \mathcal{I}_{\hat{x}} w_\gamma(x) - \int_{-|x|-\varepsilon}^{-|x|+\varepsilon} \frac{||x|+\tau|^{-\gamma} - \varepsilon^{-\gamma}}{|\tau|^{1+2s}}\, d\tau,$$

from which, by Proposition 3.7 we conclude that

$$\begin{aligned} \mathcal{I}_k^+ w(x) \geq\ & c_k(\gamma)|x|^{-(\gamma+2s)} - \int_{-|x|-\varepsilon}^{-|x|+\varepsilon} \frac{||x|+\tau|^{-\gamma} - \varepsilon^{-\gamma}}{|\tau|^{1+2s}}\, d\tau \\ =\ & |x|^{-(\gamma+2s)} \left(c_k(\gamma) - \int_{-1-\frac{\varepsilon}{|x|}}^{-1+\frac{\varepsilon}{|x|}} \frac{|1+\tau|^{-\gamma} - \left(\frac{\varepsilon}{|x|}\right)^{-\gamma}}{|\tau|^{1+2s}}\, d\tau \right). \end{aligned}$$

Let us denote I the integral term in the right-hand side of the last inequality. Using that $|x| \geq 1$ and $\varepsilon < 1/2$ we have

$$I \leq 2^{1+2s} \int_{-1-\varepsilon}^{-1+\varepsilon} |1+\tau|^{-\gamma}\, d\tau = \frac{2^{2(1+s)}}{1-\gamma}\varepsilon^{1-\gamma} \leq \frac{16}{1-\gamma}\varepsilon^{1-\gamma}.$$

Using this, we conclude that

$$\mathcal{I}_k^+ w(x) \geq\ |x|^{-(\gamma+2s)}(c_k(\gamma) - C\varepsilon^{1-\gamma}), \tag{4.10}$$

with $C = 16(1-\gamma)^{-1}$. Since $\gamma > \bar{\gamma}$ we have $c_k(\gamma) > 0$ and therefore it is sufficient to take $\varepsilon \leq \min\left\{\frac{1}{2}, \left(\frac{c_k(\gamma)}{C}\right)^{\frac{1}{1-\gamma}}\right\}$ to conclude the proof of the claim.

(*ii*) Let us consider

$$w(|x|) = \begin{cases} \varepsilon^{-\bar{\gamma}} & \text{if } |x| \leq \varepsilon \\ |x|^{-\bar{\gamma}} & \text{if } |x| > \varepsilon. \end{cases}$$

Similarly to (4.10), using the fact that $c_k(\bar{\gamma}) = 0$, we have for $|x| \geq 1$

$$\mathcal{I}_k^+ w(x) \geq -\int_{-|x|-\varepsilon}^{-|x|+\varepsilon} \frac{||x|+\tau|^{-\bar{\gamma}} - \varepsilon^{-\bar{\gamma}}}{|\tau|^{1+2s}}\, d\tau.$$

Assuming $\varepsilon \leq \frac{1}{2}$, we infer that

$$\mathcal{I}_k^+ w(x) \geq -\frac{2^{1+2s}}{|x|^{1+2s}} \int_{-|x|-\varepsilon}^{-|x|+\varepsilon} ||x|+\tau|^{-\bar{\gamma}} - \varepsilon^{-\bar{\gamma}}\, d\tau = -4^{1+s}\frac{\bar{\gamma}}{1-\bar{\gamma}}\varepsilon^{1-\bar{\gamma}}\frac{1}{|x|^{1+2s}}. \tag{4.11}$$

For any $R \geq 2^{\frac{1-\bar{\gamma}}{\bar{\gamma}}}$, the function

$$\phi(x) = m(1)\frac{w(|x|) - w(R)}{w(\varepsilon) - w(R)}$$

satisfies, for $|x| \geq 1$, the inequality

$$\mathcal{I}_k^+ \phi(x) \geq -\tilde{c}\,\varepsilon^{1-\bar{\gamma}} \frac{1}{|x|^{1+2s}}, \tag{4.12}$$

with $\tilde{c} = m(1)2^{3+2s-\bar{\gamma}} \frac{\bar{\gamma}}{1-\bar{\gamma}}$.

Now we apply (4.8) with $\gamma = \frac{1+2s}{p}$. Note that $\gamma > \bar{\gamma}$ by the assumption $p < \frac{1+2s}{\bar{\gamma}}$. From (4.7) we then obtain

$$\mathcal{I}_k^+ u(x) \leq -u^p(x) \leq -(cm(1))^p \frac{1}{|x|^{1+2s}}, \tag{4.13}$$

where c is the constant appearing in (4.8). Now, from (4.12)–(4.13), taking $\varepsilon = \varepsilon(\bar{\gamma}, p, s, m(1))$ small enough, we have

$$\mathcal{I}_k^+ u(x) \leq \mathcal{I}_k^+ \phi(x) \quad \forall |x| \geq 1.$$

Since $u \geq \phi$ for $|x| \leq 1$ and $|x| \geq R$, by comparison principle $u \geq \phi$ for $|x| \in [1, R]$. Sending $R \to +\infty$, we obtain

$$m(r) \geq m(1)\varepsilon^{\bar{\gamma}} r^{-\bar{\gamma}},$$

which is exactly (4.9) with $\bar{c} = m(1)\varepsilon^{\bar{\gamma}}$. □

Lemma 4.5. *Let $\bar{\gamma}$ be as in Proposition 3.7. Let u be a nonnegative viscosity supersolution of (4.1). Then, for any $\gamma \geq \bar{\gamma}$ there exists a positive constant $c = c(\gamma)$ such that*

$$m(R) \geq c\, m\left(\frac{R}{2}\right) \quad \forall R > 0. \tag{4.14}$$

Proof. Let $\gamma \geq \bar{\gamma}$, $R > 0$ and $R_0 = \varepsilon R$ for some $\varepsilon \in (0, 1/4)$ to be fixed.

Consider the function

$$w_R(|x|) = \begin{cases} R_0^{-\gamma} & \text{if } |x| \leq R_0 \\ |x|^{-\gamma} & \text{if } R_0 < |x|. \end{cases}$$

We claim that the function $\phi(x) = (w_R(|x|) - (2R)^{-\gamma})_+$ is a classical solution of

$$\mathcal{I}_k^+ \phi \geq 0 \quad \text{in } B_{2R} \setminus \overline{B}_{R/2}.$$

Assuming the claim is true, the function

$$\tilde{\phi}(x) = m\left(\frac{R}{2}\right) \frac{\phi(x)}{R_0^{-\gamma} - (2R)^{-\gamma}},$$

solves $\mathcal{I}_k^+ \tilde{\phi}(x) \geq 0$ for $|x| \in \left(\frac{R}{2}, 2R\right)$. Since $u(x) \geq m\left(\frac{R}{2}\right) \geq \tilde{\phi}(x)$ for $|x| \leq \frac{R}{2}$ and $u(x) \geq 0 = \tilde{\phi}(x)$ if $|x| \geq 2R$, by comparison principle we get $u(x) \geq \tilde{\phi}(x)$

for $|x| \in \left(\frac{R}{2}, 2R\right)$. In particular, we have $m(R) \geq \min_{\overline{B}_R} \tilde{\phi}$ from which we obtain

$$m(R) \geq m\left(\frac{R}{2}\right) \frac{R^{-\gamma} - (2R)^{-\gamma}}{R_0^{-\gamma} - (2R)^{-\gamma}} = c\, m\left(\frac{R}{2}\right),$$

where $c = c(\varepsilon, \gamma) := \frac{1 - 2^{-\gamma}}{\varepsilon^{-\gamma} - 2^{-\gamma}}$. Then (4.14) holds with this constant c.

Now we prove the claim. By definition, for each $|x| \in \left(\frac{R}{2}, 2R\right)$ we have

$$\mathcal{I}_k^+ \phi(x) \geq \mathcal{I}_{\hat{x}} \phi(x) + (k-1) \mathcal{I}_{x^\perp} \phi(x). \tag{4.15}$$

As in Proposition 3.7 we denote $w_\gamma(x) = |x|^{-\gamma}$. Denoting $A = \{\tau \in \mathbb{R} : |x|^2 + \tau^2 \leq (2R)^2\}$ we have $\phi(x + \tau x^\perp) = w_\gamma(x + \tau x^\perp) - (2R)^{-\gamma}$ for $\tau \in A$, while for $\tau \in A^c$ it holds that $\phi(x + \tau x^\perp) = 0$ and $w_\gamma(x + \tau x^\perp) \leq (2R)^{-\gamma}$.

Then we have

$$\begin{aligned} \mathcal{I}_{x^\perp} \phi(x) = \ & C_s \int_A [w_\gamma(x + \tau x^\perp) - w_\gamma(x)] |\tau|^{-(1+2s)} d\tau \\ & + C_s \int_{\mathbb{R} \setminus A} [(2R)^{-\gamma} - w_\gamma(x)] |\tau|^{-(1+2s)} d\tau \\ \geq \ & \mathcal{I}_{x^\perp} w_\gamma(x) \end{aligned} \tag{4.16}$$

We employ a similar argument for $\mathcal{I}_{\hat{x}} \phi(x)$. This time we denote the (disjoint) sets

$$A = [-R_0 - |x|, R_0 - |x|], \quad B = \{\tau \in \mathbb{R} : ||x| + \tau| \geq 2R\}.$$

Thus, by definition we have

$$\begin{aligned} \mathcal{I}_{\hat{x}} \phi(x) = \mathcal{I}_{\hat{x}} w_\gamma(x) + C_s & \int_A [R_0^{-\gamma} - w_\gamma(x + \tau \hat{x})] |\tau|^{-(1+2s)} d\tau \\ + C_s & \int_B [(2R)^{-\gamma} - w_\gamma(x + \tau \hat{x})] |\tau|^{-(1+2s)} d\tau \\ =: \ & \mathcal{I}_{\hat{x}} w_\gamma(x) + I_1 + I_2. \end{aligned}$$

For I_1, notice that $|\tau| \geq R/4$ for each $\tau \in A$. Then, we have

$$I_1 \geq -C_s \int_A ||x| + \tau|^{-\gamma} |\tau|^{-(1+2s)} d\tau \geq -C_s \left(\frac{4}{R}\right)^{1+2s} \int_{-R_0}^{R_0} |\tau|^{-\gamma}\, d\tau$$

from which, by the choice of R_0 we conclude

$$I_1 \geq -c_1 R^{-\gamma - 2s}, \tag{4.17}$$

where $c_1 = C_s 2^{3+4s} \frac{\varepsilon^{1-\gamma}}{1-\gamma}$. Observe that this constant tends to zero as $\varepsilon \to 0$.

As far as I_2 is concerned, notice that the integrand is nonnegative. Thus, if we denote $B' = \{\tau \in \mathbb{R} : ||x| + \tau| \geq 3R\} \subset B$ we have

$$\begin{aligned} I_2 \geq C_s (2^{-\gamma} - 3^{-\gamma}) R^{-\gamma} & \int_{B'} |\tau|^{-(1+2s)} d\tau \geq C_s (2^{-\gamma} - 3^{-\gamma}) R^{-\gamma} \\ & \int_{5R}^{+\infty} \tau^{-(1+2s)} d\tau, \end{aligned}$$

from which we get

$$I_2 \geq c_2 R^{-(\gamma+2s)}, \tag{4.18}$$

with $c_2 = C_s(2^{-\gamma} - 3^{-\gamma})\frac{5^{-2s}}{2s}$. Observe that this constant is independent of ε.

Putting together (4.17)–(4.18) into the expression of $\mathcal{I}_{\hat{x}}\phi(x)$ above, we conclude that for ε small enough we get

$$\mathcal{I}_{\hat{x}}\phi(x) \geq \mathcal{I}_{\hat{x}} w_\gamma(x),$$

and from here, replacing this and (4.16) into (4.15), we conclude the claim using Proposition 3.7. The proof is now complete. □

Lemma 4.6. *There exists a positive constant* $c = c(k, s)$ *such that the function*

$$\Gamma(x) = \frac{\ln|x|}{|x|^{\bar{\gamma}}}, \quad x \neq 0$$

satisfies

$$\mathcal{I}_k^+\Gamma(x) \geq -\frac{c}{|x|^{\bar{\gamma}+2s}}, \quad x \neq 0. \tag{4.19}$$

Proof. Let $w_{\bar{\gamma}}(x) = |x|^{-\bar{\gamma}}$. For $x \neq 0$ we have

$$\begin{aligned}\mathcal{I}_k^+\Gamma(x) &\geq \mathcal{I}_{\hat{x}}\Gamma(x) + (k-1)\mathcal{I}_{x^\perp}\Gamma(x)\\ &= \ln|x|\,\mathcal{I}_k^+ w_{\bar{\gamma}}(x) + \frac{1}{|x|^{\bar{\gamma}+2s}}\left[C_s \text{P.V.}\int_{-\infty}^{+\infty} \frac{\ln|1+\tau|}{|1+\tau|^{\bar{\gamma}}\,|\tau|^{1+2s}}\,d\tau\right.\\ &\qquad \left. + C_s(k-1)\int_0^{+\infty} \frac{\ln(1+\tau^2)}{(1+\tau^2)^{\bar{\gamma}/2}\,\tau^{1+2s}}\,d\tau\right].\end{aligned}$$

Since, by Proposition 3.7, $\bar{w}(x)$ solves $\mathcal{I}_k^+ w_{\bar{\gamma}}(x) = 0$ for $x \neq 0$, then (4.19) follows. □

Now we are in position to provide the

Proof of Theorem 4.2. The existence of nontrivial supersolutions of (4.7) when $p > 1 + \frac{2s}{\bar{\gamma}}$ is a consequence of Proposition 4.3.

Let $p \leq 1+\frac{2s}{\bar{\gamma}}$. We shall prove that $u \equiv 0$ is the only nonnegative supersolution of (4.7). By contradiction we suppose the contrary. Let u be a nontrivial supersolution of (4.7). By the strong minimum principle, see Remark 2.9, $u > 0$ in $\mathbb{R}^N$.

Let $\eta(|x|)$ be a cut-off function such that $\eta(|x|) = 0$ for $|x| \geq 1$ and $\eta(|x|) = 1$ for $|x| \leq \frac{1}{2}$. Define $\xi(x) = m\left(\frac{R}{2}\right)\eta\left(\frac{|x|}{R}\right)$. Since $\mathcal{I}_k^+\eta(x) \geq -C_\eta$, for some positive constant C_η, by scaling it turns out that

$$\mathcal{I}_k^+\xi(x) \geq -\frac{C_\eta m\left(\frac{R}{2}\right)}{R^{2s}}. \tag{4.20}$$

Moreover $u(x) \geq \xi(x)$ for $|x| \in [0, \frac{R}{2}] \cup [R, +\infty]$ and $u(x) = \xi(x)$ for some $|x| = \frac{R}{2}$. Then there exists $x_R \in \mathbb{R}^N$ such that $|x_R| \in [\frac{R}{2}, R)$ and $u(x) - \xi(x) \geq$

$u(x_R)-\xi(x_R)$ for any $x\in\mathbb{R}^N$. Then $\mathcal{I}_k^+\xi(x_R)+u^p(x_R)\le 0$ and by (4.20) we infer that

$$m^p(R)\le u^p(x_R)\le -\mathcal{I}_k^+\xi(x_R)\le \frac{C_\eta m\left(\frac{R}{2}\right)}{R^{2s}}.$$

Then, using (4.14), we have

$$m^{p-1}(R)\le \frac{C}{R^{2s}} \tag{4.21}$$

for a positive constant C.

If $p\le 1$ then

$$m^{p-1}(1)\le \frac{C}{R^{2s}} \tag{4.22}$$

for any $R\ge 1$. Letting $R\to+\infty$ in (4.22), we obtain a contradiction. In what follows, the case $p>1$ is considered.

1. Case $p<1+\frac{2s}{\bar\gamma}$. Let $\gamma>\bar\gamma$ be such that

$$\frac{2s}{p-1}-\gamma>0. \tag{4.23}$$

From (4.8) and (4.21) we have

$$m(1)\le \frac{C}{R^{\frac{2s}{p-1}-\gamma}},$$

for a positive constant C. Sending $R\to+\infty$, and using (4.23), we again reach the contradiction that $m(1)=0$.

2. Case $p=1+\frac{2s}{\bar\gamma}$. By contradiction let u be a positive supersolution of (4.7). From (4.21) we have the bound

$$m(R)R^{\bar\gamma}\le C, \tag{4.24}$$

for some $C>0$. For $x\ne 0$, let $\Gamma(|x|)=\frac{\ln|x|}{|x|^{\bar\gamma}}$. We have $\Gamma\left(e^{1/\bar\gamma}\right)=\max_{|x|>0}\Gamma(|x|)$ and, by Lemma 4.6,

$$\mathcal{I}_k^+\Gamma\ge -\frac{c}{|x|^{\bar\gamma+2s}}\qquad \text{for } x\ne 0. \tag{4.25}$$

Consider now, for $r_2>r_1>e^{1/\bar\gamma}$, the comparison function

$$\phi(x)=m(r_1)\frac{\Gamma(|x|)-\Gamma(r_2)}{\Gamma(e^{1/\bar\gamma})-\Gamma(r_2)},$$

which, by construction, satisfies $\phi(x)\le u(x)$ for $|x|\le r_1$ and $|x|\ge r_2$. Moreover, by (4.25),

$$\mathcal{I}_k^+\phi(x)\ge -\frac{cm(r_1)}{\Gamma(e^{1/\bar\gamma})-\Gamma(r_2)}\frac{1}{|x|^{\bar\gamma+2s}}\qquad \text{for } x\ne 0.$$

For r_2 sufficiently large we may further assume that $\Gamma(e^{1/\bar\gamma})-\Gamma(r_2)\ge\frac{1}{2}\Gamma(e^{1/\bar\gamma})$, so that

$$\mathcal{I}_k^+\phi(x)\ge -\frac{2cm(r_1)}{\Gamma(e^{1/\bar\gamma})}\frac{1}{|x|^{\bar\gamma+2s}}. \tag{4.26}$$

By Lemmas 4.4 and (4.9), we also have

$$\mathcal{I}_k^+ u(x) \leq -(u(x))^{1+\frac{2s}{\bar\gamma}} \leq -(m(|x|))^{1+\frac{2s}{\bar\gamma}} \leq -(\bar c)^{1+\frac{2s}{\bar\gamma}} \frac{1}{|x|^{\bar\gamma+2s}}. \qquad (4.27)$$

Since $m(r_1) \to 0$ as $r_1 \to +\infty$, in view of (4.24), we can fix r_1 large enough and use (4.26)–(4.27) to obtain that $\mathcal{I}_k^+ u(x) \leq \mathcal{I}_k^+ \phi(x)$ for any $|x| \in (r_1, r_2)$. Hence, by comparison, $u(x) \geq \phi(x)$ and passing to the limit as $r_2 \to +\infty$ we deduce that

$$m(r) r^{\bar\gamma} \geq \frac{m(r_1)}{\Gamma(e^{1/\bar\gamma})} \ln r \quad \forall r > r_1,$$

which is in contradiction to (4.24). □

4.3. Liouville-type result for the minimal operator $\mathcal{I}_k^-$ with $k < N$

When $k < N$, we infer from Theorem 3.4-(ii) (see also Remark 3.5) that for any smooth bounded radial function $u(x) = \tilde g(|x|^2)$ such that $\tilde g(r)$ is convex for $r \geq 0$, one has

$$\mathcal{I}_k^- u(x) = k\, \mathcal{I}_{x^\perp} u(x), \qquad (4.28)$$

$x^\perp$ being any unit vector orthogonal to x.

This is the key fact to conclude the following theorem

Theorem 4.7. *Let $s \in (0,1)$, $1 \leq k < N$ and consider the equation*

$$\mathcal{I}_k^- u(x) + u^p(x) = 0 \quad \text{in } \mathbb{R}^N. \qquad (4.29)$$

Then

(i) *for any $p \geq 1$ there exist positive classical solutions of (4.29);*

(ii) *for any $p \in [1-s, 1)$ there exist nonnegative viscosity solutions $u \not\equiv 0$ of (4.29);*

(iii) *for any $p \in (0, 1-s)$ there exist nonnegative viscosity supersolutions $u \not\equiv 0$ of (4.29).*

Proof. (i). We first consider the case $p > 1$. For $r \geq 0$, let

$$\tilde g(r) = \frac{\alpha}{(1+r)^{\frac{s}{p-1}}}.$$

We claim that for a suitable choice of $\alpha = \alpha(k,s,p) > 0$ the function $u(x) = \tilde g(|x|^2)$ is solution of (4.29). Since

$$\begin{aligned}
\mathcal{I}_{x^\perp} u(x) &= 2C_s\alpha \int_0^{+\infty} \left[(1+|x|^2+\tau^2)^{-\frac{s}{p-1}} - (1+|x|^2)^{-\frac{s}{p-1}}\right] \tau^{-(1+2s)}\, d\tau \\
&= 2C_s\alpha(1+|x|^2)^{-\frac{s}{p-1}} \\
&\quad \times \int_0^{+\infty} \left[\left(1+\Big(\frac{\tau}{\sqrt{1+|x|^2}}\Big)^2\right)^{-\frac{s}{p-1}} - 1\right] \tau^{-(1+2s)}\, d\tau \\
&= 2C_s\alpha(1+|x|^2)^{-\frac{sp}{p-1}} \int_0^{+\infty} \left[(1+\tau^2)^{-\frac{s}{p-1}} - 1\right] \tau^{-(1+2s)}\, d\tau\,,
\end{aligned}$$

we obtain from (4.28) that

$$\mathcal{I}_k^- u(x) = k\mathcal{I}_{x^\perp} u(x) = -\alpha\bar{c}(1+|x|^2)^{-\frac{sp}{p-1}}$$

where $\bar{c} = 2C_s k \int_0^{+\infty} \left[1-(1+\tau^2)^{-\frac{s}{p-1}}\right] \tau^{-(1+2s)}\, d\tau > 0$. Hence, we get that

$$\mathcal{I}_k^- u(x) + u^p(x) = (1+|x|^2)^{-\frac{sp}{p-1}}\Big(-\alpha\bar{c} + \alpha^p\Big),$$

from which, taking $\alpha = \bar{c}^{1/(p-1)}$ we conclude the result. Moreover, by scaling, it turns out that for any $a \neq 0$, the function

$$u(x) = \frac{\alpha}{(a^2+|x|^2)^{\frac{s}{p-1}}}$$

is again solution to (4.28) for the same choice of α as before.

In the case $p=1$ we follow a similar argument with a different radial profile. More specifically, for $\beta > 0$ to be fixed, we consider the function

$$\tilde{g}(r) = e^{-\beta r}.$$

As above, for $u(x) = \tilde{g}(|x|^2)$, we have that $\mathcal{I}_k^- u(x) = k\mathcal{I}_{x^\perp} u(x)$. It is easy to see that

$$\mathcal{I}_{x^\perp} u(x) = -e^{-\beta|x|^2} F(\beta),$$

where

$$F(\beta) = 2C_s \int_0^{+\infty} \left(1-e^{-\beta\tau^2}\right) \tau^{-(1+2s)}\, d\tau > 0.$$

Thus, we see that

$$\mathcal{I}_k^- u(x) + u(x) = -e^{-\beta|x|^2}(kF(\beta)-1).$$

By Fatou's lemma, one has

$$+\infty = \int_0^{+\infty} \tau^{-(1+2s)}\, d\tau \le \liminf_{\beta\to+\infty} \int_0^{+\infty} \left(1-e^{-\beta\tau^2}\right) \tau^{-(1+2s)}\, d\tau,$$

from which we conclude that

$$\lim_{\beta\to+\infty} F(\beta) = +\infty. \tag{4.30}$$

Moreover, for $\beta \in (0,1]$,

$$\left(1-e^{-\beta\tau^2}\right) \tau^{-(1+2s)} \le \min\left\{\frac{1}{\tau^{2s-1}}, \frac{1}{\tau^{2s+1}}\right\} \in L^1((0,+\infty))$$

and by Lebesgue's Theorem we infer that

$$\lim_{\beta\to 0^+} F(\beta) = 0. \tag{4.31}$$

Since $F(\beta)$ is continuous (again by Lebesgue's Theorem) we infer, by (4.30)–(4.31), that there exists $\bar{\beta} > 0$ such that $F(\bar{\beta}) = \frac{1}{k}$. Then $u(x) = e^{-\bar{\beta}|x|^2}$ is solution of (4.29) with $p=1$. We conclude observing that, by homogeneity, for any $b > 0$ the function $u(x) = be^{-\bar{\beta}|x|^2}$ is still a positive entire solution of (4.29).

(ii). We shall prove that the bounded and radial function

$$u(x)=\alpha(R^2-|x|^2)_+^{\frac{s}{1-p}} \tag{4.32}$$

is, for a suitable choice of $\alpha>0$ and for any $R>0$, a nonnegative viscosity solution of the equation

$$\mathcal{I}_k^- u(x)+u^p(x)=0 \quad \text{in } \mathbb{R}^N. \tag{4.33}$$

Note that $u(x)=\tilde g(|x|^2)$ with $\tilde g$ convex by the assumption $p\in[1-s,1)$. Moreover u is a smooth function for $|x|\neq R$. Then using the arguments of Lemma 3.3-(ii) and Theorem 3.4-(ii) we can still conclude that if $|x_0|\neq R$ then

$$\mathcal{I}_k^- u(x_0)=k\mathcal{I}_{x^\perp}u(x_0)\,. \tag{4.34}$$

In particular we have

$$\mathcal{I}_{x^\perp}u(x_0)=0=u^p(x_0) \quad \text{if } |x_0|>R, \tag{4.35}$$

while if $|x_0|<R$

$$\mathcal{I}_{x^\perp}u(x_0)=-\alpha c(R^2-|x_0|^2)^{\frac{ps}{1-p}}\,, \tag{4.36}$$

where $c=2C_s\displaystyle\int_0^{+\infty}\left(1-(1-\tau^2)_+^{\frac{s}{1-p}}\right)\tau^{-(1+2s)}\,d\tau>0$. Choosing $\alpha=(ck)^{-\frac{1}{1-p}}$, we infer from (4.34)–(4.35)–(4.36) that u satisfies, in the classical sense, the equation $\mathcal{I}_k^-u(x)+u^p(x)=0$ for any $|x|\neq R$.
It remains to consider the case $|x_0|=R$. To show that u is a viscosity subsolution we have to prove that if $\varphi\in C^2(B_\delta(x_0))$, $\delta>0$, is such that

$$(u-\varphi)(x)\le(u-\varphi)(x_0)=0 \quad \forall x\in B_\delta(x_0), \tag{4.37}$$

then for any $\{\xi_i\}_{i=1}^k\in\mathcal{V}_k$

$$\sum_{i=1}^k\left(\mathcal{I}_{\xi_i,\varepsilon}\varphi(x_0)+\mathcal{I}_{\xi_i}^{\varepsilon}u(x_0)\right)\ge 0 \qquad \forall\varepsilon\in(0,\delta). \tag{4.38}$$

If $\frac{s}{1-p}<2$, then there are no test functions touching u from above at x_0, so there is nothing to prove. If instead $\frac{s}{1-p}\ge 2$ and $\varphi\in C^2(B_\delta(x_0))$ satisfies (4.37), then both u and φ attains their global minimum at x_0, hence it is readily seen that (4.38) holds.
In order to prove that u is a supersolution of $\mathcal{I}_k^-u(x)+u^p(x)=0$ at $x=x_0$, let $\varphi\in C^2(B_\delta(x_0))$, $\delta>0$, be such that

$$(u-\varphi)(x)\ge(u-\varphi)(x_0)=0 \quad \forall x\in B_\delta(x_0). \tag{4.39}$$

Consider $\{\xi_i\}_{i=1}^k\in\mathcal{V}_k$ such that $\langle\xi_i,x_0\rangle=0$ for any $i=1,\dots,k$. Since

$$u(x_0\pm\tau\xi_i)=u(x_0)=0 \qquad \forall\tau\ge 0$$

and

$$\varphi(x_0\pm\tau\xi_i)\le u(x_0\pm\tau\xi_i)=\varphi(x_0) \quad \forall\tau\in[0,\delta),$$

then for any $i=1,\dots,k$ we have

$$\mathcal{I}_{\xi_i,\varepsilon}\varphi(x_0)\le 0\,, \quad \mathcal{I}_{\xi_i}^{\varepsilon}u(x_0)=0 \qquad \forall\varepsilon\in(0,\delta).$$

Thus we conclude

$$\sum_{i=1}^{k} \left(\mathcal{I}_{\xi_i,\varepsilon}\varphi(x_0) + \mathcal{I}^{\varepsilon}_{\xi_i} u(x_0) \right) \leq 0 \qquad \forall \varepsilon \in (0, \delta)\,.$$

(iii). Consider the function u defined by (4.32), with $\alpha \geq (ck)^{-\frac{1}{1-p}}$, see case (ii) for the definition of c. Differently from (4.34), we do not have a representation formula for $\mathcal{I}^-_k u(x)$, since u does not satisfy the convexity assumption of Theorem 3.4-(ii). Nevertheless using the inequality

$$\mathcal{I}^-_k u(x) \leq k\mathcal{I}_{x^\perp} u(x),$$

which holds for any admissible function u, just using the minimality of the operator $\mathcal{I}^-_k u$ among the family of k-dimensional orthonormal subsets of $\mathbb{R}^N$, we can still conclude that u is a viscosity supersolution of (4.29), using the same computations of case (ii).

□

4.4. Liouville-type theorem for the minimal operator $\mathcal{I}^-_N$

We start with the critical exponent associated to this operator. Let us remember that, by Theorem 3.4-(ii), the minimal operator $\mathcal{I}^-_N u$ coincides, within a suitable class of radial function including as the main example the function $u(x) = |x|^{-\gamma}$, with $N\mathcal{I}_{\xi^*} u$. Then a fundamental solution for the integral operator $\mathcal{I}_{\xi^*}$ is in turn a fundamental solution for $\mathcal{I}^-_N$.

Lemma 4.8. *For $s \in (0, 1)$ and $\gamma > 0$, let*

$$c(\gamma) := \int_0^{+\infty} \frac{\left(1 + \tau^2 + \frac{2}{\sqrt{N}}\tau\right)^{-\gamma/2} + \left(1 + \tau^2 - \frac{2}{\sqrt{N}}\tau\right)^{-\gamma/2} - 2}{\tau^{1+2s}}\, d\tau\,. \tag{4.40}$$

Then, there exists a unique $\tilde{\gamma} = \tilde{\gamma}(N, s) > 0$ such that $c(\gamma) < 0$ for $\gamma < \tilde{\gamma}$, $c(\tilde{\gamma}) = 0$ and $c(\gamma) > 0$ for $\gamma > \tilde{\gamma}$.

Proof. By Lebesgue's theorem we easily infer that $c(\gamma) \to 0$ as $\gamma \to 0^+$ and that

$$c'(0) = -\frac{1}{2} \int_0^{+\infty} \frac{\ln\left(1 + 2\left(1 - \frac{2}{N}\right)\tau^2 + \tau^4\right)}{\tau^{1+2s}} < 0. \tag{4.41}$$

Moreover, for any $\gamma > 0$, we have

$$c''(\gamma) = \int_0^{+\infty} \frac{f(\tau) + f(-\tau)}{\tau^{1+2s}}\, d\tau$$

where $f(\tau) = \left(1 + \tau^2 + \frac{2}{\sqrt{N}}\tau\right)^{-\gamma/2} \ln^2\left(1 + \tau^2 + \frac{2}{\sqrt{N}}\tau\right)$. Since $f(\tau) \geq 0$ for any τ, then $c(\gamma)$ is convex in $[0, +\infty)$. We claim that

$$\lim_{\gamma\to+\infty} c(\gamma) = +\infty. \tag{4.42}$$

Then, using (4.41)–(4.42), we deduce that there exists $\tilde{\gamma} = \tilde{\gamma}(N, s) > 0$ such that $c(\tilde{\gamma}) = 0$, $c(\gamma) < 0$ for $\gamma < \tilde{\gamma}$ and $c(\gamma) > 0$ for $\gamma > \tilde{\gamma}$.

To show (4.42) let

$$g(\tau)=\left(1+\tau^2+\frac{2}{\sqrt{N}}\tau\right)^{-\gamma/2}+\left(1+\tau^2-\frac{2}{\sqrt{N}}\tau\right)^{-\gamma/2}-2,$$

so that

$$\begin{aligned}c(\gamma)&=\int_0^{+\infty}\frac{g(\tau)}{\tau^{1+2s}}\,d\tau\\&=\int_0^{\frac{1}{4\sqrt{N}}}\frac{g(\tau)}{\tau^{1+2s}}\,d\tau+\int_{\frac{1}{4\sqrt{N}}}^{\frac{1}{2\sqrt{N}}}\frac{g(\tau)}{\tau^{1+2s}}\,d\tau+\int_{\frac{1}{2\sqrt{N}}}^{+\infty}\frac{g(\tau)}{\tau^{1+2s}}\,d\tau\\&=:I_1+I_2+I_3.\end{aligned}\tag{4.43}$$

We shall prove that I_1 and I_3 are bounded from below, while $I_2\to+\infty$ as $\gamma\to+\infty$.
Since $g(\tau)\geq-2$ for any $\tau>0$, we have

$$I_3\geq-2\int_{\frac{1}{2\sqrt{N}}}^{+\infty}\frac{1}{\tau^{1+2s}}\,d\tau=-\frac{(2\sqrt{N})^{2s}}{s}.$$

Moreover

$$\begin{aligned}g''(\tau)=&\,\gamma\left(1+\tau^2+\frac{2}{\sqrt{N}}\tau\right)^{-\gamma/2-2}\left((\gamma+2)\left(\tau+\frac{1}{\sqrt{N}}\right)^2-1-\tau^2-\frac{2}{\sqrt{N}}\tau\right)\\&+\gamma\left(1+\tau^2-\frac{2}{\sqrt{N}}\tau\right)^{-\gamma/2-2}\left((\gamma+2)\left(\tau-\frac{1}{\sqrt{N}}\right)^2-1-\tau^2+\frac{2}{\sqrt{N}}\tau\right).\end{aligned}$$

Then, for γ sufficiently large, $g(\tau)$ is convex in $[0,\frac{1}{4\sqrt{N}}]$. Since $g'(0)=0$ we infer that $g(\tau)\geq0$ for any $\tau\in[0,\frac{1}{4\sqrt{N}}]$. Hence $I_1\geq0$.
For $\tau\in[\frac{1}{4\sqrt{N}},\frac{1}{2\sqrt{N}}]$

$$g(\tau)\geq\left(1+\tau^2-\frac{2}{\sqrt{N}}\tau\right)^{-\gamma/2}-2\geq\left(1-\frac{7}{16N}\right)^{-\gamma/2}-2$$

and

$$I_2\geq\left(\left(1-\frac{7}{16N}\right)^{-\gamma/2}-2\right)\int_{\frac{1}{4\sqrt{N}}}^{\frac{1}{2\sqrt{N}}}\frac{1}{\tau^{1+2s}}\,d\tau\to+\infty\quad\text{as }\gamma\to+\infty.$$

□

Remark 4.9. If $N\geq3$ the value $\tilde{\gamma}$ in Lemma 4.8 is in fact strictly larger than 1. This is a consequence of the fact that the function

$$f(\tau)=\left(1+\tau^2+\frac{2}{\sqrt{N}}\tau\right)^{-1/2}+\left(1+\tau^2-\frac{2}{\sqrt{N}}\tau\right)^{-1/2}-2$$

is negative for any $\tau>0$, i.e. $c(1)<0$, which, together with the convexity of $c(\gamma)$, leads to $c(\gamma)<0$ for any $\gamma\in(0,1]$.

The main result of this subsection is the following

Theorem 4.10. (Liouville) *The equation*

$$\mathcal{I}_N^- u(x) + u^p(x) = 0, \quad u \geq 0 \ \textit{in} \ \mathbb{R}^N \tag{4.44}$$

has nontrivial viscosity supersolutions if, and only if, $p > 1 + \frac{2s}{\tilde{\gamma}}$.

As before, we divide the proof of the previous theorem in several partial results. We start with the

Proposition 4.11. *For any* $p > 1 + \frac{2s}{\tilde{\gamma}}$ *there exist positive viscosity supersolutions of the equation*

$$\mathcal{I}_N^- u(x) + u^p(x) = 0 \quad \textit{in} \ \mathbb{R}^N.$$

Proof. For $q \in \left[\frac{1}{p-1}, \frac{\tilde{\gamma}}{2s}\right)$ we consider the function

$$u(x) = \frac{1}{(1+|x|)^{2sq}}\,.$$

Using Theorem 3.4, see also Remark 3.5, for any fixed $x \in \mathbb{R}^N$, $x \neq 0$, it holds

$$\mathcal{I}_N^- u(x) = N \mathcal{I}_\xi u(x),$$

$\xi \in \mathbb{R}^N$ being a unit vector such that $\langle \hat{x}, \xi \rangle = \frac{1}{\sqrt{N}}$. Thus we have

$$\mathcal{I}_N^- u(x) = \frac{NC_s}{(1+|x|)^{2sq}} \int_0^{+\infty} \frac{\left(\frac{1+|x+\tau\xi|}{1+|x|}\right)^{-2sq} + \left(\frac{1+|x-\tau\xi|}{1+|x|}\right)^{-2sq} - 2}{\tau^{1+2s}}\, d\tau\,. \tag{4.45}$$

By the triangular inequality we have

$$\frac{1+|x \pm \tau\xi|}{1+|x|} \geq \left|\hat{x} \pm \frac{\tau}{1+|x|}\xi\right| \qquad \forall \tau \geq 0.$$

Then, by (4.45), we infer that

$$\begin{aligned}
\mathcal{I}_N^- u(x) &\leq \frac{NC_s}{(1+|x|)^{2sq}} \int_0^{+\infty} \frac{\left|\hat{x} + \frac{\tau}{1+|x|}\xi\right|^{-2sq} + \left|\hat{x} - \frac{\tau}{1+|x|}\xi\right|^{-2sq} - 2}{\tau^{1+2s}}\, d\tau \\
&= \frac{NC_s}{(1+|x|)^{2s(q+1)}} \int_0^{+\infty} \frac{|\hat{x}+\tau\xi|^{-2sq} + |\hat{x}-\tau\xi|^{-2sq} - 2}{\tau^{1+2s}}\, d\tau \\
&= \frac{NC_s}{(1+|x|)^{2s(q+1)}} c(2sq)
\end{aligned}$$

where $c(\cdot)$ is the function defined by (4.40). Using Lemma 4.8 and the assumption $2sq < \tilde{\gamma}$, we see that $c(2sq) < 0$. Let $v(x) = \varepsilon u(x)$ for $\varepsilon \in (0, (NC_s|c(2sq)|)^{1/(p-1)})$. Using $q \geq \frac{1}{p-1}$ we finally obtain

$$\begin{aligned}
\mathcal{I}_N^- v(x) + v^p(x) &\leq \frac{\varepsilon}{(1+|x|)^{2s(q+1)}} \left(NC_s c(2sq) + \frac{\varepsilon^{p-1}}{(1+|x|)^{2s(qp-q-1)}} \right) \\
&\leq \frac{\varepsilon}{(1+|x|)^{2s(q+1)}} \left(NC_s c(2sq) + \varepsilon^{p-1} \right) \leq 0\,,
\end{aligned}$$

completing the proof. □

Lemma 4.12. *Let u be a nonnegative viscosity supersolution of*

$$\mathcal{I}_N^- u(x) = 0 \quad in\ \mathbb{R}^N.$$

Then the following statements hold:

- *there exists a positive constant $a = a(\tilde{\gamma})$ such that*

$$m(r) \geq a\, m(1)\, r^{-\tilde{\gamma}} \qquad \forall r \geq 1; \tag{4.46}$$

- *for any $\gamma \geq \tilde{\gamma}$ there exists a positive constant $b = b(\gamma)$ such that*

$$m(R) \geq b\, m\left(\frac{R}{2}\right) \qquad \forall R > 0. \tag{4.47}$$

Proof. Let $\tilde{g}(|x|)$ be the radial function

$$\tilde{g}(|x|) = \begin{cases} \tilde{f}(|x|) & \text{if } |x| \leq \frac{1}{2} \\ |x|^{-\frac{\tilde{\gamma}}{2}} & \text{if } |x| > \frac{1}{2}, \end{cases} \tag{4.48}$$

where $\tilde{f}$ is defined, for $r \geq 0$, by the formula

$$f(r) = 2^{\frac{\tilde{\gamma}}{2}} \left[-\frac{1}{6}\tilde{\gamma}(\tilde{\gamma}+2)(\tilde{\gamma}+4)\left(r - \frac{1}{2}\right)^3 + \frac{1}{2}\tilde{\gamma}(\tilde{\gamma}+2)r^2 \right.$$
$$\left. -\frac{1}{2}\tilde{\gamma}(\tilde{\gamma}+4)r + 1 + \frac{1}{8}\tilde{\gamma}(\tilde{\gamma}+6)\right].$$

With the choice of such $\tilde{f}$, the function $\tilde{g}''$ is convex in $[0, +\infty)$, since the graph of $\tilde{f}''(r)$ is in fact the tangent line of the function $\left(r^{-\frac{\tilde{\gamma}}{2}}\right)'' = \frac{\tilde{\gamma}}{2}(\frac{\tilde{\gamma}}{2}+1)r^{-\frac{\tilde{\gamma}}{2}-2}$ at $r = \frac{1}{2}$.

Set $w(x) = \tilde{g}(|x|^2)$. By Theorem 3.4-(iii), for any $x \in \mathbb{R}^N$ we have

$$\mathcal{I}_N^- w(x) = N\mathcal{I}_\xi w(x),$$

where $\xi \in \mathbb{R}^N$ is an unit vector such that $\langle \hat{x}, \xi \rangle = \frac{1}{\sqrt{N}}$. Hence

$$\mathcal{I}_N^- w(x) = NC_s \int_0^{+\infty} \frac{w(|x+\tau\xi|) + w(|x-\tau\xi|) - 2w(x)}{\tau^{1+2s}}\, d\tau. \tag{4.49}$$

If $|x| \geq 1$ and $\tau > 0$ it holds that

$$|x \pm \tau\xi| \geq \sqrt{|x|^2 + \tau^2 - 2\frac{\tau|x|}{\sqrt{N}}} \geq |x|\sqrt{1 - \frac{1}{N}} \geq \frac{1}{\sqrt{2}}.$$

Then, using (4.48)–(4.49) and the definition of $\tilde{\gamma}$ given in Lemma 4.8, we infer that

$$\mathcal{I}_N^- w(x) = NC_s \int_0^{+\infty} \frac{|x+\tau\xi|^{-\tilde{\gamma}} + |x-\tau\xi|^{-\tilde{\gamma}} - 2|x|^{-\tilde{\gamma}}}{\tau^{1+2s}}\, d\tau = 0.$$

In this way the function

$$\phi(x) = m(1)\frac{w(|x|) - w(R)}{w(0) - w(R)}$$

is for any $R > 1$ a classical solution of $\mathcal{I}_N^- w(x) = 0$ for $|x| \in [1, R]$. Moreover

$$u(x) \geq m(1) \geq \phi(x) \qquad \forall |x| \leq 1$$

and

$$u(x) \geq 0 \geq \phi(x) \qquad \forall |x| \geq R.$$

Then by comparison principle, see Theorem 2.5 and Remark 2.6, we infer that $u(x) \geq \phi(x)$ for any $|x| \in [1, R]$. Letting $R \to +\infty$ we obtain

$$u(x) \geq m(1) \frac{w(|x|)}{w(0)} \quad \forall |x| \geq 1,$$

which easily imply (4.46) with

$$a = \left(2^{\frac{\tilde\gamma}{2}} \left[\frac{1}{48} \tilde\gamma(\tilde\gamma + 2)(\tilde\gamma + 4) + 1 + \frac{1}{8} \tilde\gamma(\tilde\gamma + 6) \right] \right)^{-1}.$$

The proof of (4.47) follows the same idea used before. Fix $\gamma \geq \tilde\gamma$. For $R > 0$, consider the function

$$\tilde g(|x|) = \begin{cases} \tilde f(|x|) & \text{if } |x| \leq \left(\frac{R}{2\sqrt{2}} \right)^2 \\ |x|^{-\frac{\gamma}{2}} & \text{if } |x| > \left(\frac{R}{2\sqrt{2}} \right)^2, \end{cases}$$

where

$$\tilde f(r) = \left(\frac{R}{2\sqrt{2}} \right)^{-\gamma} \left[-\frac{32}{3} \frac{\gamma(\gamma+2)(\gamma+4)}{R^6} \left(r - \frac{R^2}{8} \right)^3 + \frac{8}{R^4} \gamma(\gamma+2) r^2 \right.$$
$$\left. - \frac{2}{R^2} \gamma(\gamma+4) r + 1 + \frac{\gamma}{8}(\gamma+6) \right].$$

Set $w(x) = \tilde g(|x|^2)$. Since $\tilde g''$ is convex, we are in position to use the representation formula (4.49). Taking into account that for $|x| \geq \frac{R}{2}$

$$|x \pm \tau\xi| \geq \sqrt{|x|^2 + \tau^2 - 2\frac{\tau|x|}{\sqrt{N}}} \geq |x| \sqrt{1 - \frac{1}{N}} \geq \frac{R}{2\sqrt{2}},$$

then

$$\mathcal{I}_N^- w(x) = N C_s \int_0^{+\infty} \frac{|x + \tau\xi|^{-\gamma} + |x - \tau\xi|^{-\gamma} - |x|^{-\gamma}}{\tau^{1+2s}} \, d\tau \geq 0$$

the last inequality being a consequence of the fact that $\gamma \geq \tilde\gamma$. Consider now the function

$$\phi(x) = m\left(\frac{R}{2} \right) \frac{w(|x|) - w(2R)}{w(0) - w(2R)},$$

which is in turn a solution of $\mathcal{I}_N^- \phi(x) \geq 0$ for $|x| \in [\frac{R}{2}, 2R]$ and satisfies

$$u(x) \geq m(1) \geq \phi(x) \qquad \forall |x| \leq \frac{R}{2}$$

and

$$u(x) \geq 0 \geq \phi(x) \qquad \forall |x| \geq 2R.$$

By comparison principle we conclude

$$m(R) \geq m\left(\frac{R}{2}\right)\frac{w(R)-w(2R)}{w(0)-w(2R)} = m\left(\frac{R}{2}\right)\frac{1-2^{-\gamma}}{(2\sqrt{2})^{\gamma}\left[1+\frac{\gamma}{48}\left((\gamma+2)(\gamma+4)+6(\gamma+6)\right)\right]}.$$

□

Proof of Theorem 4.10. We shall detail the proof in the critical case $p = 1 + \frac{2s}{\tilde\gamma}$, since if $p > 1 + \frac{2s}{\tilde\gamma}$ the conclusion follows by Proposition 4.11, while the subcritical case $p < 1 + \frac{2s}{\tilde\gamma}$ can be treat in the same way as we did in the proof of Theorem 4.2, using now Lemma 4.12. When $p = 1 + \frac{2s}{\tilde\gamma}$ we need some extra work. In particular we are not in a position to use the analogous of Lemma 4.6 for the operator $\mathcal{I}_N^-$, due to the lack of validity of the representation formula for $\Gamma(|x|) = \frac{\ln|x|}{|x|^{\tilde\gamma}}$. Note that Γ doesn't even belong to $L^1_{1,2s}$ when $N \geq 3$, since $\tilde\gamma > 1$ (see Remark 4.9). Moreover moreover Γ is concave near the origin. On the other hand, for x far away the origin, we shall still obtain some useful information that are sufficient to conclude.

Let $\tilde\Gamma(|x|) = \frac{1}{2}\frac{\ln|x|}{|x|^{\tilde\gamma/2}}$. The function $\tilde\Gamma''(r)$ is convex for $r \geq r_0 := \exp\left(\frac{2}{\tilde\gamma}+\frac{2}{\tilde\gamma+2}+\frac{2}{\tilde\gamma+4}+\frac{2}{\tilde\gamma+6}\right)$. Let $\tilde f''(r) = \tilde\Gamma''(r_0) + \tilde\Gamma'''(r_0)(r-r_0)$ be the tangent line of $\tilde\Gamma''$ at $r = r_0$. By construction the function

$$\tilde g(|x|) = \begin{cases} \tilde f(|x|) & \text{if } |x| \leq r_0 \\ \tilde\Gamma(|x|) & \text{if } |x| > r_0, \end{cases}$$

where

$$\tilde f(|x|) = \frac{1}{6}\tilde\Gamma'''(r_0)(r-r_0)^3 + \frac{1}{2}\tilde\Gamma''(r_0)r^2 + \left(\tilde\Gamma'(r_0) - \tilde\Gamma''(r_0)r_0\right)r + \tilde\Gamma(r_0) - \frac{1}{2}\tilde\Gamma''(r_0)r_0^2 - \left(\tilde\Gamma'(r_0) - \tilde\Gamma''(r_0)r_0\right)r_0,$$

is such that $\tilde g''$ is convex in $[0,+\infty)$. Hence, setting $w(x) = \tilde g(|x|^2)$ and using Theorem 3.4 we have

$$\mathcal{I}_N^- w(x) = N\mathcal{I}_\xi w(x),$$

$\xi \in \mathbb{R}^N$ being an unit vector such that $\langle \hat{x}, \xi \rangle = \frac{1}{\sqrt{N}}$. Moreover for $|x| \geq \sqrt{2r_0}$ it holds that $|x \pm \tau\xi| \geq \sqrt{r_0}$ for any $\tau > 0$. Then for any $|x| \geq \sqrt{2r_0}$

$$\begin{aligned}
\mathcal{I}_N^- w(x) &= NC_s \int_0^{+\infty} \frac{\Gamma(|x+\tau\xi|) + \Gamma(|x-\tau\xi|) - 2\Gamma(|x|)}{\tau^{1+2s}}\, d\tau \\
&= NC_s \left(\ln|x| \int_0^{+\infty} \frac{|x+\tau\xi|^{-\tilde{\gamma}} + |x-\tau\xi|^{-\tilde{\gamma}} - |x|^{-\tilde{\gamma}}}{\tau^{1+2s}}\, d\tau \right. \\
&\quad \left. + \frac{1}{|x|^{\tilde{\gamma}+2s}} \int_0^{+\infty} \frac{\frac{\ln|\hat{x}+\tau\xi|}{|\hat{x}+\tau\xi|^{\tilde{\gamma}}} + \frac{\ln|\hat{x}-\tau\xi|}{|\hat{x}-\tau\xi|^{\tilde{\gamma}}}}{\tau^{1+2s}}\, d\tau \right) \\
&= NC_s \frac{1}{2|x|^{\tilde{\gamma}+2s}} \int_0^{+\infty} \left(\frac{\ln\left(1+\tau^2+\frac{2\tau}{\sqrt{N}}\right)}{\sqrt{1+\tau^2+\frac{2\tau}{\sqrt{N}}}^{\tilde{\gamma}}} + \frac{\ln\left(1+\tau^2-\frac{2\tau}{\sqrt{N}}\right)}{\sqrt{1+\tau^2-\frac{2\tau}{\sqrt{N}}}^{\tilde{\gamma}}} \right) \tau^{-(1+2s)}\, d\tau \\
&\geq -\frac{C}{|x|^{\tilde{\gamma}+2s}}
\end{aligned}$$

where $C = C(N,s)$ is a positive constant.
Now for $r_2 > r_1 > \sqrt{2r_0}$ we consider the function

$$\phi(x) = m(r_1)\frac{w(|x|) - w(r_2)}{w(0) - w(r_2)} \qquad \forall |x| \in [r_1, r_2].$$

Without loss of generality we may further assume that $w(0) - w(r_2) > \frac{1}{2}w(0)$, so that

$$\mathcal{I}_N^- \phi(x) \geq -Cm(r_1)\frac{1}{|x|^{\tilde{\gamma}+2s}} \tag{4.50}$$

where C is a positive constant depending only on N and s. In addition $u(x) \geq \phi(x)$ for any $|x| \in [0, r_1] \cup [r_2, +\infty)$.
Using the Eq. (4.44) and (4.46) we also have

$$\mathcal{I}_N^- u(x) \leq -(m(|x|))^{1+\frac{2s}{\tilde{\gamma}}} - \leq (am(1))^{1+\frac{2s}{\tilde{\gamma}}} \frac{1}{|x|^{\tilde{\gamma}+2s}} \qquad \forall |x| \geq 1. \tag{4.51}$$

Since $m(r_1) \to 0$ as $r_1 \to +\infty$, in view of the inequality

$$m(R)R^{\tilde{\gamma}} < C \qquad \forall R > 0, \tag{4.52}$$

for some positive constant C, by (4.50)–(4.51) we can then pick r_1 sufficiently large such that

$$\mathcal{I}_N^- u(x) \leq \mathcal{I}_N^- \phi(x) \qquad \forall |x| \in [r_1, r_2].$$

By comparison principle we have $u(x) \geq \phi(x)$ for any $|x| \in [r_1, r_2]$. Letting $r_2 \to +\infty$ we deduce that

$$m(r) \geq \frac{m(r_1)}{w(0)} w(r) = \frac{m(r_1)}{w(0)} \frac{\ln r}{r^{\tilde{\gamma}}} \qquad \forall r > r_1,$$

leading to a contradiction to (4.52) in the limit as $r \to +\infty$. □

5. On the operator $\mathcal{J}_k^{\pm}$

In this section we concentrate on the operators $\mathcal{J}_k^{\pm}$ defined in (1.3). We leave off the analysis the cases $k = 1$, where $\mathcal{J}_1^{\pm}$ meets $\mathcal{I}_1^{\pm}$ studied in the previous sections, and $k = N$, case in which $\mathcal{J}_k^{\pm} = -(-\Delta_{\mathbb{R}^N})^s$, already studied in [20]. For simplicity, we write $\Delta^s_{\mathbb{R}^k} = -(-\Delta_{\mathbb{R}^k})^s$, to denote the fractional Laplacian in $\mathbb{R}^k$.

The key technical result of this section is the following

Proposition 5.1. *Assume $1 < k < N$. Let $u(x) = \tilde{g}(|x|^2)$ be such that $u \in C^2(\mathbb{R}^N) \cap L^1_{k,2s}$. If $\tilde{g}$ is convex, then:*

- (i) *$\mathcal{J}_k^- u(x) = \mathcal{J}_V u(x)$, where V is any k dimensional subspace which is orthogonal to x;*
- (ii) *$\mathcal{J}_k^+ u(x) = \mathcal{J}_V u(x)$, where V is any k-dimensional subspace containing x. Moreover, defining $\tilde{u} : \mathbb{R}^k \to \mathbb{R}$ as $\tilde{u}(y) := \tilde{g}(|y|^2)$, we have*

$$\mathcal{J}_k^+ u(x) = \Delta^s_{\mathbb{R}^k} \tilde{u}(y), \tag{5.1}$$

where $y \in \mathbb{R}^k$ is such that $|y| = |x|$.

Proof. Let $x \in \mathbb{R}^N$. For k-dimensional subspace V, we choose an orthonormal basis $\{\xi_1, \ldots, \xi_k\}$ such that $\langle x, \xi_i \rangle = 0$ for $i \geq 2$. Then

$$\begin{aligned}\mathcal{J}_V u(x) = \frac{C_{k,s}}{2} \int_{\mathbb{R}^k} &[\tilde{g}(|x|^2 + |\tau|^2 + 2\tau_1 \langle x, \xi_1 \rangle) \\ &+ \tilde{g}(|x|^2 + |\tau|^2 - 2\tau_1 \langle x, \xi_1 \rangle) - 2\tilde{g}(|x|^2)]|\tau|^{-(k+2s)}\, d\tau\,.\end{aligned}$$

Assuming the convexity of $\tilde{g}$ and setting

$$h(t) = \tilde{g}(a + bt) + \tilde{g}(a - bt) \qquad \text{for } a \geq b \geq 0 \text{ and } t \in [0, 1],$$

we recall, see the proof of (3.1), that h is nondecreasing. Then $h(0) \leq h(1)$ yields $2\tilde{g}(a) \leq \tilde{g}(a + b) + \tilde{g}(a - b)$ for $a \geq b \geq 0$. Choosing

$$a = |x|^2 + |\tau|^2\,, \quad b = 2\,|\tau_1 \langle x, \xi_1 \rangle|$$

we infer that

$$2\tilde{g}(|x|^2 + |\tau|^2) \leq \tilde{g}(|x|^2 + |\tau|^2 + 2\tau_1 \langle x, \xi_1 \rangle) + \tilde{g}(|x|^2 + |\tau|^2 - 2\tau_1 \langle x, \xi_1 \rangle). \tag{5.2}$$

Consider now

$$a = |x|^2 + |\tau|^2\,, \quad b = 2\,|\tau_1|\,|x|\,.$$

For $x \neq 0$ and $\hat{x} = \frac{x}{|x|}$, the inequality $h(|\langle \hat{x}, \xi_1 \rangle|) \leq h(1)$ yields

$$\begin{aligned}&\tilde{g}(|x|^2 + |\tau|^2 + 2\tau_1 \langle x, \xi_1 \rangle) + \tilde{g}(|x|^2 + |\tau|^2 - 2\tau_1 \langle x, \xi_1 \rangle) \\ &\quad \leq \tilde{g}(|x|^2 + |\tau|^2 + 2\tau_1 |x|) + \tilde{g}(|x|^2 + |\tau|^2 - 2\tau_1 |x|)\,.\end{aligned} \tag{5.3}$$

Note that the above inequality is also true for $x = 0$. From (5.2)–(5.3) we obtain

$$\begin{aligned} 2\int_{\mathbb{R}^k} & [\tilde{g}(|x|^2+|\tau|^2)-\tilde{g}(|x|^2)]|\tau|^{-(k+2s)}\,d\tau \\ & \leq \int_{\mathbb{R}^k} [\tilde{g}(|x|^2+|\tau|^2+2\tau_1\langle x,\xi_1\rangle)+\tilde{g}(|x|^2+|\tau|^2-2\tau_1\langle x,\xi_1\rangle) \\ & \quad -2\tilde{g}(|x|^2)]|\tau|^{-(k+2s)}\,d\tau \\ & \leq \int_{\mathbb{R}^k} [\tilde{g}(|x|^2+|\tau|^2+2\tau_1|x|)+\tilde{g}(|x|^2+|\tau|^2-2\tau_1|x|)-2\tilde{g}(|x|^2)]|\tau|^{-(k+2s)}\,d\tau\,. \end{aligned}$$

Therefore if $x \in V^\perp$, then $\langle x,\xi_1\rangle = 0$, while if $x \in V$ then $|\langle x,\xi_1\rangle| = |x|$. Thus,

$$\begin{aligned} \min_{\dim V=k} \mathcal{J}_V u(x) = C_{k,s}\int_{\mathbb{R}^k} & [\tilde{g}(|x|^2+|\tau|^2) \\ & -\tilde{g}(|x|^2)]|\tau|^{-(k+2s)}\,d\tau \end{aligned}$$

which is realized when $x \in V^\perp$, and

$$\begin{aligned} \max_{\dim V=k} \mathcal{J}_V u(x) = \frac{C_{k,s}}{2}\int_{\mathbb{R}^k} & [\tilde{g}(|x|^2+|\tau|^2+2\tau_1|x|)+\tilde{g}(|x|^2+|\tau|^2-2\tau_1|x|) \\ & -2\tilde{g}(|x|^2)]|\tau|^{-(k+2s)}\,d\tau \end{aligned}$$

which is achieved when $x \in V$.

In order to prove (5.1), let $V = \langle \{\xi_i\}_{i=1}^k \rangle$ a k-dimensional subspace containing x. In this way $\mathcal{J}_k^+ u(x) = \mathcal{J}_V u(x)$ and without loss of generality we can further assume that $\langle \hat{x},\xi_i\rangle = 0$ for all $i = 2,...,k$.

Let $\{e_i\}_{i=1}^k$ the canonical basis in $\mathbb{R}^k$. Using the rotation invariance of the fractional Laplacian, for $y \in \mathbb{R}^k$ such that $|y| = |x|$ we have

$$\begin{aligned} \Delta^s_{\mathbb{R}^k}\tilde{u}(y) &= \Delta^s_{\mathbb{R}^k}\tilde{u}(|y|e_1) \\ &= \frac{C_{k,s}}{2}\,\text{P.V.}\int_{\mathbb{R}^k}\left[\tilde{g}\left(||y|e_1+\sum_{i=1}^k z_ie_i|^2\right)+\tilde{g}\left(||y|e_1-\sum_{i=1}^k z_ie_i|^2\right)\right. \\ &\quad \left. -2\tilde{g}(|y|^2)\right]|z|^{-(k+2s)}\,dz \\ &= \frac{C_{k,s}}{2}\,\text{P.V.}\int_{\mathbb{R}^k}\left[\tilde{g}(|x|^2+2|x|\tau_1+|\tau|^2)+\tilde{g}(|x|^2-2|x|\tau_1+|\tau|^2)\right. \\ &\quad \left. -2\tilde{g}(|x|^2)\right]|\tau|^{-(k+2s)}\,d\tau \\ &= \mathcal{J}_V u(x) \end{aligned}$$

as we wanted to show. □

Using known results for the fractional Laplacian (see [10,12]) and the previous proposition we get the following

Corollary 5.2. *The function $u(x) = |x|^{-(k-2s)}$ satisfies*

$$\mathcal{J}_k^+ u(x) = 0 \quad \text{for } x \in \mathbb{R}^N\setminus\{0\}.$$

Using the representation formula and Theorem 1.3 in Felmer and Quaas [20], we can get the Liouville Theorem for $\mathcal{J}_k^+$

Theorem 5.3. *Let $1 < k < N$. Then, the equation*

$$\mathcal{J}_k^+ u(x) + u^p(x) = 0 \quad \textit{in } \mathbb{R}^N \tag{5.4}$$

has nontrivial viscosity supersolutions if, and only if,

$$p > \frac{k}{k-2s}.$$

Proof. For the existence of nontrivial supersolution, we consider $\frac{1}{p-1} < q < \frac{k-2s}{2s}$ and $v(y) = c(1+|y|)^{-2sq}$, where $y \in \mathbb{R}^k$ and c is a positive constant. According to [20], if c is small enough we have

$$\mathcal{J}_k^+ v(x) + v^p = \Delta^s_{\mathbb{R}^k} v(y) + v^p \le 0.$$

On the other hand, if $p \le k/(k-2s)$ and there exists a nontrivial supersolution u for (5.4), the function

$$v(x) = \min\{u(Ox) : O \text{ is a rotation matrix in } \mathbb{R}^N\}$$

is a positive, radial supersolution for (5.4). Let $\tilde{h} : [0,+\infty) \to \mathbb{R}$ such that $v(x) = \tilde{h}(|x|)$ and denote $w(y) = \tilde{h}(|y|), y \in \mathbb{R}^k$. Then, we have

$$\begin{aligned} \Delta^s_{\mathbb{R}^k} w(y) = \frac{C_{k,s}}{2} \int_{\mathbb{R}^k} \Bigg[\tilde{h}\left(\left| \sum_{i=1}^k y_i e_i + \sum_{i=1}^k z_i e_i \right| \right) \\ + \tilde{h}\left(\left| \sum_{i=1}^k y_i e_i - \sum_{i=1}^k z_i e_i \right| \right) - 2\tilde{h}(|y|) \Bigg] |z|^{-(k+2s)}\, dz \\ \le \mathcal{J}_k^+ v\left(\sum_{i=1}^k y_i e_i \right), \end{aligned}$$

where $\{e_i\}_{i=1}^N$ is the canonical basis (we have identified $e_i \in \mathbb{R}^k$ for $i \le k$). Then, w is a nontrivial supersolution to

$$\Delta^s_{\mathbb{R}^k} w + w^p \le 0 \quad \text{in } \mathbb{R}^k,$$

which contradicts the nonexistence result in [20]. □

For $\mathcal{J}_k^-$, in analogy to Theorem 4.7 we have the following

Theorem 5.4. *Assume $1 < k < N$ and consider the equation*

$$\mathcal{J}_k^- u(x) + u^p(x) = 0 \quad \textit{in } \mathbb{R}^N. \tag{5.5}$$

Then

(i) *for any $p \ge 1$ there exist positive classical solutions of (5.5);*
(ii) *for any $p \in [1-s, 1)$ there exist nonnegative viscosity solutions $u \not\equiv 0$ of (5.5);*
(iii) *for any $p \in (0, 1-s)$ there exist nonnegative viscosity supersolutions $u \not\equiv 0$ of (5.5).*

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6. Appendix

In this section we provide a sketch of the proof of some results related to the convergence of the nonlocal operators presented here towards the local regime, that is, when $s \to 1^-$. It is worth to mention that the normalizing constant in (1.5) is given (see [18]) by

$$C_{N,s} = \left(\int_{\mathbb{R}^N} \frac{1 - \cos(z_1)}{|z|^{N+2s}} dz \right)^{-1}. \tag{6.1}$$

We start with the following convergence result that is at the core of the stability of viscosity solutions. Recall that we denote $C_s = C_{1,s}$.

Lemma 6.1. *Let $u \in C^2(\mathbb{R}^N)$ be such that*

$$\|u\|_{L^1_{1,2r_0}} := \sup_{\substack{\dim(V)=1 \\ V \text{affine}}} \left\{ \int_V \frac{|u(y)| d\mathcal{H}^1(y)}{1 + |y|^{1+2r_0}} \right\} < \infty$$

for some $r_0 \in (0,1)$.

Then, for each $x \in \mathbb{R}^N$ we have

$$\mathcal{I}_k^{\pm} u(x) \to \mathcal{P}_k^{\pm} u(x) \quad \text{as } s \to 1^-.$$

Analogously, if $u \in C^2(\mathbb{R}^N)$ and

$$\|u\|_{L^1_{k,2r_0}} := \sup_{\substack{\dim(V)=k \\ V \text{affine}}} \left\{ \int_V \frac{|u(y)| d\mathcal{H}^k(y)}{1 + |y|^{k+2r_0}} \right\} < \infty$$

for some $r_0 \in (0,1)$, *then for each* $x \in \mathbb{R}^N$ *we have*

$$\mathcal{J}_k^{\pm} u(x) \to \mathcal{P}_k^{\pm} u(x) \quad \text{as } s \to 1^-.$$

Proof. We write the proof for $\mathcal{I}_k^+$ and $\mathcal{J}_k^+$, those for $\mathcal{I}_k^-$ and $\mathcal{J}_k^-$ being similar.

Let us first show that for $r_0 \le s < 1$ and $k \ge 1$

$$\|u\|_{L^1_{k,2s}} := \sup_{\substack{\dim(V)=k \\ V \text{affine}}} \left\{ \int_V \frac{|u(y)| d\mathcal{H}^1(y)}{1+|y|^{k+2s}} \right\} \le C$$

for $C = C(k, \|u\|_{L^1_{k,2r_0}})$ independent of s. Indeed

$$\begin{aligned} \|u\|_{L^1_{k,2r_0}} &\ge \sup_{\substack{\dim(V)=k \\ V \text{ affine}}} \int_V \frac{|u(y)| d\mathcal{H}^k(y)}{(1+|y|)^{k+2r_0}} \\ &\ge \sup_{\substack{\dim(V)=k \\ V \text{ affine}}} \int_V \frac{|u(y)| d\mathcal{H}^k(y)}{(1+|y|)^{k+2s}} \\ &\ge \frac{1}{2^{k+1}} \sup_{\substack{\dim(V)=k \\ V \text{ affine}}} \int_V \frac{|u(y)| d\mathcal{H}^k(y)}{1+|y|^{k+2s}} = \frac{1}{2^{k+1}} \|u\|_{L^1_{k,2s}}. \end{aligned}$$

Let $\varepsilon > 0$. For $s \in [r_0, 1)$, there exists a frame $\{\xi_j = \xi_j(\varepsilon, s)\}_{j=1}^k \in \mathcal{V}_k$ such that

$$\begin{aligned} \mathcal{I}_k^+ u(x) - \mathcal{P}_k^+ u(x) &\le \sum_{j=1}^k \mathcal{I}_{\xi_j} u(x) - \mathcal{P}_k^+ u(x) + \varepsilon \\ &\le \sum_{j=1}^k \left(\mathcal{I}_{\xi_j} u(x) - \langle D^2 u(x) \xi_j, \xi_j \rangle \right) + \varepsilon. \end{aligned}$$

For $\delta \in (0,1)$ to be fixed, we can write for each j

$$\mathcal{I}_{\xi_j} u(x) = \frac{1}{2} C_s \int_{-\delta}^{\delta} \frac{\frac{1}{2} \langle \left(D^2 u(\tilde{x}(\xi_j(\varepsilon,s),\tau)) + D^2 u(\tilde{x}'(\xi_j(\varepsilon,s),\tau)) \right) \xi_j, \xi_j \rangle}{|\tau|^{2s-1}} d\tau + C_s O(\delta^{-2s}),$$

where $\tilde{x}(\xi_j(\varepsilon,s),\tau)$, $\tilde{x}'(\xi_j(\varepsilon,s),\tau) \in B_\delta(x)$ for all ε, j, s, τ, and $O(\delta^{-2s})$ just depend on the $\|u\|_{L^1_{2s}} \le C$ for some C independent of s and ε. Using the continuity of $D^2 u$, we can fix $\delta = \delta(\varepsilon)$ small enough in order to have

$$\left| \frac{1}{2} \left(D^2 u(\tilde{x}(\xi_j(\varepsilon,s),\tau)) + D^2 u(\tilde{x}'(\xi_j(\varepsilon,s),\tau)) \right) - D^2 u(x) \right| \le \varepsilon,$$

for all $s \in [r_0, 1), j = 1, \ldots, k$ and $|\tau| < \delta$. Then, we can write

$$\begin{aligned} \mathcal{I}_k^+ u(x) - \mathcal{P}_k^+ u(x) \le{}& \frac{C_s k \varepsilon}{2-2s} \delta^{2-2s} \\ &+ \sum_{j=1}^k \langle D^2 u(x) \xi_j, \xi_j \rangle \left(\frac{C_s}{2-2s} \delta^{2-2s} - 1 \right) + C_s O(\delta^{-2s}) + \varepsilon. \end{aligned} \tag{6.2}$$

Since

$$\frac{C_s}{2(1-s)} \to 1 \quad \text{as } s \to 1^-,$$

(see [18]), then passing to the limit in (6.2), we have

$$\limsup_{s\to 1^-} \mathcal{I}_k^+ u(x) - \mathcal{P}_k^+ u(x) \le (k+1)\varepsilon.$$

A reverse inequality can be found in the same way, and the result follows.

For $\mathcal{J}_k^\pm$ the proof is similar, so we will be sketchy. Given any $V = \{\xi_i\}_{i=1}^k \in \mathcal{V}_k$ and $\delta > 0$ we can write

$$\mathcal{J}_V u(x) = \frac{C_{k,s}}{2} \sum_{i,j=1}^k \int_{B_\delta} \frac{\frac{1}{2}\langle\left(D^2u(\tilde{x}) + D^2u(\tilde{x}')\right)\xi_i, \xi_j\rangle \tau_i\tau_j}{|\tau|^{k+2s}} d\tau + C_{k,s}\, O(\delta^{-2s}),$$

where $\tilde{x} = \tilde{x}(V,\tau)$, $\tilde{x}' = \tilde{x}'(V,\tau)$ are such that $|\tilde{x} - x| \le \delta$. Then, using the continuity of u, for each $\varepsilon > 0$ we can get $\delta(\varepsilon) > 0$ and $\{\xi_j = \xi_j(\varepsilon, s)\}_{j=1}^k \in \mathcal{V}_k$ such that

$$\mathcal{J}_V u(x) \le \frac{C_{k,s}}{2} \sum_{i,j=1}^k \left(\varepsilon + \langle D^2u(x)\xi_i, \xi_j\rangle\right) \int_{B_\delta} \tau_i\tau_j |\tau|^{-(k+2s)} d\tau + C_{k,s}\, O(\delta^{-2s}),$$

and using the symmetry of the integral term, we have

$$\begin{aligned}\int_{B_\delta} \tau_i\tau_j|\tau|^{-(k+2s)}d\tau = \delta_{ij}\int_{B_\delta}\tau_1^2|\tau|^{-(k+2s)}d\tau = \delta_{ij}k^{-1}\int_{B_\delta}|\tau|^{2-k-2s}d\tau \\ = \delta_{ij}k^{-1}|\mathbb{S}^{k-1}|\frac{\delta^{2-2s}}{2-2s}\end{aligned}$$

where δ_{ij} is the Kronecker delta and $|\mathbb{S}^{k-1}|$ denotes the $(k-1)$-dimensional measure of the unit sphere in $\mathbb{R}^k$. For $k > 1$, since we have the estimate (see Corollary 4.2 in [18])

$$\frac{C_{k,s}\,|\mathbb{S}^{k-1}|}{4k(1-s)} \to 1 \quad \text{as } s \to 1^-,$$

we obtain

$$\limsup_{s\to 1^-} \mathcal{J}_k^+ u(x) - \mathcal{P}_k^+ u(x) \le O(\varepsilon).$$

From this the result follows. □

Lemma 6.2. *Let $\bar{\gamma} = \bar{\gamma}(k,s)$ defined in Proposition 3.7. Then, $\bar{\gamma} \to 0$ as $s \to 1^-$.*

Let $\tilde{\gamma} = \tilde{\gamma}(N,s)$ defined in Lemma 4.8. Then, $\tilde{\gamma} \to N-2$ as $s \to 1^-$.

Proof. We already know that $\bar{\gamma} \in (0,1)$ and $\tilde{\gamma} > 0$. Moreover from the proof of Lemma 4.8 we can also infer that, for any $s \in (\frac{1}{2}, 1)$, $\tilde{\gamma} < c$ where c is a positive constant depending only on N. Hence both $\bar{\gamma}$ and $\tilde{\gamma}$ are uniformly bounded.

For $\bar{\gamma}$, let us first observe that, by Proposition 3.7, one has

$$\bar{\gamma}(k,s) < \bar{\gamma}(k+1,s).$$

Then it is sufficient to prove that $\bar{\gamma} \to 0$, as $s \to 1^-$, for k large, say $k \ge 4$. If not, let $\gamma_1 \in (0,1]$ be an accumulation point of $\bar{\gamma}$ as $s \to 1^-$. Then, by stability of viscosity solutions, the function $w_{\gamma_1}(x) = |x|^{-\gamma_1}$ would be a solution of $\mathcal{P}_k^+(D^2w) = 0$ for $x \ne 0$. But this contradicts the fact that the only positive exponent γ such that $w_\gamma(x) = |x|^{-\gamma}$ is solution for $\mathcal{P}_k^+$ is $\gamma = k - 2$, see [7], while $\gamma_1 < k - 2$ for $k \geq 4$. Thus, $\bar{\gamma} \to 0$ as $s \to 1^-$.

On the other hand, let $\gamma_1 \geq 0$ be an accumulation point of $\tilde{\gamma}$ as $s \to 1^-$. Using the definition of $c(\gamma)$ in (4.40), for each s we have

$$0 = C_s \int_0^{+\infty} \frac{\left(1 + \tau^2 + \frac{2}{\sqrt{N}}\tau\right)^{-\tilde{\gamma}/2} + \left(1 + \tau^2 - \frac{2}{\sqrt{N}}\tau\right)^{-\tilde{\gamma}/2} - 2}{\tau^{1+2s}} \, d\tau,$$

and from here we have

$$0 = C_s \int_0^{\frac{1}{2\sqrt{N}}} \frac{\left(1 + \tau^2 + \frac{2}{\sqrt{N}}\tau\right)^{-\tilde{\gamma}/2} + \left(1 + \tau^2 - \frac{2}{\sqrt{N}}\tau\right)^{-\tilde{\gamma}/2} - 2}{\tau^{1+2s}} \, d\tau + C_s O(1),$$

where $O(1)$ is independent of s close to 1. By a Taylor expansion, we have

$$\begin{aligned} &\left(1 + \tau^2 + \frac{2}{\sqrt{N}}\tau\right)^{-\tilde{\gamma}/2} + \left(1 + \tau^2 - \frac{2}{\sqrt{N}}\tau\right)^{-\tilde{\gamma}/2} - 2 \\ &\quad = \tilde{\gamma}\tau^2\Big(-1 + \frac{\tilde{\gamma}+2}{N}\Big) + O(\tau^3), \end{aligned}$$

where $O(\tau^3)$ is independent of s. Thus, replacing this into the integral term we get

$$0 = \frac{\tilde{\gamma}}{(4N)^{1-s}}\Big(-1 + \frac{\tilde{\gamma}+2}{N}\Big)\frac{C_s}{2-2s} + C_s O\Big(\frac{1}{3-2s}\Big) + C_s O(1),$$

from which, taking limit as $s \to 1^-$ we arrive at

$$0 = \gamma_1\Big(-1 + \frac{\gamma_1 + 2}{N}\Big),$$

for some $C > 0$, from which the result follows. If $N \geq 3$ we know that $\gamma_1 \geq 1$ (see Remark 4.9), from which the result follows. In the case $N = 2$, we see that $\gamma_1 = 0$. □

References

[1] Alvarez, O., Tourin, A.: Viscosity solutions of nonlinear integro-differential equations. Ann. Inst. H. Poincaré Anal. Non Linéaire. section C **13**(3), 293–317 (1996)

[2] Barles, G., Chasseigne, E., Georgelin, C., Jakobsen, E.: On Neumann type problems for nonlocal equations set in a half space. Trans. Am. Math. Soc. **366**(9), 4873–4917 (2014)

[3] Barles, G., Chasseigne, E., Imbert, C.: On the Dirichlet problem for second order elliptic integro-differential equations. Indiana Univ. Math. J. **57**(1), 213–246 (2008)

[4] Barles, G., Imbert, C.: Second-order Elliptic Integro-Differential Equations: Viscosity Solutions' Theory Revisited. Ann. Inst. H. Poincaré Anal. Non Linéaire, **25**(3), 567–585 (2008)

[5] Birindelli, I., Galise, G., Ishii, H.: A family of degenerate elliptic operators: maximum principle and its consequences. Ann. Inst. H. Poincaré Anal. Non Linéaire **35**, 417–441 (2018)

[6] Birindelli, I., Galise, G., Ishii, H.: Existence through convexity for the truncated Laplacians. Math. Ann. **379**, 909–950 (2021)

[7] Birindelli, I., Galise, G., Leoni, F.: Liouville theorems for a family of very degenerate elliptic nonlinear operators. Nonlinear Anal. **161**, 198–211 (2017)

[8] Birindelli, I., Galise, G., Schiera, D.: Maximum principles and related problems for a class of nonlocal extremal operators. To appear on Ann. Mat. Pura Appl., arXiv:2107.07303 (2021)

[9] Blanc, P., Rossi, Julio D.: Games for eigenvalues of the Hessian and concave/convex envelopes. J. Math. Pures Appl. (9) **127**, 192–215 (2019)

[10] Blumenthal, R.M., Getoor, R.K.: Some theorems on stable processes. Trans. Am. Math. Soc. **95**, 263–273 (1960)

[11] Caffarelli, L., Li, Y.Y., Nirenberg, L.: Some remarks on singular solutions of nonlinear elliptic equations. I J. Fixed Point Theory Appl. **5**, 353–395 (2009)

[12] Caffarelli, L., Silvestre, L.: An extension problem related to the fractional Laplacian. Commun. Partial Differ. Equ. **32**, 1245–1260 (2007)

[13] Caffarelli, L., Silvestre, L.: Regularity theory for nonlocal integro-differential equations. Commun. Pure Appl. Math **62**(5), 597–638 (2009)

[14] Capuzzo Dolcetta, I., Leoni, F., Vitolo, A.: On the inequality $F(x, D^2u) \geq f(u)+ g(u)|Du|^q$. Math. Ann. **365**(1–2), 423–448 (2016)

[15] Ciomaga, A.: On the strong maximum principle for second order nonlinear parabolic integro-differential equations. Adv. Differ. Equ. **17**, 635–671 (2012)

[16] Cutrì, A., Leoni, F.: On the Liouville property for fully nonlinear equations. Ann. Inst. H. Poincaré Anal. Non Linéaire **17**(2), 219–245 (2000)

[17] Del Pezzo, L., Quaas, A., Rossi, J.: Fractional convexity. Math. Ann. (2021). https://doi.org/10.1007/s00208-021-02254-y

[18] Di Nezza, E., Palatucci, G., Valdinoci, E.: Hitchhiker's guide to the fractional Sobolev spaces. Bull. Sci. Math. **136**(5), 521–573 (2012)

[19] Evans, L.C., Gariepy, R.F.: Measure theory and fine properties of functions. Studies in Advanced Mathematics. CRC Press, Boca Raton (1992)

[20] Felmer, P., Quaas, A.: Fundamental solutions and Liouville type theorems for nonlinear integral operators. Adv. Math. **226**, 2712–2738 (2011)

[21] Harvey, R.F., Lawson, H Blaine, Jr.: Dirichlet duality and the nonlinear Dirichlet problem. Commun. Pure Appl. Math **62**(3), 396–443 (2009)

[22] Harvey, R.F., Lawson, H Blaine, Jr.: p-convexity, p-plurisubharmonicity and the Levi problem. Indiana Univ. Math. J. **62**(1), 149–169 (2013)

[23] Kassmann, M., Rang, M., Schwab, R.: Integro-differential equations with nonlinear directional dependence. Indiana Univ. Math. J. **63**(5), 1467–1498 (2014)

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