TIDES AND DUMBBELL DYNAMICS

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Abstract. We discuss a model describing the effects of tidal dissipation on satellite's orbits. Tidal bulges are described in terms of a dumbbell, coupled to the rotation by a dissipative interaction. The assumptions on this dissipative coupling turns out to be crucial in the evolution of the system.

1. Introduction

We introduce a simple model in order to compute the effects of tidal dissipation in the two body problem. We will assume that the two bodies have very different masses, say $M \gg m$, and we call *planet* the body with mass M and *satellite* the body with mass m. We want to keep the model as simple as possible, and then we assume that the axis of the rotation of the planet and the satellite are perpendicular to the plane containing the orbit of the satellite. We study the effects of the tides formed by the satellite on the planet and, with the same approach, the effects of the tides formed by the planet on the satellite, assuming that the satellite is in $1:1$ resonance with the planet.

Our goal is twofold: first, we want to present a simplified model of the problem introducing the dumbbell dynamics: the planet and the satellite are described in terms of a point P of mass $M-\mu$ and a mechanical dumbbell centered in P, i.e., a system of two points, each having mass $\mu/2$, constrained to be at fixed mutual distance $2r$, having P as center of mass. The idea is to substitute the study of the two tidal bulges with the study of their respective centers of mass. In this way we can compute easily the potential of the tidal torque exerted on the bulges. In the appendix we will present also a direct computation of such tidal torques, obtaining in a clear way its classical expression. We will assume that the motion of the tidal dumbbell and the rotation of the related heavenly body (planet or satellite) are coupled by a dissipative friction. Note that the dynamics of the dumbbells in celestial mechanics has been already studied in different contexts (see for instance [1] and references therein) but to our knowledge the idea to apply the dumbbell dynamics to the dissipative tidal effects is not yet present in literature.

Second, we want to show that some of the common assumptions made in literature regarding the tidal dissipation should be actually specified better: as we will try to show, the problem depends strongly on the model of friction used in order to describe the motion of the tidal bulges with respect to the rotation of the related body. Such friction induces a lag between the planet-satellite direction and the actual direction of the bulges. This is the origin of the torque mentioned above. In literature, see for instance [2], this lag is computed, for the Earth–Moon system, on a local basis, assuming that the level of the sea can be described in terms of a damped and forced harmonic oscillator. An elementary computation shows that this lag is proportional to the frequency of the forcing, and it can be related to a quantity, the specific dissipation function, introduced in the study of the dissipation inside solids. A less elementary argument, that we present for completeness in the appendix and that is outlined in many classical references (see for instance [3] and [4]), give a more detailed proof of the fact that the tidal lag is equal to the inverse of the specific dissipation function in the context of a deformable body rotating under the influence of an external field, causing dissipative friction. In many references, for instance the ones mentioned above, the authors assume that in the case of the satellite-induced tides on the planet, the specific dissipation function inherits the properties computed on the basis of the experiments made on solids, assuming a solid planet, see [5]. Since for certain kind of solids the specific dissipation function does not depend on the frequency of the forcing, it has been argued that the tidal lag is fixed, and independent on the frequency of the tidal forcing. This implies many features of the system that are difficult to accept from a physical point of view: for instance the radial velocity of the satellite's orbit is not continuous in the parameters of the tidal forcing. The friction model underlying this approach is a constant solid-on-solid friction.

We will assume instead that the dissipative properties of the planet and of the satellite come from their viscous nature. We will have therefore that the friction between the dumbbells and each of the underlying bodies is velocity dependent. In classical terms, this corresponds to say that the specific dissipation function of the bodies is frequency dependent. This implies that the continuity of the orbital elements in terms of the tidal forcing parameters is restored. With this assumptions we will write the equation of motion by the aid of the Rayleigh dissipation function, obtaining in a unified and relatively standard way the equations of motions in the two cases mentioned above. We will call the study of the influence on the satellite's orbit of the tides on the planet *Earth–Moon system*, while the study of the effect of the tides on the satellite, supposed in 1 : 1 resonance, will be called *Jupyter–Io system*. All the computations will be performed to the lowest order in the small parameters (the ratio between the radius of the bodies and the orbit of the satellite and the eccentricity).

The note is organized as follows. In section 2 we present the equations of motions of the Earth–Moon system, i.e., considering the tidal torque exerted by the satellite on a dumbbell centered on the planet. Then we will evaluate the evolution of the orbital parameter of the Moon due to this interaction. We will find a quantitative estimate and we compare it with the classical expression, see for instance [2]. We will also show that this dissipation tends to circularize the orbit of the Moon. In section 3 we study the Jupyter–Io system, showing that also in this case the orbit tend to be circularized. In both cases we will show that the eccentricity tends to zero exponentially. Finally section 4 is devoted to future developments of this approach. In the appendix we compute directly the torque exerted by the dumbbell on the other body, and we describe in details the classical computations that correlate the tidal lag and the specific dissipation function.

2. Earth–Moon system.

In this section we want to study the evolution of the Earth–Moon system. We derive the equations of motion in a Lagrangian formalism.

To this end, as we show in Fig. 1, we imagine the Earth, of total mass M_E, as a sphere of radius R_E plus a symmetric dumbbell of diameter $2r$ and mass μ. We imagine the Moon, indicated with S, as a point of mass m. We indicate with ρ the distance between the Moon and the center of the Earth: we observe that in an elliptical orbit this distance varies with time. Finally, we call φ, ϑ and $(\vartheta + \varepsilon)$ the angular positions of the Earth, the Moon and the dumbbell with respect to a fixed direction

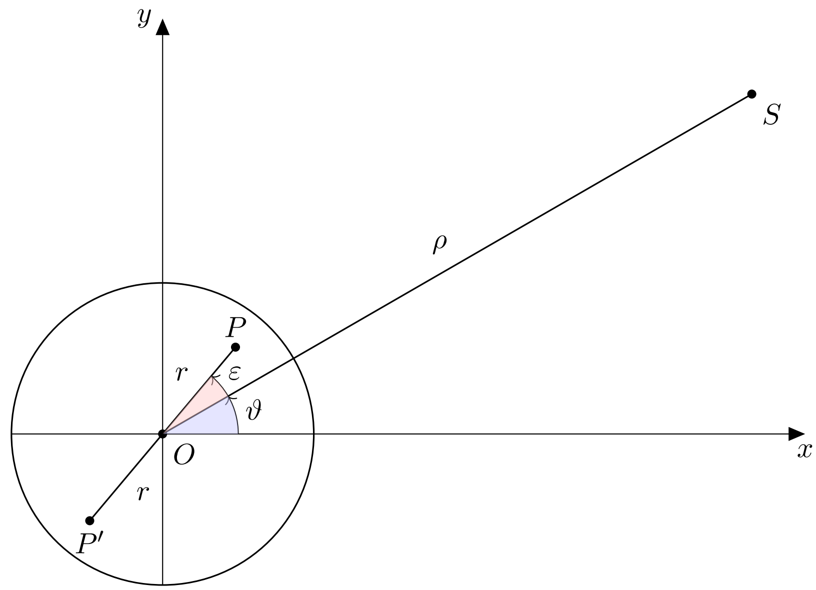

Figure 1. Earth–Moon system

(x-axis), respectively. So $\dot{\varphi}$, $\dot{\vartheta}$ and $(\dot{\vartheta}+\dot{\varepsilon})$ are the angular velocities of the Earth, the Moon and the dumbbell respectively.
Being $M_E \gg m$ we will assume the Earth to be fixed at the origin of the reference system. The total kinetic energy is the sum of the kinetic energies of the Moon, the Earth and the dumbbell:

$$\mathcal{T} = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\theta}^2) + \frac{1}{2}I\dot{\varphi}^2 + \frac{1}{2}\mu r^2(\dot{\vartheta}+\dot{\varepsilon})^2 \tag{1}$$

where I is the Earth's moment of inertia.

The potential energy is the sum of three pieces of gravitational attraction: that between the Earth (deprived of the dumbbell) and the Moon; that between point P (a bulge of the dumbbell) and the Moon and that between point P' (the other bulge of dumbbell) and the Moon:

$$\mathcal{V} = -\frac{k(M_E-\mu)m}{\rho} - \frac{k\frac{\mu}{2}m}{\sqrt{r^2+\rho^2-2r\rho\cos\varepsilon}} - \frac{k\frac{\mu}{2}m}{\sqrt{r^2+\rho^2+2r\rho\cos\varepsilon}}. \tag{2}$$

where k is the universal gravitational constant. If we now expand the potential up to the second order in $\frac{r}{\rho}$ (that is a dimensionless small parameter), we obtain the following expression:

$$\mathcal{V} = -\frac{gm}{\rho}\left[1 + \frac{\mu}{M_E}\frac{r^2}{\rho^2}\left(\frac{3}{2}\cos^2\varepsilon - \frac{1}{2}\right)\right], \tag{3}$$

with $g = kM_E$.
So, the Lagrangian of the system is, with all the aforementioned assumptions:

$$\mathcal{L} = \mathcal{T} - \mathcal{V} = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\theta}^2) + \frac{1}{2}I\dot{\varphi}^2 + \frac{1}{2}\mu r^2(\dot{\vartheta}+\dot{\varepsilon})^2 + \frac{gm}{\rho}\left[1 + \frac{\mu}{M_E}\frac{r^2}{\rho^2}\left(\frac{3}{2}\cos^2\varepsilon - \frac{1}{2}\right)\right]. \tag{4}$$

Now we want to take into account the dissipation of energy. The reasonable mechanism of such dissipation arise from the fact that a friction between the dumbbell and the underlying Earth is present. Since both the ocean and the Earth can be considered fluids (by assuming the Earth a highly viscous fluid), is reasonable to assume a Stokes-type friction both for the oceanic and the solid tides: namely a friction proportional to the difference between the angular velocity of the Earth

(that is $\dot{\varphi}$) and that of the ocean's bulges (that is $(\dot{\vartheta}+\dot{\varepsilon})$).
So we assume a frictional force [1] of the form:

$$f=-\alpha(\dot{\varphi}-\dot{\vartheta}-\dot{\varepsilon}), \tag{5}$$

with α a small friction coefficient.
A standard approach to treat a viscous friction in Lagrangian formalism is to use the Rayleigh's dissipation function R, defined as the function such that $\frac{\partial R}{\partial \dot{q}_i}=f_i$, where f_i is the frictional force acting on the i-th variable.
In our case, the Rayleigh's dissipation function assumes the form:

$$R=-\frac{1}{2}\alpha\left(\dot{\varphi}-\dot{\vartheta}-\dot{\varepsilon}\right)^2. \tag{6}$$

The Euler-Lagrange equations become:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)=\frac{\partial \mathcal{L}}{\partial q_i}+\frac{\partial R}{\partial \dot{q}_i}. \tag{7}$$

Finally, it is easy show that, being R of the form $R=-\frac{1}{2}v^2(\dot{q})$ with $v(\dot{q})=\sum_j a_j\dot{q}_j$ (that is $v(\dot{q})$ linear in $\dot{q}$), the energy dissipation rate is:

$$\dot{E}=2R=-\alpha\left(\dot{\varphi}-\dot{\vartheta}-\dot{\varepsilon}\right)^2. \tag{8}$$

Notice that in the classical literature, see for instance [2], (4.151), the dissipation is assumed to be linear in the difference of angular velocities, implying a non differentiable behavior in $\dot{\varphi}-\dot{\vartheta}-\dot{\varepsilon}$.

The Euler-Lagrange equations (7) leads to the following equations for the dynamical variables ρ, ϑ, φ, ε:

$$m\ddot{\rho}=\frac{\partial \mathcal{L}}{\partial \rho}, \tag{9}$$

$$\frac{d}{dt}\left(m\rho^2\dot{\vartheta}\right)+\mu r^2\left(\ddot{\vartheta}+\ddot{\varepsilon}\right)=\frac{\partial R}{\partial \dot{\vartheta}}=\alpha\left(\dot{\varphi}-\dot{\vartheta}-\dot{\varepsilon}\right), \tag{10}$$

$$I\ddot{\varphi}=\frac{\partial R}{\partial \dot{\varphi}}=-\alpha\left(\dot{\varphi}-\dot{\vartheta}-\dot{\varepsilon}\right), \tag{11}$$

$$\mu r^2\left(\ddot{\vartheta}+\ddot{\varepsilon}\right)=\frac{\partial \mathcal{L}}{\partial \varepsilon}+\frac{\partial R}{\partial \dot{\varepsilon}}=\frac{\partial \mathcal{L}}{\partial \varepsilon}+\alpha\left(\dot{\varphi}-\dot{\vartheta}-\dot{\varepsilon}\right). \tag{12}$$

From these four equations, we can write down two interesting relations.
First, from (10) and (11) we obtain the conservation of angular momentum:

$$\frac{d}{dt}\left(m\rho^2\dot{\vartheta}\right)+\mu r^2\left(\ddot{\vartheta}+\ddot{\varepsilon}\right)+I\ddot{\varphi}=0 \implies m\rho^2\dot{\vartheta}+\mu r^2\left(\dot{\vartheta}+\dot{\varepsilon}\right)+I\dot{\varphi}=J=const.$$

Second, from (10) and (12) we obtain the equation that determines the evolution of orbital angular momentum $J^{(O)}$:

$$\frac{d}{dt}\left(m\rho^2\dot{\vartheta}\right)=-\frac{\partial \mathcal{L}}{\partial \varepsilon}. \tag{13}$$

The explicit expression of $\frac{\partial \mathcal{L}}{\partial \varepsilon}$ and its comparison with the classical form of the tidal torque are briefly presented in the first part of the appendix.

[1] More precisely, a frictional momentum

In equations (9) to (12), there are some negligible terms. In fact, assuming small eccentricity, $\rho \sim a$, we observe that the results presented in the appendix, namely (45), imply that $\frac{\partial \mathcal{L}}{\partial \varepsilon} \propto \frac{\mu}{M_E}\frac{r^2}{a^3}$, and equation (13) becomes:

$$ma^2\ddot{\vartheta} \propto \frac{1}{a}\left(\frac{\mu}{M_E}\frac{r^2}{a^2}\right),$$

We want to study the system keeping the lowest order in $\frac{\mu}{M_E}\frac{r^2}{a^2}$, which is a very small quantity. Since:

$$\mu r^2\ddot{\vartheta} \propto \frac{M_E}{m}\frac{1}{a}\left(\frac{\mu}{M_E}\frac{r^2}{a^2}\right)^2$$

the term $\mu r^2\ddot{\vartheta}$ in (10) and (12) can be neglected. Moreover, it is also reasonable assume initial conditions such that $\ddot{\varepsilon} = O(\ddot{\vartheta})$, namely the variation of angular velocity of the bulges is of the same order than the variation of angular velocity of the Moon. Hence the term $\mu r^2\ddot{\varepsilon}$ in (10) and (12) can be neglected too.

Consequently, the simplified equations of motion are:

$$m\ddot{\rho} = \frac{\partial \mathcal{L}}{\partial \rho}, \tag{14}$$

$$\frac{d}{dt}\left(m\rho^2\dot{\vartheta}\right) = \alpha\left(\dot{\varphi} - \dot{\vartheta} - \dot{\varepsilon}\right), \tag{15}$$

$$I\ddot{\varphi} = -\alpha\left(\dot{\varphi} - \dot{\vartheta} - \dot{\varepsilon}\right), \tag{16}$$

$$\frac{\partial \mathcal{L}}{\partial \varepsilon} + \alpha\left(\dot{\varphi} - \dot{\vartheta} - \dot{\varepsilon}\right) = 0. \tag{17}$$

We call G the orbital angular momentum of the Moon:

$$G = m\rho^2\dot{\vartheta},$$

from (15) we have:

$$\dot{G} = \alpha\left(\dot{\varphi} - \dot{\vartheta} - \dot{\varepsilon}\right). \tag{18}$$

Moreover, the rate of dissipation of energy is given by (8). The energy variation of the system is made of three contributions: the variation of energy of the Earth, the variation of energy of the Moon (namely the variation of orbital energy $E^{(O)}$) and the variation of energy of the bulges. The latter can be neglected for the same reason why we have neglected the variation of the angular momentum of the bulges. So (8) becomes:

$$-\alpha\left(\dot{\varphi} - \dot{\vartheta} - \dot{\varepsilon}\right)^2 = \frac{d}{dt}\left(\frac{1}{2}I\dot{\varphi}^2\right) + \frac{dE^{(O)}}{dt}, \tag{19}$$

where $E^{(O)}$ is the orbital energy of the Moon:

$$E^{(O)} = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\theta}^2) - \frac{gm}{\rho}.$$

From (16), (18) and (19), we have:

$$-\alpha\left(\dot{\varphi} - \dot{\vartheta} - \dot{\varepsilon}\right)^2 = -\alpha\dot{\varphi}\left(\dot{\varphi} - \dot{\vartheta} - \dot{\varepsilon}\right) + \frac{dE^{(O)}}{dt} \implies \frac{dE^{(O)}}{dt} = \left(\dot{\vartheta} + \dot{\varepsilon}\right)\dot{G}. \tag{20}$$

Consider now the canonical variable L, defined as:

$$L = \frac{m^{\frac{3}{2}} g}{\sqrt{-2E^{(O)}}};$$

hence

$$\dot{L} = \frac{g\dot{E}^{(O)}}{\left(-\frac{2E^{(O)}}{m}\right)^{\frac{3}{2}}}. \tag{21}$$

Now, on a Keplerian orbit, we have:

$$E^{(O)} = -\frac{mg}{2a} \implies -\frac{2E^{(O)}}{m} = \frac{g}{a},$$

and the Keplero's third law:

$$\omega^2 a^3 = g.$$

Hence (21) becomes:

$$\dot{L} = \frac{\dot{E}^{(O)}}{\omega} = \frac{\dot{\vartheta} + \dot{\varepsilon}}{\omega}\dot{G}, \tag{22}$$

where we have used (20).
We now observe that several quantities, such as $\langle\dot{\varphi}\rangle = \Omega$, $\langle\dot{\vartheta}\rangle = \omega$ and both the orbital energy and the orbital angular momentum (and hence L and G) varies very slowly. We can therefore assume that these quantities remain constant on one orbit and receive a very small increase (or decrease) at the end of each revolution.
Therefore we can compute the average on one orbit of $\dot{G}$ and $\dot{L}$, obtaining:

$$\langle\dot{G}\rangle = \langle\alpha\left(\dot{\varphi} - \dot{\vartheta} - \dot{\varepsilon}\right)\rangle = \alpha\left(\Omega - \omega\right) \tag{23}$$

and

$$\langle\dot{L}\rangle = \langle\frac{\dot{\vartheta} + \dot{\varepsilon}}{\omega}\dot{G}\rangle = \frac{\alpha}{\omega}\langle\left(\dot{\vartheta} + \dot{\varepsilon}\right)\left(\dot{\varphi} - \dot{\vartheta} - \dot{\varepsilon}\right)\rangle. \tag{24}$$

But:

$$\vartheta = \omega t + 2e\sin(\omega t) \implies \dot{\vartheta} = \omega + 2e\omega\cos(\omega t).$$

And (see (52) for the definition of A, B and δ):

$$\varepsilon = A + eB\sin(\omega t + \delta) \implies \dot{\varepsilon} = eB\omega\cos(\omega t + \delta).$$

Therefore:

$$\langle\dot{L}\rangle = \frac{\alpha}{\omega}\langle[\omega + 2e\omega\cos(\omega t) + eB\omega\cos(\omega t + \delta)]\left[\Omega - \omega - 2e\omega\cos(\omega t) - eB\omega\cos(\omega t + \delta)\right]\rangle =$$

$$= \frac{\alpha}{\omega}\left[\omega\left(\Omega - \omega\right) + \omega e\left(\Omega - 2\omega\right)\langle 2\cos(\omega t) + B\cos(\omega t + \delta)\rangle - \omega^2 e^2\langle[2\cos(\omega t) + B\cos(\omega t + \delta)]^2\rangle\right] =$$

$$= \langle\dot{G}\rangle + \alpha e\left(\Omega - 2\omega\right)\langle 2\cos(\omega t) + B\cos(\omega t + \delta)\rangle - \alpha\omega e^2\langle[2\cos(\omega t) + B\cos(\omega t + \delta)]^2\rangle,$$

where we have used (23).
The computation of the remaining averages leads to:

$$\langle 2\cos(\omega t) + B\cos(\omega t + \delta)\rangle = 0$$

and

$$\langle[2\cos(\omega t) - B\cos(\omega t + \delta)]^2\rangle = \langle 4\cos^2(\omega t) + 4B\cos(\omega t)\cos(\omega t + \delta) + B^2\cos^2(\omega t + \delta)\rangle =$$

$$= 2 + 4B\langle\cos^2(\omega t)\cos(\delta) - \sin(\omega t)\cos(\omega t)\sin(\delta)\rangle + \frac{B^2}{2} = 2 + 2B\cos(\delta) + \frac{B^2}{2}.$$

Hence:

$$\langle \dot{L}\rangle = \langle \dot{G}\rangle - \alpha\omega e^2\left(2 + 2B\cos(\delta) + \frac{B^2}{2}\right) = \langle \dot{G}\rangle - e^2 C, \tag{25}$$

with: $C = \alpha\omega\left(2 + 2B\cos(\delta) + \frac{B^2}{2}\right) > 0$.

Now, if we assume initially $L \sim G$, with $L > G$, we have:

$$\frac{d}{dt}\frac{G^2}{L^2} = \frac{2G\dot{G}L^2 - 2L\dot{L}G^2}{L^4} = \frac{2G\dot{G}L^2 - 2LG^2(\dot{G} - e^2C)}{L^4} = \frac{2G\dot{G}L^2 - 2G\dot{G}G^2 + 2LG^2e^2C)}{L^4} =$$
$$= \frac{2G\dot{G}(L^2 - G^2) + 2LG^2e^2C}{L^4} = \frac{2G\dot{G}L^2e^2 + 2LG^2e^2C}{L^4} = 2\frac{G\dot{G}L^2 + LG^2C}{L^4}e^2 = \frac{2}{\tau_M}e^2, \tag{26}$$

with $\tau_M = \frac{L^4}{G\dot{G}L^2 + LG^2C} > 0$.
Finally:

$$\frac{d}{dt}\frac{G^2}{L^2} = \frac{d}{dt}(1 - e^2) = -2e\dot{e}. \tag{27}$$

Putting together (26) and (27), we obtain:

$$-2e\dot{e} = \frac{2}{\tau_M}e^2 \implies \dot{e} = -\frac{e}{\tau_M} \implies e(t) = e_0 \exp\left(-\frac{t}{\tau_M}\right). \tag{28}$$

Therefore the eccentricity tends exponentially to zero for $t \to \infty$: the orbit becomes circular.

Consider now the limiting case of a circular orbit: $e = 0$. In this case, on each single orbit $\dot{\varphi} = \Omega = const$ and $\dot{\vartheta} = \omega = const$ and then in equation (17) we have a constant forcing. Hence the limiting solution of such equation is also a constant: $\varepsilon = const$
From (17) and (18) we then have:

$$\dot{G} = \alpha\left(\dot{\varphi} - \dot{\vartheta}\right) = \alpha\left(\Omega - \omega\right) = -\frac{\partial\mathcal{L}}{\partial\varepsilon} = \Gamma. \tag{29}$$

We assume, as discussed before, that the very slow variation on Γ is applied as a final kick after an unperturbed Keplerian orbit on which Γ is assumed constant. We have:

$$G(T) - G(0) = \int_0^T \frac{dG}{dt}\,dt = \Gamma T. \tag{30}$$

Hence:

$$G(T) = G(0) + \Gamma T = m\sqrt{ga(0)} + \Gamma T,$$

and

$$G(T) = m\sqrt{ga(T)} = m\sqrt{g[a(0) + \dot{a}T]};$$

therefore:

$$m\sqrt{g[a(0) + \dot{a}T]} = m\sqrt{ga(0)} + \Gamma T \implies ga(0) + g\dot{a}T = ga(0) + \frac{2\Gamma}{m}T\sqrt{ga(0)} \implies \dot{a} = \frac{2\Gamma}{m}\sqrt{\frac{a(0)}{g}},$$

where we have neglected the term $\left(\frac{\Gamma}{m}T\right)^2$.
Finally, using Kepler's third law:

$$\dot{a} = \frac{2\alpha}{m\omega a}\left(\Omega - \omega\right). \tag{31}$$

Note that in (31) the dependence of $\dot{a}$ on $(\Omega - \omega)$ is regular (namely linear), while in literature, see [2] (4.160), the dependence has a singularity for $(\Omega - \omega) = 0$, since it depends on $\mathrm{sign}(\Omega - \omega)$. Note also that in this context the results one finds in literature corresponds to a different choice of the

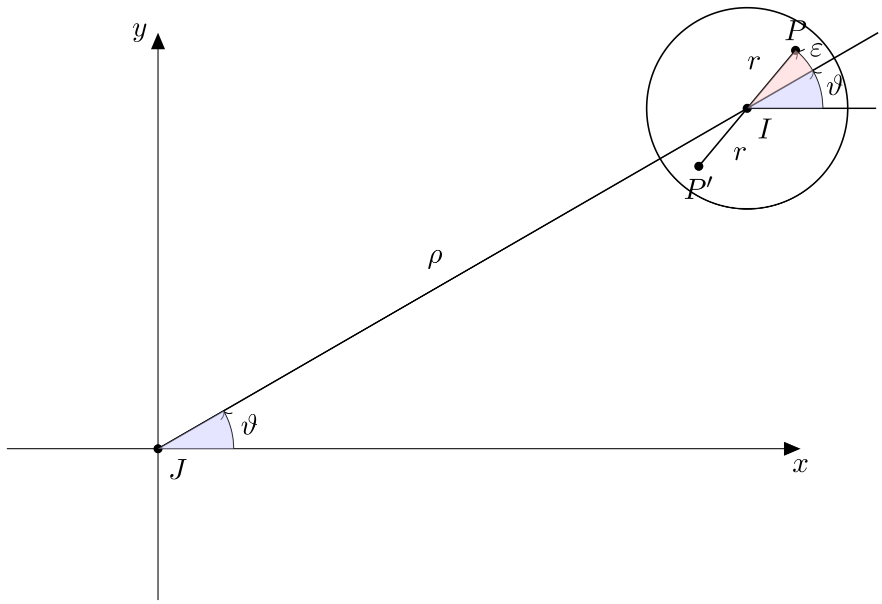

Figure 2. Jupyter–Io System

friction law in (5), i.e. the choice $f = -const$. This choice is equivalent to considering the system as if it were made up of two solid surfaces that slide over each other.

3. Jupyter–Io system.

In this section we want study the evolution of the Jupyter–Io system using the same formalism developed above.

To this end, as we show in Fig. 2, we imagine Io, of total mass m, as a sphere plus a symmetric dumbbell of diameter $2r$ and mass μ. We imagine Jupiter, indicated with J, as a point of mass M_J placed at the origin of the reference frame. We indicate with ρ the distance between Jupiter and the center of Io: we observe that in an elliptical orbit this distance varies with time. Finally, we call φ, ϑ and $(\vartheta + \varepsilon)$ the angular positions of the rotation of Io, the angular positions of the revolution of Io and the angular position of the dumbbell with respect to a fixed direction (x-axis), respectively. So $\dot{\varphi}$, $\dot{\vartheta}$ and $(\dot{\vartheta} + \dot{\varepsilon})$ are the angular velocity of the rotation of Io, the angular velocity of the revolution of Io and the angular velocity of the dumbbell respectively.

In this case, the kinetic energy of this system is the sum of kinetic energies of the dumbbell and the kinetic energy of Io, that has two pieces: one due to the revolution around Jupiter and one due to the rotation around its own axis. The potential energy is the sum of three pieces of gravitational attraction: that between Io (deprived of the dumbbell) and Jupiter; that between point P (a bulge of the dumbbell) and Jupiter and that between point P' (the other bulge of dumbbell) and Jupiter. The potential can be expanded up to the second order in $\frac{r}{\rho}$ (that is a dimensionless small parameter). So, the Lagrangian of the system is very similar to that of the previous section:

$$\mathcal{L} = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\theta}^2) + \frac{1}{2}I\dot{\varphi}^2 + \frac{1}{2}\mu r^2(\dot{\vartheta} + \dot{\varepsilon})^2 + \frac{gm}{\rho}\left[1 + \frac{\mu}{m}\frac{r^2}{\rho^2}\left(\frac{3}{2}\cos^2\varepsilon - \frac{1}{2}\right)\right], \tag{32}$$

where I represents the moment of inertia of Io, while in (4) it represents the moment of inertia of Earth. Moreover now, $g = kM_J$.

Assuming the same dissipation mechanism of previous section, justified by the fact that it is known that Io is made of molten material, we have the same Rayleigh's dissipation function of the previous section:

$$R = -\frac{1}{2}\alpha\left(\dot{\varphi} - \dot{\vartheta} - \dot{\varepsilon}\right)^2. \tag{33}$$

So the equations of motion are the same of previous section:

$$m\ddot{\rho} = \frac{\partial \mathcal{L}}{\partial \rho}, \tag{34}$$

$$\frac{d}{dt}\left(m\rho^2\dot{\vartheta}\right) = \alpha\left(\dot{\varphi} - \dot{\vartheta} - \dot{\varepsilon}\right), \tag{35}$$

$$I\ddot{\varphi} = -\alpha\left(\dot{\varphi} - \dot{\vartheta} - \dot{\varepsilon}\right), \tag{36}$$

$$\frac{\partial \mathcal{L}}{\partial \varepsilon} + \alpha\left(\dot{\varphi} - \dot{\vartheta} - \dot{\varepsilon}\right) = 0. \tag{37}$$

However in this case the initial conditions are different. In fact while in Earth–Moon system we have: $\dot{\varphi} \gg \dot{\vartheta}$ (in fact the period of rotation of Earth is much smaller than the period of revolution of Moon), in Jupyter–Io system we have: $\langle\dot{\varphi}\rangle = \langle\dot{\vartheta}\rangle = \omega$ (in fact the period of rotation of Io is the same of his period of revolution around Jupiter).
Equation (37) admits a solution of the form: $\varepsilon(t) = eB\sin(\omega t + \delta)$, with B and δ constants. This is evident if one performs all the steps seen in the appendix in the case of the Earth–Moon system. Although $\varepsilon(t)$ is different from that of the previous section (in fact here it does not contain the constant term A), all the results up to (22) remain still valid.
The average on one orbit of $\dot{G}$ and $\dot{L}$ are slightly different from those in the previous section:

$$\langle\dot{G}\rangle = 0 \tag{38}$$

and

$$\langle\dot{L}\rangle = \langle\frac{\dot{\vartheta} + \dot{\varepsilon}}{\omega}\dot{G}\rangle = \frac{\alpha}{\omega}\langle\left(\dot{\vartheta} + \dot{\varepsilon}\right)\left(\dot{\varphi} - \dot{\vartheta} - \dot{\varepsilon}\right)\rangle. \tag{39}$$

From now on, all the steps done in the previous section are the same, with $\langle\varphi\rangle = \omega$. Therefore:

$$\begin{aligned}\langle\dot{L}\rangle &= \frac{\alpha}{\omega}\langle[\omega + 2e\omega\cos(\omega t) + eB\omega\cos(\omega t + \delta)]\,[\omega - \omega - 2e\omega\cos(\omega t) - eB\omega\cos(\omega t + \delta)]\rangle = \\ &= \frac{\alpha}{\omega}\left[-\omega^2 e\langle 2\cos(\omega t) + B\cos(\omega t + \delta)\rangle - \omega^2 e^2\langle[2\cos(\omega t) + B\cos(\omega t + \delta)]^2\rangle\right] = \\ &= -\alpha\omega e^2\langle 4\cos^2(\omega t) + 4B\cos(\omega t)\cos(\omega t + \delta) + B^2\cos^2(\omega t + \delta)\rangle = \\ &= -\alpha\omega e^2\left(2 + 2B\cos(\delta) + \frac{B^2}{2}\right) = -e^2 C,\end{aligned} \tag{40}$$

with: $C = \alpha\omega\left(2 + 2B\cos(\delta) + \frac{B^2}{2}\right) > 0$.

Therefore:

$$\frac{d}{dt}\frac{G^2}{L^2} = -\frac{2L\dot{L}G^2}{L^4} = \frac{2LG^2e^2C}{L^4} = \frac{2}{\tau_I}e^2, \tag{41}$$

with $\tau_I = \frac{L^4}{LG^2C} > 0$.
Finally:

$$\frac{d}{dt}\frac{G^2}{L^2} = \frac{d}{dt}(1 - e^2) = -2e\dot{e}. \tag{42}$$

Putting together (41) and (42), we obtain:

$$-2e\dot{e} = \frac{2}{\tau_I}e^2 \implies \dot{e} = -\frac{e}{\tau_I} \implies e(t) = e_0\exp\left(-\frac{t}{\tau_I}\right). \tag{43}$$

Therefore the eccentricity tends exponentially to zero for $t \to \infty$: the orbit becomes circular even in Jupyter–Io case.

4. Conclusion

In this short note we proposed a simplified model to describe the effects of tidal dissipation on the orbital parameters of the heavenly bodies. We think that the main virtue of the model lies in the clarification of the relevance of the model of friction used in order to describe the interaction between the body and the tidal bulges. We have pointed out that the classical results assume tacitly that this friction is velocity independent, like the solid-on-solid friction, while a viscous friction seems to be more realistic and solves some difficulties present in the classical theory.

Appendix A. Computation of $\varepsilon(t)$

In this appendix we want to show the detailed computation of $\varepsilon(t)$ starting from equation (17) that we rewrite in this form:

$$\alpha\dot{\varepsilon}(t) - \frac{\partial \mathcal{L}}{\partial \varepsilon} - \alpha\left(\dot{\varphi} - \dot{\vartheta}\right) = 0. \tag{44}$$

First of all, we compute $\frac{\partial L}{\partial \varepsilon}$ from equation (4):

$$\frac{\partial \mathcal{L}}{\partial \varepsilon} = \frac{gm}{\rho}\frac{\mu}{M_E}\frac{r^2}{\rho^2}3\cos\varepsilon\sin\varepsilon \simeq 3\frac{Gm\mu}{\rho}\frac{r^2}{\rho^2}\varepsilon, \tag{45}$$

where we have used $g = GM_E$ after developing $\cos\varepsilon\sin\varepsilon$ in power series of ε and keeping only the linear terms in ε.

Recall that μ represents the mass of the ocean's bulges. These bulges can be imagined as an ellipsoid of radii R_E, R_E and $R_E + h$ deprived of a sphere of radius R_E concentric to it, being h the tidal height. Hence:

$$\mu = \delta_W \frac{4}{3}\pi R_E^2 h = \delta_W \frac{4}{3}\pi R_E^2 \frac{3}{2}\frac{m}{M_E}\left(\frac{R_E}{\rho}\right)^3 R_E = \frac{3}{2}\frac{\delta_W}{\delta_E} m \left(\frac{R_E}{\rho}\right)^3. \tag{46}$$

Finally r is the distance between the center of the Earth and the center of mass of each bulge, that is the center of mass of the aforementioned ellipsoid deprived of a sphere. It is a standard calculation to show that $r = \frac{3}{4}R_E$.

Therefore:

$$\frac{\partial \mathcal{L}}{\partial \varepsilon} = \frac{81}{32}\frac{\delta_W}{\delta_E} Gm^2 \frac{R_E^5}{\rho^6}\varepsilon = KGm^2\frac{R_E^5}{\rho^6}\varepsilon, \tag{47}$$

with $K = \frac{81}{32}\frac{\delta_W}{\delta_E}$ a dimensionless constant.

Equation (44) thus becomes:

$$\alpha\dot{\varepsilon} - KGm^2\frac{R_E^5}{\rho^6}\varepsilon - \alpha\left(\dot{\varphi} - \dot{\vartheta}\right) = 0. \tag{48}$$

Finally, we have to consider the time-dependence of ρ and ϑ. We also assume that Ω, ω and a remain constant during each revolution and change their values only at the end of each revolution. This assumption yiedls

$$\vartheta \simeq \lambda + 2e\sin\lambda = \omega t + 2e\sin(\omega t) \implies \dot{\vartheta} \simeq \omega + 2\omega e\cos(\omega t) \tag{49}$$

and

$$\rho(t) = \frac{p}{1 + e\cos(\omega t)} = \frac{a(1-e^2)}{1 + e\cos(\omega t)} \simeq a[1 - e\cos(\omega t)] \implies \frac{1}{\rho^6} \simeq \frac{1}{a^6}[1 + 6e\cos(\omega t)]. \tag{50}$$

So, equation (48) becomes:

$$\dot{\varepsilon} + \frac{\gamma_c}{\alpha}[1 + 6e\cos{(\omega t)}]\varepsilon - \Omega + \omega + 2\omega e\cos(\omega t) = 0, \tag{51}$$

where $\gamma_c = -KGm^2\frac{R_E^5}{a^6}$.
A trivial solution of this equation is:

$$\varepsilon(t) = \frac{\alpha}{\gamma_c}(\Omega - \omega) - e\frac{6\Omega - 4\omega}{\omega}\cos\delta\sin(\omega t + \delta) = A + eB\sin(\omega t + \delta), \tag{52}$$

with $A = \frac{\alpha}{\gamma_c}(\Omega - \omega), B = \frac{6\Omega - 4\omega}{\omega}\cos\delta$ and $\tan\delta = \frac{\gamma_c}{\alpha\omega}$.

Appendix B. Tidal lag and specific dissipation function.

Here we want to discuss in details the classical computations that correlates the tidal lag and the specific dissipation function, presenting it in a form that, it seems to us, is, on one hand, easier to understand, and, on the other hand, allows to grasp all the relevant physical aspects of the system.

A good model to describe the movement of the sea on the Earth is a forced harmonic oscillator in presence of friction.

The unforced oscillation is the oscillation that one could see in the absence of tides, taking as initial condition for the ocean a shape different from the sphere, for instance an ellipsoid. The forcing is due to the presence of tidal forces, that is a harmonic forcing if observed in a fixed point of the Earth.
In analogy with the harmonic oscillator, the specific dissipation function Q is defined as:

$$Q = \frac{2\pi E^*}{\Delta E}, \tag{53}$$

where E^* is the maximum potential energy stored in a cycle and ΔE is the energy dissipated in one cycle.
If we imagine the Ocean's Earth surface as an ellipsoid on which a force field of potential U acts, then the expressions of E^* and ΔE are both linked to the energy dissipated per unit time by the forcing over the whole body:

$$\frac{dE}{dt} = \int_V \rho\vec{v}(r)\cdot\vec{\nabla}U(r)\,d^3r. \tag{54}$$

But we can write:

$$\rho\vec{v}(r)\cdot\vec{\nabla}U(r) = \nabla[\rho\vec{v}(r)U(r)] - U(r)\nabla[\rho\vec{v}(r)] = \nabla[\rho\vec{v}(r)U(r)] - U(r)\frac{\partial\rho}{\partial t}, \tag{55}$$

where in the last equality we have used the continuity equation.
Since for an incompressible fluid $\frac{\partial\rho}{\partial t} = 0$, we have:

$$\frac{dE}{dt} = \int_V \nabla[U(r)\rho\vec{v}(r)]\,d^3r = \int_{\partial V} U(\sigma)\rho\vec{v}(\sigma)\cdot\vec{n}\,d\sigma, \tag{56}$$

where we have used the divergence theorem: the last integral extends over the entire ocean's surface, $d\sigma$ is an infinitesimal portion of this surface and $\vec{n}$ in the outwards normal unit vector in $d\sigma$.
An infinitesimal ocean's surface σ around the point r is subjected to the potential:

$$U(\sigma) = u(\sigma)\cos[2\nu t + \varphi(\sigma)], \tag{57}$$

where $u(\sigma)$ is the amplitude and $\varphi(\sigma)$ is the a phase. It is reasonably to think that both depend on the point r on the ocean's surface. The frequency of this potential is 2ν because in one day the

ocean's surface is subjected to two tidal cycles.
The radial displacement in the position r due to the potential $U(\sigma)$ is:

$$x_\perp(\sigma) = R + A(\sigma)\cos[2\nu t + \varphi(\sigma) + 2\varepsilon], \tag{58}$$

where 2ε is the phase lag in the response $x_\perp(\sigma)$ to the solicitation $U(\sigma)$ due to the geometrical lag ε between the direction of the Moon and the direction of the ocean's bulges.
The time derivative of $x_\perp(\sigma)$ is:

$$\vec{v}(\sigma)\cdot\vec{n} = v_\perp(\sigma) = -2\nu A(\sigma)\sin[2\nu t + \varphi(\sigma) + 2\varepsilon]. \tag{59}$$

Then the (56) becomes:

$$\begin{aligned}\frac{dE}{dt} &= -2\rho\nu\int_{\partial V} u(\sigma)A(\sigma)\cos[2\nu t+\varphi(\sigma)]\sin[2\nu t+\varphi(\sigma)+2\varepsilon]\,d\sigma\\ &= -2\rho\nu\int_{\partial V} u(\sigma)A(\sigma)\cos[2\nu t+\varphi(\sigma)]\{\sin[2\nu t+\varphi(\sigma)]\cos(2\varepsilon)+\cos[2\nu t+\varphi(\sigma)]\sin(2\varepsilon)\}\,d\sigma\\ &= -2\rho\nu\int_{\partial V} u(\sigma)A(\sigma)\{\cos[2\nu t+\varphi(\sigma)]\sin[2\nu t+\varphi(\sigma)]\cos(2\varepsilon)+\cos^2[2\nu t+\varphi(\sigma)]\sin(2\varepsilon)\}\,d\sigma.\end{aligned} \tag{60}$$

We can now compute E^* and ΔE.
Let's start with the latter. ΔE is the energy dissipated in one cycle. It is equal to the energy dissipated in unit time, $\frac{dE}{dt}$ integrated over one period, that is from 0 to $\frac{2\pi}{2\nu} = \frac{\pi}{\nu}$:

$$\Delta E = \int_0^{\frac{\pi}{\nu}} \frac{dE}{dt}\,dt. \tag{61}$$

E^*, the maximum potential energy stored in a cycle, is obtained by integrating $\frac{dE}{dt}$ from 0 to $\frac{T}{4} = \frac{2\pi}{8\nu} = \frac{\pi}{4\nu}$, indeed the energy is stored only in the first quarter of a period:

$$E^* = \int_0^{\frac{\pi}{4\nu}} \frac{dE}{dt}\,dt. \tag{62}$$

To compute this two integrals, we immagine to fix the infinitesimal portion of surface around the point r and calculate $\Delta E(\sigma)$ and $E^*(\sigma)$ locally in that point. In this case $\frac{dE}{dt}$ in (60) becomes a $\frac{dE}{dt}(\sigma)$ and corresponds to the energy dissipated per unit time locally and we should not integrate with respect to σ; moreover, once we fix the point over the surface $A(\sigma)$, $u(\sigma)$ and $\varphi(\sigma)$ are constants and in particular $\varphi(\sigma)$ can be omitted in the arguments of the goniometrical functions (indeed we can rescale the time in such a way that the phase φ becomes 0). So we have:

$$\frac{dE}{dt}(\sigma) = -2\rho\nu u(\sigma)A(\sigma)[\cos(2\nu t)\sin(2\nu t)\cos(2\varepsilon) + \cos^2(2\nu t)\sin(2\varepsilon)]. \tag{63}$$

Now, putting (63) in (61) we obtain:

$$\Delta E(\sigma) = -\pi\rho u(\sigma)A(\sigma)\sin(2\varepsilon) \tag{64}$$

While, putting (63) in (62) we obtain, up to the lowest order in ε:

$$E^*(\sigma) = -\frac{1}{2}\rho u(\sigma)A(\sigma). \tag{65}$$

So, putting (64) and (65) in (53) we obtain the relation between Q and ε, namely:

$$Q = \frac{1}{\sin(2\varepsilon)}, \tag{66}$$

that is the relation well known in literature.
This implies that ε exhibits the same dependence as Q on the frequency of the forcing.
Laboratory experiments show that in viscous solids Q depends on the frequency of the forcing [5].

So both for the oceans on the Earth and for the celestial bodies whose interior is partially molten, ε depends on the frequency of the forcing.

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