Commun. Math. Phys. 387, 397–426 (2021)
Digital Object Identifier (DOI) https://doi.org/10.1007/s00220-021-04170-3

Communications in
**Mathematical
Physics**

Loop Groups and QNEC

Lorenzo Panebianco

Dipartimento di Matematica, Università di Roma "La Sapienza", Piazzale Aldo Moro 5, 00185 Rome, Italy.
E-mail: panebianco@mat.uniroma1.it

Received: 23 November 2020 / Accepted: 5 July 2021
Published online: 23 July 2021 –

Abstract: We construct and study solitonic representations of the conformal net associated to some vacuum Positive Energy Representation (PER) of a loop group LG. For the corresponding solitonic states, we prove the Quantum Null Energy Condition (QNEC) and the Bekenstein Bound. As an intermediate result, we show that a Positive Energy Representation of a loop group LG can be extended to a PER of $H^s(S^1, G)$ for $s > 3/2$, where G is any compact, simple and simply connected Lie group. We also show the existence of the exponential map of the semidirect product $LG \rtimes R$, with R a one-parameter subgroup of $\mathrm{Diff}_+(S^1)$, and we compute the adjoint action of $H^{s+1}(S^1, G)$ on the stress energy tensor.

1. Introduction

Recently, much attention has been focused on quantum information aspects of Quantum Field Theory, which naturally takes place in the framework of quantum black holes thermodynamics [20,21]. However, more unexpected and interesting connections between the relative entropy and the stress energy tensor have arisen, and in particular it is of interest to provide and prove an axiomatic formulation of the Quantum Null Energy Condition (QNEC). In this work we prove the QNEC for some states of interest on loop group models. We refer to [26] for similar results on the Virasoro nets.

Classically, the Null Energy Condition (NEC) is a constraint on the stress energy tensor which states that $T_{ab}k^a k^b \geq 0$, where k^a is a null vector field. This constraint is motivated by the positivity of the energy and it is a necessary condition for the field k^a to have some physical meaning. However, quantum fields can violate all local energy conditions, including the NEC. At any point the energy density $\langle T_{kk}\rangle$ can be made negative, with magnitude as large as we wish, by an appropriate choice of a quantum system [13]. In the study of relativistic QFT coupled to gravity, Bousso, Fisher, Leichenauer and Wall [3] establish a new and surprising link between quantum information and the stress energy tensor. In this work, a Quantum Null Energy Condition (QNEC) is defined as a null energy lower bound which is expected to be satisfied by most reasonable quantum fields. This lower bound is defined by using the relative entropy of Araki, an object typically used in quantum information theoretical contexts.

The first non commutative entropy notion, von Neumann's quantum entropy, was originally designed as a Quantum Mechanics version of Shannon's entropy: if a normal state ψ is given by a density matrix ρ_ψ, then the von Neumann entropy is defined by

$$S(\psi) = -\mathrm{tr}\, \rho_\psi \log \rho_\psi \,.$$

However, in Quantum Field Theory local von Neumann algebras are typically factors of type III_1 [19], no trace or density matrix exists and the von Neumann entropy is undefined. Nontheless, the Tomita–Takesaki modular theory applies and one may consider the relative entropy of Araki [25]

$$S(\varphi\|\psi) = -(\xi|\log \Delta_{\eta,\xi}\xi)$$

between the normal states φ and ψ. The relative entropy generalizes the classical Kullback-Leibler divergence and measures how ψ deviates from φ. From an information theoretical viewpoint, $S(\varphi\|\psi)$ is the mean value in the state φ of the difference between the information carried by the state ψ and the state φ.

Let now $(\mathcal{A}, U, \Omega)$ be a local QFT on the Minkowski space $\mathbb{R}^{n+1}$ with vacuum state ω and stress-energy tensor T. Denote by $\mathfrak{A} = \overline{\bigcup_{\mathcal{O}} \mathcal{A}(\mathcal{O})}$ the C^*-algebra of quasi-local observables. Consider one of the two null directions u tangent to the standard Rindler wedge W. By Poincaré-covariance, we can associate a wedge W^u to any pair of a null direction u and a point x of the Minkowski space. Let $\mathcal{M}_t^u = \mathcal{A}(W_t^u)$ be the local algebra associated to $W_t^u = W^u + tu$. Given a locally normal state ψ of $\mathfrak{A}$ represented on $\mathcal{A}(W^u)$ by some vector η, consider the relative entropy $S(t) = S_{\mathcal{M}_t^u}(\psi\|\omega)$ and the averaged stress-energy density tensor $\langle T_{uu}(t)\rangle = (\eta|T_{uu}(x+tu)\eta)$. In natural units, we will say that the vector η satisfies the *Quantum Null Energy Condition (QNEC)* if for any u and x we have the null energy bound

$$\langle T_{uu}(t)\rangle \geq S''(t)/2\pi \tag{1}$$

for any $t \geq 0$, u and x, assuming this inequality to have some distributional or classical meaning. The wedges W_t^u can be replaced with arbitrary regions deformed in the null direction u [22], but here we will not work in such a generality. In this context, the convexity of the relative entropy often appears, and for this reason in other works the QNEC is just stated as the convexity of the mentioned relative entropy [8]. This convexity property has been proved in a model independent setting for a very wide class of states.

Theorem 1. [7] *Let $\mathcal{N} \subseteq \mathcal{M}$ be a $\pm$hsm inclusion with standard vector Ω giving the state ω. Denote by P the positive generator of translations and by $(\mathcal{M}_t)_{t\in\mathbb{R}}$ the associated flow of von Neumann algebras. If $\psi(x) = (\eta|x\eta)$ is a vector state with finite null energy, namely such that*

$$P_\eta = (\eta|P\eta) < +\infty \,, \tag{2}$$

then the relative entropy $S(t) = S_{\mathcal{M}_t}(\psi\|\omega)$ is convex. Furthermore, if $S(t_0)$ is finite then we have

$$-S'(t) = 2\pi \inf_{w'\in C_t'} P_{w'\eta}\,, \quad t \geq t_0 \;\; a.e.\,, \tag{3}$$

where C'_t is the set of all the isometries w' in $\mathcal{M}'_t$ such that the complement relative entropy $\bar{S}_{w'}(t) = S_{\mathcal{M}'_t}(\psi_{w'}\|\omega)$ is finite, with $\psi_{w'}(x) = (w'\eta|xw'\eta)$. The identity (3) is satisfied at each point such that $S'(t)$ exists and on such points it can be computed by

$$-S'(t) = 2\pi \inf_s P_s(t) = \pi \widehat{P}(t)\,. \tag{4}$$

In the above notation, we have

$$P_s(t) = P_{u'_s(t)\eta}\,, \quad u'_s(t) = (D\omega\colon D\psi\,;\mathcal{M}'_t)_s\,, \tag{5}$$

and $\widehat{P}(t) = P_{\widehat{\eta}_t}$, where $\widehat{\eta}_t$ is the unique vector in the natural cone of $\mathcal{M}_t$ representing the state ψ.

Actually, the provided proof refers to -hsm inclusions, but the +hsm case can be similarly proved. It is also shown that the null energies (5) are finite and that the infimum (4) is obtained as $s \to \pm\infty$ if the inclusion is $\mp$hsm. Of course, in principle such a characterization can be used for computational purposes. However, determining the Connes cocycle (5) for each deformation parameter t is a not trivial issue. For this reason, the proof of the QNEC (1) on loop group models led us to the study of Sobolev extensions of Positive Energy Representations of loop groups. We describe our work a bit more in detail.

The first result is the construction of solitonic representations σ_γ of the conformal net associated to a vacuum PER π of a loop group LG. In general, these solitons are induced by a path γ in $C^\infty([-\pi,\pi], G)$. If the path γ satisfies some periodicity conditions on its derivatives, then it can be extended to what we here call a *discontinuous loop*. Discontinuous loops are defined as elements of

$$L_hG = \left\{\zeta \in C^\infty(\mathbb{R}, G)\colon \zeta(x)^{-1}\zeta(x+2\pi) = h\right\},$$

where h is a generic element of G. If the discontinuity h of ζ is in $Z(G)$, then it is already known that the obtained soliton σ_ζ extends to a DHR representation which corresponds to a PER $\zeta_*\pi$ of same level as π [29]. What we show here is that this condition is also necessary: the soliton σ_ζ extends to a DHR representation if and only if the discontinuity h is central. The proof follows by a contradiction argument, since a locally normal DHR representation is automatically Rot-covariant [10].

Consider now the conformal net $(\mathcal{A}, U, \Omega)$ associated to some vacuum PER π of a loop group LG. In the real line picture, the von Neumann algebras associated to half-lines define a hsm inclusion. Since on these local algebras the solitonic states ω_γ of above have a representing vector satisfying (2), the convexity of the relative entropy is satisfied by Theorem 17. We then use some intermediate results used in the proof of Theorem 1 to explicitly compute the relative entropy and we show the QNEC (1) to be satisfied with an equality. Furthermore, the obtained formula allow us to prove the Bekenstein Bound in a very simple way. The purpose of doing the mentioned proof led us to the second main result of this work, that is the extension of a PER of a loop group LG to a PER of the Sobolev loop group $H^s(S^1, G)$ for $s > 3/2$. Such a Sobolev extension allows us to compute the adjoint action of $H^{s+1}(S^1, G)$ on the stress energy tensor. Similar issues involving the smearing with non-smooth functions in conformal nets contexts have been treated in [5], [23] and [31].

Unexpectedly, it turns out that our extension result follows by a more general theorem proved in [24], which implies that a Positive Energy Representation of a loop group LG can be extended to a PER of $H^1(S^1, G)$ by using holomorphic induction methods.

However, even if not completely general, the proof presented here is mathematically simpler and original. Without knowing the results of [24] we were not able to prove the QNEC (1) by directly using the expression (5), and for this reason we used an approach more similar to that ones of [17] and [26]. Nevertheless, by using explicit constructions of [30], in the case $G = SU(n)$ we are able to show that the Sobolev extension holds up to $H^s(S^1, SU(n))$ for $s > 1/2$. This technical improvement allows us to show that the identity (4) is indeed verified. The exhibited proof is not just an application of Theorem 1 or of [24] and it is thought in such a way to make this work as self contained as possible.

2. Mathematical Background

We mainly follow [29]. Let G be a compact, simple and simply connected Lie group. A *Positive Energy Representation (PER)* of the loop group $LG = C^\infty(S^1, G)$ on a separable Hilbert space $\mathcal{H}$ is a projective strongly continuous unitary representation π of $LG \rtimes \mathbb{T}$ with a commutative diagram

$$\begin{array}{ccc} LG \rtimes \mathbb{T} & \xrightarrow{\pi} & PU(\mathcal{H}) \\ \uparrow & & \uparrow \\ \mathbb{T} & \xrightarrow{U} & U(\mathcal{H}) \end{array}$$

where the torus $\mathbb{T} \cong \mathrm{Rot}$ acts on LG by rotations $R_\theta.\gamma(\phi) = \gamma(\phi - \theta)$ and U is a strongly continuous unitary representation inducing an isotypical decomposition $\mathcal{H} = \bigoplus_{n \geq n_0} \mathcal{H}(n)$ for some integer n_0. Without loss of generality, we can suppose that $n_0 = 0$ and that $\mathcal{H}(0)$ is not zero-dimensional. A PER is said to be of *finite type* if $\dim \mathcal{H}(n) < +\infty$ for every n. Irreducible PERs are of finite type.

We denote by $\mathfrak{g}_0$ the Lie algebra of G and by $\mathfrak{g}$ the complexification of $\mathfrak{g}_0$. Recall that $\mathfrak{g}_0$ is a compact Lie algebra, that is its Killing form is negative definite. In particular, there is an antilinear involution $X \mapsto X^*$ of $\mathfrak{g}$ such that

$$\mathfrak{g}_0 = \{X \in \mathfrak{g} \colon X^* = -X\}\,.$$

Let $X(n)$ be the map $\theta \mapsto Xe^{in\theta}$ for X in $\mathfrak{g}$ and n integer. Then $[X(n), Y(m)] = [X, Y](n+m)$, showing that the space spanned by these elements, which we will denote by $L^{\mathrm{pol}}\mathfrak{g}$, forms a Lie algebra. On $L^{\mathrm{pol}}\mathfrak{g}$ we can define an involution by $X(n)^* = X^*(-n)$. Moreover, if $\mathcal{H}^{\mathrm{fin}}$ is the subspace of finite energy vectors, namely the algebraic sum of the subspaces $\mathcal{H}(n)$, then we can define a projective representation π of $L^{\mathrm{pol}}\mathfrak{g}$ on $\mathcal{H}^{\mathrm{fin}}$ in such a way to verify the commutation relations ([29], Theorem 1.2.1.)

$$[\pi(X), \pi(Y)] = \pi([X, Y]) + i\ell B(X, Y)\,, \qquad B(X, Y) = \int_0^{2\pi} \langle X, \dot{Y} \rangle \frac{d\theta}{2\pi}\,.$$

We point out that the existence of such a representation of $L^{\mathrm{pol}}\mathfrak{g}$ is not a trivial issue, since these commutation relations do not uniquely determine the projective representation of $L^{\mathrm{pol}}\mathfrak{g}$, and also the representation of LG cannot be differentiated in a straightforward way as in finite dimensional cases. If d is the generator of rotations, namely $U(R_\theta) = e^{i\theta d}$, then we have that $[d, \pi(X)] = i\pi(\dot{X})$ where $\dot{X}(\theta) = \frac{d}{d\theta}X(\theta)$. The above operators are all closable and we also have the formal adjunction property $\pi(X)^* = \pi(X^*)$ on $\mathcal{H}^{\mathrm{fin}}$. Furthermore, the projective representation π of $L^{\mathrm{pol}}\mathfrak{g}$ on $\mathcal{H}^{\mathrm{fin}}$ can be lifted to a projective representation π of $L\mathfrak{g} = C^\infty(S^1, \mathfrak{g})$ on $\mathcal{H}^\infty$ in such a way to verify all the previous relations, where $\mathcal{H}^\infty$ is the Fréchet space of smooth vectors for Rot. We recall that by definition $\mathcal{H}^\infty = \bigcap_s \mathcal{H}^s$, where $s \in \mathbb{R}$ and $\mathcal{H}^s$ is the *scale space*, that is the completion of $\mathcal{H}^{\text{fin}}$ with respect to the Sobolev norm $\|\xi\|_s = \|(1+d)^s\xi\|$. Notice that the projective representation π of $L\mathfrak{g}$ is actually a representation if restricted on $\mathfrak{g}$, since the projective representation of G lifts to a unitary representation. Also, the subspaces $\mathcal{H}(n)$ are G-invariant. The adjoint action of LG on the mentioned operators is given by [29]

$$\begin{aligned} \pi(\gamma)\pi(X)\pi(\gamma)^* &= \pi(\gamma X\gamma^{-1}) + ic(\gamma, X)\,, \\ \pi(\gamma)d\pi(\gamma)^* &= d - i\pi(\dot\gamma\gamma^{-1}) + c(\gamma, d)\,, \end{aligned} \tag{6}$$

where the real constants $c(\gamma, X)$ and $c(\gamma, d)$ are explicitly given by

$$c(\gamma, X) = -\ell\int_0^{2\pi}\langle\gamma^{-1}\dot\gamma, \dot X\rangle\frac{d\theta}{2\pi}\,, \qquad c(\gamma, d) = -\frac{\ell}{2}\int_0^{2\pi}\langle\gamma^{-1}\dot\gamma, \gamma^{-1}\dot\gamma\rangle\frac{d\theta}{2\pi}\,.$$

Here $\langle\cdot,\cdot\rangle$ denotes the *basic inner product*, namely the Killing form normalized on the highest root θ in such a way that $\langle\theta,\theta\rangle = 2$. The elements $\gamma^{-1}\dot\gamma$ and $\dot\gamma\gamma^{-1}$ of $L\mathfrak{g}$ are the *left logarithmic derivative* and the *right logarithmic derivative* of γ, respectively defined by

$$\gamma^{-1}\dot\gamma(t) = \frac{d}{dh}\bigg|_{h=0}\gamma^{-1}(t)\gamma(t+h) \qquad \text{and} \qquad \dot\gamma\gamma^{-1}(t) = \frac{d}{dh}\bigg|_{h=0}\gamma(t+h)\gamma^{-1}(t)\,.$$

We will use the following notation:

$$x = \pi(X)\,, \quad x(n) = \pi(X(n))\,, \quad \langle x, y\rangle = \langle X, Y\rangle\,.$$

We can define a representation of the Virasoro algebra Vir

$$[L_n, L_m] = (n-m)L_{n+m} + \delta_{n+m,0}\frac{n(n^2-1)}{12}c\,, \tag{7}$$

by Sugawara construction, that is such a representation is given by defining on $\mathcal{H}^{\text{fin}}$ the operator

$$L_n = \frac{1}{2(\ell+g)}\sum_m :x_i(-m)x^i(m+n): \ , \tag{8}$$

where we used the Einstein convention on summations and the *normal ordering* notation, namely the symbol $:x(n)y(m):$ stands for $x(n)y(m)$ if $n \leq m$ and for $y(m)x(n)$ if $n > m$. The elements $\{x_i\}$ and $\{x^i\}$ appearing in (8) can be arbitrary dual basis with respect to the basic inner product, namely $\langle x^i, x_j\rangle = \delta_{ij}$, and g is the *dual Coxeter number*, that is

$$g = 1 + \sum a_i^\vee\,, \qquad \theta = \sum a_i^\vee\alpha_i^\vee\,,$$

where $\alpha_i^\vee$ are the simple coroots and $a_i^\vee$ are strictly positive. By the assumption $\langle\theta,\theta\rangle = 2$ it can be shown that the dual Coxeter number is half the Casimir of the adjoint representation, namely we have $[X_i, [X^i, Y]] = 2gY$ for Y in $\mathfrak{g}$. Notice that if X_i are in $\mathfrak{g}_0$ and such that $-\langle X_i, X_j\rangle = \delta_{ij}$ then $X^i = -X_i$. The constant c uniquely determined by (7) is called the *central charge* of the representation. If the PER is irreducible then we have $L_0 = d + h$ for some rational number h, where h is therefore the lowest eigenvalue of L_0 and it is called the *trace anomaly*. In any irreducible PER, the central charge and the trace anomaly are given by [15]

$$c = \frac{\ell \dim \mathfrak{g}}{\ell + g}, \qquad h = \frac{C_\lambda}{2(\ell + g)}, \tag{9}$$

where C_λ is the Casimir associated to the basic inner product $\langle \cdot, \cdot \rangle$ and to the null energy space $\mathcal{H}(0) = \mathcal{H}_\lambda$, which is the irreducible highest weight representation of $\mathfrak{g}$ associated to some dominant integral weight λ satisfying

$$\langle \lambda, \theta \rangle \leq \ell . \tag{10}$$

The set of dominant integral weights λ satisfying condition (10) is called the *level ℓ alcove*. We will say that π is a *vacuum positive energy representation*, or simply a *vacuum representation*, if $\mathcal{H}(0)$ is one-dimensional. If $\mathcal{H}(0) = \mathbb{C}\Omega$ with $(\Omega|\Omega) = 1$, then the state ω associated to Ω is called the *vacuum state*. Notice that π is a vacuum representation if and only if irreducible and with $h = 0$. More in general, if $\mathcal{H}(0) = \mathcal{H}_\lambda$ then the trace anomaly can be computed by taking in account that

$$C_\lambda = \langle \lambda, \lambda + 2\rho \rangle , \quad g = 1 + \langle \rho, \theta \rangle ,$$

where ρ is the *Weyl vector*, that is the sum of all the fundamental weights. Equivalently, the Weyl vector can be defined as half the sum of all the positive roots.

Consider now $\mathrm{Diff}_+(S^1)$, the Fréchet Lie group of the orientation preserving diffeomorphisms of the circle. The natural action of $\mathrm{Diff}_+(S^1)$ on LG is smooth. Furthermore, every PER π of LG is $\mathrm{Diff}_+(S^1)^\sim$-covariant, namely there is a projective unitary representation U of the universal covering $\mathrm{Diff}_+(S^1)^\sim$ such that $U(\tilde{\rho})\pi(\gamma)U(\tilde{\rho})^* = \pi(\rho.\gamma)$ [14,15]. Consider now $\mathcal{H}^{0,\mathrm{fin}}$, the algebraic direct sum of the eigenspaces of L_0. On the infinitesimal level, in general the space $\mathcal{H}^{0,\mathrm{fin}}$ is a direct sum of infinitely many unitary irreducible representations $V(c, h_i)$ of the Virasoro algebra. Such a representation integrates to a unitary projective representation U of $\mathrm{Diff}_+(S^1)^\sim$, and if the appearing highest weights h_i differ only by integers then U reduces to a unitary projective representation of $\mathrm{Diff}_+(S^1)$ [11,14]. Now we briefly study the irreducible unitary representations $V(c, h)$ of the Virasoro algebra appearing from an irreducible PER of level ℓ of LG. If $\ell = 0$ then $\lambda = 0$, and by $c = h = 0$ we have the trivial representation of Vir. If $\ell \geq 1$, then $V(c, h)$ belongs to the *continuous series*, namely we have $h \geq 0$ and $c \geq 1$. The estimate on the central charge follows by the inequality $g + 1 \leq \dim \mathfrak{g}$, which can be noticed by studying the following table [18]:

Dynkin diagram	A_n	B_n	C_n	D_n	E_6	E_7	E_8	F_4	G_2
Complex simple Lie algebra	$\mathfrak{sl}_{n+1}$	$\mathfrak{so}_{2n+1}$	$\mathfrak{sp}_{2n}$	$\mathfrak{so}_{2n}$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$	$\mathfrak{f}_4$	$\mathfrak{g}_2$
Complex dimension	$n^2 + 2n$	$2n^2 + n$	$2n^2 + n$	$2n^2 - n$	78	133	248	52	14
Dual Coxeter number	$n + 1$	$2n - 1$	$n + 1$	$2n - 2$	12	18	30	9	4

Lemma 2. [18] $[L_n, x(k)] = -kx(n+k)$ *on* $\mathcal{H}^{0,\mathrm{fin}}$.

As a corollary of Lemma 2, the representation of $\mathrm{Vir} = \mathbb{C} \cdot c \oplus \partial$, with ∂ the Witt algebra, extends to a representation of the semidirect product $\mathfrak{g}[t, t^{-1}] \rtimes \mathrm{Vir} \cong \tilde{\mathfrak{g}} \rtimes \partial$, with $\tilde{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C} \cdot c$. Indeed, if we set $L_n = \pi(\ell_n)$, where $\ell_n(\theta) = e^{in\theta} \frac{d}{d\theta}$, then we can define the stress energy tensor $\pi(h) = \sum_n \hat{h}_n L_n$ for any polynomial vector field h on the circle, namely a vector field which is a finite linear combination of the fields

ℓ_n. Therefore, by Lemma 2 we have $[\pi(h), \pi(X)] = \pi(h.X)$ on $\mathcal{H}^{0,\text{fin}}$ for every X in $L^{\text{pol}}\mathfrak{g}$, where $h.X(\theta) = h(\theta)\frac{d}{d\theta}X(\theta)$.

The Lie algebra $L^{\text{pol}}\mathfrak{g}$ can be completed to a Banach Lie algebra $L\mathfrak{g}_t$ for any $t \geq 0$. Indeed, given $X = \sum_k X_k(k)$ in $L^{\text{pol}}\mathfrak{g}$, we can define $L\mathfrak{g}_t$ as the completion of $L^{\text{pol}}\mathfrak{g}$ with respect to the norm

$$|X|_t = \sum_k (1+|k|)^t \|X_k\| .$$

We have norm continuous embeddings with dense range $C^{\lceil t\rceil+1}(S^1, \mathfrak{g}) \hookrightarrow L\mathfrak{g}_t \hookrightarrow C^{\lfloor t\rfloor}(S^1, \mathfrak{g})$, and for any $t \geq n$ we have $\|X^{(n)}\|_\infty \leq |X|_t$. Notice that, in general, we can similarly define the Banach Lie algebra $L\mathfrak{g}_{s,p}$ as the completion of $L^{\text{pol}}\mathfrak{g}$ with respect to the norm

$$|X|_{s,p} = \left(\sum_k (1+|k|)^{sp}\|X_k\|^p\right)^{1/p} .$$

We now set $\mathcal{S}_t = L\mathbb{C}_t$, namely the space of continuous complex functions h on S^1 satisfying

$$|h|_t = \sum_k (1+|k|)^t \|\hat{h}_k\| < +\infty .$$

Notice that we can naturally identify $\mathcal{S}_t$ with a space of Sobolev vector fields on the circle. We can define two continuous actions of $\mathcal{S}_t$ by $h.X(\theta) = h(\theta)\frac{d}{d\theta}X(\theta)$ and $hX(\theta) = h(\theta)X(\theta)$. Indeed, by noticing that $(1+|n+k|)^t \leq (1+|n|)^t(1+|k|)^t$ we have $|h.X|_{s,p} \leq |h|_s|X|_{s+1,p}$ and $|hX|_{s,p} \leq |h|_s|X|_{s,p}$.

We conclude this preliminary section with some standard notions about the stress energy tensor. Let $h(\theta)\frac{d}{d\theta}$ be a smooth vector field on the circle and define $\mathcal{V} = \bigcap_{k\geq 0}\mathcal{D}(L_0^k)$. The *stress energy tensor*

$$T(h) = \sum_n \hat{h}_n L_n ,$$

is a closable operator with $\mathcal{H}^{0,\text{fin}}$ as a dense core [5,11,17]. Moreover, by the energy bounds [16]

$$\|(1+L_0)^k L_n\xi\| \leq \sqrt{c/2}(1+|n|)^{k+3/2}\|(1+L_0)^{k+1}\xi\| ,$$

with $k \geq 0$ any natural number and ξ in $\mathcal{H}^{0,\text{fin}}$, we have that $T(h)$ is well defined and closable for every h in $\mathcal{S}_{3/2}$, since

$$\|L_0^k T(h)\xi\| \leq (c/2)^{1/2}|h|_{3/2+k}\|(1+L_0)^{k+1}\xi\|$$

for every $k \geq 0$ natural and ξ in $\mathcal{D}(L_0^{k+1})$. It follows that $\mathcal{V}$ is a dense invariant domain for $T(h)$ if h is smooth. Furthermore, we have that $T(\bar{h})^* = \overline{T(h)}$ for every h in $\mathcal{S}_{3/2}$ and therefore $T(h)$ is essentially selfadjoint if h is real valued. For further properties of the stress-energy tensor, see [14].

3. Solitonic Representations from Discontinuous Loops

In this section we follow [11] and we construct proper solitonic representations of the conformal net associated to some vacuum positive energy representation of a loop group. We begin by briefly recalling some basic definitions about conformal nets. We refer to [11,15,29] for further treatments of the topic.

Let $\mathcal{K}$ be the set of open, nonempty and non dense intervals of the circle. For I in $\mathcal{K}$, I' denotes the interior of the complement. The Möbius group Möb acts on the circle by linear fractional transformations. A *Möbius covariant net* $(\mathcal{A}, U, \Omega)$ consists of a family $\{\mathcal{A}(I)\}_{I\in\mathcal{K}}$ of von Neumann algebras acting on a separable Hilbert space $\mathcal{H}$, a strongly continuous unitary representation U of Möb and a vector Ω in $\mathcal{H}$, called the *vacuum vector*, satisfying the following properties:

(i) $\mathcal{A}(I_1) \subseteq \mathcal{A}(I_2)$ if $I_1 \subseteq I_2$ (isotony),
(ii) $\mathcal{A}(I_1) \subseteq \mathcal{A}(I_2)'$ if $I_1 \subseteq I_2'$ (locality),
(iii) $U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(g.I)$ for every g in Möb and I in $\mathcal{K}$ (Möbius covariance),
(iv) The representation U has *positive energy*, namely the generator of rotations has non-negative spectrum (positivity of the energy),
(v) Ω is cyclic for the von Neumann algebra $\bigvee_{I\in\mathcal{K}} \mathcal{A}(I)$, and up to a scalar Ω is the unique Möb-invariant vector of $\mathcal{H}$ (vacuum).

By the Howe-Moore vanishing theorem, it follows by the axioms (iv) and (v) that the vacuum vector Ω is, up to a phase, the only vector fixed by the subgroups of rotations, translations and dilations defined below (12). With these assumptions, the following properties automatically hold [11,15]:

(vi) $\mathcal{A}(I') = \mathcal{A}(I)'$ for every I in $\mathcal{K}$ (Haag duality),
(vii) $\mathcal{A}(I) \subseteq \bigvee_\alpha \mathcal{A}(I_\alpha)$ if $I \subseteq \bigcup_\alpha I_\alpha$ (additivity),
(viii) If I_+ is the upper half of the circle and Δ is the modular operator associated to $\mathcal{A}(I_+)$ and Ω, then for every t in $\mathbb{R}$ we have

$$\Delta^{it} = U(\delta_{-2\pi t}) \,, \tag{11}$$

where $\delta_t.u = e^t u$ is the one parameter group of dilations (Bisognano–Wichmann),
(ix) Each local algebra $\mathcal{A}(I)$ is a type III factor and $\bigvee_{I\in\mathcal{I}_\mathbb{R}} \mathcal{A}(I) = B(\mathcal{H})$, with $\mathcal{I}_\mathbb{R}$ the set of all the open, nonempty and non dense intervals of $S^1 \setminus \{-1\}$ (irreducibility).

Definition 3. By a *conformal net*, or also a $\mathrm{Diff}_+(S^1)$-*covariant net*, we shall mean a Möb-covariant net $(\mathcal{A}, U, \Omega)$ which satisfies the following condition:

(x) U extends to a projective unitary representation of $\mathrm{Diff}_+(S^1)$ on $\mathcal{H}$ such that

$$U(\rho)\mathcal{A}(I)U(\rho)^* = \mathcal{A}(\rho.I) \,, \quad \rho \in \mathrm{Diff}_+(S^1) \,,$$

for every I in $\mathcal{K}$. Furthermore, if $\mathrm{supp}\,\rho \subset I'$, with $\mathrm{supp}\,\rho$ the closure of the complement of the set of the points z such that $\rho(z) = z$, then we have

$$U(\rho)xU(\rho)^* = x \,, \quad x \in \mathcal{A}(I) \,,$$

A Möbius covariant net is either also conformal or not, but if it is, the extension of the of the representation to $\mathrm{Diff}_+(S^1)$ is unique [5,6]. In a conformal net, the following is automatic [23]:

(xi) if $\bar{I} \subset J$ then there is a type I factor $\mathcal{R}$ such that $\mathcal{A}(I) \subset \mathcal{R} \subset \mathcal{A}(J)$ (split property).

If the interval I does not contain -1, one can pass from the *circle picture* to the *real line picture* [11,15]. Explicitly, by the changes of variables $z = C(u)$ with $C(u) = (1+iu)/(1-iu)$ the Cayley transform, we can identify I with an interval of the real line. In the real line picture the point -1 of S^1 corresponds to ∞, with $\mathbb{R}\cup\{\infty\}$ the Alexandroff compactification of $\mathbb{R}$. With this identification, we can define the one parameters groups of rotations, dilations and translations mentioned above:

$$R_\theta.z = e^{i\theta}z\,,\ \ z \in S^1\,, \qquad \delta_t.u = e^t u\,,\ \ u \in \mathbb{R}\,, \qquad \tau_a.u = u + a\,,\ \ u \in \mathbb{R}\,. \tag{12}$$

If α is a Möbius transformation mapping the upper half of the circle I_+ onto I, then by conjugating (12) by α we can define the groups R_I, δ_I and τ_I of rotations, dilations and translations of the interval I.

Now we give some definitions about the representation theory of conformal nets. A (locally normal) *DHR (Doplicher-Haag-Roberts) representation* of a conformal net $(\mathcal{A}, U, \Omega)$ is a family $\rho = \{\rho_I\}_{I\in\mathcal{K}}$ of normal representations ρ_I of the von Neumann algebras $\mathcal{A}(I)$ on some Hilbert space $\mathcal{H}_\rho$ such that $\rho_I = \rho_J|_{\mathcal{A}(I)}$ if $I \subseteq J$. We say that two DHR representations ρ_1 and ρ_2 are *equivalent* if there is some unitary operator U from $\mathcal{H}_{\rho_1}$ to $\mathcal{H}_{\rho_2}$ such that $U\rho_{1,I}(x) = \rho_{2,I}(x)U$ for every x in $\mathcal{A}(I)$ and I in $\mathcal{K}$. The DHR representation induced by the identity is called the *vacuum representation*. A DHR representation ρ is said to be *irreducible* if $\bigvee_{I\in\mathcal{K}} \rho_I(\mathcal{A}(I)) = B(\mathcal{H}_\rho)$. If a topological group $\mathcal{G}$ acts continuously on S^1 by elements of $\mathrm{Diff}_+(S^1)$, a DHR representation ρ is said to be *$\mathcal{G}$-covariant* if there exists a strongly continuous unitary projective representation U_ρ of $\mathcal{G}$ such that

$$\mathrm{Ad}\, U_\rho(g)\cdot\rho_I(x) = \rho_{g.I}(\mathrm{Ad}\, U(\iota(g))\cdot x)\,, \quad x \in \mathcal{A}(I)\,,$$

for all g in $\mathcal{G}$ and I in $\mathcal{K}$, where $\iota\colon \mathcal{G} \to \mathrm{Diff}_+(S^1)$ is the induced homomorphism. A locally normal DHR representation is automatically Möb-covariant [10].

A similar class of representations of a conformal net is given by the so-called solitonic representations. In the following, we will denote by $\mathcal{I}_\mathbb{R}$ the set of all the open, nonempty and non dense intervals of $S^1 \setminus \{-1\}$. A (locally normal) *soliton* σ of a conformal net $(\mathcal{A}, U, \Omega)$ is a family of maps $\sigma = \{\sigma_I\}_{I\in\mathcal{I}_\mathbb{R}}$ where σ_I is a normal representation of the von Neumann algebra $\mathcal{A}(I)$ on a fixed Hilbert space $\mathcal{H}_\sigma$ such that $\sigma_I = \sigma_J|_{\mathcal{A}(I)}$ if $I \subseteq J$. Every DHR representation gives rise to a solitonic representation by restriction, while the converse is in general not true. If $\mathcal{G}$ is a topological group equipped with some homomorphism $\iota\colon \mathcal{G} \to \mathrm{Diff}_+(S^1)$, then we will say that a soliton σ is *locally $\mathcal{G}$-covariant* if there is a unitary projective continuous representation U_σ of $\mathcal{G}$ which satisfies the following property: if I is in $\mathcal{I}_\mathbb{R}$ and V is a connected neighborhood of the identity in $\mathcal{G}$ such that $g.I$ is in $\mathcal{I}_\mathbb{R}$ for every g in V, then $\mathrm{Ad}\, U_\sigma(g)\sigma_I(x) = \sigma_{\iota(g).I}(\mathrm{Ad}\, U(\iota(g))\cdot x)$ for every x in $\mathcal{A}(I)$. With $\mathbb{R}_\pm$ considered as elements of $\mathcal{I}_\mathbb{R}$, the *index* of a soliton σ is the Jones index of the inclusion $\sigma(\mathcal{A}(\mathbb{R}_+)) \subseteq \sigma(\mathcal{A}(\mathbb{R}_-))'$.

Finally, we have concluded our brief overview on conformal nets. We now consider a vacuum positive energy representation π of level ℓ of some loop group LG. We always suppose G to be simple, compact and simply connected. It can be shown that the family of von Neumann algebras

$$\mathcal{A}_\ell(I) = \{\tilde{\pi}(\gamma)\colon\ \mathrm{supp}\,\gamma \subset I\}''\,,$$

is a conformal net [15,29], where $\tilde{\pi}$ is the lift of π described below in Remark 16. We will denote by U the projective unitary continuous representation of $\mathrm{Diff}_+(S^1)$ verifying the covariance property. Consider now a smooth path $\gamma : [-\pi, \pi] \to G$. We suppose γ

to admit, at all orders, finite right derivatives in $-\pi$ and finite left derivatives in π. We then define $\sigma_\gamma = \{\sigma_\gamma^I\}_{I\in\mathcal{I}_\mathbb{R}}$ as the collection of maps given by

$$\sigma_\gamma^I : \mathcal{A}_\ell(I) \to B(\mathcal{H}), \quad \sigma_\gamma^I(x) = \mathrm{Ad}\,\pi(\gamma_I)(x), \tag{13}$$

where γ_I is a loop in LG such that $\gamma_I(\theta) = \gamma(\theta)$ for θ in I seen as a subinterval of $(-\pi, \pi)$.

Proposition 4. *σ_γ is an irreducible locally normal soliton with index* 1.

Proof. Normality on each $\mathcal{A}_\ell(I)$ follows because on these local algebras σ_γ is given by the adjoint action by a unitary operator. The compatibility property is clear, since if $I \subseteq J$ then $\pi(\gamma_I\gamma_J^{-1})$ is in $\mathcal{A}_\ell(I') = \mathcal{A}_\ell(I)'$. The index is 1 since if I is in $\mathcal{I}_\mathbb{R}$ then $\sigma_\gamma(\mathcal{A}_\ell(I)) = \mathcal{A}_\ell(I)$, and for the same reason we have that

$$\bigvee_{I\in\mathcal{I}_\mathbb{R}} \sigma_\gamma(\mathcal{A}_\ell(I)) = \bigvee_{I\in\mathcal{I}_\mathbb{R}} \mathcal{A}(I) = B(\mathcal{H})$$

since the conformal net $\mathcal{A}_\ell$ is irreducible. □

We will now focus on a smaller class of solitons. Given h in G, we define a *discontinuous loop* as an element of the group

$$L_hG = \left\{\zeta \in C^\infty(\mathbb{R}, G) : \zeta(x)^{-1}\zeta(x+2\pi) = h\right\}. \tag{14}$$

The restriction of a discontinuous loop on $[-\pi, \pi]$ clearly induces a soliton σ_ζ. In the following, we will study the equivalence classes of such solitonic representations.

Since π is irreducible if and only if it is irreducible as a projective representation of LG, then σ_ζ is irreducible if π is irreducible (see also Corollary 1.3.3. of [29]). Notice that for ζ in L_hG we have that $\zeta_t(\phi) = \zeta(\phi)\zeta(\phi - t)^{-1}$ is in LG for any t in $\mathbb{R}$. We now denote by $\mathrm{Rot}^\sim$ the universal covering of $\mathrm{Rot} \cong \mathbb{T}$, the group of rotations of the circle. If U_t is the unitary representation of Rot associated to π, then we can define $V_t^\zeta = \pi(\zeta_t)U_t$ in $PU(\mathcal{H})$ for t in $\mathbb{R}$ and notice that $V_t^\zeta V_s^\zeta = V_{t+s}^\zeta$. However, in general $V_{2\pi}^\zeta$ is not a scalar and therefore σ_ζ is not Rot-covariant but only locally $\mathrm{Rot}^\sim$-covariant. We can notice that if ζ is in L_gG and η is in L_hG then $\zeta\eta^{-1}$ is in $L_{h^{-1}g}G$ if $h^{-1}g$ is in $Z(G)$. In particular, if ζ and η are both in L_hG then $\zeta\eta^{-1}$ is in LG and σ_ζ is unitarily equivalent to σ_η.

Theorem 5. *Let π be a vacuum positive energy representation of LG of level $\ell \geq 1$. Given ζ in L_hG, the soliton σ_ζ extends to a DHR representation if and only if h is central.*

Proof. First we suppose h to be in $Z(G)$. A quick computation shows that in this case $V_{2\pi}^\zeta = \pi(h)$. By the identity $\pi(h)e^{\pi(X)}\pi(h)^* = e^{\pi(X)}$ for any X in $L\mathfrak{g}_0$ we have that $V_{2\pi}^\zeta$ is a scalar since π is irreducible. This implies that σ_ζ is locally Rot-covariant and we have that σ_ζ can be extended to a locally normal DHR representation by using the arguments of Proposition 3.8. of [11]. Now we suppose h to be not central. By absurd, σ_ζ extends to a DHR representation and thus it is Rot-covariant [10]. Denote by U_θ^ζ the corresponding intertwining projective representation of the circle. If we define the DHR representation

$$\rho_\zeta(x) = \mathrm{Ad}U_\pi \cdot \sigma_\zeta \cdot \mathrm{Ad}U_{-\pi} \cdot \sigma_{\zeta^{-1}} \cdot \mathrm{Ad}V_\pi^\zeta \cdot \mathrm{Ad}U_{-\pi}(x),$$

then by construction ρ_ζ is implemented by the unitary $U_\pi U^\zeta_{-\pi} V^\zeta_\pi U_{-\pi}$. Since σ_ζ is a locally normal DHR representation, by using the additivity property one can show that $\rho_\zeta(x) = x$ for x in $\mathcal{A}((0,\pi))$ and for x in $\mathcal{A}((0,\pi))'$. It follows that $U^\zeta_{-\pi} V^\zeta_\pi$ is a scalar and thus $V^\zeta_{2\pi}$ is a scalar. Now consider a maximal torus $T \subset G$ containing h. Since T is connected, we can suppose that $\zeta(x)$ belongs to T for any x in $\mathbb{R}$, and by commutativity we have that $V^\zeta_{2\pi} = \pi(h)$ in $PU(\mathcal{H})$. Therefore we have that h is a noncentral element acting on $\mathcal{H}$ as a scalar. If we now consider the kernel

$$N = \{g \in G \colon \pi(g) \in \mathbb{T}\}\,,$$

then N is a normal subgroup of G which is not contained in the center. But G is simple and connected, hence we have that $N = G$, which is an absurd. $\square$

We conclude this section by studying the equivalence classes of the solitons constructed above. If z is in $Z(G)$, then the DHR representations σ_ζ with ζ in $L_z G$ correspond to inequivalent irreducible positive energy representations $\zeta_* \pi$ of the same level as π (see Remark 16 and Theorem 3.2.3. of [29]). Now we pick a maximal torus T in G. Consider ζ in $L_s G$ and η in $L_t G$ for some s and t in T. We can suppose ζ and η to be both contained in T. It can be easily noticed that

$$\sigma_\zeta \cdot \sigma_\eta = \sigma_{\zeta\eta}\,, \quad \zeta\eta \in L_{st} G\,, \qquad \sigma_\zeta^{-1} = \sigma_{\zeta^{-1}}\,, \quad \zeta^{-1} \in L_{s^{-1}} G\,.$$

It follows that σ_ζ and σ_η are unitarily equivalent if and only if $s = t$, hence we have infinitely many inequivalent solitons. If we consider two maximal tori T and $T' = gTg^{-1}$, then what we can say is that we have the identity

$$\sigma_{g\zeta g^{-1}} = \mathrm{Ad}\pi(g) \cdot \sigma_\zeta \cdot \mathrm{Ad}\pi(g)^*\,,$$

that is the solitons $\sigma_{g\zeta g^{-1}}$ and σ_ζ are equivalent up to some inner automorphism.

4. Sobolev Loop Groups

We know that $LG = C^\infty(S^1, G)$ is a Fréchet Lie group if endowed with the Whitney smooth topology. Its topology is induced by the norms defined on the Banach Lie groups $L^k G = C^k(S^1, G)$. The exponential map $\exp_{LG} \colon L\mathfrak{g}_0 \to LG$ is naturally defined by $\exp_{LG}(X) = \exp_G \cdot X$ and is a local homeomorphism near the identity [27]. Here we define and describe some properties of Sobolev loop groups.

Let M be a Riemannian manifold. Suppose M to be isometrically embedded in $\mathbb{R}^\nu$ for some $\nu > 0$. Define, for $1 \le p < \infty$ and $0 \le s < \infty$, the *fractional Sobolev space* [2,12]

$$W^{s,p}(S^1, M) = \{f \in W^{s,p}(S^1, \mathbb{R}^\nu) \colon f(\theta) \in M \textit{ a.e.}\}\,.$$

Here $W^{s,p}(S^1, \mathbb{R}^\nu)$ is the completion of $C^\infty(S^1, \mathbb{R}^\nu)$ with respect to the norm $\|f\|_{s,p} = \|\Delta^{s/2} f\|_p + \|f\|_p$, where $\Delta \ge 0$ is the closure on $L^p(S^1, \mathbb{R}^\nu)$ of the laplacian seen as an operator on $C^\infty(S^1, \mathbb{R}^\nu)$ [9]. We recall that the closure of an operator between linear subspaces of Banach spaces (and not only Hilbert spaces) is its smallest closed extension, and that the fractional Laplacian Δ^α for $0 < \alpha < 1$ can be defined by the Fourier transform [12].

In the following, every compact Lie group G will be considered as a Riemannian Lie group with respect to the unique Riemannian structure extending $-\langle\cdot,\cdot\rangle$, namely the opposite of the basic inner product, and such that left and right translations are smooth isometries. We show that if G is compact and simple then every faithful unitary representation $\rho\colon G \to U(n)$ induces an isometric embedding of G in some real euclidean space. By continuity of the representation we have that G is represented as a compact embedded Lie subgroup of $U(n)$. Moreover, by simplicity of $\mathfrak{g}_0$ we have that $\lambda\mathrm{tr}(\rho(x)^*\rho(y)) = -\langle x, y\rangle$ for some $\lambda > 0$. Therefore, if we consider $M_n(\mathbb{C})$ as a real vector space with inner product $\lambda\mathrm{Re}\,\mathrm{tr}(X^*Y)$ then we have an isometric embedding $G \hookrightarrow M_n(\mathbb{C})$.

Theorem 6. *If G is a compact, simple and simply connected Lie group faithfully represented in some space of matrices, then $W^{s,p}(S^1, G)$ is an analytic Banach Lie group for p and sp in $(1, \infty)$. Its Banach Lie algebra is $W^{s,p}(S^1, \mathfrak{g}_0)$, the exponential map exists and it is a local homeomorphism. Moreover, $C^\infty(S^1, G)$ is dense in $W^{s,p}(S^1, G)$ and thus $W^{s,p}(S^1, G)$ is connected.*

Proof. First we show that $W^{s,p}(S^1, G)$ is a topological group. This can be proved by using the fact that any two functions f, g in $W^{s,p}(S^1, \mathbb{R}^\nu)$ verify, for p and sp in $(1, \infty)$, the estimate [9]

$$\|fg\|_{s,p} \leq C_{s,p}\|f\|_{s,p}\|g\|_{s,p}\,. \tag{15}$$

By this estimate and by the identity $f^{-1} - g^{-1} = f^{-1}(g - f)g^{-1}$ it follows that $W^{s,p}(S^1, G)$ is a topological group for p and sp in $(1, \infty)$, since it is clearly a Hausdorff space. Now we define the map

$$\exp_{s,p}\colon W^{s,p}(S^1, \mathfrak{g}_0) \to W^{s,p}(S^1, G)\,, \quad \exp_{s,p}(X)(z) = \exp_G(X(z))\,.$$

This map is well defined since $\exp_G \cdot X = e^X$ is an absolutely convergent series and it is also a local homeomorphism. We check that $W^{s,p}(S^1, G)$ is connected. By the density of $C^\infty(S^1, G)$ in $W^{s,p}(S^1, G)$ (see Theorem 1.1. of [2]), it suffices to prove that $C^\infty(S^1, G)$ is path connected and then connected. But a smooth homotopy between two loops in G is a path in $C^\infty(S^1, G)$, and the connectedness follows. Finally, we conclude if we prove that the group operations of inversion and multiplication are analytic. By connectedness we can reduce to prove this in an open neighborhood of the identity (see [29], Lemma 2.2.1.). The inversion $X \mapsto -X$ is clearly analytic. The analyticity of left and right multiplication follows from the Baker-Campbell-Hausdorff-Dynkin formula, where the continuity of the appearing homogeneous polynomials is guaranteed by equation (15). The theorem is proved. □

Corollary 7. *Every loop γ in $W^{s,p}(S^1, G)$ is a finite product of exponentials, since the exponential map is a local homeomorphism and $W^{s,p}(S^1, G)$ is connected.*

Remark 8. Theorem 6 still holds if the circle S^1 is replaced with a torus $\mathbb{T}^m$. This follows from the fact that the mentioned density theorem [29] is verified for a generic cube Q^m, that $\mathbb{T}^m$ can be defined as a quotient of Q^m and that the convolution with a smooth function preserves the periodicity.

We have formally defined our Sobolev loop group $W^{s,p}(S^1, G)$ and we have checked that such a space has good topological and analytical properties. Now we are finally ready to extend our PER of LG. The definition of Positive Energy Representation of a Sobolev loop group can be given just by replacing LG with $W^{s,p}(S^1, G)$ in the definition given above.

Proposition 9. *Let $\iota\colon G \to H$ and $\pi : G \to U$ be two homomorphisms of topological groups. We suppose H to be connected and $\iota(G)$ to be dense in H. Suppose the existence of a neighborhood V of the identity in H and of a continuous function $p_0\colon V \to U$ such that $\pi(g_\alpha) \to p_0(v)$ whenever $\iota(g_\alpha) \to v$, with $(g_\alpha)_{\alpha\in A}$ a net in G and v in V. Then, p_0 extends to a continuous homomorphism $p\colon H \to U$ such that $\pi = p \cdot \iota$.*

Proof. By the connectedness of H we have that $H = \cup_n V^n$. We show by induction that p can be well defined on V^n for every n. We set $p = p_0$ on V. Suppose the thesis true for V^n, and consider elements w in V^n and v in V. Pick a net $(h_\beta)_{\beta\in B}$ such that $\iota(h_\beta) \to v$. By inductive hypothesis the limit

$$p(wv) = \lim_\alpha \pi(g_\alpha) = \lim_\beta \lim_\alpha \pi(g_\alpha h_\beta^{-1}) p_0(v) = p(w)p(v)$$

is well defined and does not depend on the net $(g_\alpha)_{\alpha\in A}$ such that $\iota(g_\alpha) \to wv$. Hence p is a well defined group homomorphism. The continuity of p follows by induction as well, and the identity $\pi = p \cdot \iota$ is satisfied by construction. $\square$

Proposition 10. *Let π be a PER of a loop group LG. If X is in $W^{s,p}(S^1, \mathfrak{g}_0)$ for $1 \le p \le 2$ and $s > 3/2 + 1/p$, then $\pi(X)$ is a closable operator which is essentially skew-adjoint on any core of L_0.*

Proof. We first notice that by the Sugawara formula we have $L_0 \ge 0$, since we have $L_0 = d + h_i$ for some $h_i \ge 0$ on each irreducible summand π_i of π. If $\mathcal{H}^{0,\mathrm{fin}}$ is the algebraic direct sum of the eigenspaces of L_0, then we will denote by $\mathcal{H}^{0,s}$ the completion of $\mathcal{H}^{0,\mathrm{fin}}$ with respect to the Sobolev norm $\|\xi\|_{0,s} = \|(1 + L_0)^s\xi\|$. By Lemma 2 and Proposition 1.2.1. of [29], for ξ in $\mathcal{H}^{0,\mathrm{fin}}$ and X in $L^{\mathrm{pol}}\mathfrak{g}$ we have

$$\begin{aligned}\|\pi(X)\xi\|_{0,s} &\le \sqrt{2(\ell + g)}|X|_{|s|+1/2}\|\xi\|_{0,s+1/2} ,\\ \|[1 + L_0, \pi(X)]\xi\|_{0,s} &\le \sqrt{2(\ell + g)}|X|_{|s|+3/2}\|\xi\|_{0,s+1/2} ,\end{aligned}$$

for any s in $\mathbb{R}$. By density one extends π to $(L\mathfrak{g})_{|s|+3/2}$ in such a way to still verify the same estimates for ξ in $\mathcal{H}^{0,s+1/2}$. It follows that if X is in $L\mathfrak{g}_{3/2}$ then both $\pi(X)$ and $[1 + L_0, \pi(X)]$ are bounded operators from $\mathcal{H}^{0,1/2}$ to $\mathcal{H}$. By the Nelson commutator theorem [28, Thm. X.36] we have that if X is in $(L\mathfrak{g}_0)_{3/2}$ then the restriction of $\pi(X)$ on

$$\mathcal{D} = \left\{\psi \in \mathcal{H} \cap \mathcal{H}^{0,1/2} \colon \pi(X) \in \mathcal{H}\right\}$$

is a closable operator on $\mathcal{H}$ which is essentially skew-adjoint on any core of L_0 such as $\mathcal{H}^{0,\mathrm{fin}}$. Notice now that, by standard arguments, there is a norm continuous embedding $W^{s,p}(S^1, \mathfrak{g}) \hookrightarrow L\mathfrak{g}_{3/2}$. Indeed, if $X(\theta) = \sum_k X_k e^{ik\theta}$ then by the Hölder inequality

$$|X|_{3/2} = \sum_k (1 + |k|)^{3/2}\|X_k\| = \sum_k (1 + |k|)^{3/2-s}(1 + |k|)^s\|X_k\| \le A_{s,p}|X|_{s,p'} \le B_{s,p}\|X\|_{s,p} ,$$

where $A_{s,p}$ and $B_{s,p}$ exist and are finite by construction and by Riesz-Thorin respectively. Therefore, by the arguments given above we have that if X is in $W^{s,p}(S^1, \mathfrak{g}_0)$ then $\pi(X)$ is a skew-symmetric operator on $\mathcal{H}^{0,\mathrm{fin}}$ which is essentially skew-adjoint on any core of L_0.

Propositions 9 and 10 can be used to extend a strongly continuous projective representation of LG to a strongly continuous projective representation of $W^{s,p}(S^1, G)$. However, for convenience in the following we will focus on $H^s(S^1, G) = W^{s,2}(S^1, G)$. We show how a different approach can improve the results of Proposition 10.

Proposition 11. *If π is a PER of a loop group LG, the induced projective representation π of $L\mathfrak{g}$ can be extended to $H^s(S^1, \mathfrak{g})$ for $s > 3/2$, with $\pi(X)$ closable and such that*

$$\|\pi(X)\xi\|_{0,1/2} \le C_s |X|_{s,2} \|\xi\|_{0,1/2}\,, \qquad \xi \in \mathcal{H}^{0,1/2}\,, \tag{16}$$

for some $C_s > 0$. Moreover, $\pi(X)^* = \overline{\pi(X^*)}$, and in particular $\pi(X)$ is essentially skew-adjoint if X is skew-adjoint.

Proof. We use some techniques shown in [5]. Given $X = \sum_n X_n(n)$ in $L\mathfrak{g}_{1,1}$, the operator $\pi(X)$ is well defined on $\mathcal{H}^{0,1/2}$ and (16) follows by the previous estimates since for $t > 1/2$ and $s = 1 + t$ we have

$$|X|_1 = \sum_n (1 + |n|)\|X_n\| = \sum_n (1 + |n|)^{-t}(1 + |n|)^{1+t}\|X_n\| \le c_t |X|_{s,2}\,.$$

It is also closable since $\pi(X^*) \subseteq \pi(X)^*$. Notice also that since $\mathcal{H}^{0,\mathrm{fin}}$ is a core for $(1 + L_0)^{1/2}$ then $\pi(X^*)$ is the formal adjoint of $\pi(X)$ on the associated scale space $\mathcal{H}^{0,1/2}$ for any X in $H^{3/2}(S^1, \mathfrak{g})$. Now we define on $\mathcal{H}^{0,1/2}$ the operator

$$R_{X,\epsilon} = [\pi(X), e^{-\epsilon L_0}]\,,$$

which is well defined since $e^{-\epsilon L_0} : \mathcal{H} \to \mathcal{H}^{0,\infty} \subseteq \mathcal{H}^{0,1/2}$. By $-R_{X^*,\epsilon} \subseteq R^*_{X,\epsilon}$ we have that $R_{X,\epsilon}$ is closable. Notice that if $L_0 v_k = k v_k$ then

$$R_{x(n),\epsilon} v_k = f_{n,k}(\epsilon) x(n) v_k\,, \quad f_{n,k}(\epsilon) = e^{-\epsilon k} - e^{-\epsilon(k-n)}\,.$$

We will now show that $\|R_{x(n),\epsilon}\|^2 \le 2(\ell + g)|x(n)|_{1,1}$. The case $n = 0$ is trivial and we can suppose $n < 0$ as $-R_{X^*,\epsilon} \subseteq R^*_{X,\epsilon}$. By simple analysis techniques one can prove that

$$|f_{n,k+n}(\epsilon)|^2 \le \frac{n^2}{(k-n)^2}\,, \quad \frac{1+k}{(k-n)^2} \le \frac{1}{|n|}\,,$$

for any $\epsilon \ge 0$ and $k \ge 0$. Therefore if $v = \sum_{k\ge 0} v_k$ is in $\mathcal{H}^{0,\mathrm{fin}}$ then we have

$$\begin{aligned}
\|R_{x(n),\epsilon} v\|^2 &= \Big\| \sum_{k\ge 0} R_{x(n),\epsilon} v_k \Big\|^2 = \Big\| \sum_{k\ge 0} |f_{n,k}(\epsilon)|^2 x(n) v_k \Big\|^2 \\
&= \sum_{k\ge 0} |f_{n,k}(\epsilon)|^2 \|x(n) v_k\|^2 \\
&\le 2(\ell + g) \sum_{k\ge 0} \frac{n^2}{(k-n)^2}(1 + |n|)(1 + k)\|x\|^2 \|v_k\|^2 \\
&\le 2(\ell + g) \sum_{k\ge 0} (1 + |n|)^2 \|x\|^2 \|v_k\|^2 \\
&= 2(\ell + g)|x(n)|^2_{1,1} \|v\|^2\,.
\end{aligned}$$

It follows that $\|R_{X,\epsilon}\|^2 \leq 2(\ell + g)|X|_{1,1}$ for every X in $L\mathfrak{g}_{1,1}$ and that $R_{X,\epsilon} \to 0$ strongly as $\epsilon \to 0$. Moreover, by the identity $R^*_{X,\epsilon} = -\overline{R_{X^*,\epsilon}}$ we have that $R^*_{X,\epsilon} \to 0$ strongly as well. Now we arrive to the crucial point: if v is in $\mathcal{D}(\pi(X)^*)$ then

$$\pi(X^*)e^{-\epsilon L_0}v = \pi(X)^* e^{-\epsilon L_0}v = e^{-\epsilon L_0}\pi(X)^* v - R^*_{X,\epsilon}v \to \pi(X)^* v\,, \quad \epsilon \to 0\,,$$

and this concludes the proof since $e^{-\epsilon L_0}v \to v$ as $\epsilon \to 0$. □

Theorem 12. *Let $\pi : LG \to PU(\mathcal{H})$ be a positive energy representation of LG. Then π can be extended to a positive energy representation of $H^s(S^1, G)$ for $s > 3/2$.*

Proof. We consider an open neighborhood U in $H^s(S^1, \mathfrak{g}_0)$ on which the exponential map of $H^s(S^1, G)$ is a homeomorphism and set $V = \exp_{H^s}(U)$. For $\gamma = \exp_{H^s}(X)$ in V we define in $PU(\mathcal{H})$

$$\pi(\gamma) = e^{\pi(X)}\,, \quad X \in U\,.$$

The neighborhood V verifies Proposition 9, since if $\gamma_\alpha = \exp(X_\alpha)$ converges to $\gamma = \exp(X)$ in V then the estimate (16) implies that $\pi(X_\alpha)\xi$ is a Cauchy net for every ξ in $\mathcal{H}^{0,1/2}$. But the pointwise convergence of self-adjoint operators on a common core implies the strong resolvent convergence of such operators (Theorem VIII.25.(a) of [28]), thus π can be continuously extended. Finally, since the rotation group acts on $H^s(S^1, G)$ by continuous operators (see Lemma A.3 of [4]) and since LG is dense in $H^s(S^1, G)$, we have that π is actually a Positive Energy Representation since it is Rot-covariant. □

Proposition 13. [29] *Let $\rho_s = \exp_{\mathrm{Diff}_+(S^1)}(sh)$ be a smooth diffeomorphism of S^1, with h a smooth real vector field of the circle. Set $R_h = \{\rho_s\}_{s\in\mathbb{R}}$. Then the exponential map $L\mathfrak{g}_0 \rtimes \mathbb{R}h \to LG \rtimes R_h$ is well defined and continuous. Moreover, if $X_\alpha = \rho_\alpha.X = X\cdot\rho_\alpha^{-1}$ then*

$$\exp_{LG\rtimes R_h}(X + \alpha h) = \lim_{n\to\infty} \exp_{LG}(X/n)\exp_{LG}(X_{\alpha/n}/n)\cdots\exp_{LG}(X_{\alpha(n-1)/n}/n)\rho_\alpha\,. \tag{17}$$

Proof. To compute the exponential map, we fix $X + \alpha h$ in $L\mathfrak{g}_0 \rtimes \mathbb{R}h$ and look for $f : \mathbb{R} \to LG \rtimes R_h$ which satisfies $(X + \alpha h)f = \dot{f}$ and $f(0) = 1$. We suppose f to be of the form $f_t = \gamma^t\rho_{\phi(t)}$ with γ in LG. As a manifold, $LG \rtimes R_h$ is the product of LG and R_h, thus $s \mapsto \exp_{LG}(sX)\rho_{s\alpha}$ is the integral curve for $X + \alpha h$ at the identity. Therefore, with the notation $\gamma_s(\theta) = \gamma(\rho_s^{-1}(\theta))$ we have

$$\begin{aligned}
(X + \alpha h)f_t &= \left.\frac{d}{ds}\right|_{s=0} \exp_{LG}(sX)\rho_{s\alpha}\gamma^t\rho_{\phi(t)} = \left.\frac{d}{ds}\right|_{s=0} \exp_{LG}(sX)(\gamma^t)_{s\alpha}\rho_{s\alpha+\phi(t)}\\
&= X\gamma^t\rho_{\phi(t)} + \alpha\left.\frac{d}{ds}\right|_{s=0}(\gamma^t)_s\rho_{\phi(t)} + \alpha\gamma^t h\rho_{\phi(t)}\,,\\
\dot{f}_t &= \left(\frac{d}{dt}\gamma^t\right)\rho_{\phi(t)} + \phi'(t)\gamma^t h\rho_{\phi(t)}\,,
\end{aligned}$$

whence $\phi(t) = \alpha t$, and we must solve

$$\frac{d}{dt}\gamma^t = X\gamma^t + \alpha\left.\frac{d}{ds}\right|_{s=0}(\gamma^t)_s\,, \qquad \gamma^0 = 1\,. \tag{18}$$

Now we notice that if γ_0^t is a solution of the equation $\frac{d}{dt}\gamma_0^t = X_{-\alpha t}\gamma_0^t$ with initial condition $\gamma_0^0 = 1$, then $\gamma^t = (\gamma_0^t)_{\alpha t}$ is the solution of (18) we were looking for. Therefore, if we embed G in a space of matrices $M_m(\mathbb{C})$ and we consider LG as a closed subspace of $C^\infty(S^1, M_m(\mathbb{C}))$, then by Theorem 1.4.1. of [29] we have

$$\gamma_0^1 = \lim_{n\to\infty} \exp(X_{-\alpha}/n)\exp(X_{-\alpha(n-1)/n}/n)\cdots\exp(X_{-\alpha/n}/n)\rho_\alpha\,, \tag{19}$$

where the right side of (19) converges in each $C^k(S^1, M_m(\mathbb{C}))$ and hence in LG. Finally, equation (17) follows from $\gamma^1 = (\gamma_0^1)_\alpha$, and the continuity of $\exp_{LG\rtimes R_h}$ follows from Theorem 1.4.1. of [29]. □

Corollary 14. *The following holds in $PU(\mathcal{H})$:*

$$e^{\pi(X+i\alpha h)} = \pi(\exp_{LG\rtimes R_h}(X+\alpha h))\,.$$

Proof. By the Trotter product formula and Proposition 13 we have the following identities in $PU(\mathcal{H})$:

$$\begin{aligned} e^{\pi(X+i\alpha h)} &= \lim_{n\to\infty}(e^{\pi(X/n)}e^{i\alpha\pi(h)/n})^n = \lim_{n\to\infty}\pi(\exp_{LG}(X/n)\exp_{R_h}(\alpha h/n))^n \\ &= \lim_{n\to\infty}\pi(\exp_{LG}(X/n)\exp_{LG}(X_{\alpha/n}/n)\cdots\exp_{LG}(X_{\alpha(n-1)/n}/n)\rho_\alpha) \\ &= \pi(\exp_{LG\rtimes R_h}(X+\alpha h))\,, \end{aligned}$$

where we used the identities $e^{iT(h)} = \pi(\exp_{R_h}(h))$ and $e^{\pi(X)} = \pi(\exp_{LG}(X))$ which hold in $PU(\mathcal{H})$. □

Lemma 15. *Let $\pi : G \to PU(\mathcal{H})$ be a strongly continuous projective representation of a topological group G. Then the map*

$$G \times U(\mathcal{H}) \to U(\mathcal{H})\,, \quad (g,u) \mapsto \pi(g)u\pi(g)^*\,,$$

is well defined and strongly continuous.

Proof. The map is clearly well defined, and if g_α converges to g in G then we can choose lifts v_α and v of $\pi(g_\alpha)$ and $\pi(g)$ such that v_α converges to v in $U(\mathcal{H})$, since the short exact sequence given by $U(\mathcal{H}) \to PU(\mathcal{H})$ admits local continuous sections [1]. But in the unitary group the strong topology and the $*$-strong topology coincide and multiplication is continuous on bounded sets by the uniform boundedness principle, so the assertion follows. □

Remark 16. A continuous projective representation $\pi : G \to PU(\mathcal{H})$ can be naturally lifted to a continuous unitary representation $\tilde{\pi}$ of $\widetilde{G} = \big\{(g,u) \in G \times U(\mathcal{H}) : \pi(g) = [u]\big\}$ given by $\tilde{\pi}(g,u) = u$.

Theorem 17. *If γ is in $H^s(S^1, G)$ and X is in $H^s(S^1, \mathfrak{g}_0)$ for some $s > 3/2$, then*

$$\pi(\gamma)\pi(X)\pi(\gamma)^* = \pi(\mathrm{Ad}(\gamma)X) + ic(\gamma, X)\,, \tag{20}$$

for some continuous real function $c(\gamma, X)$. Moreover, if γ is in $H^{1+s}(S^1, G)$ and h is a real vector field $\mathcal{S}_s$, then

$$\pi(\gamma)\pi(X+ih)\pi(\gamma)^* = \pi(\mathrm{Ad}(\gamma)X) + iT(h) + \pi(h\dot{\gamma}\gamma^{-1}) + ic(\gamma, X) + ic(\gamma, h) \tag{21}$$

for some continuous real function $c(\gamma, h)$.

Proof. We first prove (21) in the smooth case. We will identify $\mathbb{R}h$ with $i\mathbb{R}h$ for formal convenience. By the previous propositions, if γ is in LG and $Y = X+ih$ is in $L\mathfrak{g}_0 \rtimes i\mathbb{R}h$, then the following identities hold in $PU(\mathcal{H})$:

$$\begin{aligned}\pi(\gamma)e^{t\pi(Y)}\pi(\gamma)^* &= \pi(\gamma)\pi(\exp_{LG\rtimes R_h}(sY))\pi(\gamma)^* \\ &= \pi(\gamma \exp_{LG\rtimes R_h}(tY)\gamma^{-1}) \\ &= \pi(\exp_{LG\rtimes R_h}(t\mathrm{Ad}(\gamma)Y)) \\ &= e^{t\pi(\mathrm{Ad}(\gamma)Y)}, \end{aligned} \tag{22}$$

and consequently $\pi(\gamma)e^{t\pi(Y)}\pi(\gamma)^* = \lambda(t)e^{t\pi(\mathrm{Ad}(\gamma)Y)}$ for some function $\lambda\colon \mathbb{R} \to \mathbb{T}$. But $\lambda\colon \mathbb{R} \to \mathbb{T}$ is a continuous homomorphism and therefore $\lambda(t) = e^{iat}$ for a unique real number $a = c(\gamma, Y)$. We point out that $\mathrm{Ad}(\gamma)$ has to be intended as the adjoint action with respect to the semidirect product $LG \rtimes R_h$. Notice also that $c(\gamma, Y)$ is linear in Y, so we can write $c(\gamma, X+ih) = c(\gamma, X)+c(\gamma, h)$, where we set $c(\gamma, ih) = c(\gamma, h)$ for simplicity. Therefore, the claimed expression follows by the Stone's theorem and by using the product rule for the derivative on the identity $1 = \gamma_t \cdot \gamma_t^{-1}$.

Now we prove (20) in the Sobolev case. Consider $(\gamma_\alpha, X_\alpha)$ in $LG \times L\mathfrak{g}_0$ converging to (γ, X) in $H^s(S^1, G) \times H^s(S^1, \mathfrak{g}_0)$. We have that both $\pi(\gamma_\alpha)e^{s\pi(X_\alpha)}\pi(\gamma_\alpha)^*$ and $e^{s\pi(\mathrm{Ad}(\gamma_\alpha)X_\alpha)}$ strongly converge to the corresponding terms in γ and X. By the argument used before we have that $e^{ic(\gamma_\alpha, X_\alpha)}$ converges to $e^{ic(\gamma,X)}$, that is $e^{ic(\gamma,X)}$ is continuous in γ and X. But continuity is a local property and the exponential map has local left inverses, thus $c(\gamma, X)$ is continuous and the first part of the theorem is proved. Now we prove (21) in the Sobolev case. Consider γ in $H^{1+s}(S^1, G)$ and h real in $\mathcal{S}_s$. Notice that $i\pi(h)$ and $\pi(h\dot\gamma\gamma^{-1})$ are both essentially skew-adjoint. Consider now smooth approximating nets $\gamma_\alpha \to \gamma$, $X_\alpha \to X$ and $h_\alpha \to h$ as before. By the previous propositions, the approximating right hand side of (21) minus $c(\gamma_\alpha, h_\alpha)$ converges in the strong resolvent sense to the corresponding term in γ, X and h since we have a net of skew-adjoint operators pointwise convergent on a common core. Similarly, $\pi(X_\alpha + ih_\alpha)$ converges in the strong resolvent sense to $\pi(X + ih)$ and therefore

$$\pi(\gamma_\alpha)e^{t\pi(X_\alpha+ih_\alpha)}\pi(\gamma_\alpha)^* \to \pi(\gamma)e^{t\pi(X+ih)}\pi(\gamma)^*$$

strongly for every t in $\mathbb{R}$. By the argument used before we have that $e^{ic(\gamma,h)}$ is continuous and thus $c(\gamma, h)$ is continuous. The thesis is proved. □

Corollary 18. *The scale space $\mathcal{H}^\alpha \subseteq \mathcal{H}$ is $H^s(S^1, G)$-invariant for $\alpha \geq 0$ and $s > 5/2$. Moreover, for any integer n such that $n \leq \lfloor s-1 \rfloor$, the corresponding map $H^s(S^1, G) \times \mathcal{H}^n \to \mathcal{H}^n/\mathbb{T}$ is continuous.*

Proof. Since $\mathcal{D}(u^*Au) = u^*\mathcal{D}(A)$ for every unitary u and every self-adjoint operator A, then

$$\begin{aligned}\mathcal{D}((1+d)^\alpha) &= \pi(\gamma)^*\mathcal{D}((1+d - i\pi(\dot\gamma\gamma^{-1}) + c(\gamma, d))^\alpha) \\ &\subseteq \pi(\gamma)^*\mathcal{D}((1+d)^\alpha)\,. \end{aligned} \tag{23}$$

Since $\mathcal{D}((1+d)^\alpha) = \mathcal{H}^\alpha$ for $\alpha \geq 0$, the $\mathcal{H}^s$-invariance follows. Now we prove the second statement, where we can suppose $n \geq 1$. By Proposition 1.5.3. of [29] we have $\|\pi(\gamma)\xi\|_n \leq (1 + M_{n-1})^n\|\xi\|_n$, where $M_p = C|\gamma^{-1}\dot\gamma|_{p+1/2} + |c(\gamma^{-1}, d)|$ for some $C > 0$, and the joint continuity can be proved as in [29]. □

Theorem 19. *With the hypotheses of Theorem 17, we have*

$$c(\gamma, X) = -\ell \int_0^{2\pi} \langle \gamma^{-1}\dot{\gamma}, X\rangle \frac{d\theta}{2\pi}\,, \quad c(\gamma, h) = -\frac{\ell}{2} \int_0^{2\pi} h\langle \gamma^{-1}\dot{\gamma}, \gamma^{-1}\dot{\gamma}\rangle \frac{d\theta}{2\pi}\,.$$

Proof. We follow Theorem 1.6.3. of [29], skipping some computations for the sake of brevity. Consider a smooth loop γ in LG and a smooth real vector field h. For Y in $L\mathfrak{g}_0 \rtimes i\mathbb{R}h$ we have

$$c(\gamma_1\gamma_2, Y) = c(\gamma_2, Y) + c(\gamma_1, \mathrm{Ad}(\gamma_2)Y)\,. \tag{24}$$

If $\gamma^t = \exp_{LG}(tX)$, then the map $t \mapsto c(\gamma^t, Y)$ is differentiable at $t = 0$ since $LG \times R_h \to LG$ is smooth. In particular, we have that

$$\partial_t\big|_{t=0} c(\gamma^t, Y) = \ell B(X, Y)\,,$$

and so

$$\partial_t\big|_{t=0} c(\gamma^t, h) = 0\,.$$

By using (24) we have that $c(\gamma^t, h)$ is differentiable everywhere, with

$$\partial_t c(\gamma^t, Y) = \ell B(X, Y) + c(\gamma^t, [X, Y])\,,$$

or more compactly

$$\dot{c}_t(Y) = i_X \ell B(Y) - (X.c_t)(Y)\,. \tag{25}$$

We naturally expect the solution of the ODE to be given by the Duhamel formula

$$c(\gamma^t, Y) = \ell B(X, \mathrm{Ad}(\gamma^t) \int_0^t \mathrm{Ad}(\gamma^{-\tau}) Y d\tau) = \ell B(X, \int_0^t \mathrm{Ad}(\gamma^s) Y ds)\,. \tag{26}$$

Using $\frac{d}{dt}\mathrm{Ad}(\gamma_t)Y = [X, \mathrm{Ad}(\gamma_t)Y]$, it is easy to verify that (26) defines a $C^1(\mathbb{R}, (L\mathfrak{g} \rtimes i\mathbb{R}h)^*)$ solution of (25) with initial condition $c_0 = 0$. The solution is unique. Finally, one can use (26) and Corollary 1.6.2. of [29] to obtain the claimed expressions in the smooth case. By the continuity of $c(\gamma, Y)$ shown in Theorem 17 the thesis is proved. □

Corollary 20. *By repeating the proof of Theorem 17, one can show that if γ is in $H^s(S^1, G)$ and X is in $H^s(S^1, \mathfrak{g}_0)$ for some $s > 3/2$, then*

$$\pi(\gamma)^*\pi(X)\pi(\gamma) = \pi(\mathrm{Ad}(\gamma^{-1})X) + ib(\gamma, X)\,, \tag{27}$$

for some continuous real function $b(\gamma, X)$. Similarly, if γ is in $H^{s+1}(S^1, G)$ and h is a real vector field in $\mathcal{S}_s$, then

$$\pi(\gamma)^*\pi(X+ih)\pi(\gamma) = \pi(\mathrm{Ad}(\gamma^{-1})X) + iT(h) - \pi(h\gamma^{-1}\dot{\gamma}) + ib(\gamma, X) + ib(\gamma, h) \tag{28}$$

for some continuous real function $b(\gamma, h)$. In particular, by $b(\gamma, Y) = c(\gamma^{-1}, Y)$ we have

$$b(\gamma, X) = -\ell \int_0^{2\pi} \langle \dot{\gamma}\gamma^{-1}, X\rangle \frac{d\theta}{2\pi}\,, \quad b(\gamma, h) = -\frac{\ell}{2} \int_0^{2\pi} h\langle \dot{\gamma}\gamma^{-1}, \dot{\gamma}\gamma^{-1}\rangle \frac{d\theta}{2\pi}\,.$$

5. Relative Entropy and QNEC

Let $\mathcal{M}$ be a von Neumann algebra in standard form, and let φ and ψ be two faithful, normal and positive linear functionals on $\mathcal{M}$ represented by vectors ξ and η in the natural cone. The relative entropy is defined by [22]

$$S(\varphi\|\psi) = -(\xi|\log\Delta_{\eta,\xi}\xi)\,,$$

where the above scalar product has to be intended by applying the spectral theorem to the relative modular operator $\Delta_{\eta,\xi}$. The relative entropy is nonnegative, convex, lower semicontinuous in the $\sigma(\mathcal{M}_*,\mathcal{M})$-topology and monotone increasing with respect to von Neumann algebras inclusions (Theorem 5.3. of [25]). If the relative entropy is finite, then

$$S(\varphi\|\psi) = i\frac{d}{dt}\varphi((D\psi:\,D\varphi)_t)\bigg|_{t=0} = -i\frac{d}{dt}\varphi((D\varphi:\,D\psi)_t)\bigg|_{t=0}\,, \tag{29}$$

where $(D\varphi:\,D\psi)_t = (D\psi:\,D\varphi)_t^*$ is the Connes cocycle.

We now denote by $\mathcal{A}_\ell = \{\mathcal{A}_\ell(I)\}_{I\in\mathcal{K}}$ the conformal net associated to a level ℓ vacuum representation π of some loop group LG. As before, we will denote by $\mathcal{K}$ the set of all the open, non empty and non dense intervals of the circle. To each interval I in $\mathcal{K}$ we associate the von Neumann algebra

$$\mathcal{A}_\ell(I) = \{\tilde{\pi}(\gamma):\ \operatorname{supp}\gamma\subset I\}''\,, \tag{30}$$

where $\tilde{\pi}$ is the lift of π described in Remark 16 and the support of a loop γ is defined by

$$\operatorname{supp}\gamma = \overline{\{z\in S^1:\ \gamma(z)\neq e\}}\,.$$

If the interval I does not contain -1, then it can be identified with an open interval of the real line through the Cayley transform. On the real line picture the stress energy tensor is given by $\Theta(f) = T(C_*f)$, with C_*f the push-forward of the vector field $f(u)\frac{d}{du}$ by the Cayley transform $C(u) = (1+iu)/(1-iu)$ [17]. Notice that even if we can identify the function f with the vector field $f(u)\frac{d}{du}$, the push-forward C_*f has to be intended as a pushforward of vector fields and not just a composition of functions.

We now come back to our work. We are interested in computing

$$S(t) = S_{\mathcal{A}_\ell(t,+\infty)}(\omega_\gamma\|\omega)\,, \tag{31}$$

where ω is the vacuum state represented by the vacuum vector Ω and $\omega_\gamma = \omega\cdot\operatorname{Ad}\pi(\gamma)^*$ is represented by $\pi(\gamma)\Omega$ for some loop γ in LG. More in general, the same result will apply to the solitonic states given by the solitons (13) of above. We introduce the groups of Sobolev loops

$$B(z_1,\ldots,z_n) = \left\{\gamma\in H^2(S^1,G):\,\gamma(z_i) = e\,,\,\dot{\gamma}(z_i) = 0\right\}. \tag{32}$$

By standard arguments, continuously differentiable and piecewise smooth loops are in $H^s(S^1,G)$ for $s<5/2$ [11,17], where we say that γ is piecewise smooth if right and left derivatives always exist and if γ is smooth except on a finite number of points. If there is no ambiguity, we will use a similar notation to denote the groups (32) in the real line picture. Consider now the interval $I = (z,w)$ of S^1 obtained by moving counterclockwise from z to w. We will denote by γ_I the map such that $\gamma_I = \gamma$ on $[z, w)$ and $\gamma_I = e$ on $[w, z)$, so that we have the identity

$$\gamma = \gamma_I \gamma_{I'} . \tag{33}$$

By Theorem 12 we have that if γ is a loop in $B(z, w)$ then in $PU(\mathcal{H})$ we have $\pi(\gamma) = \pi(\gamma_{(z,w)})\pi(\gamma_{(w,z)})$. In particular, in this case $\pi(\gamma_{(z,w)})$ is in $\mathcal{A}_\ell((z, w))$ and $\pi(\gamma_{(w,z)})$ is in $\mathcal{A}_\ell((w, z))$. We also recall that by the Bisognano-Wichmann theorem (11) we have the identity $\log \Delta = -2\pi D$ with $D = -\frac{i}{2}(L_1 - L_{-1})$, that is $\log \Delta = -2\pi T(\delta)$ with δ the vector field generating $\delta(s).u = e^s u$. Notice also that the vacuum expectation of

$$\pi(\gamma)^* T(\delta)\pi(\gamma) = T(\delta) + i\pi(\delta\gamma^{-1}\dot{\gamma}) + b(\gamma, \delta) \tag{34}$$

is given by the real constant $b(\gamma, \delta)$ described in Corollary 20.

Proposition 21. *Let γ be a loop in $H^3(S^1, G)$. Pick a non dense open interval $I = (z, w)$ of the circle and write $\gamma = \gamma_I \gamma_{I'}$ as in (33). Denote by δ_I the generator of dilations of the interval I and set $\Delta_I^{it} = e^{-2\pi i t T(\delta_I)}$. If $\dot{\gamma}$ vanishes on the boundary of I then the Connes cocycle $(D\omega_\gamma : D\omega)_t$ of $\mathcal{A}_\ell(I)$ is given by*

$$(D\omega_\gamma : D\omega)_t = e^{it(a - 2\pi c(\gamma_I, \delta_I))} e^{-2\pi t(i\pi(\delta_I) + \pi(\delta_I \dot{\gamma}_I \gamma_I^{-1}))} \Delta_I^{-it} \tag{35}$$

for some $a = a_\gamma$ in $\mathbb{R}$. In particular, a depends only on the values of γ at the boundary of I and $a_\gamma = 0$ if $\gamma(z) = e$ for z in the boundary of I.

Proof. First we check that $\delta_I \dot{\gamma}_I \gamma_I^{-1}$ is in $H^2(S^1, \mathfrak{g}_0)$ since it vanishes with its first derivative on the boundary of I. Hence the right hand side of (35), which we denote by u_t, is a well defined unitary operator which is in $\mathcal{A}_\ell(I)$ by the Trotter product formula. To prove the existence of a in $\mathbb{R}$ as in the statement it suffices to check that u_t verifies the relations

(i) $\sigma_t^\gamma(x) = u_t \sigma_t(x) u_t^* , \quad x \in \mathcal{A}_\ell(I) ,$
(ii) $u_{t+s} = u_t \sigma_t(u_s) .$

Here σ_t and σ_t^γ are the modular automorphisms associated to the states ω and ω_γ. The first relation follows by noticing that

$$\begin{aligned} \sigma_t^\gamma(x) &= \mathrm{Ad}\, \Delta_{I,\gamma}^{it}(x) = \mathrm{Ad}\, \pi(\gamma)\Delta_I^{it}\pi(\gamma)^*(x) \\ &= \mathrm{Ad}\, \pi(\gamma)\Delta_I^{it}\pi(\gamma)^*\Delta_I^{-it}\Delta_I^{it}(x) \\ &= \mathrm{Ad}\, u_t \cdot \sigma_t(x) , \end{aligned} \tag{36}$$

where we used Lemma 3.(ii) of [26] and Theorem 19. The second relation can be easily verified and thus a does exist. Now we prove that $a = a_\gamma$ depends only on the values of γ at the boundary of I. Consider η in $H^3(S^1, G)$ such that $\eta(z) = e$ and $\dot{\eta}(z) = 0$ for z in the boundary of I. Notice that $(D\omega_{\eta\gamma} : D\omega)_t = \pi(\eta)(D\omega_\gamma : D\omega)_t \sigma_t(\pi(\eta)^*)$. Therefore, with the notation of Corollary 20 we have

$$\begin{aligned} a_{\eta\gamma} + 2\pi b(\eta_I\gamma_I, \delta_I) &= -i\frac{d}{dt}\omega_{\eta\gamma}((D\omega_{\eta\gamma} : D\omega)_t)\Big|_{t=0} \\ &= -i\frac{d}{dt}\omega_{\eta\gamma}(\pi(\eta)(D\omega_\gamma : D\omega)_t\sigma_t(\pi(\eta)^*))\Big|_{t=0} \\ &= a_\gamma + 2\pi(b(\gamma_I, \delta_I) + b(\eta_I\gamma_I, \delta_I) - b(\gamma_I, \delta_I)) , \end{aligned}$$

and by the identity $a_{\eta\gamma} = a_\gamma$ the assertion is proved. If $\gamma(z) = e$ for z in the boundary of I then $\pi(\gamma_I)$ is in $\mathcal{A}_\ell(I)$ and the last statement follows by Lemma 3.(iv) of [26]. □

Remark 22. If γ is an element of $H^2([-\pi,\pi],G)$, then as in the smooth case we can consider the soliton σ_γ given above by (13). In particular, Proposition 21 still holds for the solitonic states $\omega_\gamma=\omega\cdot\sigma_\gamma^{-1}$ with γ in $H^3([-\pi,\pi],G)$. This follows from the fact that if η is a loop in $H^3(S^1,G)$ such that $\gamma=\eta$ on I, then $\omega_\eta=\omega_\gamma$ on $\mathcal{A}(I)$.

Now we arrive to the main part of this work, that is we will use the previous results to prove the QNEC (1) on loop groups models for the solitonic states $\omega_\gamma=\omega\cdot\sigma_\gamma^{-1}$ given by (13). In the real line picture, the path γ corresponds to an element of $H^2(\mathbb{R},G)$.

Theorem 23. *Let $\omega_\gamma=\omega\cdot\sigma_\gamma^{-1}$ be a solitonic state corresponding, in the real line picture, to some element γ of $H^2(\mathbb{R},G)$. Then the relative entropy (31) is finite for every t in $\mathbb{R}$ and explicitly given by*

$$S(t)=-\frac{\ell}{2}\int_t^\infty(u-t)\langle\dot\gamma\gamma^{-1},\dot\gamma\gamma^{-1}\rangle du\,. \tag{37}$$

Proof. As discussed in Remark 22, we can suppose γ to be the real line parametrization of some element of $H^2(S^1,G)$. Since the vacuum is G-invariant, we can replace γ with γg for any g in G, thus we can suppose $\gamma(\infty)=e$. By using the real line picture notation for the groups (32), we first suppose γ to be in $B(\infty)$. We point out that if $\gamma(t)=e$ and $\dot\gamma(t)=0$ then $S(t)$ is finite and given by (37) since we can use equation (29), Proposition 21 and the continuity of $\omega_\gamma((D\omega_\gamma:D\omega)_t)$ with respect to γ in $H^2(S^1,G)$. Now we prove that $S(t)$ is finite for any t real. Indeed, for any t real we can pick a smooth loop η with $\operatorname{supp}\eta\le t$ and such that $\eta(t-k)=\gamma(t-k)^{-1}$ and $(\dot{\eta\gamma})(t-k)=0$ for some $k>0$. This implies that

$$S_{\mathcal{A}_\ell(t,+\infty)}(\omega_\gamma\|\omega)=S_{\mathcal{A}_\ell(t,+\infty)}(\omega_{\eta\gamma}\|\omega)\le S_{\mathcal{A}_\ell(t-k,+\infty)}(\omega_{\eta\gamma}\|\omega)<+\infty\,,$$

where the last relative entropy is finite by the argument used above. By similar arguments we have that

$$\bar S(t)=S_{\mathcal{A}_\ell(-\infty,t)}(\omega_\gamma\|\omega) \tag{38}$$

is finite for any t real. Now we focus on the case $t=0$, since the general case follows by covariance. We suppose $\dot\gamma(0)=0$ and we write $\gamma=\gamma_+\gamma_-$, with $\gamma_+(u)=e$ for $u\le 0$ and $\gamma_-(u)=e$ for $u\ge 0$. By Proposition 21 we have

$$S(0)=a_\gamma-\frac{\ell}{2}\int_0^\infty u\langle\dot\gamma\gamma^{-1},\dot\gamma\gamma^{-1}\rangle du\,.$$

Now we emulate some techniques used in [17] and we prove that $a_\gamma=0$. Given $\lambda>0$ real, consider the function $f(u)=ue^{\lambda u}$. For $n>0$ integer, we consider a smooth diffeomorphism $\rho=\rho_{\lambda,n}$ of the circle such that, in the real line picture, it verifies $\rho(u)=f(u)$ for $0\le u\le n-\frac{1}{n}$ and $\rho(u)=f'(n)u+(f(n)-nf'(n))$ for $u\ge n$. We also suppose $\rho(u)/\rho'(u)$ to be uniformly bounded for $n-\frac{1}{n}\le u\le n$. Consider now the loop $\gamma_{\lambda,n}(u)=\gamma(\rho_{\lambda,n}^{-1}(u))$. By the identity $a_\gamma=a_{\gamma_{\lambda,n}}$ and by monotone convergence once more we have

$$0\le\inf_\lambda S_{\mathcal{A}_\ell(0,+\infty)}(\omega_{\gamma_{\lambda,n}}\|\omega)=a_\gamma-\frac{\ell}{2}\int_n^\infty(u-n)\langle\dot\gamma\gamma^{-1},\dot\gamma\gamma^{-1}\rangle du\,, \tag{39}$$

and by monotone convergence we have $a_\gamma \geq 0$. Now we prove the other inequality. Consider a smooth path ζ_n in G with extremes $\zeta(0) = e$ and $\zeta(1) = \gamma(0)$. We also suppose that $\dot{\zeta}(0) = \dot{\zeta}(1) = 0$. We now define

$$\gamma_n(u) = \begin{cases} \gamma(u) & u \geq 0\,, \\ \zeta(nu+1) & -1/n \leq u \leq 0\,, \\ e & u \leq -1/n\,. \end{cases}$$

By monotonicity $S_{\mathcal{A}_\ell(0,+\infty)}(\omega_\gamma \| \omega) = S_{\mathcal{A}_\ell(0,+\infty)}(\omega_{\gamma_n} \| \omega) \leq S_{\mathcal{A}_\ell(-1/n,+\infty)}(\omega_{\gamma_n} \| \omega)$, so that after a limit we have the inequality

$$a_\gamma \leq -\frac{\ell}{2} \int_0^1 u \langle \dot{\zeta}\zeta^{-1}, \dot{\zeta}\zeta^{-1} \rangle du\,.$$

However, if we now consider the function $g_\lambda(u) = ue^{\lambda(u-1)}$ and we define $\zeta_\lambda(u) = \zeta(g_\lambda^{-1}(u))$, then

$$a_\gamma \leq -\frac{\ell}{2} \int_0^1 u \langle \dot{\zeta_\lambda}\zeta_\lambda^{-1}, \dot{\zeta_\lambda}\zeta_\lambda^{-1} \rangle du \leq -\frac{\ell}{2\lambda} \int_0^1 u \langle \dot{\zeta}\zeta^{-1}, \dot{\zeta}\zeta^{-1} \rangle du \to 0\,, \qquad \lambda \to +\infty\,.$$

Finally, we have proved that $a_\gamma = 0$ if $\dot{\gamma}(0) = 0$. To remove this condition, we notice that if P is the generator of translations then the average energy in the state ω_γ is finite and given by

$$E_\gamma = (\pi(\gamma)\Omega | P\pi(\gamma)\Omega) = -\frac{\ell}{2} \int_{-\infty}^{+\infty} \langle \dot{\gamma}\gamma^{-1}, \dot{\gamma}\gamma^{-1} \rangle \frac{du}{2\pi}\,. \tag{40}$$

Therefore we can apply Lemma 1. of [7], namely for every t_1 and t_2 in $\mathbb{R}$ we have

$$(S(t_1) - S(t_2)) + (\bar{S}(t_2) - \bar{S}(t_1)) = (t_2 - t_1)2\pi E_\gamma\,. \tag{41}$$

This implies that $S(t)$ and $\bar{S}(t)$ are both Lipschitz functions. Consider now a smooth real function $\rho(u)$ defined on $[0, 1]$ and such that $\rho(0) = 0$ and $\rho(1) = 1$. We also suppose $\rho'(0) = \rho''(0) = 0$, $\rho'(1) = 1$ and $\rho''(1) = 0$. We define

$$\gamma_n(u) = \begin{cases} \gamma(u) & u \geq 1/n\,, \\ \gamma(\rho(nu)/n) & 0 \leq u \leq 1/n\,, \\ \eta(u) & u \leq 0\,, \end{cases}$$

where η is a smooth function such that γ_n is in $H^2(S^1, G)$. Therefore, by (41) we have

$$0 \leq S_{\mathcal{A}_\ell(0,+\infty)}(\omega_{\gamma_n} \| \omega) - S_{\mathcal{A}_\ell(1/n,+\infty)}(\omega_{\gamma_n} \| \omega) \leq \frac{2\pi}{n} E_{\gamma_n} \to 0\,,$$

and thus we have

$$\begin{aligned} S_{\mathcal{A}_\ell(0,+\infty)}(\omega_\gamma \| \omega) &= \lim_n S_{\mathcal{A}_\ell(1/n,+\infty)}(\omega_\gamma \| \omega) = \lim_n S_{\mathcal{A}_\ell(1/n,+\infty)}(\omega_{\gamma_n} \| \omega) \\ &= \lim_n S_{\mathcal{A}_\ell(1/n,+\infty)}(\omega_{\gamma_n} \| \omega) - S_{\mathcal{A}_\ell(0,+\infty)}(\omega_{\gamma_n} \| \omega) + S_{\mathcal{A}_\ell(0,+\infty)}(\omega_{\gamma_n} \| \omega) \\ &= \lim_n S_{\mathcal{A}_\ell(0,+\infty)}(\omega_{\gamma_n} \| \omega) \end{aligned}$$

$$= -\frac{\ell}{2}\int_0^\infty u\langle\dot\gamma\gamma^{-1},\dot\gamma\gamma^{-1}\rangle du\,. \tag{42}$$

The most of the work is done. Now we just have to remove the condition $\dot\gamma(\infty) = 0$. If we apply covariance to equation (42) then we have

$$S_{\mathcal{A}_\ell(-\infty,0)}(\omega_\gamma\|\omega) = -\frac{\ell}{2}\int_{-\infty}^0 u\langle\dot\gamma\gamma^{-1},\dot\gamma\gamma^{-1}\rangle du$$

for any γ in $H^2(S^1, G)$ such that $\dot\gamma(0) = 0$. But this condition can be removed as in (42), and by covariance we have that the above expression of $S(0)$ holds for a generic loop γ in $H^2(S^1, G)$. □

Corollary 24. *If E_γ is the null energy (40), then we have the Bekenstein Bound*

$$S_{\mathcal{A}_\ell(-r,r)}(\omega_\gamma\|\omega) \le \pi r E_\gamma\,, \tag{43}$$

where

$$S_{\mathcal{A}_\ell(-r,r)}(\omega_\gamma\|\omega) = -\frac{\ell}{2}\int_{-r}^r \frac{1}{2r}(r-u)(r+u)\langle\dot\gamma\gamma^{-1},\dot\gamma\gamma^{-1}\rangle du\,. \tag{44}$$

Furthermore, the complement relative entropy (38) is given by

$$\bar S(t) = -\frac{\ell}{2}\int_{-\infty}^t (t-u)\langle\dot\gamma\gamma^{-1},\dot\gamma\gamma^{-1}\rangle du\,. \tag{45}$$

Proof. As in the previous theorem, it is not restrictive to suppose γ to be in $H^2(S^1, G)$. The statement then follows by Möb-covariance, since in general we have

$$S_{\mathcal{A}_\ell(a,b)}(\omega_\gamma\|\omega) = -\frac{\ell}{2}\int_a^b D_{(a,b)}(u)\langle\dot\gamma\gamma^{-1},\dot\gamma\gamma^{-1}\rangle du\,, \tag{46}$$

for every interval (a, b) of the real line, with $D_{(a,b)}(u)$ the density of the dilation operator of (a, b). □

Now we use the previous theorem to discuss the QNEC (1). In this case we have that the second derivative $S''(t)$ exists everywhere in the classical sense with $S''(t) \ge 0$. If P is the generator of translations, then by the Sugawara formula we have $P = \Theta(\frac{d}{du})$, hence the quantity $E = E_\gamma$ given by (40) is an averaged stress energy tensor in the null direction u in the state ω_γ.

Theorem 25. *Let γ be an element of $H^2(\mathbb{R}, G)$ as in Theorem 23. If we consider the null energy density*

$$E_\gamma(t) = -\frac{\ell}{4\pi}\langle\dot\gamma\gamma^{-1},\dot\gamma\gamma^{-1}\rangle(t)\,, \tag{47}$$

then the states ω_γ verify the QNEC (1), with

$$E_\gamma(t) = S''(t)/2\pi \ge 0\,. \tag{48}$$

Similarly, $E_\gamma(t) = \bar S''(t)/2\pi \ge 0$, where $\bar S(t)$ is the complement relative entropy (38) given by (45).

This theorem is the main result of this work. However, definition (47) may seem not rigorous to the reader, since we can arbitrarily add a function with null average to the integral (40). For this reason, we will now motivate our definition of stress energy tensor density. Given h in $\mathcal{S}_{3/2}$, the notation

$$T(h) = \int_{S^1} T(z)h(z)dz\,, \quad T(z) = -\frac{1}{2\pi}\sum_{n=-\infty}^{+\infty} z^{-n-2}L_n\,, \tag{49}$$

is widely used. Therefore, in general we can consider two vectors ξ and η in $\mathcal{V} = \bigcap_{k\geq 0}\mathcal{D}(L_0^k)$, recall that

$$|(\eta|T(h)\xi)| \leq (c/2)^{1/2}|h|_{3/2}\|\eta\|\|(1+L_0)\xi\|\,,$$

and define $(\eta|T(z)\xi)$ as the kernel of some tempered distribution. This definition is consistent by defining $(\eta|T(z)\xi)$ by using (49) if ξ and η are in $\mathcal{H}^{0,\mathrm{fin}}$. By Corollary 20, in the real line picture in our case we have

$$(\pi(\gamma)\Omega|T(h)\pi(\gamma)\Omega) = -\frac{\ell}{4\pi}\int h(t)\langle\dot{\gamma}\gamma^{-1}, \dot{\gamma}\gamma^{-1}\rangle(t)dt\,,$$

for every h, and this motivates (47) since in general we expect the identity

$$(\xi|T(h)\xi) = \int_{S^1} h(z)(\xi|T(z)\xi)dz$$

to hold. In addition, we can also show a different way to recover (47) by using some results of [7].

Let $\mathcal{N} \subseteq \mathcal{M}$ be a -hsm inclusion with corresponding family of von Neumann algebras $(\mathcal{M}_t)_{t\in\mathbb{R}}$. We denote by $P \geq 0$ the generator of translations and by ω the faithful normal state given by the common standard vector Ω. Given two real parameters $t < t'$, consider a normal state ψ of $\mathcal{M}_t$ with representing vector η. If u is some isometry, then we will denote by ψ_u the vector state represented by $u\eta$. We will use the notation $P_\eta = (\eta|P\eta)$. We define

$$E_\psi(t,t') = \inf_{(w',w)\in C_t'\times C_{t'}} P_{ww'\eta}\,, \tag{50}$$

where C_t' is the family of all the isometries w' in $\mathcal{M}_t'$ such that $P_{w'\eta}$ and $S_{\mathcal{M}_t'}(\psi_{w'}\|\omega)$ are both finite, and similarly $C_{t'}$ is the family of all the isometries w in $\mathcal{M}_{t'}$ such that $P_{w\eta}$ and $S_{\mathcal{M}_{t'}}(\psi_w\|\omega)$ are finite. Notice that $E_\psi(t,t')$ is well defined as a state-dependent quantity, since any two vectors which represent ψ on $\mathcal{M}_t$ differ by an isometry of $\mathcal{M}_t'$.

Proposition 26. *Let $\mathcal{A}$ be a von Neumann algebra on $\mathcal{H}$ and let $U_s = e^{-isP}$ be a one parameter strongly continuous unitary group such that $U_{-s}\mathcal{A}U_s \subseteq \mathcal{A}$ for $s \geq 0$. If u and u' are isometries in $\mathcal{A}$ and $\mathcal{A}'$, then for every vector ξ in $\mathcal{H}$ we have*

$$P_{uu'\xi} + P_\xi = P_{u\xi} + P_{u'\xi}\,,$$

under the assumption that the quantities $|P|_{uu'\xi}$, $|P|_{u\xi}$, $|P|_{u'\xi}$, and $|P|_\xi$ are all finite.

Proof. Take $s > 0$ and consider

$$D = (\xi|U_{-s}\xi) + (uu'\xi|U_{-s}uu'\xi) - (u\xi|U_{-s}u\xi) - (u'\xi|U_{-s}u'\xi)\,.$$

Note that $(uu'\xi|U_{-s}uu'\xi) = (\xi|(u^*U_{-s}uU_{-s})(u')^*U_{-s}u'\xi)$, where we used the fact that $u^*U_{-s}uU_s$ belongs to $\mathcal{A}$ for $s > 0$. Thanks to this remark, we can write $D = D_1 + D_2 + D_3 + D_4$, where

$$\begin{aligned} D_1 &= (u^*(U_s - 1)u\xi|U_s(u')^*(U_{-s} - 1)u'\xi) \\ D_2 &= (u^*(U_s - 1)u\xi|(U_s - 1)\xi) \\ D_3 &= ((U_{-s} - 1)\xi|(u')^*(U_{-s} - 1)u'\xi) \\ D_4 &= -((U_{-s} - 1)\xi|(U_{-s} - 1)\xi)\,, \end{aligned} \tag{51}$$

and so we have the estimate $|D| \leq |D_1| + |D_2| + |D_3| + |D_4|$. We can bound all of these terms as $|D_i| \leq \|(U_s - 1)\eta_1\|\,\|(U_s - 1)\eta_2\|$, where $\eta_1, \eta_2 \in \{\xi, u\xi, u'\xi\}$. For ζ in $\{\xi, u\xi, u'\xi\}$ we can use the spectral representation of P to write, for $s > 0$, the identity

$$(\zeta|(1 - U_s)\zeta)/s = \int \frac{1 - e^{-is\lambda}}{s}\, d(\zeta|E_\lambda(P)\zeta)\,.$$

By $|1-e^{-is\lambda}|/s \leq |\lambda|$ and by the finiteness of $|P|_\zeta = (\zeta||P|\zeta)$ we can use the dominated convergence theorem, and so we have

$$\lim_{s\to 0^+}(\zeta|(1 - U_s)\zeta)/s = iP_\zeta\,.$$

It follows that

$$\lim_{s\to 0^+}\frac{\|(U_s - 1)\zeta\|^2}{s} = \lim_{s\to 0^+} 2\mathrm{Re}(\zeta|(1 - U_s)\zeta)/s = 0\,,$$

and so by the estimates above we finally obtain that $D/s \to 0$ for $s \to 0^+$. Therefore

$$0 = \lim_{s\to 0^+} D/s = P_\xi + P_{uu'\xi} - P_{u\xi} - P_{u'\xi}\,,$$

and the thesis follows. □

The previous proposition is an intermediate result used in the proof of Theorem 1. Finally, by using the proof of Theorem 1 and Proposition 26 we have the following fact.

Proposition 27. *Given two real parameters $t < t'$, consider a normal state ψ of $\mathcal{M}_t$ with representing vector η such that $P_\eta < +\infty$. Consider the Connes cocycles*

$$u'_s(t) = (D\psi : D\omega\,;\mathcal{M}'_t)_s\,,\quad u_s(t') = (D\psi : D\omega\,;\mathcal{M}_{t'})_s\,.$$

If the relative entropies $S(t) = S_{\mathcal{M}_t}(\psi\|\omega)$ and $\bar{S}(t') = S_{\mathcal{M}'_{t'}}(\psi\|\omega)$ are finite, then

$$E_\psi(t,t') = \inf_{s,s'} P_{u'_s(t)\eta} + P_{u_{s'}(t')\eta} - P_\eta = \lim_{s\to+\infty} P_{u'_s(t)\eta} + P_{u_{-s}(t')\eta} - P_\eta\,. \tag{52}$$

In other words, what we did was just to notice by the proof of Theorem 1 that, under some finiteness assumptions, the null energies of all the representing vectors for a normal state are minimized by the Connes cocycles. Notice also that by Theorem 1 and (41) we have

$$E_{\psi}(t,t') = -S'(t)/2\pi + \bar{S}'(t')/2\pi - P_{\eta}\,.$$

Finally, we can define

$$E_{\psi}(t) = \liminf_{h\to 0^+} E_{\psi}(t,t+h)/h\,. \tag{53}$$

After this premise, we can show that the density (47) is actually given by the limit (53). In this step we will use the results of [24], which ensures us that a PER of a loop group LG can be extended to a PER of $H^1(S^1, G)$. In particular, this implies that Theorem 23 and Theorem 25 are still true in this generality. The same argument applies to Proposition 31 below as well. Furthermore, as shown later in Proposition 31, we can compute (50) by using (52). By doing so we have

$$E_{\gamma}(t,t') = -\frac{\ell}{4\pi}\int_t^{t'} \langle\dot{\gamma}\gamma^{-1}, \dot{\gamma}\gamma^{-1}\rangle du\,.$$

This tells us that the null energy density (47) can be recovered by using (53).

5.1. QNEC on $LSU(n)$. In this section we focus on the case $G = SU(n)$ and we use a construction illustrated in [30] to show that a positive energy representation of $LSU(n)$ can be extended to a positive energy representation of the Sobolev group $H^s(S^1, SU(n))$ for $s > 1/2$. In particular, we will use this fact to provide a simpler proof of the QNEC (48).

We begin by considering the natural action of $G = SU(n)$ on $V = \mathbb{C}^n$ and we set $H = L^2(S^1, V)$, or equivalently $H = L^2(S^1)\otimes V$. We can naturally define a continuous action M of LG on H by $M_{\gamma} f(\phi) = \gamma(\phi) f(\phi)$. We can also define an action of Rot on H by $R_{\theta} f(\phi) = f(\phi - \theta)$ with respect to the representation of LG is covariant, that is it satisfies $R_{\theta} M_{\gamma} R_{\theta}^{-1} = M_{R_{\theta}\gamma}$. If P is the orthogonal projection onto the Hardy space H_+, namely

$$H_+ = \left\{ f \in L^2(S^1, V)\colon f(\theta) = \sum_{k\geq 0} f_k e^{ik\theta} \text{ with } f_k \in V \right\},$$

then we can define a new Hilbert space H_P which is equivalent to H as real Hilbert space, but with complex structure given by $J = iP - i(1-P)$. The Segal quantization criterion, which we now recall, allows us to define a positive energy representation of LG on the fermion Fock space $\mathcal{F}_P = \Lambda H_P$ known as the *fundamental representation* of $LSU(n)$ [27,30]. Notice that $\mathcal{F}_P(0) = \Lambda V$ is the fundamental representation of $SU(n)$. The fundamental representation of $LSU(n)$ is the direct sum of all the $n+1$ irreducible positive energy representations of $LSU(n)$ of level $\ell = 1$. The fundamental representation contains the *basic representation*, that is the unique level one vacuum representation.

Definition 28. The *restricted unitary group* is the topological group

$$U_P(H) = \left\{ u \in U(H)\colon [u, P] \in L^2(H) \right\},$$

where the considered topology is the strong operator topology combined with the metric given by the distance $d(u, v) = \|[u - v, P]\|_2$.

Any $u \in U(H)$ gives rise to an automorphism of $CAR(H)$, called *Bogoliubov automorphism*, via $a(f) \mapsto a(uf)$. For every projection P on H there is an irreducible representation of $CAR(H)$ on $\mathcal{F}_P$ which is denoted by π_P. The Bogoliubov automorphism is said to be *implemented* on $\mathcal{F}_P$ if $\pi_P(a(uf)) = U\pi_P(a(f))U^*$ for some unitary $U \in U(\mathcal{F}_P)$ [30].

Theorem 29. Segal's quantization criterion. [30] *If $[u, P]$ is a Hilbert-Schmidt operator then u is implemented on $\mathcal{F}_P$ by some unitary operator U_P. Moreover, U_P is unique up to a phase and the constructed map $U_P(H) \to PU(\mathcal{F}_P)$ is continuous.*

Proposition 30. *The fundamental representation of $LSU(n)$ can be extended to $H^s(S^1, SU(n))$ for any $s > 1/2$. In particular, every positive energy representation of $LSU(n)$ extends to a positive energy representation of $H^s(S^1, SU(n))$ for $s > 1/2$.*

Proof. Notice that since a loop γ in $LSU(n)$ is also a map from S^1 to $M_n(\mathbb{C})$, then we can write γ as a Fourier series $\gamma(z) = \sum \widehat{\gamma}_k z^k$, where $\widehat{\gamma}_k \in M_n(\mathbb{C})$. We consider on H the basis $e_j^k(z) = z^k e_j$, where (e_j) is the standard basis of $\mathbb{C}^n$. We define $M_{pq} = \widehat{\gamma}_{p-q}$ and we note that $M_\gamma e_j^k = \sum_i M_{ik} e_j^i$, so that $(e_i^p, M_\gamma e_j^q) = (e_i, M_{pq} e_j)$. So M_γ is represented by a $\mathbb{Z} \times \mathbb{Z}$ matrix (M_{pq}) of endomorphisms. We have

$$
\begin{aligned}
\|[P, M_\gamma]\|_2^2 &= \sum_{p \geq 0, q<0} \|M_{pq}\|_2^2 + \sum_{p<0, q \geq 0} \|M_{pq}\|_2^2 \\
&= \sum_{k>0} k \|\widehat{\gamma}_k\|_2^2 - \sum_{k<0} k \|\widehat{\gamma}_k\|_2^2 \\
&= \sum_{k \in \mathbb{Z}} |k| \|\widehat{\gamma}_k\|_2^2 \leq \sum_{k \in \mathbb{Z}} (1+|k|)^{2s} \|\widehat{\gamma}_k\|_2^2 ,
\end{aligned}
$$

for $s > 1/2$. It is easy to verify that the map $\gamma \mapsto M_\gamma \in U_P(H)$ is continuous. We also have that the rotation group acts on $H^s(S^1, G)$ by continuous operators (see Lemma A.3 of [4]), and by $[R_\theta, P] = 0$ we have that the projective representation of Rot is actually a strongly continuous unitary representation. Therefore, the thesis follows by the Segal quantization criterion, the complete reducibility of positive energy representations (Thm. 9.3.1. of [27]), Proposition 2.3.3. of [29] and remarks below. □

Proposition 31. *Let γ be a loop in $H^1(S^1, SU(n))$. Pick a non dense open interval $I = (z, w)$ of the circle and write $\gamma = \gamma_I \gamma_{I'}$ as in (33). Then, in $PU(\mathcal{H})$ we have*

$$
(D\omega_\gamma : D\omega)_t = \pi(\gamma_I \delta_I(-2\pi t).\gamma_I^{-1}) , \tag{54}
$$

where $(D\omega_\gamma : D\omega)_t$ is the Connes cocycle of $\mathcal{A}_\ell(I)$ and $\delta_I(t)$ denotes the dilation associated to I.

Proof. First we check that $\gamma_I \delta_I(t).\gamma_I^{-1}$ is in $H^1(S^1, SU(n))$ since it is continuous on the boundary of I, hence the right hand side of (54) is well defined. With the same computations of Proposition 21 we have that $\sigma_t^\gamma(x) = \mathrm{Ad}\,\pi(\gamma_I \delta_I(-2\pi t).\gamma_I^{-1}) \cdot \sigma_t(x)$ for x in $\mathcal{A}_\ell(I)$. Therefore, we have that $(D\omega_\gamma : D\omega)_t$ is equal to $\pi(\gamma_I \delta_I(-2\pi t).\gamma_I^{-1})$ up to a unitary V in the commutant of $\mathcal{A}_\ell(I)$, but $(D\omega_\gamma : D\omega)_t$ and $\pi(\gamma_I \delta_I(-2\pi t).\gamma_I^{-1})$ are both in $\mathcal{A}_\ell(I)$ and thus V is a scalar. □

Theorem 32. *Let γ be a loop in $H^1(S^1, SU(n))$. Suppose also that, in the real line picture, the support of γ is bounded from below. Then the relative entropy (31) is finite and given by*

$$S(t) = -\frac{\ell}{2}\int_t^{\infty}(u-t)\langle\dot\gamma\gamma^{-1}, \dot\gamma\gamma^{-1}\rangle du\,. \tag{55}$$

In particular, the QNEC (1) is satisfied as shown above in Theorem 25.

Proof. Since the vacuum is $SU(n)$-invariant, we can replace γ with γg for any g in $SU(n)$, thus we can suppose $\gamma(\infty) = e$. As above, if $\gamma(t) = e$ then $S(t)$ is finite and given by (55). We can prove that $S(t)$ is finite for any t real as in Theorem 23, and similarly we have that $\bar S(t) = S_{\mathcal{A}_\ell(-\infty,t)}(\omega_\gamma\|\omega)$ is finite for any t real. If P is the generator of translations then the average energy E_γ in the state ω_γ is finite and given by equation (40). Therefore we can apply Lemma 1. of [7], and equation (41) holds. This implies that $S(t)$ and $\bar S(t)$ are both Lipschitz functions and in particular they are absolutely continuous. The next step is an estimate of $S'(t)$. For simplicity we focus on the case $t = 0$ and we write $\gamma = \gamma_+\gamma_-$ with $\gamma_+(u) = e$ for $u \le 0$ and $\gamma_-(u) = e$ for $u \ge 0$. By Proposition 31 the Connes cocycle $u'_s = (D\omega\colon D\omega_\gamma)_s$ on $\mathcal{A}_\ell(-\infty, 0)$ is equal in $PU(\mathcal{H})$ to $\pi(\gamma_-\delta(2\pi s).\gamma_-^{-1})^*$. But also the state $\omega_\gamma\cdot \mathrm{Ad}(u'_s)^*$ verifies the finiteness conditions required to apply Lemma 1. and thus we have $-S'(0) \le 2\pi E_s$, where $E_s = (u'_s\pi(\gamma)\Omega|Pu'_s\pi(\gamma)\Omega)$ for s real. However, one can simply prove that

$$\inf_s 2\pi E_s = -\frac{\ell}{2}\int_0^{+\infty}\langle\dot\gamma\gamma^{-1}, \dot\gamma\gamma^{-1}\rangle du\,.$$

Therefore, by repeating the argument with any t in $\mathbb{R}$ we have

$$-S'(t) \le -\frac{\ell}{2}\int_t^{+\infty}\langle\dot\gamma\gamma^{-1}, \dot\gamma\gamma^{-1}\rangle du\,.$$

Finally, if we define

$$F(t) = -\frac{\ell}{2}\int_t^{\infty}(u-t)\langle\dot\gamma\gamma^{-1}, \dot\gamma\gamma^{-1}\rangle du\,,$$

then we can conclude that $S(t) = F(t)$ for any t in $\mathbb{R}$. Indeed, if the support of γ is compact then $H(t) = S(t) - F(t)$ is an absolutely continuous function with nonnegative derivative and going to 0 as $|t| \to +\infty$. If the support of γ is contained in $(k, +\infty)$ then by lower semicontinuity $S(t) \le F(t)$ for every t real, and we can similarly deduce that $H(t) = 0$ for every t real. The QNEC inequality (48) can be proved as above, with the only exception that in this case we do not have to use [24] to compute (47). $\square$

Acknowledgement. I thank Simone Del Vecchio for suggesting me the problem, and Daniela Cadamuro and Henning Bostelmann for the hospitality in the period spent in Leipzig. I also thank the reviewers for the many useful corrections.

Funding Open access funding provided by Università degli Studi di Roma La Sapienza within the CRUI-CARE Agreement.

References

1. Bargmann, V.: On unitary ray representations of continuous groups. Ann. Math. **59**, 1–46 (1954)
2. Bousquet, P., Ponce, A.C., Van Schaftingen, J.: Density of smooth maps for fractional Sobolev spaces $W^{s,p}$ into ℓ simply connected manifolds when $s \geq 1$. Conflu. Math. **5**(2), 3–22 (2013)
3. Bousso, R., Fisher, Z., Koeller, J., Leichenauer, S., Wall, A.C.: Proof of the quantum null energy condition. Phys. Rev. D **93**(2), 024017 (2016)
4. Carpi, S., et al.: Positive energy representations of Sobolev diffeomorphism groups of the circle. arXiv preprint arXiv:1808.02384 (2018)
5. Carpi, S., Weiner, M.: On the uniqueness of diffeomorphism symmetry in conformal field theory. Commun. Math. Phys. **258**(1), 203–221 (2005)
6. Carpi, S., et al.: From Vertex Operator Algebras to Conformal Nets and Back, No. 1213, vol. 254. American Mathematical Society, Providence (2018)
7. Ceyhan, F., Faulkner, T.: Recovering the QNEC from the ANEC. Commun. Math. Phys. **377**(2), 999–1045 (2020)
8. Ciolli, F., Longo, R., Ruzzi, G.: The information in a wave. Commun. Math. Phys. **379**(3), 979–1000 (2020)
9. Coulhon, T., Russ, E., Tardivel-Nachef, V.: Sobolev algebras on Lie groups and Riemannian manifolds. Am. J. Math. **123**(2), 283–342 (2001)
10. Claudio, K.F., Köster, S.: Implementation of conformal covariance by diffeomorphism symmetry. Lett. Math. Phys. **67**(3), 239–247 (2004)
11. Del Vecchio, S., Iovieno, S., Tanimoto, Y.: Solitons and nonsmooth diffeomorphisms in conformal nets. Commun. Math. Phys. **375**, 1–37 (2019)
12. Di Nezza, E., Palatucci, G., Valdinoci, E.: Hitchhiker's guide to fractional Sobolev spaces. Bull. Sci. Math. **136**(5), 521–573 (2012)
13. Epstein, H., Glaser, V., Jaffe, A.: Nonpositivity of the energy density in quantized field theories. Il Nuovo Cim. (1955–1965) **36**(3), 1016–1022 (1965)
14. Fewster, C.J., Hollands, S.: Quantum energy inequalities in two-dimensional conformal field theory. Rev. Math. Phys. **17**(05), 577–612 (2005)
15. Gabbiani, F., Fröhlich, J.: Operator algebras and conformal field theory. Commun. Math. Phys. **155**(3), 569–640 (1993)
16. Goodman, R., Wallach, N.R.: Structure and unitary cocycle representations of loop groups and the group of diffeomorphisms of the circle. J. Reine Angew. Math. **1984347**, 69–133 (1984)
17. Hollands, S.: Relative entropy for coherent states in chiral CFT. Lett. Math. Phys. **110**(4), 713–733 (2020)
18. Kac, V.G., Raina, A.K., Rozhkovskaya, N.: Bombay Lectures on Highest Weight Representations of Infinite Dimensional Lie Algebras, vol. 29. World Scientific, Singapore (2013)
19. Longo, R.: Algebraic and Modular Structure of von Neumann Algebras of Physics. Proceedings of Symposia in Pure Mathematics, vol. 38. American Mathematical Society, Providence (1982)
20. Longo, R.: An Analogue of the Kac–Wakimoto formula and black hole conditional entropy. Commun. Math. Phys. **186**(2), 451–479 (1997)
21. Longo, R., Feng, X.: Comment on the Bekenstein bound. J. Geometry Phys. **130**, 113–120 (2018)
22. Longo, R.: Entropy distribution of localised states. Commun. Math. Phys. **373**(2), 473–505 (2020)
23. Morinelli, V., Tanimoto, Y., Weiner, M.: Conformal covariance and the split property. Commun. Math. Phys. **357**(1), 379–406 (2018)
24. Neeb, K.-H.: Semibounded unitary representations of double extensions of Hilbert–loop groups. Annales de l'Institut Fourier **64**(5), 1823–1892 (2014)
25. Ohya, M., Petz, D.: Quantum Entropy and Its Use. Springer, Berlin (2004)
26. Panebianco, L.: A formula for the relative entropy in chiral CFT. Lett. Math. Phys. **110**, 2363–2381 (2019)
27. Pressley, A., Segal, G.: Loop Groups. Oxford Mathematical Monographs. Oxford University Press, Oxford (1986)
28. Reed, M., Simon, B.: Methods of Modern Mathematical Physics. Functional Analysis, vol. 1. Academic, New York (1980)
29. Toledano Laredo, V.: Fusion of positive energy representations of $LSpin_{2n}$. Dissertation University of Cambridge (1997)

30. Wassermann, A.: Operator algebras and conformal field theory III. Fusion of positive energy representations of $LSU(N)$ using bounded operators. Invent. Math. **133**(3), 467–538 (1998)
31. Weiner, M.: Conformal covariance and positivity of energy in charged sectors. Commun. Math. Phys. **265**(2), 493–506 (2006)

Communicated by H-T. Yau