Research Article

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Strong Maximum Principle for Some Quasilinear Dirichlet Problems Having Natural Growth Terms

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Abstract: In this paper, dedicated to Laurent Veron, we prove that the Strong Maximum Principle holds for solutions of some quasilinear elliptic equations having lower order terms with quadratic growth with respect to the gradient of the solution.

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1 Introduction

In this paper, dedicated to Laurent Veron, we study the so-called Maximum Principle for some quasilinear elliptic equations with lower order terms having natural (i.e., quadratic) growth with respect to the gradient.

The subject is strongly related to the papers by Laurent Veron and his coauthors [4, 5] and to a previous paper dedicated to Laurent for his sixtieth birthday (see [6]).

In an elliptic boundary value problem like

$$\begin{cases} A(u) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded open set in $\mathbb{R}^N$, A is a second order elliptic operator in divergence form and $f(x) \geq 0$ (of course not zero a.e.), the Weak Maximum Principle states that $u(x) \geq 0$, and it is zero at most in a zero measure set; whereas the Strong Maximum Principle states that the set where $u(x) = 0$ is even "smaller than a zero measure set" (e.g., empty).

In (1.1) the presence of a lower order term can destroy the Maximum Principle property (see [14] and the introduction of [12]). Nevertheless, we will prove the Strong Maximum Principle for some quasilinear elliptic equations with lower order terms having natural (i.e., quadratic) growth with respect to the gradient. One of the main motivations for the interest in quasilinear elliptic equations with lower order terms having quadratic growth with respect to the gradient comes from the Calculus of Variations (in the study of integral functionals). Indeed, consider the following examples:

$$J(v) = \frac{1}{2}\int_\Omega A(x,v)|\nabla v|^2 + \frac{\lambda}{2}\int_\Omega v^2 - \int_\Omega fv,$$

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where Ω is a bounded open subset of $\mathbb{R}^N$, $N > 2$, and $\lambda > 0$. Then the Euler–Lagrange equation for J is (at least formally) the quasilinear elliptic problem

$$\begin{cases} -\operatorname{div}(A(x,u)\nabla u) + \frac{1}{2}A'(x,u)|\nabla u|^2 + \lambda u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Similar problems (not necessarily Euler–Lagrange equations of some functional) have been studied in the literature (see, for example, [3, 7, 9, 10]), and existence of solutions has been proved under different assumptions on the datum f and on the function A. For example, if $A'(x,s)$ is a bounded function, and f belongs to $L^m(\Omega)$, with $m > \frac{N}{2}$, then there exist (see [9, 10]) solutions in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ (the presence of the lower, zero-th order term is fundamental in order to prove existence). If one assumes a *sign condition* on $A'(x,s)$, that is,

$$A'(x,s)\,\operatorname{sgn}(s) \geq \nu > 0, \tag{1.3}$$

then one can prove existence of solutions in $W_0^{1,2}(\Omega)$ under the assumption that f only belongs to $L^1(\Omega)$ (see, for example, [3, 7, 8]).

An example of function $A(x,s)$ satisfying (1.3) is

$$A(x,s) = a(x) + |s|^r,$$

where $0 < \alpha \leq a(x) \leq \beta$ and $r > 1$. In this case, problem (1.2) has been studied in [6], and becomes

$$\begin{cases} -\operatorname{div}([a(x) + |u|^r]\nabla u) + \dfrac{r}{2}u|u|^{r-2}|\nabla u|^2 = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

In this paper, we will prove that if $f \geq 0$ (and, of course, is not identically zero), then the Strong Maximum Principle holds for equations similar to (1.2) and (1.4), not necessarily Euler–Lagrange equations of some functional of the Calculus of Variations, assuming in Section 2 that the sign condition (1.3) does not hold (and we will consider "regular" data and bounded solutions), or in Section 3 that it holds (assuming $L^1(\Omega)$ data and unbounded solutions). In the final Section 4, we will prove existence of solutions, and validity of the Strong Maximum Principle, for a quasilinear elliptic equation with a "control" between the datum f and the lower order term $\lambda(x)u$.

2 General Case and Bounded Solutions

In this section we will deal with a problem similar to (1.2), under no sign condition on the lower order, quadratic gradient term, and under the assumption that $f \geq 0$ is sufficiently "regular" in order to have bounded solutions.

More precisely, let Ω be a bounded, open subset of $\mathbb{R}^N$, $N \geq 2$, and let $a(x)$, $b(x)$ be measurable functions such that

$$0 < \alpha \leq a(x) \leq \beta \tag{2.1}$$

and

$$|b(x)| \leq \gamma, \tag{2.2}$$

where α, β, γ are positive real numbers. Let also f be a function such that

$$f \in L^m(\Omega), \quad m > \frac{N}{2}. \tag{2.3}$$

In [10], existence of weak bounded solutions for general quasilinear Dirichlet problems with lower order terms having natural growth with respect to the gradient is proved. Here, in order to simplify our discussion, we confine ourselves to the simpler, but important case

$$u \in W_0^{1,2}(\Omega): \; -\operatorname{div}([a(x) + |u|^q]\nabla u) + \lambda u + b(x)|u|^p|\nabla u|^2 = f, \tag{2.4}$$

under the assumptions $\lambda > 0$, $p, q \geq 0$, (2.1), (2.2), and (2.3).

A weak solution u in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ of (2.4) is a function such that

$$\int_\Omega [a(x) + |u|^q]\nabla u \nabla \phi + \lambda \int_\Omega u\phi + \int_\Omega b(x)|u|^p|\nabla u|^2\phi = \int_\Omega f\phi, \tag{2.5}$$

for every $\phi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, and in [10] it is proved the existence of such a solution.

If, in addition, we suppose that $f \geq 0$, then $u \geq 0$.

Theorem 2.1 (Weak Maximum Principle). *If $f \geq 0$, then the weak solution u given by* [10] *is such that $u \geq 0$ almost everywhere in Ω.*

Proof. We rewrite (2.4) as

$$-\operatorname{div}([a(x) + |u|^q]\nabla u) + \lambda u + b(x)^+|u|^p|\nabla u|^2 = y(x), \tag{2.6}$$

where we have defined $y(x) = f(x) + b(x)^-|u|^p|\nabla u|^2$. In the weak formulation of (2.6), we use as test function $g(u)$, where the real function $g(t)$ is defined as

$$g(t) = \begin{cases} 0 & \text{if } t \geq 0, \\ 1 - \mathrm{e}^{-At} & \text{if } t < 0, \end{cases}$$

with $A > 0$ to be chosen later. Note that $g(u)$ is an admissible test function, since $g(u)$ belongs to $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. Then

$$\int_\Omega [a(x) + |u|^q]|\nabla u|^2 g'(u) + \lambda \int_\Omega u g(u) + \int_\Omega b(x)^+|u|^p|\nabla u|^2 g(u) = \int_\Omega y(x)g(u).$$

Since $g(u) = 0$, where $u \geq 0$, we have that

$$\int_{\{u<0\}} [a(x) + |u|^q]|\nabla u|^2 g'(u) + \lambda \int_{\{u<0\}} u g(u) + \int_{\{u<0\}} b(x)^+|u|^p|\nabla u|^2 g(u) = \int_{\{u<0\}} y(x)g(u),$$

that is, if $M = \|u\|_{L^\infty(\Omega)}$,

$$\int_{\{u<0\}} |\nabla u|^2[\alpha g'(u)] + \lambda \int_{\{u<0\}} u g(u) \leq \int_{\{u<0\}} y(x)g(u) + \int_{\{u<0\}} |\nabla u|^2[-\gamma M^p g(u)].$$

The choice $A = \frac{\gamma M^p}{\alpha}$ implies $\alpha g'(u) + \gamma M^p g(u) = 0$, so that we have

$$\lambda \int_{\{u<0\}} u g(u) \leq \int_{\{u<0\}} y(x)g(u) \leq 0,$$

which can be rewritten as

$$\lambda \int_{\{u<0\}} u \frac{\mathrm{e}^{Au} - 1}{\mathrm{e}^{Au}} \leq 0.$$

On the other hand, on the set $\{u < 0\}$, $u[\mathrm{e}^{Au} - 1]$ is positive; this implies that the set $\{u < 0\}$ must have zero measure. □

If $f \geq 0$ (and not identically zero), then the solution u is not only positive, but the Strong Maximum Principle holds.

Theorem 2.2 (Strong Maximum Principle). *If $f \geq 0$ (and not almost everywhere equal to zero), then for every set $\omega \subset\subset \Omega$, there exists $m_\omega > 0$ such that $u(x) \geq m_\omega$ almost everywhere in ω.*

Proof. By the previous result, $u \geq 0$, so it is a solution of

$$-\operatorname{div}([a(x) + u^q]\nabla u) + b(x)u^p|\nabla u|^p + \lambda u = f.$$

Let $0 \leq \phi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ and let $h(t)$ be the real decreasing function $h(t) = \mathrm{e}^{-\frac{\gamma M^p}{\alpha}t}$, where $M = \|u\|_{L^\infty(\Omega)}$.

In (2.5) (the weak formulation of (2.4)), we use $\phi h(u)$ as test function to obtain

$$\int_\Omega [a(x)+u^q]|\nabla u|^2 h'(u)\phi + \int_\Omega b(x)u^p|\nabla u|^2 h(u)\phi + \int_\Omega [a(x)+u^q]h(u)\nabla u\nabla\phi + \lambda\int_\Omega u\phi h(u) = \int_\Omega f\phi h(u).$$

Therefore (since $h(t)$ is decreasing), using (2.1) we have

$$\int_\Omega |\nabla u|^2[\alpha h'(u)]\phi + \int_\Omega |\nabla u|^2[\gamma M^p h(u)]\phi + \int_\Omega [a(x)+u^q]h(u)\nabla u\nabla\phi + \lambda\int_\Omega uh(u)\phi \ge \int_\Omega fh(u)\phi.$$

Note that $\alpha h'(u) + \gamma M^p h(u) = 0$, so that we have

$$\int_\Omega \frac{a(x)+u^q}{e^{\frac{\gamma M^p}{\alpha}u}}\nabla u\nabla\phi + \lambda\int_\Omega u\frac{1}{e^{\frac{\gamma M^p}{\alpha}u}}\phi \ge \int_\Omega f\frac{1}{e^{\frac{\gamma M^p}{\alpha}u}}\phi \ge \int_\Omega f\frac{1}{e^{\frac{\gamma M^{p+1}}{\alpha}}}\phi.$$

Observe that

$$0 \le \frac{1}{e^{\frac{\gamma M^p}{\alpha}u}} \le 1,$$

so that we have that u is a positive, bounded function such that

$$\int_\Omega \frac{a(x)+u^q}{e^{\frac{\gamma M^p}{\alpha}u}}\nabla u\nabla\phi + \lambda\int_\Omega u\frac{1}{e^{\frac{\gamma M^p}{\alpha}u}}\phi \ge \int_\Omega f\frac{1}{e^{\frac{\gamma M^{p+1}}{\alpha}}}\phi, \tag{2.7}$$

for every $\phi \ge 0$ in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$. Define

$$A(x) = \frac{a(x)+u^q}{e^{\frac{\gamma M^p}{\alpha}u}},$$

and consider the solution w in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$ of the linear Dirichlet problem

$$\int_\Omega A(x)\nabla w\nabla v + \lambda\int_\Omega wv = \int_\Omega \frac{f}{e^{\frac{\gamma M^{p+1}}{\alpha}}}v \quad \text{for all } v \in W_0^{1,2}(\Omega). \tag{2.8}$$

Since

$$\alpha_0 = \frac{\alpha}{e^{\frac{\gamma M^{p+1}}{\alpha}}} \le \frac{a(x)+u^q}{e^{\frac{\gamma M^p}{\alpha}u}} \le \beta + M^q = \beta_0,$$

the weak solution w exists, is bounded (see [13]) and satisfies the Strong Maximum Principle (see [11, 13, 14]), that is, there exists $m_\omega > 0$ such that

$$w(x) \ge m_\omega \quad \text{almost everywhere in } \omega.$$

Now we subtract the equation satisfied by w from inequality (2.7) satisfied by u, to have that

$$\int_\Omega A(x)\nabla(u-w)\nabla\phi + \lambda\int_\Omega (u-w)\phi \ge 0,$$

for every $\phi \ge 0$ in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$. The choice $\phi = (u-w)^-$, and the fact that $A(x) \ge \alpha_0 > 0$, implies that $(u-w)^- = 0$, that is, $u \ge w$. Then, for every $\omega \subset\subset \Omega$, there exists a strictly positive constant m_ω such that $u(x) \ge m_\omega > 0$ almost everywhere in ω, and the proof is complete. □

Remark 2.3. Note that, even though the equation in (2.4) is *nonlinear*, the proof of the Strong Maximum Principle uses the fact that it holds for the *linear* equation (2.8) satisfied by the function w.

Remark 2.4. Suppose now to have a positive, bounded *supersolution* of (2.4), that is, a function $w \ge 0$ in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$ such that

$$\int_\Omega [a(x)+w^q]\nabla w\nabla\phi + \int_\Omega b(x)w^p|\nabla w|^2\phi + \lambda\int_\Omega w\phi = \int_\Omega g\phi,$$

for every ϕ in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$, with $g \ge 0$ in $L^m(\Omega)$, $m > \frac{N}{2}$; then the same proof of Theorem 2.2 yields that w satisfies the Strong Maximum Principle.

3 Lower Order Terms with Sign Condition and Unbounded Solutions

In this section we will assume a sign condition on the lower order, quadratic gradient term, and will suppose that f is a positive function belonging to $L^1(\Omega)$. For $k \geq 0$ and t in $\mathbb{R}$, define

$$T_k(s) = \max(-k, \min(s, k)).$$

In [6], the existence of weak solutions for the Dirichlet problem

$$u \in W_0^{1,2}(\Omega): \ -\operatorname{div}([a(x) + |u|^q]\nabla u) + b(x)u|u|^{p-1}|\nabla u|^2 = f, \tag{3.1}$$

under various assumptions on the datum f with respect to the two positive parameters p and q, was proved. More precisely, assuming (2.1) and that $0 < \nu \leq b(x) \leq \gamma$, if f belongs to $L^m(\Omega)$, then we have the following:

(a) if $m = 1$, and $p \geq 2q$,

(b) or if $\frac{2N(q+1)}{2N+p(N-2)+4q} \leq m \leq \frac{N}{2}$, and $2q \geq p \geq q - 1$,

(c) or if $\frac{2N}{N+2} \leq m \leq \frac{N}{2}$, and $q \geq 1$, $2p \geq q - 1 \geq p$,

then there exists a weak solution of (3.1), that is, a function u in $W_0^{1,2}(\Omega)$ such that

$$b(x)|u|^p|\nabla u|^2 \in L^1(\Omega),$$

and

$$\int_\Omega [a(x) + u^q]\nabla u \nabla\varphi + \int_\Omega b(x)u^p|\nabla u|^2\varphi = \int_\Omega f\varphi, \tag{3.2}$$

for every $\varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.

As for problem (2.4), if $f \geq 0$, then $u \geq 0$.

Theorem 3.1 (Weak Maximum Principle). *If $f \geq 0$, then the solution u of* (3.1) *given by* [6] *is such that $u \geq 0$.*

Proof. In (3.2), take $-T_1(u^-)$ as test function. Then

$$-\int_\Omega [a(x) + u^q]\nabla u \nabla T_1(u^-) - \int_\Omega b(x)u|u|^{p-1}|\nabla u|^2 T_1(u^-) = -\int_\Omega f T_1(u^-),$$

that is,

$$\int_\Omega [a(x) + |u|^q]\nabla T_1(u^-)\nabla T_1(u^-) + \int_\Omega b(x)|u^-|^p|\nabla u^-|^2 T_1(u^-) \leq 0,$$

which implies $T_1(u^-) \leq 0$, for every $k > 0$; then $u^- = 0$, and so $u \geq 0$. □

As in Section 2, the Strong Maximum Principle holds.

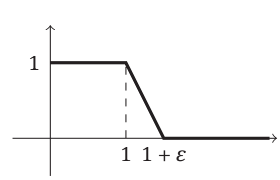
Theorem 3.2 (Strong Maximum Principle). *If $f \geq 0$ (and not almost everywhere equal to zero), then for every set $\omega \subset\subset \Omega$, there exists $m_\omega > 0$ such that $u(x) \geq m_\omega$ almost everywhere in ω.*

Proof. Since u is positive by the previous result, we have that u is a solution of

$$\int_\Omega [a(x) + u^q]\nabla u \nabla\varphi + \int_\Omega b(x)u^p|\nabla u|^2\varphi = \int_\Omega f\varphi,$$

for every φ in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. Let $\varepsilon > 0$, and let, for $t \geq 0$,

$$\rho_\varepsilon(t) = 1 - T_1\Big(\frac{(t-1)^+}{\varepsilon}\Big).$$



Let also $\phi \geq 0$ be a function in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, and choose $\varphi = \rho_\varepsilon(u)\phi$ as test function in the equation solved by u. We obtain, since $\rho_\varepsilon'(t) = -\frac{1}{\varepsilon}$ for $1 < t < 1 + \varepsilon$ and zero otherwise,

$$-\frac{1}{\varepsilon} \int\limits_{\{1 \leq u \leq 1+\varepsilon\}} [a(x) + u^q]|\nabla u|^2 \phi + \int\limits_\Omega [a(x) + u^q]\nabla u \nabla \phi \rho_\varepsilon(u) + \int\limits_\Omega b(x) u^p |\nabla u|^2 \rho_\varepsilon(u)\phi = \int\limits_\Omega f \rho_\varepsilon(u)\phi.$$

We observe now that the first term is negative, so that we have

$$\int\limits_\Omega [a(x) + u^q]\nabla u \nabla \phi \rho_\varepsilon(u) + \int\limits_\Omega b(x) u^p |\nabla u|^2 \rho_\varepsilon(u)\phi \geq \int\limits_\Omega f \rho_\varepsilon(u)\phi \geq \int\limits_\Omega T_1(f) \rho_\varepsilon(u)\phi.$$

Letting ε tend to zero, and observing that $\rho_\varepsilon(t)$ converges to the characteristic function of the set $\{0 \leq t \leq 1\}$, we thus have, by Lebesgue's theorem,

$$\int\limits_\Omega [a(x) + u^q]\nabla u \nabla \phi \chi_{\{0 \leq u \leq 1\}} + \int\limits_\Omega b(x) u^p |\nabla u|^2 \phi \chi_{\{0 \leq u \leq 1\}} \geq \int\limits_\Omega T_1(f) \phi \chi_{\{0 \leq u \leq 1\}}.$$

Since $\nabla T_1(u) = \nabla u \chi_{\{0 \leq u \leq 1\}}$, we can rewrite the above inequality as

$$\int\limits_\Omega [a(x) + T_1(u)^q]\nabla T_1(u) \nabla \phi + \int\limits_\Omega b(x) T_1(u)^p |\nabla T_1(u)|^2 \phi \geq \int\limits_\Omega g\phi,$$

where we have defined $g = T_1(f)\chi_{\{0 \leq u \leq 1\}}$. Thus, we have proved that $w = T_1(u)$ is a *bounded and positive supersolution* of the equation

$$-\operatorname{div}([a(x) + w^q]\nabla w) + b(x) w^p |\nabla w|^2 = g.$$

Thanks to Remark 2.4, w satisfies the Strong Maximum Principle. Therefore, for every $\omega \subset\subset \Omega$, there exists a constant $m_\omega > 0$ such that $w(x) \geq m_\omega$ almost everywhere in ω. Recalling that $w = T_1(u)$, we thus have that

$$u(x) \geq T_1(u(x)) = w(x) \geq m_\omega > 0 \quad \text{almost everywhere in } \omega,$$

and the proof is complete. □

4 "Controlled" Lower Order Terms

In this section (following [1] and [2]) we will consider a slightly different equation from (2.4) or (3.1). Namely, we will consider the equation

$$-\operatorname{div}([a(x) + u^q]\nabla u) + b(x) u |u|^{p-1} |\nabla u|^2 + \lambda(x) u = f, \tag{4.1}$$

with a such that (2.1) holds, $p, q \geq 0$, b a measurable function such that

$$0 < \nu \leq b(x) \leq \gamma, \tag{4.2}$$

$\lambda(x)$ a function in $L^1(\Omega)$, and f a function such that

$$0 \leq f(x) \leq Q\lambda(x), \tag{4.3}$$

for some $Q > 0$. Our result is the following.

Theorem 4.1. *Under assumptions* (2.1), (4.2) *and* (4.3), *there exists a solution u in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ of* (4.1). *Furthermore, u satisfies the Strong Maximum Principle.*

Proof. For every n in $\mathbb{N}$ there exists (see [7]) a solution $u_n \geq 0$ in $W_0^{1,2}(\Omega)$ of

$$-\operatorname{div}([a(x) + T_n(u_n)^q]\nabla u_n) + b(x) u_n^p |\nabla u_n|^p + \lambda(x) u_n = f,$$

that is, one has

$$\int_\Omega [a(x) + T_n(u_n)^q]\nabla u_n \nabla\phi + \int_\Omega b(x)u_n^p|\nabla u_n|^2\phi + \int_\Omega \lambda(x)u_n\phi = \int_\Omega f\phi,$$

for every ϕ in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$. Choosing $\phi = T_1((u_n - Q)^+)$ (which is admissible since ϕ belongs to $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$), we obtain

$$\int_{\{Q\le u_n\le Q+1\}} [a(x) + T_n(u_n)^q]|\nabla u_n|^2 + \int_\Omega b(x)u_n^p|\nabla u_n|^2 T_1((u_n - Q)^+) + \int_\Omega \lambda(x)u_n T_1((u_n - Q)^+) = \int_\Omega f T_1((u_n - Q)^+).$$

Since the first and the second term are positive, we thus have that

$$\int_\Omega \lambda(x)u_n T_1((u_n - Q)^+) \le \int_\Omega f T_1((u_n - Q)^+) \le Q\int_\Omega \lambda(x)T_1((u_n - Q)^+),$$

which can be rewritten as

$$\int_\Omega \lambda(x)(u_n - Q)T_1((u_n - Q)^+) \le 0.$$

Since $(u_n - Q)T_1((u_n - Q)^+) \ge 0$ almost everywhere, we have that

$$\int_\Omega \lambda(x)(u_n - Q)T_1((u_n - Q)^+) = 0,$$

which then implies that $0 \le u_n \le Q$ almost everywhere; that is, u_n belongs to $L^\infty(\Omega)$ and $\|u_n\|_{L^\infty(\Omega)} \le Q$. We now choose $n > Q$ and define $u = u_n$. Since $0 \le u \le Q$, we have $T_n(u) = u$, so that u is a solution in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$ of

$$-\operatorname{div}([a(x) + u^q]\nabla u) + b(x)u^p|\nabla u|^p + \lambda(x)u = f.$$

Since u is positive and bounded, we can repeat the same proof of Theorem 3.2 to prove that u satisfies the Strong Maximum Principle; that is, for every $\omega \subset\subset \Omega$, there exists $m_\omega > 0$ such that

$$u(x) \ge m_\omega \quad \text{almost everywhere in } \omega. \qquad \square$$

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