A CONSEQUENCE OF DJAIRO'S LECTURES ON THE EKELAND VARIATIONAL PRINCIPLE

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(dedicated to Djairo for his 40th birthday, twice)

1. Introduction and statement of the result

In [2] we considered some properties of the minimizing sequences for integral functionals J. Thanks to the Ekeland Lemma, the subject of the lectures given by Djairo in Bangalore (see [3]), we proved the existence of a minimizing sequence compact in $L^s(\Omega)$ or in $C^{0,\alpha}$ for functionals which do not need to have a minimum, without using the integral representation of the relaxed functional J^*.

In this paper, we improve the study done in the paper [2], under the assumption that the functional J has a minimum belonging to $L^\infty(\Omega)$. Using again Ekeland's ε-variational principle, we to prove that there exists a minimizing sequence u_n for J which uniformly converges to a minimum u.

Let us now make the precise assumptions on the functional J. Let Ω be an open, bounded subset of $\mathbb{R}^N$, $N \geq 2$, and let p be a real number, with $2 \leq p < N$. We will denote by p^* the Sobolev exponent of p, i.e., $p^* = \frac{Np}{N-p}$.

Let $j : \Omega \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function (i.e., measurable with respect to x for every $\xi \in \mathbb{R}^N$, and continuous with respect to ξ for almost every $x \in \Omega$) convex with respect to ξ, and such that

$$\alpha\,|\xi|^p \leq j(x,\xi) \leq \beta\,|\xi|^p\,, \tag{1}$$

for almost every $x \in \Omega$, for every $\xi \in \mathbb{R}^N$, where α, β are positive real numbers.

Let f in $L^m(\Omega)$, with $m \geq (p^*)'$, and let $J : W_0^{1,p}(\Omega) \to \mathbb{R}$ be defined by

$$J(v) = \int_\Omega j(x,\nabla v)\,dx - \int_\Omega f(x)\,v\,dx, \qquad v \in W_0^{1,p}(\Omega)\,.$$

Under the assumptions on f and p, J is well defined on $W_0^{1,p}(\Omega)$.

We will further assume that there exists $a(x,\xi) = j_\xi(x,\xi)$ which satisfies the classical Leray-Lions assumptions (see [8]) and the standard strong monotonicity assumption

$$[a(x,\xi) - a(x,\eta)][\xi - \eta] \geq \alpha\,|\xi - \eta|^p \qquad \forall \xi\,, \eta \in \mathbb{R}^N\,. \tag{2}$$

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Examples of functions j such that (2) holds true are $j(x,\xi) = a(x)\,|\xi|^p$, with a a measurable function such that $\alpha \leq a(x) \leq \beta$. Since the strong monotonicity condition is simpler to handle if $p \geq 2$ (the above assumption (2)), and is a little bit more involved if $1 < p < 2$, we confine ourselves to the former case.

Since J is both weakly lower semicontinuous and coercive on $W_0^{1,p}(\Omega)$, there exists a minimum u of J; we have the following results on the summability of such minima.

THEOREM 1.1. *Let u be a minimum of J on $W_0^{1,p}(\Omega)$. Then*

(i) *if $1 < m < \frac{N}{p}$, then u belongs to $L^\sigma(\Omega)$, $\sigma = \frac{(pm)^*}{p'}$ (see [1]);*
(ii) *If $m > \frac{N}{p}$, then u belongs to $L^\infty(\Omega)$ (see [9], [7]).*

Let us now recall Ekeland's ε-variational principle (see [4], [5]).

LEMMA 1.2. *Let (V,d) be a complete metric space, and let $\mathcal{F} : V \to (-\infty,+\infty]$ be a lower semicontinuous function such that $\inf_V \mathcal{F}$ is finite. Let $\varepsilon > 0$ and $u \in V$ be such that*

$$\mathcal{F}(u) \leq \inf_{v\in V} \mathcal{F}(v) + \varepsilon\,.$$

Then there exists $v \in V$ such that

(i) *$d(u,v) \leq \sqrt{\varepsilon}$;*
(ii) *$\mathcal{F}(v) \leq \mathcal{F}(u)$;*
(iii) *v minimizes the functional $\mathcal{G}(w) = \mathcal{F}(w) + \sqrt{\varepsilon}\,d(v,w)$.*

Our main result is the following.

THEOREM 1.3. *Let J be defined as above, with j satisfying (1). Let*

$$f \in L^m(\Omega) \quad m > \frac{N}{p}\,, \tag{3}$$

and let q be such that $q^* = m'$; we also suppose that J' satisfies (2). Let u be a minimum of J on $W_0^{1,p}(\Omega)$, and let $\{\bar{u}_n\}$ be any minimizing sequence for J. Then the minimizing sequence $\{u_n\}$ built after $\{\bar{u}_n\}$ using the ε-variational principle satisfies

$$\lim_{n\to+\infty} \|u_n - \bar{u}_n\|_{W_0^{1,q}(\Omega)} = 0\,, \tag{4}$$

$$\lim_{n\to\infty} \|u_n - u\|_{W_0^{1,p}(\Omega)} = 0\,, \tag{5}$$

and

$$\lim_{n\to\infty} \|u_n - u\|_{L^\infty(\Omega)} = 0\,. \tag{6}$$

The plan of the paper is as follows: we will prove Theorem 1.3 in Section 2, and in Section 3 we will show that adding a lower order term to J will allow us to prove the same result under the assumption that f belongs to $L^2(\Omega)$, and not to the possibly larger space $L^m(\Omega)$, $m > \frac{N}{p}$.

2. Proof of the main result

For $k > 0$ let us define

$$T_k(s) = \max(-k, \min(k, s))\,, \qquad G_k(s) = s - T_k(s)\,.$$

Before proving Theorem 1.3, let us note that since we know (see Theorem 1.1) that any minimum u belongs to $L^\infty(\Omega)$, there exists M such that $|u| \le M$. Since the sequence $\{u_n\}$, with $u_n = T_M(\bar{u}_n)$, satisfies

$$\int_\Omega j(x, \nabla T_M(\bar{u}_n))\,dx - \int_\Omega f(x)\,T_M(\bar{u}_n)\,dx \le \inf_{v \in W_0^{1,p}(\Omega)} J(v) + \varepsilon_n + \int_\Omega f(x)\,G_M(\bar{u}_n)\,dx\,,$$

and since

$$\lim_{n\to+\infty} \int_\Omega f(x)\,G_M(\bar{u}_n)\,dx = \int_\Omega f(x)\,G_M(u)\,dx = 0\,,$$

we have that

$$\int_\Omega j(x, \nabla u_n)\,dx - \int_\Omega f(x)\,u_n\,dx \le \inf_{v \in W_0^{1,p}(\Omega)} J(v) + \bar{\varepsilon}_n\,.$$

That is, the sequence $\{u_n\}$ is a minimizing sequence for J, and it is bounded in $L^\infty(\Omega)$.

Theorem 1.3 says more than that: thanks to the ε-variational principle, it is possible to build a minimizing sequence not only bounded in $L^\infty(\Omega)$ but also strongly convergent to u in the same space.

Proof of Theorem 1.3. Note that if q is as in the statement, the assumption $m > \frac{N}{p}$ implies that

$$1 < q < \frac{N}{N-p+1} < p\,. \tag{7}$$

Let ε_n be a sequence of positive real numbers, converging to zero, and let $\bar{u}_n$ be such that, for every $n \in \mathbb{N}$,

$$J(\bar{u}_n) \le \inf_{v \in W_0^{1,p}(\Omega)} J(v) + \varepsilon_n\,.$$

Let us now consider the complete metric space $W_0^{1,q}(\Omega)$, endowed with the distance

$$d_n(w, v) = \frac{1}{\sqrt{\varepsilon_n}} \left[\int_\Omega |\nabla w - \nabla v|^q\,dx\right]^{\frac{1}{q}}.$$

Thanks to Fatou Lemma, to the fact that $j(x, \xi) \ge 0$, and to the fact that f belongs to $W^{-1,q'}(\Omega)$ being $q^* = m'$, we have that J is strongly lower semicontinuous on $W_0^{1,q}(\Omega)$.

Thus, in view of Lemma 1.2, there exists a sequence $\{u_n\}$ in $W_0^{1,q}(\Omega)$ such that

$$\left[\int_\Omega |\nabla u_n - \nabla \bar{u}_n|^q\,dx\right]^{\frac{1}{q}} \le \sqrt{\varepsilon_n}\,,$$

which proves (4), and

$$J(u_n) \leq J(\bar{u}_n) \leq \inf_{v \in W_0^{1,p}(\Omega)} J(v) + \varepsilon_n \,, \tag{8}$$

$$J(u_n) \leq J(w) + \sqrt{\varepsilon_n} \left[\int_\Omega |\nabla u_n - \nabla w|^q \, dx \right]^{\frac{1}{q}} , \qquad \forall w \in W_0^{1,q}(\Omega) \,. \tag{9}$$

Using the growth properties of J we have that u_n is bounded in $W_0^{1,p}(\Omega)$; indeed, by (1), we have

$$\alpha \int_\Omega |\nabla u_n|^p \, dx \leq \int_\Omega j(x, \nabla u_n) \, dx \leq \int_\Omega f(x) \, u_n \, dx + \inf_{v \in W_0^{1,p}(\Omega)} J(v) + \varepsilon_n \,,$$

which implies that the sequence $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$ since f belongs to $W^{-1,p'}(\Omega)$. Thus, up to subsequences, still denoted by $\{u_n\}$, there exists a function u in $W_0^{1,p}(\Omega)$ such that

$$u_n \to u \qquad \text{weakly in } W_0^{1,p}(\Omega) \text{ and almost everywhere in } \Omega. \tag{10}$$

By the weak lower semicontinuity of J on $W_0^{1,p}(\Omega)$, and by (8), u is a minimum of J on this space.

Moreover, choosing $w = u_n - t\,\psi$ in (9), where t is a positive real number and ψ is a function in $W_0^{1,p}(\Omega)$, we obtain

$$J(u_n - t\,\psi) - J(u_n) + \sqrt{\varepsilon_n}\, t \left[\int_\Omega |\nabla \psi|^q \, dx \right]^{\frac{1}{q}} \geq 0 \,.$$

Dividing by t, and letting t tend to zero, we get, since J is differentiable,

$$-\langle J'(u_n), \psi \rangle + \sqrt{\varepsilon_n} \left[\int_\Omega |\nabla \psi|^q \, dx \right]^{\frac{1}{q}} \geq 0 \,,$$

so that

$$\langle J'(u_n), \psi \rangle \leq \sqrt{\varepsilon_n} \left[\int_\Omega |\nabla \psi|^q \, dx \right]^{\frac{1}{q}} . \tag{11}$$

Recalling that $J'(u) = 0$ since u is a minimum, we have

$$\langle J'(u_n) - J'(u), \psi \rangle \leq \sqrt{\varepsilon_n} \left[\int_\Omega |\nabla \psi|^q \, dx \right]^{\frac{1}{q}} ,$$

for every ψ in $W_0^{1,p}(\Omega)$. Observe that

$$\langle J'(u_n), \psi \rangle = \int_\Omega a(x, \nabla u_n) \nabla \psi \, dx - \int_\Omega f(x) \psi \, dx. \tag{12}$$

Choosing $\psi = u_n - u$, it is easy to prove (5) using (2). In order to prove (6), let $k > 0$, define $A_{k,n} = \{|u_n - u| \geq k\}$, and choose $\psi = G_k(u_n - u)$; we obtain, by (2), and by Hölder inequality,

$$
\begin{aligned}
\alpha \int_\Omega |\nabla G_k(u_n - u)|^p \, dx &\leq \sqrt{\varepsilon_n} \left[\int_\Omega |\nabla G_k(u_n - u)|^q \, dx \right]^{\frac{1}{q}} \\
&\leq \sqrt{\varepsilon_n} \left[\left(\int_\Omega |\nabla G_k(u_n - u)|^p \, dx \right)^{\frac{q}{p}} \operatorname{meas}(A_{k,n})^{1-\frac{q}{p}} \right]^{\frac{1}{q}} \\
&= \sqrt{\varepsilon_n} \left[\int_\Omega |\nabla G_k(u_n - u)|^p \, dx \right]^{\frac{1}{p}} \operatorname{meas}(A_{k,n})^{\frac{1}{q}-\frac{1}{p}} ,
\end{aligned}
$$

which in turn yields

$$
\alpha \left(\int_\Omega |\nabla G_k(u_n - u)|^p \, dx \right)^{\frac{1}{p'}} \leq \sqrt{\varepsilon_n} \operatorname{meas}(A_{k,n})^{\frac{1}{q}-\frac{1}{p}} .
$$

Using Sobolev inequality, and choosing $h > k$ we arrive after straightforward passages, to

$$
(h-k)^p \operatorname{meas}(A_{h,n})^{\frac{p}{p^*}} \leq C_1 \, \varepsilon_n^{\frac{p'}{2}} \operatorname{meas}(A_{k,n})^{(\frac{1}{q}-\frac{1}{p})p'} ,
$$

which implies

$$
\operatorname{meas}(A_{h,n}) \leq C_2 \, \frac{\varepsilon_n^{\frac{p^*}{p}\frac{p'}{2}}}{(h-k)^{p^*}} \operatorname{meas}(A_{k,n})^{\frac{p^*}{p}(\frac{1}{q}-\frac{1}{p})p'} .
$$

Note that (7) implies that

$$
\frac{p^*}{p} \left(\frac{1}{q} - \frac{1}{p} \right) p' > 1 ,
$$

so that, by Lemme 4.1 of [9],

$$
\|u_n - u\|_{L^\infty(\Omega)} \leq C_3 \, \varepsilon_n^A ,
$$

for some positive constant A depending only on p and N. Recalling that ε_n converges to zero, we have the result. □

Remark 2.1. Assumption (3) was used only to ensure that the functional J is lower semicontinuos on $W_0^{1,q}(\Omega)$. Since the terms with f "cancel out" when calculating $J'(u_n) - J'(u)$, the summability of f is not necessary to prove that $u_n - u$ belongs to $L^\infty(\Omega)$.

Remark 2.2. We remark that from (11), choosing ψ and $-\psi$ it follows that u_n satisfies

$$
-\sqrt{\varepsilon_n} \left[\int_\Omega |\nabla \psi|^q \, dx \right]^{\frac{1}{q}} \leq \langle J'(u_n), \psi \rangle \leq \sqrt{\varepsilon_n} \left[\int_\Omega |\nabla \psi|^q \, dx \right]^{\frac{1}{q}} . \tag{13}
$$

Thus,

$$
\langle J'(u_n) - J'(u_m), \psi \rangle \leq \sqrt{\varepsilon_n} \left[\int_\Omega |\nabla \psi|^q \, dx \right]^{\frac{1}{q}} + \sqrt{\varepsilon_m} \left[\int_\Omega |\nabla \psi|^q \, dx \right]^{\frac{1}{q}}
$$

The choice of $\psi = G_k(u_n - u_m)$, and the same steps in the proof of Theorem 1.3, yield

$$\|u_n - u_m\|_{L^\infty(\Omega)} \le c\,(\varepsilon_n + \varepsilon_m)^A\,. \tag{14}$$

Note that we cannot say that $\{u_n\}$ is a Cauchy sequence in $L^\infty(\Omega)$, since the functions u_n may not belong to $L^\infty(\Omega)$, even if the difference of two of them is bounded. However, passing to the limit in (14) as m tends to infinity, the almost everywhere convergence (10) implies that

$$\|u_n - u\|_{L^\infty(\Omega)} \le c\,\varepsilon_n^A\,, \tag{15}$$

which implies that the functions u_n belong to $L^\infty(\Omega)$, since $u \in L^\infty(\Omega)$, and that the sequence $\{u_n\}$ uniformly converges to u. In other words, Theorem 1.3 can also be proved starting from (14).

3. The impact of a lower order term

Let the integral functional J be defined now by

$$J(v) = \int_\Omega j(x, \nabla v)\,dx + \frac{1}{2}\int_\Omega [f(x) - v]^2\,dx, \quad v \in W_0^{1,p}(\Omega) \cap L^2(\Omega)\,, \tag{16}$$

where

$$f \in L^2(\Omega). \tag{17}$$

Note that $W_0^{1,p}(\Omega) \cap L^2(\Omega) = W_0^{1,p}(\Omega)$ if $p \ge \frac{2N}{N+2}$. Since both $j(x, \nabla v)$ and $[f(x) - v]^2$ are positive, J is lower semicontinuous on $W_0^{1,q}(\Omega)$, for every $q \ge 1$.

Note that any minimum u of J does not belong to $L^\infty(\Omega)$, if $2 < \frac{N}{p}$, i.e., if $p < \frac{N}{2}$.

Then the minimizing sequence $\{u_n\}$ built after $\{\bar u_n\}$ using the ε-variational principle satisfies (11) with $q = 1$:

$$\langle J'(u_n), \psi\rangle \le \sqrt{\varepsilon_n}\int_\Omega |\nabla \psi|\,dx, \qquad \forall\, w \in W_0^{1,1}(\Omega)\,.$$

Observe that now

$$\langle J'(u_n), \psi\rangle = \int_\Omega a(x, \nabla u_n)\nabla\psi\,dx + \int_\Omega u_n(x)\psi\,dx - \int_\Omega f(x)\psi\,dx. \tag{18}$$

We can follow the same steps as in Remark 2.2 in order to prove inequalities (14) and (15), but now the assumption on f does not imply that $u \in L^\infty(\Omega)$. Therefore, in (15) the function u_n and its limit u may not belong to $L^\infty(\Omega)$; nevertheless their difference belongs to $L^\infty(\Omega)$ and tends to zero in that space.

References

[1] L. Boccardo, D. Giachetti: A nonlinear interpolation result with application to the summability of minima of some integral functionals; Discrete and Continuous Dynamical Systems, Series B, **11** (2009), 31–42.

[2] L. Boccardo, V. Ferone, N. Fusco, L. Orsina: Regularity of minimizing sequences for functionals of the calculus of variations via the Ekeland principle; Differential Integral Equations **12** (1999), 119–135.

[3] D.G. de Figueiredo: Lectures on the Ekeland variational principle with applications and detours; Tata Institute of Fundamental Research Lectures on Mathematics and Physics, **81**, Springer-Verlag, Berlin, 1989.

[4] I. Ekeland: Nonconvex minimization problems; Bull. Am. Math. Soc., **1** (1979), 443–474.

[5] I. Ekeland: On the variational principle; J. Math. Anal. Appl, **47** (1974) 324–353.

[6] I. Ekeland, R. Temam: Analyse convexe et problèmes variationnels, Dunod & Gauthier-Villars, Paris, 1974.

[7] O.A. Ladyzenskaya, N.N. Uralceva: Equations aux dérivées partielles de type elliptique; Dunod, Paris, 1968.

[8] J. Leray, J. L. Lions: Quelques résultats de Visik sur les problèmes aux limites non linéaires, Dunod & Gauthier-Villars, Paris, 1969.

[9] G. Stampacchia: Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus; Ann. Inst. Fourier (Grenoble), **15** (1965), 189–258.