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Estimating the interaction graph of stochastic neuronal dynamics by observing only pairs of neurons

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Abstract

We address the questions of identifying pairs of interacting neurons from the observation of their spiking activity. The neuronal network is modeled by a system of interacting point processes with memory of variable length. The influence of a neuron on another can be either excitatory or inhibitory. To identify the existence and the nature of an interaction we propose an algorithm based only on the observation of joint activity of the two neurons in successive time slots. This reduces the amount of computation and storage required to run the algorithm, thereby making the algorithm suitable for the analysis of real neuronal data sets. We obtain computable upper bounds for the probabilities of false positive and false negative detection. As a corollary we prove the consistency of the identification algorithm.

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Keywords: Neuronal networks; Multivariate point processes; Stochastic processes with memory of variable length; Interaction graphs; Statistical model selection

1. Introduction

We address the question of inferring the interactions in a system of spiking neurons modeled as follows. The spiking activity of each neuron is modeled as a point process whose intensity depends on the previous activity of a set of neighbors, henceforth called its presynaptic neurons. Each neuron i is affected by the spiking activity of its presynaptic neurons taking place after

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the last spiking time of i. This means that a neuron resets its memory after each of its spikes. This biologically motivated feature implies that this system of interacting point processes has a memory of variable length. This observation is at the origin of the article by Galves and Löcherbach (2013) [17], in a discrete time framework. The variable length memory feature was subsequently modeled in a continuous time framework by De Masi et al. (2015) [12], and many other articles including [1–3,14,16,24,26], and [28]. For a self-contained presentation of this class of variable length memory models for system of spiking neurons, both in discrete and continuous time, we refer the reader to Galves et al. (2021) [18].

The idea of considering discrete time chains on variable length was introduced in the pioneering article by Rissanen (1983) [27], and then popularized in the statistical community by Bühlmann and Wyner (1999) [10].

In the present article we introduce a new statistical procedure to infer for each pair of neurons, whether one is presynaptic to the other.

As far as we know, the problem of inferring the graph of interactions for such a kind of models has been addressed only in Duarte et al. (2019). Given a sample of the spiking activity of a large set of neurons, they propose a pruning procedure to retrieve the set of presynaptic neurons of a fixed neuron i. First the algorithm assumes that all the remaining neurons are presynaptic to i. Then the nonparametric maximum likelihood estimates of the spiking probabilities of i are computed, as a function of the spiking activities of the other neurons after the last spiking time of i. The same procedure is repeated by excluding the candidate presynaptic neuron j. The criterion to prune or not j from the estimated set of presynaptic neurons of i is the following. For any fixed observed history of the activities of the other neurons, the difference between the estimated probabilities, with or without the information concerning j, is computed. If the maximum of these differences is below a given threshold, then j is pruned. Otherwise, j is kept in the estimated set of presynaptic neurons of i. Under suitable assumptions, in [13] the consistency of the procedure has been obtained: upper bounds are provided for the probabilities of false positive and false negative detection, that converge to 0 as the length of the sample increases.

Despite the clear mathematical interest of the result obtained in Duarte et al. [13], the proposed procedure presents drawbacks when used to analyze real neuronal data. First of all, it requires extremely lengthy data sets in order to observe the possible histories of a big set of neurons for a sufficiently large number of times. Moreover, the required computations cannot be localized in the observed set of neurons. In fact, to obtain the required estimates, the histories of all the neurons have to be taken into account at the same time. Even more questionable is the assumption that all the neurons of the system can be observed. Actually, the activity of many inhibitory neurons can hardly be observed directly, by means of the actual spike sorting procedures.

The algorithm introduced in the present article aims to overcome these drawbacks. To guess the influence of neuron j on neuron i, only the spiking activities of these two neurons are considered. The observation time of the system is divided into short time slots. Then the following two probabilities are compared: (a) the probability of a spike of neuron i, following another spike of i, observed in the previous time slot; (b) the probability of a spike of i, following a spike of j and one of i, observed in the two previous time slots (first i and then j). For sufficiently small time slots, the difference between the latter and the former probabilities reveals the eventual presence of j in the set of presynaptic neurons of i. Under suitable assumptions, in the limit as the length of the time slots decreases to 0, if j has an excitatory, respectively inhibitory, effect on i, then this difference becomes positive, respectively negative.

If j is not in the set of presynaptic neurons of i, then the limit of this difference is 0. These asymptotic results provide the basis for the statistical algorithm considered in the present article.

To recover the limiting behavior described above from the observation of a large but finite sample, the length of the time slot must be sufficiently small. On the other hand, we need to observe a sufficiently large number of times the event that two spikes of i, with a spike of j in between, occur in three consecutive time slots. Therefore the length of the time slot cannot be too small. This fact is reminiscent of the familiar bias–variance tradeoff. As a matter of fact, in our proof we estimate the maximal length of the time slots for which our consistency proof works.

The rationale behind the algorithm considered here is simple to explain. The algorithm considers the spiking activity of neurons i and j, in three successive time slots, starting with a spike of i, in order to take advantage of the reset feature of the neuronal activity. Indeed, after each spike, a neuron resets its memory by forgetting the previous history of the system. Therefore, if we know that there was a spike of i in the first time slot, the reset property helps detecting if a spike of neuron j, occurring in the second time slot, influences or not the activity of i in the third time slot.

The idea of modeling system of spiking neurons as interacting point processes was pioneered by Brillinger and coauthors. See, for instance, [6,9], and [7]. As a matter of fact our procedure is reminiscent of the first and second order conditional rate functions introduced in [8].

In neuroscience the problem of inference of the graph of interactions has a specific feature which makes the statistical problem particularly hard. This feature is the impossibility of observing the totality of the system. As a matter of fact experimental procedure can only record an extremely tiny portion of the global neural activity. The treatment of multi unitary neural data sets thus requires algorithms which are suited for partially observed systems.

In general, this kind of statistical difficulty appears in the analysis of many big systems with interactions in time and space. The problem of identifying interactions in a multivariate process has a long history, starting at least with Granger (1969) [20]. An account of the general theory of graphical models can be found in Lauritzen (1996) [25]. Problems of graph identification have been the main source of motivation for the design of the so called *reversible jump algorithms* (see Green (1995) [21]).

Since then, the problem of statistical selection of graphical models has been discussed in several articles. Some of them consider the case in which the samples are i.i.d. @realizations of the same law, see for instance Csiszár and Talata (2006) [11], Bresler at al. (2013) [5], Galves et al. (2015) [19], Hamilton et al. (2017) [23]. Others address such a problem when the sample is constituted by a single observation of the time evolution of a stochastic system like in Brillinger and coauthors seminal articles. For instance, the problem of identifying pairs of interacting components in a different class of multivariate point processes, namely Hawkes processes, was addressed by Eichler, Dahlhaus, Dueck (2017) [15].

In this framework, we thank an anonymous referee for attracting our attention to the article by Bresler et al. (2018) [4] which is devoted to the estimation of the interacting pairs of the Ising Glauber dynamics. As a matter of fact the algorithm considered in the above article is also reminiscent of Brillinger et al. (1976) approach. Nevertheless there are substantial differences between their and our article. The dynamics they consider is reversible, which is not our case. Moreover Bresler et al. (2018) observation model is statistically artificial since they consider that all the events of the underlying Poisson point process are observed even when they do not lead to a modification of the system.

The structure of the paper is the following. In Section 2 the model is introduced and the two main results, Theorems 1 and 2 are stated, whose proofs are given in Sections 3 and 4, respectively.

2. Definitions and main result

We start by introducing the multivariate point process modeling the system of spiking neurons considered here. The main ingredients used to define the process are the following:

- a finite set I, henceforth called the set of *neurons*;
- a matrix $(w_{j\to i} \in \mathbb{R} : (j,i) \in I^2)$, henceforth called the matrix of *synaptic weights*;
- a family of simple point processes $\{\left(T_n^i\right)_{n\geq 1} : i \in I\}$, with $0 < T_1^i < T_2^i < \cdots$, denoting the successive spiking times of neuron i;
- a family of non-decreasing functions $\phi_i : \mathbb{R} \to [0, +\infty[$, henceforth called *spiking rate functions*.

If $w_{j\to i} > 0$ (respectively $w_{j\to j} < 0$), we say that the neuron j has an *excitatory* (respectively *inhibitory*) effect on neuron i. In case $w_{j\to i} = 0$, we say that neuron j does not affect neuron i. We assume that there is no self-interaction and therefore $w_{j\to j} = 0$, for all j. The reason for this terminology will be readily clarified (see (1) and (2)).

The set $\mathcal{V}^i = \{j : w_{j\to i} \neq 0\}$, is called the set of *presynaptic neurons* of i. Obviously

$$\mathcal{V}^i = \mathcal{V}^i_+ \cup \mathcal{V}^i_- ,$$

where V^i_+ and V^i_- are the sets of excitatory and inhibitory ones,

$$\mathcal{V}^i_+ = \left\{j : w_{j\to i} > 0\right\}, \qquad \mathcal{V}^i_- = \left\{j : w_{j\to i} < 0\right\},$$

respectively.

We suppose that a bound d for the maximal cardinality of the sets of presynaptic neurons is known

$$\max\{|\mathcal{V}^i| : i \in I\} \leq d .$$

For any neuron i, we define the spike counting measure N^i as follows. For any subset $A \subset \mathbb{R}^+$,

$$N^i(A) = \sum_{n\geq 1} \mathbf{1}_{\{T_n^i \in A\}}.$$

For any positive real number t, when the event $\{T_1^i \leq t\}$ is realized, we define $L^i(t)$ as the last spiking time of neuron i occurring before time t

$$L^i(t) = \sup\{T_n^i : n \geq 1, T_n^i \leq t\} .$$

This definition allows to introduce the *membrane potential* $U^i(t)$ of neuron i at time t as follows

$$U^i(t) = \begin{cases} U^i(0) + \sum_{j\in\mathcal{V}^i} w_{j\to i} N^j(0,t], & \text{if } 0 \leq t < T_1^i, \\ \sum_{j\in\mathcal{V}^i} w_{j\to i} N^j(L^i(t),t], & \text{if } T_1^i \leq t, \end{cases} \tag{1}$$

where $U^i(0)$ denotes the initial value of the membrane potential.

We will denote by $U(t)$ the vector of the membrane potentials of all the neurons at time t

$$U(t) = \left(U^i(t) : i \in I\right).$$

In what follows, the initial value $U(0)$ of the vector of membrane potentials is chosen in an arbitrary way. We are not assuming the stationarity of the processes.

Finally, for any positive real number t, we define $\mathcal{F}_t$ as the $\sigma-$algebra generated by the family of spike counting measures $\left(N^i(A) : i \in I, A \subset [0,t]\right)$, together with the initial vector of membrane potentials $U(0) = \left(U^i(0) : i \in I\right)$.

With this notation, we can now formally relate the elements of the model in the following way. For any neuron i and any pair of positive real number $t < t'$, we require the spike counting measures $N^i(t,t']$, $i \in I$, to satisfy the equation

$$\mathbb{E}\left(N^i(t,t'] \,|\, \mathcal{F}_t\right) = \mathbb{E}\left(\int_t^{t'} \phi_i(U^i(r-))dr|\mathcal{F}_t\right). \tag{2}$$

Informally this condition can be stated as

$$\mathbb{P}\left(N^i(t,t+dt] = 1 \,|\, \mathcal{F}_t\right) = \phi_i(U^i(t-))dt + o(dt).$$

Assumption 1. The spiking rate functions $\phi_i : \mathbb{R} \to (0, +\infty)$ are

1. nondecreasing,
2. bounded away from 0, with
$$\min_{i\in I} \inf_{u\in\mathbb{R}} \phi_i(u) \ge \alpha > 0,$$
3. bounded from above, with
$$\max_{i\in I} \sup_{u\in\mathbb{R}} \phi_i(u) \le \beta < +\infty,$$
4. and satisfy
$$\min_{i\in I}\{|\phi_i(w_{j\to i}) - \phi_i(0)| : j \in \mathcal{V}^i\} \ge \delta > 0,$$

for some known positive constants α, β, δ.

Observe that α, β and δ are such that $\alpha + \delta \le \beta$. In the sequel we shall use the shorthand notation

$$s = \frac{\alpha}{\beta} \text{ and } \tau = \frac{\delta}{\beta} \in (0,1), \tag{3}$$

with the constraint $s + \tau \le 1$.

Before defining the estimation algorithm, we need to introduce the following events, depending on a parameter $\Delta > 0$ to be chosen in a suitable way. Given two neurons $i \in I$ and $j \in I$, with $i \ne j$, we denote

$$A^i(\Delta) = \{N^i(0,\Delta] > 0\}, \tag{4}$$

$$B^i(\Delta) = A^i(\Delta) \cap \{N^i(\Delta, 2\Delta] > 0\}, \tag{5}$$

$$C^{j\to i}(\Delta) = A^i(\Delta) \cap \{N^j(\Delta, 2\Delta] > 0\}, \tag{6}$$
$$D^{j\to i}(\Delta) = C^{j\to i}(\Delta) \cap \{N^i(2\Delta, 3\Delta] > 0\}. \tag{7}$$

In the following, we are going to use the notation $\mathbb{P}_u(\cdot)$ instead of $\mathbb{P}(\cdot \mid U(0) = u)$.

Theorem 1. *Suppose that the family of spiking rate functions $\{\phi_i : i \in I\}$ satisfy Assumption 1. Let $\Delta_0 = \frac{s^2}{5d\beta}$. For any $\Delta \le \Delta_0$ any fixed pair of neurons i and j with $i \neq j$, and any pair of vectors of membrane potentials u and u', the following inequalities hold.*

If $j \notin \mathcal{V}^i$, then

$$-\zeta_1(\Delta) < \frac{\mathbb{P}_u(D^{j\to i}(\Delta))}{\mathbb{P}_u(C^{j\to i}(\Delta))} - \frac{\mathbb{P}_{u'}(B^i(\Delta))}{\mathbb{P}_{u'}(A^i(\Delta))} < \zeta_2(\Delta). \tag{8}$$

If $j \in \mathcal{V}^i_-$, then

$$\frac{\mathbb{P}_u(D^{j\to i}(\Delta))}{\mathbb{P}_u(C^{j\to i}(\Delta))} - \frac{\mathbb{P}_{u'}(B^i(\Delta))}{\mathbb{P}_{u'}(A^i(\Delta))} \le -\zeta^-(\Delta). \tag{9}$$

If $j \in \mathcal{V}^i_+$, then

$$\zeta^+(\Delta) \le \frac{\mathbb{P}_u(D^{j\to i}(\Delta))}{\mathbb{P}_u(C^{j\to i}(\Delta))} - \frac{\mathbb{P}_{u'}(B^i(\Delta))}{\mathbb{P}_{u'}(A^i(\Delta))}, \tag{10}$$

where

$$\zeta_1(\Delta) = \frac{9}{s^2}\, d(\beta\Delta)^2, \qquad \zeta_2(\Delta) = \frac{5+3s^2}{s^3}\, d(\beta\Delta)^2, \tag{11}$$

$$\zeta^-(\Delta) = -\zeta_1(\Delta) + \left(1 + \frac{5d\beta\Delta}{s^3}\right)\tau\beta\Delta, \tag{12}$$
$$\zeta^+(\Delta) = -\zeta_2(\Delta) + \left(1 - \frac{5d\beta\Delta}{s^2}\right)\tau\beta\Delta. \tag{13}$$

Since the quantities

$$\frac{\mathbb{P}_u(D^{j\to i}(\Delta))}{\mathbb{P}_u(C^{j\to i}(\Delta))} - \frac{\mathbb{P}_{u'}(B^i(\Delta))}{\mathbb{P}_{u'}(A^i(\Delta))}$$

will be estimated from data, convenient error terms are added to $\zeta_1(\Delta)$ and $\zeta_2(\Delta)$, whereas they are subtracted to $\zeta^+(\Delta)$ and $\zeta^-(\Delta)$, defining

$$\xi_1(\Delta) = \zeta_1(\Delta) + \frac{\tau}{10}\beta\Delta\left(2 - \frac{d\beta\Delta}{s^2}\right), \tag{14}$$
$$\xi_2(\Delta) = \zeta_2(\Delta) + \frac{\tau}{10}\beta\Delta\left(2 + \frac{5-3s^2}{s^3}d\beta\Delta\right), \tag{15}$$
$$\xi^-(\Delta) = \zeta^-(\Delta) - \frac{\tau}{10}\beta\Delta\left(2 - \tau + \frac{5(1-\tau)-3s^2}{s^3}d\beta\Delta\right), \tag{16}$$
$$\xi^+(\Delta) = \zeta^+(\Delta) - \frac{\tau}{10}\beta\Delta\left(2 + \tau - \frac{1+5\tau}{s^2}d\beta\Delta\right). \tag{17}$$

It is easily seen that all the quantities in brackets are positive for $\Delta \in (0, \Delta_0]$; indeed the reader can easily verify that they are bounded from below by 1.

Proposition 1. *Let* $\Delta^* = \frac{s^3\tau}{34d\beta}$. *For any* $\Delta \in (0, \Delta^*]$

$$\xi_1(\Delta) \leq \xi^-(\Delta), \qquad \xi_2(\Delta) \leq \xi^+(\Delta). \tag{18}$$

Since $\Delta^* < \Delta_0$, Proposition 1 and (14)–(17) imply the following inequalities

$$0 < \zeta_1(\Delta) < \xi_1(\Delta) \leq \xi^-(\Delta) < \zeta^-(\Delta), \tag{19}$$

$$0 < \zeta_2(\Delta) < \xi_2(\Delta) \leq \xi^+(\Delta) < \zeta^+(\Delta). \tag{20}$$

The usefulness of the above inequalities will became clear in Theorem 2.

The above results suggest the following estimation algorithm, which is based on a partition of the observation time in slots of fixed length Δ^*.

Given two neurons $i \in I$ and $j \in I$, with $i \neq j$, we set

$$A_1^i = A^i(\Delta^*), \quad B_1^i = B^i(\Delta^*), \quad C_1^{j\to i} = C^{j\to i}(\Delta^*), \quad D_1^{j\to i} = D^{j\to i}(\Delta^*),$$

and likewise, for any positive integer $k > 1$, we define the events

$$A_k^i = \{N^i((2k-2)\Delta^*, (2k-1)\Delta^*] > 0\},$$

$$B_k^i = A_k^i \cap \{N^i((2k-1)\Delta^*, 2k\Delta^*] > 0\},$$

$$C_k^{j\to i} = \{N^i((3k-3)\Delta^*, (3k-2)\Delta^*] > 0,\ N^j((3k-2)\Delta^*, (3k-1)\Delta^*] > 0\},$$

$$D_k^{j\to i} = C_k^{j\to i} \cap \{N_i((3k-1)\Delta^*, 3k\Delta^*] > 0\}.$$

For any integer $n \geq 1$, we define

$$S^{A^i}(n) = \sum_{k=1}^{n} \mathbf{1}_{A_k^i}, \quad S^{B^i}(n) = \sum_{k=1}^{n} \mathbf{1}_{B_k^i},$$

$$S^{C^{j\to i}}(n) = \sum_{k=1}^{n} \mathbf{1}_{C_k^{j\to i}}, \quad S^{D^{j\to i}}(n) = \sum_{k=1}^{n} \mathbf{1}_{D_k^{j\to i}},$$

and, for any integer $m \geq 1$, we define

$$K_m^i = \inf\{n \geq 1 : S^{A^i}(n) = m\},$$

$$H_m^{j\to i} = \inf\{n \geq 1 : S^{C^{j\to i}}(n) = m\}.$$

Next define

$$t_n = \lceil \alpha\Delta^* n \rceil,\ m_n = \left\lceil \frac{19}{20}\alpha^2(\Delta^*)^2\Big(1 - \frac{\tau}{10}\sqrt{\alpha\Delta^*}\Big)n \right\rceil. \tag{21}$$

Once these parameters are set, we can define the empirical ratios

$$R^i(n) = \frac{S^{B^i}(K_{m_n}^i \wedge t_n)}{S^{A^i}(K_{m_n}^i \wedge t_n)} = \begin{cases} \dfrac{S^{B^i}(K_{m_n}^i)}{m_n}, & \text{if } K_{m_n}^i \leq t_n \\[2ex] \dfrac{S^{B^i}(t_n)}{S^{A^i}(t_n)}, & \text{if } K_{m_n}^i > t_n, \end{cases} \tag{22}$$

and

$$G^{j\to i}(n)=\frac{S^{D^{j\to i}}(H^{j\to i}_{m_n}\wedge n)}{S^{C^{j\to i}}(H^{j\to i}_{m_n}\wedge n)}=\begin{cases}\dfrac{S^{D^{j\to i}}(H^{j\to i}_{m_n})}{m_n}, & \text{if } H^{j\to i}_{m_n}\le n\\[2ex] \dfrac{S^{D^{j\to i}}(n)}{S^{C^{j\to i}}(n)}, & \text{if } H^{j\to i}_{m_n}> n.\end{cases} \tag{23}$$

For any pair of neurons $i \neq j$, the statistics $R^i(n)$ and $G^{j\to i}(n)$ will be used to identify whether j is presynaptic to i or not. They are ratio estimators that are stopped once their denominator reaches the level m_n, but differently from $G^{j\to i}(n)$, which is allowed to reach this level up to the time $3\Delta^* n$, the estimator $R^i(n)$ is stopped at most after the time $2\Delta^* t_n$ is expired. The latter time is much smaller than the former, since a rough upper bound for $\alpha\Delta^*$ is $\frac{1}{34d}$. Therefore a much smaller interval of time is sufficient for $R^i(n)$ to reach the same accuracy as $G^{j\to i}(n)$.

Finally we define the estimated sets $\hat{\mathcal{V}}^i_+(n)$, $\hat{\mathcal{V}}^i_-(n)$, and $\hat{\mathcal{V}}^i(n)$, as follows

$$\hat{\mathcal{V}}^i_-(n)=\{j\in I\setminus\{i\}: G^{j\to i}(n)-R^i(n)\le -\xi_1(\Delta^*)\} \tag{24}$$

$$\hat{\mathcal{V}}^i_+(n)=\{j\in I\setminus\{i\}: G^{j\to i}(n)-R^i(n)\ge \xi_2(\Delta^*)\}, \tag{25}$$

$$\hat{\mathcal{V}}^i(n)=\hat{\mathcal{V}}^i_+(n)\cup\hat{\mathcal{V}}^i_-(n). \tag{26}$$

We can now state our main theorem, in which $\mathbb{P}$ stands for a probability measure on the spiking processes with an arbitrary law of the initial vector of the membrane potentials.

Theorem 2. *For any observation time $T>0$,define $n(T):=\lfloor\frac{T}{3\Delta^*}\rfloor$, where $\Delta^*=\frac{s^3\tau}{34d\beta}$. Let $R^i(n)$, $G^{j\to i}(n)$, $\hat{\mathcal{V}}^i{}_-(n)$, $\hat{\mathcal{V}}^i{}_+(n)$, and $\hat{\mathcal{V}}^i(n)$ be defined as in (22)–(26). Then the following inequalities hold:*

$$\text{if } j\notin\mathcal{V}^i,\ \text{then } \mathbb{P}\Big(j\notin\hat{\mathcal{V}}^i\big(n(T)\big)\Big)\ge 1-2Ce^{-\omega T},$$

$$\text{if } j\in\mathcal{V}^i_-,\ \text{then } \mathbb{P}\Big(j\in\hat{\mathcal{V}}^i_-\big(n(T)\big)\Big)\ge 1-Ce^{-\omega T},$$

$$\text{if } j\in\mathcal{V}^i_+,\ \text{then } \mathbb{P}\Big(j\in\hat{\mathcal{V}}^i{}_+\big(n(T)\big)\Big)\ge 1-Ce^{-\omega T},$$

where

$$\omega=\vartheta_0\frac{\tau^4s^9\beta}{d^2},\qquad C=4e^{\omega\frac{s^3\tau}{10d\beta}},$$

ϑ_0 being a computable universal constant.

3. Proofs of Theorem 1 and Proposition 1

The proof of Theorem 1 is based on a particular construction of the counting measures N^i, and the membrane potential processes U^i, $i\in I$, in such a way that (1) and (2) hold.

We consider a Poisson measure on $[0,\infty)\times[0,\beta]\times I$,

$$\mathcal{N}(dt,dx,dz)=\sum_{k\ge1}\delta_{(\mathcal{T}_k,X_k,Z_k)}(dt,dx,dz),$$

with intensity measure

$$\mu_{\mathcal{N}}(dt \times dx \times \{i\}) = dt \times dx, \quad i \in I.$$

Without loss of generality we assume that $0 < \mathcal{T}_1 < \mathcal{T}_2 < \cdots$. Observe that the marks X_k, $k \geq 1$, are independent and uniform in $[0, \beta]$. The $\mathcal{T}_k$'s are candidates to be spiking times for neurons: $\mathcal{T}_1$ is accepted as a spike for neuron j if and only if

$$Z_1 = j, \quad X_1 \leq \phi_j(U^j(0)).$$

If this is the case then $T_1^j = \mathcal{T}_1$, and the potential vector is updated in the interval $[\mathcal{T}_1, \mathcal{T}_2)$ according to

$$U^j(t) = 0, \quad U^i(t) = U^i(0) + w_{j\to i}, \quad i \neq j;$$

otherwise $U(t) = U(0)$, and the construction proceeds with the next candidate time.

Recursively in k the candidate time $\mathcal{T}_k$ is accepted as a spike for neuron Z_k if and only if

$$X_k \leq \phi_{Z_k}(U^{Z_k}(\mathcal{T}_k-)),$$

If this is the case the potential vector is updated in the interval $[\mathcal{T}_k, \mathcal{T}_{k+1})$ according to

$$U^{Z_k}(t) = 0, \quad U^i(t) = U^i(\mathcal{T}_k-) + w_{Z_k\to i}, \quad i \neq Z_k;$$

otherwise $U(t) = U(\mathcal{T}_k-)$, and the construction proceeds with the next candidate time.

Next, for each neuron $i \in I$, and $x \in (0, \beta]$, define the Poisson measures $N^{i,x}(t, t'] := \mathcal{N}((t, t'] \times [0, x] \times \{i\})$, with intensity measure $x\,dt$, on $(0, \infty)$.

From now on we set

$$\overline{N}^i = N^{i,\beta}, \quad \underline{N}^i = N^{i,\alpha}$$

so that for any $i \in I$

$$\underline{N}^i(t, t'] \leq N^i(t, t'] \leq \overline{N}^i(t, t'], \quad 0 \leq t < t'.$$

It is also convenient to use the notation

$$N^{\mathcal{W}}(t, t'] = \sum_{j\in\mathcal{W}} N^j(t, t'], \qquad 0 \leq t < t', \quad \mathcal{W} \subset I,$$

and the same notation for the measures $\underline{N}^j$, and $\overline{N}^j$.

For each pair of neurons $i \neq j$, and for each positive real number Δ we now define the events

$$\tilde{A}^i(\Delta) = A^i(\Delta) \cap \left\{N^{\mathcal{V}^i}(0, 2\Delta] = 0\right\},$$

$$\tilde{B}^i(\Delta) = B^i(\Delta) \cap \left\{N^{\mathcal{V}^i}(0, 2\Delta] = 0\right\},$$

$$\tilde{C}^{j\to i}(\Delta) = C^{j\to i}(\Delta) \cap \left\{N^{\mathcal{V}^i}(0, \Delta] = N^{\mathcal{V}^i\setminus\{j\}}(\Delta, 2\Delta] = N^{\mathcal{V}^i}(2\Delta, 3\Delta] = 0\right\},$$

$$\tilde{D}^{j\to i}(\Delta) = D^{j\to i}(\Delta) \cap \left\{N^{\mathcal{V}^i}(0, \Delta] = N^{\mathcal{V}^i\setminus\{j\}}(\Delta, 2\Delta] = N^{\mathcal{V}^i}(2\Delta, 3\Delta] = 0\right\},$$

where $A^i(\Delta)$, $B^i(\Delta)$, $C^{j\to i}(\Delta)$, and $D^{j\to i}(\Delta)$ have been defined in (4)–(7).

Lemma 1. *Irrespectively of the vector u of membrane potentials, for any $\Delta > 0$, the following inequalities hold.*

$$\frac{\mathbb{P}_u(\tilde{B}^i(\Delta))}{\mathbb{P}_u(\tilde{A}^i(\Delta))}\left(1-\frac{1-e^{-2d\beta\Delta}}{s}\right) \le \frac{\mathbb{P}_u(B^i(\Delta))}{\mathbb{P}_u(A^i(\Delta))} \le \left(1+\frac{e^{2d\beta\Delta}-1}{s^2}\right)\frac{\mathbb{P}_u(\tilde{B}^i(\Delta))}{\mathbb{P}_u(\tilde{A}^i(\Delta))} \tag{27}$$

$$\begin{aligned}\frac{\mathbb{P}_u(\tilde{D}^{j\to i}(\Delta))}{\mathbb{P}_u(\tilde{C}^{j\to i}(\Delta))}\left(1-\frac{1-e^{-3d\beta\Delta}}{s^2}\right) &\le \frac{\mathbb{P}_u(D^{j\to i}(\Delta))}{\mathbb{P}_u(C^{j\to i}(\Delta))}\\ &\le \left(1+\frac{e^{3d\beta\Delta}-1}{s^3}\right)\frac{\mathbb{P}_u(\tilde{D}^{j\to i}(\Delta))}{\mathbb{P}_u(\tilde{C}^{j\to i}(\Delta))}\end{aligned} \tag{28}$$

Proof. First observe that for any triple of events E, F and G such that $E \subset F$ the following inequalities hold

$$\frac{\mathbb{P}(E\cap G)}{\mathbb{P}(F\cap G)}\left(1-\frac{\mathbb{P}(F\setminus G)}{\mathbb{P}(F)}\right) \le \frac{\mathbb{P}(E)}{\mathbb{P}(F)} \le \left(1+\frac{\mathbb{P}(E\setminus G)}{\mathbb{P}(E\cap G)}\right)\frac{\mathbb{P}(E\cap G)}{\mathbb{P}(F\cap G)}.$$

Choosing

$$E = B^i(\Delta), \quad F = A^i(\Delta), \quad G = \{N^{\mathcal{V}^i}(0, 2\Delta] = 0\}$$

first, and then

$$E = D^{j\to i}(\Delta),\ F = C^{j\to i}(\Delta),$$

$$G = \{N^{\mathcal{V}^i}(0,\Delta] = N^{\mathcal{V}^i\setminus\{j\}}(\Delta, 2\Delta] = N^{\mathcal{V}^i}(2\Delta, 3\Delta] = 0\},$$

the inequalities (27) and (28) are consequences of the bounds

$$\frac{\mathbb{P}_u(A^i(\Delta)\setminus\tilde{A}^i(\Delta))}{\mathbb{P}_u(A^i(\Delta))} \le \frac{(1-e^{-2d\beta\Delta})(1-e^{-\beta\Delta})}{1-e^{-\alpha\Delta}} \le \frac{\beta}{\alpha}(1-e^{-2d\beta\Delta}),$$

and

$$\frac{\mathbb{P}_u(B^i(\Delta)\setminus\tilde{B}^i(\Delta))}{\mathbb{P}_u(\tilde{B}^i(\Delta))} \le e^{2d\beta\Delta}\left(\frac{1-e^{-\beta\Delta}}{1-e^{-\alpha\Delta}}\right)^2(1-e^{-2d\beta\Delta}) \le \left(\frac{\beta}{\alpha}\right)^2(e^{2d\beta\Delta}-1).$$

The former is due to

$$A^i(\Delta)\setminus\tilde{A}^i(\Delta) \subset \{\overline{N}^i(0,\Delta] > 0, \overline{N}^{\mathcal{V}^i}(0,2\Delta] > 0\},\ \ \{\underline{N}^i(0,\Delta] > 0\} \subset A^i(\Delta),$$

and the majorization

$$\frac{1-e^{-\beta x}}{1-e^{-\alpha x}} \le \frac{\beta}{\alpha} = \frac{1}{s}, \quad x \ge 0,$$

which holds since $\beta \ge \alpha$, in view of the fact that the function $x \mapsto \frac{e^x-1}{x}$ is increasing in $\mathbb{R}$. The latter is due to

$$B^i(\Delta)\setminus\tilde{B}^i(\Delta) \subset \{\overline{N}^i(0,\Delta] > 0, \overline{N}^i(\Delta, 2\Delta] > 0, \overline{N}^{\mathcal{V}^i}(0,2\Delta] > 0\}$$

and

$$\{\overline{N}^{\mathcal{V}^i}(0,2\Delta] = 0,\ \ \underline{N}^i(0,\Delta] > 0, \underline{N}^i(\Delta, 2\Delta] > 0\} \subset \tilde{B}^i(\Delta).$$

With similar arguments one proves that

$$\frac{\mathbb{P}_u(C^{j\to i}(\Delta)\setminus \tilde{C}^{j\to i}(\Delta))}{\mathbb{P}_u(C^{j\to i}(\Delta))} \le \frac{(1-e^{-\beta\Delta})^2(1-e^{-3d\beta\Delta})}{(1-e^{-\alpha\Delta})^2} \le \left(\frac{\beta}{\alpha}\right)^2 (1-e^{-3d\beta\Delta})$$

$$\begin{aligned}\frac{\mathbb{P}_u(D^{j\to i}(\Delta)\setminus \tilde{D}^{j\to i}(\Delta))}{\mathbb{P}_u(\tilde{D}^{j\to i}(\Delta))} &\le e^{3d\beta\Delta}\left(\frac{1-e^{-\beta\Delta}}{1-e^{-\alpha\Delta}}\right)^3 (1-e^{-3d\beta\Delta})\\ &\le \left(\frac{\beta}{\alpha}\right)^3 (e^{3d\beta\Delta}-1).\end{aligned}$$

Lemma 2. *Irrespectively of the vector u of membrane potentials, for any $\Delta > 0$, one has*

$$\big(1-e^{-\phi_i(0)\Delta}\big)e^{-d(\beta-\alpha)\Delta} \le \frac{\mathbb{P}_u\big(\tilde{B}^i(\Delta)\big)}{\mathbb{P}_u\big(\tilde{A}^i(\Delta)\big)} \le \big(1-e^{-\phi_i(0)\Delta}\big)e^{d(\beta-\alpha)\Delta}. \tag{29}$$

Moreover, provided $j \notin \mathcal{V}^i$

$$\big(1-e^{-\phi_i(0)\Delta}\big)\, e^{-d(\beta-\alpha)\Delta} \le \frac{\mathbb{P}_u\big(\tilde{D}^{j\to i}(\Delta)\big)}{\mathbb{P}_u\big(\tilde{C}^{j\to i}(\Delta)\big)} \le \big(1-e^{-\phi_i(0)\Delta}\big)\, e^{d(\beta-\alpha)\Delta}, \tag{30}$$

whereas if $j \in \mathcal{V}^i_+$

$$\frac{\mathbb{P}_u\big(\tilde{D}^{j\to i}(\Delta)\big)}{\mathbb{P}_u\big(\tilde{C}^{j\to i}(\Delta)\big)} \ge (1-e^{-(\phi_i(0)+\delta)\Delta})e^{-d(\beta-\alpha)\Delta}, \tag{31}$$

and if $j \in \mathcal{V}^i_-$

$$\frac{\mathbb{P}_u\big(\tilde{D}^{j\to i}(\Delta)\big)}{\mathbb{P}_u\big(\tilde{C}^{j\to i}(\Delta)\big)} \le (1-e^{-(\phi_i(0)-\delta)\Delta})e^{d(\beta-\alpha)\Delta}. \tag{32}$$

Proof. It is convenient to split the events $\tilde{A}^i(\Delta)$ and $\tilde{C}^{j\to i}(\Delta)$ in the following way: $\tilde{A}^i(\Delta) = A^i_- \cap A^i_+$ and $\tilde{C}^{j\to i}(\Delta) = C^{ji}_- \cap C^i_+$, where

$$A^i_- = \{N^i(0,\Delta] > 0, N^{\mathcal{V}^i}(0,\Delta] = 0\},\ A^i_+ = \{N^{\mathcal{V}^i}(\Delta, 2\Delta] = 0\},$$

$$C^{ji}_- = \{N^i(0,\Delta] > 0, N^j(\Delta,2\Delta] > 0, N^{\mathcal{V}^i}(0,\Delta] = N^{\mathcal{V}^i\setminus\{j\}}(\Delta,2\Delta] = 0\},$$

$$C^i_+ = \{N^{\mathcal{V}^i}(2\Delta, 3\Delta] = 0\}.$$

Since for any triple of events E, F_+, F_-, such that $E \subset F = F_+ \cap F_-$

$$\frac{\mathbb{P}(E)}{\mathbb{P}(F)} = \mathbb{P}(E|F) = \mathbb{P}(E|F_-\cap F_+) = \frac{\mathbb{P}(E\cap F_+|F_-)}{\mathbb{P}(F_+|F_-)},$$

we get

$$\frac{\mathbb{P}_u\big(\tilde{B}^i(\Delta)\big)}{\mathbb{P}_u\big(\tilde{A}^i(\Delta)\big)} = \frac{\mathbb{P}_u\big(\tilde{B}^i(\Delta)\cap A^i_+|A^i_-\big)}{\mathbb{P}_u\big(A^i_+|A^i_-\big)},$$

$$\frac{\mathbb{P}_u\big(\tilde{D}^{j\to i}(\Delta)\big)}{\mathbb{P}_u\big(\tilde{C}^{j\to i}(\Delta)\big)} = \frac{\mathbb{P}_u\big(\tilde{D}^{j\to i}(\Delta)\cap C^i_+|C^{ji}_-\big)}{\mathbb{P}_u\big(C^i_+|C^{ji}_-\big)}.$$

Next we proceed to bound the right hand side in the previous formulas.

By writing $\mathbb{P}_v^{k\Delta}(\cdot) = \mathbb{P}(\cdot|U(k\Delta) = v)$ for $k = 1, 2$, then by the Markov property we have

$$\frac{\inf\limits_{v:v^i=0} \mathbb{P}_v^{\Delta}\left(N^i(\Delta, 2\Delta] > 0, A_+^i\right)}{\sup\limits_{v:v^i=0} \mathbb{P}_v^{\Delta}\left(A_+^i\right)} \leq \frac{\mathbb{P}_u\left(\tilde{B}^i(\Delta) \cap A_+^i | A_-^i\right)}{\mathbb{P}_u\left(A_+^i | A_-^i\right)} \leq \frac{\sup\limits_{v:v^i=0} \mathbb{P}_v^{\Delta}\left(N^i(\Delta, 2\Delta] > 0, A_i^+\right)}{\inf\limits_{v:v^i=0} \mathbb{P}_v^{\Delta}\left(A_i^+\right)} \tag{33}$$

since A_-^i implies $U^i(\Delta) = 0$. Likewise

$$\frac{\inf\limits_{v:v^i\in I_{ij}} \mathbb{P}_v^{2\Delta}\left(N^i(2\Delta, 3\Delta] > 0, C_+^i\right)}{\sup\limits_{v:v^i\in I_{ij}} \mathbb{P}_v^{2\Delta}\left(C_+^i\right)} \leq \frac{\mathbb{P}_u\left(\tilde{D}^{j\to i}(\Delta) \cap C_+^i | C_-^{ji}\right)}{\mathbb{P}_u\left(C_+^i | C_-^{ji}\right)} \leq \frac{\sup\limits_{v:v^i\in I_{ij}} \mathbb{P}_v^{2\Delta}\left(N^i(2\Delta, 3\Delta] > 0, C_+^i\right)}{\inf\limits_{v:v^i\in I_{ij}} \mathbb{P}_v^{2\Delta}\left(C_+^i\right)} \tag{34}$$

where $I_{ij} = \{0\}$ if $j \notin \mathcal{V}_i$, $I_{ij} = [w_{j\to i}, +\infty)$ if $j \in \mathcal{V}_i^+$ and $I_{ij} = (-\infty, w_{j\to i}]$ if $j \in \mathcal{V}_i^-$. Indeed C_-^{ji} implies: $U^i(\Delta) = 0$ in first case, $U^i(\Delta) \geq w_{j\to i}$ in the second case, and $U^i(\Delta) \leq w_{j\to i}$ in the third case.

Since, when $j \notin \mathcal{V}^i$, conditionally to $U^i(\Delta) = 0$

$$\{N^{i,\phi_i(0)}(\Delta, 2\Delta] > 0, \overline{N}^{\mathcal{V}^i}(\Delta, 2\Delta]\} \subset \{N^i(\Delta, 2\Delta] > 0, A_+^i\} \tag{35}$$

we have, for any v with $v_i = 0$,

$$\begin{aligned} \mathbb{P}_v^{\Delta}\left(N^i(\Delta, 2\Delta] > 0, A_+^i\right) &\geq \mathbb{P}\left(N^{i,\phi_i(0)}(\Delta, 2\Delta] > 0, \overline{N}^{\mathcal{V}^i}(\Delta, 2\Delta] = 0\right) \\ &= (1 - e^{-\phi_i(0)\Delta})\mathbb{P}\left(\overline{N}^{\mathcal{V}^i}(\Delta, 2\Delta] = 0\right) \end{aligned} \tag{36}$$

On the other hand $\mathbb{P}_v^{\Delta}\left(A_+^i\right) \leq \mathbb{P}\left(\underline{N}^{\mathcal{V}^i}(\Delta, 2\Delta] = 0\right)$, from which, by taking (33) into account, together with the obvious inequality

$$\frac{\mathbb{P}\left(\overline{N}^{\mathcal{V}^i}(\Delta, 2\Delta] = 0\right)}{\mathbb{P}\left(\underline{N}^{\mathcal{V}^i}(\Delta, 2\Delta] = 0\right)} = e^{-(\beta-\alpha)|\mathcal{V}^i|\Delta} \geq e^{-(\beta-\alpha)d\Delta}, \tag{37}$$

the leftmost inequality in (29) is obtained. The same argument can be used to prove that, when $j \notin \mathcal{V}^i$

$$\mathbb{P}_v^{2\Delta}\left(N^i(2\Delta, 3\Delta] > 0, C_+^i\right) \geq (1 - e^{-\phi_i(0)\Delta})\mathbb{P}\left(\overline{N}^{\mathcal{V}^i}(2\Delta, 3\Delta] = 0\right)$$

and together with $\mathbb{P}_v^{2\Delta}(C_+^i) \leq \mathbb{P}\left(\underline{N}^{\mathcal{V}^i}(2\Delta, 3\Delta] = 0\right)$, the leftmost inequality in (30) is established.

On the other side, always conditionally to $U^i(\Delta) = 0$, from the inclusions

$$\begin{aligned} &\{N^i(\Delta, 2\Delta] > 0, A_+^i\} \subset \{N^{i,\phi_i(0)}(\Delta, 2\Delta] > 0, \underline{N}^{\mathcal{V}^i}(\Delta, 2\Delta] = 0\}, \\ &\{\overline{N}^{\mathcal{V}^i}(\Delta, 2\Delta] > 0\} \subset A_+^i \end{aligned} \tag{38}$$

and the rightmost inequality in (33), one obtains

$$\frac{\mathbb{P}_u\big(\tilde{B}^i(\Delta)\big)}{\mathbb{P}_u\big(\tilde{A}^i(\Delta)\big)} \leq \mathbb{P}(N^{i,\phi_i(0)}(\Delta, 2\Delta) > 0)\, \frac{\mathbb{P}(\underline{N}^{\mathcal{V}^i}(\Delta, 2\Delta] = 0)}{\mathbb{P}(\overline{N}^{\mathcal{V}^i}(\Delta, 2\Delta] = 0)}$$

which, thanks to (37), implies the rightmost inequality in (29).

Again, the same argument can be used to prove that

$$\frac{\mathbb{P}_u\big(\tilde{D}^{j\to i}(\Delta)\big)}{\mathbb{P}_u\big(\tilde{C}^{j\to i}(\Delta)\big)} \leq \mathbb{P}(N^{i,\phi_i(0)}(2\Delta, 3\Delta) > 0)\, \frac{\mathbb{P}(\underline{N}^{\mathcal{V}^i}(2\Delta, 3\Delta) = 0)}{\mathbb{P}(\overline{N}^{\mathcal{V}^i}(2\Delta, 3\Delta] = 0)}$$

which immediately implies the rightmost inequality in (30).

For the proof of (31) observe that by the Markov property C^{ji}_- implies $U^i(2\Delta) \geq w_{j\to i} > 0$, hence for any v with $v_i \geq w_{j\to i}$

$$\mathbb{P}^{2\Delta}_v\left(N^i(2\Delta, 3\Delta] > 0, C^i_+\right) \geq (1 - e^{-(\phi_i(0)+\delta)\Delta})\, \mathbb{P}(\overline{N}^{\mathcal{V}^i}(2\Delta, 3\Delta] = 0).$$

Indeed, conditionally to $U(2\Delta) = v$,

$$\phi_i(U^i(2\Delta)) \geq \phi_i(w_{j\to i}) \geq \phi_i(0) + \delta,$$

and therefore

$$\begin{aligned}&\{N^i(2\Delta, 3\Delta] > 0, N^{\mathcal{V}^i}(2\Delta, 3\Delta] = 0\}\\ &\qquad \supset \{N^{i,\phi_i(0)+\delta}(2\Delta, 3\Delta] > 0, \overline{N}^{\mathcal{V}^i}(2\Delta, 3\Delta] = 0\}.\end{aligned}$$

With the upper bound

$$\mathbb{P}^{2\Delta}_v(C^i_+) \leq \mathbb{P}(\underline{N}^{\mathcal{V}^i}(2\Delta, 3\Delta] = 0)$$

the proof of (31) is finished.

For proving (32), observe that by the Markov property this time C^{ji}_- implies $U^i(2\Delta) \leq w_{j\to i} < 0$, hence by using the rightmost inequality of (34)

$$\frac{\mathbb{P}_u\big(\tilde{D}^{j\to i}(\Delta)\big)}{\mathbb{P}_u\big(\tilde{C}^{j\to i}(\Delta)\big)} \leq \mathbb{P}(N^{i,\phi_i(0)-\delta}(2\Delta, 3\Delta] > 0)\, \frac{\mathbb{P}(\underline{N}^{\mathcal{V}^i}(2\Delta, 3\Delta] = 0)}{\mathbb{P}(\overline{N}^{\mathcal{V}^i}(2\Delta, 3\Delta] = 0)}.$$

Indeed, conditionally to $U(2\Delta) = v$, with $v^i \leq w_{j\to i} < 0$

$$\phi_i(U^i(2\Delta)) \leq \phi_i(w_{j\to i}) \leq \phi_i(0) - \delta,$$

and therefore

$$\{N^i(2\Delta, 3\Delta] > 0, C^i_+\} \subset \{N^{i,\phi_i(0)-\delta}(2\Delta, 3\Delta] > 0, \underline{N}^{\mathcal{V}^i}(2\Delta, 3\Delta] = 0\},$$

With the lower bound

$$\mathbb{P}^{2\Delta}_v(C^i_+) \geq \mathbb{P}(\overline{N}^{\mathcal{V}^i}(2\Delta, 3\Delta] = 0)$$

the proof of (32) is finished.

Collecting together the results of the previous two lemmas we arrive to the following

Lemma 3. *Irrespectively of the vector u of membrane potentials, for $0 < \Delta < \Delta_0 = \frac{s^2}{5d\beta}$, it holds*

$$\left(1 - \frac{3d\beta\Delta}{s}\right)\phi_i(0)\Delta \leq \frac{\mathbb{P}_u(B^i(\Delta))}{\mathbb{P}_u(A^i(\Delta))} \leq \left(1 + \frac{4d\beta\Delta}{s^2}\right)\phi_i(0)\Delta. \tag{39}$$

Furthermore, for $j \notin \mathcal{V}^i$, it holds

$$\left(1-\frac{5d\beta\Delta}{s^2}\right)\phi_i(0)\Delta \leq \frac{\mathbb{P}_u(D^{j\to i}(\Delta))}{\mathbb{P}_u(C^{j\to i}(\Delta))} \leq \left(1+\frac{5d\beta\Delta}{s^3}\right)\phi_i(0)\Delta, \tag{40}$$

whereas, for $j \in \mathcal{V}^i_+$, it holds

$$\left(1-\frac{5d\beta\Delta}{s^2}\right)(\phi_i(0)+\delta)\Delta \leq \frac{\mathbb{P}_u(D^{j\to i}(\Delta))}{\mathbb{P}_u(C^{j\to i}(\Delta))}, \tag{41}$$

and finally, for $j \in \mathcal{V}^i_-$, it holds

$$\frac{\mathbb{P}_u(D^{j\to i}(\Delta))}{\mathbb{P}_u(C^{j\to i}(\Delta))} \leq \left(1+\frac{5d\beta\Delta}{s^3}\right)(\phi_i(0)-\delta)\Delta\,. \tag{42}$$

Proof. As far as the lower bounds are concerned, observe that

$$\begin{aligned}\left(1-\frac{1-e^{-2d\beta\Delta}}{s}\right)e^{-d(\beta-\alpha)\Delta} &\geq \left(1-\frac{2d\beta}{s}\Delta\right)(1-d\beta(1-s)\Delta)\\ &\geq 1-d\beta\left(\frac{2}{s}+1-s\right)\Delta \geq 1-\frac{9d\beta}{4s}\Delta > 0,\end{aligned}$$

and

$$\begin{aligned}\left(1-\frac{1-e^{-3d\beta\Delta}}{s^2}\right)e^{-d(\beta-\alpha)\Delta} &\geq \left(1-\frac{3d\beta}{s^2}\Delta\right)(1-d(\beta-\alpha)\Delta)\\ &\geq 1-d\beta\left(\frac{3}{s^2}+1-s\right)\Delta \geq 1-\frac{85d\beta}{27s^2}\Delta > 0,\end{aligned}$$

where these inequalities are guaranteed since $\Delta_0 < \frac{27s^2}{85d\beta}$.

Next observe that

$$1-e^{-x} \geq x\,(1-x/2) > 0,\;\; x \in (0,2), \tag{43}$$

so that, for $\epsilon \in \{0,1\}$

$$\begin{aligned}1-e^{-(\phi_i(0)+\epsilon\delta)\Delta} &\geq (\phi_i(0)+\epsilon\delta)\Delta\left(1-(\phi_i(0)+\epsilon\delta)\frac{\Delta}{2}\right)\\ &\geq (\phi_i(0)+\epsilon\delta)\Delta\left(1-\beta\frac{\Delta}{2}\right) > 0\end{aligned}$$

which are guaranteed since $\Delta_0 < \frac{2}{\beta}$ and $\phi_i(0)+\delta \leq \beta$. As a consequence, given that

$$\frac{3d\beta}{s} \geq \frac{9d\beta}{4s}+\frac{\beta}{2},\; \frac{5d\beta}{s^2} \geq \frac{85d\beta}{27s^2}+\frac{\beta}{2},$$

from the l.h.s.'s of (27) and (29) we have

$$\begin{aligned}\frac{\mathbb{P}_u(B^i(\Delta))}{\mathbb{P}_u(A^i(\Delta))} &\geq \left(1-\frac{1-e^{-2d\beta\Delta}}{s}\right)e^{-d(\beta-\alpha)\Delta}\left(1-e^{-\phi_i(0)\Delta}\right)\\ &\geq \left(1-\frac{9d\beta}{4s}\Delta\right)\left(1-\frac{\beta\Delta}{2}\right)\phi_i(0)\Delta \geq \left(1-\frac{3d\beta\Delta}{s}\right)\phi_i(0)\Delta > 0,\end{aligned}$$

and, from the l.h.s.'s of (28) and (30), and from (31) we have

$$\begin{aligned}\frac{\mathbb{P}_u(D^{j\to i}(\Delta))}{\mathbb{P}_u(C^{j\to i}(\Delta))} &\ge \left(1-\frac{1-e^{-3d\beta\Delta}}{s^2}\right)e^{-d(\beta-\alpha)\Delta}\left(1-e^{-(\phi_i(0)+\epsilon\delta)\Delta}\right)\\ &\ge \left(1-\frac{85d\beta}{27s^2}\Delta\right)\left(1-\frac{\beta}{2}\Delta\right)(\phi_i(0)+\epsilon\delta)\Delta\\ &\ge \left(1-\frac{5d\beta\Delta}{s^2}\right)(\phi_i(0)+\epsilon\delta)\,\Delta>0,\end{aligned}$$

where $\epsilon=0$ if $j\notin\mathcal{V}^i$, and $\epsilon=1$ if $j\in\mathcal{V}^i_+$, for $\Delta<\Delta_0=\frac{s^2}{5d\beta}$.

From the r.h.s.'s of (27) and (29), using the bound $1-e^{-x}\le x$ one gets

$$\begin{aligned}\frac{\mathbb{P}_u(B^i(\Delta))}{\mathbb{P}_u(A^i(\Delta))} &\le e^{d\beta(1-s)\Delta}\Big(1+\frac{e^{2d\beta\Delta}-1}{s^2}\Big)\phi_i(0)\Delta\\ &=\frac{e^{d\beta(3-s)\Delta}-(1-s^2)e^{d\beta(1-s)\Delta}}{s^2}\phi_i(0)\Delta,\end{aligned}\tag{44}$$

and from the r.h.s.'s of (28) and (30), and from (32) one gets

$$\begin{aligned}\frac{\mathbb{P}_u(D^{j\to i}(\Delta))}{\mathbb{P}_u(C^{j\to i}(\Delta))} &\le e^{d\beta(1-s)\Delta}\Big(1+\frac{e^{3d\beta\Delta}-1}{s^3}\Big)(\phi_i(0)-\epsilon\delta)\Delta\\ &=\frac{e^{d\beta(4-s)\Delta}-(1-s^3)e^{d\beta(1-s)\Delta}}{s^3}(\phi_i(0)-\epsilon\delta)\Delta,\end{aligned}\tag{45}$$

where $\epsilon=0$ if $j\notin\mathcal{V}^i$, and $\epsilon=1$ if $j\in\mathcal{V}^i_-$.

Notice that all the exponential rates appearing in (44) and (45) are bounded from above, uniformly in $0<s<1$, by

$$d\beta(3-s)\Delta_0=\frac{s^2(3-s)}{5}\le\frac{2}{5},\quad d\beta(4-s)\Delta_0=\frac{s^2(4-s)}{5}\le\frac{3}{5}.$$

The function $x\to\frac{e^x-1}{x}$ being increasing, it holds that, for any fixed $\zeta>0$,

$$e^x\le 1+\frac{e^\zeta-1}{\zeta}x,\qquad 0<x\le\zeta.$$

Therefore, by taking into account that both $\frac{5}{2}(e^{2/5}-1)$ and $\frac{5}{3}(e^{3/5}-1)$ do not exceed $3/2$, the fraction appearing in the expression (44) is bounded from above by

$$\begin{aligned}&\frac{1+\frac{3}{2}d\beta(3-s)\Delta}{s^2}-\frac{(1-s^2)(1+d\beta(1-s)\Delta)}{s^2}\\ &=1+\frac{d\beta\Delta}{s^2}\left\{\frac{3}{2}(3-s)-(1-s)(1-s^2)\right\}\le 1+4\frac{d\beta\Delta}{s^2},\end{aligned}$$

and the expression (45) is bounded from above by

$$\begin{aligned}&\frac{1+\frac{3}{2}d\beta(4-s)\Delta}{s^3}-\frac{(1-s^3)(1+d\beta(1-s)\Delta)}{s^3}\\ &=1+\frac{d\beta\Delta}{s^3}\left\{\frac{3}{2}(4-s)-(1-s)(1-s^3)\right\}\le 1+5\frac{d\beta\Delta}{s^3},\end{aligned}$$

for any $s\in(0,1)$.

At this point the proof of Theorem 1 is readily completed.

Proof of Theorem 1. Thanks to Lemma 3 the thresholds $\zeta_1(\Delta)$ and $\zeta_2(\Delta)$ defined in (11) are obtained by taking the difference between the upper bound in (39) and the lower bound in (40), and the difference between the upper bound in (40) and the lower bound in (39), respectively, and finally replacing $\phi_i(0)$ with its upper bound β. The thresholds $\zeta^-(\Delta)$ and $\zeta^+(\Delta)$ defined in (12) and (13) are similarly obtained by using the bounds in (39), (41), and (42).

Remark 1. Since the differences

$$\frac{\mathbb{P}_u(D^{j\to i}(\Delta))}{\mathbb{P}_u(C^{j\to i}(\Delta))} - \frac{\mathbb{P}_{u'}(B^i(\Delta))}{\mathbb{P}_{u'}(A^i(\Delta))}$$

have to be estimated from data, the upper bounds and the lower bounds established in Lemma 3 have to be multiplied, say by $1+\frac{\tau}{10}$ and by $1-\frac{\tau}{10}$, respectively. The resulting "statistical" thresholds $\xi_1(\Delta)$, $\xi_2(\Delta)$, $\xi^-(\Delta)$, and $\xi^+(\Delta)$ defined in (14)–(17) are then obtained with the same procedure used to derive the "probabilistic" thresholds $\zeta_1(\Delta)$, $\zeta_2(\Delta)$, $\zeta^-(\Delta)$, and $\zeta^+(\Delta)$.

Proof of Proposition 1. To begin with, observe that

$$\xi^+(\Delta) = -\xi_1(\Delta) + \tau\lambda_1(\Delta), \qquad \xi^-(\Delta) = -\xi_2(\Delta) + \tau\lambda_2(\Delta) \tag{46}$$

where

$$\lambda_1(\Delta) = \beta\Delta\Big(1 - \frac{5d\beta\Delta}{s^2}\Big)\Big(1-\frac{\tau}{10}\Big), \quad \lambda_2(\Delta) = \beta\Delta\Big(1 + \frac{5d\beta\Delta}{s^3}\Big)\Big(1+\frac{\tau}{10}\Big). \tag{47}$$

With these positions the inequalities in (18) become

$$\xi_1(\Delta) \le -\xi_2(\Delta) + \tau\lambda_2(\Delta), \quad \xi_2(\Delta) \le -\xi_1(\Delta) + \tau\lambda_1(\Delta),$$

or equivalently

$$\xi_1(\Delta) + \xi_2(\Delta) \le \tau\ \min(\lambda_1(\Delta), \lambda_2(\Delta)) = \tau\,\lambda_1(\Delta).$$

The last inequality can be explicitly rewritten as

$$(10+\tau)\Big(2 + (4s+5)\frac{d\beta\Delta}{s^3}\Big) \le (10-\tau)\Big\{\Big(2-(3s+5)\frac{d\beta\Delta}{s^2}\Big) + \tau\Big(1 - 5\frac{5d\beta\Delta}{s^2}\Big)\Big\},$$

which is satisfied for

$$0 < \Delta \le \frac{s^3\tau(6-\tau)}{d\beta\{(4s+5)(10+\tau) + s(10-\tau)(3s+5(1+\tau))\}}.$$

In order to prove that the right hand side is bounded from below by Δ^*, notice that the numerator is bounded from below by $5s^3\tau$, whereas 170 is the maximum value of the expression within brackets at the denominator for $s,\tau \in [0,1]$, with $s+\tau \le 1$, attained at $s=1, \tau=0$.

4. Proof of Theorem 2

In the proof of Theorem 2, we will use the following two inequalities.

Proposition 2. *Let X be a random variable with binomial distribution, with parameters n and p, and let $0<\gamma<1$. Then*

$$\mathbb{P}(X \le np(1-\gamma)) \le e^{-np\gamma^2/2}$$

$$\mathbb{P}(X \ge np(1+\gamma)) \le e^{-np\gamma^2/3}$$

For the proof of Proposition 2 we refer the reader to [22].

Corollary 1. *Let $Y_1, \dots, Y_n$ be Bernoulli random variables with the property*

$$\mathbb{P}(Y_{k+1} = 1 | Y_1, \dots, Y_k) \geq c, \quad k = 0, \dots, n-1, \tag{48}$$

for some constant $c > 0$. Then for any $0 < \gamma < 1$

$$\mathbb{P}(Y_1 + \dots + Y_n \leq nc(1-\gamma)) \leq e^{-nc\gamma^2/2}. \tag{49}$$

When $Y_1, \dots, Y_n$ have the property

$$\mathbb{P}(Y_{k+1} = 1 | Y_1, \dots, Y_k) \leq C, \quad k = 0, \dots, n-1, \tag{50}$$

for some constant $C < 1$, then for any $0 < \gamma < 1$

$$\mathbb{P}(Y_1 + \dots + Y_n \geq nC(1+\gamma)) \leq e^{-nC\gamma^2/3}. \tag{51}$$

Proof. Let us denote

$$p_{k+1}(y_1, \dots, y_k) = \mathbb{P}(Y_{k+1} = 1 | Y_1 = y_1, \dots, Y_k = y_k), \quad k = 0, \dots, n-1,$$

where $y_i \in \{0, 1\}$, $i = 1, \dots, k$.

With $U_1, \dots, U_n$ independent and uniformly distributed in $(0, 1)$, the random variables Y'_k, $k = 1, \dots, n$, are constructed recursively as

$$\begin{cases} Y'_{k+1} = \mathbf{1}_{[0, p_{k+1}(Y'_1, \dots, Y'_k)]}(U_{k+1}), & k = 1, \dots, n-1, \\ Y'_1 = \mathbf{1}_{[0, p_1]}(U_1), \end{cases}$$

so that $(Y'_1, \dots, Y'_n)$ and $(Y_1, \dots, Y_n)$ share the same distribution. Moreover (48) implies $Y'_k \geq \mathbf{1}_{[0,c]}(U_k)$, whereas (50) implies $Y'_k \leq \mathbf{1}_{[0,C]}(U_k)$, for $k = 1, \dots, n$. The proof is completed by observing the random variables $\mathbf{1}_{[0,v]}(U_k)$, $k = 1, \dots, n$ are Bernoulli i.i.d., for any fixed $v \in (0, 1)$, and the application of Proposition 2: in the former case the law of $Y_1 + \dots + Y_n$ stochastically dominates the binomial distribution with parameters n and c, and in the latter is stochastically dominated by the binomial distribution with parameters n and C.

We can now prove Theorem 2 through a series of lemmas. For the first one we recall the definitions (21)

$$t_n = \lceil \alpha \Delta^* n \rceil, \quad m_n = \left\lceil \frac{19}{20} n \alpha^2 \Delta^{*2} (1 - \frac{\tau}{10} \sqrt{\alpha \Delta^*}) \right\rceil.$$

Lemma 4. *For $i, j \in I$, and any integer $n \geq 1$ the following inequalities hold*

$$\mathbb{P}\left(S^{A_i}(t_n) < m_n\right) \leq \rho(n), \tag{52}$$

$$\mathbb{P}\left(S^{C^{j \to i}}(n) < m_n\right) \leq \rho(n), \tag{53}$$

where

$$\rho(n) = e^{-\frac{19}{4 \times 10^3} \alpha^3 \Delta^{*3} \tau^2 n}. \tag{54}$$

Proof. Since for any interval I, $N^i(I) \geq \underline{N}^i(I)$, we have the following lower bounds

$$S^{A_i}(t_n) \geq \sum_{k=1}^{t_n} \mathbf{1}_{\{\underline{N}^i((2k-2)\Delta, (2k-1)\Delta] > 0\}} = \underline{S}^{A_i}(t_n),$$

$$S^{C^{j\to i}}(n) \geq \sum_{k=1}^{\ell} \mathbf{1}_{\{\underline{N}^i((3k-3)\Delta,(3k-2)\Delta]>0\}} \mathbf{1}_{\{\underline{N}^j((3k-2)\Delta,(3k-1)\Delta]>0\}} = \underline{S}^{C^{j\to i}}(n),$$

where the two variables at the r.h.s. are binomial with t_n trials and n trials, respectively, and probabilities of success bounded from below as follows

$$1 - e^{-\alpha\Delta^*} \geq \left(1 - \frac{\alpha\Delta^*}{2}\right)\alpha\Delta^* = \left(1 - \frac{s^4\tau}{68d}\right)\alpha\Delta^* \geq \frac{67}{68}\alpha\Delta^* > \frac{19}{20}\alpha\Delta^*, \tag{55}$$

$$(1 - e^{-\alpha\Delta^*})^2 \geq \left(\frac{67}{68}\right)^2 \alpha^2\Delta^{*2} > \frac{19}{20}\alpha^2\Delta^{*2}, \tag{56}$$

respectively, by using (43). The bounds (52) and (53) are then obtained by applying Proposition 2. Indeed

$$\begin{aligned}
&\mathbb{P}\left(S^{A_i}(t_n) < m_n\right) \leq \mathbb{P}\left(S^{A_i}(t_n) \leq \frac{19}{20}\alpha^2\Delta^{*2}(1 - \frac{\tau}{10}\sqrt{\alpha\Delta^*}) \cdot n\right)\\
\leq&\mathbb{P}\left(S^{A_i}(t_n) \leq \frac{19}{20}\alpha\Delta^*(1 - \frac{\tau}{10}\sqrt{\alpha\Delta^*}) \cdot t_n\right)\\
\leq&\mathbb{P}\left(\underline{S}^{A_i}(t_n) \leq \frac{19}{20}\alpha\Delta^*(1 - \frac{\tau}{10}\sqrt{\alpha\Delta^*}) \cdot t_n\right)\\
\leq&e^{-\frac{19}{4\times 10^3}\alpha^2\Delta^{*2}\tau^2 t_n} \leq e^{-\frac{19}{4\times 10^3}\alpha^3\Delta^{*3}\tau^2 n},
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{P}\left(S^{C^{j\to i}}(n) < m_n\right) \leq \mathbb{P}\left(S^{C^{j\to i}}(n) \leq \frac{19}{20}\alpha^2\Delta^{*2}(1 - \frac{\tau}{10}\sqrt{\alpha\Delta^*}) \cdot n\right)\\
\leq&\mathbb{P}\left(\underline{S}^{C^{j\to i}}(n) \leq \frac{19}{20}\alpha^2\Delta^{*2}(1 - \frac{\tau}{10}\sqrt{\alpha\Delta^*}) \cdot n\right) \leq e^{-\frac{19}{4\times 10^3}\alpha^3\Delta^{*3}\tau^2 n}.
\end{aligned}$$

Lemma 5. *For any positive integer n define*

$$\sigma(n) = e^{-\frac{19^2}{116\times 10^3}\alpha^3\Delta^{*3}\tau^2 n}. \tag{57}$$

Then

$$\mathbb{P}\left(S^{B^i}(K^i_{m_n}) \leq m_n\phi_i(0)\Delta^*\left(1 - \frac{3d\beta\Delta^*}{s}\right)\left(1 - \frac{\tau}{10}\right)\right) \leq \sigma(n), \tag{58}$$

$$\mathbb{P}\left(S^{B^i}(K^i_{m_n}) \geq m_n\phi_i(0)\Delta^*\left(1 + \frac{4d\beta\Delta^*}{s^2}\right)\left(1 + \frac{\tau}{10}\right)\right) \leq \sigma(n); \tag{59}$$

moreover, if $j \notin \mathcal{V}^i$

$$\mathbb{P}\left(S^{D^{j\to i}}(H^{j\to i}_{m_n}) \leq m_n\phi_i(0)\Delta^*\left(1 - \frac{5d\beta\Delta^*}{s^2}\right)\left(1 - \frac{\tau}{10}\right)\right) \leq \sigma(n), \tag{60}$$

$$\mathbb{P}\left(S^{D^{j\to i}}(H^{j\to i}_{m_n}) \geq m_n\phi_i(0)\Delta^*\left(1 + \frac{5d\beta\Delta^*}{s^3}\right)\left(1 + \frac{\tau}{10}\right)\right) \leq \sigma(n); \tag{61}$$

whereas if $j \in \mathcal{V}^-_i$

$$\mathbb{P}\left(S^{D^{j\to i}}(H^{j\to i}_{m_n}) \geq m_n(\phi_i(0) - \delta)\Delta^*\left(1 + \frac{5d\beta\Delta^*}{s^3}\right)\left(1 + \frac{\tau}{10}\right)\right) \leq \sigma(n); \tag{62}$$

and if $j \in \mathcal{V}_i^+$

$$\mathbb{P}\left(S^{D^{j\to i}}(H^{j\to i}_{m_n}) \le m_n(\phi_i(0)+\delta)\Delta^*\left(1-\frac{5d\beta\Delta^*}{s^2}\right)\left(1-\frac{\tau}{10}\right)\right) \le \sigma(n). \tag{63}$$

Proof. The estimates are obtained by means of Corollary 1 with the choices

$$Y'_h = \mathbf{1}_{B^i_{K^i_h}},\ h = 1, 2, \dots, m_n, \qquad Y''_k = \mathbf{1}_{D^{j\to i}_{H^{j\to i}_k}},\ k = 1, 2, \dots, m_n,$$

respectively. Indeed, observe that

$$S^{B^i}(K^i_{m_n}) = \sum_{h=1}^{m_n} Y'_h, \quad \text{and} \quad S^{D^{j\to i}}(H^{j\to i}_{m_n}) = \sum_{k=1}^{m_n} Y''_k.$$

For any non negative integer h and for any value of $j_1, \dots, j_h \in \{0, 1\}$

$$\begin{aligned}&\mathbb{P}(Y'_{h+1} = 1|Y'_1 = j_1, \dots, Y'_h = j_h)\\ &= \sum_{\ell=h+1}^{\infty} \mathbb{P}(K^i_{h+1} = \ell|Y'_1 = j_1, \dots, Y'_h = j_h)\mathbb{P}(B^i_\ell|Y'_1 = j_1, \dots, Y'_h = j_h, K^i_{h+1} = \ell).\end{aligned}$$

Notice that

$$\{K^i_{h+1} = \ell\} = \{K^i_h \le \ell - 1, K^i_{h+1} > \ell - 1, A^i_\ell\}$$

and that the event

$$F = \{Y'_1 = j_1, \dots, Y'_h = j_h, K^i_h \le \ell - 1, K^i_{h+1} > \ell - 1\}$$

is $\mathcal{F}_{2(\ell-1)\Delta^*}$-measurable, so that we get

$$\begin{aligned}&\mathbb{P}(B^i_\ell|Y'_1 = j_1, \dots, Y'_h = j_h, K^i_{h+1} = \ell) = \mathbb{P}(B^i_\ell|F \cap A^i_\ell)\\ &= \frac{\mathbb{E}\big[\mathbb{P}(B^i_\ell|\mathcal{F}_{2(\ell-1)\Delta^*})\mathbf{1}_F\big]}{\mathbb{E}\big[\mathbb{P}(A^i_\ell|\mathcal{F}_{2(\ell-1)\Delta^*})\mathbf{1}_F\big]}.\end{aligned}$$

Using again the notation $\mathbb{P}^{k\Delta^*}_v(\cdot) = \mathbb{P}(\cdot|U(k\Delta^*) = v)$, for $k \ge 1$, we have

$$\mathbb{P}(B^i_\ell|\mathcal{F}_{2(\ell-1)\Delta^*}) = \mathbb{P}^{2(\ell-1)\Delta^*}_{U(2(\ell-1)\Delta^*)}(B^i_\ell) = \mathbb{P}_{U(2(\ell-1)\Delta^*)}(B^i_1),$$

$$\mathbb{P}(A^i_\ell|\mathcal{F}_{2(\ell-1)\Delta^*}) = \mathbb{P}^{2(\ell-1)\Delta^*}_{U(2(\ell-1)\Delta^*)}(A^i_\ell) = \mathbb{P}_{U(2(\ell-1)\Delta^*)}(A^i_1).$$

As a consequence the bounds in Lemma 3 can be applied, from which, for $0 \le h \le m_n - 1$,

$$\begin{aligned}\left(1-\frac{3d\beta\Delta^*}{s}\right)\phi_i(0)\Delta^* &\le \mathbb{P}(Y'_{h+1} = 1|Y'_1 = j_1, \dots, Y'_h = j_h)\\ &\le \left(1+\frac{4d\beta\Delta^*}{s^2}\right)\phi_i(0)\Delta^*,\end{aligned} \tag{64}$$

and when $j \notin \mathcal{V}^i$

$$\begin{aligned}\left(1-\frac{5d\beta\Delta^*}{s}\right)\phi_i(0)\Delta^* &\le \mathbb{P}(Y''_{h+1} = 1|Y''_1 = j_1, \dots, Y''_h = j_h)\\ &\le \left(1+\frac{5d\beta\Delta^*}{s^2}\right)\phi_i(0)\Delta^*,\end{aligned} \tag{65}$$

whereas when $j \in \mathcal{V}^i_+$

$$\left(1-\frac{5d\beta\Delta^*}{s}\right)(\phi_i(0)+\delta)\Delta^* \le \mathbb{P}(Y''_{h+1}=1|Y''_1=j_1,\dots,Y''_h=j_h) \tag{66}$$

and finally, when $j \in \mathcal{V}^i_-$

$$\mathbb{P}(Y''_{h+1}=1|Y''_1=j_1,\dots,Y''_h=j_h) \le \left(1+\frac{5d\beta\Delta^*}{s^2}\right)(\phi_i(0)-\delta)\Delta^*. \tag{67}$$

Now one applies Corollary 1 to all these bounds, with m_n in place of n, and $\gamma = \frac{\tau}{10}$.

Beginning with the leftmost inequality in (64), with $c = (1-3d\beta\Delta^*/s)\phi_i(0)\Delta^*$ in (49), we obtain

$$\begin{aligned}
&\mathbb{P}\left(S^{B^i}(K^i_{m_n}) \le m_n\phi_i(0)\Delta^*\left(1-\frac{3d\beta\Delta^*}{s}\right)\left(1-\frac{\tau}{10}\right)\right)\\
&\le e^{-\frac{1}{2}m_n\phi_i(0)\Delta^*(1-\frac{3d\beta\Delta^*}{s})(\frac{\tau}{10})^2} e^{-\frac{1}{2}m_n\alpha\Delta^*(1-\frac{3d\beta\Delta^*}{s})(\frac{\tau}{10})^2}\\
&\le e^{-n\alpha^3\Delta^{*3}(1-\frac{\tau}{10}\sqrt{\alpha\Delta^*})(\frac{\tau}{10})^2\frac{19\times31}{40\times34}} \le e^{-n\alpha^3\Delta^{*3}\tau^2\frac{3\times19^2\times31}{4\times34\times58\times10^3}} = \sigma(n)^{\frac{31}{34}\times\frac{3}{2}}.
\end{aligned}$$

In the first inequality at the last line, after replacing m_n with the argument of the integer part, we have taken into account that

$$\frac{d\beta\Delta^*}{s} = \frac{s^2\tau}{34} \le \frac{1}{34} \quad\Rightarrow\quad 1-\frac{3d\beta\Delta^*}{s} \ge \frac{31}{34},$$

and in the second inequality that

$$\alpha\Delta^* = \frac{s^4\tau}{34d} \le \frac{1}{34} \quad\Rightarrow\quad 1-\frac{\tau}{10}\sqrt{\alpha\Delta^*} \ge 1-\frac{1}{10\sqrt{34}} > \frac{57}{58} = \frac{3\times19}{58}.$$

For the rightmost inequality in (64) choose $C = (1+4d\beta\Delta^*/s^2)\phi_i(0)\Delta^*$ in (51) obtaining

$$\begin{aligned}
&\mathbb{P}\left(S^{B^i}(K^i_{m_n}) \ge m_n\phi_i(0)\Delta^*\left(1+\frac{4d\beta\Delta^*}{s^2}\right)\left(1+\frac{\tau}{10}\right)\right)\\
&\le e^{-\frac{1}{3}m_n\phi_i(0)\Delta^*(1+\frac{4d\beta\Delta^*}{s^2})(\frac{\tau}{10})^2} \le e^{-\frac{1}{3}m_n\alpha\Delta^*(1+\frac{4d\beta\Delta^*}{s^2})(\frac{\tau}{10})^2}\\
&\le e^{-n\alpha^3\Delta^{*3}(1-\frac{\tau}{10}\sqrt{\alpha\Delta^*})(\frac{\tau}{10})^2\frac{19}{60}} \le e^{-n\alpha^3\Delta^{*3}\tau^2\frac{19^2}{116\times10^3}} = \sigma(n).
\end{aligned} \tag{68}$$

Taking into account that $\frac{31}{34}\times\frac{3}{2} > 1$, the estimates (58) and (59) are obtained.

Analogously, from Corollary 1 with $c = (1-5d\beta\Delta^*/s)\phi_i(0)\Delta^*$ in (49), for the leftmost inequality in (65) one obtains

$$\begin{aligned}
&\mathbb{P}\left(S^{D^{j\to i}}(H^{j\to i}_{m_n}) \le m_n\phi_i(0)\Delta^*\left(1-\frac{5d\beta\Delta^*}{s}\right)\left(1-\frac{\tau}{10}\right)\right)\\
&\le e^{-n\alpha^3\Delta^{*3}\tau^2\frac{3\times19^2\times29}{4\times34\times58\times10^3}} = \sigma(n)^{\frac{29}{34}\times\frac{3}{2}}
\end{aligned}$$

and for the rightmost one, with $C = (1+5d\beta\Delta^*/s)\phi_i(0)\Delta^*$ in (51), one obtains

$$\begin{aligned}
&\mathbb{P}\left(S^{D^{j\to i}}(H^{j\to i}_{m_n}) \ge m_n\phi_i(0)\Delta^*\left(1+\frac{5d\beta\Delta^*}{s^2}\right)\left(1+\frac{\tau}{10}\right)\right)\\
&\le e^{-n\alpha^3\Delta^{*3}\tau^2\frac{19^2}{116\times10^3}} = \sigma(n).
\end{aligned}$$

Since $\frac{29}{34}\times\frac{3}{2} > 1$ the estimates (60) and (61) are obtained. The bounds (62) and (63) are obtained in a completely analogous way, taking into account that $\phi_i(0)\pm\delta \ge \alpha$.

With the help of the previous results we are in a position to control the behavior of the estimators $R^i(n)$ and $G^{j\to i}(n)$ defined in (22) and (23), respectively.

Lemma 6. *For any positive integer n, the following inequalities hold, with $\sigma(n)$ defined in (57),*

$$\mathbb{P}\left(R^i(n) \le \phi_i(0)\Delta^*\left(1-\frac{3d\beta\Delta^*}{s}\right)\left(1-\frac{\tau}{10}\right)\right) \le 2\sigma(n), \tag{69}$$

$$\mathbb{P}\left(R^i(n) \ge \phi_i(0)\Delta^*\left(1+\frac{4d\beta\Delta^*}{s^2}\right)\left(1+\frac{\tau}{10}\right)\right) \le 2\sigma(n); \tag{70}$$

moreover:

if $j \notin \mathcal{V}^i$

$$\mathbb{P}\left(G^{j\to i}(n) \le \phi_i(0)\Delta^*\left(1-\frac{5d\beta\Delta^*}{s^2}\right)\left(1-\frac{\tau}{10}\right)\right) \le 2\sigma(n), \tag{71}$$

$$\mathbb{P}\left(G^{j\to i}(n) \ge \phi_i(0)\Delta^*\left(1+\frac{5d\beta\Delta^*}{s^3}\right)\left(1+\frac{\tau}{10}\right)\right) \le 2\sigma(n); \tag{72}$$

if $j \in \mathcal{V}^i_-$

$$\mathbb{P}\left(G^{j\to i}(n) \ge (\phi_i(0)-\delta)\Delta^*\left(1+\frac{5d\beta\Delta^*}{s^3}\right)\left(1+\frac{\tau}{10}\right)\right) \le 2\sigma(n); \tag{73}$$

if $j \in \mathcal{V}^i_+$

$$\mathbb{P}\left(G^{j\to i}(n) \le (\phi_i(0)+\delta)\Delta^*\left(1-\frac{5d\beta\Delta^*}{s^2}\right)\left(1-\frac{\tau}{10}\right)\right) \le 2\sigma(n). \tag{74}$$

Proof. We are going to prove only (69) in detail, since the other inequalities (70)–(74) need completely similar arguments. So, observe that

$$\begin{aligned}
&\left\{R^i(n) \le \phi_i(0)\Delta^*\left(1-\frac{3d\beta\Delta^*}{s}\right)\left(1-\frac{\tau}{10}\right)\right\}\\
&\quad\subset \left\{K^i_{m_n} > t_n\right\} \cup \left\{S^{B_i}(K^i_{m_n}) \le m_n\phi_i(0)\Delta^*\left(1-\frac{3d\beta\Delta^*}{s}\right)\left(1-\frac{\tau}{10}\right)\right\}\\
&\subset \left\{S^{A_i}(t_n) < m_n\right\} \cup \left\{S^{B_i}(K^i_{m_n}) \le m_n\phi_i(0)\Delta^*\left(1-\frac{3d\beta\Delta^*}{s}\right)\left(1-\frac{\tau}{10}\right)\right\}.
\end{aligned}$$

Since the probabilies of the two events have been bounded from above by $\rho(n)$ and $\sigma(n)$ in (52) and (58), respectively, then

$$\mathbb{P}\left(R^i(n) \le \phi_i(0)\Delta^*\left(1-\frac{\tau}{10}\right)\left(1-\frac{3d\beta\Delta^*}{s}\right)\right) \le \rho(n)+\sigma(n)\,.$$

Since $\rho(n) = \sigma(n)^{19/29} < \sigma(n)$, see (54) and (57), the proof of (69) is concluded.

To conclude the proof of Theorem 2 we need to deduce suitable bounds for the difference $G^{j\to i}(n) - R^i(n)$.

Lemma 7. *For any positive integer n the following inequalities hold:*

if $j \notin \mathcal{V}^i$

$$\mathbb{P}\left(G^{j\to i}(n) - R^i(n) \le -\xi_1(\Delta^*)\right) \le 4\sigma(n), \tag{75}$$

$$\mathbb{P}\left(G^{j\to i}(n) - R^i(n) \geq \xi_2(\Delta^*)\right) \leq 4\sigma(n), \tag{76}$$

from which

$$\mathbb{P}\left(-\xi_1(\Delta^*) < G^{j\to i}(n) - R^i(n) < \xi_2(\Delta^*)\right) \geq 1 - 8\sigma(n); \tag{77}$$

if $j \in \mathcal{V}^i_-$

$$\mathbb{P}\left(G^{j\to i}(n) - R^i(n) < -\xi^-(\Delta^*)\right) \geq 1 - 4\sigma(n); \tag{78}$$

if $j \in \mathcal{V}^i_+$

$$\mathbb{P}\left(G^{j\to i}(n) - R^i(n) > \xi^+(\Delta^*)\right) \geq 1 - 4\sigma(n). \tag{79}$$

Proof. First of all recall that $\xi_1(\Delta^*)$, $\xi_2(\Delta^*)$, $\xi^-(\Delta^*)$, and $\xi^+(\Delta^*)$ are defined in (14)–(17), and that (see (46) and (47))

$$-\xi^-(\Delta^*) = \xi_2(\Delta^*) - \tau\lambda_2(\Delta^*) \quad \text{and} \quad \xi^+(\Delta^*) = -\xi_1(\Delta^*) + \tau\lambda_1(\Delta^*).$$

Furthermore observe that

$$\xi_1(\Delta^*) = \beta\Delta^* \left[\tfrac{\tau}{5} + \left(9 - \tfrac{\tau}{10}\right)\frac{d\beta\Delta^*}{s^2}\right],$$

$$\xi_2(\Delta^*) = \beta\Delta^* \left\{\tfrac{\tau}{5} + \left[5 + 3s^2 + \tfrac{\tau(5-3s^2)}{10}\right]\frac{d\beta\Delta^*}{s^3}\right\}.$$

The following argument is based on the inequality

$$\mathbb{P}(X - Y \geq a - b) \leq \mathbb{P}(\{X \geq a\} \cup \{Y \leq b\}) \leq \mathbb{P}(X \geq a) + \mathbb{P}(Y \leq b),$$

which holds for any pair of random variables X and Y, and for any $a, b \in \mathbb{R}$.

Now let $\chi = 0$ if $j \notin \mathcal{V}^i$ and $\chi = 1$ if $j \in \mathcal{V}^i_+$, then

$$\begin{aligned}
&\mathbb{P}\left(G^{j\to i}(n) - R^i(n) \leq -\xi_1(\Delta^*) + \chi\tau\lambda_1(\Delta^*)\right)\\
\leq&\mathbb{P}\Big(G^{j\to i}(n) - R^i(n) \leq -\frac{\phi_i(0)}{\beta}\xi_1(\Delta^*) + \chi\tau\lambda_1(\Delta^*)\Big)\\
\leq&\mathbb{P}\left(G^{j\to i}(n) \leq \phi_i(0)\Delta^*\left(1 - \frac{5d\beta\Delta^*}{s^2}\right)\left(1 - \tfrac{\tau}{10}\right) + \chi\tau\lambda_1(\Delta^*)\right)\\
+&\mathbb{P}\left(R^i(n) \geq \phi_i(0)\Delta^*\left(1 + \frac{4d\beta\Delta^*}{s^2}\right)\left(1 + \tfrac{\tau}{10}\right)\right),
\end{aligned}$$

since $-\frac{\phi_i(0)}{\beta}\xi_1(\Delta^*)$ coincides with

$$\phi_i(0)\Delta^*\left(1 - \frac{5d\beta\Delta^*}{s^2}\right)\left(1 - \tfrac{\tau}{10}\right) - \phi_i(0)\Delta^*\left(1 + \frac{4d\beta\Delta^*}{s^2}\right)\left(1 + \tfrac{\tau}{10}\right).$$

As a consequence (75) and (79) are established by using the bounds (71), (70) and (74).

Analogously let $\chi = 0$ if $j \notin \mathcal{V}^i$ and $\chi = 1$ if $j \in \mathcal{V}^i_-$, then

$$\begin{aligned}
&\mathbb{P}\left(G^{j\to i}(n) - R^i(n) \geq \xi_2(\Delta^*) - \chi\tau\lambda_2(\Delta^*)\right)\\
\leq&\mathbb{P}\Big(G^{j\to i}(n) - R^i(n) \geq \frac{\phi_i(0)}{\beta}\xi_2(\Delta^*) - \chi\tau\lambda_2(\Delta^*)\Big)
\end{aligned}$$

$$\leq \mathbb{P}\left(G^{j\to i}(n) \geq \phi_i(0)\Delta^*\left(1+\frac{5d\beta\Delta^*}{s^3}\right)\left(1+\tfrac{\tau}{10}\right) - \chi\tau\lambda_2(\Delta^*)\right)$$
$$+\mathbb{P}\left(R^i(n) \leq \phi_i(0)\Delta^*\left(1-\frac{3d\beta\Delta^*}{s}\right)\left(1-\tfrac{\tau}{10}\right)\right),$$

since $\frac{\phi_i(0)}{\beta}\xi_2(\Delta^*)$ coincides with

$$\phi_i(0)\Delta^*\left(1+\frac{5d\beta\Delta^*}{s^3}\right)\left(1+\tfrac{\tau}{10}\right) - \phi_i(0)\Delta^*\left(1-\frac{3d\beta\Delta^*}{s}\right)\left(1-\tfrac{\tau}{10}\right).$$

As a consequence (76) and (78) are established by using the bounds (72), (69) and (73). Moreover (77) is trivially obtained by (75) and (76).

At this point the proof of Theorem 2 is readily completed.

Proof of Theorem 2. The proof follows directly from Proposition 1 and (77), (78), and (79), of Lemma 7. Then, setting $n = n(T)$ in the expression of $\sigma(n)$, and substituting $\alpha\Delta^* = \frac{s^4\tau}{34d}$, we obtain

$$\sigma\big(n(T)\big) = e^{-\lfloor \frac{T}{3\Delta^*}\rfloor s^{12}\tau^5\vartheta_0} \leq e^{\omega\frac{s^3\tau}{10d\beta}} e^{-\omega T},$$

where

$$\vartheta_0 = \frac{19^2}{3\times 116\times 34^2\times 10^3}, \quad \omega = \vartheta_0\,\frac{\tau^4 s^9\beta}{d^2}.$$

To conclude the proof of Theorem 2 is enough to recall that $C = 4e^{\omega\frac{s^3\tau}{10d\beta}}$.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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