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010

# FROM VORTEX LAYERS TO VORTEX SHEETS*

D. BENEDETTO† AND M. PULVIRENTI‡

**Abstract**. This paper shows that the solution of the Birkhoff–Rott equation for the vortex sheet can be approximated, for short times, by the solutions of the Euler equation for a thin vortex layer of vorticity, when its thickness vanishes and its vorticity intensity diverges suitably. The result is obtained in an analytical setup, and an example seems to indicate that this is indeed necessary.

**Key words.** vortex layer, vortex sheet

**AMS(MOS) subject classification.** 76C05

**1. Statement of the problem.** A two-dimensional vortex sheet is a curve in the plane in which the tangential component of the velocity field is discontinuous. Such a field is assumed irrotational elsewhere. In other terms, the vorticity is concentrated in a curve as a delta function. Denoting by $y = \phi(x)$ the equation of the sheet and $\eta = \eta(x,t)$ the vorticity intensity, the general laws of dynamics of incompressible, nonviscous flows yields the following equations:

$$\partial_t\phi(x,t) + (u\,\partial_x\phi)(x,t) = v(x,t), \tag{1.1a}$$

$$\partial_t\eta(x,t) + \partial_x(u\,\eta)(x,t) = 0, \tag{1.1b}$$

where

$$\begin{aligned} \mathbf{u}(x,\phi(x,t),t) &= \begin{pmatrix} u \\ v \end{pmatrix}(x,t) \\ &= \frac{1}{2\pi}\fint_{-\infty}^{+\infty} dx'\,\eta(x',t)\frac{\begin{pmatrix} -(\phi(x)-\phi(x')) \\ x-x' \end{pmatrix}}{(x-x')^2+(\phi(x)-\phi(x'))^2} \end{aligned} \tag{1.2}$$

denotes the velocity field computed on the sheet, and $\fint$ denotes the Cauchy principal value integral. Since $\mathbf{u}$ is discontinuous on $\phi$, the expression computed by (1.2) means that

$$\mathbf{u}(x,\phi(x,t),t) = \frac{\mathbf{u}^+(x,\phi(x,t),t) + \mathbf{u}^-(x,\phi(x,t),t)}{2}, \tag{1.3}$$

where $\mathbf{u}^+$ and $\mathbf{u}^-$ denote the upper and lower limits, respectively. The jump discontinuity of $\mathbf{u}$ takes the form

$$\mathbf{j} = -\frac{1}{2}\begin{pmatrix} 1 \\ \partial_x\phi \end{pmatrix}\frac{\eta}{1+(\partial_x\phi)^2}. \tag{1.4}$$

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† Dipartimento di Matematica, Pura ed Applicata, Università dell'Aquila, 67100 L'Aquila, Italy.

‡ Dipartimento di Matematica, Università di Roma "La Sapienza," Piazzale A. Moro 2, 00185 Rome, Italy.

From a dynamical point of view, (1.1) expresses the convection of the interface and the conservation of the vorticity, respectively.

Other parametrizations of the curve $y = \phi(x,t)$ are obviously possible, yielding similar descriptions. A special and useful parametrization, in terms of the circulation variable for which (1.1) reduces to a single complex equation, is called the Birkhoff–Rott equation [1]. However, for future convenience, we prefer to parametrize the interface in terms of the $x$ variable.

It is well known that the stationary solution $\phi = 0$, $\eta$ =const is unstable (Kelvin–Helmholtz instability). Actually, a small periodic disturbance of wavenumber $k$ may grow exponentially in time, as $e^{|k|t}$, which follows by the linear analysis. Due to this fact, the initial value problem associated with (1.1) is naturally described in analytical functions spaces, and, in this framework, an existence theorem for short times can be proved [14]. On the other hand, the formation of singularities, while evident from numerical simulations, has been analytically obtained in [2], [3], [6], while the illposedness of the problem in some Sobolev spaces is discussed in [3] and [7].

The initial value problem associated with (1.1) may be regularized by considering a small thickness for the vortex sheet. More precisely, consider an initial vorticity profile of the form

$$\omega_\varepsilon(x,y) = \frac{1}{\varepsilon}\,\mathcal{X}_{\Lambda_\varepsilon}(x,y), \tag{1.5}$$

where $\mathcal{X}_{\Lambda_\varepsilon}$ is the characteristic function of the set

$$\Lambda_\varepsilon = \left\{(x,y) \,\middle|\, \phi_\varepsilon^-(x) < y < \phi_\varepsilon^+(x)\right\}. \tag{1.6}$$

The sequence of functions $\phi_\varepsilon^\pm$ are chosen in such a way that

$$\lim_{\varepsilon\to 0} \frac{\phi_\varepsilon^+ - \phi_\varepsilon^-}{\varepsilon} = \eta, \tag{1.7}$$

$$\lim_{\varepsilon\to 0} \phi_\varepsilon^\pm = \phi \tag{1.8}$$

Under such hypotheses, $\omega_\varepsilon$ converges locally, in the sense of the weak convergence of the measures, to the measure

$$\omega(x,y) = \delta(y - \phi(x))\,\eta(x)\,dx\,dy. \tag{1.9}$$

It is well known that an evolution according to the Euler equation for the initial value (1.5) can be given globally in time [15]. It is of the form

$$\omega_\varepsilon(x,y,t) = \frac{1}{\varepsilon}\,\mathcal{X}_{\Lambda_\varepsilon(t)}(x,y), \tag{1.10}$$

where $\Lambda_\varepsilon(t)$ moves from $\Lambda_\varepsilon$ according to the current lines. It is natural to expect that, for short times, $\omega_\varepsilon(x,y,t)$ converges weakly to

$$\delta(y - \phi(x,t))\,\eta(x,t)\,dx\,dy, \tag{1.11}$$

where $\phi$ and $\eta$ solve (1.1). The aim of the present paper is to give a proof of this fact.

The interest of such a result is twofold. From one side, it provides a rigorous justification of (1.1) (or, equivalently, of the Birkhoff–Rott equation). From a practical point of view, it shows that the thin-layer method used to simulate the vortex sheet dynamics (see [12], [13]) is a convergent algorithm. In this context, we mention that point-vortex and blob-vortex methods can also be used for approximating the vortex sheet evolution (see [9], [10]). A convergence proof of these methods is given in [4].

The plan of the paper is the following. In the §2 we establish and prove our main result. The proof is based on some estimates, which are proved in §3, where the convergence result also somehow improved. Finally, in §4 we discuss the nature of the approximation at time zero. Namely, we conjecture that nonanalytical approximations of the vortex sheet at time zero could give different solutions of the vortex sheet equation.

**2. The main result.** Consider $X_\rho$, $\rho > 0$, the Banach space of all analytic $2\pi$-periodic functions with the norm

$$\|\phi\|_\rho = \sum_{k=-\infty}^{+\infty} e^{\rho|k|} \left|\hat{\phi}(k)\right|. \tag{2.1}$$

Here, as usual, $\hat{\phi}$ denotes the Fourier transform of $\phi$.

The initial value $(\phi, \eta)$ of the evolution problem (1.1) is assumed in $X_\rho \times X_\rho$. Moreover, $\phi$ and $\eta$ are a small perturbation of the flat sheet of uniform vorticity: $\eta = 1 + \mu$, $\|\mu\|_\rho$, $\|\phi\|_\rho \ll 1$.

A vorticity profile of the type (1.5), (1.6), whose support $\Lambda_\varepsilon$ is enclosed between two smooth functions $\phi^\pm$, generates a velocity field given by

$$\mathbf{u}_\varepsilon(x,y) = \begin{pmatrix} u_\varepsilon \\ v_\varepsilon \end{pmatrix}(x,y) = \frac{1}{2\pi\varepsilon} \int_{-\infty}^{+\infty} dx' \int_{\phi_\varepsilon^-(x')}^{\phi_\varepsilon^+(x')} dy' \frac{\begin{pmatrix} -(y-y') \\ x-x' \end{pmatrix}}{(x-x')^2 + (y-y')^2}. \tag{2.2}$$

The Euler evolution for such a vortex layer reduces to the following equations:

$$\partial_t \phi_\varepsilon^\pm(x,t) + (u_\varepsilon^\pm\, \partial_x \phi_\varepsilon^\pm)(x,t) = v_\varepsilon^\pm(x,t), \tag{2.3}$$

where

$$\begin{pmatrix} u_\varepsilon^\pm \\ v_\varepsilon^\pm \end{pmatrix}(x,t) = \mathbf{u}_\varepsilon(x, \phi_\varepsilon^\pm(x,t)). \tag{2.4}$$

Defining the approximate vorticity intensity by

$$\eta_\varepsilon = \frac{\phi_\varepsilon^+ - \phi_\varepsilon^-}{\varepsilon} \tag{2.5}$$

as a consequence of (2.3), we have that
(2.6a)

$$\partial_t \eta_\varepsilon(x,t) = -\partial_x \left( \frac{1}{\varepsilon} \int_{\phi_\varepsilon^-(x,t)}^{\phi_\varepsilon^+(x,t)} dy\, u_\varepsilon(x,y) \right) + \frac{1}{\varepsilon} \int_{\phi_\varepsilon^-(x,t)}^{\phi_\varepsilon^+(x,t)} dy\, \left(\partial_x u_\varepsilon(x,y) + \partial_y v_\varepsilon(x,y)\right).$$

By virtue of the incompressibility condition, the last term of (2.6a) vanishes, so that

$$\begin{aligned}\partial_t\eta_\varepsilon(x,t) &= -\,\partial_x\left(\frac{1}{\varepsilon}\int_{\phi_\varepsilon^-(x,t)}^{\phi_\varepsilon^+(x,t)} dy\, u_\varepsilon(x,y)\right)\\ &= -\,\partial_x\left(\eta_\varepsilon(x,t)\int_0^1 d\lambda\, u_\varepsilon(x,\lambda\phi_\varepsilon^+(x,t)+(1-\lambda)\phi_\varepsilon^-(x,t))\right),\end{aligned} \tag{2.6b}$$

and the continuity equation (1.1b) is recovered in the formal limit $\varepsilon \to 0$. Actually, the vorticity intensity $\eta$ is conserved only with respect to the mean velocity. Note that the integral in the right-hand side of (2.6b) is an average of the velocity of $u_\varepsilon$. We will later show that this average converges to $\frac{1}{2}(u^+ + u^-)$, and thus the correct limiting equation is obtained. As we noted in the previous section, the Euler solution for the initial vortex layer is known to exist globally in time. However, its equivalence with (2.3) cannot be established for all times. Actually, $\partial\Lambda_\varepsilon(t)$ may lose its smoothness even if it is assumed at time zero, so that it could be difficult to give a sense to (2.3). Numerical simulation and analytical considerations seem to indicate this (see [5] and the references therein). Moreover, $\partial\Lambda_\varepsilon(t)$ could not be described in terms of function of $x$ after a finite time. In all cases, in view of the asymptotic behavior, we prove that no smoothness uniform in $\varepsilon$ is expected to hold after a critical time. In this context, it is natural to set the initial value problem associated with (2.3) in the scale of Banach spaces $X_\rho$, for which we need a Cauchy–Kowalevski-type of theorem. The following theorem is very well known (see, e.g., [8]).

THEOREM 2.1. *Let $\{X_\rho\}_{\rho\geq 0}$ be a scale of Banach spaces satisfying $X_{\rho'} \subset X_\rho$, $\|\cdot\|_\rho \leq \|\cdot\|_{\rho'}$ for $\rho < \rho'$, where $\|\cdot\|_\rho$ denotes the norm in $X_\rho$. Consider the Cauchy problem*

$$\xi(t) = \xi_0 + \int_0^t ds\, F(\xi(s)). \tag{2.7}$$

*Suppose that there exists $\rho_0$ and $R > 0$ such that $F : \{\xi \,|\, \xi \in X_{\rho'},\, \|\xi\|_{\rho'} < R\} \to X_\rho$ is a continuous mapping satisfying*

$$\|F(\xi)\|_\rho \leq C\frac{\|\xi\|_{\rho'}}{\rho'-\rho}, \tag{2.8}$$

$$\|F(\xi_1) - F(\xi_2)\|_\rho \leq C\frac{\|\xi_1-\xi_2\|_{\rho'}}{\rho'-\rho} \tag{2.9}$$

*for all $\rho < \rho' < \rho_0$, $\xi$, $\xi_1$, $\xi_2$ $\in X_{\rho'}$, $\|\xi\|_{\rho'}$, $\|\xi_1\|_{\rho'}$, $\|\xi_2\|_{\rho'} < R$, and $C$ is some positive constant depending on $R$. Then, for $\xi_0 \in X_{\rho_0}$ with $\|\xi_0\|_{\rho_0} < R_0 < R$, there exists a unique continuous solution $\xi(t) \in X_\rho$, satisfying $\|\xi(t)\|_\rho < R$ for $t \in [0, a(\rho_0-\rho))$, with $a$ chosen suitably small.*

*Remark.* A possible choice for $a$ is $a = (1/4C)((\sqrt{R}-\sqrt{R_0})^2/R)$.

In view of the application of Theorem 2.1 to our purposes, it is convenient to introduce, from (2.2), the complex velocity field $U_\varepsilon(z) = u_\varepsilon(z) + iv_\varepsilon(z)$, $z = x+iy$ as

$$\begin{aligned}U_\varepsilon(z) &= \frac{1}{2\pi\varepsilon i}\int_{-\infty}^{+\infty} d\alpha \int_{\phi_\varepsilon^-(x+\alpha)}^{\phi_\varepsilon^+(x+\alpha)} \frac{dy'}{\alpha + i(y-y')}\\ &= \frac{1}{2\pi i}\int_{-\infty}^{+\infty} d\alpha\, \eta_\varepsilon(x+\alpha)\int_0^1 \frac{d\gamma}{\alpha + i\,(y - \varepsilon\gamma\eta_\varepsilon(x+\alpha) - \phi_\varepsilon^-(x+\alpha))}.\end{aligned} \tag{2.10}$$

For $y = \operatorname{Im} z = \lambda\phi_\varepsilon^+ + (1-\lambda)\phi_\varepsilon^-$, we have that

$$\begin{aligned} U_\varepsilon(z) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} d\alpha\ (1+\mu_\varepsilon(x+\alpha)) \\ \cdot\int_0^1 \frac{d\gamma}{\alpha + i\ (\lambda a^+ + (1-\lambda)a^- + \varepsilon(\lambda-\gamma)(1+\mu_\varepsilon(x+\alpha)))}, \end{aligned} \tag{2.11}$$

where $a^\pm = \phi_\varepsilon^\pm(x) - \phi_\varepsilon^\pm(x+\alpha)$ and $\mu_\varepsilon = \eta_\varepsilon - 1$.

Consider now the above expression as an operator of $\phi^+$, $\phi^-$, $\mu$ as independent functions, forgetting the link expressed by (2.5), as follows:

$$\begin{aligned} U_\varepsilon[\lambda;\mu,\phi^+,\phi^-](x) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} d\alpha\,(1+\mu(x+\alpha)) \\ \cdot\int_0^1 \frac{d\gamma}{\alpha + i\ (\lambda\Delta\phi^+(x,\alpha) + (1-\lambda)\Delta\phi^-(x,\alpha) + \varepsilon(\lambda-\gamma)(1+\mu(x+\alpha)))}, \end{aligned} \tag{2.12}$$

where $\Delta\phi^\pm(x,\alpha) = \phi^\pm(x) - \phi^\pm(x+\alpha)$. For $\lambda = 1$ and $\lambda = 0$, such an operator does not depend on $\phi^-$ and $\phi^+$, respectively. Define

$$\begin{aligned} u_\varepsilon^+[\mu,\phi^+] &= \operatorname{Re} U_\varepsilon[1;\mu,\phi^+,\phi^-], \\ v_\varepsilon^+[\mu,\phi^+] &= \operatorname{Im} U_\varepsilon[1;\mu,\phi^+,\phi^-], \\ u_\varepsilon^-[\mu,\phi^-] &= \operatorname{Re} U_\varepsilon[0;\mu,\phi^+,\phi^-], \\ v_\varepsilon^-[\mu,\phi^-] &= \operatorname{Im} U_\varepsilon[0;\mu,\phi^+,\phi^-], \end{aligned} \tag{2.13}$$

and

$$F_\varepsilon^\pm[\mu,\phi^\pm] = -u_\varepsilon^\pm[\mu,\phi^\pm]\,\partial_x\phi^\pm + v_\varepsilon^\pm[\mu,\phi^\pm], \tag{2.14}$$

$$G_\varepsilon[\mu,\phi^+,\phi^-] = -\partial_x\left((1+\mu)\int_0^1 d\lambda\, u_\varepsilon[\lambda;\mu,\phi^+,\phi^-]\right), \tag{2.15}$$

where $u_\varepsilon[\lambda;\mu,\phi^+,\phi^-] = \operatorname{Re} U_\varepsilon[\lambda;\mu,\phi^+,\phi^-]$. Consider the Cauchy problem

$$\frac{d}{dt}\xi(t) = F_\varepsilon(\xi(t)), \qquad \xi(0) = \xi_0, \tag{2.16}$$

where

$$\xi = \begin{pmatrix} \phi^+ \\ \phi^- \\ \mu \end{pmatrix}, \qquad F_\varepsilon(\xi) = \begin{pmatrix} F_\varepsilon^+[\mu,\phi^+] \\ F_\varepsilon^-[\mu,\phi^-] \\ G_\varepsilon^+[\mu,\phi^+,\phi^-] \end{pmatrix}. \tag{2.17}$$

Note that the above Cauchy problem is *not* equivalent to that associated with (2.3). Here we have a third equation (expressing the conservation of the vorticity), which is considered as independent of the first two. Actually, we are interested in solutions of (2.16) for which

$$1 + \mu = \varepsilon^{-1}(\phi^+ - \phi^-). \tag{2.18}$$

The basic technical step in the present paper is summarized in the following proposition.

**Proposition 2.1.** *Suppose that* $\|\mu_j\|_\rho < \frac{1}{4}$, $\|\partial_x \phi_j^{\pm}\|_\rho < \frac{1}{4}$, $j = 1, 2$. *Then there exists a constant* $C$ *(independent of* $\lambda$ *and* $\varepsilon$*) for which*

$$\|U_\varepsilon[\lambda; \mu_j, \phi_j^+, \phi_j^-]\|_\rho \leq C, \tag{2.19}$$

$$\begin{aligned} &\|U_\varepsilon[\lambda; \mu_1, \phi_1^+, \phi_1^-] - U_\varepsilon[\lambda; \mu_2, \phi_2^+, \phi_2^-]\|_\rho \\ &\qquad \leq C\left(\lambda\|\partial_x(\phi_1^+ - \phi_2^+)\|_\rho + (1-\lambda)\|\partial_x(\phi_1^- - \phi_2^-)\|_\rho + \|\mu_1 - \mu_2\|_\rho\right). \end{aligned} \tag{2.20}$$

We show the proof of Proposition 2.1 in the next section. We are now in a position to apply Theorem 2.1. Define

$$\|\xi\|_\rho = \|\phi^+\|_\rho + \|\partial_x \phi^+\|_\rho + \|\phi^-\|_\rho + \|\partial_x \phi^-\|_\rho + \|\mu\|_\rho. \tag{2.21}$$

Then inequalities (2.8) and (2.9) are easy consequences of (2.19) and (2.20), provided that $\|\partial_x \phi^{\pm}\|_{\rho'}$ and $\|\mu\|_{\rho'}$ are sufficiently small. In fact, (2.9) follows from (2.19) and (2.20) via the obvious inequality $\|\partial_x \psi\|_\rho \leq \|\psi\|_{\rho'}/(\rho' - \rho)$. Finally, using the fact that $F(0) = 0$ and (2.9), we also obtain (2.8). Therefore, choosing $\xi_0$ such that $\|\xi_0\|_{\rho_0} < R_0 < \frac{1}{4}$, for a given $\rho_0 > 0$, we can conclude the existence of an unique solution $\xi(t) \in X_\rho$, for $t \in [0, a(\rho_0 - \rho))$ solving the integral version of (2.16) and satisfying the bound $\|\xi(t)\|_\rho < \frac{1}{4}$.

However, we have not yet solved the Cauchy problem for (2.3) because the solution we have interested in must satisfy (2.18). This is an easy consequence of the following consideration. Note that Theorem 2.1 can be proved by showing the convergence of the iteration scheme

$$\xi^0(t) = \xi_0, \qquad \xi^n(t) = \xi_0 + \int_0^t ds\, F_\varepsilon(\xi^{n-1}(s)). \tag{2.22}$$

By direct inspection, condition (2.18) is satisfied at level $n$, provided that it is satisfied at level $n-1$. Since the initial state obeys to such a constraint, we conclude that the limit does it. Thus the solution of the initial value problem (2.16) coincides with that of the vortex layer dynamics, provided that

$$\xi_0 = \begin{pmatrix} \phi_0^+ \\ \phi_0^- \\ \mu_0 \end{pmatrix}$$

satisfies (2.18).

At this point we can prove our main theorem.

THEOREM 2.2. *Let*

$$\omega_\varepsilon(x, y) = \frac{1}{\varepsilon}\, \mathcal{X}_{\Lambda_\varepsilon}(x, y), \tag{2.23}$$

$$\Lambda_\varepsilon = \left\{(x, y) \mid \phi_\varepsilon^-(x) < y < \phi_\varepsilon^+(x)\right\} \tag{2.24}$$

*be an initial family of profiles of vorticity,* $\varepsilon \in (0, \varepsilon_0)$. *Denote by* $\omega_\varepsilon(x, y, t)$ *and* $\Lambda_\varepsilon(t)$ *the corresponding quantities evolved according to the Euler equation.*

*Suppose also that, initially,*

$$\|\xi\|_{\rho_0} = \|\phi_\varepsilon^+\|_{\rho_0} + \|\partial_x \phi_\varepsilon^+\|_{\rho_0} + \|\phi_\varepsilon^-\|_{\rho_0} + \|\partial_x \phi_\varepsilon^-\|_{\rho_0} + \|\mu_\varepsilon\|_{\rho_0} < R_0 < \tfrac{1}{4}, \tag{2.25}$$

*where*

$$\mu_\varepsilon = \eta_\varepsilon - 1, \qquad \eta_\varepsilon = \frac{\phi_\varepsilon^+ - \phi_\varepsilon^-}{\varepsilon}. \tag{2.26}$$

*Then* (i) *there exists* $a > 0$*, independent of* $\varepsilon$*, such that*

$$\Lambda_\varepsilon(t) = \{(x,y) \,|\, \phi_\varepsilon^-(x,t) < y < \phi_\varepsilon^+(x,t)\} \tag{2.27}$$

*with* $\phi_\varepsilon^\pm(\cdot\,,t) \in X_\rho$ *for* $t \in [0,\, a(\rho_0 - \rho))$*, and* $\phi_\varepsilon^\pm(x,t)$ *satisfy* (2.3).

*In addition, the bound*

$$\|\xi\|_\rho = \|\phi_\varepsilon^+\|_\rho + \|\partial_x \phi_\varepsilon^+\|_\rho + \|\phi_\varepsilon^-\|_\rho + \|\partial_x \phi_\varepsilon^-\|_\rho + \|\mu_\varepsilon\|_\rho < \tfrac{1}{4}$$

*holds.*

(ii) *For all* $t \in [0,\, a(\rho_0 - \rho))$*,*

$$\lim_{\varepsilon \to 0} \omega_\varepsilon(\cdot\,,t) = \omega(\cdot\,,t), \tag{2.28}$$

*with*

$$\omega(dx,\, dy,\, t) = \delta\,(y - \phi(x,t))\; \eta(x,t)\, dx\, dy \tag{2.29}$$

*and where the limit* (2.28) *holds in the sense of the weak convergence of the measures. Moreover,* $\phi(\cdot\,,t)$*,* $\eta(\cdot\,,t) \in X_\rho$*, and uniquely solve* (*in* $X_\rho$) *the vortex sheet equations* (1.1) *for* $t \in [0,\, a(\rho_0 - \rho))$.

*Proof.* We give a short proof of Theorem 2.2. A constructive proof and additional considerations is developed in the next section, where the structure of the velocity field generated by a thin vortex layer is analyzed to prove Proposition 2.1. We first remark that (i) has already been proved as consequence of Theorem 2.1. To prove (ii), we first find a sequence $\{\phi_n^\pm,\, \mu_n\} \subset \{\phi_\varepsilon^\pm,\, \mu_\varepsilon\}_{\varepsilon \in (0,\varepsilon_0)}$ converging in $X_{\rho'}$ for all $t \in [0,\, a(\rho_0 - \rho'))$, $\rho' < \rho$ to $\phi^\pm$, $\mu \in X_{\rho'}$. Such a sequence exists by standard compactness arguments, and, by the usual diagonal trick, we can find a single sequence for all $\rho' < \rho$. Since $\eta_n = 1 + \mu_n \to \eta = 1 + \mu$ in $X_{\rho'}$, it follows that $\phi^+ = \phi^- = \phi$.

The sequence of measures $\omega_n(dx,\, dy) = (1/\varepsilon)\mathcal{X}_{\Lambda_n(t)}\, dx\, dy$ cannot fail to converge weakly to the right-hand side of (2.29). The corresponding velocity field $\mathbf{u}_n(t) = \nabla^\perp \triangle^{-1} \omega_n$, where

$$\nabla^\perp = \begin{pmatrix} -\partial_y \\ \partial_x \end{pmatrix},$$

converge locally in $\mathbf{L}_1$ to the velocity field $\mathbf{u}$.

Obviously, $\mathbf{u}_n$ satisfies the Euler equation in a weak form, i.e.,

$$< \mathbf{u}_n,\, \nabla f >= 0, \qquad < \mathbf{u}_n,\, \partial_t \mathbf{w} > + \sum_{i=1,2} < (\mathbf{u}_n)_i\, \mathbf{u}_n,\, \partial_i \mathbf{w} >= 0, \tag{2.30}$$

where $f$ and $w_i$ are $x$-periodic smooth test functions of compact support in $y$ and $t$, and $< \ ,\ >$ denotes the scalar product in the $x$, $y$, $t$ variables. Since $\|\mathbf{u}_n(t)\|_\infty < C$, for $t \in [0, a\rho_0)$, we have no problem with the nonlinear term to prove that $\mathbf{u}$ also is a solution of the Euler equation in the weak form (2.30). Finally, since we know that any weak solution $\mathbf{u}$ of the Euler equation, whose vorticity is concentrated on a smooth curve $\phi(\cdot\,,t)$ with smooth intensity $\eta(\cdot\,,t)$, forces $\phi$ and $\eta$ to satisfy the vortex sheet equation (see [14]), we conclude the proof by the uniqueness of the vortex sheet dynamics in $X_\rho$. □

Roughly speaking, in the layer dynamics, $\phi_\varepsilon^+$ and $\phi_\varepsilon^-$ slip in the opposite direction. In the vortex sheet equation, $\phi$ moves convected by the mean field $\frac{1}{2}(\mathbf{u}^+ + \mathbf{u}^-)$. However, defining in the limit situation $\phi^+$ and $\phi^-$ as the profiles convected by $\mathbf{u}^+$ and $\mathbf{u}^-$, the continuity of the normal component of the velocity implies that $\phi^+ = \phi^- = \phi$. On the other hand, it is absolutely essential that the mean field appears in the continuity equation for $\eta$. This shows why the limits hold and explains why we find it convenient to consider three equations in the approximating problem (2.16). As we see in the next section, this remark is essential for a more explicit evaluation of convergence. On the other hand, if a Lagrangian representation of $\phi_\varepsilon^\pm$ were chosen, then we would note that the Lagrangian points would be moving in opposite directions, depending on whether they were on the upper or lower interface. This would pose a problem in what is meant by convergence. By using a Eulerian reference frame, we avoid this problem. A modified Eulerian frame was used in numerical calculations by Baker and Shelley [13].

## 3. Technicalities and additional comments.

*Proof of Proposition* 2.1. Denote $\psi(x) = \lambda\phi^+(x) + (1-\lambda)\phi^-(x)$, for fixed $\lambda \in [0,1]$, and $\Delta\psi = \psi(x) - \psi(x+\alpha)$. From (2.12),

(3.1)
$$\begin{aligned}
U_\varepsilon(x) \equiv & U_\varepsilon\left[\lambda;\mu,\phi^+,\phi^-\right](x)\\
= & \frac{1}{2\pi i}⨍_{-\infty}^{+\infty} d\alpha\ (1+\mu(x+\alpha)) \int_0^1 \frac{d\gamma}{\alpha + i\ (\Delta\psi + \varepsilon(\lambda-\gamma) + \varepsilon(\lambda-\gamma)\,\mu(x+\alpha))}\\
= & \frac{1}{2\pi i}⨍_{-\infty}^{+\infty} d\alpha\ (1+\mu(x+\alpha)) \int_0^1 \frac{d\gamma}{(\alpha + i\,\varepsilon(\lambda-\gamma))\left(1 + i\,\frac{\Delta\psi + \varepsilon(\lambda-\gamma)\mu(x+\alpha)}{\alpha + i\,\varepsilon(\lambda-\gamma)}\right)}\\
= & \frac{1}{2\pi i}\sum_{n=0}^{\infty}(-i)^n \sum_{l=0}^{n}\binom{n}{l}⨍_{-\infty}^{+\infty} d\alpha\ (1+\mu(x+\alpha))\\
& \int_0^1 d\gamma\, \frac{(\Delta\psi)^l\,(\varepsilon(\lambda-\gamma))^{n-l}\mu(x+\alpha)^{n-l}}{(\alpha + i\,\varepsilon(\lambda-\gamma))^{n+1}}.
\end{aligned}$$

Note that the last principal value symbol applies only for the term $n=0$. Henceforth, we often skip this symbol, being evident from the context when the integrals must be understood in the sense of Cauchy principal value.

The last step in (3.1) is justified by the inequality

$$\left|\frac{\Delta\psi + \varepsilon\,(\lambda-\gamma)\,\mu(x+\alpha)}{\alpha + i\,\varepsilon(\lambda-\gamma)}\right| < 1. \tag{3.2}$$

Indeed, the left-hand side of (3.2) is bounded by

$$\left|\frac{\Delta\psi}{\alpha}\right| + |\mu| < \frac{1}{2}, \tag{3.3}$$

provided that $\|\partial_x\phi^\pm\|_\rho < \frac{1}{4}$ and $\|\mu\|_\rho < \frac{1}{4}$.

Taking the Fourier transform of (3.1),

(3.4)

$$\hat{U}_\varepsilon(k) = \frac{1}{2\pi i}\sum_{n=0}^{\infty}(-i)^n\sum_{l=0}^{n}\binom{n}{l}\sum_{\substack{h\\ r_1,\dots,r_l\\ s_1,\dots,s_{n-l}:\\ h+\sum r_j+\sum s_j=k}}\Bigg((\widehat{1+\mu})(h)\prod_{j=1}^{l}\hat{\psi}(r_j)\prod_{m=1}^{n-l}\hat{\mu}(s_m)$$

$$\cdot\int_{-\infty}^{+\infty}d\alpha\int_0^1 d\gamma\,\frac{(\varepsilon\,(\lambda-\gamma))^{n-l}}{(\alpha+i\,\varepsilon(\lambda-\gamma))^{n+1}}e^{ih\alpha}\prod_{j=1}^{l}(1-e^{ir_j\alpha})\prod_{m=1}^{n-l}e^{is_m\alpha}$$

$$=\frac{1}{2\pi i}\sum_{n=0}^{\infty}(-i)^n\sum_{l=0}^{n}\binom{n}{l}\sum_{\substack{h,r_j,s_m:\\ h+\sum r_j+\sum s_j=k}}(\widehat{1+\mu})(h)\prod_{j=1}^{l}r_j\,\hat{\psi}(r_j)\prod_{m=1}^{n-l}\hat{\mu}(s_m)$$

$$\cdot(-i)^l\int_0^1 dt_1\cdots\int_0^1 dt_l\int_0^1 d\gamma\int_{-\infty}^{+\infty}d\alpha\frac{(\varepsilon\,(\lambda-\gamma))^{n-l}\,\alpha^l\,e^{iT\alpha}}{(\alpha+i\,\varepsilon(\lambda-\gamma))^{n+1}}\Bigg),$$

where

$$T=\sum_{j=1}^{l}r_jt_j+\sum_{m=1}^{n-l}s_m+h. \tag{3.5}$$

Denoting by $I$ the last integral in the right-hand side of (3.4), we have that

(3.6)

$$I=\sum_{b=0}^{l}\binom{l}{b}\int_{-\infty}^{+\infty}d\alpha\,(\varepsilon\,(\lambda-\gamma))^{n-l}\,\frac{(\alpha+i\,\varepsilon(\lambda-\gamma))^{b}\,(-i\,\varepsilon(\lambda-\gamma))^{l-b}}{(\alpha+i\,\varepsilon(\lambda-\gamma))^{n+1}}\,e^{iT\alpha}$$

$$=2\pi i\sum_{b=0}^{l}\binom{l}{b}(-i)^{l-b}\,(\varepsilon(\lambda-\gamma))^{n-b}\,\mathrm{sgn}(T)\,\mathcal{X}\,(T\cdot(\lambda-\gamma)<0)\,\frac{(iT)^{n-b}}{(n-b)!}e^{-\varepsilon|T|\,|\lambda-\gamma|},$$

after having used the formula

$$\int_{-\infty}^{+\infty}d\alpha\frac{e^{iT\alpha}}{(\alpha+i\xi)^n}=\mathrm{sgn}(T)\,\mathcal{X}\,(T\cdot\xi<0)\,\frac{2\pi i}{(n-1)!}\,(iT)^{n-1}\,e^{-|T|\,|\xi|}, \tag{3.7}$$

and we denote by $\mathcal{X}\,(T\cdot\xi<0)$ the characteristic function of the set $\{T\cdot\xi<0\}$.

Inserting (3.6) in (3.4), we finally have that

(3.8)

$$\hat{U}_\varepsilon(k)=\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}\sum_{\substack{h,r_j,s_m:\\ h+\sum r_j+\sum s_j=k}}\sum_{b=0}^{l}\binom{l}{b}(-1)^l$$

$$\cdot\Bigg((\widehat{1+\mu})(h)\prod_{j=1}^{l}r_j\,\hat{\psi}(r_j)\prod_{m=1}^{n-l}\hat{\mu}(s_m)\int_0^1 dt_1\cdots\int_0^1 dt_l\int_0^1 d\gamma$$

$$\cdot\,\mathrm{sgn}(T)\,\mathcal{X}\,(T\cdot(\lambda-\gamma)<0)\,\frac{(\varepsilon T(\lambda-\gamma))^{n-b}}{(n-b)!}e^{-\varepsilon|T|\,|\lambda-\gamma|}\Bigg).$$

The above quantity can now be easily estimated as
(3.9)

$$\left|\hat{U}_\varepsilon(k)\right| \leq \sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l} \sum_{\substack{h,r_j,s_m: \\ h+\sum r_j+\sum s_j=k}} 2^l \left|\widehat{(1+\mu)}(h)\right| \prod_{j=1}^{l}|r_j| \left|\hat{\psi}(r_j)\right| \prod_{m=1}^{n-l}|\hat{\mu}(s_m)|.$$

Finally,
(3.10)

$$\|U_\varepsilon\|_\rho \leq \sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}\|1+\mu\|_\rho \left(2\|\partial_x\psi\|_\rho\right)^l \|\mu\|_\rho^{n-l} = \frac{\|1+\mu\|_\rho}{1-\left(2\|\partial_x\psi\|_\rho+\|\mu\|_\rho\right)}.$$

This achieves the first part of Proposition 2.1.

The Lipschitz estimate follows the same lines. Fixed $\mu$, we take a variation with respect to $\psi$. We have that
(3.11)

$$\begin{aligned}\delta U_\varepsilon(x) =& \frac{1}{2\pi i}\sum_{n=0}^{\infty}(-i)^n\sum_{l=0}^{n}\binom{n}{l}\sum_{p=0}^{l-1} \\ &\cdot\int_{-\infty}^{+\infty} d\alpha \left( (1+\mu(x+\alpha)) \int_0^1 d\gamma\, (\Delta\psi_1-\Delta\psi_2)(\Delta\psi_1)^p(\Delta\psi_2)^{l-p-1}\right. \\ &\left.\cdot\frac{(\varepsilon(\lambda-\gamma))^{n-l}\mu(x+\alpha)^{n-l}}{(\alpha+i\,\varepsilon(\lambda-\gamma))^{n+1}}\right).\end{aligned}$$

Proceeding as above, we obtain that
(3.12)

$$\begin{aligned}\widehat{\delta U_\varepsilon}(k) =& \frac{1}{2\pi i}\sum_{n=0}^{\infty}(-i)^n\sum_{l=0}^{n}\binom{n}{l}\sum_{p=0}^{l-1}{\sum}' \\ &\cdot\left(\widehat{(1+\mu)}(h_1)\left(\hat{\psi}_1-\hat{\psi}_2\right)(h_2)\,h_2\prod_{j=1}^{p}q_j\,\hat{\psi}_1(q_j)\prod_{j=1}^{l-p-1}r_j\,\hat{\psi}_2(r_j)\right. \\ &\left.\cdot\prod_{j=1}^{n-l}\hat{\mu}(s_j)\,(-i)^l\int_{[0,1]^l}dt_1\cdots dt_l\int_0^1 d\gamma\int_{-\infty}^{+\infty}d\alpha\frac{(\varepsilon(\lambda-\gamma))^{n-l}\alpha^l}{(\alpha+i\,\varepsilon(\lambda-\gamma))^{n+1}}e^{iT\alpha}\right),\end{aligned}$$

where $\sum'$ means that we are summing over all

$$h_1,\, h_2,\, q_1,\,\cdots,\, q_p,\, r_1,\,\cdots,\, r_{l-p-1},\, s_1,\,\cdots,\, s_{n-l}$$

such that $h_1+h_2+\sum q_j+\sum r_j+\sum s_j=k$, and

$$T = h_1 + t_l h_2 + \sum_{j=1}^{p} t_j q_j + \sum_{j=1}^{l-p-1} t_j r_j + \sum_{j=1}^{n-l} s_j. \tag{3.13}$$

Evaluating the integral in the right-and side of (3.12) as before, we finally get that

$$\begin{aligned}\|\delta U_\varepsilon\|_\rho \leq & \|1+\mu\|_\rho \|\partial_x(\psi_1-\psi_2)\|_\rho \\ & \cdot \sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}\sum_{p=0}^{l-1}\|\partial_x\psi_1\|_\rho^p\|\partial_x\psi_2\|_\rho^{l-p-1}2^l\|\mu\|_\rho^{n-l} \\ \leq & 4\|1+\mu\|_\rho\|\partial_x(\psi_1-\psi_2)\|_\rho \sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}\frac{l}{2^l}\frac{1}{4^{n-l}} \\ = & C\|1+\mu\|_\rho\|\partial_x(\psi_1-\psi_2)\|_\rho,\end{aligned} \tag{3.14}$$

where we have used $\|\mu\|_\rho$, $\|\partial_x\psi_1\|_\rho$, $\|\partial_x\psi_2\|_\rho < \frac{1}{4}$. Variations with respect to $\mu$ can be dealt in the same way. □

We now want to investigate more explicitly the convergence $\varepsilon \to 0$. The field $U_\varepsilon$ acting on the upper boundary of the layer takes the form (recall (2.12) with $\lambda = 1$)

$$\begin{aligned}U_\varepsilon^+[\mu,\phi](x) = & \frac{1}{2\pi i}\int_{-\infty}^{+\infty} d\alpha \Bigg((1+\mu(x+\alpha)) \\ & \cdot \int_0^1 \frac{d\gamma}{\alpha + i(\phi(x)-\phi(x+\alpha)+\varepsilon(1-\gamma)(1+\mu(x+\alpha)))}\Bigg).\end{aligned} \tag{3.15}$$

We consider $U_\varepsilon[\mu,\phi]$ a complex operator of $\mu$ and $\phi$ (note that $\phi$ refers to the upper boundary, while the lower boundary is given by the expression $\phi - \varepsilon(1+\mu)$).

The limit field is given by (see (1.2))

$$U[\mu,\phi](x) = \frac{1}{2\pi i}\int_{\infty}^{+\infty} d\alpha \, \frac{1+\mu(x+\alpha)}{\alpha + i(\phi(x)-\phi(x+\alpha))}. \tag{3.16}$$

Obviously, $U_\varepsilon^+$ does not converge to $U$ as $\varepsilon \to 0$ (for fixed $\mu$ and $\phi$). What is expected to be true is that

$$\lim_{\varepsilon\to 0} U_\varepsilon^+[\mu,\phi](x) = U^+[\mu,\phi](x), \tag{3.17}$$

where

$$U^+[\mu,\phi](x) = U[\mu,\phi] + \tfrac{1}{2}J[\mu,\phi](x), \tag{3.18}$$

and $J$ is the jump discontinuity of the sheet in the point x. This can be seen by either using classical arguments in potential theory or by looking at the Fourier transforms.

From (3.8),

$$\begin{aligned}\hat{U}_\varepsilon^+[\mu,\phi](k) = & -\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}\sum_{\substack{h,r_j,s_m:\\ h+\sum r_j+\sum s_j=k}}\sum_{b=0}^{l}\binom{l}{b}(-1)^l \\ & \cdot\Bigg((\widehat{1+\mu})(h)\prod_{j=1}^{l} r_j\,\hat{\phi}(r_j)\prod_{m=1}^{n-l}\hat{\mu}(s_m)\int_0^1 dt_1\cdots \\ & \cdot\int_0^1 dt_l\int_0^1 d\gamma\mathcal{X}(T<0)\frac{(\varepsilon T\gamma)^{n-b}}{(n-b)!}e^{-\varepsilon|T|\gamma}\Bigg).\end{aligned} \tag{3.19}$$

The sum of the terms corresponding to $l = n, \ \ b = l$ is

$$\hat{\mathcal{S}}_k = -\sum_{n=0}^{\infty}(-1)^n \sum_{\substack{h,r_j \\ h+\sum r_j=k}} \Bigg( (\widehat{1+\mu})(h) \prod_{i=1}^{n} r_j\, \hat{\phi}(r_j) \cdot \int_0^1 dt_1 \cdots \int_0^1 dt_n \int_0^1 d\gamma\, \mathcal{X}\,(T<0)\, e^{-\varepsilon|T|\gamma} \Bigg). \tag{3.20}$$

Using the identity

$$-\mathcal{X}\,(T<0) = \tfrac{1}{2}\mathrm{sgn}(T) - \tfrac{1}{2} \tag{3.21}$$

in expression (3.20) for $\varepsilon = 0$, we realize that the term $\frac{1}{2}\mathrm{sgn}(T)$ gives the Fourier transform of $U[\mu, \phi]$, while the contribution $\frac{1}{2}$ leads to $\frac{1}{2}J$. Namely, the identities

$$\hat{U}[\mu,\phi](k) = \sum_{n=0}^{\infty}(-1)^n \sum_{\substack{h,r_j: \\ h+\sum r_j=k}} \Bigg( (\widehat{1+\mu})(h) \prod_{j=1}^{l} r_j\, \hat{\phi}(r_j) \cdot \frac{1}{2}\int_0^1 dt_1 \cdots \int_0^1 dt_n\, \mathrm{sgn}\left(h + \sum_{j=1}^{n} r_j t_j\right)\Bigg) \tag{3.22}$$

and

$$\hat{J}[\mu,\phi] = -\sum_{n=0}^{\infty}(-1)^n \sum_{\substack{h,r_j: \\ h+\sum r_j=k}} (\widehat{1+\mu})(h) \prod_{j=1}^{l} r_j\, \hat{\phi}(r_j) \tag{3.23a}$$

follow rather straightforwardly from the analysis developed up to this point, and by the definition

$$J = -(1+\mu)\frac{1+i\,\partial_x\phi}{1+(\partial_x\phi)^2} = -(1+\mu)\frac{1}{1-i\,\partial_x\phi}. \tag{3.23b}$$

In fact, expanding the exponential in the right-hand side of (3.20), we find that

$$\|U^+[\mu,\phi] - \mathcal{S}\|_\rho \le C\frac{\varepsilon}{\rho'-\rho} \tag{3.24}$$

for $\rho < \rho'$, assuming that $\mu$ and $\phi \in X_{\rho'}$, with $\|\mu\|_{\rho'}$, $\|\partial_x\phi\|_{\rho'} < \frac{1}{4}$. If $n - b > 0$, the right-hand side of (3.19) can be estimated analogously. Finally,

$$\big\|U^+[\mu,\phi] - U_\varepsilon^+[\mu,\phi]\big\|_\rho \le C\frac{\varepsilon}{\rho'-\rho}. \tag{3.25}$$

Consider now the field $U_\varepsilon[\lambda; \mu, \phi^+, \phi^-]$ given by (2.12) and assume it a functional of $\phi = \phi^+$ and $\mu$ only ($\phi^-$ being recovered by the relation $\phi^- = \phi^+ - \varepsilon(1+\mu)$).

The averaged field

$$\overline{U}_\varepsilon[\mu,\phi] = \int_0^1 d\lambda\, U_\varepsilon[\lambda,\mu,\phi,\phi - \varepsilon\,(1+\mu)] \tag{3.26}$$

is that field appearing in the approximated continuity equation

$$\partial_t \mu_\varepsilon + \partial_x \left((1+\mu_\varepsilon)\mathrm{Re}\,\overline{U}_\varepsilon[\mu,\phi]\right) = 0. \tag{3.27}$$

We now want to exploit the limit $\varepsilon \to 0$ for $\overline{U}_\varepsilon$. It is not surprising that the jump here disappears. In fact, expanding as before, $\hat{\overline{U}}_\varepsilon[\mu,\phi](k)$, and, computing for $\varepsilon = 0$ the sum of the terms corresponding to $l = n,\ \ b = l$, we find that

$$\begin{aligned}\hat{\overline{\mathcal{S}}}_k\Big|_{\varepsilon=0} = &\sum_{n=0}^{\infty}(-1)^n \sum_{\substack{h,\,r_j:\\ h+\sum r_j = k}} \Bigg( \widehat{(1+\mu)}(h) \prod_{i=1}^{n} r_j\, \hat{\phi}(r_j) \\ &\cdot \int_0^1 dt_1 \cdots \int_0^1 dt_n \int_0^1 d\gamma \int_0^1 d\lambda\, \mathrm{sgn}(T)\, \mathcal{X}\left(T(\lambda-\gamma)<0\right)\Bigg).\end{aligned} \tag{3.28}$$

Finally, using the identity

$$\int_0^1 d\gamma \int_0^1 d\lambda\, \mathrm{sgn}(T)\, \mathcal{X}\left(T(\lambda-\gamma)<0\right) = \frac{1}{2}\mathrm{sgn}(T), \tag{3.29}$$

we arrive easily at the estimate

$$\left\| U[\mu,\phi] - \overline{U}_\varepsilon[\mu,\phi] \right\|_\rho \le C\frac{\varepsilon}{\rho' - \rho}. \tag{3.30}$$

Before approaching the convergence problem, we first realize that (1.1a) can be rewritten as

$$\partial_t \phi + u^+[\mu,\phi]\,\partial_x\phi = v^+[\mu,\phi], \tag{3.31}$$

where $U^+ = u^+ + i\,v^+$. In fact, the jump discontinuity is directed along the tangent to $\phi$ (see (3.23b)), so it does not change the time variation of the shape of $\phi$.

Subtracting now the evolution equation for $\phi_\varepsilon^+$ (2.3) to that for $\phi$ (2.31) and (3.27) to the continuity equation (1.1b), we obtain

$$\begin{aligned}\partial_t(\phi - \phi_\varepsilon^+) =& u^+[\mu,\phi]\,\partial_x(\phi_\varepsilon^+ - \phi) \\ &+ \left(u_\varepsilon^+[\mu,\phi] - u^+[\mu,\phi]\right)\,\partial_x\phi_\varepsilon^+ \\ &+ \left(u_\varepsilon^+[\mu_\varepsilon,\phi_\varepsilon^+] - u_\varepsilon^+[\mu,\phi]\right)\,\partial_x\phi_\varepsilon^+ \\ &+ \left(v^+[\mu,\phi] - v_\varepsilon^+[\mu,\phi]\right) \\ &+ \left(v_\varepsilon^+[\mu,\phi] - v_\varepsilon^+[\mu_\varepsilon,\phi_\varepsilon^+]\right),\end{aligned} \tag{3.32}$$

$$\begin{aligned}\partial_t(\mu - \mu_\varepsilon) =& \partial_x\left((\mu_\varepsilon - \mu)\,\overline{u}_\varepsilon[\mu_\varepsilon,\phi_\varepsilon^+]\right) \\ &+ \partial_x\left((1+\mu)\left(\overline{u}_\varepsilon[\mu_\varepsilon,\phi_\varepsilon^+] - \overline{u}_\varepsilon[\mu,\phi]\right)\right) \\ &+ \partial_x\left((1+\mu)\left(\overline{u}_\varepsilon[\mu,\phi] - u[\mu,\phi]\right)\right).\end{aligned} \tag{3.33}$$

Define

$$\xi_\varepsilon(\rho,t) = \|\phi - \phi_\varepsilon^+\|_\rho + \|\partial_x(\phi - \phi_\varepsilon^+)\|_\rho + \|\mu - \mu_\varepsilon\|_\rho. \tag{3.34}$$

We want to obtain an integral inequality for $\xi_\varepsilon(\rho,t)$, and, to do this, we must estimate the $\rho$-norm of the right-hand side of (3.23) and its first derivative, and the $\rho$-norm of the right-hand side of (3.33). We do not do it in detail, but limit ourselves to considering

$$v^+[\mu,\phi] - v_\varepsilon^+[\mu,\phi] \tag{3.35}$$

and

$$\partial_x\left((1+\mu)\left(\overline{u}_\varepsilon[\mu_\varepsilon,\phi_\varepsilon^+] - \overline{u}_\varepsilon[\mu,\phi]\right)\right) \tag{3.36}$$

only. All the other term can be dealt with analogously. We first observe that, according to Theorem 2.1, setting $\overline{\rho}(s) = \rho_0 - s/a$, the $\overline{\rho}(s)$-norms of $\mu_\varepsilon(s)$, $\phi_\varepsilon^+(s)$, $\mu(s)$, $\phi(s)$ are estimated by a constant for $s < a\rho_0$. For a positive small $\delta > 0$, consider $t < a(\rho_0 - \delta)$, $s < t$, $\rho < \rho(s) < \overline{\rho}(s) - \delta$, where $\rho(s) = \frac{1}{2}(\rho_0 + \rho - \delta - s/a)$. We replace $\rho_0$ by $\rho_0 - \delta$ to take advantage of the a priori estimate to bound singular terms. In fact, we have, from (3.25), that

$$\begin{aligned}
&\|v^+[\mu,\phi] - v_\varepsilon^+[\mu,\phi]\|_\rho + \|\partial_x\left(v^+[\mu,\phi] - v_\varepsilon^+[\mu,\phi]\right)\|_\rho \\
&\le \frac{1}{\rho(s)-\rho}\left(C\varepsilon + \|v^+[\mu,\phi] - v_\varepsilon^+[\mu,\phi]\|_{\rho(s)}\right) \\
&\le \frac{\varepsilon C}{\rho(s)-\rho}\left(1 + \frac{1}{\overline{\rho}(s)-\rho(s)}\right)
\end{aligned} \tag{3.37}$$

and, from definition (3.26) and estimates of the type (3.30),

$$\begin{aligned}
&\|\partial_x\left((1+\mu)\left(\overline{u}_\varepsilon[\mu_\varepsilon,\phi_\varepsilon^+] - \overline{u}_\varepsilon[\mu,\phi]\right)\right)\|_\rho \\
&\le \frac{C}{\rho(s)-\rho}\|\overline{u}_\varepsilon[\mu_\varepsilon,\phi_\varepsilon^+] - \overline{u}_\varepsilon[\mu,\phi]\|_{\rho(s)} \\
&\le \frac{C}{\rho(s)-\rho}\left(\|\partial_x(\phi-\phi_\varepsilon^+)\|_{\rho(s)} + \|\mu-\mu_\varepsilon\|_{\rho(s)} + \varepsilon\|\partial_x(\mu-\mu_\varepsilon)\|_{\rho(s)}\right) \\
&\le \frac{C}{\rho(s)-\rho}\left(\|\partial_x(\phi-\phi_\varepsilon^+)\|_{\rho(s)} + \|\mu-\mu_\varepsilon\|_{\rho(s)}\right) + \frac{\varepsilon C}{\rho(s)-\rho}\,\frac{1}{\overline{\rho}(s)-\rho(s)}.
\end{aligned} \tag{3.38}$$

In conclusion, all terms we must estimate can be bounded either by

$$\frac{\varepsilon C}{\rho(s)-\rho}\,\frac{1}{\overline{\rho}(s)-\rho(s)} \le \frac{\varepsilon C}{\delta(\rho(s)-\rho)} \tag{3.39}$$

or by

$$\frac{C\xi_\varepsilon(\rho(s),s)}{\rho(s)-\rho}. \tag{3.40}$$

Thus, for $\rho < \rho_0 - \delta$ and $t < a(\rho_0 - \delta - \rho)$,

$$\xi_\varepsilon(\rho,t) \le \xi_\varepsilon(\rho,0) + \frac{\varepsilon C}{\delta}\int_0^t ds\,\frac{1}{\rho(s)-\rho} + C\int_0^t ds\,\frac{\xi_\varepsilon(\rho(s),s)}{\rho(s)-\rho}. \tag{3.41}$$

We now proceed as in the proof of Theorem 2.1. Setting

$$\mathcal{M}_\varepsilon = \sup_{\substack{0<\rho<\rho_0-\delta \\ t\in[0,\,a(\rho_0-\delta-\rho))}} \xi_\varepsilon(\rho,t)\left(1-\frac{t}{a(\rho_0-\rho)}\right), \tag{3.42}$$

we get that

$$\begin{aligned} \xi_\varepsilon(\rho,t) \leq & \xi_\varepsilon(\rho,0) + a\frac{\varepsilon C}{\delta}\ln\left(1+\frac{t}{a(\rho_0-\rho)-t}\right) \\ & + C\,\mathcal{M}_\varepsilon \int_0^t ds\,\frac{1}{(\rho(s)-\rho)\left(1-\frac{s}{a(\rho_0-\rho(s))}\right)}, \end{aligned} \tag{3.43}$$

and, evaluating the last integral, we finally obtain that

$$\mathcal{M}_\varepsilon \leq \xi_\varepsilon(\rho_0,0) + a\frac{\varepsilon C}{\delta} + Ca\mathcal{M}_\varepsilon,$$

from which $\mathcal{M}_\varepsilon \to 0$ as $\varepsilon \to 0$, provided that $a$ is small enough.

We note that we have convergence in $X_\rho$ for $t < a(\rho_o - \rho)$, since $\delta$ is arbitrary. However, the constant $a$ appearing here could be smaller than the first choice of $a$ after Theorem 2.1. Since we are, presumably, still far from the critical time in which singularities can occur, a more accurate analysis of the constants is not very meaningful.

**4. Concluding remarks.** The convergence result that we have discussed in the previous sections works in spaces of analytical functions. The following question arises quite naturally. Even assuming that $\phi$, $\eta \in X_{\rho_0}$ for some $\rho_0$ at time zero, if $\omega_\varepsilon$ is *any* sequence of vorticity profiles weakly converging to $\eta(x)\,\delta(y-\phi(x))$ at time zero, does $\omega_\varepsilon(t)$ converge to $\eta(x,t)\,\delta(y-\phi(x,t))$, for small positive $t$, where $\eta(\cdot\,,t)$ and $\phi(\cdot\,,t)$ solve the vortex sheet equation?

We conjecture that the answer is no. The reason lies in the following example. Consider the flat sheet of constant intensity $\phi = 0$, $\eta = 1$. This is the only analytical solution of the initial value problem. Suppose that we approximate the sheet by

$$\begin{aligned} &\phi_\varepsilon^\pm = \pm\frac{\varepsilon}{2} \quad \text{if } x \in (-\infty, -\alpha(\varepsilon)] \cup [0,+\infty), \\ &\phi_\varepsilon^\pm = 0 \quad \text{if } x \in (-\alpha(\varepsilon), 0). \end{aligned} \tag{4.1}$$

Obviously, $\omega_\varepsilon \to \omega(x,y)\,dx\,dy = \delta(y)\,dx\,dy$ in the sense of weak convergence of measures for all $\alpha(\varepsilon) \to 0$. Now compute the vertical component of the velocity field at the origin. We have that

$$\begin{aligned} -\frac{1}{2\pi\varepsilon}\int_0^{\alpha(\varepsilon)} dx \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} dy\,\frac{x}{x^2+y^2} &= -\frac{1}{\pi\varepsilon}\int_0^{\alpha(\varepsilon)} dx\,\text{arctg}\left(\frac{\varepsilon}{2x}\right) \\ &= -\frac{1}{\pi}\int_0^{\frac{\alpha(\varepsilon)}{\varepsilon}} d\xi\,\text{arctg}\left(\frac{1}{2\xi}\right). \end{aligned} \tag{4.2}$$

From this expression, it follows that, when $\alpha(\varepsilon)/\varepsilon \to \text{const}$, the field is not vanishing and it may even diverge when $\alpha(\varepsilon)/\varepsilon \to \infty$. This indicates that hardly to the flat sheet in this case. These kinds of pathologies are perhaps related to the existence of many, piecewise analytic, solutions to the vortex sheet equation. In particular, two-branched vortex sheets are studied in [11].

A physically reasonable way to choose a solution among the very many arising in the context of the vortex sheet dynamics is to study the vanishing viscosity limit for such a problem. It is natural to conjecture that the analytic solution is the only stable with respect to viscous perturbations.

*Note added in proof.* Before Theorem 2.1 of this paper, it is stated that we expect appearance of singularities in a finite time of boundary. The referee informed us that, in the recent paper [J.-Y. Chemin, *Compte Rendu Academie Sciences*, 312 (1991), pp. 803–806], the regularity of the boundary of a vortex patch has been proved globally in time.

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