Breaking the gauge symmetry in lattice gauge-invariant models

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We consider the role that gauge symmetry breaking terms play on the continuum limit of gauge theories in three dimensions. As a paradigmatic example we consider scalar electrodynamics in which N_f complex scalar fields interact with a U(1) gauge field. We discuss under which conditions a gauge-symmetry breaking term destabilizes the critical behavior (continuum limit) of the gauge-invariant theory. We find that the gauge symmetry is robust at transitions at which gauge fields are not critical. At charged transitions, where gauge fields are critical, gauge symmetry is lost as soon as the perturbation is added.

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1. Introduction

Gauge symmetries play a fundamental role in the description of microscopic phenomena, both in high-energy [1] and condensed-matter physics [2, 3]. While, in the first case, models enjoy an exact gauge invariance—the existence of an exact gauge symmetry is a basic tenet in the description of fundamental interactions—in the second case, it may happen that the symmetry is not an exact property of the microscopic system. It emerges at continuous transitions and it only characterizes the long-distance (or the low-energy in the quantum setting) behavior of the system. Of course, this is possible only if the microscopic gauge-symmetry breaking (GSB) terms are irrelevant, in the renormalization-group sense, at the transition. For this reason, it is important to understand the role that GSB terms play when added to gauge-invariant models. This issue is also crucial in the context of analog quantum simulations, when the interactions in atomic systems are engineered to effectively reproduce the dynamics of gauge-symmetric models, see Refs. [4, 5] and references therein.

In this talk, we will discuss recent results [6, 7] on the effects of GSB terms in 3D lattice gauge models with U(1) Abelian gauge invariance. Some of the considerations presented here, however, apply also to non-Abelian models.

2. Critical transitions in lattice gauge models

In this section we will briefly discuss the role that a local gauge symmetry plays at 3D continuous transitions. The considerations are general and apply both to Abelian and non-Abelian gauge models. Let Φ^A be a complex scalar field that transforms as $\Phi^A \to \tilde{W}^{AB}(g)\Phi^B$ under a unitary representation W of a group G, g being an element of G. A G-invariant scalar lattice model can be defined by the Hamiltonian (action)

$$H = \text{Re} \sum_{x\mu, A} \Phi_x^{A*}\Phi_{x+\hat{\mu}}^A + \sum_x V(\Phi_x^2), \tag{1}$$

where the first sum is over all lattice links (μ labels the lattice directions), $\Phi_x^2 = \sum_A \Phi_x^{A*}\Phi_x^A$, and $V(x)$ is a generic potential. By construction, this Hamiltonian is invariant under global G transformations.

A gauge model is obtained by selecting a subgroup $G' \subset G$ and by associating group elements $U_{x,\mu} \in G'$ to each link. If $\tilde{U}_{x,\mu}$ corresponds to $U_{x,\mu}$ in the representation under which Φ transforms, we obtain the Hamiltonian

$$H = \text{Re} \sum_{x\mu, AB} \Phi_x^{A*}\tilde{U}_{x,\mu}^{AB}\Phi_{x+\hat{\mu}}^B + \sum_x V(\Phi_x^2) + \gamma\text{Re} \sum_{x,\mu<\nu} \Pi_{x,\mu\nu}, \tag{2}$$

where $\Pi_{x,\mu\nu}$ is the plaquette in the $\mu\nu$ plane built in terms of the elements $U_{x,\mu}$. The new model is invariant under local transformations belonging to the group G'. As as example, in Sec. 3 we will consider the compact Abelian-Higgs model, in which G is the U(N_f) group, G' is the U(1) subgroup, and the fields transform under the fundamental representation of U(1). In this case $U_{x,\mu} = \exp(i\theta_{x,\mu})$, where $\theta_{x,\mu}$ is a real number in $[0, 2\pi[$ and $\tilde{U}_{x,\mu} = U_{x,\mu}$.

Our extensive work on models with Hamiltonian (2) shows [8–14] that phase transitions occurring in gauge models can be divided into two broad classes. First of all, there are transitions

where only scalar-matter correlations are critical. Gauge variables do not display long-range correlations, although their presence is crucial to identify the gauge-invariant scalar-matter critical degrees of freedom. At these transitions, gauge fields prevent non-gauge invariant scalar correlators from acquiring nonvanishing vacuum expectation values and developing long-range order: the gauge symmetry hinders some scalar degrees of freedom—those that are not gauge invariant— from becoming critical. The lattice Abelian-Higgs model with compact gauge fields and unit-charge N_f-component scalar fields is an example of this type of behavior [8].

A second class of transitions is instead characterized by the presence of long-range gauge correlations. They are expected to correspond to the stable fixed points with nonvanishing gauge couplings (we will name them charged fixed points) that occur in the statistical field theories that are obtained in the formal continuum limit, i.e., that have the same field content and the same global and local symmetries. At present, this type of transitions have been observed in Abelian models—specifically, in the lattice Abelian-Higgs model with noncompact fields [13] or with compact doubly-charged fields [12]—and in an SU(2) gauge model with SU(N_f) global invariance [14].

At transitions that occur for $\gamma = 0$, i.e., in the absence of the plaquette term in Eq. (2), gauge fields are noncritical. Indeed, for $\gamma = 0$, gauge fields can be exactly integrated out. If we define

$$e^{-\beta G(\Phi_1, \Phi_2; \beta)} = \int d\tilde{U} \exp\left[-\beta \text{Re} \sum_{AB} \Phi_1^{A*} \tilde{U}^{AB} \Phi_2^B\right],\tag{3}$$

the model with Hamiltonian

$$H = \sum_{x\mu} G(\Phi_x, \Phi_{x+\hat{\mu}}; \beta) + \sum_x V(\Phi_x^2),\tag{4}$$

is equivalent to the original one, as long as we consider observables that only depend on the scalar field. Gauge invariance is still present—the function $G(\Phi_1, \Phi_2; \beta)$ does not vary if we perform gauge transformations on Φ_1 and Φ_2. This is due to the fact that the nearest-neighbor coupling $G(\Phi_1, \Phi_2; \beta)$ can be expressed in terms of gauge-invariant combinations of the local fields that play the role of order parameters. For instance, in the Abelian-Higgs U(1) case we mentioned above, in the London limit ($\Phi_x^2 = 1$), we obtain

$$\int d\theta \exp\left[-\beta \text{Re}(\Phi_1^* e^{i\theta} \Phi_2)\right] = I_0(\beta\sqrt{X}) \qquad X = \sum_{AB} Q_1^{AB} Q_2^{BA} + 1/N_f,\tag{5}$$

Here $I_0(x)$ is a modified Bessel function, which satisfies $I_0(x) = 1 + x^2/4 + O(x^4)$ for small x, and Q^{AB} is a gauge-invariant bilinear operator

$$Q^{AB} = \Phi^{A*}\Phi^B - \frac{1}{N_f}\delta^{AB}.\tag{6}$$

Thus, the original model is equivalent to a model with

$$H_1 = -\frac{1}{\beta} \sum_{x\mu} \ln I_0\left[\beta \sum_{AB} Q_x^{AB} Q_{x+\mu}^{BA} + \frac{\beta}{N_f}\right].\tag{7}$$
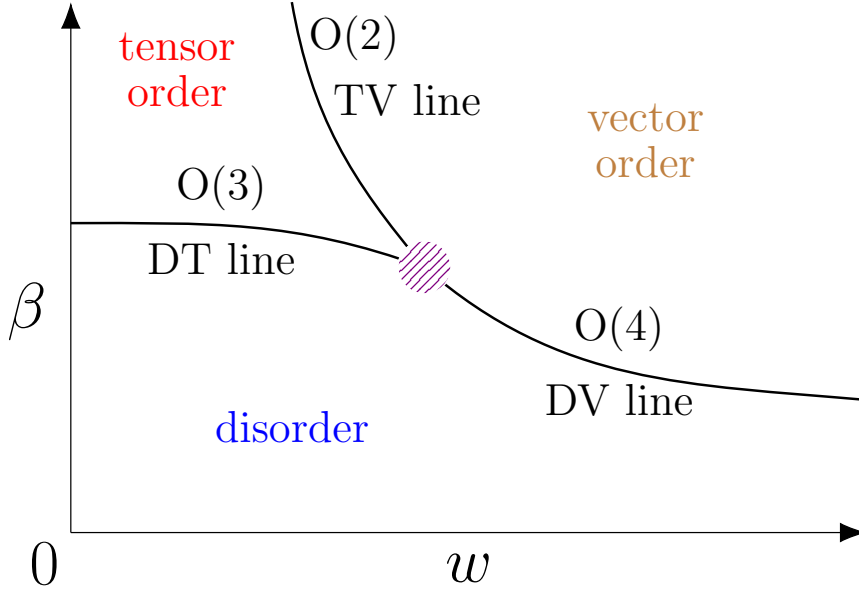
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Figure 1: Sketch of the phase diagram of the compact lattice Abelian Higgs model with $N_f = 2$, in the presence of the GSB term $H_{GSB} = -w \sum_{x,\mu} \text{Re} \, U_{x,\mu}$ for a fixed value of γ. The phase diagram is characterized by three different phases: a disordered phase (small β), a tensor-ordered phase where the tensor operator Q condenses (large β and small w), and a vector-ordered phase where the vector field Φ_x condenses (large β and w). These phases are separated by the disordered-tensor (DT), disordered-vector (DV), and (tensor-vector) TV transition lines, where $CP^1/O(3)$, O(4) vector, and O(2) vector critical behaviors are observed.

Exact gauge invariance is due to the fact that the Hamiltonian only depends on the gauge-invariant operator Q. In this case the critical behavior or continuum limit is driven by the condensation of the Q operators that play the role of fundamental fields in the Landau-Ginzburg-Wilson theory that should provide an effective description of the critical dynamics [8]. In the effective model, no gauge fields are considered.

3. The role of GSB terms at transitions with noncritical gauge fields

Let us now consider the role played by GSB terms, considering the Abelian-Higgs model with N_f flavors in the London limit ($\Phi_x^2 = 1$). As discussed in Ref. [8], in this model gauge fields are never critical: for instance, the phase behavior is independent of the value of γ. For each γ, two different phases occur as β is varied: for small β there is a disordered phase, while at large β there is an ordered phase, in which the bilinear gauge-invariant field Q defined in Eq. (6) condenses. We call this phase tensor-ordered. Because of gauge invariance, vector correlations of the fundamental field are ultralocal, i.e., $\langle \Phi_x^{A*} \Phi_y^A \rangle = \delta_{x,y}$, so that there is no vector order.

Let us now add the GSB term

$$H_{\text{GSB}} = -w \sum_{x\mu} \text{Re} \, U_{x\mu} \tag{8}$$

to the Hamiltonian. The model was studied in Ref. [7] for $N_f = 2$, obtaining the phase diagram shown in Fig. 1. For small w we have a low-temperature that only displays tensor order as for

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$w = 0$. The corresponding order-disorder transition is the same as in the gauge-invariant model. Only for large values of β does the nature of the low-temperature change. In this case, we have vector order, i.e., vector correlations of the fundamental field are long-ranged. Tha analysis of Ref. [7] shows therefore that the GSB term is irrelevant, in the renormalization-group sense, at the transitions occurring in the gauge-invariant model: the gauge-invariant behavior is robust under small perturbations.

We wish now to present an argument that shows that this results is a general property of GSB perturbations at transitions where gauge fields are not critical. Indeed, let us consider the partition function of a generic model with a GSB perturbation that only depends on the gauge fields:

$$Z = \int [dU_{x\mu} d\Phi_x] \exp\left(-\beta H - \beta H_{\text{GSB}}[\{U_{x\mu}\}]\right), \tag{9}$$

where H is gauge invariant. We now perform a change of variables—therefore Z does not change—on the scalar and gauge fields that corresponds to a gauge transformation. In particular, we redefine $U_{x\mu} \to V_x U_{x\mu} V_{x+\mu}^\dagger$. As H is gauge invariant, the partition function becomes

$$Z = \int [dU_{x\mu} d\Phi_x] \exp\left(-\beta H - \beta H_{\text{GSB}}[\{V_x U_{x\mu} V_{x+\mu}^\dagger\}]\right). \tag{10}$$

The partition function does not depend on the set of variables V_x and thus we can integrate over them without changing the partition function. We define

$$e^{-H_2[(\{U_{x\mu}\}]} = \int [dV_x] \exp[-\beta H_{\text{GSB}}[\{V_x U_{x\mu} V_{x+\mu}^\dagger\}], \tag{11}$$

and a new Hamiltonian $H' = H + H_2$. The new Hamiltonian is gauge invariant and equivalent to the original one, if we consider the partition function and, more generally, any gauge-invariant correlator. The Hamiltonian H_2 contains interactions between fields $U_{x,\mu}$ and $U_{y,\nu}$ at any distance $|x - y|$. However, for small βw these interactions are exponentially suppressed for $|x - y| \to \infty$, and thus H' represents a gauge-invariant model with short-range interactions. To prove this crucial point, note that, if βw is small, one can compute H_2 by performing a strong-coupling expansion. In this way, H_2 is written as a sum of lattice loops. In the expansion, a lattice loop of length L is weighted by a factor that behaves as $(\beta w)^L$ for $\beta w \to 0$. For instance, the leading term is the plaquette, with a weight of order $(\beta w)^4$, which renormalizes the value of γ. The next term corresponds to the 2×1 plaquette, with a coefficient proportional to $(\beta w)^6$, and so on. Couplings therefore scale as $\exp[-a|x - y|]$, with $a \sim -\log(\beta w)$, proving the short-range nature of the interactions.

This argument proves that, for small values of w, the partition function and gauge-invariant correlations can be computed in an equivalent gauge-invariant theory, without GSB terms, with short-range interactions. Finally, to conclude the argument, let us note that we are considering a model in which gauge fields do not play any role, i.e., the critical behavior is independent of the gauge-field interactions: it is the same as in the original model with $\gamma = 0$, finally proving that the phase structure is independent of w. Note that the argument does not rely on the Abelian nature of the theory, and thus is should also hold in non-Abelian models.

4. The role of GSB terms at charged transitions

We will now discuss the role of GSB terms at charged transitions [6], considering the non-compact Abelian-Higgs model with U(N_f) global invariance. The fundamental gauge field is a real field $A_{\boldsymbol{x}\mu}$ defined on the lattice links. In the London limit $\Phi_{\boldsymbol{x}}^2 = 1$, the Hamiltonian is

$$H_{nc} = \text{Re} \sum_{\boldsymbol{x}\mu} \Phi_{\boldsymbol{x}}^* \cdot \Phi_{\boldsymbol{x}+\hat{\mu}} U_{\boldsymbol{x},\mu} + \gamma \sum_{x,\mu<\nu} (\nabla_\mu A_{\boldsymbol{x}\nu} - \nabla_\nu A_{\boldsymbol{x}\mu})^2, \tag{12}$$

where $U_{\boldsymbol{x},\mu} = \exp(iA_{\boldsymbol{x}\mu})$, $\nabla_\mu f(\boldsymbol{x}) = f(\boldsymbol{x}+\hat{\mu}) - f(\boldsymbol{x})$, and $\Phi_{\boldsymbol{x}}$ is an N_f-dimensional unit-length complex vector as before. For $N_f \geq N_f^*$, $N_f^* = 7(2)$, and for a sufficiently small gauge coupling (i.e., for γ large enough), the model undergoes a transition that is associated with the charged fixed point of the corresponding field theory [13]. As expected, such a transition is not present for small values of γ, i.e., when gauge fields are supposed to play no role (as we already stressed, for $\gamma = 0$ they can be integrated out).

The noncompact nature of the fields and of the gauge invariance group (the additive group of the real numbers replaces here the compact U(1) group) makes the discussion more complex than for the compact model. Indeed, since the fields are unbounded, in the gauge-invariant model only gauge invariant correlations are well-defined. Therefore, one cannot study the question of the relevance of the GSB perturbations directly in the nonperturbed model. The way out of this problem is well known: a gauge fixing should be added to make all correlations well defined. Therefore, in the noncompact model one should consider both gauge-fixing terms and generic GSB perturbations.

In Ref. [6] we studied the effects of adding the perturbation

$$P_M = \frac{r}{2} \sum_{\boldsymbol{x}\mu} A_{\boldsymbol{x}\mu}^2 \tag{13}$$

to the Hamiltonian H_{nc} in the presence of two different gauge fixings. We considered the axial gauge fixing (AGF) $A_{\boldsymbol{x}3} = 0$, and a soft Lorentz gauge fixing (LGF), obtained by adding $H_{LGF} = \sum_{\boldsymbol{x}} \exp[-a(\sum \nabla_\mu A_{\boldsymbol{x}\mu})^2]$ to the Hamiltonian.

To characterize the strength of the perturbation P_M, we computed its RG dimension $y_r > 0$. This exponent provides information on how to scale r to keep GSB effects small. Indeed, when the correlation length ξ increases, approaching the continuum limit, one should decrease r faster than ξ^{-y_r} to ensure that GSB effects are negligible.

A numerical finite-size scaling study shows that the perturbation (13) is relevant at the charged fixed point occurring for $N_f \geq N_f^*$. This is not unexpected, as this term drastically changes the long-distance properties of the gauge-field correlations. In particular, the Coulomb phase that is present in these models disappears when P_M is added, since its addition corresponds to adding a photon mass to the model. Therefore, as soon as the perturbation is turned on ($r > 0$), the system flows out of the charged Abelian-Higgs fixed point.

However, the numerical estimates of the exponent y_r showed an unexpected dependence on the gauge-fixing procedure. For $N_f = 25$, we found $y_r = 2.55(5)$ for the model with AGF, and $y_r = 1.4(1)$ for the model with LGF [for two values of a, $a = 1$ and $a = 10$]. The dependence of the results on the gauge fixing is puzzling and is presently under investigation. One possibility is that the different results are not due to the fact that we are considering two different gauge fixings, the

AGF and the LGF. Rather, they may be the result of the different procedure used. In the axial case, field configurations satisfy the condition $A_{\boldsymbol{x},3} = 0$, while in the Lorentz case, the gauge-fixing term is added to the Hamiltonian, as usually done in perturbation theory, without requiring the stronger condition $\sum_\mu(\nabla_\mu A_{\boldsymbol{x}\mu}) = 0$, which would correspond to $a = \infty$. Although this possibility might seem unlikely to perturbation-theory practitioners, the noncommutativity of the infinite-volume limit and of the limit $a \to \infty$ was already noticed in Ref. [15]. They considered the one-component Abelian Higgs model and proved that the infinite-volume average value of the scalar field—this is the expected order parameter—in the Lorentz gauge behaves differently for finite a and for $a = \infty$.

5. Conclusions

In this talk we have presented our recent results on the role of GSB perturbations in gauge-invariant systems. At transitions in which gauge fields are not critical, the gauge symmetry is robust against GSB perturbations. If the GSB coupling is small, we still observe the same critical behavior as in the gauge-invariant model. In particular, the transition is still driven by the condensation of gauge invariant observables that play the role of effective order parameters.

At charged transitions (the ones where gauge fields are critical), instead, GSB perturbations are relevant. The addition of a GSB term drives the system out of the charged fixed point. We have studied this issue in the noncompact Abelian Higgs model, in which gauge-dependent observables can only be computed once a proper gauge fixing is added. The unexpected result is that the renormalization-group dimension of the GSB perturbation depends on the gauge fixing procedure. This issue clearly requires additional work, that we hope to present at the next-year Lattice conference.

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