Sapienza
Università di Roma

Quantum Null Energy Condition, Loop Groups and Modular Nuclearity

Facoltà di Scienze Matematiche, Fisiche e Naturali
Dottorato di Ricerca in Matematica – XXXIV Ciclo

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A thesis submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Matematica

2021/2022

Thesis defended on March 2, 2022
in front of a Board of Examiners composed by:

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Quantum Null Energy Condition, Loop Groups and Modular Nuclearity
Ph.D. thesis. Sapienza – University of Rome

This thesis has been typeset by LaTeX and the Sapthesis class.

Version: April 5, 2022

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God may or may not play dice but she sure loves a von Neumann algebra

Vaughan Jones

Acknowledgments

Quello della ricerca è, come tanti altri, un mondo di contraddizioni. È un mondo che vuole idealmente porsi l'obiettivo di incrementare la conoscenza e quindi il benessere, ma in cui i ricercatori influenzano dati e risultati, e non solo perché costretti dal bisogno di costante produttività. È un mondo in cui si auspica l'utilizzo di un linguaggio universale, ma in cui è richiesta una specializzazione tale da rendere sfidante la comunicazione anche tra esperti dello stesso campo. È un mondo di eccellenze ma anche di tantissimo precariato, in cui ottenere una posizione lavorativa stabile prima dei 35 anni è un lusso, e in cui spesso prima di allora bisogna sottostare a contratti che non garantiscono dei diritti essenziali come l'indennità per congedo di maternità. È un mondo di solitudine, rimpianti e frustrazione, in cui non si è mai soddisfatti né dei propri risultati né dell'ambito di ricerca in cui si lavora, in cui sono richieste enormi capacità di adattabilità e autosufficienza, e in cui la sindrome dell'impostore sembra quasi essere un requisito necessario. Alla luce di quest'amara verità, le uniche armi in nostro possesso sono la solidarietà e l'affetto, ed è la riconoscenza per simili gesti e sentimenti a rendere importante la sezione dei ringraziamenti.

Il mio primo ringraziamento va al mio relatore Roberto Longo, per avermi concesso l'opportunità di lavorare sotto la sua supervisione. Ringrazio poi le persone con cui ho collaborato o anche solo discusso di matematica in questi tre anni: Henning Bostelmann, Daniela Cadamuro, Simone Del Vecchio, Alessio Ranallo, Yoh Tanimoto, Benedikt Wegener, solo per dirne alcuni. Ringrazio i miei genitori e mia sorella Valentina per il supporto e l'incoraggiamento, specialmente all'inizio di questo dottorato. Ringrazio i miei compagni di dottorato dell'Università di Roma "La Sapienza", per avermi aiutato a trascorrere con più leggerezza questi tre anni. Infine, per tutto il tempo trascorso insieme, ringrazio tutti i miei amici più stretti conosciuti all'infuori dell'ambito matematico: le persone che frequento sin dai tempi del liceo, tutti quelli di "Osservatorio" con inclusi gli altri compagni di Dungeons & Dragons, e tutte le altre persone a me care che ho conosciuto tramite queste. Se non vi nomino individualmente è solo per pigrizia, in quanto sarebbe una lista piacevolmente lunga.

Contents

Introduction

Entropy is a concept almost as old as thermodynamics, a quantity that measures the "disorder" and hence our "ignorance" of a given state of the system. When dealing with Quantum Information objects in Quantum Field Theory (QFT), the algebraic approach seems a good choice, since the tools regarding quantum entropy can be formulated very precisely in terms of operator algebras. Quantum information aspects of QFT naturally take place in the framework of quantum black holes thermodynamics, as for example the Bekenstein Bound and the Landauer's principle show [69, 70]. However, more unexpected and interesting connections between the relative entropy and the stress energy tensor have arisen, and in particular it is of interest to provide and prove an axiomatic formulation of the Quantum Null Energy Condition (QNEC).

Classically, the *Null Energy Condition (NEC)* is a constraint on the stress energy tensor stating that $T_{ab}k^a k^b \geq 0$, where k^a is a null vector field. This constraint is motivated by the positivity of the energy, and it is a necessary condition for the field k^a to have some physical meaning. However, quantum fields can violate all local energy conditions, including the NEC. At any point the energy density $\langle T_{kk}\rangle$ can be made negative, with magnitude as large as we wish, by an appropriate choice of a quantum system [43]. In the study of relativistic QFT coupled to gravity, Bousso, Fisher, Leichenauer and Wall [14] establish a new and surprising link between Quantum Information and the stress energy tensor. In this work, a Quantum Null Energy Condition is defined as a null energy lower bound which is expected to be satisfied by most reasonable quantum fields. Informally, this formulation of the QNEC can be described as follows. Given a null plane N and Cauchy surface C in the Minkowski space, denote by $\mathcal{R}$ one half of C obtained as the linear manifold N "cuts" the surface C. For a null direction v of N, one can deform $\mathcal{R}$ in the v-direction and define a family of regions $\mathcal{R}_t$, $t \in \mathbb{R}$. Denote by $S(t)$ the von Neumann entropy of some state ψ restricted to the region $\mathcal{R}_t$. The QNEC [14] states that, in natural units, every physical state ψ shall verify the inequality

$$\langle T_{vv}(t)\rangle \geq \frac{1}{2\pi} S''(t)\,. \tag{0.1}$$

Here T is the stress-energy tensor, $\langle T_{vv}(t)\rangle = \langle T_{vv}(p_t)\rangle_\psi$ is its density at some point p_t in $\mathcal{R}_t \cap N$ in the state ψ and $S''(t)$ is the second derivative of the von Neumann entropy of ψ with respect to the deformation parameter. However, this statement lacks mathematical rigour and a different entropy-type state functional is required in order to properly formulate the QNEC.

The first non-commutative entropy notion, von Neumann's quantum entropy, was originally designed as a Quantum Mechanics version of Shannon's entropy: if a normal state ψ is given by a density matrix ρ_ψ, then the von Neumann entropy is defined by

$$S(\psi) = -\mathrm{tr}\,\rho_\psi \log \rho_\psi\,.$$

However, in Quantum Field Theory local von Neumann algebras are typically factors of type III_1, hence no trace or density matrix exists. The von Neumann entropy can still be defined on these von Neumann algebras, but by the Connes-Störmer homogeneity theorem it results to be infinite for every state. Nontheless, the Tomita-Takesaki modular theory applies and if $\mathcal{M}$ is a von Neumann algebra in standard form then one may consider the relative entropy of Araki between two normal states φ and ψ, namely [82]

$$S(\varphi\|\psi) = -(\xi|\log \Delta_{\eta,\xi}\xi)\,.$$

Here ξ and η are the representing vectors in the natural cone of φ and ψ respectively and $\Delta_{\eta,\xi}$ is the relative modular operator. The relative entropy generalizes the classical Kullback-Leibler divergence and measures how ψ deviates from φ. From an information theoretical viewpoint, $S(\varphi\|\psi)$ is the mean value in the state φ of the difference between the information carried by the state ψ and the state φ. By using the Araki's relative entropy, a rigorous statement of the QNEC can be given as follows. Let $(\mathcal{A}, U, \Omega)$ be a local QFT on the Minkowski space $\mathbb{R}^{n+1}$ with vacuum state ω and C^*-algebra of quasi-local observables $\mathfrak{A}$. Consider unique the future-pointing null directions u tangent to the Rindler wedge W of equation $x_1 > |x_0|$. More generally, we can replace W with some deformed wedge W_V, where V is some non-negative continuous function on $\mathbb{R}^{n-1}$ (see Section 2.1), and then apply a Poincaré transformation g in order to define the deformed wedge $gW_V = g(W_V)$. If $V_t = (1+t)V$ and $\mathcal{M}_t = \mathcal{A}(gW_{V_t})$, we will say that a state ψ of $\mathfrak{A}$ satisfies the *Quantum Null Energy Condition (QNEC)* if the relative entropy $S(t) = S_{\mathcal{M}_t}(\psi\|\omega)$ is convex for any such couple (g, V). This formulation of the QNEC does not involve null energy lower bounds, but it has been found in [58] by physical arguments as a condition equivalent to (0.1).

The first chapter of this Ph.D. thesis is a collection of mathematical preliminaries about Operator Algebras and the general structure of the algebraic formulation of local QFT. This chapter contains a few personal remarks like Lemma 5, identity (1.8) and Proposition 24, but it is mostly just a summary of well known results.

In the second chapter we describe a particular class of local QFTs, namely that of $1+1$-dimensional chiral Conformal Field Theories (CFTs). The study of these models can be reduced to the study of local nets of von Neumann algebras parametrized by open intervals of a lightray. By Cayley transform, these models can be defined as local nets on the circle which are known as *conformal nets*. In this Ph.D. thesis we prove the QNEC for some solitonic states constructed in [34] on a generic conformal net. This is the main theorem of the second chapter. The proof relies on explicit computations on the Virasoro nets and on the use of vacuum preserving conditional expectations. In order to provide this proof, partial results of [51] have been of relevance. This theorem can be considered as a particular case of a very much more general theorem proved in [27], but our explicit proof allows us to add explicit formulae and other intermediate results of interest. For example, we can show that the solitonic states of [34] verify the Bekenstein Bound [69].

The third chapter is the one containing the most of the results of this Ph.D. thesis. We focus on conformal nets induced by vacuum Positive Energy Representations (PERs) π of a loop group $LG = C^\infty(S^1, G)$, where G is a compact, simple and simply connected Lie group. The first result of this chapter is the construction of some solitonic representations σ_γ of such conformal nets. In general, these solitons are induced by a path γ in $C^\infty([-\pi,\pi], G)$. If the path γ satisfies some periodicity conditions on its derivatives, then it can be extended to what we call a *discontinuous loop*. Discontinuous loops are defined as elements of

$$L_h G = \{\zeta \in C^\infty(\mathbb{R}, G) \colon \zeta(x)^{-1}\zeta(x+2\pi) = h\}\,,$$

where h is a generic element of G. If the discontinuity h of ζ is in the center $Z(G)$, then it is already known that the obtained soliton σ_ζ extends to a DHR representation which corresponds to a PER $\zeta_*\pi$ of same level as π [93]. What we show here is that this condition is also necessary: the soliton σ_ζ extends to a DHR representation if and only if the discontinuity h is central. The proof follows by a contradiction argument, since a locally normal DHR representation is automatically Rot-covariant [33]. The other main result is about Sobolev extensions of Positive Energy Representations. More specifically, we show that any PER of a loop group LG can be extended to a PER of the Sobolev loop group $H^s(S^1, G)$ for $s > 3/2$. In the case $G = SU(n)$, we can show that this is true even for $s > 1/2$ by using explicit constructions of [95]. Actually, results of this type had already been achieved in [80] in the more general context of semibounded representations, but the proof here presented is completely original. Such a Sobolev extension allows us to compute the adjoint action of $H^{s+1}(S^1, G)$ on the stress energy tensor, which is one more interesting result of this chapter. These technical results, together with some intermediate lemmas of [27], are used to prove the QNEC for the above solitonic states by explicitly computing the relative entropy. As in the previous chapter, the obtained formula allows us to prove the Bekenstein Bound in a very simple way. Finally, an alternative and simpler proof of the QNEC is provided in the case $G = SU(n)$.

The topic of the fourth and last chapter is *entanglement*. Entanglement is a typical quantum mechanical phenomenon giving rise to some randomness of the state on a bipartite system even without "lack of knowledge" as the state is restricted to its subsystems. States exhibiting such a behavior are called *entangled*. Entanglement has been profoundly investigated as a means of probing the very foundations of quantum mechanics (as in the EPR paradox and Bell's inequalities) as well as a resource for quantum information theory. In the operator algebraic language, an entanglement measure for a bipartite system is a state functional that vanishes on separable states and that does not increase under separable operations. For pure states, essentially all entanglement measures are equal to the von Neumann entropy of the reduced state, but for mixed states this uniqueness is lost. The role of entanglement in QFT is more recent and increasingly important [73]. It appears in relations with several primary research topics in theoretical physics as area theorems [52], c-theorems [25] and quantum null energy inequalities [76, 85].

Several approaches towards a rigorous definition of some entanglement entropy rely on some nuclearity conditions of the system. Explicitly, let $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O}) \subseteq B(\mathcal{H})$ be some local Haag net on the Minkowski space. Denote by Ω the vacuum vector and by ω the corresponding vacuum state. Given an inclusion $\mathcal{O} \subset \widetilde{\mathcal{O}}$ of spacetime regions, one says that the *modular nuclearity condition* holds if the map [19, 20]

$$\Xi\colon \mathcal{A}(\mathcal{O}) \to \mathcal{H}\,, \quad \Xi(x) = \Delta^{1/4} x\Omega\,, \tag{0.2}$$

is nuclear, with Δ the modular operator of the bigger local algebra $\mathcal{A}(\widetilde{\mathcal{O}})$ with respect to Ω. If the map (0.2) is p-nuclear then one will say that the *modular p-nuclearity condition* is satisfied. If modular p-nuclearity holds for some $0 < p \leq 1$ then the modular nuclearity condition is satisfied, and if so then it is well known that the *split property* holds, namely there is an intermediate type I factor $\mathcal{A}(\mathcal{O}) \subset \mathcal{F} \subset \mathcal{A}(\widetilde{\mathcal{O}})$. The existence of such an intermediate type I factor implies some statistical independence of the local algebras $\mathcal{A}(\mathcal{O})$ and $\mathcal{A}(\widetilde{\mathcal{O}}') \subseteq \mathcal{A}(\widetilde{\mathcal{O}})'$, since one has a spatial isomorphism $\mathcal{A}(\mathcal{O}) \vee \mathcal{A}(\widetilde{\mathcal{O}})' \cong \mathcal{A}(\mathcal{O}) \otimes \mathcal{A}(\widetilde{\mathcal{O}})'$ [71]. In the last chapter, based on [86], we gather and prove some results on this topic. In particular, we prove that if modular p-nuclearity holds for some $0 < p < 1$ then the mutual information is finite. In general, the mutual information for a bipartite system $A \otimes B$ is given by

$$E_I(\omega) = S(\omega \| \omega_A \otimes \omega_B)\,.$$

In the notation above, bipartite systems in QFT contexts are given by setting $A = \mathcal{A}(\mathcal{O})$ and $B = \mathcal{A}(\widetilde{\mathcal{O}}')$. Inspired by [83], we also prove a similar result for a different entanglement entropy. We then add a few additional remarks concerning area laws by applying results of [52]. Finally, we apply these considerations to a wide family of $1+1$-dimensional integrable models with factorizing S-matrices [60]. These models provide a very interesting example of local quantum field theories for which modular p-nuclearity holds for wedge algebras inclusion, which is no more true in higher dimension. In this context, in which nuclear norms have been estimated very sharply [2, 62], we briefly investigate the asymptotic behaviour of different entanglement measures as the distance between two causally disjoint wedges diverges.

Chapter 1

Mathematical background

1.1 Modular theory

This section aims to provide a very concise overview of some important results of the modular theory of operator algebras. The Tomita-Takesaki theorem is a powerful result which states that any von Neumann algebra $\mathcal{M}$ can be represented on a Hilbert space $\mathcal{H}$ on which $\mathcal{M}$ is anti-isomorphic to its commutant. The Tomita-Takesaki theory was successfully used by A. Connes to classify type III factors [29].

Theorem 1. *Let $\mathcal{M}$ be a σ-finite von Neumann algebra acting on a Hilbert space $\mathcal{H}$. If there is a standard vector Ω in $\mathcal{H}$, then there is an anti-unitary involution J such that*

$$J\mathcal{M}J = \mathcal{M}' \,, \quad JxJ = x^* \ \text{ if } x \in Z(\mathcal{M}) \,.$$

Explicitly, J is given by the polar decomposition $S = J\Delta^{1/2}$, where S is the closure of $S_0(x\Omega) = x^*\Omega$ for x in $\mathcal{M}$. Furthermore,

$$\sigma_t(x) = \Delta^{it} x \Delta^{-it} \,, \quad \sigma_t(x) = x \ \text{ if } x \in Z(\mathcal{M}) \,,$$

is a group of automorphisms of both $\mathcal{M}$ and $\mathcal{M}'$.

The operators J and Δ^{it} are respectively called *modular conjugation* and *modular operator*. The most general statement of the Tomita-Takesaki theorem is given by using the language of Hilbert algebras, and we refer to [90] for a full treatment of the topic. Without using all such an equipment, we can simply talk about standard forms. More precisely, if $\mathcal{M}$ is a von Neumann algebra then a *standard form* for $\mathcal{M}$ is a quadruple $(\mathcal{H}, \pi, \mathcal{P}, J)$ where $\mathcal{H}$ is a Hilbert space, π is a faithful normal representation of $\mathcal{M}$ on $\mathcal{H}$, J is an antiunitary involution and $\mathcal{P}$ is a closed cone of $\mathcal{H}$, such that the conditions

(i) $J\pi(\mathcal{M})J = \pi(\mathcal{M})'$,

(ii) $JxJ = x^*$, $x \in Z(\pi(\mathcal{M}))$,

(iii) $J\xi = \xi$, $\xi \in \mathcal{P}$,

(iv) $JxJx\mathcal{P} \subseteq \mathcal{P}$, $x \in \pi(\mathcal{M})$,

are satisfied. We will refer to J as the *modular conjugation* and to $\mathcal{P}$ as the *natural positive cone* of the standard form.

The standard form $(\mathcal{H}, \pi, \mathcal{P}, J)$ of $\mathcal{M}$ is unique in the following sense: if $(\widetilde{\mathcal{H}}, \widetilde{\pi}, \widetilde{\mathcal{P}}, \widetilde{J})$ is another standard form for $\mathcal{M}$, then there is a unique unitary operator u from $\mathcal{H}$ to $\widetilde{\mathcal{H}}$ which intertwines π with $\widetilde{\pi}$, J with $\widetilde{J}$ and such that $u\mathcal{P} = \widetilde{\mathcal{P}}$. If p is a projection of $\mathcal{M}$ and $q = pj(p)$, then $(q\mathcal{H}, \mathrm{Ad}\, q \cdot \pi, q\mathcal{P}, qJq)$ is a standard form of the reduced algebra $p\mathcal{M}p$. Below we describe the relations which connect these objects one each other and doing this we shall identify $\mathcal{M}$ with $\pi(\mathcal{M})$. Every positive normal functional φ of $\mathcal{M}$ has a unique representative vector ξ_φ in $\mathcal{P}$, and the map $\varphi \mapsto \xi_\varphi$ is norm continuous as shown by the estimate [90]

$$\|\xi_\varphi - \xi_\omega\|^2 \le \|\varphi - \omega\| \le \|\xi_\varphi - \xi_\omega\| \|\xi_\varphi + \xi_\omega\| \, .$$

Since the conjugate-linear $*$-isomorphism $j(x) = JxJ$ maps $\mathcal{M}$ onto its commutant $\mathcal{M}'$, a vector ξ in $\mathcal{P}$ is cyclic if and only if it is separating. Whenever φ is faithful, the set of vectors $xj(x)\xi_\varphi$ with x in $\mathcal{M}$ is dense in $\mathcal{P}$. Finally, the natural positive cone is self-dual in the following sense:

$$\xi \in \mathcal{P} \quad \text{if and only if} \quad (\xi|\eta) \ge 0 \text{ for every } \eta \in \mathcal{P} \, .$$

We now move to define and study the modular operators as in [27]. We consider two vectors ξ and η in $\mathcal{H}$, generally not in the natural cone, and we denote by φ and ψ the corresponding normal vector states. The supports of φ and ψ on $\mathcal{M}$ are given by $s(\varphi) = [\mathcal{M}'\xi]$ and $s(\psi) = [\mathcal{M}'\eta]$, while on the commutant we have $s'(\varphi) = [\mathcal{M}\xi]$ and $s'(\psi) = [\mathcal{M}\eta]$. We define the *Tomita relative operator*

$$S_{\xi,\eta}(x\eta + \zeta) = s(\psi)x^*\xi \, , \quad x \in \mathcal{M} \, , \; \zeta \in [\mathcal{M}\eta]^\perp \, .$$

This densely defined conjugate-linear operator is closable. Its closure will be equally denoted and its polar decomposition is given by

$$S_{\xi,\eta} = J_{\xi,\eta}\Delta_{\xi,\eta}^{1/2} \, ,$$

where

$$\operatorname{supp} \Delta_{\xi,\eta} = s(\varphi)s'(\psi) \, , \quad J_{\xi,\eta}^* J_{\xi,\eta} = s(\varphi)s'(\psi) \, , \quad J_{\xi,\eta}J_{\xi,\eta}^* = s'(\varphi)s(\psi) \, .$$

In the case $\xi = \eta$ we will write $S_\xi = S_{\xi,\xi}$, and similarly $J_\xi = J_{\xi,\xi}$ and $\Delta_\xi = \Delta_{\xi,\xi}$. If ξ and η are both in the natural cone, then we also have the polar decomposition

$$S_{\xi,\eta} = J\Delta_{\xi,\eta}^{1/2} \, ,$$

with J the modular conjugation. Finally, by using apices to denote the modular operators of the commutant, we have the identities

$$J_{\xi,\eta}\Delta_{\xi,\eta}^{1/2}J_{\xi,\eta} = \Delta_{\eta,\xi}^{-1/2} \, , \quad J'_{\xi,\eta} = J_{\eta,\xi} \, , \quad (\Delta'_{\eta,\xi})^z = \Delta_{\xi,\eta}^{-z} \, .$$

We now give a few very simple lemmas.

Lemma 2. *Let $\mathcal{M}$ be a von Neumann algebra on $\mathcal{H}$ and $u \in U(\mathcal{H})$ a unitary operator. Consider two vectors ξ and η of $\mathcal{H}$.*

(i) If ξ is standard for $\mathcal{M}$ then $u\xi$ is standard for $u\mathcal{M}u^*$.

(ii) $\Delta^{u\mathcal{M}u^*}_{u\xi,u\eta} = u\Delta^{\mathcal{M}}_{\xi,\eta}u^*$.

(iii) If $u = vv'$, with v and v' unitary operators in $\mathcal{M}$ and $\mathcal{M}'$ respectively, then $\Delta^{\mathcal{M}}_{\xi,u\eta} = v'\Delta^{\mathcal{M}}_{\xi,\eta}v'^*$.

(iv) If $u = vv'$ as in (iii), then $\Delta^{\mathcal{M}}_{u\xi,\eta} = v\Delta^{\mathcal{M}}_{\xi,\eta}v^*$.

(v) If v and w are isometries in $\mathcal{M}'$, then $\Delta^{\mathcal{M}}_{v\xi,w\eta} = w\Delta^{\mathcal{M}}_{\xi,\eta}w^*$.

Proof. (i) $\overline{u\mathcal{M}u^*(u\xi)} = \overline{\mathcal{M}\xi}$, so $u\xi$ is cyclic and the same holds for $u\eta$. Since the commutant of $u\mathcal{M}u^*$ is $u\mathcal{M}'u^*$, the assertion follows. (ii) The proof is standard: one first proves that $S^{u\mathcal{M}u^*}_{u\xi,u\eta} = uS_{\xi,\eta}u^*$ and then uses the fact that $\Delta_{\xi,\eta} = S^*_{\xi,\eta}S_{\xi,\eta}$. (iii) In this case we have that $u\mathcal{M}u^* = \mathcal{M}$. The thesis follows by noticing that $S_{\xi,u\eta} = vS_{\xi,\eta}v'^*$ and applying the definition of $\Delta_{\xi,u\eta}$. (iv) By applying (ii) and (iii), the statement follows. (v) As before, one can check the identity $S^{\mathcal{M}}_{v\xi,w\eta} = vS^{\mathcal{M}}_{\xi,\eta}w^*$, so that $\Delta^{\mathcal{M}}_{v\xi,w\eta} = w\Delta^{\mathcal{M}}_{\xi,\eta}w^*$ follows by definition. □

Lemma 3. *Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$. Let φ and ψ be two vector states on $B(\mathcal{H})$ respectively represented by the vectors ξ and η. Suppose that $\varphi|_{\mathcal{M}'} = \psi|_{\mathcal{M}'}$. Then there is an isometry u in $\mathcal{M}$ such that $u\xi = \eta$. Moreover, if $\varphi|_{\mathcal{M}}$ is faithful then u is unitary if and only if $\psi|_{\mathcal{M}}$ is faithful too.*

Proof. By hypothesis we have that $x'\xi + y \mapsto x'\eta + y$, with $x' \in \mathcal{M}'$ and y orthogonal to $\mathcal{M}'\xi$, is a well defined isometric map u. By construction $u\xi = \eta$ and $u \in \mathcal{M}'' = \mathcal{M}$. Suppose now $\varphi|_{\mathcal{M}}$ to be faithful, so that ξ is cyclic for $\mathcal{M}'$. By explicit computation one can notice that $u^*x'\eta = x'\xi$ for $x' \in \mathcal{M}'$. Therefore $uu^* = 1$ on $\mathcal{M}'\eta$ and the assertion follows. □

Corollary 4. *Let φ be a positive normal functional of a standard von Neumann algebra $\mathcal{M}$ represented by a vector ξ in the natural cone $\mathcal{P}$. If η is another vector representing φ then $\eta = v\xi$ for some isometry v in $\mathcal{M}'$, and if φ is faithful then v is unitary if and only if η is cyclic for $\mathcal{M}$.*

Proof. Just recall that the representing vectors in the natural positive cone $\mathcal{P}$ are cyclic if and only if they are separating and apply the previous lemma. □

Consider now two normal positive functionals ω and ψ of $\mathcal{M}$ respectively represented by vectors ξ and η of $\mathcal{H}$. Relative modular operators can be used to explicit compute the Connes cocycle between two normal states. Suppose $\mathcal{M}$ to be σ-finite and let φ be a normal faithful state represented by a vector ζ. The operator

$$\Delta^{z}_{\eta,\zeta}\Delta^{-z}_{\xi,\zeta}\,, \quad z \in \mathbb{C}\,,$$

is independent by the choice of the standard vector ζ representing φ. In our convention if A is a positive operator then A^z stands for the usual power A^z on $\operatorname{supp} A = (\ker A)^{\perp}$ and for 0 on $\ker A$. The *Connes cocycle* between ψ and ω is then defined by

$$(D\psi\colon D\omega)_t = \Delta^{it}_{\eta,\zeta}\Delta^{-it}_{\xi,\zeta}\,, \quad t \in \mathbb{R}\,.$$

If $s(\psi)$ and $s(\omega)$ commute then $(D\psi\colon D\omega)_t$ is a partial isometry with initial and final projection $s(\psi)s(\omega)$. Clearly $(D\psi\colon D\omega)_t^* = (D\omega\colon D\psi)_t$, and $(D\psi\colon D\omega)_t$ is a family of unitary operators if ω and ψ are both faithful. Always for ζ standard we have that

$$\sigma_t^\omega(x) = \Delta_{\xi,\zeta}^{it} x \Delta_{\xi,\zeta}^{-it}\,, \quad x \in \mathcal{M}\,, \tag{1.1}$$

maps $\mathcal{M}$ onto the reduced algebra $\mathcal{M}_{s(\omega)} = s(\omega)\mathcal{M}s(\omega)$. The family of operators σ^ω defines a group of automorphisms of $\mathcal{M}_{s(\omega)}$ and is called the *modular group* of the normal state ω. The group (1.1) extends the modular automorphism of $\mathcal{M}_{s(\omega)}$ mentioned in Theorem 1, where ω is faithful on $\mathcal{M}_{s(\omega)}$ by construction. Finally, we notice the cocycle relation

$$\sigma_s^\psi(x) = \mathrm{Ad}\,(D\psi\colon D\omega)_s \cdot \sigma_s^\omega(x)\,, \quad x \in \mathcal{M}\,.$$

As mentioned above, modular theory has been used by A. Connes to classify III-type factors into the III_λ-type factors, since III_λ-type factors exhibit a different S-invariant for different values of λ in $[0,1]$ [29, 90]. This phenomenon does not appear in I-type and II-type factors as in these cases the modular group σ_t^φ is inner for each parameter t and each normal faithful state (n.f.s.) φ. However, the S-invariant is not the only property to differently characterize III_λ-type factors. For example, one more phenomenon that shows up only in III_1-type factors is the existence of a n.f.s. with trivial centralizer, namely a n.f.s. φ such that the only elements fixed by σ^φ are the scalars. To be more precise, if a von Neumann algebra different from $\mathbb{C}$ admits such a state then it must be a III_1-type factor [71]. The converse implication, up to now, is not known to hold or not. The hyperfinite III_1-type factor admits infinitely many n.f.ss. with trivial centralizer which are dense in norm in the convex set of all the normal states. To show this one first shows the existence of one n.f.s. with trivial centralizer (namely the vacuum state on the von Neumann algebra generated by the CCR relations [16]), then applies the Connes-Störmer homogeneity theorem (Thm. 5.12. of [90]) and then recalls that n.f.ss. are norm dense in the space of normal states if the von Neumann algebra is properly infinite [37].

It can be shown that if ψ is a n.f.s. with trivial centralizer, then it is the only σ^ψ-invariant n.f.s. ([90], Corollary 3.6). Therefore, on such a space we have that the action of σ^ψ is ergodic. With the following lemma we further characterize this ergodicity property by showing that ψ is attractive with respect to the modular dynamics.

Lemma 5. *Consider two positive normal functionals φ and ψ on a von Neumann algebra $\mathcal{M}$. If ψ has trivial centralizer, then*

$$\varphi \cdot \sigma_t^\psi(x) \to \varphi(s(\psi))\psi(x)$$

for every $x \in \mathcal{M}$ as $|t| \to +\infty$.

Proof. We first notice that the restriction of φ on $\mathcal{M}_s = s\mathcal{M}s$, with $s = s(\psi)$, is still normal. Therefore, if the assertion holds with ψ faithful then the general case follows by noticing that

$$\varphi \cdot \sigma_t^\psi(x) = \varphi \cdot \sigma_t^\psi(sxs) \to \varphi(s)\psi(sxs) = \varphi(s)\psi(x)\,.$$

We now suppose ψ to be faithful, we assume $\mathcal{M}$ to be in standard form on a Hilbert space $\mathcal{H}$ and we denote by η the standard vector in the natural cone representing ψ. Set $\Delta^{it} = \Delta_\eta^{it}$. Since ψ has trivial centralizer, the vectors in $\mathbb{C}\eta$ are the only Δ^{it}-invariant vectors (see [50], Thm. §2). Therefore, if P is the orthogonal projection onto $\mathbb{C}\eta$ then by the Howe-Moore vanishing theorem $\Delta^{it} \to P$ in the weak operator topology as $|t| \to +\infty$. Now consider an element y in $\mathcal{M}'$ and notice that for every x in $\mathcal{M}$ we have

$$(\sigma_t^\psi(x) - \psi(x))y\eta = y(\sigma_t^\psi(x) - \psi(x))\eta \to y(Px - \psi(x))\eta = 0\,.$$

By the faithfulness of ψ we have that $\mathcal{M}'\eta$ is dense in $\mathcal{H}$ and therefore $\sigma_t^\psi(x) \to \psi(x)$ weakly as $|t| \to +\infty$. By averaging on the vector of $\mathcal{H}$ representing φ we have the thesis. □

1.2 Standard subspaces

Standard subspaces arise naturally in the modular theory of von Neumann algebras and are widely used in local QFT contexts. In this section we summarize some general facts about this unexpectedly rich theory. We follow [68].

Let $\mathcal{H}$ be a complex Hilbert space and $H \subseteq \mathcal{H}$ a real linear subspace. The *symplectic complement* H' of H is the real Hilbert subspace

$$H' = \{\xi \in \mathcal{H} \colon \mathrm{Im}(\xi|\eta) = 0\,, \eta \in H\}\,.$$

Clearly $H' = (iH)^\perp$, where $\perp$ denotes the real orthogonal complement in $\mathcal{H}$, namely the orthogonal complement with respect to the real scalar product $\mathrm{Re}(\cdot|\cdot)$. Therefore $H \subseteq H''$ and $\overline{H} = H''$. Moreover, $H_1' \supseteq H_2'$ if $H_1 \subseteq H_2$. A closed real subspace H is called *cyclic* if $H + iH$ is dense in $\mathcal{H}$ and *separating* if $H \cap iH = (0)$. It is easy to check that H is cyclic if and only if H' is separating. A *standard* subspace H of $\mathcal{H}$ is a closed, real linear subspace of $\mathcal{H}$ which is both cyclic and separating. Clearly a closed subspace H is standard if and only if H' is.

If M is a von Neumann algebra acting on $\mathcal{H}$ and $\xi \in \mathcal{H}$ is a standard vector for M, then the map $M \to \mathcal{H}$ given by $x \mapsto x\xi$ is injective and

$$H_M = \overline{\{x\xi \colon x = x^*\,, x \in M\}}$$

is a standard subspace of $\mathcal{H}$. Conversely, there is a natural way to associate a von Neumann algebra in the bosonic and fermionic Fock space of $\mathcal{H}$ to every standard subspace $H \subseteq \mathcal{H}$, and this assignment has many nice properties [67, 68, 79]. This establishes a direct connection between standard subspaces and pairs $(\mathcal{M}, \xi)$ of von Neumann algebras with standard vectors.

Let H be a standard subspace of $\mathcal{H}$. On the domain $D = H + iH$ we define the anti-linear operator $S = S_H$ by $S(\xi + i\eta) = \xi - i\eta$. The operator S is well-defined, densely defined and clearly satisfies $S^2 = 1$.

Proposition 6. *[68] The map $H \mapsto S_H$ is a bijection between the set of standard subspaces of $\mathcal{H}$ and the set of closed, densely defined, anti-linear involutions of $\mathcal{H}$. The inverse map is given by $S \mapsto \ker(1-S)$. Moreover, this map is order preserving, namely $H_1 \subseteq H_2$ if and only if $S_{H_1} \subseteq S_{H_2}$, and we have $S_H^* = S_{H'}$.*

The closable operator S_H can be used to define some modular theory in analogy to that of σ-finite von Neumann algebras in standard form. Let

$$S_H = J_H \Delta_H^{1/2}$$

be the polar decomposition of $S = S_H$. Also, set $J = J_H$ and $\Delta = \Delta_H$. Then J is an anti-unitary involution, namely $J = J^* = J^{-1}$. The operator $\Delta = S^*S$ is positive, non-singular and such that $J\Delta J = \Delta^{-1}$. It follows that [68]

$$Jf(\Delta)J = \bar{f}(\Delta^{-1})$$

for every complex Borel function f on $\mathbb{R}$, and in particular J commutes with Δ^{it}. Also, $J_{H'} = J_H$ and $\Delta_{H'} = \Delta_H^{-1}$. Finally, if U is some unitary operator on $\mathcal{H}$ then $UH = H$ if and only if $U\Delta_H U^* = \Delta_H$ and $UJ_HU^* = J_H$. The operator Δ_H is called *modular operator* and J_H is called *modular conjugation* of H. The following theorem is the real Hilbert subspace (easier) version of the fundamental Tomita-Takesaki theorem for von Neumann algebras.

Theorem 7. *[68] With $\Delta = \Delta_H$ and $J = J_H$ as above, we have for all $t \in \mathbb{R}$:*

$$\Delta^{it}H = H\,, \quad JH = H'\,.$$

Proof. The proof is provided in [68] and we write it here just for the sake of completeness. Δ^{it} commutes with $\Delta^{1/2}$ and J, thus with S. Therefore, the first relation follows because if $\xi \in H$ then $S\Delta^{it}\xi = \Delta^{it}S\xi = \Delta^{it}\xi$, namely $\Delta^{it}H \subseteq H$ for any t in $\mathbb{R}$, hence $\Delta^{it}H = H$. Concerning the second relation, notice that if ξ is in H then $(J\xi|\xi) = (JS\xi|\xi) = (\Delta^{1/2}\xi|\xi)$ belongs to $\mathbb{R}$, thus $(J(\xi+\eta)|\xi+\eta)$ is real for all $\xi, \eta \in H$. It follows that $\mathrm{Im}(J\xi|\eta) = 0$, namely $JH \subseteq H'$. As $J_H = J_{H'}$, we also have $JH' \subseteq H'' = H$ and therefore $JH = H'$. □

Corollary 8. *[68] If $\mathcal{H}$ is a Hilbert space, then there is a bijective correspondence between standard subspaces H of $\mathcal{H}$ and pairs (A, J) where A is a selfadoint linear operator on H, J is some anti-unitary involution on $\mathcal{H}$ and $JAJ = -A$.*

Proof. The proof is provided in [68] and we write it here just for completeness. Given H standard, the corresponding pair is $(\log \Delta_H, J_H)$. Conversely, given (A, J) then $S = Je^{A/2}$ is an anti-linear closed involution and one gets a standard subspace H by the mentioned above procedure $H = \ker(1 - S)$. Clearly these constructions are one the inverse of the other. □

Let H be a real linear subspace of $\mathcal{H}$ and V a one-parameter unitary group of $\mathcal{H}$ leaving H globally invariant. We now consider the following *(one particle) KMS condition* at inverse temperature $\beta > 0$: for every $\xi, \eta \in H$ there exists a function

$F = F_{\xi,\eta}$ which is bounded and continuous on the strip $\overline{S_\beta} = \{z \in \mathbb{C} \colon 0 \le \mathrm{Im} z \le \beta\}$, analytic in the interior S_β of $\overline{S_\beta}$ and such that

$$F(t) = (V(t)\xi|\eta)\,, \quad F(t+i\beta) = (\eta|V(t)\xi)\,.$$

As the uniform limit of a net of holomorphic functions is a holomorphic function, it follows that if the KMS condition holds for H then it holds for $\overline{H}$. If H is standard, then Δ_H^{-it} and H satisfy the KMS condition at inverse temperature 1. Conversely, if $V(t)$ and H as above satisfy the KMS condition at inverse temperature $\beta = 1$ then H is a standard subspace of $\mathcal{H}$ and $V(t) = \Delta_H^{-it}$ [68]. Similarly, the modular conjugation J_H can be characterized as follows.

Proposition 9. *[68] Let H be standard. Then J_H is the unique anti-unitary involution J of $\mathcal{H}$ such that $JH \supseteq H'$ and $(J\xi|\xi) \ge 0$ for all $\xi \in H$.*

Proof. As above, this fact is already proved in [68] and we write it here just for completeness. The positivity property holds for J_H because $(J_H\xi|\xi) = (\Delta_H^{1/2}\xi|\xi) \ge 0$ for all $\xi \in H$. On the other hand, let J be some anti-unitary involution that satisfies the above positivity condition. Then for all $\xi, \eta \in H$ we have that $(J(\xi+\eta)|\xi+\eta)$ is real, so $(J\xi|\eta)$ is real. Therefore $JH \subseteq H'$. Assuming that $JH \supseteq H'$ we then have $JH = H'$. Moreover, for all $\xi + i\eta$ in $H + iH$ we have

$$(JS_H(\xi+i\eta)|\xi+i\eta) = (J\xi|\xi) + (J\eta|\eta) \ge 0\,.$$

So there is a canonical, positive selfadjoint operator Δ on $\mathcal{H}$, with $D(\Delta^{1/2}) \supseteq D(S_H)$ (use the Friederich extension) such that

$$(JS_H\xi|\xi) = (\Delta^{1/2}\xi|\xi)\,, \quad \xi \in D(\Delta_H^{1/2}) = D(S_H)\,. \tag{1.2}$$

Now Δ_H^{it} commutes with S_H and with J (because $JH = H'$) so with JS_H. Therefore Δ_H^{it} commutes with $\Delta^{1/2}$. By functional calculus it follows that $\Delta_H^{1/2}$ commutes with $\Delta^{1/2}$, thus they have a common core, so $\Delta^{1/2}$ is selfadjoint on $D(S_H)$. By (1.2) we then have $JS_H = \Delta^{1/2}$, and by the uniqueness of the polar decomposition $S_H = J\Delta^{1/2}$ we have $\Delta = \Delta_H$ and $J = J_H$. □

We conclude this section by mentioning the relation between the theory of standard subspaces and the classical modular theory, which is our motivational setting. Let M be a von Neumann algebra on a Hilbert space $\mathcal{H}$ and $\Omega \in \mathcal{H}$ a vector. Clearly

$$H_M = \overline{M_{\mathrm{sa}}\Omega}$$

is a real Hilbert subspace of $\mathcal{H}$, where M_{sa} denotes the selfadjoint part of M. It follows from the definitions that Ω is cyclic if and only if H_M is cyclic, and Ω is separating if and only if H_M is separating as well.

We now assume Ω to be standard for some von Neumann algebra M. The map $M \mapsto H_M$ is not injective. However, H_M gives the full knowledge of the modular operator and the modular conjugation of M since

$$\Delta_M = \Delta_{H_M}\,, \quad J_M = J_{H_M}\,,$$

because of the KMS property and Proposition 9. In particular, $H'_M = H_{M'}$ because $H'_M = J_{H_M} H_M = J_M \overline{M_{\mathrm{sa}}\Omega} = \overline{M'_{\mathrm{sa}}\Omega} = H_{M'}$. It is also easy to verify that if N_1, N_2 are two von Neumann subalgebras of M then $N_1 \subseteq N_2$ if $H_{N_1} \subseteq H_{N_2}$. As a corollary, if N commutes with M and $H_N = H'_M$, then $N = M'$.

1.3 Quantum channels

The purpose of this section is to provide a few basic definitions, examples and theorems about completely positive maps and conditional expectations.

The quantum operation formalism is a general tool for describing the evolution of a quantum system in a wide variety of circumstances. If we describe quantum states by a density matrix ρ, then a *quantum operation*, or also a *quantum channel* in Quantum Information Theory, will be represented by a linear operator which maps positive operators in positive operators. However, this positivity property is not mathematically sufficient to define a quantum process in a sufficiently satisfactory way. The desire to apply the same experimental manipulations independently to n copies of the same system motivates the definition of complete positivity.

Definition 10. A linear map $\mathcal{F}\colon \mathfrak{A}_2 \to \mathfrak{A}_1$ between two C^*-algebras is called *positive (p)* if $\mathcal{F}(a)$ is positive whenever a is. $\mathcal{F}$ is called *completely positive (cp)* if $\mathbb{1}_n \otimes \mathcal{F}$ is positive as a map $M_n(\mathbb{C}) \otimes \mathfrak{A}_2 \to M_n(\mathbb{C}) \otimes \mathfrak{A}_1$ for every $n \geq 1$, with $M_n(\mathbb{C}) \otimes \mathfrak{A}_i$ the algebraic tensor product. A (completely) positive map between unital C^*-algebras is called *unital* if $\mathcal{F}(1) = 1$. A positive unital map $\mathcal{F}$ gives rise to a map $\mathcal{F}^*\colon \mathfrak{A}_1^* \to \mathfrak{A}_2^*$ defined by $(\mathcal{F}^*\omega)(a) = \omega(\mathcal{F}(a))$. A positive unital map $\mathcal{F}$ is called *normal* whenever $\mathcal{F}^*$ maps normal states to normal states.

In the operator formalism, quantum channels are usually defined as completely positive unital (cpu) maps. Depending on the context, other requirements can be added to the definition [70]. In addition to a quantum channel, one could perform measurements and post-select a sub-ensemble according to the results. For a von Neumann measurement, mathematically given by a collection of orthogonal projections P_k of a von Neumann algebra $\mathfrak{A}$ which sum up to 1, we note that the maps $\mathcal{F}_k\colon \mathfrak{A} \to \mathfrak{A}$ defined by $\mathcal{F}_k(a) = P_k a P_k$ are cp. By the measurement on a state ω we obtain a new state $\omega_k = \mathcal{F}_k^*\omega/\omega(P_k)$ with probability $\omega(P_k)$, when $\omega(P_k) > 0$. A combination of quantum channels and measurements is called an *operation*. It is described by a family $\mathcal{F}_k\colon \mathfrak{A}_2 \to \mathfrak{A}_1$ of cp maps with $\sum_k \mathcal{F}_k(1) = 1$, which transform a state ω on $\mathfrak{A}_1$ into $\omega_k = \mathcal{F}_k^*/p_k$ with probability $p_k = \omega(\mathcal{F}_k(1))$ when $p_k > 0$.

Definition 11. Let $\mathfrak{B} \subseteq \mathfrak{A}$ be an inclusion of C^*-algebras and let $\varepsilon\colon \mathfrak{A} \to \mathfrak{B}$ be a linear mapping. If ω is state on $\mathfrak{A}$, then ε is said to be *ω-preserving* if $\omega \cdot \varepsilon = \omega$. The map ε is said to be a *projection* if $\varepsilon(b) = b$ for every b in $\mathfrak{B}$, while it is said to be $\mathfrak{B}$*-linear* if $\varepsilon(ab) = \varepsilon(a)b$ and $\varepsilon(ba) = b\varepsilon(a)$ for every a in $\mathfrak{A}$ and b in $\mathfrak{B}$. A $\mathfrak{B}$-linear positive projection $\varepsilon\colon \mathfrak{A} \to \mathfrak{B}$ is said to be a *conditional expectation*. Finally, ε is called a *Schwarz mapping* if $\varepsilon(a)^*\varepsilon(a) \leq \varepsilon(a^*a)$ for all a in $\mathfrak{A}$.

It follows by the definition that any projection ε has norm $\|\varepsilon\| \geq 1$. We now mention a few standard facts related to conditional expectations, the interested reader can consult [89] for details. Every conditional expectation $\varepsilon\colon \mathfrak{A} \to \mathfrak{B}$ is a completely positive projection of unital norm, and if $\mathfrak{A}$ is unital then $\mathfrak{B}$ is also unital and $\varepsilon(1_{\mathfrak{A}}) = 1_{\mathfrak{B}}$. A Schwarz mapping which is also a projection is a conditional expectation and every conditional expectation is a Schwarz mapping. Finally, every projection of unital norm is a conditional expectation.

Example 1. *Let us give some examples of cp maps and conditional expectations.*

(i) *Trivially, any (unit preserving) $*$-homomorphism between C^*-algebras is a (unital) cp map. Furthermore, every conditional expectation is a cp map.*

(ii) *If $V\colon \mathcal{H} \to \mathcal{K}$ is a bounded linear map between Hilbert spaces, then $\mathcal{F}\colon B(\mathcal{K}) \to B(\mathcal{H})$ given by $\mathcal{F}(a) = V^*aV$ is a cp map. Furthermore, any state of a C^*-algebra is a cpu map.*

(iii) *Let $\mathcal{H}$ be a Hilbert space carrying a continuous unitary representation U of some compact Lie group G. The map $\mathcal{F}\colon B(\mathcal{H}) \to B(\mathcal{H})$ given by*

$$\mathcal{F}(x) = \int_G dg\, U(g)xU(g)^*\,,$$

with dg the normalized Haar measure, is a cpu map. It is also a conditional expectation onto the commutant of the representation U.

(iv) *If A and B are von Neumann algebras, then $\mathcal{F}\colon A \to A \otimes B$ given by $\mathcal{F}(a) = a \otimes \mathbb{1}$ is a cpu map.*

(v) *Given a state ω on B then the map $\mathcal{F}\colon A \otimes B \to A$ given by $\mathcal{F}(a \otimes b) = a\omega(b)$ is a cpu map. It is also a conditional expectation onto A.*

A general result due to Stinespring shows that, given a C^*-algebra $\mathfrak{A}$, all cp maps $\mathcal{F}\colon \mathfrak{A} \to B(\mathcal{H})$ can be written as $\mathcal{F}(a) = V^*\pi(a)V$, where π is a representation of $\mathfrak{A}$ on some Hilbert space $\mathcal{K}$ and $V\colon \mathcal{H} \to \mathcal{K}$ is bounded. When $\mathcal{F}$ is unital, one can suppose V to be an isometry. Furthermore, if $\mathfrak{A}$ already acts on $\mathcal{H}$ and π is a countable direct sum of identity representations, one recovers a formulation in terms of *Kraus operators* (or *noise operators* in Quantum Information Theory) for $\mathcal{F}$ as follows

$$\mathcal{F}(a) = \sum_j V_j^* a V_j\,, \quad \sum_j V_j^* V_j = 1\,.$$

It follows from standard properties of finite type I factors, that in this case all cp maps arise in this way. On a much more general class of von Neumann algebras, a generalization of the Stinespring dilation theorem can be given as follows.

Let $\alpha\colon \mathcal{N} \to \mathcal{M}$ be a normal cpu map between von Neumann algebras. A pair (ρ, v) with $\rho\colon \mathcal{N} \to \mathcal{M}$ a homomorphism and $v \in \mathcal{M}$ an isometry such that $\alpha(n) = v^*\rho(n)v$ for n in $\mathcal{N}$ will be called a *dilation pair* for α, and ρ a *dilation homomorphism*. With

(ρ, v) a dilation pair, the subspace $\rho(\mathcal{N})v\mathcal{H}$ of the underlying Hilbert space $\mathcal{H}$ is clearly both $\rho(\mathcal{N})$-invariant and $\mathcal{M}'$-invariant, thus the projection e onto $\overline{\rho(\mathcal{N})v\mathcal{H}}$ belongs to $\rho(\mathcal{N})' \cap \mathcal{M}$. We shall say that (ρ, v) is *minimal* if $e = 1$.

Theorem 12. *[70] Let $\alpha\colon \mathcal{N} \to \mathcal{M}$ be a normal cpu map between two properly infinite von Neumann algebras. Then there exists a minimal dilation pair (ρ, v) for α. Furthermore, if (ρ', v') is another minimal dilation pair for α, then there exists a unique unitary u in $\mathcal{M}$ such that $u\rho(n) = \rho'(n)u$ and $v' = uv$.*

Corollary 13. *Let $\mathfrak{A}$ be a unital C^*-algebra and $\Phi\colon \mathfrak{A} \to \mathcal{M}$ a cpu map, with $\mathcal{M}$ a properly infinite von Neumann algebra. Then there exist an isometry $v \in \mathcal{M}$ and a representation ρ of $\mathfrak{A}$ on $\mathcal{H}$ with $\rho(\mathfrak{A}) \subseteq \mathcal{M}$ such that $\Phi(x) = v^*\rho(x)v$.*

Proof. This fact has been proved in [70], here we provide the proof just for completeness. Let ψ be a faithful normal state of $\mathcal{M}$ and $\varphi = \psi \cdot \Phi$ its pullback to a state of $\mathfrak{A}$. Then Φ factors through the GNS representation π_φ of $\mathfrak{A}$ given by φ, namely we have $\Phi = \Phi_0 \cdot \pi_\varphi$ with $\Phi_0\colon \mathcal{N} \to \mathcal{M}$ a completely positive map and $\mathcal{N} = \pi_\varphi(\mathfrak{A})''$. Indeed, if $a \in \mathfrak{A}$ then we have

$$\pi_\varphi(a) = 0 \Rightarrow \varphi(a^*a) = 0 \Rightarrow \psi \cdot \Phi(a^*a) = 0 \Rightarrow \Phi(a^*a) = 0 \Rightarrow \Phi(a) = 0\,,$$

since $\Phi(a)^*\Phi(a) \leq \Phi(a^*a)$. As $\psi \cdot \Phi_0$ is normal on $\mathcal{N}$, it follows that Φ_0 is normal too. We now want to apply Theorem 12. If $\mathcal{N}$ is properly infinite then we are done. In general, we may consider the spatial tensor product $\mathcal{N} \otimes \mathcal{F}$, with $\mathcal{F}$ a type I_∞ factor, and a faithful normal conditional expectation $\varepsilon\colon \mathcal{N} \otimes \mathcal{F} \to \mathcal{N}$. Therefore we can apply Theorem 12 to $\Phi_0 \cdot \varepsilon$, and by the commutative diagram

$$\begin{tikzcd}
\mathfrak{A} \arrow[r, "\Phi"] \arrow[d, "\pi_\varphi"'] & \mathcal{M} & \mathcal{M} \arrow[l, "\mathrm{Ad}\, v"'] \\
\mathcal{N} \arrow[ur, "\Phi_0"'] \arrow[r, "i"'] & \mathcal{N} \otimes \mathcal{F} \arrow[u, "\Phi_0 \cdot \varepsilon"'] \arrow[ur, "\rho"'] &
\end{tikzcd}$$

the thesis follows. □

The following corollary extends to the infinite-dimensional case the known construction of Kraus operators.

Corollary 14. *Let $\alpha\colon \mathcal{F} \to \mathcal{F}$ be a normal cpu map with $\mathcal{F}$ a type I_∞ factor. Then there exist a sequence of elements $T_i \in \mathcal{F}$ with $\sum_i T_i T_i^* = 1$ such that*

$$\alpha(x) = \sum_i T_i x T_i^*\,, \quad x \in \mathcal{F}\,.$$

Proof. This proof has been provided in [70] and we write it here just for completeness. Write $\alpha(\cdot) = v^*\rho(\cdot)v$ by Theorem 12, with (ρ, v) a minimal dilation pair for α. As shown in [64], every endomorphism of a type I factor is inner, namely there exist a sequence of isometries $v_i \in \mathcal{F}$ with $\sum_i v_i v_i^* = 1$ such that

$$\rho(x) = \sum_i v_i x v_i^*\,, \quad x \in \mathcal{F}\,,$$

where the sum is meant to be strongly convergent. Thus

$$\alpha(x) = v\rho(x)v^* = \sum_i v^* v_i x v_i^* v_i = \sum_i T_i x T_i^* , \quad x \in \mathcal{F} ,$$

with $T_i = v^* v_i$. □

We conclude this section by mentioning a different context in which cp maps arise. Let $\mathcal{M}$ be a von Neumann algebra with a normal state φ. Given a von Neumann subalgebra $\mathcal{M}_0$ of $\mathcal{M}$, we denote by φ_0 the restriction of φ on $\mathcal{M}_0$ and we set $p_0 = \operatorname{supp}\varphi_0$. Let $(\mathcal{H}, \pi, \mathcal{P}, J)$ and $(\mathcal{H}_0, \pi_0, \mathcal{P}_0, J_0)$ be the standard representations of $\mathcal{M}$ and $\mathcal{M}_0$, and denote by ξ and ξ_0 the representing vectors for φ and φ_0 in $\mathcal{P}$ and $\mathcal{P}_0$ respectively. We define a partial isometry V from $\mathcal{H}_0$ to $\mathcal{H}$ with initial projection $[\pi_0(\mathcal{M}_0)\xi_0]$ by the formula $V\pi_0(x)\xi_0 = \pi(x)\xi$ for x in $\mathcal{M}_0$. One can check that $V^*\pi(\mathcal{M})'V \subseteq \pi_0(\mathcal{M}_0)'$. Hence, there is a unique element $\varepsilon(x)$ in $p_0\mathcal{M}_0 p_0$ such that $\pi_0(\varepsilon(x)) = J_0 V^* J\pi(x) JV J_0$ for a fixed x in $\mathcal{M}$. The just constructed map $\varepsilon\colon \mathcal{M} \to p_0\mathcal{M}p_0$, originally introduced in [1], is called a *generalized conditional expectation.* By construction, the generalized conditional expectation is a cp mapping such that $\operatorname{supp}\varepsilon = \operatorname{supp}\varphi$, $\varphi \cdot \varepsilon = \varphi$ and $\varepsilon(\operatorname{supp}\varphi) = p_0$ [82]. In particular, by the invariance property $\varphi \cdot \varepsilon = \varphi$ it follows that ε is faithful and normal if φ is faithful and normal. In local QFT contexts, generalized conditional expectations typically appear as *canonical endomorphisms* [71].

1.4 Quantum entropy basics

The first non-commutative entropy notion, von Neumann's quantum entropy, was originally designed as a Quantum Mechanics version of Shannon's entropy: if a state ψ of $B(\mathcal{H})$ has density matrix ρ_ψ then the von Neumann entropy is given by

$$S(\psi) = -\operatorname{tr}\rho_\psi \log \rho_\psi .$$

The von Neumann entropy can be viewed as the lack of information about the system in the state ψ, assuming that the observer has, in principle, access to all observables in $B(\mathcal{H})$. This interpretation is in accord for instance with the facts that $S(\psi) \geq 0$ and that a pure state has vanishing von Neumann entropy.

A related notion is that of relative entropy. On a type I factor $B(\mathcal{H})$, it is defined for two normal states ω and φ with density matrices ρ_ω and ρ_φ by

$$S(\omega\|\varphi) = \operatorname{tr}\rho_\omega(\ln\rho_\omega - \ln\rho_\varphi) \tag{1.3}$$

if $\operatorname{supp}\rho_\varphi \geq \operatorname{supp}\rho_\omega$ and by $S(\omega\|\varphi) = +\infty$ otherwise. If $\mathcal{H} = \mathcal{H}_n$ is finite dimensional then $S(\omega) = -S(\omega\|\mathrm{Tr})$, with Tr the unnormalized trace of $B(\mathcal{H}_n)$. The relative entropy $S(\omega\|\varphi)$ generalizes the classical Kullback-Leibler divergence and measures how φ deviates from ω. From an informational theoretical viewpoint, $S(\omega\|\varphi)$ is the mean value in the state ω of the difference between the information carried by the state φ and the state ω. However, in Quantum Field Theory local von Neumann algebras are typically factors of type III_1, no trace or density matrix exists and the von Neumann entropy is undefined [63]. A generalization of the relative entropy to a generic von Neumann algebra was found by Araki [4, 5].

Definition 15. Let $\mathcal{M}$ be a von Neumann algebra in standard form on $\mathcal{H}$ and let φ, ψ be two normal positive linear functionals on $\mathcal{M}$ represented by two vectors ξ, η. The *relative entropy* between φ and ψ is defined by

$$S(\varphi\|\psi) = -(\xi|\log\Delta_{\eta,\xi}\xi) \tag{1.4}$$

if $s(\varphi) \leq s(\psi)$, otherwise $S(\varphi\|\psi) = +\infty$ by definition.

We will write $S(\varphi\|\psi) = S_{\mathcal{M}}(\varphi\|\psi)$ if we want to stress the dependence on $\mathcal{M}$. As follows by Lemma 2 and Corollary 4, equation (1.4) does not depend on the choice of the representing vectors. If $\mathcal{M}$ is not in standard form, then equation (1.4) holds if the relative modular operator is replaced with a spatial derivative [82]. The motivation of this definition comes from the well known modular theory of type I factors, as the following example shows.

Example 2. *For a type I factor $\mathcal{M} = B(\mathcal{H})$, normal faithful states ω and φ correspond to density matrices ρ_ω and ρ_φ. The relative modular operator $\Delta_{\varphi,\omega}$ corresponds to $\rho_\varphi \otimes \rho_\omega$ in the GNS representation of $\mathcal{M}$ on $\mathcal{H}\otimes\mathcal{H}$ with respect to ω. In this representation ω is represented by the vector $\Omega = \rho_\omega^{1/2}$, and the definition* (1.4) *gives*

$$S(\omega\|\varphi) = -(\Omega|(\log\rho_\omega \otimes 1 - 1\otimes\log\rho_\varphi)\Omega) = -\mathrm{tr}\rho_\omega(\ln\rho_\varphi - \ln\rho_\omega)\,,$$

and therefore we recover the classical relative entropy (1.3)*.*

The scalar product (1.4) has to be intended by applying the spectral theorem to the relative modular operator $\Delta_{\eta,\xi}$, namely we have

$$S(\varphi\|\psi) = -\int_0^1 \log\lambda\, d(\xi|E_{\eta,\xi}(\lambda)\xi) - \int_1^\infty \log\lambda\, d(\xi|E_{\eta,\xi}(\lambda)\xi)\,, \tag{1.5}$$

where the second integral is always finite by the estimate $\log\lambda \leq \lambda$. In particular, $S(\varphi\|\psi)$ is finite if and only if the first integral appearing in (1.5) is finite. By this remark it follows that [82]

$$S(\varphi\|\psi) = i\frac{d}{dt}\varphi((D\psi\colon D\varphi)_t)\Big|_{t=0} = -i\frac{d}{dt}\varphi((D\varphi\colon D\psi)_t)\Big|_{t=0}\,, \tag{1.6}$$

where $(D\varphi\colon D\psi)_t = (D\psi\colon D\varphi)_t^*$ is the Connes cocycle. Identity (1.6) can be proved by using the dominated convergence theorem if $S(\varphi\|\psi)$ is finite and by the Fatou's lemma if $S(\varphi\|\psi) = +\infty$ [28]. We recall some properties of the relative entropy [82].

(r0) $S(\varphi\|\psi) \geq \varphi(I)(\log\varphi(I) - \log\psi(I))$, and $S(\lambda\varphi\|\mu\psi) = \lambda S(\varphi\|\psi) - \lambda\varphi(I)\log(\mu/\lambda)$ for any $\lambda, \mu \geq 0$. Moreover, $S(\varphi\|\psi) \geq \|\varphi-\psi\|^2/2$, so that $S(\varphi\|\psi) = 0$ if and only if $\varphi = \psi$.

(r1) $S(\varphi\|\psi)$ is lower semi-continuous in the $\sigma(\mathcal{M}_*, \mathcal{M})$-topology.

(r2) $S(\varphi\|\psi)$ is convex in both its variables. By (r0) this is equivalent to the subadditivity of $S(\varphi\|\psi)$ in both its variables.

(r3) $S(\varphi\|\psi)$ is superadditive in its first argument. Furthermore, $S(\varphi\|\psi) \leq S(\varphi'\|\psi')$ if $\psi \geq \psi'$ and $\varphi \geq \varphi'$ with $\|\varphi\| = \|\varphi'\|$.

(r4) If $\alpha\colon \mathcal{M}_1 \to \mathcal{M}_2$ is a Schwarz mapping such that $\varphi_2 \cdot \alpha \leq \varphi_1$ and $\psi_2 \cdot \alpha \leq \psi_1$, then $S_{\mathcal{M}_1}(\varphi_1\|\psi_1) \leq S_{\mathcal{M}_2}(\varphi_2\|\psi_2)$. In particular, $S(\varphi\|\psi)$ is monotone increasing with respect to inclusions of von Neumann algebras.

(r5) Let $(\mathcal{M}_i)_i$ be an increasing net of von Neumann subalgebras of $\mathcal{M}$ with the property $(\cup_i \mathcal{M}_i)'' = \mathcal{M}$. Then the increasing net $S_{\mathcal{M}_i}(\varphi\|\psi)$ converges to $S(\varphi\|\psi)$.

(r6) Let $\varepsilon\colon \mathcal{M} \to \mathcal{N}$ be a faithful normal conditional expectation. If φ and ψ are normal states on $\mathcal{M}$ and $\mathcal{N}$ respectively, then $S_{\mathcal{M}}(\varphi\|\psi \cdot \varepsilon) = S_{\mathcal{N}}(\varphi\|\psi) + S_{\mathcal{M}}(\varphi\|\varphi \cdot \varepsilon)$.

(r7) Let φ be a normal state on the spatial tensor product $\mathcal{M}_1 \otimes \mathcal{M}_2$ with partials $\varphi_i = \varphi|_{\mathcal{M}_i}$. Consider then normal states ψ_i on $\mathcal{M}_i$. As a corollary of (r6), we have $S(\varphi\|\psi_1 \otimes \psi_2) = S(\varphi_1\|\psi_1) + S(\varphi_2\|\psi_2) + S(\varphi\|\varphi_1 \otimes \varphi_2)$.

By using the universal representation, the relative entropy can be defined on a generic C^*-algebra. If we replace the strong closure with the norm closure in (r5) and the $\sigma(\mathcal{M}_*, \mathcal{M})$-topology with the weak topology in (r1), then properties from (r0) to (r5) still hold in the C^*-algebraic setting [82]. We now provide a few little personal remarks on the relative entropy. Further results can be found in [38] and related works. As shown in [82], if ψ is a positive normal functional of $\mathcal{M}$ and $t \in \mathbb{R}$, then the sublevel

$$\mathcal{K}(\psi, t) = \{\varphi \in \mathcal{M}^*_+ \colon S(\varphi\|\psi) \leq t\} \tag{1.7}$$

consists of normal functionals and it is a convex compact set with respect to the $\sigma(\mathcal{M}_*, \mathcal{M})$-topology.

Lemma 16. *$\mathcal{K}(\psi, t)$ is sequentially $\sigma(\mathcal{M}_*, \mathcal{M})$-compact and its set of extremal points is*

$$\mathcal{E}(\psi, t) = \{\varphi \in \mathcal{M}^*_+ \colon S(\varphi\|\psi) = t\}\,. \tag{1.8}$$

Moreover, after a restriction to $\mathcal{M}_{s(\psi)}$, the union $\mathcal{K}(\psi) = \bigcup_t \mathcal{K}(\psi, t)$ is norm dense in the set of normal positive functionals of the reduced von Neumann algebra $\mathcal{M}_{s(\psi)}$.

Proof. The first two claims follow by the Eberlein-Smulian theorem and Donald's identity ([82], Proposition 5.23). The last point holds if ψ is faithful, since in this case the set of positive normal functionals φ such that $\varphi \leq \alpha\psi$ for some $\alpha > 0$ is norm dense in $\mathcal{M}^+_*$ ([15], Theorem 2.3.19). The general case follows by noticing that $S_{\mathcal{M}}(\varphi\|\psi) = S_{\mathcal{M}_{s(\psi)}}(\varphi\|\psi)$ if $s(\varphi) \leq s(\psi)$, which is a necessary condition for $S_{\mathcal{M}}(\varphi\|\psi)$ to be finite. $\square$

Definition 17. If φ is a state on a C^*-algebra A, then the *von Neumann entropy* of φ is defined by

$$S_A(\varphi) = \sup\Big\{ \sum_i \lambda_i S(\varphi_i\|\varphi) \colon \sum_i \lambda_i \varphi_i = \varphi \Big\}\,,$$

where the supremum is over all decompositions of φ into finite (or equivalently countable) convex combinations of other states. If A is clear, we will simply write $S_A(\varphi) = S(\varphi)$.

Some properties of $S(\varphi)$ are immediate from those of the relative entropy: $S(\varphi)$ is nonnegative, vanishes if and only if φ is a pure state and it is weakly lower semicontinuous. On type I factors, the von Neumann entropy of a normal state φ with density matrix ρ is given by $S(\varphi) = -\mathrm{tr}\,\rho \ln \rho$. We now list a few properties of the von Neumann entropy [82] (the notation $\eta(t) = -t \ln t$ is standard in information theory):

(s0) (concavity) Given states φ and ω, then $\lambda S(\varphi) + (1-\lambda)S(\omega) \leq S(\lambda\varphi + (1-\lambda)\omega) \leq \lambda S(\varphi) + (1-\lambda)S(\omega) + H(\lambda, 1-\lambda)$ for $0 < \lambda < 1$, where $H(\lambda, 1-\lambda) = \eta(\lambda) + \eta(1-\lambda)$.

(s1) (strong subadditivity) On a three-fold-product $B(\mathcal{H}_1) \otimes B(\mathcal{H}_2) \otimes B(\mathcal{H}_3)$, a normal state ω_{123} with marginal states ω_{ij} satisfies $S(\omega_{123}) + S(\omega_2) \leq S(\omega_{12}) + S(\omega_{23})$.

(s2) $S(\psi) = \inf\{-\sum_i \eta(\lambda_i)\}$, with $\eta(t) = -t \log t$ and where the infimum is taken over all the possible decompositions into pure states.

(s3) (tensor product) On the projective tensor product $A \otimes B$, we have the identity $S(\varphi_1 \otimes \varphi_2) = S(\varphi_1) + S(\varphi_2)$.

Definition 18. Consider an inclusion of C^*-algebras $A \subseteq B$ and a state φ on B. The *subalgebra entropy of φ with respect to A* is

$$H_\varphi^B(A) = \sup\Big\{\sum_i \lambda_i S_A(\varphi_i \| \varphi) \colon \varphi = \sum_i \lambda_i \varphi_i\Big\}, \tag{1.9}$$

where the supremum is over all finite (countable) convex linear decompositions of φ on into states of B.

The subalgebra entropy is actually a particular case of what is known as *conditional entropy*. If there is no ambiguity about the bigger C^*-algebra B, we will enlighten the notation by setting $H_\varphi^B(A) = H_\varphi(A)$. We list a few of its properties [31, 82].

(c0) (monotonicity) $H_\varphi^{\overline{B}}(A) \leq H_\varphi^B(\overline{A})$ if $A \subseteq \overline{A}$ and $B \subseteq \overline{B}$.

(c1) (semicontinuity) $\varphi \mapsto H_\varphi^B(A)$ is weakly lower semicontinuous.

(c2) (martingale property) $\lim_i H_\varphi^B(A_i) = H_\varphi^B(A)$ if $(A_i)_i$ is an increasing net of C^*-subalgebras of B whose union is norm dense in A.

(c3) (concavity) $\lambda H_{\varphi_1}^B(A) + (1-\lambda)H_{\varphi_2}^B(A) \leq H_\varphi^B(A) \leq \lambda H_{\varphi_1}^B(A) + (1-\lambda)H_{\varphi_2}^B(A) + H(\lambda, 1-\lambda)$ for $\varphi = \lambda\varphi_1 + (1-\lambda)\varphi_2$ on B and λ in $(0,1)$, $H(\lambda, 1-\lambda) = \eta(\lambda) + \eta(1-\lambda)$.

In (c2), the union $\cup_i A_i$ can be strongly dense if all the C^*-algebras are replaced with von Neumann algebras and the state φ is normal. We point out that the concavity of $H_\varphi(A)$ mentioned in (c3) certainly holds whenever A is AF ([82], Theorem 5.29 and Proposition 10.6), but the general case is a bit unclear to the author [31]. What is clear instead, is the following original simple lemma which says whenever the inequality (c0) reduces to an equality in the case $A = \overline{A}$.

Lemma 19. *Consider the C^*-algebras inclusions $A \subseteq B \subseteq \overline{B}$ and $A \subseteq \overline{A} \subseteq \overline{B}$. Let φ be a state on $\overline{B}$. If there is a φ-preserving conditional expectation $\varepsilon \colon \overline{B} \to B$, then*

$$H_\varphi^B(A) \leq H_\varphi^{\overline{B}}(\overline{A}).$$

Proof. We can follow Proposition 6.7 of [82]. Indeed, if ψ is a state of B then $\psi \cdot \varepsilon$ is a state of $\overline{B}$. Therefore, if $\varphi = \sum_i \lambda_i \varphi_i$ on B for some states φ_i of B then $\varphi = \sum_i \lambda_i \varphi_i \cdot \varepsilon$ is a decomposition of φ into states of $\overline{B}$. The rest follows from $S_A(\varphi_i \| \varphi) \leq S_{\overline{A}}(\varphi_i \cdot \varepsilon \| \varphi)$. $\square$

1.5 Half sided modular inclusions

J. Bisognano and E. Wichmann [10] made a discovery about the connection of the modular operator and the modular conjugation for the von Neumann algebra generated by quantum fields in a wedge region of the Minkowski space-time. H. J. Borchers [11] formulates an important feature of this connection in the abstract setting of a pair of von Neumann algebras $\mathcal{N} \subseteq \mathcal{M}$ with a common cyclic and separating vector Ω and a one-parameter group of unitaries U_t having a positive generator. A further development has been achieved by by H. W. Wiesbrock [96], who introduces the notion of half sided modular inclusion and obtains the underlying group structure. In this section we define and describe some properties of this purely operator algebraic object.

Let $\mathcal{N} \subseteq \mathcal{M}$ be an inclusion of σ-finite von Neumann algebras on a Hilbert space $\mathcal{H}$ and ω a faithful normal state given by a unit vector ω in $\mathcal{H}$ which is standard for both $\mathcal{N}$ and $\mathcal{M}$. We shall say that the inclusion $\mathcal{N} \subseteq \mathcal{M}$ is $\pm$*half-sided modular ($\pm$hsm)* with respect to ω if

$$\sigma_s^\omega(\mathcal{N}) \subseteq \mathcal{N}\,, \quad \pm s \geq 0\,,$$

where $\sigma_s^\omega = \operatorname{Ad}\Delta_{\mathcal{M}}^{is}$ is the modular operator and $\Delta_{\mathcal{M}}^{is} = \Delta_\Omega^{is}$. For simplicity, in the following we will only consider -hsm inclusions, yet every statement will have a dual statement for +hsm inclusions.

Theorem 20. *[6, 12] Let $\mathcal{N} \subseteq \mathcal{M}$ be a -hsm inclusion of σ-finite von Neumann algebras as above. Denote the corresponding modular operators and conjugations by $\Delta_{\mathcal{M}}$, $J_{\mathcal{M}}$ and $\Delta_{\mathcal{N}}$, $J_{\mathcal{N}}$ respectively. Then*

$$P = \frac{1}{2\pi}\big(\log \Delta_{\mathcal{N}} - \log \Delta_{\mathcal{M}}\big) \tag{1.10}$$

is an essentially self-adjoint operator with positive closure still denoted by P. If $U_s = e^{isP}$ for $s \in \mathbb{R}$, then we have the following:

(i) $\Delta_{\mathcal{M}}^{-it} U_s \Delta_{\mathcal{M}}^{it} = \Delta_{\mathcal{N}}^{-it} U_s \Delta_{\mathcal{N}}^{it} = U_{e^{2\pi t}s}$,

(ii) $J_{\mathcal{M}} U_s J_{\mathcal{M}} = J_N U_s J_{\mathcal{N}} = U_{-s}$,

(iii) $U_{1-e^{2\pi t}} = \Delta_{\mathcal{N}}^{-is} \Delta_{\mathcal{M}}^{is}$ and $\Delta_{\mathcal{N}}^{is} = U_1 \Delta_{\mathcal{M}}^{is} U_1^*$,

(iv) $U_2 = J_{\mathcal{N}} J_M$ and $J_{\mathcal{N}} = U_1 J_{\mathcal{M}} U_1^*$,

(v) $\mathcal{N} = U_1 \mathcal{M} U_1^*$ and $U_s \mathcal{M} U_s^* \subseteq \mathcal{M}$ for $s \geq 0$.

Furthermore, U_s can be strongly continuously extended to the complex half-plane $\operatorname{Im} s \leq 0$ *where it is bounded by* 1 *in norm and analytic in the interior.*

It can be noted by the identity $\Delta_{\mathcal{M}}^{-it} U_s \Delta_{\mathcal{M}}^{it} = U_{e^{2\pi t}s}$ that the modular operator $\Delta_{\mathcal{M}}^{it}$ and the unitary group U_s define a unitary representation of the $ax+b$ group. For this reason, the operator (1.10) is often referred to as the generator of translations. It is also of interest to notice that if Ω is, up to a phase, the unique vector fixed by the translation group U_s, then ω has trivial centralizer and hence $\mathcal{M}$ must be a III$_1$-type factor, unless of course $\mathcal{M} = \mathbb{C}$ and $\mathcal{H}$ is one dimensional. Furthermore, it has been recently shown in [72] that the 2×2 Connes' matrix trick can be used in order to provide a relative analogue of Theorem 20. We describe how.

Let $\mathcal{N} \subseteq \mathcal{M}$ be a -hsm inclusion of von Neumann algebras with respect to φ, a faithful normal state represented by a standard vector ξ in the natural cone. We then have the translation tunnel $\mathcal{M}_t = U_t \mathcal{M} U_t^*$, with $\mathcal{N} = \mathcal{M}_1$. Let ψ be another faithful normal state on $\mathcal{M}$ given by a standard vector η in the natural cone which is standard for both $\mathcal{M}$ and $\mathcal{N}$. We will assume the Connes Radon-Nikodym unitary cocycle to be localized as follows:

$$w_s = (D\psi \colon D\varphi)_s \in \mathcal{M}_R \,, \quad s \leq 0 \,, \tag{1.11}$$

for some $R \geq 1$. We then have:

Lemma 21. *[72] $\mathcal{N} \subseteq \mathcal{M}$ is -hsm with respect to ψ. Moreover, we have $\sigma_s^\psi(\mathcal{M}_t) = \mathcal{M}_{te^{-2\pi s}}$ for $t \leq R$ and $s \geq 0$.*

Proof. This lemma, proved in [72], is here written just for the sake of completeness. The first claim is immediate by the identity $\sigma_s^\psi(\mathcal{N}) = w_s \sigma_s^\varphi(\mathcal{N}) w_s^* \subseteq w_s \mathcal{N} w_s^* = \mathcal{N}$, which holds for $s \leq 0$. To prove the other assertion, we notice that $1 = w_{s-s} = w_s \sigma_s^\varphi(w_{-s})$, namely $w_s = \sigma_s^\varphi(w_{-s}^*)$. Let $s \geq 0$. Since $w_{-s} \in \mathcal{M}_R$, it follows that $w_s \in \sigma_s^\varphi(\mathcal{M}_R) = \mathcal{M}_{Re^{-2\pi s}}$. Therefore, for $t \leq R$ we have

$$\sigma_s^\psi(\mathcal{M}_t) = w_s \sigma_s^\varphi(\mathcal{M}_t) w_s^* = w_s \mathcal{M}_{te^{-2\pi s}} w_s^* = \mathcal{M}_{te^{-2\pi s}} \,.$$

because $w_s \in \mathcal{M}_{Re^{-2\pi s}} \subseteq \mathcal{M}_{te^{-2\pi s}}$. $\square$

Consider now the 2×2 matrix algebras over $\mathcal{N}$ and $\mathcal{M}$, namely

$$\widetilde{\mathcal{N}} = \mathcal{N} \otimes \mathrm{Mat}_2(\mathbb{C}) \,, \quad \widetilde{\mathcal{M}} = \mathcal{M} \otimes \mathrm{Mat}_2(\mathbb{C}) \,,$$

and denote by ϑ the positive linear functional on $\widetilde{\mathcal{M}}$ given by

$$\vartheta(x) = \varphi(x_{11}) + \psi(x_{22}) \,, \quad x = (x_{ij}) \in \widetilde{\mathcal{M}} \,.$$

Corollary 22. *[72] The inclusion $\widetilde{\mathcal{N}} \subseteq \widetilde{\mathcal{M}}$ is -hsm with respect to ϑ. Moreover, $\sigma_s^\vartheta(\mathcal{M}_t \otimes \mathrm{Mat}_2(\mathbb{C})) = \mathcal{M}_{te^{-2\pi s}} \otimes \mathrm{Mat}_2(\mathbb{C})$ for $s \geq 0$ and $t \leq R$.*

Proof. As above, here we write a fact already proved in [72] just for the sake of completeness. We have [90]

$$\sigma_s^\vartheta \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} \sigma_s^\varphi(x_{11}) & \sigma_s^\varphi(x_{12}) w_s^* \\ w_s \sigma_s^\varphi(x_{21}) & \sigma_s^\psi(x_{22}) \end{pmatrix} \,,$$

thus the assertions follows by the previous lemma. $\square$

Theorem 23. *[72] Let $\mathcal{N} \subseteq \mathcal{M}$ be a -hsm inclusion with respect to ξ and η with the property* (1.11) *as above. Then*

$$P = \frac{1}{2\pi}\big(\log \Delta_{\eta,\xi,\mathcal{N}} - \log \Delta_{\eta,\xi}\big)$$

is an essentially self-adjoint operator with positive closure still denoted by P. The one parameter group U generated by P satisfies

$$U_t \mathcal{M} U_{-t} = \mathcal{M}_t \,, \quad U_t \log \Delta_{\eta,\xi} U_t^* = \log \Delta_{\eta,\xi,\mathcal{M}_t} \,, \quad t \leq R \,.$$

Furthermore, U_t and $\Delta_{\eta,\xi}^{is}$ provide a representation of the $ax+b$ group, namely

$$\Delta_{\eta,\xi}^{is} U_t \Delta_{\eta,\xi}^{-is} = U_{te^{-2\pi s}} \,, \quad s,t \in \mathbb{R} \,.$$

Proof. This theorem, proved in [72], is here written just for the sake of completeness. The idea is to apply Theorem 20 to the -hsm inclusion given by Corollary 22 and then to restrict the operators in order to have the claimed relations. The GNS Hilbert space of ϑ is $\widetilde{\mathcal{H}} = \bigoplus_{ij} \mathcal{H}_{ij}$, with $\mathcal{H}_{ij} = \mathcal{H}$ and where ϑ is given by the vector $\theta = \xi \oplus \eta$ in $\mathcal{H}_{11} \oplus \mathcal{H}_{22}$. The modular operator $\Delta_\theta = \Delta_{\theta,\widetilde{\mathcal{M}}}$ decomposes as $\Delta_\theta = \bigoplus_{ij} \Delta_{ij}$, with

$$\Delta_{11} = \Delta_{\xi,\mathcal{M}} \,, \quad \Delta_{22} = \Delta_{\eta,\mathcal{M}} \,, \quad \Delta_{12} = \Delta_{\xi,\eta,\mathcal{M}} \,, \quad \Delta_{21} = \Delta_{\eta,\xi,\mathcal{M}} \,.$$

Then, by applying Theorem 20 we have that

$$\widetilde{P} = \frac{1}{2\pi}\big(\log \Delta_{\theta,\widetilde{\mathcal{N}}} - \log \Delta_\theta\big)$$

is an essentially self-adjoint operator with positive closure. It follows that the generated one parameter group decomposed as $\widetilde{U} = \bigoplus_{ij} U_{ij}$ as well, where U_{11} and U_{22} are the generators of translations of the -hsmi $\mathcal{N} \subseteq \mathcal{M}$ with respect to ξ and η respectively. Finally, the restriction of $\widetilde{P}$ on the subspace $\mathcal{H}_{21}$ gives identity (23), while the other relations follow by the decomposition of $\widetilde{U}$ and by the identity $\widetilde{\mathcal{M}_t} = \mathcal{M}_t \otimes \mathrm{Mat}_2(\mathbb{C})$ which holds for $t \leq R$. □

With the aim of studying the Quantum Null Energy Condition in a model independent setting, half sided modular inclusions have been recently studied in [27]. Here we exhibit some intermediate results. Hereafter, we will use the notation $P_\xi = (\xi|P\xi)$ where ξ is some vector of $\mathcal{H}$ and P is some self-adjoint operator. By the spectral theorem, the scalar product $(\xi|P\xi)$ is a well defined finite quantity if $|P|_\xi$ is finite, namely if ξ is in the domain of $|P|^{1/2}$.

Proposition 24. *Let $\mathcal{M}$ be a von Neumann algebra on $\mathcal{H}$ and let $U_s = e^{-isP}$ be a one parameter strongly continuous unitary group such that $U_{-s}\mathcal{M}U_s \subseteq \mathcal{M}$ for $s \geq 0$. If u and u' are isometries in $\mathcal{M}$ and $\mathcal{M}'$, then for every vector ξ in $\mathcal{H}$ we have*

$$P_{uu'\xi} + P_\xi = P_{u\xi} + P_{u'\xi} \,,$$

under the assumption that the quantities $|P|_{uu'\xi}$, $|P|_{u\xi}$, $|P|_{u'\xi}$, and $|P|_\xi$ are all finite.

Proof. This proposition is a very little variation of a proposition proved in [26], a few personal notes auxiliaries to [27]. Take $s > 0$ and consider

$$D = (\xi|U_{-s}\xi) + (uu'\xi|U_{-s}uu'\xi) - (u\xi|U_{-s}u\xi) - (u'\xi|U_{-s}u'\xi)\,.$$

Note that $(uu'\xi|U_{-s}uu'\xi) = (\xi|(u^*U_{-s}uU_{-s})(u')^*U_{-s}u'\xi)$, where we used the fact that $u^*U_{-s}uU_s$ belongs to $\mathcal{M}$ for $s > 0$. Thanks to this remark, we can write $D = D_1 + D_2 + D_3 + D_4$, where

$$\begin{aligned}
D_1 &= (u^*(U_s - 1)u\xi|U_s(u')^*(U_{-s} - 1)u'\xi)\,,\\
D_2 &= (u^*(U_s - 1)u\xi|(U_s - 1)\xi)\,,\\
D_3 &= ((U_{-s} - 1)\xi|(u')^*(U_{-s} - 1)u'\xi)\,,\\
D_4 &= -((U_{-s} - 1)\xi|(U_{-s} - 1)\xi)\,,
\end{aligned}$$

and so we have the estimate $|D| \le |D_1| + |D_2| + |D_3| + |D_4|$. We can bound all of these terms as $|D_i| \le \|(U_s - 1)\eta_1\|\,\|(U_s - 1)\eta_2\|$, where $\eta_1, \eta_2 \in \{\xi, u\xi, u'\xi\}$. For ζ in $\{\xi, u\xi, u'\xi\}$ we can use the spectral representation of P to write, for $s > 0$, the identity

$$(\zeta|(1 - U_s)\zeta)/s = \int \frac{1 - e^{-is\lambda}}{s}\, d(\zeta|E_\lambda(P)\zeta)\,.$$

By $|1 - e^{-is\lambda}|/s \le |\lambda|$ and by the finiteness of $|P|_\zeta$ we can use the dominated convergence theorem, and so we have

$$\lim_{s\to 0^+} (\zeta|(1 - U_s)\zeta)/s = iP_\zeta\,.$$

It follows that

$$\lim_{s\to 0^+} \frac{\|(U_s - 1)\zeta\|^2}{s} = \lim_{s\to 0^+} 2\mathrm{Re}(\zeta|(1 - U_s)\zeta)/s = 0\,,$$

and so by the estimates above we finally obtain that $D/s \to 0$ for $s \to 0^+$. Therefore

$$0 = \lim_{s\to 0^+} D/s = P_\xi + P_{uu'\xi} - P_{u\xi} - P_{u'\xi}\,,$$

and the thesis follows. □

We now introduce some identities used in [27]. Let $\mathcal{N} \subseteq \mathcal{M}$ be a -hsm inclusion with common standard vector ξ giving a normal state ω and positive generator of translations P. If ψ is another normal state given by a vector η, then we can consider the modular cocycles

$$u_s = (D\omega\colon D\psi)_s = \Delta_\xi^{is}\Delta_{\eta,\xi}^{-is}\,, \quad u'_s = (D\omega\colon D\psi)'_s = (\Delta'_\xi)^{is}(\Delta'_{\eta,\xi})^{-is}\,.$$

We can similarly define the conjugation cocycles

$$\Theta = J_\xi J_{\eta,\xi} \in \mathcal{M}\,, \quad \Theta' = J'_\xi J'_{\eta,\xi} \in \mathcal{M}'\,.$$

If we define

$$\Theta_s = \Delta_\xi^{is}\Theta\Delta_{\eta,\xi}^{-is}\,, \quad \Theta'_s = (\Delta'_\xi)^{is}\Theta'(\Delta'_{\eta,\xi})^{-is}\,,$$

then we have the relations

$$\Delta_\xi^{is} u'_s = u_s \Delta_\eta^{is}\,, \qquad \Delta_\xi^{is} u'_s \Delta_\xi^{-is} = (u'_s)^*\,,$$
$$J_\xi \Delta_\xi^{is} \Theta'_s = \Theta'_{-s} J_\eta \Delta_\eta^{is}\,, \qquad J_\xi \Delta_\xi^{is} \Theta'_s J_\xi \Delta_\xi^{-is} = \Theta_s^*\,,$$

along with the commutant version of these. Now consider the vector state ψ, $\widehat{\psi}$ and $\widetilde{\psi}$ represented by the vectors η, $\widehat{\eta} = \Theta'\eta$ and $\widetilde{\eta} = \Theta\eta$ respectively. We define the families of states

$$\psi_s(x) = (\eta_s|x\eta_s)\,, \quad \eta_s = u'_s\eta\,, \quad u'_s = (D\omega\colon D\psi)'_s\,,$$
$$\widehat{\psi}_s(x) = (\widehat{\eta}_s|x\widehat{\eta}_s)\,, \quad \widehat{\eta}_s = \widehat{u}'_s\eta = \Theta'_s\eta\,, \quad \widehat{u}'_s = (D\omega\colon D\widehat{\psi})'_s\,,$$
$$\widetilde{\psi}_s(x) = (\widetilde{\eta}_s|x\widetilde{\eta}_s)\,, \quad \widetilde{\eta}_s = \widetilde{u}'_s\eta = \Delta_\Omega^{is}\Theta_s\eta\,, \quad \widetilde{u}'_s = (D\omega\colon D\widetilde{\psi})'_s\,.$$

Similarly, if $P_\eta = (\eta|P\eta)$, $\widehat{P} = (\widehat{\eta}|P\widehat{\eta})$ and $\widetilde{P} = (\widetilde{\eta}|P\widetilde{\eta})$, then we can consider

$$P_s = (\eta_s|P\eta_s)\,, \quad \widehat{P}_s = (\widehat{\eta}_s|P\widehat{\eta}_s)\,, \quad \widetilde{P}_s = (\widetilde{\eta}_s|P\widetilde{\eta}_s)\,.$$

Theorem 25. *[27] If $P_\eta < +\infty$, then P_s, $\widehat{P}_s$ and $\widetilde{P}_s$ are finite for all s in $\mathbb{R}$. Moreover, we have*

$$\begin{aligned} 2P_s &= \widehat{P} + \widetilde{P}e^{-2\pi s}\,, \\ 2\widehat{P}_s &= (1+e^{-2\pi s})\widehat{P}\,, \\ 2\widetilde{P}_s &= (1+e^{-2\pi s})\widetilde{P}\,. \end{aligned} \tag{1.12}$$

Proof. (Sketch) This last theorem is very technical but is one of the most interesting results of [27]. The idea of the proof is to study analytic continuations of the function

$$g(s,t) = (\psi_s|U_{-a}\psi_s)\,, \quad U_{-a} = e^{iaP}\,, \quad a = \epsilon e^{2\pi t}\,, \quad s,t \in \mathbb{R}\,,$$

for a fixed $\epsilon > 0$. By the Vitali-Porter theorem one is able to study the limit $\epsilon \to 0$, and by doing so one can show that if P_η is finite then P_s is finite for all s in $\mathbb{R}$. Similar but more tricky techniques of complex analysis show that if P_η is finite then so do $\widehat{P}$ and $\widetilde{P}$, hence all the quantities appearing in (1.12) are finite. Finally, Proposition 24 and some simple algebraic manipulations imply the relations (1.12). □

The previous theorem is a technical result used to study the relative entropy $S(t) = S_{\mathcal{M}_t}(\psi\|\omega)$ in the general context of hsm inclusions. Here ω is the state represented by the standard vector Ω, $P \geq 0$ is the generator of translations and $\mathcal{M}_t = U_t\mathcal{M}U_t^*$ with $U_t = e^{itP}$. The main assumption is the requirement for the state ψ to be represented, at least for $t \geq c$ for some $c \in \mathbb{R}$, by some vector η in $\mathcal{H}$ with finite null energy, namely such that $P_\eta < +\infty$. One more partial result that we will use later is the following.

Lemma 26. *[27] Let $(\mathcal{M}_t)_{t\in\mathbb{R}}$ be a decreasing family of von Neumann algebras associated to some -hsm inclusion. Let ψ be a vector state represented by a vector η in $\mathcal{H}$ and consider the relative entropies $S(t) = S_{\mathcal{M}_t}(\psi\|\omega)$ and $\overline{S}(t) = S_{\mathcal{M}'_t}(\psi\|\omega)$. We assume P_η, $S(c)$ and $\overline{S}(c)$ to be finite for some $c \in \mathbb{R}$. Then, for all t_1, t_2 in $\mathbb{R}$ we have*

$$(S(t_1) - S(t_2)) + (\overline{S}(t_2) - \overline{S}(t_1)) = 2\pi(t_2 - t_1)P_\eta\,.$$

Corollary 27. *$S(t)$ and $\overline{S}(t)$ are everywhere finite and Lipschitz continuous.*

1.6 Algebraic formulation of Quantum Field Theory

In this section we recall the framework of Algebraic Quantum Field Theory (AQFT). An introduction to the subject can be found in [7, 48]. We begin with some geometric preliminaries. We consider the Minkowski space $\mathbb{R}^{n+1}$ with inner product

$$x \cdot y = x_0 y_0 - (x_1 y_1 + \cdots + x_n y_n) , \quad n = 0, 1, 2, 3 . \tag{1.13}$$

Notice that in the case $n = 0$ we have the real line with standard inner product. Hereafter we will work with natural units $c = \hbar = 1$. The subregions of points x with $x \cdot x > 0$, $x \cdot x < 0$ and $x \cdot x = 0$ are called *timelike*, *spacelike* and *lightlike*, respectively. The set of lightlike vectors is called the *light cone*. The sets

$$V_+ = \{x \in M : (x, x) > 0,\, x_0 > 0\} , \quad V_- = \{x \in M : (x, x) > 0,\, x_0 < 0\} ,$$

are called the *future cone* and the *past cone*. A *double cone* $\mathcal{O}$ is defined as a non-empty intersection of a forward cone $x + V_+$ and a backward cone $y + V_-$, with x and y in $\mathbb{R}^{n+1}$.

The symmetry group of this space is the Poincaré group $\mathcal{P}$, which is generated by translations, time-fixing isometries, time inversion and boost transformations. The connected component of the identity is called the *proper orthocronus Poincaré group* and it is denoted by $\mathcal{P}_+^\uparrow$. In the $n = 1$ case, boost transformations correspond to the matrices

$$\Lambda(\lambda) \colon x \mapsto \begin{pmatrix} \cosh\lambda & \sinh\lambda \\ \sinh\lambda & \cosh\lambda \end{pmatrix} x , \quad \lambda \in \mathbb{R} . \tag{1.14}$$

For the localization of physical observables, different regions of $\mathbb{R}^{n+1}$ will be of interest for us. We adopt the convention to work with open regions only, and we define the *spacelike complement* $\mathcal{O}'$ of a region $\mathcal{O} \subset \mathbb{R}^{n+1}$ as the interior of $\{x \in \mathbb{R}^{n+1} \colon (x - y)^2 < 0 , y \in \overline{\mathcal{O}}\}$. Of particular significance for us is the family of *wedges*, which is defined as follows. The so-called *right wedge* is the set

$$W_R = \{x \in \mathbb{R}^{n+1} \colon x_1 > |x_0|\} ,$$

and the *left wedge* is $W_L = W_R' = -W_R$. An arbitrary wedge is defined to be a set of the form gW_R, where $g \in \mathcal{P}$ is a Poincaré transformation. The set of all wedges will be denoted by $\mathcal{W}$. The wedges W_R and W_L are invariant under the action of the boost transformations, hence $\mathcal{W}$ is given by the translated left wedges $W_L + x$ and by the translated right wedges $W_R + x$, with $x \in \mathbb{R}^{n+1}$. In the $n = 1$ case, a non-empty intersection of a right wedge and a left wedge is a double cone.

We now define the formalism for a relativistic quantum theory on the Minkowski space $\mathbb{R}^{d+1}$. In the algebraic approach to Quantum Field Theory, a model is characterized in terms of a C^*-algebra $\mathfrak{A}$ of quasi-local observables, which are given by bounded self-adjoint operators on a fixed Hilbert space $\mathcal{H}$. All the physical information of the theory is encoded in a map

$$\mathbb{R}^{n+1} \supseteq \mathcal{O} \mapsto \mathcal{A}(\mathcal{O}) \subseteq B(\mathcal{H}) , \tag{1.15}$$

where $\mathcal{O}$ is an open region of $\mathbb{R}^{n+1}$ and $\mathcal{A}(\mathcal{O})$ is a von Neumann algebra on a fixed Hilbert space $\mathcal{H}$. The von Neumann algebras $\mathcal{A}(\mathcal{O})$ are called *local algebras*. The generated C^*-algebra, denoted by $\mathfrak{A} = \overline{\bigcup_{\mathcal{O}} \mathcal{A}(\mathcal{O})}$, is called the algebra of *quasi-local observables*.

Definition 28. A *local quantum field theory* on the Minkowski space $\mathbb{R}^{n+1}$, or shortly a *local net* of von Neumann algebras, is a map (1.15) satisfying the following axioms:

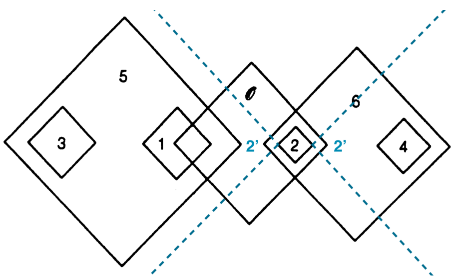
(a0) (Isotony) $\mathcal{A}(\mathcal{O}_1) \subseteq \mathcal{A}(\mathcal{O}_2)$ if $\mathcal{O}_1 \subseteq \mathcal{O}_2$.

(a1) (Causality) $\mathcal{A}(\mathcal{O}_1) \subseteq \mathcal{A}(\mathcal{O}_2)'$ if $\mathcal{O}_1 \subseteq \mathcal{O}_2'$.

(a2) (Covariance) For each g in $\widetilde{\mathcal{P}}_+^\uparrow$, there is an automorphism α_g of $\mathfrak{A}$ such that $\alpha_g(\mathcal{A}(\mathcal{O})) = \mathcal{A}(g.\mathcal{O})$. Such an automorphism is implemented by a strongly continuous unitary representation U of $\widetilde{\mathcal{P}}_+^\uparrow$ on $\mathcal{H}$.

(a3) (Vacuum) There is a vector Ω of $\mathcal{H}$, called *vacuum vector*, which is cyclic for $\mathfrak{A}$. Furthermore, Ω is the only vector fixed by U up to a phase.

(a4) (Spectrum condition) The representation U must be with *positive energy*, namely if $U(x) = e^{ix\cdot P}$ is the unitary associated to the translation of $x \in \mathbb{R}^{n+1}$ then the *energy momentum spectrum*, namely the joint spectrum of $P = (P^\nu)$, must be contained in the closed future cone $\overline{V_+}$.



In the following, local quantum field theories will be shortly denoted by the triple $(\mathcal{A}, U, \Omega)$. Two local theories will be said to be *equivalent* if there is a unitary operator intertwining the local net $\mathcal{A}$, the unitary representation U and the vacuum vector Ω. The state ω induced by Ω is called *vacuum state*. Notice also that, by assumption, $\mathfrak{A}$ is faithfully represented in its GNS representation induced by the vacuum state ω, and hence its identity representation will be called the *vacuum representation*. Notice that the isotony property implies that (1.15) is a functor. The causality axiom is also called *locality* axiom. A further requirement could be some tighter connection between the local algebras associated to double cones.

(a5) (Additivity) $\mathcal{A}(\widetilde{\mathcal{O}}) = \bigvee_{\mathcal{O}\subset\widetilde{\mathcal{O}}} \mathcal{A}(\mathcal{O})$ for every double cone $\widetilde{\mathcal{O}}$.

This additivity axiom can be reformulated by replacing $\widetilde{\mathcal{O}}$ with a generic wedge W. Additivity for wedges has been used in [60] to transport the Reeh-Schlieder property from wedge algebras to double cone algebras. Furthermore, with a very reasonable assumption of *weak additivity* it can be proved the following property [52]:

(a6) (Reeh-Schlieder) The vacuum vector Ω is cyclic for each local algebra $\mathcal{A}(\mathcal{O})$, with $\mathcal{O} \subseteq \mathbb{R}^{n+1}$ open and not empty.

Notice that, by the causality and the Reeh-Schlieder properties, Ω is a standard vector for $\mathfrak{A}$ and hence ω is faithful. The condition of locality can be strengthened by requiring (a7) (Haag duality) $\mathcal{A}(\mathcal{O}') = \mathcal{A}(\mathcal{O})'$ for every double cone $\mathcal{O}$.

This property is often satisfied by local quantum field theories describing bosonic particles. In other cases, Haag duality can be replaced by some twisted analogue [74]. Finally, the following property sets up a connection between the modular theory and the boost symmetries [10]:

(a8) (Bisognano-Wichmann) $\Delta_W^{-is} = U(\Lambda_W(2\pi s))$.

The Bisognano-Wichmann theorem can be proved under very general assumptions which include Reeh-Schlieder and Haag duality for the wedges W and W'. Here W is the right wedge (1.6), Δ_W is the modular operator of $\mathcal{A}(W)$ associated to the vacuum vector Ω and Λ_W is the one-parameter group of boosts preserving W (given by equation (1.14) in dimension $1+1$).

One last mathematical object concerning the algebraic formulation of quantum field theory is the *stress-energy tensor*. The stress-energy tensor of a local quantum field theory $(\mathcal{A}, U, \Omega)$ on $\mathbb{R}^{n+1}$ is a local object which is not included in the basic axioms of algebraic quantum field theory. However, the common mathematical structure is more or less the one depicted in [43]. In general, it corresponds to a family $T_{\mu\nu}(f)$ of sesquilinear forms on some dense domain $\mathcal{D}$, where f is a test function on $\mathbb{R}^{n+1}$ and $\mu, \nu = 0, \ldots, n$. More explicitly, one has a map

$$T_{\mu\nu} \colon \mathcal{D}(\mathbb{R}^{n+1}) \to \mathcal{Q}\,,$$

with $\mathcal{Q}$ the space of sesquilinear forms on $\mathcal{D}$, a dense U-invariant common core for the generators of translations $P = (P^\nu)_\nu$ containing the vacuum vector Ω. For each real test function f one has that $T(f)$ is an hermitian form, and the notation $(\xi|T_{\mu\nu}(f)\eta) = T_{\mu\nu}(f)(\xi, \eta)$ is commonly used. Also, we expect the identities $(\Omega|T_{\mu\nu}(\cdot)\Omega) = 0$, $T_{\mu\nu} = T_{\nu\mu}$ and $\partial_\mu T^{\mu\nu} = 0$ to be satisfied. Finally, if σ is a space-like plane then we expect that

$$\lim_n (\xi|T_{\mu\nu}(f_n)\eta) = (\xi|P^\nu\eta)$$

for some sequence $0 \leq f_n \leq 1$ of real test functions such that $\cap_n \mathrm{supp} f_n \subseteq \sigma$ and $f_n|_\sigma \rightharpoonup 1$, namely $f_n \to 1$ in the distributional sense when restricted to σ [43]. Finally, one could ask the stress-energy tensor to be a tempered distribution, so that it would make sense to talk about *stress-energy tensor density* $T_{\mu\nu}(x)$.

For a complete exposition, we conclude this section by describing the free scalar Klein-Gordon field of mass $m \geq 0$ on a $(n+1)$-dimensional Minkowski space. Free scalar fields on the Minkowski space are the simplest example of local quantum field theories and, as the name suggests, these theories describe noninteracting particles. There exist actually several formulations of the theory of free fields [7, 52, 88]. One way to proceed is the following [7].

We consider the Minkowski space $\mathbb{R}^{n+1}$ with inner product (1.13). Denote by $\mathcal{S}$ the space of Schwartz functions and by H_m^+ the *Lorentz hyperboloid* of mass $m \geq 0$, namely

H_m^+ is the manifold

$$H_m^+ = \{p \in \mathbb{R}^{n+1} \colon p^2 = m^2\,,\ p_0 > 0\}\,.$$

This manifold, also called *upper mass shell*, is endowed with a Lorentz invariant measure given, in local coordinates, by $d\mu(p) = (p^2+m^2)^{-1/2}dp$. The Hilbert space of this theory is the *bosonic Fock space* [3]

$$\Gamma(\mathcal{H}_1) = \mathbb{C}\Omega \oplus \bigoplus_{n>0} E_n \mathcal{H}_1^{\otimes n}\,,$$

with $\mathcal{H}_1 = L^2(H_m^+, d\mu)$ the *one particle space*. Here E_n is the projection onto the subspace of $E_n\mathcal{H}_1^{\otimes n}$ of totally symmetric elements. On this Fock space one can define the *creation operator* $a^*(\chi)$ and the *annihilation operator* $a(\chi)$, with χ in $\mathcal{H}_1$. Explicitly, for any $\Psi_n = E_n(\psi_1 \otimes \cdots \otimes \psi_n)$ one defines

$$\begin{aligned} a^*(\chi)\Psi_n &= (n+1)^{1/2} E_{n+1}(\chi \otimes \Psi_n)\,, \\ a(\chi)\Psi_n &= n^{-1/2} \sum_{j=1}^{n} (\chi|\psi_j) E_{n-1}(\psi_1 \otimes \cdots \otimes \widehat{\psi_j} \otimes \cdots \otimes \psi_n)\,, \end{aligned} \tag{1.16}$$

where by definition $E_0\Omega = \Omega$ and $a(\chi)\Omega = 0$, with Ω a vector of unital norm by definition called *vacuum vector*. These operators are closable, verify $\overline{a(\chi)} = a^*(\chi)^*$ and satisfy the *Canonical Commutation Relations (CCR)* $[a(\chi), a^*(\chi')] = (\chi|\chi')$ on their common core given by the algebraic sum of the subspaces $\mathcal{H}_n = E_n\mathcal{H}_1^{\otimes n}$. Notice that $a^*(\chi)$ is complex-linear in χ, while $a(\chi)$ is conjugate-linear in χ. Consider now a real valued Schwartz function f in $\mathcal{S}_\mathbb{R}$. Its Fourier transform is naturally given by

$$\widetilde{f}(p) = (2\pi)^{-(n+1)/2} \int e^{ip\cdot x} f(x) dx\,. \tag{1.17}$$

We then define an operator $E \colon \mathcal{S}_\mathbb{R} \to \mathcal{H}_1$ with dense range by restriction of $\widetilde{f}$ on H_m^+, that is $Ef = \widetilde{f}|_{H_m^+}$. The *Segal field* associated to f is the densely defined operator

$$\phi(f) = a^*(Ef) + a(Ef)\,.$$

If $W(f) = e^{i\phi(f)}$ is the associated *Weyl operator*, then we have the *Weyl relations*

$$W(f)W(g) = W(f+g)e^{-i\sigma(f,g)}\,, \quad W(f)^* = W(-f)\,,$$

where $\sigma(f,g) = \operatorname{Im}(Ef|Eg)$. Notice that the natural action u of $\mathcal{P}_+^\uparrow$ on $\mathcal{S}$ induces a unitary action U on $L^2(H_m^+, d\mu)$ by using the intertwining property of the Fourier transform (1.17). Therefore, the Fock functor $\Gamma(U)$ induces a unitary representation of $\widetilde{\mathcal{P}}_+^\uparrow$ on $\mathcal{H}$ satisfying the covariance property $\Gamma(U)W(f)\Gamma(U)^* = W(u.f)$. Finally, the local net of this theory is given by

$$\mathcal{A}(\mathcal{O}) = \{W(f) \colon f \in \mathcal{S}_\mathbb{R}\,,\ \mathrm{supp} f \subset \mathcal{O}\}''\,.$$

For different values of the mass $m \geq 0$ these theories are inequivalent in the sense that there is not a unitary map preserving all the field theory structures. These models verify

Haag duality and energy nuclearity on double cones [20, 60].

Lastly, we describe the stress energy tensor of this theory [28]. Let $\phi \in \mathcal{S}'$ be a solution of the Klein-Gordon equation, namely ϕ is a tempered distribution such that

$$(\Box + m^2)\phi = 0\,, \tag{1.18}$$

where $\Box = \partial_0^2 - \partial_1^2 - \cdots - \partial_n^2$. Then, the Fourier transform $\widetilde{\phi}$ of ϕ is supported on the closure $H_m = \overline{H_m^+ \cup -H_m^+}$. If ϕ is real, then $\widetilde{\phi}(-p) = \overline{\widetilde{\phi}(p)}$, so $\widetilde{\phi}$ is determined by its restriction to H_m^+. As known, if f, g are real Schwartz functions on $\mathbb{R}^n$, there is a unique smooth real solution ϕ of (1.18) with Cauchy data $\phi|_{x_0=0} = f$ and $\partial_0\phi|_{x_0=0} = g$. This solution is explicitly given by an integral formula and the partial functions $\phi(x_0, \cdot)$ are in the Schwartz space $\mathcal{S}$. We will call *wave* any real smooth solution of the Klein-Gordon equation with compactly supported Cauchy data, and denote by $\mathcal{T}$ the real linear space of such waves. The map $E\colon \mathcal{T} \to \mathcal{H}_1$ given by $E\phi = \widetilde{\phi}|_{H_m^+}$ is well defined, injective and its range is a real linear total subspace of $\mathcal{H}_1$. Let now $g = (g^{\mu\nu}) = (g_{\mu\nu})$ be the Lorentz metric. If ∂_μ is the partial derivative with respect to x_μ, then we will set $\partial^\mu = g^{\mu\nu}\partial_\nu$. Notice that, in our convention, $\partial^0 = \partial_0$. The stress-energy tensor density $(E\phi|T_{\mu\nu}(\cdot)E\phi)$ associated with a wave $\phi \in \mathcal{T}$ is then given by the function [28]

$$T_{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - g^{\mu\nu}\mathcal{L}\,, \quad \mathcal{L} = \frac{1}{2}\Big(g_{\mu\nu}\partial^\mu\phi\partial^\nu\phi - m^2\phi^2\Big)\,, \tag{1.19}$$

where $\mathcal{L}$ is the Lagrangian density. In particular,

$$T_{00} = \frac{1}{2}\Big(\sum_{\mu=0}^{n}(\partial^\mu\phi)^2 + m^2\phi^2\Big)\,, \quad T_{0\nu} = \partial^0\phi\partial^\nu\phi\,, \quad \nu = 1, \dots, n\,.$$

1.7 DHR charges

DHR theory was historically developed in [41]. The starting point of this successful theory is the choice of a selection criterion for the representations ρ of a local theory $\mathfrak{A}$ which are of interest in elementary particle physics. The motivation for this criterion is the idealization of the absence of matter at infinity. Of course, such a mathematical issue appears since in general $\mathfrak{A}$ has infinitely many inequivalent irreducible representations. In this section we illustrate a brief overview of this topic [72].

Let $(\mathcal{A}, U, \Omega)$ be a local QFT on the Minkowski space of spacetime dimension $n+1$ on the vacuum Hilbert space $\mathcal{H}$. Let $\mathfrak{A}$ be the C^*-algebra of quasi-local observables. More in general, given a spacetime region F, let $\mathfrak{A}(F)$ be the C^*-algebra generated by all the von Neumann algebras $\mathcal{A}(\mathcal{O})$ where $\mathcal{O}$ runs on the double cones contained in F. We denote by $\mathcal{A}(F)$ the weak closure $\mathfrak{A}(F)''$ of $\mathfrak{A}(F)$ and we assume weak additivity, so that the Reeh-Schlieder theorem holds and the vacuum vector is standard for F whenever F and F' have non-empty interiors. We also assume Haag duality.

Let now ρ be a covariant representation of $\mathfrak{A}$ on a Hilbert space $\mathcal{H}_\rho$, so there exists a positive energy unitary representation U_ρ of $\widetilde{\mathcal{P}}_+^\uparrow$ on $\mathcal{H}_\rho$ such that

$$\rho(U(g)xU(g)^*) = U_\rho(g)\rho(x)U_\rho(g)^*\,, \quad x \in \mathfrak{A}\,,\, g \in \widetilde{\mathcal{P}}_+^\uparrow\,. \tag{1.20}$$

Assume for the moment that U_ρ is *massive*, namely the energy-momentum spectrum has an isolated lower mass shell. Then, for any spacelike cone $\mathcal{S}$ in the Minkowski spacetime, the restriction $\rho|_{\mathfrak{A}(\mathcal{S}')}$ is unitarily equivalent to $\mathrm{id}|_{\mathfrak{A}(\mathcal{S}')}$, with id the vacuum representation. Thus, up to unitary equivalence, we may choose a spacelike cone $\mathcal{S}_0$, identify $\mathcal{H}_\rho$ with $\mathcal{H}$ and assume that $\rho(x) = x$ for x in $\mathfrak{A}(\mathcal{S}_0')$. We then say that ρ is *localised* in $\mathcal{S}_0$. This motivating discussion motivates the DHR selection criterion and leads to the following definition.

Definition 29. Let $(\mathcal{A}, U, \Omega)$ be a local QFT on the Minkowski space. A *DHR charge*, or also a *sector*, is the unitary equivalence class $[\rho]$ of a representation ρ of $\mathfrak{A}$ such that $\rho|_{\mathfrak{A}(\mathcal{O}')}$ is equivalent to $\mathrm{id}|_{\mathfrak{A}(\mathcal{O}')}$ for some double cone $\mathcal{O}$. Furthermore, this equivalence is supposed to still hold if $\mathcal{O}$ is replaced with $g.\mathcal{O}$ for some g is in $\mathcal{P}_+^\uparrow$. The corresponding endomorphisms of $\mathfrak{A}$ are called *localized endomorphisms*, where we will say that ρ is *localized*, or also *supported*, in $\mathcal{O}$ if $\rho(x) = x$ for x in $\mathfrak{A}(\mathcal{O}')$.

We now want to further describe the structure of DHR charges. If we assume that Haag duality holds for spacelike cones and that ρ is localized in $\mathcal{S}_0$, then ρ maps $\mathfrak{A}(\mathcal{S}_0)$ to $\mathcal{A}(\mathcal{S}_0)$. Now, $\rho|_{\mathfrak{A}(\mathcal{S}_0)}$ is normal because $\mathcal{S}_0 \subset \mathcal{S}'$ for some spacelike cone $\mathcal{S}$. So $\rho|_{\mathfrak{A}(\mathcal{S}_0)}$ extends to a normal endomorphism $\rho_{\mathcal{S}_0}$ of $\mathcal{A}(\mathcal{S}_0)$, and similarly to a normal endomorphism ρ_W of $\mathcal{A}(W)$ if $W \supset \mathcal{S}_0$ is a wedge. We may loosely say that $\rho_{\mathcal{S}_0}$ and ρ_W are the restrictions of ρ on $\mathcal{A}(\mathcal{S}_0)$ and $\mathcal{A}(W)$ and still denote them simply by ρ if it is clear from the context that we are dealing with restrictions.

The important fact of the DHR criterion is that, by identifying sectors with localized endomorphisms, they can be composed. This allows us to define the composition of sectors by $[\rho_1][\rho_2] = [\rho_1\rho_2]$. This induces a structure of monoidal C^*-category on the DHR charges. Explicitly, we have a category with localized endomorphisms as objects, intertwiners as arrows and composition as tensor product. Such a category, denoted by $\mathcal{T}_\mathcal{A}$, is called *superselection theory*. We refer to Appendix A for further details.

Now consider an irreducible DHR charge ρ. Suppose the covariance property (32) to be satisfied for space-time translations. We also assume, as above, the corresponding representation U_ρ to satisfy the spectrum condition and to be massive. Then, it can be shown the existence of a finite integer number $d(\rho)$ called the *statistical dimension* of ρ. Allowing the case $d(\rho) = \infty$, the statistical dimension can be defined for any localized endomorphism [48]. We will provide further details of this dimension function in Appendix A. The statistical dimension $d(\rho)$ is a natural number uniquely determined by the sector $[\rho]$, it is multiplicative with respect to the composition of sectors and it is additive with respect to direct sums of representations. If ρ is localized in a double cone $\mathcal{O}$, the DHR dimension $d(\rho)$ of ρ turns out to be the square root of the minimal

Jones index $[\mathcal{A}(\mathcal{O})\colon \rho(\mathcal{A}(\mathcal{O}))]$ [65]. If ρ has finite Jones index, then there exists a standard left inverse $\Phi = \Phi_\rho$ of ρ, namely a completely positive map $\Phi\colon \mathfrak{A} \to \mathfrak{A}$ such that $\Phi\cdot\rho = \mathrm{id}$. Indeed, $\Phi = \rho^{-1}\cdot\varepsilon$ with $\varepsilon\colon \mathfrak{A} \to \rho(\mathfrak{A})$ the minimal conditional expectation [65].

Given an endomorphism ρ localized in $\mathcal{O}$ as above, we shall consider the charged state $\psi = \psi_\rho$ given by $\psi_\rho = \omega \cdot \Phi_\rho$, with ω the vacuum state. Note that ψ is localised in $\mathcal{O}$ as above, namely $\psi = \omega$ on $\mathfrak{A}(\mathcal{O}')$, and that, by composing ψ with the adjoint action of a localised unitary, we get a state localised in any given double cone. If W is a wedge region containing $\mathcal{O}$, then $\psi|_{\mathfrak{A}(W)}$ extends to a normal faithful state of $\mathcal{A}(W)$ (ψ is inner automorphism equivalent to a state localised in W') that we denote by ψ_W, and similarly $\omega_W = \omega|_{\mathcal{A}(W)}$.

Let now $u_g^\rho = U_\rho(g)U(g)^*$, with g in $\widetilde{\mathcal{P}}_+^\uparrow$, be the *covariance cocycle* of some covariant localised endomorphism ρ with finite index. Thus

$$\rho(x) = u_g^\rho \rho_g(x) u_g^{\rho *}\,, \qquad \rho_g(x) = U(g)\rho(U(g)^* x U(g))U(g)^*\,.$$

If ρ is localised in $\mathcal{O}$ then the charge ρ_g is localised in $g.\mathcal{O}$. If $\widetilde{\mathcal{O}}$ is a double cone containing both $\mathcal{O}$ and $g.\mathcal{O}$, then then both ρ and ρ_g act identically on $\mathfrak{A}(\widetilde{\mathcal{O}}')$, and thus by Haag duality we have that u_g^ρ is in $\mathcal{A}(\mathcal{O})$. With W a wedge region and Λ_W the corresponding boost one parameter group, let ρ be localised in W. Then $u^\rho_{\Lambda_W(s)}$ is in $\mathcal{A}(W)$ for any s in $\mathbb{R}$. More specifically, we have the relation [66]

$$u^\rho_{\Lambda_W(s)} = d(\rho)^{is}(D\psi_W\colon D\omega_W)_s\,, \quad s \in \mathbb{R}\,. \tag{1.21}$$

Thus, while the Bisognano-Wichmann theorem sets up a connection between the modular theory and the vacuum boost symmetries, formula (1.21) sets up a connection between the relative modular operator and the boost symmetries in the charged representation. Indeed, this formula is equivalent to

$$2\pi K_{\rho,W} = -\log \Delta_{\eta,\xi,W} - \log d(\rho)\,,$$

where ξ is the vacuum vector, η is any standard vector for $\mathcal{A}(W)$ representing ψ_W and $K_{\rho,W}$ is the *modular hamiltonian*, namely the selfadjoint generator of the boost one parameter unitary group $U_\rho(\Lambda_W(\cdot))$. From a physical point of view, $E_\rho = (\Omega|K_{\rho,W}\Omega)$ is the is mean vacuum energy for a Rindler observer.

Chapter 2

Quantum Null Energy Condition on conformal nets

2.1 Statement of the Quantum Null Energy Condition

This section aims to provide a brief overview about the Quantum Null Energy Condition (QNEC). The QNEC was originally stated in [14], then reformulated in [58] and finally rigorously treated in [27, 72]. As mentioned in the introduction, in [14] a Quantum Null Energy Condition (QNEC) is defined as a stress-energy tensor density lower bound which is expected to be satisfied by most reasonable quantum fields. More specifically, consider a local QFT on the Minkowski space $\mathbb{R}^{n+1}$ with stress-energy tensor T. Then, one uses a function V in order to "cut" a null plane and define a region $\mathcal{R}$ as in figure 2.1.

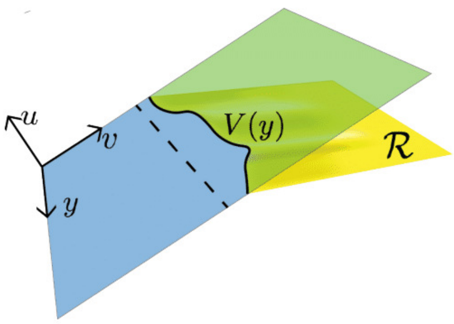


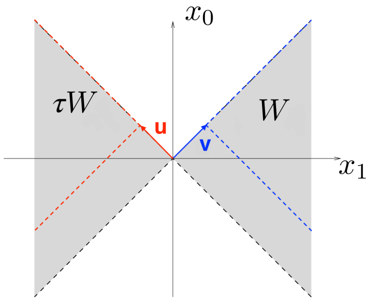
Figure 2.1. This image depicts a section of the plane $u = t - x = 0$. The region $\mathcal{R}$ is defined to be one side of a Cauchy surface split by the codimension-two entangling surface $\partial\mathcal{R} = \{u = 0, v = V(y)\}$. The dashed line corresponds to a flat cut of the null plane.

Then, by considering $V_t = (1 + t)V$ one obtains a one-parameter family of regions $\mathcal{R}_t$. If we denote by $S(t)$ the von Neumann entropy of a state ψ restricted to $\mathcal{R}_t$, or rather to some von Neumann algebra $\mathcal{A}(\mathcal{R}_t)$. Then, the QNEC [14] states that

$$\langle T_{vv}(y)\rangle \geq \frac{\hbar}{2\pi} S''(y)\,, \tag{2.1}$$

where $\langle T_{vv}(y)\rangle = \langle T_{vv}(y)\rangle_\psi$ and $S''(y)$ is defined after a limit procedure on $S''(t)$ as V approaches some delta distribution supported on y [58]. However, this statement of the QNEC lacks mathematical rigour. For example, the von Neumann entropy $S(t)$ has to be constantly infinite as long as we expect the identity $\mathcal{A}(\mathcal{R}_t) = \mathcal{A}(\mathcal{R}_t'')$ to hold [76].

In order to fix this problem, a rigorous statement of the QNEC can be given as follows [72]. Let $(\mathcal{A}, U, \Omega)$ be a local QFT on the Minkowski space $\mathbb{R}^{n+1}$ with vacuum state ω and C^*-algebra of quasi-local observables $\mathfrak{A}$.



Let W be the right wedge region $x_1 > |x_0|$. We will use the coordinates $u = (x_0 - x_1)/\sqrt{2}$, $v = (x_0 + x_1)/\sqrt{2}$ and $y_k = x_k$ for $k > 1$. Let $V(y)$ be a non-negative continuous function of $y = (y_2, \ldots, y_n)$ and define

$$W_V = \{(u, v, y) \colon u < 0\,, v > V(y)\}\,. \tag{2.2}$$

We now set $V_t = (1+t)V$ for t in $\mathbb{R}$ and $\mathcal{M}_t = \mathcal{A}(W_{V_t})$. More generally, after a Poincaré transformation g one can associate a family of deformed wedges $gW_{V_t} = g(W_{V_t})$ and define $\mathcal{M}_t = \mathcal{A}(gW_{V_t})$. Notice that $W' = \tau W$, with τ the reflection with respect to the x_0 axis.

Definition 30. We will say that a state ψ of $\mathfrak{A}$ satisfies the *Quantum Null Energy Condition (QNEC)* if $S(t) = S_{\mathcal{M}_t}(\psi\|\omega)$ is convex for any couple (g, V) as above.

According to Definition 30, the QNEC does not involve any stress-energy tensor and it is not formulated as a null energy lower bound. However, this formulation of the QNEC is motivated by some physical arguments of [58], in which the inequality (2.1) is (not rigorously) shown to be equivalent to the positivity of the second derivative of some relative entropy. The convexity of the relative entropy has been rigorously proved in a model independent setting for a very wide class of states.

Theorem 31. *[27] Let $\mathcal{N} \subseteq \mathcal{M}$ be a $\pm$hsm inclusion with standard vector Ω giving the state ω. Denote by P the positive generator of translations and by $(\mathcal{M}_t)_{t\in\mathbb{R}}$ the associated flow of von Neumann algebras. If $\psi(x) = (\eta|x\eta)$ is a vector state with finite null energy, namely such that*

$$P_\eta = (\eta|P\eta) < +\infty\,, \tag{2.3}$$

then the relative entropy $S(t) = S_{\mathcal{M}_t}(\psi\|\omega)$ is convex. Furthermore, if $S(t_0)$ is finite then we have

$$-S'(t) = 2\pi \inf_{w'\in C_t'} P_{w'\eta}\,, \quad t \geq t_0 \quad a.e.\,, \tag{2.4}$$

where C_t' is the set of all the isometries w' in $\mathcal{M}_t'$ such that the complement relative entropy $\bar{S}_{w'}(t) = S_{\mathcal{M}_t'}(\psi_{w'}\|\omega)$ is finite, with $\psi_{w'}(x) = (w'\eta|xw'\eta)$. Identity (2.4) is satisfied at each point such that $S'(t)$ exists and on such points it can be computed by

$$-S'(t) = 2\pi \inf_s P_s(t) = \pi\widehat{P}(t)\,. \tag{2.5}$$

In the above notation, we have

$$P_s(t) = P_{u'_s(t)\eta}\,, \quad u'_s(t) = (D\omega\colon D\psi\,; \mathcal{M}'_t)_s\,, \tag{2.6}$$

and $\widehat{P}(t) = P_{\widehat{\eta}_t}$*, where* $\widehat{\eta}_t$ *is the unique vector in the natural cone of* $\mathcal{M}_t$ *representing the state* ψ*.*

Actually, the provided proof refers to -hsm inclusions, but the +hsm case can be similarly proved. It is also shown that the null energies (2.6) are finite and that the infimum (2.5) is obtained as $s \to \pm\infty$ if the inclusion is $\mp$hsm.

In this chapter, based on the paper [84], we study the QNEC on a generic conformal net. Our proof is based on a direct computation on the Virasoro nets and then on the use of some conditional expectations in order to extend the result to a generic conformal net.

2.2 Conformal nets

Chiral Conformal Field Theories (CFTs) describe one chiral half of a conformal field theory in $1+1$-dimensions and are well-investigated. They are described in the algebraic setting by *conformal nets*, namely local nets of von Neumann algebras parametrized by open intervals of the circle. We now briefly recall some basic definitions about conformal nets. We refer to [34, 45, 93] for further treatments of the topic.

Let $\mathcal{K}$ be the family of all the open, nonempty and non dense intervals of the circle. For I in $\mathcal{K}$, I' denotes the interior of the complement. The Möbius group Möb acts on the circle by linear fractional transformations. A *Möbius covariant net* $(\mathcal{A}, U, \Omega)$ consists of a family $\{\mathcal{A}(I)\}_{I\in\mathcal{K}}$ of von Neumann algebras acting on a separable Hilbert space $\mathcal{H}$, a strongly continuous unitary representation U of Möb and a vector Ω in $\mathcal{H}$, called the *vacuum vector*, satisfying the following properties:

(i) $\mathcal{A}(I_1) \subseteq \mathcal{A}(I_2)$ if $I_1 \subseteq I_2$ (isotony),
(ii) $\mathcal{A}(I_1) \subseteq \mathcal{A}(I_2)'$ if $I_1 \subseteq I_2'$ (locality),
(iii) $U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(g.I)$ for every g in Möb and I in $\mathcal{K}$ (Möbius covariance),
(iv) the representation U has *positive energy*, namely the generator of rotations has non-negative spectrum (positivity of the energy),
(v) Ω is cyclic for the von Neumann algebra $\bigvee_{I\in\mathcal{K}} \mathcal{A}(I)$, and up to a scalar Ω is the unique Möb-invariant vector of $\mathcal{H}$ (vacuum).

By the Howe-Moore vanishing theorem, it follows by the axioms (iv) and (v) that the vacuum vector Ω is, up to a phase, the only vector fixed by the subgroups of rotations, translations and dilations defined below (2.9). With these assumptions, the following properties automatically hold [34, 45]:

(vi) $\mathcal{A}(I') = \mathcal{A}(I)'$ for every I in $\mathcal{K}$ (Haag duality),

(vii) $\mathcal{A}(I) \subseteq \bigvee_\alpha \mathcal{A}(I_\alpha)$ if $I \subseteq \bigcup_\alpha I_\alpha$ (additivity),

(viii) if I_+ is the upper half of the circle and Δ is the modular operator associated to $\mathcal{A}(I_+)$ and Ω, then for every t in $\mathbb{R}$ we have

$$\Delta^{it} = U(\delta_{-2\pi t}), \tag{2.7}$$

where δ is the one parameter group of dilations (Bisognano-Wichmann),

(ix) each local algebra $\mathcal{A}(I)$ is a type III factor and $\bigvee_{I \in \mathcal{I}_\mathbb{R}} \mathcal{A}(I) = B(\mathcal{H})$, with $\mathcal{I}_\mathbb{R}$ the set of all the open, nonempty and non dense intervals of $S^1 \setminus \{-1\}$ (irreducibility).

Definition 32. By a *conformal net*, or also a $\mathrm{Diff}_+(S^1)$-*covariant net*, we shall mean a Möb-covariant net $(\mathcal{A}, U, \Omega)$ which satisfies the following condition:

(x) U extends to a projective unitary representation of $\mathrm{Diff}_+(S^1)$, the Fréchet Lie group of the orientation preserving diffeomorphisms of the circle, such that

$$U(\rho)\mathcal{A}(I)U(\rho)^* = \mathcal{A}(\rho.I), \quad \rho \in \mathrm{Diff}_+(S^1), \ I \in \mathcal{K}.$$

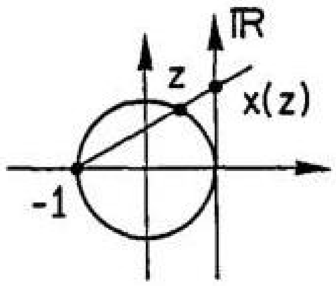
Furthermore, we require that

$$U(\rho)xU(\rho)^* = x, \quad x \in \mathcal{A}(I) \tag{2.8}$$

if $\operatorname{supp}\rho \subset I'$, with $\operatorname{supp}\rho$ the *support* of ρ, namely the closure of the complement of the set of the points z such that $\rho(z) = z$.

A Möbius covariant net is either also conformal or not, but if it is, the extension of the representation to $\mathrm{Diff}_+(S^1)$ is unique [22, 23]. It is known that to any representation U as above satisfying (iv) and (x), it can uniquely assigned a real number $c \geq 0$ called the *central charge* of the representation. This number will be called the *central charge* of the conformal net. In a conformal net, the following is automatic [75]:

(xi) if $\bar{I} \subset J$ then there is a type I factor $\mathcal{R}$ such that $\mathcal{A}(I) \subset \mathcal{R} \subset \mathcal{A}(J)$ (split property).



If the interval I does not contain the point -1, it is a common procedure to pass from the *circle picture* to the *real line picture* [34, 45]. Namely, one can change variables $z = C(x)$ by using the Cayley transform

$$C(x) = (1 + ix)/(1 - ix),$$

so that I can be identified with a proper open interval of the real line. In the real line picture the point -1 of S^1 corresponds to ∞, where $\mathbb{R} \cup \{\infty\}$ is the classical Alexandroff compactification of the real line. Thanks to this identification, we can easily define the one parameters groups of rotations, dilations and translations mentioned above:

$$R_\theta.z = e^{i\theta}z, \ z \in S^1, \qquad \delta_t.x = e^t x, \ x \in \mathbb{R}, \qquad \tau_a.x = x + a, \ x \in \mathbb{R}. \tag{2.9}$$

By using the real line picture, conformal nets can be used to describe chiral CFTs on the $1+1$ dimensional Minkowski space. To be more explicit, consider a conformal net $(\mathcal{A}, U, \Omega)$ in its real line picture. If we denote by (x_0, x_1) our spacetime coordinates, then by passing to light ray coordinates $\xi_\pm = x_0 \pm x_1$ we can denote by $\mathcal{L}_\pm = \{\xi \colon \xi_\pm = 0\}$ the two light ray lines. If $I_\pm \subseteq \mathcal{L}_\pm$ are open intervals, then for the region $\mathcal{O} = I_+ \times I_-$ we can define the corresponding local algebra by

$$\mathcal{A}(\mathcal{O}) = \mathcal{A}(I_+) \otimes \mathcal{A}(I_-)\,, \tag{2.10}$$

where the tensor product is the spatial tensor product of von Neumann algebras. Similar techniques can be used to define other types of Conformal Field Theory [9, 56].

We provide a few more definitions. Let $(\mathcal{A}, U, \Omega)$ be a Möbius covariant net on a Hilbert space $\mathcal{H}$. We call a family $\mathcal{B} = \{\mathcal{B}(I)\}_{I \in \mathcal{K}}$ of von Neumann subalgebras $\mathcal{B}(I) \subseteq \mathcal{A}(I)$ a *subnet of* $\mathcal{A}$ if it satisfies isotony and Möbius covariance with respect to U. We will use the notation $\mathcal{B} \subseteq \mathcal{A}$ to denote the subnets $\mathcal{B}$ of $\mathcal{A}$. If $\mathcal{A}(I)' \cap \mathcal{B}(I) = \mathbb{C}$ for one (and hence for all) interval I in $\mathcal{K}$, then the inclusion $\mathcal{B} \subseteq \mathcal{A}$ is said to be *irreducible.* If we denote by $e = [\mathcal{H}_\mathcal{B}]$ the orthogonal projection onto $\mathcal{H}_\mathcal{B} = \overline{\bigvee_{I \in \mathcal{K}} \mathcal{B}(I)\Omega}$, then it is easy to notice that e is in the commutant of all the von Neumann algebras $\mathcal{B}(I)$ and that it commutes with U. Then $\mathcal{B}$ induces a Möbius covariant net on $e\mathcal{H}$ by considering the induced von Neumann algebras $e\mathcal{B}(I)e$ with unitary representation the restriction of U on $e\mathcal{H}$. By the Reeh-Schlieder property Ω is standard for all the $\mathcal{B}(I)$ and hence e is actually the Jones projection of each inclusion $\mathcal{B}(I) \subseteq \mathcal{A}(I)$ (see Proposition 3.1.4. of [54]). If we have an inclusion of nets $\mathcal{B} \subseteq \mathcal{A}$, then for each interval I there is a canonical faithful normal conditional expectation $\varepsilon_I \colon \mathcal{A}(I) \to \mathcal{B}(I)$ which preserves the vacuum state ω by Bisognano-Wichmann and the Takesaki's theorem ([90], Theorem IX.4.2.).

Proposition 33. *Let $\mathcal{B} \subseteq \mathcal{A}$ be an inclusion of a Möbius covariant nets. Then all the canonical conditional expectations $\varepsilon_I \colon \mathcal{A}(I) \to \mathcal{B}(I)$ extend to a unique vacuum-preserving conditional expectation $\varepsilon \colon \mathfrak{A} \to \mathfrak{B}$, with $\mathfrak{A}$ and $\mathfrak{B}$ the C^*-algebras generated by the local algebras $\mathcal{A}(I)$ and $\mathcal{B}(I)$ respectively, namely $\mathfrak{A}$ is the norm closure of the union of the local algebras $\mathcal{A}(I)$ and similarly for $\mathfrak{B}$.*

Proof. Denote by $\omega(\cdot) = (\Omega| \cdot \Omega)$ the vacuum state and by e the orthogonal projection onto $\mathcal{H}_\mathcal{B} = \overline{\bigvee_{I \in \mathcal{K}} \mathcal{B}(I)\Omega}$. As mentioned above, for each interval I in $\mathcal{K}$ we have that $e(x\Omega) = \varepsilon_I(x)\Omega$ for x in $\mathcal{A}(I)$. As the vacuum is locally faithful by Reeh-Schlieder, this implies that the conditional expectations are compatible, namely ε_J is an extension of ε_I whenever $I \subseteq J$. Therefore, for x in the union $\bigcup_{I \in \mathcal{K}} \mathcal{A}(I)$ we can define $\varepsilon(x)$ by setting $\varepsilon(x) = \varepsilon_I(x)$ whenever x is in $\mathcal{A}(I)$. The map ε is bounded since every ε_I has unital norm, hence we can continuously extend ε to $\mathfrak{A}$ (this procedure is sometimes known as the BLT theorem [88]). Finally, ε is a conditional expectation since by continuity ε is a positive $\mathfrak{B}$-linear projection, and the identity $\omega = \omega \cdot \varepsilon$ follows as well. □

The *index* $[\mathcal{A} \colon \mathcal{B}]$ of the inclusion $\mathcal{B} \subseteq \mathcal{A}$ is defined as the Jones index $[\mathcal{A}(I) \colon \mathcal{B}(I)]$ with respect to the conditional expectation ε_I [59]. Such an index does not depend on I. If the index is finite, then the inclusion is irreducible. More in general, this index can be defined whenever a Möbius covariant net $\mathcal{A}$ is an *extension* of a Möbius covariant net $\mathcal{B}$, namely if $\mathcal{B}$ is unitarily equivalent to a subnet of $\mathcal{A}$.

We now provide some notions about the representation theory of conformal nets. A (locally normal) *DHR (Doplicher-Haag-Roberts) representation* of a conformal net $(\mathcal{A}, U, \Omega)$ is a family $\rho = \{\rho_I\}_{I\in\mathcal{K}}$ of normal representations ρ_I of the von Neumann algebras $\mathcal{A}(I)$ on some Hilbert space $\mathcal{H}_\rho$ such that $\rho_I = \rho_J|_{\mathcal{A}(I)}$ if $I \subseteq J$. We say that two DHR representations ρ_1 and ρ_2 are *equivalent* if there is some unitary operator U from $\mathcal{H}_{\rho_1}$ to $\mathcal{H}_{\rho_2}$ such that $U\rho_{1,I}(x) = \rho_{2,I}(x)U$ for every x in $\mathcal{A}(I)$ and I in $\mathcal{K}$. The DHR representation induced by the identity is called the *vacuum representation.* A DHR representation ρ is said to be *irreducible* if $\bigvee_{I\in\mathcal{K}} \rho_I(\mathcal{A}(I)) = B(\mathcal{H}_\rho)$. If a topological group $\mathcal{G}$ acts continuously on S^1 by elements of $\mathrm{Diff}_+(S^1)$, a DHR representation ρ is said to be $\mathcal{G}$*-covariant* if there exists a strongly continuous unitary projective representation U_ρ of $\mathcal{G}$ such that

$$\operatorname{Ad} U_\rho(g) \cdot \rho_I(x) = \rho_{g.I}(\operatorname{Ad} U(\iota(g)) \cdot x)\,, \quad x \in \mathcal{A}(I)\,,$$

for all g in $\mathcal{G}$ and I in $\mathcal{K}$, where $\iota\colon \mathcal{G} \to \mathrm{Diff}_+(S^1)$ is the induced homomorphism. A locally normal DHR representation is automatically Möb-covariant [33].

A larger class of representations of a conformal net is given by the so-called solitonic representations. In the following, we will denote by $\mathcal{I}_\mathbb{R}$ the set of all the open, nonempty and non dense intervals of $S^1 \setminus \{-1\}$. A (locally normal) *soliton* σ of a conformal net $(\mathcal{A}, U, \Omega)$ is a family of maps $\sigma = \{\sigma_I\}_{I\in\mathcal{I}_\mathbb{R}}$ where σ_I is a normal representation of the von Neumann algebra $\mathcal{A}(I)$ on a fixed Hilbert space $\mathcal{H}_\sigma$ such that $\sigma_I = \sigma_J|_{\mathcal{A}(I)}$ if $I \subseteq J$. If $\mathcal{G}$ is a topological group equipped with some homomorphism $\iota\colon \mathcal{G} \to \mathrm{Diff}_+(S^1)$, then we will say that a soliton σ is *locally* $\mathcal{G}$*-covariant* if there is a unitary projective continuous representation U_σ of $\mathcal{G}$ which satisfies the following property: if I is in $\mathcal{I}_\mathbb{R}$ and V is a connected neighborhood of the identity in $\mathcal{G}$ such that $g.I$ is in $\mathcal{I}_\mathbb{R}$ for every g in V, then $\operatorname{Ad} U_\sigma(g)\sigma_I(x) = \sigma_{\iota(g).I}(\operatorname{Ad} U(\iota(g)) \cdot x)$ for every x in $\mathcal{A}(I)$. With $\mathbb{R}_\pm$ considered as elements of $\mathcal{I}_\mathbb{R}$, the *index* of a soliton σ is the Jones index of the inclusion $\sigma(\mathcal{A}(\mathbb{R}_+)) \subseteq \sigma(\mathcal{A}(\mathbb{R}_-))'$.

We now illustrate a class of proper solitonic representations of a conformal net constructed in [34]. Let $\mathrm{Diff}_+(S^1, -1)$ be the class of orientation preserving homeomorphisms ν of S^1 which have the following properties:

(i) $\nu(-1) = -1$,

(ii) ν is smooth on $S^1 \setminus \{-1\}$, the left and right derivative at all orders exist at the point -1, and the first left and right derivatives are nonzero.

Let now $(\mathcal{A}, U, \Omega)$ be a conformal net on S^1 on some Hilbert space $\mathcal{H}$. For ν in $\mathrm{Diff}_+(S^1, -1)$ and for every interval I in $\mathcal{I}_\mathbb{R}$ we choose ν_I in $\mathrm{Diff}_+(S^1)$ which agrees with ν on I (there is such ν_I even if one of the endpoints of I is -1 [34]). We denote by σ_ν the family of maps

$$\sigma_\nu^I\colon \mathcal{A}(I) \to B(\mathcal{H})\,, \quad \sigma_\nu^I(x) = \operatorname{Ad}U(\nu_I)(x)\,,$$

where $\nu \in \mathrm{Diff}_+(S^1,-1)$ and $I \in \mathcal{I}_{\mathbb{R}}$.

Proposition 34. *[34] For ν in $\mathrm{Diff}_+(S^1,-1)$, σ_ν is an irreducible soliton of the conformal net $\mathcal{A}$ with index* 1*.*

As for DHR representations, solitons can be used to construct *solitonic states.* In this case, as the Jones index is 1, the solitonic state associated to some homeomorphism ν in $\mathrm{Diff}_+(S^1,-1)$ is given by

$$\omega_\nu = \omega \cdot \sigma_\nu^{-1}\,. \tag{2.11}$$

By use of the modular theory, and in particular by using the fact that on III_1-type factors the modular group σ_t is inner if and only if $t=0$, it can be shown that if ν has different left and right derivatives then σ_ν is a proper soliton [34]. Let us introduce the notation for left and right derivatives:

$$\partial_\pm \nu(-1) = -i \lim_{\theta\to 0^\pm} \frac{\nu(-e^{i\theta})-\nu(-1)}{\theta}\,.$$

We will denote their ratio by $r(\nu) = \partial_+\nu(-1)/\partial_-\nu(-1)$. Furthermore, we will denote by $\mathrm{Diff}_+^{1,\mathrm{ps}}(S^1)$ the group of piecewise smooth C^1-diffeomorphisms of S^1, namely γ belongs to $\mathrm{Diff}_+^{1,\mathrm{ps}}(S^1)$ if it is a C^1 diffeomorphism and S^1 can be decomposed into finitely many closed intervals on each of which γ is smooth in the interior and has derivatives of all orders at the end points. Finally, we will denote by $\mathrm{Diff}_{+,0}^{1,\mathrm{ps}}(S^1)$ the subgroup of $\mathrm{Diff}_+^{1,\mathrm{ps}}(S^1)$ given by all the homeomorphisms γ fixing -1 and by $\mathrm{Diff}_{+,1}^{1,\mathrm{ps}}(S^1)$ the subgroup of $\mathrm{Diff}_{+,0}^{1,\mathrm{ps}}(S^1)$ of all the elements γ such that $\gamma'(-1)=1$.

Theorem 35. *[34] If ν in $\mathrm{Diff}_+(S^1,-1)$ verifies $r(\nu)\neq 1$, then σ_ν is a proper, irreducible $\mathrm{Diff}_{+,0}^{1,\mathrm{ps}}(S^1)$-covariant soliton of $\mathcal{A}$. Furthermore, σ_{ν_1} and σ_{ν_2} are unitarily equivalent if and only if $r(\nu_1)=r(\nu_2)$.*

Finally, we end this section by describing the most simple example of conformal net. We consider an irreducible, unitary projective representation U of $\mathrm{Diff}_+(S^1)$. We require U to be with positive energy, namely we assume the group of rotations to have a positive generator (that is the induced unitary representation of its universal covering has a positive generator). Such a representation is uniquely determined by a couple of values (c,h), where c is called the *central charge* and h is called the *trace anomaly.* More precisely, $h\geq 0$ is the lowest eigenvalue of the generator of rotations and $c \geq 0$ uniquely determines the 2-cocycle

$$U(\rho_1)U(\rho_2) = e^{icB(\rho_1,\rho_2)}U(\rho_1\rho_2)\,, \tag{2.12}$$

with B the *Bott cocycle* [44]

$$B(\rho_1,\rho_2) = -\frac{1}{48\pi}\mathrm{Re}\int \ln((\rho_1\rho_2)'(z))\frac{d}{dz}\ln(\rho_2'(z))dz\,. \tag{2.13}$$

It can be shown that the Bott cocycle lifts to a cocycle of the universal covering $\mathrm{Diff}_+(S^1)^\sim$ of $\mathrm{Diff}_+(S^1)$, and a projective unitary representation U of $\mathrm{Diff}_+(S^1)^\sim$ satisfying (2.12) is called a *multiplier representation.* The mentioned irreducible positive energy representation U of $\mathrm{Diff}_+(S^1)$ exists if and only if $c \geq 1$ and $h \geq 0$ (continuous series representation), or $(c,h) = (c(m), h_{p,q}(m))$, where [55]

$$c(m) = 1 - \frac{6}{(m+2)(m+3)}\,, \quad h_{p,q}(m) = \frac{(p(m+1)-qm)^2 - 1}{4m(m+1)}\,,$$

with m, p and q integers such that $m \geq 3$ and $1 \leq q \leq p \leq m-1$ (discrete series representation). If $h = 0$, then U is called a *vacuum representation*. Since it can be shown the existence of a vector Ω of norm one which is, up to a phase, the unique vector fixed by the rotation group [45], by the Howe-Moore theorem Ω is also, up to a phase, the unique vector fixed by Möb $\subseteq \mathrm{Diff}_+(S^1)$.

Definition 36. If U is a vacuum representation of $\mathrm{Diff}_+(S^1)$, then the *Virasoro net of central charge* c is the conformal net $(\mathcal{V}_c, U, \Omega)$ with local algebras

$$\mathcal{V}_c(I) = \{U(\rho) \colon \operatorname{supp}\rho \subset I\}''\,.$$

Also in this case, these models are endowed with a stress energy tensor density described in detail in the next chapter. Virasoro nets have a very different nature for different values of the central charge $c < 1$ and $c \geq 1$.

Definition 37. A conformal net $(\mathcal{A}, U, \Omega)$ is said to be *strongly additive* if, for $I_1, I_2 \in \mathcal{K}$ adjacent intervals and $I = (I_1 \cup I_2)'' = \overline{I_1 \cup I_2}^{\circ}$, the identity $\mathcal{A}(I_1) \vee \mathcal{A}(I_2) = \mathcal{A}(I)$ holds. The *μ-index* of $\mathcal{A}$, denoted by $\mu(\mathcal{A})$, is the minimal Jones index of the inclusion $\mathcal{A}(I_1) \vee \mathcal{A}(I_3) \subseteq (\mathcal{A}(I_2) \vee \mathcal{A}(I_4))'$, where $I_i \in \mathcal{K}$ are disjoint intervals in a clockwise order whose union is dense in S^1. A conformal net is *completely rational* if it strongly additive and with finite μ-index.

For $c < 1$ it is known that $\mathcal{V}_c$ is completely rational, while for $c \geq 1$ the μ-index is not finite and $\mathcal{V}_c$ is not even strongly additive [91]. The Virasoro nets are the operator algebraic version of the so called *minimal models*. Such theories describe discrete statistical models at their critical points. For example, the case $c = 1$ corresponds to the critical Ising model. For a conventional description of such models, see [36]. The importance of the Virasoro nets lies in the fact that any $\mathrm{Diff}_+(S^1)$-covariant conformal net contains a Virasoro net as a subnet, as simply follows by (2.8). This fact has been exploited in the classification of conformal nets with charge $c < 1$ [57]. A deeper analysis of the Virasoro nets will be provided in the next section.

2.3 Analysis on Virasoro nets

We begin by describing our notation and some basic facts about the structure of a Virasoro net. The material is standard and more details may be found e.g. in [44]. The starting point is the *Virasoro algebra*, that is the infinite dimensional Lie algebra Vir with generators $\{L_n, c\}_{n\in\mathbb{Z}}$ obeying the relations

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{1}{12}n(n^2-1)\delta_{n,-m}c\,, \quad [L_n, c] = 0\,. \tag{2.14}$$

A *(unitary) positive energy representation* of Vir on a Hilbert space $\mathcal{H}$ is a representation such that

(i) $L_n^* = L_{-n}$,

(ii) L_0 is diagonalizable with non-negative eigenvalues of finite multiplicity,

(iii) the central element is represented by $c\mathbb{1}$.

From now, we assume such a positive energy representation on the infinite dimensional separable Hilbert space $\mathcal{H}$. We assume furthermore that $\mathcal{H}$ contains a vector Ω annihilated by L_{-1}, L_0, L_{+1} ($\mathfrak{sl}(2,\mathbb{R})$-invariance) which is a highest weight vector of weight 0, that is $L_n\Omega = 0$ for all $n > 0$. In [18], [23], [46] and [47] one can find the proof of the bound

$$\|(1+L_0)^k L_n \Psi\| \leq \sqrt{c/2}(|n|+1)^{k+3/2}\|(1+L_0)^{k+1}\Psi\| \tag{2.15}$$

for $\Psi \in \mathcal{V} = \bigcap_{k\geq 0}\mathcal{D}(L_0^k)$ and $k \geq 0$ integer. Given a smooth function $f(z)$ on the circle, one defines the *stress energy tensor*

$$T(f) = -\frac{1}{2\pi}\sum_{n=-\infty}^{+\infty}\left(\int_{S^1} f(z)z^{-n-2}dz\right)L_n\,.$$

Notice that $T(f)$ has zero expectation on the vacuum, that is $(\Omega|T(f)\Omega) = 0$. This follows by the commutation relations of the Virasoro algebra (2.14), since $L_{-n}\Omega$ is an n-eigenvalue of the conformal hamiltonian L_0. The notation

$$T(f) = \int_{S^1} T(z)f(z)dz\,, \quad T(z) = -\frac{1}{2\pi}\sum_{n=-\infty}^{+\infty} z^{-n-2}L_n\,, \tag{2.16}$$

is widely used. Moreover, the estimate (2.15) shows that $T(f)$ is well defined and closable for any function f in the Sobolev space $\mathcal{S}_{3/2} = W^{3/2,1}(S^1)$, since

$$\|L_0^k T(f)\xi\| \leq (c/2)^{1/2}\|f\|_{3/2+k,1}\|(1+L_0)^{k+1}\xi\| \tag{2.17}$$

for every $k \geq 0$ natural and ξ in $\mathcal{D}(L_0^{k+1})$. It follows that $\mathcal{V}$ is a dense invariant domain for $T(f)$ if f is smooth. We recall that the norm of $W^{s,p}$ is

$$\|f\|_{s,p} = \Big(\sum_n |\hat{f}_n|^p(1+|n|)^{ps}\Big)^{1/p}\,,$$

where $\hat{f}_n$ is the n-th Fourier coefficient. If we now define $\Gamma f(z) = -z^2\overline{f(z)}$, then the stress-energy tensor is an essentially self-adjoint operator on any core of L_0 (such as $\mathcal{V}$) for any function $f \in \mathcal{S}_{3/2}$ obeying the reality condition

$$\Gamma f = f\,. \tag{2.18}$$

More in general, one has that $T(f)^* = \overline{T(\Gamma f)}$ as shown in [22]. We also point out that $T(f)$ must be thought of as an operator depending not on the function $f(z)$, but rather on the vector field $f(z)\frac{d}{dz}$. In particular, we have that

$$L_n = iT(l_n)\,, \quad l_n = z^{n+1}\frac{d}{dz}\,. \tag{2.19}$$

Notice that by changing variables $z = e^{i\theta}$, the stress energy tensor may be written as

$$T(f) = \sum_n \hat{f}_n L_n \,,$$

with $f = f(\theta)$. As a little remark, we can notice by the relations (2.14) and the hypothesis $L_0\Omega = 0$ that $T(f)\Omega = 0$ if and only if f is in the Hardy space $H^2(S^1)$, that is if and only if $\hat{f}_n = 0$ for every $n < 0$. Furthermore, let $\mathcal{H}^{\mathrm{fin}}$ be the dense subspace of *finite energy vectors*, that is the algebraic direct sum of the eigenspaces of L_0. Then the vectors $\Psi \in \mathcal{H}^{\mathrm{fin}}$ are entire analytic vectors for the stress energy tensor. This indeed can be easily proved for eigenvectors for L_0 by using the estimate (2.15) and by a simple induction. For further properties of the stress-energy tensor, see [44].

We now make the connection with the representation of the diffeomorphism group on the circle. To do this, given a function $f \in C^\infty(S^1)$ real in the sense of equation (2.18), we denote by $\mathrm{Exp}(tf) = \rho_t \in \mathrm{Diff}_+(S^1)$ the 1-parameter flow of orientation preserving diffeomorphisms generated by the vector field f. In other words, ρ_t is uniquely determined by the conditions

$$\frac{\partial}{\partial t}\rho_t(z) = f(\rho_t(z)) \,, \quad \rho_0 = \mathrm{id} \,. \tag{2.20}$$

Notice that ρ_t acts as the identity for all $t \in \mathbb{R}$ outside the support of f. The unitary operators $W(f) = e^{iT(f)}$ can be thought of as representers of the diffeomorphisms $\mathrm{Exp}(f)$. More precisely, there exists a strongly continuous unitary projective representation $\mathrm{Diff}_+(S^1) \ni \rho \mapsto V(\rho) \in U(\mathcal{H})$ satisfying:

(v0) V leaves invariant $\mathcal{V}$,

(v1) V is a multiplier representation,

(v2) $\frac{d}{dt}V(\mathrm{Exp}(tf)) = itT(f)$ on any core of $T(f)$. In particular, we have that $e^{iT(f)} = e^{i\alpha(t)}V(\rho_t)$, with $\alpha'(0) = 0$.

We now describe the commutation rules between two operators $e^{iT(f)}$ and $e^{iT(g)}$. For a smooth diffeomorphism ρ on the circle, the *Schwarzian derivative* is defined by

$$S\rho(z) = \left(\frac{\rho''(z)}{\rho'(z)}\right)' - \frac{1}{2}\left(\frac{\rho''(z)}{\rho'(z)}\right)^2 .$$

It has been shown in [44], which uses results of [46] and [92], that on the domain $\mathcal{V}$ we have the relations

$$V(\rho)T(f)V(\rho)^* = T(\rho_* f) + \beta(\rho, f)\mathbb{1} \,, \tag{2.21}$$

$$i[T(f), T(g)] = T(f'g - g'f) + c\omega(f,g)\mathbb{1} \,, \tag{2.22}$$

meaning that (2.21) holds on $\mathcal{V}$ and (2.22) on $\mathcal{V} \cap D(T(f)T(g)) \cap D(T(g)T(f))$. Here $\rho_* g$ is the push-forward of the vector field $g(z)\frac{d}{dz}$ through ρ and

$$\beta(\rho, f) = -\frac{c}{24\pi}\int_{S^1} f(z)S\rho(z)dz \,, \quad \omega(f,g) = -\frac{c}{48\pi}\int_{S^1}(f(z)g'''(z) - f'''(z)g(z))dz \,.$$

Equation (2.21) implies that we have the commutation relations

$$W(f)W(g) = e^{i\beta(\rho,g)}W(\rho_* g)W(f)\,,$$

with $W(h) = e^{iT(h)}$ and $\rho = \mathrm{Exp}(f)$. We also notice that, by comparing the identities (2.16) and (2.21) for each test function f we have

$$(V(\rho)^*\Omega|T(z)V(\rho)^*\Omega) = -\frac{c}{24\pi}S\rho(z)\,,$$

where $T(z)$ has to be meant as the density of the distribution $f \mapsto T(f)$. Notice that by (2.17) we already knew the stress-energy tensor to be a tempered distribution. In the real line picture, the stress energy tensor Θ on $\mathbb{R}$ is defined by the formula $\Theta(f) = T(C_* f)$, with $C_* f$ the pushforward of the vector field on the real line $f(u)\frac{d}{du}$ through the Cayley trasform $C(u) = (1+iu)/(1-iu)$. By definition, the stress energy tensor on the real line is then

$$\Theta(u) = \left(\frac{dC(u)}{du}\right)^2 T(C(u)) = -\frac{4}{(1-iu)^4}T(C(u))\,. \tag{2.23}$$

Using the equations (2.23) and (2.19), we obtain an expression for the generators of $\mathfrak{sl}(2,\mathbb{R})$. In particular, the generator $D = -\frac{i}{2}(L_1 - L_{-1})$ of dilations is given by

$$D = \int_{-\infty}^{+\infty} u\Theta(u)du\,.$$

We recall that, by the Bisognano-Wichmann theorem, we have $\log\Delta = -2\pi D$: the modular dynamic is implemented by boost transformations.

We end this section with a few remarks about the representation V of $\mathrm{Diff}_+(S^1)$ and the stress energy tensor. Given n points on S^1, say z_i with $i = 1,\dots,n$, we denote by $B(z_1,\dots,z_n)$ the group of all the piecewise smooth and C^1 diffeomorphisms ρ of S^1 which are smooth except that on the points z_i and such that $\rho(z_i) = z_i$ and $\rho'(z_i) = 1$. By piecewise smooth we mean that left and right derivatives exist at all orders at every point, hence we have $B(-1) = \mathrm{Diff}^{1,\mathrm{ps}}_{+,1}(S^1)$ in the notation of Section 2.2. A similar notation will be used in the real line picture, where in this case we will be particularly interested in $B(\infty)$. The unitary projective representation V can be extended to $B = \mathrm{Diff}^{1,\mathrm{ps}}_{+}(S^1)$ in such a way that the properties (v0), (v1) and (v2) are still satisfied. For details, see [51] and the appendix of [34]. The relation (v2) is then satisfied by any real valued C^1 function f on S^1 which is smooth except that on a finite number of points z_i such that $f(z_i) = 0$ and $f'(z_i) = 0$. We precise that if g is a piecewise smooth, real, compactly supported C^1-function on S^1, then by standard arguments $g \in W^{s,1}$ for any $s < 2$ (see Lemma 3.1. of [34]). Therefore, $T(g)$ is a closable essentially self-adjoint operator and (v2) is verified. Furthermore, in [22] it is proved that if $g_n \to g$ in $W^{3/2,1}$ then $e^{iT(g_n)} \to e^{iT(g)}$ in the strong operator topology.

The groups $B(z,z')$ are of interest also for the following reason. Given two points z and z' of the circle, consider a diffeomorphism ρ in $B(z,z')$, that is a diffeomorphism in $\mathrm{Diff}_+(S^1)$ fixing z and z' and with unital derivative in such points. Given an interval

I in $\mathcal{K}$, we will say that a diffeomorphism ρ in $\mathrm{Diff}_+(S^1)$ is *localized in* I if $\rho(z) = z$ for z in I'. Define $I = (z, z')$, where the interval is obtained moving counterclockwise from z to z'. Then it is possible to define a diffeomorphism ρ_+ localized in I and a diffeomorphism ρ_- localized in I' such that $\rho = \rho_+\rho_- = \rho_-\rho_+$. If $\rho = \mathrm{Exp}(f)$, then this is possible if f and its derivative vanish at the points z and z'. This splitting property of a diffeomorphism ρ in $B(z, z')$ will be of interest in the next section.

2.4 QNEC and Bekenstein bound for solitonic states

In this section we will prove the QNEC on a generic conformal net for the solitonic states (2.11). The proof relies on explicit computations on the Virasoro nets and on the use of Proposition 33. As an intermediate result, we show that the same states verify the Bekenstein Bound [69].

Let $(\mathcal{A}, V, \Omega)$ be a conformal net of central charge c on a Hilbert space $\mathcal{H}$. In order to fix some notation, we set $\omega_U = \omega \cdot \mathrm{Ad}U^*$, where ω is the vacuum state and U is some unitary operator on $\mathcal{H}$. In particular, we notice that if $\rho = \mathrm{Exp}(f)$ for some real smooth function f and $W(f) = \exp(iT(f))$ then $\omega_{W(f)} = \omega_{V(\rho)}$. Finally, if ω_ρ is a solitonic state associated to some homeomorphism ρ in $\mathrm{Diff}_+(S^1, -1)$ then we will denote by $S_I(\omega_\rho \| \omega)$ the relative entropy $S(\omega_\rho \| \omega)$ on the local algebra $\mathcal{A}(I)$ for some interval I in $\mathcal{I}_\mathbb{R}$. In this section we will first compute $S_I(\omega_\rho \| \omega)$ when I is a bounded interval of the real line, and this will imply the Bekenstein Bound. Then, by Möb-covariance we will be able to compute $S_I(\omega_\rho \| \omega)$ whenever I is unbounded and check the QNEC.

We begin by making a few considerations about $S_I(\omega_{V(\rho)} \| \omega)$ and the Connes cocycle $(D\omega_{V(\rho)} \colon D\omega)_t$ of $\mathcal{A}(I)$. Working in the real line picture, by Möb-covariance we can assume $I = (0, +\infty)$. We consider a diffeomorphism ρ in $B(0, \infty)$, so that $\rho(0) = 0$ and $\rho'(0) = 1$. We also have $\rho(u) \to 0$ and $\rho'(u) \to 1$ if $u \to \infty$. It then follows that $V(\rho) = V(\rho_+)V(\rho_-)$ up to a phase, where $V(\rho_+)$ belongs to $\mathcal{A}(0, +\infty)$ and $V(\rho_-)$ belongs to $\mathcal{A}(-\infty, 0)$. Notice that the same properties hold for the map

$$\eta = \rho^{-1} = \mathrm{Exp}(-f)\,. \tag{2.24}$$

Therefore, it follows by (2.21) that

$$S_{(0,+\infty)}(\omega_{V(\rho)} \| \omega) = -\frac{c}{12} \int_0^{+\infty} u S\eta(u) du\,. \tag{2.25}$$

Notice that the integral on the r.h.s. is finite, since through the Cayley transform it reduces to an integral of a bounded continuous function on the upper half circle. To prove this one also has to take advantage of the chain rule for the Schwarzian derivative

$$S(f \cdot g)(z) = g'(z)^2 Sf(g(z)) + Sg(z)\,.$$

Therefore, integrating by parts we obtain the expression

$$S_{(0,+\infty)}(\omega_{V(\rho)} \| \omega) = \frac{c}{24} \int_0^{+\infty} u \left(\frac{\eta''(u)}{\eta'(u)} \right)^2 du\,. \tag{2.26}$$

This formula holds if ρ is in $B(0,\infty)$, or equivalently if ρ is in $B(\infty) \cong \mathrm{Diff}^{1,\mathrm{ps}}_{+,1}(S^1)$ verifies $\rho(0)=0$ and $\rho'(0)=1$. To remove these boundary conditions, we need to prove the ansatz on the Connes cocycle

$$(D\omega_{V(\rho)}\colon D\omega)_t = V(\rho_+ \cdot \delta_t \cdot \rho_+^{-1} \cdot \delta_{-t})e^{ia(t)}\,, \tag{2.27}$$

with $\delta_t(u) = e^{-2\pi t}u$ and $a(t)\in\mathbb{R}$ to be determined.

Proposition 38. *[51] If $\rho(0)=0$ and $\rho(\infty)=\infty$, then equation (2.27) has a solution on the Virasoro nets.*

Proof. Let $(\mathcal{V}_c, V, \Omega)$ be the Virasoro net of central charge c. We denote by u_t the r.h.s. of equation (2.27), with $a(t)$ to be determined. Notice that even though ρ_+ is not globally C^1, (2.27) is well defined since the combination $[\rho,\delta_t] = \rho_+ \cdot \delta_t \cdot \rho_+^{-1} \cdot \delta_{-t}$ is globally C^1, using the usual notation $[g_1,g_2] = g_1g_2g_1^{-1}g_2^{-1}$ for the commutator in a group. Moreover the diffeomorphism $\alpha = \rho_+ \cdot \delta_t \cdot \rho_+^{-1} \cdot \delta_{-t}$ belongs to $B(0,\infty)$, hence u_t belongs to $\mathcal{V}_c(0,+\infty)$ for every $t\in\mathbb{R}$. Therefore, the thesis follows if we find a function $a(t)$ such that u_t verifies the relations

(i) $\sigma^t_{\omega_{V(\rho)}}(x) = u_t\sigma^t_\omega(x)u_t^*\,,\quad x\in\mathcal{V}_c(0,+\infty)\,,$

(ii) $u_{t+s} = u_t\sigma^t_\omega(u_s)\,.$

Note that the first relation suffices to be verified for $x = V(\tau)$, with $\tau = \mathrm{Exp}(g)$ and $\mathrm{supp}\, g \subseteq (0,+\infty)$. Notice also that, since $\rho(0)=0$, we can apply Lemma 2. Therefore, by noticing that $\rho\cdot\delta_t\cdot\rho^{-1}\cdot\tau\cdot\rho\cdot\delta_{-t}\cdot\rho^{-1} = \rho_+\cdot\delta_t\cdot\rho_+^{-1}\cdot\tau\cdot\rho_+\cdot\delta_{-t}\cdot\rho_+^{-1}$ and by the explicit expression of the Bott 2-cocycle (2.13) we have

$$\begin{aligned}\sigma^t_{\omega_{V(\rho)}}(V(\tau)) &= V(\rho)V(\delta_t)V(\rho)^*V(\tau)V(\rho)V(\delta_t^{-1})V(\rho^*)\\ &= V(\rho_+)V(\delta_t)V(\rho_+^{-1})V(\tau)V(\rho_+)V(\delta_t^{-1})V(\rho_+^{-1})\\ &= V(\rho_+\cdot\delta_t\cdot\rho_+^{-1}\cdot\delta_{-t})V(\delta_t)V(\tau)V(\delta_t^{-1})V(\rho_+\cdot\delta_t\cdot\rho_+^{-1}\cdot\delta_{-t})^*\\ &= u_t\sigma^t_\omega(V(\tau))u_t^*\,.\end{aligned}\tag{2.28}$$

We now study the condition (ii). This is equivalent to

$$a(t)+a(s)-a(t+s) = b(t,s)\,, \tag{2.29}$$

where

$$b(t,s) = cB([\rho_+,\delta_t], \delta_t\cdot[\rho_+,\delta_s]\cdot\delta_t^{-1})\,.$$

We can rewrite this condition as **b**a=b, where **b** is the cocycle operator on the additive group $\mathbb{R}$. Since there are not non-trivial 2-cocycles on this group, solutions a of (2.29) can be found provided that **b**b=0, with

$$\mathbf{b}b(t,s,r) = b(t,s) - b(t+s,r) + b(t,r+s) - b(s,r)\,.$$

By using the identity $\mathbf{b}B(g_1,g_2,g_3)=0$ one can directly verify that $\mathbf{b}b(t,s,r)=0$ (see Lemma 2. of [51] for the explicit computation). This concludes the proof. □

Remark 39. Notice that the proposition can be easily adapted to a generic bounded interval (a,b) of the real line, provided that $\rho(a)=a$ and $\rho(b)=b$.

Now we assume I to be a bounded interval of the real line. First of all we notice that the dilation operator of $I=(a,b)$ can be computed as $D_{(a,b)}=\Theta(D_{(a,b)}(u))$, with

$$D_{(a,b)}(u)=\frac{1}{b-a}(b-u)(u-a)\,.$$

We proceed by cases. Suppose that ρ belongs to $B(a,b)$, that is ρ fixes a and b and has unital derivative in such points. As in the case of the half-line we have that $V(\rho)=V(\rho_+)V(\rho_-)$ up to a phase, with $V(\rho_+)$ in $\mathcal{V}_c(a,b)$ and $V(\rho_-)$ in $\mathcal{V}_c(a,b)'$. Therefore, we can take advantage of the formula (2.25), and integrating by parts we obtain

$$\begin{aligned}S_{(a,b)}(\omega\|\omega_{V(\rho)})&=\frac{c}{24}\int_a^b D_{(a,b)}(u)\left(\frac{\rho''(u)}{\rho'(u)}\right)^2du+\frac{c}{12}\int_a^b D'_{(a,b)}(u)\left(\frac{\rho''(u)}{\rho'(u)}\right)du\\&=\frac{c}{24}\int_a^b D_{(a,b)}(u)\left(\frac{\rho''(u)}{\rho'(u)}\right)^2du+\frac{c/6}{b-a}\int_a^b\log\rho'(u)du\,.\end{aligned}$$

Now we generalize the previous equation to the case in which $\rho'(a)$ and $\rho'(b)$ are generic. Given $r>0$, consider the sequence of functions

$$h_n(u)=(n\log r)^{-1}(e^{n(\log r)u}-1)\,. \tag{2.30}$$

Notice that $h_n(0)=0$, $h_n(1/n)\to 0$ if $n\to+\infty$, $h_n'(0)=1$ and $h_n'(1/n)=r$. Notice also that

$$\int_0^{1/n}u\left(\frac{h_n''(u)}{h_n'(u)}\right)^2du=\frac{(\log r)^2}{2}\,. \tag{2.31}$$

If we denote the function (2.30) by h_n^r and we set $r_a=\rho'(a)$, $r_b=\rho'(b)$ then we can define

$$h_n^1(u)=a+h_n^{r_a}(u)\,,\quad h_n^2(u)=b-h_n^{r_b}(b-u)\,.$$

We now consider the following maps: given to intervals $[a,b]$ and $[c,d]$, let $g_{[a,b]}^{[c,d]}(u)=mu+q$ be the affine function mapping $[a,b]$ to $[c,d]$. If $a_n=a+1/n$ and $b_n=b-1/n$, then we define

$$g_n^1=g_{[a,b]}^{[h_n^1(a_n),h_n^2(b_n)]}\,,\quad g_n^2=g_{[a_n,b_n]}^{[a,b]}\,.$$

Finally, we consider the following sequence of functions:

$$\rho_n(u)=\begin{cases}u & u\le a\,,\ u\ge b\\ h_n^1(u) & a\le u\le a_n\\ g_n^1\rho g_n^2(u) & a_n\le u\le b_n\\ h_n^2(u) & b_n\le u\le b\end{cases}\,.$$

Up to mollify a bit ρ_n in a_n and b_n, we have that ρ_n is a sequence of C^1 functions such that $\rho_n'(a)=\rho_n'(b)=1$. Moreover, by (2.31) one can notice that

$$\int_a^b D_{(a,b)}(u)S\rho_n(u)du\to-\frac{(\log r_a)^2+(\log r_b)^2}{4}+\int_a^b D_{(a,b)}(u)S\rho(u)du\,. \tag{2.32}$$

Now we arrive to the crucial point of the proof. The idea is to approximate $S_{(a,b)}(\omega\|\omega_{V(\rho)})$ with $S_{(a,b)}(\omega\|\omega_{V(\rho_n)})$, since for the functions ρ_n formula (2.4) holds. Unfortunately, the relative entropy does not behave well in the limit. However, by studying the Bott 2-cocycle (2.13) it is shown in [51] that $S_{(a,b)}(\omega\|\omega_{V(\rho)})$ and $\lim_n S_{(a,b)}(\omega\|\omega_{V(\rho_n)})$ are both solutions of an equation whose solutions are unique up to a constant term m_ρ. More precisely, this term depends only on the derivatives $r_a = \rho'(a)$ and $r_b = \rho'(b)$. Therefore by (2.32) we obtain that

$$S_{(a,b)}(\omega\|\omega_{V(\rho)}) = \nu(r_a, r_b) + \frac{c}{24}\int_a^b D_{(a,b)}(u)\left(\frac{\rho''(u)}{\rho'(u)}\right)^2 du + \frac{c/6}{b-a}\int_a^b \log\rho'(u)du$$

for some function $\nu(r_a, r_b)$ which we are now going to prove is zero. To do this, we will construct sequences of functions ρ_n with the same derivatives as ρ at $u = a$ and $u = b$. For simplicity we consider $(a, b) = (0, 3)$. The general case will follow by covariance, that is by noticing that

$$S_{(a,b)}(\omega\|\omega_{V(\rho)}) = S_{(0,3)}(\omega\|\omega_{V(\alpha\rho\alpha^{-1})})\,,$$

with $\alpha(u) = cu + d$ in the Moebius group mapping $(0, 3)$ in (a, b). We start by proving that $0 \leq \nu(r_0, r_3)$. Given $r > 0$, consider the sequence of functions

$$\sigma_n(u) = \frac{\log n}{\log(n/r)}\left[(u + 1/n)^{\log(n/r)/\log(n)} - (1/n)^{\log(n/r)/\log(r)}\right]\,. \tag{2.33}$$

We notice that $\sigma_n(0) = 0$, $\sigma_n(1 - 1/n) = \frac{\log n}{\log(n/r)}\left(1 - \frac{r}{n}\right) \to 1$ and $\sigma_n'(1 - 1/n) = 1$. If we denote the function (2.33) by σ_n^r, then we define

$$\rho_n(u) = \begin{cases} \sigma_n^{r_0}(u) & 0 \leq u \leq 1 - 1/n \\ 3 - \sigma_n^{r_3}(3 - u) & 2 + \frac{1}{n} \leq u \leq 3 \\ \gamma_n(u) & \text{otherwise} \end{cases}\,,$$

with γ_n a smooth function such that ρ_n is C^1. Moreover, since $\rho_n(1 - 1/n) \to 1$ and $\rho_n(2 + 1/n) \to 2$, we can suppose that γ_n converges uniformly with its derivatives (up to the second order) to the identity function on $[1, 2]$. In particular we can assume that

$$\int_{1-1/n}^{2+1/n} \rho_n(u)du \to 0 \quad \text{if } n \to \infty\,.$$

Therefore, by the positivity of the relative entropy we have

$$\begin{aligned}
0 &\leq \nu(r_0, r_3) + \frac{c}{24}\int_0^3 D_{(0,3)}(u)\left(\frac{d}{du}\log\rho_n'(u)\right)^2 du + \frac{c}{18}\int_0^3 \log\rho_n'(u)du \\
&\sim \nu(r_0, r_3) + \frac{c}{24}\left[\left(\frac{\log r_0}{\log n}\right)^2\int_0^{1-1/n}\frac{D_{(0,3)}(u)du}{(u+1/n)^4} + \left(\frac{\log r_3}{\log n}\right)^2\int_{2+1/n}^3 \frac{D_{(0,3)}(u)du}{(3-u+1/n)^4}\right] \\
&+ \frac{c}{18}\left[\frac{\log r_0}{\log n}\int_0^{1-1/n}\frac{du}{u+1/n} + \frac{\log r_3}{\log n}\int_{2+1/n}^3 \frac{du}{(3-u+1/n)}\right] \\
&\to \nu(r_0, r_3)\,,
\end{aligned}$$

where $\sim$ means the equality up to a term going to zero. This proves that $\nu \geq 0$. Now we prove the other inequality. Given $r > 0$, consider

$$\zeta_n(u) = -\frac{1}{n} + \int_{-1/n}^u \exp[(\log r)(ns + 1)^{1/n}]ds\,. \tag{2.34}$$

Notice that $\zeta_n(-1/n) = -1/n$, $\zeta_n'(-1/n) = 1$ and $\zeta_n'(0) = r$. Notice also that $\zeta_n(0) \to 0$ and $\frac{d}{du} \log \zeta_n'(u) = (\log r)(1+nu)^{1/n-1}$. Always in the case $I = (0,3)$, if we denote the function (2.34) by ζ_n^r then we can define

$$\rho_n(u) = \begin{cases} \zeta_n^{r_0}(u) & -1/n \leq u \leq 0 \\ \sigma_n^{r_0}(u) + c_n & 0 \leq u \leq 1-1/n \\ 3 - \sigma_n^{r_3}(3-u) + d_n & 2+1/n \leq u \leq 3 \\ 3 - \zeta_n^{r_3}(3-u) & 3 \leq u \leq 3+1/n \\ \gamma_n(u) & \text{otherwise} \end{cases} ,$$

with γ_n, c_n and d_n such that ρ_n is C^1. Notice that $c_n \to 0$ and $d_n \to 0$, so that we can suppose that $\gamma_n(u) \to u$ in $[1,2]$ as before. Moreover, if we mollify ζ_n at $u = 0$ then by monotonicity we get

$$S_{(0,3)}(\omega \| \omega_{V(\rho_n)}) \leq S_{(-1/n,3+1/n)}(\omega \| \omega_{V(\rho_n)}) . \tag{2.35}$$

Notice that on the right side of (2.35) the term $\nu(r_0, r_3)$ does not appear. Therefore, up to a term going to zero we have

$$\nu(r_0, r_3) \leq \frac{c}{24} I_n + \frac{c/6}{3+2/n} J_n ,$$

with

$$I_n = d_n (\log r_0)^2 \int_{-1/n}^{0} (1+nu)^{-2(1-1/n)} du + d_n (\log r_3)^2 \int_{3}^{3+1/n} (1+n(3-u))^{-2(1-1/n)} du ,$$

$$J_n = (\log r_0) \int_{-1/n}^{0} (1+nu)^{1/n} du + (\log r_3) \int_{3}^{3+1/n} (1+n(3-u))^{1/n} du .$$

But by direct computation and by the estimate $D_{(-1/n,3+1/n)}(u) \leq u + 1/n$ one has that $I_n \to 0$ and $J_n \to 0$, and so $\nu(r_0, r_3) \leq 0$, as required. We can then conclude with the following formula: if $\rho(a) = a$ and $\rho(b) = b$, then

$$S_{(a,b)}(\omega \| \omega_{V(\rho)}) = \frac{c}{24} \int_a^b D_{(a,b)}(u) \left(\frac{\rho''(u)}{\rho'(u)} \right)^2 du + \frac{c/6}{b-a} \int_a^b \log \rho'(u) du \tag{2.36}$$

$$= -\frac{c}{12} \int_a^b D_{(a,b)}(u) S\rho(u) du + \frac{c}{12} \log \rho'(a) \rho'(b) , \tag{2.37}$$

where the second expression is obtained by integration by parts. This expression allows us to compute the relative entropy with switched states. Indeed, if we consider a transformation α in the Möbius group such that $\eta\alpha$ fixes $\eta(a)$ and $\eta(b)$, then we have

$$S_{(a,b)}(\omega_{V(\rho)} \| \omega) = S_{\eta(a,b)}(\omega \| \omega_{V(\eta)}) = S_{(\eta(a),\eta(b))}(\omega \| \omega_{V(\eta\alpha)}) . \tag{2.38}$$

Therefore, to generalize the formula (2.37) to a generic diffeomorphism ρ in $\mathrm{Diff}_+(S^1)$ fixing ∞ it suffices to find a diffeomorphism α in the Möbius group such that $\rho\alpha(a) = a$, $\rho\alpha(b) = b$ and $\alpha(\infty) = \infty$. Finally, by Proposition 33 we have the following.

Theorem 40. *If ρ is a diffeomorphism in $\mathrm{Diff}_+(S^1)$ such that $\rho(\infty) = \infty$ in the real line picture, then on a generic conformal net of central charge c we have*

$$\begin{aligned} S_{(a,b)}(\omega\|\omega_{V(\rho)}) = &-\frac{c}{12}\int_{\rho^{-1}(a)}^{\rho^{-1}(b)} D_{\rho^{-1}(a,b)} S\rho(u)du \\ &+ \frac{c}{12}\log\rho'(\rho^{-1}(a))\rho'(\rho^{-1}(b)) + \frac{c}{6}\log\left(\frac{\rho^{-1}(b)-\rho^{-1}(a)}{b-a}\right). \end{aligned} \tag{2.39}$$

for any bounded interval (a,b). Similarly, by applying $\eta = \rho^{-1}$ we have that

$$\begin{aligned} S_{(a,b)}(\omega_{V(\rho)}\|\omega) = &-\frac{c}{12}\int_a^b D_{(a,b)} S\eta(u)du + \frac{c}{12}\log\eta'(a)\eta'(b) \\ &- \frac{c}{6}\log\left(\frac{\eta(b)-\eta(a)}{b-a}\right). \end{aligned} \tag{2.40}$$

Theorem 41. *Let $(\mathcal{A}, V, \Omega)$ be a conformal net of central charge c. Given some homeomorphism ν in $\mathrm{Diff}_+(S^1, -1)$, denote by $\omega_\nu = \omega \cdot \sigma_\nu^{-1}$ the associated solitonic state. Define*

$$S(r) = S_{(-r,r)}(\omega\|\omega_\nu), \quad \bar{S}(r) = S_{(-r,r)}(\omega_\nu\|\omega).$$

Then, we have the Bekenstein Bounds

$$S(r) \leq \pi r \inf_\rho E_\rho, \quad \bar{S}(r) \leq \pi r \inf_\rho \bar{E}_\rho, \tag{2.41}$$

where the infima are over all the diffeomorphisms ρ in $\mathrm{Diff}_+(S^1)$ such that $\rho = \nu$ on $(-r,r)$, $E_\rho = (\Omega|H_\tau\Omega)$ is the mean vacuum energy in the representation $\tau = \mathrm{Ad}V(\rho)$ and $\bar{E}_\rho$ is the mean vacuum energy in the conjugate representation $\bar{\tau} = \mathrm{Ad}V(\rho)^*$.

Proof. We prove the Bekenstein Bound involving $S(r)$, the other case can be equally proved. Since σ_ν is locally implemented by some unitary operator, we can assume ν to have trivial ratio $r(\nu) = 1$. We then replace ν with some diffeomorphism ρ in $\mathrm{Diff}_+(S^1)$ as $S_{(-r,r)}(\omega\|\omega_\nu) = S_{(-r,r)}(\omega\|\omega_\rho)$ for such a diffeomorphism ρ. By Proposition 33 and property (r6) of the relative entropy mentioned in Section 1.4 we can actually assume $(\mathcal{A}, V, \Omega)$ to be a Virasoro net of same central charge c. Furthermore, as the vacuum is Möb-invariant we can replace ρ with $\rho\alpha$ where α is some Möbius transformation such that $\rho\alpha$ fixes $\pm r$. We also do the following remark: if γ is an orientation-preserving diffeomorphism fixing a and b, then by the Jensen inequality

$$\frac{1}{b-a}\int_a^b \log\gamma'(u)du \leq \log\left(\frac{1}{b-a}\int_a^b \gamma'(u)du\right) = 0. \tag{2.42}$$

Therefore, if $\eta = \rho^{-1}$ then by (2.39) we have

$$S(r) \leq \frac{cr}{48}\int_{-r}^{r}\left(\frac{\rho''(u)}{\rho'(u)}\right)^2 du$$

and the first bound follows, as

$$E_\rho = (\Omega|H_\tau\Omega) = (\Omega|V(\rho)HV(\rho)^*\Omega) = \frac{c}{48\pi}\int_{-\infty}^{+\infty}\left(\frac{\rho''(u)}{\rho'(u)}\right)^2 du.$$

We recall that $H = \int \Theta(u)du$ is the generator of translations, while H_τ is the generator of translations in the representation τ (see Section 1.7). Notice also that the conjugate charge of τ is $\bar{\tau} = \mathrm{Ad}V(\rho)^*$: the conjugate equation is trivially satisfied and $d(\tau) = 1$ (see Appendix A). Hence in this case the mean vacuum energy is

$$\bar{E}_\rho = (\Omega|H_{\bar{\tau}}\Omega) = \frac{c}{48\pi}\int_{-\infty}^{+\infty}\left(\frac{\eta''(u)}{\eta'(u)}\right)^2 du\,,$$

and the thesis follows. □

Our next step will be the proof of the main theorem of this chapter, namely the QNEC for solitonic states. However, before doing that we want to do a few remarks about the notion of energy density. The infima appearing in (2.41) suggest us some model independent definition of energy density of a DHR state on some subregion, namely one defines it as the infimum on a proper family of DHR representations of all the associated vacuum mean energies. In the particular case of above, we have

$$E_\rho(t,t') = \frac{c}{48\pi}\int_t^{t'}\left(\frac{\rho''(u)}{\rho'(u)}\right)^2 du\,.$$

This approach seems reasonable if we want to define the energy as an extensive quantity. If we then want to define a punctual energy density, then a reasonable definition would imply that

$$E_\rho(t) = \frac{c}{48\pi}\left(\frac{\rho''(t)}{\rho'(t)}\right)^2\,. \tag{2.43}$$

The definition we will use here is simply

$$E_\rho(t) = \liminf_{t'\to t^+}\frac{E_\rho(t,t')}{t'-t}\,,$$

and similarly for $\bar{E}_\rho(t)$. This topic will be further treated in Section 3.5 of the next chapter. We point out that this (null) energy density must not be confused with the stress-energy tensor density, which in the real line picture is given by

$$(\Omega|V(\rho)\Theta(t)V(\rho)^*\Omega) = -\frac{c}{48\pi}S\rho(t)\,.$$

Theorem 42. *Let $(\mathcal{A}, U, \Omega)$ be a conformal net of central charge c. Given some homeomorphism ν in $\mathrm{Diff}_+(S^1,-1)$, denote by $\omega_\nu = \omega\cdot\sigma_\nu^{-1}$ the associated solitonic state and set $\eta = \nu^{-1}$. Define*

$$S(t) = S_{(t,+\infty)}(\omega_\nu\|\omega)\,,\quad \bar{S}(t) = S_{(-\infty,t)}(\omega_\nu\|\omega)\,.$$

Then $S(t)$ and $\bar{S}(t)$ are both finite and explicitly given by

$$S(t) = \frac{c}{24}\int_t^{+\infty}(u-t)\left(\frac{\eta''(u)}{\eta'(u)}\right)^2 du\,,\quad \bar{S}(t) = \frac{c}{24}\int_{-\infty}^{t}(t-u)\left(\frac{\eta''(u)}{\eta'(u)}\right)^2 du\,. \tag{2.44}$$

In particular, the QNEC is satisfied with the equalities

$$\bar{E}_\rho(t) = S''(t)/2\pi \geq 0\,,\quad \bar{E}_\rho(t) = \bar{S}''(t)/2\pi \geq 0\,. \tag{2.45}$$

Proof. As in the proof of Theorem 41 we can replace ν with some diffeomorphism ρ in $\mathrm{Diff}_+(S^1)$ and the generic conformal net with a Virasoro net $(\mathcal{A}, V, \Omega)$ of same central charge. Equations (2.44) are implied by (2.39) and 2.40 after a limit step thanks to the monotonicity property of the relative entropy (property (r5) in the Section 1.4 notation). The QNEC (2.45) follows. □

Remark 43. Thanks to property (r7) of the relative entropy (see Section 1.4) all the theorems of this section have a natural generalization to $(1+1)$-dimensional chiral CFT (2.10) for locally normal states of the type $\omega_{\nu_1} \otimes \omega_{\nu_2}$.

2.5 Additional remarks

Despite this remark may lack of any physical meaning, it is natural to investigate the convexity of the relative entropy $S_\nu(t) = S_{(t,+\infty)}(\omega \| \omega_\nu)$. In this section we show a counterexample to this property.

We first notice that an explicit expression for $S_\nu(t)$ can be found by proceeding as in (2.38). Then, for the sake of simplicity, we replace ν with some diffeomorphism ρ in $\mathrm{Diff}_+(S^1)$. We also assume that $\rho = \mathrm{Exp}(f)$ for some smooth and compactly supported vector field f. Clearly $S_\rho(\rho(t))$ has negative derivative, and so $S_\rho(t)$ is decreasing since ρ is increasing. In particular, we have

$$\begin{aligned} S'_\rho(\rho(t))\rho'(t) &= -\frac{c}{24}\int_t^{+\infty}\left(\frac{\rho''(u)}{\rho'(u)}\right)^2 du\,, \\ S''_\rho(\rho(t))\rho'(t)^2 &= \frac{c}{24}\left(\frac{\rho''(t)}{\rho'(t)}\right)^2 - S'_\rho(\rho(t))\rho''(t) \\ &= \frac{c}{24}\frac{\rho''(t)}{\rho'(t)}\left(\frac{\rho''(t)}{\rho'(t)} + \int_t^{+\infty}\left(\frac{\rho''(u)}{\rho'(u)}\right)^2 du\right)\,. \end{aligned} \tag{2.46}$$

In this case we can notice that the relative entropy is convex in the average, that is if $[a,b]$ contains the support of f then

$$\int_a^b S''_\rho(t)dt = \frac{c}{24}\int_a^b\left(\frac{\rho''(t)}{\rho'(t)}\right)^2 du \geq 0\,,$$

where this identity follows from (2.46) and from the fact that $\rho(u) = u$ outside $[a,b]$. However, in this case the second derivative is not always positive, as shown by the following counterexample. Let us consider the function

$$f(x) = \begin{cases} \frac{1}{1+\tan(x)^2} & -\pi/2 \leq x \leq \pi/2 \\ 0 & \text{otherwise} \end{cases}\,.$$

This is a C^1 function with compact support and smooth except that on the points $\pm\pi/2$. We now compute its exponential map ρ. Clearly $\rho(u) = u$ outside the interval $[-\pi/2, \pi/2]$, so we can suppose $u \in (-\pi/2, \pi/2)$. Notice that the equation (2.20) can be seen as a family of Cauchy problems

$$\frac{d}{dt}\rho^u(t) = f(\rho^u(t))\,, \quad \rho^u(0) = u\,,$$

with $\rho^u(t) = \rho_t(u)$. If $f(u) \neq 0$ then $\rho^u(t) = F_u^{-1}(t)$, with

$$F_u(s) = \int_u^s \frac{dv}{f(v)} = \tan(s) - \tan(u)\,. \tag{2.47}$$

It then follows that $\rho_t(u) = F_u^{-1}(t) = \arctan(\tan(u)+t)$ and hence $\rho(u) = \arctan(\tan(u)+1)$. In particular, we have $\rho''(0)/\rho'(0) = -1$. Moreover, by numerical integration one obtains that

$$\int_0^{\pi/2} \left(\frac{\rho''(u)}{\rho'(u)}\right)^2 du \sim 1.4\,.$$

Therefore, by (2.46) we obtain

$$S''_\rho(\pi/4)/4 \sim -c/60\,,$$

as announced before.

One more little remark we will add in this section is about some extensive property of the relative entropy. For the sake of simplicity, as above we will consider a state ω_f induced by some diffeomorphism of the type $\rho = \mathrm{Exp}(f)$. It can be noticed by the formulae given above that if $S_f = S_I(\omega_f \| \omega)$ for some interval I, then $S_{f_1+f_2} = S_{f_1} + S_{f_2}$ if the supports of f_1 and f_2 are disjoint (up to a set of zero measure). Clearly if this is not the case then this fact is no longer true. Therefore, if we define $S_f(t) = S_{(t,+\infty)}(\omega_{V(\rho)}\|\omega)$ then we will have

$$S_{f_1+f_2}(t) = S_{f_1}(t) + S_{f_2}(t) + s_t(f_1, f_2)$$

for some term $s_t(f_1, f_2)$. Here we give an estimate of $s_t(f_1, f_2)$.

Let $[a_1, b_1]$ and $[a_2, b_2]$ be the supports of f_1 and f_2, with $b_1 \leq b_2$. Clearly $s_t(f_1, f_2) = 0$ if $t \geq b_1$, since for such values of t we have that f_1 does not contribute to the relative entropy. We can then assume $t \leq b_1$. Before proceeding, we now make a general remark: consider a real function f with compact support, and recall that if $f(u) \neq 0$ then the exponential flow $\rho_t(u)$ of f is obtained by inverting the function $F_u(s)$ defined in (2.47). Then by deriving the relation $t = F_u(\rho_t(u))$ with respect to the variable u we have

$$\partial_u \rho_t(u) = \frac{f(\rho_t(u))}{f(u)}\,.$$

Deriving again and applying the obtained formulae to $\rho_{-1} = \rho^{-1} = \eta$ and $f = f_1 + f_2$ one obtains

$$\frac{\eta''}{\eta'} = \frac{\eta_1''}{\eta_1'} + \frac{\eta_2''}{\eta_2'} + \delta(f_1, f_2)\,,$$

with

$$\delta(f_1, f_2)(u) = f'(\eta(u)) - f_1'(\eta_1(u)) - f_2'(\eta_2(u))\,.$$

Notice that if $\mathrm{supp} f_1 \cup \mathrm{supp} f_2 \subseteq [a, b]$ then

$$\|\delta(f_1, f_2)\|_\infty \leq |b-a| \cdot \|f_1''\|_\infty + |b-a| \cdot \|f_2''\|_\infty\,.$$

We use this fact to estimate

$$\frac{c}{24}\int_t^{\infty}(u-t)\delta(f_1,f_2)^2du \leq \frac{c}{24}\left(\|f_1''\|_\infty+\|f_2''\|_\infty\right)^2(b-a)^2\frac{(b_1-t)^2}{2}=\epsilon_0(t)\,.$$

Moreover, by applying Cauchy-Schwarz with respect to the measure $(u-t)du$ on $(t,+\infty)$ we have

$$\frac{c}{12}\int_t^{\infty}(u-t)\frac{\eta_i''(u)}{\eta_i'(u)}\delta(f_1,f_2)(u)du \leq \frac{c}{12}\sqrt{S_{f_i}(t)\epsilon_0(t)}=\epsilon_i(t)\,,$$

for $i=1,2$. Always by Cauchy-Schwarz we have

$$\frac{c}{12}\int_t^{\infty}(u-t)\frac{\eta_1''\eta_2''}{\eta_1'\eta_2'}du \leq 2\sqrt{S_{f_1}(t)S_{f_2}(t)}=\epsilon_3(t)\,,$$

and therefore we can conclude that $|s_t(f_1,f_2)|\leq\epsilon(t)$, with

$$\epsilon(t)=\epsilon_0(t)+\epsilon_1(t)+\epsilon_2(t)+\epsilon_3(t)\,.$$

We conclude by noticing that, since $S_{f_1}(t)$ vanishes with its first derivative at $t=b_1$, then $\epsilon(t)\leq C(b_1-t)$ for t near to b_1 for some $C>0$.

Chapter 3

Positive Energy Representations of Loop Groups

3.1 Infinite dimensional Lie algebras

In this chapter, based on the paper [85], we emulate what we did on the Virasoro nets in [84] but we focus on loop group models. In order to do so, we define solitonic states by following [34] and we then follow [93] in such a way to explicitly compute the needed algebraic relations. The other main result of this work is about Sobolev extensions of Positive Energy Representations of Loop Groups. We begin this chapter with some general notions of infinite dimensional Lie algebras [53, 55].

Let $\mathfrak{g}$ be a simple complex Lie algebra. We fix the notation: $\mathfrak{h}$ is a Cartan subalgebra, or equivalently a maximal toral subalgebra, Φ is the relative root system with highest root θ and Δ is a set of simple roots, with $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ if $\mathfrak{g}$ has rank l. We then denote by $\langle \cdot, \cdot \rangle$ the normalized Killing form, namely we normalize it in such a way to have a bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{h}^*$ satisfying $\langle \theta, \theta \rangle = 2$.

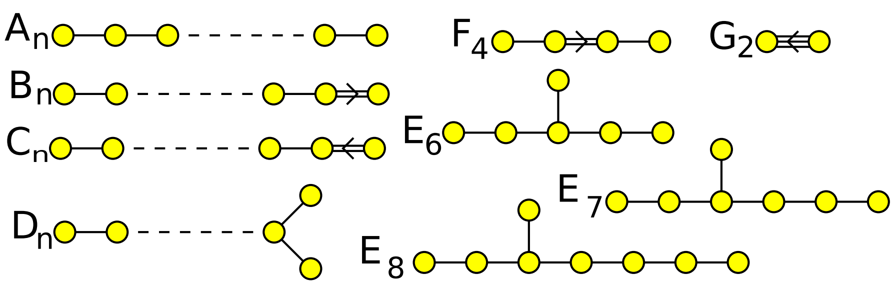


Figure 3.1. Simple Dynkin diagrams

Let $\mathbb{C}[t, t^{-1}]$ be the algebra of complex Laurent polynomials in t, that is the set of the formal series $f = \sum_{k \in \mathbb{Z}} c_k t^k$ where all but a finite number of c_k are zero. We define the *residue* of a Laurent polynomial f by $\mathrm{Res} f = c_{-1}$. Clearly Res is a linear functional on $\mathbb{C}[t, t^{-1}]$ and it satisfies $\mathrm{Res} f' = 0$ where $f' = \frac{df}{dt}$ is the formal derivative of f. We observe that $\mathfrak{g}[t, t^{-1}] \cong \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]$ is a natural Lie algebra with bracket given by

$[x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{n-m}$. It is a subalgebra of the rational maps from $\mathbb{C}^*$ to $\mathfrak{g}$, hence the *evaluation of* $x \in \mathfrak{g}$ on S^1 is then obtained by replacing t by $e^{i\theta}$. We will often use the notation $x(n)$ for $x \otimes t^n$ and identify $\mathfrak{g}$ with $\mathfrak{g} \otimes 1$ in $\mathfrak{g}[t, t^{-1}]$.

Let Ω be the bilinear form on $\mathfrak{g}[t, t^{-1}]$ defined by $\Omega(f, g) = \operatorname{Res} \langle f', g \rangle$. Observe that for x, y in $\mathfrak{g}$ we have

$$\Omega(x(n), y(m)) = n\delta_{n,-m} \langle x, y \rangle \ .$$

One easily checks that Ω is a two-cocycle on $\mathfrak{g}[t, t^{-1}]$ and hence we have a central extension $\widetilde{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C} \cdot c$.

Consider the Lie algebra $\partial = \mathbb{C}[z, z^{-1}]\frac{d}{dz}$. It is a subalgebra of the Lie algebra of rational vector fields on $\mathbb{C}$ and it is called the *Witt algebra*. The natural basis of ∂ is given by $d_n = t^{n+1}\frac{d}{dt}$ where $n \in \mathbb{Z}$. The commutation relations of ∂ are then

$$[d_n, d_m] = (n - m)d_{n+m}\,, \quad n, m \in \mathbb{Z}\,.$$

The Witt algebra naturally acts on $\mathfrak{g}[t, t^{-1}]$ by derivations, and this action lifts to $\widetilde{\mathfrak{g}}$ by trivial action on the central element c. Explicitly one has

$$d_n x(m) = -mx(n + m)\,, \qquad d_n c = 0\,.$$

Definition 44. The *Virasoro algebra* Vir is the central extension of the Witt algebra ∂ given by the two-cocycle

$$\omega(d_n, d_m) = \frac{1}{12}(n^3 - n)\delta_{n,-m}\,, \quad n, m \in \mathbb{Z}\,.$$

If $\mathrm{Vir} = \partial \oplus \mathbb{C} \cdot \kappa$, then κ is called the *central charge* or the *conformal anomaly.*

Since κ is central, the Lie bracket of Vir is uniquely determined by

$$[d_n, d_m] = (n - m)d_{n+m} + \frac{1}{12}(n^3 - n)\delta_{n,-m}\kappa\,. \tag{3.1}$$

Up to equivalence, Vir is the unique central extension of the Witt algebra. The action of ∂ on $\widetilde{\mathfrak{g}}$ on can be extended to an action of Vir by letting κ acting trivially. We now show that this action allows us to further extend the Lie algebra $\widetilde{\mathfrak{g}}$.

Definition 45. Let $\mathfrak{g}$ and $\mathfrak{h}$ be two Lie algebras and let $\rho\colon \mathfrak{g} \to \operatorname{Der}(\mathfrak{h})$ be a homomorphism. The *semidirect product* $\mathfrak{g} \ltimes \mathfrak{h}$ is the Lie algebra with underlying vector space $\mathfrak{g} \oplus \mathfrak{h}$ and whose Lie bracket is given by

$$[x_1 + h_1, x_2 + h_2] = [x_1, x_2] + \rho(x_1)(h_2) - \rho(x_2)(h_1)\,.$$

Clearly $\mathfrak{g} \ltimes \mathfrak{h}$ depends on ρ. We note that $\mathfrak{h}$ is an ideal of $\mathfrak{g} \ltimes \mathfrak{h}$. Indeed $\mathfrak{g} \ltimes \mathfrak{h}$ is an extension of $\mathfrak{g}$ by $\mathfrak{h}$, that is we have a short exact sequence

$$0 \to \mathfrak{h} \to \mathfrak{g} \ltimes \mathfrak{h} \to \mathfrak{g} \to 0\,.$$

Let $G \ltimes H$ be a semidirect product induced by $\tau\colon G \to \operatorname{Aut}(H)$. We remember that $\operatorname{Lie}(\operatorname{Aut}(H)) \subseteq \operatorname{Der}(\mathfrak{h})$, since by derivation we can consider $\operatorname{Aut}(H)$ as a subgroup of $\operatorname{Aut}(\mathfrak{h})$ and $\operatorname{Lie}(\operatorname{Aut}(\mathfrak{h})) = \operatorname{Der}(\mathfrak{h})$. Hence $\tilde{\tau}(g) = d_{e_H}\tau(g)$ belongs to $\operatorname{Aut}(\mathfrak{h})$ for each $g \in G$, where d_{e_H} is the differential at the identity of H. As the notation suggests, the Lie algebra of $G \ltimes H$ is the semidirect product $\mathfrak{g} \ltimes \mathfrak{h}$ induced by $\rho = d_{e_G}\tilde{\tau}$.

Definition 46. Let $\mathfrak{g}$ be a complex simple Lie algebra. The *(Kac-Moody) affine Lie algebra* of $\mathfrak{g}$ is the central extension $\hat{\mathfrak{g}} = \widetilde{\mathfrak{g}} \rtimes \mathbb{C}d$ induced by the derivation $d = -d_0$ of the Virasoro algebra.

We characterize Vir and $\hat{\mathfrak{g}}$ in relation with an involution of $\mathfrak{g}$. Let $\mathfrak{g}_k$ be a real form of $\mathfrak{g}$ induced by a Chevalley basis. Let τ be the conjugation of $\mathfrak{g}$ given by $\mathfrak{g}_k$ and define the involution $x^* = -\tau(x)$ for $x \in \mathfrak{g}$, so that $\mathfrak{g}_k = \{x \in \mathfrak{g} \colon x^* = -x\}$. The map $x \mapsto x^*$ extends to an involution of $\widetilde{\mathfrak{g}}$ by defining $c^* = c$ and $x(n)^* = x^*(-n)$. The *compact form* of $\widetilde{\mathfrak{g}}_k$ is the real form

$$\widetilde{\mathfrak{g}}_k = \{x \in \widetilde{\mathfrak{g}} \colon x^* = -x\} .$$

We define an involution on Vir by setting $d_n^* = d_{-n}$ and $\kappa^* = \kappa$. This extension is compatible with the action of Vir on $\widetilde{\mathfrak{g}}$, that is it naturally extends to an involution of $m = \widetilde{\mathfrak{g}} \rtimes \mathrm{Vir}$. Note $\hat{\mathfrak{g}}$ is a subalgebra of m close under involution. Furthermore,

$$\partial_k = \{d \in \partial \colon d^* = -d\}$$

is a real form of ∂ which coincides with the Lie algebra of real polynomial vector fields on S^1. Indeed, by evaluating on S^1 we can identify $d_n^* - d_n$ with $2 \sin n\theta \frac{d}{d\theta}$ and $(id_n)^* - id_n$ with $2 \cos n\theta \frac{d}{d\theta}$. Since $(dx)^* = -d^* x^*$ for each $d \in \partial$ and $x \in \widetilde{\mathfrak{g}}$, then $\widetilde{\mathfrak{g}}_k$ is stable under the action of ∂_k.

We further examine the structure of $\hat{\mathfrak{g}}$. The Lie algebra $\hat{\mathfrak{g}}$ is neither semisimple nor finite-dimensional, but we can equally describe a root space decomposition relative to the abelian subalgebra $\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$ as follows. A weight λ in $\mathfrak{h}^*$ can be extended to $\hat{\mathfrak{h}}$ by setting $\lambda(c) = \lambda(d) = 0$. Define $\delta \in \hat{\mathfrak{h}}^*$ by setting $\delta(\mathfrak{h}) = 0$, $\delta(c) = 0$ and $\delta(d) = 1$. Since for each root $\alpha \in \Phi$ we have

$$[h + d, x_\alpha(n)] = (\alpha(h) + n) x_\alpha(n) ,$$

where $x_\alpha \in \mathfrak{g}_\alpha$ and $h \in \mathfrak{h}$, the root system of $\hat{\mathfrak{g}}$ with respect to $\hat{\mathfrak{h}}$ is

$$\widehat{\Phi} = \{k\delta + \alpha \,|\, k \in \mathbb{Z}, \alpha \in \Phi\} \cup \{k\delta \,|\, k \in \mathbb{Z}^*\} .$$

We now construct a Chevalley basis of $\mathfrak{g}$ in the following way. We denote by $\mathfrak{g}_\alpha$ the root space associated to some root $\alpha \in \Phi$, we define a *root vector* as a nonzero vector of a root space and then we choose a set of root vectors $e_\alpha \in \mathfrak{g}_\alpha$ for each $\alpha \in \Phi^+$. For each positive root $\alpha \in \Phi^+$ we find a root vector $e_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that $\langle e_\alpha, e_{-\alpha} \rangle = 2/\langle \alpha, \alpha \rangle$. It is easy to check that $[e_\alpha, e_{-\alpha}] = h_\alpha$ where $h_\alpha = 2\eta(\alpha)/\langle \alpha, \alpha \rangle = \eta(\alpha^\vee)$ and $\eta \colon \mathfrak{h}^* \to \mathfrak{h}$ is the isomorphism induced by $\langle \cdot, \cdot \rangle$. If we define $h_i = h_{\alpha_i}$ where α_i is a simple root, then we have just constructed a basis of $\mathfrak{g}$ given by

$$\{e_\alpha \, , \; \alpha \in \Phi \, ; \;\; h_i \, , \; 1 \le i \le l\} . \tag{3.2}$$

Clearly since $\mathfrak{h}$ is maximal toral then $[h_i, h_j] = 0$ and $[h_i, e_\alpha] = \alpha(h_i) e_\alpha = C(\alpha_i, \alpha) e_\alpha$, where $C(\alpha_i, \alpha) = 2 \langle \alpha_i, \alpha \rangle / \langle \alpha_i, \alpha_i \rangle$ is the Cartan integer. Note that $[e_\alpha, e_{-\alpha}] = h_\alpha$ is a linear combination with integral coefficient of the h_i. Moreover, given $\alpha, \beta \in \Phi$ we have $[e_\alpha, e_\beta] = c_{\alpha,\beta} e_{\alpha+\beta}$ if $\alpha + \beta$ is a root.

We now come back to study our affine Lie algebra $\hat{\mathfrak{g}}$. If we pick a Chevalley basis associated to $\mathfrak{h}$ as in (3.2), then the corresponding root spaces are $\hat{\mathfrak{g}}_{\alpha+n\delta} = \mathbb{C}e_\alpha(n)$ and $\hat{\mathfrak{g}}_{n\delta} = \{h(n) \mid h \in \mathfrak{h}\}$. If we set

$$\widehat{\Phi}^+ = \{\alpha + n\delta \mid \alpha \in \Phi,\, n > 0\} \cup \{n\delta \mid n > 0\} \cup \Phi^+ ,$$

then a base of positive simple roots is $\hat{\Delta} = \{\alpha_0, \dots, \alpha_l\}$ where $\Delta = \{\alpha_1, \dots, \alpha_l\}$ is simple roots system of $\mathfrak{g}$ and $\alpha_0 = \delta - \theta$. Note that by the condition $\langle\theta,\theta\rangle = 2$ we have that each root $\alpha \in \Phi$ equals its coroot $\alpha^\vee = 2\alpha/\langle\alpha,\alpha\rangle$. Hence through the isomorphism $\lambda \mapsto h_\lambda$ the fundamental weights of $\mathfrak{g}$ are given by $\Lambda_i(h_{\alpha_j}) = \delta_{ij}$. We define the *coroots* of $\hat{\mathfrak{g}}$ as the elements in $\hat{\mathfrak{h}}$ given by $\hat{h}_{\alpha_0} = c - h_\theta$ and $\hat{h}_{\alpha_i} = h_{\alpha_i}$ for $i = 1, \dots, l$. The *fundamental weights* $\hat{\Lambda}_0, \dots, \hat{\Lambda}_l$ are then defined by

$$\hat{\Lambda}_i(\hat{h}_{\alpha_j}) = \delta_{ij} \quad \text{and} \quad \hat{\Lambda}_i(d) = 0 \,.$$

Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}_- \oplus \mathfrak{n}_+$ be a triangular decomposition of $\mathfrak{g}$. Defining $\hat{\mathfrak{n}}_- = \mathfrak{n}_- \oplus t^{-1}\mathfrak{g}[t^{-1}]$ and $\mathfrak{n}_+ = \hat{\mathfrak{n}}_+ \oplus t\mathfrak{g}[t]$, then we have the following triangular decomposition of $\hat{\mathfrak{g}}$:

$$\hat{\mathfrak{g}} = \hat{\mathfrak{n}}_- \oplus \hat{\mathfrak{h}} \oplus \hat{\mathfrak{n}}_+ \,.$$

The restriction of $\hat{\Lambda}_1, \dots, \hat{\Lambda}_l$ to $\mathfrak{h}$ are the fundamental weights for $\mathfrak{g}$. Moreover, if a_i are the integers such that $\theta = a_1\alpha_i + \dots + a_l\alpha_l$ then we have $\hat{\Lambda}_i = \Lambda_i + a_i\hat{\Lambda}_0$. If $\rho = \sum_{i=1}^l \Lambda_i$, then the *dual Coxeter number* of $\mathfrak{g}$ is

$$g = 1 + \sum_{i=1}^{l} a_i \,.$$

Note that $g = 1 + \langle\rho,\theta\rangle$. If we set $\hat{\rho} = \sum_{i=0}^l \hat{\Lambda}_i$ then $\hat{\rho} = \rho + g\Lambda_0$. The *weight lattice* of $\hat{\mathfrak{g}}$ is $\widehat{\Pi} = \sum_{i=0}^l \mathbb{Z}\hat{\Lambda}_i$ and the set of *dominant integral weights* is $\widehat{\Pi}_+ = \sum_{i=0}^l \mathbb{Z}_+\hat{\Lambda}_i$ where $\mathbb{Z}_+ = \{0, 1, \dots\}$. The *level* of $\hat{\Lambda} \in \widehat{\Pi}_+$ is the positive number $\hat{\Lambda}(c)$. Levels of dominant integral weights are integers, since $\hat{\Lambda}_0(c) = 1$ and $\hat{\Lambda}_i(c) = a_i$ for $i = 1, \dots, l$. Given $m \in \mathbb{Z}_+$, denote by $\widehat{\Pi}_+^{(m)}$ the set of dominant integral weights of level m. This is a finite set for each $m \in \mathbb{Z}_+$ and $\widehat{\Pi}_+^{(0)} = \{0\}$. We extend the bilinear form $\langle\cdot,\cdot\rangle$ on $\mathfrak{h}^*$ to a symmetric bilinear form on $\hat{\mathfrak{h}}^*$ by

$$\langle\mathfrak{h}^*, \mathbb{C}\delta + \mathbb{C}\hat{\Lambda}_0\rangle = 0\,, \quad \langle\delta,\delta\rangle = \langle\hat{\Lambda}_0,\hat{\Lambda}_0\rangle = 0\,, \quad \langle\delta,\hat{\Lambda}_0\rangle = 1\,.$$

Notice that $\hat{\Lambda}(c) = \langle\hat{\Lambda},\delta\rangle$.

We now provide some basic definitions about the representations theory of $\hat{\mathfrak{g}}$. For a detailed study the reader can consult [55] or find other references in [45].

Definition 47. Let $\hat{\Lambda} \in \hat{\mathfrak{h}}$ be a weight of $\hat{\mathfrak{g}}$. A $\hat{\mathfrak{g}}$-module (V,π) is a *highest weight representation* with highest weight $\hat{\Lambda}$ if there exists a cyclic vector $v_{\hat{\Lambda}} \in V$ such that

$$\pi(\hat{\mathfrak{n}}_+)v_{\hat{\Lambda}} = 0\,, \quad \pi(h)v_{\hat{\Lambda}} = \hat{\Lambda}(h)v_{\hat{\Lambda}}\,, \quad \pi(\mathcal{U}(\hat{\mathfrak{g}}))v_{\hat{\Lambda}} = V\,,$$

for $h \in \hat{\mathfrak{h}}$ and where $\mathcal{U}(\hat{\mathfrak{g}})$ is the universal enveloping algebra of $\hat{\mathfrak{g}}$.

Like for the highest weight representations of finite-dimensional semisimple Lie algebras, a way to exhibit a highest weight representation with highest weight $\hat{\Lambda}$ is to prove the existence of the Verma module $M(\hat{\Lambda})$. The proof is analogous, since we still have a triangular decomposition. Hence for each $\hat{\Lambda} \in \mathfrak{h}^*$ there exists a unique irreducible highest weight representation of $M(\hat{\Lambda})$ that we'll denote by $(L(\hat{\Lambda}), \pi_{\hat{\Lambda}})$. If $\hat{\Lambda} \in \widehat{\Pi}_+$ is a dominant integral weight, then $L(\hat{\Lambda})$ is *unitary*, that is there exists a positive definite hermitian form $(\cdot|\cdot)$ on $L(\hat{\Lambda})$ such that $(\pi_{\hat{\Lambda}}(x)u|v) = (u|\pi_{\hat{\Lambda}}(x^*)v)$ for all u, v in $L(\hat{\Lambda})$ and x in $\hat{\mathfrak{g}}$. Such a hermitian form is said to be *contravariant* with respect to the $*$-conjugation of $\hat{\mathfrak{g}}$.

3.2 Loop groups

Let G be a Lie group. The *loop group* of G is $LG = C^\infty(S^1, G)$. It is a group with respect to the pointwise composition in G. We see that it is a smooth manifold as follows. If $\mathfrak{g}$ is the Lie algebra of G, then we define on $L\mathfrak{g} = C^\infty(S^1, \mathfrak{g})$ a topology of separable Fréchet space saying that $\{f_k\}$ converges to f in $L\mathfrak{g}$ if $d^n f_k/d\theta^n$ converges uniformly to $d^n f/d\theta^n$ for each n. Let now U be an open neighborhood of the identity element in G which is homeomorphic by the exponential map to an open set $\check{U}$ of the Lie algebra $\mathfrak{g}_0$ of G. We prescribe $\mathcal{U} = C^\infty(S^1, U)$ to be open and homeomorphic to $\check{\mathcal{U}} = C^\infty(S^1, \check{U})$. So LG is a Fréchet manifold, and moreover it is a infinite dimensional Fréchet group [87]. The Lie algebra of LG is $L\mathfrak{g}_0$. The exponential map is well defined and it is a local homeomorphism near the identity. If G has a complexification $G_\mathbb{C}$ then LG ha a complexification $LG_\mathbb{C} = C^\infty(S^1, G_\mathbb{C})$ [87].

Through constant loops, we can think to G as a subgroup of LG. If G is simply connected then LG is connected. We now consider the evaluation map $e_1 \colon LG \to G$ given by $f \mapsto f(1)$. We define its kernel the *based loop group* ΩG. It is a normal closed subgroup of LG. The just defined two maps give rise to a split exact sequence

$$1 \to \Omega G \to LG \to G \to 1\,,$$

so that we can claim the isomorphism of Lie groups $LG \cong G \ltimes \Omega G$ to be true. As compact Lie groups are studied together with unitary representations, loop groups are studied together with the so called Positive Energy Representations.

We mainly follow [93]. Let G be a compact, simple and simply connected Lie group. A *Positive Energy Representation (PER)* of the loop group $LG = C^\infty(S^1, G)$ on a separable Hilbert space $\mathcal{H}$ is a projective strongly continuous unitary representation π of $LG \rtimes \mathbb{T}$ with a commutative diagram

$$\begin{tikzcd} LG \rtimes \mathbb{T} \arrow[r, "\pi"] & PU(\mathcal{H}) \\ \mathbb{T} \arrow[u] \arrow[r, "U"] & U(\mathcal{H}) \arrow[u] \end{tikzcd} \tag{3.3}$$

where the torus $\mathbb{T} \cong \mathrm{Rot}$ acts on LG by rotations $R_\theta.\gamma(\phi) = \gamma(\phi - \theta)$ and U is a strongly continuous unitary representation inducing an isotypical decomposition $\mathcal{H} = \bigoplus_{n \geq n_0} \mathcal{H}(n)$ for some integer n_0. Without loss of generality, we can suppose that $n_0 = 0$ and that $\mathcal{H}(0)$ is not zero-dimensional. A PER is said to be of *finite type* if $\dim \mathcal{H}(n) < +\infty$ for every n. Irreducible PERs are of finite type.

We denote by $\mathfrak{g}_0$ the Lie algebra of G and by $\mathfrak{g}$ the complexification of $\mathfrak{g}_0$. Recall that $\mathfrak{g}_0$ is a compact Lie algebra, that is its Killing form is negative definite. In particular, there is an antilinear involution $X \mapsto X^*$ of $\mathfrak{g}$ such that

$$\mathfrak{g}_0 = \{X \in \mathfrak{g} \colon X^* = -X\}\,.$$

Let $X(n)$ be the map $\theta \mapsto Xe^{in\theta}$ for X in $\mathfrak{g}$ and n integer. Then $[X(n), Y(m)] = [X, Y](n+m)$, showing that the space spanned by these elements, which we will denote by $L^{\mathrm{pol}}\mathfrak{g}$, forms a Lie algebra. On $L^{\mathrm{pol}}\mathfrak{g}$ we can define an involution by $X(n)^* = X^*(-n)$. Moreover, if $\mathcal{H}^{\mathrm{fin}}$ is the subspace of finite energy vectors, namely the algebraic sum of the subspaces $\mathcal{H}(n)$, then we can define a projective representation π of $L^{\mathrm{pol}}\mathfrak{g}$ on $\mathcal{H}^{\mathrm{fin}}$ in such a way to verify the commutation relations ([93], Theorem 1.2.1.)

$$[\pi(X), \pi(Y)] = \pi([X, Y]) + i\ell B(X, Y)\,, \qquad B(X, Y) = \int_0^{2\pi} \langle X, \dot{Y}\rangle \, \frac{d\theta}{2\pi}\,.$$

We point out that the existence of such a representation of $L^{\mathrm{pol}}\mathfrak{g}$ is not a trivial issue, since these commutation relations do not uniquely determine the projective representation of $L^{\mathrm{pol}}\mathfrak{g}$, and also the representation of LG cannot be differentiated in a straightforward way as in finite dimensional cases. If d is the generator of rotations, namely $U(R_\theta) = e^{i\theta d}$, then we have that $[d, \pi(X)] = i\pi(\dot{X})$ where $\dot{X}(\theta) = \frac{d}{d\theta}X(\theta)$. The above operators are all closable and we also have the formal adjunction property $\pi(X)^* = \pi(X^*)$ on $\mathcal{H}^{\mathrm{fin}}$. Furthermore, the projective representation π of $L^{\mathrm{pol}}\mathfrak{g}$ on $\mathcal{H}^{\mathrm{fin}}$ can be lifted to a projective representation π of $L\mathfrak{g} = C^\infty(S^1, \mathfrak{g})$ on $\mathcal{H}^\infty$ in such a way to verify all the previous relations, where $\mathcal{H}^\infty$ is the Fréchet space of smooth vectors for Rot. We recall that by definition $\mathcal{H}^\infty = \bigcap_s \mathcal{H}^s$, where $s \in \mathbb{R}$ and $\mathcal{H}^s$ is the *scale space*, that is the completion of $\mathcal{H}^{\mathrm{fin}}$ with respect to the Sobolev norm $\|\xi\|_s = \|(1+d)^s\xi\|$. Notice that the projective representation π of $L\mathfrak{g}$ is actually a representation if restricted on $\mathfrak{g}$, since the projective representation of G lifts to a unitary representation. Also, the subspaces $\mathcal{H}(n)$ are G-invariant. The adjoint action of LG on the mentioned operators is given by [93]

$$\begin{aligned} \pi(\gamma)\pi(X)\pi(\gamma)^* &= \pi(\gamma X \gamma^{-1}) + ic(\gamma, X)\,, \\ \pi(\gamma) d \pi(\gamma)^* &= d - i\pi(\dot{\gamma}\gamma^{-1}) + c(\gamma, d)\,, \end{aligned} \tag{3.4}$$

where the real constants $c(\gamma, X)$ and $c(\gamma, d)$ are explicitly given by

$$c(\gamma, X) = -\ell \int_0^{2\pi} \langle \gamma^{-1}\dot{\gamma}, \dot{X}\rangle \, \frac{d\theta}{2\pi}\,, \qquad c(\gamma, d) = -\frac{\ell}{2}\int_0^{2\pi} \langle \gamma^{-1}\dot{\gamma}, \gamma^{-1}\dot{\gamma}\rangle \, \frac{d\theta}{2\pi}\,.$$

Here $\langle \cdot, \cdot \rangle$ denotes the *basic inner product*, namely the Killing form normalized on the highest root θ in such a way that $\langle \theta, \theta\rangle = 2$. The elements $\gamma^{-1}\dot{\gamma}$ and $\dot{\gamma}\gamma^{-1}$ of $L\mathfrak{g}$ are the *left logarithmic derivative* and the *right logarithmic derivative* of γ, respectively defined by [93]

$$\gamma^{-1}\dot{\gamma}(t) = \frac{d}{dh}\bigg|_{h=0} \gamma^{-1}(t)\gamma(t+h) \qquad \text{and} \qquad \dot{\gamma}\gamma^{-1}(t) = \frac{d}{dh}\bigg|_{h=0} \gamma(t+h)\gamma^{-1}(t)\,.$$

We will use the following notation:

$$x = \pi(X)\,, \quad x(n) = \pi(X(n))\,, \quad \langle x, y\rangle = \langle X, Y\rangle\ .$$

We can define a representation of the Virasoro algebra Vir

$$[L_n, L_m] = (n-m)L_{n+m} + \delta_{n+m,0}\frac{n(n^2-1)}{12}c\,, \tag{3.5}$$

by Sugawara construction, that is such a representation is given by defining on $\mathcal{H}^{\text{fin}}$ the operator

$$L_n = \frac{1}{2(\ell+g)}\sum_m : x_i(-m)x^i(m+n):\ , \tag{3.6}$$

where we used the Einstein convention on summations and the *normal ordering* notation, namely the symbol $:x(n)y(m):$ stands for $x(n)y(m)$ if $n \leq m$ and for $y(m)x(n)$ if $n > m$. The elements $\{x_i\}$ and $\{x^i\}$ appearing in (3.6) can be arbitrary dual basis with respect to the basic inner product, namely $\langle x^i, x_j\rangle = \delta_{ij}$, and g is the *dual Coxeter number*, that is

$$g = 1 + \sum a_i^\vee\,, \qquad \theta = \sum a_i^\vee \alpha_i^\vee\,,$$

where $\alpha_i^\vee$ are the simply coroots and $a_i^\vee$ are strictly positive. By the assumption $\langle\theta,\theta\rangle = 2$ it can be shown that the dual Coxeter number is half the Casimir of the adjoint representation, namely we have $[X_i, [X^i, Y]] = 2gY$ for Y in $\mathfrak{g}$. Notice that if X_i are in $\mathfrak{g}_0$ and such that $-\langle X_i, X_j\rangle = \delta_{ij}$ then $X^i = -X_i$. The constant c uniquely determined by (3.5) is called the *central charge* of the representation. If the PER is irreducible then we have $L_0 = d + h$ for some rational number h, where h is therefore the lowest eigenvalue of L_0 and it is called the *trace anomaly*. In any irreducible PER, the central charge and the trace anomaly are given by [45]

$$c = \frac{\ell \dim \mathfrak{g}}{\ell + g}\,, \qquad h = \frac{C_\lambda}{2(\ell+g)}\,, \tag{3.7}$$

where C_λ is the Casimir associated to the basic inner product $\langle\cdot,\cdot\rangle$ and to the null energy space $\mathcal{H}(0) = \mathcal{H}_\lambda$, which is the irreducible highest weight representation of $\mathfrak{g}$ associated to some dominant integral weight λ satisfying

$$\langle\lambda,\theta\rangle \leq \ell\,. \tag{3.8}$$

The set of dominant integral weights λ satisfying condition (3.8) is called the *level ℓ alcove*. We will say that π is a *vacuum positive energy representation*, or simply a *vacuum representation*, if $\mathcal{H}(0)$ is one-dimensional. If $\mathcal{H}(0) = \mathbb{C}\Omega$ with $(\Omega|\Omega) = 1$, then the state ω associated to Ω is called the *vacuum state*. Notice that π is a vacuum representation if and only if irreducible and with $h = 0$. More in general, if $\mathcal{H}(0) = \mathcal{H}_\lambda$ then the trace anomaly can be computed by taking in account that

$$C_\lambda = \langle \lambda, \lambda + 2\rho \rangle \ , \quad g = 1 + \langle \rho, \theta \rangle \ ,$$

where ρ is the *Weyl vector*, that is the sum of all the fundamental weights. Equivalently, the Weyl vector can be defined as half the sum of all the positive roots.

Consider now $\mathrm{Diff}_+(S^1)$, the Fréchet Lie group of the orientation preserving diffeomorphisms of the circle. The natural action of $\mathrm{Diff}_+(S^1)$ on LG is smooth. Furthermore, every PER π of LG is $\mathrm{Diff}_+(S^1)^\sim$-covariant, namely there is a projective unitary representation U of the universal covering $\mathrm{Diff}_+(S^1)^\sim$ such that $U(\tilde\rho)\pi(\gamma)U(\tilde\rho)^* = \pi(\rho.\gamma)$ [44, 45]. Consider now $\mathcal{H}^{0,\mathrm{fin}}$, the algebraic direct sum of the eigenspaces of L_0. On the infinitesimal level, in general the space $\mathcal{H}^{0,\mathrm{fin}}$ is a direct sum of infinitely many unitary irreducible representations $V(c, h_i)$ of the Virasoro algebra. Such a representation integrates to a unitary projective representation U of $\mathrm{Diff}_+(S^1)^\sim$, and if the appearing highest weights h_i differ only by integers then U reduces to a unitary projective representation of $\mathrm{Diff}_+(S^1)$ [34, 44]. Now we briefly study the irreducible unitary representations $V(c, h)$ of the Virasoro algebra appearing from an irreducible PER of level ℓ of LG. If $\ell = 0$ then $\lambda = 0$, and by $c = h = 0$ we have the trivial representation of Vir. If $\ell \geq 1$, then $V(c, h)$ belongs to the *continuous series*, namely we have $h \geq 0$ and $c \geq 1$. The estimate on the central charge follows by the inequality $g + 1 \leq \dim \mathfrak{g}$, which can be noticed by studying the following table [55]:

Dynkin diagram	Simple Lie algebra	Complex dimension	Dual Coxeter number
A_n	$\mathfrak{sl}_{n+1}$	$n^2 + 2n$	$n + 1$
B_n	$\mathfrak{so}_{2n+1}$	$2n^2 + n$	$2n - 1$
C_n	$\mathfrak{sp}_{2n}$	$2n^2 + n$	$n + 1$
D_n	$\mathfrak{so}_{2n}$	$2n^2 - n$	$2n - 2$
E_6	$\mathfrak{e}_6$	78	12
E_7	$\mathfrak{e}_7$	133	18
E_8	$\mathfrak{e}_8$	248	30
F_4	$\mathfrak{f}_4$	52	9
G_2	$\mathfrak{g}_2$	14	4

Lemma 48. $[L_n, x(k)] = -kx(n+k)$ *on* $\mathcal{H}^{0,\mathrm{fin}}$.

Proof. This is actually a well known lemma proved in [55], here we show a more basic and personal proof following [87]. Let $\{x_i\}$ be a orthonormal basis in $\mathfrak{g}_0$, so that its dual basis is given by $x^i = -x_i$. We have the following relations:

$$\begin{aligned}
[xy, z] &= x[y, z] + [x, z]y \, , \\
[x(a), [y, z](b)] &= [[z, x], y](a + b) + [[x, y](a), z(b)] \, , \\
[x_i(a), [x^i, z](b)] &= [x_i, [x^i, z]](a + b) = 2gz(a + b) \, ,
\end{aligned}$$

for any integers a and b. Notice also that on $\mathcal{H}^{0,\mathrm{fin}}$ we have

$$L_n = \frac{1}{2(\ell + g)} \sum_{m \geq -n/2} (2 - \delta_{-m,n/2}) x_i(-m) x^i(m + n) \, .$$

Therefore we can compute

$$\begin{aligned}[x_i(-m)x^i(m+n),x_j(k)] &= x_i(-m)[x^i(m+n),x_j(k)] + [x_i(-m),x_j(k)]x^i(m+n) \\ &= -\ell k\delta_{i,j}(\delta_{k,-m-n}+\delta_{k,m})x_j(n+k) \\ &\quad + x^i(-m)[x_i,x_j](m+n+k) + [x_i,x_j](-m+k)x^i(m+n)\,.\end{aligned}$$

The structure constants $\{c_{ij}^h\}$ relative to $\{x_i\}$, that is the constants determined by $[x_i,x_j]=c_{ij}^h x_h$, verify the relations $c_{ij}^h = c_{jh}^i = -c_{hj}^i$. It follows that

$$\begin{aligned}[x_i,x_j](a)x^i(b) &= -x^i(a)[x_i,x_j](b)\,, \\ x^i(a)[x_i,x_j](b)+x^i(b)[x_i,x_j](a) &= 2gx_j(a+b)\,, \\ x^i(a)[x_i,x_j](a) &= gx_j(2a)\,.\end{aligned}$$

In particular, if $X_m = x^i(-m)[x_i,x_j](m+n+k)$ then $X_m + X_{-m-n-k} = 2gx_j(n+k)$. We now assume $k\le 0$, since the general case easily follows by $[L_n,x_j(k)]^* = [L_{-n},x_j(-k)]$. Thus, by explicit computation one can prove that

$$\begin{aligned}[L_n,x_j(k)] &= -\frac{k\ell}{(\ell+g)}x_j(n+k) + \frac{1}{2(\ell+g)}\sum_{m\ge -n/2}(2-\delta_{-m,n/2})(X_m - X_{m-k}) \\ &= -kx_j(n+k)\,,\end{aligned}$$

and then we have $[L_n,x(k)] = -kx(n+k)$ on $\mathcal{H}^{0,\mathrm{fin}}$ for every x in $\mathfrak{g}$ and k,n in $\mathbb{Z}$. $\square$

As a corollary of Lemma 48, the representation of $\mathrm{Vir} = \mathbb{C}\cdot c\oplus\partial$, with ∂ the Witt algebra, extends to a representation of the semidirect product $\mathfrak{g}[t,t^{-1}]\rtimes\mathrm{Vir}\cong\tilde{\mathfrak{g}}\rtimes\partial$, with $\tilde{\mathfrak{g}} = \mathfrak{g}[t,t^{-1}]\oplus\mathbb{C}\cdot c$. Indeed, if we set $L_n = \pi(\ell_n)$, where $\ell_n(\theta) = e^{in\theta}\frac{d}{d\theta}$, then we can define the stress energy tensor $\pi(h) = \sum_n \hat{h}_n L_n$ for any polynomial vector field h on the circle, namely a vector field which is a finite linear combination of the fields ℓ_n. Therefore, by Lemma 48 we have $[\pi(h),\pi(X)] = \pi(h.X)$ on $\mathcal{H}^{0,\mathrm{fin}}$ for every X in $L^{\mathrm{pol}}\mathfrak{g}$, where $h.X(\theta) = h(\theta)\frac{d}{d\theta}X(\theta)$.

The Lie algebra $L^{\mathrm{pol}}\mathfrak{g}$ can be completed to a Banach Lie algebra $L\mathfrak{g}_t$ for any $t\ge 0$. Indeed, given $X=\sum_k X_k(k)$ in $L^{\mathrm{pol}}\mathfrak{g}$, we can define $L\mathfrak{g}_t$ as the completion of $L^{\mathrm{pol}}\mathfrak{g}$ with respect to the norm

$$|X|_t = \sum_k (1+|k|)^t\|X_k\|\,.$$

We have norm continuous embeddings with dense range $C^{\lceil t\rceil+1}(S^1,\mathfrak{g})\hookrightarrow L\mathfrak{g}_t\hookrightarrow C^{\lfloor t\rfloor}(S^1,\mathfrak{g})$, and for any $t\ge n$ we have $\|X^{(n)}\|_\infty\le |X|_t$. Notice that, in general, we can similarly define the Banach Lie algebra $L\mathfrak{g}_{s,p}$ as the completion of $L^{\mathrm{pol}}\mathfrak{g}$ with respect to the norm

$$|X|_{s,p} = \left(\sum_k (1+|k|)^{sp}\|X_k\|^p\right)^{1/p}\,.$$

We now set $\mathcal{S}_t = L\mathbb{C}_t$, namely the space of continuous complex functions h on S^1 satisfying

$$|h|_t = \sum_k (1+|k|)^t\|\hat{h}_k\| < +\infty\,.$$

Notice that we can naturally identify $\mathcal{S}_t$ with a space of Sobolev vector fields on the circle. We can define two continuous actions of $\mathcal{S}_t$ by $h.X(\theta) = h(\theta)\frac{d}{d\theta}X(\theta)$ and $hX(\theta) = h(\theta)X(\theta)$. Indeed, by noticing that $(1+|n+k|)^t \leq (1+|n|)^t(1+|k|)^t$ we have $|h.X|_{s,p} \leq |h|_s|X|_{s+1,p}$ and $|hX|_{s,p} \leq |h|_s|X|_{s,p}$.

3.3 Solitonic representations from discontinuous loops

In this section we follow [34] and we construct proper solitonic representations of the conformal net associated to some vacuum positive energy representation of a loop group. We begin by briefly recalling some basic definitions about conformal nets. We refer to [34, 45, 93] for further treatments of the topic.

Let $\mathcal{K}$ be the set of open, nonempty and non dense intervals of the circle. For I in $\mathcal{K}$, I' denotes the interior of the complement. We consider a vacuum positive energy representation π of level ℓ of some loop group LG. We always suppose G to be simple, compact and simply connected. It can be shown that

$$\mathcal{A}_\ell(I) = \{\tilde{\pi}(\gamma)\colon \ \operatorname{supp}\gamma \subset I\}'' \,,$$

is a conformal net [45, 93], where $\tilde{\pi}$ is the lift of π described below in Remark 61. We will denote by U the projective unitary continuous representation of $\mathrm{Diff}_+(S^1)$ verifying the covariance property. Consider now a smooth path $\gamma\colon [-\pi,\pi] \to G$. We suppose γ to admit, at all orders, finite right derivatives in $-\pi$ and finite left derivatives in π. We then define $\sigma_\gamma = \{\sigma_\gamma^I\}_{I\in\mathcal{I}_\mathbb{R}}$ as the collection of maps given by

$$\sigma_\gamma^I\colon \mathcal{A}_\ell(I) \to B(\mathcal{H})\,, \quad \sigma_\gamma^I(x) = \operatorname{Ad}\pi(\gamma_I)(x)\,, \tag{3.9}$$

where γ_I is a loop in LG such that $\gamma_I(\theta) = \gamma(\theta)$ for θ in I seen as a subinterval of $(-\pi,\pi)$.

Proposition 49. *σ_γ is an irreducible locally normal soliton with index* 1*. Furthermore, σ_γ is $\mathrm{Diff}^{1,\mathrm{ps}}_{+,1}(S^1)$-covariant.*

Proof. Normality on each $\mathcal{A}_\ell(I)$ follows because on these local algebras σ_γ is given by the adjoint action by a unitary operator. The compatibility property is clear, since if $I \subseteq J$ then $\pi(\gamma_I\gamma_J^{-1})$ is in $\mathcal{A}_\ell(I') = \mathcal{A}_\ell(I)'$. The index is 1 since if I is in $\mathcal{I}_\mathbb{R}$ then $\sigma_\gamma(\mathcal{A}_\ell(I)) = \mathcal{A}_\ell(I)$, and for the same reason we have that

$$\bigvee_{I\in\mathcal{I}_\mathbb{R}} \sigma_\gamma(\mathcal{A}_\ell(I)) = \bigvee_{I\in\mathcal{I}_\mathbb{R}} \mathcal{A}(I) = B(\mathcal{H})$$

since the conformal net $\mathcal{A}_\ell$ is irreducible. The last statement follows by Theorem 3.4. of [34]. □

We will now focus on a smaller class of solitons. Given h in G, we define a *discontinuous loop* as an element of the group

$$L_hG = \{\zeta \in C^\infty(\mathbb{R}, G)\colon \zeta(x)^{-1}\zeta(x+2\pi) = h\}\,. \tag{3.10}$$

The restriction of a discontinuous loop on $[-\pi, \pi]$ clearly induces a soliton σ_ζ. In the following, we will study the equivalence classes of such solitonic representations.

Since π is irreducible if and only if it is irreducible as a projective representation of LG, then σ_ζ is irreducible if π is irreducible (see also Corollary 1.3.3. of [93]). Notice that for ζ in L_hG we have that $\zeta_t(\phi) = \zeta(\phi)\zeta(\phi - t)^{-1}$ is in LG for any t in $\mathbb{R}$. We now denote by $\mathrm{Rot}^\sim$ the universal covering of $\mathrm{Rot} \cong \mathbb{T}$, the group of rotations of the circle. If U_t is the unitary representation of Rot associated to π, then we can define $V_t^\zeta = \pi(\zeta_t)U_t$ in $PU(\mathcal{H})$ for t in $\mathbb{R}$ and notice that $V_t^\zeta V_s^\zeta = V_{t+s}^\zeta$. However, in general $V_{2\pi}^\zeta$ is not a scalar and therefore σ_ζ is not Rot-covariant but only locally $\mathrm{Rot}^\sim$-covariant. We can notice that if ζ is in L_gG and η is in L_hG then $\zeta\eta^{-1}$ is in $L_{h^{-1}g}G$ if $h^{-1}g$ is in $Z(G)$. In particular, if ζ and η are both in L_hG then $\zeta\eta^{-1}$ is in LG and σ_ζ is unitarily equivalent to σ_η.

Theorem 50. *Let π be a vacuum positive energy representation of LG of level $\ell \geq 1$. Given ζ in L_hG, the soliton σ_ζ extends to a DHR representation if and only if h is central.*

Proof. First we suppose h to be in $Z(G)$. A quick computation shows that in this case $V_{2\pi}^\zeta = \pi(h)$. By the identity $\pi(h)e^{\pi(X)}\pi(h)^* = e^{\pi(X)}$ for any X in $L\mathfrak{g}_0$ we have that $V_{2\pi}^\zeta$ is a scalar since π is irreducible. This implies that σ_ζ is locally Rot-covariant and we have that σ_ζ can be extended to a locally normal DHR representation by using the arguments of Proposition 3.8. of [34]. Now we suppose h to be not central. By absurd, σ_ζ extends to a DHR representation and thus it is Rot-covariant [33]. Denote by U_θ^ζ the corresponding intertwining projective representation of the circle. If we define the DHR representation

$$\rho_\zeta(x) = \mathrm{Ad}U_\pi \cdot \sigma_\zeta \cdot \mathrm{Ad}U_{-\pi} \cdot \sigma_{\zeta^{-1}} \cdot \mathrm{Ad}V_\pi^\zeta \cdot \mathrm{Ad}U_{-\pi}(x)\,,$$

then by construction ρ_ζ is implemented by the unitary $U_\pi U_{-\pi}^\zeta V_\pi^\zeta U_{-\pi}$. Since σ_ζ is a locally normal DHR representation, by using the additivity property one can show that $\rho_\zeta(x) = x$ for x in $\mathcal{A}((0,\pi))$ and for x in $\mathcal{A}((0,\pi))'$. It follows that $U_{-\pi}^\zeta V_\pi^\zeta$ is a scalar and thus $V_{2\pi}^\zeta$ is a scalar. Now consider a maximal torus $T \subset G$ containing h. Since T is connected, we can suppose that $\zeta(x)$ belongs to T for any x in $\mathbb{R}$, and by the commutativity of T we have that $V_{2\pi}^\zeta = \pi(h)$ in $PU(\mathcal{H})$. Therefore we have that h is a noncentral element acting on $\mathcal{H}$ as a scalar. If we now consider the kernel

$$N = \{g \in G \colon \pi(g) \in \mathbb{T}\}\,,$$

then N is a normal subgroup of G which is not contained in the center. But G is simple and connected, hence we have that $N = G$, which is an absurd. □

We conclude this section by studying the equivalence classes of the solitons constructed above. If z is in $Z(G)$, then the DHR representations σ_ζ with ζ in L_zG correspond to inequivalent irreducible positive energy representations $\zeta_*\pi$ of the same level as π (see Remark 61 and Theorem 3.2.3. of [93]). Now we pick a maximal torus T in

G. Consider ζ in L_sG and η in L_tG for some s and t in T. We can suppose ζ and η to be both contained in T. It can be easily noticed that

$$\sigma_\zeta \cdot \sigma_\eta = \sigma_{\zeta\eta}\,, \quad \zeta\eta \in L_{st}G\,, \qquad \sigma_\zeta^{-1} = \sigma_{\zeta^{-1}}\,, \quad \zeta^{-1} \in L_{s^{-1}}G\,.$$

It follows that σ_ζ and σ_η are unitarily equivalent if and only if $s = t$, hence we have infinitely many inequivalent solitons. If we consider two maximal tori T and $T' = gTg^{-1}$, then what we can say is that we have the identity

$$\sigma_{g\zeta g^{-1}} = \operatorname{Ad}\pi(g) \cdot \sigma_\zeta \cdot \operatorname{Ad}\pi(g)^*\,,$$

that is the solitons $\sigma_{g\zeta g^{-1}}$ and σ_ζ are equivalent up to some inner automorphism.

3.4 Sobolev loop groups

We know that $LG = C^\infty(S^1, G)$ is a Fréchet Lie group if endowed with the Whitney smooth topology. Its topology is induced by the norms defined on the Banach Lie groups $L^kG = C^k(S^1, G)$. The exponential map $\exp_{LG}\colon L\mathfrak{g}_0 \to LG$ is naturally defined by $\exp_{LG}(X) = \exp_G \cdot X$ and is a local homeomorphism near the identity [87]. Here we define and describe some properties of Sobolev loop groups.

Let M be a Riemannian manifold. Suppose M to be isometrically embedded in $\mathbb{R}^\nu$ for some $\nu > 0$. Define, for $1 \leq p < \infty$ and $0 \leq s < \infty$, the *fractional Sobolev space* [13, 35]

$$W^{s,p}(S^1, M) = \{f \in W^{s,p}(S^1, \mathbb{R}^\nu)\colon f(\theta) \in M \text{ a.e.}\}\,.$$

Here $W^{s,p}(S^1, \mathbb{R}^\nu)$ is the completion of $C^\infty(S^1, \mathbb{R}^\nu)$ with respect to the norm $\|f\|_{s,p} = \|\Delta^{s/2}f\|_p + \|f\|_p$, where $\Delta \geq 0$ is the closure on $L^p(S^1, \mathbb{R}^\nu)$ of the laplacian seen as an operator on $C^\infty(S^1, \mathbb{R}^\nu)$ [32]. We recall that the closure of an operator between linear subspaces of Banach spaces is its smallest closed extension, and that the fractional Laplacian Δ^α for $0 < \alpha < 1$ can be defined by the Fourier transform [35].

Hereafter, every compact Lie group G will be considered as a Riemannian Lie group with respect to the unique Riemannian structure extending $-\langle\cdot,\cdot\rangle$, namely the opposite of the basic inner product, and such that left and right translations are smooth isometries. We show that if G is compact and simple then every faithful unitary representation $\rho\colon G \to U(n)$ induces an isometric embedding of G in some real euclidean space. By continuity of the representation we have that G is represented as a compact embedded Lie subgroup of $U(n)$. Moreover, by simplicity of $\mathfrak{g}_0$ we have that $\lambda\mathrm{tr}(\rho(x)^*\rho(y)) = -\langle x, y\rangle$ for some $\lambda > 0$. Therefore, if we consider $M_n(\mathbb{C})$ as a real vector space with inner product $\lambda\mathrm{Re}\,\mathrm{tr}(X^*Y)$ then we have an isometric embedding $G \hookrightarrow M_n(\mathbb{C})$.

Theorem 51. *If G is a compact, simple and simply connected Lie group faithfully represented in some space of matrices, then $W^{s,p}(S^1, G)$ is an analytic Banach Lie group for p and sp in $(1, \infty)$. Its Banach Lie algebra is $W^{s,p}(S^1, \mathfrak{g}_0)$, the exponential map exists and it is a local homeomorphism. Moreover, $C^\infty(S^1, G)$ is dense in $W^{s,p}(S^1, G)$ and thus $W^{s,p}(S^1, G)$ is connected.*

Proof. First we show that $W^{s,p}(S^1, G)$ is a topological group. This can be proved by using the fact that any two functions f, g in $W^{s,p}(S^1, \mathbb{R}^\nu)$ verify, for p and sp in $(1, \infty)$, the estimate [32]

$$\|fg\|_{s,p} \leq C_{s,p}\|f\|_{s,p}\|g\|_{s,p}\,. \tag{3.11}$$

By this estimate and by the identity $f^{-1} - g^{-1} = f^{-1}(g - f)g^{-1}$ it follows that $W^{s,p}(S^1, G)$ is a topological group for p and sp in $(1, \infty)$, since it is clearly a Hausdorff space. Now we define the map

$$\exp_{s,p}\colon W^{s,p}(S^1, \mathfrak{g}_0) \to W^{s,p}(S^1, G)\,, \quad \exp_{s,p}(X)(z) = \exp_G(X(z))\,.$$

This map is well defined since $\exp_G \cdot X = e^X$ is an absolutely convergent series and it is also a local homeomorphism. We check that $W^{s,p}(S^1, G)$ is connected. By the density of $C^\infty(S^1, G)$ in $W^{s,p}(S^1, G)$ (see Theorem 1.1. of [13]), it suffices to prove that $C^\infty(S^1, G)$ is path connected and then connected. But a smooth homotopy between two loops in G is a path in $C^\infty(S^1, G)$, and the connectedness follows. Finally, we conclude if we prove that the group operations of inversion and multiplication are analytic. By connectedness we can reduce to prove this in an open neighborhood of the identity (see [93], Lemma 2.2.1.). The inversion $X \mapsto -X$ is clearly analytic. The analyticity of left and right multiplication follows from the Baker-Campbell-Hausdorff-Dynkin formula, where the continuity of the appearing homogeneous polynomials is guaranteed by equation (3.11). The theorem is proved. □

Corollary 52. *Every loop γ in $W^{s,p}(S^1, G)$ is a finite product of exponentials, since the exponential map is a local homeomorphism and $W^{s,p}(S^1, G)$ is connected.*

Remark 53. Theorem 51 still holds if the circle S^1 is replaced with a torus $\mathbb{T}^m$. This follows from the fact that the mentioned density theorem [13] is verified for a generic cube Q^m, that $\mathbb{T}^m$ can be defined as a quotient of Q^m and that the convolution with a smooth function preserves the periodicity.

We have formally defined our Sobolev loop group $W^{s,p}(S^1, G)$ and we have checked that such a space has good topological and analytical properties. Now we are finally ready to extend our PER of LG. The definition of Positive Energy Representation of a Sobolev loop group can be given just by replacing LG with $W^{s,p}(S^1, G)$ in the definition given above in (3.3).

Proposition 54. *Let $\iota\colon G \to H$ and $\pi\colon G \to U$ be two homomorphisms of topological groups. We suppose H to be connected and $\iota(G)$ to be dense in H. Suppose the existence of a neighborhood V of the identity in H and of a continuous function $p_0\colon V \to U$ such that $\pi(g_\alpha) \to p_0(v)$ whenever $\iota(g_\alpha) \to v$, with $(g_\alpha)_{\alpha \in A}$ a net in G and v in V. Then, p_0 extends to a continuous homomorphism $p\colon H \to U$ such that $\pi = p \cdot \iota$.*

Proof. By the connectedness of H we have that $H = \cup_n V^n$. We show by induction that p can be well defined on V^n for every n. We set $p = p_0$ on V. Suppose the thesis true for V^n, and consider elements w in V^n and v in V. Pick a net $(h_\beta)_{\beta \in B}$ such that $\iota(h_\beta) \to v$. By inductive hypothesis the limit

$$p(wv) := \lim_\alpha \pi(g_\alpha) = \lim_\alpha \lim_\beta \pi(g_\alpha h_\beta^{-1}) p_0(v) = \lim_{(\alpha,\beta)} \pi(g_\alpha h_\beta^{-1}) p_0(v) = p(w)p(v)$$

is well defined and does not depend on the net $(g_\alpha)_{\alpha\in A}$ such that $\iota(g_\alpha)\to wv$. Notice that, in order to properly apply the inductive hypothesis, we considered $A\times B$ as a directed set by using the lexicographic order. Hence p is a well defined group homomorphism. The continuity of p follows by induction as well, and the identity $\pi = p\cdot\iota$ is satisfied by construction. □

Proposition 55. *Let π be a PER of a loop group LG. If X is in $W^{s,p}(S^1,\mathfrak{g}_0)$ for $1\le p\le 2$ and $s>3/2+1/p$, then $\pi(X)$ is a closable operator which is essentially skew-adjoint on any core of L_0.*

Proof. We first notice that by the Sugawara formula we have $L_0\ge 0$, since we have $L_0 = d + h_i$ for some $h_i\ge 0$ on each irreducible summand π_i of π. If $\mathcal{H}^{0,\mathrm{fin}}$ is the algebraic direct sum of the eigenspaces of L_0, then we will denote by $\mathcal{H}^{0,s}$ the completion of $\mathcal{H}^{0,\mathrm{fin}}$ with respect to the Sobolev norm $\|\xi\|_{0,s} = \|(1+L_0)^s\xi\|$. By Lemma 48 and Proposition 1.2.1. of [93], for ξ in $\mathcal{H}^{0,\mathrm{fin}}$ and X in $L^{\mathrm{pol}}\mathfrak{g}$ we have

$$\|\pi(X)\xi\|_{0,s}\le\sqrt{2(\ell+g)}|X|_{|s|+1/2}\|\xi\|_{0,s+1/2}\,,$$
$$\|[1+L_0,\pi(X)]\xi\|_{0,s}\le\sqrt{2(\ell+g)}|X|_{|s|+3/2}\|\xi\|_{0,s+1/2}\,,$$

for any s in $\mathbb{R}$. By density one extends π to $(L\mathfrak{g})_{|s|+3/2}$ in such a way to still verify the same estimates for ξ in $\mathcal{H}^{0,s+1/2}$. It follows that if X is in $L\mathfrak{g}_{3/2}$ then both $\pi(X)$ and $[1+L_0,\pi(X)]$ are bounded operators from $\mathcal{H}^{0,1/2}$ to $\mathcal{H}$. By the Nelson commutator theorem ([88], Thm. X.36) we have that if X is in $(L\mathfrak{g}_0)_{3/2}$ then the restriction of $\pi(X)$ on

$$\mathcal{D} = \{\psi\in\mathcal{H}\cap\mathcal{H}^{0,1/2}\colon \pi(X)\in\mathcal{H}\}$$

is a closable operator on $\mathcal{H}$ which is essentially skew-adjoint on any core of L_0 such as $\mathcal{H}^{0,\mathrm{fin}}$. Notice now that, by standard arguments, there is a norm continuous embedding $W^{s,p}(S^1,\mathfrak{g})\hookrightarrow L\mathfrak{g}_{3/2}$. Indeed, if $X(\theta)=\sum_k X_k e^{ik\theta}$ then by the Hölder inequality

$$|X|_{3/2} = \sum_k (1+|k|)^{3/2}\|X_k\| = \sum_k (1+|k|)^{3/2-s}(1+|k|)^s\|X_k\|\le A_{s,p}|X|_{s,p'}\le B_{s,p}\|X\|_{s,p}\,,$$

where $A_{s,p}$ and $B_{s,p}$ exist and are finite by construction and by Riesz-Thorin respectively. Therefore, by the arguments given above we have that if X is in $W^{s,p}(S^1,\mathfrak{g}_0)$ then $\pi(X)$ is a skew-symmetric operator on $\mathcal{H}^{0,\mathrm{fin}}$ which is essentially skew-adjoint on any core of L_0. □

Propositions 54 and 55 can be used to extend a strongly continuous projective representation of LG to a strongly continuous projective representation of $W^{s,p}(S^1,G)$. However, for convenience in the following we will focus on $H^s(S^1,G) = W^{s,2}(S^1,G)$. We show how a different approach can improve the results of Proposition 55.

Proposition 56. *If π is a PER of a loop group LG, the induced projective representation π of $L\mathfrak{g}$ can be extended to $H^s(S^1,\mathfrak{g})$ for $s>3/2$, with $\pi(X)$ closable and such that*

$$\|\pi(X)\xi\|_{0,1/2}\le C_s|X|_{s,2}\|\xi\|_{0,1/2}\,,\qquad \xi\in\mathcal{H}^{0,1/2}\,,\tag{3.12}$$

for some $C_s>0$. Moreover, $\pi(X)^* = \overline{\pi(X^*)}$, and in particular $\pi(X)$ is essentially skew-adjoint if X is skew-adjoint.

Proof. We use some techniques shown in [22]. Given $X = \sum_n X_n(n)$ in $L\mathfrak{g}_{1,1}$, the operator $\pi(X)$ is well defined on $\mathcal{H}^{0,1/2}$ and (3.12) follows by the previous estimates since for $t > 1/2$ and $s = 1 + t$ we have

$$|X|_1 = \sum_n (1 + |n|)\|X_n\| = \sum_n (1 + |n|)^{-t}(1 + |n|)^{1+t}\|X_n\| \leq c_t|X|_{s,2} .$$

It is also closable since $\pi(X^*) \subseteq \pi(X)^*$. Notice also that since $\mathcal{H}^{0,\mathrm{fin}}$ is a core for $(1 + L_0)^{1/2}$ then $\pi(X^*)$ is the formal adjoint of $\pi(X)$ on the associated scale space $\mathcal{H}^{0,1/2}$ for any X in $H^{3/2}(S^1, \mathfrak{g})$. Now we define on $\mathcal{H}^{0,1/2}$ the operator

$$R_{X,\epsilon} = [\pi(X), e^{-\epsilon L_0}] ,$$

which is well defined since $e^{-\epsilon L_0} \colon \mathcal{H} \to \mathcal{H}^{0,\infty} \subseteq \mathcal{H}^{0,1/2}$. By $-R_{X^*,\epsilon} \subseteq R^*_{X,\epsilon}$ we have that $R_{X,\epsilon}$ is closable. Notice that if $L_0 v_k = k v_k$ then

$$R_{x(n),\epsilon} v_k = f_{n,k}(\epsilon) x(n) v_k , \quad f_{n,k}(\epsilon) = e^{-\epsilon k} - e^{-\epsilon(k-n)} .$$

We will now show that $\|R_{x(n),\epsilon}\|^2 \leq 2(\ell + g)|x(n)|_{1,1}$. The case $n = 0$ is trivial and we can suppose $n < 0$ as $-R_{X^*,\epsilon} \subseteq R^*_{X,\epsilon}$. By simple analysis techniques one can prove that

$$|f_{n,k+n}(\epsilon)|^2 \leq \frac{n^2}{(k-n)^2} , \quad \frac{1+k}{(k-n)^2} \leq \frac{1}{|n|} ,$$

for any $\epsilon \geq 0$ and $k \geq 0$. Therefore if $v = \sum_{k\geq 0} v_k$ is in $\mathcal{H}^{0,\mathrm{fin}}$ then we have

$$\begin{aligned}
\|R_{x(n),\epsilon} v\|^2 &= \Big\| \sum_{k\geq 0} R_{x(n),\epsilon} v_k \Big\|^2 = \Big\| \sum_{k\geq 0} |f_{n,k}(\epsilon)|^2 x(n) v_k \Big\|^2 \\
&= \sum_{k\geq 0} |f_{n,k}(\epsilon)|^2 \|x(n) v_k\|^2 \\
&\leq 2(\ell + g) \sum_{k\geq 0} \frac{n^2}{(k-n)^2} (1 + |n|)(1 + k)\|x\|^2\|v_k\|^2 \\
&\leq 2(\ell + g) \sum_{k\geq 0} (1 + |n|)^2 \|x\|^2 \|v_k\|^2 \\
&= 2(\ell + g)|x(n)|^2_{1,1}\|v\|^2 .
\end{aligned}$$

It follows that $\|R_{X,\epsilon}\|^2 \leq 2(\ell+g)|X|_{1,1}$ for every X in $L\mathfrak{g}_{1,1}$ and that $R_{X,\epsilon} \to 0$ strongly as $\epsilon \to 0$. Moreover, by the identity $R^*_{X,\epsilon} = -\overline{R_{X^*,\epsilon}}$ we have that $R^*_{X,\epsilon} \to 0$ strongly as well. Now we arrive to the crucial point: if v is in $\mathcal{D}(\pi(X)^*)$ then

$$\pi(X^*) e^{-\epsilon L_0} v = \pi(X)^* e^{-\epsilon L_0} v = e^{-\epsilon L_0} \pi(X)^* v - R^*_{X,\epsilon} v \to \pi(X)^* v , \quad \epsilon \to 0 ,$$

and this concludes the proof since $e^{-\epsilon L_0} v \to v$ as $\epsilon \to 0$. □

Theorem 57. *Let $\pi \colon LG \to PU(\mathcal{H})$ be a positive energy representation of LG. Then π can be extended to a positive energy representation of $H^s(S^1, G)$ for $s > 3/2$.*

Proof. We consider an open neighborhood U in $H^s(S^1, \mathfrak{g}_0)$ on which the exponential map of $H^s(S^1, G)$ is a homeomorphism and set $V = \exp_{H^s}(U)$. For $\gamma = \exp_{H^s}(X)$ in V we define in $PU(\mathcal{H})$

$$\pi(\gamma) = e^{\pi(X)}, \quad X \in U.$$

The neighborhood V verifies Proposition 54, since if $\gamma_\alpha = \exp(X_\alpha)$ converges to $\gamma = \exp(X)$ in V then the estimate (3.12) implies that $\pi(X_\alpha)\xi$ is a Cauchy net for every ξ in $\mathcal{H}^{0,1/2}$. But the pointwise convergence of self-adjoint operators on a common core implies the strong resolvent convergence of such operators (Theorem VIII.25.(a) of [88]), thus π can be continuously extended. Finally, since the rotation group acts on $H^s(S^1, G)$ by continuous operators (see Lemma A.3 of [24]) and since LG is dense in $H^s(S^1, G)$, we have that π is actually a Positive Energy Representation since it is Rot-covariant. □

Proposition 58. *Let $\rho_s = \exp_{\mathrm{Diff}_+(S^1)}(sh)$ be a smooth diffeomorphism of S^1, with h a smooth real vector field of the circle. Set $R_h = \{\rho_s\}_{s\in\mathbb{R}}$. Then the exponential map $L\mathfrak{g}_0 \rtimes \mathbb{R}h \to LG \rtimes R_h$ is well defined and continuous. Moreover, if $X_\alpha = \rho_\alpha . X = X \cdot \rho_\alpha^{-1}$ then*

$$\exp_{LG\rtimes R_h}(X + \alpha h) = \lim_{n\to\infty} \exp_{LG}(X/n) \exp_{LG}(X_{\alpha/n}/n) \cdots \exp_{LG}(X_{\alpha(n-1)/n}/n)\rho_\alpha . \tag{3.13}$$

Proof. We follow [93]. To compute the exponential map, we fix $X + \alpha h$ in $L\mathfrak{g}_0 \rtimes \mathbb{R}h$ and look for $f : \mathbb{R} \to LG \rtimes R_h$ which satisfies $(X + \alpha h)f = \dot{f}$ and $f(0) = 1$. We suppose f to be of the form $f_t = \gamma^t \rho_{\phi(t)}$ with γ in LG. As a manifold, $LG \rtimes R_h$ is the product of LG and R_h, thus $s \mapsto \exp_{LG}(sX)\rho_{s\alpha}$ is the integral curve for $X + \alpha h$ at the identity. Therefore, with the notation $\gamma_s(\theta) = \gamma(\rho_s^{-1}(\theta))$ we have

$$\begin{aligned}
(X + \alpha h)f_t &= \frac{d}{ds}\Big|_{s=0} \exp_{LG}(sX)\rho_{s\alpha}\gamma^t\rho_{\phi(t)} = \frac{d}{ds}\Big|_{s=0} \exp_{LG}(sX)(\gamma^t)_{s\alpha}\rho_{s\alpha+\phi(t)} \\
&= X\gamma^t\rho_{\phi(t)} + \alpha\frac{d}{ds}\Big|_{s=0} (\gamma^t)_s\rho_{\phi(t)} + \alpha\gamma^t h\rho_{\phi(t)} , \\
\dot{f}_t &= \left(\frac{d}{dt}\gamma^t\right)\rho_{\phi(t)} + \phi'(t)\gamma^t h\rho_{\phi(t)} ,
\end{aligned}$$

whence $\phi(t) = \alpha t$, and we must solve

$$\frac{d}{dt}\gamma^t = X\gamma^t + \alpha\frac{d}{ds}\Big|_{s=0} (\gamma^t)_s , \qquad \gamma^0 = 1 . \tag{3.14}$$

Now we notice that if γ_0^t is a solution of the equation $\frac{d}{dt}\gamma_0^t = X_{-\alpha t}\gamma_0^t$ with initial condition $\gamma_0^0 = 1$, then $\gamma^t = (\gamma_0^t)_{\alpha t}$ is the solution of (3.14) we were looking for. Therefore, if we embed G in a space of matrices $M_m(\mathbb{C})$ and we consider LG as a closed subspace of $C^\infty(S^1, M_m(\mathbb{C}))$, then by Theorem 1.4.1. of [93] we have

$$\gamma_0^1 = \lim_{n\to\infty} \exp(X_{-\alpha}/n) \exp(X_{-\alpha(n-1)/n}/n) \cdots \exp(X_{-\alpha/n}/n)\rho_\alpha , \tag{3.15}$$

where the right side of (3.15) converges in each $C^k(S^1, M_m(\mathbb{C}))$ and hence in LG. Finally, equation (3.13) follows from $\gamma^1 = (\gamma_0^1)_\alpha$, and the continuity of $\exp_{LG\rtimes R_h}$ follows from Theorem 1.4.1. of [93]. □

Corollary 59. *The following holds in $PU(\mathcal{H})$:*

$$e^{\pi(X+i\alpha h)} = \pi(\exp_{LG\rtimes R_h}(X+\alpha h))\,.$$

Proof. By the Trotter product formula and Proposition 58 we have the following identities in $PU(\mathcal{H})$:

$$\begin{aligned} e^{\pi(X+i\alpha h)} &= \lim_{n\to\infty}(e^{\pi(X/n)}e^{i\alpha\pi(h)/n})^n = \lim_{n\to\infty}\pi(\exp_{LG}(X/n)\exp_{R_h}(\alpha h/n))^n \\ &= \lim_{n\to\infty}\pi(\exp_{LG}(X/n)\exp_{LG}(X_{\alpha/n}/n)\cdots\exp_{LG}(X_{\alpha(n-1)/n}/n)\rho_\alpha) \\ &= \pi(\exp_{LG\rtimes R_h}(X+\alpha h))\,, \end{aligned}$$

where we used the identities $e^{iT(h)} = \pi(\exp_{R_h}(h))$ and $e^{\pi(X)} = \pi(\exp_{LG}(X))$ which hold in $PU(\mathcal{H})$. □

Lemma 60. *Let $\pi\colon G\to PU(\mathcal{H})$ be a strongly continuous projective representation of a topological group G. Then the map*

$$G\times U(\mathcal{H})\to U(\mathcal{H})\,,\quad (g,u)\mapsto \pi(g)u\pi(g)^*\,,$$

is well defined and strongly continuous.

Proof. The map is clearly well defined, and if g_α converges to g in G then we can choose lifts v_α and v of $\pi(g_\alpha)$ and $\pi(g)$ such that v_α converges to v in $U(\mathcal{H})$, since the short exact sequence given by $U(\mathcal{H})\to PU(\mathcal{H})$ admits local continuous sections [8]. But in the unitary group the strong topology and the $*$-strong topology coincide and multiplication is continuous on bounded sets by the uniform boundedness principle, so the assertion follows. □

Remark 61. A continuous projective representation $\pi\colon G\to PU(\mathcal{H})$ can be naturally lifted to a continuous unitary representation $\tilde\pi$ of $\widetilde{G} = \{(g,u)\in G\times U(\mathcal{H})\colon \pi(g)=[u]\}$ given by $\tilde\pi(g,u)=u$.

Theorem 62. *If γ is in $H^s(S^1,G)$ and X is in $H^s(S^1,\mathfrak{g}_0)$ for some $s>3/2$, then*

$$\pi(\gamma)\pi(X)\pi(\gamma)^* = \pi(\mathrm{Ad}(\gamma)X) + ic(\gamma,X)\,, \tag{3.16}$$

for some continuous real function $c(\gamma,X)$. Moreover, if γ is in $H^{1+s}(S^1,G)$ and h is a real vector field $\mathcal{S}_s$, then

$$\pi(\gamma)\pi(X+ih)\pi(\gamma)^* = \pi(\mathrm{Ad}(\gamma)X) + iT(h) + \pi(h\dot\gamma\gamma^{-1}) + ic(\gamma,X) + ic(\gamma,h) \tag{3.17}$$

for some continuous real function $c(\gamma,h)$.

Proof. We first prove (3.17) in the smooth case. We will identify $\mathbb{R}h$ with $i\mathbb{R}h$ for formal convenience. By the previous propositions, if γ is in LG and $Y = X+ih$ is in $L\mathfrak{g}_0\rtimes i\mathbb{R}h$, then the following identities hold in $PU(\mathcal{H})$:

$$\begin{aligned} \pi(\gamma)e^{t\pi(Y)}\pi(\gamma)^* &= \pi(\gamma)\pi(\exp_{LG\rtimes R_h}(sY))\pi(\gamma)^* \\ &= \pi(\gamma\exp_{LG\rtimes R_h}(tY)\gamma^{-1}) \\ &= \pi(\exp_{LG\rtimes R_h}(t\mathrm{Ad}(\gamma)Y)) \\ &= e^{t\pi(\mathrm{Ad}(\gamma)Y)}\,, \end{aligned} \tag{3.18}$$

and consequently $\pi(\gamma)e^{t\pi(Y)}\pi(\gamma)^* = \lambda(t)e^{t\pi(\mathrm{Ad}(\gamma)Y)}$ for some function $\lambda\colon \mathbb{R} \to \mathbb{T}$. But $\lambda\colon \mathbb{R} \to \mathbb{T}$ is a continuous homomorphism and therefore $\lambda(t) = e^{iat}$ for a unique real number $a = c(\gamma, Y)$. We point out that $\mathrm{Ad}(\gamma)$ has to be intended as the adjoint action with respect to the semidirect product $LG \rtimes R_h$. Notice also that $c(\gamma, Y)$ is linear in Y, so we can write $c(\gamma, X + ih) = c(\gamma, X) + c(\gamma, h)$, where we set $c(\gamma, ih) = c(\gamma, h)$ for simplicity. Therefore, the claimed expression follows by the Stone's theorem and by using the product rule for the derivative on the identity $1 = \gamma_t \cdot \gamma_t^{-1}$.

Now we prove (3.16) in the Sobolev case. Consider $(\gamma_\alpha, X_\alpha)$ in $LG \times L\mathfrak{g}_0$ converging to (γ, X) in $H^s(S^1, G) \times H^s(S^1, \mathfrak{g}_0)$. We have that both $\pi(\gamma_\alpha)e^{s\pi(X_\alpha)}\pi(\gamma_\alpha)^*$ and $e^{s\pi(\mathrm{Ad}(\gamma_\alpha)X_\alpha)}$ strongly converge to the corresponding terms in γ and X. By the argument used before we have that $e^{ic(\gamma_\alpha, X_\alpha)}$ converges to $e^{ic(\gamma,X)}$, that is $e^{ic(\gamma,X)}$ is continuous in γ and X. But continuity is a local property and the exponential map has local left inverses, thus $c(\gamma, X)$ is continuous and the first part of the theorem is proved. Now we prove (3.17) in the Sobolev case. Consider γ in $H^{1+s}(S^1, G)$ and h real in $\mathcal{S}_s$. Notice that $i\pi(h)$ and $\pi(h\dot{\gamma}\gamma^{-1})$ are both essentially skew-adjoint. Consider now smooth approximating nets $\gamma_\alpha \to \gamma$, $X_\alpha \to X$ and $h_\alpha \to h$ as before. By the previous propositions, the approximating right hand side of (3.17) minus $c(\gamma_\alpha, h_\alpha)$ converges in the strong resolvent sense to the corresponding term in γ, X and h since we have a net of skew-adjoint operators pointwise convergent on a common core. Similarly, $\pi(X_\alpha + ih_\alpha)$ converges in the strong resolvent sense to $\pi(X + ih)$ and therefore

$$\pi(\gamma_\alpha)e^{t\pi(X_\alpha + ih_\alpha)}\pi(\gamma_\alpha)^* \to \pi(\gamma)e^{t\pi(X+ih)}\pi(\gamma)^*$$

strongly for every t in $\mathbb{R}$. By the argument used before we have that $e^{ic(\gamma,h)}$ is continuous and thus $c(\gamma, h)$ is continuous. The thesis is proved. □

Corollary 63. *The scale space $\mathcal{H}^\alpha \subseteq \mathcal{H}$ is $H^s(S^1, G)$-invariant for $\alpha \geq 0$ and $s > 5/2$. Moreover, for any integer n such that $n \leq \lfloor s - 1 \rfloor$, the corresponding map $H^s(S^1, G) \times \mathcal{H}^n \to \mathcal{H}^n/\mathbb{T}$ is continuous.*

Proof. Since $\mathcal{D}(u^*Au) = u^*\mathcal{D}(A)$ for every unitary u and every self-adjoint operator A, then

$$\begin{aligned}\mathcal{D}((1+d)^\alpha) &= \pi(\gamma)^*\mathcal{D}((1 + d - i\pi(\dot{\gamma}\gamma^{-1}) + c(\gamma, d))^\alpha) \\ &\subseteq \pi(\gamma)^*\mathcal{D}((1+d)^\alpha)\,.\end{aligned} \tag{3.19}$$

Since $\mathcal{D}((1 + d)^\alpha) = \mathcal{H}^\alpha$ for $\alpha \geq 0$, the $\mathcal{H}^s$-invariance follows. Now we prove the second statement, where we can suppose $n \geq 1$. By Proposition 1.5.3. of [93] we have $\|\pi(\gamma)\xi\|_n \leq (1 + M_{n-1})^n \|\xi\|_n$, where $M_p = C|\gamma^{-1}\dot{\gamma}|_{p+1/2} + |c(\gamma^{-1}, d)|$ for some $C > 0$, and the joint continuity can be proved as in [93]. □

Theorem 64. *With the hypotheses of Theorem 62, we have*

$$c(\gamma, X) = -\ell \int_0^{2\pi} \langle \gamma^{-1}\dot{\gamma}, X\rangle \,\frac{d\theta}{2\pi}\,, \quad c(\gamma, h) = -\frac{\ell}{2}\int_0^{2\pi} h\,\langle \gamma^{-1}\dot{\gamma}, \gamma^{-1}\dot{\gamma}\rangle\,\frac{d\theta}{2\pi}\,.$$

Proof. We follow Theorem 1.6.3. of [93], skipping some computations for the sake of brevity. Consider a smooth loop γ in LG and a smooth real vector field h. For Y in $L\mathfrak{g}_0 \rtimes i\mathbb{R}h$ we have

$$c(\gamma_1\gamma_2, Y) = c(\gamma_2, Y) + c(\gamma_1, \mathrm{Ad}(\gamma_2)Y)\,. \tag{3.20}$$

If $\gamma^t = \exp_{LG}(tX)$, then the map $t \mapsto c(\gamma^t, Y)$ is differentiable at $t = 0$ since $LG \times R_h \to LG$ is smooth. In particular, we have that

$$\partial_t\big|_{t=0} c(\gamma^t, Y) = \ell B(X, Y)\,,$$

and so

$$\partial_t\big|_{t=0} c(\gamma^t, h) = 0\,.$$

By using (3.20) we have that $c(\gamma^t, h)$ is differentiable everywhere, with

$$\partial_t c(\gamma^t, Y) = \ell B(X, Y) + c(\gamma^t, [X, Y])\,,$$

or more compactly

$$\dot{c}_t(Y) = i_X \ell B(Y) - (X.c_t)(Y)\,. \tag{3.21}$$

We naturally expect the solution of the ODE to be given by the Duhamel formula

$$c(\gamma^t, Y) = \ell B(X, \mathrm{Ad}(\gamma^t) \int_0^t \mathrm{Ad}(\gamma^{-\tau}) Y d\tau) = \ell B(X, \int_0^t \mathrm{Ad}(\gamma^s) Y ds)\,. \tag{3.22}$$

Using $\frac{d}{dt}\mathrm{Ad}(\gamma_t)Y = [X, \mathrm{Ad}(\gamma_t)Y]$, it is easy to verify that (3.22) defines a $C^1(\mathbb{R}, (L\mathfrak{g} \rtimes i\mathbb{R}h)^*)$ solution of (3.21) with initial condition $c_0 = 0$. The solution is unique. Finally, one can use (3.22) and Corollary 1.6.2. of [93] to obtain the claimed expressions in the smooth case. By the continuity of $c(\gamma, Y)$ shown in Theorem 62 the thesis is proved. □

Corollary 65. *By repeating the proof of Theorem 62, one can show that if γ is in $H^s(S^1, G)$ and X is in $H^s(S^1, \mathfrak{g}_0)$ for some $s > 3/2$, then*

$$\pi(\gamma)^* \pi(X) \pi(\gamma) = \pi(\mathrm{Ad}(\gamma^{-1})X) + ib(\gamma, X)\,, \tag{3.23}$$

for some continuous real function $b(\gamma, X)$. Similarly, if γ is in $H^{s+1}(S^1, G)$ and h is a real vector field in $\mathcal{S}_s$, then

$$\pi(\gamma)^* \pi(X + ih) \pi(\gamma) = \pi(\mathrm{Ad}(\gamma^{-1})X) + iT(h) - \pi(h\gamma^{-1}\dot{\gamma}) + ib(\gamma, X) + ib(\gamma, h) \tag{3.24}$$

for some continuous real function $b(\gamma, h)$. In particular, by $b(\gamma, Y) = c(\gamma^{-1}, Y)$ we have

$$b(\gamma, X) = -\ell \int_0^{2\pi} \langle \dot{\gamma}\gamma^{-1}, X \rangle \frac{d\theta}{2\pi}\,, \quad b(\gamma, h) = -\frac{\ell}{2} \int_0^{2\pi} h \langle \dot{\gamma}\gamma^{-1}, \dot{\gamma}\gamma^{-1} \rangle \frac{d\theta}{2\pi}\,.$$

3.5 QNEC on loop group models

We now denote by $\mathcal{A}_\ell = \{\mathcal{A}_\ell(I)\}_{I \in \mathcal{K}}$ the conformal net associated to a level ℓ vacuum representation π of some loop group LG. As before, we will denote by $\mathcal{K}$ the set of all the open, non empty and non dense intervals of the circle. To each interval I in $\mathcal{K}$ we associated the von Neumann algebra

$$\mathcal{A}_\ell(I) = \{\tilde{\pi}(\gamma) \colon \ \mathrm{supp}\, \gamma \subset I\}''\,, \tag{3.25}$$

where $\tilde{\pi}$ is the lift of π described in Remark 61 and the support of a loop γ is defined by

$$\mathrm{supp}\, \gamma = \overline{\{z \in S^1 \colon \ \gamma(z) \neq e\}}\,.$$

We are interested in computing

$$S(t) = S_{\mathcal{A}_\ell(t,+\infty)}(\omega_\gamma \| \omega)\,, \tag{3.26}$$

where ω is the vacuum state represented by the vacuum vector Ω and $\omega_\gamma = \omega \cdot \operatorname{Ad}\pi(\gamma)^*$ is represented by $\pi(\gamma)\Omega$ for some loop γ in LG. More in general, the same result will apply to the solitonic states given by the solitons (3.9) of above. We introduce the groups of Sobolev loops

$$B(z_1,\dots,z_n) = \{\gamma \in H^2(S^1,G)\colon \gamma(z_i) = e\,, \dot\gamma(z_i) = 0\}\,. \tag{3.27}$$

By standard arguments, continuously differentiable and piecewise smooth loops are in $H^s(S^1,G)$ for $s < 5/2$ [34, 51], where we say that γ is piecewise smooth if right and left derivatives always exist and if γ is smooth except on a finite number of points. If there is no ambiguity, we will use a similar notation to denote the groups (3.27) in the real line picture. Consider now the interval $I = (z,w)$ of S^1 obtained by moving counterclockwise from z to w. We will denote by γ_I the map such that $\gamma_I = \gamma$ on $[z,w)$ and $\gamma_I = e$ on $[w,z)$, so that we have write the identity

$$\gamma = \gamma_I \gamma_{I'}\,. \tag{3.28}$$

By Theorem 57 we have that if γ is a loop in $B(z,w)$ then in $PU(\mathcal{H})$ we have $\pi(\gamma) = \pi(\gamma_{(z,w)})\pi(\gamma_{(w,z)})$. In particular, in this case $\pi(\gamma_{(z,w)})$ is in $\mathcal{A}_\ell((z,w))$ and $\pi(\gamma_{(w,z)})$ is in $\mathcal{A}_\ell((w,z))$. We also recall that by the Bisognano-Wichmann theorem (2.7) we have the identity $\log\Delta = -2\pi D$ with $D = -\frac{i}{2}(L_1 - L_{-1})$, that is $\log\Delta = -2\pi T(\delta)$ with δ the vector field generating $\delta(s).u = e^s u$. Notice also that the vacuum expectation of

$$\pi(\gamma)^* T(\delta)\pi(\gamma) = T(\delta) + i\pi(\delta\gamma^{-1}\dot\gamma) + b(\gamma,\delta) \tag{3.29}$$

is given by the real constant $b(\gamma,\delta)$ described in Corollary 65.

Proposition 66. *Let γ be a loop in $H^3(S^1,G)$. Pick a non dense open interval $I = (z,w)$ of the circle and write $\gamma = \gamma_I\gamma_{I'}$ as in (3.28). Denote by δ_I the generator of dilations of the interval I and set $\Delta_I^{it} = e^{-2\pi i t T(\delta_I)}$. If $\dot\gamma$ vanishes on the boundary of I then the Connes cocycle $(D\omega_\gamma\colon D\omega)_t$ of $\mathcal{A}_\ell(I)$ is given by*

$$(D\omega_\gamma\colon D\omega)_t = e^{it(a - 2\pi c(\gamma_I,\delta_I))} e^{-2\pi t(i\pi(\delta_I) + \pi(\delta_I\dot\gamma_I\gamma_I^{-1}))}\Delta_I^{-it} \tag{3.30}$$

for some $a = a_\gamma$ in $\mathbb{R}$. In particular, a depends only on the values of γ at the boundary of I and $a_\gamma = 0$ if $\gamma(z) = e$ for z in the boundary of I.

Proof. First we check that $\delta_I\dot\gamma_I\gamma_I^{-1}$ is in $H^2(S^1,\mathfrak{g}_0)$ since it vanishes with its first derivative on the boundary of I. Hence the right hand side of (3.30), which we denote by u_t, is a well defined unitary operator which is in $\mathcal{A}_\ell(I)$ by the Trotter product formula. To prove the existence of a in $\mathbb{R}$ as in the statement it suffices to check that u_t verifies the relations

(i) $\sigma_t^\gamma(x) = u_t\sigma_t(x)u_t^*\,, \quad x \in \mathcal{A}_\ell(I)\,,$
(ii) $u_{t+s} = u_t\sigma_t(u_s)\,.$

Here σ_t and σ_t^γ are the modular automorphisms associated to the states ω and ω_γ. The first relation follows by noticing that

$$\begin{aligned}\sigma_t^\gamma(x) &= \operatorname{Ad}\Delta_{I,\gamma}^{it}(x) = \operatorname{Ad}\pi(\gamma)\Delta_I^{it}\pi(\gamma)^*(x)\\ &= \operatorname{Ad}\pi(\gamma)\Delta_I^{it}\pi(\gamma)^*\Delta_I^{-it}\Delta_I^{it}(x)\\ &= \operatorname{Ad}u_t\cdot\sigma_t(x)\,,\end{aligned} \tag{3.31}$$

where we used Lemma 2.(ii) and Theorem 64. The second relation can be easily verified and thus a does exist. Now we prove that $a = a_\gamma$ depends only on the values of γ at the boundary of I. Consider η in $H^3(S^1,G)$ such that $\eta(z) = e$ and $\dot\eta(z) = 0$ for z in the boundary of I. Notice that $(D\omega_{\eta\gamma}\colon D\omega)_t = \pi(\eta)(D\omega_\gamma\colon D\omega)_t\sigma_t(\pi(\eta)^*)$. Therefore, with the notation of Corollary 65 we have

$$\begin{aligned}a_{\eta\gamma} + 2\pi b(\eta_I\gamma_I,\delta_I) &= -i\frac{d}{dt}\omega_{\eta\gamma}((D\omega_{\eta\gamma}\colon D\omega)_t)\Big|_{t=0}\\ &= -i\frac{d}{dt}\omega_{\eta\gamma}(\pi(\eta)(D\omega_\gamma\colon D\omega)_t\sigma_t(\pi(\eta)^*))\Big|_{t=0}\\ &= a_\gamma + 2\pi(b(\gamma_I,\delta_I) + b(\eta_I\gamma_I,\delta_I) - b(\gamma_I,\delta_I))\,,\end{aligned}$$

and by the identity $a_{\eta\gamma} = a_\gamma$ the assertion is proved. If $\gamma(z) = e$ for z in the boundary of I then $\pi(\gamma_I)$ is in $\mathcal{A}_\ell(I)$ and the last statement follows by Lemma 2. □

Remark 67. If γ is an element of $H^2([-\pi,\pi],G)$, then as in the smooth case we can consider the soliton σ_γ given above by (3.9). In particular, Proposition 66 still holds for the solitonic states $\omega_\gamma = \omega\cdot\sigma_\gamma^{-1}$ with γ in $H^3([-\pi,\pi],G)$. This follows from the fact that if η is a loop in $H^3(S^1,G)$ such that $\gamma = \eta$ on I, then $\omega_\eta = \omega_\gamma$ on $\mathcal{A}(I)$.

Now we arrive to the main part of this chapter, that is we will use the previous results to prove the QNEC on loop groups models for the solitonic states $\omega_\gamma = \omega\cdot\sigma_\gamma^{-1}$ given by (3.9). In the real line picture, the path γ corresponds to an element of $H^2(\mathbb{R},G)$.

Theorem 68. *Let $\omega_\gamma = \omega\cdot\sigma_\gamma^{-1}$ be a solitonic state corresponding, in the real line picture, to some element γ of $H^2(\mathbb{R},G)$. Then the relative entropy* (3.26) *is finite for every t in $\mathbb{R}$ and explicitly given by*

$$S(t) = -\frac{\ell}{2}\int_t^\infty (u-t)\,\langle\dot\gamma\gamma^{-1},\dot\gamma\gamma^{-1}\rangle\,du\,. \tag{3.32}$$

Proof. As discussed in Remark 67, we can suppose γ to be the real line parametrization of some element of $H^2(S^1,G)$. Since the vacuum is G-invariant, we can replace γ with γg for any g in G, thus we can suppose $\gamma(\infty) = e$. By using the real line picture notation for the groups (3.27), we first suppose γ to be in $B(\infty)$. We point out that if $\gamma(t) = e$ and $\dot\gamma(t) = 0$ then $S(t)$ is finite and given by (3.32) since we can use equation (2.46), Proposition 66 and the continuity of $\omega_\gamma((D\omega_\gamma\colon D\omega)_t)$ with respect to γ in $H^2(S^1,G)$. Now we prove that $S(t)$ is finite for any t real. Indeed, for any t real we can pick a smooth loop η with $\operatorname{supp}\eta \leq t$ and such that $\eta(t-k) = \gamma(t-k)^{-1}$ and $\dot{(\eta\gamma)}(t-k) = 0$ for some $k > 0$. This implies that

$$S_{\mathcal{A}_\ell(t,+\infty)}(\omega_\gamma\|\omega) = S_{\mathcal{A}_\ell(t,+\infty)}(\omega_{\eta\gamma}\|\omega) \leq S_{\mathcal{A}_\ell(t-k,+\infty)}(\omega_{\eta\gamma}\|\omega) < +\infty\,,$$

where the last relative entropy is finite by the argument used above. By similar arguments we have that

$$\bar{S}(t) = S_{\mathcal{A}_\ell(-\infty,t)}(\omega_\gamma\|\omega) \tag{3.33}$$

is finite for any t real. Now we focus on the case $t=0$, since the general case follows by covariance. We suppose $\dot\gamma(0)=0$ and we write $\gamma=\gamma_+\gamma_-$, with $\gamma_+(u)=e$ for $u\le 0$ and $\gamma_-(u)=e$ for $u\ge 0$. By Proposition 66 we have

$$S(0) = a_\gamma - \frac{\ell}{2}\int_0^\infty u\,\langle\dot\gamma\gamma^{-1},\dot\gamma\gamma^{-1}\rangle\,du\,.$$

Now we emulate some techniques used in [51] and we prove that $a_\gamma=0$. Given $\lambda>0$ real, consider the function $f(u)=ue^{\lambda u}$. For $n>0$ integer, we consider a smooth diffeomorphism $\rho=\rho_{\lambda,n}$ of the circle such that, in the real line picture, it verifies $\rho(u)=f(u)$ for $0\le u\le n-\frac{1}{n}$ and $\rho(u)=f'(n)u+(f(n)-nf'(n))$ for $u\ge n$. We also suppose $\rho(u)/\rho'(u)$ to be uniformly bounded for $n-\frac{1}{n}\le u\le n$. Consider now the loop $\gamma_{\lambda,n}(u)=\gamma(\rho^{-1}_{\lambda,n}(u))$. By the identity $a_\gamma=a_{\gamma_{\lambda,n}}$ and by monotone convergence once more we have

$$0\le\inf_\lambda S_{\mathcal{A}_\ell(0,+\infty)}(\omega_{\gamma_{\lambda,n}}\|\omega) = a_\gamma - \frac{\ell}{2}\int_n^\infty (u-n)\,\langle\dot\gamma\gamma^{-1},\dot\gamma\gamma^{-1}\rangle\,du\,, \tag{3.34}$$

and by monotone convergence we have $a_\gamma\ge 0$. Now we prove the other inequality. Consider a smooth path ζ_n in G with extremes $\zeta(0)=e$ and $\zeta(1)=\gamma(0)$. We also suppose that $\dot\zeta(0)=\dot\zeta(1)=0$. We now define

$$\gamma_n(u) = \begin{cases} \gamma(u) & u\ge 0\,,\\ \zeta(nu+1) & -1/n\le u\le 0\,,\\ e & u\le -1/n\,.\end{cases}$$

By monotonicity $S_{\mathcal{A}_\ell(0,+\infty)}(\omega_\gamma\|\omega)=S_{\mathcal{A}_\ell(0,+\infty)}(\omega_{\gamma_n}\|\omega)\le S_{\mathcal{A}_\ell(-1/n,+\infty)}(\omega_{\gamma_n}\|\omega)$, so that after a limit we have the inequality

$$a_\gamma \le -\frac{\ell}{2}\int_0^1 u\,\langle\dot\zeta\zeta^{-1},\dot\zeta\zeta^{-1}\rangle\,du\,.$$

However, if we now consider the function $g_\lambda(u)=ue^{\lambda(u-1)}$ and we define $\zeta_\lambda(u)=\zeta(g_\lambda^{-1}(u))$, then

$$a_\gamma \le -\frac{\ell}{2}\int_0^1 u\,\langle\dot\zeta_\lambda\zeta_\lambda^{-1},\dot\zeta_\lambda\zeta_\lambda^{-1}\rangle\,du \le -\frac{\ell}{2\lambda}\int_0^1 u\,\langle\dot\zeta\zeta^{-1},\dot\zeta\zeta^{-1}\rangle\,du\to 0\,,\qquad \lambda\to+\infty\,.$$

Finally, we have proved that $a_\gamma=0$ if $\dot\gamma(0)=0$. To remove this condition, we notice that if P is the generator of translations then the average energy in the state ω_γ is finite and given by

$$E_\gamma = (\pi(\gamma)\Omega|P\pi(\gamma)\Omega) = -\frac{\ell}{2}\int_{-\infty}^{+\infty}\langle\dot\gamma\gamma^{-1},\dot\gamma\gamma^{-1}\rangle\,\frac{du}{2\pi}\,. \tag{3.35}$$

Therefore we can apply Lemma 26, namely for every t_1 and t_2 in $\mathbb{R}$ we have

$$(S(t_1)-S(t_2)) + (\bar S(t_2)-\bar S(t_1)) = (t_2-t_1)2\pi E_\gamma\,. \tag{3.36}$$

This implies that $S(t)$ and $\bar{S}(t)$ are both Lipschitz functions. Consider now a smooth real function $\rho(u)$ defined on $[0,1]$ and such that $\rho(0)=0$ and $\rho(1)=1$. We also suppose $\rho'(0)=\rho''(0)=0$, $\rho'(1)=1$ and $\rho''(1)=0$. We define

$$\gamma_n(u) = \begin{cases} \gamma(u) & u \geq 1/n\,, \\ \gamma(\rho(nu)/n) & 0 \leq u \leq 1/n\,, \\ \eta(u) & u \leq 0\,, \end{cases}$$

where η is a smooth function such that γ_n is in $H^2(S^1, G)$. Therefore, by (3.36) we have

$$0 \leq S_{\mathcal{A}_\ell(0,+\infty)}(\omega_{\gamma_n}\|\omega) - S_{\mathcal{A}_\ell(1/n,+\infty)}(\omega_{\gamma_n}\|\omega) \leq \frac{2\pi}{n} E_{\gamma_n} \to 0\,,$$

and thus we have

$$\begin{aligned} S_{\mathcal{A}_\ell(0,+\infty)}(\omega_\gamma\|\omega) &= \lim_n S_{\mathcal{A}_\ell(1/n,+\infty)}(\omega_\gamma\|\omega) = \lim_n S_{\mathcal{A}_\ell(1/n,+\infty)}(\omega_{\gamma_n}\|\omega) \\ &= \lim_n S_{\mathcal{A}_\ell(1/n,+\infty)}(\omega_{\gamma_n}\|\omega) - S_{\mathcal{A}_\ell(0,+\infty)}(\omega_{\gamma_n}\|\omega) + S_{\mathcal{A}_\ell(0,+\infty)}(\omega_{\gamma_n}\|\omega) \\ &= \lim_n S_{\mathcal{A}_\ell(0,+\infty)}(\omega_{\gamma_n}\|\omega) \\ &= -\frac{\ell}{2}\int_0^\infty u\,\langle \dot\gamma\gamma^{-1}, \dot\gamma\gamma^{-1}\rangle\, du\,. \end{aligned} \tag{3.37}$$

The most of the work is done. Now we just have to remove the condition $\dot\gamma(\infty)=0$. If we apply covariance to equation (3.37) then we have

$$S_{\mathcal{A}_\ell(-\infty,0)}(\omega_\gamma\|\omega) = -\frac{\ell}{2}\int_{-\infty}^0 u\,\langle \dot\gamma\gamma^{-1}, \dot\gamma\gamma^{-1}\rangle\, du$$

for any γ in $H^2(S^1, G)$ such that $\dot\gamma(0)=0$. But this condition can be removed as in (3.37), and by covariance the above expression of $S(0)$ holds for all γ in $H^2(S^1,G)$. □

Corollary 69. *Let η be a loop in $H^2(S^1,G)$ such that $\eta=\gamma$ on $(-r,r)$ in the real line picture. If E_η is the null energy* (3.35)*, then we have the Bekenstein Bound*

$$S_{\mathcal{A}_\ell(-r,r)}(\omega_\gamma\|\omega) \leq \pi r \inf_\eta E_\eta\,, \tag{3.38}$$

where the infimum is over all such η. Furthermore,

$$S_{\mathcal{A}_\ell(-r,r)}(\omega_\gamma\|\omega) = -\frac{\ell}{2}\int_{-r}^r \frac{1}{2r}(r-u)(r+u)\,\langle \dot\gamma\gamma^{-1}, \dot\gamma\gamma^{-1}\rangle\, du\,, \tag{3.39}$$

and the complement relative entropy (3.33) *is given by*

$$\bar{S}(t) = -\frac{\ell}{2}\int_{-\infty}^t (t-u)\,\langle \dot\gamma\gamma^{-1}, \dot\gamma\gamma^{-1}\rangle\, du\,. \tag{3.40}$$

Proof. As in the previous theorem, it is not restrictive to suppose γ to be in $H^2(S^1,G)$. The statement then follows by Möb-covariance, since in general we have

$$S_{\mathcal{A}_\ell(a,b)}(\omega_\gamma\|\omega) = -\frac{\ell}{2}\int_a^b D_{(a,b)}(u)\,\langle \dot\gamma\gamma^{-1}, \dot\gamma\gamma^{-1}\rangle\, du\,, \tag{3.41}$$

for every interval (a,b) of the real line, with $D_{(a,b)}(u)$ the density of the dilation operator of (a,b). □

We now discuss the QNEC. If P is the generator of translations, then by the Sugawara formula we have $P = \Theta(\frac{d}{du})$, hence the quantity $E = E_\gamma$ given by (3.35) is an averaged stress energy tensor in the null direction u in the state ω_γ.

Theorem 70. *Let γ be an element of $H^2(\mathbb{R}, G)$ as in Theorem 68. If we consider the null energy density*

$$E_\gamma(t) = -\frac{\ell}{4\pi}\left\langle \dot\gamma\gamma^{-1}, \dot\gamma\gamma^{-1}\right\rangle(t)\,, \tag{3.42}$$

then the states ω_γ verify the QNEC with the equality

$$E_\gamma(t) = S''(t)/2\pi \geq 0\,. \tag{3.43}$$

Similarly, $E_\gamma(t) = \bar S''(t)/2\pi \geq 0$, where $\bar S(t)$ is the complement relative entropy (3.33) *given by* (3.40).

This theorem is the main result of this chapter. However, definition (3.42) may seem not rigorous to the reader, since we can arbitrarily add a function with null average to the integral (3.35). For this reason, we will now motivate our definition of null energy tensor density (as anticipated in the previous chapter). In particular, the following argument will show a model-independent way to recover (3.42) by using some intermediate results of [27].

Let $\mathcal{N} \subseteq \mathcal{M}$ be a -hsm inclusion with corresponding family of von Neumann algebras $(\mathcal{M}_t)_{t\in\mathbb{R}}$. We denote by $P \geq 0$ the generator of translations and by ω the faithful normal state given by the common standard vector Ω. Given two real parameters $t < t'$, consider a normal state ψ of $\mathcal{M}_t$ with representing vector η. If u is some isometry, then we will denote by ψ_u the vector state represented by $u\eta$. We will use the notation $P_\eta = (\eta|P\eta)$. We define

$$E_\psi(t,t') = \inf_{(w',w)\in C'_t\times C_{t'}} P_{ww'\eta}\,, \tag{3.44}$$

where C'_t is the family of all the isometries w' in $\mathcal{M}'_t$ such that $P_{w'\eta}$ and $S_{\mathcal{M}'_t}(\psi_{w'}\|\omega)$ are both finite, and similarly $C_{t'}$ is the family of all the isometries w in $\mathcal{M}_{t'}$ such that $P_{w\eta}$ and $S_{\mathcal{M}_{t'}}(\psi_w\|\omega)$ are finite. Notice that $E_\psi(t,t')$ is well defined as a state-dependent quantity, since any two vectors which represent ψ on $\mathcal{M}_t$ differ by an isometry of $\mathcal{M}'_t$. Finally, by using the proof of Theorem 31 and Proposition 24 we have the following fact.

Proposition 71. *Given two real parameters $t < t'$, consider a normal state ψ of $\mathcal{M}_t$ with representing vector η such that $P_\eta < +\infty$. Consider the Connes cocycles*

$$u'_s(t) = (D\psi\colon D\omega\,;\mathcal{M}'_t)_s\,,\quad u_s(t') = (D\psi\colon D\omega\,;\mathcal{M}_{t'})_s\,.$$

If the relative entropies $S(t) = S_{\mathcal{M}_t}(\psi\|\omega)$ and $\bar S(t') = S_{\mathcal{M}'_{t'}}(\psi\|\omega)$ are finite, then

$$E_\psi(t,t') = \inf_{s,s'} P_{u'_s(t)\eta} + P_{u_{s'}(t')\eta} - P_\eta = \lim_{s\to+\infty} P_{u'_s(t)\eta} + P_{u_{-s}(t')\eta} - P_\eta\,. \tag{3.45}$$

In other words, what we did was just to notice by the proof of Theorem 31 that, under some finiteness assumptions, the null energies of all the representing vectors for a normal state are minimized by the Connes cocycles. Notice also that by Theorem 31 and (3.36) we have

$$E_\psi(t,t') = -S'(t)/2\pi + \bar{S}'(t')/2\pi - P_\eta\,.$$

Finally, we can define

$$E_\psi(t) = \liminf_{h\to 0^+} E_\psi(t,t+h)/h\,. \tag{3.46}$$

After this premise, we can show that the density (3.42) is actually given by the limit (3.46). In this step we will use the results of [80], which ensures us that a PER of a loop group LG can be extended to a PER of $H^1(S^1,G)$. In particular, this implies that Theorem 68 and Theorem 70 are still true in this generality. The same argument applies to Proposition 75 below as well. Furthermore, as shown later in Proposition 75, we can compute (3.44) by using (3.45). By doing so we have

$$E_\gamma(t,t') = -\frac{\ell}{4\pi}\int_t^{t'} \langle \dot\gamma\gamma^{-1}, \dot\gamma\gamma^{-1}\rangle\, du\,,$$

and this tells us that the null energy density (3.42) can be recovered by using (3.46).

We conclude this section by noticing that, for the loop states studied in this chapter, the null energy density is equal to the stress-energy tensor density. We recall that, given h in $\mathcal{S}_{3/2}$, in general we can consider two vectors ξ and η in $\mathcal{V} = \bigcap_{k\geq 0}\mathcal{D}(L_0^k)$, recall that

$$|(\eta|\Theta(h)\xi)| \leq (c/2)^{1/2}|h|_{3/2}\|\eta\|\|(1+L_0)\xi\|\,,$$

and define $(\eta|\Theta(u)\xi)$ as the kernel of the tempered distribution $h\mapsto\Theta(h)$. In our case, by Corollary 65 in the real line picture we have

$$(\pi(\gamma)\Omega|\Theta(h)\pi(\gamma)\Omega) = -\frac{\ell}{4\pi}\int h(t)\,\langle\dot\gamma\gamma^{-1},\dot\gamma\gamma^{-1}\rangle\,(t)dt\,,$$

for every h, and this tells us that in this case we have the identity

$$(\pi(\gamma)\Omega|\Theta(t)\pi(\gamma)\Omega) = E_\gamma(t)\,.$$

3.6 QNEC on $LSU(n)$

In this section we focus on the case $G = SU(n)$ and we use a construction illustrated in [95] to show that a Positive Energy Representation of $LSU(n)$ can be extended to a Positive Energy Representation of the Sobolev group $H^s(S^1, SU(n))$ for $s > 1/2$. In particular, we will use this fact to provide a simpler proof of the QNEC (3.43).

We begin by considering the natural action of $G = SU(n)$ on $V = \mathbb{C}^n$ and we set $H = L^2(S^1,V)$, or equivalently $H = L^2(S^1)\otimes V$. We can naturally define a continuous action M of LG on H by $M_\gamma f(\phi) = \gamma(\phi)f(\phi)$. We can also define an action of Rot on H by $R_\theta f(\phi) = f(\phi-\theta)$ with respect to the representation of LG is covariant, that is it satisfies $R_\theta M_\gamma R_\theta^{-1} = M_{R_\theta \gamma}$. If P is the orthogonal projection onto the Hardy space H_+, namely

$$H_+ = \left\{ f \in L^2(S^1, V) \colon f(\theta) = \sum_{k \geq 0} f_k e^{ik\theta} \text{ with } f_k \in V \right\},$$

then we can define a new Hilbert space H_P which is equivalent to H as real Hilbert space, but with complex structure given by $J = iP - i(1-P)$. The Segal quantization criterion, which we now recall, allows us to define a positive energy representation of LG on the fermion Fock space $\mathcal{F}_P = \Lambda H_P$ known as the *fundamental representation* of $LSU(n)$ [87, 95]. Notice that $\mathcal{F}_P(0) = \Lambda V$ is the fundamental representation of $SU(n)$. The fundamental representation of $LSU(n)$ is the direct sum of all the $n+1$ irreducible positive energy representations of $LSU(n)$ of level $\ell = 1$. The fundamental representation contains the *basic representation*, that is the unique level one vacuum representation.

Definition 72. The *restricted unitary group* is the topological group

$$U_P(H) = \left\{ u \in U(H) \colon [u, P] \in L^2(H) \right\},$$

where the considered topology is the strong operator topology combined with the metric given by the distance $d(u, v) = \|[u - v, P]\|_2$.

Any $u \in U(H)$ gives rise to an automorphism of $CAR(H)$, called *Bogoliubov automorphism*, via $a(f) \mapsto a(uf)$. For every projection P on H there is an irreducible representation of $CAR(H)$ on $\mathcal{F}_P$ which is denoted by π_P. The Bogoliubov automorphism is said to be *implemented* on $\mathcal{F}_P$ if $\pi_P(a(uf)) = U\pi_P(a(f))U^*$ for some unitary $U \in U(\mathcal{F}_P)$ [95].

Theorem 73. Segal's quantization criterion. *[95] If $[u, P]$ is a Hilbert-Schmidt operator then u is implemented on $\mathcal{F}_P$ by some unitary operator U_P. Moreover, U_P is unique up to a phase and the constructed map $U_P(H) \to PU(\mathcal{F}_P)$ is continuous.*

Proposition 74. *The fundamental representation of $LSU(n)$ can be extended to a PER of $H^s(S^1, SU(n))$ for any $s > 1/2$. In particular, every positive energy representation of $LSU(n)$ extends to a positive energy representation of $H^s(S^1, SU(n))$ for $s > 1/2$.*

Proof. Notice that since a loop γ in $LSU(n)$ is also a map from S^1 to $M_n(\mathbb{C})$, then we can write γ as a Fourier series $\gamma(z) = \sum \widehat{\gamma}_k z^k$, where $\widehat{\gamma}_k \in M_n(\mathbb{C})$. We consider on H the basis $e_j^k(z) = z^k e_j$, where (e_j) is the standard basis of $\mathbb{C}^n$. We define $M_{pq} = \widehat{\gamma}_{p-q}$ and we note that $M_\gamma e_j^k = \sum_i M_{ik} e_j^i$, so that $(e_i^p, M_\gamma e_j^q) = (e_i, M_{pq} e_j)$. So M_γ is represented by a $\mathbb{Z} \times \mathbb{Z}$ matrix (M_{pq}) of endomorphisms. We have

$$\begin{aligned}
\|[P, M_\gamma]\|_2^2 &= \sum_{p \geq 0, q < 0} \|M_{pq}\|_2^2 + \sum_{p < 0, q \geq 0} \|M_{pq}\|_2^2 \\
&= \sum_{k > 0} k \|\widehat{\gamma}_k\|_2^2 - \sum_{k < 0} k \|\widehat{\gamma}_k\|_2^2 \\
&= \sum_{k \in \mathbb{Z}} |k| \|\widehat{\gamma}_k\|_2^2 \leq \sum_{k \in \mathbb{Z}} (1 + |k|)^{2s} \|\widehat{\gamma}_k\|_2^2,
\end{aligned}$$

for $s > 1/2$. It is easy to verify that the map $\gamma \mapsto M_\gamma \in U_P(H)$ is continuous. We also have that the rotation group acts on $H^s(S^1, G)$ by continuous operators (see Lemma A.3 of [22]), and by $[R_\theta, P] = 0$ we have that the projective representation of Rot is actually a strongly continuous unitary representation. Therefore, the thesis follows by Theorem 73, the complete reducibility of Positive Energy Representations (Thm. 9.3.1. of [87]), Proposition 2.3.3. of [93] and remarks below. □

Proposition 75. *Let γ be a loop in $H^1(S^1, SU(n))$. Pick a non dense open interval $I = (z, w)$ of the circle and write $\gamma = \gamma_I \gamma_{I'}$ as in (3.28). Then, in $PU(\mathcal{H})$ we have*

$$(D\omega_\gamma \colon D\omega)_t = \pi(\gamma_I \delta_I(-2\pi t).\gamma_I^{-1}), \tag{3.47}$$

where $(D\omega_\gamma \colon D\omega)_t$ is the Connes cocycle of $\mathcal{A}_\ell(I)$ and $\delta_I(t)$ denotes the dilation associated to I.

Proof. First we check that $\gamma_I \delta_I(t).\gamma_I^{-1}$ is in $H^1(S^1, SU(n))$ since it is continuous on the boundary of I, hence the right hand side of (3.47) is well defined. With the same computations of Proposition 66 we have that $\sigma_t^\gamma(x) = \operatorname{Ad}\pi(\gamma_I \delta_I(-2\pi t).\gamma_I^{-1}) \cdot \sigma_t(x)$ for x in $\mathcal{A}_\ell(I)$. Therefore, we have that $(D\omega_\gamma \colon D\omega)_t$ is equal to $\pi(\gamma_I \delta_I(-2\pi t).\gamma_I^{-1})$ up to a unitary V in the commutant of $\mathcal{A}_\ell(I)$, but $(D\omega_\gamma \colon D\omega)_t$ and $\pi(\gamma_I \delta_I(-2\pi t).\gamma_I^{-1})$ are both in $\mathcal{A}_\ell(I)$ and thus V is a scalar. □

Theorem 76. *Let γ be a loop in $H^1(S^1, SU(n))$. Suppose also that, in the real line picture, the support of γ is bounded from below. Then the relative entropy (3.26) is finite and given by*

$$S(t) = -\frac{\ell}{2} \int_t^\infty (u - t) \, \langle \dot\gamma\gamma^{-1}, \dot\gamma\gamma^{-1} \rangle \, du \,. \tag{3.48}$$

In particular, the QNEC is satisfied as shown above in Theorem 70.

Proof. Since the vacuum is $SU(n)$-invariant, we can replace γ with γg for any g in $SU(n)$, thus we can suppose $\gamma(\infty) = e$. As above, if $\gamma(t) = e$ then $S(t)$ is finite and given by (3.48). We can prove that $S(t)$ is finite for any t real as in Theorem 68, and similarly we have that $\bar S(t) = S_{\mathcal{A}_\ell(-\infty,t)}(\omega_\gamma \| \omega)$ is finite for any t real. If P is the generator of translations then the average energy E_γ in the state ω_γ is finite and given by equation (3.35). Therefore we can apply Lemma 26, and equation (3.36) holds. This implies that $S(t)$ and $\bar S(t)$ are both Lipschitz functions and in particular they are absolutely continuous. The next step is an estimate of $S'(t)$. For simplicity we focus on the case $t = 0$ and we write $\gamma = \gamma_+ \gamma_-$ with $\gamma_+(u) = e$ for $u \leq 0$ and $\gamma_-(u) = e$ for $u \geq 0$. By Proposition 75 the Connes cocycle $u'_s = (D\omega \colon D\omega_\gamma)_s$ on $\mathcal{A}_\ell(-\infty, 0)$ is equal in $PU(\mathcal{H})$ to $\pi(\gamma_- \delta(2\pi s).\gamma_-^{-1})^*$. But also the state $\omega_\gamma \cdot \operatorname{Ad}(u'_s)^*$ verifies the finiteness conditions required to apply Lemma 1. and thus we have $-S'(0) \leq 2\pi E_s$, where $E_s = (u'_s \pi(\gamma)\Omega | P u'_s \pi(\gamma)\Omega)$ for s real. However, one can simply prove that

$$\inf_s 2\pi E_s = -\frac{\ell}{2} \int_0^{+\infty} \langle \dot\gamma\gamma^{-1}, \dot\gamma\gamma^{-1} \rangle \, du \,.$$

Therefore, by repeating the argument with any t in $\mathbb{R}$ we have

$$-S'(t) \leq -\frac{\ell}{2} \int_t^{+\infty} \langle \dot\gamma\gamma^{-1}, \dot\gamma\gamma^{-1} \rangle \, du \,.$$

Finally, if we define

$$F(t) = -\frac{\ell}{2} \int_t^{\infty} (u - t) \, \langle \dot\gamma\gamma^{-1}, \dot\gamma\gamma^{-1} \rangle \, du \,,$$

then we can conclude that $S(t) = F(t)$ for any t in $\mathbb{R}$. Indeed, if the support of γ is compact then $H(t) = S(t) - F(t)$ is an absolutely continuous function with nonnegative derivative and going to 0 as $|t| \to +\infty$. If the support of γ is contained in $(k, +\infty)$ then by lower semicontinuity $S(t) \leq F(t)$ for every t real, and we can similarly deduce that $H(t) = 0$ for every t real. Finally, the first identity appearing in (3.43) can be proved as above, with the only exception that in this case we do not have to use [80] in order to compute (3.42). $\square$

Chapter 4

Nuclearity as an entanglement measure

4.1 Modular nuclearity conditions

Let A, B be a couple of commuting von Neumann algebras on some Hilbert space $\mathcal{H}$. We shall say that the pair (A, B) is *split* if there exists a von Neumann algebra isomorphism $\phi\colon A \vee B \to A \otimes B$ such that $\phi(ab) = a \otimes b$. If $A \vee B$ is σ-finite, then the pair (A, B) is split if and only if for any given normal states φ_A on A and φ_B on B there exists a normal state φ on $A \vee B$ such that $\varphi(ab) = \varphi_A(a)\varphi_B(b)$ [71].

Following standard arguments, we further characterize a split pair (A, B). We assume A, B and $A \vee B$ to be in standard form with respect to some state ω given by some vector Ω of $\mathcal{H}$. We denote by $J_A = J_{A,\Omega}$, $J_B = J_{B,\Omega}$ and $J_{A\vee B} = J_{A\vee B,\Omega}$ the corresponding modular conjugations. As $A \otimes B$ is in standard form with respect to the state $\omega \otimes \omega$, the isomorphism $\phi\colon A \vee B \to A \otimes B$ has a *canonical implementation*, namely is uniquely implemented by some unitary U which maps the natural cone of $A \vee B$ onto the natural cone of $A \otimes B$ [39]. It can also be shown that $J_A \otimes J_B = UJ_{A\vee B}U^{-1}$. The *canonical intermediate type I factors* are $F = U^{-1}(B(\mathcal{H}) \otimes \mathbb{1})U$ and $F' = U^{-1}(\mathbb{1} \otimes B(\mathcal{H}))U$. By construction, F is the unique $J_{A\vee B}$-invariant type I factor $A \subseteq F \subseteq B'$, and similarly for F'. If A and B are both factors then $F = A \vee JAJ = B' \cap JB'J$, with $J = J_{A\vee B}$, and therefore $F' = B \vee JBJ = A' \cap JA'J$ [39].

Definition 77. Let $N \subseteq M$ be an inclusion of von Neumann algebras on some Hilbert space $\mathcal{H}$. We shall say that $N \subseteq M$ is a *split inclusion* if there exists an intermediate type I factor $N \subseteq F \subseteq M$.

Clearly the trivial inclusion $N = M$ is split if and only if N is a type I factor. The inclusion $N \subseteq M$ is said to be *standard* if there is a vector Ω which is standard for N, M and the relative commutant $N' \cap M$. If $N \subseteq M$ is a standard split inclusion then each intermediate type I factor is σ-finite and hence separable, therefore the Hilbert space $\mathcal{H}$ has to be separable as $F\Omega$ is dense in $\mathcal{H}$. If $N' \cap M$ has a cyclic and separating vector, then the pair (N, M') is split if and only if $N \subseteq M$ is split [71].

Definition 78. Consider an inclusion $N \subseteq M$ of von Neumann algebras on a Hilbert space $\mathcal{H}$. Assume the existence of a standard vector Ω for M and denote by Δ the corresponding modular operator. We will say that the inclusion $N \subseteq M$ satisfies the *modular nuclearity condition* if the map

$$\Xi\colon N \to \mathcal{H}\,, \quad \Xi(x) = \Delta^{1/4} x\Omega\,, \tag{4.1}$$

is nuclear.

A modular nuclear inclusion of factors is split, and a split inclusion of factors implies the compactness of the map (4.1) [19]. This motivates the interest in the split property in local quantum field theory contexts, where the split property amounts to some form of statistical independence between causally disjoint spacetime regions [52, 60]. We will develop this topic in Section 4.4.

A similar nuclearity condition can be given by use of the language of standard subspaces. A closed real subspace H of a complex Hilbert space $\mathcal{H}$ is called *standard* if $H \cap iH = (0)$ and $H + iH$ is dense in $\mathcal{H}$. Standard subspaces arise naturally in the modular theory of von Neumann algebras. If M is a von Neumann algebra acting on $\mathcal{H}$ and $\xi \in \mathcal{H}$ is a standard vector for M, then the map $M \to \mathcal{H}$ given by $x \mapsto x\xi$ is injective and

$$H_M = \overline{\{x\xi\colon x = x^*\,, x \in M\}} \tag{4.2}$$

is a standard subspace of $\mathcal{H}$. Conversely, there is a natural way to associate to every standard subspace $H \subseteq \mathcal{H}$ a von Neumann algebra in the bosonic and fermionic Fock space of $\mathcal{H}$, and this assignment has many nice properties [67, 79].

Definition 79. Let $\mathcal{H}$ be a complex Hilbert space and $K \subseteq H$ an inclusion of standard subspaces of $\mathcal{H}$. We shall say that $K \subseteq H$ satisfies the *modular nuclearity condition* if the operator

$$\Xi = \Delta_H^{1/4} P_K \tag{4.3}$$

is nuclear, where P_K is the real linear orthogonal projection onto K.

It is easy to check that if $N \subseteq M$ is a standard inclusion of von Neumann algebras, then this inclusion satisfies modular nuclearity if the corresponding inclusion $H_N \subseteq H_M$ of standard subspaces given by (4.2) satisfies modular nuclearity [68]. If $K' \cap H$ is a standard space and $J = J_{K'\cap H}$ then on bosonic models the subspace $F = K \vee JK = H \cap JH$ is a standard space called the *canonical intermediate type I standard subspace* [73].

In the perspective of proving our main theorem, we now try to introduce a new nuclearity condition. Given a pair (A, B) of commuting von Neumann algebras, assume A and B to be both in standard form with respect to a normal state ω represented by a standard vector Ω. Denote by Δ_A and Δ_B the corresponding modular operators. We will say that the pair (A, B) satisfies the *modular nuclearity condition* if at least one of the two maps

$$\Xi_A(b) = \Delta_{A'}^{1/4} b\Omega\,, \quad \Xi_B(a) = \Delta_{B'}^{1/4} a\Omega\,, \tag{4.4}$$

is nuclear. In order to motivate our definition, we notice that if (A, B) satisfies the modular nuclearity condition then the pair (A, B) is split.

The previous nuclearity conditions can be easily generalized as follows. Consider a linear map $\Theta\colon \mathcal{E} \to \mathcal{F}$ between Banach spaces. The map Θ is said to be of *type* l^p, $p > 0$, if there exists a sequence of linear mappings $\Theta_i\colon \mathcal{E} \to \mathcal{F}$ of rank i such that [20]

$$\sum_i \|\Theta - \Theta_i\|^p < +\infty\,.$$

The map Θ will be said to be of *type s* if it is of type l^p for any $p > 0$. Each mapping Θ of type l^p for some $0 < p \leq 1$ is nuclear. Indeed, there are sequences of linear functionals $e_i \in \mathcal{E}^*$ and of elements f_i in $\mathcal{F}$ such that

$$\Theta(x) = \sum_i e_i(x) f_i\,, \quad x \in \mathcal{E}\,,$$

is an absolutely convergent series for each x in $\mathcal{E}$, with

$$\Theta(x) = \sum_i e_i(x) f_i\,, \quad \Theta(x) = \sum_i \|e_i\|^p \|f_i\|^p < +\infty\,.$$

The induced quasi-norm, also called *p-norm*, is given by

$$\|\Theta\|_p = \inf\Big(\sum_i \|e_i\|^p \|f_i\|^p\Big)^{1/p}\,,$$

where the infimum is taken over all possible representations of Θ of the form (4.1). The above conditions of nuclearity can be then rephrased as *modular p-nuclearity conditions* if the maps (4.1), (4.3) or (4.4) are of type l^p for some $0 < p \leq 1$.

4.2 Entanglement Measures

In this section we discuss entanglement in a general setting and we review some quantitative measures of entanglement and their properties [52].

Let A, B be a couple of commuting von Neumann algebras. We will refer to the spatial tensor product $A \otimes B$ as a *bipartite system*. A state ω on the bipartite system $A \otimes B$ is said to be *separable* if there are positive normal functionals φ_j on A and ψ_j on B such that $\omega = \sum_j \varphi_j \otimes \psi_j$, where the sum is assumed to be norm convergent. Separable states are normal. A normal state which is not separable is called *entangled*. In quantum field theory, bipartite systems are associated to causally disjoint regions. Therefore, an entanglement measure for a bipartite system should be a state functional that vanishes on separable states.

Definition 80. The *relative entanglement entropy* of a normal state ω on a bipartite system $A \otimes B$ is given by

$$E_R(\omega) = \inf\{S(\omega\|\sigma)\colon \sigma \text{ is a separable state}\}\,. \tag{4.5}$$

The *mutual information* $E_I(\omega)$ is given by

$$E_I(\omega) = S(\omega\|\omega_A \otimes \omega_B)\,.$$

where $\omega_A = \omega|_A$ and similarly for B.

Clearly $E_R(\omega) \leq E_I(\omega)$. As an example, let us consider a bipartite system given by $A = B(\mathcal{H})$ and $B = B(\mathcal{H}')$, with $\mathcal{H}$ and $\mathcal{H}'$ finite dimensional Hilbert spaces. The mutual information is given by

$$E_I(\omega) = S(\omega_A) + S(\omega_B) - S(\omega)\,. \tag{4.6}$$

We point out that, without any finiteness assumption, on hyperfinite type I factors we can only write $E_I(\omega) + S(\omega) = S(\omega_A) + S(\omega_B)$. It is an easy remark to notice that, always by assuming A and B to be finite dimensional type I factors, if $\omega = \sum_j \lambda_j \omega_j$ is a convex decomposition of a state ω in states ω_j, then

$$\textstyle\sum_j \lambda_j E_I(\omega_j) - \sum_j \eta(\lambda_j) \leq E_I(\omega) \leq \sum_j \lambda_j E_I(\omega_j) + 2\sum_j \eta(\lambda_j)\,.$$

By monotonicity of the relative entropy, the same inequalities hold if ω is normal and A and B are both hyperfinite type I factors. Furthermore, if ω is pure then $E_I(\omega) = 2S(\omega_A) = 2S(\omega_B)$ (Proposition 6.5. of [82]) while the relative entanglement entropy between A and B is [94]

$$E_R(\omega) = S(\omega_A) = S(\omega_B)\,.$$

Definition 81. A cp map $\mathcal{F}\colon A_1 \otimes B_1 \to A_2 \otimes B_2$ between two bipartite systems will be called *local* if it is of the form

$$\mathcal{F}(a \otimes b) = \mathcal{F}_A(a) \otimes \mathcal{F}_B(b)\,,$$

where $\mathcal{F}_A$ and $\mathcal{F}_B$ are normal cp maps. More generally, a *separable operation* is by definition a family of normal, local cp maps $\mathcal{F}_j$ such that $\sum_j \mathcal{F}_j(1) = 1$. We think of such an operation as mapping a state ω with probability $p_j = \omega(\mathcal{F}_j(1))$ to $\mathcal{F}_j^* \omega / p_j$.

Separable operations map separable states to separable states. The relative entanglement entropy (4.5) of a bipartite system $A \otimes B$ has the following properties [52].

(e0) (symmetry) $E_R(\omega)$ is independent of the order of the systems A and B.[1]

(e1) (non-negative) $E_R(\omega) \in [0, \infty]$, with $E_R(\omega) = 0$ if ω is separable and $E_R(\omega) = \infty$ when ω is not normal. Furthermore, if $E_R(\omega) = 0$ then ω is norm limit of separable states.

(e2) (continuity) Let $(\mathfrak{A}_i)_i$ and $(\mathfrak{B}_i)_i$ be two increasing nets of subalgebras of A and B respectively, with $\mathfrak{A}_i \cong \mathfrak{B}_i \cong M_{n_i}(\mathbb{C})$. Let ω_i and ω_i' be normal states on $\mathfrak{A}_i \otimes \mathfrak{B}_i$ such that $\lim_i \|\omega_i - \omega_i'\| = 0$. Then

$$\lim_{i \to \infty} \frac{E_R(\omega_i') - E_R(\omega_i)}{\ln n_i} = 0\,.$$

(e3) (convexity) E_R is convex.

[1] More precisely, we should require that $E_R(\omega) = E_R(\omega \cdot \pi)$, with $E_R(\omega \cdot \pi)$ the relative entanglement entropy on $B \otimes A$ and π the natural permutation isomorphism.

(e4) (monotonicity under separable definitions) Consider a separable operation described by cp maps $\mathcal{F}_j$ with $\sum_j \mathcal{F}_j(1) = 1$. Then

$$\sum_j p_j E_R(\mathcal{F}_j^* \omega / p_j) \leq E_R(\omega) ,$$

where the sum is over all j with $p_j = \omega(\mathcal{F}_j(1)) > 0$.

(e5) (tensor products) Let $A_i \otimes B_i$ with $i = 1, 2$ be two bipartite systems, and let ω_i be states on $A_i \otimes B_i$. Then

$$E_R(\omega_1 \otimes \omega_2) \leq E_R(\omega_1) + E_R(\omega_2) .$$

The mutual information (4.6) clearly satisfies (e0) and (e5), and it is shown in [52] that it also satisfies properties (e2) and (e4). Property (e1) does not follow in a straightforward way from the definitions, as for separable states $\omega = \sum_j \lambda_j \varphi_j \otimes \psi_j$ we can only show that $E_I(\omega) \leq \sum_j \eta(\lambda_j)$, with $\eta(t) = -t \ln t$ the information function. Property (e3) does not hold in general.

We now describe one more entanglement measure. This notion of entanglement, known as *entanglement of formation*, is a well-known bipartite entanglement measure with an operational meaning that asymptotically quantifies how many Bell states are needed to prepare the given state using local quantum operations and classical communications. This last entanglement measure has already been studied in [78], and here we discuss its properties in a personal fashion only for the sake of clarity and completeness.

Definition 82. [78] Let ω be a state on a finite dimensional bipartite system $S = A \otimes B$. The *entanglement of formation* of ω is defined by

$$E_F^S(\omega) = \inf \{ \sum_i \lambda_i S_A(\omega_i) \} = \inf \{ \sum_i \lambda_i S_B(\omega_i) \} , \tag{4.7}$$

where the infima are over all finite (countable) convex linear combinations $\omega = \sum_i \lambda_i \omega_i$ on $A \otimes B$. More generally, given a bipartite system $S = A \otimes B$ one defines

$$E_F^S(\omega) = \sup_s E_F^s(\omega) ,$$

where the supremum is over all finite dimensional bipartite subsystems s of S.

The entanglement of formation is well defined because by the concavity of the von Neumann entropy the above infima are obtained for convex linear combinations $\omega = \sum_i \lambda_i \omega_i$ in pure states, for which $S_A(\omega_i) = S_B(\omega_i)$ (Lemma 6.4 of [82] is still true for finite dimensional von Neumann algebras). If the system $S = A \otimes B$ is clear from the context, we will simply write $E_F^S(\omega) = E_F(\omega)$. The entanglement of formation is non-negative and satisfies properties (e0), (e3) and (e5). If A and B are both finite dimensional factors, by exploiting the definitions (1.9) and (4.7) one also finds that [78]

$$E_F(\omega) + H_\omega(A) = S_A(\omega) , \quad E_F(\omega) + H_\omega(B) = S_B(\omega) , \tag{4.8}$$

where $H_\omega(A) = H_\omega^{A\otimes B}(A)$ and $H_\omega(B) = H_\omega^{A\otimes B}(B)$. By using equation (4.8), it is easy to prove that if $B \subseteq \overline{B}$ and $A \subseteq \overline{A}$, then $E_F^{A\otimes B}(\omega) \leq E_F^{\overline{A}\otimes\overline{B}}(\overline{\omega})$. One more result of [78] is the following martingale property.

Lemma 83. *[78] Let ω be a normal state on a bipartite system $S = A \otimes B$. We assume A and B to be hyperfinite factors. Let $(A_i)_i$ and $(B_i)_i$ be two increasing nets of finite dimensional subalgebras with $\cup_i A_i$ and $\cup_i B_i$ strongly dense in A and B respectively. If $E_F^{s_i}(\omega)$ is the entanglement of formation of ω on the subsystem $s_i = A_i \otimes B_i$, then*

$$E_F(\omega) = \lim_i E_F^{s_i}(\omega) = \sup_i E_F^{s_i}(\omega) \,.$$

This lemma says that the entanglement of formation is well-behaved on hyperfinite factors. In particular, always by assuming A and B to be hyperfinite factors, the entanglement of formation vanishes on separable states, as the following original simple lemma implies.

Lemma 84. *Given two von Neumann algebras A and B, consider a state ω on the bipartite system $A \otimes B$. If ω is separable, then*

$$S_A(\omega) = H_\omega^{A \otimes B}(A) \,.$$

Proof. If $\omega = \sum_j \phi_j \otimes \psi_j$ and π is the GNS representation of A associated to the marginal state $\omega_A = \omega|_A$, then we have an equivalence of GNS representations $\pi \cong \oplus_j \pi_j$, where π_j is the GNS representation of A given by $\psi_j(1)\phi_j$. We can define a cpu map $\varepsilon_j \colon A \otimes B \to \pi_j(A)$ by $\varepsilon_j(a \otimes b) = \pi_j(a)\psi_j(b)/\psi_j(1)$, and this lead us to define a conditional expectation $\varepsilon \colon A \otimes B \to pAp$ by $\varepsilon = \pi^{-1} \cdot \oplus_j \varepsilon_j$, where p is the support projection of ω_A. The claim follows from the identity $S_A(\omega) = S_{pAp}(\omega)$ (see the proof of Proposition 6.8 of [82]) and Lemma 19. □

Always by assuming A and B to be hyperfinite factors one can show that [78]

$$E_F(\omega) + \eta(\lambda) + \eta(1-\lambda) \geq \lambda E_F(\omega_1) + (1-\lambda) E_F(\omega_2) \,,$$

with ω_1 and ω_2 states such that $\omega = \lambda\omega_1 + (1-\lambda)\omega_2$ and λ in $(0,1)$. If A and B are hyperfinite type I factors, then by equation (4.6) and Lemma 83 it is easy to notice that

$$E_I(\omega) \leq E_F(\omega) + \min\{S(\omega_A), S(\omega_B)\} \,.$$

Moreover, under the assumption of Lemma 83 one has [78]

$$E_F(\omega) \leq -(c-1)\ln(c-1) + c \ln c \tag{4.9}$$

whether $\omega \leq \sigma$ for some separable functional $\sigma = \sum_j \varphi_j \otimes \psi_j$ with norm $\|\sigma\| = c$.

4.3 Modular nuclearity and entanglement

Given a split pair (A, B) of von Neumann algebras on some Hilbert space $\mathcal{H}$, assume A, B and $A \vee B$ to be in standard form with respect to some state ω given by a standard vector Ω. As in Section 4.1, we will denote by $A \subseteq F \subseteq B'$ the canonical intermediate type I factor with respect to the inclusion $A \subseteq B'$.

Definition 85. The *canonical entanglement entropy* of the bipartite system $A \otimes B$ is

$$E_C(\omega) = S_F(\omega) = S_{F'}(\omega)\,,$$

namely the von Neumann entropy of ω on the canonical intermediate type I factor F.

By construction, since ω is a pure state on $B(\mathcal{H})$ then we have the identity [94]

$$E_C(\omega) = \inf\{S(\omega\|\sigma \cdot \mathrm{Ad}\,U) \colon \sigma \text{ is separable on } B(\mathcal{H}) \otimes B(\mathcal{H})\}\,,$$

where U is the unitary canonically implementing the isomorphism $A \vee B \cong A \otimes B$, so that $B(\mathcal{H}) = F \vee F' \cong B(\mathcal{H}) \otimes B(\mathcal{H})$.

Conjecture 1. The canonical entanglement entropy is finite if the split pair (A, B) satisfies the modular p-nuclearity condition for some $0 < p < 1$.

This conjecture has been verified on the free Fermi net in [73], but up to now a general proof on a model independent ground is lacking. In this chapter we provide a few partial results in this direction.

Definition 86. Let (A, B) be a split pair of von Neumann algebras as above. Denote by Ξ_A and Ξ_B the two maps (4.4). Given $p > 0$ we define the *p-partition function* as

$$z_p = \min\{\|\Xi_A\|_p, \|\Xi_B\|_p\}\,. \tag{4.10}$$

Lemma 87. *Let (A, B) be a split pair as above. We assume modular p-nuclearity to hold for some $0 < p \leq 1$, namely (4.10) is finite for some $0 < p \leq 1$. Given $\epsilon > 0$, there are sequences of normal linear functionals ϕ_j on A and ψ_j on B such that*

$$\omega(ab) = \sum_j \phi_j(a)\psi_j(b)\,, \quad a \in A\,, \;\; b \in B\,, \tag{4.11}$$

and $\sum_j \|\phi_j\|^p \|\psi_j\|^p < z_p^p + \epsilon$.

Proof. We follow [52, Lemma 3]. We can assume $z_p = \|\Xi_B\|_p$. Using the commutativity of A and B, we note that

$$\begin{aligned}\omega(ab) = (\Omega|ab\Omega) &= ((\Delta^{1/4} + \Delta^{-1/4})^{-1}(1 + \Delta^{-1/2})b^*\Omega|\Delta^{1/4}a\Omega)\\ &= ((\Delta^{1/4} + \Delta^{-1/4})^{-1}(b^* + JbJ)\Omega|\Xi_B(a))\,,\end{aligned}$$

where we set $\Delta = \Delta_{B'}$. If z_p is finite and $\epsilon > 0$, then there are sequences of positive normal functionals ϕ_j on A and vectors ξ_j in $\mathcal{H}$ such that

$$\Xi_B(a) = \sum_j \phi(a)\xi_j\,, \quad a \in A\,,$$

and $\sum_j \|\phi_j\|^p \|\xi_j\|^p < z_p^p + \epsilon$. Define now normal functionals ψ_j on B by

$$\psi_j(b) = ((\Delta^{1/4} + \Delta^{-1/4})^{-1}(b^* + JbJ)\Omega|\xi_j)\,,$$

and note that $\|\psi_j\| \leq \|\xi_j\|$ because of the estimate $\|(\Delta^{1/4} + \Delta^{-1/4})^{-1}\| \leq 1/2$ and the spectral calculus. Putting both paragraphs together we find the conclusion. □

Corollary 88. *With the notation of the previous lemma, for every $\epsilon > 0$ we can write $\omega = (1+\lambda)\omega_+ - \lambda\omega_-$, where $\omega_\pm$ are separable states and $(1+\lambda)^p \leq 4(z_p^p + \epsilon)$.*

Proof. We decompose $\phi_j = \sum_{k=0}^3 (i)^\alpha \phi_j^\alpha$ and $\psi_j = \sum_{k=0}^3 (i)^\alpha \psi_j^\alpha$ in four positive normal functionals. One also has that $\|\phi_j^\alpha\| \leq \phi_j$ holds, and similarly for ψ_j. Since ω is positive, then after the identification $A \vee B \cong A \otimes B$ we find

$$\omega = \sum_j \sum_{\alpha=0}^{3} \phi_j^\alpha \otimes \psi_j^{4-\alpha} - \sum_j \sum_{\alpha=0}^{3} \phi_j^\alpha \otimes \psi_j^{2-\alpha},$$

namely ω is difference of two separable functionals. The thesis follows. □

Lemma 89. *With the hypotheses of Lemma 87, assume ω to have an expression like in* (4.11) *and assume $\mu_p = \sum_j \|\phi_j\|^p \|\psi_j\|^p$ to be finite for some $0 < p \leq 1$. Then there is a separable positive linear functional σ such that $\sigma \geq \omega$ on $A \vee B$ and $\|\sigma\|^p = \mu_1^p \leq \mu_p$.*

Proof. We follow [52, Lemma 4]. By polar decomposition there are partial isometries u_j in A such that $\phi(u_j \cdot) \geq 0$ on A and $\phi_j(u_j u_j^* \cdot) = \phi_j$. It follows in particular that $\phi_j(u_j) = \|\phi_j\|$ and

$$\bar{\phi}_j(a) = \overline{\phi_j(u_j u_j^* a^*)} = \phi_j(u_j(u_j^* a^*)^*) = \phi_j(u_j a u_j)$$

for all a in A, where we used the fact that $\phi_j(u_j \cdot)$ is hermitian (here $\bar{\psi}(a) = \overline{\psi(a^*)}$). Similarly, there are partial isometries v_j in B such that $\psi_j(v_j \cdot) \geq 0$ and $\psi_j(v_j v_j^* \cdot) = \psi_j$. Note that the positive linear functional $\rho_j = \phi_j(u_j \cdot) \otimes \psi_j(v_j \cdot)$ is separable. Writing $w_j = u_j \otimes v_j$ we then define

$$\sigma_j(\cdot) = \frac{1}{2}\rho_j(\cdot) + \frac{1}{2}\rho_j(w^* \cdot w),$$

which is also separable, because w is a simple tensor product. Furthermore,

$$\|\sigma_j\| = \sigma_j(1) = \rho_j(1) = \|\phi_j\| \|\psi_j\|,$$

and also

$$0 \leq \frac{1}{2}\rho_j((1-w^*) \cdot (1-w)) = \sigma_j - \frac{1}{2}(\phi_j \otimes \psi_j + \bar{\phi}_j \otimes \bar{\psi}_j).$$

We conclude that $\sigma = \sum_j \sigma_j$ is a separable positive linear functional with

$$\sigma \geq \frac{1}{2}\sum_j (\phi_j \otimes \psi_j + \bar{\phi}_j \otimes \bar{\psi}_j) = \frac{1}{2}(\omega + \overline{\omega}) = \omega.$$

and $\|\sigma\|^p = \left(\sum_j \|\sigma_j\|\right)^p \leq \sum_j \|\sigma_j\|^p = \mu_p$. □

Remark 90. Notice that, by the previous lemma, we have $\nu_p \geq 1$.

Theorem 91. *Let (A, B) be a split pair of hyperfinite factors. Assume the p-partition function* (4.10) *to be finite for some $0 < p < 1$. Then the mutual information is finite, with*

$$E_I(\omega) \leq c_p z_p + \eta(z_p - 1) - \eta(z_p), \tag{4.12}$$

where $c_p = \frac{1}{(1-p)e}$ and $\eta(t) = -t \ln t$.

Proof. We begin the proof with a general remark. Consider a state ω on $A \otimes B$, with A and B finite dimensional type I factors. If $\omega = \sum_j \lambda_j \omega_j$ is a convex decomposition in states, then by (4.6) we have

$$E_I(\omega) \geq \sum_j \lambda_j E_I(\omega_j) - \sum_j \eta(\lambda_j)\,.$$

By monotonicity of the relative entropy, the same expression holds if A and B are hyperfinite factors. Therefore, if (A, B) is a split pair as in the statement, then by the previous lemmas for every $\epsilon > 0$ we have a separable state $\sigma \geq \omega$ such that $\|\sigma\|^p \leq z_p^p + \epsilon$. By setting $\hat{\sigma} = \sigma/\|\sigma\|$ we can write $\omega = \|\sigma\|\hat{\sigma} - \|\tau\|\hat{\tau}$ and notice that

$$\|\sigma\| E_I(\hat{\sigma}) \geq E_I(\omega) + \eta(\|\sigma\|) - \eta(\|\sigma\| - 1)\,,$$

where we used the positivity property $E_I(\hat{\tau}) \geq 0$. The claimed estimate follows by the inequalities $\eta(t) \leq c_p t^p$ for $p < 1$, $E_I(\hat{\sigma}) \leq \|\sigma\|^{-p} c_p(z_p^p + \epsilon)$, and the monotonicity of $\eta(s-1) - \eta(s)$ for $s \geq 1$. □

Remark 92. Due to the general inequality $S(\varphi\|\omega) \geq \|\varphi - \omega\|^2/2$ [82], we can use the previous result to estimate the distance between the states ω and $\omega \otimes \omega$. See [75] for related issues concerning the split property.

We now follow [83] and we show the finiteness of some "tailored" entanglement entropy under the assumption of modular p-nuclearity for some $0 < p < 1$. The result is not strong enough in order to prove the Conjecture 1, but we find of interest to point out a different approach.

Definition 93. [83] Let (A, B) be a split pair of von Neumann algebras on a Hilbert space $\mathcal{H}$. Assume A, B and $A' \cap B$ to be in standard form with respect to some vector Ω. If $u\colon \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ is a unitary implementing the natural isomorphism $A \vee B' \cong A \otimes B'$, then we will denote by $R_u = u^{-1}(B(\mathcal{H}) \otimes \mathbb{1})u$ the corresponding type I factor. We define an *intermediate pair* any such pair (u, R_u).

Definition 94. [83] Let (A, B) be a split pair of von Neumann algebras as in the previous definition. Given a state ψ on $B(\mathcal{H})$, we call the *Otani's entanglement entropy* of ψ the functional

$$OH(\psi) = \sup_{(u,R_u)} \inf_{\phi,\lambda} \frac{1}{\lambda} S(\phi)\,,$$

where supremum is over all intermediate pairs, the infimum is over all states ϕ on $B(\mathcal{H})$ and real numbers $0 < \lambda \leq 1$ such that $\phi \geq \lambda\psi$ on $A \vee B'$ and $S(\phi)$ is the von Neumann entropy of ϕ on the intermediate type I factor R_u.

Theorem 95. *Let $A \subseteq B$ be a standard split inclusion w.r.t. some vector Ω. Denote by z_p the p-partition function* (4.10)*. If z_p is finite for some $0 < p < 1$, then the Otani's entanglement entropy is finite. Explicitly,*

$$OH(\omega) \leq z_p \ln z_p + c_p z_p^p\,. \tag{4.13}$$

Proof. The proof consists of a computation that does not depend on the choice of the intermediate pair, which is therefore implicit in what follows. We will identify A with $A \otimes \mathbb{1}$ and B with $B(\mathcal{H}) \otimes B$. Lemma 89 gives a separable dominating normal functional $\sigma \geq \omega$ with $\|\sigma\|^p \leq z_p^p + \epsilon$ for $\epsilon > 0$ arbitrarily small. We utilize the separability of $\hat{\sigma} = \sigma/\|\sigma\|$ over the bipartite system $A \vee B' \cong A \otimes B'$ and decompose it into positive, normal functionals, say $\hat{\sigma} = \sum_j \phi_j \otimes \psi_j$. Without loss of generality we can assume ϕ_j to be states on A. Now we notice that $\phi_j \otimes \psi_j$ is a normal positive functional on $A \otimes B'$, hence it can be extended by taking a representative vectors. Since such extension has same norm, we can extend σ to a separable positive functional on $B(\mathcal{H}) \otimes B(\mathcal{H})$ in such a way that still $\|\sigma\|^p \leq z_p^p + \epsilon$. We introduce some further notation by setting $\eta(t) = -t \ln t$ and $1/c_p = (1-p)e$. Therefore, we have

$$\|\sigma\| S(\hat{\sigma}) \leq \|\sigma\| \ln \|\sigma\| + \sum_j \eta(\|\psi_j\|) \leq \|\sigma\| \ln \|\sigma\| + c_p \sum_j \|\psi_j\|^p \,,$$

and the claimed estimate follows by the arbitrarity of σ. □

4.4 Application to local quantum field theory

Definition 96. A local quantum field theory $(\mathcal{A}, U, \Omega)$ on the Minkowski space is said to satisfy the *split property (for double cones)* if the inclusion $\mathcal{A}(\mathcal{O}_1) \subseteq \mathcal{A}(\mathcal{O}_2)$ is split whenever $\mathcal{O}_1 \subset \mathcal{O}_2$ is an inclusion of double cones such that $\overline{\mathcal{O}_1} \subset \mathcal{O}_2$.

As mentioned in Section 4.1, the split property ensures some statistical independence property of the considered model. The split property does not hold for unbouded regions like wedges in more than two spacetime dimensions [17, 60].

In the literature there are several criteria which are known to imply the split property. Many of them are formulated in terms of nuclear maps, and are therefore formulated as "nuclearity conditions". The first nuclearity condition was invented as some additional requirement which ensures the theory to have a particle interpretation. In particular, a theory complying with such a requirement should exhibit the thermodynamical behavior expected from a theory of particles. In order to formulate this condition within the theory of a local net on a d-dimensional Minkowski space, one considers a region $\mathcal{O} \subseteq \mathbb{R}^d$ and a parameter $\beta > 0$ representing the inverse temperature. In analogy with the form of Gibbs equilibrium states in statistical mechanics, one defines the map

$$\Theta_{\beta,\mathcal{O}} \colon \mathcal{A}(\mathcal{O}) \to \mathcal{H} \,, \quad \Theta_{\beta,\mathcal{O}}(A) = e^{-\beta H} A \Omega \,, \tag{4.14}$$

where $H = P_0$ denotes the Hamiltonian with respect to the time direction x_0.

Definition 97. A local quantum field theory on the Minkowski space $\mathbb{R}^d$ is said to satisfy the *energy nuclearity condition* if the maps (4.14) are nuclear for any bounded region $\mathcal{O}$ and any inverse temperature $\beta > 0$. Moreover, there must exist constants $\beta_0, n > 0$ depending on $\mathcal{O}$ such that the nuclear norm of $\Theta_{\beta,\mathcal{O}}$ is bounded by

$$\|\Theta_{\beta,\mathcal{O}}\|_1 \leq e^{(\beta_0/\beta)^n} \,, \quad \beta \to 0 \,.$$

Definition 98. A local quantum field theory on the Minkowski space is said to satisfy the *modular nuclearity condition (for double cones)* if the inclusion $\mathcal{A}(\mathcal{O}_1) \subseteq \mathcal{A}(\mathcal{O}_2)$ is modular nuclear whenever $\mathcal{O}_1 \subset \mathcal{O}_2$ is an inclusion of double cones such that $\overline{\mathcal{O}_1} \subset \mathcal{O}_2$.

A modular nuclear inclusion of factors is split, and a split inclusion of factors implies the compactness of the map (4.1) [19]. As in Section (4.1), the previous nuclearity conditions can be rephrased as *modular p-nuclearity conditions* by requiring to the considered maps to be of type l^p for some $0 < p \leq 1$. Modular p-nuclearity has been proved in the theory of a scalar free field for any $p > 0$ [61] and holds on conformal nets satisfying the trace class property [21].

It has been shown in [20] that in application to the local algebras of a quantum field theory, the energy and the modularity conditions are essentially equivalent. However, the argument used for the equivalence of the two conditions breaks down when applied to local algebras associated to unbounded regions. In particular, this opens up the possibility for the modular nuclearity condition to hold for inclusions of wedge algebras in two spacetime dimensions [60].

More specifically, it has been shown in [60] how to construct a wide class of integrable models in 1+1 dimensional Minkowski space by proving the modular nuclearity condition for wedge inclusions. The only input in this construction, apart from the value $m > 0$ of the mass of the basic particle, is a 2-body scattering matrix. Here we review the structure and some property of such models.

Definition 99. A *(2-body) scattering function* is an analytic function $S_2 \colon S(0, \pi) \to \mathbb{C}$ which is bounded and continuous on the closure of this strip and satisfies the equations

$$\overline{S_2(\theta)} = S_2(\theta)^{-1} = S_2(-\theta) = S_2(\theta + i\pi)\,, \quad \theta \in \mathbb{R}\,.$$

The set of all the scattering functions will be denoted by $\mathcal{S}$. For S_2 in $\mathcal{S}$, we define

$$\kappa(S_2) = \inf\{\operatorname{Im}\zeta \colon \zeta \in S(0, \pi/2)\,, \quad S_2(\zeta) = 0\}\,.$$

The subfamily $\mathcal{S}_0 \subset \mathcal{S}$ consists of those scattering functions S_2 with $\kappa(S_2) > 0$ and for which

$$\|S_2\|_\kappa = \sup\left\{|S_2(\zeta)| \colon \zeta \in \overline{S(-\kappa, \pi + \kappa)}\right\} < +\infty\,, \quad \kappa \in (0, \kappa(S_2))\,.$$

The families of scattering functions $\mathcal{S}$ and $\mathcal{S}_0$ can then be divided into "bosonic" and "fermionic" classes according to

$$\begin{aligned} \mathcal{S}^{\pm} &= \left\{S_2 \in \mathcal{S} \colon S_2(0) = \pm 1\right\}, \quad \mathcal{S} = \mathcal{S}^+ \cup \mathcal{S}^-\,, \\ \mathcal{S}_0^{\pm} &= \left\{S_2 \in \mathcal{S}_0 \colon S_2(0) = \pm 1\right\}, \quad \mathcal{S} = \mathcal{S}_0^+ \cup \mathcal{S}_0^-\,. \end{aligned}$$

The full S-matrix of an interacting quantum field theory is a very complicated object. Indeed, if the structure of the two-particle S-matrix has been studied extensively, much less is known about the higher S-matrix elements $S_{n,m}$ with $n, m > 2$. However, the interesting point in two-spacetime dimensions is that there do exist S-matrices, called *factorizing S-matrices*, which are completely determined by the two-particle S-matrix.

A factorizing S-matricx has the two following properties: $S_{n,m}$ vanishes for $n \neq m$ and $S_n = S_{n,n}$ factorizes into a product of several two-particle S-matrices. We now briefly describe the structure of these models.

As for free fields, the single particle space $\mathcal{H}_1$ of the theory can be identified with the space $L^2(\mathbb{R}, d\mu)$ of square integrable momentum wavefunctions on the upper mass shell $H_m^+ = \{((p^2+m^2)^{1/2}, p) \colon p \in \mathbb{R}\}$, where $d\mu(p) = (p^2+m^2)^{-1/2}dp$ is the usual Lorentz invariant measure. However, on two-dimensional Minkowski space it is more convenient to use as a variable the *rapidity* instead of the momentum. The rapidity $\theta \in \mathbb{R}$ is a particular parametrization of the upper mass shell H_m^+ given by $p(\theta) = m(\cosh\theta, \sinh\theta)$. In the rapidity space, the single particle space is $\mathcal{H}_1 = L^2(\mathbb{R}, d\theta)$.

The construction of the net $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$ corresponding to a given S_2 matrix starts by considering an "S_2-symmetric" Fock-space over $\mathcal{H}_1 = L^2(\mathbb{R}, d\theta)$. This Fock space is a direct sum $\mathcal{H} = \mathbb{C}\Omega \oplus_{n\geq 1} \mathcal{H}_n$ of n-particle spaces. By contrast to the case of the bosonic Fock-space, $\mathcal{H}_n$ is not obtained by applying a symmetrization projection to $\mathcal{H}_1^{\otimes n}$. Rather, one applies a projection E based on S_2. For that, let τ_i be an elementary transposition of the elements i and $i+1$ in the symmetric group $\mathfrak{S}_n$. Define an exchange operator $D_n(\tau_i)$ on $\mathcal{H}_1^{\otimes n}$, identified with $L^2(\mathbb{R}^n, d^n\theta)$, as

$$(D_n(\tau_i)\Psi_n)(\theta_1, \dots, \theta_i, \theta_{i+1}, \dots, \theta_n) = S_2(\theta_{i+1} - \theta_i)\Psi(\theta_1, \dots, \theta_{i+1}, \theta_i, \dots, \theta_n)\,.$$

This exchange operator gives a unitary representation of $\mathfrak{S}_n$ on $\mathcal{H}_1^{\otimes n}$. Define an S_2-symmetric projection $E_n = (1/n!)\sum_{\sigma\in\mathfrak{S}_n} D_n(\sigma)$, define $\mathcal{H}_n = E_n\mathcal{H}_1^{\otimes n}$ and define $\mathcal{H} = \mathbb{C}\Omega \oplus_{n\geq 1} \mathcal{H}_n$. On this space one can define the *creation operator* $z^\dagger(\chi)$ and the *annihilation operator* $z(\chi)$, with χ in $\mathcal{H}_1$. Explicitly, for any $\Psi_n = E_n(\psi_1 \otimes \cdots \otimes \psi_n)$ one defines

$$\begin{aligned} z^\dagger(\chi)\Psi_n &= (n+1)^{1/2}E_{n+1}(\chi \otimes \Psi_n)\,, \\ z(\chi)\Psi_n &= n^{-1/2}\sum_{j=1}^{n}(\chi|\psi_j)E_{n-1}(\psi_1 \otimes \cdots \otimes \widehat{\psi_j} \otimes \cdots \otimes \psi_n)\,, \end{aligned}$$

where by definition $E_0\Omega = \Omega$ and $z(\chi)\Omega = 0$, with Ω a vector of unital norm called *vacuum vector*. These operators are closable and have a common core, namely the algebraic sum of the subspaces $\mathcal{H}_n = E_n\mathcal{H}_1^{\otimes n}$. Notice that $z^\dagger(\chi)$ is complex-linear in χ, while $z(\chi)$ is conjugate-linear in χ.

If we write informally $z^\dagger(\Psi) = \int d\theta\Psi(\theta)z^\dagger(\theta)$ and similarly for $z(\Psi)$, then these operators satisfy the relations of the *Zamolodchikov-Faddeev (ZF) algebra*, namely

$$z(\theta)z^\dagger(\theta') - S_2(\theta - \theta')z^\dagger(\theta')z(\theta) = \delta(\theta - \theta')\cdot 1\,, \quad z(\theta)z(\theta') - S_2(\theta' - \theta)z(\theta')z(\theta) = 0\,.$$

Furthermore, we can define an operator J on $\mathcal{H}$ by

$$(J\Psi)_n(\theta_1, \dots, \theta_n) = \overline{\Psi_n(\theta_n, \dots, \theta_1)}\,.$$

Consider now a Schwartz function $f \in \mathcal{S}(\mathbb{R}^2)$ and set

$$f^\pm(\theta) = \frac{1}{2\pi}\int dx f(\pm x)e^{ip(\theta)\cdot x}\,, \quad p(\theta) = m(\cosh\theta, \sinh\theta)\,.$$

The *field operators* $\phi(f)$ and $\phi'(f)$ are then defined as

$$\phi(f) = z^\dagger(f^+) + z(f^-)\,, \quad \phi'(f) = J\phi(f^*)J\,,$$

where $f^*(x) = \overline{f(-x)}$. These operators are closable and essentially self-adjoint if f is real valued. Furthermore, if f and g are Schwartz functions with f supported in the right wedge W_R given by $x_1 > |x_0|$ and g supported in the left wedge $W_L = W_R'$, then we have $[\phi'(f), \phi(g)]\Psi = 0$ for Ψ in a common dense core $\mathcal{D}$. Finally, all this construction gives rise to a local net $W \mapsto \mathcal{A}(W)$ of wedge algebras defined by

$$\begin{aligned}\mathcal{A}(W_L + x) &= \{e^{i\phi(f)} \colon f \in \mathcal{S}(W_L + x) \text{ real}\}''\,,\\ \mathcal{A}(W_R + x) &= \{e^{i\phi'(f)} \colon f \in \mathcal{S}(W_R + x) \text{ real}\}''\,.\end{aligned}$$

The algebra of observables localized in a double cone $\mathcal{O} = W_1 \cap W_2$ is defined as

$$\mathcal{A}(W_1 \cap W_2) = \mathcal{A}(W_1) \cap \mathcal{A}(W_2)\,,$$

and for arbitrary open regions $\mathcal{Q} \subseteq \mathbb{R}^2$ we put $\mathcal{A}(\mathcal{Q})$ as the von Neumann algebra generated by all the local algebras $\mathcal{A}(\mathcal{O})$ with $\mathcal{O} \subseteq \mathcal{Q}$. In [60], the author shows the above models with scattering function S_2 in $\mathcal{S}_0^-$ to be well-posed, namely they satisfy the basic axioms of algebraic QFT as long as the additivity property, the Reeh-Schlieder property, Haag duality and Bisognano-Wichmann. In particular, the proof relies on the following technical result.

Theorem 100. *[2, 60, 62] Let $(\mathcal{A}, U, \Omega)$ be an integrable quantum field theory on $\mathbb{R}^2$ given by some factorizing S-matrix $S_2 \in \mathcal{S}$. Define*

$$\Xi(s) \colon \mathcal{A}(W_R) \to \mathcal{H}\,, \quad \Xi(s)A = \Delta^{1/4}U(\mathbf{s})A\Omega\,, \quad s > 0\,, \tag{4.15}$$

where $U(\mathbf{s})$ is the unitary associated to the translation of $(s, 0)$ and Δ is the modular operator of $(\mathcal{A}(W_R), \Omega)$. If $S_2 \in \mathcal{S}_0^-$, then there is some splitting distance $s_{min} < \infty$ such that $\Xi(s)$ is p-nuclear for all $p > 0$ and $s > s_{min}$. Moreover, $\|\Xi(s)\|_p \to 1$ as $s \to +\infty$.

By this theorem it follows that Ω is cyclic for the local algebra associated to the double cone $\mathcal{O}_{a,b} = (W_R + a) \cap (W_L - b)$ if $b - a \in W_R$ and $-(b-a)^2 > s_{\min}$. In particular, local algebras associated to the double cones $\mathcal{O}_{a,b}$ mentioned above are hyperfinite III_1-factors in standard form. A proof of this result was originally given in [60], but due to some incorrect estimated a new proof has been provided in [2] and [62].

Conjecture 2. By studying carefully [2], [60] and [62], it should also follow that the maps

$$\Sigma(s) \colon \mathcal{A}(W_R)_{\mathrm{sa}}\Omega \to \mathcal{H}\,, \quad \Sigma(s)A\Omega = \Delta^{1/4}U(\mathbf{s})A\Omega\,, \quad s > 0\,,$$

are p-nuclear for all $p > 0$ and $s > s_{\min}$.

4.5 Conclusions

We close this chapter with some additional results that might be useful for future research in this area.

Theorem 91 can be applied in any local QFT satisfying the Reeh-Schlieder axiom by setting $A = \mathcal{A}(\mathcal{O})$ and $B = \mathcal{A}(\widetilde{\mathcal{O}}')$, with $\mathcal{O} \subset \widetilde{\mathcal{O}}$ an inclusion of double cones satisfying modular p-nuclearity condition for some $0 < p < 1$. We point out that we are not assuming Haag duality ([19], Lemma 2.4).

Corollary 101. *Let $(\mathcal{A}, U, \Omega)$ be a conformal covariant local QFT. Assume A and B to be local algebras associated to causally disjoint spacetime regions $\mathcal{S}_A$ and $\mathcal{S}_B$. Denote by d the distance between these two regions. Then, by the results of [52, Section 5] it follows that the mutual information is finite and satisfies lower bounds of area law type in the limit $d \to 0$.*

Due to the monotonicity of the relative entanglement entropy, one can also claim that the canonical entanglement entropy $E_C(\omega)$ satisfies lower bounds of area law type in the limit $d \to 0$ in conformal covariant local QFT. However, up to now a general proof of the finiteness of this entanglement measure is missing [72].

We now investigate a bit the asymptotic behaviour of these two different entanglement measures as $s \to +\infty$ in the setting of $1+1$-dimensional integrable models. If we denote by $E_R(s)$ the vacuum relative entropy of entanglement corresponding to the wedge inclusion mentioned in Theorem 100, then by Lemma 87 and Lemma 89 it is easy to notice that [52]

$$E_R(s) \leq \ln \|\Xi(s)\|_1 \to 0\,, \quad s \to +\infty\,.$$

However, if we denote by $E_I(s)$ the associated mutual information and we apply the estimate (4.12) then we can state at most that

$$\limsup_{s\to+\infty} E_C(s) \leq 1/e\,.$$

A similar remark can be done with the Otani's entanglement entropy by applying (4.13). About this point, the author is not sure if this depends on some nonoptimality of the provided bounds or rather on some intrinsically different behaviour of these entanglement measures.

The techniques of Section 4.3 rely on the presence of a separable state σ on the bipartite system $A \otimes B'$ that dominates ω. If F is an intermediate type I factor $A \subseteq F \subseteq B$ arising from the natural isomorphism $A \vee B' \cong A \otimes B'$ as in Definition 93, then it is possible to construct a separable functional on $B(\mathcal{H}) \otimes B(\mathcal{H}) \cong F \vee F'$ that dominates ω on F and on F' by use of generalized conditional expectations.

More specifically, in the situation on 4.3 one has two ω-preserving generalized conditional expectations, say ε and ε', induced by the inclusions $A \subseteq F$ and $B' \subseteq F'$

respectively. By the isomorphism $B(\mathcal{H}) \otimes B(\mathcal{H}) \cong F \vee F' = B(\mathcal{H})$ we can then define a map $\varepsilon \otimes \varepsilon'$ on $B(\mathcal{H})$ extending both ε and ε'. If σ is the dominating separable functional from Lemma 89, then $\sigma_0 = \sigma \cdot (\varepsilon \otimes \varepsilon')$ dominates $\omega_0 = \omega \cdot (\varepsilon \otimes \varepsilon')$. Notice that $\omega = \omega_0$ on F and on F', but in general not on $B(\mathcal{H})$. The functional $\sigma_0 = \sum_j \phi_j \cdot \varepsilon \otimes \psi_j \cdot \varepsilon'$ is separable with $\sum_j \|\phi_j \cdot \varepsilon\|^p \|\psi_j \cdot \varepsilon'\|^p = \mu_p$ finite (cf. Lemma 89 for notation), and in the notation of Theorem 91 we have

$$E_I(\omega_0) = S(\omega_0 \| \omega_F \otimes \omega_{F'}) \leq c_p \nu_p + \eta(\nu_p - 1) - \eta(\nu_p)\,,$$

where the r.h.s. is finite under the assumption of modular p-nuclearity for some $0 < p < 1$. Unfortunately, this does not imply the finiteness of the canonical entanglement entropy since ω_0 is not a pure state on $B(\mathcal{H})$. But we can make use of generalized conditional expectations to give an equivalent description of the canonical entanglement entropy. In particular, by use of equation (4.6) and Lemma 84 we can claim that

$$2E_C(\omega) = S_{B(\mathcal{H})}(\omega \| \omega_F \otimes \omega_{F'}) = 2S_{B(\mathcal{H})}(\omega_0) = 2H_{\omega_0}(F)\,,$$

with $H_{\omega_0}(F) = H_{\omega_0}^{B(\mathcal{H})}(F)$ the conditional entropy. The authors of [42] argued on grounds of physical arguments that

$$E_C(\omega) \approx E_I^{F \vee B'}(\omega) = S(\omega \| \omega_F \otimes \omega_{B'})\,,$$

and indeed it is reasonable to expect that the results of this chapter can be properly strengthened.

For example, Theorem 91 implies that $E_I^{F \vee B'}(\omega)$ is finite if the ω-preserving generalized conditional expectation from $F \vee B'$ onto $A \vee B'$ is a separable operation in the terminology of Definition 81. Another natural strategy could be that of estimating the entanglement entropy of some energy cutoff of the vacuum state like in [83] and then to operate some limit procedure. A different approach is the one of [73], in which the authors use the language of standard subspaces. Unfortunately, even if completely rigorous, this last work heavily depends on the structure of the free Fermi nets, and a generalization of it seems quite challenging up to now. In this context, the author expects the p-nuclearity of (4.3) with $0 < p < 1$ to be a good assumption to start with.

Appendix A

DR categories

The purpose of this appendix is to shed light on the mathematical structure of DR categories in order to define the statistical dimension [7, 40, 48, 81].

Consider a local QFT $(\mathcal{A}, U, \Omega)$ on the Minkowski space $\mathbb{R}^{n+1}$ satisfying Haag duality. With the notation of Section 1.7, $\mathcal{A}(F) = \mathfrak{A}(F)''$ and $\mathfrak{A} = \mathfrak{A}(\mathbb{R}^{n+1})$ denotes the C^*-algebra of quasi-local observables. Denote by Δ the semigroup of all the localized endomorphisms. An intertwiner T between two localized endomorphisms ρ and ρ', namely an operator T such that $\rho'(x)T = T\rho(x)$ for x in $\mathfrak{A}$, will be denoted by $\mathbf{T} = (\rho'|T|\rho)$. The hom-set of intertwiners of this type will be denoted by (ρ', ρ). If ρ and ρ' are localized in a double cone $\mathcal{O}$, then T belongs to $\mathcal{A}(\mathcal{O}')' = \mathcal{A}(\mathcal{O})$. If T is also unitary then ρ and ρ' belong to the same sector (see Definition 29). We can then define *conjugation*, *product* and *crossed product* of intertwiners via

$$(\rho'|T|\rho)^* = (\rho|T^*|\rho')\,,$$
$$(\rho''|T_2|\rho') \circ (\rho'|T_1|\rho) = (\rho''|T_2T_1|\rho)\,,$$
$$(\rho_2'|T_2|\rho_2) \times (\rho_1'|T_1|\rho_1) = (\rho_2'\rho_1'|T_2\rho_2(T_1)|\rho_2\rho_1)\,.$$

By a simple calculation we have the following properties:

$$\mathbf{T}_3 \times (\mathbf{T}_2 \times \mathbf{T}_1) = (\mathbf{T}_3 \times \mathbf{T}_2) \times \mathbf{T}_1\,,$$
$$(\mathbf{T}_2 \times \mathbf{T}_1)^* = \mathbf{T}_2^* \times \mathbf{T}_1^*\,,$$
$$(\mathbf{T}_2 \circ \mathbf{T}_1)^* = \mathbf{T}_1^* \circ \mathbf{T}_2^*\,,$$
$$(\mathbf{T}_2' \circ \mathbf{T}_2) \times (\mathbf{T}_1' \circ \mathbf{T}_1) = (\mathbf{T}_2' \times \mathbf{T}_1') \circ (\mathbf{T}_2 \times \mathbf{T}_1)\,.$$

If a support of ρ_1 is spacelike to a support of ρ_2 and a support of ρ_1' is spacelike to a support of ρ_2', then $\mathbf{T}_1 = (\rho_1'|T_1|\rho_1)$ and $\mathbf{T}_1 = (\rho_2'|T_2|\rho_2)$ are said to be *causally disjoint*. If this is the case, then

$$\mathbf{T}_1 \times \mathbf{T}_2 = \mathbf{T}_2 \times \mathbf{T}_1\,.$$

We now discuss permutations of n excitations. For $\rho_1, \dots, \rho_n$ in Δ choose $\tilde{\rho}_j$ equivalent to ρ_j such that the closures of the supports of $\tilde{\rho}_j$ lie spacelike to each other for different indices j and fix an intertwiner $\mathbf{U}_j = (\tilde{\rho}_j|U_j|\rho_j)$ for each ρ_j and $\tilde{\rho}_j$. Let $\mathfrak{S}_n$ be the permutation group of n elements, and let e be its unit element. Given p in $\mathfrak{S}_n$, we write

$$\mathbf{U}(p) = \mathbf{U}_{p^{-1}(1)} \times \cdots \times \mathbf{U}_{p^{-1}(n)}\,.$$

Since supports of $\tilde{\rho}_j$ lie mutually spacelike, we obtain

$$\tilde{\rho}_{p^{-1}(1)} \cdots \tilde{\rho}_{p^{-1}(n)} = \tilde{\rho}_1 \cdots \tilde{\rho}_n \,.$$

Thus, we can define an intertwiner of the form

$$\boldsymbol{\varepsilon}_p = (\rho_{p^{-1}(1)} \cdots \rho_{p^{-1}(n)} | \varepsilon_p | \rho_1 \dots \rho_n)$$

by the product

$$\boldsymbol{\varepsilon}_p(\rho_1 \cdots \rho_n) = \mathbf{U}(p)^* \circ \mathbf{U}(e) \,. \tag{A.1}$$

Theorem 102. *[7] The intertwiner $\boldsymbol{\varepsilon}_p$ is well defined, that is it depends neither on $\tilde{\rho}_j$ nor on $\mathbf{U}_j$. Moreover, it has the following properties:*

(i) If the supports of the ρ_j have spacelike closures, then

$$\boldsymbol{\varepsilon}_p(\rho_1 \cdots \rho_n) = \mathbf{1} \,.$$

(ii) For each p, q in $\mathfrak{S}_n$ it holds

$$\boldsymbol{\varepsilon}_q(\rho_{p^{-1}(1)} \cdots \rho_{p^{-1}(n)}) \circ \boldsymbol{\varepsilon}_p(\rho_1 \cdots \rho_n) = \boldsymbol{\varepsilon}_{pq}(\rho_1 \cdots \rho_n) \,.$$

(iii) Given $m < n$, let τ_m be the permutation of m and $m+1$. Then

$$\boldsymbol{\varepsilon}_{\tau_m}(\rho_1 \cdots \rho_n) = \mathbf{1}_{\rho_1} \times \cdots \times \mathbf{1}_{\rho_{m-1}} \times \boldsymbol{\varepsilon}_{\tau_m}(\rho_m \rho_{m+1}) \times \mathbf{1}_{\rho_{m+1}} \times \cdots \times \mathbf{1}_{\rho_n} \,.$$

Notice that given n intertwiners $\mathbf{T}_j = (\rho'_j | T_j | \rho_j)$, we can define as before the permutated crossed product

$$\mathbf{T}(p) = \mathbf{T}_{p^{-1}(1)} \times \cdots \times \mathbf{T}_{p^{-1}(n)} \,.$$

Clearly if ρ'_j and ρ_j are two collections of endomorphisms with mutually spacelike supports, then we obtain $\mathbf{T}(p) = \mathbf{T}(e)$. More in general, the following formula holds:

$$\mathbf{T}(p) \circ \boldsymbol{\varepsilon}_p(\rho_1 \cdots \rho_n) = \boldsymbol{\varepsilon}_p(\rho'_1 \cdots \rho'_n) \circ \mathbf{T}(e) \,.$$

For the special case $\rho_1 = \cdots = \rho_n = \rho$, we write

$$\boldsymbol{\varepsilon}_p(\rho \cdots \rho) = \boldsymbol{\varepsilon}_\rho^{(n)}(p) = (\rho^n | \varepsilon_\rho^{(n)}(p) | \rho^n) \,,$$

and call $\varepsilon_\rho^{(n)}(p)$ the *permutation operator*.

Corollary 103. *The map $p \mapsto \varepsilon_\rho^{(n)}(p)$ is a unitary representation of the permutation group $\mathfrak{S}_n$ and its equivalence class is determined solely by the equivalence class of ρ. Moreover, $\varepsilon_\rho^{(n)}(p)$ belongs to $\rho^n(\mathcal{A})'$ for each p.*

The permutation represented by $\varepsilon_\rho^{(n)}$ can be interpreted as a permutation of n spacelike separated excitations of the same kind. In particular, if $\mathfrak{S}_2 = \{e, \tau\}$ then the operator

$$\varepsilon_\rho = \varepsilon_\rho^{(2)}(\tau) \,,$$

called the *statistics operator*, is of interest. Its properties are described in [7]. We now give a couple of definitions. We define a *left inverse* of a localized endomorphism ρ as a positive linear map ϕ from $\mathfrak{A}$ into itself such that $\phi(x\rho(y)) = \phi(x)y$ and $\phi(\rho(x)y) = x\phi(y)$ for any x and y in $\mathfrak{A}$. Furthermore, we denote by Δ^{irr} the family of all the irreducible localized endomorphisms.

Theorem 104. *[7] Given ρ in Δ^{irr}, let ϕ be a left inverse of it. Then*
(i) $\phi(\varepsilon_\rho) = \lambda\mathbb{1}$ for some $\lambda \in \mathbb{C}$.
(ii) λ is zero or $\pm d^{-1}$, where d is a natural number.
(iii) The value of λ uniquely determines the sector $[\rho]$.

The parameter $\lambda = \lambda_\rho$ of the previous theorem is called the *statistical parameter* of the sector $[\rho]$. Such a parameter describes the behavior under permutations of a number of excitations $[\rho]$ in vacuum. The number $d = d(\rho)$ is called the *statistical dimension* of the sector $[\rho]$, with the convention that $d = \infty$ if $\lambda = 0$. If $d(\rho)$ is finite then we say that $[\rho]$ has *finite statistics*, otherwise we say that $[\rho]$ has *infinite statistics*.

The above analysis as been described for the case of irreducible ρ in Δ^{irr}. For a general $\rho \in \Delta$, there exists a φ among the left inverses ϕ of ρ such that $\phi(\varepsilon_\rho)^2$ is a constant multiple of the identity operator, and this ϕ is called a *standard left inverse*. For a standard left inverse ϕ, ρ is said to have *infinite statistics* if $\phi(\varepsilon_\rho) = 0$ and *finite statistics* otherwise. A necessary and sufficient condition for ρ to have finite statistics is that it has a decomposition as a direct sum of a finite number of irreducible $\rho_i \in \Delta^{\mathrm{irr}}$ with finite statistics [7, 48].

Theorem 105. *[7] For $\rho \in \Delta^{\mathrm{irr}}$ with finite statistics, there is $\bar\rho \in \Delta^{\mathrm{irr}}$ with finite statistics such that $\bar\rho\rho$ contains the vacuum representation ι, and $[\bar\rho]$ is uniquely determined by $[\rho]$. In this case, $\bar\rho\rho$ contains ι with multiplicity 1 and $\lambda_\rho = \lambda_{\bar\rho}$.*

The sector $[\bar\rho]$ in this theorem is called the *charge conjugate sector* of $[\rho]$. Since $[\bar\rho\rho] = [\rho\bar\rho]$, from the uniqueness of charge conjugate sector we have $[\bar{\bar\rho}] = [\rho]$.

Now recall that, in our notation, the *superselection theory* of $\mathcal{A}$ is the category $\mathcal{T}_\mathcal{A}$ with localized endomorphisms as objects and intertwiners as arrows. With this last theorem, we have finally described the main properties of the superselection theory with finite statistics $\mathcal{T}_\mathcal{A}^{\mathrm{fin}}$, namely the full subcategory of $\mathcal{T}_\mathcal{A}$ whose objects are the localized endomorphisms with finite statistics. More in general, these properties can be formulated in the language of category theory.

Definition 106. A category $\mathcal{C}$ is a *C^*-category* if:
(i) each hom-set is a complex Banach space such that the composition of morphisms $(S,T) \to ST$ is a bilinear map with $\|ST\| \leq \|S\|\|T\|$,
(ii) there exists an antilinear contravariant functor $^*\colon \mathcal{T} \to \mathcal{T}$ which is the identity map on objects and such that $T^{**} = T$ for each morphism T. Furthermore, it is required that $\|T^*T\| = \|T\|^2$ for each morphism T, and in particular $\mathrm{End}(U) = \mathrm{Mor}(U,U)$ is a unital C^*-algebra.

Using the *-operation we can define notions of projection, unitary, partial isometry, etc., for morphisms. For example, a morphism u is called *unitary* if $u^*u = 1$. A C^*-algebra is a C^*-category with a single object. A first example of C^*-category is clearly **Hilb**, the category of Hilbert spaces with bounded linear operators as morphisms. One fact about C^*-categories is that, as every C^*-algebra can be faithfully represented on a Hilbert space, for every C^*-category there is a faithful functor on **Hilb**.

Definition 107. A C^*-category is called a *C^*-tensor category*, or also a *monoidal C^*-category*, if in addition are given a bilinear bifunctor $\otimes\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ and natural unitary isomorphisms

$$\alpha_{U,V,W}\colon (U\otimes V)\otimes W \to U\otimes(V\otimes W)$$

called *associativity morphisms*, an object $\mathbb{1}$, called *unit object*, and natural unitary isomorphisms

$$\lambda_U\colon \mathbb{1}\otimes U\to U\,,\quad \rho_U\colon U\otimes\mathbb{1}\to U\,,$$

such that

(iii) the *pentagonal diagram*

$$\begin{tikzcd}
 & ((U\otimes V)\otimes W)\otimes X \arrow[dl, "\alpha\otimes\iota"'] \arrow[dr, "\alpha_{12,3,4}"] & \\
(U\otimes(V\otimes W))\otimes X \arrow[d, "\alpha_{1,23,4}"'] & & (U\otimes V)\otimes(W\otimes X) \arrow[d, "\alpha_{1,2,34}"] \\
U\otimes((V\otimes W)\otimes X) \arrow[rr, "\iota\otimes\alpha"] & & U\otimes(V\otimes(W\otimes X))
\end{tikzcd}$$

commutes, where the leg-numbering notation for the associativity morphisms stands for $\alpha_{12,3,4}=\alpha_{U\otimes V,W,X}$ etc.,

(iv) $\lambda_{\mathbb{1}}=\rho_{\mathbb{1}}$, and

$$\begin{tikzcd}
(U\otimes\mathbb{1})\otimes V \arrow[rr, "\alpha"] \arrow[dr, "\rho\otimes\iota"'] & & U\otimes(\mathbb{1}\otimes V) \arrow[dl, "\iota\otimes\lambda"] \\
 & U\otimes V &
\end{tikzcd}$$

(v) $(S\otimes T)^*=S^*\otimes T^*$ for any morphisms S and T.

Definition 108. Let $\mathcal{C}$ be a monoidal C^*-category. We will say that

(vi) the category $\mathcal{C}$ is *closed under finite direct sums* if for any objects U and V there exist an object W and isometries $u\in\mathrm{Mor}(U,W)$ and $v\in\mathrm{Mor}(V,W)$ such that $uu^*+vv^*=1$.

(vii) the category $\mathcal{C}$ is *closed under subobjects* if for any projection p in $\mathrm{End}(U)$ there exist an object V and an isometry $v\in\mathrm{Mor}(V,U)$ such that $vv^*=p$. Note that an object defined by the zero projection $0\in\mathrm{End}(U)$ is a *zero object*, that is an object $\mathbf{0}$ such that $\mathrm{Mor}(\mathbf{0},W)=0$ and $\mathrm{Mor}(W,\mathbf{0})=0$ for any W,

(viii) the unit object $\mathbb{1}$ is *simple*, that is $\mathrm{End}(\mathbb{1})=\mathbb{C}$,

(ix) the category $\mathcal{C}$ is *strict* if $(U\otimes V)\otimes W=U\otimes(V\otimes W)$, $\mathbb{1}\otimes U=U\otimes\mathbb{1}=U$, and α, λ and ρ are the identity morphisms.

Definition 109. A *braiding* on a C^*-tensor category $\mathcal{C}$ is a collection of natural isomorphisms $\sigma_{U,V}\colon U\otimes V\to V\otimes U$ such that the *hexagon diagram*

$$\begin{tikzcd}
& (U \otimes V) \otimes W \arrow[dl, "\sigma\otimes\iota"'] \arrow[dr, "\alpha"] & \\
(V \otimes U) \otimes W \arrow[d, "\alpha"'] & & U \otimes (V \otimes W) \arrow[d, "\sigma_{1,23}"] \\
V \otimes (U \otimes W) \arrow[dr, "\iota\otimes\sigma"'] & & (V \otimes W) \otimes U \arrow[dl, "\alpha"] \\
& V \otimes (W \otimes U) &
\end{tikzcd}$$

and the same diagram with σ replace by σ^{-1} both commute. When a braiding is fixed, the category $\mathcal{C}$ is called *braided.* If in addition $\sigma^2 = \iota$, then $\mathcal{C}$ is called *symmetric.*

Definition 110. Let $\mathcal{C}$ be a strict C^*-tensor category. An object $\bar{U}$ is said to be *conjugate* to an object U in $\mathcal{C}$ if there exist morphisms $R\colon \mathbb{1} \to \bar{U} \otimes U$ and $\bar{R}\colon \mathbb{1} \to U \otimes \bar{U}$ such that the *conjugate equations*

$$(\bar{R}^* \otimes \iota)(\iota \otimes R) = \iota\,, \quad (R^* \otimes \iota)(\iota \otimes \bar{R}) = \iota\,,$$

are satisfied. If every object has a conjugate, then $\mathcal{C}$ is said to be a *rigid* C^*-tensor category, or a C^*-tensor category *with conjugates.*

Note that this definition is symmetric in U and $\bar{U}$, so U is conjugate to $\bar{U}$.

Proposition 111. *[81] For any object U in $\mathcal{C}$ a conjugate object, if it exists, is uniquely determined up to an isomorphism. More precisely, if $(R, \bar{R})$ is a solution of the conjugate equations for U and $\bar{U}$, and $(R', \bar{R}')$ is a solution of the conjugate equations for U and $\bar{U}'$, then*

$$T = (\iota_{\bar{U}} \otimes \bar{R}'^*)(R \otimes \iota_{\bar{U}'}) \in Mor(\bar{U}', \bar{U})$$

is invertible with inverse $S = (\iota_{\bar{U}'} \otimes \bar{R}^*)(R' \otimes \iota_{\bar{U}})$, and

$$R' = (T^{-1} \otimes \iota)R\,, \quad \bar{R}' = (\iota \otimes T^*)\bar{R}\,.$$

As a corollary, if U is a simple object, $\bar{U}$ is a conjugate object to U and $(R, \bar{R})$ is a solution of the conjugate equations for U and $\bar{U}$, then any other solution has form $R' = \bar{\lambda} R$ and $\bar{R}' = \lambda^{-1}\bar{R}$ for some $\lambda \in \mathbb{C}^*$. In particular, $\|R\| \cdot \|\bar{R}\|$ is independent of the solution.

Definition 112. Let $\mathcal{C}$ be a strict C^*-tensor category. If U is a simple object with a conjugate $\bar{U}$, the number

$$d_i(U) = \|R\| \cdot \|\bar{R}\|$$

is called the *intrinsic dimension* of U. For a general U admitting a conjugate object $\bar{U}$, decompose U into a direct sum of simple objects $U = \oplus_k U_k$ and put $d_i(U) = \sum_k d_i(U_k)$.

Note that $d_i(\mathbb{1}) = 1$, hence the intrinsic dimension is always a natural number. If an object U has a conjugate, then $\mathrm{End}(U)$ is finite dimensional (Proposition 2.2.8. of [81]). In particular, every such object can be decomposed into a finite direct sum of simple objects, so the above definition is well posed. The class of objects of $\mathcal{C}$ that have conjugates form a C^*-tensor subcategory of $\mathcal{C}$. The intrinsic dimension for nonsimple objects can be expressed in terms of solutions of the conjugate equations. In order to see this, decompose an object U into a direct sum of simple objects, say $U = \oplus_k U_k$. This decompositions means that we have isometries $w_k \in \mathrm{Mor}(U_k, U)$ such that $\sum_k w_k^* w_k = 1$. For every k choose a conjugate $\bar{U}_k$ to U_k. Let $\bar{U}$ be the direct sum of the $\bar{U}_k$, and let $\bar{w}_k \in \mathrm{Mor}(\bar{U}_k, \bar{U})$ be the corresponding isometries. If $(R_k, \bar{R}_k)$ are solutions of the conjugate equations for U_k and $\bar{U}_k$, then $R = \sum_k (\bar{w}_k \otimes w_k) R_k$ and $\bar{R} = \sum_k (\bar{w}_k \otimes w_k) \bar{R}_k$ is a solution of the conjugate equations for U and $\bar{U}$.

Definition 113. A solution of the conjugate equation for U and $\bar{U}$ of the form

$$R = \sum_k (\bar{w}_k \otimes w_k) R_k \,, \quad \bar{R} = \sum_k (\bar{w}_k \otimes w_k) \bar{R}_k \,,$$

with U_k simple and $\|R_k\| = \|\bar{R}_k\| = d_i(U_k)^{1/2}$ for all k, is called *standard.*

The standard solution is unique in the following natural sense. If $(R, \bar{R})$ and $(R', \bar{R}')$ are standard solutions of the conjugate equations for $(U, \bar{U})$ and $(U, \bar{U}')$ respectively, then there exists a unitary $T \in \mathrm{Mor}(\bar{U}, \bar{U}')$ such that $R' = (T \otimes \iota) R$ and $\bar{R}' = (T \otimes \iota) \bar{R}$.

Definition 114. Let $\mathcal{C}$ be a strict C^*-tensor category with conjugates. A *dimension function* on $\mathcal{C}$ is a nonnegative number $d(U)$ to every object U in $\mathcal{C}$ such that $d(U) > 0$ if U is nonzero, $d(U) = d(V)$ if $U \cong V$,

$$d(U \oplus V) = d(U) + d(V) \,, \quad d(U \otimes V) = d(U) d(V) \,, \ \text{ and } \ d(\bar{U}) = d(U) \,.$$

Note that since $\mathbb{1} = \mathbb{1} \otimes \mathbb{1}$ and $\mathbb{1}$ is a subobject of $U \otimes \bar{U}$ for every nonzero U, we automatically have $d(1) = 1$ and $d(U) \geq 1$. The intrinsic dimension is an example of dimension function. Finally, we now have all the ingredients in order to provide the definition which names this appendix.

Definition 115. A strict symmetric monoidal C^*-category which is closed with respect direct sums and subobjects, rigid and with simple unit object is called a *DR-category.*

The basic example of DR-category is $\mathcal{T}_G$, the category of finite-dimensional, continuous, unitary representations of a compact group. Notice that in this case we also have a dimension function in a natural way. The other important example of DR-category is the superselection theory with finite statistics $\mathcal{T}_{\mathcal{A}}^{\mathrm{fin}}$. Explicitly, the monoidal structure is given by the composition of localized endomorphisms and the crossed product of intertwiners, the braiding is given by (A.1) and the conjugate equations read

$$\bar{R}_\rho^* \rho(R_\rho) = 1 \,, \quad R_\rho^* \bar{\rho}(\bar{R}_\rho) = 1 \,. \tag{A.2}$$

Furthermore, as mentioned above $\mathcal{T}_{\mathcal{A}}^{\mathrm{fin}}$ is characterized by the existence of a dimension function, namely the statistical dimension. The statistical dimension corresponds the the intrinsic dimension in the sense mentioned above.

We conclude this appendix by mentioning one of the most important results emerged in the study of Doplicher and Roberts on superselection theory. A classical result in representation theory is the *Tannaka-Krein duality*. Tannaka's theorem provides a way to reconstruct the compact group G from its category of representations. Krein's theorem shows necessary and sufficient conditions for a category to be the dual object of a compact group. Therefore, the Tannaka-Krein duality finds necessary conditions for a subcategory of **Hilb** to be the representation category of some compact group. The stunning result proved in [40] is the characterization of all such subcategories: every DR-category is isomorphic to a category $\mathcal{T}_G$ for a compact group G which is unique up to isomorphism. This result has added a new chapter to the mathematical theory of group duality, and allow us to define the gauge group G of a superselection theory $\mathcal{T}_{\mathcal{A}}^{\text{fin}}$ as its dual object.

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