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On Panyushev's Rootlets for Infinitesimal Symmetric Spaces

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Introduction

Let $\mathfrak{g}$ be a complex finite semisimple Lie algebra endowed with an involution σ. The map σ induces a $\mathbb{Z}_2$-gradation on $\mathfrak{g}$ that we can express as $\mathfrak{g} = \mathfrak{g}^{\bar{0}} \oplus \mathfrak{g}^{\bar{1}}$ with $\sigma(x) = x$ for every $x \in \mathfrak{g}^{\bar{0}}$ and $\sigma(x) = -x$ for every $x \in g^{\bar{1}}$. The subspace $\mathfrak{g}^{\bar{0}}$ turns out to be a reductive Lie algebra, so we can fix a Borel subalgebra $\mathfrak{b}^{\bar{0}} \subset \mathfrak{g}^{\bar{0}}$. In this work we are going to explore the poset of all the abelian subalgebras of $\mathfrak{g}^{\bar{1}}$ which are stable under the action of $\mathfrak{b}^{\bar{0}}$. We will do it by decomposing this poset in special subposets with remarkable properties, by means of an extension of the so called *Panyushev rootlets*, used in the past to prove, in the case of a complex simple Lie algebra $\mathfrak{g}$ and a Borel subalgebra $\mathfrak{b}$, the surprising correspondence between maximal abelian ideals of $\mathfrak{b}$ and long simple roots in the corresponding set of roots $\Delta_{\mathfrak{g}}$. This work is organized as follows:

Chapter 1. In the first chapter we will review some of the mathematics which led to study the set of abelians ideals of $\mathfrak{g}$. We will start from Kostant's results [10], which created a link between these ideals and the eigenvalues of the Laplacian in the setting of Lie algebra cohomology. The chapter will continue exploring a paper of Kostant [12] which presents unpublished results by Peterson. He translated the problem of studying the set of abelian ideals in $\mathfrak{b}$ into a combinatorial problem, giving an explicit isomorphism with the subset of the affine Weyl group $\widehat{W}$ of $\mathfrak{g}$ consisting of the so called *minuscule* elements of $\widehat{W}$. This implies the surprising result known as *Peterson's $2^{rk(\mathfrak{g})}$ Theorem* which counted with the elegant formula $2^{rk(\mathfrak{g})}$ the number of abelian ideals in $\mathfrak{b}$. We will continue looking at Panyushev's paper [16] in which the so called rootlets were introduced and the proof of the correspondence between maximal abelian ideals of $\mathfrak{b}$ and long simple roots was given.

The chapter will go on following the historical path, presenting the $\mathbb{Z}_2$-graded case, and showing the reasons behind it due again to Panyushev in [17]. The problem of studying the abelian subalgebras of $g^{\bar{1}}$ which are $\mathfrak{b}^{\bar{0}}$-stable was translated again into a combinatorial problem thanks to Cellini, Möseneder Frajria and Papi in [4] in 2004. Indeed they showed that this poset is isomorphic to a subset of an affine Weyl group associated to a certain Kac-Moody algebra, called the set of σ*-minuscule* elements, and computed its cardinality providing general formulas. We will also explore a later work [6] from the same authors in 2012. Indeed they defined special subposets of the set of σ-minuscule elements in order to study the maximal elements of the poset. The outcome was a complete parametrization of the set of these maximal elements, and general formulas to compute the dimension of the corresponding maximal abelian subalgebras of $g^{\bar{1}}$ which are stable under the action of $\mathfrak{b}^{\bar{0}}$. The chapter ends with the discussion of some well known results on Weyl groups and root systems that will be required in the following chapters.

Chapter 2. In this chapter we will give new proofs of results on the abelian ideals of $\mathfrak{b}$. Indeed we will decompose the set of minuscule elements in special subsets that will have the properties of having a unique minimum, a unique maximum, and of being complete, meaning that if $w_1 < w < w_2$ and w_1, w_2 belong to the poset then also w belongs to it. We will prove that they are isomorphic to the poset of minimal right coset representatives for a pair of certain suitable Weyl groups. This will be used to prove again the correspondence between maximal abelian ideals of $\mathfrak{b}$ and long simple roots.

Chapter 3. This final chapter will be the core of this work, presenting the use of the rootlets in the framework of the $\mathbb{Z}_2$-graded case. Consider the set of σ-minuscule elements $\mathcal{W}_\sigma^{ab}$, it is a peculiar subset of the affine Weyl group $\widehat{W}$ of $\widehat{L}(\mathfrak{g}, \sigma)$, a specific Kac-Moody algebra associated to the pair $(\mathfrak{g}, \sigma)$ that will be defined in Chapter 1. $\mathcal{W}_\sigma^{ab}$ is a finite set, that can be seen as a poset when endowed with the weak Bruhat order. We will decompose the finite poset $\mathcal{W}_\sigma^{ab}$, in the semisimple cases in both the twisted and untwisted case, into special subposets $\mathcal{I}_{\alpha,\mu}$ with α a positive root called *rootlet*, and μ one of the roots inside the so called set of *walls*. We will give necessary and sufficient conditions for the sets $\mathcal{I}_{\alpha,\mu}$ to be non-empty. We will show that when non-empty these posets posses a unique minimum, and are complete. Moreover we will explicitly show their structure, and will prove that they are isomorphic to the poset of minimal right coset representatives for some suitable Weyl groups, with few remarkable exceptions.

Application and open problems

Spherical nilpotent orbits

Let G be a connected simply connected semisimple complex algebraic group with Lie algebra $\mathfrak{g}$. Let B be a Borel subgroup, and set $\mathfrak{b} = \mathrm{Lie}B$. Recall that a G-variety X is called *G-spherical* if it possesses an open B-orbit. The relationships between spherical nilpotent orbits and abelian ideals of $\mathfrak{b}$ have been first investigated in [21]. There it is shown that if $\mathfrak{a}$ is an abelian ideal of $\mathfrak{b}$, then any nilpotent orbit meeting $\mathfrak{a}$ is a G-spherical variety and $G\mathfrak{a}$ is the closure of a spherical nilpotent orbit. In particular, B acts on $\mathfrak{a}$ with finitely many orbits.

Subsequently, Panyushev [19] dealt with similar questions in the $\mathbb{Z}_2$-graded case. Let σ be an involution of G and $\mathfrak{g} = \mathfrak{g}^{\bar{0}} \oplus \mathfrak{g}^{\bar{1}}$ be the corresponding eigenspace decomposition at the Lie algebra level. Let G_0 be the connected subgroup of G corresponding to $\mathfrak{g}^{\bar{0}}$ and $B_0 \subset G_0$ a Borel subgroup of G_0 corresponding to the Borel subalgebra $\mathfrak{b}^{\bar{0}} \subset \mathfrak{g}^{\bar{0}}$. The "graded" analog of the set of abelian ideals of $\mathfrak{b}$ is our set $\mathcal{I}_{ab}^{\sigma}$ of (abelian) $\mathfrak{b}^{\bar{0}}$-stable subalgebras of $\mathfrak{g}^{\bar{1}}$. We say that $\mathfrak{a} \in \mathcal{I}_{ab}^{\sigma}$ is *G-spherical* (resp. *G_0-spherical*) if all orbits Gx, $x \in \mathfrak{a}$ are G-spherical (resp. if all orbits G_0x, $x \in \mathfrak{a}$ are G_0-spherical).

Panyushev [18] started the classification of the spherical nilpotent G_0-orbits in $\mathfrak{g}^{\bar{1}}$. The classification of the spherical nilpotent G_0-orbits in $\mathfrak{g}^{\bar{1}}$ was then completed by King [11] (see also [1], where the classification is reviewed and a missing case is pointed out). Shortly afterwards, Panyushev [19]

- noticed the emergence of non-spherical subalgebras $\mathfrak{a} \in \mathcal{I}_{ab}^{\sigma}$;
- classified the involutions σ for which these subalgebras exist;

- he also found that an element $\mathfrak{a} \in \mathcal{I}_{ab}^{\sigma}$ is G-spherical if and only if it is G_0-spherical.

In [14] the authors proved

i) B_0 acts on $\mathfrak{a}$ with finitely many orbits, independently of its sphericity. Orbits are parametrized via orthogonal set of weights of $\mathfrak{a}$.

ii) Assume that there exist non-spherical subalgebras. A construction of a canonical non-spherical subalgebra $\mathfrak{a}_p$ was provided.

iii) A simple criterion to decide whether $\mathfrak{a}$ is spherical or not was given: there exists $\overline{\mathfrak{a}} \in \mathcal{I}_{ab}^{\sigma}$ such that $\mathfrak{a}$ is non-spherical if and only if $\mathfrak{a} \supset \overline{\mathfrak{a}}$.

It would be interesting to study the interplay of the posets $\mathcal{I}_{\alpha,\mu}^{\sigma}$, which are the core of our investigation, and of their intersection with the above results. Our main conjecture is the following: write $\mathcal{M}_\sigma = \{\mu_1, \dots, \mu_s\}$ for the set of walls, and for $\alpha_1, \dots, \alpha_s \in \widehat{\Delta}^+$ set

$$\mathcal{I}_{\alpha_1,\dots\alpha_s} = \mathcal{I}_{\alpha_1,\mu_1} \cap \dots \cap \mathcal{I}_{\alpha_s,\mu_s}.$$

We can write

$$\mathcal{W}_{\sigma}^{ab} = \bigsqcup_{\alpha_1,\dots,\alpha_s \in \widehat{\Delta}^+} \mathcal{I}_{\alpha_1,\dots,\alpha_s}$$

with some possibly empty sets.

Conjecture. *For every $\alpha_1, \dots, \alpha_s \in \widehat{\Delta}^+$, one of the following holds:*

- $\forall \mathfrak{a} \in \mathcal{I}_{\alpha_1,\dots,\alpha_s}$, $\mathfrak{a}$ *is G-spherical.*
- $\forall \mathfrak{a} \in \mathcal{I}_{\alpha_1,\dots,\alpha_s}$, $\mathfrak{a}$ *is non G-spherical.*

It looks clear that this is true every time one of the sets $\mathcal{I}_{\alpha_i,\mu_i}$ is a singleton, which occurs many times as shown in the main Theorem 3.1.1 of this work.

Hermitian symmetric case

For the Hermitian symmetric case, where Π_1 is composed by 2 simple roots, we expect most of the results to hold in a very similar fashion to those in the semisimple case. The vast majority of the techniques appear to work without much trouble. What's hindering the progress in the Hermitian symmetric case are difficulties encountered in few particular cases, especially the ones related to the affine diagram $E_6^{(1)}$. Indeed in this case, some subsets $\mathcal{I}_{\gamma,\mu}$ appear to be non empty for unexpected pairs of positive roots $\gamma \in \widehat{\Delta}^+$ and walls $\mu \in \mathcal{M}_\sigma$. Further investigation is still needed.

Contents

Chapter 1

Preliminaries

1.1 Abelian ideals of Borel subalgebras

1.1.1 Motivations

The interest in abelian subalgebras of Borel subalgebras in semisimple Lie algebras has been alive for a long while up to now, so that we need to dig a bit deep to find where everything had started. Let's make a step back in the past, and shed a light on why the study of these abelian subalgebras began in the first place. Let $\mathfrak{g}$ be a complex finite dimensional simple Lie algebra. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra, let Δ be the corresponding root system and W the associated Weyl group. Choose a positive root system Δ^+ in Δ. For $\alpha \in \Delta^+$, let L_α be the root space in $\mathfrak{g}$ corresponding to α, and $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^+} L_\alpha$ be the associated Borel subalgebra. Let $\langle \cdot, \cdot \rangle$ be the Killing form, and choose a basis $x_1, \cdots, x_n$ for $\mathfrak{g}$ and the dual basis $x'_1, \cdots, x'_n$ w.r.t. the Killing form, i.e. $\langle x_i, x'_j \rangle = \delta_{i,j}$. The Casimir operator in the universal enveloping algebra $U(\mathfrak{g})$ is given by

$$C = \sum_{i}^{n} x_i \cdot x'_i$$

and it can be shown that it doesn't depend on the basis we chose. The Casimir operator acts on the exterior algebra $\bigwedge \mathfrak{g}$ via the adjoint representation on $\mathfrak{g}$ extended to the wedge product as a derivation. Recall that the action of C on every finite dimensional irreducible representation of $\mathfrak{g}$ is scalar. Writing $\pi : \mathfrak{g} \to End(V_\lambda)$ for the irreducible representation V_λ associated to the highest weight λ, we have

$$\pi(C) = \langle \lambda, \lambda + 2\rho \rangle I_{V_\lambda}$$

with $\rho = \frac{1}{2}\sum_{\gamma \in \Delta^+} \gamma$. Since $\mathfrak{g}$ is simple we could be interested in the eigenvalues appearing in the action of C on $\bigwedge \mathfrak{g}$, decomposed in its sum of irreducible representations. Indeed, define the coboundary operator d on $\bigwedge \mathfrak{g}$

$$d = \frac{1}{2}\sum_{i=1}^{n} \epsilon(x_i')ad_{x_i}$$

where ϵ is the left wedge product, given by

$$\epsilon(v_0)(v_1 \wedge \cdots \wedge v_k) = v_0 \wedge v_1 \wedge \ldots v_k$$

and ∂ is its adjoint operator, given by

$$\partial(v_1 \wedge \cdots \wedge v_n) = \sum_{i<j} (-1)^{i+j+1}[v_i, v_j] \wedge v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_n$$

for every $v_1, \ldots, v_n \in \mathfrak{g}$. It is shown in [13] that the Laplace operator $L = d\partial + \partial d$ satisfies

$$L = \frac{1}{2}ad(C).$$

This result gives an actual reason to be interested in the eigenvalues of the Casimir operator acting on $\bigwedge \mathfrak{g}$. Now a result from Kostant in [10] provides the link between these eigenvalues and abelian subalgebras of semisimple Lie algebras. Let A be the set of abelian subalgebras of $\mathfrak{g}$, and A_k the subspace generated by the elements $\bigwedge^k \mathfrak{a}$ with $\mathfrak{a} \in A$ a k-dimensional commutative subalgebra.

Theorem 1.1.1 (Kostant). *Let m_k be the maximal eigenvalue of C on $\bigwedge^k \mathfrak{g}$, then*

$$m_k \leq k.$$

Moreover equality holds if and only if there exists a commutative subalgebra of $\mathfrak{g}$ of dimension k. In this case the eigenspace associated to m_k is A_k, and every decomposable element of A_k corresponds to a commutative subalgebra of $\mathfrak{g}$.

Furthermore in [10] Kostant shows that the focus can be restricted to the case of abelian subalgebras contained in $\mathfrak{b}$, and with the property of being $\mathfrak{b}$-stable.

Theorem 1.1.2 (Kostant)**.**

1) Let V be a $\mathfrak{g}$-module and $W \subset \bigwedge^k V$ an irreducible module. Then W is generated by decomposable vectors if and only if it has a decomposable highest weight vector.

2) If M_k is the eigenspace corresponding to the maximal eigenvalue of the action of C on $\bigwedge^k V$, then it is a sum of irreducible $\mathfrak{g}$-submodules generated by decomposable vectors.

This result proved that it sufficed to study the abelian ideals of Borel subalgebras of semisimple Lie algebras.

1.1.2 A combinatorial problem

Note that every abelian ideal $\mathfrak{i}$ of a Borel subalgebra $\mathfrak{b}$ is necessarily a direct sum of root spaces, i.e.

$$\mathfrak{i} = \bigoplus_{\alpha \in \Phi} L_\alpha$$

for some $\Phi \subset \Delta^+$. This shows that the set $\mathcal{I}_{ab}$ of the abelian ideals of $\mathfrak{b}$ is a finite set, and can be regarded as a graded poset, the poset structure given by the mutual inclusion of the ideals, and the grading given by the dimension of the ideals. The poset $\mathcal{I}_{ab}$ has a unique minimum which is the zero ideal. This shows that the problem of studying these abelian ideals, their dimensions and inclusions can be translated into a combinatorial problem. One of the first and surprising results around this topic is reported by Kostant in [12] and attributed to Peterson. Indeed, he counted the number of abelian ideal of $\mathfrak{b}$, i.e. the cardinality of the poset, in an uniform way providing a closed and very elegant formula. He found a one-to-one correspondence between $\mathcal{I}_{ab}$ and a certain set of combinatorial items. Let $\Pi = \{\alpha_1, \cdots, \alpha_n\}$ be the set of simple roots in Δ^+ and define $V = \mathfrak{h}^*_{\mathbb{R}} = \oplus_{i=1}^n \mathbb{R}\alpha_i$, and $(\cdot, \cdot)$ for the symmetric positive bilinear form induced on V by the Killing form. Extend V and its product to $\hat{V} = V \oplus \mathbb{R}\delta \oplus \mathbb{R}\lambda$ with $(\delta, \delta) = (\lambda, V) = (\lambda, \lambda) = (\lambda, V) = 0$ and $(\lambda, \delta) = 1$. The affine root system is given by $\widehat{\Delta} = \Delta + \mathbb{Z}\delta$, while the positive roots are given by $\widehat{\Delta}^+ = (\Delta^+ + \mathbb{N}\delta) \cup (-\Delta^+ + \mathbb{Z}_+\delta)$. Moreover if we write θ for the highest root in Δ and with $\alpha_0 = \delta - \theta$, we can consider the set of affine simple roots $\widehat{\Pi} = \{\alpha_0, \alpha_1, \cdots, \alpha_n\}$ and the associated Coxeter group $\widehat{W}$ generated by the reflections s_{α_i} with $\alpha_i \in \widehat{\Pi}$. There is a natural isomorphism between $\widehat{W}$ and W_{af}, the group of affine transformations of V generated by the reflections with respect to the hyperplanes of V given by $H_{\alpha,k} = \{x \in V | (x, \alpha) = k\}$ for $\alpha \in \Delta^+$ and $k \in \mathbb{Z}$. Let A be the foundamental alcove, i.e. the polytope bounded by the hyperplanes

$$A = \{x \in V | (x, \alpha) > 0 \ \forall \alpha \in \Pi, (x, \theta) < 1\}.$$

For $w \in \widehat{W}$ let's define the inversion set

$$N(w) = \{\alpha \in \widehat{\Delta}^+ | w^{-1}(\alpha) \in -\widehat{\Delta}^+\}.$$

These sets, key parts in our work, have remarkable and well-known properties [3] that we will discuss in more detail in Chapter 1.3.

1) $N(w_1) = N(w_2) \iff w_1 = w_2$ for every $w_1, w_2 \in \widehat{W}$.

2) They are biconvex, i.e. both $N(w)$ and its complementary set $\widehat{\Delta}^+ \setminus N(w)$ are closed with respect to the sum in the root system.

3) Unless there is a connected component of the Dynkin diagram of $\mathfrak{g}$ of type A_1, every subset of $\widehat{\Delta}^+$ finite and biconvex is of the type $N(w)$ for a unique $w \in \widehat{W}$.

Definition 1.1.1. We call **minuscule** the elements $w \in \widehat{W}$ such that

$$N(w) = \{\delta - \gamma | \gamma \in S\}$$

for some $S \subset \Delta^+$. We write $\mathcal{W}^{ab}$ for the set of minuscule elements.

We have the following key proposition due to Peterson; we give a proof proposed in [3] by Cellini and Papi.

Proposition 1.1.3. *The map $\mathcal{I}_{ab} \to \mathcal{W}^{ab}$ given by*

$$\mathfrak{i} = \bigoplus_{\alpha \in \Phi} L_\alpha \mapsto w_\mathfrak{i}$$

where $w_\mathfrak{i}$ is the unique element such that $N(w_\mathfrak{i}) = \{\delta - \Phi\}$, is an order preserving bijection between the poset of abelian ideals of $\mathfrak{b}$ and the poset $\mathcal{W}^{ab}$ of the minuscule elements endowed with the weak Bruhat order ($w_i \leq w_j \iff N(w_i) \subset N(w_j)$).

Proof. Let $\mathfrak{i} = \bigoplus_{\alpha \in \Phi} L_\alpha$ be an abelian ideal in $\mathfrak{b}$. Define

$$N_\mathfrak{i} = \bigcup_{k \geq 1} (-\Phi^k + k\delta)$$

where $\Phi^k = (\Phi^{k-1} + \Phi) \cap \Delta$. We see that $\Phi^k = 0$ for $k \geq 2$ since $\mathfrak{i}$ is abelian. This implies that $N_\mathfrak{i}$ is closed, and also that its complementary set is closed, indeed otherwise we can find $\alpha_1, \alpha_2 \in \Delta^+ \setminus \Phi$ such that $\alpha_1 - \alpha_2 \in \Phi$, against the fact that $\mathfrak{i}$ is an ideal. Thanks to property (3) of the inversion sets, there exists $w \in \widehat{W}$ such that $N(w) = N_\mathfrak{i}$. The converse is trivial. □

The problem of studying the poset of the abelian ideals of $\mathfrak{b}$ has been completely transformed into a combinatorial one, namely the problem of studying the structure of the poset $\mathcal{W}^{ab}$. Define the polytope

$$D = \bigcup_{w \in \mathcal{W}^{ab}} wA.$$

It turns out it is just $2A$, i.e. twice the fundamental alcove [2]. This simplex is paved by $2^{rk(\mathfrak{g})}$ tiles each of them congruent to A, moreover the action is faithful, giving as the remarkable result:

Theorem 1.1.4 (Peterson)**.** *The number of abelian ideals of $\mathfrak{b}$ is $2^{rk(\mathfrak{g})}$.*

Let's see an example of this.

Example 1.1.2. Consider the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$ and its root system A_2 generated by its simple roots α and β. Let's consider also $\Delta^+ = \{\alpha, \beta, \alpha + \beta\}$ and $\mathfrak{b} = \mathfrak{h} \oplus L_\alpha \oplus L_\beta \oplus L_{\alpha+\beta}$. The abelian ideals of $\mathfrak{b}$ are clearly just 0, and those corresponding to the sets of roots $\{\alpha+\beta\}$, $\{\alpha+\beta,\beta\}$ and $\{\alpha+\beta,\alpha\}$. The ideal $\mathcal{I}_{\alpha+\beta}$ corresponds to $\{\delta-\alpha-\beta\} = N(s_0)$, so $\mathcal{I}_{\alpha+\beta} \mapsto s_0$. The ideal $\mathcal{I}_{\alpha+\beta,\alpha}$ corresponds to $\{\delta-\alpha-\beta, \delta-\alpha\} = N(s_0 s_\beta)$, so $\mathcal{I}_{\alpha+\beta,\alpha} \mapsto s_0 s_\beta$. The ideal $\mathcal{I}_{\alpha+\beta,\beta}$ corresponds to $\{\delta-\alpha-\beta, \delta-\beta\} = N(s_0 s_\alpha)$, so $\mathcal{I}_{\alpha+\beta,\beta} \mapsto s_0 s_\alpha$. Moreover $D = A \cup s_0 A \cup s_0 s_\alpha A \cup s_0 s_\beta A = 2A$ as shown in Figure 1.1. Of course as expected from Peterson's Theorem, the number of abelian ideals is $2^{rk(\mathfrak{sl}_3)} = 2^2 = 4$.

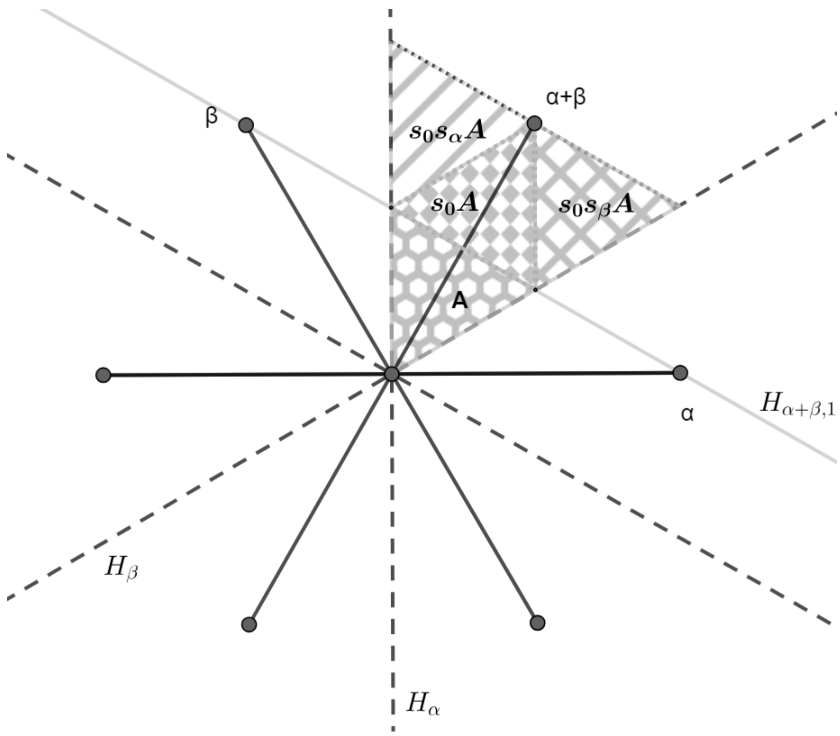

Figure 1.1: A_2 - Alcove A and simplex $D = 2A$.

1.1.3 A complete solution

A breakthrough to the problem of studying the poset of minuscule elements of the affine Weyl group of a semisimple Lie algebra was found by Panyushev in 2002. Indeed in [16] he showed a one-to-one correspondence between the set $\mathcal{I}_{max}$ of maximal minuscule elements of the poset, i.e. the maximal abealian ideals of the

Borel subalgebra $\mathfrak{b}$, and the set Π_l of long simple roots in Δ. Let's write $\mathcal{I}_w$ for the abelian ideal in $\mathfrak{b}$ associated to the minuscule element w, Δ_l^+ for the set of long positive roots, and $\mathcal{I}_{ab}^0 = \mathcal{I}_{ab} \setminus \{0\}$. Then the following result holds

Theorem 1.1.5 (Panyushev)**.** *The map $\tau : \mathcal{I}_{ab}^0 \to \Delta_l^+$ given by*

$$\mathcal{I}_w \mapsto w^{-1}(\alpha_0) + \delta$$

is well defined and surjective. If $\tau(\mathcal{I}_w)$ is not simple then $\mathcal{I}_w$ is not maximal. Moreover if $\mu \in \Delta_l^+$, the fiber $\tau^{-1}(\mu)$ is a complete subposet of $\mathcal{I}_{ab}^0$, meaning that if $w_1 < w < w_2$ and $w_1, w_2 \in \tau^{-1}(\mu)$ then also $w \in \tau^{-1}(\mu)$, and has a unique minimum and a unique maximum.

An immediate consequence of this theorem is the following remarkable corollary:

Corollary 1.1.6. *The restriction of τ to $\mathcal{I}_{max}$ is a bijection*

$$\overline{\tau} : \mathcal{I}_{max} \to \Pi_l$$

between the set maximal abelian ideals of $\mathfrak{b}$ and the long positive simple roots of Δ.

Let's first see an example.

Example 1.1.3. Consider the algebra $\mathfrak{sl}_4(\mathbb{C})$ and the associated root system $\Delta^+ = \{\alpha, \beta, \gamma, \alpha+\beta, \beta+\gamma, \alpha+\beta+\gamma\}$, with simple roots $\Pi = \{\alpha, \beta, \gamma\}$. The set of non zero abelian ideals of $\mathfrak{b}$ is

$$\mathcal{I}_{ab}^0 = \{\{\alpha+\beta+\gamma\}, \{\alpha+\beta+\gamma, \alpha+\beta\}, \{\alpha+\beta+\gamma, \beta+\gamma\}, \{\alpha+\beta+\gamma, \alpha+\beta, \alpha\},$$
$$\{\alpha+\beta+\gamma, \beta+\gamma, \alpha+\beta\}, \{\alpha+\beta+\gamma, \beta+\gamma, \gamma\}, \{\alpha+\beta+\gamma, \beta+\gamma, \alpha+\beta, \beta\}\}.$$

The corresponding minuscule elements are

$$\mathcal{W}^{ab} = \{\{s_0\}, \{s_0 s_\gamma\}, \{s_0 s_\alpha\}, \{s_0 s_\gamma s_\beta\}, \{s_0 s_\alpha s_\gamma\}, \{s_0 s_\alpha s_\beta\}, \{s_0 s_\alpha s_\gamma s_0\}\}.$$

The corresponding images through the map τ are given by

$$\tau(\mathcal{W}^{ab}) = \{\{\alpha+\beta+\gamma\}, \{\alpha+\beta\}, \{\beta+\gamma\}, \{\alpha\}, \{\beta\}, \{\gamma\}, \{\beta\}\}.$$

The situation is shown in Figure 1.2. Note that the 3 maximal abelian ideals correspond to the 3 long simple roots α, β and γ.

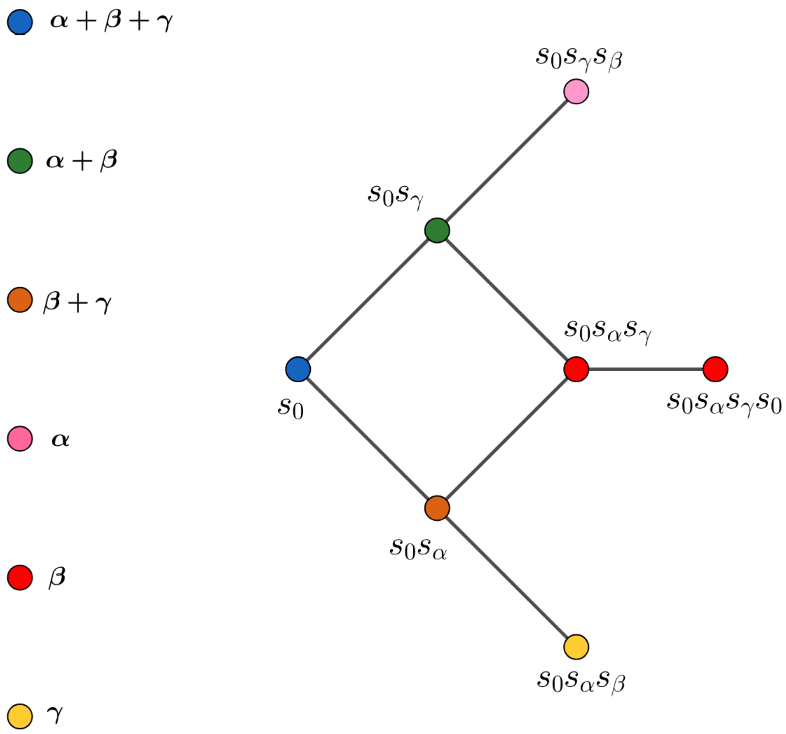

Figure 1.2: $\mathfrak{sl}_4$ - Decomposition of $\mathcal{W}^{ab}$.

Panyushev also found the dimension of a minimal subalgebra in a fiber $\tau^{-1}(\mu)$ to be equal to $1 + (\rho, \theta^\vee - \mu^\vee)$, and proved conditions controlling the cardinality of a fiber. He also described, using a case by case argument, the structure of the fiber as a poset.

Definition 1.1.4. If $A \subset \widehat{\Pi}$ we denote by $W(A)$ the subgroup of $\widehat{W}$ generated by $s_\alpha, \alpha \in A$. Given $\mu \in \Delta^+$ set

$$\widehat{\Pi}_\mu = \widehat{\Pi} \cap \mu^\perp, \quad \widehat{W}_{\perp\mu} = W(\widehat{\Pi}_\mu),$$
$$\widehat{\Pi}_{\mu,\delta+\alpha_0} = \widehat{\Pi}_\mu \setminus \{\alpha_0\},$$
$$\widehat{W}_{\perp\mu,\delta+\alpha_0} = W(\widehat{\Pi}_{\mu,\alpha_0}).$$

Proposition 1.1.7. *When non empty, the fiber $\tau^{-1}(\mu)$ is isomorphic as a poset to the set of minimal right coset representatives $\widehat{W}_{\perp\mu,\delta+\alpha_0}\backslash\widehat{W}_{\perp\mu}$ equipped with the weak Bruhat order of $\widehat{W}$.*

More details about sets of minimal right coset representatives will be given in Chapter 1.3. New proofs of these results, and a general proof for the last proposition were given in [3] by Cellini and Papi in 2004. We won't prove these results here, because we will give a new proof in the next chapter, involving techniques also required in the main last chapter, in a simpler fashion for this classical case. Moreover they were able to describe the minimal and maximal elements of the fibers $\tau^{-1}(\mu)$, and gave another proof of the results by Suter [22] describing the dimension of the maximal abelian ideal $\mathfrak{m}(\mu)$ corresponding to the maximal element in $\tau^{-1}(\mu)$:

$$dim(\mathfrak{m}(\mu)) = g - 1 + \frac{1}{2}(|\langle\widehat{\Pi}_\mu\rangle| - |\langle\Pi_\mu\rangle|),$$

where $\langle\Pi_\mu\rangle$ is the root system generated by $\Pi_\mu = \Pi \cap \mu^\perp$ and g is the dual Coxeter number of Δ. They also found a uniform version of the Mal'cev's formulas [15] for the global maximal dimension of a commutative subalgebra in $\mathfrak{g}$, indeed if d denotes such dimension, we have

$$d = \dim(\mathfrak{m}(\bar{\mu})),$$

where $\bar{\mu}$ is a long simple root having maximal distance from α_0 in the Dynkin diagram of $\widehat{\Delta}$. Write $\mathfrak{m}_j(\mu)$ for a subalgebra in the fiber $\tau^{-1}(\mu)$ with distance $j-1$ from the minimum of the fiber according to the poset structure. For every h such that $1 \leq h \leq k(\mu)$ where $k(\mu)$ represents the position of the maximal element in the fiber, they associated a certain finite irreducible subsystem $\widehat{\Delta}_h(\mu)$ of $\widehat{\Delta}$ and proved that

$$\dim(\mathfrak{m}_j(\mu)) = g - 1 + \sum_{h=1}^{j-1}(g_h(\mu) - 1),$$

where $g_h(\mu)$ is the dual Coxeter number of $\widehat{\Delta}_h(\mu)$. What people had tried so far is to bring these remarkable results about $\mathcal{W}^{ab}$ and the abelian ideal of $\mathfrak{b}$ in the wider setting of $\mathbb{Z}_2$-graded Lie algebras as we will discuss in the next section.

1.2 Abelian subalgebras in $\mathbb{Z}_2$-graded Lie algebras

1.2.1 Motivations

Let $\mathfrak{g}$ be a semisimple finite dimensional complex Lie algebra and σ an indecomposable involution of $\mathfrak{g}$. Recall that σ is indecomposable if $\mathfrak{g}$ has no nontrivial σ-invariant ideals. Let $(\cdot,\cdot)$ be the Killing form of $\mathfrak{g}$. For $j \in \mathbb{Z}$ set $\bar{j} = j + 2\mathbb{Z}$, and let $\mathfrak{g}^{\bar{j}} = \{X \in \mathfrak{g} \mid \sigma(X) = (-1)^j X\}$, so that we have $\mathfrak{g} = \mathfrak{g}^{\bar{0}} \oplus \mathfrak{g}^{\bar{1}}$. Choose a basis of $\mathfrak{g}$ of eigenvectors of σ, $\{x_1, \ldots, x_N\}$. Then we see that

$$d = d_0 + d_1, \quad d_i = \frac{1}{2} \sum_{j : x_j \in \mathfrak{g}^{\bar{i}}} \epsilon(x_j) ad_{x_j}$$

$$C = C_0 + C_1, \quad C_i = \frac{1}{2} \sum_{j : x_j \in \mathfrak{g}^{\bar{i}}} x_j \cdot x_j'.$$

Note that C_0 is the Casimir operator of $\mathfrak{g}^{\bar{0}}$ w.r.t. the restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{g}^{\bar{0}}$. Similar to the results for the classical case, Panyushev in [17] showed the following link between the eigenvalues of this Casimir operator and the abelian subalgebras of $\mathfrak{g}^{\bar{1}}$.

Theorem 1.2.1 (Panyushev)**.** *If l_k is the maximal eigenvalue of C_0 acting on $\bigwedge^k \mathfrak{g}^{\bar{1}}$ then*

$$l_k \leq \frac{k}{2}.$$

Moreover the equality holds if and only if $\mathfrak{g}^{\bar{1}}$ contains a k-dimensional abelian subalgebra. In this case the eigenspace associated to l_k is generated by $\bigwedge^k \mathfrak{a}$ where $\mathfrak{a}$ runs over all the k-dimensional abelian subalgebras of $\mathfrak{g}^{\bar{1}}$.

As in the classical case, Panyushev showed as well that it is possible to restrict the attention to the abelian subalgebras of $\mathfrak{g}^{\bar{1}}$ which are $\mathfrak{b}^{\bar{0}}$-stable. Those recalled above were some of the results that created interest in these abelian subalgebras and made researchers start investigating their structure and properties. The problem became then to study $\mathcal{I}_{ab}^{\sigma}$, the set of abelian subalgebras of $\mathfrak{g}^{\bar{1}}$ stable under the action of $\mathfrak{b}^{\bar{0}}$. We will now look into a link between $\mathcal{I}_{ab}^{\sigma}$ and a subset of some Weyl group of a suitable Dynkin diagram.

1.2.2 A combinatorial problem

We let $\widehat{L}(\mathfrak{g},\sigma)$ be the affine Kac-Moody Lie algebra associated to σ in [8, Section 8.2]. Let $\mathfrak{h}_0$ be a Cartan subalgebra of $\mathfrak{g}^{\bar{0}}$. As shown in [8, Chapter 8], $\mathfrak{h}_0$ contains a regular element h_{reg} of $\mathfrak{g}$. In particular the centralizer $Cent(\mathfrak{h}_0)$ of $\mathfrak{h}_0$ in $\mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}$ and h_{reg} defines a set of positive roots in the set of roots of $(\mathfrak{g}, Cent(\mathfrak{h}_0))$ and a set Δ_0^+ of positive roots in the set Δ_0 of roots for $(\mathfrak{g}^{\bar{0}},\mathfrak{h}_0)$. Since σ fixes h_{reg}, we see that the action of σ on the positive roots defines, once Chevalley generators are fixed, a diagram automorphism η of $\mathfrak{g}$ that, clearly, fixes $\mathfrak{h}_0$. Set, using the notation of [8], $\widehat{\mathfrak{h}} = \mathfrak{h}_0 \oplus \mathbb{C}K \oplus \mathbb{C}d$. Recall that d is the element of $\widehat{L}(\mathfrak{g},\sigma)$ acting on $\widehat{L}(\mathfrak{g},\sigma) \cap (\mathbb{C}[t,t^{-1}] \otimes \mathfrak{g})$ as $t\frac{d}{dt}$, while K is a central element. Define $\delta' \in \widehat{\mathfrak{h}}^*$ by setting $\delta'(d) = 1$ and $\delta'(\mathfrak{h}_0) = \delta'(K) = 0$ and let $\lambda \mapsto \overline{\lambda}$ be the restriction map $\widehat{\mathfrak{h}} \to \mathfrak{h}_0$. There is a unique extension, still denoted by $(\cdot,\cdot)$, of the Killing form of $\mathfrak{g}$ to a nondegenerate symmetric bilinear invariant form on $\widehat{L}(\mathfrak{g},\sigma)$. Let $\nu : \widehat{\mathfrak{h}} \to \widehat{\mathfrak{h}}^*$ be the isomorphism induced by the form $(\cdot,\cdot)$, and denote again by $(\cdot,\cdot)$ the form induced on $\widehat{\mathfrak{h}}^*$. One has $(\delta',\delta') = (\delta',\mathfrak{h}_0^*) = 0$.

We let $\widehat{\Delta}$ be the set of $\widehat{\mathfrak{h}}$-roots of $\widehat{L}(\mathfrak{g},\sigma)$. We can choose as set of positive roots $\widehat{\Delta}^+ = \Delta_0^+ \cup \{\alpha \in \widehat{\Delta} \mid \alpha(d) > 0\}$. We let $\widehat{\Pi} = \{\alpha_0,\dots,\alpha_n\}$ be the corresponding set of simple roots. It is known that n is the rank of $\mathfrak{g}^{\bar{0}}$. Recall that any $\widehat{L}(\mathfrak{g},\sigma)$ is a Kac-Moody Lie algebra $\mathfrak{g}(A)$ defined by generator and relations starting from a generalized Cartan matrix A of affine type. These matrices are classified by means of Dynkin diagrams listed in [8]. Given a Dynkin diagram of type $X_N^{(k)}$ in the classification of affine Kac-Moody Lie algebras given in [8, pp.53-55] in table k with $k = 1,2,3$, it is possible to associate an automorhpism of $\mathfrak{g}$ to each $(n+1)$-tuple $s = (s_0,\dots,s_n)$ of non-negative coprime integers. We will say that this automorphism is of type $(s;k)$ and write $\sigma_{s,k}$. We can now recall Kac's classification of finite order automorphisms [8]:

Theorem 1.2.2.

a) If a_i denote the coefficients associated to the simple roots in the diagram of $X_N^{(k)}$, then the order of $\sigma_{s,k}$ is $m = k(\sum_{i=0}^n a_i s_i)$ and thus it's finite.

b) In the group of automorphisms of $\mathfrak{g}$, every element of order m is conjugated to some $\sigma_{s,k}$.

c) Two automorphisms $\sigma_{s,k}$ and $\sigma_{s',k'}$ are conjugated if and only if $k = k'$ and s can be transformed into s' by applying an automorphism of the Dynkin diagram of $X_N^{(k)}$.

In the case of our Lie algebra $\mathfrak{g}$ of type X_N endowed with a $\mathbb{Z}_2$-gradation, our automorphism is associated to an $(n+1)-$*tuple* $\{s_0, \dots, s_n\}$ and an integer $k = 1, 2$ such that $k(\sum_{i=0}^{n} a_i s_i) = 2$ with a_i the coefficients of the diagram $X_N^{(k)}$ in table k. There can be three cases depending on k and the number of s_i's that are not 0.

1) $k = 1$ and there are two indices $p \neq q$ such that $s_p = s_q = 1$ corresponding to coefficients $a_p = a_q = 1$, and $s_i = 0$ for every $i \neq p, q$.

2) $k = 1$ and there is an index p such that $s_p = 1$ and $a_p = 2$, and $s_i = 0$ for every $i \neq p$.

3) $k = 2$ and there is an index p such that $s_p = 1$ and $a_p = 1$, and $s_i = 0$ for every $i \neq p$.

The first case is called *Hermitian symmetric case*, while the other two are called *semisimple cases*. For a generic pair $(\mathfrak{g}, \sigma)$ of a semisimple Lie algebra of type X_N and its finite order automorphism we write $\widehat{\Delta}$ for the root system associated to the affine Kac-Moody algebra $\widehat{L}(\mathfrak{g}, \sigma)$ corresponding to the diagram $X_N^{(k)}$, and $\widehat{W}$ for the associated Weyl group. The set of simple roots $\widehat{\Pi} = \{\alpha_0, \dots, \alpha_n\}$ has $\Pi_0 = \{\alpha_i | s_i = 0\}$ as a subset corresponding to the root system of $\mathfrak{g}^{\bar{0}}$ and $\Pi_1 = \widehat{\Pi} \backslash \Pi_0$. In the root system $\widehat{\Delta}$ we define a σ-*height* in the following way. Given $\alpha \in \widehat{\Delta}$, we write $\alpha = \sum_{i=0}^{n} c_i \alpha_i$, then

$$h_\sigma(\alpha) = \sum_{i=0}^{n} c_i s_i,$$

and consider the sets $\widehat{\Delta}_i = \{\gamma \in \widehat{\Delta}^+ | h_\sigma(\gamma) = i\}$ for $i \in \mathbb{Z}$.

Definition 1.2.1. We call ***σ-minuscule*** the elements in $\widehat{W}$ such that $N(w) \subset \widehat{\Delta}_1$. We write $\mathcal{W}_\sigma^{ab}$ for the set of σ-minuscule elements, and we see it as a poset with the order given by the weak Bruhat order.

Cellini, Möseneder Frajria and Papi proved in [4] the link between this poset and the set of the $\mathfrak{b}^{\bar{0}}$-stable abelian subalgebras of $\mathfrak{g}^{\bar{1}}$:

Theorem 1.2.3 (Cellini-Möseneder Frajria-Papi). *Let $w \in \mathcal{W}_\sigma^{ab}$ and $N(\beta) = \{\beta_1, \ldots, \beta_k\}$. Then the map $\mathcal{W}_\sigma^{ab} \to \mathcal{I}_{ab}^\sigma$ defined by*

$$w \mapsto \bigoplus_{i=1}^{k} \mathfrak{g}^{\bar{1}}_{-\bar{\beta}_i}$$

is a poset isomorphism.

Once again the algebraic problem related to abelian subalgebras has been transformed into the combinatorial problem of studying the structure of the poset $\mathcal{W}_\sigma^{ab}$ and its elements. Let's first see an example of such a poset in the case of $G_2^{(1)}$.

Example 1.2.2. Consider the Lie algebra $\mathfrak{g}$ associated to the root system G_2. Write α and β for its simple roots, β is the long root. Fix the following $\mathbb{Z}_2$-gradation of $\mathfrak{g}$ given by

$$\mathfrak{g}^{\bar{0}} = L_\alpha \oplus L_{-\alpha} \oplus L_{3\alpha+2\beta} \oplus L_{-3\alpha-2\beta} \oplus \mathfrak{h}$$

$$\mathfrak{g}^{\bar{1}} = L_\beta \oplus L_{-\beta} \oplus L_{\beta+\alpha} \oplus L_{-\beta-\alpha} \oplus L_{\beta+2\alpha} \oplus L_{-\beta-2\alpha} \oplus L_{\beta+3\alpha} \oplus L_{-\beta-3\alpha}.$$

The corresponding diagram is $G_2^{(1)}$, the only one existing for the G type, and it's given in Figure 1.3. The only admissible $(n+1)$-tuple is $s = (0, 1, 0)$, so $\Pi_0 = \{\alpha_0, \alpha\}$

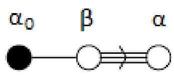

Figure 1.3: Dynkin diagram for $G_2^{(1)}$.

with $\alpha_0 = \delta - 3\alpha - 2\beta$, corresponding to $A_1 \oplus A_1$. The poset of abelian subalgebras of $\mathfrak{g}^{\bar{1}}$ and $\mathfrak{b}^{\bar{0}}$-stable is given in Figure 1.4. The elements with σ-height equal 1 are $\widehat{\Delta}_\sigma^1 = \{\beta, \beta+\alpha, \beta+2\alpha, \beta+3\alpha, \delta-3\alpha-\beta, \delta-2\alpha-\beta, \delta-\alpha-\beta, \delta-\beta\}$. The biconvex subsets contained in $\widehat{\Delta}_\sigma^1$ are $\{\beta\}, \{\beta\, \beta+\alpha\}, \{\beta, \delta-3\alpha-\beta\}, \{\beta, \delta-3\alpha-\beta, \beta+\alpha\}$. The corresponding σ-minuscule elements are given by $\{s_\beta, s_\beta s_\alpha, s_\beta s_0, s_\beta s_0 s_\alpha\} \subset \mathcal{W}_\sigma^{ab}$. As

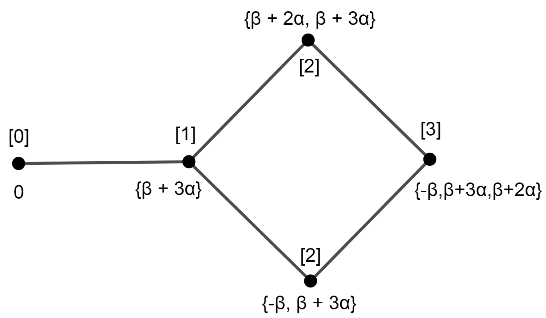

Figure 1.4: Abelian $\mathfrak{b}^{\bar{0}}$-stable subalgebras of $\mathfrak{g}^{\bar{1}}$.

we see, adding the unity 1 we get the same poset given by the subalgebras as in Figure 1.4.

In the same article [4] the authors compute the cardinality of the set $\mathcal{W}_\sigma^{ab}$ of σ-minuscule elements in a general way. Write W_σ for the Weyl group associated to the root system $\widehat{\Delta}_0$, and W_f for the Weyl group associated to the root system generated by $\Pi_f = \{\alpha_1, \dots, \alpha_n\}$. Consider the *Hermitian symmetric case.* We can assume that $p = 0$ so we may see W_σ as a subgroup of W_f. Write ℓ_σ for the connection index of W_σ, and ℓ_f for the connection index of W_f, then the following holds.

Proposition 1.2.4. *In the Hermitian symmetric case*

$$|\mathcal{W}_\sigma^{ab}| = \frac{|W_f|}{|W_\sigma|}\Big(1 + \frac{\ell_\sigma}{\ell_f}\Big).$$

In the *semisimple case* instead, write χ_ℓ for the truth function on $\widehat{\Delta}$ which is 1 if the argument is long and 0 otherwise, and L for the number of long simple roots in Π_f, then we have the following result.

Proposition 1.2.5. *In the semisimple case*

$$|\mathcal{W}_\sigma^{ab}| = a_0(\chi_\ell(\alpha_p) + 1)k^{n-L}\frac{|W_f|}{|W_\sigma|} - \chi_\ell(\alpha_p).$$

Both these formulas are proven considering again the fundamental alcove A as in the classical case, computing the volume of the polytope

$$D_\sigma = \bigcup_{w \in W^\sigma_{ab}} wA$$

and the ratio between D_σ and A.

1.2.3 Maximal elements

In a later work [6], Cellini, Möseneder Frajria, Papi and Pasquali made a breakthrough in the study of the poset $\mathcal{W}^{ab}_\sigma$, fully describing its maximal elements and the dimensions of the corresponding maximal $\mathfrak{b}^{\bar{0}}$-stable abelian subalgebras of $\mathfrak{g}^{\bar{1}}$. Note that Π_0 could be a disconnected subdiagram of $\widehat{\Pi}$. Let's write $\Sigma|\Pi_0$ to mean that Σ is a connected component of Π_0, θ_Σ for its highest root, and let a be the square of the norm of the length of a root of maximal length in $\widehat{\Delta}$. Also set $\delta = \sum_{i=0}^n a_i\alpha_i$ and $\Pi_1 = \widehat{\Pi} \setminus \Pi_0$. We define $\widehat{\Pi}_0^* = \Pi_0 \cup \{r\delta - \theta_\Sigma | a \leq 2||\theta_\Sigma||^2\}$ and $\Phi_\sigma = \widehat{\Pi}_0^* \cup \{\alpha + r\delta | \alpha \in \Pi_1, \alpha \text{ is long}\}$. In particular it is shown that the polytope D_σ can be obtained as the intersection of the hyperplanes corresponding to the roots of Φ_σ. Finally define the set of walls

$$\mathcal{M}_\sigma = \Phi_\sigma \setminus (\widehat{\Pi} \cap \Phi_\sigma).$$

The following proposition was the starting point for the study of some special subsets of $\mathcal{W}^{ab}_\sigma$, which represent a core element to describe the maximal elements of $\mathcal{W}^{ab}_\sigma$.

Proposition 1.2.6. *If $w \in \mathcal{W}^{ab}_\sigma$ is maximal, then there exist $\alpha \in \widehat{\Pi}$ and $\mu \in \mathcal{M}_\sigma$ such that $w(\alpha) = \mu$.*

This proposition makes clear that in order to study the maximal elements in $\mathcal{W}^{ab}_\sigma$, the main point was to study the following subposets: given $\alpha \in \widehat{\Pi}$ and $\mu \in \mathcal{M}_\sigma$, define

$$\mathcal{I}_{\alpha,\mu} = \{w \in \mathcal{W}^{ab}_\sigma \mid w(\alpha) = \mu\}. \tag{1.1}$$

Let's first have a look to an example to have a clearer picture of the situation.

Example 1.2.3. Let's consider again G_2. D_σ is the polytope given in Figure 1.5. We also have that $\Phi_\sigma = \{\alpha_0, \alpha, 2\beta + 3\alpha, \delta + \beta\}$, note that the hyperplanes of reflection represented by its elements mark the perimeter of D_σ. We also have that $\mathcal{M}_\sigma = \{2\beta+3\alpha, \delta+\beta\}$, and recall that the unique maximal σ-minuscule element is $s_\beta s_0 s_\alpha$. Then if we take $\beta \in \widehat{\Pi}, \delta + \beta \in \mathcal{M}_\sigma$ we find $s_\beta s_0 s_\alpha(\beta) = \delta + \beta$, and so $s_\beta s_0 s_\alpha \in \mathcal{I}_{\beta,\delta+\beta}$.

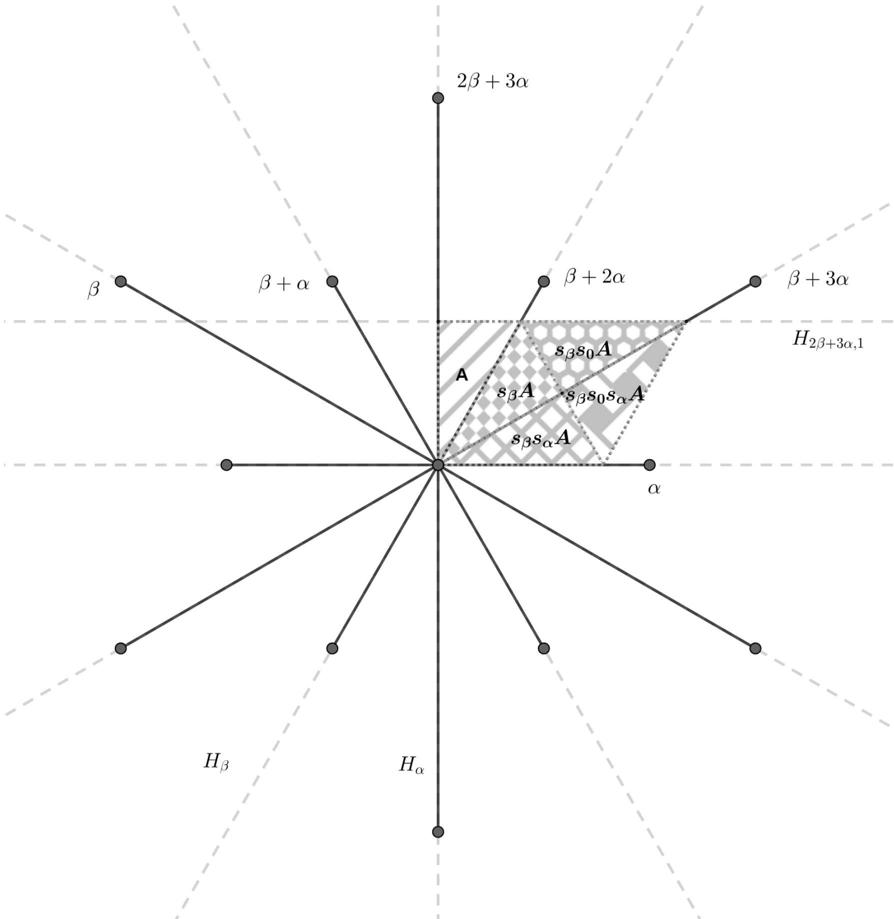

Figure 1.5: $G2$ - Alcove A and D_σ.

In [6] the authors find conditions under which the posets $\mathcal{I}_{\alpha,\mu}$ are non empty. They showed that when non empty, the posets $\mathcal{I}_{\alpha,\mu}$ have a unique minimum element, they are complete, and are isomorphic to the set of minimal right coset representatives for a suitable pair of subgroups of $\widehat{W}$. We won't give more details here because these results are proved again in Chapter 3 in a more general setting. The structure of their intersections is also given in the article. As a main result they gave a parametrization of all the maximal $\mathfrak{b}^{\bar{0}}$-stable abelian subalgebras of $\mathfrak{g}^{\bar{1}}$, and found formulas to compute their dimensions. In order to recall the main result we need some definitions. If S is a connected subset of the set of simple roots, we denote by S_ℓ the set of elements of S of the same length of its highest root θ_S.

Definition 1.2.4. A real root α is *noncompact* if $\mathfrak{g}_{\bar{\alpha}} \subset \mathfrak{g}^{\bar{1}}$, *compact* if $\mathfrak{g}_{\bar{\alpha}} \subset \mathfrak{g}^{\bar{0}}$, and *complex* otherwise. We say that a root is of *type 1* if it is long and non complex, of *type 2* otherwise. We also write Π_1^1 for the roots of type 1 in Π_1.

Definition 1.2.5. Let $\Sigma|\Pi_0$ and consider the subgraph of $\widehat{\Pi}$ given by the vertices $\{\alpha \in \widehat{\Pi} | (\alpha, \theta_\Sigma) \leq 0\}$. We define $A(\Sigma)$ to be the union of the connected components of this subgraph that contain at least one simple root from Π_1. Moreover we define

$$\Gamma(\Sigma) = A(\Sigma) \cap \Sigma.$$

Note that if $|\Pi_1| = 1$, then $A(\Sigma)$ is connected. Let's also recall the following useful proposition.

Proposition 1.2.7. *[6, Lemma 4.4] Assume $\Sigma|\Pi_0$, $k\delta - \theta_\Sigma \in \mathcal{M}_\sigma$, $\alpha \in \widehat{\Pi}$, and $\|\alpha\| = \|\theta_\Sigma\|$.*

1. *If θ_Σ is of type 1, let u_α^Σ be the element of minimal length such that $u_\alpha^\Sigma(\alpha) = k\delta - \theta_\Sigma$ $\alpha \in A(\Sigma)$. Then $u_\alpha^\Sigma \in \mathcal{W}_\sigma^{ab}$.*
2. *If θ_Σ is of type 2, $\alpha \in \Sigma$, v_α is the element of minimal length in $W(\Sigma)$ such that $v_\alpha(\alpha) = \theta_\Sigma$, and s is the element of minimal length in $\widehat{W}$ such that $s(\theta_\Sigma) = k\delta - \theta_\Sigma$, then $sv_\alpha \in \mathcal{W}_\sigma^{ab}$. Moreover, $\ell(sv_\alpha) = \ell(s) + \ell(v_\alpha)$ and sv_α is the element of minimal length in $\widehat{W}$ that maps α to $k\delta - \theta_\Sigma$.*

The main goal of [6] was to determine a parameter space for maximal $\mathfrak{b}^{\bar{0}}$ -stable abelian subalgebras of $\mathfrak{g}^{\bar{1}}$. The following main result holds.

Theorem 1.2.8. *The maximal $\mathfrak{b}^{\bar{0}}$-stable abelian subalgebras of $\mathfrak{g}^{\bar{1}}$ are parametrized by the set*

$$\mathcal{M} = \left(\bigcup_{\substack{\Sigma|\Pi_0 \\ \Sigma \textit{ of type } 1}} \Gamma(\Sigma)_\ell \right) \cup \left(\bigcup_{\substack{\Sigma|\Pi_0 \\ \Sigma \textit{ of type } 2}} \Sigma_\ell \right) \cup \left(\bigcup_{\substack{\Sigma,\Sigma'|\Pi_0, \Sigma \prec \Sigma' \\ \Sigma,\Sigma' \textit{ of type } 1}} \Sigma_\ell \times \Sigma'_\ell \right) \cup \Pi_1^1.$$

1.3 Results on Coxeter groups

We collect here some well known facts about several tools involved in our work.

1.3.1 Combinatoric of inversion sets

Recall that we define for $w \in \widehat{W}$ its inversion set

$$N(w) = \{\alpha \in \widehat{\Delta}^+ | w^{-1}(\alpha) \in -\widehat{\Delta}^+\}.$$

For a real root $\alpha \in \widehat{\Delta}^+$ we write s_α for the associated reflection. For a simple root α_i we write s_i in place of s_{α_i}. We present the most important facts, that are proven in [3]:

(1) $N(w_1) = N(w_2) \iff w_1 = w_2$.

(2) If $w = s_{i_1} \cdots s_{i_m}$ is in reduced form, then

$$N(w) = \{\alpha_{i_1}, s_{i_1}(\alpha_{i_2}), \ldots, s_{i_1} \cdots s_{i_{m-1}}(\alpha_m)\}.$$

Moreover if $\tau_j = s_{i_1} \cdots s_{i_{j-1}}(\alpha_j)$ for $1 \leq j \leq m$, then

$$w = s_{\tau_m} \cdots s_{\tau_1}.$$

(3) $N(w)$ is biconvex, which means that both $N(w)$ and its complementary set $\widehat{\Delta}^+ \setminus N(w)$ are closed with respect to the sum in $\widehat{\Delta}^+$. Vice versa unless there is a connected component of the Dynkin diagram of $\mathfrak{g}$ of type A_1, every subset of $\widehat{\Delta}^+$ finite and biconvex is of the type $N(w)$ for a unique $w \in \widehat{W}$.

(4) Let $\leq$ be the weak left Bruhat order, i.e. $w_1 \leq w_2$ if there is a reduced form for w_1 which is the initial part of a reduced form for w_2. Then the following holds

$$w_1 < w_2 \iff N(w_1) \subset N(w_2).$$

(5) Set $N^{\pm}(w) = N(w) \cup -N(w)$. Then, $N^{\pm}(w_1 w_2) = N^{\pm}(w_1 w_2) \dotplus w_1(N^{\pm}(w_2))$ with $\dotplus$ used to denote the symmetric difference. The following facts are equivalent:

(a) $N(w_1 w_2) = N(w_1) \cup w_1(N(w_2))$,

(b) $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$,

(c) $w_1(N(w_2)) \subset \widehat{\Delta}^+$.

We can define left and right descent sets for any $w \in \widehat{W}$ as follows:

$$L(w) = \{\alpha \in \widehat{\Pi} | \ell(s_\alpha w) < \ell(w)\},$$
$$R(w) = \{\alpha \in \widehat{\Pi} | \ell(w s_\alpha) < \ell(w)\}.$$

It can be proved that $L(w) = \widehat{\Pi} \cap N(w)$ and $R(w) = \widehat{\Pi} \cap N(w^{-1})$.

Remark 1.3.1 (A remark on notation)**.** Coxeter groups will play a major role in the following. However, to avoid overloading notation, we will not fix once for all the notation for the simple reflections. So we will freely use notation as $u = u_1 \cdots u_n$, $u = s_1 \cdots s_k$ and so on to denote reduced expressions.

1.3.2 Reflection subgroups and coset representatives

These results are a key component to understand the structure of the posets $\mathcal{I}_{\alpha,\mu}$ in both the classical and the $\mathbb{Z}_2$-graded case. Let G be a reflection group with S as a set of generating simple reflections and let ℓ be the associated length function. Let R be the associated root system, Π_R a set of simple roots and R^+ the set of positive roots. Given a subgroup G' of G generated by reflections and considering the subset R' of R made of roots α such that $s_\alpha \in G'$, it can be proved that R' is a root system as well, and that a set of simple roots is given by

$$\Pi_{R'} = \{\alpha \in R^+ | N(s_\alpha) \cap R' = \{\alpha\}\}$$

with corresponding set of positive roots given by $R' \cap R^+$. If $g \in G$ we call $w \in G'g$ a minimal right coset representative if among the elements in $G'g$ it is of minimal length. It is known [7] that such an element is unique for every $G'g$ and it is characterized by the property that

$$w^{-1}(\alpha) > 0 \quad \forall \alpha \in R'^+.$$

We write $G' \backslash G$ for the set of minimal right coset representatives and we see it as a poset with the induced partial order given by the weak Bruhat order on G. Take a root $\alpha \in R$ and consider G' the stabilizer of α in G, then the minimal right coset representative for $G'g$ is the unique minimal length element that maps $g^{-1}\alpha$ to α, and it's characterized by

$$w^{-1}(\beta) > 0 \quad \forall \beta \in R^+ \text{orthogonal to } \alpha.$$

If $\Pi_{R'} \subseteq \Pi_R$ we say that G' is a standard parabolic subgroup, and if $g \in G$ and w is the minimal right coset representative of $G'g$ we have that $g = g'w$ with $g' \in G'$ and $\ell(g) = \ell(g') + \ell(w)$, and $N(g) \cap R' = N(g')$. It is also known that in this setting

$$G' \backslash G = \{w \in G | L(w) \subseteq \Pi_R \setminus \Pi_{R'}\}.$$

When G is a finite group then $G' \backslash G$ has a unique minimum and a unique maximum, in particular the identity 1 is the minimum and $w_0' w_0$ is the unique maximum, where w_0 is the longest element of G and w_0' is the longest element of G'. Its length is given by

$$\ell(w_0' w_0) = |\Delta^+(R)| - |\Delta^+(R')|.$$

Chapter 2

The case of abelian ideals: a new proof

2.1 Results

In this chapter we want to provide new proofs to the main results from Panyushev in [16] and from Cellini and Papi in [3]. This also serves to make the reader familiar with some of the techniques that will be used in the next chapter. Let $\mathfrak{g}$ be a simple Lie algebra of type X_n. Write Δ for the associated root system with $\Pi = \{\alpha_1, \dots, \alpha_n\}$ as the set of simple roots, and W for its Weyl group. Consider the corresponding affine Dynkin diagram $X_n^{(1)}$ as in [8] and write $\widehat{\Delta}$ for the associated root system with $\widehat{\Pi} = \{\beta, \alpha_1, \dots, \alpha_n\}$ as the set of simple roots, and $\widehat{W}$ for its Weyl group. Recall that $\delta = \beta + \theta$ where θ is the highest root in Δ. We define the *α-height* $c_\alpha(\gamma)$ for a simple root $\alpha \in \widehat{\Pi}$ of a root $\gamma \in \widehat{\Delta}$ in this way: if $\gamma = \sum_{\tau \in \widehat{\Pi}} b_\tau \tau$, then $c_\alpha(\gamma) = b_\alpha$. Also recall the main definition

Definition 2.1.1. We call **minuscule** the elements $w \in \widehat{W}$ such that

$$N(w) = \{\delta - \gamma | \gamma \in S\}$$

for some $S \subset \Delta^+$. We write $\mathcal{W}^{ab}$ for the set of minuscule elements.

Note that this is equivalent to say that w is minuscule iff for every $\tau \in N(w)$ we have $0 < \tau < \delta$ and $c_\beta(\tau) = 1$. Note that $\tau < \delta$ can be dropped because if $x \in \Delta^+$ then $w^{-1}(\delta + x) = \delta + w^{-1}(x) > \delta > 0$. Clearly also $0 < \tau$ can be dropped. In the end w is minuscule iff for every $\tau \in N(w)$ we have $c_\beta(\tau) = 1$. For a given root $\gamma \in \widehat{\Delta}^+$ we define the set

$$\mathcal{I}_{\gamma,\delta+\beta} = \{w \in \mathcal{W}^{ab} | w(\gamma) = \delta + \beta\}.$$

Definition 2.1.2. We call **rootlet** any root $\gamma \in \widehat{\Delta}$ such that $\gamma = w^{-1}(\delta + \beta)$ for some $w \in \mathcal{W}^{ab}$.

In other words a rootlet is a root $\gamma \in \widehat{\Delta}$ such that $\mathcal{I}_{\gamma,\delta+\beta} \neq \emptyset$. We decompose $\mathcal{W}^{ab}$ as the disjoint union of possibly empty sets

$$\mathcal{W}^{ab} = \bigsqcup_{\gamma \in \widehat{\Delta}^+} \mathcal{I}_{\gamma,\delta+\beta}.$$

Note that it's enough to use $\gamma \in \widehat{\Delta}^+$ because if $w \in \mathcal{W}^{ab}$ then $\gamma = w^{-1}(\delta + \beta) > 0$ since $c_\beta(\delta + \beta) = 2 \neq 1$. We collect what we are going to prove in the next theorem and corollary, then we go through the required proofs.

Theorem 2.1.1. *$\mathcal{I}_{\gamma,\delta+\beta}$ is non empty if and only if $\gamma \in \Delta_\ell^+$ or $\gamma = \delta + \beta$. When non empty, it is a connected subposet, meaning that if $w_1 < w < w_2$ and $w_1, w_2 \in \mathcal{I}_{\gamma,\delta+\beta}$ then also $w \in \mathcal{I}_{\gamma,\delta+\beta}$. Moreover it has a unique minimum and a unique maximum, and it is isomorphic as a poset to the set of minimal right coset representatives $\widehat{W}_{\perp\gamma,\delta+\beta} \backslash \widehat{W}_{\perp\gamma}$ equipped with the weak Bruhat order of $\widehat{W}$.*

Corollary 2.1.2. *There is a one-to-one correspondence between maximal elements of $\mathcal{W}^{ab}$ and long simple roots of Δ.*

2.2 Proof of the results

In every case except for A_n, write α_β for the unique simple root in $\widehat{\Pi}$ connected to β. For A_1 there is nothing to prove. For A_n with $n > 1$ there are two simple roots connected to β in the Dynkin diagram, let's say α_1 and α_n. In this case define for every $\gamma \in \widehat{\Delta}$, $c_{\alpha_\beta}(\gamma) := c_{\alpha_1}(\gamma) + c_{\alpha_n}(\gamma)$; the following results hold in this case as well. Note that $c_{\alpha_\beta}(\theta) = 2$, that is easily seen because $s_\beta(\theta) = \delta + \beta$ and β is a long root. For $x, y \in \widehat{\Delta}^+$ we write $x \subseteq y$ iff writing $x = \sum_{\tau \in \widehat{\Pi}} b_\tau \tau$ and $y = \sum_{\tau \in \widehat{\Pi}} c_\tau \tau$ we have $b_\tau \leq c_\tau$ for every $\tau \in \widehat{\Pi}$. Notice that $x \subseteq \delta$ is equivalent to $x \leq \delta$ for every $x \in \widehat{\Delta}^+$.

Lemma 2.2.1. *Let $\tau \in \Delta$ with $c_{\alpha_\beta}(\tau) = 2$, then $\tau = \theta$.*

Proof. $s_\beta(\tau) = \tau + 2\beta \supseteq \delta + \beta$ because $c_\beta(\tau + 2\beta) = 2$, so $\tau + \beta \supseteq \delta$ implies $\tau = \theta$. □

Lemma 2.2.2. *Let $\gamma \in \Delta_l^+$, then $\mathcal{I}_{\gamma, \delta+\beta} \neq \emptyset$.*

Proof. Let $u_\gamma \in W$ be an element of shortest length such that $u_\gamma(\gamma) = \theta$. Then $s_\beta u_\gamma(\gamma) = \delta + \beta$. We want to prove that $s_\beta u_\gamma$ is minuscule. Let $\tau \in N(s_\beta u_\gamma(\gamma)) = \{\beta\} \cup s_\beta(N(u_\gamma))$; then we only have to check that $c_\beta(\tau) = 1$. If $\gamma = \theta$ there is nothing to prove. Write $u_\gamma = s_1 \dots s_n$ in reduced form, with $s_i = s_{\alpha_i}$, and $q_k = s_1 \dots s_{k-1}(\alpha_k)$. Note that, for $1 \leq k \leq n$, we have $q_k > 0$ and thus $c_{\alpha_\beta}(q_k) \geq 0$. Moreover $q_k \in \Delta$, so $c_{\alpha_\beta}(q_k) \leq 2$. We want to prove that $c_{\alpha_\beta}(q_k) = 1$ for all k. Write

$$s_k(s_{k+1} \dots s_n(\gamma)) = s_{k+1} \dots s_n(\gamma) + a_k \alpha_k,$$

so that, multiplying by $s_1 \cdots s_k$, we have

$$s_1 \dots s_{k-1} s_{k+1} \dots s_n(\gamma) = \theta - a_k q_k.$$

Notice that $a_k \neq 0$ by minimality of u_γ. Moreover since $\theta - a_k q_k \in \Delta$ we have $a_k > 0$ by maximality of θ, and also $c_{\alpha_\beta}(q_k) \neq 0$, thanks to 2.2.1 because $\theta - a_k q_k \subset \theta$. If $c_\beta(q_k) = 2$, by lemma 2.2.1, $q_k = \theta$. So $s_1 \cdots s_{k-1}(\alpha_k) = \theta$, but also $s_1 \cdots s_{k-1}(s_k \cdots s_n)(\gamma) = \theta$, and combining these relations we get $s_k \cdots s_n(\gamma) = \alpha_k$. Finally, applying s_k, we have

$$-\alpha_k = s_{k+1} \cdots s_n(\gamma) \geq \gamma > 0$$

since $a_i > 0$ for every i, but this is absurd. We conclude that $c_{\alpha_\beta}(q_k) = 1$. □

Lemma 2.2.3. *Let $\gamma \in \Delta_l^+$, then $s_\beta u_\gamma = \min \mathcal{I}_{\gamma, \delta - \theta_\Sigma}$.*

Proof. Let $w \in \mathcal{I}_{\gamma,\delta+\beta}$ be minimal, $w \neq s_\beta u_\gamma$, and write $s_\beta u_\gamma = u_1 \cdots u_n$ in reduced form ($u_1 = s_\beta$). Consider $N(s_\beta u_\gamma) \cap N(w) = \Psi$. If $\alpha, \tau \in \Psi$ then $\alpha, \tau \in N(s_\beta u_\gamma)$ and $\alpha, \tau \in N(w)$; since these are inversion sets we have $\alpha + \tau \in N(s_\beta u_\gamma), N(w)$, and thus $\alpha + \tau \in \Psi$. On the other hand if $\alpha + \tau \in \Psi$, then $\alpha + \tau \in N(s_\beta u_\gamma), N(w)$. Since these are inversion sets of minuscule elements, exactly one among α and τ can have $c_\beta = 1$, let's say α, so $\alpha \in N(s_\beta u_\gamma)$, $\alpha \in N(w)$ and thus $\alpha \in \Psi$. So Ψ turns out to be an inversion set, and we can write

$$N(s_\beta u_\gamma) \cap N(w) = N(u_1 \cdots u_k)$$

for some $k = 1, \cdots, n$. Let's call γ_i the rootlet of $u_1 \cdots u_i$, i.e. $\gamma_i = u_{i+1} \cdots u_n(\gamma)$. We have seen in Lemma 2.2.2 that $\gamma_{i-1} = u_i(\gamma_i) = \gamma_i + a_i \alpha_i > \gamma_i$ because $a_i > 0$ for every i. Hence

$$\delta + \beta = \gamma_0 > \gamma_1 > \cdots > \gamma_n = \gamma.$$

The rootlet of $w = u_1 \cdots u_k t_1 \cdots t_m$ (which is written in reduced form) is γ, thus all the simple reflections in $u_{k+1}, \cdots, u_n$ must appear at least once in $t_1, \cdots, t_m$; in particular u_{k+1} appears. Let's say $t_j = u_{k+1}$ for some j, and assume that $t_i \neq u_{k+1}$ for all $i < j$. We have $u_1 \cdots u_k t_1 \cdots t_{j-1} u_{k+1} \in \mathcal{W}^{ab}$. Call τ_i the simple root corresponding to t_i, write $q_i = u_1 \cdots u_k t_1 \cdots t_{i-1}(\tau_i) \in N(w)$ and $t_i(\tau_{k+1}) = \tau_{k+1} + b_i \tau_i$ for some $b_i \geq 0$ since $t_i \neq u_{k+1}$, all of them for every $i = 1, \ldots, m$. Then the elements

$$u_1 \cdots u_k t_1 \cdots t_{j-1}(\alpha_{k+1}) = u_1 \cdots u_k t_1 \cdots t_{j-2}(\alpha_{k+1} + b_{j-1}\tau_{j-1}) =$$

$$= b_{j-1} q_{j-1} + u_1 \cdots u_k t_1 \cdots t_{j-2}(\alpha_{k+1}) = \cdots = \sum_{i=1}^{j-1} b_i q_i + u_1 \cdots u_k(\alpha_{k+1})$$

must have c_β equal 1. Since all the q_i's and $u_1 \cdots u_k(\alpha_{k+1})$ have $c_\beta = 1$ and the b_i's are non negative, then $b_i = 0$ for all $i = 1, \ldots, j-1$ and

$$t_i(\alpha_{k+1}) = \alpha_{k+1}$$

for all $i = 1, \ldots, j-1$. So $t_i u_{k+1} = u_{k+1} t_i$ for all $i = 1, \ldots, j-1$ and thus we can write

$$w = u_1 \cdots u_k u_{k+1} t_1 \cdots t_{j-1} t_{j+1} \cdots t_m$$

against the fact that $N(s_\beta u_\gamma) \cap N(w) = N(u_1 \cdots u_k)$. $\square$

Lemma 2.2.4. *Let $\gamma \in \Delta_l^+$ and $s_\beta u_\gamma s_1 \cdots s_n \in \mathcal{I}_{\gamma,\delta+\beta}$ written in reduced form. Then $s_i(\gamma) = \gamma$ for every $i = 1, \dots, n$.*

Proof. Suppose there is an s_i such that $s_i(\gamma) \neq \gamma$. Then there exists a rootlet γ', a simple root q and a minuscule element vs_q with $\ell(vs_q) = \ell(v) + 1$, such that $vs_q(\gamma') = \delta + \beta$ and $s_q(\gamma') = \gamma' - aq$ for some positive a, so

$$v(\gamma') = \delta + \beta + av(q)$$

implying $\beta + av(q) \in \widehat{\Delta}$. Since vs_q is minuscule, we have $c_\beta(av(q)) = a \geq 1$, moreover $\beta \in N(vs_q)$ and $v(q) \in N(vs_q)$ so $\beta + av(q) \in N(vs_q)$, but $c_\beta(\beta + av(q)) = 1 + a \geq 2$ which is absurd. □

Lemma 2.2.5. *Let $\gamma \in \Delta_l^+$. Then $s_q(\gamma) = \gamma \iff s_\beta u_\gamma(q) = u_\gamma(q) \quad \forall q \in \widehat{\Pi}$.*

Proof. Assume first that $q \neq \beta$. Suppose $s_q(\gamma) = \gamma$ and $s_\beta u_\gamma(q) \neq u_\gamma(q)$, then $s_\beta u_\gamma(q) = u_\gamma(q) + a\beta \in \widehat{\Delta}$, and so a and $u_\gamma(q)$ have the same sign. Moreover

$$u_\gamma s_q {u_\gamma}^{-1}(u_\gamma(q) + a\beta) = u_\gamma s_q(q + a\delta - a\gamma) = u_\gamma(-q + a\delta - a\gamma) = -u_\gamma(q) + a\beta \in \widehat{\Delta}$$

implying that a and $u_\gamma(q)$ have opposite sign, absurd. Suppose $s_q(\gamma) \neq \gamma$ and $s_\beta u_\gamma(q) = u_\gamma(q)$, then $s_q(\gamma) = \gamma + aq$ with $a \neq 0$. So

$$u_\gamma(\gamma + aq) = \delta - \beta + au_\gamma(q) \in \widehat{\Delta}$$

implying that a and $u_\gamma(q)$ have opposite sign. But also

$$s_\beta(\delta - \beta + au_\gamma(q)) = \delta + \beta + au_\gamma(q) \in \widehat{\Delta}$$

implying that a and $u_\gamma(q)$ have the same sign, absurd. Let's now assume $q = \beta$. Suppose $s_\beta(\gamma) = \gamma$ and $s_\beta u_\gamma(\beta) \neq u_\gamma(\beta)$, then $s_\beta u_\gamma(\beta) = u_\gamma(\beta) + a\beta$ with $a \neq 0$. Thus

$$-s_\beta u_\gamma s_\beta {u_\gamma}^{-1}(u_\gamma(\beta) + a\beta) = -s_\beta u_\gamma s_\beta(\beta + a\delta - a\gamma) =$$
$$= -s_\beta u_\gamma(-\beta + a\delta - a\gamma) = s_\beta(u_\gamma(\beta) - a\beta) = u_\gamma(\beta) + 2a\beta \in \widehat{\Delta}.$$

Moreover also

$$-u_\gamma s_\beta {u_\gamma}^{-1}(u_\gamma(\beta) + 2a\beta) = -u_\gamma s_\beta(\beta + 2a\delta - 2a\gamma) =$$

$$= -u_\gamma(2a\delta - \beta - 2a\gamma) = u_\gamma(\beta) - 2a\beta \in \widehat{\Delta}.$$

Without loss of generality we can take $a > 0$, then $u_\gamma(\beta) - 2a\beta < 0$ since $c_\beta(u_\gamma(\beta)) = 1$, then $u_\gamma(\beta) = \beta$, but $u_\gamma(\beta) + 2a\beta = (2a+1)\beta \in \widehat{\Delta}$ which is absurd. Suppose $s_\beta(\gamma) \neq \gamma$ and $s_\beta u_\gamma(\beta) = u_\gamma(\beta)$, then $s_\beta(\gamma) = \gamma + a\beta$ with $a \neq 0$. Thus

$$-s_\beta {u_\gamma}^{-1} s_\beta u_\gamma(-2\delta + \gamma + a\beta) = -s_\beta {u_\gamma}^{-1} s_\beta(-\delta - \beta + a u_\gamma(\beta)) =$$

$$= -s_\beta {u_\gamma}^{-1}(-\delta + \beta + a u_\gamma(\beta)) = -s_\beta(-\gamma + a\beta) = \gamma + 2a\beta \in \widehat{\Delta}.$$

Moreover also

$$-{u_\gamma}^{-1} s_\beta u_\gamma(-2\delta+\gamma+2a\beta) = -{u_\gamma}^{-1} s_\beta(-\delta-\beta+2a u_\gamma(\beta)) = -{u_\gamma}^{-1}(-\delta+\beta+2a u_\gamma(\beta)) =$$

$$= \gamma - 2a\beta \in \widehat{\Delta}.$$

Without loss of generality we can take $a > 0$, then $\gamma - 2a\beta < 0$ since $c_\beta(\gamma) = 1$, so $\gamma = \beta$. But then

$$s_\beta u_\gamma(\beta) = s_\beta u_\gamma(\gamma) = \delta + \beta \neq \delta - \beta = u_\gamma(\gamma) = u_\gamma(\beta)$$

which is absurd. □

Lemma 2.2.6. *Let $\gamma \in \Delta_l^+$. Suppose $u_\gamma w$ is such that $\ell(u_\gamma w) = \ell(u_\gamma) + \ell(w)$ and write $w = s_1 \cdots s_n$ in reduced form. Then $u_\gamma w \in \mathcal{I}_{\gamma,\delta+\beta} \iff w \in \mathcal{W}_\sigma^{ab}$ and $s_i(\gamma) = \gamma$ for every $i = 1, \ldots, n$. Moreover $\mathcal{I}_{\gamma,\delta+\beta}$ is isomorphic as a poset to $\widehat{W}_{\perp\gamma,\delta+\beta} \backslash \widehat{W}_{\perp\gamma}$.*

Proof. Suppose $s_i(\gamma) = \gamma$ for every $i = 1, \ldots, n$ and write α_i for the simple root associated to s_i. Then by Lemma 2.2.5 $s_\beta u_\gamma(q) = u_\gamma(q)$ for every $q \in \widehat{\Pi}$. If $q \neq \beta$ then $c_\beta(s_\beta u_\gamma(q)) = c_\beta(u_\gamma(q)) = 0$, if $q = \beta$ then $c_\beta(s_\beta u_\gamma(\beta)) = c_\beta(u_\gamma(\beta)) = 1$. Now just consider $s_\beta u_\gamma(s_1 \cdots s_{k-1}(\alpha_k)) \in N(u_\gamma w)$ for every $k = 1, \ldots, n$, we have $c_\beta(s_\beta u_\gamma(s_1 \cdots s_{k-1}(\alpha_k))) = c_\beta(s_1 \cdots s_{k-1}(\alpha_k))$ and the equivalence follows. To prove the second claim just notice that if $u \in \widehat{W}_{\perp\gamma,\delta+\beta} \backslash \widehat{W}_{\perp\gamma}$ then for every $\tau \in N(u)$ we have $c_\beta(\tau) \geq 1$ by definition of $\widehat{W}_{\perp\gamma,\delta+\beta} \backslash \widehat{W}_{\perp\gamma}$, moreover since $\widehat{\Pi}_\gamma$ is a finite diagram and $c_\beta(\delta) = 1$ we also have $c_\beta(\tau) \leq 1$, and so $c_\beta(\gamma) = 1$. □

Lemma 2.2.7. *If $\tau \in \widehat{\Delta}$ is such that $\tau \notin \Delta_l^+$, then we have*

$$\mathcal{I}_{\tau,\delta+\beta} = \emptyset$$

or $\tau = \delta + \beta$ and $\mathcal{I}_{\delta+\beta,\delta+\beta} = \{1\}$.

Proof. Suppose that there is $\tau \in \widehat{\Delta}^+$ and $\tau \notin \Delta_l^+$ for which there is a $w \in \mathcal{I}_{\tau,\delta+\beta}$, $w \neq 1$. Write $w = s_\beta s_2 \ldots s_n$ in reduced form. Since $c_\beta(\tau) \neq 0$ and $c_\beta(s_2 \ldots s_n(\tau)) = c_\beta(\delta - \beta) = 0$, there must be an index $k \in [2, n]$ such that $s_k = s_\beta$ is the last simple reflection in w that changes the β-height of τ applying the sequence of simple reflections $s_2 \ldots s_n$. So $\gamma = s_\beta s_{k+1} \ldots s_n(\tau)$ is such that $c_\beta(\gamma) = 0$ and

$$s_\beta s_2 \ldots s_{k-1} \in \mathcal{I}_{\gamma,k\delta+\beta}.$$

Thanks to Lemma 2.2.4, since $s_\beta(\gamma) \neq \gamma$, $s_\beta s_2 \ldots s_{k-1}$ is the minimum in the poset $\mathcal{I}_{\gamma,\delta+\beta}$, so $s_\beta s_2 \ldots s_{k-1} = s_\beta u_\gamma$. But then $s_\beta u_\gamma s_\beta$ can't be minuscule due to Lemma 2.2.5, since $s_\beta u_\gamma(\beta) \neq u_\gamma(\beta)$ and thus $c_\beta(s_\beta s_2 \ldots s_{k-1}(\beta)) = c_\beta(s_\beta u_\gamma(\beta)) \neq 1$, absurd. To prove the final statement suppose that there is a minuscule element $w \in \widehat{W}$, $w \neq 1$, such that $w(\delta + \beta) = \delta + \beta$. Then $w^{-1}(\beta) = \beta > 0$ which is absurd so $\mathcal{I}_{\delta+\beta,\delta+\beta} = \{1\}$.

□

We give now a direct proof of the existence of a unique maximum in $\mathcal{I}_{\gamma,\delta+\beta}$; this statement might be deduced by the fact that $\widehat{W}_{\perp\gamma}$ is a finite Weyl group and $\widehat{W}_{\perp\gamma,\delta+\beta}$ is a standard parabolic subgroup.

Lemma 2.2.8. *Let $\gamma \in \Delta_l^+$. Then $\mathcal{I}_{\gamma,\delta+\beta}$ has a unique maximum.*

Proof. Every element in $\mathcal{I}_{\gamma,\delta+\beta}$ can be built up by taking u_γ and adding a block in reduced form $s_1 \cdots s_n$ such that $s_i(\gamma) = \gamma$ for every $i = 1, \ldots, n$ and $s_1 \ldots s_n \in \mathcal{W}^{ab}$. Then consider the finite subdiagram of $\widehat{\Pi}$ made of the simple roots associated to simple reflections fixing γ, and consider its connected component containing β and call it $\mathcal{B}$. If w_1, w_2 are minuscule elements in the subgroup $W(\mathcal{B})$ generated by the simple reflections associated to the simple roots contained in $\mathcal{B}$, then if we prove that also $N(w_1) \cup N(w_2)$ is biconvex then there would exist $w \in W(\mathcal{B})$ such that

$N(w_1) \cup N(w_2) = N(w)$, and we could prove our Lemma taking the union on all the inversion sets of minuscule elements contained in $W(\mathcal{B})$. To prove the claim just note that if $\tau_1 \in N(w_1)$ and $\tau_2 \in N(w_2)$ then $\tau_1+\tau_2$ is not a root since $c_\beta(\tau_1+\tau_2) = 2$ and $\tau_1 + \tau_2 \in \langle \mathcal{B} \rangle$ which is a finite diagram, so every root is strictly contained in δ and $c_\beta(\delta) = 1$. □

We now give a proof to Corollary 2.1.2 on the one-to-one correspondence between maximal elements of $\mathcal{W}^{ab}$ and long simple roots of Δ.

Proof. Given a simple long root α in Π we associate to it the unique maximal element w in $\mathcal{I}_{\alpha,\delta+\beta}$. We need to prove it is a maximal element in $\mathcal{W}^{ab}$. Suppose it is not, so there exists a simple reflection s_q such that $\ell(ws_q) > \ell(w)$, $c_\beta(w(q)) = 1$ and $s_q(\alpha) = \alpha + aq$ for some $a > 0$ because of the maximality of w in $\mathcal{I}_{\alpha,\delta+\beta}$ (and $q \neq \alpha$ because $c_\beta(w(\alpha)) = 2 \neq 1$). But then $w(\alpha) = ws_q(\alpha+aq) = \delta+\beta$ but $\ell(ws_q) > \ell(w)$ and also their rootlets verify $\alpha + aq \supseteq \alpha$ against the argument in 2.2.4 (that it's not possible to both reduce the length of a minuscule element and decrease the rootlet with respect to the partial order induced by $\subset$). Conversely given a maximal element w in $\mathcal{W}^{ab}$ we associate to it its corresponding rootlet $\alpha = w^{-1}(\delta + \beta)$. We need to check that it is simple. Suppose it is not. Since $w \neq 1$ we excluded $\alpha = \delta + \beta$ so this implies $\alpha \in \Delta_l^+$. Then we can write $\alpha = \gamma + aq$ with $\gamma \in \widehat{\Delta}^+$, $q \in \Pi$, $a > 0$ and $s_q(\gamma) = \gamma + aq$ because Δ is a finite root system. We have $w(\gamma + aq) = ws_q(\gamma) = \theta$. If $w(q) < 0$ then $\ell(ws_q) < \ell(w)$ and so $ws_q \in \mathcal{W}^{ab}$, but this is against the argument in 2.2.4 because $\ell(ws_q) < \ell(w)$ and also their rootlets satisfy $\gamma \subseteq \gamma + aq$. Then $w(q) > 0$ and $\ell(ws_q) > \ell(w)$. We write

$$w(\gamma) = \delta + \beta - aw(q)$$

and since $\beta - aw(q) \in \widehat{\Delta}$ then $c_\beta(w(q)) \geq 1$ otherwise $c_\beta(\beta - aw(q)) > 0$ but there would be another simple root τ such that $c_\tau(\beta - aw(q)) < 0$. Suppose $c_\beta(w(q)) \geq 2$. Then $c_\beta(w(\gamma)) \leq 0$ and since $\delta + \beta$ is the smallest root with $c_\beta = 2$ we also have $w(\gamma) < 0$. Write $w(\gamma) = -(-\delta - \beta + aw(q))$ with $(-\delta - \beta + aw(q)) \in \widehat{\Delta}^+$, then $w^{-1}(-\delta - \beta + aw(q)) = -\gamma < 0$ and so $c_\beta(-\delta - \beta + aw(q)) = 1$ and $c_\beta(aw(q)) = 3$ forcing $a = 1$ and $c_\beta(w(q)) = 3$. But then $-\delta - \beta + w(q) \in N(w)$

and of course $\beta \in N(w)$ so also $(-\delta - \beta + w(q)) + (\beta) = -\delta + w(q) \in N(w)$ but $c_\beta(-\delta + w(q)) = 2 \neq 1$. In the end $c_\beta(w(q)) = 1$, but this is against the maximality of w, so α must be a simple root. $\square$

Chapter 3

A rootlets theory for $\mathfrak{b}^{\bar{0}}$-stable abelian subspaces

In this chapter we will present the main results concerning the decomposition of $\mathcal{W}_\sigma^{ab}$ in the semisimple case in both the twisted and untwisted case. In the first part we divide the possible outcomes in several subcases and state the main theorem of this work. In the second part we give proofs for every single case and prove the main theorem. In the last part we show tables which summarize all the findings.

3.1 A rootlets theory for $\mathfrak{b}^{\bar{0}}$-stable abelian subspaces

Assume $\Pi_1 = \{\beta\}$, hence from now on we will not consider the Hermitian symmetric case. Given $\alpha \in \widehat{\Delta}^+$, $\mu \in \mathcal{M}_\sigma$, set, extending (1.1)

$$\mathcal{I}_{\alpha,\mu} = \{w \in \mathcal{W}_\sigma^{ab} \mid w(\alpha) = \mu\}. \tag{3.1}$$

Fix $\mu \in \mathcal{M}_\sigma$. Then, clearly

$$\mathcal{W}_\sigma^{ab} = \bigsqcup_{\alpha \in \widehat{\Delta}^+} \mathcal{I}_{\alpha,\mu}.$$

Write $\mathcal{M}_\sigma = \{\mu_1, \dots, \mu_s\}$ and set

$$\mathcal{I}_{\alpha_1,\dots\alpha_s} = \mathcal{I}_{\alpha_1,\mu_1} \cap \dots \cap \mathcal{I}_{\alpha_s,\mu_s}.$$

Then

$$\mathcal{W}_\sigma^{ab} = \bigsqcup_{\alpha_1,\dots,\alpha_s \in \widehat{\Delta}^+} \mathcal{I}_{\alpha_1,\dots,\alpha_s}. \tag{3.2}$$

Our main problem is to establish when the r.h.s. of (3.1) is non-empty and to understand the structure of the corresponding poset.

We will use the following notation

$$\langle A \rangle = \widehat{\Delta}^+ \cap \mathbb{Z}A, \quad A \subset \widehat{\Pi},$$
$$\langle A \rangle_l = \widehat{\Delta}_l^+ \cap \mathbb{Z}A, \quad A \subset \widehat{\Pi},$$
$$\widehat{\Delta}_\eta^i = \{\gamma \in \widehat{\Delta}^+ \mid c_\eta(\gamma) = i\},$$
$$\alpha^{\leq} = \{\gamma \in \widehat{\Delta}_{re}^+ \mid \gamma \leq \alpha\}, \quad \alpha \in \widehat{\Delta}_{re}^+,$$

$\alpha_\Sigma \in \Sigma$ is the only root in Σ connected to β, i.e. $s_{\alpha_\Sigma}(\beta) \neq \beta$,

Σ_β is the connected component containing β in $\langle \{\alpha \in \widehat{\Pi} : |\alpha| = |\beta|\} \rangle$.

Let ordinary be the walls of type $k\delta - \theta_\Sigma$, let special be the walls of type $k\delta + \beta$. Also recall that in our case a root is said to be of type 1 if it is long, of type 2 otherwise. The possibilities are listed in the following table.

Name	Type of wall	Length of β	Type of θ_Σ	$\lvert\Sigma\rvert$
a	ordinary	long	1	> 1
b	ordinary	long	1	1
c	ordinary	long	2	1
d	special	long		
e	ordinary	long	2	> 1
f	ordinary	short		
g	special	short		

Set

$$B_\mu = \begin{cases} \{\gamma \in \widehat{\Pi} \mid (\gamma, \theta_\Sigma^\vee) = 1\} & \text{if } \mu = k\delta - \theta_\Sigma \text{ and } \theta_\Sigma \text{ is of type 1,} \\ \Pi_1 & \text{if } \mu = k\delta - \theta_\Sigma \text{ and } \theta_\Sigma \text{ is of type 2,} \\ \beta & \text{if } \mu = k\delta + \beta \text{ and } \beta \in \Pi_1. \end{cases}$$

Definition 3.1.1. Given $\alpha \in \widehat{\Delta}^+_{re}$ and $\mu \in \mathcal{M}_\sigma$ such that $\mathcal{I}_{\alpha,\mu} \neq \emptyset$, we set

$$\begin{aligned}
&\widehat{\Pi}_\alpha = \widehat{\Pi} \cap \alpha^\perp, \quad \widehat{W}_{\perp\alpha} = W(\widehat{\Pi}_\alpha),\\
&\widehat{\Pi}_{\alpha,\mu} = \widehat{\Pi}_\alpha \setminus B_\mu,\\
&\widehat{\Pi}^*_{\alpha,\mu} = \begin{cases} \widehat{\Pi}_{\alpha,\mu} \cup \{\theta_\Sigma\} & \text{if } \mu = k\delta - \theta_\Sigma, \theta_\Sigma \text{ of type 1, } |\Sigma| > 1,\\ & \alpha \in \langle A(\Sigma) \setminus (\Sigma \cup \Pi_1)\rangle,\\ \widehat{\Pi}_{\alpha,\mu} & \text{in all other cases;} \end{cases}\\
&\widehat{W}_{\perp\alpha,\mu} = W(\widehat{\Pi}^*_{\alpha,\mu}).
\end{aligned}$$

Let's see an example for each case. For short we write, e.g., $D_4 = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ to mean that the root subsystem of $\widehat{\Pi}$ generated by $\alpha_2, \alpha_3, \alpha_4, \alpha_5$ is of type D_4.

Example 3.1.2.

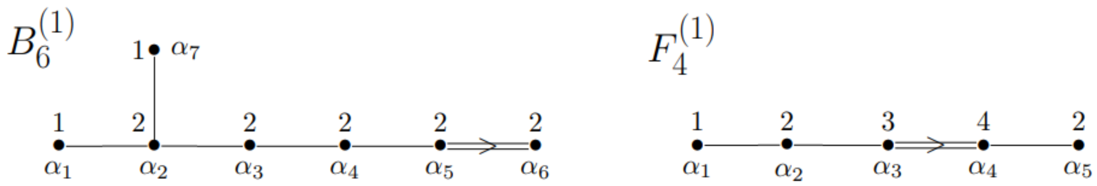

Figure 3.1: Affine Dynkin diagrams $B_6^{(1)}$ and $F_4^{(1)}$.

(a) Consider $B_6^{(1)}, p = 4$, choose $\Sigma \simeq D_4 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_7\}$, so that $A(\Sigma) \simeq B_4 = \{\alpha_3, \alpha_4, \alpha_5, \alpha_6\}, \Gamma(\Sigma) = \{\alpha_3\}$, and take $\mu = \delta - \theta_\Sigma$. Let's take $\alpha = \alpha_5$, note that $\alpha_5 \in \langle A(\Sigma) \setminus (\Sigma \cup \Pi_1)\rangle = \langle\{\alpha_5, \alpha_6\}\rangle$. We have $\widehat{\Pi}_\alpha = \{\alpha_1, \alpha_2, \alpha_3, \alpha_7\}$, and $\widehat{\Pi}^*_{\alpha,\mu} = \{\alpha_1, \alpha_3, \alpha_7, \theta_\Sigma\}$. Let's take $\alpha = \alpha_4$ instead, note that $\alpha_4 \notin \langle A(\Sigma) \setminus (\Sigma \cup \Pi_1)\rangle = \langle\{\alpha_5, \alpha_6\}\rangle$. We have $\widehat{\Pi}_\alpha = \{\alpha_1, \alpha_2, \alpha_6, \alpha_7\}$, and $\widehat{\Pi}^*_{\alpha,\mu} = \{\alpha_1, \alpha_6, \alpha_7\}$.

(b) Consider $B_6^{(1)}, p = 2$, choose $\Sigma \simeq A_1 = \{\alpha_1\}$, so that $A(\Sigma) \simeq A_6 = \{\alpha_7, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}, \Gamma(\Sigma) = \emptyset$, and take $\mu = \delta - \alpha_1$. Let's take $\alpha = \alpha_3 + \alpha_4 + \alpha_5$. We have $\widehat{\Pi}_\alpha = \{\alpha_1, \alpha_4, \alpha_7\}$, and $\widehat{\Pi}^*_{\alpha,\mu} = \{\alpha_1, \alpha_4, \alpha_7\}$.

(c) Consider $B_6^{(1)}, p = 5$, and choose $\Sigma \simeq A_1 = \{\alpha_6\}$, so that $A(\Sigma) \simeq D_6 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_7\}, \Gamma(\Sigma) = \emptyset$, and take $\mu = \delta - \alpha_6$. Let's take $\alpha = \alpha_6$. We have $\widehat{\Pi}_\alpha = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_7\}$, and $\widehat{\Pi}^*_{\alpha,\mu} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_7\}$.

(d) Consider $B_6^{(1)}, p = 4, \mu = \delta + \alpha$. Let's take $\alpha = \alpha_3 + \alpha_4 + \alpha_5$. We have $\widehat{\Pi}_\alpha = \{\alpha_1, \alpha_4, \alpha_7\}$, and $\widehat{\Pi}^*_{\alpha,\mu} = \{\alpha_1, \alpha_7\}$.

(e) Consider $F_4^{(1)}, p = 3, \Sigma \simeq A_2 = \{\alpha_4, \alpha_5\}, A(\Sigma) \simeq A_3 = \{\alpha_1, \alpha_2, \alpha_3\}, \Gamma(\Sigma) = \emptyset$, and $\mu = \delta - \alpha_4 - \alpha_5$. Let's take $\alpha = \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$. We have $\widehat{\Pi}_\alpha = \{\alpha_3\}$, and $\widehat{\Pi}^*_{\alpha,\mu} = \emptyset$.

(f) Consider $B_6^{(1)}, p = 6, \Sigma \simeq D_6 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_7\}, A(\Sigma) \simeq B_2 = \{\alpha_5, \alpha_6\}, \Gamma(\Sigma) = \{\alpha_5\}$, and take $\mu = \delta - \alpha_6$. Let's take $\alpha = \alpha_5$. We have $\widehat{\Pi}_\alpha = \{\alpha_1, \alpha_2, \alpha_3, \alpha_7\}$, and $\widehat{\Pi}^*_{\alpha,\mu} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_7\}$.

(g) Consider $B_6^{(1)}, p = 6, \mu = \delta + \alpha_6$. Let's take $\alpha = \alpha_5 + \alpha_6$. We have $\widehat{\Pi}_\alpha = \{\alpha_1, \alpha_2, \alpha_3, \alpha_6, \alpha_7\}$, and $\widehat{\Pi}^*_{\alpha,\mu} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_7\}$.

Let s be the number of components of Π_0, let $W_0 = W(\Pi_0)$ and w_0 its longest element. Consider the set of simple roots $\Phi = \{\alpha \in \Pi_0 | (\alpha, \beta) = 0\}$, write $W_{0,\beta} = W(\Phi)$, $w_{0,\beta}$ for its longest element, and define $w_\beta = s_\beta w_{0,\beta} w_0$. The following theorem summarizes our results on the structure of the posets $\mathcal{I}_{\alpha,\mu}$. It will be proven in Section 3.2, by looking at each case individually.

Theorem 3.1.1. *Assume $\Pi_1 = \{\beta\}$. Let $\alpha \in \widehat{\Delta}^+_{re}$ and $\mu \in \mathcal{M}_\sigma$ be such that $\mathcal{I}_{\alpha,\mu} \neq \emptyset$. Assume β is long.*
(a). If $\mu = k\delta - \theta_\Sigma$ with θ_Σ of type 1, then

1. *Assume $\Gamma(\Sigma) = \{\alpha_\Sigma\}$. Then if $c_{\alpha_\Sigma}(\alpha) = 0$ the map $w_{\alpha,\mu} u \mapsto u$ is a poset isomorphism between $\mathcal{I}_{\alpha,\mu}$ and $\widehat{W}_{\perp\alpha,\mu} \backslash \widehat{W}_{\perp\alpha}$; if instead $\alpha \neq \alpha_\Sigma$ and $c_{\alpha_\Sigma}(\alpha) \neq 0$, then the map $x \mapsto x w_{\alpha,\mu}$ is a poset isomorphism between $\mathcal{I}_{\mu,\mu} \to \mathcal{I}_{\alpha,\mu}$, where $\mathcal{I}_{\mu,\mu}$ is the doubleton defined in Lemma 3.2.12.*
2. *If $|\Gamma(\Sigma)| \neq 1$ then the map $u \mapsto w_{\alpha,\mu} u$ is a poset isomorphism between $\mathcal{I}_{\alpha,\mu}$ and $\widehat{W}_{\perp\alpha,\mu} \backslash \widehat{W}_{\perp\alpha}$ unless $\alpha = \alpha_\Sigma + \beta + \eta$, $\alpha_\Sigma \in \Gamma(\Sigma)$, $(\alpha_\Sigma, \beta) \neq 0$, and $\eta \in$*

$\{0\}\cup\{\tau\in\Pi_0|(\tau,\beta)\neq 0,\tau\neq\alpha_\Sigma\}$; in the latter case $\mathcal{I}_{\alpha,\mu}\cong\widehat{W}_{\perp\alpha,\mu}\backslash\widehat{W}_{\perp\alpha}\sqcup\{\overline{u}\}$, via $w_{\alpha,\mu}u\mapsto u$ where

$$\overline{u}=\min(\widehat{W}_{\perp\alpha,\mu}\backslash\widehat{W}_{\perp\alpha})\cdot\max\mathcal{P}$$

is an absolute maximum, $\Gamma(\Sigma)_\perp$ is the component of $\Gamma(\Sigma)$ orthogonal to $\theta_{\Gamma(\Sigma)}$ and containing α_Σ and

$$\mathcal{P}=\begin{cases}W(\Gamma(\Sigma)_\perp)/W(\Gamma(\Sigma)_\perp\setminus\{\alpha_\Sigma\}) & \text{if } s=1,\\ W(\Gamma(\Sigma))/W(\Gamma(\Sigma)\setminus\{\alpha_\Sigma\}) & \text{otherwise.}\end{cases}$$

Moreover, $\overline{u}=w_\beta$, $w_\beta s_\eta$ according to whether $\eta=0,\eta\neq 0$.

(b). If $\mu=\delta-\theta_\Sigma,\theta_\Sigma\in\widehat{\Pi}_l$, then $\mathcal{I}_{\alpha,\mu}$ is a singleton.
(c). If $\mu=k\delta-\theta_\Sigma,\theta_\Sigma\in\widehat{\Pi}_s$, then in case of a double link between β and θ_Σ the poset $\mathcal{I}_{\alpha,\mu}$ is a doubleton unless $\alpha=\theta_\Sigma$, in which case it is a singleton. In case of a triple or quadruple link $\mathcal{I}_{\alpha,\mu}$ is a singleton.
(d). If $\mu=k\delta+\beta$ let $u\in W_0$ be of minimal length such that $u(\alpha)=k\delta-\beta$, then the map $s_\beta uv\mapsto v$ is a poset isomorphism between $\mathcal{I}_{\alpha,\mu}$ and $\widehat{W}_{\perp\alpha,\mu}\backslash\widehat{W}_{\perp\alpha}$.
(e). If $\alpha\in\widehat{\Delta}^0_{\alpha_\Sigma}\cup\widehat{\Delta}^1_{\alpha_\Sigma}$, $|\alpha|=|\theta_\Sigma|$ and $\alpha\neq\alpha_\Sigma+\beta$, or if $\alpha=\delta+\alpha_\Sigma$, or if $\alpha=\delta+\alpha_\Sigma+\beta$ the map $u^\Sigma_\alpha v\mapsto v$ is a poset isomorphism between $\mathcal{I}_{\alpha,\mu}$ and $\widehat{W}_{\perp\alpha,\mu}\backslash\widehat{W}_{\perp\alpha}$. If $\alpha=\alpha_\Sigma+\beta$ and u is the shortest element such that $u(\alpha_\Sigma+\beta)=\mu$ then $\mathcal{I}_{\alpha_\Sigma+\beta,\mu}=\{u,us_{\alpha_\Sigma}\}$.
Assume β is short.
(f). If $\mu=k\delta-\theta_\Sigma$ with θ_Σ of type 1, then the map $w_{\alpha,\mu}u\mapsto u$ is a poset isomorphism between $\mathcal{I}_{\alpha,\mu}$ and $\widehat{W}_{\perp\alpha,\mu}\backslash\widehat{W}_{\perp\alpha}$.
(g). If $\mu=k\delta+\beta$, then the map $w_{\alpha,\mu}u\mapsto u$ is a poset isomorphism between $\mathcal{I}_{\alpha,\mu}$ and $\widehat{W}_{\perp\alpha,\mu}\backslash\widehat{W}_{\perp\alpha}$.

Define $\widehat{\Delta}_\mu$ according to the following table.

Case	$\widehat{\Delta}_\mu$
a	$\langle A(\Sigma)\rangle_l$
b	$\langle A(\Sigma)\rangle_l \cup (\{\delta - \langle A(\Sigma)\rangle_l\})$
c	$\widehat{\Delta}^1_{\theta_\Sigma}$ if $\beta <-> \theta_\Sigma$ is a double link, $\{\gamma \in (k\delta)^< : \lvert\gamma\rvert = \lvert\theta_\Sigma\rvert, \gamma \neq \theta_\Sigma\}$ otherwise
d	$\widehat{\Delta}^1_\beta \cup \{k\delta + \beta\}$
e	$\{\gamma \in \widehat{\Delta}^0_{\alpha_\Sigma} \cup \widehat{\Delta}^1_{\alpha_\Sigma} : \lvert\gamma\rvert = \lvert\theta_\Sigma\rvert\} \cup \{\delta + \alpha_\Sigma\} \cup \{\delta + \alpha_\Sigma + \beta\}$
f	$\langle A(\Sigma)\rangle_l$
g	$\{\gamma \in \widehat{\Delta}^1_\beta : \gamma = k\delta - \tau, \tau \in \langle\Sigma_\beta\rangle\} \cup \{k\delta + \beta\}$

Note that case b might be also displayed as

$$\langle A(\Sigma)\rangle_l \cup (\{\delta - \langle A(\Sigma)\rangle_l\} \setminus \{\theta_\Sigma\}) = (\widehat{\Delta}^0_{\theta_\Sigma} \cup \widehat{\Delta}^1_{\theta_\Sigma}) \setminus \{\theta_\Sigma\}.$$

Corollary 3.1.2. *Assume $\mu \in \mathcal{M}_\sigma$ and $\alpha \in \widehat{\Delta}^+_{re}$. Then $\mathcal{I}_{\alpha,\mu} \neq \emptyset$ if and only if $\alpha \in \widehat{\Delta}_\mu$.*

3.2 Proof of the Main Theorem

We start proving some general facts.

Lemma 3.2.1.

$$w_0(\beta) = k\delta - \beta.$$

Proof. Recall that $k\delta - \beta = \beta + k\sum_{i=1}^n a_i\alpha_i$, where α_i runs over all the simple roots of the diagram but β.

$$k\delta - w_0(\beta) = w_0(k\delta - \beta) = w_0(\beta + k\sum_i a_i\alpha_i) = w_0(\beta) + k\sum_i a_i w_0(\alpha_i) =$$

$$= w_0(\beta) - k\sum_i a_i\alpha_{\sigma(i)} = w_0(\beta) - k\sum_i a_{\sigma(i)}\alpha_i$$

with σ the permutation associated to w_0 acting on the several components $\Sigma|\Pi$. Then

$$w_0(\beta) = \frac{1}{2}(k\delta + k\sum_i a_{\sigma(i)}\alpha_i) = \frac{1}{2}(2\beta + k\sum_i a_i\alpha_i + k\sum_i a_{\sigma(i)}\alpha_i) = \beta + k\sum_i \frac{a_i + a_{\sigma(i)}}{2}\alpha_i.$$

But then $\frac{a_i+a_{\sigma(i)}}{2} \leq a_i \ \forall i = 1, \cdots, n$, i.e. $a_{\sigma(i)} \leq a_i \ \forall i = 1, \cdots, n$. Since of course $\sum_i a_{\sigma(i)} = \sum_i a_i$ we must have $a_{\sigma(i)} = a_i \ \forall i = 1, \cdots, n$. So

$$w_0(\beta) = \beta + k \sum_i a_i \alpha_i = k\delta - \beta.$$

□

Corollary 3.2.2. *Let $\alpha \in \Pi$ be a simple root connected to β in the Dynkin diagram, then $w_0(\alpha) = -\alpha$.*

Proof. $w_0(\alpha) \in -\Pi$ and belongs to the same component $\Sigma|\Pi$ of α. Since β is long, $w_0(\beta) = k\delta - \beta$ and $k\delta - \alpha \in \widehat{\Delta}$, we have

$$w_0 s_\beta(k\delta - \alpha) = k\delta - w_0 s_\beta(\alpha) = k\delta - w_0(\alpha+\beta) = k\delta - w_0(\alpha) - k\delta + \beta = \beta - w_0(\alpha) \in \widehat{\Delta}$$

forcing $w_0(\alpha) = -\alpha$. □

Lemma 3.2.3. *If β is long, then $c_{\alpha_\Sigma}(\theta_\Sigma) = 1$.*

Proof. Note that

$$c_{\alpha_\Sigma}(\theta_\Sigma) = c_\beta(s_\beta(\theta_\Sigma)) = c_\beta(\theta_\Sigma - (\beta^\vee, \theta_\Sigma)\beta) = -(\beta^\vee, \theta_\Sigma).$$

Now applying the Cauchy-Schwartz inequality to the non linearly dependent vectors $\beta^\vee$ and θ_Σ, we get

$$|(\beta^\vee, \theta_\Sigma)| < |\beta^\vee| \cdot |\theta_\Sigma| = 2\frac{|\theta_\Sigma|}{|\beta|} \leq 2,$$

so $|(\beta^\vee, \theta_\Sigma)| \leq 1$ and of course $c_{\alpha_\Sigma}(\theta_\Sigma) = 1$. □

Note that in every case $A(\Sigma)$ is a diagram of finite type, because it is obtained removing at least one simple root from the original affine diagram.

Case a.

We assume that β is a long root, and we consider $\mu = k\delta - \theta_\Sigma$, with $|\Sigma| > 1$ and θ_Σ of type 1. Recall that in this case $k\delta - \theta_\Sigma \in \langle A(\Sigma) \rangle$ and it's its highest root. A proof of this is given in [6, Lemma 4.1 and Proposition 4.2]. After proving some basic facts, we show in Lemmas 3.2.7 and 3.2.8 that if $\gamma \in \langle A(\Sigma)\rangle_l$, then the minimal element u_γ^Σ in $W(A(\Sigma))$ such that $u_\gamma^\Sigma(\gamma) = k\delta - \theta_\Sigma$ is also such that $u_\gamma^\Sigma = \min \mathcal{I}_{\gamma,\mu}$, proving in particular that $\mathcal{I}_{\gamma,\mu} \neq \emptyset$. After some technicalities, we find in Corollary 3.2.10 conditions under which we can add chains of simple reflections fixing γ to u_γ^Σ, in order to find other elements in $\mathcal{I}_{\gamma,\mu}$. Then we split the remaining work in two cases: $|\Gamma(\Sigma)| = 1$ and $|\Gamma(\Sigma)| > 1$. Note that $c_{\alpha_\Sigma}(k\delta) \leq 4$, because $c_\beta(k\delta) = 2$ and $s_\beta(k\delta) = k\delta$, and $c_{\alpha_\Sigma}(k\delta) \geq 2$, because $|\Sigma| > 1$. We look separately at the cases $c_{\alpha_\Sigma}(k\delta) = 3, 4$ and $c_{\alpha_\Sigma}(k\delta) = 2$, which is equivalent to $\Gamma(\Sigma) = \{\alpha_\Sigma\}$ as we prove in Lemma 3.2.5. For $c_{\alpha_\Sigma}(k\delta) = 3, 4$ we show in Lemma 3.2.11 that only for at most two specific roots $\gamma \in \langle A(\Sigma)\rangle_l$ we are able to find one new element in $\mathcal{I}_{\gamma,\mu}$ that cannot be obtained adding a chain of simple roots fixing γ to u_γ^Σ, concluding on determining the structure of $\mathcal{I}_{\gamma,\mu}$. For $c_{\alpha_\Sigma}(k\delta) = 2$, we show in Lemma 3.2.12 that $\mathcal{I}_{\mu,\mu} = \{1, s_\beta w_{0,\alpha_\Sigma} w_0 s_\beta\}$, then we prove in Lemmas 3.2.13 and 3.2.14 that right multiplication by u_γ^Σ is a poset isomorphism between $\mathcal{I}_{\mu,\mu}$ and $\mathcal{I}_{\gamma,\mu} = \{u_\gamma^\Sigma, s_\beta w_{0,\alpha_\Sigma} w_0 s_\beta u_\gamma^\Sigma\}$. We conclude showing in Proposition 3.2.15 that if $\gamma \notin \langle A(\Sigma)\rangle_l$, then $\mathcal{I}_{\gamma,\mu} = \emptyset$; this proves $\widehat{\Delta}_\mu = \langle A(\Sigma)\rangle_l$.

Lemma 3.2.4. *$\Gamma(\Sigma) \neq \emptyset$.*

Proof. Assume $\Gamma(\Sigma) = \emptyset$. This implies $s_{\alpha_\Sigma}(\theta_\Sigma) = \theta_\Sigma - \alpha_\Sigma$ and so $\Sigma \simeq A_n$ for some n. Write $\alpha_1, \ldots, \alpha_n$ for the simple roots in Σ, ordered in the way such that $\alpha_n = \alpha_\Sigma$, and write s_i for the associated simple reflections. Then we can write

$$k\delta = \sum_{i=1}^{n} a_i \alpha_i + 2\beta + R$$

with R a sum of other simple roots in the affine diagram. We get

$$\begin{aligned} a_1 &= c_{\alpha_1}(k\delta) = c_{\alpha_1}(s_1(k\delta)) = -a_1 + a_2, \\ a_i &= c_{\alpha_i}(k\delta) = c_{\alpha_i}(s_i(k\delta)) = -a_i + a_{i-1} + a_{i+1} \quad 2 \leq i \leq n-1, \\ a_n &= c_{\alpha_n}(k\delta) = c_{\alpha_n}(s_n(k\delta)) = -a_n + a_{n-1} + 2. \end{aligned}$$

For $n = 2$ we solve the system and get $a_i \notin \mathbb{N}$. For $n > 2$ we sum the equations and simplify, and get $a_1 = -a_1 - a_n + 2$. This can be expressed as

$$a_n = 2 - 2a_1 \leq 0$$

which is absurd. This proves $\Gamma(\Sigma) \neq \emptyset$. □

Lemma 3.2.5. $\Gamma(\Sigma) = \{\alpha_\Sigma\} \iff c_{\alpha_\Sigma}(k\delta) = 2$.

Proof. Suppose $\Gamma(\Sigma) = \{\alpha_\Sigma\}$. Note that $c_\beta(k\delta - \theta_\Sigma) = 2$, $k\delta - \theta_\Sigma$ is the highest root in $\langle A(\Sigma)\rangle = \langle (\Pi \setminus \Sigma) \cup \{\alpha_\Sigma\}\rangle$, α_Σ is long and is at the edge of $A(\Sigma)$ next to β, and $s_{\alpha_\Sigma}(k\delta - \theta_\Sigma) = k\delta - \theta_\Sigma$, then $c_{\alpha_\Sigma}(k\delta - \theta_\Sigma) = 1$ and so $c_{\alpha_\Sigma}(k\delta) = 2$. For the converse suppose $c_{\alpha_\Sigma}(k\delta) = 2$, then $c_{\alpha_\Sigma}(k\delta - \theta_\Sigma) = 1$. $\alpha_\Sigma \in \mathit{Supp}\,(k\delta - \theta_\Sigma)$ so $\alpha_\Sigma \in \Gamma(\Sigma)$; this implies $s_{\alpha_\Sigma}(k\delta - \theta_\Sigma) = k\delta - \theta_\Sigma$, and since $c_\beta(k\delta - \theta_\Sigma) = 2$ any other simple root in Σ cannot be in $\mathit{Supp}\,(k\delta - \theta_\Sigma) = A(\Sigma)$.

□

Lemma 3.2.6. *Assume $\gamma \in \langle A(\Sigma)\rangle$ with $c_\beta(\gamma) = 2$. Then $\gamma = k\delta - \theta_\Sigma$.*

Proof. Suppose there is a root $\gamma \in A(\Sigma), \gamma \neq k\delta - \theta_\Sigma$ with $c_\beta(\gamma) = 2$; then write $\gamma + R = k\delta - \theta_\Sigma$ with R a (non zero) sum of simple roots in $A(\Sigma) \setminus \{\beta\}$. So $\gamma = k\delta - \theta_\Sigma - R$ but $\theta_\Sigma + R$ is not a root because $c_\beta(\theta_\Sigma + R) = 0$ against the maximality of θ_Σ. □

Lemma 3.2.7. *Let $\gamma \in \langle A(\Sigma)\rangle_l$, then $u_\gamma^\Sigma \in \mathcal{W}_\sigma^{ab}$.*

Proof. Write $u_\gamma^\Sigma = s_1 \dots s_n$ in reduced form, with $s_i = s_{\alpha_i}$, and $q_j = s_1 \dots s_{j-1}(\alpha_j)$. Note that, for $1 \leq j \leq n$, we have $q_j > 0$ and thus $c_\beta(q_j) \geq 0$. Moreover $q_j \in \langle A(\Sigma)\rangle$, so $c_\beta(q_j) \leq 2$. We want to prove that $c_\beta(q_j) = 1$ for all j. Write

$$s_j(s_{j+1} \dots s_n(\gamma)) = s_{j+1} \dots s_n(\gamma) + a_j\alpha_j,$$

so that, multiplying by $s_1 \cdots s_j$, we have

$$s_1 \dots s_{j-1}s_{j+1} \dots s_n(\gamma) = k\delta - \theta_\Sigma - a_jq_j.$$

Notice that $a_j \neq 0$ by minimality of u_γ^Σ. Moreover since $k\delta - \theta_\Sigma - a_j q_j \in \langle A(\Sigma)\rangle$ we have $a_j > 0$ by maximality of $k\delta - \theta_\Sigma$. If $c_\beta(q_j) = 0$, since $a_j > 0$ and $\theta_\Sigma + a_j q_j$ is a root, we have $q_j \in \langle\Sigma\rangle$ but θ_Σ is maximal in $\langle\Sigma\rangle$. If $c_\beta(q_j) = 2$, by lemma 3.2.6, $q_j = k\delta - \theta_\Sigma$. So $s_1 \cdots s_{j-1}(\alpha_j) = k\delta - \theta_\Sigma$, but also $s_1 \cdots s_{j-1}(s_j \cdots s_n)(\gamma) = k\delta - \theta_\Sigma$, and together they give us $s_j \cdots s_n(\gamma) = \alpha_j$ and finally

$$-\alpha_j = s_{j+1} \cdots s_n(\gamma) \geq \gamma > 0$$

since $a_i > 0$ for every i, but this is absurd. So $c_\beta(q_j) = 1$ for every j and the claim follows. □

The proof of the following lemma is similar to that of Lemma 2.2.3. We include it for completeness.

Lemma 3.2.8. *Let $\gamma \in \langle A(\Sigma)\rangle_l$, then $u_\gamma^\Sigma = \min \mathcal{I}_{\gamma, k\delta - \theta_\Sigma}$.*

Proof. Let $w \in \mathcal{I}_{\gamma, k\delta-\theta_\Sigma}$ be minimal, $w \neq u_\gamma^\Sigma$. Consider $N(u_\gamma^\Sigma) \cap N(w) = \Psi$. If $\alpha, \tau \in \Psi$ then $\alpha, \tau \in N(u_\gamma^\Sigma)$ and $\alpha, \tau \in N(w)$, since those are inversion sets $\alpha + \tau \in N(u_\gamma^\Sigma), N(w)$, and thus $\alpha + \tau \in \Psi$. On the other hand if $\alpha + \tau \in \Psi$, then $\alpha + \tau \in N(u_\gamma^\Sigma), N(w)$. Since those are inversion sets of σ-minuscule elements, exactly one among α and τ can have $c_\beta = 1$, let's say α, so $\alpha \in N(u_\gamma^\Sigma)$, $\alpha \in N(w)$ and thus $\alpha \in \Psi$. Hence Ψ turns out to be an inversion set, so we can write

$$N(u_\gamma^\Sigma) \cap N(w) = N(u_1 \cdots u_l)$$

for some $l = 0, \cdots, n-1$. Let's call γ_i the rootlet of $u_1 \cdots u_i$. We have seen in Lemma 3.2.7 that $\gamma_{i-1} = u_i(\gamma_i) = \gamma_i + a_i\alpha_i > \gamma_i$ because $a_i > 0$ for every i. Hence

$$k\delta - \theta_\Sigma = \gamma_0 > \gamma_1 > \cdots > \gamma_n = \gamma.$$

The rootlet of $w = u_1 \cdots u_l t_1 \cdots t_m$ (which is written in reduced form) is γ, thus all the simple reflections in $u_{l+1}, \cdots, u_n$ must appear at least once in $t_1, \cdots, t_m$, in particular u_{l+1}. Let's say $t_j = u_{l+1}$ for some j, and assume that $t_i \neq u_{l+1}$ for all $i < j$. We have $u_1 \cdots u_l t_1 \cdots t_{j-1} u_{l+1} \in \mathcal{W}_\sigma^{ab}$. Call τ_i the simple root corresponding to t_i, write $q_i = u_1 \cdots u_l t_1 \cdots t_{i-1}(\tau_i) \in N(w)$ and $t_i(\tau_{l+1}) = \tau_{l+1} + b_i \tau_i$ for some $b_i \geq 0$ since $t_i \neq u_{l+1}$, all of them for every $i = 1, \ldots, m$. Then

$$u_1 \cdots u_l t_1 \cdots t_{j-1}(\alpha_{l+1}) = u_1 \cdots u_l t_1 \cdots t_{j-2}(\alpha_{l+1} + b_{j-1}\tau_{j-1}) =$$

$$= b_{j-1}q_{j-1} + u_1 \cdots u_l t_1 \cdots t_{j-2}(\alpha_{l+1}) = \cdots = \sum_{i=1}^{j-1} b_i q_i + u_1 \cdots u_l(\alpha_{l+1})$$

must have c_β equal 1. Since all the q_i's and $u_1 \cdots u_l(\alpha_{l+1})$ have $c_\beta = 1$ and the b_i's are non negative, then $b_i = 0$ for all $i = 1, \ldots, j-1$ and

$$t_i(\alpha_{l+1}) = \alpha_{l+1}$$

for all $i = 1, \ldots, j-1$. So $t_i u_{l+1} = u_{l+1} t_i$ for all $i = 1, \ldots, j-1$ and thus we can write

$$w = u_1 \cdots u_l u_{l+1} t_1 \cdots t_{j-1} t_{j+1} \cdots t_m$$

against the fact that $N(u_\gamma^\Sigma) \cap N(w) = N(u_1 \cdots u_l)$. $\square$

Lemma 3.2.9. *Let $\gamma \in \langle A(\Sigma) \rangle_l$ and let q be a simple root such that $s_q(\gamma) = \gamma$. Then*

(1) If $q \in A(\Sigma)$, then $c_\beta(u_\gamma^\Sigma(q)) = 0$.

(2) If $q \notin A(\Sigma)$ and is not connected to $A(\Sigma)$ in the Dynkin diagram, then $u(q) = q$ and $c_\beta(u_\gamma^\Sigma(q)) = 0$.

(3) If $q \notin A(\Sigma)$ and it is connected to $A(\Sigma)$ in the Dynkin diagram, then $c_\beta(u_\gamma^\Sigma(q)) = 1$.

Proof. Notice first that if $s_q(\gamma) = \gamma$ then $u_\gamma^\Sigma(q) > 0$, otherwise u_γ^Σ has a reduced form ending in s_q, and the remaining element is still σ-minuscule, against the minimality of u_γ^Σ in $\mathcal{I}_{\gamma,k\delta-\theta_\Sigma}$.

(1). Notice that $u_\gamma^\Sigma(q) \in \langle A(\Sigma) \rangle$ and thus $0 \leq c_\beta(u_\gamma^\Sigma(q)) \leq 2$. Suppose $c_\beta(u_\gamma^\Sigma(q)) = 2$, then, by Lemma 3.2.6, $u_\gamma^\Sigma(q) = k\delta - \theta_\Sigma$ and so $\gamma = q$ which is against $s_q(\gamma) = \gamma$. Suppose now $c_\beta(u_\gamma^\Sigma(q)) = 1$, then there exists $v \in W(A(\Sigma) \setminus \{\beta\})$

such that $v(\beta) = u_\gamma^\Sigma(q)$ and by definition $v(\theta_\Sigma) = \theta_\Sigma$. Then $v(\theta_\Sigma + \beta) = u_\gamma^\Sigma(q) + \theta_\Sigma$ is a root. We get

$$u_\gamma^\Sigma s_q (u_\gamma^\Sigma)^{-1}(u_\gamma^\Sigma(q) + \theta_\Sigma) = u_\gamma^\Sigma s_q(q + k\delta - \gamma) = u_\gamma^\Sigma(-q + k\delta - \gamma) = -u_\gamma^\Sigma(q) + \theta_\Sigma.$$

On the other hand $-u_\gamma^\Sigma(q) + \theta_\Sigma$ is not a root, because $c_\beta(-u_\gamma^\Sigma(q) + \theta_\Sigma) = -1$ and $c_\xi(-u_\gamma^\Sigma(q) + \theta_\Sigma) > 0$, ξ being a simple root in Σ not orthogonal to θ_Σ (in particular, $\xi \notin A(\Sigma)$).

(2). Obvious.

(3). Let's divide the proof into two cases. Assume first that $A(\Sigma) \cup \{q\}$ is not the whole Dynkin diagram (e.g. in type $E_6^{(1)}$). Consider the unique path of simple roots connecting the support of γ to q in the Dynkin diagram and call the simple roots $\alpha_1, \ldots, \alpha_n, q$ and their simple reflections $s_1, \ldots, s_n, s_q$. Set for short $u = u_\gamma^\Sigma, A = A(\Sigma)$. We can compute $s_q s_n \cdots s_2(\alpha_1) = \alpha_1 + b_2\alpha_2 + \cdots + b_n\alpha_n + b_q q$ with $b_i > 0$, and $s_1(\gamma) = \gamma + a_1\alpha_1$ with $a_1 > 0$, and finally

$$us_q s_n \cdots s_2 s_1(\gamma) = k\delta - \theta_\Sigma + a_1(u(\alpha_1) + b_2 u(\alpha_2) + \cdots + b_n u(\alpha_n) + b_q u(q)).$$

Thanks to part (1) we have

$$c_\beta(u(\alpha_i)) = 0 \quad i = 2, \ldots, n. \tag{3.3}$$

To compute $c_\beta(u(\alpha_1))$ we just observe that $us_1(\gamma) = k\delta - \theta_\Sigma + a_1 u(\alpha_1)$ and since $a_1 > 0$ and $u(\alpha_1) \in \langle A \rangle$, by the definition of A as in the previous lemma we have $u(\alpha_1) < 0$ and thus $c_\beta(u(\alpha_1)) \leq -1$ (and of course at least -2). If $c_\beta(u(\alpha_1)) = -2$, by Lemma 3.2.6 we have $u(\alpha_1) = -k\delta + \theta_\Sigma$ and so $u(-\alpha_1) = k\delta - \theta_\Sigma$ which is absurd because $-\alpha_1 = \gamma$ and rootlets must be positive. Hence

$$c_\beta(u(\alpha_1)) = -1. \tag{3.4}$$

Note now that

$$u(\alpha_1) + b_2 u(\alpha_2) + \cdots + b_n u(\alpha_n) + b_q u(q) = u(\alpha_1 + b_2\alpha_2 + \cdots + b_n\alpha_n + b_q q) \in \widehat{\Delta}^+ \tag{3.5}$$

since $q \notin A$ and $\alpha_i \in A$ and $u \in W(A(\Sigma))$. Evaluating $c_\beta(u(q))$ from (3.5) and using (3.3), (3.4) we get $c_\beta(u(q)) \geq 1$. On the other hand if $c_\beta(u(q))$ were greater than 1, we would have $c_\beta(\theta_\Sigma - a_1 u(\alpha_1 + b_2\alpha_2 + \cdots + b_n\alpha_n + b_q q)) < 0$ and thus $\theta_\Sigma - a_1 u(\alpha_1 + b_2\alpha_2 + \cdots + b_n\alpha_n + b_q q) < 0$ which is not possible by the assumption that $A \cup \{q\}$ is not the whole Dynkin diagram (so some simple roots in Σ are missing in $u(\alpha_1 + b_2\alpha_2 + \cdots + b_n\alpha_n + b_q q)$). Therefore $c_\beta(u(q)) = 1$.

Suppose now $A \cup \{q\}$ is the whole Dynkin diagram (e.g. $F_4^{(1)}$, $p = 2$). If $k\delta - q \in \langle A \rangle$ then since $c_\beta(k\delta - q) = 2$ we have $k\delta - q = k\delta - \theta_\Sigma$ and so $q = \theta_\Sigma$ which is against $|\Sigma| > 1$. This implies $u(q) < k\delta$ and thus $c_\beta(u(q)) \leq 2$. Suppose $c_\beta(u(q)) = 2$. By inspection $c_q(k\delta) \leq 2$, and so $b_q \leq 2$ because $\alpha_1 + b_2\alpha_2 + \cdots + b_n\alpha_n + b_q q < k\delta$. Indeed $c_q(k\delta) = 2$, since if $c_q(k\delta) = 1$, then $k\delta - q \in A$, which is excluded. If $b_q = 2$ then $c_q(u(\alpha_1 + b_2\alpha_2 + \cdots + b_n\alpha_n + b_q q)) = 2$ and $c_\beta(u(\alpha_1 + b_2\alpha_2 + \cdots + b_n\alpha_n + b_q q)) = 3$, and so $u(\alpha_1 + b_2\alpha_2 + \cdots + b_n\alpha_n + b_q q) = k\delta + \tau$ with $\tau \in \langle A \rangle$. But then $k\delta - \theta_\Sigma + a_1(k\delta + \tau) = (1 + a_1)k\delta - \theta_\Sigma + a_1\tau$ is a root, hence $k\delta - \theta_\Sigma + a_1\tau$ is a root in $\langle A \rangle$, which is absurd since it is greater than the highest root. If $b_q = 1$ and $a_1 = 1$ then $c_q(u(\alpha_1 + b_2\alpha_2 + \cdots + b_n\alpha_n + b_q q)) = 1$ and $c_q(\theta_\Sigma) = 1$ and $c_\beta(u(\alpha_1 + b_2\alpha_2 + \cdots + b_n\alpha_n + b_q q)) = 1$, so $-\theta_\Sigma + u(\alpha_1 + b_2\alpha_2 + \cdots + b_n\alpha_n + b_q q) > 0$. On the other hand $c_q(-\theta_\Sigma + u(\alpha_1 + b_2\alpha_2 + \cdots + b_n\alpha_n + b_q q)) > 0$, so $c_q(\theta_\Sigma) = 1$, which is not possible since $k\delta - \theta_\Sigma \in \langle A(\Sigma) \rangle$.

If $b_q = 1$ and $a_1 = 2$ we have $c_\beta(-\theta_\Sigma + 2u(\alpha_1 + b_2\alpha_2 + \cdots + b_n\alpha_n + b_q q)) = 2$ and $c_q(-\theta_\Sigma + 2u(\alpha_1 + b_2\alpha_2 + \cdots + b_n\alpha_n + b_q q)) = 0$ since again $c_q(\theta_\Sigma) = 2$, so $-\theta_\Sigma + 2u(\alpha_1 + b_2\alpha_2 + \cdots + b_n\alpha_n + b_q q) \in \langle A \rangle$ with $c_\beta = 2$ and thus we must have $-\theta_\Sigma + 2u(\alpha_1 + b_2\alpha_2 + \cdots + b_n\alpha_n + b_q q) = k\delta - \theta_\Sigma$ implying $u(\alpha_1 + b_2\alpha_2 + \cdots + b_n\alpha_n + b_q q) = \frac{k\delta}{2}$, which is not a root, contradiction. □

Corollary 3.2.10. *Let* $\gamma \in \langle A(\Sigma) \rangle_l$ *and* $w \in \widehat{W}$ *be a product of simple reflections fixing* γ*. Suppose that* $u_\gamma^\Sigma w$ *is in reduced form. Let* Ψ *be the set of simple roots in* $\Sigma \setminus \Gamma(\Sigma)$ *connected to* $A(\Sigma)$*. Then* $u_\gamma^\Sigma w \in \mathcal{I}_{\gamma, k\delta - \theta_\Sigma}$ *if and only if for every* $\tau \in N(w)$ *we have* $\sum_{h \in \Psi} c_h(\tau) = 1$*.*

Proof. Assume $\sum_{h \in \Psi} c_h(\tau) = 1$ for every $\tau \in N(w)$. Since $N(u_\gamma^\Sigma w) = N(u_\gamma^\Sigma) \sqcup u_\gamma^\Sigma(N(w))$, it suffices to show that $c_\beta(u_\gamma^\Sigma(\tau)) = 1$ for all $\tau \in N(w)$. By assumption τ is a sum of simple roots only one of which, say $\bar{\tau}$, is in Ψ. Notice that $u_\gamma^\Sigma(\bar{\tau}) > 0$. By Lemma 3.2.9, $c_\beta(u_\gamma^\Sigma(\tau)) = 1$. The converse is similar. □

For $c_{\alpha_\Sigma}(k\delta) = 3, 4$:

Lemma 3.2.11. *Let $w' \in \mathcal{W}_\sigma^{ab}$. Suppose we can write $w' = ws_q$ in reduced form such that $s_q(\gamma) = \gamma - aq$ for some positive a, a simple root q and a rootlet γ. If $c_{\alpha_\Sigma}(k\delta) = 4$ then $w' = s_\beta w_{0,\beta} w_0$ and $w' \in \mathcal{I}_{\alpha_\Sigma+\beta,k\delta-\theta_\Sigma}$. If $c_{\alpha_\Sigma}(k\delta) = 3$ let x be the simple root connected to β and not in Σ, then $w' = s_\beta w_{0,\beta} w_0$ and $w' \in \mathcal{I}_{\alpha_\Sigma+\beta,k\delta-\theta_\Sigma}$, or $w' = s_\beta w_{0,\beta} w_0 s_x$ and $w' \in \mathcal{I}_{\alpha_\Sigma+\beta+x,k\delta-\theta_\Sigma}$.*

Proof. We claim that $a = 1$. We have $w(\gamma - aq) = ws_q(\gamma) = k\delta - \theta_\Sigma$ and so

$$w(\gamma) = k\delta - \theta_\Sigma + aw(q).$$

Note that $c_\beta(w(q)) = 1$ since $ws_q \in \mathcal{W}_\sigma^{ab}$, and so $-\theta_\Sigma + aw(q) > 0$.

$$w^{-1}(-\theta_\Sigma + aw(q)) = -w^{-1}(\theta_\Sigma) + aq < 0$$

because $w \in \mathcal{W}_\sigma^{ab}$, unless $w^{-1}(\theta_\Sigma) = q$, which is not possible because then $w(q) = \theta_\Sigma$ and $c_\beta(w(q)) = 0$. Thus we get $-\theta_\Sigma + aw(q) \in N(w)$ and finally $a = 1$. We claim that $\theta_\Sigma + \beta \in N(ws_q)$. Write $w(q) = \theta_\Sigma + \beta + R$ with R a sum of simple roots not containing β, and in general not a root. Then

$$w^{-1}(\theta_\Sigma + \beta) = q - w^{-1}(R) < 0$$

unless $R = 0$, in which case $w(q) = \theta_\Sigma + \beta \in N(ws_q)$ already, otherwise $\theta_\Sigma + \beta \in N(w) \subset N(ws_q)$. Our claim implies that ws_q can be written in reduced form starting with $s_\beta w_{0,\alpha_\Sigma} w_0$. In the case $c_{\alpha_\Sigma}(k\delta) = 4$ we have $w_{0,\alpha_\Sigma} = w_{0,\beta}$ and so $ws_q = s_\beta w_{0,\beta} w_0 = w_\beta$ since it is maximal in $\mathcal{W}_\sigma^{ab}$. We have $w_\beta(\beta) = k\delta + \beta$ and $w_\beta(\alpha_\Sigma) = s_\beta w_{0,\beta}(\alpha_\Sigma) = -s_\beta(\theta_S) = -\theta_\Sigma - \beta$, so $w_\beta(\alpha_\Sigma + \beta) = k\delta + \beta - \theta_\Sigma - \beta = k\delta - \theta_\Sigma$ and $w_\beta \in \mathcal{I}_{\alpha_\Sigma+\beta,k\delta-\theta_\Sigma}$. In the case $c_{\alpha_\Sigma}(k\delta) = 3$ we have $w_{0,\alpha_\Sigma} = w_{0,\beta} s_x$ and so ws_q starts with $s_\beta w_{0,\beta} s_x w_0 = s_\beta w_{0,\beta} w_0 s_x = w_\beta s_x < w_\beta$. There is no simple root $y \neq x$ such that $w_\beta s_x s_y > w_\beta s_x$ and $w_\beta s_x s_y \in \mathcal{W}_\sigma^{ab}$, indeed if $y \neq \beta$ then $w_\beta s_x s_y = w_\beta s_y s_x$ and $w_\beta s_y \in \mathcal{W}_\sigma^{ab}$ but w_β is maximal, if $y = \beta$ we get $w_\beta s_x(\beta) = w_\beta(x+\beta) = w_\beta(x) + k\delta + \beta = s_\beta w_{0,\beta} w_0(x) + k\delta + \beta = s_\beta w_{0,\beta}(-x) + k\delta + \beta = s_\beta(-x) + k\delta + \beta = -x - \beta + \beta + k\delta = k\delta - x$ with $c_\beta(k\delta - x) = 2 \neq 1$. Finally we see that $w_\beta s_x(\alpha_\Sigma+\beta+x) = w_\beta(\alpha_\Sigma+\beta) = k\delta-\theta_\Sigma$ and $w_\beta s_x \in \mathcal{I}_{\alpha_\Sigma+\beta+x,k\delta-\theta_\Sigma}$, and as before $w_\beta \in \mathcal{I}_{\alpha_\Sigma+\beta,k\delta-\theta_\Sigma}$. We claim that all these elements are not the minimum in their posets, nor they can be obtained extending the minimum word in their posets using simple reflections fixing their rootlets, indeed these are new elements. They can't be minimal elements because w_β contains simple reflections related to simple roots not in $\langle A(\Sigma)\rangle$ because $w_\beta(\beta) = k\delta+\beta > k\delta$, and for $w_\beta s_x < w_\beta$, $s_x \in W(A(\Sigma))$ anyway. They are not even minimal elements with added simple reflections fixing γ, indeed they end in s_{α_Σ} because $w_\beta s_x(\alpha_\Sigma) = w_\beta(\alpha_\Sigma) = -\theta_\Sigma - \beta < 0$, and $s_{\alpha_\Sigma}(\alpha_\Sigma+\beta) = \beta \neq \alpha_\Sigma+\beta$, $s_{\alpha_\Sigma}(\alpha_\Sigma+\beta+x) = \beta+x \neq \alpha_\Sigma+\beta+x$. This proves we found new elements. □

For $c_{\alpha_\Sigma}(k\delta) = 2$:

Lemma 3.2.12. $\mathcal{I}_{k\delta-\theta_\Sigma,k\delta-\theta_\Sigma} = \{1, s_\beta w_{0,\alpha_\Sigma} w_0 s_\beta\}$.

Proof. We start proving the following relations:

(1) $w_{0,\alpha_\Sigma}(\theta_\Sigma) = \alpha_\Sigma$,

(2) $w_{0,\alpha_\Sigma}(\alpha_\Sigma) = \theta_\Sigma$,

(3) $w_{0,\alpha_\Sigma}(\beta) = k\delta - \beta - \theta_\Sigma - \alpha_\Sigma$.

To prove (1) we show that $w_{0,\alpha_\Sigma}(\theta_\Sigma)$ is a simple root. Suppose that there exists a simple root q and $a \in \mathbb{N}$ such that $w_{0,\alpha}(\theta_\Sigma) - aq$ is a root, then applying $w_{0,\alpha}$ we get that $\theta_\Sigma - aw_{0,\alpha}(q)$ is a root as well. $q \notin \Sigma \setminus \{\alpha_\Sigma\}$ otherwise $w_{0,\alpha_\Sigma}(q) < 0$ against the fact that θ_Σ is the highest root in $\langle\Sigma\rangle$. So $q = \alpha_\Sigma$. If $a = 2$, since $c_{\alpha_\Sigma}(\theta_\Sigma) = 1$, $c_{\alpha_\Sigma}(\theta_\Sigma - 2w_{0,\alpha_\Sigma}(\alpha_\Sigma)) = -1$ and thus $\theta_\Sigma = \alpha_\Sigma$ against the fact that $|\Sigma| > 1$. If $a = 1$ then $w_{0,\alpha_\Sigma}(\theta_\Sigma) - \alpha_\Sigma > 0$ and $w_{0,\alpha_\Sigma}(\theta_\Sigma) - \alpha_\Sigma \in \Sigma \setminus \{\alpha_\Sigma\}$, so applying w_{0,α_Σ} we get $\theta_\Sigma - w_{0,\alpha_\Sigma}(\alpha_\Sigma) < 0$ which is absurd since θ_Σ is the highest root in $\langle\Sigma\rangle$ and $w_{0,\alpha_\Sigma}(\alpha_\Sigma) \in \Sigma$. We conclude that $w_{0,\alpha_\Sigma}(\theta_\Sigma)$ is a simple root, and since $c_{\alpha_\Sigma}(w_{0,\alpha_\Sigma}(\theta_\Sigma)) = 1$ we must have $w_{0,\alpha_\Sigma}(\theta_\Sigma) = \alpha_\Sigma$. The second identity follows from the first one applying w_{0,α_Σ}. To prove (3), recall by Lemma 3.2.1 that $w_0(\beta) = k\delta-\beta$. Since w_0 can be decomposed into commuting subwords according to the connected componets of the Dynkin diagram, as well as w_{0,α_Σ}, and they coincide

everywhere but on the component Σ, we get that $w_{0,\alpha_\Sigma}(\beta) = k\delta - \beta - k\delta_\Sigma$, where $k\delta = \sum_\tau a_\tau \tau$ and $k\delta_\Sigma = \sum_{\tau\in\Sigma} a_\tau \tau$ ($k\delta_\Sigma$ is not necessarily a root). Consider the root $s_{\alpha_\Sigma} s_\beta(k\delta - \theta_\Sigma) = k\delta - \theta_\Sigma - \alpha_\Sigma - \beta$. Since $c_{\alpha_\Sigma}(k\delta - \theta_\Sigma - \alpha_\Sigma - \beta) = 0$ and $c_\beta(k\delta - \theta_\Sigma - \alpha_\Sigma - \beta) = 1$ its support is completely contained outside of Σ and so $k\delta_\Sigma = \theta_\Sigma + \alpha_\Sigma$, then $w_{0,\alpha_\Sigma}(\beta) = k\delta - \beta - \alpha_\Sigma - \theta_\Sigma$.
Let's go back to the main claim. We first check that $s_\beta w_{0,\alpha_\Sigma} w_0 s_\beta(\theta_\Sigma) = \theta_\Sigma$. We have

$$s_\beta w_{0,\alpha_\Sigma} w_0 s_\beta(\theta_\Sigma) = s_\beta w_{0,\alpha_\Sigma} w_0(\theta_\Sigma + \beta) = s_\beta w_{0,\alpha_\Sigma}(k\delta - \beta - \theta_\Sigma) = s_\beta(\beta + \theta_\Sigma) = \theta_\Sigma.$$

We now need to check that $s_\beta w_{0,\alpha_\Sigma} w_0 s_\beta \in \mathcal{W}_\sigma^{ab}$.

$$N(s_\beta w_{0,\alpha_\Sigma} w_0 s_\beta) = \{\beta\} \cup s_\beta N(w_{0,\alpha_\Sigma} w_0) \cup s_\beta w_{0,\alpha_\Sigma} w_0(\beta).$$

$N(w_{0,\alpha_\Sigma} w_0)$ contains exactly all the roots in Σ with $c_{\alpha_\Sigma} = 1$, so $c_\beta(s_\beta N(w_{0,\alpha_\Sigma} w_0)) = 1$. For the last element

$$s_\beta w_{0,\alpha_\Sigma} w_0(\beta) = s_\beta w_{0,\alpha_\Sigma}(k\delta - \beta) = s_\beta(\alpha_\Sigma + \beta + \theta_\Sigma) = \alpha + \beta + \theta_\Sigma$$

and so $c_\beta = 1$. We now want to prove that every element $w \in \mathcal{I}_{k\delta-\theta_\Sigma, k\delta-\theta_\Sigma}$ such that $w \neq 1$ must start with $s_\beta w_{0,\alpha_\Sigma} w_0 s_\beta$; this will end the proof, since we will also show that this element is maximal in $\mathcal{W}_\sigma^{ab}$. We need to prove that $\theta_\Sigma + \beta \in N(w)$. We have $w^{-1}(\theta_\Sigma + \beta) = \theta_\Sigma + w^{-1}(\beta)$, if it is positive, then $w^{-1}(\beta) \in \Sigma$. To find a contradiction we just need to prove that w^{-1} maps Σ to Σ, because it is invertible and Σ is a finite set. Consider any $\tau \in \Sigma$, then $w^{-1}(\tau) + w^{-1}(\theta_\Sigma - \tau) = \theta_\Sigma$, $\theta_\Sigma - \tau$ could be not a root, but both addends are positive because w is σ-minuscule, then $w^{-1}(\tau) \in \Sigma$ and the claim follows. In the end $w^{-1}(\beta) \in \Sigma$ is impossible and $w^{-1}(\theta_\Sigma + \beta) < 0$, in particular $c_\beta(w^{-1}(\beta)) < 0$. Now for every other root $\tau \in \Sigma$ with $c_{\alpha_\Sigma}(\tau) = 1$ we have $w^{-1}(\tau + \beta) < 0$ since $c_\beta(w^{-1}(\beta)) < 0$ and so w must start with $s_\beta w_{0,\alpha_\Sigma} w_0$. For the last element we have $w^{-1}(\alpha + \beta + \theta_\Sigma) < 0$ since $c_\beta(w^{-1}(\beta)) < 0$, $w^{-1}(\alpha_\Sigma), w^{-1}(\theta_\Sigma) \in \Sigma$. So w must start with $s_\beta w_{0,\alpha_\Sigma} w_0 s_\beta$.
It remains to prove that $s_\beta w_{0,\alpha_\Sigma} w_0 s_\beta$ is maximal in $\mathcal{W}_\sigma^{ab}$. We try to add s_{α_Σ}, s_x for every simple root x linked to β and $x \neq \alpha_\Sigma$, and s_y for any other simple root y.

$$s_\beta w_{0,\alpha_\Sigma} w_0 s_\beta(\alpha_\Sigma) = s_\beta w_{0,\alpha_\Sigma} w_0(\alpha + \beta) = s_\beta w_{0,\alpha_\Sigma}(k\delta - \alpha_\Sigma - \beta) = s_\beta(\alpha_\Sigma + \beta) = \alpha_\Sigma,$$

$$s_\beta w_{0,\alpha_\Sigma} w_0 s_\beta(x) = s_\beta w_{0,\alpha_\Sigma} w_0(x+\beta) = s_\beta(\alpha_\Sigma + \beta + \theta_\Sigma + x) = \alpha_\Sigma + 2\beta + \theta_\Sigma + x,$$

$$s_\beta w_{0,\alpha_\Sigma} w_0 s_\beta(y) = s_\beta w_{0,\alpha_\Sigma} w_0(y) = s_\beta(y') = y'$$

where y' is a positive simple root not β and not linked to β. In each case the σ-height of the resulting element is not 1, thus the element $s_\beta w_{0,\alpha_\Sigma} w_0 s_\beta$ is maximal in $\mathcal{W}_\sigma^{ab}$. □

Lemma 3.2.13. *Let $v \in \mathcal{I}_{\gamma,k\delta-\theta_\Sigma}$, $\gamma \neq \alpha_\Sigma$, then $s_\beta w_{0,\alpha} w_0 s_\beta v \in \mathcal{I}_{\gamma,k\delta-\theta_\Sigma}$.*

Proof. We just need to check that $s_\beta w_{0,\alpha} w_0 s_\beta v$ is σ-minuscule. Call the eventual simple roots not in Σ connected to β, x_1 and x_2, with simple reflections s_1 and s_2, or just x if there is just 1. Write $v = v_1 \cdots v_n$ in reduced form, and $\tau_j = v_1 \cdots v_{j-1}(\alpha_j) = \beta + a\alpha_\Sigma + bx_1 + cx_2 + R$ with R a sum of other simple roots. Note that $a+b+c \leq 4$. In particular we claim that $a+b+c \leq 2$, indeed if $a+b+c=4$ then $c_\beta(s_\beta(\tau_j)) = 3$ and so $s_\beta(\tau_j) = k\delta + \beta$ and $\tau_j = k\delta - \beta$, which is not possible because otherwise $k\delta = (k\delta - \beta) + \beta \in N(v)$; if $a+b+c = 3$ then $s_\beta(\tau_j) = \tau_j + \beta \in N(v)$ but $c_\beta(\tau_j + \beta) = 2$ and the claim is proved. If both $b \neq 0$ and $c \neq 0$ we have $\tau_j = \beta + x_1 + x_2$ and $v = s_\beta s_1 s_2 s_\beta$, and in this case $s_\beta w_{0,x} w_0 s_\beta v$ is not σ-minuscule because $s_\beta w_{0,x} w_0 s_1 s_2(\beta) = s_\beta w_{0,x} w_0(\beta + x_1 + x_2) = s_\beta(\beta + \theta_\Sigma + \alpha_\Sigma + x_1 + x_2) = k\delta + \beta$ with $c_\beta = 3$. In this case its rootlet is $s_\beta s_2 s_1 s_\beta(k\delta - \theta_\Sigma) = \alpha_\Sigma$. In every other case we can just write $\tau_j = \beta + a\alpha_\Sigma + bx + R$ with $a + b \leq 2$ and compute $s_\beta w_{0,\alpha} w_0 s_\beta(\tau_j) = s_\beta w_{0,\alpha_\Sigma} w_0(-\beta + a\alpha_\Sigma + a\beta + bx + b\beta + R) = s_\beta w_{0,\alpha} w_0((a+b-1)\beta + a\alpha_\Sigma + bx + R) = s_\beta w_{0,\alpha}((a+b-1)(k\delta - \beta) - a\alpha_\Sigma - bx - R') = s_\beta((a+b-1)(\alpha_\Sigma + \beta + \theta_\Sigma) - a\theta_\Sigma + bx + R'') = (2b-1)\beta + (a+b-1)\alpha_\Sigma + (b-1)\theta_\Sigma + bx + R''$. In the end we need to look at the element

$$(2b-1)\beta + (a+b-1)\alpha_\Sigma + (b-1)\theta_\Sigma + bx + R''.$$

If $b = 0$ it is negative, and if $b = 1$ the $c_\beta = 1$, in both cases we are done. If $b = 2$ then $a = 0$ and the root is

$$3\beta + \alpha_\Sigma + \theta_\Sigma + 2x + R = k\delta + \beta$$

(for example because it is greater than $k\delta$ but applying s_β its c_β becomes 1). So $s_\beta w_{0,\alpha} w_0 s_\beta(\tau_j) = k\delta + \beta$ and $\tau_j = s_\beta w_0 w_{0,\alpha} s_\beta(k\delta + \beta) = k\delta - \alpha_\Sigma - \beta - \theta_\Sigma$. In case

$c_x(k\delta) = 2$ we have $\tau_j = k\delta - \alpha_\Sigma - \beta - \theta_\Sigma = \beta + x + \theta_2$ and so $v = s_\beta w_{0,x} w_0 s_\beta$ and its rootlet is $s_\beta w_0 w_{0,x} s_\beta(k\delta - \theta_\Sigma) = \alpha_\Sigma$. If otherwise $c_x(k\delta) = 1$ we have $\tau_j = k\delta - \alpha_\Sigma - \beta - \theta_\Sigma = \beta + x_1 + x_2$ we have already discussed. □

Note that $s_\beta s_1 s_2 s_\beta$ and $s_\beta w_{0,\alpha} w_0 s_\beta$ are both maximal in $\mathcal{W}_\sigma^{ab}$, so $\mathcal{I}_{\alpha_\Sigma, k\delta-\theta_\Sigma}$ is a singleton.

Lemma 3.2.14. *Let $\gamma \in \langle A(\Sigma)\rangle_l$. If $u_\gamma^\Sigma w \in \mathcal{I}_{\gamma,k\delta-\theta_\Sigma}$ and $c_{\alpha_\Sigma}(\gamma) = 0$ then w can be written as a product of simple reflections fixing γ. If $c_{\alpha_\Sigma}(\gamma) \neq 0$ and $\gamma \neq \alpha_\Sigma$, then $\mathcal{I}_{\gamma,k\delta-\theta_\Sigma} = \{u_\gamma^\Sigma, s_\beta w_{0,\alpha_\Sigma} w_0 s_\beta u_\gamma^\Sigma\}$.*

Proof. Suppose there is $w' \in \mathcal{I}_{\gamma,k\delta-\theta_\Sigma}$ such that it is u_γ^Σ extended with a block of simple reflections not all fixing γ. At some point starting from right there must be a simple reflection s_q for which $s_q(\gamma') = \gamma' - aq$ for some positive a and some rootlet γ', $w \in \mathcal{W}_\sigma^{ab}$ and $ws_q \in \mathcal{I}_{\gamma',k\delta-\theta_\Sigma}$; then as in Lemma 3.2.11 $a = 1$, $\theta_\Sigma + \beta \in N(ws_q)$ and ws_q starts with $s_\beta w_{0,\alpha_\Sigma} w_0$, and so does w', $w' = s_\beta w_{0,\alpha} w_0 v$ with $l(s_\beta w_{0,\alpha_\Sigma} w_0 v) = l(s_\beta w_{0,\alpha_\Sigma} w_0) + l(v)$ and v in reduced form. Left multiplication by $s_\beta w_{0,\alpha} w_0 s_\beta$ gives us $s_\beta v$ which is in $\mathcal{I}_{\gamma,k\delta-\theta_\Sigma}$ thanks to Lemma 3.2.13. Note that $s_\beta v$ cannot start with $s_\beta w_{0,\alpha} w_0$ and so $s_\beta v = u_\gamma^\Sigma f$ where f can be written as a product of simple reflections fixing γ and $l(u_\gamma^\Sigma f) = l(u_\gamma^\Sigma) + l(f)$. We claim that $s_\beta w_{0,\alpha_\Sigma} w_0 = s_\beta s_{\alpha_\Sigma} s s_{\alpha_\Sigma}$ where s is the shortest element such that $s(\alpha_\Sigma) = \theta_\Sigma$. We show that they have the same inversion set, indeed

$$N(s_\beta s_{\alpha_\Sigma} s s_{\alpha_\Sigma}) = \{\beta\} \cup \{\alpha_\Sigma + \beta\} \cup s_\beta s_{\alpha_\Sigma} N(s) \cup \{s_\beta s_{\alpha_\Sigma} s(\alpha_\Sigma)\}.$$

We have $s_\beta s_{\alpha_\Sigma} s(\alpha_\Sigma) = s_\beta s_{\alpha_\Sigma}(\theta_\Sigma) = s_\beta(\theta_\Sigma) = \beta + \theta_\Sigma$ because $\alpha_\Sigma \in \langle A(\Sigma)\rangle$. Since there is $\beta + \theta_\Sigma$ we just need to prove that $N(s)$ contains only roots in $\langle \Sigma \setminus \{\alpha_\Sigma\}\rangle$ with coefficient 1 for one simple root $z \in \Sigma$ connected to α_Σ. The first part is clear by minimality of s. Write $s = s_1 \cdots s_n$ in reduced form, and $\tau_j = s_1 \cdots s_{j-1}(\alpha_j) \in N(s)$. Then

$$s_1 \cdots s_{j-1} s_{j+1} \cdots s_n(\alpha_\Sigma) = \theta_\Sigma - a_j \tau_j$$

for some positive a_j since θ_Σ is maximal and s is minimal. Then since $k\delta - \theta_\Sigma$ is the highest root in $\langle A(\Sigma)\rangle$, τ_j must contain in its support a simple root z linked to $A(\Sigma)$, and so to α_Σ since $c_{\alpha_\Sigma}(k\delta) = 2$ is equivalent to $\Gamma(\Sigma) = \{\alpha_\Sigma\}$ thanks to lemma 3.2.5 (exactly one otherwise the support of τ_j is disconnected by α_Σ). If $c_z(\tau_j) = b > 1$ then $s_{\alpha_\Sigma}(\tau_j) = \tau_j + b\alpha_\Sigma$ with $c_{\alpha_\Sigma}(\tau_j + b\alpha_\Sigma) = b > 1$ but $\tau_j + b\alpha_\Sigma \in \langle\Sigma\rangle$ which is absurd. In conclusion $s_\beta w_{0,\alpha_\Sigma} w_0 = s_\beta s_{\alpha_\Sigma} s s_{\alpha_\Sigma}$. If $c_{\alpha_\Sigma}(\gamma) \neq 0$, since $c_{\alpha_\Sigma}(k\delta - \theta_\Sigma) = 1$ then also $c_{\alpha_\Sigma}(\gamma) = 1$. By minimality of u_γ^Σ it can be written itself in reduced form with just one simple reflection s_{α_Σ} and so of course we can write $u_\gamma^\Sigma = s_\beta s_{\alpha_\Sigma} \bar{u}$ with $\bar{u} \in W(A(\Sigma) \setminus \{\alpha_\Sigma\})$ and so $s_\beta v = u_\gamma^\Sigma f = s_\beta s_{\alpha_\Sigma} \bar{u} f$. We get $w' = s_\beta w_{0,\alpha_\Sigma} w_0 v = s_\beta s_{\alpha_\Sigma} s s_{\alpha_\Sigma} s_\beta s_\beta v = s_\beta s_{\alpha_\Sigma} s s_{\alpha_\Sigma} s_\beta u_\gamma^\Sigma f = s_\beta s_{\alpha_\Sigma} s s_{\alpha_\Sigma} s_\beta s_\beta s_{\alpha_\Sigma} \bar{u} f = s_\beta s_{\alpha_\Sigma} s \bar{u} f$. Now since $s \in W(\Sigma \setminus \{\alpha_\Sigma\})$ and $\bar{u} \in W(A(\Sigma) \setminus \{\alpha_\Sigma\})$ we get $s\bar{u} = \bar{u}s$ and s is made of simple reflections fixing $\gamma \in \langle A(\Sigma)\rangle$. In the end $w' = s_\beta w_{0,\alpha_\Sigma} w_0 v = s_\beta s_{\alpha_\Sigma} \bar{u} s f = u_\gamma^\Sigma s f$ which is against the assumptions. If otherwise $c_{\alpha_\Sigma}(\gamma) = 1$, $\gamma \neq \alpha_\Sigma$, then thanks to the previous lemmas $s_\beta v = u_\gamma^\Sigma$ because it can't be extended with simple reflections fixing γ. We conclude that $w' = s_\beta w_{0,\alpha_\Sigma} w_0 s_\beta u_\gamma^\Sigma \neq u_\gamma^\Sigma$. □

Proposition 3.2.15. *If $\mathcal{I}_{\gamma,k\delta-\theta_\Sigma} \neq \emptyset$, then $\gamma \in \langle A(\Sigma)\rangle_l = \widehat{\Delta}_\mu$.*

Proof. Suppose $v \in \mathcal{I}_{\gamma,k\delta-\theta_\Sigma}$ and $\gamma \notin \langle A(\Sigma)\rangle_l$. Then we can write $v = s_1 \cdots s_n$ with simple reflections in reduced form, and for some $j, 1 \leq j \leq n$ we have $s_{j+1} \cdots s_n(\gamma) \notin \langle A(\Sigma)\rangle_l$ and $\bar{\gamma} = s_j s_{j+1} \cdots s_n(\gamma) \in \langle A(\Sigma)\rangle_l$ definitively. Let's call q_j the simple root associated to s_j and $\bar{u}$ the shortest element in $\mathcal{I}_{\bar{\gamma},k\delta-\theta_\Sigma}$. As we already know, since $s_1 \cdots s_{j-1} \in \mathcal{I}_{\bar{\gamma},k\delta-\theta_\Sigma}$ in general we must have $s_1 \cdots s_{j-1} = \bar{u}w$ with $l(\bar{u}w) = l(\bar{u}) + l(w)$, w made of simple roots fixing $\bar{\gamma}$ and for every $\tau \in N(w)$ we have $\sum_{h\in\Psi} c_h(\tau) = 1$. We want to compute $c_\beta(\bar{u}w(q_j))$. w is made of simple reflections associated to simple roots that can be divided in connected components containing at least one root connected to $A(\Sigma)$, moreover every root is not in *Supp* $(\bar{\gamma})$, so there is exactly one root connected to $A(\Sigma)$ in every connected component. There can't be a connected component containing q_j since $s_j(\bar{\gamma}) \neq \bar{\gamma}$. Then we get $w(q_j) = q_j$. We are just required to compute $c_\beta(\bar{u}(q_j))$. $\bar{u}s_j(\bar{\gamma}) = k\delta - \theta_\Sigma + a\bar{u}(q_j)$ is a root for some $a > 0$, and so is $-\theta_\Sigma + a\bar{u}(q_j)$. If $A(\Sigma) \cup \{q_j\}$ is not the whole Dynkin diagram, then $-\theta_\Sigma + a\bar{u}(q_j) < 0$ and $c_\beta(\bar{u}(q_j)) = 0$. If otherwise $A(\Sigma) \cup \{q_j\}$ is the whole Dynkin diagram, then as we have seen in Lemma 3.2.9, $c_{q_j}(k\delta) = c_{q_j}(\theta_\Sigma) = 2$. If $a = 1$ then $c_{q_j}(-\theta_\Sigma + a\bar{u}(q_j)) = -1$ and so $c_\beta(\bar{u}(q_j)) = 0$. If $a = 2$ then

$c_{q_j}(-\theta_\Sigma+a\bar{u}(q_j))=0$ and $c_\beta(-\theta_\Sigma+a\bar{u}(q_j))=2$ since $\bar{u}s_j$ is σ-minuscule ($ws_j=s_jw$), and so $-\theta_\Sigma+2\bar{u}(q_j)=k\delta-\theta_\Sigma$ and $\bar{u}(q_j)=\frac{k\delta}{2}$ which is not possible. In every case we find a contradiction, or $c_\beta(\bar{u}w(q_j))\neq 1$, so $\bar{u}ws_j=s_1\cdots s_j$ is not σ-minuscule, which is absurd. We are left to check less common possibilities.

If $c_{\alpha_\Sigma}(k\delta)=3,4$ we could be in case $s_1\cdots s_{j-1}=s_\beta w_{0,\beta}w_0$ or $s_1\cdots s_{j-1}=s_\beta w_{0,\beta}w_0s_x$. In ther first case $w_{0,\beta}w_0(q_j)$ is a positive simple root not connected to β since $q_j\neq\alpha_\Sigma$, so $c_\beta(s_\beta w_{0,\beta}w_0(q_j))=0$. In the second case $s_\beta w_{0,\beta}w_0s_x(q_j)=s_\beta w_{0,\beta}w_0(q_j)$ so we can argue as in the first case.

If $c_{\alpha_\Sigma}(k\delta)=2$ we could be in case $s_1\cdots s_{j-1}=s_\beta w_{0,\alpha_\Sigma}w_0s_\beta\bar{u}$ and $s_1\cdots s_j=s_\beta w_{0,\alpha_\Sigma}w_0s_\beta\bar{u}s_j$. From the previous lemmas $s_\beta w_{0,\alpha_\Sigma}w_0s_\beta\bar{u}s_j\in\mathcal{W}^{ab}_\sigma$ iff $\bar{u}s_j\in\mathcal{W}^{ab}_\sigma$, but as we have seen earlier $c_\beta(\bar{u}(q_j))\neq 1$. □

Case b.

We assume that β is a long root, and we consider $\mu=k\delta-\theta_\Sigma$, with θ_Σ of type 1 and $|\Sigma|=1$. Note that necessarily $k=1$ and $c_{\theta_\Sigma}(\delta)=1$. We write W_{θ_Σ} for the Coxeter group associated to the finite Dynkin diagram obtained removing θ_Σ, which is a simple root, from the original diagram. We start showing in Lemmas 3.2.16 and 3.2.17 that $\widehat{\Delta}_\mu\subseteq\langle A(\Sigma)\rangle_l\cup(\{\delta-\langle A(\Sigma)\rangle_l\}\setminus\{\theta_\Sigma\})$. Then we break $\langle A(\Sigma)\rangle_l\cup(\{\delta-\langle A(\Sigma)\rangle_l\}\setminus\{\theta_\Sigma\})$ into its two components. For $\gamma\in\langle A(\Sigma)\rangle_l$, let u^Σ_γ be the shortest element in $W(A(\Sigma))$ such that $u^\Sigma_\gamma(\gamma)=\delta-\theta_\Sigma$, then it is σ-minuscule and is the minimum of $\mathcal{I}_{\gamma,\delta-\theta_\Sigma}$ as in Lemmas 3.2.7 and 3.2.8. We show that if $\gamma\in\langle A(\Sigma)\rangle_l$, then $\mathcal{I}_{\gamma,\mu}$ is a singleton in Lemma 3.2.20, and that if $\gamma\in(\{\delta-\langle A(\Sigma)\rangle_l\}\setminus\{\theta_\Sigma\})$, then $\mathcal{I}_{\gamma,\mu}$ is a singleton in Lemma 3.2.21. This also shows that $\widehat{\Delta}_\mu=\langle A(\Sigma)\rangle_l\cup(\{\delta-\langle A(\Sigma)\rangle_l\}\setminus\{\theta_\Sigma\})$. At the end of the section we give a closed formula to compute $|\mathcal{W}^{ab}_\sigma|$.

Lemma 3.2.16. $\mathcal{I}_{\theta_\Sigma,\delta-\theta_\Sigma}=\emptyset$.

Proof. Suppose $w\in\mathcal{I}_{\theta_\Sigma,\delta-\theta_\Sigma}$, then $w(\theta_\Sigma)=\delta-\theta_\Sigma$.

$$w^{-1}(\theta_\Sigma+\beta)=\delta-\theta_\Sigma+w^{-1}(\beta)>0$$

because $\theta_\Sigma - w^{-1}(\beta) > \delta \iff -w^{-1}(\beta) > \delta$ since θ_Σ is simple, if and only if $w^{-1}(\delta+\beta) < 0$, which is not possible since w is σ-minuscule. Now for every root τ with $c_\beta(\tau) = c_{\theta_\Sigma}(\tau) = 1$, $\tau = \beta + \theta_\Sigma + R$ we get

$$w^{-1}(\tau) = w^{-1}(\beta + \theta_\Sigma) + w^{-1}(R) > 0$$

since $w^{-1}(R) > 0$. Then for every $\tau \in N(w)$ we have $c_{\theta_\Sigma}(\tau) = 0$ and thus $w \in W_{\theta_\Sigma}$, but this is against the assumptions since

$$1 = c_{\theta_\Sigma}(\theta_\Sigma) = c_{\theta_\Sigma}(w(\theta_\Sigma)) = c_{\theta_\Sigma}(\delta - \theta_\Sigma) = 0.$$

□

Lemma 3.2.17. *If $\mathcal{I}_{\gamma,\delta-\theta_\Sigma} \neq \emptyset$ then $\gamma \in \langle A(\Sigma)\rangle_l \cup (\{\delta - \langle A(\Sigma)\rangle_l\} \setminus \{\theta_\Sigma\})$.*

Proof. Let w be in $\mathcal{I}_{\gamma,\delta-\theta_\Sigma}$, and notice that γ is long. $\gamma = w^{-1}(\delta - \theta_\Sigma)$ implies $c_{\theta_\Sigma}(\gamma) \geq 0$, and $\gamma = \delta - w^{-1}(\theta_\Sigma)$ implies $c_{\theta_\Sigma}(\gamma) \leq 1$. If $c_{\theta_\Sigma}(\gamma) = 0$ then $\gamma \in \langle A(\Sigma)\rangle_l$, if $c_{\theta_\Sigma}(\gamma) = 1$ then $\delta - \gamma = a \in \langle A(\Sigma)\rangle_l$ and so $\gamma = \delta - a \in \{\delta - \langle A(\Sigma)\rangle_l\}$. Thanks to the previous lemma we can cut out θ_Σ. □

For $\gamma \in \langle A(\Sigma)\rangle_l$, let $u_\gamma^\Sigma \in W_{\theta_\Sigma}$ be the shortest element such that $u_\gamma^\Sigma(\gamma) = \delta - \theta_\Sigma$, then it is σ-minuscule and is the minimum of $\mathcal{I}_{\gamma,\delta-\theta_\Sigma}$ as in Lemmas 3.2.7 and 3.2.8.

Lemma 3.2.18. *If $v \in \mathcal{I}_{\gamma,\delta-\theta_\Sigma}$, then $s_\beta w_{0,\beta} w_0 s_\beta v \in \mathcal{I}_{\delta-\gamma,\delta-\theta_\Sigma}$ unless $v = 1$.*

Proof. Let's write $v = v_1 \cdots v_n$ in reduced form and

$$\tau = v_1 v_2 \cdots v_{k-1}(\alpha_k) = \beta + \sum_i a_i \alpha_i + R \in N(v)$$

with α_k the simple root associated to v_k, α_i the simple roots connected to β in the Dynkin diagram, and R a sum of other simple roots. Recall that in general $\sum_i a_i \leq 4$, and since $\tau \in N(v)$ and v is σ-minuscule, in particular $\sum_i a_i \leq 2$. Let's compute $s_\beta w_{0,\beta} w_0 s_\beta(\tau) = -(\delta+\beta) + \sum_i [a_i(\delta - \alpha_i - R_1)] + R_2 = \delta(\sum_i a_i - 1) - \beta - \sum_i a_i \alpha_i + R'$ thus

$$c_\beta(s_\beta w_{0,\beta} w_0 s_\beta(\tau)) = 2\left(\sum_i a_i - 1\right) - 1 = 2\sum_i a_i - 3.$$

For $\sum_i a_i \leq 1$ we get $c_\beta < 0$, for $\sum_i a_i = 2$ we get $c_\beta = 4 - 3 = 1$. Finally note that if $v = 1$ then $s_\beta w_{0,\beta} w_0(\beta) = \delta + \beta$ with $c_\beta = 3$. □

Lemma 3.2.19. $\mathcal{I}_{\delta-\theta_\Sigma,\delta-\theta_\Sigma} = \{1\}$.

Proof. Suppose there is $y \in \mathcal{I}_{\delta-\theta_\Sigma,\delta-\theta_\Sigma}$, $y \neq 1$. Then

$$s_\beta w_{0,\beta} w_0 s_\beta y \in \mathcal{I}_{\theta_\Sigma,\delta-\theta_\Sigma} = \emptyset.$$

□

Lemma 3.2.20. *If* $\gamma \in \langle A(\Sigma)\rangle_l$, *then* $\mathcal{I}_{\gamma,\delta-\theta_\Sigma} = \{u_\gamma^\Sigma\}$.

Proof. Since u_γ^Σ is the minimum in $\mathcal{I}_{\gamma,k\delta-\theta_\Sigma}$, every other element $v \in \mathcal{I}_{\gamma,k\delta-\theta_\Sigma}$ can be expressed as $v = u_\gamma^\Sigma s$ with $l(u_\gamma^\Sigma s) = l(u_\gamma^\Sigma) + l(s)$ and $s(\gamma) = \gamma$. This implies that v can be rewritten as $v = y u_\gamma^\Sigma$ with $y = u_\gamma^\Sigma s (u_\gamma^\Sigma)^{-1}$ and $y(\theta_\Sigma) = \theta_\Sigma$. We want to prove y is σ-minuscule, so that $y = 1$. Consider then $\tau = u_\gamma^\Sigma s u_n \cdots u_{k+1}(\alpha_k)$ with α_k the simple root associated to u_k. Since $u_\gamma^\Sigma s (u_\gamma^\Sigma)^{-1}(\theta_\Sigma) = \theta_\Sigma$, we can rewrite $u_\gamma^\Sigma s u_n \cdots u_{k+1}(u_{k-1} \cdots u_1(\theta_\Sigma) + a_k\alpha_k) = \theta_\Sigma$ and

$$u_\gamma^\Sigma s u_n \cdots u_{k+1} u_{k-1} \cdots u_1(\theta_\Sigma) = \theta_\Sigma - a_k\tau.$$

Note that since $(u_\gamma^\Sigma)^{-1}(\theta_\Sigma) = \delta - \gamma$ we have $a_k > 0$. When $\tau < 0$ there is nothing to prove, suppose then $\tau > 0$. Suppose for now $0 < \tau < \delta$. We have in this case $c_\beta(\tau) \leq 2$. We see that if $\tau \in \widehat{\Pi}$ then $\tau = \theta_\Sigma$ and $a_k = 2$, but then $u_\gamma^\Sigma s u_n \cdots u_{k+1} u_{k-1} \cdots u_1(\theta_\Sigma) = -\theta_\Sigma$ and

$$u_n \cdots u_{k+1} u_{k-1} \cdots u_1(\theta_\Sigma) = -s^{-1}(u_\gamma^\Sigma)^{-1}(\theta_\Sigma) = \gamma - \delta.$$

Note that $u_\gamma^\Sigma \in W_{\theta_\Sigma}$ since it is minimal, thus c_{θ_Σ} must stay the same, but

$$1 = c_{\theta_\Sigma}(u_n \cdots u_{k+1} u_{k-1} \cdots u_1(\theta_\Sigma)) \neq c_{\theta_\Sigma}(a - \delta) = -1.$$

We conclude that if $\tau \in N(y)$ then $c_\beta(\tau) > 0$. Suppose now $c_\beta(\tau) = 2$. If $c_{\theta_\Sigma}(\tau) = 0$, then $\delta - \tau = \theta_\Sigma$ and so $\tau = \delta - \theta_\Sigma$, thus $y^{-1}(\tau) = \delta - \theta_\Sigma > 0$ but $\tau \in N(y)$. If $c_{\theta_\Sigma}(\tau) = 1$, then $\tau = 2\beta + \theta_\Sigma + R$. We write

$$u s u_n \cdots u_{k+1} u_{k-1} \cdots u_1(\theta_\Sigma) = (1 - a_k)\theta_\Sigma - 2a_k\beta - a_k R.$$

If $a_k = 2$ then the right hand side becomes $-\theta_\Sigma - 4\beta - 2R$ which is impossible because we must have $-\theta_\Sigma - 4\beta - 2R = 2\delta - \theta_\Sigma$ that is clear adding 2δ and checking its new c_β and c_{θ_Σ}. Then

$$u_n \cdots u_1(\theta_\Sigma) = 2\delta - s^{-1}(u_\gamma^\Sigma)^{-1}(\theta_\Sigma) = 2\delta - (\delta - \gamma) = \delta + \gamma.$$

This is the same as $u_1 \cdots u_n(\gamma) = \theta_\Sigma - \delta < 0$ which is absurd because $u_1 \cdots u_n(\gamma) = \delta - \theta_\Sigma > 0$. If $a_k = 1$ then $usu_n \cdots u_{k+1}u_{k-1} \cdots u_1(\theta_\Sigma) = -2\beta - R = \theta_\Sigma - \delta$, so

$$u_n \cdots u_{k+1}u_{k-1} \cdots u_1(\theta_\Sigma) = s^{-1}(u_\gamma^\Sigma)^{-1}(\theta_\Sigma - \delta) = -\gamma.$$

Again since $u_\gamma^\Sigma \in W_{\theta_\Sigma}$ then c_{θ_Σ} must stay the same, but

$$1 = c_{\theta_\Sigma}(u_n \cdots u_{k+1}u_{k-1} \cdots u_1(\theta_\Sigma)) \neq c_{\theta_\Sigma}(-\gamma) = 0.$$

Note also that since $\theta_\Sigma - a_k\tau$ is an actual root, $c_{\theta_\Sigma}(\tau) = 1$. In conclusion if $0 < \tau < \delta$ then $c_\beta(\tau) = 1$, $c_{\theta_\Sigma}(\tau) = 1$. Note that every root greater than δ can be written as $j\delta + x$ with $0 < x < \delta$, indeed for all the roots of the form $i\delta - x'$ with $0 < x' < \delta$ we can rewrite $i\delta - x' = i\delta - x' + \delta - \delta = (i-1)\delta + (\delta - x')$. Then if $\tau = j\delta + x \in N(y)$ then $x \in N(y)$ because $y^{-1}(j\delta) = j\delta$. But then $c_\beta(x) = 1$ and exactly as we have just seen $c_{\theta_\Sigma}(x) = 1$, so $\tau = j\delta + \beta + \theta_\Sigma + R$. We can expand $\tau = u_\gamma^\Sigma s u_n \cdots u_{k+1}(\alpha_k) = j\delta + \beta + \theta_\Sigma + R$ and multiplying by $s^{-1}(u_\gamma^\Sigma)^{-1}$

$$u_n \cdots u_{k+1}(\alpha_k) = j\delta + s^{-1}(u_\gamma^\Sigma)^{-1}(\beta) + \delta - \gamma + s^{-1}(u_\gamma^\Sigma)^{-1}(R). \tag{3.6}$$

Note that since $u_\gamma^\Sigma s$ is σ-minuscule then $s^{-1}(u_\gamma^\Sigma)^{-1}(R) > 0$ and $p := c_{\theta_\Sigma}(s^{-1}(u_\gamma^\Sigma)^{-1}(R)) \geq 0$, on the other hand $s^{-1}(u_\gamma^\Sigma)^{-1}(\beta) < 0$ and for $n := c_{\theta_\Sigma}(s^{-1}(u_\gamma^\Sigma)^{-1}(\beta))$ we have $-1 \leq n \leq 0$. Recalling that $u_\gamma^\Sigma \in W_{\theta_\Sigma}$, we compute c_{θ_Σ} on both sides of (3.6) $0 = j + n + 1 - 0 + p$ and so

$$j = -n - p - 1 \leq 1 + 0 - 1 = 0.$$

In the end y is σ-minuscule, $y(\delta - \theta_\Sigma) = \delta - \theta_\Sigma$ and so $y = 1$. □

Lemma 3.2.21. *If $\gamma \in \langle A(\Sigma) \rangle_l$, $\gamma \neq \delta - \theta_\Sigma$, then $\mathcal{I}_{\delta-\gamma, \delta-\theta_\Sigma} = \{s_\beta w_{0,\beta} w_0 s_\beta u_\gamma^\Sigma\}$.*

Proof. Since left multiplication by $s_\beta w_{0,\beta} w_0 s_\beta$ is an invertible map between $\mathcal{I}_{\gamma,\delta-\theta_\Sigma}$ and $\mathcal{I}_{\delta-\gamma,\delta-\theta_\Sigma}$, we get the result. $\square$

Corollary 3.2.22. *If $\gamma \neq \delta - \theta_\Sigma$, left multiplication by $s_\beta w_{0,\beta} w_0 s_\beta$ induces an isomorphism of posets between $\mathcal{I}_{\gamma,\delta-\theta_\Sigma}$ and $\mathcal{I}_{\delta-\gamma,\delta-\theta_\Sigma}$*

Corollary 3.2.23. *If any component Σ of the Dynkin diagram satisfies $|\Sigma| = 1$ and θ_Σ is of type 1, then we can use the following formula to compute the cardinality of the set of σ-minuscule elements. Let L be the number of long positive roots $\tau \in \widehat{\Delta}$ with $\tau < \delta$, then*

$$|\mathcal{W}_\sigma^{ab}| = L - 1.$$

Case c.

We assume that β is a long root, and we consider $\mu = k\delta - \theta_\Sigma$, with θ_Σ of type 2 and $|\Sigma| = 1$. We denote by W_{θ_Σ} the Coxeter group associated to the diagram of finite type obtained removing θ_Σ from the original diagram. Note that $k\delta - \theta_\Sigma$ is not contained in a diagram of finite type and is not the highest root of any such diagram, so most of the previous techniques will not work in this case. Of course $\Gamma(\Sigma) = \emptyset$. We divide the arguments according to the type of link between β and θ_Σ, it can be double, triple or quadruple. Indeed it is given by the coefficient a in $s_{\theta_\Sigma}(\beta) = \beta + a\theta_\Sigma$. If $a > 4$ then $s_\beta(\beta + a\theta_\Sigma) = (a-1)\beta + a\theta_\Sigma$ with $c_\beta \geq 4$, since $\beta + a\theta_\Sigma < k\delta$ then $(a-1)\beta + a\theta_\Sigma = k\delta + (a-3)\beta$ with $(a-3) \geq 2$ which is absurd. For the double link case we first prove that $\widehat{\Delta}_\mu \subseteq \widehat{\Delta}^1_{\theta_\Sigma}$ in Lemma 3.2.24. Then we show in Lemmas 3.2.27 and 3.2.28 that if $\gamma \in \widehat{\Delta}^1_{\theta_\Sigma}$, then the minimal element u_γ^Σ in W_{θ_Σ} such that $u_\gamma^\Sigma(\gamma) = k\delta - \theta_\Sigma$ is such that $u_\gamma^\Sigma = \min \mathcal{I}_{\gamma,\mu}$. We also show in Lemma 3.2.29 that $\mathcal{I}_{\mu,\mu} = \{1, s_\beta s_{\theta_\Sigma} s_\beta\}$ and in Lemma 3.2.30 that $\mathcal{I}_{\theta_\Sigma,\mu} = \{s_\beta w_{0,\beta} w_0 s_{\theta_\Sigma} s_\beta\}$. For the other roots in $\widehat{\Delta}^1_{\theta_\Sigma}$ we show in Lemma 3.2.31 that another element in $\mathcal{I}_{\gamma,\mu}$ other than u_γ^Σ can be found via left multiplication by $s_\beta s_{\theta_\Sigma} s_\beta$, i.e. $s_\beta s_{\theta_\Sigma} s_\beta u_\gamma^\Sigma \in \mathcal{I}_{\gamma,\mu}$. In Lemma 3.2.32 we prove that $\mathcal{I}_{\gamma,\mu} = \{u_\gamma^\Sigma, s_\beta s_{\theta_\Sigma} s_\beta u_\gamma^\Sigma\}$. This also implies that $\widehat{\Delta}_\mu = \widehat{\Delta}^1_{\theta_\Sigma}$. At the end of the section we give a closed formula to compute $|\mathcal{W}_\sigma^{ab}|$. For the triple and quadruple link cases, we are able to show that they are necessary associated to the root systems $G_2^{(1)}$ and $A_1^{(2)}$ respectively, and we exhibit explicit realizations in Lemmas 3.2.36 and 3.2.37.

Let's start looking at the double link case.

Lemma 3.2.24. *If $\mathcal{I}_{\gamma,k\delta-\theta_\Sigma} \neq \emptyset$, then $c_{\theta_\Sigma}(\gamma) = 1$.*

Proof. Let w be in $\mathcal{I}_{\gamma,k\delta-\theta_\Sigma}$. Then $w(\gamma) = k\delta - \theta_\Sigma$ and thus we have $\gamma = w^{-1}(k\delta - \theta_\Sigma) > 0$ since $c_\beta(k\delta - \theta_\Sigma) = 2 \neq 1$ implying $c_{\theta_\Sigma}(\gamma) \geq 0$, and also $\gamma = k\delta - w^{-1}(\theta_\Sigma)$ implying $c_{\theta_\Sigma}(\gamma) \leq 2$ since $c_\beta(\theta_\Sigma) = 0 \neq 1$. We prove that the parity of c_{θ_Σ} of any root can never change applying elements of $\widehat{W}$. Of course it can change only when we apply s_{θ_Σ}, consider then any root $\tau = d\theta_\Sigma + a\beta + R$ with R a sum of other simple roots, we have $s_{\theta_\Sigma}(\tau) = s_{\theta_\Sigma}(d\theta_\Sigma + a\beta + R) = -d\theta_\Sigma + a\beta + 2a\theta_\Sigma + R = (2a-d)\theta_\Sigma + a\beta + R$ and $(2a-d)$ has the same parity as d. Now since $w(\gamma) = k\delta - \theta_\Sigma$ and $c_{\theta_\Sigma}(k\delta - \theta_\Sigma) = 1$ we get that $c_{\theta_\Sigma}(\gamma)$ is odd and thus $c_{\theta_\Sigma}(\gamma) = 1$. ☐

Corollary 3.2.25. *If θ_2 is another short root, then for every $w \in \widehat{W}$ we have $w(\theta_\Sigma) \neq \theta_2$.*

Proof. $c_{\theta_\Sigma}(\theta_\Sigma) = 1$ and its parity cannot change applying w. Indeed writing $w = s_1 \cdots s_n$ in reduced form, we see that applying simple reflections $s_i \neq s_{\theta_\Sigma}$ the value of c_{θ_Σ} for the resulting root doesn't change. As seen in Lemma 3.2.24, consider then any root $\tau = d\theta_\Sigma + a\beta + R$ with R a sum of other simple roots, we have $s_{\theta_\Sigma}(\tau) = (2a-d)\theta_\Sigma + a\beta + R$, and $(2a-d)$ has the same parity as d. This proves that for every $w \in \widehat{W}$ we get that $c_{\theta_\Sigma}(w(\theta_\Sigma))$ is odd and can never be 0. ☐

Lemma 3.2.26. *If $c_{\theta_\Sigma}(\gamma) = 1$ then there exists $u \in W_{\theta_\Sigma}$ such that $u(\gamma) = \theta_\Sigma$. In particular γ is a short root.*

Proof. Consider the height map $h : \Delta^+ \to \mathbb{N}$ defined by $h(\sum_i b_i\alpha_i) = \sum_i b_i$. Consider the set

$$\Gamma = \{\tau \in \widehat{\Delta}^1_{\theta_\Sigma} | w(\tau) \neq \theta_\Sigma \text{ for all } w \in W_{\theta_\Sigma}\}.$$

Assume Γ non empty and let $\tau \in \Gamma$ be an element of minimal image through h. Then for every simple reflection $s \in \widehat{\Pi} \setminus \{\theta_\Sigma\}$ we have $h(s(\tau)) \geq h(\tau)$ since $s(\tau) \in \Gamma$, implying that $h(s_{\theta_\Sigma}(\tau)) < h(\tau)$, otherwise $h(w(\tau)) \geq h(\tau)$ for every $w \in \widehat{W}$, against the fact that every real root is $\widehat{W}$-connected to a simple root. But $c_{\theta_\Sigma}(\tau) = 1$ and its parity can never change, so there can only be one possibility: $\tau = \theta_\Sigma$ and $s_{\theta_\Sigma}(\theta_\Sigma) = -\theta_\Sigma$, but this is against $\tau \in \Gamma$. We conclude that $\Gamma = \emptyset$. In particular any $\gamma \in \widehat{\Delta}^1_{\theta_\Sigma}$ is short since there is $u \in W_{\theta_\Sigma}$ such that $u(\gamma) = \theta_\Sigma$, which is short. $\square$

Notice that the map that associates γ to $k\delta - \gamma$ is an involution of $\widehat{\Delta}^1_{\theta_\Sigma}$. We call $u^\Sigma_\gamma \in W_{\theta_\Sigma}$ the shortest element such that $u^\Sigma_\gamma(\gamma) = k\delta - \theta_\Sigma$ for any γ with $c_{\theta_\Sigma}(\gamma) = 1$.

Lemma 3.2.27. *If $c_{\theta_\Sigma}(\gamma) = 1$ then $u^\Sigma_\gamma \in \mathcal{I}_{\gamma,k\delta-\theta_\Sigma}$.*

Proof. Write $u^\Sigma_\gamma = u_1 \cdots u_n$ in reduced form. Then for every $j = 1, \cdots, n-1$ setting $\tau_j = u_1 \cdots u_{j-1}(\alpha_j)$ we have

$$u_1 \cdots u_{j-1} u_{j+1} u_n(\gamma) = k\delta - \theta_\Sigma - a_j \tau_j.$$

We see that $a_j \neq 0$, otherwise $u_1 \cdots u_{j-1}(\tau_{j+1}) = \theta_\Sigma$ against the minimality of u^Σ_γ. Moreover, since $u_i \neq s_{\theta_\Sigma}$ for every i and $\alpha_j \neq \theta_\Sigma$ we have $a_j > 0$ and $c_\beta(\tau_j) \geq 1$. Moreover $c_{\theta_\Sigma}(\tau_j) = 0$ so $c_\beta(\tau_j) \leq 2$. If $c_\beta(\tau_j) = 2$ then $c_\beta(k\delta - \tau_j) = 0$ and $c_{\theta_\Sigma}(k\delta - \tau_j) = 2$, which is absurd; so $c_\beta(\tau_j) = 1$ and the claim is proved. $\square$

Lemma 3.2.28. *If $c_{\theta_\Sigma}(\gamma) = 1$, then $u^\Sigma_\gamma = \min \mathcal{I}_{\gamma,k\delta-\theta_\Sigma}$.*

Proof. Write $u^\Sigma_\gamma = u_1 \cdots u_n$ in reduced form and call γ_j the rootlet of $u_1 \cdots u_j$ for $j = 0, \cdots, n$. As we have seen in the previous lemma $a_j > 0$ for every j and so

$$\gamma = \gamma_n < \gamma_{n-1} < \cdots < \gamma_0 = k\delta - \theta_\Sigma.$$

Then the claim follows as in Lemma 3.2.8. $\square$

Lemma 3.2.29. $\mathcal{I}_{k\delta-\theta_\Sigma,k\delta-\theta_\Sigma} = \{1, s_\beta s_{\theta_\Sigma} s_\beta\}$.

Proof. We only need to prove that there are no other elements in $\mathcal{I}_{k\delta-\theta_\Sigma,k\delta-\theta_\Sigma}$. Let $y \in \mathcal{I}_{k\delta-\theta_\Sigma,k\delta-\theta_\Sigma}$, $y \neq 1$. Of course $y(\theta_\Sigma) = \theta_\Sigma$. Since $y^{-1}(\beta) < 0$ and $y^{-1}(\beta) \neq -\theta_\Sigma = y^{-1}(-\theta_\Sigma)$ we have that

$$y^{-1}(\beta + 2\theta_\Sigma) = y^{-1}(\beta) + 2\theta_\Sigma < 0$$

thus $\beta + 2\theta_\Sigma \in N(y)$. Writing $\beta + 2\theta_\Sigma = (\beta + \theta_\Sigma) + \theta_\Sigma$, and since $y^{-1}(\theta_\Sigma) = \theta_\Sigma > 0$ we see that also $\beta + \theta_\Sigma \in N(y)$, and for the same reason also $\beta \in N(y)$. We can write $y = s_\beta s_{\theta_\Sigma} s_\beta s_1 \cdots s_n$ in reduced form. But then we notice that for every $s_1 \in \widehat{\Pi}$ such that $l(s_\beta s_{\theta_\Sigma} s_\beta s_1) = 4$ we have that $s_\beta s_{\theta_\Sigma} s_\beta s_1 \notin \mathcal{W}_\sigma^{ab}$. In fact $s_\beta s_{\theta_\Sigma} s_\beta(\theta_\Sigma) = \theta_\Sigma$ with $c_\beta \neq 1$, if $\alpha \neq \theta_\Sigma$, α connected to β in the diagram, $s_\beta s_{\theta_\Sigma} s_\beta(\alpha) = \alpha + 2\beta + 2\theta_\Sigma$ with $c_\beta \neq 1$, and if x is any simple root not connected to β in the diagram we have $s_\beta s_{\theta_\Sigma} s_\beta(x) = x$ with $c_\beta \neq 1$. In the end $y = s_\beta s_{\theta_\Sigma} s_\beta$. □

Lemma 3.2.30. $\mathcal{I}_{\theta_\Sigma, k\delta - \theta_\Sigma} = \{s_\beta w_{0,\beta} w_0 s_{\theta_\Sigma} s_\beta\}$.

Proof. First we check that $s_\beta w_{0,\beta} w_0 s_{\theta_\Sigma} s_\beta \in \mathcal{I}_{\theta_\Sigma, k\delta - \theta_\Sigma}$. We already know that $s_\beta w_{0,\beta} w_0 s_{\theta_\Sigma}$ is σ-minuscule since $l(s_\beta w_{0,\beta} w_0 s_{\theta_\Sigma}) < l(s_\beta w_{0,\beta} w_0)$. We then compute $s_\beta w_{0,\beta} w_0 s_{\theta_\Sigma}(\beta) = s_\beta w_{0,\beta} w_0(\beta + 2\theta_\Sigma) = k\delta + \beta + 2(-\theta_\Sigma - \beta) = k\delta - \beta - 2\theta_\Sigma$, which has $c_\beta = 1$. We check that $s_\beta w_{0,\beta} w_0 s_{\theta_\Sigma} s_\beta \in \mathcal{I}_{\theta_\Sigma, k\delta - \theta_\Sigma}$:

$$\begin{aligned} &s_\beta w_{0,\beta} w_0 s_{\theta_\Sigma} s_\beta(\theta_\Sigma) = s_\beta w_{0,\beta} w_0(\beta + \theta_\Sigma) = s_\beta w_{0,\beta}(k\delta - \beta - \theta_\Sigma) = \\ &= s_\beta(k\delta - \beta - \theta_\Sigma) = k\delta - \theta_\Sigma. \end{aligned}$$

Let v be in $\mathcal{I}_{\theta_\Sigma, k\delta - \theta_\Sigma}$ its minimum (so v doesn't contain s_{θ_Σ}), then

$$v^{-1}(k\delta - 2\theta_\Sigma - \beta) = k\delta - 2(k\delta - \theta_\Sigma) - v^{-1}(\beta) < 0$$

since $c_\beta(v^{-1}(\beta)) \geq -1$ because $-v^{-1}(\beta)$ is in $\langle \widehat{\Pi} \setminus \{\theta_\Sigma\} \rangle$, whose highest root is $k\delta - 2\theta_\Sigma - \beta$ with $c_\beta = 1$, thus $k\delta - 2\theta_\Sigma - \beta \in N(v)$. Moreover this root contains the highest root of the connected component Σ_2 not containing θ_Σ, i.e. $k\delta - 2\theta_\Sigma - \beta = \theta_{\Sigma_2} + \beta + R$ with R a sum of simple roots in Σ_2. Since $v^{-1}(\theta_{\Sigma_2} + \beta + R) < 0$ and $v^{-1}(R) > 0$ we see that $v^{-1}(\theta_{\Sigma_2} + \beta) < 0$. Then for every decomposition of $\theta_{\Sigma_2} = \xi_1 + \xi_2$ we have that exactly one of them has $\beta + \xi_i \in N(v)$. Note that the longest element with support in Σ_2 is $w_0 s_{\theta_\Sigma}$, and the longest element with support in $\widehat{\Pi} \setminus \{s_\beta, s_{\theta_\Sigma}\}$ is $w_{0,\beta}$, so $s_\beta N(w_{0,\beta} w_0) \subset N(v)$. Because $N(s_\beta w_{0,\beta} w_0 s_{\theta_\Sigma} s_\beta) = \beta \cup s_\beta N(w_{0,\beta} w_0) \cup \{k\delta - 2\theta_\Sigma - \beta\}$ we have that $s_\beta w_{0,\beta} w_0 s_{\theta_\Sigma} s_\beta \leq v$. In conclusion $s_\beta w_{0,\beta} w_0 s_{\theta_\Sigma} s_\beta$ is the minimum of $\mathcal{I}_{\theta_\Sigma, k\delta - \theta_\Sigma}$. We now check that for every simple reflection s for which $l(s_\beta w_{0,\beta} w_0 s_{\theta_\Sigma} s_\beta s) = l(s_\beta w_{0,\beta} w_0 s_{\theta_\Sigma} s_\beta) + 1$ we have $s_\beta w_{0,\beta} w_0 s_{\theta_\Sigma} s_\beta s \notin \mathcal{W}_\sigma^{ab}$. For θ_Σ we get $s_\beta w_{0,\beta} w_0 s_{\theta_\Sigma} s_\beta(\theta_\Sigma) = s_\beta w_{0,\beta} w_0(\beta+\theta_\Sigma) = s_\beta(k\delta - \beta - \theta_\Sigma) = k\delta - \theta_\Sigma$ with $c_\beta = 2 \neq 1$. For α a simple root not θ_Σ linked to β in the diagram, we get

$$\begin{aligned} & s_\beta w_{0,\beta} w_0 s_{\theta_\Sigma} s_\beta(\alpha) = s_\beta w_{0,\beta} w_0(\alpha+\beta+2\theta_\Sigma) = s_\beta w_{0,\beta}(k\delta - \alpha - \beta - 2\theta_\Sigma) = \\ & = s_\beta(k\delta - \alpha - \beta - 2\theta_\Sigma - R) = k\delta - \alpha - 2\beta - 2\theta_\Sigma - R \end{aligned}$$

with $c_\beta = 0 \neq 1$. For any other simple root $x \in \widehat{\Pi}$ we get $s_\beta w_{0,\beta} w_0 s_{\theta_\Sigma} s_\beta(x) = s_\beta w_{0,\beta} w_0(x) = s_\beta(-x') = -x'$ with x' a simple root that is not β, α nor θ_Σ. The claim follows. □

Lemma 3.2.31. *If $v \in \mathcal{I}_{\gamma, k\delta-\theta_\Sigma}$ and $\gamma \neq \theta_\Sigma$, then $s_\beta s_{\theta_\Sigma} s_\beta v \in \mathcal{I}_{\gamma, k\delta-\theta_\Sigma}$.*

Proof. Write $v = v_1 \cdots v_n$ in reduced form, α_j the simple root associated to the reflection v_j. Let's use α_1 and α_2 to indicate the simple roots (which are at most 2) connected to β that are not θ_Σ. We have

$$\tau := v_1 \cdots v_{j-1}(\alpha_j) = a_1\alpha_1 + a_2\alpha_2 + \beta + b\theta_\Sigma + R$$

with R a sum of other simple roots. Recall that we always have $a_1 + a_2 + b \leq 4$. On the other hand $a_1 + a_2 + b \neq 3$ otherwise $s_\beta(\tau) = \tau + \beta \in \widehat{\Delta}$, and since both τ and $\beta \in N(v)$ we would have $\tau + \beta \in N(v)$ which is impossible because $c_\beta(\tau + \beta) = 2$. Moreover $a_1 + a_2 + b \neq 4$ otherwise $s_\beta(\tau) = \tau + 2\beta = k\delta + \beta$ and so $\tau = k\delta - \beta$, but then $\tau + \beta = k\delta \in N(v)$ which is impossible. In conclusion $a_1 + a_2 + b \leq 2$. Let's compute

$$s_\beta s_{\theta_\Sigma} s_\beta(\tau) = a_1\alpha_1 + a_2\alpha_2 + (2a_1 + 2a_2 - 1)\beta + (2a_1 + 2a_2 + b - 2)\theta_\Sigma + R.$$

For $a_1 + a_2 \leq 1$ the root is negative or $c_\beta = 1$ and the claim follows. When $a_1 + a_2 = 2$ we have $b = 0$, so $c_{\alpha_1} + c_{\alpha_2} = 2$, $c_{\theta_\Sigma} = 2$ and $c_\beta = 3$ thus $s_\beta s_{\theta_\Sigma} s_\beta(\tau) = k\delta + \beta$ and so $\tau = s_\beta s_{\theta_\Sigma} s_\beta(k\delta + \beta) = k\delta - \beta - 2\theta_\Sigma \in N(v)$. We claim that $k\delta - \beta - 2\theta_\Sigma \in N(v)$ implies $v = s_\beta w_{0,\beta} w_0 s_{\theta_\Sigma} s_\beta$ and $\gamma = \theta_\Sigma$. Indeed $k\delta - \beta - 2\theta_\Sigma$ is the highest root in the diagram obtained by removing θ_Σ form the original diagram, and so if $k\delta - \beta - 2\theta_\Sigma \in N(v)$ then every root in such a diagram with $c_\beta = 1$ must be in $N(v)$, so $N(s_\beta w_{0,\beta} w_0 s_{\theta_\Sigma} s_\beta) \subset N(v)$. As we have seen in Lemma 3.2.30 $s_\beta w_{0,\beta} w_0 s_{\theta_\Sigma} s_\beta$ is maximal so $s_\beta w_{0,\beta} w_0 s_{\theta_\Sigma} s_\beta = v$ and $\gamma = \theta_\Sigma$. □

Lemma 3.2.32. *If $v \in \mathcal{I}_{\gamma,k\delta-\theta_\Sigma}$, then there exists $y \in \mathcal{I}_{k\delta-\theta_\Sigma,k\delta-\theta_\Sigma}$ such that $v = yu_\gamma^\Sigma$.*

Proof. Since u_γ^Σ is the minimum in $\mathcal{I}_{\gamma,k\delta-\theta_\Sigma}$, we can write $v = u_\gamma^\Sigma s$ with $l(u_\gamma^\Sigma s) = l(u_\gamma^\Sigma) + l(s)$, and rewrite it as $v = u_\gamma^\Sigma s = yu_\gamma^\Sigma$ with $y = u_\gamma^\Sigma s(u_\gamma^\Sigma)^{-1}$. We want to prove that $u_\gamma^\Sigma s(u_\gamma^\Sigma)^{-1} = 1$ or $u_\gamma^\Sigma s(u_\gamma^\Sigma)^{-1} = s_\beta s_{\theta_\Sigma} s_\beta$. Let y be of smallest length such that $y(\theta_\Sigma) = \theta_\Sigma$, $yu_\gamma^\Sigma \in \mathcal{I}_{\gamma,k\delta-\theta_\Sigma}$, $y \neq 1$, $y \neq s_\beta s_{\theta_\Sigma} s_\beta$. Since $l(u_\gamma^\Sigma s(u_\gamma^\Sigma)^{-1}) \geq l(u_\gamma^\Sigma) + l(s) - l(u_\gamma^\Sigma) = l(s) > 0$ and $\beta \in N(u_\gamma^\Sigma s) = N(v)$ is the only simple root in this inversion set, we see that $\beta \in N(y) = N(u_\gamma^\Sigma s) \dotplus usN((u_\gamma^\Sigma)^{-1})$. Then

$$y^{-1}(\beta + 2\theta_\Sigma) = y^{-1}(\beta) + 2\theta_\Sigma < 0$$

since $y^{-1}(\beta) < 0$ and $y^{-1}(\beta) \neq -\theta_\Sigma = y^{-1}(-\theta_\Sigma)$. $y^{-1}(\beta + \theta_\Sigma) = y^{-1}(\beta) + \theta_\Sigma < 0$ as well, so we can write $y = s_\beta s_{\theta_\Sigma} s_\beta y'$ in reduced form. But then thanks to Lemma 3.2.31 we see that or $y = 1$, or $s_\beta s_{\theta_\Sigma} s_\beta y = (s_\beta s_{\theta_\Sigma} s_\beta)(s_\beta s_{\theta_\Sigma} s_\beta) y' = y'$ and $y' u_\gamma^\Sigma \in \mathcal{I}_{\gamma,k\delta-\theta_\Sigma}$. Of course $y'(\theta_\Sigma) = \theta_\Sigma$ and $l(y') < l(y)$, but then $y' = 1$ or $y' = s_\beta s_{\theta_\Sigma} s_\beta$ with respectively $y = s_\beta s_{\theta_\Sigma} s_\beta$ or $y = 1$ against the assumptions. □

Lemma 3.2.33. *If $v \in \mathcal{I}_{\gamma,k\delta-\theta_\Sigma}$, $v \neq s_\beta s_{\theta_\Sigma} s_\beta$, then $s_\beta w_{0,\beta} w_0 s_{\theta_\Sigma} s_\beta v \in \mathcal{I}_{k\delta-\gamma,k\delta-\theta_\Sigma}$. If $v \in W_{\theta_\Sigma}$ then $s_\beta w_{0,\beta} w_0 s_{\theta_\Sigma} s_\beta v \in W_{\theta_\Sigma}$.*

Proof. Write $v = v_1 \cdots v_n$ in reduced form, and let α_j be the simple root associated to the reflection v_j. Let α_1 and α_2 be as in Lemma 3.2.31 the simple roots connected to β that are not θ_Σ. We have

$$\tau := v_1 \cdots v_{j-1}(\alpha_j) = a_1\alpha_1 + a_2\alpha_2 + \beta + b\theta_\Sigma + R$$

with R a sum of other simple roots. Recall that we always have $a_1 + a_2 + b \leq 2$. We compute $s_\beta w_{0,\beta} w_0 s_{\theta_\Sigma} s_\beta(\tau) = k\delta(-1 + a_1 + a_2 + b) + (-2a + 1)\beta + (-2a_1 - 2a_2 + 2 - b)\theta_\Sigma - a_1\alpha - a_2\alpha_2 + R'$, thus

$$c_\beta = 2(-1 + a_1 + a_2 + b) + 1 - 2a_1 - 2a_2 = 2b - 1,$$

$$c_{\theta_\Sigma} = 2(-1 + a_1 + a_2 + b) - 2 - 2a_1 - 2a_2 - b = b,$$

$$c_{\alpha_1} + c_{\alpha_2} = 2(-1 + a_1 + a_2 + b) - a_1 - a_2 = a_1 + a_2 + 2b - 2.$$

For $b = 0$ we get $c_\beta < 0$ and for $b = 1$ we get $c_\beta = 1$. When $b = 2$ then $a_1 = a_2 = 0$, so we find $c_\beta = 3$, $c_{\theta_\Sigma} = 2$, $c_{\alpha_1} + c_{\alpha_2} = 2$, thus $s_\beta w_{0,\beta} w_0 s_{\theta_\Sigma} s_\beta(\tau) = k\delta + \beta$. But then

$$\tau = s_\beta s_{\theta_\Sigma} w_0 w_{0,\beta} s_\beta(k\delta + \beta) = k\delta - k\delta + \beta + 2\theta_\Sigma = \beta + 2\theta_\Sigma$$

and we conclude that since $\beta + 2\theta_\Sigma \in N(v)$ we have $v = s_\beta s_{\theta_\Sigma} s_\beta$ because it is maximal. The final statement is trivial. □

Corollary 3.2.34. *If $\gamma \neq \theta_\Sigma, k\delta - \theta_\Sigma$, then left multiplication by $s_\beta w_{0,\beta} w_0 s_{\theta_\Sigma} s_\beta$ induces an isomorphism of posets between $\mathcal{I}_{\gamma,k\delta-\theta_\Sigma}$ and $\mathcal{I}_{k\delta-\gamma,k\delta-\theta_\Sigma}$.*

Corollary 3.2.35. *If any component Σ of the diagram satisfies $|\Sigma| = 1$ and θ_Σ is of type 2, then we can use the following formula to compute the cardinality of the set of σ-minuscule elements. Let C be the number of roots in $\widehat{\Delta}$ with $c_{\theta_\Sigma} = 1$, then*

$$|\mathcal{W}_\sigma^{ab}| = 2C - 1.$$

Let's move on now to the triple link case.

Lemma 3.2.36. *If β is long, $|\Sigma| = 1$ and θ_Σ has a triple link with β, then the following holds: the system is $G_2^{(1)}$; writing x for the remaining simple root we have*

$$\mathcal{W}_\sigma^{ab} = \{1, s_\beta, s_\beta s_{\theta_\Sigma}, s_\beta s_x, s_\beta s_{\theta_\Sigma} s_x\}$$

and in particular

$$\begin{aligned}
\mathcal{I}_{\delta-\theta_\Sigma,\delta-\theta_\Sigma} &= \{1\}, \\
\mathcal{I}_{x+\beta+2\theta_\Sigma,\delta-\theta_\Sigma} &= \{s_\beta\}, \\
\mathcal{I}_{\beta+2\theta_\Sigma,\delta-\theta_\Sigma} &= \{s_\beta s_x\}, \\
\mathcal{I}_{x+\beta+\theta_\Sigma,\delta-\theta_\Sigma} &= \{s_\beta s_{\theta_\Sigma}\}, \\
\mathcal{I}_{\beta+\theta_\Sigma,\delta-\theta_\Sigma} &= \{s_\beta s_{\theta_\Sigma} s_x\}.
\end{aligned}$$

Proof. Since β and θ_Σ form a diagram of type G_2, there must be another simple root x in the diagram connected to β. Let's write $s_x(\beta) = \beta + ax$ and $s_\beta(x) = x + j\beta$ with $a, j \geq 1$. We now compute $s_x s_\beta s_{\theta_\Sigma}(\beta) = 2ax + 2\beta + 3\theta_\Sigma$. We claim that it is greater than $k\delta$. Indeed if it is smaller than $k\delta$ then we apply s_β obtaining $s_\beta(2ax+2\beta+3\theta_\Sigma) = 2ax+(2aj+1)\beta+3\theta_\Sigma$ which is greater than $k\delta$ since its $c_\beta \geq 3$. Indeed $2ax + (2aj+1)\beta + 3\theta_\Sigma = k\delta + (2aj-1)\beta$ and so $(2aj-1) = 1$ implying $a = j = 1$ and $|x| = |\beta|$. In the end $k\delta = 2x+2\beta+3\theta$ but $s_x(k\delta) = k\delta - 2x \neq k\delta$ which is absurd. We conlude $2ax+2\beta+3\theta_\Sigma > k\delta$, but then it must be $2ax+2\beta+3\theta_\Sigma = k\delta+x$ so $k\delta = (2a-1)x + 2\beta + 3\theta_\Sigma$ forcing $k = 1$. Moreover since $s_\beta(\delta) = \delta$ they must have the same c_β, so $1 + (2a-1)j = 2$ which implies $a = j = 1$ and $|x| = |\beta|$, so the diagram is just $G_2^{(1)}$. The other statements are trivial. □

Let's move on now to the quadruple link case.

Lemma 3.2.37. *If β is long, $|\Sigma| = 1$ and θ_Σ has a quadruple link with β, then the following holds: the system is $A_1^{(2)}$,*

$$\mathcal{W}_\sigma^{ab} = \{1, s_\beta, s_\beta s_{\theta_\Sigma}\}$$

and in particular

$$\begin{aligned}
\mathcal{I}_{2\delta-\theta_\Sigma,2\delta-\theta_\Sigma} &= \{1\},\\
\mathcal{I}_{\beta+3\theta_\Sigma,2\delta-\theta_\Sigma} &= \{s_\beta\},\\
\mathcal{I}_{\beta+\theta_\Sigma,2\delta-\theta_\Sigma} &= \{s_\beta s_{\theta_\Sigma}\}.
\end{aligned}$$

Proof. We have $s_{\theta_\Sigma}(\beta) = \beta + 4\theta_\Sigma < k\delta$ because $c_\beta < 2$. So $s_\beta(\beta+4\theta_\Sigma) = 3\beta + 4\theta_\Sigma = k\delta + \beta$ and $k\delta = 2\beta + 4\theta_\Sigma$. This shows that $k = 2$ and $\delta = \beta + 2\theta_\Sigma$ proving that the diagram is $A_1^{(2)}$. The other statements are trivial. □

Case d.

We assume that β is a long root, and we consider $\mu = k\delta + \beta$. This case is very similar to the case of abelian ideals in Chapter 2. We show in Lemma 3.2.39 that if $\gamma \in \widehat{\Delta}_\beta^1$, then if u_γ^β is the minimal element in W_0 such that $u_\gamma^\beta(\gamma) = k\delta - \beta$, then $s_\beta u_\gamma^\beta = \min \mathcal{I}_{\gamma,\mu}$. After some technicalities, we find in Lemma 3.2.41 conditions under which we can add chains of simple reflections fixing γ to $s_\beta u_\gamma^\beta$ in order to find other elements in $\mathcal{I}_{\gamma,\mu}$, and that every element in $\mathcal{I}_{\gamma,\mu}$ can be written adding a chain of simple reflections fixing γ to $s_\beta u_\gamma^\beta$. We conclude showing in Lemma 3.2.42 that if $\gamma \notin \widehat{\Delta}_\beta^1 \cup \{k\delta + \beta\}$, then $\mathcal{I}_{\gamma,\mu} = \emptyset$, proving that $\widehat{\Delta}_\mu = \widehat{\Delta}_\beta^1 \cup \{k\delta + \beta\}$.

Let's denote $W_{\alpha_1,\dots,\alpha_n}$ the parabolic subgroup of $\widehat{W}$ generated by the simple reflections different from $s_1, \dots, s_n$. Note that $W_\beta = W_0$.

Lemma 3.2.38. *Let $\gamma \in \widehat{\Delta}$ be a long root such that $c_\beta(\gamma) = 1$, then there exists $u \in W_0$ such that*

$$u(\gamma) = \beta.$$

Proof. Let's consider the set

$$\Gamma = \{\tau \in \widehat{\Delta} | \tau \text{ is long}, c_\beta(\tau) = 1, w(\tau) \neq \beta \ \forall w \in W_0\}$$

and suppose it is not empty. Consider also the height map $h : \Gamma \to \mathbb{N}$ defined by $\gamma = \sum_i a_i \alpha_i \mapsto \sum_i a_i$, and pick $\gamma \in \Gamma$ such that $h(\gamma)$ realizes a minimum on Γ. Thus $\forall w \in W_0$ we have $h(w(\gamma)) \geq h(\gamma)$, otherwise $w(\gamma) \notin \Gamma$ implying that there exists $v \in W_0$ such that $vw(\gamma) = \beta$ and $vw \in W_0$. Then $h(s_\beta(\gamma)) < h(\gamma)$, otherwise $h(g(\gamma)) \geq h(\gamma) > 1 \ \forall g \in \widehat{W}$ since $\gamma \neq \beta$, against the fact that every root is $\widehat{W}$-connected to a simple root. We must have $s_\beta(\gamma) = \gamma - j\beta$ and the only possibility is $j = 1$, since if $\gamma - j\beta < 0$ then γ is a positive multiple of β, i.e. $\gamma = \beta \notin \Gamma$. If α_1 is the only simple root connected to β in *Supp* (γ), we must have $c_\alpha(\gamma) = c_\alpha(\gamma - \beta) = 1$. For all $w \in W_{\beta,\alpha_1}$ we have

$$h(ws_\beta(\gamma)) = h(w(\gamma - \beta)) = h(w(\gamma)) - h(w(\beta)) \geq h(\gamma) - h(\beta) = h(s_\beta(\gamma))$$

thus again, since for all $w \in W_{\alpha_1}$ $h(ws_\beta(\gamma)) \geq h(s_\beta(\gamma))$ we have $h(s_{\alpha_1} s_\beta(\gamma)) < h(s_\beta(\gamma))$ forcing $s_{\alpha_1} s_\beta(\gamma) = \gamma - \beta - \alpha_1$ or $s_{\alpha_1} s_\beta(\gamma) = \gamma - \beta - 2\alpha_1$. The former implies *Supp* $(s_{\alpha_1} s_\beta(\gamma)) \subset$ *Supp* $(s_\beta(\gamma))$ and the root connected to α_1, let's say α_2, satisfies $c_{\alpha_2}(\gamma) = c_{\alpha_2}(\gamma - \beta - \alpha_1) = 1$ and α_1 is not shorter than α_2. In the latter case $\gamma - \beta - 2\alpha_1 < 0$ implies $\gamma = \beta + \alpha_1$ and α_1 is long since $\alpha_1 = s_\beta(\gamma)$. Now let $\beta + \alpha_1 + \alpha_2 + \cdots + \alpha_n$ the longest stretch of connected simple roots of height 1 in γ that can be removed applying s_i, i.e. $s_n \dots s_1 s_\beta(\gamma) = \gamma - \beta - \alpha_1 - \cdots - \alpha_n$. Let's call α_{n+1} the only remaining root that was linked to α_n. Then for all $w \in W_{\beta,\alpha_1,\dots,\alpha_n,\alpha_{n+1}}$ we have $h(ws_n \dots s_1 s_\beta(\gamma)) = h(w(\gamma)) - n - 1 \geq h(\gamma) - n - 1 = h(\gamma - \beta - \alpha_1 - \cdots - \alpha_n) =$

$h(s_n \dots s_1 s_\beta(\gamma))$ thus for all $w \in W_{\alpha_{n+1}}$, $h(ws_n \dots s_1 s_\beta(\gamma)) \geq h(s_n \dots s_1 s_\beta(\gamma))$ implying that $h(s_{n+1}s_n \dots s_1 s_\beta(\gamma)) < h(s_n \dots s_1 s_\beta(\gamma))$. But then

$$s_{n+1}s_n \dots s_1 s_\beta(\gamma) - \gamma - \beta - \alpha_1 - \cdots - \alpha_n - 2\alpha_{n+1} < 0$$

and so $\gamma = \beta + \alpha_1 + \alpha_2 + \cdots + \alpha_n + \alpha_{n+1}$ and $\alpha_{n+1} = s_n \dots s_1 s_\beta(\gamma)$ is long. But then all the α_i for $i = 1, \dots, n$ are long as well, because $\gamma = s_\beta s_1 \cdots s_n(\alpha_{n+1})$ and $c_{\alpha_i}(\gamma) = 1$ for every $i = 1, \dots, n$. In the end

$$s_{n+1}s_n \dots s_2 s_1(\beta) = \gamma$$

which is absurd, thus $\Gamma = \emptyset$. □

Using this lemma we can give a new proof of Corollary 3.2.2, i.e. that $w_0(\beta) = k\delta - \beta$. Let $u \in W_0$ be of maximal length such that $u(\beta) = k\delta - \beta$. Then for every simple root τ not connected to β in the diagram, since $s_\tau u(k\delta - \beta) = s_\tau(\beta) = \beta$ we have $l(s_\tau u) < l(u)$. Let's pick α connected to β, then since $us_\alpha(k\delta - \beta) = u(k\delta - \beta - a\alpha) = \beta - au(\alpha) \in \widehat{\Delta}$ with $a > 0$, we have that $u(\alpha) < 0$, thus $l(s_\alpha u) < l(u)$. In the end for every simple root $\tau \in W_0$ we have $l(s_\tau u) < l(u)$ forcing $u = w_0$.

If γ is long and $c_\beta(\gamma) = 1$, then also $k\delta - \gamma$ has the same properties, and the map $\gamma \mapsto (k\delta - \gamma)$ is invertible. We call u_γ^β the shortest element in W_0 such that $u_\gamma^\beta(\gamma) = k\delta - \beta$ for a given long root γ with $c_\beta(\gamma) = 1$.

Lemma 3.2.39. *$s_\beta u_\gamma^\beta \in \mathcal{I}_{\gamma, k\delta+\beta}$ and is its minimum.*

Proof. Write $u_\gamma^\beta = s_1 s_2 \dots s_n$ in reduced form, α_j the simple root associated to s_j for every j, and $N(u_\gamma^\beta) = \{\alpha_1, s_1(\alpha_2), \dots, s_1 s_2 \dots s_{n-1}(\alpha_n)\}$. We want to show that every root in $N(u_\gamma^\beta)$ has in its support exactly one root linked to β in the diagram. Write $\tau_j = s_1 \cdots s_{j-1}(\alpha_j)$ and

$$s_1 \cdots s_{j-1}s_{j+1} \cdots s_n(\gamma) = k\delta - \beta - a_j\tau_j.$$

We have $c_\beta(\tau_j) = 0$ by construction of u_γ^β and $a_j \neq 0$ for the minimality of u_γ^β, so $a_j > 0$. Since $\tau_j \in \Sigma_s$ for some s, the root linked to β has height at most 1 and the claim follows. Now call γ_i the rootlet associated to $s_1 \cdots s_i$, i.e. $s_1 \cdots s_i(\gamma_i) = k\delta + \beta$. Since $a_j > 0$ for every j, we see that

$$k\delta + \beta = \gamma_0 > \gamma_1 > \cdots > \gamma_n = \gamma$$

and the second claim follows as in Lemma 3.2.8. □

The proof of the following lemma is similar to that of Lemma 2.2.5. We include it for completeness.

Lemma 3.2.40. *Let γ be a long root such that $c_\beta(\gamma) = 1$. Then*

$$s_q(\gamma) = \gamma \iff s_\beta u^\beta_\gamma(q) = u^\beta_\gamma(q) \quad \forall q \in \Pi.$$

Proof. Assume first that $q \neq \beta$. Suppose $s_q(\gamma) = \gamma$ and $s_\beta u^\beta_\gamma(q) \neq u^\beta_\gamma(q)$, then $s_\beta u^\beta_\gamma(q) = u^\beta_\gamma(q) + a\beta$ with $a = \pm 1$ since the simple root connected to β in $u^\beta_\gamma(q)$ must have coefficient ± 1. Thus

$$u^\beta_\gamma s_q (u^\beta_\gamma)^{-1}(u^\beta_\gamma(q)+a\beta) = u^\beta_\gamma s_q(q+ak\delta-a\gamma) = u^\beta_\gamma(-q+ak\delta-a\gamma) = -u^\beta_\gamma(q)+a\beta \in \widehat{\Delta}$$

which is absurd because a and $u^\beta_\gamma(q)$ have the same sign. Suppose $s_q(\gamma) \neq \gamma$ and $s_\beta u^\beta_\gamma(q) = u^\beta_\gamma(q)$, then $s_q(\gamma) = \gamma + aq$ with $a \neq 0$. So

$$u^\beta_\gamma(\gamma + aq) = k\delta - \beta + au^\beta_\gamma(q) \in \widehat{\Delta}$$

implying that a and $u^\beta_\gamma(q)$ have opposite signs. But also

$$s_\beta(k\delta - \beta + au^\beta_\gamma(q)) = k\delta + \beta + au^\beta_\gamma(q) \in \widehat{\Delta}$$

implying a and $u^\beta_\gamma(q)$ have the same sign, absurd. Let's now assume $q = \beta$. Suppose $s_\beta(\gamma) = \gamma$ and $s_\beta u^\beta_\gamma(\beta) \neq u^\beta_\gamma(\beta)$, then $s_\beta u^\beta_\gamma(\beta) = u^\beta_\gamma(\beta) + a\beta$ with $a \neq 0$. Thus

$$-s_\beta u^\beta_\gamma s_\beta (u^\beta_\gamma)^{-1}(u^\beta_\gamma(\beta) + a\beta) = -s_\beta u^\beta_\gamma s_\beta(\beta + ak\delta - a\gamma) =$$

$$= -s_\beta u^\beta_\gamma(-\beta + ak\delta - a\gamma) = s_\beta(u^\beta_\gamma(\beta) - a\beta) = u^\beta_\gamma(\beta) + 2a\beta \in \widehat{\Delta}.$$

Moreover also

$$-u^\beta_\gamma s_\beta (u^\beta_\gamma)^{-1}(u^\beta_\gamma(\beta) + 2a\beta) = -u^\beta_\gamma s_\beta(\beta + 2ak\delta - 2a\gamma) =$$

$$= -u_\gamma^\beta(2ak\delta - \beta - 2a\gamma) = u_\gamma^\beta(\beta) - 2a\beta \in \widehat{\Delta}.$$

Without loss of generality we can take $a > 0$, then $u_\gamma^\beta(\beta) - 2a\beta < 0$ since $c_\beta(u_\gamma^\beta(\beta)) = 1$, then $u_\gamma^\beta(\beta) = \beta$, but $u_\gamma^\beta(\beta) + 2a\beta = (2a+1)\beta \in \widehat{\Delta}$ which is absurd. Suppose $s_\beta(\gamma) \neq \gamma$ and $s_\beta u_\gamma^\beta(\beta) = u_\gamma^\beta(\beta)$, then $s_\beta(\gamma) = \gamma + a\beta$ with $a \neq 0$. Thus

$$-s_\beta(u_\gamma^\beta)^{-1} s_\beta u_\gamma^\beta(-2k\delta + \gamma + a\beta) = -s_\beta(u_\gamma^\beta)^{-1} s_\beta(-k\delta - \beta + au_\gamma^\beta(\beta)) =$$

$$= -s_\beta(u_\gamma^\beta)^{-1}(-k\delta + \beta + au_\gamma^\beta(\beta)) = -s_\beta(-\gamma + a\beta) = \gamma + 2a\beta \in \widehat{\Delta}.$$

Moreover also

$$-(u_\gamma^\beta)^{-1} s_\beta u_\gamma^\beta(-2k\delta + \gamma + 2a\beta) = -(u_\gamma^\beta)^{-1} s_\beta(-k\delta - \beta + 2au_\gamma^\beta(\beta)) =$$

$$= -(u_\gamma^\beta)^{-1}(-k\delta + \beta + 2au_\gamma^\beta(\beta)) = \gamma - 2a\beta \in \widehat{\Delta}.$$

Without loss of generality we can take $a > 0$, then $\gamma - 2a\beta < 0$ since $c_\beta(\gamma) = 1$, so $\gamma = \beta$. But then

$$s_\beta u_\gamma^\beta(\beta) = s_\beta u_\gamma^\beta(\gamma) = k\delta + \beta \neq k\delta - \beta = u_\gamma^\beta(\gamma) = u_\gamma^\beta(\beta)$$

which is absurd. □

Lemma 3.2.41. *Suppose $s_\beta u_\gamma^\beta w$ is such that $l(s_\beta u_\gamma^\beta w) = l(s_\beta u_\gamma^\beta) + l(w)$ and write $w = s_1 \cdots s_n$ in reduced form. Then $s_\beta u_\gamma^\beta w \in \mathcal{I}_{\gamma, k\delta+\beta} \iff w \in \mathcal{W}_\sigma^{ab}$ and $s_i(\gamma) = \gamma$ for every $i = 1, \ldots, n$.*

Proof. Suppose $s_i(\gamma) = \gamma$ for every $i = 1, \ldots, n$ and write α_i for the simple root associated to s_i. Then by Lemma 3.2.40 $s_\beta u_\gamma^\beta(q) = u_\gamma^\beta(q)$ for every $q \in \Pi$. If $q \neq \beta$ then $c_\beta(s_\beta u_\gamma^\beta(q)) = c_\beta(u_\gamma^\beta(q)) = 0$, if $q = \beta$ then $c_\beta(s_\beta u_\gamma^\beta(\beta)) = c_\beta(u_\gamma^\beta(\beta)) = 1$. Now just consider $s_\beta u_\gamma^\beta(s_1 \cdots s_{j-1}(\alpha_j)) \in N(u_\gamma^\beta w)$ for every j and the equivalence follows. Suppose now there is an s_i such that $s_i(\gamma) \neq \gamma$. Then there exists a rootlet γ', a simple root q and a σ-minuscule element vs_q with $l(vs_q) = l(v) + 1$, such that $vs_q(\gamma') = k\delta + \beta$ and $s_q(\gamma') = \gamma' - aq$ for some positive a, so

$$v(\gamma') = k\delta + \beta + av(q).$$

Since vs_q is σ-minuscule $c_\beta(av(q)) = a \geq 1$, moreover $\beta \in N(vs_q)$ and $v(q) \in N(vs_q)$ so $\beta + av(q) \in N(vs_q)$, but $c_\beta(\beta + av(q)) = 1 + a \geq 2$ which is absurd. □

Lemma 3.2.42. *If $\tau \in \widehat{\Delta}$ is such that $c_\beta(\tau) \neq 1$, then*

$$\mathcal{I}_{\tau,k\delta+\beta} = \emptyset$$

or $\tau = k\delta + \beta$ and $\mathcal{I}_{\tau,k\delta+\beta} = \{1\}$.

Proof. Suppose that there is $\tau \in \widehat{\Delta}$ with $c_\beta(\tau) \neq 1$ for which there is a $w \in \mathcal{I}_{\tau,k\delta+\beta}$, $w \neq 1$. Write $w = s_\beta s_2 \dots s_n$ in reduced form. Since $c_\beta(\tau) \neq 1$ and $c_\beta(s_2 \dots s_n(\tau)) = c_\beta(k\delta - \beta) = 1$, there must be an index $j \in [2, n]$ such that $s_j = s_\beta$ is the last simple reflection in w that changes the β-coefficient of τ in the sequence of simple reflections $s_2 \dots s_n$. So $\gamma := s_\beta s_{j+1} \dots s_n(\tau)$ is such that $c_\beta(\gamma) = 1$ and

$$s_\beta s_2 \dots s_{j-1} \in \mathcal{I}_{\gamma,k\delta+\beta}.$$

Thanks to Lemma 3.2.41, since $s_\beta(\gamma) \neq \gamma$, $s_\beta s_2 \dots s_{j-1}$ is the minimum in the poset $\mathcal{I}_{\gamma,k\delta+\beta}$, so $s_\beta s_2 \dots s_{j-1} = s_\beta u^\beta_\gamma$. But then $s_\beta u^\beta_\gamma s_\beta$ can't be σ-minuscule due to Lemma 3.2.40, since $s_\beta u^\beta_\gamma(\beta) \neq u^\beta_\gamma(\beta)$ and thus $c_\beta(s_\beta s_2 \dots s_{j-1}(\beta)) = c_\beta(s_\beta u^\beta_\gamma(\beta)) \neq 1$. □

Corollary 3.2.43. *For every γ long root such that $c_\beta(\gamma) = 1$, $\mathcal{I}_{\gamma,k\delta+\beta}$ and $\mathcal{I}_{k\delta-\gamma,k\delta+\beta}$ are isomorphic as posets. The isomorphism is given by left multiplication by $s_\beta w_{0,\beta} w_0 s_\beta$.*

Proof. If s is a simple reflection, $s(\gamma) = \gamma$ if and only if $s(k\delta - \gamma) = k\delta - \gamma$, thus we can attach to the minima of the two posets the same σ-minuscule elements. We just need to prove that $s_\beta w_{0,\beta} w_0 s_\beta s_\beta u^\beta_\gamma = s_\beta w_{0,\beta} w_0 u^\beta_\gamma \in \mathcal{I}_{k\delta-\gamma,k\delta+\beta}$. Indeed

$$s_\beta w_{0,\beta} w_0 u^\beta_\gamma (k\delta - \gamma) = s_\beta w_{0,\beta} w_0(\beta) = k\delta + \beta.$$

To see that it is σ-minuscule write

$$N(s_\beta w_{0,\beta} w_0 u^\beta_\gamma) = N(s_\beta w_{0,\beta} w_0) \dot{+} s_\beta w_{0,\beta} w_0 N(u^\beta_\gamma).$$

We know that $N(u^\beta_\gamma)$ contains only roots contained in some Σ_j, with one simple root linked to β in the diagram. w_0 makes them negative with the same property, $w_{0,\beta}$ doesn't change coefficients of simple roots linked to β, so it stays negative, and s_β just adds $-\beta$, so the final root is always negative and the claim follows. □

Case e.

We assume that β is a long root, and we consider $\mu = k\delta - \theta_\Sigma$, with $|\Sigma| > 1$ and θ_Σ of type 2. Note that there can be two simple roots $\bar{\alpha}_1$ and $\bar{\alpha}_2$ in the diagram adjacent to α_Σ, and since $\Gamma(\Sigma) = \{\alpha_\Sigma\}$, this situation occurs exactly when there are 4 simple roots in the diagram, i.e. $\{\beta\} \cup A_3 = A_5^{(2)}$. We write $\bar{\alpha}$ for a generic simple root adjacent to α_Σ, and with $c_{\bar{\alpha}}$ we mean in this case $c_{\bar{\alpha}_1} + c_{\bar{\alpha}_2}$. After some technicalities, we show in Lemmas 3.2.46 and 3.2.47 that if γ is such that $|\gamma| = |\theta_\Sigma|$ and $\gamma \in \widehat{\Delta}^0_{\alpha_\Sigma} \cup \widehat{\Delta}^1_{\alpha_\Sigma} \cup \{\delta + \alpha_\Sigma\} \cup \{\delta + \alpha_\Sigma + \beta\}$, then u^Σ_γ, as defined in Lemma 3.2.46, is such that $u^\Sigma_\gamma = \min \mathcal{I}_{\gamma,\mu}$. Then we show in Corollary 3.2.49 conditions under which we can add chains of simple reflections fixing γ to u^Σ_γ, in order to find other elements in $\mathcal{I}_{\gamma,\mu}$. In Lemma 3.2.50 we show that every element in $\mathcal{I}_{\gamma,\mu}$ can be written in such way. Finally, in Lemma 3.2.53, we show that $\widehat{\Delta}_\mu = \{\gamma \in \widehat{\Delta}^0_{\alpha_\Sigma} \cup \widehat{\Delta}^1_{\alpha_\Sigma} : |\gamma| = |\theta_\Sigma|\} \cup \{\delta + \alpha_\Sigma\} \cup \{\delta + \alpha_\Sigma + \beta\}$.

Lemma 3.2.44. *The following relations hold:* $c_{\alpha_\Sigma}(\theta_\Sigma) = 1$, $s_{\alpha_\Sigma}(\theta_\Sigma) = \theta_\Sigma$, $k = 2$, $\delta = \theta_\Sigma + \alpha_\Sigma + \beta$ *and* $\Gamma(\Sigma) = \{\alpha_\Sigma\}$.

Proof. $c_{\alpha_\Sigma}(\theta_\Sigma) = 1$ because β is long . For the second part suppose $s_{\alpha_\Sigma}(\theta_\Sigma) = \theta_\Sigma - \alpha_\Sigma$, then there is only one simple root α_1 adjacent to α_Σ in θ_Σ and $c_{\alpha_1}(\theta_\Sigma) = 1$. Repeating the argument on $\theta_\Sigma - \alpha_\Sigma$ as the highest root in $\Sigma \setminus \{\alpha_\Sigma\}$, we see by induction that $\Sigma = A_n$ for some $n > 1$. Then $s_{\theta_\Sigma}(\beta) = \beta + 2\theta_\Sigma$ is the highest root of the diagram of finite type $\Sigma \cup \{\beta\} = C_n$ and thus there must be another simple root adjacent to β in the affine diagram, let's call it x. So $\tau = s_\beta s_{\theta_\Sigma} s_\beta(x) = x + 2\beta + 2\theta_\Sigma$ has $c_\beta(\tau) = 2$, and is such that there exists a simple reflection $s_q \in W(\Sigma)$ such that $s_q(\tau) < \tau$, so $k\delta - \tau \in \Sigma$. Moreover for every simple reflection $s \in W(\Sigma)$ we see that $s(\tau) \leq \tau$, so for every simple reflection $s \in W(\Sigma)$ we have $s(k\delta - \tau) \geq k\delta - \tau$ which is absurd. In conclusion $s_{\alpha_\Sigma}(\theta_\Sigma) = \theta_\Sigma$. For the third and forth claims just compute $s_{\alpha_\Sigma} s_{\theta_\Sigma}(\beta) = \beta + 2\theta_\Sigma + 2\alpha_\Sigma$, since it has $c_{\alpha_\Sigma} = 4$ then $k\delta - (\beta + 2\theta_\Sigma + 2\alpha_\Sigma) = \beta$ and so $k\delta = 2\beta + 2\theta_\Sigma + 2\alpha_\Sigma$ which implies $k = 2$ and $\delta = \beta + \theta_\Sigma + \alpha_\Sigma$. For the last claim pick a simple reflection $\alpha_1 \in \Sigma$ adjacent to α_Σ, then $s_{\alpha_1}(\theta_\Sigma) = s_{\alpha_1}(\delta - \beta - \alpha_\Sigma) = \delta - \beta - \alpha_\Sigma - a\alpha_1 = \theta_\Sigma - a\alpha_1$ with $a \neq 0$. $\square$

Lemma 3.2.45. *Let $\tau \in \langle\Sigma\rangle$, $|\tau| = |\theta_\Sigma|$, and $\bar{\alpha}$ be a simple root adjacent to $\alpha_\Sigma \in \Sigma$. If $c_{\bar{\alpha}}(\tau) = 2$, then $\tau = \theta_\Sigma$.*

Proof. If $\tau \neq \gamma$ then take v to be the shortest element in $W(\Sigma)$ such that $v(\tau) = \theta_\Sigma$. Write $v = s_1 \cdots s_n$ in reduced form; then $s_2 \cdots s_n(\tau) = \theta_\Sigma - a_1\alpha_1$. Consider the root $\delta - \theta_\Sigma + a_1\alpha_1 = \alpha_\Sigma + \beta + a_1\alpha_1$, thanks to the previous lemma. Since $s_{\alpha_\Sigma}(\theta_\Sigma) = \theta_\Sigma$, $\alpha_1 \neq \alpha_\Sigma$ and so α_1 is adjacent to α_Σ and $c_{\bar{\alpha}}(\tau) < 2$. $\square$

Lemma 3.2.46. *If $|\gamma| = |\theta_\Sigma|$ and $\gamma \in \widehat{\Delta}^0_{\alpha_\Sigma} \cup \widehat{\Delta}^1_{\alpha_\Sigma} \cup \{\delta + \alpha_\Sigma\} \cup \{\delta + \alpha_\Sigma + \beta\}$, then $\mathcal{I}_{\gamma,2\delta-\theta_\Sigma} \neq \emptyset$.*

Proof. Assume first $\gamma \in \widehat{\Delta}^0_{\alpha_\Sigma} \cup \widehat{\Delta}^1_{\alpha_\Sigma}$, $|\gamma| = |\theta_\Sigma|$ and $\gamma \in \langle\Sigma\rangle$. Note that $\gamma \in \widehat{\Delta}^0_{\alpha_\Sigma} \cup \widehat{\Delta}^1_{\alpha_\Sigma}$ implies $c_\tau(\gamma) \leq c_\tau(\delta - \alpha_\Sigma) = c_\tau(\beta + \theta_\Sigma)$ for every $\tau \in \widehat{\Pi}$. Take v as the shortest element in $W(\Sigma)$ such that $v(\gamma) = \theta_\Sigma$. If $\tau_k \in N(v)$, from the previous lemma it follows that $c_{\bar{\alpha}}(\tau_k) \geq 1$. Moreover $c_{\bar{\alpha}}(\tau_k) \leq 2$, if $c_{\bar{\alpha}}(\tau_k) = 2$ then $\tau_k = \theta_\Sigma$ again for the previous lemma, so $\gamma = v^{-1}(\theta_\Sigma) = v^{-1}(\tau_k) < 0$ but $\gamma > 0$. We see then that $c_{\bar{\alpha}}(\tau_k) = 1$. Consider $u^\Sigma_\gamma = s_\beta s_{\alpha_\Sigma} s_\beta v$. We have $u^\Sigma_\gamma(\gamma) = s_\beta s_{\alpha_\Sigma} s_\beta(\theta_\Sigma) = \theta_\Sigma + 2\beta + 2\alpha_\Sigma = 2\delta - \theta_\Sigma$. Moreover

$$N(u^\Sigma_\gamma) = \{\beta, \beta + \alpha_\Sigma, \beta + 2\alpha_\Sigma\} \cup s_\beta s_{\alpha_\Sigma} s_\beta N(v).$$

If $c_{\alpha_\Sigma}(\tau_k) = 0$ then $s_\beta s_{\alpha_\Sigma} s_\beta(\tau_k) = s_\beta s_{\alpha_\Sigma}(\tau_k) = \alpha_\Sigma + \tau_k + \beta$ with $c_\beta = 1$. If otherwise $c_{\alpha_\Sigma}(\tau_k) = 1$ then $s_\beta s_{\alpha_\Sigma} s_\beta(\tau_k) = s_\beta s_{\alpha_\Sigma}(\tau_k + \beta) = s_\beta(\tau_k + \beta + \alpha_\Sigma) = \alpha_\Sigma + \tau_k + \beta$ with $c_\beta = 1$. In the end $u^\Sigma_\gamma \in \mathcal{W}^{ab}_\sigma$. Assume now $\gamma \in \widehat{\Delta}^0_{\alpha_\Sigma} \cup \widehat{\Delta}^1_{\alpha_\Sigma}$, $|\gamma| = |\theta_\Sigma|$ and $\gamma = \beta + \tau$ with $\tau \in \langle\Sigma\rangle$. Take v as the shortest element in $W(\Sigma)$ such that $v(\tau) = \theta_\Sigma$. Consider $u^\Sigma_\gamma = s_\beta s_{\alpha_\Sigma} v$. Note that since $c_{\alpha_\Sigma}(\tau) = c_{\alpha_\Sigma}(\theta_\Sigma) = 1$, v can be written in reduced form without using s_{α_Σ}, and so $v(\beta) = \beta$ and $v(\gamma) = v(\beta + \tau) = \beta + \theta_\Sigma$. We have $u^\Sigma_\gamma(\gamma) = s_\beta s_{\alpha_\Sigma} v(\gamma) = s_\beta s_{\alpha_\Sigma}(\beta + \theta_\Sigma) = s_\beta(\beta + 2\alpha_\Sigma + \theta_\Sigma) = 2\beta + 2\alpha_\Sigma + \theta_\Sigma = 2\delta - \theta_\Sigma$. Moreover

$$N(u^\Sigma_\gamma) = \{\beta, \beta + \alpha_\Sigma\} \cup s_\beta s_{\alpha_\Sigma} N(v).$$

Again since v can be written in reduced form without using s_{α_Σ}, $c_{\alpha_\Sigma}(\tau_k) = 0$ and again from the previous lemma $c_{\bar{\alpha}}(\tau_k) = 1$, so $s_\beta s_{\alpha_\Sigma}(\tau_k) = \tau_k + \alpha_\Sigma + \beta$ with $c_\beta = 1$. We conclude that when $\gamma \in \widehat{\Delta}^0_{\alpha_\Sigma} \cup \widehat{\Delta}^1_{\alpha_\Sigma}$, $|\gamma| = |\theta_\Sigma|$, then $u^\Sigma_\gamma \in \mathcal{W}^{ab}_\sigma$. For $\gamma = \delta + \alpha_\Sigma$ we take $u^\Sigma_\gamma = s_\beta$. For $\gamma = \delta + \alpha_\Sigma + \beta$ we take $u^\Sigma_\gamma = 1$. $\square$

Lemma 3.2.47. *Let γ and u^{Σ}_{γ} be as in the previous lemma, then* $u^{\Sigma}_{\gamma} = \min \mathcal{I}_{\gamma, 2\delta - \theta_{\Sigma}}$.

Proof. Write $u^{\Sigma}_{\gamma} = s_1 \cdots s_n$ in reduced form and consider the rootlet γ_k of $s_1 \cdots s_k$ for every k. Then in every case we see that

$$\gamma = \gamma_n < \gamma_{n-1} < \cdots < \gamma_0 = 2\delta - \theta_{\Sigma}$$

so the proof follows as in Lemma 3.2.8. □

Lemma 3.2.48. *Let γ and u^{Σ}_{γ} be as in the previous lemma, and s_q a simple reflection associated to the simple root q with $s_q(\gamma) = \gamma$. Then*

(1) if $q = \beta$ then $c_{\beta}(u^{\Sigma}_{\gamma}(\beta)) = 1$,

(2) if $q \neq \alpha_{\Sigma}, \beta$ then $c_{\beta}(u^{\Sigma}_{\gamma}(q)) = 0$,

(3) if $q = \alpha_{\Sigma}$ and $\gamma \neq \alpha_{\Sigma} + \beta$ then $c_{\beta}(u^{\Sigma}_{\gamma}(\alpha_{\Sigma})) = 0$.

Proof. Notice first that if $s_q(\gamma) = \gamma$ then $u^{\Sigma}_{\gamma}(q) > 0$, otherwise u^{Σ}_{γ} has a reduced form ending in s_q, and the remaining element is still σ-minuscule, against the minimality of u^{Σ}_{γ} in $\mathcal{I}_{\gamma, 2\delta - \theta_{\Sigma}}$. For $\gamma = \delta + \alpha_{\Sigma}, \gamma = \delta + \alpha_{\Sigma} + \beta$ the claims are obvious. Consider then $|\gamma| = |\theta_{\Sigma}|$ and $\gamma \in \widehat{\Delta}^{0}_{\alpha_{\Sigma}} \cup \widehat{\Delta}^{1}_{\alpha_{\Sigma}}$.

(1) If $s_{\beta}(\gamma) = \gamma$, then $\gamma \in \langle \Sigma \rangle$ and $c_{\alpha_{\Sigma}}(\gamma) = 0$ (recall that our γ's have $c_{\alpha_{\Sigma}}(\gamma) \leq 1$). Then $u^{\Sigma}_{\gamma}(\beta) = s_{\beta} s_{\alpha_{\Sigma}} s_{\beta} v(\beta) = s_{\beta} s_{\alpha_{\Sigma}} s_{\beta}(\beta + 2\alpha_{\Sigma} + 2R)$ with R a sum of other simple roots, because there is exactly one $s_{\alpha_{\Sigma}}$ in a reduced form of v, and all the other roots can be added or removed only with an even coefficient. $R \neq 0$ because otherwise $v(\beta) = \beta + 2\alpha$ which implies $v^{-1}(\beta) = \beta - 2v^{-1}(\alpha_{\Sigma})$ but $v^{-1}(\alpha_{\Sigma}) > 0$ becasue for every $\tau_k \in N(v)$ we have $c_{\bar{\alpha}} = 1$, and $c_{\beta}(v^{-1}(\alpha_{\Sigma})) = 0$ so $v^{-1}(\beta)$ can't exist which is absurd. We can go on calculating $u^{\Sigma}_{\gamma}(\beta) = s_{\beta} s_{\alpha_{\Sigma}} s_{\beta}(\beta + 2\alpha_{\Sigma} + 2R) = s_{\beta} s_{\alpha_{\Sigma}}(\beta + 2\alpha_{\Sigma} + 2R)$. Since $R \neq 0$ we can have $s_{\alpha_{\Sigma}}(\beta + 2\alpha_{\Sigma} + 2R) = \beta + 2\alpha_{\Sigma} + 2R$ or $s_{\alpha_{\Sigma}}(\beta + 2\alpha_{\Sigma} + 2R) = \beta + 4\alpha_{\Sigma} + 2R$ (and no more because otherwise subracting 2δ we get a root with different signs in c_{β} and $c_{\alpha_{\Sigma}}$). If $s_{\alpha_{\Sigma}}(\beta + 2\alpha_{\Sigma} + 2R) = \beta + 4\alpha_{\Sigma} + 2R = 2\delta - \beta$ which is clear subtratting 2δ, then $v(\beta) = s_{\alpha_{\Sigma}}(2\delta - \beta) = 2\delta - \beta - 2\alpha_{\Sigma} =$

$\beta+2\theta_\Sigma$, but this is impossible because it implies $v^{-1}(\beta)=\beta-2\gamma$ but $\gamma>0$ and $\gamma\in\langle\Sigma\rangle$. In conclusion

$$u_\gamma^\Sigma(\beta)=s_\beta s_{\alpha_\Sigma}(\beta+2\alpha_\Sigma+2R)=s_\beta(\beta+2\alpha_\Sigma+2R)=\beta+2\alpha_\Sigma+2R$$

with $c_\beta=1$.

(2) $v(q)\in\langle\Sigma\rangle$, we claim $v(q)>0$. If $v(q)<0$ then since $u_\gamma^\Sigma(\beta)>0$ we must have $v(q)\in\langle\alpha_\Sigma,\beta\rangle$ and thus $v(q)=-\alpha_\Sigma$ and so $v^{-1}(\alpha_\Sigma)=-q<0$ that as we know it's not possible.

- If $c_{\alpha_\Sigma}(v(q))=c_{\bar\alpha}(v(q))=0$ then $s_\beta s_{\alpha_\Sigma}(s_\beta(v(q)))=v(q)$ with $c_\beta=0$.
- If $c_{\alpha_\Sigma}(v(q))=1$ and $c_{\bar\alpha}(v(q))=0$ then $v(q)=\alpha_\Sigma$ but in case $c_\beta(\gamma)=1$ $s_\beta s_{\alpha_\Sigma}(\alpha_\Sigma)<0$ against $u_\gamma^\Sigma(q)>0$, and in case $c_\beta(\gamma)=0$ $s_\beta s_{\alpha_\Sigma}s_\beta(\alpha_\Sigma)=\alpha_\Sigma$ with $c_\beta=0$.
- If $c_{\alpha_\Sigma}(v(q))=0$ and $c_{\bar\alpha}(v(q))=1$ then (in both cases for $c_\beta(\gamma)=0,1$) $s_\beta s_{\alpha_\Sigma}v(q)=s_\beta s_{\alpha_\Sigma}s_\beta v(q)=v(q)+\alpha_\Sigma+\beta$. But this root cannot exist because, indeed write $\gamma=a\beta+\tau$ with $a=0,1$ and $\tau\in\langle\Sigma\rangle$. Then $v(q)+\alpha_\Sigma+\beta=v(q)+\delta-\theta_\Sigma$ and thus $v^{-1}(v(q)+\delta-\theta_\Sigma)=q+\delta-\tau$. This implies that also $s_q(\tau-q)=\tau+q$ is a root ($s_q(\tau)=\tau$ because $q\neq\alpha_\Sigma,\beta$) and so also $v(\tau+q)=\theta_\Sigma+v(q)$ which is in $\langle\Sigma\rangle$ and $v(q)>0$, absurd.
- If $c_{\alpha_\Sigma}(v(q))=1$ and $c_{\bar\alpha}(v(q))=1$ then if $c_\beta(\gamma)=0$ we have $u_\gamma^\Sigma(q)=s_\beta s_{\alpha_\Sigma}s_\beta v(q)=v(q)+\beta+\alpha_\Sigma$ which is again impossible, and if $c_\beta(\gamma)=1$ we have $u_\gamma^\Sigma(q)=s_\beta s_{\alpha_\Sigma}v(q)=v(q)-\alpha_\Sigma$ with $c_\beta=0$.
- If $c_{\alpha_\Sigma}(v(q))=1$ and $c_{\bar\alpha}(v(q))=2$ then $v(q)=\theta_\Sigma$ from Lemma 3.2.45 and so $q=\gamma$ but $s_q(\gamma)=s_q(q)\neq q=\gamma$.

(3) If $s_{\alpha_\Sigma}(\gamma)=\gamma$ then $\gamma=\beta+\alpha_\Sigma$ or $\gamma=\theta_\Sigma$ or $\gamma\in\langle\Sigma\rangle$ and $c_{\alpha_\Sigma}(\gamma)=c_{\bar\alpha}(\gamma)=0$. If $\gamma=\theta_\Sigma$ then $v=1$ and $u_\gamma^\Sigma(\alpha_\Sigma)=s_\beta s_{\alpha_\Sigma}s_\beta(\alpha_\Sigma)=\alpha_\Sigma$ with $c_\beta=0$. Finally if $\gamma\in\langle\Sigma\rangle$ and $c_{\alpha_\Sigma}(\gamma)=c_{\bar\alpha}(\gamma)=0$ then following the steps as in part (2) we

only need to check what happens if $\alpha_\Sigma + \beta + v(\alpha_\Sigma)$ is a root. In this case $\alpha_\Sigma + \beta + v(\alpha_\Sigma) = \delta - \theta_\Sigma + v(\alpha_\Sigma)$ and so $v^{-1}(\delta - \theta_\Sigma + v(\alpha_\Sigma)) = \delta - \gamma + \alpha_\Sigma$ but $\gamma - \alpha_\Sigma$ can't be a root due to $\gamma \in \langle\Sigma\rangle$ and $c_{\alpha_\Sigma}(\gamma) = c_{\bar\alpha}(\gamma) = 0$. In the end if $\gamma \neq \alpha_\Sigma + \beta$ then $c_\beta(u^\Sigma_\gamma(\alpha_\Sigma)) = 0$.

□

Corollary 3.2.49. *Let $|\gamma| = |\theta_\Sigma|$ and $\gamma \in \widehat\Delta^0_{\alpha_\Sigma} \cup \widehat\Delta^1_{\alpha_\Sigma}$ with $\gamma \neq \alpha_\Sigma + \beta$, or $\gamma = \delta + \alpha_\Sigma$ or $\gamma = \delta + \alpha_\Sigma + \beta$, and let $u^\Sigma_\gamma = \min \mathcal{I}_{\gamma,2\delta-\theta_\Sigma}$. If s can be written in reduced form with simple reflections fixing γ, then $u^\Sigma_\gamma s \in \mathcal{I}_{\gamma,2\delta-\theta_\Sigma} \iff s \in \mathcal{W}^{ab}_\sigma$.*

Lemma 3.2.50. *Let $|\gamma| = |\theta_\Sigma|$ and $\gamma \in \widehat\Delta^0_{\alpha_\Sigma} \cup \widehat\Delta^1_{\alpha_\Sigma}$ or $\gamma = \delta + \alpha_\Sigma$ or $\gamma = \delta + \alpha_\Sigma + \beta$, and let $u^\Sigma_\gamma = \min \mathcal{I}_{\gamma,2\delta-\theta_\Sigma}$. If $u^\Sigma_\gamma s \in \mathcal{I}_{\gamma,2\delta-\theta_\Sigma}$ then s can be written in reduced form with simple reflections fixing γ.*

Proof. Suppose there are any $w \in \mathcal{W}^{ab}_\sigma$, s_q a simple reflection and $\gamma' \in \widehat\Delta^+$ with $s_q(\gamma') = \gamma' - aq$ for some $a > 0$, $l(ws_q) = l(w) + 1$, such that $ws_q \in \mathcal{W}^{ab}_\sigma$ and $ws_q(\gamma') = 2\delta - \theta_\Sigma$. Then as in Lemma 3.2.11 $\beta + \theta_\Sigma \in N(ws_q)$ and thus ws_q starts with $s_\beta w_{0,\alpha_\Sigma} w_0 = s_\beta w_{0,\beta} w_0$, which is maximal in $\mathcal{W}^{ab}_\sigma$, so $ws_q = s_\beta w_{0,\beta} w_0 \in \mathcal{I}_{\alpha_\Sigma+\beta,2\delta-\theta_\Sigma}$. Then $q = \beta$, the only removable simple root in $\alpha_\Sigma + \beta$. This implies $w = s_\beta w_{0,\beta} w_0 s_\beta \notin \mathcal{W}^{ab}_\sigma$, absurd. □

Corollary 3.2.51. *Let $|\gamma| = |\theta_\Sigma|$ and $\gamma \in \widehat\Delta^0_{\alpha_\Sigma} \cup \widehat\Delta^1_{\alpha_\Sigma}$ with $\gamma \neq \alpha_\Sigma + \beta$, or $\gamma = \delta + \alpha_\Sigma$ or $\gamma = \delta + \alpha_\Sigma + \beta$, and let $u^\Sigma_\gamma = \min \mathcal{I}_{\gamma,2\delta-\theta_\Sigma}$. $u^\Sigma_\gamma s \in \mathcal{I}_{\gamma,2\delta-\theta_\Sigma}$ iff $s \in \mathcal{W}^{ab}_\sigma$ and can be written in reduced form with simple reflections fixing γ.*

Corollary 3.2.52. $\mathcal{I}_{\alpha_\Sigma+\beta,2\delta-\theta_\Sigma} = \{s_\beta w_{0,\beta} w_0 s_{\alpha_\Sigma}, s_\beta w_{0,\beta} w_0\}$.

Proof. As usual write v for the shortest element it $W(\Sigma)$ such that $v(\alpha_\Sigma) = \theta_\Sigma$. We claim that $w_{0,\beta} w_0 = s_{\alpha_\Sigma} v s_{\alpha_\Sigma}$. Indeed we show that they have the same inversion set, recall that if $\tau_k \in N(v)$ then $c_{\bar\alpha}(\tau_k) = 1$ and since $c_{\alpha_\Sigma}(\alpha_\Sigma) = c_{\alpha_\Sigma}(\theta_\Sigma) = 1$ then $c_{\alpha_\Sigma}(\tau_k) = 0$.

$$N(s_{\alpha_\Sigma} v s_{\alpha_\Sigma}) = \{\alpha_\Sigma\} \cup \{\alpha_\Sigma + \tau_k\}_k \cup \{s_{\alpha_\Sigma} v(\alpha_\Sigma) = \theta_\Sigma\}.$$

Since $\theta_\Sigma \in N(s_{\alpha_\Sigma} v s_{\alpha_\Sigma})$ and α_Σ is its unique simple root, every root $\tau \in \langle \Sigma \rangle$ with $c_{\alpha_\Sigma}(\tau) = 1$ is such that $\tau \in N(s_{\alpha_\Sigma} v s_{\alpha_\Sigma})$. To see this just write $\theta_\Sigma = \tau + R$ and $v^{-1}(\theta_\Sigma) = v^{-1}(\tau) + v^{-1}(R) < 0$ and recall that $v^{-1}(R) > 0$. The claim follows. In the end we get $u^\Sigma_{\alpha_\Sigma+\beta} = s_\beta s_{\alpha_\Sigma} v = s_\beta w_{0,\beta} w_0 s_{\alpha_\Sigma}$. The only simpe reflection that can extend $s_\beta w_{0,\beta} w_0 s_{\alpha_\Sigma}$ and fixes $\alpha_\Sigma + \beta$ is s_{α_Σ}, which indeed gives $s_\beta w_{0,\beta} w_0 \in \mathcal{W}^{ab}_\sigma$. □

Lemma 3.2.53. *If $\mathcal{I}_{\gamma, 2\delta-\theta_\Sigma} \neq \emptyset$, then $|\gamma| = |\theta_\Sigma|$ and $\gamma \in \widehat{\Delta}^0_{\alpha_\Sigma} \cup \widehat{\Delta}^1_{\alpha_\Sigma} \cup \{\delta + \alpha_\Sigma\} \cup \{\delta + \alpha_\Sigma + \beta\}$.*

Proof. Suppose there is a root γ outside of our set of rootlets, and let $w \in \mathcal{I}_{\gamma, 2\delta-\theta_\Sigma}$. Then write $w = s_1 \cdots s_n$ in reduced form, and γ_k for the rootlet of $s_1 \cdots s_k$. There must be an index i such that γ_i is not in our set of rootlets, and γ_{i-1} is in it instead. Lemma 3.2.50 shows that in any case $\gamma_{i-1} \geq \gamma_i$, and our assumption gives of course $\gamma_{i-1} > \gamma_i$. But then if $\gamma_{i-1} \leq \delta - \alpha_\Sigma$ then also $\gamma_i < \gamma_{i-1} \leq \delta - \alpha_\Sigma$, and so it is in our set, absurd. If $\gamma_{i-1} = \delta + \alpha_\Sigma$ then $\gamma_i = \delta - \alpha_\Sigma$ which is in our set, absurd, and if $\gamma_{i-1} = \delta + \alpha_\Sigma + \beta$ then $\gamma_i = \delta + \alpha_\Sigma$ which is again in our set, absurd. □

Case f.

We assume β is a short root, and we consider $\mu = k\delta - \theta_\Sigma$. After some technicalities, we show in Lemma 3.2.56 that if $\gamma \in \langle A(\Sigma) \rangle_l$, then the minimal element u^Σ_γ in $W(A(\Sigma))$ such that $u^\Sigma_\gamma(\gamma) = k\delta - \theta_\Sigma$ is such that $u^\Sigma_\gamma = \min \mathcal{I}_{\gamma,\mu}$. Then we find in Lemma 3.2.57 conditions under which we can add chains of simple reflections fixing γ to u^Σ_γ, in order to find other elements in $\mathcal{I}_{\gamma,\mu}$, and that every element in $\mathcal{I}_{\gamma,\mu}$ can be written in such way. Finally, in Lemma 3.2.58, we show that $\widehat{\Delta}_\mu = \langle A(\Sigma) \rangle_l$.

Lemma 3.2.54. *If β is short then θ_Σ is long.*

Proof. Suppose it is short. Then every simple root in Σ is short. Pick a closest long simple root q to β in the diagram. Consider the path of simple short roots from q to β and write $\alpha_1, \ldots, \alpha_n$, and $s_1, \ldots, s_n$ for their reflections. Then

$$\xi := s_\beta s_n \cdots s_1(q) = q + 2\sum_{i=1}^{n} \alpha_i + 2\beta = k\delta - \theta_\Sigma$$

because $c_\beta(\xi) = 2$, $Supp\,(\xi) \cap \Sigma = \emptyset$, $k\delta - \theta_\Sigma \in \widehat{\Delta}$ and θ_Σ is maximal in $\langle \Sigma \rangle$. This is a contradiction because q is long and $k\delta - \theta_\Sigma$ is short. $\square$

Lemma 3.2.55. *If β is short, then $k\delta - \theta_\Sigma \in \langle A(\Sigma) \rangle$ and it's its highest root.*

Proof. Since θ_Σ is long, $\tau := s_\beta(k\delta - \theta_\Sigma) = k\delta - \theta_\Sigma - 2\beta$ and the claim follows because $c_\beta(\tau) = 0$, unless $\tau \in \langle \Sigma \rangle$. For every simple reflection $s_q \neq s_{\alpha_\Sigma}$ with $q \in \Sigma$ we have $s_q(k\delta - \theta_\Sigma - 2\beta) = k\delta - \theta_\Sigma - 2\beta$ becuase θ_Σ is maximal in $\langle \Sigma \rangle$. Moreover if $s_{\alpha_\Sigma}(\theta_\Sigma) = \theta_\Sigma - a\alpha_\Sigma$ with $a \geq 0$, we have $s_{\alpha_\Sigma}(k\delta - \theta_\Sigma - 2\beta) = k\delta - \theta_\Sigma - 2\beta + a\alpha_\Sigma - 2\alpha_\Sigma \leq k\delta - \theta_\Sigma - 2\beta$ because $a \leq 2$ (since $s_\beta(\theta_\Sigma) = \theta_\Sigma + 2\beta$). This implies $k\delta - \theta_\Sigma - 2\beta = \theta_\Sigma$ because it is the highest root in $\langle \Sigma \rangle$. Then $k\delta = 2\theta_\Sigma + 2\beta$, forcing $k = 2$ and $\delta = \theta_\Sigma + \beta = s_{\theta_\Sigma}(\beta)$, which is absurd and the claim follows. $\square$

For every $\gamma \in \langle A(\Sigma) \rangle_l$ write u^Σ_γ for an element of shortest length in $W(A(\Sigma))$ such that $u^\Sigma_\gamma(\gamma) = k\delta - \theta_\Sigma$.

Lemma 3.2.56. *u^Σ_γ is σ-minuscule, in particular $u^\Sigma_\gamma = \min \mathcal{I}_{\gamma, k\delta - \theta_\Sigma}$.*

Proof. It follows from the previous lemmas, as in Lemmas 3.2.7 and 3.2.8. $\square$

Lemma 3.2.57. *Suppose β is short and let $\gamma \in \langle A(\Sigma) \rangle_l$. Let Ψ be the set of simple roots in $\Sigma \setminus \Gamma(\Sigma)$ connected to $A(\Sigma)$. Then $u^\Sigma_\gamma v \in \mathcal{I}_{\gamma, k\delta - \theta_\Sigma}$ iff v can be written as a product of simple reflections fixing γ and for every $\tau \in N(v)$ we have $\sum_{h \in \Psi} c_h(\tau) = 1$.*

Proof. Suppose there is $w' \in \mathcal{I}_{\gamma, k\delta - \theta_\Sigma}$ such that it is u^Σ_γ extended with a block of simple reflections not all fixing γ. At some point starting from right there must be a simple reflection s_q for which $s_q(\gamma') = \gamma' - aq$ for some positive a and some rootlet γ', $w \in \mathcal{W}^{ab}_\sigma$ and $ws_q \in \mathcal{I}_{\gamma', k\delta - \theta_\Sigma}$, then as in Lemma 3.2.11 $a = 1$ and $w^{-1}(\theta_\Sigma + \beta) < 0$ even though it is not necessarely a root a priori. This immediately implies anyway that $w^{-1}(\alpha_\Sigma + \beta) < 0$ which is always a root, and α_Σ is short, because otherwise even $s_\beta s_{\alpha_\Sigma} \notin \mathcal{W}^{ab}_\sigma$ because $s_\beta(\alpha_\Sigma) = \alpha_\Sigma + 2\beta$. But then $s_{\theta_\Sigma}(\beta) = \theta_\Sigma + \beta$ is a root, and $s_\beta(\theta_\Sigma) = \theta_\Sigma + 2\beta = (\theta_\Sigma + \beta) + \beta \in N(ws_q)$ is also a root, which is absurd because $c_\beta = 2$. Since there cannot be non fixing reflections in v, the remaining claims follows as in Lemma 3.2.9 and Corollary 3.2.10. $\square$

Lemma 3.2.58. *If $\mathcal{I}_{\gamma,k\delta-\theta_\Sigma} \neq \emptyset$, then $\gamma \in \langle A(\Sigma)\rangle_l$.*

Proof. It follows as in Lemma 3.2.15. □

Remark 3.2.1. Note that when β is short, σ-minuscule elements can only be made up of simple reflections associated to short roots, indeed otherwise taking the first long simple root α_j then $s_1 \cdots s_{j-1}(\alpha_j)$ is long and must have an even c_β. This makes immediately clear that for $B_n^{(1)}$ we have $\mathcal{W}_\sigma^{ab} = \{1, s_\beta\}$ and for $F_4^{(1)}$ we have $\mathcal{W}_\sigma^{ab} = \{1, s_\beta, s_\beta s_{\alpha_\Sigma}\}$. Note that in some cases as C_n^1 we have $\Gamma(\Sigma) = \emptyset$. For $C_n^{(1)}$ in the case in which θ_Σ is a simple long root, the diagram is $A(\Sigma) \cup \{\theta_\Sigma\}$, but $u_\gamma^\Sigma(\theta_\Sigma)$ is long and so has an even c_β, thus u_γ^Σ can't be extended. The only interesting cases for β short appear for $C_n^{(1)}$ with $|\Sigma| > 1$.

Case g.

We assume β is a short root, and we consider $\mu = k\delta + \beta$. We write Σ_β for the connected component of simple short roots containing β. We show in Lemma 3.2.59 that if $\tau \in \langle \Sigma_\beta \rangle$ with $c_\beta(\tau) = 1$, and we write $\gamma = k\delta - \tau$, then $\mathcal{I}_{\gamma,\delta+\beta} \neq \emptyset$. We prove in Lemma 3.2.60 that $s_\beta u_\gamma^\Sigma = \min \mathcal{I}_{\gamma,\mu}$, where u_γ^Σ is defined as in Lemma 3.2.59. Then we find in Lemma 3.2.61 conditions under which we can add chains of simple reflections fixing γ to u_γ^Σ, in order to find other elements in $\mathcal{I}_{\gamma,\mu}$, and we prove that every element in $\mathcal{I}_{\gamma,\mu}$ can be written in such way. Finally, in Lemma 3.2.62, we show that $\widehat{\Delta}_\mu = \{\gamma \in \widehat{\Delta}_\beta^1 : \gamma = k\delta - \tau, \tau \in \langle \Sigma_\beta \rangle\} \cup \{k\delta + \beta\}$.

Lemma 3.2.59. *Let $\tau \in \langle \Sigma_\beta \rangle$ with $c_\beta(\tau) = 1$, and write $\gamma = k\delta - \tau$. Then $\mathcal{I}_{\gamma,k\delta+\beta} \neq \emptyset$.*

Proof. Let's write θ_β for the highest root in the diagram of finite type determined by Σ_β. We claim that $c_\beta(\theta_\beta) = 1$. Indeed $c_\beta(\theta_\beta) \leq 2$ since $\theta_\beta < k\delta$. Suppose $c_\beta(\theta_\beta) = 2$. Then take q a long simple root connected to Σ_β, and compute $s_{\theta_\beta}(q) = q + 2\theta_\beta$. Since $c_\beta(q + 2\theta_\beta) = 4$ and $c_q(q + 2\theta_\beta) = 1$ we have $q + 2\theta_\beta = k\delta + \bar{\theta}$ with $\bar{\theta} \in \langle \Sigma_\beta \rangle$, which is a short root, but $s_{\theta_\beta}(q)$ is a long root. This contradiction proves our claim.

We claim that there exists $v \in W(\Sigma_\beta \setminus \{\beta\})$ such that $v(\tau) = \beta$. Indeed in every diagram of finite type if y is a root and θ is the highest root with $|y| = |\theta|$, then the shortest element $v = s_1 \cdots s_n$ in reduced form with $v(y) = \theta$ is such that for every i, $s_i(s_{i+1} \cdots s_n(y)) = s_{i+1} \cdots s_n(y) + a_i\alpha_i$ with α_i the simple root associated to s_i and $a_i > 0$. This is because if otherwise $a_i < 0$ for some i ($a_i \neq 0$ due to minimality) $s_1 \cdots s_{i-1}s_{i+1} \cdots s_n(y) = \theta - a_i\alpha_i > \theta$ belongs to the diagram of finite type. This implies in our case that since $c_\beta(\beta) = c_\beta(\theta_\beta) = 1$ there is an element $v_1 \in W(\Sigma_\beta \setminus \{\beta\})$ such that $v_1(\beta) = \theta_\beta$, and since $c_\beta(\tau) = c_\beta(\theta_\beta) = 1$ there is an element $v_2 \in W(\Sigma_\beta \setminus \{\beta\})$ such that $v_2(\tau) = \theta_\beta$. So taking $v = v_1^{-1}v_2$ we have $v(\tau) = v_1^{-1}v_2(\tau) = \beta$ and $v \in W(\Sigma_\beta \setminus \{\beta\})$. Let $u_\gamma^\beta \in W(\Sigma_\beta \setminus \{\beta\})$ be an element of shortest length such that $u_\gamma^\beta(\tau) = \beta$. We see that

$$s_\beta u_\gamma^\beta(\gamma) = s_\beta u_\gamma^\beta(k\delta - \tau) = s_\beta(k\delta - \beta) = k\delta + \beta$$

so we only need to check the set

$$N(s_\beta u_\gamma^\beta) = \{\beta\} \cup s_\beta N(u_\gamma^\beta).$$

Since $u_\gamma^\beta \in W(\Sigma_\beta \setminus \{\beta\})$, if $\tau_j \in N(u_\gamma^\beta)$ then $c_\beta(\tau_j) = 0$. Writing $u_\gamma^\beta = s_1 \cdots s_n$ in reduced form, we see that

$$s_1 \cdots s_{j-1}s_{j+1} \cdots s_n(\tau) = \beta - a_j\tau_j$$

so for exactly one simple root q in Σ_β linked to β we have $c_q(\tau_j) \geq 1$. Finally $c_q(\tau_j) = 1$ since otherwise $s_\beta(\tau_j) = \tau_j + a\beta$ with $a > 1$ is in $\langle \Sigma_\beta \rangle$ but $c_\beta(\theta_\beta) = 1$. This implies $c_\beta(s_\beta N(u_\gamma^\beta)) = 1$. □

Lemma 3.2.60. $s_\beta u_\gamma^\beta = \min \mathcal{I}_{\gamma,k\delta+\beta}$.

Proof. Write γ_i for the rootlet associated to $s_1 \cdots s_i$, i.e. $s_1 \cdots s_i(\gamma_i) = k\delta + \beta$. We see that

$$k\delta + \beta = \gamma_0 > \gamma_1 > \cdots > \gamma_n = \gamma$$

so the claim follows as in Lemma 3.2.8. □

Lemma 3.2.61. *Suppose $u_\gamma^\beta w$ is such that $l(u_\gamma^\beta w) = l(u_\gamma^\beta) + l(w)$ and write $w = s_1 \cdots s_n$ in reduced form. Then $u_\gamma^\beta w \in \mathcal{I}_{\gamma,k\delta+\beta} \iff w \in \mathcal{W}_\sigma^{ab}$ and $s_i(\gamma) = \gamma$ for every $i = 1, \ldots, n$.*

Proof. It follows from Lemmas 3.2.40 and 3.2.41, as in Lemma 3.2.41. □

Lemma 3.2.62. *If γ doesn't belong to the set of roots that can be expressed as $k\delta - \tau$ with $\tau \in \langle \Sigma_\beta \rangle$ and $c_\beta(\tau) = 1$, then $\mathcal{I}_{\gamma,k\delta+\beta} = \emptyset$, or $\gamma = k\delta + \beta$ and $\mathcal{I}_{\gamma,k\delta+\beta} = \{1\}$.*

Proof. Suppose there are w and γ against our claim. As we have pointed out in Remark 3.2.1, when σ-minuscule elements are written in reduced form, they cannot have simple reflections associated to long roots. Moreover it's clear that the short simple reflections must all be contained in Σ_β. In addition $\gamma = w^{-1}(k\delta + \beta) = k\delta + w^{-1}(\beta) < k\delta$ so $c_\beta(\gamma) \leq 2$. Summing up these findings we can write $\gamma = k\delta - \tau$ with $\tau \in \Sigma_\beta$, $\tau > 0$. Suppose $c_\beta(\tau) \neq 1$, then the contradiction follows as in Lemma 3.2.42. □

3.3 Data

We collect here some useful data. We number Dynkin diagrams as in Bourbaki, and, for short we write, e.g., $D_4 = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ to mean that the root subsystem of $\widehat{\Pi}$ generated by $\alpha_2, \alpha_3, \alpha_4, \alpha_5$ is of type D_4. Let us display all possible (non Hermitian) cases.

Untwisted type	α_p	Σ_1	$A(\Sigma_1)$	$\Gamma(\Sigma_1)$	Σ_2	$A(\Sigma_2)$	$\Gamma(\Sigma_2)$
B_n	2	A_1	B_n	$\emptyset$	B_{n-p}	D_{p+2}	$A_1 = \{\alpha_{p+1}\}$
B_n	p	D_p	B_{n-p+2}	$A_1 = \{\alpha_{p-1}\}$	B_{n-p}	D_{p+2}	$A_1 = \{\alpha_{p+1}\}$
D_n	2	A_1	D_n	$\emptyset$	D_{n-p}	D_4	$A_1 = \{\alpha_{p+1}\}$
D_n	p	D_p	D_{n-p+2}	$A_1 = \{\alpha_{p-1}\}$	D_{n-p}	D_{n-p+2}	$A_1 = \{\alpha_{p+1}\}$
C_n	p	C_p	C_{n-p}	$\emptyset$	C_{n-p}	C_p	$\emptyset$
E_6	2	A_5	D_5	$A_3 = \{\alpha_3, \alpha_4, \alpha_5\}$	A_1	E_6	$\emptyset$
E_7	2	A_7	E_6	$A_5 = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$			
E_7	6	D_6	D_6	$D_4 = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$	A_1	E_7	$\emptyset$
E_8	1	D_8	E_7	$D_6 = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$			
E_8	8	E_7	D_8	$D_6 = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$	A_1	E_8	$\emptyset$
F_4	2	A_1	F_4	$\emptyset$	C_3	B_4	$B_2 = \{\alpha_2, \alpha_3\}$
F_4	4	B_4	C_3	$\emptyset$	B_4	C_3	$B_2 = \{\alpha_2, \alpha_3\}$

Here we number Dynkin diagrams as in [8].

Twisted type	α_p	Σ_1	$A(\Sigma_1)$	$\Gamma(\Sigma_1)$	Σ_2	$A(\Sigma_2)$	$\Gamma(\Sigma_2)$
A_{2n}	n	B_n	C_2	$A_1 = \{\alpha_{n-1}\}$			
A_{2n-1}	n	D_n	C_3	$A_2 = \{\alpha_{n-1}\}$			
A_{2n-1}	1	C_n	C_n	C_{n-1}			
D_{n+1}	p	B_p	B_{n-p+2}	$A_1 = \{\alpha_{p-1}\}$	B_{n-p}	B_{p+2}	$A_1 = \{\alpha_{p+1}\}$
E_6	0	F_4	C_4	$C_3 = \{\alpha_1, \alpha_2, \alpha_3\}$			
E_6	4	C_4	F_4	$C_3 = \{\alpha_1, \alpha_2, \alpha_3\}$			

For the diagram of type $A_{2l}^{(2)}$, where 3 different root lengths appear, we denote them as long (l), medium (m) and short (s).

Type	$\beta = \alpha_k$	Length of β	Type of Π_0	Lengths of $\prod_\Sigma \theta_\Sigma$
$B_l^{(1)}$	$2 \leq k \leq l-2$	long	$D_k \times B_{l-k}$	(l, l)
$B_l^{(1)}$	$k = l-1$	long	$D_{l-1} \times A_1$	(l, s)
$C_l^{(1)}$	$1 \leq k \leq l-1$	short	$C_k \times C_{l-k}$	(l, l)
$D_l^{(1)}$	$2 \leq k \leq l-2$	long	$D_k \times D_{l-k}$	(l, l)
$G_2^{(1)}$	$k = 1$	long	$A_1 \times A_1$	(l, s)
$F_4^{(1)}$	$k = 1$	long	$A_1 \times C_3$	(l, l)
$F_4^{(1)}$	$k = 4$	short	B_4	l
$E_6^{(1)}$	$k = 2$	long	$A_1 \times A_5$	(l, l)
$E_7^{(1)}$	$k = 2$	long	A_7	l
$E_7^{(1)}$	$k = 3$	long	$A_1 \times D_6$	(l, l)
$E_8^{(1)}$	$k = 1$	long	$A_1 \times E_7$	(l, l)
$E_8^{(1)}$	$k = 8$	long	D_8	l

Type	$\beta = \alpha_k$	Length of β	Type of Π_0	Lengths of $\prod_\Sigma \theta_\Sigma$
$A_{2l}^{(2)}$	$k = l$	long	B_l	m
$A_{2l-1}^{(2)}$	$k = l$	long	D_l	s
$A_{2l-1}^{(2)}$	$k = 0$	short	C_l	l
$D_{l+1}^{(2)}$	$k = 0$	short	B_l	l
$D_{l+1}^{(2)}$	$1 \leq k \leq l-1$	long	$B_k \times B_{l-k}$	(l, l)
$E_6^{(2)}$	$k = 0$	short	F_4	l
$E_6^{(2)}$	$k = 4$	long	C_4	l

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