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The adiabatic groupoid and the Higson–Roe exact sequence

Vito Felice Zenobi

Abstract. Let $\widetilde{X}$ be a smooth Riemannian manifold equipped with a proper, free, isometric, and cocompact action of a discrete group Γ. In this paper, we prove that the analytic surgery exact sequence of Higson–Roe for $\widetilde{X}$ is isomorphic to the exact sequence associated to the adiabatic deformation of the Lie groupoid $\widetilde{X} \times_\Gamma \widetilde{X}$. We then generalize this result to the context of smoothly stratified manifolds. Finally, we show, by means of the aforementioned isomorphism, that the ϱ-classes associated to a metric with a positive scalar curvature defined by Piazza and Schick (2014) correspond to the ϱ-classes defined by Zenobi (2019).

1. Introduction

Let $\widetilde{X}$ be a proper metric space equipped with a proper and cocompact action of a discrete group Γ. In [39], Roe proved that the assembly map can be realized as the boundary map in K-theory associated to the short exact sequence of C^*-algebras

$$0 \to C^*(\widetilde{X})^\Gamma \to D^*(\widetilde{X})^\Gamma \to D^*(\widetilde{X})^\Gamma / C^*(\widetilde{X})^\Gamma \to 0. \tag{1.1}$$

We will call it the *coarse* assembly map. In their seminal papers [24–26], Higson and Roe constructed a map from the surgery exact sequence of Browder, Novikov, Sullivan, and Wall

$$\cdots \to L_*(\mathbb{Z}\Gamma) \to S_*(X) \to \mathcal{N}_*(X) \to \cdots \tag{1.2}$$

to

$$\cdots \to K_*\big(C^*(\widetilde{X})^\Gamma\big) \to K_*\big(D^*(\widetilde{X})^\Gamma\big) \to K_*\big(D^*(\widetilde{X})^\Gamma / C^*(\widetilde{X})^\Gamma\big) \to \cdots \tag{1.3}$$

which was called the *analytic surgery exact sequence*, in analogy with its topological counterpart (1.2).

In [34], Piazza and Schick use index theoretic techniques to map the Stolz positive scalar curvature sequence to (1.3). In [35], they then revisit the mapping from (1.2) to (1.3). The main results of those papers are the definition of certain K-theoretic secondary invariants and the proof of the *delocalized APS index theorem*.

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The papers of Higson and Roe stimulated a fervent activity resulting in a number of different realizations of the analytic surgery exact sequence. In what follows we list a few of the main contributions. In [50], the author of the present paper uses Lipschitz structures to generalize the results of [35] from the setting of smooth manifolds to the one of topological manifolds. In the same paper, a new exact sequence is introduced, isomorphic to (1.3). This new realization was then used for proving product formulas for secondary invariants. The group $\mathcal{S}^{\Gamma}_*(\widetilde{X})$, which corresponds to $K_*(D^*(\widetilde{X})^{\Gamma})$, is given roughly speaking by the homotopy fiber of the Kasparov assembly map. Let us point out that if Γ is a topological groupoid acting on a topological space X and A is a Γ-algebra, one also has a more general definition of $\mathcal{S}^{\Gamma}_*(\widetilde{X})$ which fits into the following exact sequence:

$$\cdots \to KK_*\big(\mathbb{C}, C_0(0,1) \otimes A \rtimes \Gamma\big) \to \mathcal{S}^{\Gamma}_*(\widetilde{X}; A) \to KK^{\Gamma}_*\big(C_0(\widetilde{X}), A\big) \to \cdots, \tag{1.4}$$

involving the assembly map for a groupoid action with coefficients in the C^*-algebra A; see [29,42]. In their two recent works [6,7], Benameur and Roy introduce the Higson–Roe exact sequence for the action of a transformation groupoid.

In [48], Yu introduces the so-called *localization algebras* and another assembly-type map (the *local index map*), which is proved to be equivalent to the coarse assembly map; see also [38]. Using the localization algebras, many authors have contributed to the study of K-theoretic analytic invariants associated to the surgery theory and metrics of positive scalar curvature; see [45–47, 49]. In [19–21], Deeley and Goffeng produce a geometrical version of the analytic surgery exact sequence in the spirit of Baum's geometric K-homology theory.

A further way of implementing the index map was given in [10] by Alain Connes, where he gave the definition of the *tangent groupoid* associated to a smooth manifold, by now also called *adiabatic groupoid*. In [51], the author of the present paper used the group $K_*(C^*_r(G^{[0,1)}_{\mathrm{ad}}))$ appearing in the exact sequence

$$\cdots \to K_*\big(C^*_r\big(G \times (0,1)\big)\big) \to K_*\big(C^*_r\big(G^{[0,1)}_{\mathrm{ad}}\big)\big) \xrightarrow{\mathrm{ev}_0} K_*\big(C^*_r(\mathfrak{A}G)\big) \to \cdots \tag{1.5}$$

as a receptacle for K-theoretic secondary invariants. Here G is a Lie groupoid, $\mathfrak{A}G$ is its Lie algebroid, and $G^{[0,1)}_{\mathrm{ad}}$ is its adiabatic deformation. The case of a smooth manifold $\widetilde{X}$ with a proper and free Γ action is realized by the particular groupoid $\widetilde{X} \times_{\Gamma} \widetilde{X}$. But this approach can also be applied to other geometric situation encoded by a general Lie groupoid, such as foliations. We refer the reader to [37, Subsection 1.3] for a short explanation of the advantages of the Lie groupoid approach. Here it is worth to mention that an important feature of the groupoid C^*-algebras is that they are smaller and easier to work with than the C^*-algebras arising in the coarse setting. In particular, this approach (actually the exact sequence involving pseudodifferential operators of Remark 4.6, obtained as a by-product of the proof of Theorem 1.1) was a key, in the recent paper [36] of the author with Paolo Piazza and Thomas Schick, to construct a mapping from the Higson–Roe exact sequence to the non-commutative de Rham homology of any Frèchet completion of $\mathbb{C}\Gamma$. In turn, this allowed to construct a well-defined pairing of the analytic structure group with delocalized cyclic cocycles.

The main results of this paper are the following.

Theorem 1.1. *Let $\widetilde{X}$ be a smooth Riemannian manifold equipped with a proper, free, isometric, and cocompact action of a discrete group Γ. Let G be the Lie groupoid $\widetilde{X} \times_\Gamma \widetilde{X}$. Then there exists a commutative diagram*

$$\begin{tikzcd}
\cdots \arrow[r] & K_*(C^*_r(G) \otimes C_0(0,1)) \arrow[r] \arrow[d] & K_*(C^*_r(G^{[0,1)}_{\mathrm{ad}})) \arrow[r] \arrow[d] & K_*(C^*_r(TX)) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & K_{*+1}(C^*(\widetilde{X})^\Gamma) \arrow[r] & K_{*+1}(D^*(\widetilde{X})^\Gamma) \arrow[r] & K_{*+1}(D^*(\widetilde{X})^\Gamma / C^*(\widetilde{X})^\Gamma) \arrow[r] & \cdots
\end{tikzcd} \tag{1.6}$$

such that the vertical arrows are isomorphisms.

The vertical maps are given by the composition of the Connes–Thom isomorphism and the vertical maps of diagram (4.1).

In [37], the methods from [51] are used to define secondary invariants associated to metrics with a positive scalar curvature on stratified manifolds and other singular situations such as foliations which degenerate on the boundary. In order to deal with the singularities, a slightly different exact sequence of groupoid C^*-algebras is used. The proof of Theorem 1.1 can be easily adapted to the context of stratified manifolds and we obtain the following result.

Theorem 1.2. *Let ${}^{\mathrm{S}}X$ be a Thom–Mather stratified space with a fundamental group Γ and let $\widetilde{{}^{\mathrm{S}}X}$ be its universal covering with the associated Γ-equivariant stratification. Let the regular part of $\widetilde{{}^{\mathrm{S}}X}$ be equipped with an incomplete iterated edge metric, then there exists a commutative diagram*

$$\begin{tikzcd}
\cdots \arrow[r] & K_*(C^*_r(G_{|\overset{\circ}{X}}) \otimes C_0(0,1)) \arrow[r] \arrow[d] & K_*(C^*_r(G^{[0,1)}_{\mathrm{ad}})) \arrow[r] \arrow[d] & K_*(C^*_r(T^{\mathrm{NC}}_\varphi X)) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & K_{*+1}(C^*(\widetilde{{}^{\mathrm{S}}X})^\Gamma) \arrow[r] & K_{*+1}(D^*(\widetilde{{}^{\mathrm{S}}X})^\Gamma) \arrow[r] & K_{*+1}(D^*(\widetilde{{}^{\mathrm{S}}X})^\Gamma / C^*(\widetilde{{}^{\mathrm{S}}X})^\Gamma) \arrow[r] & \cdots
\end{tikzcd} \tag{1.7}$$

such that the vertical arrows are isomorphisms.

The vertical maps of this diagram are defined as the composition of the Connes–Thom isomorphism and the vertical maps of diagram (5.3). The definition of the groupoids in the first row will be recalled in Section 5.3.

Here it is worth noticing that in the first row we make use of non-compact manifolds equipped with complete metrics, whereas the second row is constructed from compact stratified manifolds equipped with a metric which is non-complete on the regular stratum, and the two rows are related by a conformal change of metrics.

In the last section of the paper, we will consider the ϱ-class associated to a metric with positive scalar curvatures $\varrho(g)$ defined in [34] and $\varrho^{\mathrm{ad}}(g)$ defined in [51], respectively. The precise relation between these two classes remained the subject of an open question. Thanks to Theorem 1.1, we will prove the following.

Theorem 1.3. *The classes $\varrho(g)$ and $\varrho^{\mathrm{ad}}(g)$ correspond to each other through the central vertical isomorphism in diagram* (1.6).

2. Roe's algebras

In this section, we are going to recall the fundamental definitions and results about coarse geometry, coarse C^*-algebras, and coarse index theory. We will not enter into the details of the proofs, which one can easily find in the literature. See, for instance, [23, 26, 41].

Let X be any set. If $A \subset X \times X$ and $B \subset X \times X$, we will use the following notation:

$$A^{-1} := \{(y, x) \mid (x, y) \in A\}$$

and

$$A \circ B := \{(x, z) \mid \exists\, y \in X : (x, y) \in A \text{ and } (y, z) \in B\}.$$

Definition 2.1. A coarse structure on X is a collection of subsets of $X \times X$, called entourages, that have the following properties:

- for any entourages A and B, the subsets A^{-1}, $A \circ B$, and $A \cup B$ are entourages;
- every finite subset of $X \times X$ is an entourage;
- any subset of an entourage is an entourage.

If $\{(x, x) \mid x \in X\}$ is an entourage, then the coarse structure is said to be unital.

Definition 2.2. Let (X, d) be a metric space and let S be any set. Two functions $f_1, f_2\colon S \to X$ are said to be close if $\{d(f_1(s), f_2(s)) : s \in S\}$ is a bounded set of $\mathbb{R}$.

Definition 2.3. Let (X, d) be a metric space. A subset $E \subset X \times X$ is said to be controlled if the restriction of the projection maps $\pi_1, \pi_2\colon X \times X \to X$ to E is close.

The controlled sets are the ones that are contained in a uniformly bounded neighborhood of the diagonal. The *metric coarse structure* on (X, d) is given by the collection of all controlled subsets of $X \times X$.

Let $\widetilde{X}$ be a proper metric space equipped with a free and proper action of a countable discrete group Γ of isometries of $\widetilde{X}$.

Definition 2.4. Let H be a Hilbert space equipped with a representation

$$\rho\colon C_0(\widetilde{X}) \to \mathbb{B}(H)$$

and a unitary representation

$$U\colon \Gamma \to \mathbb{B}(H)$$

such that $U(\gamma)\rho(f) = \rho(\gamma^{-1} f)U(\gamma)$ for every $\gamma \in \Gamma$ and $f \in C_0(\widetilde{X})$. We will call such a triple (H, U, ρ) a Γ-equivariant $C_0(\widetilde{X})$-module.

Exemple 2.5. Let us set $H = L^2(\widetilde{X}, \mu)$, where μ is a Γ-invariant Borel measure on $\widetilde{X}$. Put

- $\rho\colon C_0(\widetilde{X}) \to \mathbb{B}(H)$ the representation given by multiplication operators and

- $U\colon \Gamma \to \mathbb{B}(H)$ the representation given by translation $U_\gamma \varphi(x) = \varphi(\gamma^{-1}x)$ for every $x \in X$.

Then (H, U, ρ) is a Γ-equivariant $C_0(\widetilde{X})$-module.

Definition 2.6. Let A be a C^*-algebra and let H be a Hilbert space. A representation $\rho\colon A \to \mathbb{B}(H)$ is said to be ample if

- ρ is non-degenerate, meaning $\rho(A)H$ is dense in H and
- $\rho(a)$ is compact for $a \in A$ if and only if $a = 0$.

Moreover, we will say that a representation $\rho\colon A \to \mathbb{B}(H)$ is very ample if it is the countable direct sum of a fixed ample representation.

If H is equipped with a unitary representation U of Γ, then we say that an operator $T \in \mathbb{B}(H)$ is Γ-equivariant if $U_\gamma T U_{\gamma^{-1}} = T$ for all $\gamma \in \Gamma$.

2.1. Controlled operators

Definition 2.7. Let X and Y be two proper metric spaces. Let $\rho_X\colon C_0(X) \to \mathbb{B}(H_X)$ and $\rho_Y\colon C_0(Y) \to \mathbb{B}(H_Y)$ be two representations on separable Hilbert spaces.

- The support of an element $\xi \in H_X$ is the set $\mathrm{supp}(\xi)$ of all $x \in X$ such that for every open neighborhood U of x there is a function $f \in C_0(U)$ with $\rho_X(f)\xi \neq 0$.
- The support of an operator $T \in \mathbb{B}(H_X, H_Y)$ is the set $\mathrm{supp}(T)$ of all $(y, x) \in Y \times X$ such that for all open neighborhoods $U \ni y$ and $V \ni x$, there exist $f \in C_0(U)$ and $g \in C_0(Y)$ such that $\rho_Y(f) T \rho_X(g) \neq 0$.
- An operator $T \in \mathbb{B}(H_X, H_Y)$ is properly supported if the slices $\{y \in Y : (y, x) \in \mathrm{supp}(T)\}$ and $\{x \in X : (y, x) \in \mathrm{supp}(T)\}$ are compact sets.

Definition 2.8. Let X be as in the previous definition. An operator $T \in \mathbb{B}(H_X)$ is said to be controlled if its support is a controlled subset of $X \times X$.

This means that an operator is controlled if it is supported in a uniformly bounded neighborhood of the diagonal of $X \times X$. These operators are also said to have finite propagation.

Proposition 2.9. *The set of all controlled operators for $\rho_X\colon C_0(X) \to H_X$ is a unital $*$-algebra of $\mathbb{B}(H_X)$.*

2.2. The C^*-algebras $C^*(\widetilde{X})^\Gamma$ and $D^*(\widetilde{X})^\Gamma$

Let $(H_{\widetilde{X}}, U, \rho)$ be an ample Γ-equivariant $C_0(\widetilde{X})$-module.

Definition 2.10. We define the C^*-algebra $C^*(\widetilde{X})^\Gamma$ as the closure in $\mathbb{B}(H_{\widetilde{X}})$ of the $*$-algebra of all Γ-equivariant operators T such that

- T has finite propagation, i.e., there is an $R > 0$ such that $\rho(\varphi) T \rho(\psi) = 0$ for all $\varphi, \psi \in C_0(\widetilde{X})$ with $d(\mathrm{supp}(\varphi), \mathrm{supp}(\psi)) > R$;

- T is locally compact, i.e., $T\rho(\varphi)$ and $\rho(\varphi)T$ are compact operators for all $\varphi \in C_0(\widetilde{X})$.

Definition 2.11. Let $H_{\widetilde{X}}$ be a very ample Γ-equivariant $\widetilde{X}$-module. The algebra $D^*(\widetilde{X})^\Gamma$ is the closure in $\mathbb{B}(H_{\widetilde{X}})$ of the $*$-algebra of all Γ-equivariant operators T such that

- T has finite propagation;
- T is pseudolocal, i.e., $[T, \rho(\varphi)]$ is a compact operator for any $\varphi \in C_0(\widetilde{X})$.

If Γ is the trivial group, then we will suppress it from the notation and write $C^*(\widetilde{X})$ and $D^*(\widetilde{X})$.

Remark 2.12. The C^*-algebra $D^*(\widetilde{X})^\Gamma$ is a $*$-subalgebra of the multiplier algebra of $C^*(\widetilde{X})^\Gamma$.

Remark 2.13. The algebras $C^*(\widetilde{X})^\Gamma$ and $D^*(\widetilde{X})^\Gamma$ depend on the $C_0(\widetilde{X})$-module used to construct it, but one can prove that their K-theory does not.

2.3. Paschke duality

Since $C^*(\widetilde{X})^\Gamma$ is a two-sided $*$-ideal in $D^*(\widetilde{X})^\Gamma$, we can consider the quotient C^*-algebra $D^*(\widetilde{X})^\Gamma/C^*(\widetilde{X})^\Gamma$. By a truncation argument one can prove the following isomorphism of C^*-algebras:

$$D^*(\widetilde{X})^\Gamma/C^*(\widetilde{X})^\Gamma \cong D^*(X)/C^*(X), \tag{2.1}$$

where X is the quotient space $\widetilde{X}/\Gamma$.

The Paschke duality is the isomorphism

$$\mathcal{P}: K_0\big(D^*(X)/C^*(X)\big) \to KK_1\big(C(X), \mathbb{C}\big) \tag{2.2}$$

that sends the projection $p \in D^*(X)/C^*(X)$ to the Kasparov bimodule $(H, \rho, 2p-1)$, where $\rho: C(X) \to H$ is the $C(X)$-module used to define $D^*(X)/C^*(X)$. See, for instance, [26, Proposition 1.3].

We can see $\mathcal{P}$ as an asymptotic morphism in $E^1(D^*(X)/C^*(X) \otimes C(X), \mathbb{C})$; see [38]. Indeed consider the generator u of $E^1(\mathbb{Q}(H), \mathbb{C})$, where $\mathbb{Q}(H)$ is the Calkin algebra of H. It is the class associated with the following extension of C^*-algebra:

$$0 \to \mathbb{K}(H) \to \mathbb{B}(H) \to \mathbb{Q}(H) \to 0$$

and u is given by the boundary map of the long exact sequence in E-theory associated to the previous exact sequence.

Let μ: $D^*(X)/C^*(X) \otimes C(X) \to \mathbb{Q}(H)$ be the $*$-homomorphism given by

$$T \otimes f \mapsto M_f T,$$

where M_f is the multiplication operator. It is a well-defined $*$-homomorphism because $D^*(X)/C^*(X)$ and $C(X)$ commute in $\mathbb{Q}(H)$. Then $\mathcal{P}$ is a map of K-groups given by the product with

$$\mu^*(u) \in E^1\big(D^*(X)/C^*(X) \otimes C(X), \mathbb{C}\big). \tag{2.3}$$

More precisely, $\mu^*(u)$ arises from the pull-back extension

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{K}(H) \arrow[r] \arrow[d, "="] & E \arrow[r] \arrow[d] & D^*(X)/C^*(X) \otimes C(X) \arrow[r] \arrow[d, "\mu"] & 0 \\ 0 \arrow[r] & \mathbb{K}(H) \arrow[r] & \mathbb{B}(H) \arrow[r] & \mathbb{Q}(H) \arrow[r] & 0 \end{tikzcd}$$

as the boundary morphism of the long exact sequence in E-theory associated to the top row.

2.4. The analytic surgery exact sequence

Let $\widetilde{X}$ be a proper metric space such that the countable discrete group Γ acts properly, freely, and isometrically on it. The algebras defined in the previous section fit in the following exact sequence:

$$\cdots \to K_*\big(C^*(\widetilde{X})^\Gamma\big) \to K_*\big(D^*(\widetilde{X})^\Gamma\big) \to K_*\big(D^*(\widetilde{X})^\Gamma / C^*(\widetilde{X})^\Gamma\big) \to \cdots, \tag{2.4}$$

called the Higson–Roe analytic surgery exact sequence. Notice that $K_*(C^*(\widetilde{X})^\Gamma)$ is isomorphic to $K_*(C_r^*(\Gamma))$ and recall that $K_*(D^*(\widetilde{X})^\Gamma / C^*(\widetilde{X})^\Gamma)$ is isomorphic to $KK_{*-1}(C_0(X), \mathbb{C})$ by Paschke duality. In [39], Roe proves that the boundary morphism of (2.4) is equivalent to the assembly map. In other words, the diagram

$$\begin{tikzcd} K_{*+1}\big(D^*(\widetilde{X})^\Gamma / C^*(\widetilde{X})^\Gamma\big) \arrow[r, "\partial"] \arrow[d, "\mathcal{P}"] & K_*\big(C^*(\widetilde{X})^\Gamma\big) \arrow[d, "\cong"] \\ K_*^\Gamma(\widetilde{X}) \arrow[r, "\mu_X^\Gamma"] & K_*\big(C_r^*(\Gamma)\big) \end{tikzcd} \tag{2.5}$$

is commutative. Here we used the fact that, because the action of Γ on $\widetilde{X}$ is proper and free, the equivariant K-homology group $KK_*^\Gamma(C_0(\widetilde{X}), \mathbb{C})$ is isomorphic to $KK_*(C_0(X), \mathbb{C})$.

3. The adiabatic groupoid and the gauge adiabatic groupoid

We refer the reader to [14] and the bibliography inside it for the notations and a detailed overview about groupoids and index theory. A more recent overview about the subject can be found in [18].

3.1. Lie groupoids and Lie algebroids

Definition 3.1. Let G and $G^{(0)}$ be two sets. A groupoid structure on G over $G^{(0)}$ is given by the following morphisms.

- An injective map $u : G^{(0)} \to G$, called the unit map. We can identify $G^{(0)}$ with its image in G.

- Two surjective maps $r, s : G \to G^{(0)}$, which are, respectively, the range and source map.
- An involution $i : G \to G$, $\gamma \mapsto \gamma^{-1}$, called the inverse map. It satisfies $s \circ i = r$.
- A map $p : G^{(2)} \to G$, $(\gamma_1, \gamma_2) \mapsto \gamma_1 \cdot \gamma_2$, called the product, where the set

$$G^{(2)} := \{(\gamma_1, \gamma_2) \in G \times G \mid s(\gamma_1) = r(\gamma_2)\}$$

is the set of composable pair. Moreover, for $(\gamma_1, \gamma_2) \in G^{(2)}$, we have $r(\gamma_1 \cdot \gamma_2) = r(\gamma_1)$ and $s(\gamma_1 \cdot \gamma_2) = s(\gamma_2)$.

The following properties must be fulfilled.

- The product is associative: for any $\gamma_1, \gamma_2, \gamma_3$ in G, such that $s(\gamma_1) = r(\gamma_2)$ and $s(\gamma_2) = r(\gamma_3)$, the following equality holds:

$$(\gamma_1 \cdot \gamma_2) \cdot \gamma_3 = \gamma_1 \cdot (\gamma_2 \cdot \gamma_3).$$

- For any γ in G, $r(\gamma) \cdot \gamma = \gamma \cdot s(\gamma) = \gamma$ and $\gamma \cdot \gamma^{-1} = r(\gamma)$.

We denote a groupoid structure on G over $G^{(0)}$ by $G \rightrightarrows G^{(0)}$, where the arrows stand for the source and target maps.

We will adopt the following notations:

$$G_A := s^{-1}(A), \quad G^B = r^{-1}(B), \quad \text{and} \quad G_A^B = G_A \cap G^B$$

in particular if $x \in G^{(0)}$, where the s-fiber (respectively, the r-fiber) of G over x is $G_x = s^{-1}(x)$ (respectively, $G^x = r^{-1}(x)$).

Definition 3.2. We call G a Lie groupoid when G and $G^{(0)}$ are second-countable smooth manifolds with $G^{(0)}$ Hausdorff, the structural homomorphisms are smooth, u is an embedding, r and s are submersions, and i is a diffeomorphism.

Definition 3.3. A Lie algebroid $\mathfrak{A} = (p : \mathfrak{A} \to TM, [\,,\,]_{\mathfrak{A}})$ on a smooth manifold M is a vector bundle $\mathfrak{A} \to M$ equipped with a bracket $[\,,\,]_{\mathfrak{A}} : \Gamma(\mathfrak{A}) \times \Gamma(\mathfrak{A}) \to \Gamma(\mathfrak{A})$ on the module of sections of $\mathfrak{A}$, together with a homomorphism of fiber bundle $p : \mathfrak{A} \to TM$ from $\mathfrak{A}$ to the tangent bundle TM of M, called the anchor map, fulfilling the following conditions:

- the bracket $[\,,\,]_{\mathfrak{A}}$ is $\mathbb{R}$-bilinear, antisymmetric and satisfies the Jacobi identity;
- $[X, fY]_{\mathfrak{A}} = f[X, Y]_{\mathfrak{A}} + p(X)(f)Y$ for all $X, Y \in \Gamma(\mathfrak{A})$ and f is a smooth function of M;
- $p([X, Y]_{\mathfrak{A}}) = [p(X), p(Y)]$ for all $X, Y \in \Gamma(\mathfrak{A})$.

The tangent space to s-fibers, that is $T_sG := \ker ds = \bigcup_{x \in G^{(0)}} TG_x$, has the structure of a Lie algebroid on $G^{(0)}$, with an anchor map given by dr. It is denoted by $\mathfrak{A}G$ and we call it the Lie algebroid of G. One can prove that it is isomorphic to the normal bundle of the inclusion $G^{(0)} \hookrightarrow G$; see [8, p. 43].

3.2. The adiabatic groupoid and the gauge adiabatic groupoid

Let M_0 be a smooth compact submanifold of a smooth manifold M with normal bundle $\mathcal{N}$. As a set, the deformation to the normal cone is

$$\mathrm{DNC}(M_0, M) := \mathcal{N} \times \{0\} \sqcup M \times \mathbb{R}. \tag{3.1}$$

In order to recall its smooth structure, we fix an exponential map, which is a diffeomorphism θ from a neighborhood V' of the zero section M_0 in N to a neighborhood V of M_0 in M. We may cover $\mathrm{DNC}(M_0, M)$ with two open sets: $M \times \mathbb{R}^*$, with the product differentiable structure, and $W = \mathcal{N} \times 0 \sqcup V \times \mathbb{R}^*$, endowed with the differentiable structure for which the map

$$\Psi\colon \big\{(m,\xi,t) \in \mathcal{N} \times \mathbb{R} \mid (m,t\xi) \in V'\big\} \to W \tag{3.2}$$

given by $(m,\xi,t) \mapsto (\theta(m,t\xi),t)$, for $t \neq 0$, and by $(m,\xi,0) \mapsto (m,\xi,0)$, for $t = 0$, is a diffeomorphism. One can verify that the transition map on the overlap of these two charts is smooth; see, for instance, [27, Section 3.1].

Definition 3.4. The adiabatic groupoid $G_{\mathrm{ad}}^{[0,1]}$ is given by the groupoid

$$\mathfrak{A}G \times \{0\} \cup G \times (0,1] \rightrightarrows G^{(0)} \times [0,1],$$

with the smooth structure given by the deformation to the normal cone associated to the embedding $G^{(0)} \hookrightarrow G$. We will use the notation $G_{\mathrm{ad}}^{[0,1)}$ for the restriction of the adiabatic groupoid to the interval open at 1, given by

$$\mathfrak{A}G \times \{0\} \cup G \times (0,1) \rightrightarrows G^{(0)} \times [0,1).$$

Remark 3.5. Let $\mathrm{ev}_0\colon C_r^*(G_{\mathrm{ad}}^{[0,1]}) \to C_r^*(\mathfrak{A}G)$ be the evaluation at $t = 0$, then the associated KK-element is a KK-equivalence. Indeed notice that $C_r^*(\mathfrak{A}G)$ is nuclear and that the kernel of ev_0 is isomorphic to the contractible C^*-algebra $C_r^*(G) \otimes C_0((0,1])$. Then $[\mathrm{ev}_0]\colon KK(A, C_r^*(G_{\mathrm{ad}}^{[0,1]})) \to KK(A, C_r^*(\mathfrak{A}G))$, understood as the Kasparov product with $[\mathrm{ev}_0]$ on the right, is an isomorphism for all C^*-algebras A. This implies that there exists an element $[\mathrm{ev}_0]^{-1} \in KK(C_r^*(\mathfrak{A}G), C_r^*(G_{\mathrm{ad}}^{[0,1]}))$ such that $[\mathrm{ev}_0]^{-1} \otimes [\mathrm{ev}_0] = 1_{C_r^*(\mathfrak{A}G)}$ and $[\mathrm{ev}_0] \otimes [\mathrm{ev}_0]^{-1} = 1_{C_r^*(G_{\mathrm{ad}}^{[0,1]})}$.

Now we recall a definition from [16, Section 2.3]. We have a natural action of the group $\mathbb{R}_+^*$ compatible with the groupoid structure on $G_{\mathrm{ad}}^{[0,1)}$ defined as follows. Define

$$\alpha\colon (0,1] \times \mathbb{R}_+^* \to (0,1), \quad \alpha(t,\lambda) = \frac{2}{\pi}\arctan\Big(\lambda \tan\Big(\frac{\pi}{2}t\Big)\Big).$$

Then one can easily check that $\alpha(\alpha(t,\lambda),\lambda') = \alpha(t,\lambda\lambda')$.

Thus we have that the map defined by $(\gamma,t;\lambda) \mapsto (\gamma,\alpha(t,\lambda))$ for $t \neq 0$ and $(x,V,0;\lambda) \mapsto (x,\frac{1}{\lambda}V,0)$ gives an action of $\mathbb{R}_+^*$ on $G_{\mathrm{ad}}^{[0,1)}$. Notice that this action is isomorphic to the action on G_{ad} from [16, Section 2.3].

Definition 3.6. The gauge adiabatic groupoid $G_{\mathrm{ga}}^{[0,1)}$ is the Lie groupoid obtained as the crossed product of this action:

$$G_{\mathrm{ga}}^{[0,1)} := G_{\mathrm{ad}}^{[0,1)} \rtimes \mathbb{R}_+^* \rightrightarrows G^{(0)} \times [0,1).$$

3.3. Groupoid C^*-algebras

We can associate to a Lie groupoid G the $*$-algebra $C_c^\infty(G, \Omega^{1/2}(\ker ds \oplus \ker dr))$ of the compactly supported sections of the half densities bundle associated to $\ker ds \oplus \ker dr$, with

- the involution given by $f^*(\gamma) = \overline{f(\gamma^{-1})}$ and
- the product defined as $f * g(\gamma) = \int_{\eta \in G_{s(\gamma)}} f(\gamma\eta^{-1})g(\eta)$.

For all $x \in G^{(0)}$, the algebra $C_c^\infty(G, \Omega^{1/2}(\ker ds \oplus \ker dr))$ can be represented on $L^2(G_x, \Omega^{1/2}(G_x))$ by

$$\lambda_x(f)\xi(\gamma) = \int_{\eta \in G_x} f(\gamma\eta^{-1})g(\eta),$$

where $f \in C_c^\infty(G, \Omega^{1/2}(\ker ds \oplus \ker dr))$ and $\xi \in L^2(G_x, \Omega^{1/2}(G_x))$.

Definition 3.7. The reduced C^*-algebra of a Lie groupoid G, denoted by $C_r^*(G)$, is the completion of $C_c^\infty(G, \Omega^{1/2}(\ker ds \oplus \ker dr))$ with respect to the norm

$$\|f\|_r = \sup_{x \in G^{(0)}} \|\lambda_x(f)\|.$$

The full C^*-algebra of G is the completion of $C_c^\infty(G, \Omega^{1/2}(\ker ds \oplus \ker dr))$ with respect to the norm

$$\|f\|_{full} = \sup_\pi \|\pi(f)\|,$$

where π ranges over all non-degenerate $*$-representation of $C_c^\infty(G, \Omega^{1/2}(\ker ds \oplus \ker dr))$ on Hilbert spaces.

Definition 3.8. Let G be a Lie groupoid, then we associate to it a short exact sequence of C^*-algebras:

$$0 \to C_r^*\big(G \times (0,1)\big) \to C_r^*\big(G_{\mathrm{ad}}^{[0,1)}\big) \xrightarrow{\mathrm{ev}_0} C_r^*(\mathfrak{A}G) \to 0 \tag{3.3}$$

and we are going to call the long exact sequence in K-theory

$$\cdots \to K_*\big(C_r^*\big(G \times (0,1)\big)\big) \to K_*\big(C_r^*\big(G_{\mathrm{ad}}^{[0,1)}\big)\big) \xrightarrow{\mathrm{ev}_0} K_*\big(C_r^*(\mathfrak{A}G)\big) \to \cdots \tag{3.4}$$

the (reduced) adiabatic exact sequence of G.

The boundary map of (3.4) is given by the composition of the KK-element

$$[\mathrm{ev}_0]^{-1} \otimes [\mathrm{ev}_1] \in KK\big(C_r^*(\mathfrak{A}G), C_r^*(G)\big), \tag{3.5}$$

and the suspension isomorphism S. Finally, notice that there is an analogous extension for the full groupoid C^*-algebras.

It is worth to point out that the Connes–Thom isomorphism [9, 22] gives a natural isomorphism of long exact sequences of KK-groups:

$$\begin{tikzcd} \cdots \arrow[r] & KK^*(A, C_r^*(G)\otimes C_0(0,1)) \arrow[r] \arrow[d, "\mathrm{TC}"] & KK^*(A, C_r^*(G_{\mathrm{ad}}^{[0,1)})) \arrow[r] \arrow[d, "\mathrm{TC}"] & KK^*(A, C_r^*(\mathfrak{A}G)) \arrow[r] \arrow[d, "\mathrm{TC}"] & \cdots \\ \cdots \arrow[r] & KK^{*+1}(A, C_r^*(G)\otimes \mathbb{K}) \arrow[r] & KK^{*+1}(A, C_r^*(G_{\mathrm{ga}}^{[0,1)})) \arrow[r] & KK^{*+1}(A, C_r^*(\mathfrak{A}G)\rtimes \mathbb{R}_+^*) \arrow[r] & \cdots \end{tikzcd} \tag{3.6}$$

for any separable C^*-algebra A, where the vertical arrows are given by the Kasparov product by the element constructed in [22, Proposition 1 (i)]. Notice that the map $\mathbb{R}_+^* \times \mathbb{R}_+^* \to \mathbb{R}_+^* \rtimes \mathbb{R}_+^*$ given by $(x_1, x_2) \mapsto (x_2, \frac{x_1}{x_2})$ gives an isomorphism of Lie groupoids over $\mathbb{R}_+^*$. Now, up to identify $\mathbb{R}_+^*$ and $(0, 1)$, we have that the last isomorphism of groupoids induces the following isomorphism of C^*-algebras:

$$C_r^*(G) \otimes C_0(0,1) \rtimes \mathbb{R}_+^* \cong C_r^*(G) \otimes \mathbb{K}. \tag{3.7}$$

3.4. The Lie groupoid $\widetilde{X} \times_\Gamma \widetilde{X}$

Let $\pi\colon \widetilde{X} \to X$ be a Galois Γ-covering. Then the diagonal action of Γ on $\widetilde{X} \times \widetilde{X}$ is proper and free. Let $G = \widetilde{X} \times_\Gamma \widetilde{X}$ be the quotient of this action. It has a Lie groupoid structure over X described as follows:

- the source and the range are given by $s([\tilde{x}_1, \tilde{x}_2]) = \pi(\tilde{x}_2)$ and $r([\tilde{x}_1, \tilde{x}_2]) = \pi(\tilde{x}_1)$;
- the product of $[\tilde{x}_1, \tilde{x}_2]$ and $[\tilde{x}_3, \tilde{x}_4]$ is given by $[\tilde{x}_1, \gamma(\tilde{x}_2, \tilde{x}_3) \cdot \tilde{x}_4]$, where $\gamma(\tilde{x}_2, \tilde{x}_3)$ is the element of Γ that sends $\tilde{x}_3$ to $\tilde{x}_2$.

The reduced C^*-algebra $C_r^*(\widetilde{X} \times_\Gamma \widetilde{X})$ is the C^*-closure of the $C_c^\infty(\widetilde{X} \times_\Gamma \widetilde{X})$ with respect to the reduced norm. One can see this $*$-algebra simply as Γ-equivariant smoothing kernels on the universal covering of X. It is easy to prove that $C_r^*(\widetilde{X} \times_\Gamma \widetilde{X})$ is Morita equivalent to $C_r^*(\Gamma)$.

The Lie algebroid of G is isomorphic to TX, the tangent bundle of X, and the anchor map is given by the identity. The reduced C^*-algebra of the tangent bundle $C_r^*(TX)$ is isomorphic to the C^*-algebra $C_0(T^*X)$ of continuous function vanishing at infinity. This isomorphism is given by the fiber-wise Fourier transform.

Let us denote by $\partial_X^\Gamma \in KK(C_0(T^*X), C_r^*(\Gamma))$ the element defined in (3.5) (up to Fourier transform and Morita equivalences). In [31], it is proved that the map induced by the Kasparov product with ∂_X^Γ is the Γ-equivariant analytical index of Atiyah–Singer.

3.5. Poincaré duality

Let us consider the Lie groupoid of the pairs $X \times X$ over X. The groupoid C^*-algebra $C_r^*(X \times X)$ is isomorphic to $\mathbb{K}(L^2(X))$, the algebra of compact operators on $L^2(X)$. Its

Lie algebroid is still TX and

$$\cdot \otimes \partial_X \colon KK^*\big(\mathbb{C}, C_0(T^*X)\big) \to KK^*(\mathbb{C}, \mathbb{C}) \tag{3.8}$$

is equivalent to the analytic index of Atiyah–Singer.

Now, let $m\colon C_0(T^*X) \otimes C(X) \to C_0(T^*X)$ be the morphism given by

$$\sigma \otimes f \mapsto \sigma \cdot \pi^* f,$$

where π is the bundle projection. Then the so-called *Dirac element*

$$D_X := [m] \otimes_{C_0(T^*X)} \partial_X \in KK\big(C_0(T^*X) \otimes C(X), \mathbb{C}\big)$$

implements, by the Kasparov product, the Poincaré duality

$$\cdot \otimes D_X \colon KK\big(\mathbb{C}, C_0(T^*X)\big) \to KK\big(C(X), \mathbb{C}\big) \tag{3.9}$$

whose inverse is given by the principal symbol map. See [11, 12, 28] for a detailed proof.

3.6. Pseudodifferential operators

Let us recall the definition of a pseudodifferential G-operator. We refer the reader to [32, 43, 44] for pseudodifferential calculus on Lie groupoids. All along the paper, we will consider classical pseudodifferential operators and classical symbols, without specifying it anymore.

Definition 3.9. A linear G-operator is a continuous linear map

$$P\colon C_c^\infty(G, \Omega^{1/2}) \to C^\infty(G, \Omega^{1/2})$$

such that

- P restricts to a continuous family $(P_x)_{x\in G^{(0)}}$ of linear operators $P_x\colon C_c^\infty(G_x, \Omega^{1/2}) \to C^\infty(G_x, \Omega^{1/2})$ such that

$$Pf(\gamma) = P_{s(\gamma)} f_{s(\gamma)}(\gamma) \quad \forall f \in C_c^\infty(G, \Omega^{1/2}),$$

  where f_x denotes the restriction of f to G_x;
- the following equivariance property holds:

$$U_\gamma P_{s(\gamma)} = P_{r(\gamma)} U_\gamma,$$

  where U_γ is the map induced on functions by the right multiplication by γ.

  A linear G-operator P is pseudodifferential of order m if
- its Schwartz kernel k_P is smooth outside $G^{(0)}$ and

- for every distinguished chart $\psi : U \subset G \to \Omega \times s(U) \subset \mathbb{R}^{n-p} \times \mathbb{R}^p$ of G:

$$\begin{tikzcd} U \arrow[rr, "\psi"] \arrow[dr, "s"'] & & \Omega \times s(U) \arrow[dl, "p_2"] \\ & s(U) & \end{tikzcd}$$

  the operator $(\psi^{-1})^* P \psi^* : C_c^\infty(\Omega \times s(U)) \to C_c^\infty(\Omega \times s(U))$ is a smooth family parametrized by $s(U)$ of pseudodifferential operators of order m on Ω.

  We say that P is smoothing if k_P is smooth and that P is compactly supported if k_P is compactly supported on G.

The space $\Psi_c^*(G)$ of the compactly supported pseudodifferential G-operators is an involutive algebra. Observe that a pseudodifferential G-operator induces a family of pseudodifferential operators on s-fibers. So we can define the principal symbol of a pseudodifferential G-operator P as a function $\sigma(P)$ on $\mathfrak{S}^*G$, the cosphere bundle associated to the Lie algebroid $\mathfrak{A}G$ by

$$\sigma(P)(x,\xi) = \sigma(P_x)(x,\xi),$$

where $\sigma(P_x)$ is the principal symbol of the pseudodifferential operator P_x on the manifold G_x. Conversely, given a symbol f of order m on $\mathfrak{A}^*G$ together with the following data:

(1) a smooth embedding $\theta : U \to \mathfrak{A}G$, where U is an open set in G containing $G^{(0)}$, such that $\theta(G^{(0)}) = G^{(0)}$, $(d\theta)|_{G^{(0)}} = \mathrm{Id}$, and $\theta(\gamma) \in \mathfrak{A}_{s(\gamma)}G$ for all $\gamma \in U$;

(2) a smooth compactly supported map $\phi : G \to \mathbb{R}_+$ such that $\phi^{-1}(1) = G^{(0)}$;

then a G-pseudodifferential operator $P_{f,\theta,\phi}$ is obtained by the formula

$$P_{f,\theta,\phi}u(\gamma) = \int_{\gamma' \in G_{s(\gamma)},\, \xi \in \mathfrak{A}^*_{r(\gamma)}G} e^{-i\theta(\gamma'\gamma^{-1})\cdot\xi} f\big(r(\gamma),\xi\big)\phi(\gamma'\gamma^{-1})u(\gamma')$$

with $u \in C_c^\infty(G, \Omega^{1/2})$. The principal symbol of $P_{f,\theta,\phi}$ is just the leading part of f.

The principal symbol map respects a point-wise product while the product law for total symbols is much more involved. An operator is *elliptic* when its principal symbol never vanishes and in that case, as in the classical situation, it has a parametrix inverting it modulo $\Psi_c^{-\infty}(G) = C_c^\infty(G)$.

Remark 3.10. All these definitions and properties immediately extend to the case of operators acting between sections of bundles on $G^{(0)}$ pulled back to G with the range map r. The space of compactly supported pseudodifferential operators on G acting on sections of r^*E and taking values in sections of r^*F will be noted $\Psi_c^*(G,E,F)$. If $F = E$, we get an algebra denoted by $\Psi_c^*(G,E)$.

The operators of zero order $\Psi_c^0(G)$ form a subalgebra of the multiplier algebra $M(C_r^*(G))$ and we will denote by $\Psi^0(G)$ its closure in the C^*-norm. Moreover, the closure of the operators of negative order is $C_r^*(G)$.

From now on, G will be the Lie groupoid $\widetilde{X} \times_\Gamma \widetilde{X} \rightrightarrows X$, where Γ acts on $\widetilde{X}$ freely and properly with $X = \widetilde{X}/\Gamma$.

Remark 3.11. In our particular case, it turns out that the algebra of 0-order pseudodifferential G-operators is nothing but the algebra $\Psi^0_{\Gamma,\mathrm{prop}}(\widetilde{X})$ of a properly supported Γ-invariant pseudodifferential operator on $\widetilde{X}$; see [32, Example 4.4]. We will denote by $\Psi^0_\Gamma(\widetilde{X})$ its C^*-closure.

As in the classical case, one has the following pseudodifferential extension:

$$0 \to C^*_r(G) \to \Psi^0(G) \xrightarrow{\sigma} C(\mathfrak{S}^*G) \to 0,$$

where the role of compact operators is played by the groupoid C^*-algebra and symbols are functions on the cosphere bundle of the Lie algebroid.

If we take the pseudodifferential extension of the adiabatic groupoid, we have the following short exact sequence:

$$0 \to C^*_r\big(G^{[0,1)}_{\mathrm{ad}}\big) \to \Psi^0\big(G^{[0,1)}_{\mathrm{ad}}\big) \xrightarrow{\sigma} C\big(\mathfrak{S}^*\big(G^{[0,1)}_{\mathrm{ad}}\big)\big) \to 0. \tag{3.10}$$

Since $\mathfrak{A}(G^{[0,1)}_{\mathrm{ad}})$ is isomorphic to $\mathfrak{A}(G) \times [0,1)$, it follows that $K_*(C(\mathfrak{S}^*(G^{[0,1)}_{\mathrm{ad}})))$ is trivial and then, by exactness, one has that the first arrows of (3.10) induce the isomorphism

$$K_*\big(C^*_r\big(G^{[0,1)}_{\mathrm{ad}}\big)\big) \cong K_*\big(\Psi^0\big(G^{[0,1)}_{\mathrm{ad}}\big)\big). \tag{3.11}$$

Let us investigate then more closely the algebra $\Psi^0(G^{[0,1)}_{\mathrm{ad}})$. It is a $C_0([0,1))$-algebra such that

- at $t \neq 0$ we have the algebra $\Psi^0(G)$, that is isomorphic to $\Psi^0_\Gamma(\widetilde{X})$; so for $t \in (0,1)$ we have a path P_t of Γ-equivariant operators on $\widetilde{X}$ such that $P_1 = 0$ and the propagation of P_t goes to 0 as t goes to 0 (recall the differential structure of the adiabatic deformation given by (3.2));
- at $t = 0$ we have $\Psi^0(TX)$, where we are seeing $TX \rightrightarrows X$ as a Lie groupoid. Since the source and the target maps are the same for TX, it turns out that an element in $\Psi^0(TX)$ is a family of $\mathbb{R}^k$-invariant pseudodifferential operators on $\mathbb{R}^k$, with $k = \dim X$. Since a pseudodifferential operator is uniquely determined by its total symbol and since we are considering polyhomogeneous symbols, it is easy to check that $\Psi^0_{\mathbb{R}^k}(\mathbb{R}^k)$ is isomorphic to the closure of $S^0(\mathbb{R}^k)^{\mathbb{R}^k}$, the algebra of the $\mathbb{R}^k$-equivariant symbols on $\mathbb{R}^k$. But this algebra is isomorphic to the algebra of continuous functions on the closed unit ball B^k. Hence at $t = 0$ we have the algebra $C(\mathfrak{B}^*X)$ of the continuous functions on the co-disk bundle of X.

Consider the map $\mathfrak{m}\colon C(X) \to \Psi^0_\Gamma(\widetilde{X})$ which associates to a function f the operator $\mathfrak{m}(f)$ of multiplication by f. The mapping cone C^*-algebra of $\mathfrak{m}$ is given by

$$\mathcal{C}_{\mathfrak{m}} := \big\{(f, P_t) \in C(X) \oplus \Psi^0_\Gamma(\widetilde{X})[0,1) \mid P_0 = \mathfrak{m}(f)\big\}.$$

Observe that such a path P_t defines an element in $\Psi^0(G^{[0,1)}_{\mathrm{ad}})$, thus inducing a $*$-homomorphism $\eta\colon \mathcal{C}_{\mathfrak{m}} \to \Psi^0(G^{[0,1)}_{\mathrm{ad}})$.

Lemma 3.12. *The $*$-homomorphism η induces an isomorphism*

$$[\eta]\colon K_*(\mathcal{C}_{\mathfrak{m}}) \to K_*\big(\Psi^0\big(G_{\mathrm{ad}}^{[0,1)}\big)\big).$$

Proof. The commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Psi^0_\Gamma(\widetilde{X}) \otimes C_0(0,1) \arrow[r] \arrow[d] & \mathcal{C}_{\mathfrak{m}} \arrow[r, "\mathrm{ev}_0"] \arrow[d, "\eta"] & C(X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Psi^0_\Gamma(\widetilde{X}) \otimes C_0(0,1) \arrow[r] & \Psi^0\big(G_{\mathrm{ad}}^{[0,1)}\big) \arrow[r, "\mathrm{ev}_0"] & \Psi^0(TX) \arrow[r] & 0
\end{tikzcd}$$

has exact rows. Moreover, up to the isomorphism between $\Psi^0(TX)$ and $C(\mathcal{B}^*X)$, the right vertical arrow is exactly given by the pull-back of functions induced by $\pi\colon \mathcal{B}^*X \to X$. Since π is a homotopy equivalence, π^* induces an isomorphism in K-theory. By the Five Lemma, it follows that η induces an isomorphism of K-groups. ∎

4. The main theorem

From now on, let $G \rightrightarrows X$ be the Lie groupoid $\widetilde{X} \times_\Gamma \widetilde{X}$ of Subsection 3.4. In this section, we are going to compare the adiabatic exact sequence (3.4) associated to G and the analytic surgery exact sequence (2.4) for $\widetilde{X}$ and we establish an explicit isomorphism between them.

4.1. First approach

First consider the Hilbert space $\mathcal{H} := L^2\big(\widetilde{X} \times (0,1)\big)$. It is a very ample Γ-equivariant $C_0(\widetilde{X})$-module. Now observe that the essential $*$-ideal $C_r^*(G) \otimes C_0(0,1) \rtimes \mathbb{R}_+^*$ of $C_r^*(G_{\mathrm{ga}}^{[0,1)})$ is isomorphic to the subalgebra $C^*(\widetilde{X})^\Gamma$ of $\mathbb{B}(\mathcal{H})$. This implies that the algebra $C_r^*(G_{\mathrm{ga}}^{[0,1)})$ is faithfully represented on $\mathcal{H}$ through a $*$-homomorphism

$$\iota\colon C_r^*\big(G_{\mathrm{ga}}^{[0,1)}\big) \to \mathbb{B}(\mathcal{H}).$$

Remark 4.1. One can see the C^*-algebra

$$C_r^*(G) \otimes C_0(0,1) \rtimes \mathbb{R}_+^* \cong C_r^*(G) \otimes \mathbb{K}\big(L^2(0,1)\big)$$

(remember (3.7) for this isomorphism) as the Γ-equivariant elements of a subalgebra sitting inside the multipliers of the groupoid C^*-algebra of $\widetilde{G} := \widetilde{X} \times \widetilde{X} \times (0,1) \times (0,1) \rightrightarrows \widetilde{X}$. Notice that, although one is tempted to say that $C_r^*(G) \otimes \mathbb{K}(L^2(0,1))$ is the Γ-equivariant part of $C_r^*(\widetilde{G})$ itself, this is not true: indeed $C_r^*(\widetilde{G})$ is defined as the closure of compactly supported elements and it is isomorphic to $\mathbb{K}(\mathcal{H})$, whereas the equivariant lifts of elements in $C_r^*(G) \otimes \mathbb{K}(L^2(0,1))$ are supported near the diagonal: in other words they are properly supported, but not compactly supported. The same reasoning holds for $C_r^*(G_{\mathrm{ga}}^{[0,1)})$, which is the Γ-equivariant part of a subalgebra in the multipliers of $C_r^*(\widetilde{G}_{\mathrm{ga}}^{[0,1)})$.

Finally, observe that if $\widetilde{\xi} \in C^\infty(\widetilde{G}_{\mathrm{ga}}^{[0,1)})$ is the Γ-equivariant lift of an element $\xi \in C_c^\infty(G_{\mathrm{ga}}^{[0,1)})$, then $\iota(\xi)$ is the image of the lift $\widetilde{\xi}$ through the extension to multiplier algebras of the morphism $\widetilde{\iota}\colon C_r^*(\widetilde{G}) \otimes C_0(0,1) \rtimes \mathbb{R}_+^* \to \mathbb{K}(\mathcal{H})$.

Lemma 4.2. *The image of ι is contained in $D^*(\widetilde{X})^\Gamma$.*

Proof. By Remark 4.1, we deduce that $f \cdot \iota(\xi) = \widetilde{\iota}(r^* f \cdot \widetilde{\xi})$ and $\iota(\xi) \cdot f = \widetilde{\iota}(s^* f \cdot \widetilde{\xi})$ for all $f \in C_0(\widetilde{X})$ and for all $\xi \in C_r^*(G_{\mathrm{ga}}^{[0,1)})$. Since at the parameter 0 the range and the source maps coincide, we have that $r^* f = s^* f$ and then that, at the parameter 0, $[\iota(\xi), f]$ vanishes. Therefore, for all $\xi \in C_r^*(G_{\mathrm{ga}}^{[0,1)})$ and $f \in C_0(\widetilde{X})$, $[\iota(\xi), f]$ is not only in the image of $C_r^*(G) \otimes \mathbb{K}(L^2(0,1))$, but in $\mathbb{K}(L^2(\widetilde{X}) \otimes L^2(0,1))$ because f is a limit of compactly supported functions.

Finally, observe that the image of $C_r^*(G_{\mathrm{ga}}^{[0,1)})$ into $\mathbb{B}(\mathcal{H})$ is the closure of a $*$-algebra of Γ-equivariant operators of finite propagation. It follows that $\iota(C_r^*(G_{\mathrm{ga}}^{[0,1)}))$ is contained in $D^*(\widetilde{X})^\Gamma$. ∎

As a consequence of the previous lemma, we have the commutative diagram of C^*-algebras

$$\begin{tikzcd}
0 \arrow[r] & C_r^*(G) \otimes \mathbb{K} \arrow[r] \arrow[d, "\iota|_{C_r^*(G)\otimes\mathbb{K}}"] & C_r^*\left(G_{\mathrm{ga}}^{[0,1)}\right) \arrow[r] \arrow[d, "\iota"] & C_r^*(TX \rtimes \mathbb{R}_+^*) \arrow[r] \arrow[d, "\bar{\iota}"] & 0 \\
0 \arrow[r] & C^*(\widetilde{X})^\Gamma \arrow[r] & D^*(\widetilde{X})^\Gamma \arrow[r] & D^*(\widetilde{X})^\Gamma / C^*(\widetilde{X})^\Gamma \arrow[r] & 0
\end{tikzcd} \tag{4.1}$$

where the first and the third vertical arrows are the restriction of ι to $C_r^*(G) \otimes \mathbb{K}$ and the well-defined map induced on the quotient by ι, respectively.

Theorem 4.3. *The vertical arrows of diagram* (4.1) *induce isomorphisms in K-theory.*

Proof. Obviously $\iota\colon C_r^*(G) \otimes \mathbb{K} \to C^*(\widetilde{X})^\Gamma$ induces an isomorphism. So if we prove that $[\bar{\iota}]\colon K_*(C_r^*(TX \rtimes \mathbb{R}_+^*)) \to K_*(D^*(\widetilde{X})^\Gamma / C^*(\widetilde{X})^\Gamma)$ is an isomorphism, thanks to the Five Lemma, we obtain the desired result.

First of all recall that, by using isomorphism (2.1), we can replace $D^*(\widetilde{X})^\Gamma / C^*(\widetilde{X})^\Gamma$ with $D^*(X)/C^*(X)$. Since Paschke duality is an isomorphism, it follows that proving that $\mathcal{P} \circ [\iota]\colon K_*(C_r^*(TX \rtimes \mathbb{R}_+^*)) \to KK^{*+1}(C(X), \mathbb{C})$ is an isomorphism is equivalent to prove that $[\bar{\iota}]$ is so.

Recall that Paschke duality is given by the asymptotic morphism $\mu^*(u)$ in (2.3), hence $\mathcal{P} \circ [\bar{\iota}]$ is given by the asymptotic morphism

$$(\bar{\iota} \otimes \mathrm{id}_{C(X)})^* \mu^*(u) \in E^1\big(C_r^*(TX \rtimes \mathbb{R}_+^*) \otimes C(X), \mathbb{C}\big). \tag{4.2}$$

Observe that, since $C_r^*(TX \rtimes \mathbb{R}_+^*) \otimes C(X)$ is nuclear, $(\bar{\iota} \otimes \mathrm{id}_{C(X)})^* \mu^*(u)$ is an element of $KK^1(C_r^*(TX \rtimes \mathbb{R}_+^*) \otimes C(X), \mathbb{C})$. Moreover, $(\iota \otimes \mathrm{id}_{C(X)})^* \mu^*(u) = (\mu \circ (\bar{\iota} \otimes \mathrm{id}_{C(X)}))^*(u)$ and $\mu \circ (\bar{\iota} \otimes \mathrm{id}_{C(X)}) = \bar{\iota} \circ \bar{\mu}$, where

$$\bar{\mu}\colon C_r^*(TX \rtimes \mathbb{R}_+^*) \otimes C(X) \to C_r^*(TX \rtimes \mathbb{R}_+^*)$$

is the $*$-homomorphism of C^*-algebras given by $\xi \otimes f \mapsto \xi \cdot r^* f$ (notice that $\xi \cdot r^* f = s^* f \cdot \xi$ so that $\bar{\mu}$ is well defined, with the source and the target maps of $TX \rtimes \mathbb{R}_+^*$ being equal).

Hence $\mathcal{P} \circ [\bar{\iota}]$ is given by the KK-element $\bar{\mu}^* \bar{\iota}^*(u)$. But $\bar{\iota}^*(u)$ is exactly the boundary map of the second row of (3.6) for $G = X \times X$ and then for $C_r^*(G) \cong \mathbb{K}(L^2(X))$. So $TC \circ \bar{\iota}^*(u) \circ TC^{-1}$ is equal to the composition of the KK-element ∂_X in (3.8) and the suspension isomorphism S. Moreover, since $C_0(0,1) \rtimes \mathbb{R}_+ \cong \mathbb{K}$, the composition of the suspension isomorphism and TC corresponds to the Morita equivalence $KK(\mathbb{C}, A) \cong KK(\mathbb{C}, A \otimes \mathbb{K})$.

Finally, observe that $TC \circ [\bar{\mu}] \circ TC^{-1}$ is equal to $[m]$, the morphism used in Section 3.5 to define the KK-element D_X. It follows that $TC \circ \mathcal{P} \circ [\bar{\iota}] \circ TC^{-1} \circ S^{-1}$ is equal to D_X, which defines the Poincaré duality (3.9). In conclusion, we have that

$$[\bar{\iota}] = \mathcal{P}^{-1} \circ TC^{-1} \circ D_X \tag{4.3}$$

and consequently that $[\bar{\iota}]$ is an isomorphism. ∎

4.2. Second approach

Let $D^*(\widetilde{X})^\Gamma$ be the structure Roe algebra associated to the very ample Γ-equivariant $C_0(\widetilde{X})$-module $L^2(\widetilde{X}) \otimes H$, with $H = l^2(\mathbb{N})$. Let us consider the subalgebra

$$L^\infty(X) \otimes \mathbb{B}(H) \cong L^\infty(\widetilde{X})^\Gamma \otimes \mathbb{B}(H) \subset \mathbb{B}(L^2(\widetilde{X}) \otimes H);$$

it is immediate to see that $L^\infty(X) \otimes \mathbb{B}(H)$ is contained in $D^*(\widetilde{X})^\Gamma$.

Lemma 4.4. *Let j be the inclusion $L^\infty(X) \otimes \mathbb{B}(H) \to D^*(\widetilde{X})^\Gamma$ and let $SD^*(\widetilde{X})^\Gamma$ denote the suspension of $D^*(\widetilde{X})^\Gamma$. Then the inclusion of $SD^*(\widetilde{X})^\Gamma$ into $\mathcal{C}_j$, the mapping cone C^*-algebra of j, induces an isomorphism in K-theory.*

Proof. Consider the following exact sequence:

$$0 \to SD^*(\widetilde{X})^\Gamma \to \mathcal{C}_j \to L^\infty(X) \otimes \mathbb{B}(H) \to 0.$$

By the Künneth theorem, since $K_*(\mathbb{B}(H))$ is trivial, the K-theory of $L^\infty(X) \otimes \mathbb{B}(H)$ is trivial too. Then the desired isomorphism follows. ∎

Theorem 4.5. *The following maps induce isomorphisms in K-theory:*

$$SD^*(\widetilde{X})^\Gamma \to \mathcal{C}_j \leftarrow \mathcal{C}_{\mathfrak{m}} \to \Psi^0\big(G_{\mathrm{ad}}^{[0,1)}\big) \leftarrow C_r^*\big(G_{\mathrm{ad}}^{[0,1)}\big). \tag{4.4}$$

Proof. The first arrow induces the isomorphism stated in Lemma 4.4, the third arrow induces the isomorphism of Lemma 3.12, and the last arrow gives the isomorphism in (3.11). The only isomorphism to check is the one induced by the second arrow. The diagram

$$
\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
SC^*(\widetilde{X})^\Gamma \arrow[r] \arrow[d] & SC^*(\widetilde{X})^\Gamma \arrow[d] & SC^*_r(G) \arrow[l] \arrow[r] \arrow[d] & SC^*_r(G) \arrow[d] & SC^*_r(G) \arrow[l] \arrow[d] \\
SD^*(\widetilde{X})^\Gamma \arrow[r] \arrow[d] & \mathcal{C}_j \arrow[d] & \mathcal{C}_{\mathfrak{m}} \arrow[l] \arrow[r] \arrow[d] & \Psi^0(G^{[0,1)}_{\mathrm{ad}}) \arrow[d] & C^*_r(G^{[0,1)}_{\mathrm{ad}}) \arrow[l] \arrow[d] \\
S\big(D^*(X)/C^*(X)\big) \arrow[r] \arrow[d] & \mathcal{C}_{j'} \arrow[d] & \mathcal{C}_{\pi^*} \arrow[l] \arrow[r] \arrow[d] & \Psi^0(G^{[0,1)}_{\mathrm{ad}})/SC^*_r(G) \arrow[d] & C^*_r(TX) \arrow[l] \arrow[d] \\
0 & 0 & 0 & 0 & 0
\end{tikzcd}
\tag{4.5}
$$

is commutative with exact columns and, using the Five Lemma, one can prove that all the horizontal arrows but $\mathcal{C}_{\mathfrak{m}} \to \mathcal{C}_j$ and $\mathcal{C}_{\pi^*} \to \mathcal{C}_{j'}$ induce isomorphisms in K-theory.

Here $\mathcal{C}_{\pi^*}$ is the mapping cone of $\pi^*\colon C(X) \to C(S^*X)$ and $\mathcal{C}_{j'}$ is the mapping cone of $j'\colon L^\infty(X) \otimes B(H) \to D^*(X)/C^*(X)$. Notice that here we freely identify the algebra $D^*(\widetilde{X})^\Gamma/C^*(\widetilde{X})^\Gamma$ with the algebra $D^*(X)/C^*(X)$.

In order to apply the Five Lemma for the second and third columns, we are proving that $\mathcal{C}_{\pi^*} \to \mathcal{C}_{j'}$ induces an isomorphism. To that aim we are going to use, as in the proof of Theorem 4.3, the naturality of Paschke and Poincaré duality.

First notice that $C(X)$ is a subalgebra of the multipliers of all the algebras in the bottom row of diagram (4.5). Moreover, $C(X)$ commutes with all of them into their multipliers algebra: $\mathcal{C}_{\pi^*}$, $\Psi^0(G^{[0,1)}_{\mathrm{ad}})/SC^*_r(G)$ and $C^*_r(TX)$ are commutative, so for them it is obvious; for $D^*(X)/C^*(X)$ it is true by definition of pseudolocality; finally $C(X)$ commutes with $L^\infty(X) \otimes B(H)$ because it acts only on the factor $L^\infty(X)$. Let A denote any among the algebras in the bottom row of diagram (4.5), then we have a well defined $*$-homomorphism $m_A\colon C(X) \otimes A \to A$, because of the commutativity just observed.

Secondly observe that, if we substitute $C^*(\widetilde{X})^\Gamma$ with $C^*(X)$, $D^*(\widetilde{X})^\Gamma$ with $D^*(X)$, and G with the groupoid $X \times X$, by considering the diagram analogous to (4.5) (that we will call the *non-equivariant* diagram), in the bottom row we obtain the same algebras we have in the bottom row of diagram (4.5), and in the first row we have the suspension of compact operators $\mathbb{K}(L^2(X))$ everywhere.

As before, let A denote any among the algebras in the bottom row of diagram (4.5), then let us denote by $\partial_A \in E(A, \mathbb{C}) \cong E(A, \mathbb{K})$ the E-theory element associated to the boundary map of the column in the new diagram corresponding to A.

So for each A, by pulling-back through m_A, we obtain E-theory elements $m_A^*(\partial_A) \in E(A \otimes C(X), \mathbb{C})$, which give in turn group morphisms $m_A^*(\partial_A)\colon K_*(A) \to KK^*(C(X), \mathbb{C})$. By the commutativity of the non-equivariant diagram we deduce that

$$\begin{tikzcd} K_*(A) \arrow[rr, "\alpha"] \arrow[dr, "m_A^*(\partial_A)"'] & & K_*(A') \arrow[dl, "m_{A'}^*(\partial_{A'})"] \\ & KK^*\big(C(X), \mathbb{C}\big) & \end{tikzcd} \tag{4.6}$$

is commutative, for any arrow $\alpha\colon A \to A'$ in the bottom row of diagram (4.5). We obtain the following isomorphisms.

- If $\alpha\colon A \to A'$ is given by $S(D^*(X)/C^*(X)) \to \mathcal{C}_{j'}$, we have that $m_A^*(\partial_A)$ is the Paschke duality, and in Lemma 4.4 it was proven that the map in K-theory induced by α is an isomorphism: by the commutativity of (4.6), $m^*_{\mathcal{C}_{j'}}(\partial_{\mathcal{C}_{j'}})$ is an isomorphism.
- Analogously, if $\alpha\colon A \to A'$ is given by $C_r^*(TX) \to \Psi^0(G_{\mathrm{ad}}^{[0,1)})/SC_r^*(G)$, we have that $m_A^*(\partial_A)$ is the Poincaré duality and we deduce by (3.11) and the Five Lemma that the map in K-theory induced by α is an isomorphism. Then in this case also $m_{A'}^*(\partial_{A'})$ is an isomorphism.
- In the exactly same way, if $\alpha\colon A \to A'$ is given by $\mathcal{C}_{\pi^*} \to \Psi^0(G_{\mathrm{ad}}^{[0,1)})/SC_r^*(G)$, we deduce from Lemma 3.12 and the Five Lemma that α induces an isomorphism in K-theory and then that $m^*_{\mathcal{C}_{\pi^*}}(\partial_{\mathcal{C}_{\pi^*}})$ is an isomorphism.
- Finally, since $m^*_{\mathcal{C}_{\pi^*}}(\partial_{\mathcal{C}_{\pi^*}})$ and $m^*_{\mathcal{C}_{j'}}(\partial_{\mathcal{C}_{j'}})$ are isomorphisms, then by the commutativity of (4.6) we deduce that the map in K-theory induced by $\mathcal{C}_{\pi^*} \to \mathcal{C}_{j'}$ is an isomorphism.

Thanks to the last isomorphism we can use the Five Lemma to deduce that $\mathcal{C}_{\mathfrak{m}} \to \mathcal{C}_j$ induces an isomorphism and this completes the proof of the theorem. ∎

Remark 4.6. As a by-product of this proof, we obtain that the Higson–Roe exact sequence associated to the Γ-space $\widetilde{X}$ is isomorphic to the long exact sequence in K-theory induced by the following short exact sequence:

$$0 \to SC_r^*(G) \to \mathcal{C}_{\mathfrak{m}} \to \mathcal{C}_{\pi^*} \to 0.$$

This exact sequence is fundamental in the recent work of the author with Paolo Piazza and Thomas Schick [36].

5. Stratified spaces

In this section, we are going to see that the previous results apply without much more effort to the context of smoothly stratified spaces. For the comfort of the reader, it seems suitable to treat first the non-singular case and then to explain why it works in the same way for the singular context. This allows to separate the difficulties of the proof (which is the same in both settings) from the issues arising when one treats stratified spaces.

5.1. Blow-up groupoid

We quickly recall the blow-up construction in the groupoid context from [17]. Let Y be a smooth compact manifold, let X be a submanifold of Y, and let $\mathrm{DNC}(Y,X)$ be the associated deformation to the normal cone; see Section 3.2. The group $\mathbb{R}^*$ acts on $\mathrm{DNC}(Y,X)$ by

$$\lambda\cdot\big((x,\xi),0\big)=\big((x,\lambda^{-1}\xi),0\big),\quad \lambda\cdot(y,t)=(y,\lambda t)\quad \text{with } (x,\xi)\in N_X^Y,\ (y,t)\in Y\times\mathbb{R}^*.$$

Given a commutative diagram of smooth maps

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "f"] & Y \arrow[d, "f"] \\ X' \arrow[r, hook] & Y' \end{tikzcd}$$

where the horizontal arrows are inclusions of submanifolds, we naturally obtain a smooth map $\mathrm{DNC}(f)\colon \mathrm{DNC}(Y,X)\to\mathrm{DNC}(Y',X')$.

This map is defined by

$$\begin{aligned} \mathrm{DNC}(f)(y,\lambda)&=(f(y),\lambda), && \text{for } y\in Y \text{ and } \lambda\in\mathbb{R},\\ \mathrm{DNC}(f)(x,\xi,0)&=\big(f(x),f_N(\xi),0\big) && \text{for } (x,\xi)\in N_X^Y, \end{aligned}$$

where $f_N\colon N_X^Y\to N_{X'}^{Y'}$ is the linear map induced by the differential df. Moreover, it is equivariant with respect to the action of $\mathbb{R}^*$.

The action of $\mathbb{R}^*$ is free and locally proper on $\mathrm{DNC}(Y,X)\setminus X\times\mathbb{R}$ and we define $\mathrm{Blup}(Y,X)$ as the quotient space of this action.

If $H\rightrightarrows H^{(0)}$ is a closed subgroupoid of a Lie groupoid $G\rightrightarrows G^{(0)}$, then $\mathrm{DNC}(G,H)$ is a Lie groupoid over $\mathrm{DNC}(G^{(0)},H^{(0)})$, where the source and target maps are simply given by $\mathrm{DNC}(s)$ and $\mathrm{DNC}(r)$ as defined above. On the other hand, $\mathrm{Blup}(G,H)$ is not a Lie groupoid over $\mathrm{Blup}(G^{(0)},H^{(0)})$, since the Blup construction is not functorial.

Definition 5.1. The blow-up groupoid of H in G is defined as the dense open subset of $\mathrm{Blup}(G,H)$ given by

$$\begin{aligned} &\mathrm{Blup}_{r,s}(G,H)\\ &:=\big(\mathrm{DNC}(G,H)\setminus\big(H\times\mathbb{R}\cup\mathrm{DNC}(s)^{-1}(H^{(0)}\times\mathbb{R})\cup\mathrm{DNC}(r)^{-1}(H^{(0)}\times\mathbb{R})\big)\big)/\mathbb{R}^*; \end{aligned}$$

it is a Lie groupoid over $\mathrm{Blup}(G^{(0)},H^{(0)})$.

We shall be also interested in a variant of this construction: we consider

$$\mathrm{DNC}(G,H)\rightrightarrows\mathrm{DNC}(G^{(0)},H^{(0)})$$

and define $\mathrm{DNC}^+(G,H)$ as its restriction to $(N_{H^{(0)}}^{G^{(0)}})^+\times\{0\}\cup G^{(0)}\times\mathbb{R}_+^*$ with $(N_{H^{(0)}}^{G^{(0)}})^+$ denoting the positive normal bundle, where, for $h\in H^{(0)}$, $(N_{H^{(0)}}^{G^{(0)}})_h^+$ is defined by $(\mathbb{R}^n)_+:=$

$\mathbb{R}^n_+$ once we fix a linear isomorphism $(N_{H^{(0)}}^{G^{(0)}})_h$ with $\mathbb{R}^n$. We also define $\mathrm{Blup}^+(G,H)$ as the quotient of $\mathrm{DNC}^+(G,H) \setminus H \times \mathbb{R}_+$ by the action of $\mathbb{R}^*_+$. We obtain in this way the groupoid

$$\mathrm{Blup}^+_{r,s}(G,H) \rightrightarrows \mathrm{Blup}^+(G^{(0)}, H^{(0)}).$$

5.2. Manifolds with fibered corners and iterated edge metrics

Let us recall the notion of a manifold with fibered corners, due to Melrose.

Definition 5.2. Let X be a compact manifold with corners and let $H_1, \dots, H_k$ be an exhaustive list of its set of boundary hypersurfaces $M_1 X$. Suppose that each boundary hypersurface H_i is the total space of a smooth fibration $\phi_i \colon H_i \to S_i$, where the base S_i is also a compact manifold with corners. The collection of fibrations $\phi = (\phi_1, \dots, \phi_k)$ is said to be an iterated fibration structure if there is a partial order on the set of hypersurfaces such that

(1) for any subset $I \subset \{1, \dots, k\}$ with $\bigcap_{i \in I} H_i \neq \emptyset$, the set $\{H_i \mid i \in I\}$ is totally ordered;

(2) if $H_i < H_j$, then $H_i \cap H_j \neq \emptyset$, $\phi_i \colon H_i \cap H_j \to S_i$ is a surjective submersion, and $S_{ji} := \phi_j(H_i \cap H_j) \subset S_j$ is a boundary hypersurface of the manifold with corners S_j. Moreover, there is a surjective submersion $\phi_{ji} \colon S_{ji} \to S_i$ such that on $H_i \cap H_j$ we have $\phi_{ji} \circ \phi_j = \phi_i$;

(3) the boundary hypersurfaces of S_j are exactly the S_{ji} with $H_i < H_j$. In particular, if H_i is minimal, then S_i is a closed manifold.

Denote by Z_j the fiber of the fibration $\phi_j \colon H_j \to S_j$.

The quotient space ${}^{\mathrm{S}}X = X/\sim$, where

$$x \sim y \quad \Leftrightarrow \quad x = y \text{ or } \exists i \text{ such that } x, y \in H_i \text{ with } \varphi_i(x) = \varphi_i(y),$$

is a so-called Thom–Mather stratified space with strata $\{S_1, \dots, S_k\}$; see [30]. In turn, X is called a resolution of ${}^{\mathrm{S}}X$.

Recall from [1–3] that an incomplete iterated edge metric g (shortly an iie-metric) is a metric on $\mathring{X}$ such that in a collar neighborhood of a hypersurface H_i it takes the form

$$dx_i^2 + x_i^2 g_{Z_i} + \varphi^* g_{S_i},$$

where x_i is a boundary defining function of H_i and g_{Z_j} and g_{S_j} are metrics with the same structure on Z_j and S_j. In particular, an iie-metric on $\mathring{X}$ induces a Riemannian metric on each stratum of ${}^{\mathrm{S}}X$ and that these metrics fit together continuously.

In particular, by [33, Theorem 2.4.7], the topology on ${}^{\mathrm{S}}X$ is that of the metric space with distance between two points given by taking the minimum over rectifiable curves joining them. As a metric space, ${}^{\mathrm{S}}X$ is complete and locally compact [33, Theorem 2.4.17] and hence a length space.

Remark 5.3. Consider a Galois Γ-covering ${}^{\mathrm{S}}\widetilde{X}$ of ${}^{\mathrm{S}}X$ and its resolution $\widetilde{X}$. They come with Galois Γ-coverings $\widetilde{H}_i$ and $\widetilde{S}_i$ over H_i and S_i, respectively for all i. Moreover, we have a Γ-equivariant lift $\widetilde{\varphi}_i\colon \widetilde{H}_i \to \widetilde{S}_i$ of φ_i such that the links are still Z_i; see, for instance, [37, Section 2.4]. Finally, from an iie-metric g on ${}^{\mathrm{S}}X$ we obtain a Γ-equivariant iie-metric $\widetilde{g}$ on ${}^{\mathrm{S}}\widetilde{X}$.

As in [4, Section 3.5], we can consider the analytic surgery sequence of Higson and Roe for Thom–Mather spaces, induced by the exact sequence of C^*-algebras

$$\cdots \to K_*(C^*({}^{\mathrm{S}}\widetilde{X})^\Gamma) \to K_*(D^*({}^{\mathrm{S}}\widetilde{X})^\Gamma) \to K_*(D^*({}^{\mathrm{S}}\widetilde{X})^\Gamma / C^*({}^{\mathrm{S}}\widetilde{X})^\Gamma) \to \cdots, \quad (5.1)$$

and, as before, we have that $K_*(D^*({}^{\mathrm{S}}\widetilde{X})^\Gamma / C^*({}^{\mathrm{S}}\widetilde{X})^\Gamma) \simeq KK_{*+1}(C({}^{\mathrm{S}}X), \mathbb{C})$ by Paschke duality.

5.3. Poincaré duality for stratified spaces

We can associate a Lie groupoid to a manifold with fibered corners in the following way. Let $\{H_1, \ldots, H_k\}$ be a list such that if $H_i < H_j \Rightarrow i < j$ and observe that if $H_i < H_j$, then $H_i \times_{S_i} H_i \rightrightarrows H_i$ is a closed Lie subgroupoid of $H_j \times_{S_j} H_j$.

Definition 5.4. Let $G(X, \varphi) \rightrightarrows X$ be the Lie groupoid

$$\mathrm{Blup}^+_{r,s}\big(\cdots\big(\mathrm{Blup}^+_{r,s}\big(\mathrm{Blup}^+_{r,s}(X \times X, H_1 \times_{S_1} H_1), H_2 \times_{S_2} H_2\big)\cdots, H_k \times_{S_k} H_k\big).$$

Notice that in this definition the order of the blow-ups is important: if $H_i < H_j$, then there is no immersion of $H_i \times_{S_i} H_i$ into the blow-up of $H_j \times_{S_j} H_j$ into $X \times X$. As a set $G(X, \varphi)$ is given by

$$\mathring{X} \times \mathring{X} \cup \bigsqcup_{j=0}^{k} (H_j \times_{S_j} TS_j \times_{S_j} H_j)_{|X_j} \rtimes \mathbb{R},$$

where $X_j = H_j \setminus (H_j \cap \bigcup_{i>j} H_i)$.

Definition 5.5. Consider the adiabatic deformation groupoid $G(X, \varphi)^{[0,1)}_{\mathrm{ad}} \rightrightarrows X \times [0, 1)$. Set $X_\partial := \mathring{X} \cup (\partial X \times [0, 1))$ and define the non-commutative tangent bundle of X as the Lie groupoid

$$T^{\mathrm{NC}}_\varphi X := \big(G(X, \varphi)^{[0,1)}_{\mathrm{ad}}\big)_{|X_\partial} \rightrightarrows X_\partial.$$

As a set $T^{\mathrm{NC}}_\varphi X$ is equal to $T\mathring{X} \cup \bigsqcup_{j=0}^{k} ((H_j \times_{S_j} TS_j \times_{S_j} H_j)_{|X_j} \rtimes \mathbb{R})^{[0,1)}_{\mathrm{ad}}$.

We thus obtain an exact sequence of C^*-algebras analogous to (3.3):

$$0 \to C^*_r\big(\mathring{X} \times \mathring{X} \times (0, 1)\big) \to C^*_r\big(G(X, \varphi)^{[0,1)}_{\mathrm{ad}}\big) \to C^*_r\big(T^{\mathrm{NC}}_\varphi X\big) \to 0. \quad (5.2)$$

Denote by $\partial_{(X,\varphi)}\colon KK(\mathbb{C}, C^*_r(T^{\mathrm{NC}}_\varphi X)) \to KK(\mathbb{C}, C^*_r(\mathring{X} \times \mathring{X}))$ the boundary map associated to (5.2), up to a suspension isomorphism.

Denote by $q\colon X_\partial \to {}^{\mathrm{S}}X$ the obvious quotient map. Observe that

$$q \circ s = q \circ r\colon T_\varphi^{\mathrm{NC}} X \to {}^{\mathrm{S}}X;$$

this implies that the morphism ${}^{\mathrm{S}}m\colon C_r^*(T_\varphi^{\mathrm{NC}}X) \otimes C({}^{S}X) \to C_r^*(T_\varphi^{\mathrm{NC}}X)$ given by $\xi \otimes f \mapsto \xi \cdot (q \circ s)^* f$ is well defined.

Definition 5.6. The *Dirac element* is defined as the following KK-element:

$${}^{\mathrm{S}}D_X := [{}^{\mathrm{S}}m] \otimes_{C_r^*(T_\varphi^{\mathrm{NC}}X)} \partial_{(X,\varphi)} \in KK\big(C_r^*(T_\varphi^{\mathrm{NC}}X) \otimes C({}^{S}X), \mathbb{C}\big).$$

Theorem 5.7 (Poincaré duality [13,15]). *Let ${}^{S}X$ be a Thom–Mather stratified space, then ${}^{\mathrm{S}}D_X$ gives a Poincaré duality. In particular, the Kasparov product with ${}^{\mathrm{S}}D_X$ induces the following isomorphism of KK-groups:*

$$KK\big(\mathbb{C}, C_r^*(T_\varphi^{\mathrm{NC}}X)\big) \cong KK\big(C({}^{S}X), \mathbb{C}\big).$$

Remark 5.8. Observe that the Lie algebroid $\mathfrak{A}G(X,\varphi)$ of $G(X,\varphi)$ is non-canonically isomorphic to TX and the anchor map $\mathfrak{A}G(X,\varphi) \to TX$ is an isomorphism over $\mathring{X}$ and it is the projection onto the kernel of $d\varphi_i$ over H_i. In particular, we have that the continuous sections of $\mathfrak{A}G(X,\varphi)$ are given by the Lie subalgebra of vector fields over X

$$\mathcal{V}_{\mathrm{e}}(X) = \big\{\xi \in \mathcal{V}_b(X);\ \xi|_{H_i} \text{ is tangent to the fibers of } \varphi_i : H_i \to S_i\ \forall i\big\},$$

where

$$\mathcal{V}_b(X) = \big\{\xi \in C^\infty(X, TX);\ \xi x_i \in x_i C^\infty(X)\ \forall i\big\}.$$

In particular, a continuous metric g_e for $\mathfrak{A}G(X,\varphi)$ is given by a so-called iterated edge metric, which is defined as $\rho^2 g$, where g is an iie-metric and the conformal factor is given by $\rho \in C^\infty(X)$, the product of all the boundary defining functions x_i, with $i \in \{1,\dots,k\}$.

It is worth to point out that the proof of Poincaré duality in [15, Theorem 11.1] takes place in the context of *iterated fibered corners* metrics which are associated to a slightly different Lie groupoid: as a set it is the same, whereas the smooth structure is different. Nevertheless, one can rewrite the entire work in [15, Section 11] by using iterated fibered corners metrics and the proof of Theorem 5.7 goes, word by word, exactly in the same way. To have a further insight of why that is true, observe that the proof of Poincaré duality in [15] corresponds up to KK-equivalence to the one in [13], see [15, Corollary 11.5], but the last one does not depend on the metric we choose to put on ${}^{S}X$.

5.4. The main theorem: the stratified case

Let $\widetilde{X}$ be as in Remark 5.3 and let us denote $\widetilde{X} \setminus \partial\widetilde{X}$ by $\widetilde{X}^{\mathrm{reg}}$. Consider the Lie groupoid $G(\widetilde{X},\widetilde{\varphi}) \rightrightarrows \widetilde{X}$ given by

$$\widetilde{X}^{\mathrm{reg}} \times \widetilde{X}^{\mathrm{reg}} \cup \bigsqcup_{j=0}^{k} \big(\widetilde{H}_j \times_{\widetilde{S}_j} T\widetilde{S}_j \times_{\widetilde{S}_j} \widetilde{H}_j\big)_{|\widetilde{X}_j} \rtimes \mathbb{R}$$

and observe that there is a proper and free action of Γ on $G(\widetilde{X},\widetilde{\varphi})$ through groupoid automorphisms: let g be an element of Γ and let (x,y) be in $\widetilde{X}^{\mathrm{reg}} \times \widetilde{X}^{\mathrm{reg}}$, then $g \cdot (x,y) = (g \cdot x, g \cdot y)$; if instead (x,ξ,y,t) is an element over the boundary, then $g \cdot (x,\xi,y,t) = (g \cdot x, dg \cdot \xi, g \cdot y, t)$.

Definition 5.9. Define the groupoid $\mathscr{G} \rightrightarrows X$ as the quotient of $G(\widetilde{X},\widetilde{\varphi}) \rightrightarrows \widetilde{X}$ by the action of Γ. As a set, it is given by

$$\widetilde{X}^{\mathrm{reg}} \times_\Gamma \widetilde{X}^{\mathrm{reg}} \cup \bigsqcup_{j=0}^{k} (H_j \times_{S_j} TS_j \times_{S_j} H_j)_{|X_j} \rtimes \mathbb{R}.$$

Remark 5.10. The fact that, in the restriction of $\mathscr{G}$ to the boundary, no product over Γ appears is explained in a detailed way in [37, Section 5].

Consequently, we have the following exact sequence of C^*-algebras:

$$0 \to C^*_r\big(\widetilde{X}^{\mathrm{reg}} \times_\Gamma \widetilde{X}^{\mathrm{reg}} \times (0,1)\big) \to C^*_r\big(\mathscr{G}^{[0,1)}_{\mathrm{ad}}\big) \to C^*_r\big(T^{\mathrm{NC}}_\varphi X\big) \to 0.$$

Notice that $C^*_r(\widetilde{X}^{\mathrm{reg}} \times_\Gamma \widetilde{X}^{\mathrm{reg}} \times (0,1) \rtimes \mathbb{R}_+)$ is a subalgebra of $\mathscr{H}' := L^2(\widetilde{X}^{\mathrm{reg}} \times (0,1), g')$, where we endow $\widetilde{X}$ with a complete iterated edge metric g'. Through the multiplication by the total boundary function ρ we get an isomorphism $m(\rho)\colon \mathscr{H} \to \mathscr{H}'$ with $\mathscr{H} := L^2(\widetilde{X}^{\mathrm{reg}} \times (0,1), g)$, where $g := \rho^{-2} g'$ is an iie-metric. It is a Γ-equivariant $C_0({}^S\widetilde{X})$-module and one can immediately see that the conjugation by $m(\rho)$ maps $C^*_r(\widetilde{X}^{\mathrm{reg}} \times_\Gamma \widetilde{X}^{\mathrm{reg}} \times (0,1) \rtimes \mathbb{R}_+)$ isomorphically onto $C^*({}^S\widetilde{X})^\Gamma$. As in Section 4.1, this isomorphism extends to an injective map ${}^S\iota\colon C^*_r(\mathscr{G}^{[0,1)}_{\mathrm{ga}}) \to \mathbb{B}(\mathscr{H})$.

Lemma 5.11. *The image of ${}^S\iota$ is contained in $D^*({}^S\widetilde{X})^\Gamma$.*

Proof. The proof is similar to the one of Lemma 4.2: one checks in the exactly same way that the commutator of $f \in C_0({}^S\widetilde{X})$ and an element of ${}^S\iota(C^*_r(\mathscr{G}^{[0,1)}_{\mathrm{ga}}))$ is zero at the parameter 0 of the deformation. The only additional thing to check is that the commutator of f is in $C_0({}^S\widetilde{X})$ and that an element of ${}^S\iota(C^*_r(\mathscr{G}^{[0,1)}_{\mathrm{ga}}))$ is zero on the singular part, which is a necessary condition for being locally compact: indeed compact operators are in the closure of smoothing operators whose kernel is compactly supported on $\widetilde{X}^{\mathrm{reg}} \times \widetilde{X}^{\mathrm{reg}} \times (0,1) \times (0,1)$. Recall, from the discussion in Remark 4.1, that we can see $C^*_r(\mathscr{G}^{[0,1)}_{\mathrm{ga}})$ as a subalgebra in the multiplier algebra of $C^*_r(G(\widetilde{X},\widetilde{\varphi}))$ generated by properly supported Γ-equivariant elements. Let $\widetilde{q} \circ pr\colon \widetilde{X} \times [0,1) \to {}^S\widetilde{X}$ be the composition of the projection to $\widetilde{X}$ and the quotient map. Then it follows that $\widetilde{q}^* f$ is constant along the fibers of φ_i for all $i = 1,\dots,k$ and this implies that $r^*(\widetilde{q} \circ pr)^* f = s^*(\widetilde{q} \circ pr)^* f$ is constant on $G(\widetilde{X},\widetilde{\varphi})_{|\partial\widetilde{X}}$. Consequently, $[f, {}^S\iota(x)] = {}^S\iota((r^*(\widetilde{q} \circ pr)^* f - s^*(\widetilde{q} \circ pr)^* f)x)$ is zero on the singular part of ${}^S\widetilde{X}$. Then, since f is a limit of compactly supported functions, $[f, {}^S\iota(x)]$ is compact. Finally, observe that the image of ${}^S\iota$ in $\mathbb{B}(\mathscr{H})$ is the closure of a $*$-algebra of Γ-equivariant operators of finite propagation, which implies that the image of ${}^S\iota$ is contained in $D^*({}^S\widetilde{X})^\Gamma$. ∎

Now we are able to state the main result of this section whose proof follows exactly the proof of Theorem 4.3.

Theorem 5.12. *There exists a commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & C_r^*(\widetilde{X}^{\mathrm{reg}} \times_\Gamma \widetilde{X}^{\mathrm{reg}}) \otimes \mathbb{K} \arrow[r] \arrow[d, "s_{\iota_\circ}"] & C_r^*(\mathscr{G}_{\mathrm{ga}}^{[0,1)}) \arrow[r] \arrow[d, "s_\iota"] & C_r^*(T_\varphi^{\mathrm{NC}} X \rtimes \mathbb{R}_+^*) \arrow[r] \arrow[d, "s_{\bar\iota}"] & 0 \\ 0 \arrow[r] & C^*({}^S\widetilde{X})^\Gamma \arrow[r] & D^*({}^S\widetilde{X})^\Gamma \arrow[r] & D^*({}^S\widetilde{X})^\Gamma / C^*({}^S\widetilde{X})^\Gamma \arrow[r] & 0 \end{tikzcd} \tag{5.3}$$

such that the vertical arrows induce isomorphisms in K-theory.

Remark 5.13. Let us highlight that one can also follow the second approach in Section 4.2, since Lemma 3.12 holds also for $\mathscr{G}$. More precisely, we obtain that the analogous of the middle column of (4.5) is given by

$$0 \to C_r^*(\mathscr{G}_{|X^{\mathrm{reg}}}) \otimes C_0(0,1) \to \mathscr{C}\big(C(X) \xrightarrow{\mathfrak{m}} \Psi^0(\mathscr{G})\big) \to \mathscr{C}\big(C(X) \xrightarrow{\mathfrak{m}} \Sigma_{nc}(X)\big) \to 0, \tag{5.4}$$

where $\Sigma_{nc}(X) := \Psi^0(\mathscr{G})/C_r^*(\mathscr{G}_{|X^{\mathrm{reg}}})$ is the C^*-algebra of non-commutative symbols. This C^*-algebra is given by the following pull-back:

$$\begin{tikzcd} \Sigma_{nc}(X) \arrow[r] \arrow[d] & \Psi^0\big((\mathscr{G}_{|\partial X})_{\mathrm{ad}}^{[0,1)}\big) \arrow[d] \\ C(\mathfrak{S}^*\mathscr{G}) \arrow[r] & C(\mathfrak{S}^*\mathscr{G}_{|\partial X}) \end{tikzcd} \tag{5.5}$$

Remark 5.14. Notice that Theorem 5.12 reveals the correspondence between K-theoretic invariants associated to incomplete metrics and the complete metrics obtained under a conformal change of the metric. Indeed in the first row, complete metrics are used to define the C^*-algebras, whereas in the second row the metrics are incomplete.

6. Comparing secondary invariants

In this section, we shall give the proof of Theorem 1.3. In particular, we are going to prove that the isomorphism $K_{*+1}(D^*(\widetilde{X})^\Gamma) \cong K_*(C_r^*(G_{\mathrm{ad}}^{[0,1)}))$, induced by (4.5), puts in correspondence the ϱ-classes associated to a metric g with a positive scalar curvature, defined in [34, 51], respectively.

Let $\widetilde{X}$ be a smooth spin manifold with a free, proper, and isometric action of Γ. Let $\not{S}$ denote the spinor bundle over $\widetilde{X}$. Let g be a Γ-invariant complete metric on $\widetilde{X}$ and assume that the scalar curvature of g is positive everywhere on $\widetilde{X}$. The Lichnerowicz formula implies that the Dirac operator $\not{D}$ associated to g is invertible.

Denote by $\chi\colon \mathbb{R} \to \mathbb{R}$ the sign function and by $\psi\colon \mathbb{R} \to \mathbb{R}$ the chopping function $t \mapsto \frac{t}{\sqrt{1+t^2}}$. There is a path of functions $\psi_s\colon t \mapsto \psi(\frac{t}{1-s})$ such that $\psi_0 = \psi$ and $\psi_1 = \chi$

(actually it is a continuous path of continuous functions on $\mathbb{R} \setminus \{0\}$, where continuity here is with respect to the sup norm).

6.1. Coarse invariants

Let us recall the definition of the ϱ-classes of Piazza and Schick in [34].

Definition 6.1. Let $\dim(\widetilde{X})$ be odd. Since $\not{D}$ is invertible, the operator $\chi(\not{D})$ is a symmetry in $D^*(\widetilde{X})^\Gamma$. Then we can define $\varrho(g)$ as the class

$$\left[\frac{1}{2}(\chi(\not{D}) + 1)\right] \in K_0\big(D^*(\widetilde{X})^\Gamma\big).$$

Here $D^*(\widetilde{X})^\Gamma$ is represented on the very ample Γ-equivariant $C_0(\widetilde{X})$-module $L^2(\not{S}) \otimes l^2\mathbb{N}$ and $\frac{1}{2}(\chi(\not{D}) + 1)$ is intended as the infinite matrix with $\frac{1}{2}(\chi(\not{D}) + 1)$ in the top left corner and the identity along all the diagonal.

Remark 6.2. Notice that, in the odd dimensional case, $\frac{1}{2}(\chi(\not{D}) + 1)$ is exactly $\mathscr{P}_>$, the projection on the positive part of the spectrum of $\not{D}$. Consequently, $\varrho(g)$ is the image of $[\mathscr{P}_>]$ through the map $K_0(\Psi^0_\Gamma(\widetilde{X})) \to K_0(D^*(\widetilde{X})^\Gamma)$.

Let us now consider the even dimensional context. In this case, the spinor bundle is graded by the chirality element and it splits in the following way: $\not{S} = \not{S}_+ \oplus \not{S}_-$. In turn, the Dirac operator is odd with respect to the grading and it is of the following matrix form: $\begin{bmatrix} 0 & \not{D}_+ \\ \not{D}_- & 0 \end{bmatrix}$.

Notice that, even though $\not{S}_+$ and $\not{S}_-$ are not isomorphic as smooth bundles, there exists an isometric Γ-equivariant isomorphism $u\colon \not{S}_- \to \not{S}_+$ of measurable bundles, which is given by the Clifford multiplication by any Γ-invariant vector field whose zero set is of a measure equal to zero. It induces the unitary Γ-equivariant maps $U\colon L^\infty(\not{S}_-) \to L^\infty(\not{S}_+)$ and $U\colon L^2(\not{S}_-) \to L^2(\not{S}_+)$.

Definition 6.3. Let $\chi_+(\not{D})$ be the positive part of $\chi(\not{D})$. Then $\varrho(g)$ is defined by the class

$$\big[U\chi_+(\not{D})\big] \in K_1\big(D^*(\widetilde{X})^\Gamma\big).$$

Here $D^*(\widetilde{X})^\Gamma$ is represented on the Γ-equivariant $C_0(\widetilde{X})$-module $L^2(\not{S}_+) \otimes l^2\mathbb{N}$. Moreover, notice that the definition does not depend on the choice of U; see [35, Section 2B2].

6.2. Adiabatic invariants

Since $\widetilde{X}$ is spin, the Lie algebroid of the adiabatic deformation of $G = \widetilde{X} \times_\Gamma \widetilde{X}$, which is $TX \times [0,1]$, is obviously spin. So we can consider $\not{D}_{\mathrm{ad}}$, the Dirac operator of $G^{[0,1]}_{\mathrm{ad}}$, defined on the $C^\infty_c(G^{[0,1]}_{\mathrm{ad}})$-module $C^\infty_c(G^{[0,1]}_{\mathrm{ad}}, r^*\not{S} \otimes \Omega^{1/2})$; see [51, Definition 3.21] for the definition of the Dirac operator on a Lie groupoid with a spin Lie algebroid. Let us denote by $\mathcal{E}^{[0,1]}_{\mathrm{ad}}$ its $C^*_r(G^{[0,1]}_{\mathrm{ad}})$-completion and let us denote by $\mathcal{E}$ its restriction at $t = 1$.

As explained in Section 3.6, we can consider it as a field of operators such that at $t=1$ it is the Γ-equivariant Dirac operator $\not{D}$ of $\widetilde{X}$ and at $t=0$ it is the one given by the Fourier transform of its symbol, namely by Clifford multiplication.

Notice that $\psi(\not{D}_{\mathrm{ad}})$ belongs to $\Psi^0(G_{\mathrm{ad}}^{[0,1]})$. Moreover, since the restriction of $\not{D}_{\mathrm{ad}}$ at $t=1$ is invertible, we have a continuous path of operators $\psi_s(\not{D})$ from $\psi(\not{D})$ to $\chi(\not{D})$.

Definition 6.4. Let us define

$$\varrho^{\mathrm{ad}}(g) = \big[\mathcal{E}_{\mathrm{ad}}^{[0,1)}, \psi_{\mathrm{ad}}^{[0,1)}(\not{D})\big]$$

as the class in $KK^*(\mathbb{C}, C_r^*(G_{\mathrm{ad}}^{[0,1]}))$ given by the concatenation of the Kasparov bimodules $[\mathcal{E}_{\mathrm{ad}}^{[0,1]}, \psi(\not{D}_{\mathrm{ad}})]$ and $[\mathcal{E} \otimes C_0[0,1), \psi_s(\not{D})]$, after a suitable reparametrization.

This definition makes sense in both the odd and the even dimensional cases, because the definition of KK-groups takes the grading of the spinor bundle into account.

6.3. Comparison of ϱ-classes

Let us first calculate the image of $\varrho^{\mathrm{ad}}(g)$ in $KK^*(\mathbb{C}, \Psi^0(G_{\mathrm{ad}}^{[0,1)}))$ under the inclusion $C_r^*(G_{\mathrm{ad}}^{[0,1)}) \hookrightarrow \Psi^0(G_{\mathrm{ad}}^{[0,1)})$. From the definition of $\mathcal{E}_{\mathrm{ad}}^{[0,1)}$ it is easy to see that

$$\mathcal{E}_{\mathrm{ad}}^{[0,1)} \otimes_{\Psi^0(G_{\mathrm{ad}}^{[0,1)})} \Psi^0\big(G_{\mathrm{ad}}^{[0,1)}\big) \cong \Psi^0\big(G_{\mathrm{ad}}^{[0,1)}, \not{S}\big),$$

whereas the operator can be seen as unchanged.

We are going to treat the odd and the even dimensional cases separately. Let us start with the odd case and let us denote by Λ the isomorphism

$$K_1\big(SD^*(\widetilde{X})^\Gamma\big) \to K_1\big(C_r^*(G_{\mathrm{ad}}^{[0,1)})\big).$$

Recall that in the odd dimensional case $\varrho(g)$ is the image of $[\mathcal{P}_>]$ through the inclusion of $\Psi_\Gamma^0(\widetilde{X})$ into $D^*(\widetilde{X})^\Gamma$. Since the following triangle

$$\begin{tikzcd} & K_1\big(S\Psi_\Gamma^0(\widetilde{X})\big) \arrow[dl] \arrow[dr] & \\ K_1\big(SD^*(\widetilde{X})^\Gamma\big) \arrow[rr, "\Lambda"] & & K_1\big(\Psi^0(G_{\mathrm{ad}}^{[0,1)})\big) \end{tikzcd}$$

is obviously commutative, it is enough to compare the image of the suspension of $[\mathcal{P}_>]$ and the image of $\varrho^{\mathrm{ad}}(g)$ inside $K_1(\Psi^0(G_{\mathrm{ad}}^{[0,1)}))$. The suspension of $[\mathcal{P}_>]$ is given by the path of unitaries

$$\exp(2\pi i t\mathcal{P}_>) = e^{2\pi i t}\mathcal{P}_> + (1-\mathcal{P}_>) \in C_0(0,1) \otimes \Psi_\Gamma^0(\widetilde{X}, \widetilde{\not{S}}).$$

First observe that the identification $KK^1(\mathbb{C}, A) \cong K_1(A)$ is given by the map

$$[H, F] \mapsto \big[\exp(2\pi i P)\big],$$

where $P = \frac{F+1}{2}$. An easy calculation shows that $[\exp(2\pi i t\mathcal{P}_{>})]$ corresponds to the Kasparov bimodule

$$\left[\mathcal{H}, t\chi(\not{D}) + (t-1)\right] \in KK^1\left(\mathbb{C}, C_0(0,1) \otimes \Psi^0_\Gamma(\widetilde{X})\right),$$

where $\mathcal{H}$ is the $C_0(0,1) \otimes \Psi^0_\Gamma(\widetilde{X})$-module $C_0(0,1) \otimes \Psi^0_\Gamma(\widetilde{X}, \widetilde{\not{S}})$. The operator $t\chi(\not{D}) + (t-1)$ extends to the $\Psi^0(G^{[0,1)}_{\mathrm{ad}})$-module $\Psi^0(G^{[0,1)}_{\mathrm{ad}}, \not{S})$ and we obtain the corresponding element in $KK(\mathbb{C}, \Psi^0(G^{[0,1)}_{\mathrm{ad}}))$, obtained through the inclusion $C_0(0,1) \otimes \Psi^0_\Gamma(\widetilde{X}) \hookrightarrow \Psi^0(G^{[0,1)}_{\mathrm{ad}})$.

Finally, observe that $t\chi(\not{D}) + (t-1)$ and $\psi^{[0,1)}_{\mathrm{ad}}(\not{D})$, both of them operators on $\Psi^0(G^{[0,1)}_{\mathrm{ad}}, \not{S})$, commute. Then by [40, Lemma 11] there is a homotopy connecting them, hence the images of $\varrho(g)$ and $\varrho^{\mathrm{ad}}(g)$ coincide in $KK^1(\mathbb{C}, \Psi^0(G^{[0,1)}_{\mathrm{ad}}))$.

Let us now pass to the even dimensional case. We refer the reader to [5, Sections 2.2 and 2.3] for a detailed account of the isomorphism between the relative K-group of a morphism and the K-theory group of its mapping cone C^*-algebra. In this case, we are going to start with the class of $K_1(D^*(\widetilde{X})^\Gamma)$ induced by the unitary $U\chi(\not{D})_+$ of Definition 6.3. Following the arrows in (4.5), we see that it induces the class $[L^\infty(\widetilde{X}, \widetilde{\not{S}}_+), L^\infty(\widetilde{X}, \widetilde{\not{S}}_+), U\chi(\not{D})_+]$ in the relative group $K_0(j)$. Using any path of unitaries from U to the identity and the fact that $U^{-1}L^\infty(\widetilde{X}, \widetilde{\not{S}}_+) = L^\infty(\widetilde{X}, \widetilde{\not{S}}_-)$, we see that the last class is equal to $[L^\infty(\widetilde{X}, \widetilde{\not{S}}_+), L^\infty(\widetilde{X}, \widetilde{\not{S}}_-), \chi(\not{D})_+]$, which in turn is clearly the image of $[C(\widetilde{X}, \widetilde{\not{S}}_+), C(\widetilde{X}, \widetilde{\not{S}}_-), \chi(\not{D})_+] \in K_0(\mathfrak{m})$ through the second arrows in (4.5). Now observe that $\mathfrak{m}$ is injective and, as explained in [5, Section 2], one can easily see that the realization of our class in $K_0(C_{\mathfrak{m}})$ is given by

$$\left[\begin{pmatrix} \cos^2\left(\frac{\pi}{2}t\right)1_+ & \cos\left(\frac{\pi}{2}t\right)\sin\left(\frac{\pi}{2}t\right)\chi(\not{D})_+ \\ \cos\left(\frac{\pi}{2}t\right)\sin\left(\frac{\pi}{2}t\right)\chi(\not{D})_- & \sin^2\left(\frac{\pi}{2}t\right)1_- \end{pmatrix}\right] - \left[\begin{pmatrix} 0 & 0 \\ 0 & 1_- \end{pmatrix}\right] \quad t \in [0,1], \tag{6.1}$$

where $1_\pm$ is the identity of the $\Psi^0_\Gamma(\widetilde{X})$-module $C(\widetilde{X}, \widetilde{\not{S}}_\pm) \otimes_{C(X)} \Psi^0_\Gamma(\widetilde{X}) = \Psi^0_\Gamma(\widetilde{X}, \not{S}_\pm)$ and the second matrix is meant to denote the constant path. The first term of (6.1) is obtained by conjugating $\begin{pmatrix} 1_+ & 0 \\ 0 & 0 \end{pmatrix}$ with the path of invertible matrices

$$\begin{pmatrix} \cos\left(\frac{\pi}{2}t\right) & \sin\left(\frac{\pi}{2}t\right)\chi(\not{D})_+ \\ \sin\left(\frac{\pi}{2}t\right)\chi(\not{D})_- & \cos\left(\frac{\pi}{2}t\right) \end{pmatrix} \tag{6.2}$$

and the last path of invertible martices is homotopic, through paths of invertible elements, to

$$\begin{pmatrix} \cos^2\left(\frac{\pi}{2}t\right) & \sin\left(\frac{\pi}{2}t\right)\chi(\not{D})_+ \\ \sin\left(\frac{\pi}{2}t\right)\chi(\not{D})_- & \cos^2\left(\frac{\pi}{2}t\right) \end{pmatrix}. \tag{6.3}$$

Now conjugating $\left(\begin{smallmatrix} 1_+ & 0 \\ 0 & 0 \end{smallmatrix}\right)$ by (6.3) instead of (6.2), we obtain exactly the image of $[\Psi^0_\Gamma(\widetilde{X}, \not{S}_+) \oplus \Psi^0_\Gamma(\widetilde{X}, \not{S}_-), F]$ by means of the standard identification of $KK(\mathbb{C}, C_{\mathfrak{m}})$ and $K_0(C_{\mathfrak{m}})$; here the operator in the Kasparov bimodule is given by

$$F = \begin{pmatrix} 0 & \sin\left(\frac{\pi}{2}t\right)\chi(\not{D})_+ \\ \sin\left(\frac{\pi}{2}t\right)\chi(\not{D})_- & 0 \end{pmatrix}.$$

Finally, if we move to $KK(\mathbb{C}, \Psi^0(G^{[0,1)}_{\mathrm{ad}}))$ through the map η from Lemma 3.12, we obtain the class $[\Psi^0(G^{[0,1)}_{\mathrm{ad}}, \not{S}), F]$ which is, by [40, Lemma 11], operatorially homotopic to the image of $\varrho^{\mathrm{ad}}(g)$ in $KK(\mathbb{C}, \Psi^0(G^{[0,1)}_{\mathrm{ad}}))$. Observe that this is true because the identity is a compact operator on the module $\Psi^0(G^{[0,1]}_{\mathrm{ad}})$. This completes the proof of Theorem 1.3.

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Vito Felice Zenobi
Dipartimento di Matematica, Sapienza Università di Roma, Piazzale Aldo Moro 5, 00185 Roma, Italy; vitofelice.zenobi@uniroma1.it