

## Global symmetry breaking in gauge theories: the case of multiflavor scalar chromodynamics

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Universal features of continuous phase transitions can be investigated by studying the  $\phi^4$  field theory with the corresponding global symmetry breaking pattern. When gauge symmetries are present, the same technique is usually applied to a gauge-invariant order parameter field, as in the Pisarski-Wilczek analysis of the QCD chiral phase transition. Gauge fields are thus assumed to be irrelevant in the effective critical model, a fact that is however far from trivial. We will investigate the validity of this approach using three-dimensional scalar lattice models with non-abelian global and local symmetries, for which critical exponents and scaling functions can be numerically determined with high accuracy.

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## 1. Introduction

In this proceeding we report on our ongoing research project aimed at a better understanding of continuous phase transitions in three dimensional (3D) gauge theories. It is well known that, when a spontaneous symmetry breaking is associated with a continuous phase transition, universal properties emerge, which are encoded in the  $\phi^4$  field theory with the same global symmetry breaking pattern. The main goal of our project is to understand to what extent this approach can be applied to systems with local gauge symmetries.

In order to explain more in detail the motivations of our work, and to introduce the subject, it will be useful to review in some detail a specific example likely familiar to the reader, that of the chiral phase transition in QCD. The by now classical analysis by Pisarski and Wilczek of the finite  $T$  chiral phase transition [1] goes as follows:

1. first of all, we assume the transition to be continuous;
2. we model the four-dimensional finite- $T$  transition by a three-dimensional effective field theory;
3. we assume the chiral phase transition to be described by an effective model written using a gauge-invariant order parameter, the simplest choice being the chiral condensate matrix;
4. we write down the most general  $\phi^4$  effective Lagrangian for the order parameter compatible with the assumed chiral symmetries, and we study its renormalization group (RG) flow;
5. if infrared (IR) stable fixed points (FPs) of the RG-flow exist, we conclude that the phase transition can be a continuous one, otherwise it has to be discontinuous.

In the first point we have to assume the phase transition to be continuous since universality arguments can only be applied to continuous transitions, and a discontinuous one can *never* be excluded by such an argument (see also point 5). It is instead possible to exclude (modulo the assumption 3 to be discussed in a moment) the presence of a continuous transition if no IR-stable FPs of the effective model exist. We however have to be cautious in drawing such a conclusion, since the absence of IR-stable FPs could also be a consequence of the approximation scheme adopted to study the RG flow: for example the leading order  $\epsilon$ -expansion computation performed in [1] found no IR-stable FPs when the  $U_A(1)$  symmetry is explicitly broken, FPs that were instead identified by subsequent more refined analyses (see e. g. [2]).

The fundamental hypothesis in the Pisarski-Wilczek analysis is assumption 3: they assume that the effective theory is associated with a gauge-invariant order parameter. Of course, an operator has to be gauge invariant to have a non-vanishing expectation value, e. g., in the low-temperature phase (like the chiral condensate), but here the question is different: can the effective theory describing the transition be defined using only gauge-invariant (composite) operators? If the effective Lagrangian is written by using only gauge-invariant local composite operators, then (gauge-invariant) gauge field correlators are noncritical, so the assumption 3 of the previous list is equivalent to assuming the irrelevance of the gauge modes at the transition. The possibility of unconventional critical points, in which gauge fields develop critical correlations, has been put forward in the condensed-matter community (see, e. g., [3]), and the only way of deciding which case is physically realized (i.e., if

gauge fields are relevant or irrelevant) is to compare the predictions of the universality arguments with the results of numerical simulations or experiments.

A precise numerical estimate of the critical exponents of the chiral phase transition in QCD is an extremely complicated task, and in some cases even the nature of the transition (i.e., continuous or discontinuous) is still debated, see, e. g., [4]. To gain some insight into this class of problems, it is thus convenient to switch to a simpler class of systems. We have considered 3D multiflavor scalar models. For these models universality arguments analogous to those by Pisarski and Wilczek can be carried out, and simulations can be easily performed by using local Monte Carlo algorithms, providing precise numerical estimates of the critical properties.

## 2. The lattice model

The basic variables of the model we will study are  $N_c \times N_f$  complex matrices  $Z_x^{cf}$ , where the first index is a ‘‘color’’ index and the second is a ‘‘flavor’’ one. For concreteness, in this proceeding we will present results only for the simplest maximally symmetric case, whose action is written in the form ( $\mu = 1, 2, 3$ )

$$S_g = -\beta N_f \sum_{x,\mu} \text{Re Tr} \left[ Z_x^\dagger U_{x,\hat{\mu}} Z_{x+\hat{\mu}} \right] - \frac{\beta_g}{N_c} \sum_{x,\mu>\nu} \text{Re Tr} \square_{x,\mu\nu}, \quad \text{Tr} Z_x^\dagger Z_x = 1, \quad (1)$$

where  $x$  stands for the site of a 3D cubic lattices,  $U_{x,\hat{\mu}} \in \text{SU}(N_c)$  is the lattice gauge fields and the symbol  $\square_{x,\mu\nu}$  denotes the plaquette in position  $x$  laying in the  $(\mu, \nu)$  plane. The symmetry of this model is maximal, meaning that in the ungauged limit  $U_{x,\hat{\mu}} \rightarrow 1$  (i.e., for  $\beta_g \rightarrow \infty$  in the thermodynamic limit) the action is  $\text{O}(2N_c N_f)$  symmetric, as can be seen by writing explicitly the real and imaginary parts of  $Z_x^{cf}$ . Note, however, that, to obtain an  $\text{SU}(N_c)$  gauge theory, it is sufficient to start from a scalar model with  $\text{U}(N_c) \times \text{U}(N_f)$  symmetry, which can be obtained by adding to  $S_g$  a quartic term proportional to  $\text{Tr} (Z_x^\dagger Z_x)^2$ ; we will comment in the final section on the results obtained when such a term is also present.

The action  $S_g$  is invariant under the local transformation  $Z_x \rightarrow G_x Z_x$ ,  $U_{x,\hat{\mu}} \rightarrow G_x U_{x,\hat{\mu}} G_{x+\hat{\mu}}^\dagger$ , where  $G_x \in \text{SU}(N_c)$ , and under the global transformation  $Z_x \rightarrow Z_x M$ ,  $U_{x,\hat{\mu}} \rightarrow U_{x,\hat{\mu}}$  where  $M \in \text{U}(N_f)$ . The two-color case is somehow peculiar for what concerns the global symmetry: since  $\text{SU}(2)$  is pseudo-real, it can be shown that for  $N_c = 2$  the global symmetry group of  $S_g$  is not  $\text{U}(N_f)$  but the symplectic group  $\text{Sp}(N_f)$  (subgroup of  $\text{U}(2N_f)$ ), see [5] for more details.

The global symmetry  $\text{U}(N_f)$  (or  $\text{Sp}(N_f)$ ) of this model can be spontaneously broken, and to identify an effective model for the transition we follow the Pisarski-Wilczek analysis of the finite- $T$  chiral transition. The simplest gauge-invariant order parameter for the  $\text{U}(N_f)$  symmetry is

$$Q_x^{fg} = \sum_a \bar{Z}_x^{af} Z_x^{ag} - \frac{1}{N_f} \delta^{fg}, \quad (2)$$

which is an hermitian traceless matrix, and under the global symmetry transforms according to  $Q_x \rightarrow M^\dagger Q_x M$ . In fact  $Q_x$  is an order parameter for the breaking of  $\text{SU}(N_f)$  and not  $\text{U}(N_f)$ , since it is blind with respect to the global  $\text{U}(1)$ , see [5] for a thorough discussion of the remaining  $\text{U}(1)$  symmetry. For the particular case  $N_c = 2$  one can introduce an order parameter for  $\text{Sp}(N_f)$  which is very similar to  $Q_x$ , see [5] for its explicit expression and its relation to  $Q_x$  and to the  $\text{U}(1)$

	$N_c = 2$	$N_c > 2$
$N_f = 2$	O(5) or 1 <sup>st</sup> order	O(3) or 1 <sup>st</sup> order
$N_f > 2$	1 <sup>st</sup> order	1 <sup>st</sup> order

**Table 1:** Universality class predicted by the effective model with gauge invariant order parameter Eq. (3).

symmetry. Note that for  $N_f = 1$  the order parameter  $Q_x$  identically vanishes due to the fixed length constraint of the scalar fields; this is consistent with know rigorous results stating that for  $N_f = 1$  a single thermodynamic phase is present in the model [6].

We can now write the most general Lagrangian containing up to 4<sup>th</sup>-order powers of  $Q_x$  (more precisely of its coarse-grained continuum counterpart  $Q(x)$ ) and invariant under the global symmetry:

$$\mathcal{L} = \text{Tr}(\partial_\mu Q)^2 + r \text{Tr} Q^2 + w \text{Tr} Q^3 + u (\text{Tr} Q^2)^2 + v \text{Tr} Q^4. \quad (3)$$

For  $N_f > 2$ , a cubic term is present,  $\text{Tr} Q^3 \neq 0$ , so that a first-order phase transition is expected. For  $N_f = 2$  and  $N_c > 3$ ,  $\mathcal{L}$  reduces to the effective action of the 3D O(3) universality class. Finally for  $N_f = 2$  and  $N_c = 2$  we obtain (using the appropriate order parameter, see [5]) the effective action of the 3D O(5) universality class; this is consistent with the isomorphism  $\text{SO}(5) = \text{Sp}(2)/\mathbb{Z}_2$ . The universality class predicted by the gauge-invariant order parameter effective Lagrangian are thus the ones reported in Tab. 1.

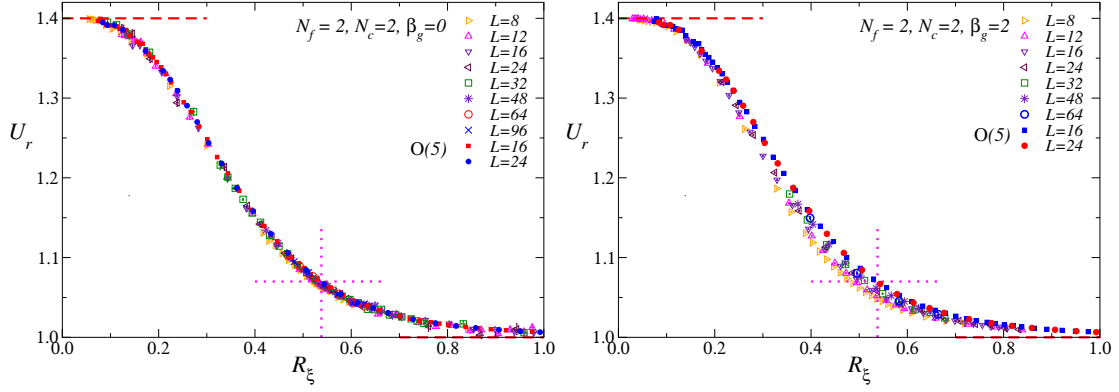
### 3. Numerical results

To identify the universality class of the transition of the model in Eq. (1) for some values of the parameters  $N_c$  and  $N_f$ , we performed finite-size scaling (FSS) analyses of observables related to the order parameter  $Q_x$  introduced in Eq. (2). In particular, using the notation  $G(\mathbf{x} - \mathbf{y}) = \langle \text{Tr}(Q_x Q_y) \rangle$  for the two-point function, we monitored the susceptibility  $\chi = \sum_x G(\mathbf{x})$ , the second-moment finite-volume correlation length  $\xi$  and the Binder cumulant  $U$ , defined by

$$\xi^2 = \frac{1}{4 \sin^2(\pi/L)} \frac{\tilde{G}(\mathbf{0}) - \tilde{G}(\mathbf{p}_m)}{\tilde{G}(\mathbf{p}_m)}, \quad U = \frac{\langle \mu_2^2 \rangle}{\langle \mu_2 \rangle^2}, \quad \mu_2 = \frac{1}{V^2} \sum_{x,y} \text{Tr} Q_x Q_y, \quad (4)$$

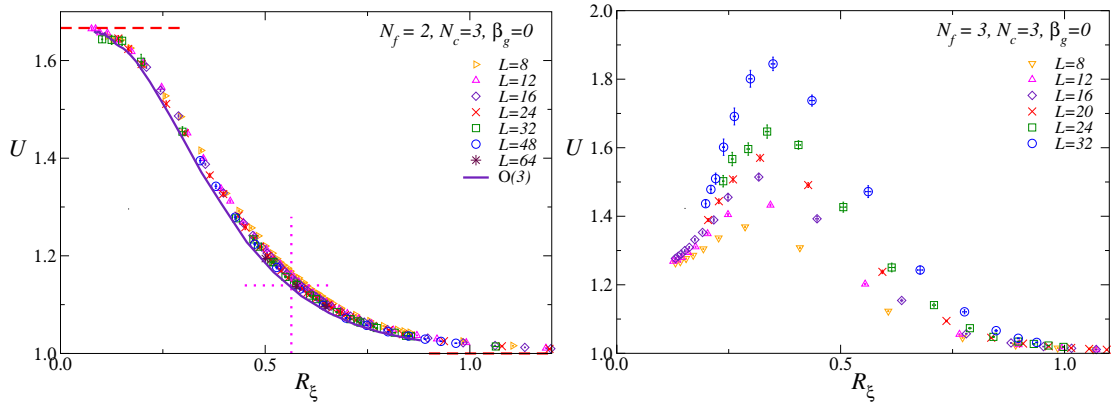
where  $\tilde{G}$  denotes the Fourier transform,  $L$  is the lattice size and  $\mathbf{p}_m$  is the minimum value of the momentum consistent with the periodic boundary conditions.

Keeping  $\beta_g$  fixed, we scanned in  $\beta$  the phase diagram of the model. Since the quantities  $U$  and  $R_\xi = \xi/L$  are RG invariants, close to a continuous transition they scale as  $\approx f_{U/R_\xi}(X)$ , where  $X = (\beta - \beta_c)L^{1/\nu}$ ,  $\beta_c$  is the critical value of the coupling  $\beta$ ,  $\nu$  is the thermal critical exponent and  $f_{U/R_\xi}$  is a function universal up to a multiplicative rescaling of its argument (we neglected scaling corrections for the sake of the simplicity). Since we aim to test the predictions in Tab. 1 against numerical results, it is particularly convenient to plot  $U$  as a function of  $R_\xi$  instead of  $\beta$  (or  $X$ ): in this way we obtain the scaling law  $U(\beta, L) \approx F_U(R_\xi)$ , where  $F_U$  is an universal function independent of any non-universal rescaling factor. It is thus easy to compare in a completely unbiased way the data obtained by simulating the model in Eq. (1) with those of the expected universality class.



**Figure 1:** Comparison of the scaling of the Binder cumulant (for the Sp(2) order parameter, denoted by  $U_r$ ) as a function of  $R_g$  for the  $N_c = 2$ ,  $N_f = 2$  model and the O(5) model. (left)  $\beta_g = 0$  and (right)  $\beta_g = 2$ . Dotted lines denote the known critical values of  $U$  and  $R_g$  for the O(5) model.

Such a comparison is performed in Fig. 1 between the data of the model with  $N_c = 2$ ,  $N_f = 2$  (for two different values of  $\beta_g$ ) and those of the 3D O(5) model; note that in this case the order parameter for Sp(2) has to be used. Both for  $\beta_g = 0$  and  $\beta_g = 2$  the data of the gauge model approach those of the expected universality class as the lattice size increases. An analogous comparison is reported in Fig. 2 (left), from which we infer that the transition of the model with  $N_c = 3$  and  $N_f = 2$  belongs to the 3D O(3) universality class, as expected from Eq. (3). More tests are discussed in [5], which fully support the hypothesis that the transition of the model with  $N_f = 2$  is continuous and belongs to the O(5) and O(3) universality classes for  $N_c = 2$  and  $N_c = 3$ , respectively (for all  $\beta_g$  values investigated). Finally, for  $N_f = 3$  we found indications of a discontinuous transition both for  $N_c = 2$  and  $N_c = 3$ . While the latent heat is too small to clearly identify the typical linear dependence on the volume of the susceptibilities, a strong indication favoring the discontinuous nature of the transition is the absence of scaling of  $U$  versus  $R_g$ , with  $U$  that seems to diverge as the lattice size is increased, see Fig. 2 (right) for  $N_c = 3$ ,  $N_f = 3$ .



**Figure 2:** (left) Comparison of the scaling of the Binder cumulant as a function of  $R_g$  for the  $N_c = 3$ ,  $N_f = 2$  model and the O(3) model. Dotted lines denote the known critical values of  $U$  and  $R_g$  for the O(3) model. (right) Behaviour of  $U$  as a function of  $R_g$  for the  $N_c = 3$ ,  $N_f = 3$  model.

#### 4. Conclusions and perspectives

We discussed the universality class of the transitions of the lattice 3D multiflavor (i.e.  $N_f > 1$ ) scalar gauge models. We first of all presented the results of a theoretical analysis similar to the one by Pisarski and Wilczek [1] for the chiral phase transition in QCD, emphasizing the assumptions that are implicit in such an analysis. We then examined the numerical lattice data obtained by simulating the lattice action in Eq. (1), which provide compelling numerical evidence for the correctness of the predictions based on the effective action in Eq. (3) for the cases studied. All the data presented in this proceeding refers to the lattice action in Eq. (1), in which scalar fields transform according to the fundamental representation of the gauge group and no quartic coupling is present.

A natural question to ask is whether the conclusions reached in this case remain valid also in more general contexts. The type of analysis discussed in the previous sections has been extended also to other cases, and specifically to the case of scalar fields transforming in the adjoint representation of the gauge group [7] and to the case in which a quartic term is also present (both in the adjoint [7] and in the fundamental [8] representation). In a subset of the region where  $\beta_g$  and the quartic coupling are positive (in the fixed-length limit negative couplings are also allowed) something different was in fact found: in the adjoint case with  $N_c = 2$  and  $N_f = 4$  a continuous transition incompatible with the expected 3D O(4) universality class was found [7], while in the fundamental case with  $N_c = 2$  and  $N_f = 40$ , where Eq. (3) would predict a first-order transition, a continuous transition was identified [8].

Which universality classes describe the critical behaviours associated with these transitions? Since they are not compatible with the predictions based on the effective Lagrangian written in terms of gauge-invariant operators, Eq. (3), one expects them to be associated with critical gauge fields. The simplest guess for the effective action is then the *continuum* field theory for  $SU(N_c)$  gauge fields coupled to  $N_f$  scalar fields, transforming under the appropriate representation of the gauge group. Indeed, at leading order in the  $\epsilon$ -expansion, such a theory has an IR stable charged (i.e., with a nonvanishing gauge coupling) fixed point for a large enough number of scalar flavors (see [7, 8]). While the identification of these unconventional critical behaviours with the charged fixed points of the corresponding continuum theory seems the most natural choice (also in view of the analogous results for the U(1) gauge case [9]), further studies are required to support this identification, putting it on more solid ground, from both the theoretical and numerical point of view.

Finally, the relevance of these results for the case of theories with fermions, and in particular for the finite-temperature chiral QCD transition, needs to be better understood. A leading-order computation in the  $\epsilon$ -expansion shows that the continuous field theory of  $SU(3)$  gauge fields coupled to  $N_f$  Dirac fermions has an IR-stable fixed point for  $N_f > N_f^* = 33/2$ . However, these leading-order estimates typically largely overestimate the critical number of flavors  $N_f^*$ . For instance, in the Abelian case, a charged fixed point exists for  $N_f \gtrsim 183$ , close to four dimensions, while in three dimensions  $N_f^*$  is significantly smaller: numerically one finds  $N_f^*(3D) = 7(2)$  (see [9]). Could it be that the continuum three-dimensional QCD has an IR-stable fixed point relevant for the finite-temperature transition of four-dimensional quantum chromodynamics?

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