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ORIGINAL RESEARCH

Minimal winning coalitions and orders of criticality

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Abstract
In this paper, we analyze the order of criticality in simple games, under the light of minimal winning coalitions. The order of criticality of a player in a simple game is based on the minimal number of other players that have to leave so that the player in question becomes pivotal. We show that this definition can be formulated referring to the cardinality of the minimal blocking coalitions or minimal hitting sets for the family of minimal winning coalitions; moreover, the blocking coalitions are related to the winning coalitions of the dual game. Finally, we propose to rank all the players lexicographically accounting the number of coalitions for which they are critical of each order, and we characterize this ranking using four independent axioms.

Keywords Order of criticality · Hitting set · Dual game · Axiomatic approach

1 Introduction

This paper deals with the concept of criticality for players in simple games, originally introduced in Dall'Aglio et al. (2016). Simple games are used for representing those situations in which the outcome may be a success or a failure; when a group of agents succeeds, it forms a winning coalition, otherwise it forms a losing coalition. The notion of criticality arises from the willingness to measure the possibility of success of the agents in a decision situation.

In the classical literature, an agent is called critical for a winning coalition when she/he is able to make the coalition lose, i.e., after her/his leaving, the winning coalition becomes a losing one. In Dall'Aglio et al. (2016), the definition of criticality was extended, in order to

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analyze the power of an agent in conjunction with others; in particular, the classical notion of criticality is called first order criticality, while higher orders of criticality of an agent i are defined on the basis of the minimal number of agents that have to leave a winning coalition so that i is able to make the coalition lose. Note that the extended definition refers to a fixed winning coalition. In Dall'Aglio et al. (2019a), the analysis is performed taking into account the ordering of the entering players in a winning coalition or simply the set of players in the winning coalition; the former case leads to the indices of criticality *à la* Shapley, while the latter one leads to the indices of criticality *à la* Banzhaf (see Dall'Aglio et al. (2019a)). More specifically, in Dall'Aglio et al. (2019a) the authors introduce an index which measures, for any player, her/his power in being critical of a fixed order or of being never critical of any order, and a collective index that is given by a weighted sum of the criticality indices for all the possible orders of criticality.

The higher orders of criticality may be useful when the situation under investigation is unstable. For instance, we may think of a political majority in a Parliament where some parties are critical of the first order, i.e. they are able to reject a proposal of the government if they vote against it, while the parties that have a higher order of criticality have this possibility only if they join to other parties; in a particular unstable situation, it can be useful to assess the effective influence of these parties that may have the chance to destroy the majority. Another example arises from connection situations: let us consider a computer network in which the connections among the nodes of the network have a low reliability, so that connections (edges) that have a higher order of criticality may fail and compromise the connection of the network.

In this paper, we propose a direct criterion to find the order of criticality of any player in a simple game, based on the cardinality of the minimal blocking coalitions, i.e those coalitions whose complements lose (see Dubey and Shapley (1979), Burgin and Shapley (2000)). It turns out that the minimal blocking coalitions coincide with the minimal hitting sets for the family of minimal winning coalitions, i.e. those sets that contain at least one agent for each minimal winning coalition. Moreover, we illustrate the equivalence between the notion of blocking coalitions (or hitting sets) of a simple game and winning coalitions of the dual game. Therefore, we can adapt an algorithm from the rather rich literature on minimal hitting sets generation (see Gainer-Dewar and Vera-Licona (2017)) to compute the order of criticality of players. If, on the one hand, the rich literature on minimal hitting set generation can be useful to find a ranking among the players, on the other hand, it may be worthy to define our ranking in the models for which the algorithms were originally designed. A notable example is given by the metabolic reaction networks. In these models the vertices represent metabolites, and the connections represent biochemical reactions that consume or produce metabolites until a steady state is reached, i.e. a set of reactions that maintains those metabolites in our setting a winning coalition. In metabolic engineering, it is possible to focus on blocking a targeting reaction through cut sets. So, it is possible to construct a set family whose elements are the reactions of the original network and whose sets are the relevant elementary flux modes. The minimal hitting sets of this family represent the reactions to be cancelled. In Sajitz-Hermstein and Nikoloski (2012, 2013) the authors propose a game-theoretic framework in the metabolic reaction networks, investigating the contribution of each single reaction through the concept of the Shapley value. In the same spirit, the order of criticality of a reaction can represent its power in the metabolic reaction network and the criticality-based power ranking can give an ordering on the family of the reactions.

The equivalence between the notion of blocking coalitions of a simple game and winning coalitions of its dual is the starting point to introduce some properties that a ranking aimed at comparing agents' power should satisfy. Therefore, in this paper, we also investigate the use of the notion of orders of criticality (for the grand coalition) to rank players according to their “power” to block decisions in a simple game. To be more specific, we propose four properties that a solution (i.e., a map that associates to any simple game a total preorder, or ranking, over the individual players) should satisfy. Along the lines of the example of a parliament introduced earlier (where the agents are political parties), the first property we introduce (*players’ anonymity*) is a classical anonymity property, saying that the name of the parties should not affect the final ranking. The second property (*dual coalitional anonymity*) considers two distinct parliaments and two parties which belong to the same number of minimal blocking coalitions of same size in the two parliaments: then the two parties should have the same relative power under the two parliaments. The third axiom (*dual monotonicity*) states that if two parties share the same position of a ranking in a given parliament, increasing the number of minimal blocking coalitions containing only one of the two parties should break the tie in favour of the party that now has more possibility to block a decision. Finally, the last property (*independence of higher cardinalities*), is a coherence principle affirming that, once a party is considered strictly more powerful than another one in a parliament, adding coalitions which are minimal, but larger than those already present in the government, should not affect the relative ranking of the two parties.

We then show that a solution satisfies these four properties if and only if it is the *criticality-based ranking*, which ranks parties according to a lexicographic comparison of vectors whose components represent the number of times each party is critical of any order among the minimal winning coalitions. So, any party critical of order one is more important than any party critical, at most, of order two, which is, in turn, more important than any party critical, at most, of order three, etc. It is worthy to note that the four properties used to axiomatically characterize the criticality-based ranking are similar, from a technical point of view, to those used in Bernardi et al. (2019) to axiomatically characterize the *lex-cel* social ranking solution in a completely different domain, but they suggest a very different interpretation.

The paper is structured as follows. In Sect. 2, we recall the basic elements about simple games and criticality. In Sect. 3, we provide a criterion for computing the order of criticality of a player analysing the relation among criticality, blocking coalitions and hitting sets. We provide a short survey of the types of algorithms for hitting sets and we find the orders of criticality in an example presented in Zhao et al. (2018). In Sect. 4, we introduce the notion of ranking of the players and we give the formal definition of four properties for ranking solutions. In Sect. 5, we define a new ranking solution and characterize it via the four properties introduced in the previous section, and we prove that these properties are logically independent. In Sect. 6, we discuss the main notions and solutions provided in the previous section applied to a specific class of simple games on graphs. Section 7 concludes. In Appendix A, we prove the uniqueness of the solution satisfying the four axioms (Theorem 2), and in Appendix B, we describe the algorithm originally presented in Zhao et al. (2018) we used in the examples.

2 Preliminaries

A *simple cooperative game with transferable utility* (TU-game) is a pair (N, v), where $N = \{1, 2, \ldots, n\}$ denotes the finite set of players and $v : 2^n \to \{0, 1\}$ is the *characteristic function*, with $v(\emptyset) = 0$, $v(S) \leq v(T)$ for all S, T subsets of N such that $S \subseteq T$ and $v(N) = 1$.

Given a coalition $S \subseteq N$, if $v(S) = 0$ then S is a *losing* coalition, while if $v(S) = 1$, then S is a *winning* coalition. Given a winning coalition S, if $S \setminus \{i\}$ is losing, then $i \in S$ is a *critical player for* S. When a coalition S contains at least one critical player for it, S is a *quasi-minimal winning* coalition; when all the players of S are critical, it is a *minimal winning* coalition.

Given a TU-game (N, v), we recall the definition of order of criticality given in Dall'Aglio et al. (2016).

Definition 1 Let $k \geq 0$ be an integer, let $M \subseteq N$, with $|M| \geq k + 1$, be a winning coalition. We say that a player i is *critical of order* $k + 1$ for coalition M, and write $\rho(i, M) = k + 1$, if k is the minimum integer such that there exists a coalition $K \subseteq M \setminus \{i\}$ of cardinality k with

$$v\big(M \setminus K\big) - v\big(M \setminus (K \cup \{i\})\big) = 1. \tag{1}$$

Remark 1 In the above definition, the fact that K is a coalition of minimal cardinality k satisfying (1), implies that

$$v(M \setminus T) = 0 \text{ or } v\,(M \setminus (T \cup \{i\}) = 1 \text{ for any } T \subseteq M \setminus \{i\} \text{ with } |T| < k \tag{2}$$

must hold.

If a player i is critical of no order, i.e. $v(M \setminus T) - v(M \setminus (T \cup \{i\})) = 0$ for all $T \subseteq M \setminus \{i\}$ then the player i is called *null* for M. Note that when in Definition 1 $k = 0$, i.e. $K = \varnothing$, we obtain the classical definition of critical player.

For any given winning coalition $M \subseteq N$, let $\mathcal{W}^M$ be the set of winning coalitions in M

$$\mathcal{W}^M = \{S \subseteq M : v(S) = 1\}$$

and let $\mathcal{W}^M_{\min}$ be the set of minimal winning coalitions in M

$$\mathcal{W}^M_{\min} = \text{Min } \mathcal{W}^M$$

where, for any family of sets $\mathcal{F}$, the Min operator on $\mathcal{F}$ removes all non-inclusion-minimal sets of $\mathcal{F}$:

$$\text{Min } \mathcal{F} = \{F \in \mathcal{F} | \nexists G \in \mathcal{F} : G \subset F\}.$$

3 A criterion for the criticality order

We now provide a simple operational criterion to compute the order of criticality of any player in a winning coalition M, once the minimal winning coalitions for M are given.

Definition 2 Given a simple TU-game (N, v) and a winning coalition $M \subseteq N$, a coalition $B \subseteq M$ is called *blocking coalition* for M if $v(M \setminus B) = 0$. Let $\mathcal{B}^M$ be the set of all blocking coalitions for M in the game (N, v) and let $\mathcal{B}^M_{\min}$ be the set of all minimal blocking coalitions, i.e. $\mathcal{B}^M_{\min} = \text{Min } \mathcal{B}^M$.

Example 1 Consider a TU-game (N, v) and let $M = \{1, 2, 3, 4\} \subseteq N$ be a winning coalition with $\mathcal{W}^M_{\min} = \{\{1, 2, 3\}, \{1, 3, 4\}\}$. Then $\mathcal{B}^M_{\min} = \{\{1\}, \{3\}, \{2, 4\}\}$[1].

[1] Clearly, $\mathcal{B}^M$ contains all supersets of $\mathcal{B}^M_{\min}$. Being a rather large family, we omit listing its elements.

Remark 2 In examining the existing literature on coalition formation there is no general agreement on the definition of a blocking coalition. In Carreras (2009), a coalition $S \subseteq M$ is blocking when it and its complementary coalition $M \setminus S$ are both losing. In our paper a coalition is blocking whenever its complementary coalition is losing. According to Example 1 we have that coalition $\{1\}$ is blocking but not winning and coalition $\{1, 2, 3\}$ is blocking and winning[2].

We now show that the family of blocking coalitions coincides with that of the hitting sets, a notion widely employed in hypergraph theory.

Definition 3 Given a simple TU-game (N, v) and a winning coalition $M \subseteq N$. Let us call $\mathcal{H}^M$ the set of all the *hitting sets* of $\mathcal{W}^M_{\min}$, i.e. the subsets of M that intersect all the minimal winning coalitions in M:

$$\mathcal{H}^M = \{H \subseteq M : H \cap W \neq \varnothing, \ \forall W \in \mathcal{W}^M_{\min}\}.$$

Moreover let $\mathcal{H}^M_{\min} = \text{Min}\, \mathcal{H}^M$ be the set of all minimal hitting sets.

Example 2 Consider a TU-game (N, v) and a winning coalition $M = \{1, 2, 3, 4, 5\} \subset N$ with $\mathcal{W}^M_{\min} = \{\{1, 2\}, \{2, 3\}, \{1, 5\}, \{3, 5\}\}$. The set $\{1, 3, 4\} \in \mathcal{H}^M$ and the set of all minimal hitting sets is given by $\mathcal{H}^M_{\min} = \{\{1, 3\}, \{2, 5\}\}$.

Proposition 1 *Given a TU-game (N, v) and a winning coalition M then $\mathcal{H}^M = \mathcal{B}^M$ and so $\mathcal{H}^M_{\min} = \mathcal{B}^M_{\min}$.*

Proof Let $H \in \mathcal{H}^M$. For all $W \in \mathcal{W}^M_{\min}$, we have $H \cap W \neq \varnothing$; so, $W \nsubseteq (M \setminus H)$. Then, $v(M \setminus H) = 0$, which means that $H \in \mathcal{B}^M$. Let $B \in \mathcal{B}^M$. We have $v(M \setminus B) = 0$. So, for all $W \in \mathcal{W}^M_{\min}$, $W \nsubseteq (M \setminus B)$; then, $B \cap W \neq \varnothing$, which means that $B \in \mathcal{H}^M$.

The familiy of blocking coalitions/hitting sets provides an easy criterion to find the order of criticality of each player in a winning coalition: The order is given by the cardinality of the smallest blocking coalition/hitting set containing that player, while the player is critical of no order if it does not belong to any such sets.

Proposition 2 *Let $M \subseteq N$ be a winning coalition. The order of criticality for a player $i \in M$ is given by*

$$\rho(i, M) = \begin{cases} 0 & \text{if } i \notin B, \ \forall B \in \mathcal{B}^M_{\min}; \\ \min\{|B| : i \in B, \ B \in \mathcal{B}^M_{\min}\} & \text{otherwise.} \end{cases}$$

Proof If i does not belong to any set in $\mathcal{B}^M_{\min}$, then i is critical of no order, and $\rho(i, M) = 0$. Conversely, fix $i \in M$, and define

$$r_i^M = \min\{|B| : i \in B, \ B \in \mathcal{B}^M_{\min}\}.$$

Denote as B^* one of the sets attaining minimal cardinality. Defining $K^* = B^* \setminus \{i\}$, $v(M \setminus K^*) = 1$ holds because $K^* \notin \mathcal{B}^M$, and $v(M \setminus (K^* \cup \{i\})) = 0$, because $K^* \cup \{i\} \in \mathcal{B}^M_{\min}$. Thus K^* satisfies (1) and $\rho(i, M) \leq r_i^M$.

To prove that equality holds, argue by contradiction, and suppose that there exists $T \subseteq M \setminus \{i\}$ with $|T| < r_i^M - 1$ such that $v(M \setminus T) = 1$ and $v(M \setminus (T \cup \{i\})) = 0$, with $T \cup \{i\} \notin \mathcal{B}^M_{\min}$. There are two cases.

[2] In a *proper* game, a coalition S and its complementary coalition $M \setminus S$ cannot be both winning, while all the other possibilities are allowed.

First, $T \cup \{i\} \notin \mathcal{B}^M$. Since $T \cup \{i\}$ is not blocking, then $v(M \setminus (T \cup \{i\})) = 1$, but this is a contradiction with the above assumptions on T.

Second, $T \cup \{i\} \in \mathcal{B}^M \setminus \mathcal{B}^M_{\min}$. Let $R \subset T \cup \{i\}$ such that $R \in \mathcal{B}^M_{\min}$. If $i \in R$, then by minimality of r_i^M and our assumptions on T we have $r_i^M \leq |R| < |T \cup \{i\}| < r_i^M$, a contradiction. If $i \notin R$, then $R \subseteq T$, and hence $v(M \setminus R) = 1$ by our assumptions on T and the fact that (N, v) is a simple game. But this means that $R \notin \mathcal{B}^M_{\min}$, again a contradiction. Therefore, we must conclude that $\rho(i, M) = r_i^M$.

The minimal hitting set generation problem is of interest in combinatorics and in Boolean algebra. In the first topic, it has at least three equivalent formulations: i) finding the minimal transversal of a hypergraph (see Berge (1984)), ii) enumerate the maximum independent set of a hypergraph, or iii) enumerate the minimal set covers of a set family. The same problem arises in many applied fields: model-fault diagnosis, computational biology, data mining and even Sudoku problems (see Gainer-Dewar and Vera-Licona (2017)).

The problem's pervasiveness generated a rich list of algorithms to compute all or some of the minimal hitting sets. Starting from Berge's proposal for the hypergraphs (Berge (1984)), more than 20 different algorithms to find all minimal hitting sets have been proposed. In a recent review paper, Gainer-Dewar and Vera-Licona (2017) gather those proposals into 4 groups which differ on their approach:

Set Iteration Approach which works through the input set family, one set at a time, building minimal hitting sets as they go;

Divide-And-Conquer Approach which partitions the input family into disjoint subfamilies, finds their minimal hitting sets separately, and then combines them;

Buildup Approach which builds candidate minimal hitting sets one element at a time, keeping track of un-hit sets as they go;

Full Cover Approach which improves on the divide-and-conquer approach with a technical hypergraph lemma that allows more efficient recombination.

The review contains a brief description of each algorithm, together with an analysis of their efficiency. The problem of determining whether a given set family has a hitting set of size no greater than some k is NP-complete. Consequently, none of the proposed algorithms can achieve the task in subexponential time for every instance.

Several tests have been proposed to identify problems that stand between the tractable (polynomial time) and the intractable (fully exponential time) ones. We refer to output-polynomial, incremental-polynomial and polynomial delay running time tests (see Johnson et al. (1988) for details). Unfortunately, as reported in Gainer-Dewar and Vera-Licona (2017), none of the mentioned algorithms meet those tests, but algorithms based on the Divide-and-Conquer and the Buildup approach are usually faster than those based on the other approaches.

For this reason, to find the family of minimal blocking coalitions $\mathcal{B}^M_{\min}$ for a given set of minimal winning coalitions $\mathcal{W}^M_{\min}$ we used an algorithm belonging to the class "Divide and Conquer". It was presented in Zhao et al. (2018) in the context of model-based diagnosis, where all the hitting sets are seen as candidate diagnosis of a given family of conflict sets.

More details on the algorithm are given in Appendix B. In the next example, the algorithm is used to find the orders of criticality in a ten-players' example.

Example 3 [Adapted from Zhao et al. (2018)] Let $N = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and let

$$\mathcal{W}^N_{\min} = \{\{1, 2\}, \{2, 3, 4\}, \{4, 5\}, \{5, 6\}, \{3, 8, 9\}, \{6, 9\}, \{7, 8\}, \{7, 10\}\}.$$

We have

$$\mathcal{B}^N_{\min} = \{\{1, 3, 4, 6, 7\}, \{1, 3, 5, 6, 7\}, \{1, 3, 5, 6, 8, 10\}, \{1, 3, 5, 7, 9\}, \{1, 3, 5, 8, 9, 10\}, \{1, 4, 5, 7, 9\}, \{1, 4, 5, 8, 9, 10\}, \{1, 4, 6, 7, 8\}, \{1, 4, 6, 7, 9\}, \{1, 4, 6, 8, 10\}, \{2, 3, 4, 6, 7\}, \{2, 3, 5, 6, 7\}, \{2, 4, 6, 7, 8\}, \{2, 4, 6, 7, 9\}, \{2, 4, 6, 8, 10\}, \{2, 5, 6, 7, 8\}, \{2, 5, 6, 8, 10\}, \{2, 5, 7, 9\}, \{2, 5, 8, 9, 10\}\}.$$

We can easily compute the order of criticality of each player in the grand coalition

$$\rho(2, N) = \rho(5, N) = \rho(7, N) = \rho(9, N) = 4,$$
$$\rho(1, N) = \rho(3, N) = \rho(4, N) = \rho(6, N) = \rho(8, N) = \rho(10, N) = 5.$$

4 Duality and properties for power rankings

We start this section investigating the relationship between the set of blocking coalitions, where $M = N$, an assumption which holds true from here on, and the corresponding *dual game* (N, v^*), defined by

$$v^*(S) = v(N) - v(N \setminus S), \tag{3}$$

for each coalition $S \in 2^N$. The dual v^* of a simple game v is also a simple game, which can be interpreted as representing the "blocking" mechanism of a voting system, i.e., the winning coalitions in v^* are precisely those that prevent their complements to win in v, i.e. the blocking coalitions in v (see Weber 1994)).

Proposition 3 *Let v be a simple game with $\mathcal{W}^v_{\min}$ as the set of minimal winning coalitions. Then its dual v^* is such that the set of minimal winning coalitions is $\mathcal{W}^{v^*}_{\min} = \mathcal{B}^N_{\min}$.*

Proof If $S \in \mathcal{B}^N$ by definition $v(N \setminus S) = 0$. Then by relation (3), $v^*(S) = 1$. On the other hand, if $S \notin \mathcal{B}^N$, then $v(N \setminus S) = 1$ and by relation (3), $v^*(S) = 0$. So, $\mathcal{B}^N$ is the set of all winning coalitions in v^* (blocking coalitions in v), and $\mathcal{B}^N_{\min}$ is the set of minimal winning coalitions of v^*.

Remark 3 It is straightforward to see that $v = (v^*)^*$ (the dual game of the dual of a game v is game v itself). So, in view of Proposition 3, we immediately have

$$\mathcal{B}^{v^*} = \{T \subseteq N : T \cap S \neq \emptyset, \ \forall S \in \mathcal{B}^N_{\min}\};$$

and,

$$\mathcal{W}^v_{\min} = \{T \in \mathcal{B}^{v^*} : \nexists Q \subseteq \mathcal{B}^{v^*} \text{ s.t. } Q \subset T\}.$$

In other words, we can define $\mathcal{B}^N_{\min}$ from $\mathcal{W}^v_{\min}$, as done in Sect. 1, and conversely, $\mathcal{W}^v_{\min}$ from $\mathcal{B}^N_{\min}$, as specified above.

Now, we introduce some properties for methods aimed at ranking players of a simple game according to their power to block a decision in the grand-coalition N. Let us start recalling that a binary relation on N is a subset of $N \times N$. A reflexive, transitive and total binary relation on N is a *total preorder* (also called, a *ranking*) on N. We denote by $\mathcal{T}^N$ the set of all total preorders on N. Let $\mathcal{SG}^N$ be the class of all simple games with N as the set of players. We define a *power ranking solution* or, simply, a *solution*, as a map $R : \mathcal{SG}^N \to \mathcal{T}^N$ that associates to each simple game $v \in \mathcal{SG}^N$ a total preorder on N. The value assumed by a map

R on a simple game v is the ranking on N denoted by R^v. We use the notation $i R^v j$ to say that $(i, j) \in R^v$, and it means that "i is at least as important as j according to ranking R^v", for all $i, j \in N$.

We denote by I^v the symmetric part of R^v, i.e. $i I^v j$ means that $(i, j) \in R^v$ and $(j, i) \in R^v$ (i and j are equivalent), and by P^v its asymmetric part, i.e. $i P^v j$ means that $(i, j) \in R^v$ and $(j, i) \notin R^v$ (i is strictly more important than j).

We introduce some properties for solutions (inspired by those introduced in Bernardi et al. (2019)).

The first axiom is a classical anonymity axiom and states that a social ranking should not depend on the identity of players.

Axiom 1 [Players' Anonymity] Let σ be a bijection on N. For any $v \in \mathcal{SG}^N$, let $v_\sigma \in \mathcal{SG}^N$ be a simple game defined by

$$v_\sigma(S) = v(\sigma^{-1}(S)).$$

A solution R satisfies the *players' anonymity* property if

$$i R^v j \Leftrightarrow \sigma(i) R^{v_\sigma} \sigma(j).$$

Example 4 (Owen (1995)) Consider a company with four stockholders $N = \{1, 2, 3, 4\}$ having 40%, 30%, 20% and 10% shares of the stock, respectively. Suppose that any coalition of stakeholders holding a simple majority of the shares can approve a motion. We can model such a situation as a simple game with minimal winning coalitions $\{\{1, 2\}, \{1, 3\}, \{2, 3, 4\}\}$. Suppose now that stockholder 1 sells 30% of her stock to stockholder 4. In the new situation, stockholder 1 and 4 have exchanged their respective stocks (in terms of Axiom 1, the new situation can be obtained from the original one, taking a bijection σ such that $\sigma(1) = 4$, $\sigma(2) = 2$, $\sigma(3) = 3$ and $\sigma(4) = 1$). So, Axiom 1 says that the ranking of 1 and 4 in the new situation should be the ranking in the original one with 1 in the role of 4 and 4 in the role of 1.

The following three axioms deal with the notion of blocking coalitions introduced in the previous section. The next property says that blocking coalitions of the same size should have the same impact on the ranking, independently of their members.

Axiom 2 [Dual Coalitional Anonymity] Let $i, j \in N$, $v, v_\pi \in \mathcal{SG}^N$ (with their respective dual games v^* and v^*_π) and let π be a bijection on $2^{N\setminus\{i,j\}}$ with $|\pi(S)| = |S|$ and such that

$$S \cup \{i\} \in \mathcal{W}^{v^*}_{\min} \Leftrightarrow S \cup \{i\} \in \mathcal{W}^{v^*_\pi}_{\min}$$

and

$$S \cup \{j\} \in \mathcal{W}^{v^*}_{\min} \Leftrightarrow \pi(S) \cup \{j\} \in \mathcal{W}^{v^*_\pi}_{\min},$$

for all $S \in 2^{N\setminus\{i,j\}}$. A solution R satisfies the *dual coalitional anonymity* property if

$$i R^v j \Leftrightarrow i R^{v_\pi} j.$$

Example 5 Consider again Example 4. Notice that any motion can be blocked by coalitions $\{1, 2\}, \{1, 3\}, \{2, 3\}$ or $\{1, 4\}$. Suppose now that 4 sells his share of 10% of the stock to 3: now, the minimal winning coalitions are $\{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ and any motion can be blocked by supersets of coalitions $\{1, 2\}$, $\{1, 3\}$ or $\{2, 3\}$. However, comparing stockholders 2 and 3, nothing has changed with respect to their effective possibilities to block a motion (to see this in terms of Axioms 2, take the identity bijection π on $2^{\{1,4\}}$ when comparing 2 and 3 in the role of i and j, respectively). So, Axiom 2 states that the relative ranking R between 2 and 3 in this new situation should be precisely the same of these two players in the original situation, regardless of the number of minimal blocking coalitions containing both 2 and 3 (or neither of them).

Moreover, notice that, due to Axiom 2, the relative ranking between 2 and 3 would remain unchanged even if, for some reasons, the new minimal blocking coalitions were $\{1, 2\}$, $\{3, 4\}$, $\{2, 3\}$ and $\{1, 4\}$ (which corresponds, with respect to the simple game in Example 4, to a new game v^*_π with the bijection π on $2^{\{1,4\}}$ such that $\pi(\{1\}) = \{4\}$ and $\pi(\{4\}) = \{1\}$). In other words, Axiom 2 also states that the relative ranking between two players i and j is not affected by the identity of the other players in the minimal blocking coalitions, but only the size of such coalitions counts.

Remark 4 While Axiom 1 is a fundamental property representing the well-established principle that the relative ranking of two players should not depend on their names, but only on the essential information provided by the characteristic function of a game, Axiom 2 represents a more sophisticated notion of anonymity for coalitions which deserves some further clarification. By the definition of bijection π in Axiom 2, and as shown in the previous examples, it follows that minimal winning coalitions of the same size count equally to determine the relative ranking of two players, independently of the identity of the coalitions' members. Such a property can be recast as an assumption of equal probability to form minimal winning coalitions of equal size, a standard property implicitly required by many classical power indices, like the Shapley-Shubik index (see Shapley and Shubik (1954)) or the Banzhaf index (see Banzhaf (1965)) and, more in general, by all semivalues (see Dubey et al. (1981)).

Remark 5 Comparing our approach with the one introduced in the paper Bernardi et al. (2019), dealing with the problem of ranking the elements of a set N starting from an arbitrary preference relation over the subsets of N, we notice that a simple game can be interpreted as a dichotomous preference relation over coalitions with only two indifference classes (such that all minimal winning coalitions belong to the best indifference class). As a consequence, it is legitimate to compare Axiom 2 in our paper, with the similar Coalitional Anonymity axiom introduced in Bernardi et al. (2019) over the domain of dichotomous preference relations. On such a domain, it is immediate to see that a solution R satisfying the Coalitional Anonymity axiom introduced in Bernardi et al. (2019) (which applies to any bijection among coalitions) automatically satisfies also Axiom 2 (which only applies to bijections among coalitions of the same size). Instead, a solution R that satisfies Axiom 2, does not necessarily satisfy the Coalitional Anonimity axiom in Bernardi et al. (2019) on the corresponding dichotomous rankings. To see this, just consider two dual games v^* and $\widehat{v}^*$ with minimal winning coalitions, respectively, $\{2\}$ and $\{1, 3\}$ for v^*, and $\{2\}$ and $\{3\}$ for $\widehat{v}^*$. There is no reason to affirm that a solution R satisfying Axiom 2 ranks players 2 and 3 in the same way on games v^* and $\widehat{v}^*$. Instead, considering the corresponding dichotoumous ranking $\succeq_{v^*}$ (where $\{2\}$ and $\{1, 3\}$ are in the best indifference class) and $\succeq_{\widehat{v}^*}$ (where $\{2\}$ and $\{3\}$ are in the best indifference class), a solution R satisfying the Coalitional Anonimity axiom in Bernardi et al. (2019) should rank 2 and 3 in the same way both in $\succeq_{v^*}$ and $\succeq_{\widehat{v}^*}$ (to see this according to the definition of the Coalitional Anonymity axiom in Bernardi et al. (2019), just consider a bijection that assigns the set $\{1\}$ to the empty set $\varnothing$).

The next axiom is a monotonicity axiom used to break ties: if two players i and j are equivalent, adding some blocking coalitions containing i but not j should break the tie in favour of i.

Axiom 3 [Dual Monotonicity] Let $i, j \in N$. For any $v \in \mathcal{SG}^N$ (with its dual game v^*), let $\mathcal{S}^i$ be a collection of (minimal) coalitions of equal size containing i but not j , i.e., $\mathcal{S}^i = \{S_1, \ldots, S_r\}$ such that $S_k \subseteq N \setminus \{j\}$, $i \in S_k$ and there is no $Q \in \mathcal{W}^{v^*}_{\min} \cup \mathcal{S}^i$ with $Q \subset S_k$, for all $k \in \{1, \ldots, r\}$. A solution R satisfies the *dual monotonicity* property if

$$i I^v j \Rightarrow i P^{v'} j,$$

where v' is a simple game such that the set of minimal winning coalitions of its dual v'^* is $\mathcal{W}^{v'^*}_{\min} = \mathcal{W}^{v^*}_{\min} \cup \mathcal{S}^i$.

Example 6 Consider the second situation of Example 5 with the four stockholders sharing respectively 40%, 30%, 30%, and 0 shares of the stock. So, the minimal blocking coalitions are $\{1, 2\}$, $\{1, 3\}$ and $\{2, 3\}$. Suppose that the power ranking rule says that 1 an 3 have equal power (on the other hand, their effective possibilities to block a motion are the same for 1 and 3). Now, consider a new situation where 3 sells back 10% of the stock to 4: now the minimal winning coalitions are again $\{1, 2\}$, $\{1, 3\}$ and $\{2, 3, 4\}$, and the minimal blocking coalitions are $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$ and $\{1, 4\}$. So, stockholder 1 now has one more minimal blocking coalition than 3 and, therefore, Axiom 3 suggests to break the tie between 1 and 3 in favour of 1.

The last axiom says that, once a solution exists, in which a player i is more important than a player j, adding new "larger" blocking coalitions should not affect the relative ranking between i and j.

Axiom 4 [Independence of Higher Cardinalities] Let $i, j \in N$. For any $v \in \mathcal{SG}^N$ (with its dual game v^*), let $h = \max\{|S| : S \in \mathcal{W}^{v^*}_{\min}$ and $S \cap \{i, j\} \neq \varnothing\}$ be the highest cardinality of coalitions in the set $\mathcal{W}^{v^*}_{\min}$ containing either i or j. Let $\mathcal{S}_h$ be a collection of (minimal) coalitions with cardinality strictly larger than h, i.e., $\mathcal{S}_h = \{S_1, \ldots, S_r\}$ such that $S_k \subseteq N$, $|S_k| > h$ and there is no $Q \in \mathcal{W}^{v^*}_{\min} \cup \mathcal{S}_h$ with $Q \subset S_k$, for all $k \in \{1, \ldots, r\}$. A solution R satisfies the *independence of higher cardinalities* property if

$$i P^v j \Rightarrow i P^{v''} j,$$

where v'' is a simple game such that the set of minimal winning coalitions of its dual v''^* is $\mathcal{W}^{v''^*}_{\min} = \mathcal{W}^{v^*}_{\min} \cup \mathcal{S}_h$.

Example 7 Consider a company with four stockholders and where the minimal blocking coalitions are $\{1, 2\}$ and $\{1, 3\}$ and suppose that the power ranking rule says that 1 is strictly more powerful than 2. Now suppose that, after some trading among the stockholders, coalition $\{2, 3, 4\}$ is added to the list of minimal blocking coalitions. Notice that this coalition contains 2 but not 1. However, since the size of this new minimal blocking coalition is larger than the size of those in the original situation, Axiom 4 states that 1 should keep a position in the ranking strictly higher than 2 also in the new situation.

Axiom 4 reflects a criterion for solutions aimed at rewarding the membership to minimal blocking coalitions of small size. Such a criterion is particularly suitable in various international assemblies where, due to the underlying rules governing the collective decision-making process and to "psycological" factors like the actions of mediators or a rotating presidency, building large blocking coalitions is quite difficult (as argued, for instance, in Kleinowski (2019) for the European Council). In these practical situations, it seems natural to prioritize the impact of minimal blocking coalitions of small cardinality on the players' ranking and to keep a strict ranking unchanged after the addition of large, and consequently less likely to act, blocking coalitions. For example, a veto player, being a singleton blocking coalition, is considered more relevant than any other non-veto player, regardless of the number of minimal blocking coalitions any other non-veto player may form.

5 A criticality-based power ranking

Given a simple game (N, v) and $i \in N$, we denote by i_k the number of minimal blocking coalitions of cardinality k that include agent i, so $i_k = |\{S \in \mathcal{B}^N_{\min} : i \in S, |S| = k\}|$[3] for all $k \in \{1, \ldots, n\}$. For each $i \in N$, let $\theta_v(i)$ be the n-dimensional vector $\theta_v(i) = (i_1, \ldots, i_n)$ associated to v.

Consider the lexicographic order among vectors of real numbers:

$$\mathbf{x} \geq_L \mathbf{y} \quad \text{if either } \mathbf{x} = \mathbf{y} \quad \text{or } \exists k : x_t = y_t, t = 1, \ldots, k-1 \text{ and } x_k > y_k.$$

Definition 4 The *criticality-based solution* is the function $R_c : \mathcal{SG}^N \longrightarrow \mathcal{T}^N$ defined for any simple game $v \in \mathcal{SG}^N$ as

$$i \; R^v_c \; j \quad \text{if} \quad \theta_v(i) \;\geq_L\; \theta_v(j).$$

(Following the same convention as before, I^v_c and P^v_c stand for the symmetric part and the asymmetric part of R^v_c, respectively.)

Example 8 Consider the simple game of Example 3 then we have that

$$\begin{aligned}
\theta_v(1) &= \{0,0,0,0,7,3,0,0,0,0\}, & \theta_v(2) &= \{0,0,0,1,8,0,0,0,0,0\},\\
\theta_v(3) &= \{0,0,0,0,5,2,0,0,0,0\}, & \theta_v(4) &= \{0,0,0,0,9,1,0,0,0,0\},\\
\theta_v(5) &= \{0,0,0,1,7,3,0,0,0,0\}, & \theta_v(6) &= \{0,0,0,0,12,1,0,0,0,0\},\\
\theta_v(7) &= \{0,0,0,1,11,0,0,0,0,0\}, & \theta_v(8) &= \{0,0,0,0,7,3,0,0,0,0\},\\
\theta_v(9) &= \{0,0,0,1,5,2,0,0,0,0\}, & \theta_v(10) &= \{0,0,0,0,4,3,0,0,0,0\}.
\end{aligned}$$

Then the criticality-based solution ranks

$$7 \; P^v_c \; 2 \; P^v_c \; 5 \; P^v_c \; 9 \; P^v_c \; 6 \; P^v_c \; 4 \; P^v_c \; 1 \; I^v_c \; 8 \; P^v_c \; 3 \; P^v_c \; 10.$$

Example 9 Consider again Example 4 with $\mathcal{B}^N_{\min} = \mathcal{W}^{v^*}_{\min} = \{\{1,2\},\{1,3\},\{2,3\},\{1,4\}\}$. We have that $\theta_v(1) = (0,3,0,0)$, $\theta_v(2) = (0,2,0,0)$, $\theta_v(3) = (0,2,0,0)$, $\theta_v(4) = (0,1,0,0)$. Then the criticality-based solution ranks $1 \; P^v_c \; 2 \; I^v_c \; 3 \; P^v_c \; 4$.

We prove the next results following an approach similar to the one used for the axiomatic characterization of the lexicographic-excellent ranking solution provided in Bernardi et al. (2019).

Proposition 4 *Let R be a solution satisfying Axioms 1 and 2. Then for any simple game v and $i, j \in N$ such that $\theta_v(i) = \theta_v(j)$ we have that $i I^v j$.*

Proof Since $\theta_v(i) = \theta_v(j)$, we have that $i_k = j_k$ for all $k \in \{1, \ldots, n\}$. Define a bijection π on $2^{N \setminus \{i,j\}}$ such that for each $k \in \{1, \ldots, n-1\}$ and for each coalition $S \in 2^{N \setminus \{i,j\}}$ of size $k-1$ with $S \cup \{j\} \in \mathcal{W}^{v^*}_{\min}$, $\pi(S) = T$, where $T \in 2^{N \setminus \{i,j\}}$ is a coalition of size $k-1$,

[3] Equivalently, $i_k = |\{S \in \mathcal{W}^{v^*}_{\min} : i \in S, |S| = k\}|$ for all $k \in \{1, \ldots, n\}$.

with $T \cup \{i\} \in \mathcal{W}^{v^*}_{\min}$. Consider a game v^*_π such that $S \cup \{i\} \in \mathcal{W}^{v^*}_{\min} \Leftrightarrow S \cup \{i\} \in \mathcal{W}^{v^*_\pi}_{\min}$ and $S \cup \{j\} \in \mathcal{W}^{v^*}_{\min} \Leftrightarrow \pi(S) \cup \{j\} \in \mathcal{W}^{v^*_\pi}_{\min}$.

Notice that i, j, π, v and v_π (and their dual games v^* and v^*_π), satisfy the conditions for bijections demanded in the statement of Axiom 2, with v_π such that the set of minimal winning coalitions of the dual v^*_π is

$$\mathcal{W}^{v^*_\pi}_{\min} = \bigcup_{T \in 2^{N\setminus\{i,j\}} \text{ s.t. } T\cup\{i\}\in\mathcal{W}^{v^*}_{\min}} \{T \cup \{i\}, T \cup \{j\}\}$$

(notice that the minimality of the elements in $\mathcal{W}^{v^*_\pi}_{\min}$ is guaranteed by the minimality of the elements in $\mathcal{W}^{v^*}_{\min}$). So, since R satisfies Axiom 2, we have that

$$i R^v j \Leftrightarrow i R^{v_\pi} j. \tag{4}$$

Moreover, since $S \cup \{i\} \in \mathcal{W}^{v^*_\pi}_{\min}$ if and only if $S \cup \{j\} \in \mathcal{W}^{v^*_\pi}_{\min}$, we immediately have that $v_\pi = (v_\pi)_\sigma$, where σ is a bijection on N such that $\sigma(i) = j$, $\sigma(j) = i$ and $\sigma(t) = t$ for all $t \in N \setminus \{i, j\}$. Then, by the fact that R satisfies Axiom 1, we have that

$$i R^{v_\pi} j \Leftrightarrow j R^{v_\pi} i. \tag{5}$$

Since R^v is a total relation, then by relation (5), it immediately follows that $i I^{v_\pi} j$, and by relation (4), we have proven that $i I^v j$.

Theorem 1 *The criticality-based solution R_c fulfills Axioms 1, 2, 3 and 4.*

Proof *Axiom 1.*

Let σ be a bijection on N. For any $v \in \mathcal{SG}^N$, let $v_\sigma \in \mathcal{SG}^N$ be a simple game such that $v_\sigma(S) = v(\sigma^{-1}(S))$. Note that $\theta_v(i) = \theta_{v_\sigma}(\sigma(i))$ for all $i \in N$, and by Definition 4 we directly have that R_c satisfies Axiom 1.

Axiom 2.

Let $i, j \in N$, $v, v_\pi \in \mathcal{SG}^N$ (and their respective duals v^* and v^*_π) and let π be a bijection on $2^{N\setminus\{i,j\}}$ such that $|\pi(S)| = |S|$ for all $S \in 2^{N\setminus\{i,j\}}$ and such that

$$S \cup \{i\} \in \mathcal{W}^{v^*}_{\min} \Leftrightarrow S \cup \{i\} \in \mathcal{W}^{v^*_\pi}_{\min}$$

and

$$S \cup \{j\} \in \mathcal{W}^{v^*}_{\min} \Leftrightarrow \pi(S) \cup \{j\} \in \mathcal{W}^{v^*_\pi}_{\min},$$

for all $S \in 2^{N\setminus\{i,j\}}$. Then, $\theta_v(i) = \theta_{v_\pi}(i)$, for all $i \in N$, and by Definition 4 we directly have that R_c satisfies Axiom 2.

Axiom 3.

Let $i, j \in N$. For any $v \in \mathcal{SG}^N$ and its dual v^*. Suppose $i I^v_c j$. Then, by Definition 4, $\theta_v(i) = \theta_v(j)$. Now, let $\mathcal{S}^i$ be a collection of minimal winning coalitions in $\mathcal{W}^{v^*}_{\min}$ with the same cardinality containing i but not j, i.e., $\mathcal{S}^i = \{S_1, \ldots, S_r\}$ such that $S_k \subseteq N \setminus \{j\}, i \in S_k$ and there is no $Q \in \mathcal{W}^{v^*}_{\min} \cup \mathcal{S}^i$ with $Q \subset S_k$, for all $k \in \{1, \ldots, r\}$. Consider the simple game v' such that $\mathcal{W}^{v'^*}_{\min} = \mathcal{W}^{v^*}_{\min} \cup \mathcal{S}^i$. Note that $\theta_{v'}(j) = \theta_v(j)$, whereas $\theta_{v'}(i) >_L \theta_v(i)$ (only some i_k strictly increases). So, $\theta_{v'}(i) >_L \theta_v(i) = \theta_v(j) = \theta_{v'}(j)$. Therefore, we have $i P^{v'}_c j$, and R_c satisfies Axiom 3.

Axiom 4.

Let $i, j \in N$, $v \in \mathcal{SG}^N$ (and its dual v^*) be such that $i P_c^v j$. This means that, by Definition 4, $\theta_v(i) >_L \theta_v(j)$, that is, there exists k with $i_t = j_t, t = 1, \ldots, k-1$ and $i_k > j_k$. Let $h = \max\{|S| : S \in \mathcal{W}_{\min}^{v^*} \text{ and } S \cap \{i, j\} \neq \varnothing\}$ be the highest cardinality of coalitions in the set $\mathcal{W}_{\min}^{v^*}$ containing either i or j. Let $\mathcal{S}_h$ be a collection of coalitions with cardinality larger than h, i.e., $\mathcal{S}_h = \{S_1, \ldots, S_r\}$ such that $S_l \subseteq N$, $|S_l| > h$, and there is no $Q \in \mathcal{W}_{\min}^{v^*} \cup \mathcal{S}_h$ with $Q \subset S_l$, for all $l \in \{1, \ldots, r\}$. Let v'' be a simple game such that $\mathcal{W}_{\min}^{v''*} = \mathcal{W}_{\min}^{v^*} \cup \mathcal{S}_h$. Note that $h \geq k$, so we have that $\theta_{v''}(i)_t = \theta_v(i)_t$ and $\theta_{v''}(j)_t = \theta_v(j)_t$ for all $t \leq k$. So, $\theta_{v''}(i) >_L \theta_{v''}(j)$ and therefore $i P_c^{v''} j$, which proves that R_c satisfies Axiom 4.

The next theorem shows that the four axioms characterize the solution R_c.

Theorem 2 *The criticality-based solution R_c is the unique that fulfills Axioms 1, 2, 3 and 4.*

Proof The proof is analogue to the one presented in Bernardi et al. (2019) and we postpone it in Appendix A.

We conclude this section discussing the logical independence of the properties of Axioms 1, 2, 3 and 4.

Example 10 [No Axiom 1] Given $i, j \in N$, consider the ranking solution R_N defined by

$$i \; R_N \; j \quad \text{if} \quad i < j.$$

This solution satisfies all the axioms but Axiom 1.

Example 11 [No Axiom 2] Consider the ranking solution R_{DCA} equivalent to R_c^v except that, when $\theta_v(i) = \theta_v(j)$ for some $i, j \in N$, then the agent that belongs to a minimal winning coalition including a lower index player is ranked first between the two. This solution satisfies all the axioms but Axiom 2.

Example 12 [No Axiom 3] Consider the ranking solution R_{DM} that equally ranks all the agents. This solution satisfies all the axioms but Axiom 3.

Example 13 [No Axiom 4] For each $i \in N$, let $\overline{\theta}_v(i)$ be the n-dimensional vector $\overline{\theta}_v(i) = (i_n, \ldots, i_1)$ associated to v. Given $i, j \in N$, consider the vector ranking solution R_{IHC} defined by

$$i \; R_{IHC} \; j \quad \text{if} \quad \overline{\theta}_v(i) \geq_L \overline{\theta}_v(j).$$

This solution satisfies all the axioms but Axiom 4.

6 Connection games

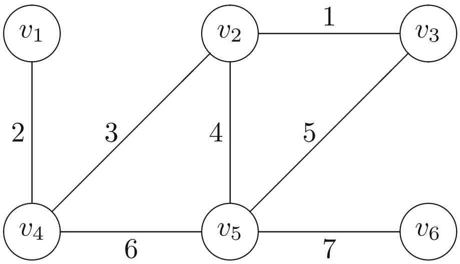
In Dall'Aglio et al. (2019b), the authors studied connection games on graphs $G = (V; E)$, where the players are edges, $E = N$, and the minimal winning coalitions are spanning trees. According to this notation a blocking coalition is a set of edge-cuts in G.

Proposition 5 *Let v be a connection game on a graph G, as defined in Dall'Aglio et al. (2019b)), and v^* its dual. Then,*

$$\mathcal{W}_{\min}^{v^*} = \mathcal{B}_{\min}^{N} = \{S \subseteq N : S \text{ is a minimal (w.r.t. inclusion) edge-cut in } G\}.$$

Proof It is well known that a graph is connected if and only if it has a spanning tree. Then, a coalition of edges is an edge-cut if and only if graph G minus edges in S contains no spanning tree, which means that every spanning tree has some edges in common with S; so $S \in \mathcal{B}^N$ and the proof follows by Proposition 3.

Example 14 Consider Example 2 in Dall'Aglio et al. (2019b).



The minimal winning coalitions are already listed explicitly, and they are: {1, 2, 3, 4, 7}, {1, 2, 3, 5, 7}, {1, 2, 3, 6, 7}, {1, 2, 4, 6, 7}, {1, 2, 5, 6, 7}, {2, 3, 4, 5, 7}, {2, 3, 5, 6, 7}, {2, 4, 5, 6, 7}. Looking at the graph, it is quite straightforward to verify that

$$\mathcal{B}^N_{\text{min}} = \{\{4, 5, 6\}, \{1, 3, 4\}, \{2\}, \{1, 5\}, \{3, 6\}, \{7\}, \{1, 4, 6\}, \{3, 4, 5\}\}$$

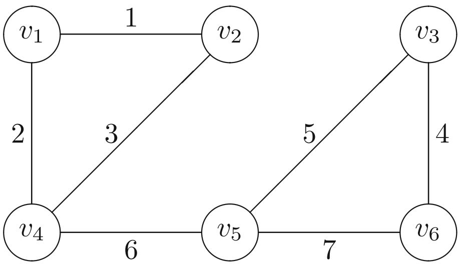
then

$$\begin{aligned}
&\theta_v(1) = \{0, 1, 2, 0, 0, 0, 0\},\ \theta_v(2) = \{1, 0, 0, 0, 0, 0, 0\},\ \theta_v(3) = \{0, 1, 2, 0, 0, 0, 0\},\\
&\theta_v(4) = \{0, 0, 4, 0, 0, 0, 0\},\ \theta_v(5) = \{0, 1, 2, 0, 0, 0, 0\},\\
&\theta_v(6) = \{0, 1, 2, 0, 0, 0, 0\},\ \theta_v(7) = \{1, 0, 0, 0, 0, 0, 0\},
\end{aligned}$$

and, therefore

$$2\ I_c^v\ 7\ P_c^v\ 1\ I_c^v\ 3\ I_c^v\ 5\ I_c^v\ 6\ P_c^v\ 4.$$

Example 15 Consider Example 3 in Dall'Aglio et al. (2019b).



Looking at the graph, it is easy to infer that

$$\mathcal{B}^N_{\text{min}} = \{\{6\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{4, 5\}, \{4, 7\}, \{5, 7\}\}$$

then

$$\begin{aligned}
&\theta_v(1) = \{0, 2, 0, 0, 0, 0, 0\},\ \theta_v(2) = \{0, 2, 0, 0, 0, 0, 0\},\ \theta_v(3) = \{0, 2, 0, 0, 0, 0, 0\},\\
&\theta_v(4) = \{0, 2, 0, 0, 0, 0, 0\},\ \theta_v(5) = \{0, 2, 0, 0, 0, 0, 0\},\\
&\theta_v(6) = \{1, 0, 0, 0, 0, 0, 0\},\ \theta_v(7) = \{0, 2, 0, 0, 0, 0, 0\},
\end{aligned}$$

and, therefore

$$6 \; P_c^v \; 1 \; I_c^v \; 2 \; I_c^v \; 3 \; I_c^v \; 4 \; I_c^v \; 5 \; I_c^v \; 7.$$

7 Conclusions and further research

We have set a systematic method for determining the order of criticality of any player in a winning coalition of a simple game. This result is made possible by pointing out the connection between the problem at hand and the generation of the minimal hitting set of a given family of subsets. The rich collection of algorithms available for the combinatorics' problem offers a vast array of codes that can be easily adapted for our purposes. The algorithms retain the computational hardness of their sources, but this is not a problem for the typical game theoretic applications. For instance, the number of political parties in a present-day parliament rarely exceeds a single digit.

Moreover, keeping the grand coalition as reference, we have defined and axiomatized a ranking of the players according to their capability to disrupt the current partnership, alone or in conjunction with other players in the same coalition. The result is based on the notion of dual game that signals blocking coalitions.

As it is well known in the combinatorics literature (see Gainer-Dewar and Vera-Licona (2017)), applying the minimal hitting set generation procedure twice to a set family which is minimal over inclusion, returns the set family itself. Therefore, the main results in the present work are based on two distinct but related notions of duality. A connection between them has already been established in Proposition 2, where it was shown that the set of minimal winning coalitions for the dual game coincides with the minimal hitting sets of the minimal winning coalitions for the original game. Duality is so important here because we see two different problems deeply mirrored into each other. We may exemplify them as two different moments in the life of a parliamentary legislature: immediately after an election, political parties begin negotiations to form a government alliance. Here, minimal winning coalitions represent the closest achievable goals. In a later phase, allied parties may be tempted to break a governing alliance, and they may ask for help to other partners. Now, the minimal blocking coalitions are the closest achievable goals. We hope that the dual nature of the two problems will help to get further insights on both issues.

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Appendix A: Proof of Theorem 2

Proof To show that R_c is the unique solution fulfilling Axioms 1, 2, 3 and 4, we need to prove that, if a solution $R : \mathcal{SG}^N \to \mathcal{T}^N$ satisfies Axioms 1, 2, 3 and 4, then $i R_c^v j \Leftrightarrow i R^v j$ or, equivalently, $i P_c^v j \Leftrightarrow i P^v j$ and $i I_c^v j \Leftrightarrow i I^v j$.

We first prove that $i P_c^v j \Leftrightarrow i P^v j$:

$(\Rightarrow)$

Let $i P_c^v j$. By Definition 4, let k' be the smallest integer in $\{1, \dots, n\}$ with $i_{k'} > j_{k'}$. Let $s = i_{k'} - j_{k'}$ and $\mathcal{S}_{k'}^i = \{S \in \mathcal{W}_{\min}^{v^*} : |S| = k' \text{ and } S \cap \{i, j\} = i\}$ be a subset of coalitions in $\mathcal{W}_{\min}^{v^*}$ of size k' containing i but not j such that $|\mathcal{S}_{k'}^i| = s$. Moreover, let $\Sigma = \{S \in \mathcal{W}_{\min}^{v^*} : |S| > k'\}$ be the set of coalitions in $\mathcal{W}_{\min}^{v^*}$ with cardinality strictly larger than k'. Now consider a new simple game $\widehat{v}$ such that $\mathcal{W}_{\min}^{\widehat{v}^*} = \mathcal{W}_{\min}^{v^*} \setminus (\mathcal{S}_{k'}^i \cup \Sigma)$. Notice that $\theta_{\widehat{v}}(i) = \theta_{\widehat{v}}(j)$. Then, by Proposition 4, we have that $i I^{\widehat{v}} j$. Now, consider a new game $\widehat{v}'$ such that $\mathcal{W}_{\min}^{\widehat{v}'^*} = \mathcal{W}_{\min}^{\widehat{v}^*} \cup \mathcal{S}_{k'}^i$ (notice that the minimality of elements in $\mathcal{W}_{\min}^{\widehat{v}'^*}$ is satisfied, as it already was in $\mathcal{W}_{\min}^{v^*}$). Then, by Axiom 3 on R (with $\widehat{v}$ in the role of v in the statement of Axiom 3), we have that $i P^{\widehat{v}'} j$. Finally, by Axiom 4 on R (with $\widehat{v}'$ in the role of v in the statement of Axiom 4), we have that $i P^v j$, as $\mathcal{W}_{\min}^{v^*} = \mathcal{W}_{\min}^{\widehat{v}'^*} \cup \Sigma$.

$(\Leftarrow)$

Let $i P^v j$. Suppose that $i I_c^v j$. Then, by Definition 4, $\theta_{\widehat{v}}(i) = \theta_{\widehat{v}}(j)$. So, by Proposition 4, $i I^v j$, which yields a contradiction with $i P^v j$. Since it can't even be $j P_c^v i$ (by the other implication proved above), and by the fact that P_c^v is a total relation, it must be $i P_c^v j$.

We now prove that $i I_c^v j \Leftrightarrow i I^v j$:

$(\Rightarrow)$

Let $i I_c^v j$. Then, by Definition 4, $\theta_{\widehat{v}}(i) = \theta_{\widehat{v}}(j)$. So, by Proposition 4 and the fact that R^v satisfies Axioms 1 and 2, $i I^v j$.

$(\Leftarrow)$

Let $i I^v j$. As we have shown previously, $i P_c^v j \Leftrightarrow i P^v j$. So it is not possible that $i P_c^v j$ or $j P_c^v i$. Since P_c is a total relation, it must be $i I_c^v j$, which concludes the proof.

Appendix B: Divide and conquer algorithm for the hitting set problem

The basic idea of the Divide and Conquer algorithm class is to divide the family of sets $\mathcal{F}$ in disjoint subfamilies, to find all the minimal hitting sets of these families and to compute the cross-product between them. The family thus obtained can include non-minimal hitting sets, then a procedure of minimization is required, and it represents a costly step in terms of efficiency of the algorithm. The contribution of Zhao et al. (2018) is the reduction of the number of minimization procedures in the recursion steps in this class of algorithms.

In more details, let MHS be the operator that associates to a family of sets $\mathcal{F}$ its family of minimal hitting sets, i.e. $MHS(\mathcal{F})$. Firstly the authors divide the family $\mathcal{F}$ in two sub-families $\mathcal{F}_1$, $\mathcal{F}_2$ with no common subsets and such that $\mathcal{F}_1 \cup \mathcal{F}_2 = \mathcal{F}$. They generate two sub-families of minimal hitting sets, $\mathcal{M}_1 = MHS(\mathcal{F}_1)$ and $\mathcal{M}_2 = MHS(\mathcal{F}_2)$. These families can contain sets sharing some element and to avoid redundant sets, each family is divided into two subfamilies $\mathcal{M}_{11}, \mathcal{M}_{12}$ and $\mathcal{M}_{21}, \mathcal{M}_{22}$ respectively, where each $M_{11} \in \mathcal{M}_{11}$ and $M_{21} \in \mathcal{M}_{21}$ are disjoint. Then, taking the cross-product of these two subfamilies $\mathcal{M}_{11}$, $\mathcal{M}_{21}$, they obtain a subset of $MHS(\mathcal{F})$. The remaining hitting sets are found by the cross-product of $\mathcal{M}_{11}$ with $\mathcal{M}_{22}$, of $\mathcal{M}_{12}$ with $\mathcal{M}_{21}$, and of $\mathcal{M}_{12}$ and $\mathcal{M}_{22}$ once redundant sets are removed.

They start observing that some elements in $\mathcal{F}_1$ can be covered by elements in $\mathcal{M}_{22}$. Then, removing these kind of sets from $\mathcal{F}_1$, they define a new family $\mathcal{F}_1'$. By a minimization procedure on the set $\mathcal{M}_{11}$, they define $\mathcal{M}_{11}' = MHS(\mathcal{F}_1')$. At this point, instead of merging $\mathcal{M}_{11}$ with $\mathcal{M}_{22}$, they directly take the cross-product of $\mathcal{M}_{11}'$ with $\mathcal{M}_{22}$. A similar argument can be applied for the remaining two cases, and they write the $MHS(\mathcal{F})$ taking the union of the obtained families.

The code of the algorithm is available from the authors upon request.

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