Large blow-up sets for the prescribed Q-curvature equation in the Euclidean space

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Abstract

Let $m \geq 2$ be an integer. For any open domain $\Omega \subset \mathbb{R}^{2m}$, non-positive function $\varphi \in C^\infty(\Omega)$ such that $\Delta^m \varphi \equiv 0$, and bounded sequence $(V_k) \subset L^\infty(\Omega)$ we prove the existence of a sequence of functions $(u_k) \subset C^{2m-1}(\Omega)$ solving the Liouville equation of order $2m$

$$(-\Delta)^m u_k = V_k e^{2mu_k} \quad \text{in } \Omega, \quad \limsup_{k\to\infty} \int_\Omega e^{2mu_k} dx < \infty,$$

and blowing up exactly on the set $S_\varphi := \{x \in \Omega : \varphi(x) = 0\}$, i.e.

$$\lim_{k\to\infty} u_k(x) = +\infty \text{ for } x \in S_\varphi \text{ and } \lim_{k\to\infty} u_k(x) = -\infty \text{ for } x \in \Omega \setminus S_\varphi,$$

thus showing that a result of Adimurthi, Robert and Struwe is sharp. We extend this result to the boundary of Ω and to the case $\Omega = \mathbb{R}^{2m}$. Several related problems remain open.

1 Introduction and main results

In several nonlinear elliptic problems of second order and "critical type", sequences of solutions are not always compact, as they can blow up at finitely many points, see e.g [2], [4], [5], [11], [24], [25], [26]. For instance, as shown by H. Brézis and F. Merle in [5]:

Theorem A ([5]) *Given a sequence $(u_k)_{k\in\mathbb{N}}$ of solutions to the Liouville equation*

$$-\Delta u_k = V_k e^{2u_k} \quad \textit{in } \Omega \subset \mathbb{R}^2, \tag{1}$$

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with $\|V_k\|_{L^\infty} \le C$ *and* $\|e^{2u_k}\|_{L^1} \le C$ *for some* C *independent of* k*, there exists a finite (possibly empty) set* $S_1 = \{x^{(1)}, \dots, x^{(I)}\} \subset \Omega$ *such that, up to extracting a subsequence one of the following alternatives holds:*

- (i) (u_k) *is bounded in* $C^{1,\alpha}_{\mathrm{loc}}(\Omega \setminus S_1)$*.*
- (ii) $u_k \to -\infty$ *locally uniformly in* $\Omega \setminus S_1$*.*

A similar behaviour is also found on manifolds, or in higher order and higher dimensional problems, see e.g. [21], [27], or even in 1-dimensional situations involving the operator $(-\Delta)^{\frac{1}{2}}$, see [9], [10]. Now consider the problem

$$(-\Delta)^m u_k = V_k e^{2mu_k} \quad \text{in } \Omega \subset \mathbb{R}^{2m} \tag{2}$$

$$\limsup_{k\to\infty} \int_\Omega e^{2mu_k}\,dx < \infty, \quad \limsup_{k\to\infty} \|V_k\|_{L^\infty(\Omega)} < \infty, \tag{3}$$

We recall that (2) is a special case of the prescribed Q-curvature equation on a Riemannian manifold (M, g) of dimension $2m$

$$P_g u + Q_g = Q_{g_u} e^{2mu} \quad \text{in } M, \tag{4}$$

where $P_g = (-\Delta_g)^m + l.o.t.$ and Q_g are the GJMS-operator of order $2m$ and the Q-curvature of the metric g, respectively, and Q_{g_u} is the Q-curvature of the conformal metric $g_u := e^{2u}g$. In this sense every solution u_k to (2) gives rise to a metric $e^{2u_k}|dx|^2$ on Ω with Q-curvature V_k, and volume $\int_\Omega e^{2mu_k}dx$.

Since blow-up at finitely many points appears in many problems with various critical nonlinearities and also of higher order, one might suspect that this is a general feature also holding for (2). On the other hand Adimurthi, Robert and Struwe [1] found an example of solutions to (2)-(3) for $m = 2$ that blow up on a hyperplane, and showed in general that the blow-up set of a sequence (u_k) of solutions to (2)-(3) can be of Hausdorff dimension 3. This was generalized to the case of arbitrary m in [19]. More precisely for a finite set $S_1 \subset \Omega \subset \mathbb{R}^{2m}$ let us introduce

$$\mathcal{K}(\Omega, S_1) := \{\varphi \in C^\infty(\Omega \setminus S_1) : \varphi \le 0,\, \varphi \not\equiv 0,\, \Delta^m \varphi \equiv 0\}, \tag{5}$$

and for a function $\varphi \in \mathcal{K}(\Omega, S_1)$ set

$$S_\varphi := \{x \in \Omega \setminus S_1 : \varphi(x) = 0\}. \tag{6}$$

Theorem B ([1, 19]) *Let* (u_k) *be a sequence of solutions to* (2)-(3) *for some* $m \ge 1$*. Then the set*

$$S_1 := \left\{ x \in \Omega : \lim_{r\downarrow 0} \limsup_{k\to\infty} \int_{B_r(x)} |V_k| e^{2mu_k} dy \ge \frac{\Lambda_1}{2} \right\}, \quad \Lambda_1 := (2m-1)!\mathrm{vol}(S^{2m})$$

is finite (possibly empty) and up to a subsequence either *(i)* (u_k) *is bounded in* $C^{2m-1,\alpha}_{\mathrm{loc}}(\Omega \setminus S_1)$*, or*

(ii) *there exists a function* $\varphi \in \mathcal{K}(\Omega, S_1)$ *and a sequence* $\beta_k \to \infty$ *as* $k \to +\infty$ *such that*

$$\frac{u_k}{\beta_k} \to \varphi \text{ locally uniformly in } \Omega \setminus S_1.$$

In particular $u_k \to -\infty$ *locally uniformly in* $\Omega \setminus (S_\varphi \cup S_1)$.

Notice that Theorem B contains Theorem A since when $m = 1$ we have $S_\varphi = \emptyset$ for every $\varphi \in \mathcal{K}(\Omega, S_1)$ by the maximum principle. In fact the more complex blow-up behaviour of (2) when $m > 1$ can be seen as a consequence of the size of $\mathcal{K}(\Omega, S_1)$. A way of recovering a finite blow-up behaviour for (2)-(3) was given by F. Robert [23] when $m = 2$ and generalized by the third author [20] when $m \geq 3$, by additionally assuming

$$\|\Delta u_k\|_{L^1(B_r(x))} \leq C \quad \text{on some ball } B_r(x) \subset \Omega,$$

which is sufficient to control the "polyharmonic part" of u_k.

The first result that we will prove shows that the condition given in [1] and [19] on the set S_φ above is sharp, at least when $S_1 = \emptyset$. In fact we shall consider a slightly stronger result, by defining

$$S^*_\varphi := S_\varphi \cup \{x \in \partial\Omega : \lim_{\Omega \ni y \to x} \varphi(y) = 0\}, \tag{7}$$

namely we add to S_φ the points on $\partial\Omega$ where φ can be continuously extended to 0. Then we have

Theorem 1 *Let* $\Omega \subset \mathbb{R}^{2m}$*,* $m \geq 2$*, be an open (connected) domain and let* $(V_k) \subset L^\infty(\Omega)$ *be bounded. Then for every* $\varphi \in \mathcal{K}(\Omega, \emptyset)$ *there exists a sequence* (u_k) *of solutions to* (2) *with*

$$\int_\Omega e^{2mu_k}\, dx \to 0, \tag{8}$$

such that as $k \to \infty$

$$u_k \to -\infty \text{ loc. unif. in } \Omega \setminus S_\varphi, \quad u_k \to +\infty \text{ loc. unif. on } S^*_\varphi, \tag{9}$$

where S_φ *and* S^*_φ *are as in* (6) *and* (7)*. The same result holds if* $m = 1$ *and* Ω *is smoothly bounded.*

The proof of Theorem 1 is based on a Schauder's fixed-point argument. The case when Ω is smoothly bounded is very elementary, as one looks for solutions of the form

$$u_k = c_k\varphi + k + v_k, \quad c_k \to \infty,$$

where v_k is a small correction term.

The general case is a priori more rigid. For instance in the case $m = 1$, when $V_k \equiv 1$ there are few solutions to (2)-(3) when $\Omega = \mathbb{R}^2$ (see [8]) and many more when Ω is bounded (see [7]). To treat the general case we will borrow ideas from [28] (see also [14]) and suitably prescribe the asymptotic behavior of u_k at infinity. More precisely we will look for solutions of the form

$$u_k = c_k\varphi + k - \alpha_k \log(1+|x|^2) - \beta|x|^2 + v_k,$$

for some $c_k \to \infty$, $\alpha_k \to 0$, $\beta > 0$, and a function $v_k \to 0$ uniformly. If $\varphi(x) \to -\infty$ sufficiently fast as $|x| \to \infty$, or when Ω is bounded, one can choose $\beta = 0$, but the case $\Omega = \mathbb{R}^{2m}$, $\varphi(x_1, \ldots, x_{2m}) = -x_1^2$ shows that β in general must be positive when

$$\liminf_{x\in\Omega, |x|\to\infty} \varphi(x) > -\infty,$$

otherwise the condition (3) might fail to be satisfied.

The simplicity of the proof of Theorem 1 comes at the cost of not being able to prescribe the total Q-curvature of the metric $g_{u_k} := e^{2u_k}|dx|^2$, which will necessarily go to zero, together with the volume of g_{u_k}. Resting on variational methods from [15] going back to [6], we can extend Theorem 1 to the case in which we prescribe both the blow-up set S_φ and the total curvature of the metrics g_{u_k}. This time, though, we will have to restrict to non-negative functions V_k.

Theorem 2 *Let $0 < \Lambda < \Lambda_1/2$, $\Omega \subset \mathbb{R}^{2m}$ open, $m \geq 2$, $\varphi \in \mathcal{K}(\Omega, \emptyset)$, and let S_φ be as in* (6)*. Let further $V_k : \Omega \to \mathbb{R}$ be functions for which there exists $x_0 \in S_\varphi^*$ such that*

$$\liminf_{k\to+\infty} \int_{B_\varepsilon(x_0)\cap\Omega} V_k\, dx > 0, \quad \text{for every } \varepsilon > 0, \quad 0 \leq V_k \leq b < \infty. \tag{10}$$

Then there exists a sequence $(u_k)_{k\in\mathbb{N}}$ of solutions to (2) *with*

$$\int_\Omega V_k e^{2mu_k} dx = \Lambda, \tag{11}$$

such that (9) *holds.*

The integral assumption in (10) is crucial. In fact, for any $\varphi \in \mathcal{K}(\Omega, \emptyset)$ there are functions V_k satisfying $0 \leq V_k \leq b < \infty$, such that for every $\Lambda > 0$ there exists no sequence (u_k) of solution to (2) satisfying (9) and (11) (see Proposition 12).

As we shall see, Theorems 1 and 2 give several examples of solutions blowing-up on the boundary, already in dimension 2.

Corollary 3 *Let $\Omega \subset \mathbb{R}^{2m}$ with $m \geq 1$ be a bounded domain with smooth boundary and let $\Gamma \subset \partial\Omega$ be a proper closed subset. Let (V_k) be as in Theorem 1. Then we can find solutions $u_k : \Omega \to \mathbb{R}$ to* (2) *such that the conclusion of Theorem 1 holds with $S_\varphi^* = \Gamma$ for some $\varphi \in \mathcal{K}(\Omega, \emptyset)$. If $m \geq 2$ and (V_k) additionally satisfies* (10) *for some $x_0 \in \Gamma$, then we can prescribe* (11) *instead of* (8).

Open problem 1 *Can one remove the assumption* $\Lambda < \frac{\Lambda_1}{2}$ *in Theorem 2? In the radially symmetric case this appears to be the case, as the following result shows.*

Theorem 4 *Let* $\Omega = B_{R_2} \setminus B_{R_1} \subset \mathbb{R}^{2m}$ *and* $\varphi \in \mathcal{K}(\Omega, \emptyset)$ *be radially symmetric. Let* $\Lambda > 0$ *and let* (V_k) *be radially symmetric satisfying* (10). *Then there exists a sequence of radially symmetric solutions* (u_k) *to* (2) *such that* (9) *and* (11) *hold. For* $\Omega = B_R$ *the same conclusion holds if in addition we have* $\Delta\varphi(0) > 0$ *and* $V_k \to 1$ *in* $L^\infty(B_\delta(0))$ *for some* $\delta > 0$.

Gluing open problems

We have worked under the assumption $S_1 = \emptyset$. What happens if we drop it?

Open problem 2 *Can one have both* $S_1 \neq \emptyset$ *and* $S_\varphi \neq \emptyset$ *in Theorem B? Or when* $m = 1$ *can one have* $S_1 \neq \emptyset$ *and* $S_\varphi^* \neq \emptyset$*?*

This can be considered as a gluing problem. For instance, can one glue a standard bubble of the form

$$u_{x_0,\lambda}(x) := \log \frac{2\lambda}{1 + \lambda^2|x - x_0|^2}, \quad \text{for some } \lambda > 0,\ x_0 \in \mathbb{R}^{2m} \tag{12}$$

solving

$$(-\Delta)^m u_{x_0,\lambda} = (2m-1)! e^{2m u_{x_0,\lambda}}, \quad (2m-1)! \int_{\mathbb{R}^{2m}} e^{2m u_{x_0,\lambda}} dx = \Lambda_1, \tag{13}$$

to one of the solutions provided by Theorems 1 and 2?

Moreover, as shown by Chang-Chen [6], when $m \geq 2$ problem (13) has several solutions which are not of the form (12). Such solutions behave polynomially at infinity, as shown in [17, 18] (see also [12, 16] for similar results in odd dimension). Let us call v such a solution and

$$v_{x_1,\mu}(x) := v(\mu(x - x_1)) + \log \mu, \quad \text{for some } \mu > 0,\ x_1 \in \mathbb{R}^{2m}.$$

Open problem 3 *Can one glue a spherical solution* $u_{x_0,\lambda}$ *to a non-spherical solution* $v_{x_1,\mu}$ *as above* ($x_1 \neq x_0$)*? More precisely, can one find a sequence of solutions* (u_k) *to* (2)-(3) *with* $u_k = u_{x_0,\lambda_k} + w_k$ *suitably close to* x_0 *and* $u_k = v_{x_1,\mu_k} + w_k$ *suitably close to* x_1*, with an error term* w_k *bounded and* $\lambda_k, \mu_k \to \infty$*?*

This problem can be seen in terms of gradient estimates or estimates for Δu_k. Indeed on any fixed ball B one has

$$\|\Delta u_{\lambda,x_0}\|_{L^1(B)} = O(1), \quad \|\Delta v_{\lambda,x_1}\|_{L^1(B)} \to \infty, \quad \text{as } \lambda \to \infty$$

(see Theorems 1 and 2 in [18]). This is consistent with a result of F. Robert [23], extended in [20], stating that in a region Ω_0 such that $\|\Delta u_k\|_{L^1(\Omega_0)} \leq C$, u_k has a bubbling behaviour leading to solutions of the form (12).

It was open whether there exists a sequence (u_k) of solutions to (2)-(3) on some domain Ω in $\mathbb{R}^{2m}$ with 2 open regions $\Omega_0, \Omega_1 \subset \Omega$ such that

$$\|\Delta u_k\|_{L^1(\Omega_0)} = O(1), \quad \|\Delta u_k\|_{L^1(\Omega_1)} \to \infty.$$

We will prove that this is actually possible.

Theorem 5 *On* $\Omega = B_2 \subset \mathbb{R}^{2m}$ *for any* $\Lambda \in (0, \Lambda_1)$ *we can find a sequence* (u_k) *of solutions to* (2)-(3) *with* $V_k \equiv 1$ *such that*

$$\int_{B_2} e^{2mu_k} dx = \Lambda, \tag{14}$$

and

$$\int_{B_1} |\Delta u_k| dx \leq C, \quad \int_{B_2} (\Delta u_k)^- dx \xrightarrow{k\to\infty} \infty. \tag{15}$$

It remains open whether in the situation of Theorem 5 one can also have blow-up in B_1, in $B_2 \setminus B_1$, or in both regions.

In what follows we will denote by C a generic positive constant that can change its value from line to line.

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2 Proof of Theorem 1

In order to clarify the simple idea behind the proof we start considering the easier case when Ω is bounded and has regular boundary. The proof in the general case is more complex and only works when $m \geq 2$ (easy counterexamples can be found when $m = 1$, $\Omega = \mathbb{R}^2$, $V_k \equiv 1$, using the classification result from [8]).

2.1 Case Ω smoothly bounded

In this case we can assume $m \geq 1$. The proof will be based on an application of a fixed-point argument. Consider the Banach space

$$X := C^0(\bar{\Omega}), \quad \|v\|_X = \max_{x\in\bar{\Omega}} |v(x)|.$$

For each $k \in \mathbb{N}$ choose $c_k \geq k^2$ such that

$$\|e^{2mc_k\varphi}\|_{L^2(\Omega)} \leq e^{-3mk}.$$

For $k \in \mathbb{N}$ consider the operator $T_k : X \to X$ defined by $T(v) = \bar{v}$ where $\bar{v}$ is the unique solution of

$$\begin{cases} (-\Delta)^m \bar{v} = V_k e^{2m(k+c_k\varphi+v)} & \text{in } \Omega \\ \bar{v} = \Delta \bar{v} = \cdots = \Delta^{m-1}\bar{v} = 0 & \text{on } \partial\Omega. \end{cases}$$

From elliptic estimates, the Sobolev embedding and Ascoli-Arzelà's theorem it follows that T_k is compact. Moreover, for every $v \in X$ we have

$$\|\bar{v}\|_X \leq C_1 \|\Delta^m \bar{v}\|_{L^2(\Omega)} \leq C_2 M e^{2mk} \|e^{2mv}\|_X \|e^{2mc_k\varphi}\|_{L^2(\Omega)}, \quad \|V_k\|_{L^\infty} \leq M.$$

This shows that

$$\|T_k(v)\|_X \leq C_3 e^{2mk} e^{-3mk}, \quad \text{for } \|v\|_X \leq 1, \quad C_3 := C_2 M. \tag{16}$$

Therefore $T_k(\bar{B}_1) \subset \bar{B}_{\frac{1}{2}}$ for k large enough (here B_r is a ball in X), and hence T_k has a fixed point in X. We denote it by v_k. Notice that $\|v_k\|_X \leq Ce^{-mk} \to 0$ as $k \to \infty$. Moreover, by Hölder's inequality,

$$\int_\Omega e^{2mk} e^{2mc_k\varphi} e^{2mv_k} dx \leq e^{2mk} \sqrt{|\Omega|} \|e^{2mc_k\varphi}\|_{L^2(\Omega)} \xrightarrow{k\to\infty} 0.$$

We set

$$u_k := v_k + k + c_k\varphi.$$

Then u_k satisfies

$$(-\Delta)^m u_k = V_k e^{2mu_k} \quad \text{in } \Omega, \qquad \int_\Omega e^{2mu_k} dx \xrightarrow{k\to\infty} 0.$$

Moreover

$$\inf_{x \in S_\varphi^*} u_k = o(1) + k \xrightarrow{k\to\infty} \infty.$$

Finally, for any compact subset $K \Subset \Omega \setminus S_\varphi$, using that $c_k \geq k^2$, we obtain

$$\max_{x\in K} u_k = o(1) + k + c_k \max_{x\in K} \varphi \leq k - \varepsilon k^2 \xrightarrow{k\to\infty} -\infty,$$

where $\varepsilon > 0$ is such that $\max_{x\in K} \varphi < -\varepsilon$. This completes the proof.

2.2 General case

In the genaral case we need to assume $m \geq 2$. We will use many ideas from [14] and [28]. Let $\varphi \in \mathcal{K}(\Omega, \emptyset)$. Fix $u_0 \in C^\infty(\mathbb{R}^{2m})$, $u_0 > 0$, such that $u_0(x) = \log|x|$ for $|x| \geq 2$, and notice that integration by parts yields

$$\int_{\mathbb{R}^{2m}} (-\Delta)^m u_0 \, dx = -\gamma_{2m}, \tag{17}$$

where γ_{2m} is defined by

$$(-\Delta)^m \log \frac{1}{|x|} = \gamma_{2m} \delta_0 \text{ in } \mathbb{R}^{2m}, \text{ i.e. } \gamma_{2m} = \frac{\Lambda_1}{2}. \tag{18}$$

We will work in weighted spaces.

Definition 6 *For $k \in \mathbb{N}$, $\delta \in \mathbb{R}$ and $p \geq 1$ we set $M^p_{k,\delta}(\mathbb{R}^{2m})$ to be the completion of $C^\infty_c(\mathbb{R}^{2m})$ in the norm*

$$\|f\|_{M^p_{k,\delta}} := \sum_{|\beta| \leq k} \|(1+|x|^2)^{\frac{(\delta+|\beta|)}{2}} D^\beta f\|_{L^p(\mathbb{R}^{2m})}.$$

We also set $L^p_\delta(\mathbb{R}^{2m}) := M^p_{0,\delta}(\mathbb{R}^{2m})$. Finally we set

$$\Gamma^p_\delta(\mathbb{R}^{2m}) := \left\{ f \in L^p_{2m+\delta}(\mathbb{R}^{2m}) : \int_{\mathbb{R}^{2m}} f dx = 0 \right\},$$

whenever $\delta p > -2m$, so that $L^p_{2m+\delta}(\mathbb{R}^{2m}) \subset L^1(\mathbb{R}^{2m})$ and the above integral is well defined.

Lemma 7 (Theorem 5 in [22]) *For $1 < p < \infty$ and $\delta \in \left(-\frac{2m}{p}, -\frac{2m}{p} + 1\right)$, the operator $(-\Delta)^m$ is an isomorphism from $M^p_{2m,\delta}(\mathbb{R}^{2m})$ to $\Gamma^p_\delta(\mathbb{R}^{2m})$.*

Lemma 8 (Lemma 2.3 in [14]) *For $\delta > -\frac{2m}{p}$, $p \geq 1$, the embedding*

$$E : M^p_{2m,\delta}(\mathbb{R}^{2m}) \hookrightarrow C_0(\mathbb{R}^{2m})$$

is compact.

We will construct a sequence $(u_k)_{k\in\mathbb{N}}$ of solutions to (2)-(8) of the form

$$u_k = -\beta|x|^2 + c_k\varphi - \alpha_k u_0 + k + v_k, \quad \text{in } \Omega, \tag{19}$$

for some $\beta \geq 0$ and $v_k \in C^{2m-1}(\mathbb{R}^{2m})$ such that as $k \to \infty$

$$\sup_\Omega |v_k| \to 0, \quad c_k \to \infty, \quad \alpha_k \to 0.$$

In general $\beta > 0$ is an arbitrary fixed constant, but if φ satisfies

$$\int_\Omega e^{2m\varphi}|x|^{2s}\,dx < \infty, \quad \text{for some } s > 0, \tag{20}$$

then we can take $\beta = 0$ as well. If there exists $s > 0$ such that (20) holds then we set $q = s$, otherwise we take $\beta > 0$ and set $q = 1$.

We consider

$$X := C_0(\mathbb{R}^{2m}) := \left\{ v \in C^0(\mathbb{R}^{2m}) : \lim_{|x|\to\infty} v(x) = 0 \right\}, \quad \|v\|_X = \sup_{x\in\mathbb{R}^{2m}} |v(x)|.$$

For $c \in \mathbb{R}$ we set

$$F_{k,c} = \begin{cases} V_k e^{2mk} e^{-2m\beta|x|^2} e^{2mc\varphi} & \text{in } \Omega \\ 0 & \text{in } \mathbb{R}^{2m}\setminus\Omega. \end{cases}$$

Let $\varepsilon_1 \in (0, \frac{q}{8m})$ (to be fixed later). We fix $p > 1$ and $\delta \in (-\frac{2m}{p}, \frac{2m}{p} + 1)$ such that $p(2m+\delta) < \frac{q}{4}$. For each $k \in \mathbb{N}$ we choose $c_k \geq k^2$ so that

$$\int_{\mathbb{R}^{2m}} |F_{k,c_k}(x)|(M+|x|)^q dx \leq \varepsilon_1 e^{-k} e^{-2m}, \tag{21}$$

$$\|F_k(M+|x|)^{\frac{q}{4}}\|_{L^p_{2m+\delta}} \leq \varepsilon_1 e^{-k}, \quad F_k := F_{k,c_k}, \tag{22}$$

$$\int_\Omega e^{2m(c_k\varphi+k)}(M+|x|)^q\,dx \leq e^{-k}, \tag{23}$$

where q is defined as above and $M > 0$ is such that $e^{u_0} \leq M$ on B_2. For each $k \in \mathbb{N}$, define a continuous function I_k on $X \times (-\frac{q}{2m}, \frac{q}{2m})$ given by

$$I_k(v,\alpha) = \frac{1}{\gamma_{2m}} \int_{\mathbb{R}^{2m}} F_k e^{-2m\alpha u_0} e^{2mv} dx.$$

If $I_k(v,0) > 0$ then

$$\lim_{\alpha\to 0^+} \frac{I_k(v,\alpha)}{\alpha} = \infty, \quad \frac{I_k(v,\varepsilon_1 e^{-k})}{\varepsilon_1 e^{-k}} \leq 1, \quad \|v\|_X \leq 1,$$

and hence there exists $\alpha \in (0, \varepsilon_1 e^{-k}]$ such that $I_k(v,\alpha) = \alpha$. Notice that

$$\sup_{\alpha\in[-\frac{q}{4m},0]} |I_k(v,\alpha)| \leq e^{-k}\varepsilon_1, \quad \text{for } \|v\|_X \leq 1.$$

Thus, if $I_k(v,0) < 0$ then

$$\lim_{\alpha\to 0^-} \frac{I_k(v,\alpha)}{\alpha} = \infty, \quad \frac{|I_k(v,-\varepsilon_1 e^{-k})|}{\varepsilon_1 e^{-k}} \leq 1, \quad \|v\|_X \leq 1,$$

and hence there exists $\alpha \in [-\varepsilon_1 e^{-k}, 0)$ such that $I_k(v,\alpha) = \alpha$. For $\|v\|_X \le 1$ we define

$$\alpha_{k,v} := \begin{cases} \inf\{\alpha > 0 : \alpha = I_k(v,\alpha)\} & \text{if } I_k(v,0) > 0 \\ \sup\{\alpha < 0 : \alpha = I_k(v,\alpha)\} & \text{if } I_k(v,0) < 0 \\ 0 & \text{if } I_k(v,0) = 0. \end{cases}$$

From the continuity of I_k it follows that $\alpha_{k,v} = I_k(v, \alpha_{k,v})$.

Lemma 9 *There exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ and for every $v \in B_1$ if*

$$I_k(v,\alpha_v) = \alpha_v \quad \text{for some } |\alpha_v| < \frac{q}{4m},$$

then for every $w \in B_{\varepsilon^2}(v) \cap B_1$ there exists $\alpha_w \in (\alpha_v - \varepsilon, \alpha_v + \varepsilon)$ such that

$$I_k(w, \alpha_w) = \alpha_w.$$

Moreover, the map $v \mapsto \alpha_{k,v}$ is continuous on B_1.

Proof. Let $R > 0$ be such that $R^q = \frac{1}{\varepsilon^2}$. With this particular choice of R we have

$$\int_{B_R^c} |F_k| \, (1 + |x|)^q \, dx \le C\varepsilon^2.$$

Now for $|\alpha_v - \alpha|(2m \log R)^2 < \frac{1}{2}$ we have

$$\begin{aligned}
&\frac{1}{\gamma_{2m}} \int_{B_R} F_k e^{-2m\alpha u_0} e^{2mw} \, dx \\
&= \frac{1}{\gamma_{2m}} \int_{B_R} F_k e^{-2m\alpha_v u_0} e^{2mv} e^{2m(w-v)} e^{2m(\alpha_v - \alpha)u_0} \, dx \\
&= \frac{1}{\gamma_{2m}} \int_{B_R} F_k e^{-2m\alpha_v u_0} e^{2mv} \left(1 + 2m(\alpha_v - \alpha)u_0 + O\left(\alpha_v - \alpha\right)\right) \left(1 + O(\varepsilon^2)\right) \, dx \\
&= I_k(v, \alpha_v) + \frac{2m(\alpha_v - \alpha)}{\gamma_{2m}} (1 + O(\varepsilon^2)) \int_{B_R} F_k e^{-2m\alpha_v u_0} e^{2mv} u_0 \, dx \\
&\quad + O\left(\alpha_v - \alpha\right) \int_{B_R} F_k e^{-2m\alpha_v u_0} e^{2mv} \, dx + O(\varepsilon^2) \\
&=: I_k(v, \alpha_v) + \frac{2m(\alpha_v - \alpha)}{\gamma_{2m}} (1 + O(\varepsilon^2)) J_1 + O\left(\alpha_v - \alpha\right) J_2 + O(\varepsilon^2).
\end{aligned}$$

Using (21) we get

$$\begin{aligned}
|J_1| &\le e^{2m} \int_{B_R} |F_k| e^{-2m\alpha_v u_0} u_0 \, dx \le e^{2m} \int_{B_R} |F_k| (M + |x|)^{\frac{q}{2}} u_0 \, dx \\
&\le C(q) e^{2m} \int_{B_R} |F_k| (M + |x|)^q \, dx \le C(q) \varepsilon_1,
\end{aligned}$$

and $J_2 = O(\varepsilon_1)$. Let $\alpha = \alpha_v + \rho$, with $|\rho| \leq \frac{1}{2(2m\log R)^2}$. Then

$$I_k(w, \alpha_v + \rho) - (\alpha_v + \rho) = \rho + O(\varepsilon^2) + \rho O(\varepsilon_1).$$

We fix $\varepsilon_0 > 0$ and $\varepsilon_1 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ we have $|O(\varepsilon^2)| \leq \frac{\varepsilon}{4}$ and $|O(\varepsilon_1)| \leq \frac{1}{4}$. Then we can choose $\bar{\rho} \in (-\varepsilon, \varepsilon)$ such that

$$|\bar{\rho}| \leq \frac{1}{2(2m\log R)^2}, \quad \bar{\rho} + O(\varepsilon^2) + \bar{\rho} O(\varepsilon_1) = 0,$$

concluding the first part of the lemma.

Now we prove the continuity of the map $v \mapsto \alpha_{k,v}$ from B_1 to $\mathbb{R}$.

For $v_n \to v \in B_1$ it follows that (at least) for large n, $|\alpha_{k,v_n}| < \frac{q}{4m}$ and $|\alpha_{k,v}| < \frac{q}{4m}$. First we consider the case $\alpha_{k,v} = 0$. Then for any $\varepsilon > 0$ one has $I_k(v_n, \alpha_{v_n}) = \alpha_{v_n}$ for some $\alpha_{v_n} \in (-\varepsilon, \varepsilon)$ where $\|v - v_n\|_X < \varepsilon^2$. This follows from the first part of the lemma. Since $|\alpha_{k,v_n}| \leq |\alpha_{v_n}|$, we have the continuity.

Now we consider $\alpha_{k,v} > 0$ (negative case is similar). Then $I_k(v, 0) > 0$, and hence $\alpha_{k,v_n} \geq 0$ for large n. We set $\alpha_\infty := \lim_{n\to\infty} \alpha_{k,v_n}$ (this limit exists at least for a subsequence). From the continuity of the map I_k it follows that $I_k(v, \alpha_\infty) = \alpha_\infty$. Since $\alpha_\infty \geq 0$ and $I_k(v, 0) > 0$, we must have $\alpha_\infty > 0$. From the definition of $\alpha_{k,v}$ we deduce that $\alpha_{k,v} \leq \alpha_\infty$. We fix $\varepsilon \in (0, \frac{\alpha_{k,v}}{2})$. Then by the first part of the lemma there exists $\alpha_{v_n} \in (\alpha_{k,v} - \varepsilon, \alpha_{k,v} + \varepsilon)$ such that $I_k(v_n, \alpha_{v_n}) = \alpha_{v_n}$ for every $\|v - v_n\|_X < \varepsilon^2$. Since $\alpha_{k,v_n} \leq \alpha_{v_n}$ and $\alpha_{k,v_n} \to \alpha_\infty$, we have for n large

$$\alpha_{k,v} \leq \alpha_\infty \leq \alpha_{k,v_n} + \varepsilon \leq \alpha_{v_n} + \varepsilon \leq \alpha_{k,v} + 2\varepsilon.$$

We conclude the lemma. □

Proof of Theorem 1 We define $T_k : B_1 \subset X \to X$, $v \mapsto \bar{v}$, where

$$\bar{v}(x) := \frac{1}{\gamma_{2m}} \int_{\mathbb{R}^{2m}} \log\left(\frac{1}{|x-y|}\right) F_k(y) e^{-2m\alpha_{k,v}u_0 + 2mv(y)}\, dy + \alpha_{k,v} u_0,$$

that is $\bar{v}$ solves

$$(-\Delta)^m \bar{v} = F_k e^{-2m\alpha_{k,v}u_0 + 2mv} + \alpha_{k,v}(-\Delta)^m u_0.$$

Notice that arguing as in [14] one gets $\bar{v} \in X$. Using (17) and our choice of $\alpha_{k,v}$ we have

$$\int_{\mathbb{R}^{2m}} (-\Delta)^m \bar{v}\, dx = 0.$$

With our choice of δ and p we have $\bar{v} \in M^p_{2m,\delta}(\mathbb{R}^{2m})$. For $v \in \bar{B}_1 \subset X$ we bound with Lemma 7, Lemma 8 and (22)

$$\begin{aligned}\|T_k(v)\|_X &\leq C_1 \|T_k(v)\|_{M^p_{2m,\delta}} \leq C_1 \|(-\Delta)^m \bar{v}\|_{\Gamma^p_\delta}, \\ &\leq C_1 \|e^{-2m\alpha_{k,v}u_0} F_k\|_{L^p_{2m+\delta}} + C_1 |\alpha_{k,v}| \|(-\Delta)^m u_0\|_{L^p_{2m+\delta}} \xrightarrow{k\to\infty} 0.\end{aligned}$$

Therefore, for ε_1 small enough, $\|T_k(v)\|_X \leq \frac{1}{2}$ and there exists a fixed point v_k for every k. Hence, thanks to (23), the sequence

$$u_k(x) = -\beta|x|^2 - \alpha_{k,v_k} u_0(x) + c_k\varphi(x) + k + v_k(x), \quad x \in \Omega,$$

is a sequence of solutions with the stated properties. $\square$

3 Proof of Theorem 2 and Corollary 3

A slightly different version of the following proposition appears in [15]. For the sake of completeness we give a sketch of the proof.

Proposition 10 *Let $w_0(x) = \log \frac{2}{1+|x|^2}$ and consider two functions $K, f : \mathbb{R}^{2m} \to \mathbb{R}$ such that*

$$K \geq 0, \quad K \not\equiv 0, \quad Ke^{-2mw_0} \in L^\infty(\mathbb{R}^{2m})$$

and

$$fe^{-2mw_0} \in L^\infty(\mathbb{R}^{2m}), \quad \Lambda := \int_{\mathbb{R}^{2m}} f dx \in (0, \Lambda_1).$$

Then there exists a function $w \in C^{2m-1}(\mathbb{R}^{2m})$ and a constant c_w such that

$$(-\Delta)^m w = Ke^{2m(w+c_w)} - f \quad \text{in } \mathbb{R}^n, \quad \int_{\mathbb{R}^{2m}} Ke^{2m(w+c_w)} dx = \Lambda, \tag{24}$$

and $\lim_{|x|\to\infty} w(x) \in \mathbb{R}$. Moreover, if f is of the form $f = (-\Delta)^m g$ for some $g \in C^{2m}(\mathbb{R}^{2m})$ with $g(x) = O(\log|x|)$ at infinity, then w satisfies

$$w(x) = \frac{1}{\gamma_{2m}} \int_{\mathbb{R}^{2m}} \log\left(\frac{1+|y|}{|x-y|}\right) K(y) e^{2m(w(y)+c_w)} dy - g(x) + C,$$

for some $C \in \mathbb{R}$.

Proof. Let π be the stereographic projection from S^{2m} to $\mathbb{R}^{2m}$. We define the functional J on $H^m(S^{2m})$ given by

$$J(u) = \int_{S^{2m}} \left(\frac{1}{2}|(P^{2m}u)^{\frac{1}{2}}|^2 + \tilde{f}_1 u\right) dV_0 - \frac{\Lambda}{2m} \log\left(\int_{S^{2m}} \tilde{K} e^{-2mw_0\circ\pi} e^{2mu} dV_0\right),$$

where $f_1 := fe^{-2mw_0}$, $\tilde{f}_1 := f_1 \circ \pi$, $\tilde{K} := K \circ \pi$ and P^{2m} is the Paneitz operator of order $2m$ with respect to the standard metric on S^{2m} and dV_0 is the volume element on S^{2m}. Following the arguments in [15] one can show that there exists $u \in H^{2m}(S^{2m})$ such that

$$P^{2m}u = \frac{\Lambda \tilde{K} e^{-2mw_0\circ\pi} e^{2mu}}{\int_{S^{2m}} \tilde{K} e^{-2mw_0\circ\pi} e^{2mu} dV_0} - \tilde{f}_1 =: C_0 \tilde{K} e^{-2mw_0\circ\pi} e^{2mu} - \tilde{f}_1.$$

Notice that $P^{2m}u \in L^\infty(S^{2m})$, thanks to the embedding $H^{2m}(S^{2m}) \hookrightarrow C^0(S^{2m})$, and hence $u \in C^{2m-1}(S^{2m})$.

We set $w = u \circ \pi^{-1}$. Then $w \in C^{2m-1}(\mathbb{R}^{2m})$ and $\lim_{|x|\to\infty} w(x) \in \mathbb{R}$. Using the following identity of Branson (see [3])

$$(-\Delta)^m (v \circ \pi^{-1}) = e^{2mw_0}(P^{2m}v) \circ \pi^{-1}, \quad \text{for every } v \in C^\infty(S^{2m}),$$

and by an approximation argument, we have that

$$(-\Delta)^m w = C_0 K e^{2mw} - f =: K e^{2m(w+c_w)} - f, \quad \text{in } \mathbb{R}^{2m}.$$

Now we set

$$\tilde{w}(x) := \frac{1}{\gamma_{2m}} \int_{\mathbb{R}^{2m}} \log\left(\frac{1+|y|}{|x-y|}\right) K(y) e^{2m(w(y)+c_w)}\, dy - g(x).$$

Then $\Delta^m(w-\tilde{w}) = 0$ in $\mathbb{R}^{2m}$ and $(w-\tilde{w})(x) = O(\log|x|)$ at infinity. Therefore, $w = \tilde{w} + C$ for some $C \in \mathbb{R}$.

This finishes the proof of the proposition. $\square$

Proof of Theorem 2 Let $\varphi \in \mathcal{K}(\Omega, \emptyset)$ and let $u_0 \in C^\infty(\mathbb{R}^{2m})$ be such that $u_0 = -\log|x|$ on B_1^c. We set $f = \frac{2\Lambda}{\Lambda_1}(-\Delta)^m u_0$. For each $k \in \mathbb{N}$ we set

$$K = K_k := V_k e^{2m(-\beta|x|^2 + k\varphi + \alpha u_0)}, \quad \alpha := \frac{2\Lambda}{\Lambda_1}, \quad \beta > 0,$$

and we extend K_k by 0 outside Ω. Then by Proposition 10 there exists a sequence of functions (w_k) satisfying

$$w_k(x) = \frac{1}{\gamma_{2m}} \int_{\mathbb{R}^{2m}} \log\left(\frac{1+|y|}{|x-y|}\right) K_k(y) e^{2m(w_k(y)+c_{w_k})} dy - \frac{2\Lambda}{\Lambda_1} u_0 + a_k,$$

for some $a_k \in \mathbb{R}$. We set

$$u_k(x) := w_k + c_{w_k} - \beta|x|^2 + k\varphi(x) + \frac{2\Lambda}{\Lambda_1} u_0(x), \quad x \in \Omega \cup S_\varphi^*.$$

Then u_k satisfies

$$u_k(x) = \frac{1}{\gamma_{2m}} \int_\Omega \log\left(\frac{1+|y|}{|x-y|}\right) V_k e^{2mu_k(y)} dy - \beta|x|^2 + k\varphi(x) + c_k$$

and also (11), where $c_k := a_k + c_{w_k}$. We conclude the proof with Lemma 11. $\square$

Lemma 11 *Let Ω be a domain in $\mathbb{R}^{2m}$. Let φ and V_k as in Theorem 2. Let (u_k) be a sequence of solutions to*

$$u_k(x) = \frac{1}{\gamma_{2m}} \int_\Omega \log\left(\frac{1+|y|}{|x-y|}\right) V_k e^{2mu_k(y)} dy - \beta|x|^2 + k\varphi(x) + c_k, \quad x \in \Omega \cup S_\varphi^*,$$

for some $\beta > 0$. Assume that

$$\int_\Omega V_k e^{2mu_k(y)} dy = \Lambda < \frac{\Lambda_1}{2}.$$

Then $c_k \to \infty$, $c_k = o(k)$ and

$$I_k(x) := \frac{1}{\gamma_{2m}} \int_\Omega \log\left(\frac{1+|y|}{|x-y|}\right) V_k e^{2mu_k(y)} dy, \quad x \in \mathbb{R}^{2m},$$

is locally uniformly bounded from above on $\Omega \setminus S_\varphi$, and locally uniformly bounded from below on $\mathbb{R}^{2m}$. In particular, $u_k \to \infty$ on S_φ^* and $u_k \to -\infty$ locally uniformly on $\Omega \setminus S_\varphi$.

Proof. For any fixed $R > 0$ and $x \in B_R$ we bound

$$\begin{aligned} I_k(x) &= \int_{|y|\le 2R,\, y\in\Omega} \log\left(\frac{1+|y|}{|x-y|}\right) V_k e^{2mu_k(y)} dy + \int_{|y|>2R,\, y\in\Omega} \log\left(\frac{1+|y|}{|x-y|}\right) V_k e^{2mu_k(y)} dy \\ &\ge -C(R) + \int_{|y|>2R,\, y\in\Omega} \log\left(\frac{1}{2} + \frac{1}{2|y|}\right) V_k e^{2mu_k(y)} dy \\ &\ge -C(R). \end{aligned}$$

Since $\Lambda < \frac{\Lambda_1}{2}$, using Jensens inequality we obtain for some $p < 2m$

$$e^{2mu_k(x)} \le e^{2mc_k} e^{-2m\beta|x|^2 + 2mk\varphi(x)} \int_{\mathbb{R}^{2m}} \left(\frac{1+|y|}{|x-y|}\right)^p V_k(y) e^{2mu_k(y)} dy.$$

Using that

$$\int_\Omega \left(\frac{1+|y|}{|x-y|}\right)^p e^{-2m\beta|x|^2 + 2mk\varphi(x)} dx \xrightarrow{k\to\infty} 0,$$

and together with Fubini theorem, one has

$$\int_\Omega V_k(x) e^{2mu_k(x)} dx = e^{2mc_k} o(1), \quad \text{as } k \to \infty.$$

Now $\Lambda > 0$ implies that $c_k \to \infty$.

We assume by contradiction that $c_k \ne o(k)$. Then for some $\varepsilon > 0$ we have $\frac{c_k}{k} \ge 2\varepsilon$ for k large. Let $x_0 \in S_\varphi^*$ be such that (10) holds. Let $\delta > 0$ be such that $\varphi(x) > -\varepsilon$ for $x \in B_\delta(x_0) \cap \Omega$. Therefore

$$u_k(x) \ge -C - k\varepsilon + c_k \ge -C + k\varepsilon, \quad x \in B_\delta(x_0) \cap \Omega,$$

and hence

$$\int_\Omega V_k e^{2mu_k} dx \ge e^{-C+k\varepsilon} \int_{B_\delta(x_0)} V_k dx \xrightarrow{k\to\infty} \infty,$$

a contradiction.

Now we prove that I_k is locally uniformly bounded from above on $\Omega\backslash S_\varphi$. For $\tilde{\Omega} \Subset \Omega\backslash S_\varphi$ we have

$$k\varphi + c_k \to -\infty \quad \text{uniformly on } \tilde{\Omega}.$$

Using Jensens inequality one can show that $\|e^{2mu_k}\|_{L^p(\Omega_1)} \leq C$ for some $p > 1$, where $\tilde{\Omega} \Subset \Omega_1 \Subset \Omega \setminus S_\varphi$. Let p' be the conjugate exponent of p. For $x \in \tilde{\Omega}$ we obtain by Hölder inequality

$$\begin{aligned} I_k(x) &= \frac{1}{\gamma_{2m}} \int_{\Omega_1^c \cap \Omega} \log\left(\frac{1+|y|}{|x-y|}\right) V_k e^{2mu_k(y)} dy + \frac{1}{\gamma_{2m}} \int_{\Omega_1 \cap \Omega} \log\left(\frac{1+|y|}{|x-y|}\right) V_k e^{2mu_k(y)} dy \\ &\leq C + C \| \log|x - \cdot\,| \|_{L^{p'}(\Omega_1)} \|e^{2mu_k}\|_{L^p(\Omega_1)} \\ &\leq C. \end{aligned}$$

The remaining part of the lemma follows immediately. □

Proof of Corollary 3. Let $g \in C^\infty(\partial\Omega)$ be such that $g \leq 0$, $g \not\equiv 0$ on $\partial\Omega$ and $g = 0$ on Γ. Let φ be the solution to

$$\begin{cases} (-\Delta)^m \varphi = 0 & \text{in } \Omega, \\ (-\Delta)^j \varphi = 0 & \text{on } \partial\Omega, \quad j = 1, \ldots, m-1 \\ \varphi = g & \text{on } \partial\Omega. \end{cases}$$

Then by maximum principle $\varphi < 0$ in Ω and hence $S_\varphi^* = \Gamma$. Then the conclusion follows by Theorem 1 and 2. □

Proposition 12 *Let Ω be a domain in $\mathbb{R}^{2m}$. Let $\varphi \in \mathcal{K}(\Omega, \emptyset)$. Let $\tilde{\Omega} \Subset \Omega \setminus S_\varphi$ be an open set. Let V_k be such that $V_k \equiv 0$ on $\tilde{\Omega}^c$ and $V_k \equiv 1$ on $\tilde{\Omega}$. Then for any $\Lambda > 0$ there exists no sequence (u_k) of solutions to* (2) *satisfying* (9) *and* (11).

Proof. We assume by contradiction that the statement of the proposition is not true. Then there exists a sequence of solutions (u_k) to (2) satisfying (9) and (11) for some $\Lambda > 0$. Therefore, by (9), $u_k \to -\infty$ uniformly in $\tilde{\Omega}$ and hence

$$\Lambda = \int_\Omega V_k e^{2mu_k} dx = \int_{\tilde{\Omega}} e^{2mu_k} dx \xrightarrow{k\to\infty} 0,$$

a contradiction. □

4 Proof of Theorem 4

4.1 The case Ω is an annulus.

Let $\Omega = B_{R_2} \setminus B_{R_1}$ be an annulus. Let $X = C^0_{rad}(\bar{\Omega})$. We fix $\Lambda \in (0, \infty)$. For $k \in \mathbb{N}$ and $v \in X$ we choose $c_v = c(v,k) \in \mathbb{R}$ so that

$$\int_\Omega V_k e^{2m(v+c_v)} dx = \Lambda.$$

Let $\varphi \in \mathcal{K}(\Omega, \emptyset)$ be radially symmetric. For $k \in \mathbb{N}$ we define an operator $T_k : X \to X$, $v \mapsto \bar{v}$ where

$$\bar{v} := \tilde{v} + k\varphi(x), \quad \tilde{v}(x) = \int_\Omega G(x,y) V_k(y) e^{2m(v(y)+c_v)} dy,$$

and G is the Green function of $(-\Delta)^m$ on Ω with the Navier boundary conditions.

Lemma 13 *Let $k \in \mathbb{N}$ be fixed. Let $(v,t) \in X \times (0,1]$ satisfies $v = tT_k(v)$. Then there exists $M > 0$ such that $\|v\|_X \le M$ for all such (v,t).*

Proof. We have

$$v(x) = t\int_\Omega G(x,y) V_k(y) e^{2m(v(y)+c_v)} dy + tk\varphi(x) \ge -C(k) \quad \text{in } \Omega.$$

Hence from the definition of c_v we get

$$\Lambda = \int_\Omega V_k e^{2m(v+c_v)}\, dx \ge e^{2m(-C(k)+c_v)} \int_\Omega V_k\, dx > a e^{2m(-C(k)+c_v)}$$

hence $c_v \le C(k)$. Define the cone $\mathcal{C}$ as the set

$$\mathcal{C} := \{x \in \Omega \colon |\bar{x}| \le \rho x_1\}, \quad \text{with } x = (x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^{2m-1}, \tag{25}$$

for some $\rho > 0$ to be fixed later. For some finite $M = M(\rho)$ we can write Ω as a union of (not necessarily disjoint) cones $\{\mathcal{C}_i\}_{i=1}^M$ such that for each such cone $\mathcal{C}_i$ we have

- (i) $\mathcal{C}_i$ is congruent to $\mathcal{C}$,
- (ii) $\int_{N(\mathcal{C}_i)} V_k(y) e^{2m(v(y)+c_v)} dy \le \frac{\Lambda_1}{4}, \quad N(\mathcal{C}_i) := \cup_{\mathcal{C}_i \cap \mathcal{C}_j \neq \emptyset} \mathcal{C}_j$

and we fix ρ such that (ii) holds. Notice that there exists $\delta > 0$ such that $dist(\mathcal{C}_i, N(\mathcal{C}_i)^c) \ge \delta$ for $i = 1, \dots, M$. Therefore, for $x \in \mathcal{C}_1$

$$v(x) \le t\int_{N(\mathcal{C}_1)} G(x,y) V_k(y) e^{2m(v(y)+c_v)} dy + tk\varphi(x) + C(\delta),$$

and together with Jensen's inequality, for some $p > 1$ we get

$$\int_\Omega e^{p2m(v+c_v)}dx \leq M \int_{\mathcal{C}_1} e^{p2m(v+c_v)}dx \leq C.$$

Since φ is radially symmetric and polyharmonic we have $\varphi \in C^{2m}(\bar{\Omega})$, and therefore by elliptic estimates and Sobolev embeddings

$$\|v - tk\varphi\|_X \leq C\|v - tk\varphi\|_{W^{2m,p}(\Omega)} \leq C\|(-\Delta)^m v\|_{L^p(\Omega)} \leq C,$$

concluding the proof. □

A consequence of Lemma 13 is that for every $k \in \mathbb{N}$, the operator T_k has a fixed point $v_k \in X$. We set $u_k = v_k + c_{v_k}$. Then

$$u_k(x) = \int_\Omega G(x,y)V_k e^{2mu_k(y)}dy + k\varphi(x) + c_{v_k}, \quad \int_\Omega V_k e^{2mu_k(y)}dx = \Lambda. \tag{26}$$

We claim that $c_{v_k} \to \infty$.

Again writing Ω as a union of cones and using Jensen's inequality we obtain

$$\int_\Omega e^{2mu_k}dx \leq Ce^{2mc_{v_k}} \int_\Omega e^{2mu_k(y)}dy \int_\Omega \frac{e^{2mk\varphi(x)}}{|x-y|^p}dx,$$

for some $p < 2m$. Hence, if $c_{v_k} \leq C$, then

$$\int_\Omega V_k e^{2mu_k}dx \leq Cb \int_\Omega e^{2mu_k(y)}dy \int_\Omega \frac{e^{2mk\varphi(x)}}{|x-y|^p}dx \xrightarrow{k\to\infty} 0,$$

a contradiction. Thus $c_{v_k} \to \infty$, and hence $u_k \to \infty$ on S_φ^*.

It remains to show that $u_k \to -\infty$ in $C^0_{loc}(\Omega \setminus S_\varphi)$. Arguing as in Lemma 11 we conclude the proof. □

4.2 The case Ω is a ball

We consider

$$X = C^2_{rad}(\bar{B}_R), \quad \|v\|_X := \max_{\bar{B}_R}(|v(x)| + |v'(x)| + |v''(x)|).$$

Let $\Lambda > 0$. We fix $k \in \mathbb{N}$. For $v \in X$ define $c_v \in \mathbb{R}$ given by

$$\int_\Omega V_k e^{2m(v+c_v)}dx = \Lambda.$$

We define $T_k : X \to X$ given by $v \mapsto \bar{v}$ where

$$\bar{v}(x) = \frac{1}{\gamma_{2m}} \int_\Omega \log\left(\frac{1}{|x-y|}\right) V_k(y)e^{2m(v(y)+c_v)}dy + \left(k + \frac{|\Delta v(0)|}{2\Delta\varphi(0)}\right)\varphi(x).$$

Arguing as in [13] one can show that the operator T_k has a fixed point, say v_k. We set $u_k = v_k + c_{v_k}$. Then

$$u_k(x) = \frac{1}{\gamma_{2m}} \int_\Omega \log\left(\frac{1}{|x-y|}\right) V_k(y) e^{2mu_k(y)} dy + \left(k + \frac{|\Delta v_k(0)|}{2\Delta\varphi(0)}\right)\varphi(x) + c_{v_k},$$

and

$$\int_\Omega V_k e^{2mu_k} dx = \Lambda.$$

Again as in [13] one can show that there exists $C > 0$ such that $u_k \le C$ on B_ε for some $\varepsilon > 0$. Using this, and as in the annulus domain case, one can show that $c_{v_k} \to \infty$. Thus $u_k(x) \to \infty$ for every $x \in S_\varphi^*$. Finally, similar to the annulus domain case, it follows that $u_k \to -\infty$ locally uniformly in $\Omega \setminus S_\varphi$. □

5 Proof of Theorem 5

Let $m \ge 2$. We set

$$\varphi_k(r,\theta) := r^k \cos(k\theta), \quad 0 \le r \le 2,\ 0 \le \theta \le 2\pi.$$

We extend φ_k on $B_2 \subset \mathbb{R}^{2m}$ as a function of only two variables, that is, $\varphi_k(x) := \varphi_k(r,\theta)$ for $x \in B_2$, where (r,θ) is the polar coordinate of $\Pi(x)$ and $\Pi : \mathbb{R}^{2m} \to \mathbb{R}^2$ is the projection map. Then φ_k is a harmonic function on B_2. Let Φ_k be the solution to the equation

$$\begin{cases} -\Delta\Phi_k = \varphi_k & \text{in } B_2, \\ \Phi_k = 0 & \text{on } \partial B_2. \end{cases}$$

We fix $0 < \Lambda < \Lambda_1$. Then by Proposition 10 there exists a sequence of solutions (w_k) to (24) with

$$f := \frac{2\Lambda}{\Lambda_1}(-\Delta)^m u_0, \quad K_k := \begin{cases} e^{2m(\Phi_k + \frac{2\Lambda}{\Lambda_1}u_0)} & \text{on } B_2 \\ 0 & \text{on } B_2^c, \end{cases}$$

where $u_0 \in C^\infty(\mathbb{R}^{2m})$ with $u_0 = -\log|x|$ on B_1^c. Then

$$u_k := w_k + c_{w_k} + \Phi_k + \frac{2\Lambda}{\Lambda_1}u_0$$

satisfies (14) and u_k is given by

$$u_k(x) = \frac{1}{\gamma_{2m}} \int_{B_2} \log\left(\frac{1+|y|}{|x-y|}\right) e^{2mu_k(y)} dy + \Phi_k(x) + c_k,$$

for some $c_k \in \mathbb{R}$. Moreover,

$$\Delta u_k = -\varphi_k + e_k,$$

where

$$|e_k(x)| \le C \int_{B_2} \frac{e^{2mu_k(y)}}{|x-y|^2} dy.$$

Integrating, using Fubini's theorem and (14) we obtain $\|e_k\|_{L^1(B_2)} \le C$. Then (15) follows at once from the definition of φ_k. $\square$

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