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On the number of critical points of the second eigenfunction of the Laplacian in convex planar domains ☆

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A B S T R A C T

In this paper we consider the second eigenfunction of the Laplacian with Dirichlet boundary conditions in convex domains. If the domain has *large eccentricity* then the eigenfunction has *exactly* two nondegenerate critical points (of course they are one maximum and one minimum). The proof uses some estimates proved by Jerison ([13]) and Grieser-Jerison ([10]) jointly with a topological degree argument. Analogous results for higher order eigenfunctions are proved in rectangular-like domains considered in [11].

1. Introduction and main results

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded and smooth domain. Assume that u is a classical solution of the following problem

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$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $f : [0, +\infty) \to [0, +\infty)$ is a smooth function.

It is known that the shape of the solution u is strongly influenced by the geometry of the domain Ω and by the nonlinearity f. In particular a classical problem concerns the study of the number of critical points of solutions of problem (1.1).

If u is a positive solution a lot of results can be found in literature. We are going to recall some of them. The uniqueness of the critical point can be recovered in any dimension and for any locally Lipschitz nonlinearity under symmetry assumptions: this is a consequence of the celebrated results [8] if we ask Ω to be convex and symmetric with respect to all directions.

Under the only convexity assumption of the domain Ω, the uniqueness of the critical point can be proved only in special cases. If we consider the torsion problem, i.e. $f \equiv 1$, Makar-Limanov [16] proved uniqueness and nondegeneracy of the critical point when $N = 2$. Moreover he showed that u is quasiconcave, that is all the superlevel sets are convex. Then, the same result has been obtained in the case of the first Dirichlet eigenfunction in any dimension, namely $f(u) = \lambda_1 u$, see [3,1].

In dimension $N = 2$, without any symmetry assumption Cabré and Chanillo in [4] proved that u possesses exactly one nondegenerate maximum point, provided that the curvature of the boundary of Ω is strictly positive and u is semi-stable i.e. if for all $\varphi \in \mathcal{C}_0^\infty(\Omega)$ it holds

$$\int_\Omega |\nabla\varphi|^2 - \int_\Omega f'(u)|\varphi|^2 \geq 0.$$

The result has been recently extended to domains with nonnegative curvature in [7].

We point out that the convexity assumption can not be dropped, indeed for $N = 2$, for any $k \in \mathbb{N}$ it is possible to find a smooth "almost convex" domain Ω such that the solution of the torsion problem has at least k critical point, see [9] (see also [6] for a generalization).

In this paper we are interested in the study of the number of critical points in the case of sign-changing solutions. To our knowledge there are no results in the literature. So our starting point is the classical problem of the second Dirichlet eigenfunction of the Laplacian in dimension $N = 2$, that is we consider the following eigenvalue problem

$$\begin{cases} -\Delta u = \lambda_2 u & \text{in } \Omega \subset \mathbb{R}^2 \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where λ_2 is the second eigenvalue of the Laplace operator and u a corresponding eigenfunction. It is known that u must change sign and the geometry and location of its nodal line $\Lambda = \overline{\{(x,y) \in \Omega : u(x,y) = 0\}}$ has addressed a lot of interest. A longstanding conjecture is the following

(C) *For which domains $\Omega \subset \mathbb{R}^2$ does the nodal line Λ touch $\partial\Omega$ at exactly two points?*

In [18] it was conjectured that it happens for any bounded domain and in [17] it was proved in convex domains, as conjectured in [20] (for other works about this conjecture, see for instance [15,19,2,5]). The conjecture is not true in any domain: in [12] it was given an example of a domain with a lot of holes where the nodal line of the second eigenfunction does not touch the boundary. In the same paper it was conjectured that (C) holds in planar simply-connected domains.

Of course the computation of the critical points of eigenfunctions to (1.2) is strongly influenced by the geometry of the nodal line. If it is a closed curve contained in Ω we expect at least 3 critical points, otherwise 2 is the minimum number. For this reason we restrict our interest to the case of *convex* domains but, even in this case, there are not sufficient qualitative information on the eigenfunction. So we have to consider a suitable subset of convex domains, namely those with *large eccentricity*. Let us recall that the eccentricity of a planar domain is defined as

$$\mathrm{ecc}(\Omega) = \frac{\text{diameter } \Omega}{\text{inradius } \Omega}$$

where inradius Ω is the radius of the largest circle contained in Ω. These domains were considered by Jerison ([13]) and Grieser-Jerison ([10]) where the location of the nodal line Λ was characterized. In order to state their result we need to normalize the domain Ω in an appropriate way. First let us rotate Ω so that its projection on the y-axis has the shortest possible length, and then dilate so that this projection has length 1. Denote by N the length of the projection of Ω on the x-axis. Then $N \geq 1$, and N is essentially the diameter of Ω. From now we denote by Ω_N a domain satisfying the previous properties and accordingly by u_N a solution to (1.2) in $\Omega = \Omega_N$ with Λ_N its nodal line.

Note that in this setting the domain Ω_N is close to the strip (in a suitable way) $\Omega_\infty = \{\, (x,y) \in \mathbb{R}^2 : 0 < y < 1 \,\}$. We have the following result.

Theorem 1.1 *([10, Theorem 1]). There is an absolute constant C_0 such that the width of the nodal line Λ_N is at most C_0/N. In other words, up to translate Ω_N, one has*

$$(x,y) \in \Lambda_N \implies |x| < \frac{C_0}{N}.$$

This result is our starting point to compute the number of critical points of u_N in Ω_N. We have the following theorem.

Theorem 1.2. *For N large enough, u_N has exactly two critical points $P_N, Q_N \in \Omega_N$ (Fig. 1). Moreover P_N (say) is a nondegenerate maximum point while Q_N is a nondegenerate minimum. Finally $|P_N|, |Q_N| \to +\infty$ as $N \to +\infty$.*

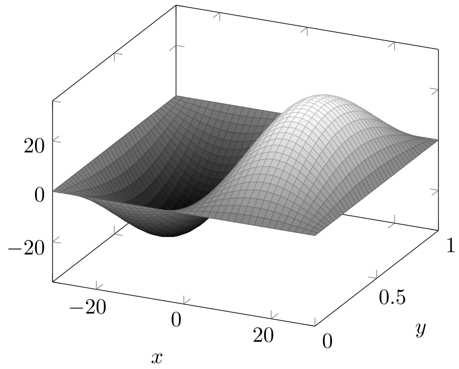

Fig. 1. A graph of u_N for N large.

The proof of the previous theorem is splitted in two parts. In the first one we deduce, up to a suitable normalization, the convergence on compact sets of the eigenfunction u_N to the "limit" function $u_\infty(x, y) = A_0 x \sin(\pi y)$ where A_0 is a nonzero constant. This will be done combining some results in [10] and [11]. We stress that the choice of the normalization of the eigenfunction u_N is not a trivial issue, as already discussed in [13] and [10].

The second part of the proof involves a topological argument: we introduce the vector field $T : \Omega_N \cap \left\{x > \frac{1}{2}\right\} \to \mathbb{R}^2$

$$T(q) = (u_{yy}(q)u_x(q) - u_{xy}(q)u_y(q), u_{xx}(q)u_y(q) - u_{xy}(q)u_x(q)),$$

$q \in \Omega_N \cap \left\{x > \frac{1}{2}\right\}$, which allows to "count" the critical points of u_N. It will be proved that the vector field T is homotopic to the map $I - (x_0, y_0)$ with $(x_0, y_0) \in \Omega_N \cap \{\, x > \frac{1}{2} \,\}$ (the same will be done in $\Omega_N \cap \{\, x < -\frac{1}{2} \,\}$). This result, jointly with some properties of the zeros of the vector field T, will give the uniqueness and nondegeneracy of the critical point of u_N in the set where $u_N > 0$ and $u_N < 0$ respectively.

All these computations strongly use the convexity of the domain Ω_N and the convergence of u_N to u_∞. We stress that, although this convergence is only on compact sets, it will be enough to handle the computations in *all* Ω_N.

In the last part of the paper we deal with a particular class of convex domain not included in the previous section, which are perturbation of rectangles which still converge to the strip. This family of domains has been studied in [11] where they give a full asymptotic expansion for the m-th Dirichlet eigenvalue and for the associated eigenfunction (see Theorem 5.1).

Let $\varphi : [0, 1] \to [0, \infty)$ be a Lipschitz and concave function and for $N \in [0, \infty)$ set

$$\mathcal{R}_N := \left\{ \, (x, y) \in \mathbb{R}^2 \;\middle|\; 0 < y < 1, \; -\varphi(y) < x < N \, \right\}. \tag{1.3}$$

Let $u_{m,N} \in \mathcal{C}^\infty(\mathcal{R}_N)$ be the m-th Dirichlet eigenfunction in $\mathcal{R}_N$ which solves

$$\begin{cases} -\Delta u_{m,N} = \lambda_{m,N} u_{m,N} & \text{in } \mathcal{R}_N \\ u_{m,N} = 0 & \text{on } \partial\mathcal{R}_N, \end{cases}$$

where $\lambda_{m,N}$ is the m-th eigenvalue. In next theorem we prove the existence of exactly m critical points for $u_{m,N}$ in $\mathcal{R}_N$.

Theorem 1.3. *For N large enough, $u_{m,N}$ has exactly m nondegenerate critical points in the set $\mathcal{R}_N$. Moreover all of them are maxima and minima.*

Unlike Theorem 1.2, the proof of Theorem 1.3 is much easier and it strongly follows by the estimates proved in [11].
The paper is organized as follows: in the next section we recall some notations and some results for the second eigenfunction on convex domain with high eccentricity from the papers of Jerison and Grieser, Jerison and in the next one we extrapolate the local convergence of u_N to u_∞ (see Proposition 3.1). Section 4 is devoted to the topological argument where we perform the computations involving the vector field T and we prove Theorem 1.2. Finally, in the last section we investigate the eigenfunctions on convex perturbations of long rectangles, proving Theorem 1.3.

2. Preliminary results

In this section we collect some results proved in [13,10] (see also [14] for an overview of the problem). As we pointed out in the Introduction, let us rotate Ω_N so that its projection on the y-axis has the shortest possible length, then dilate so that this projection has length 1. Denote by N the length of the projection of the domain on the x-axis, then $N \geq 1$. Hence, we write

$$\Omega_N = \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; f_{1,N}(x) < y < f_{2,N}(x), \quad x \in (a_N, b_N) \right\},$$

where $b_N - a_N = N$, $0 \leq f_{1,N} \leq f_{2,N} \leq 1$, and the height function of Ω_N is $h_N := f_{2,N} - f_{1,N}$. We require that

$$f_{1,N} \to 0 \text{ and } f_{2,N} \to 1 \text{ in } C^\infty_{loc}(\mathbb{R}) \text{ as } N \to +\infty.$$

By the convexity of Ω_N we have that $f''_{1,N} \geq 0$ and $f''_{2,N} \leq 0$. Our assumptions imply that the set Ω_N "converges" to the strip

$$\Omega_\infty := \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; 0 < y < 1 \right\}.$$

More precisely we have that for all compact sets $K \subset \mathbb{R}^2$ one has $|(\Omega_N \Delta \Omega_\infty) \cap K| \to 0$. As we recalled in the Introduction we know that the nodal line

$$\Lambda_N := \overline{\{\, (x,y) \in \Omega_N \mid u_N(x,y) = 0 \,\}},$$

is close to the straight line $\{x = 0\}$, up to a translation (see Theorem 1.1 in the Introduction above). Finally let $u_N \in \mathcal{C}^\infty(\Omega_N)$ be the solution of

$$\begin{cases} -\Delta u = \lambda_{2,N} u & \text{in } \Omega_N \\ u = 0 & \text{on } \partial\Omega_N, \end{cases} \tag{2.1}$$

and for all $(x_0, y_0) \in \Lambda_N \cap \Omega_N$ we can assume that $u_N(x_0+1, y_0) > 0$ and $u_N(x_0-1, y_0) < 0$, that is $u_N > 0$ on the right of the nodal line and u_N is negative on the left.

Finally, let L_N be the length of the longest interval $I_{L_N} \subset (a_N, b_N)$ such that

$$h_N(x) = f_{2,N} - f_{1,N} \geq 1 - \frac{1}{L_N^2}, \quad \text{in } I_{L_N}.$$

The number L_N is related to the length of the rectangle contained in Ω_N with lowest first eigenvalue and it satisfies the following bounds (see [10,14])

$$N^{1/3} \leq L_N \leq N. \tag{2.2}$$

For future convenience, we introduce for $k \in \mathbb{R}$ the sets

$$\Omega_N^k := \{\, (x,y) \in \Omega_N \mid -k < x < k \,\},$$

and

$$\Omega_\infty^k := \left\{\, (x,y) \in \mathbb{R}^2 \;\middle|\; -k < x < k,\ 0 < y < 1 \,\right\},$$

where we remember that $\Omega_\infty = \mathbb{R} \times (0,1)$ is the infinite strip of height 1. Since $0 \leq f_{1,N} \leq f_{2,N} \leq 1$, we have that the continuous embedding $H_0^1(\Omega_N) \hookrightarrow H_0^1(\Omega_\infty)$ holds true by means of zero extension outside Ω_N.

An important step to deduce good estimates for the eigenfunction u_N is to choose a correct normalization. So let us define $\widehat{u}_N$ as

$$\widehat{u}_N := L_N \frac{u_N}{||u_N||_\infty}.$$

With a little abuse of notation, in the following we will set

$$\widehat{u}_N = u_N.$$

From the results in [10] we will deduce the following lemma.

Lemma 2.1. *There exists a positive constant C independent of N, such that*

$$|u_N(x,y)| \le C(1+|x|), \quad \forall (x,y) \in \Omega_N, \tag{2.3}$$

and

$$|u_N(\pm 1, 1/2)| \ge \frac{1}{C}. \tag{2.4}$$

Proof. The first estimate (2.3) is proved in [10, Theorem 4].

To prove (2.4), still recalling [10], define the following function

$$\tilde{u}_N(x,y) := \psi_N(x)\sqrt{\frac{2}{h_N(x)}}\sin\left(\pi\frac{y-f_{1,N}(x)}{h_N(x)}\right),$$

where

$$\psi_N(x) := \sqrt{\frac{2}{h_N(x)}}\int_{f_{1,N}(x)}^{f_{2,N}(x)} \sin\left(\pi\frac{y-f_{1,N}(x)}{h_N(x)}\right) u_N(x,y)\,dy.$$

Note that $\tilde{u}_N(x,y) \sim \sqrt{2}\sin(\pi y)\psi_N(x)$ and $\psi_N(x) \sim \sqrt{2}\int_0^1 \sin(\pi y) u_N(x,y)\,dy$ if x is bounded. Finally, let $v_N := u_N - \tilde{u}_N$.

Now, let $C > 0$ be any positive constant independent from N which may vary in the rest of the proof and recall the following estimates. [10, Equation (26)] tells us

$$|\psi_N(x)| \ge C|x|, \quad -2 < x < 2,$$

and [10, Lemma 5] gives for all $(x,y) \in \Omega_N^2$

$$\begin{aligned}
&|v_N(x,y)| \\
&\le \sqrt{\frac{2}{h_N(x)}}\sin\left(\pi\frac{y-f_{1,N}(x)}{h_N(x)}\right)\left(1+|x|\left|\log\left(\sqrt{\frac{2}{h_N(x)}}\sin\left(\pi\frac{y-f_{1,N}(x)}{h_N(x)}\right)\right)\right|\right)L_N^{-3} \\
&\le \frac{C}{L_N^3}.
\end{aligned}$$

Hence for $(x,y) \in \Omega_N^2$ one has

$$\begin{aligned}
|u_N(x,y)| &= |\tilde{u}_N(x,y) + v_N(x,y)| \\
&\ge \left|\psi_N(x)\sqrt{\frac{2}{h_N(x)}}\sin\left(\pi\frac{y-f_{1,N}(x)}{h_N(x)}\right)\right| - |v_N(x,y)| \\
&\ge |\psi_N(x)|\sin\left(\pi\frac{y-f_{1,N}(x)}{h_N(x)}\right) - \frac{C}{L_N^3}
\end{aligned}$$

$$\geq C|x|\sin\left(\pi\frac{y-f_{1,N}(x)}{h_N(x)}\right)-\frac{C}{L_N^3}.$$

Finally, since for $N\to+\infty$ from (2.2) also $L_N\to+\infty$, one has $(\pm1,(f_{1,N}(\pm1)+f_{2,N}(\pm1))/2)\to(\pm1,1/2)$, and then we have

$$\begin{aligned}|u_N(\pm1,1/2)|&=|u_N(\pm1,(f_{1,N}(\pm1)+f_{2,N}(\pm1))/2)|+o(1)\\&\geq C|\pm1|(1+o(1))\geq\frac{C}{2}.\quad\square\end{aligned}$$

Remark 2.2. From (2.3) one has

$$\|u_N\|_{L^\infty(\Omega_\infty^k)}\leq C(1+k),\quad\forall k\in\mathbb{N}. \tag{2.5}$$

The following lemma follows by the standard elliptic regularity theory.

Lemma 2.3. *For $m\in\mathbb{N}$, $f\in H^m(\Omega_N^{k+1})$, let $u\in H^1(\Omega_N^{k+1})$ be a weak solution of*

$$\begin{cases}-\Delta u=f & \text{in } \Omega_N^{k+1}\\ u=0 & \text{on } \partial\Omega_N^{k+1}\setminus\{\,x=\pm(k+1)\,\}.\end{cases}$$

Then for $\delta\in(0,1)$ it holds

$$u\in H^{m+2}(\Omega_N^{k+\delta}),$$

with the estimate

$$\|u\|_{H^{m+2}(\Omega_N^{k+\delta})}\leq C\left(\|f\|_{H^m(\Omega_N^{k+1})}+\|u\|_{L^2(\Omega_N^{k+1})}\right),$$

for some $C>0$ independent from N.

We point out that the independence from N follows from the convergence of Ω_N to Ω_∞, that is the fact that $|(\Omega_N\Delta\Omega_\infty)\cap K|\to0$, for all compact sets $K\subset\mathbb{R}^2$.

3. The asymptotic behavior of u_N

In this section we study the limiting behavior of the solution u_N on compact sets. In particular, u_N converges to a function which is a solution in the whole strip Ω_∞.

Proposition 3.1. *Up to renormalize u_N, we have that for all multiindices α, with $|\alpha|\leq2$ and fixed $k\in\mathbb{N}$, it holds*

$$\sup_{\overline{\Omega}_N\cap\{-k\leq x\leq k\}}\left|D^\alpha\big(u_N-A_0x\sin(\pi y)\big)\right|=o(1),\quad \text{for } N\to+\infty, \tag{3.1}$$

for some suitable constant $A_0\neq0$.

The proof of the previous proposition is a consequence of the next two lemmas.

Lemma 3.2. *We have that there exists $u_\infty : \Omega_\infty \to \mathbb{R}$ such that for all multiindices α, with $|\alpha| \le 2$ and fixed $k \in \mathbb{N}$, it holds up to subsequences*

$$\sup_{\overline{\Omega}_N \cap \{-k \le x \le k\}} \left| D^\alpha \big(u_N - u_\infty\big) \right| = o(1), \quad \textit{for } N \to +\infty, \tag{3.2}$$

and u_∞ solves

$$\begin{cases} -\Delta u_\infty = \pi^2 u_\infty & \textit{in } \Omega_\infty \\ u_\infty = 0 & \textit{for } y = 0, 1. \end{cases}$$

Proof. In the proof of the lemma, convergence will be understood up to subsequences.

Fix $k \in \mathbb{N}$. From (2.5) and Lemma 2.3 we have

$$\|u_N\|_{H^2\left(\Omega_\infty^{k+\frac{1}{2}}\right)} \le C(k),$$

for some $C(k) > 0$ and so there exists $u_\infty^k \in H^1\left(\Omega_\infty^{k+\frac{1}{2}}\right)$ such that

$$u_N \rightharpoonup u_\infty^k \quad \text{weakly in } H^1\left(\Omega_\infty^{k+\frac{1}{2}}\right).$$

Let us show that in $\Omega_\infty^{k+\frac{1}{2}}$ we have that $-\Delta u_\infty^k = \pi^2 u_\infty^k$ in weak sense. Indeed, for all $\varphi \in \mathcal{C}_0^\infty\left(\Omega_\infty^{k+\frac{1}{2}}\right)$ one has

$$\begin{aligned} \int_{\Omega_\infty^{k+\frac{1}{2}}} \nabla u_\infty^k \nabla \varphi &= \int_{\Omega_\infty^{k+\frac{1}{2}}} (\nabla u_\infty^k \nabla \varphi + u_\infty^k \varphi) - \int_{\Omega_\infty^{k+\frac{1}{2}}} u_\infty^k \varphi \\ &= \lim_N \int_{\Omega_\infty^{k+\frac{1}{2}}} (\nabla u_N \nabla \varphi + u_N \varphi) - \lim_N \int_{\Omega_\infty^{k+\frac{1}{2}}} u_N \varphi \\ &= \lim_N \int_{\Omega_\infty^{k+\frac{1}{2}}} \nabla u_N \nabla \varphi \\ &= \lim_N \lambda_{2,N} \int_{\Omega_\infty^{k+\frac{1}{2}}} u_N \varphi \\ &= \pi^2 \int_{\Omega_\infty^{k+\frac{1}{2}}} u_\infty^k \varphi. \end{aligned}$$

Moreover, it is not difficult to see that

$$u_\infty^k = 0, \quad \text{on } \partial\Omega_\infty^{k+\frac{1}{2}} \setminus \{\, x = \pm(k+\frac{1}{2}) \,\},$$

and by Lemma 2.3 we obtain that $u_\infty^k \in \mathcal{C}^\infty\left(\Omega_\infty^{k+\frac{1}{3}}\right)$.

By (2.4) we deduce that $u_\infty^k \not\equiv 0$ in Ω_∞^k, and from the assumptions on the nodal lines of u_N one has $u_\infty^k(0,y) = 0$ for all $y \in (0,1)$.

Next we show the $\mathcal{C}^2$ convergence up to the boundary of Ω_N^k. Let us start by fixing a point $(x,0)$ with $-k < x < k$. From the assumption on Ω_N we can define the set

$$B(N) := \Omega_N \cap B_r(x,0) = \{\, (x,y) \in B_r(x,0) \mid y > f_{1,N}(x) \,\},$$

for some $r > 0$ suitably small. Then, from the standard regularity theory we deduce that

$$\left\| u_N - u_\infty^k \right\|_{C^2(B_{1/2}(N))} \to 0, \quad \text{for } N \to +\infty,$$

where $B_{1/2}(N) := \Omega_N \cap B_{r/2}(x,0)$. To show $\mathcal{C}^2$ convergence in the whole Ω_∞^k it is enough to cover the segments $(-k,k)\times\{0\}$ and $(-k,k)\times\{1\}$ with finitely many balls.
Thus we have proved that for all $k \in \mathbb{N}$ we can find a function $u_\infty^k \in \mathcal{C}^\infty(\Omega_\infty^k)$ such that $u_N \to u_\infty^k$ in $\mathcal{C}^2(\Omega_\infty^k)$ and u_∞^k solves

$$\begin{cases} -\Delta u_\infty^k = \pi^2 u_\infty^k & \text{in } \Omega_\infty^k \\ u_\infty^k = 0 & \text{for } y = 0, 1. \end{cases}$$

By uniqueness of the limit we have $u_\infty^{k+1} = u_\infty^k$ in Ω_∞^k, and this allows us to define a $\mathcal{C}^2$ function in the whole strip Ω_∞ given by

$$u_\infty(x,y) := u_\infty^k(x,y), \quad \text{for } (x,y) \in \Omega_\infty^k,$$

which is a solution of

$$\begin{cases} -\Delta u_\infty = \pi^2 u_\infty & \text{in } \Omega_\infty \\ u_\infty = 0 & \text{for } y = 0, 1. \end{cases} \tag{3.3}$$

Moreover, from the corresponding properties of u_∞^k, note that $u_\infty(0,y) = 0$ for all $y \in (0,1)$ and $|u_\infty(\pm 1, 1/2)| > 0$. □

To conclude the proof of Proposition 3.1 we must prove that $u_\infty(x,y) = A_0 x \sin(\pi y)$ for some $A_0 > 0$. This is a consequence of the next lemma.

Lemma 3.3. *The functions $u(x,y) = Ax\sin(\pi y)$ are the unique solutions of the problem*

$$\begin{cases} -\Delta u = \pi^2 u & \text{in } \Omega_\infty \\ u(0,y) = 0 & \text{for any } y \in [0,1] \\ u(x,0) = u(x,1) = 0 & \text{for any } x \in \mathbb{R} \\ |u(x,y)| \le C(1+|x|) & \text{for some constant } C > 0, \end{cases} \tag{3.4}$$

for any $A \in \mathbb{R}$.

Proof. Here we follow [11, Lemma 6]. Let $u(x,y)$ be a solution to (3.4). Then for each fixed x its Fourier series is given by

$$u(x,y) = \sum_{j=1}^{\infty} A_j(x)\sin(j\pi y),$$

where

$$A_j(x) := 2\int_0^1 u(x,t)\sin(j\pi t)\,dt, \tag{3.5}$$

that is $A_1(x) = c_1 x + d_1$ and

$$A_j(x) = c_j e^{-\sqrt{j^2-1}\pi x} + d_j e^{\sqrt{j^2-1}\pi x}, \quad \text{for } j \ge 2,$$

with $c_j, d_j \in \mathbb{R}$ for all $j \ge 1$, see [11, Lemma 6] for more details.

Then we evaluate (3.5) for $x = 0$ and taking into account that $u(0,y) = 0$ for all $y \in [0,1]$ we have

$$d_1 = A_1(0) = 2\int_0^1 u(0,y)\sin(\pi y)\,dy = 0,$$

and

$$c_j + d_j = A_j(0) = 2\int_0^1 u(0,y)\sin(j\pi y)\,dy = 0, \tag{3.6}$$

for $j \ge 2$.

By the definition of $A_j(x)$ and since u has growth at most linear we have that $d_j = 0$ for all $j \ge 2$. Hence (3.6) implies $c_j = 0$ for all $j \ge 2$ and then

$$
\begin{aligned}
u(x,y) &= \sum_{j=1}^{\infty} A_j(x)\sin(j\pi y) = A_1(x)\sin(\pi y) \\
&= (c_1 x + d_1)\sin(\pi y) = c_1 x \sin(\pi y),
\end{aligned}
$$

and the claim follows. □

Now we are in the position to give the proof of Proposition 3.1.

Proof of Proposition 3.1. By Lemma 3.2 u_N converges up to a subsequence to u_∞, let us show that $u_\infty(x,y) = A_0 x \sin(\pi y)$. First we observe that from inequality (2.3) in Lemma 2.1 we know that u_∞ has growth at most linear for $x \to \pm\infty$. Hence Lemma 3.3 applies and so $u_\infty(x,y) = Ax\sin(\pi y)$. Finally $A = A_0 = u_\infty(1, 1/2) > 0$. To conclude the proof we need to show that, up to renormalize some u_N the convergence holds for the whole sequence. By contradiction, assume that we can find a subsequence $(u_{N_m})_m \subset (u_N)_N$ not converging to u_∞ and $C > 0$ such that

$$
\|u_{N_m} - A_0 x \sin(\pi y)\|_{L^\infty(\Omega_{N_m} \cap \{-k<x<k\})} \geq C.
$$

Now, we can apply Lemma 3.2, and in turn Lemma 3.3, to the sequence $(u_{N_m})_m$ to find that, up to subsequences

$$
\|u_{N_m} - A_1 x \sin(\pi y)\|_{L^\infty(\Omega_{N_m} \cap \{-k<x<k\})} \to 0, \quad \text{for } m \to +\infty,
$$

for some $A_1 > 0$. Hence, up to multiply u_{N_m} by A_0/A_1 we get $u_{N_m} \to u_\infty$, a contradiction. □

Remark 3.4. A consequence of (3.1) is that $\nabla u \neq 0$ in $\Omega_N \cap \{-1 < x < 1\}$. Note also that by the previous lemmas it is possible to deduce that in $\Lambda_N \cap \partial\Omega_N$ there are two nondegenerate saddle points. Indeed, from Theorem 1.1 the nodal line is contained in $\Omega_N \cap \{-1 < x < 1\}$ and [15, Lemma 1.2] tells us that the two points in $\Lambda_N \cap \partial\Omega_N$ are critical points. Moreover, setting $\Lambda_N \cap \partial\Omega_N = \{q_1, q_2\}$ we have $q_1 = (o(1), 1 + o(1))$ and $q_2 = (o(1), o(1))$ and then from Proposition 3.1, writing $q_i := (x_{q_i}, y_{q_i})$, we get for $i = 1, 2$

$$
\partial_{xx} u_N(q_i) = \partial_{xx}\left(A_0 x \sin(\pi y_{q_i})\right) + o(1) = 0 + o(1) = o(1),
$$

and similarly one has

$$
\begin{aligned}
\partial_{xy} u_N(q_i) &= \partial_{xy}\left(A_0 x \sin(\pi y_{q_i})\right) + o(1) \\
&= A_0 \pi \cos(\pi y_{q_i}) + o(1) = (-1)^i A_0 \pi + o(1), \\
\partial_{yy} u_N(q_i) &= \partial_{yy}\left(A_0 x_{q_i} \sin(\pi y_{q_i})\right) + o(1) = -A_0 \pi^2 x_{q_i} \sin(\pi y_{q_i}) + o(1) = o(1).
\end{aligned}
$$

This yields to

$$\det \mathrm{Hess}_u(q_i) = o(1) - ((-1)^i A_0 \pi)^2 < 0,$$

and the claim follows.

4. The topological argument

Up to the end of this section let us write u instead of u_N for brevity. Let us recall some notations and some results from [4] and [7].

For every $\theta \in [0, \pi)$ we write $e_\theta := (\cos\theta, \sin\theta)$ and we set

$$\begin{aligned} u_\theta &:= \langle \nabla u, e_\theta \rangle = \frac{\partial u}{\partial e_\theta}, \\ N_\theta &:= \{\, p \in \overline{\Omega}_N \mid u_\theta(p) = 0 \,\} \ \text{(the nodal set of } u_\theta), \\ M_\theta &:= \{\, p \in N_\theta \mid \nabla u_\theta(p) = \mathbf{0} \,\} \ \text{(the singular points of } u_\theta). \end{aligned}$$

Let us point out that u_θ clearly solves $-\Delta u_\theta = \lambda_{2,N} u_\theta$ in Ω_N. Moreover, if the set $\{\, u = c \,\}$ is smooth then its curvature is given by

$$\mathfrak{K} := -\frac{u_{yy}u_x^2 - 2u_{xy}u_x u_y + u_{xx}u_y^2}{|\nabla u|^3}.$$

Consider

$$\Omega'_N := \{\, (x, y) \in \Omega_N \mid x > 1/2 \,\}.$$

In the next proposition we recall some properties of the sets M_θ and N_θ in Ω'_N.

Proposition 4.1. *We have that for every $\theta \in [0, \pi)$,*

- *(i) around any $p \in (N_\theta \cap \Omega'_N) \setminus M_\theta$ the nodal set N_θ is a smooth curve;*
- *(ii) if $p \in M_\theta \cap \Omega'_N$, then N_θ consists of at least two smooth curves intersecting transversally at p;*
- *(iii) from the domain monotonicity for Dirichlet eigenvalues there is no nonempty domain $H \subset \Omega'_N$ such that $\partial H \subset N_\theta$ (where the boundary of H is considered as a subset of $\mathbb{R}^2$);*
- *(iv) if $p \in \big(N_\theta \cap \partial(\Omega'_N \cap \Omega_N)\big) \setminus M_\theta$ by the implicit function theorem one has that around p, N_θ is a smooth curve intersecting $\partial\Omega'_N$ transversally in p.*

Proof. See [4]. □

The following result tells us that for each $\theta \in [0, \pi)$ the nodal set of u_θ is a smooth curve without self intersection and every critical point of u is nondegenerate.

Proposition 4.2. *For N large enough and for every $\theta \in [0, \pi)$, the nodal set N_θ of the partial derivative u_θ is a smooth curve in $\overline{\Omega}'_N$ without self-intersection which hits $\partial\Omega'_N$ exactly at two points. Moreover at any critical point of u in Ω'_N the Hessian matrix has rank 2.*

Proof. The proof uses Proposition 4.1 jointly with Proposition 3.1.

From the previous points, if we prove that
$a)$ $M_\theta = \emptyset$ on $N_\theta \cap \partial\Omega'_N$,
and
$b)$ $N_\theta \cap \partial\Omega'_N = \{p_1, p_2\}$,
we have the claim. Indeed if $a)$ and $b)$ hold then we cannot have self-intersections of N_θ otherwise (iii) of Proposition 4.1 fails. So $M_\theta = \emptyset$ and this fact jointly with (i) of Proposition 4.1 gives the smoothness of N_θ in Ω'_N. In order to prove $a)$ and $b)$ we will show that the following scenario holds:

- If θ is far away from 0 and π then N_θ intersect $\partial\Omega'_N$ exactly at *two* points, one of them belonging to $\partial\Omega_N$ and the other on the straight line $x = \frac{1}{2}$.
- If θ is close to 0 and π then N_θ intersect $\partial\Omega'_N$ exactly at *two* points, both belonging to the straight line $x = \frac{1}{2}$.
- In both cases N_θ intersect $\partial\Omega'_N$ transversely.

Now let us consider the two different situations.

Case 1: $a)$ and $b)$ hold for θ far away from 0 and π.
From the assumptions on Ω_N and taking into account that the curvature $\mathfrak{K}$ is positive, there exist $\delta_i := \delta_i(N) > 0$, with $\delta_i \to 0$ as $N \to +\infty$, for $i = 1, 2$, such that for $\theta \in (\delta_1(N), \pi - \delta_2(N))$ there exists a unique p_1 on $\partial\Omega_N$ with $x > 1/2$ such that the tangent vector of $\partial\Omega'_N$ at p_1 is parallel to e_θ.

It follows that $p_1 \in N_\theta$ and from $\mathfrak{K} > 0$ we get $p_1 \notin M_\theta$. Indeed

$$u_{\theta\theta}(p_1) = u_{tt}(p_1) = \mathfrak{K}(p_1)u_\nu(p_1) \neq 0,$$

where t denotes the unit tangent normal vector, ν the unit exterior vector and $u_\nu(p_1) \neq 0$ by the Hopf boundary lemma. Hence $p \in \big(N_\theta \cap \partial(\Omega'_N \cap \Omega_N)\big) \setminus M_\theta$ and (iv) of Proposition 4.1 implies that N_θ is a smooth curve intersecting $\partial(\Omega'_N \cap \Omega_N)$ transversely in p_1.

Next let us show that for $\theta \in (\delta_1(N), \pi - \delta_2(N))$ and $p = (1/2, y)$ we have that N_θ is a singleton. Taking into account (3.1), one has

$$\begin{aligned} 0 = u_\theta &= \cos\theta\partial_x u + \sin\theta\partial_y u \\ &= \cos\theta\partial_x\left(A_0 x\sin(\pi y)\right) + \sin\theta\partial_y\left(A_0 x\sin(\pi y)\right) + o(1) \end{aligned}$$

$$= A_0 \cos\theta \sin(\pi y) + A_0 \frac{\pi}{2} \sin\theta \cos(\pi y) + o(1),$$

if and only if

$$\cot\theta = -\frac{\pi}{2} \cot(\pi y)(1 + o(1)),$$

which tells us that, for N sufficiently large, there exists exactly one point $p_2 = (1/2, y_\theta)$ such that $u_\theta(p_2) = 0$. Uniqueness of p_2 follows from $\mathcal{C}^1$ convergence of u_θ given by Proposition 3.1. Moreover similar computations show that $p_2 \notin M_\theta$, indeed

$$\begin{aligned}\partial_x u_\theta &= \cos\theta \partial_{xx} u + \sin\theta \partial_{xy} u \\ &= \cos\theta \partial_{xx} (A_0 x \sin(\pi y)) + \sin\theta \partial_{xy} (A_0 x \sin(\pi y)) + o(1) \\ &= A_0 \pi \sin\theta \cos(\pi y) + o(1) \neq 0,\end{aligned}$$

for $y \neq 1/2 + o(1)$. If $y = 1/2 + o(1)$ one has

$$\begin{aligned}\partial_y u_\theta &= A_0 \pi \cos\theta \cos(\pi y) - A_0 \frac{\pi^2}{2} \sin\theta \sin(\pi y) + o(1) \\ &= -A_0 \frac{\pi^2}{2} \sin\theta + o(1) \neq 0.\end{aligned}$$

So $N_\theta \cap \partial\Omega'_N = \{p_1, p_2\}$ and $p_i \notin M_\theta$ for $i = 1, 2$; hence $a)$ and $b)$ hold for $\theta \in (\delta_1(N), \pi - \delta_2(N))$.

Case 2: $a)$ and $b)$ hold for θ close to 0 and π.
According to the notations of the previous case let us consider $\theta \in [0, \delta_1(N)) \cup (\pi - \delta_2(N), \pi)$. So in this case either $\theta \to 0$ or $\theta \to \pi$ as $N \to +\infty$.

Note that here we have that $N_\theta \cap \partial\Omega_N \cap \partial\Omega'_N = \emptyset$ and then we only have to study what happens on the straight line $x = \frac{1}{2}$. Moreover, Remark 3.4 implies the existence of at least a critical point in Ω'_N and then $N_\theta \cap \Omega'_N \neq \emptyset$. Since there are no intersections of N_θ with $\Omega_N \cap \partial\Omega'_N$ then necessarily N_θ intersects the straight line $x = \frac{1}{2}$, otherwise N_θ is a closed curve contained in Ω'_N, a contradiction with $iii)$ in Proposition 4.1.

Next let us study the intersection of N_θ with $x = \frac{1}{2}$. Recalling that $u(x, y) \sim A_0 x \sin(\pi y)$ we get that $u_\theta(1/2, y) = 0$ if and only if

$$0 = u_\theta(1/2, y) = A_0 \underbrace{\cos\theta}_{\to \pm 1} \sin(\pi y) + \frac{A_0}{2} \underbrace{\sin\theta}_{= o(1)} \cos(\pi y) + o(1),$$

that implies

$$\sin(\pi y) + o(1) = 0,$$

and hence we have two solutions $y_1 = o(1)$ and $y_2 = 1 + o(1)$. Observe that the last equation admits *exactly* two solution by the $\mathcal{C}^1$ convergence of u_θ to $\partial_\theta (A_0 x \sin(\pi y))$.

Finally let us show that both points $p_1 = \left(\frac{1}{2}, y_1\right)$ and $p_2 = \left(\frac{1}{2}, y_2\right)$ do not belong to M_θ. Indeed, for N large enough

$$\partial_y u_\theta(p_1) = \frac{A_0}{2}\pi + o(1) \neq 0 \quad \text{and} \quad \partial_y u_\theta(p_2) = -\frac{A_0}{2}\pi + o(1) \neq 0,$$

which shows that $p_1, p_2 \notin M_\theta$ and as before the implicit function theorem tells us that if $x = 1/2$ the nodal set N_θ is a smooth curve intersecting transversely the line $\{x = 1/2\}$ at p_1 and p_2. This ends the *Case* 2.

Hence we proved $a)$ and $b)$ for all $\theta \in [0, \pi)$.

Finally at any critical point of u we have that the Hessian matrix is nondegenerate otherwise we deduce that there exists θ such that $M_\theta \neq \emptyset$ contradicting $a)$. □

For u solution of (2.1), consider the vector field $T : \overline{\Omega'_N} \to \mathbb{R}^2$ given by

$$T(q) := (u_{yy}(q)u_x(q) - u_{xy}(q)u_y(q), u_{xx}(q)u_y(q) - u_{xy}(q)u_x(q)), \quad q \in \Omega'_N.$$

By the smoothness of u we have that T is of class $\mathcal{C}^1$. In next lemmas we recall some important properties of the vector field T, proved in [7].

Lemma 4.3 *([7, Lemma 2]). If $q \in \Omega'_N$ is such that $T(q) = \mathbf{0}$ then either*

$$q \textit{ is a critical point for } u,$$

or

$$\det \text{Hess}\left(u(q)\right) = 0 \textit{ and for } \cos\theta = \frac{u_x(q)}{\sqrt{u_x^2(q) + u_y^2(q)}} \textit{ we have that } q \in M_\theta.$$

From now if q is an isolated zero of T, for $r > 0$ small enough, we denote by $\text{ind}(T, q) := \deg\left(T, B(q, r), \mathbf{0}\right)$ where *deg* denotes the standard Brower degree.

Lemma 4.4 *([7, Lemma 3]). Let $q \in \Omega'_N$ be such that $T(q) = \mathbf{0}$. Then we have that*

(i) if q is a nondegenerate critical point for u, then $\text{ind}(T, q) = 1$;
(ii) if q is a singular point belonging to M_θ for some $\theta \in [0, \pi)$ and it is a nondegenerate critical point for u_θ then $\text{ind}(T, q) = -1$.

Next corollary was proved in [7, Corollary 1] but we prefer to repeat here the proof.

Corollary 4.5 *([7, Corollary 1]). Let $D \subset \overline{\Omega'_N}$ be such that $M_\theta \cap D = \emptyset$ for all $\theta \in [0, \pi)$ and $\mathbf{0} \notin T(\partial D)$. If $\deg(D, T, \mathbf{0}) = 1$, then u has exactly one critical point in D which is a maximum with negative definite Hessian.*

Proof. Since $\mathbf{0} \notin T(\partial D)$ the degree of T is well posed. Moreover since $M_\theta \cap D = \emptyset$ we have no singular points and moreover all critical points are nondegenerate. So we have finitely many critical points and

$$1 = \deg(D, T, \mathbf{0}) = \sum_{q \in \{\text{ critical points of } u\}} \operatorname{ind}(T, q) = \sharp \{\text{ critical points of } u\},$$

which gives the claim. □

Next we prove the uniqueness of critical point in Ω'_N.

Proposition 4.6. *For N large enough u_N has exactly one critical point in the set Ω'_N. In particular it is a nondegenerate maximum point.*

Proof. We want to apply Corollary 4.5. First of all note that $T \neq \mathbf{0}$ on $\partial\Omega'_N$. Indeed, in $\partial\Omega'_N \cap \partial\Omega_N$, $T = \mathbf{0}$ implies

$$\begin{aligned}-|\nabla u|^3 \mathfrak{K} &= u_{yy}u_x^2 - 2u_{xy}u_xu_y + u_{xx}u_y^2 \\ &= u_x\left(u_{yy}u_x - u_{xy}u_y\right) + u_y\left(u_{xx}u_{xy} - u_{xy}u_x\right) = 0,\end{aligned}$$

a contradiction with the Hopf boundary lemma and the assumption $\mathfrak{K} > 0$ on $\partial\Omega_N$. On the other hand, for $p = (1/2, y)$, using (3.1), we have

$$\begin{aligned}u_xu_{yy} - u_yu_{xy} &= \partial_x\left(A_0 x \sin(\pi y)\right)\partial_{yy}\left(A_0 x \sin(\pi y)\right) + \\ &\quad - \partial_y\left(A_0 x \sin(\pi y)\right)\partial_{xy}\left(A_0 x \sin(\pi y)\right) + o(1) \\ &= -\frac{A_0^2\pi^2}{2}(1 + o(1)),\end{aligned} \tag{4.1}$$

and then $T \neq \mathbf{0}$.

So the degree of T is well defined and if for $p_0 := \left(1, \frac{1}{2}\right)$ the homotopy

$$\begin{aligned}H : [0,1] \times \overline{\Omega'}_N &\to \mathbb{R}^2 \\ (t, q) &\mapsto tT(q) + (1-t)(q - p_0),\end{aligned}$$

is admissible then we deduce

$$\deg(\Omega'_N, T, \mathbf{0}) = \deg(\Omega'_N, I - p_0, \mathbf{0}) = 1.$$

Assume, by contradiction, that the homotopy H is not admissible. Hence, there exist $\tau \in [0,1]$ and $q := (x_q, y_q) \in \partial\Omega'_N$ such that $H(\tau, q) = \mathbf{0}$, i.e.

$$\begin{cases}\tau(u_{yy}(q)u_x(q) - u_{xy}(q)u_y(q)) = (\tau - 1)(x_q - 1) \\ \tau(u_{xx}(q)u_y(q) - u_{xy}(q)u_x(q)) = (\tau - 1)(y_q - 1/2).\end{cases} \tag{4.2}$$

Then, multiplying the first equation by $u_x(q)$, the second by $u_y(q)$ and summing we get

$$-\tau\mathfrak{K}(q)|\nabla u(q)|^3 = (\tau-1)[(x_q-1)u_x(q)+(y_q-1/2)u_y(q)]. \tag{4.3}$$

We want to show that (4.3) leads to a contradiction. First assume that $q \in \partial\Omega'_N \cap \partial\Omega_N$.

For $(x,y) \in \partial\Omega'_N \cap \partial\Omega_N$ denote by $\nu = (\nu_x, \nu_y)$ the unit normal exterior vector at q (consider ν as the exterior normal to $\partial\Omega_N$ if $x_q = 1/2$). Using that Ω'_N is star-shaped with respect to p_0 and the Hopf boundary lemma we have

$$(x_q-1)u_x(q)+(y_q-1/2)u_y(q) = u_\nu(q)[(x_q-1)\nu_x+(y_q-1/2)\nu_y] < 0.$$

Since $\mathfrak{K} > 0$ on $\partial\Omega'_N \cap \partial\Omega_N$, from (4.3) we get a contradiction. It follows that $q \notin \partial\Omega'_N \cap \partial\Omega_N$ and then $q = (1/2, y_q)$. From (4.1) and the first line of (4.2) we get

$$-\frac{A_0^2\pi^2}{2}\tau(1+o(1)) = (\tau-1)(1/2-1) = \frac{1-\tau}{2},$$

again a contradiction.

So $\deg(\Omega'_N, T, \mathbf{0}) = 1$ and by Corollary 4.5 we get that there exists exactly one critical point in Ω'_N: a maximum with negative definite Hessian. □

Similarly we can prove the following proposition.

Proposition 4.7. *For N big enough, u_N has exactly one critical point in the set $\{(x,y) \in \Omega_N | x < -1/2\}$. In particular, it is a nondegenerate minimum point.*

Finally the proof of Theorem 1.2 easily follows.

Proof of Theorem 1.2. The proof follows from Remark 3.4, Proposition 4.6 and Proposition 4.7. Observe that by the local convergence of u_N to the function $u_\infty(x,y) = A_0 x\sin(\pi y)$ we get that $|P_N|, |Q_N| \to +\infty$. □

5. Convex perturbations of rectangles: proof of Theorem 1.3

We start recalling the asymptotic expansion of $u_{N,m}$ given in [11].

Theorem 5.1 *([11, Theorem 1]). There is a number $a := a(\varphi) \in [0, \max\varphi]$ such that for each $m \in \mathbb{N}$ the m-th Dirichlet eigenvalue of $\mathcal{R}_N$ (see (1.3)) satisfies*

$$\lambda_{m,N} = \pi^2 + \frac{m^2\pi^2}{(N+a(\varphi))^2} + O(N^{-5}), \quad N \to \infty.$$

In particular, the eigenvalues $\lambda_{1,N}, \dots, \lambda_{m,N}$ of $\mathcal{R}_N$ are simple for N sufficiently large. The suitably rescaled eigenfunction $u_{m,N}$ satisfies, for all multiindices α,

$$\sup_{\substack{x>3\log N\\0<y<1}} |D^\alpha \left(u_{m,N}(x,y) - v_m(x,y)\right)| = O(N^{-3}), \tag{5.1}$$

where

$$v_m(x,y) := \sin\left(m\pi \frac{x+a(\varphi)}{N+a(\varphi)}\right)\sin\left(\pi y\right),$$

and

$$\sup_{\substack{x\le 3\log N\\0<y<1}} |u_{m,N}(x,y)| = O(N^{-1}\log N).$$

We prove Theorem 1.3 for $m = 2$, the general case is a simple generalization as will be clear from the proof, see also Remark 5.3. We write $u_N = u_{2,N}$ and $v = v_2$ for brevity.

For future convenience let us set

$$\begin{aligned} x_N &:= \frac{1}{2}(N+a) - a,\\ x_N^+ &:= \frac{1}{4}(N+a) - a,\\ x_N^- &:= \frac{3}{4}(N+a) - a,\\ x_N' &:= \frac{1}{12}(N+a) - a. \end{aligned}$$

Proposition 5.2. *For N big enough, the eigenfunction u_N has exactly one nondegenerate maximum point and one nondegenerate minimum point in the set $\mathcal{R}_N \cap \{x > 3\log N\}$.*

Proof. From (5.1) easily follows that u_N has a maximum point close to $(x_N^+, 1/2)$ and a minimum point close to $(x_N^-, 1/2)$. To show that they are the only ones and are nondegenerate, let $p := (x_p, y_p) \in \mathcal{R}_N \cap \{x > 3\log N\}$ be a critical point for u_N.

Then (5.1) implies that there exist a continuous and decreasing function $h : (0, +\infty) \to (0, +\infty)$ such that $\lim_{N\to+\infty} h(N) = 0$ and one of the following occurs

$$p \in B_{h(N)}(x_N^+, 1/2), \tag{5.2}$$

$$p \in B_{h(N)}(x_N^-, 1/2), \tag{5.3}$$

$$p \in B_{h(N)}(x_N, 0) \cap \Omega_N, \tag{5.4}$$

$$p \in B_{h(N)}(x_N, 1) \cap \Omega_N, \tag{5.5}$$

$$p \in B_{h(N)}(N, 0) \cap \Omega_N, \tag{5.6}$$

$$p \in B_{h(N)}(N, 1) \cap \Omega_N. \tag{5.7}$$

Assume (5.2), then from (5.1) one has

$$\partial_{xx}u_N(p) = \partial_{xx}v(p) + O(N^{-3})$$
$$= -\frac{4\pi^2}{(N+a)^2}\sin(\pi/2)\sin(\pi/2)(1+o(1)) = -\frac{4\pi^2}{(N+a)^2}(1+o(1)),$$

and similarly

$$\partial_{xy}u_N(p) = o(N^{-1}) \quad \text{and} \quad \partial_{yy}u_N(p) = -\pi^2(1+o(1)).$$

Hence p is a nondegenerate maximum point. Moreover, we can find $r > 0$ independent from N such that the following homotopy

$$H : [0,1] \times \overline{B_r(x_N^+, 1/2)} \to \mathbb{R}^2$$
$$(t,q) \mapsto t\nabla u_N(q) + (1-t)\nabla v(q),$$

is admissible for N big enough. Then

$$\deg(B_r(x_N^+, 1/2), \nabla u_N, \mathbf{0}) = \deg(B_r(x_N^+, 1/2), \nabla v, \mathbf{0}) = 1,$$

shows that there is exactly one critical point satisfying (5.2). If we assume (5.3), by similar computations, we obtain the existence of exactly one nondegenerate minimum point in $B_{h(N)}(x_N^-, 1/2)$.

Now assume (5.4) i.e. $p \in B_{h(N)}(x_N, 0) \cap \mathcal{R}_N$. Then the same computation as before tell us that p is a nondegenerate saddle point, indeed one has

$$\partial_{xx}u_N(p) = o(N^{-2}), \quad \partial_{xy}u_N(p) = -\frac{2\pi^2}{N+a}(1+o(1)), \quad \partial_{yy}u_N(p) = o(1). \tag{5.8}$$

Now, if $\Lambda_N := \overline{\{(x,y) \in \mathcal{R}_N | u_N(x,y) = 0\}}$ is the nodal line of u_N, let $p_N := (\tilde{x}_N, 0) \in \partial\mathcal{R}_N \cap \Lambda_N$. Since $\mathcal{R}_N$ is convex we know from [2, Theorem 1] that Λ_N intersects $\partial\mathcal{R}_N$ transversally at p_N. In particular $\partial_y u_N(p_N) = 0$ and then p_N is a critical point for u and (5.8) shows that it is a nondegenerate saddle point. Since both p and p_N are nondegenerate we can find $g(N) \in (0, h(N))$ such that $p \in B_{h(N)}(x_N, 0) \setminus \overline{B_{g(N)}(x_N, 0)}$, and for $r > 0$ suitably small and N big enough, since in every critical point in $\omega_N := B_r(x_N, 0) \setminus \overline{B_{g(N)}(x_N, 0)} \cap \Omega_N$ one has

$$\det \mathrm{Hess}\, u_N = -\left(\frac{2\pi^2}{N+a}\right)^2 (1+o(1)) < 0,$$

thanks to (5.8), and since at least p belongs to ω_N it follows $\deg(\omega_N, \nabla u_N, \mathbf{0}) \le -1$ and then

$$-1 \ge \deg(\omega_N, \nabla u_N, \mathbf{0}) = \deg(\omega_N, \nabla v, \mathbf{0}) = 0,$$

a contradiction.

The same argument shows that (5.5), (5.6) and (5.7) cannot occur and the proof is complete. □

Remark 5.3. In case $m > 2$, [2, Theorem 1] still ensures that the nodal line intersects the boundary $\partial\mathcal{R}_N$ transversally at $2m$ different points

Proposition 5.4. *For N big enough, u_N has no critical point in the set*

$$\mathcal{R}'_N := \{ (x, y) \in \mathcal{R}_N \mid x < x'_N \} .$$

Proof. Let us point out that, from the estimate (5.1) and since $x'_N < x_N$, it follows $u_N > 0$ in $\mathcal{R}'_N$. By the domain monotonicity for Dirichlet eigenvalues one has $\lambda_1(\mathcal{R}'_N) > \lambda_{2,N}$ and then the operator $-\Delta - \lambda_{2,N}$ satisfies the maximum principle in $\mathcal{R}'_N$. From (5.1) one has for all $y \in (0, 1)$

$$\partial_x u_N(x'_N, y) = \frac{2\pi}{N + a} \cos(\pi/6) \sin(\pi y) (1 + o(1)) \geq 0.$$

Therefore, $\partial_x u_N \geq 0$ on $\partial\mathcal{R}'_N$ and then the maximum principle gives $\partial_x u_N > 0$ on $\mathcal{R}'_N$. □

Proof of Theorem 1.3. The proof is an obvious consequence of Proposition 5.2 and Proposition 5.4. □

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