

Research Article

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Relaxation of functionals with linear growth: Interactions of emerging measures and free discontinuities

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Abstract: For an integral functional defined on functions $(u, v) \in W^{1,1} \times L^1$ featuring a prototypical strong interaction term between u and v , we calculate its relaxation in the space of functions with bounded variations and Radon measures. Interplay between measures and discontinuities brings various additional difficulties, and concentration effects in recovery sequences play a major role for the relaxed functional even if the limit measures are absolutely continuous with respect to the Lebesgue one.

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1 Introduction

Oscillations and/or concentrations appear naturally in many problems in the calculus of variations, partial differential equations, and optimal control theory due to the lack of convexity properties and/or compactness. Concentrations usually do not play such a prominent role in minimization problems for integral functionals with superlinear growth because of various decomposition lemmas (see, e.g., [22]) allowing us to show that the integrand is equi-integrable along minimizing sequences. However, concentrations are a key issue in problems with only linear coercivity. This phenomenon is intimately connected with nonreflexivity of the underlying spaces L^1 and $W^{k,1}$ where such problems are usually formulated.

Minimization problems for a functional

$$F : W^{1,p}(\Omega; \mathbb{R}^m) \times L^q(\Omega; \mathbb{R}^d) \rightarrow \mathbb{R},$$

where $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain, were considered in, e.g., [16] with $p = 1$, $q = +\infty$, in [11, 12] with $p = 1$, $q \in (1, \infty]$ and in [17] with $p, q > 1$ for functionals modeling energy of multiphase materials. There,

$$F(u, v) := \int_{\Omega} \psi(\nabla u(x), v(x)) \, dx,$$

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with ψ a material stored energy density, u an elastic deformation and v denoting a chemical composition. Other examples include, e.g., magnetoelasticity [15, 17] where u is again a deformation mapping and v is a magnetization vector, models of particle inclusions in elastic matrices [9], or elastoplasticity where v plays the role of the plastic strain. In this article, we focus on $p = q = 1$ and study a simplified model problem for the general case:

$$F(u, v) := \int_{\Omega} \psi(x, u(x), \nabla u(x), v(x)) \, dx, \quad (1.1)$$

whose main feature is a “strong” interaction term between u and v of the form $f_1(u)f_2(v)$ where f_2 has positive linear growth.

The linear growth $p = q = 1$ combines oscillation effects that can be caused by nonconvexity of the energy density in ∇u and v with possible concentrations of minimizing sequences related to the lack of weak compactness of $W^{1,1}(\Omega; \mathbb{R}^m) \times L^1(\Omega; \mathbb{R}^d)$. The latter inevitably calls for an extension of our minimization problem to a large space possessing better compactness properties. A natural choice is the space of functions of bounded variations $BV(\Omega; \mathbb{R}^m)$ for the variable u , and the Radon measures on $\bar{\Omega}$ and $\mathcal{M}(\bar{\Omega}; \mathbb{R}^d)$ for v . Correspondingly, F must be extended to $BV(\Omega; \mathbb{R}^m) \times \mathcal{M}(\bar{\Omega}; \mathbb{R}^d)$, which we perform by calculating an explicit representation of $\bar{F} := \Gamma\text{-}\liminf F$ with respect to the weak*-topology in $BV \times L^1$; cf. [14]. This extension procedure is called *relaxation* and provides us with a weak*-lower semicontinuous functional. Relaxation results for ψ quasiconvex and independent of v and u were first proven by Ambrosio and Dal Maso in [2] and then extended by Fonseca and Müller to the case $\psi = \psi(x, u, \nabla u)$ and $\psi(x, u, \cdot)$ quasiconvex in [20, 21]. A new proof under weaker assumptions was recently given by Rindler and Shaw in [29].

If the variable v is included in ψ and we consider F from (1.1), then we refer to [27, 28] for relaxation results if $p = 1$ and $q \geq 1$. However, this result uses the L^1 -weak topology for approximation of v , thereby completely ruling out concentrations and avoiding general measures ν outside of L^1 . In the context of dimension reduction, related results for the case without explicit dependence on u can be found in [4].

Interesting phenomena are expected to occur if concentrations are combined with discontinuities, because discontinuous functions do not belong to the predual space of Radon measures. In particular, such phenomena naturally appear in impulse control. For instance, application of a drug may lead to an “instant” change of conditions of a patient, the ignition of the engine makes the space shuttle “suddenly” change its position. Besides, interactions of oscillations, concentrations and discontinuities lead to interesting questions and challenges for mathematical research; see, e.g., [23, 24] for some recent results. For suitable examples, the relaxation results presented below allow us to observe the natural formation of all three phenomena just described in minimizing sequences.

In the following, $\Omega \subset \mathbb{R}^n$ is a bounded open set with Lipschitz boundary. For every

$$(u, v) \in W^{1,1}(\Omega; \mathbb{R}^m) \times L^1(\Omega; \mathbb{R}^d),$$

we consider the functional

$$F(u, v) := \int_{\Omega} f_1(u)f_2(v) \, dx + \int_{\Omega} W(\nabla u) \, dx, \quad (1.2)$$

where $f_1 : \mathbb{R}^m \rightarrow \mathbb{R}$, $f_2 : \mathbb{R}^d \rightarrow \mathbb{R}$ and $W : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$ are continuous functions such that the following conditions hold:

(H1) There exist $C_2 > C_1 > 0$ such that for every $a \in \mathbb{R}^m$,

$$C_1 \leq f_1(a) \leq C_2.$$

(H2) There exists $K > 0$, such that for every $b \in \mathbb{R}^d$,

$$K^{-1}|b| \leq f_2(b) \leq K(1 + |b|).$$

(H3) There exists $\kappa > 0$ such that for every $\xi \in \mathbb{R}^{m \times n}$,

$$\kappa^{-1}|\xi| \leq W(\xi) \leq \kappa(1 + |\xi|).$$

We aim at giving an integral representation of the sequentially lower semicontinuous envelope of F with respect to the $BV \times \mathcal{M}$ weak* convergence, namely

$$\begin{aligned} \bar{\mathcal{F}}(u, v) := \inf_{k \rightarrow +\infty} \{ \liminf F(u_k, v_k) \mid (u_k, v_k) \in W^{1,1}(\Omega; \mathbb{R}^m) \times L^1(\Omega; \mathbb{R}^d), \\ (u_k, v_k) \xrightarrow{*} (u, v) \text{ in } BV(\Omega; \mathbb{R}^m) \times \mathcal{M}(\bar{\Omega}; \mathbb{R}^d) \}. \end{aligned} \quad (1.3)$$

Making use of the Radon–Nikodým decomposition of a measure $\mu = \mu^a + \mu^s$ given in (2.4), we prove the following two main results.

Theorem 1.1 (Relaxation theorem. The case $n \geq 2$). *Let $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Under the assumptions (H1)–(H3), we have that*

$$\begin{aligned} \bar{\mathcal{F}}(u, v) = \int_{\Omega} \Omega W(\nabla u) \, dx + \int_{\Omega} (\Omega W)^{\infty} \left(\frac{dDu^s}{d|Du^s|} \right) d|Du^s| \\ + \int_{\Omega} g\left(u, \frac{dv^a}{d\mathcal{L}^n}\right) dx + \int_{\Omega} f_1^{\min}(f_2^{**})^{\infty} \left(\frac{dv^s}{d|v^s|} \right) d|v^s| \end{aligned} \quad (1.4)$$

for every $u \in BV(\Omega; \mathbb{R}^m)$ and $v \in \mathcal{M}(\bar{\Omega}; \mathbb{R}^d)$. Here, ΩW denotes the quasiconvex envelope of W (see Section 2 below),

$$g(a, b) := \min \{ f_1(a) f_2^{**}(b_1) + f_1^{\min}(f_2^{**})^{\infty}(b_2) \mid b_1, b_2 \in \mathbb{R}^d, b_1 + b_2 = b \}, \quad (1.5)$$

f_2^{**} denotes the bipolar function of f_2 , which, in view of (H2), coincides with the greatest lower semicontinuous and convex function below f_2 , the superscript “ ∞ ” denotes the recession functions of the above-mentioned envelopes, and

$$f_1^{\min} := \inf_{a \in \mathbb{R}^m} \{ f_1(a) \} \geq C_1. \quad (1.6)$$

For the one-dimensional case, yet another decomposition of measures is relevant, namely $\mu = \mu^0 + \mu^{\text{diff}}$ decomposed into its atomic part μ^0 and the “diffuse” rest μ^{diff} ; cf. (4.1).

Theorem 1.2 (Relaxation theorem: The case $n = 1$). *Suppose $\Omega = (\alpha, \beta)$ is a bounded open interval, suppose $u \in BV(\Omega; \mathbb{R}^m)$, $v \in \mathcal{M}(\bar{\Omega}; \mathbb{R}^d)$, and suppose that (H1)–(H3) hold. Then we have that*

$$\begin{aligned} \bar{\mathcal{F}}(u, v) = \int_{\Omega} (f_1(u(x)) df_2^{**}(v^{\text{diff}})(x) + dW^{**}(Du^{\text{diff}})(x)) + \sum_{x \in S^0} f_W^0(u(x^+), u(x^-), v^0(\{x\})) \\ + \inf_{z \in \mathbb{R}^m} f_W^0(u(\alpha^+), z, v^0(\{\alpha\})) + \inf_{z \in \mathbb{R}^m} f_W^0(z, u(\beta^-), v^0(\{\beta\})). \end{aligned} \quad (1.7)$$

Here, f_2^{**} and W^{**} are the convex hulls of f_2 and W , respectively,

$$S^0 := \{x \in \Omega \mid |v|(\{x\}) + |Du|(\{x\}) \neq 0\}$$

is the (at most countable) set charged by atomic contributions in the interior of Ω , and the associated density $f_W^0 : \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^d \rightarrow \mathbb{R}$ is given by

$$f_W^0(a^+, a^-, b) := \inf_{\substack{u \in W^{1,1}((-1,1); \mathbb{R}^m) \\ v \in L^1((-1,1); \mathbb{R}^d) \\ u(-1) = a^-, u(1) = a^+ \\ \int_{-1}^1 v \, dx = b}} \left\{ \int_{-1}^1 (f_1(u)(f_2^{**})^{\infty}(v) + (W^{**})^{\infty}(u')) \, dx \right\}. \quad (1.8)$$

In addition, for the diffuse contributions we used the following abbreviations for nonlinear transformations of measures defined with the help of recession functions (cf. Section 2.4):

$$\begin{aligned} df_2^{**}(v^{\text{diff}})(x) &= f_2^{**} \left(\frac{dv^{\text{diff},a}}{dx}(x) \right) dx + (f_2^{**})^{\infty} \left(\frac{dv^{\text{diff},s}}{d|v^{\text{diff},s}|}(x) \right) d|v^{\text{diff},s}|(x), \\ dW^{**}(Du^{\text{diff}})(x) &= W^{**}(u'(x)) dx + (W^{**})^{\infty} \left(\frac{Du^c}{d|Du^c|}(x) \right) d|Du^c|(x), \end{aligned}$$

where

$$v^{\text{diff}} = v^{\text{diff},a} + v^{\text{diff},s} \quad \text{and} \quad Du^{\text{diff}} = Du^{\text{diff},a} + Du^{\text{diff},s} = \nabla u \mathcal{L}^1 + Du^c$$

are split into absolutely continuous and singular parts with respect to the Lebesgue measure \mathcal{L}^1 . Notice that, in case of Du^{diff} , this can be expressed using the approximate gradient u' and the Cantor part Du^c of Du . Moreover, $u(\alpha^+)$ and $u(\beta^-)$ denote the traces of u at the boundary, and similarly $u(x^+)$ and $u(x^-)$ denote the traces of u from the right and from the left, respectively, at an interior point x .

Remark 1.3. Both relaxation theorems have easily deduced variants in the space $BV(\Omega; \mathbb{R}^m) \times \mathcal{M}(\Omega; \mathbb{R}^d)$ (measures on Ω instead of $\bar{\Omega}$). The integral representations for the relaxed functionals in that case are obtained by simply dropping all contributions on $\partial\Omega$ in (1.4) and (1.7), respectively.

Remark 1.4. Both relaxation theorems still hold if we impose an additional mass constraint on v , i.e., $\int_{\Omega} v(x) dx = M$ with a constant M , which turns into $v(\bar{\Omega}) = M$ for the relaxation. To see this, the recovery sequences in the proofs (for the upper bound) have to be modified slightly, artificially correcting the mass, which is easily done with negligible error for the energy. This is actually the reason we chose to study the relaxation problem with the measures v on the closure $\bar{\Omega}$, as a mass constraint is in general lost along weak*-convergence in $\mathcal{M}(\Omega; \mathbb{R}^d)$.

Remark 1.5. In essence, the difference between the cases $n = 1$ and $n \geq 2$ arises from the fact that points have 1-capacity zero for $n \geq 2$, but not for $n = 1$. As a consequence, for $n \geq 2$, it is possible to change the value of u to optimize $f_1(u)$ locally near points that get charged by v or its recovery sequence, without noticeable energy cost paid in W . This ultimately leads to the natural appearance of f_1^{\min} in the relaxation formula for $n \geq 2$ and can in fact make concentration effects in the recovery sequence for v energetically favorable even if the limit state v is absolutely continuous with respect to the Lebesgue measure; see Example 3.3 and Remark 3.4.

Remark 1.6. For $n \geq 2$, Theorem 1.1 can be easily modified to include the case $1 < p \leq n$, with $u \in W^{1,p}$ and using $W^{1,p} \ni u_k \rightharpoonup u$ weakly in $W^{1,p}$ in (1.3) (instead of $W^{1,1} \ni u_k \rightharpoonup^* u$ weakly* in BV). In that scenario, the correct relaxed energy density is simply the restriction of (1.4) to $u \in W^{1,p}$, i.e., the term with $(\Omega W)^{\infty}$ is removed as $Du^s = 0$. Apart from a few simplifications natural in $W^{1,p}$, the proof can essentially be followed step by step. The only point where one has to be careful is the upper bound (cf. Section 3.2), more precisely, the function φ , which now has to be $W^{1,p}$ to still allow the cut-off arguments based on it. This is possible only for $p \leq n$, as for large p any function in $W^{1,p}$ is locally bounded by embedding. (The relaxation for $p > n$ is a much more classic problem where $u_k \rightarrow u$ uniformly and the variable for u in the relaxation formula simply gets frozen instead of the more complex phenomena we observe here.)

The plan of the rest of the paper is as follows. First, we introduce necessary notation in Section 2. A proof of Theorem 1.1 is contained in Section 3 together with a few remarks. We also indicate there possible generalizations and give an example where the relaxed functional is calculated explicitly. Finally, a proof of Theorem 1.2 can be found in Section 4.

2 Notation and preliminary results

In the following, Ω is a bounded open set of \mathbb{R}^n , $n \geq 2$ and $\mathcal{A}(\Omega)$ stands for the family of open subsets of Ω .

2.1 Radon measures

By $\mathcal{M}(\Omega)$ we denote the set of signed Radon measures on Ω , for the \mathbb{R}^d -valued we use the symbol $\mathcal{M}(\Omega; \mathbb{R}^d)$. We write $\text{Supp } \mu$ to denote the support of $\mu \in \mathcal{M}(\Omega; \mathbb{R}^d)$. Any element $\mu \in \mathcal{M}(\Omega; \mathbb{R}^d)$ can be decomposed as

$$\mu := \mu^a + \mu^s, \tag{2.1}$$

where μ^a is absolutely continuous with respect to the n -dimensional Lebesgue measure \mathcal{L}^n , and μ^s is the singular part. A particularly important example of a scalar-valued, purely singular measure is the *Dirac mass* δ_z charging a single point $z \in \mathbb{R}^n$. For any Borel set $A \subset \mathbb{R}^n$, it is defined by $\delta_z(A) := 1$ if $z \in A$, and $\delta_z(A) := 0$ otherwise.

If $\mu \in \mathcal{M}(\Omega; \mathbb{R}^d)$ and $\lambda \in \mathcal{M}(\Omega)$ is a nonnegative Radon measure, we denote by $\frac{d\mu}{d\lambda}$ the Radon–Nikodým derivative of μ with respect to λ . By a generalization of the Besicovitch differentiation theorem (see [19, Theorem 1.153 and related results in Sections 1.2.1 and 1.2.2]), it can be proved that there exists a Borel set $E \subset \Omega$ such that $\lambda(E) = 0$ and

$$\frac{d\mu}{d\lambda}(x) = \lim_{\varepsilon \rightarrow 0^+} \frac{\mu(x + \varepsilon C)}{\lambda(x + \varepsilon C)}$$

for all $x \in \text{Supp } \mu \setminus E$ and any open convex set C containing the origin. We recall that the exceptional set E does not depend on C . Also recall that almost every point in \mathbb{R}^n is a Lebesgue point if $f \in L^1_{\text{loc}}(\mathbb{R}^n, \mu)$ and μ is a Radon measure, i.e.,

$$\lim_{\varepsilon \rightarrow 0^+} \frac{1}{\mu(x + \varepsilon C)} \int_{x + \varepsilon C} |f(y) - f(x)| d\mu(y) = 0$$

for μ -a.e. $x \in \mathbb{R}^n$ and for every bounded, convex, open set C containing the origin.

Analogous results hold for Radon measures on the compact set $\bar{\Omega}$, and we use analogous notation for this case, consistently replacing Ω by $\bar{\Omega}$ above. If necessary, every measure in $\mathcal{M}(\bar{\Omega})$ is understood to be extended by zero outside of $\bar{\Omega}$.

In what follows, for every measure $\mu \in \mathcal{M}(\Omega; \mathbb{R}^d)$ we will identify the measure μ^a with its density $\frac{d\mu^a}{d\mathcal{L}^n}$.

2.2 Functions of bounded variation

By $BV(\Omega; \mathbb{R}^m)$ we denote the space of functions with bounded variation, i.e., the set of $L^1(\Omega; \mathbb{R}^m)$ functions whose distributional gradient Du is a bounded Radon measure on Ω (i.e., an element of $\mathcal{M}(\Omega; \mathbb{R}^d)$) with values in the space $M^{m \times n}$ of $m \times n$ matrices. For every $u \in BV(\Omega; \mathbb{R}^m)$, its distributional derivative Du can be decomposed as $Du^a + Du^s$, where Du^a is the absolutely continuous part and the singular part with respect to the Lebesgue measure \mathcal{L}^n , respectively. The function u is approximately differentiable \mathcal{L}^n -a.e. in Ω and its approximate gradient ∇u belongs to $L^1(\Omega; M^{m \times n})$ and coincides \mathcal{L}^n -a.e. with $\frac{dDu^a}{d\mathcal{L}^n}$. Moreover, S_u coincides with the complement of the set of Lebesgue points of u , up to a set of null \mathcal{H}^{n-1} measure; see [3, Sections 3.5 and 3.6] for more details.

More precisely, the following decomposition holds:

$$Du = \nabla u \mathcal{L}^n + Du^s = \nabla u \mathcal{L}^n + Du^j + Du^c, \quad (2.2)$$

where Du^j is concentrated on S_u and the latter term of (2.2) is the part of Du^s concentrated on the set $\{x \in \Omega \mid (2.3) \text{ holds}\}$, where

$$\lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon^n} \int_{Q(x, \varepsilon)} |u(y) - u(x)| dy = 0 \quad (2.3)$$

holds. Here, and in what follows, given $x \in \mathbb{R}^n$ and $\varepsilon > 0$,

$$Q(x, \varepsilon) = \prod_{i=1}^n \left(x_i - \frac{\varepsilon}{2}, x_i + \frac{\varepsilon}{2} \right).$$

According to decomposition (2.2), we can further specialize (2.1), i.e., we can decompose any measure $\nu \in \mathcal{M}(\bar{\Omega}; \mathbb{R}^d)$ into three mutually orthogonal measures

$$\nu = \nu^a + \nu^{|Du^s|} + \nu^\sigma, \quad (2.4)$$

i.e., $\nu^s = \nu^{|Du^s|} + \nu^\sigma$, whereas $\nu^{|Du^s|}$ is absolutely continuous with respect to $|Du^s|$ and ν^σ is singular with respect to $|Du|$.

2.3 Convex envelopes and recession functions

Let $W_1 : \mathbb{R}^m \times \mathbb{R}^d \times \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$ be defined by

$$W_1(a, b, \xi) := f_1(a)f_2(b) + W(\xi),$$

where f_1, f_2 and W are the functions introduced above and satisfying (H1)–(H3). This entails, in particular, that W_1 has linear growth with respect to the last two variables.

We recall that the *convex envelope* of a function $f_2 : \mathbb{R}^d \rightarrow \mathbb{R}$ is the greatest convex function which is below f_2 , and as already observed, by virtue of (H2), it coincides with the biconjugate of f_2 ; cf. [13]. In the same way, the *quasiconvex envelope* of a function $W : \mathbb{R}^{d \times n} \rightarrow \mathbb{R}$ is the greatest quasiconvex function which is below W and it admits the following representation:

$$\mathcal{Q}W(\xi) = \inf \left\{ \frac{1}{|D|} \int_D W(\xi + \nabla \phi(y)) \, dy \mid \phi \in W_0^{1,\infty}(D; \mathbb{R}^m) \right\},$$

with D being a bounded Lipschitz domain. This definition is independent of D and we say that W is quasiconvex if $W = \mathcal{Q}W$.

Integral functionals with such integrands of linear growth can be extended to measures in a meaningful way if we assume that there is a well defined recession function. Here, following [26, 29], we define the (generalized) recession function of a function $h : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$ by

$$h^\infty(z) := \limsup_{t \rightarrow +\infty, z' \rightarrow z} \frac{h(tz')}{t}. \quad (2.5)$$

We recall that if h is quasiconvex with linear growth, in particular if $h = \mathcal{Q}W$, then h^∞ inherits these properties, both h and h^∞ are globally Lipschitz continuous and the lim sup in (2.5) is a limit along all rank-1 lines, i.e., if both z and z' are restricted to the class of matrices of rank 1. In case of a convex function $h : \mathbb{R}^m \rightarrow \mathbb{R}$ with linear growth, for instance $h = f_2^{**}$, its recession function automatically exists as the limit

$$h^\infty(b) = \lim_{t \rightarrow +\infty} \frac{h(tb)}{t} \quad \text{locally uniformly in } b \in \mathbb{R}^m. \quad (2.6)$$

Another technically useful property equivalent to (2.6) is

$$\left| \frac{1}{t} h(tb) - h^\infty(b) \right| \leq \sigma(t)(|b| + 1) \quad \text{for all } t > 0, b \in \mathbb{R}^m, \text{ where } \sigma(t) \xrightarrow{t \rightarrow \infty} 0. \quad (2.7)$$

Here,

$$\sigma : (0, \infty) \rightarrow (0, \infty)$$

is a non-increasing, continuous function independent of b ; for instance, given (2.6), one may choose

$$\sigma(t) := \sup \left\{ \left| h^\infty(b) - \frac{1}{s} h(sb) \right| \mid |b| \leq 1, s \geq t \right\}.$$

Due to the special form of W_1 , the recession function (with respect to the variable $(b, \xi) \in \mathbb{R}^d \times \mathbb{R}^{m \times n}$) of

$$\mathcal{C}\mathcal{Q}W_1(a, b, \xi) := f_1(a)f_2^{**}(b) + \mathcal{Q}W(\xi)$$

can be expressed as

$$(\mathcal{C}\mathcal{Q}W_1)^\infty(a, b, \xi) = f_1(a)(f_2^{**})^\infty(b) + (\mathcal{Q}W)^\infty(\xi)$$

for every $(a, b, \xi) \in \mathbb{R}^m \times \mathbb{R}^d \times \mathbb{R}^{m \times n}$.

2.4 Nonlinear transformation of measures

Throughout, we will use the following notation for scalar-, vector- or matrix-valued measures transformed by a nonlinear function. Let $\nu \in \mathcal{M}(\Lambda; \mathbb{R}^M)$ be such a measure on a Borel set $\Lambda \subset \mathbb{R}^n$ (e.g., $\Lambda = \bar{\Omega}$), and let

$f : \Lambda \times \mathbb{R}^M \rightarrow \mathbb{R}$ with recession function $f^\infty(x, \cdot) := (f(x, \cdot))^\infty$ in the second variable, for fixed $x \in \Lambda$. For every Borel set $A \subset \Lambda$, we then define

$$f(v)(A) := \int_A df(x, v)(x), \quad df(x, v)(x) := f\left(x, \frac{dv^a}{d\mathcal{L}^n}(x)\right) dx + f^\infty\left(x, \frac{dv^s}{d|v|^s}(x)\right) d|v|^s(x),$$

where $v = v^a + v^s$ is the Radon–Nikodým decomposition of v into an absolutely continuous and a singular component with respect to the Lebesgue measure \mathcal{L}^n . Obviously, this definition requires f to be regular enough so that the integral above is well defined, which will always be the case below.

2.5 Continuous extension of functionals with respect to area-strict convergence

The most natural way to extend a functional on $W^{1,1}$ or L^1 to BV or \mathcal{M} , respectively, is by continuous extension with respect to area-strict convergence, following [26]. Here, given a Borel set Λ which is either open or compact, we say that a sequence $(v_k) \subset \mathcal{M}(\Lambda; \mathbb{R}^d)$ converges to $v \in \mathcal{M}(\Lambda; \mathbb{R}^d)$ *area-strictly in* $\mathcal{M}(\Lambda; \mathbb{R}^d)$ if

$$v_k \xrightarrow{*} v \quad \text{in } \mathcal{M}(\Lambda; \mathbb{R}^d) \quad \text{and} \quad \int_\Lambda da(v_k)(x) \xrightarrow{k \rightarrow \infty} \int_\Lambda da(v)(x), \quad \text{where } a(\cdot) := \sqrt{1 + |\cdot|^2}$$

is the density of the area functional. Correspondingly, for a sequence $(u_k) \subset \text{BV}(\Omega; \mathbb{R}^m)$ and a function $u \in \text{BV}(\Omega; \mathbb{R}^m)$, we say that $u_k \rightarrow u$ *area-strictly in* $\text{BV}(\Omega; \mathbb{R}^m)$ if $u_k \rightarrow u$ in $L^1(\Omega; \mathbb{R}^m)$ and $Du_k \rightarrow Du$ area-strictly in $\mathcal{M}(\Omega; \mathbb{R}^{m \times n})$. The more classical, slightly weaker notion of *strict convergence* is recovered if we replace a by $\tilde{a}(\cdot) := |\cdot|$ above.

An important feature is that $L^1(\Lambda; \mathbb{R}^d)$ is dense in $\mathcal{M}(\Lambda; \mathbb{R}^d)$ with respect to area-strict convergence in $\mathcal{M}(\Lambda; \mathbb{R}^d)$, $\Lambda = \Omega, \bar{\Omega}$ (the slightly more subtle latter case is shown in Lemma 2.5), and $W^{1,1}(\Omega; \mathbb{R}^m)$ is dense in $\text{BV}(\Omega; \mathbb{R}^m)$ with respect to area-strict convergence in $\text{BV}(\Omega; \mathbb{R}^m)$ (by mollifying as in, e.g., [30, Section 5.3]). As a consequence, the following continuity result of [26] for functionals on \mathcal{M} and BV, respectively, implies that these are actually the unique continuous extensions of their restrictions to L^1 and $W^{1,1}$, with respect to area-strict convergence.

Proposition 2.1 (cf. [26, Theorems 3 and 4]). *Let f_2 and W be continuous functions satisfying (H2) and (H3). Then the functionals*

$$\begin{aligned} v &\mapsto \int_{\bar{\Omega}} df_2^{**}(v)(x), & \mathcal{M}(\bar{\Omega}; \mathbb{R}^d) &\rightarrow \mathbb{R}, \\ v &\mapsto \int_{\Omega} df_2^{**}(v)(x), & \mathcal{M}(\Omega; \mathbb{R}^d) &\rightarrow \mathbb{R}, \\ u &\mapsto \int_{\Omega} dQW(Du)(x), & \text{BV}(\Omega; \mathbb{R}^m) &\rightarrow \mathbb{R}, \end{aligned}$$

are sequentially continuous with respect to area-strict convergence in $\mathcal{M}(\bar{\Omega}; \mathbb{R}^d)$, $\mathcal{M}(\Omega; \mathbb{R}^d)$ and $\text{BV}(\Omega; \mathbb{R}^m)$, respectively.

We remark that the convex or quasiconvex envelopes cannot be dropped in Proposition 2.1, but convexity is exploited there only to ensure that the recession functions exist in a strong enough sense, ensuring continuity of the integrands at infinity.

2.6 Auxiliary results concerning sequences and approximation

The following result (cf. [19, Lemma 2.31]) will be used in the proof of the lower bound of Theorem 1.1.

Lemma 2.2 (Decomposition lemma in L^1). *Let $(z_k)_k \subset L^1(\Omega; \mathbb{R}^d)$ be bounded. Then, up to a not relabelled subsequence, z_k can be decomposed as*

$$z_k = z_k^{\text{osc}} + z_k^{\text{conc}},$$

with two bounded sequences $(z_k^{\text{osc}})_k, (z_k^{\text{conc}})_k \subset L^1(\Omega; \mathbb{R}^d)$ such that $|z_k^{\text{osc}}|$ is equi-integrable and z_k^{conc} is the purely concentrated part in the sense that $z_k^{\text{conc}} \rightarrow 0$ in measure. In fact, one may even assume that

$$\mathcal{L}^n(\{z_k^{\text{conc}} \neq 0\}) \rightarrow 0 \quad \text{as } k \rightarrow +\infty.$$

Moreover, whenever $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is globally Lipschitz,

$$\|f(z_k) - f(z_k^{\text{osc}}) - f(z_k^{\text{conc}}) + f(0)\|_{L^1(\Omega)} \rightarrow 0 \quad \text{as } k \rightarrow +\infty.$$

In the one-dimensional case, we also use the following lemma – here stated for any dimension – which allows us to manipulate boundary values.

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $A \Subset \Omega$ be an open subset with Lipschitz boundary. Let $f_1 : \mathbb{R}^m \rightarrow \mathbb{R}, f_2 : \mathbb{R}^d \rightarrow \mathbb{R}$ and $W : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$ be continuous functions satisfying (H1)–(H3). Consider $(u, v) \in \text{BV}(\Omega; \mathbb{R}^m) \times \mathcal{M}(\Omega; \mathbb{R}^d)$ such that $|v|(\partial A) = 0$ and assume that $(u_k, v_k) \subset W^{1,1}(A; \mathbb{R}^m) \times L^1(A; \mathbb{R}^d)$ is a sequence satisfying $u_k \xrightarrow{*} u$ in $\text{BV}(A; \mathbb{R}^m), v_k \xrightarrow{*} v$ in $\mathcal{M}(A; \mathbb{R}^d)$ and*

$$\lim_{k \rightarrow +\infty} \int_A (f_1(u_k) f_2(v_k) + W(\nabla u_k)) \, dx = l,$$

for some $l < +\infty$. Then there exists a sequence $(\bar{u}_k, \bar{v}_k) \subset W^{1,1}(A; \mathbb{R}^m) \times L^1(A; \mathbb{R}^d)$ such that $\bar{u}_k = u$ on ∂A (in the sense of trace), $\bar{u}_k \xrightarrow{*} u$ in $\text{BV}(A; \mathbb{R}^m), \bar{v}_k \xrightarrow{*} v$ in $\mathcal{M}(A; \mathbb{R}^d)$, and

$$\limsup_{k \rightarrow +\infty} \int_A (f_1(\bar{u}_k) f_2(\bar{v}_k) + W(\nabla \bar{u}_k)) \, dx \leq l.$$

The proof is omitted, it is similar to the proofs of [8, Lemma 2.2] and [4, Lemmas 4.4 and 4.5].

The following, essentially well-known, approximation of an absolutely continuous measure by a purely concentrating sequence is a key ingredient for the construction of a recovery sequence in Theorem 1.1.

Lemma 2.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $\sigma \in \mathcal{M}(\bar{\Omega}; \mathbb{R}^d)$ be absolutely continuous. There is a sequence $(v_\varepsilon)_\varepsilon \subset L^1(\Omega; \mathbb{R}^d)$ converging to zero in measure such that $v_\varepsilon \rightarrow \sigma$ weakly* and strictly in $\mathcal{M}(\bar{\Omega}; \mathbb{R}^d)$, as $\varepsilon \rightarrow 0$. In fact, v_ε can be constructed as a function supported on balls of radius $r(\varepsilon)$ centered at finitely many points $x_j^\varepsilon, j = 1, \dots, J(\varepsilon)$, such that $r(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$.*

Proof. We partition $\bar{\Omega}$ into a collection of mutually disjoint sets, i.e., $P_\ell = \{\Omega_j^\ell\}_{j=1}^{J(\ell)}$ such that $\text{diam}(\Omega_j^\ell) < \frac{1}{\ell}$ and $\Omega_j^\ell \cap \Omega_k^\ell = \emptyset$ if $j \neq k$. Take $x_j^\ell \in \text{int}(\Omega_j^\ell)$,

$$z_{j\varepsilon}^\ell = \begin{cases} \sigma(\Omega_j^\ell) / (|\sigma(\Omega_j^\ell)| \varepsilon) & \text{if } |\sigma(\Omega_j^\ell)| \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

and for every $j \in \{1, \dots, J(\ell)\}$ such that $|\sigma(\Omega_j^\ell)| \neq 0$ consider $x_j^\ell \in \text{int} \Omega_j^\ell$ and define

$$N_{j\varepsilon}^\ell = \left\{ x \in \Omega \mid |x - x_j^\ell|^\ell < \frac{|\sigma(\Omega_j^\ell)|}{|z_{j\varepsilon}^\ell| \mathcal{L}^n(B(0, 1))} \right\}.$$

If j is such that $|\sigma(\Omega_j^\ell)| = 0$, we set $N_{j\varepsilon}^\ell = \emptyset$. Put

$$w_\varepsilon^\ell(x) = \begin{cases} z_{j\varepsilon}^\ell & \text{if } x \in N_{j\varepsilon}^\ell, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that

$$\int_\Omega |w_\varepsilon^\ell(x)| \, dx = \sum_j \mathcal{L}^n(N_{j\varepsilon}^\ell) |z_{j\varepsilon}^\ell| = \sum_j |\sigma(\Omega_j^\ell)| = |\sigma|(\bar{\Omega}) < +\infty \quad (2.8)$$

and that $w_{j\varepsilon}^\ell \rightarrow 0$ in measure as $\varepsilon \rightarrow 0$.

If $\varphi \in C(\bar{\Omega}; \mathbb{R}^m)$, we get

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} w_{\varepsilon}^{\ell} \varphi(x) dx = \lim_{\varepsilon \rightarrow 0} \sum_{j=1}^{J(\ell)} \int_{N_{j\varepsilon}^{\ell}} z_{j\varepsilon}^{\ell} \varphi(x_j^{\ell}) dx + \lim_{\varepsilon \rightarrow 0} \sum_{j=1}^{J(\ell)} \int_{N_{j\varepsilon}^{\ell}} z_{j\varepsilon}^{\ell} (\varphi(x) - \varphi(x_j^{\ell})) dx = \int_{\bar{\Omega}} \varphi(x) d\sigma^{\ell}(x),$$

where $\sigma^{\ell} = \sum_{j=1}^{J(\ell)} \sigma(\Omega_j^{\ell}) \delta_{x_j^{\ell}}$. Above, we exploited that

$$\lim_{\varepsilon \rightarrow 0} \sum_{j=1}^{J(\ell)} \int_{N_{j\varepsilon}^{\ell}} z_{j\varepsilon}^{\ell} (\varphi(x) - \varphi(x_j^{\ell})) dx = 0$$

because φ is uniformly continuous on $\bar{\Omega}$ and $\mathcal{L}^n(N_{j\varepsilon}^{\ell}) = |\sigma(\Omega_j^{\ell})|/|z_{j\varepsilon}^{\ell}|$.

Altogether, we have that $w * -\lim_{\varepsilon \rightarrow 0} w_{\varepsilon}^{\ell} = \sigma^{\ell}$.

On the other hand, $w^* \text{-}\lim_{\ell \rightarrow \infty} \sigma^{\ell} = \sigma$. A diagonalization argument with a suitable $\ell = \ell(\varepsilon)$ gives us a sequence $v_{\varepsilon} := w_{\varepsilon}^{\ell(\varepsilon)}$ converging to zero in measure as well as weakly* to σ as $\varepsilon \rightarrow 0$. Combining this with (2.8), we also obtain that $v_{\varepsilon} \rightarrow \sigma$ strictly. \square

Strict approximation of general measures by L^1 functions is also well known, but we would like to stress here that this is possible even when boundary points can be charged.

Lemma 2.5. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded. There exists a family of linear operators*

$$(I_j)_{j \in \mathbb{N}} : \mathcal{M}(\bar{\Omega}; \mathbb{R}^d) \rightarrow L^1(\Omega; \mathbb{R}^d)$$

with $I_j \circ I_j = I_j$ such that for all $v \in \mathcal{M}(\bar{\Omega}; \mathbb{R}^d)$,

$$\int_{\Omega} |I_j v| dx \leq \|v\|_{\mathcal{M}(\bar{\Omega}; \mathbb{R}^d)}, \quad \int_{\Omega} a(I_j v) dx \leq \int_{\Omega} a(v) \quad \text{with } a(\cdot) := \sqrt{1 + |\cdot|^2}, \quad (2.9)$$

and $I_j v \rightarrow v$ weakly*, strictly and area-strictly in $\mathcal{M}(\bar{\Omega}; \mathbb{R}^d)$ as $j \rightarrow \infty$. If $v \in L^1(\Omega; \mathbb{R}^d)$, then $I_j v \rightarrow v$ strongly in $L^1(\Omega; \mathbb{R}^d)$.

Proof. The operator I_j is defined as a piecewise constant interpolation on a suitable cubical grid of grid size $\frac{1}{j}$. For each $j \in \mathbb{N}$ and $z \in \mathbb{Z}^n$, let

$$\mathcal{G}_j := \{Q(j, z) \mid z \in \mathbb{Z}^n\}, \quad \text{with } Q(j, z) := \frac{1}{j}(z + [0, 1)^n).$$

While these cubes form a pairwise disjoint covering of \mathbb{R}^n , it is possible that certain cubes intersect $\bar{\Omega}$ “from the outside” in the sense that $Q \cap \Omega = \emptyset \neq Q \cap \bar{\Omega}$ for some $Q \in \mathcal{G}_j$. If v charges $Q \cap \partial\Omega$, this contribution would be lost in our construction below. We avoid the issue by slightly changing some cubes on the surface: If necessary, reassign surface points from an “outer” cube intersecting $\bar{\Omega}$ as above to a neighbor cube intersecting Ω . This leads to

$$\tilde{\mathcal{G}}_j := \{\tilde{Q}(j, z) \mid z \in \mathbb{Z}^n\},$$

where the $\tilde{Q}(j, z)$ are chosen as pairwise disjoint Borel sets such that $Q(j, z) \subset \tilde{Q}(j, z) \subset \overline{Q(j, z)}$, $\mathbb{R}^n = \bigcup \tilde{\mathcal{G}}_j$ and

$$\tilde{Q}(j, z) \cap \bar{\Omega} \neq \emptyset \quad \text{implies} \quad \tilde{Q}(j, z) \cap \Omega \neq \emptyset. \quad (2.10)$$

With the grid given by $\tilde{\mathcal{G}}_j$, we define the piecewise constant interpolation $I_j : \mathcal{M}(\bar{\Omega}; \mathbb{R}^d) \rightarrow L^1(\Omega; \mathbb{R}^d)$ by

$$v \mapsto I_j v \quad \text{with} \quad I_j v|_{\Omega \cap Q} := \frac{v(\bar{\Omega} \cap Q)}{\mathcal{L}^n(\Omega \cap Q)} \quad \text{for all } Q \in \tilde{\mathcal{G}}_j.$$

As constructed, I_j is a linear, continuous projection satisfying (2.9) (cube by cube, by convexity of $|\cdot|$ and a , using Jensen’s inequality), and $I_j v \rightharpoonup^* v$ in $\mathcal{M}(\bar{\Omega}; \mathbb{R}^d)$. To see the latter, take a test function $\varphi \in C(\bar{\Omega}; \mathbb{R}^d)$. Hence, for any $Q = Q(j, z) \in \tilde{\mathcal{G}}_j$ with $Q \cap \bar{\Omega} \neq \emptyset$, and any fixed, arbitrarily chosen $x_Q \in Q \cap \bar{\Omega}$,

$$\int_{\bar{\Omega} \cap Q(j, z)} |\varphi(x) - \varphi(x_{Q(j, z)})| dx \leq \sigma_j := \sup_{z \in \mathbb{Z}^n} \sup_{x \in Q(j, z) \cap \bar{\Omega}} |\varphi(x) - \varphi(x_{Q(j, z)})| \xrightarrow{j \rightarrow \infty} 0.$$

As a consequence, if we consider the equality

$$\int_{\Omega \cap Q} I_j v \cdot \varphi(x_Q) dx = v(\overline{\Omega} \cap Q) I_j v|_Q \cdot \varphi(x_Q),$$

valid by definition of I_j (where, due to (2.10), both sides vanish if $Q \cap \Omega = \emptyset$), sum it over all $Q = Q(j, z)$ and pass to the limit as $j \rightarrow \infty$, we obtain that

$$\int_{\Omega} (I_j v)(x) \varphi(x) dx \rightarrow \int_{\overline{\Omega}} \varphi(x) dv(x),$$

i.e., $I_j v \rightharpoonup^* v$ in $\mathcal{M}(\overline{\Omega}; \mathbb{R}^d)$, as claimed.

Combined with (2.9), weak* convergence of $I_j w$ to w implies that $I_j w \rightarrow w$ strictly and area-strictly in $\mathcal{M}(\overline{\Omega}; \mathbb{R}^d)$. In case $w \in L^1(\Omega; \mathbb{R}^d)$ with $|w|(\partial\Omega) = 0$, we get

$$\limsup_j \int_{\Omega} |I_j w| dx \leq \int_{\Omega} |w| dx$$

and $I_j w \rightharpoonup^* w$ in $\mathcal{M}(\Omega; \mathbb{R}^d)$, which implies that $I_j w \rightarrow w$ strongly in L^1 . \square

2.7 The localized energy and its relaxation

Let $\mathcal{A}_r(\overline{\Omega})$ be the family of all subsets of $\overline{\Omega}$ which are open with respect to the relative topology of $\overline{\Omega}$. For each $A \in \mathcal{A}_r(\overline{\Omega})$, we introduce the localized energy

$$\mathcal{F}(\cdot, \cdot, A) : \text{BV}(\Omega \cap A; \mathbb{R}^m) \times \mathcal{M}(A; \mathbb{R}^d) \rightarrow [0, +\infty]$$

defined by

$$\mathcal{F}(u, v, A) := \begin{cases} \int_{\Omega \cap A} (f_1(u)f_2(v) + W(\nabla u)) dx & \text{if } (u, v) \in W^{1,1}(\Omega \cap A; \mathbb{R}^n) \times L^1(\Omega \cap A; \mathbb{R}^m), \\ +\infty & \text{otherwise.} \end{cases} \quad (2.11)$$

Its *relaxation*, i.e., its lower semicontinuous envelope with respect to weak* convergence, is the functional

$$\overline{\mathcal{F}}(\cdot, \cdot, A) : \text{BV}(\Omega \cap A; \mathbb{R}^m) \times \mathcal{M}(A; \mathbb{R}^d) \rightarrow [0, +\infty]$$

defined by

$$\overline{\mathcal{F}}(u, v, A) := \inf_{k \rightarrow +\infty} \{ \liminf \mathcal{F}(u_k, v_k, A) \mid (u_k, v_k) \in W^{1,1}(\Omega \cap A; \mathbb{R}^m) \times L^1(\Omega \cap A; \mathbb{R}^d), \\ (u_k, v_k) \xrightarrow{*} (u, v) \text{ in } \text{BV}(\Omega \cap A; \mathbb{R}^m) \times \mathcal{M}(A; \mathbb{R}^d) \}. \quad (2.12)$$

Here recall that, for $A \in \mathcal{A}_r(\overline{\Omega})$, $\mathcal{M}(A; \mathbb{R}^d)$ is the dual space of the continuous functions $\varphi : A \rightarrow \mathbb{R}^d$ with $\varphi = 0$ on $\partial A \cap \Omega$. Also notice that (2.12) extends the notation $\overline{\mathcal{F}}(u, v)$ of (1.4), in the sense that

$$\overline{\mathcal{F}}(u, v) = \overline{\mathcal{F}}(u, v, \overline{\Omega}).$$

There are several other, equivalent representations of $\overline{\mathcal{F}}$ that will be useful to simplify our proofs. For now, we only collect the statements of these results. Their proofs are presented in Section A.

A first observation is that in (2.12), f_2 and W can be replaced by f_2^{**} and QW , respectively.

Proposition 2.6. *Let f_1, f_2 and W be continuous functions satisfying (H1)–(H3), let $A \in \mathcal{A}_r(\overline{\Omega})$ and consider the corresponding functional \mathcal{F} in (2.11). Consider furthermore the relaxed functionals (1.3) and*

$$\overline{\mathcal{F}}_{**}(u, v, A) := \inf \left\{ \liminf_{k \rightarrow +\infty} \int_{\Omega \cap A} f_1(u_k) f_2^{**}(v_k) dx + \int_{\Omega \cap A} QW(\nabla u_k) dx \mid \right. \\ \left. (u_k, v_k)_k \subset W^{1,1}(\Omega \cap A; \mathbb{R}^m) \times L^1(\Omega \cap A; \mathbb{R}^d), \right. \\ \left. (u_k, v_k) \xrightarrow{*} (u, v) \text{ in } \text{BV}(\Omega \cap A; \mathbb{R}^m) \times \mathcal{M}(A; \mathbb{R}^d) \right\}. \quad (2.13)$$

Then $\overline{\mathcal{F}}(\cdot, \cdot, \cdot)$ coincides with $\overline{\mathcal{F}}_{**}(\cdot, \cdot, \cdot)$ in $\text{BV}(\Omega, \mathbb{R}^m) \times \mathcal{M}(\overline{\Omega}; \mathbb{R}^d) \times \mathcal{A}_r(\overline{\Omega})$.

In addition, $u_k \in W^{1,1}(\Omega; \mathbb{R}^m)$ can be replaced by $u_k \in BV(\Omega; \mathbb{R}^m)$ in the definition of $\overline{\mathcal{F}}_{**}$ if we use an appropriate extension of \mathcal{F}_{**} .

Proposition 2.7. *In the situation of Proposition 2.6, we have*

$$\begin{aligned} \overline{\mathcal{F}}_{**}(u, v, A) = \inf \left\{ \liminf_{k \rightarrow +\infty} \int_{\Omega \cap A} f_1(u_k) f_2^{**}(v_k) dx + \int_{\Omega \cap A} d\Omega W(Du_k)(x) \right. \\ \left. (u_k, v_k)_k \subset BV(\Omega \cap A; \mathbb{R}^m) \times L^1(\Omega \cap A; \mathbb{R}^d), \right. \\ \left. (u_k, v_k) \xrightarrow{*} (u, v) \text{ in } BV(\Omega \cap A; \mathbb{R}^m) \times \mathcal{M}(A; \mathbb{R}^d) \right\}. \end{aligned} \quad (2.14)$$

A key ingredient to obtain an integral representation of $\overline{\mathcal{F}}$ via the blow-up method is the following preliminary result, which shows that the localized relaxed functional is the restriction of a Radon measure on $\mathcal{A}_r(\overline{\Omega})$.

Lemma 2.8. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with Lipschitz boundary and let f_1, f_2 and W be as in (1.2), satisfying (H1)–(H3). For every $(u, v) \in BV(\Omega; \mathbb{R}^m) \times \mathcal{M}(\overline{\Omega}; \mathbb{R}^d)$, the set function $\overline{\mathcal{F}}(u, v, \cdot)$ in (2.12) is the trace of a Radon measure absolutely continuous with respect to $\mathcal{L}^n + |Du| + |v|$.*

3 Proof of Theorem 1.1

We start observing that the growth assumptions (H1) and the structure of the functional (1.2) ensure that sequences $(u_k, v_k)_k \subset W^{1,1}(\Omega; \mathbb{R}^m) \times \mathcal{M}(\overline{\Omega}; \mathbb{R}^d)$ with $\sup_k F(u_k, v_k) < +\infty$ have (non-re-labeled) subsequences

$$\nabla u_k \xrightarrow{*} m \in \mathcal{M}(\Omega; \mathbb{R}^m) \quad \text{and} \quad v_k \xrightarrow{*} v$$

in $\mathcal{M}(\overline{\Omega}; \mathbb{R}^d)$. Up to imposing some boundary conditions or removing constants by working in a quotient space, we can assume without loss of generality that $u_k \xrightarrow{*} u$ in $BV(\Omega; \mathbb{R}^m)$.

The proof of Theorem 1.1 will be achieved in two main steps, first we find a lower bound and then we prove that it is sharp.

3.1 Lower bound

We will now show that

$$\begin{aligned} \overline{\mathcal{F}}(u, v) \geq \int_{\Omega} \Omega W(\nabla u) dx + \int_{\Omega} (\Omega W)^{\infty} \left(\frac{dDu^s}{d|Du^s|} \right) d|Du^s| \\ + \int_{\Omega} g \left(u, \frac{dv^a}{d\mathcal{L}^n} \right) dx + \int_{\overline{\Omega}} f_1^{\min}(f_2^{**})^{\infty} \left(\frac{dv^s}{d|v^s|} \right) d|v^s|. \end{aligned} \quad (3.1)$$

Proof of the lower bound. Take

$$\begin{aligned} (u_k, v_k) &\in W^{1,1}(\Omega; \mathbb{R}^m) \times L^1(\Omega; \mathbb{R}^d), \\ u_k &\xrightarrow{*} u && \text{in } BV(\Omega; \mathbb{R}^m), \\ v_k &\xrightarrow{*} v && \text{in } \mathcal{M}(\overline{\Omega}; \mathbb{R}^d). \end{aligned}$$

We prove individual lower bounds for the two parts of \mathcal{F} . Due to [2] and the trivial estimate $W \geq \Omega W$, we have that

$$\liminf_k \int_{\Omega} W(\nabla u_k) dx \geq \int_{\Omega} \Omega W(\nabla u) dx + \int_{\Omega} (\Omega W)^{\infty} \left(\frac{dDu^s}{d|Du^s|} \right) d|Du^s|.$$

It remains to show that

$$\liminf_k \int_{\Omega} f_1(u_k) f_2(v_k) dx \geq \int_{\Omega} g \left(u, \frac{dv^a}{d\mathcal{L}^n} \right) dx + \int_{\overline{\Omega}} f_1^{\min}(f_2^{**})^{\infty} \left(\frac{dv^s}{d|v^s|} \right) d|v^s|.$$

For v_k , we use the decomposition lemma in L^1 (cf. Lemma 2.2): $v_k = v_k^{\text{osc}} + v_k^{\text{conc}}$, where $(v_k^{\text{osc}})_k$ is equi-integrable and $|\{v_k^{\text{conc}} \neq 0\}| \rightarrow 0$. Up to a subsequence,

$$v_k^{\text{osc}} \rightharpoonup^* v_{\text{osc}}, \quad v_k^{\text{conc}} \rightharpoonup^* v_{\text{conc}} \quad \text{in } \mathcal{M}(\bar{\Omega}; \mathbb{R}^d).$$

The equi-integrability of $(v_k^{\text{osc}})_k$ and the Dunford–Pettis theorem (see [19, Theorem 2.54]) ensure that

$$v_k^{\text{osc}} \rightharpoonup v_{\text{osc}} \quad \text{in } L^1(\Omega; \mathbb{R}^d).$$

We decompose $v_{\text{conc}} = v_{\text{conc}}^a + v_{\text{conc}}^s$ into an absolutely continuous and a singular part with respect to the Lebesgue measure. Accordingly, the Lebesgue decomposition of the original limit measure $\nu = \nu^a + \nu^s$ is given by

$$\nu^a = \nu_{\text{osc}} + \nu_{\text{conc}}^a \quad \text{and} \quad \nu^s = \nu_{\text{conc}}^s.$$

With a slight abuse of notation, we identify the absolutely continuous measures ν_{osc} and ν_{conc}^a with their respective densities

$$\nu_{\text{osc}} = \frac{d\nu_{\text{osc}}}{d\mathcal{L}^n} \quad \text{and} \quad \nu_{\text{conc}}^a = \frac{d\nu_{\text{conc}}^a}{d\mathcal{L}^n}$$

in $L^1(\Omega; \mathbb{R}^d)$ below. Observe that

$$\begin{aligned} \liminf_k \int_{\Omega} f_1(u_k) f_2(v_k) dx &\geq \liminf_k \int_{\Omega} f_1(u_k) f_2^{**}(v_k) dx \\ &\geq \liminf_k \int_{\{v_k^{\text{conc}}=0\}} f_1(u_k) f_2^{**}(v_k^{\text{osc}}) dx + \liminf_k \int_{\{v_k^{\text{conc}} \neq 0\}} f_1(u_k) f_2^{**}(v_k^{\text{osc}} + v_k^{\text{conc}}) dx. \end{aligned} \quad (3.2)$$

We will estimate these two terms separately. For the first term, by [19, Corollary 7.9], exploiting the convexity of f_2^{**} , we get that

$$\liminf_k \int_{\{v_k^{\text{conc}}=0\}} f_1(u_k) f_2^{**}(v_k^{\text{osc}}) dx \geq \int_{\Omega} f_1(u) f_2^{**}(v_{\text{osc}}) dx. \quad (3.3)$$

For the second term, we claim that

$$\begin{aligned} \liminf_k \int_{\{v_k^{\text{conc}} \neq 0\}} f_1(u_k) f_2^{**}(v_k^{\text{osc}} + v_k^{\text{conc}}) dx &\geq \liminf_k \int_{\{v_k^{\text{conc}} \neq 0\}} f_1^{\min} f_2^{**}(v_k^{\text{conc}}) dx \\ &\geq \liminf_k \int_{\Omega} f_1^{\min} (f_2^{**})^{\infty}(v_k^{\text{conc}}) dx \\ &\geq \int_{\Omega} f_1^{\min} (f_2^{**})^{\infty}(v_{\text{conc}}^a) dx + \int_{\bar{\Omega}} f_1^{\min} (f_2^{**})^{\infty} \left(\frac{d\nu_{\text{conc}}^s}{d|\nu_{\text{conc}}^s|} \right) d|\nu_{\text{conc}}^s|(x), \end{aligned} \quad (3.4)$$

To see that the first line of (3.4) holds, notice that $f_1(u_k) \geq f_1^{\min} > 0$ and

$$\int_{\{v_k^{\text{conc}} \neq 0\}} |f_2^{**}(v_k^{\text{osc}} + v_k^{\text{conc}}) - f_2^{**}(v_k^{\text{conc}})| dx \leq C \int_{\{v_k^{\text{conc}} \neq 0\}} |v_k^{\text{osc}}| dx \xrightarrow{k \rightarrow \infty} 0,$$

because convex functions with linear growth are Lipschitz continuous; see, for instance, [13, Proposition 2.32]. The inequality in the second line of (3.4) (actually an equality, as a matter of fact) is essentially due to (2.6) and the fact that (v_k^{conc}) is purely concentrating. More precisely, we exploit that

$$\alpha_t := \inf_{|y|=1} \frac{f_2^{**}(ty)}{t(f_2^{**})^{\infty}(y)} \xrightarrow{t \rightarrow +\infty} 1$$

by (2.6) and

$$\left| \int_{\{|v_k^{\text{conc}}| \leq t\}} (f_2^{**})^{\infty}(v_k^{\text{conc}}) dx \right| \leq (1+K)t\mathcal{L}^n(\{v_k^{\text{conc}} \neq 0\}) \xrightarrow{k \rightarrow \infty} 0 \quad \text{for every } t > 0$$

(using (H2), $(f_2^{**})^\infty(0) = 0$ and the fact that $\mathcal{L}^n(\{v_k^{\text{conc}} \neq 0\}) \rightarrow 0$ by Lemma 2.2), whence

$$\begin{aligned} \liminf_k \int_{\{v_k^{\text{conc}} \neq 0\}} f_2^{**}(v_k^{\text{conc}}) dx &\geq \lim_{t \rightarrow +\infty} \liminf_k \int_{\{|v_k^{\text{conc}}| > t\}} \alpha_t (f_2^{**})^\infty(v_k^{\text{conc}}) dx \\ &= \liminf_k \int_{\Omega} (f_2^{**})^\infty(v_k^{\text{conc}}) dx. \end{aligned}$$

Finally, in the last line of (3.4), we applied the weak*-lower semicontinuity of convex integral functionals on measures (see, e.g., [19, Theorem 5.27]).

Combining (3.2)–(3.4), we conclude that

$$\begin{aligned} \liminf_k \int_{\Omega} f_1(u_k) f_2(v_k) dx &\geq \int_{\Omega} f_1(u) f_2^{**}(v_{\text{osc}}) dx + \int_{\Omega} f_1^{\min}(f_2^{**})^\infty(v_{\text{conc}}^a) dx + \int_{\Omega} f_1^{\min}(f_2^{**})^\infty\left(\frac{dv_{\text{conc}}^s}{d|v_{\text{conc}}^s|}\right) d|v_{\text{conc}}^s|(x) \\ &\geq \int_{\Omega} g(u, v^a) dx + \int_{\Omega} f_1^{\min}(f_2^{**})^\infty\left(\frac{dv_{\text{conc}}^s}{d|v_{\text{conc}}^s|}\right) d|v_{\text{conc}}^s|(x), \end{aligned} \quad (3.5)$$

because at each x , $b_1 := v_{\text{osc}}(x)$ and $b_2 := v_{\text{conc}}^a(x)$ are admissible for the minimization in (1.5), the definition of g , where $b := v^a(x) = v_{\text{osc}}(x) + v_{\text{conc}}^a(x)$ for \mathcal{L}^n -a.e. x . \square

3.2 Upper bound

By Proposition 2.6 and Proposition 2.7, we may assume that f_2 is convex and W is quasiconvex. We must show that the opposite inequality to (3.1) holds.

The proof will be divided into two steps. First, we assume that $v \in L^1(\Omega; \mathbb{R}^d)$, and the general case $v \in \mathcal{M}(\bar{\Omega}; \mathbb{R}^d)$ will be considered afterwards.

First step: Let $u \in \text{BV}(\Omega; \mathbb{R}^m)$ and $v \in L^1(\Omega; \mathbb{R}^d)$. Fix $\eta > 0$ and choose, for \mathcal{L}^n a.e. $x \in \Omega$, a decomposition of v into two functions

$$v(x) = v_{\text{osc}}^\eta(x) + v_{\text{conc}}^\eta(x) \quad (3.6)$$

almost optimal for the definition of g , i.e., such that

$$g(u(x), v(x)) + \eta \geq f_1(u(x)) f_2(v_{\text{osc}}^\eta(x)) + f_1^{\min} f_2^{\text{co}}(v_{\text{conc}}^\eta(x)). \quad (3.7)$$

Here, notice that for each x , the set of admissible choices for $v_{\text{osc}}^\eta(x)$ (thus fixing $v_{\text{conc}}^\eta(x) = v(x) - v_{\text{osc}}^\eta(x)$) is always non-empty (as $\eta > 0$) and open (by continuity of f_2 and f_2^{co}). In addition, u and v are measurable and g and f_1 are continuous. As a consequence, it is possible to choose v_{osc}^η as a measurable function (cf., e.g., [18, Lemma 3.10]). By the coercivity of f_2 assumed in (H2), inequality (3.7) then implies that $v_{\text{osc}}^\eta, v_{\text{conc}}^\eta \in L^1(\Omega; \mathbb{R}^d)$.

By Lemma 2.4, we can find a sequence of functions $(v_{\text{conc}}^{\eta, \varepsilon})_\varepsilon \subset L^1(\Omega; \mathbb{R}^d)$ such that

$$v_{\text{conc}}^{\eta, \varepsilon} \xrightarrow{\varepsilon \rightarrow 0} v_{\text{conc}}^\eta \quad \text{strictly (and therefore also weakly*) in } \mathcal{M}(\bar{\Omega}; \mathbb{R}^d), \quad (3.8)$$

as well as $v_{\text{conc}}^{\eta, \varepsilon} \rightarrow 0$ in measure.

Let $\delta > 0$ and choose $u_{\text{min}}^\delta \in \mathbb{R}^m$ such that

$$f_1^{\min} \leq f_1(u_{\text{min}}^\delta) \leq f_1^{\min} + \delta,$$

where $f_1^{\min} = \inf f_1$ as defined in (1.6). In the following, we will modify u near the (small) sets where $v_{\text{conc}}^{\eta, \varepsilon} \neq 0$, there replacing its value by u_{min}^δ . For that purpose, we use that in dimension $n \geq 2$ there exists a function

$$\varphi \in C^\infty(\bar{B}_1(0) \setminus \{0\}) \cap W_0^{1,1}(B_1(0)) \quad \text{such that } \varphi(y) \rightarrow +\infty \text{ as } y \rightarrow 0,$$

for instance, $\varphi(y) = \log(1 - \log|y|)$. For $s > 0$, let

$$\varphi_s(y) := \begin{cases} 0 & \text{if } \varphi(y) \leq \frac{1}{s}, \\ s\varphi(y) - 1 & \text{if } \frac{1}{s} < \varphi(y) < \frac{2}{s}, \\ 1 & \text{if } \varphi(y) \geq \frac{2}{s}. \end{cases}$$

Notice that $0 \leq \varphi_s \leq 1$, and the support of φ_s shrinks to 0 as $s \rightarrow 0$, since $\varphi(x) \rightarrow +\infty$ as $x \rightarrow 0$. In particular, $\|\nabla\varphi_s\|_{L^1} \rightarrow 0$ as $s \rightarrow 0$. We define

$$\tilde{u}_{\varepsilon,\delta}(x) := (1 - h_\varepsilon(x))u^{[1/\delta]}(x) + h_\varepsilon(x)u_{\min}^\delta,$$

where $u^{[1/\delta]}$ is the component-wise truncation of u on the level $1/\delta$, i.e., for $u = (u^{(1)}, \dots, u^{(m)})$,

$$u^{[1/\delta]} = (u^{[1/\delta],(1)}, \dots, u^{[1/\delta],(m)})$$

is defined by

$$u^{[1/\delta],(i)}(x) := \begin{cases} u^{(i)}(x) & \text{if } |u^{(i)}(x)| \leq \frac{1}{\delta}, \\ \frac{1}{\delta} \frac{u^{(i)}(x)}{|u^{(i)}(x)|} & \text{if } |u^{(i)}(x)| > \frac{1}{\delta}, \end{cases} \quad i = 1, \dots, m,$$

and, with some $s(\varepsilon) > 0$ and $(x_j^\varepsilon)_{j,\varepsilon} \subset \Omega$, $J(1/\varepsilon) \in \mathbb{N}$ and $r(\varepsilon)$ given by Lemma 2.4 when we applied it to get (3.8), we obtain

$$h_\varepsilon(x) := \sum_{1 \leq j \leq J(1/\varepsilon)} \varphi_{s(\varepsilon)}(x - x_j^\varepsilon). \quad (3.9)$$

Here, we choose $s(\varepsilon) > 0$ such that $s(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$, but still slow enough so that $\varphi(y) \geq 2/s(\varepsilon)$ for all $|y| \leq r(\varepsilon)$, whence

$$\varphi_{s(\varepsilon)}(y) = 1 \quad \text{for all } |y| \leq r(\varepsilon) \quad (3.10)$$

by construction of φ_s . For $\varepsilon > 0$ small enough, h_ε has support contained in a union of disjoint balls centered at x_j^ε with vanishing radii (as $\varepsilon \rightarrow 0$). Moreover, $0 \leq h_\varepsilon(x) \leq 1$ and $\|\nabla h_\varepsilon\|_{L^1} \rightarrow 0$. Consequently, for fixed δ , we get that

$$\lim_{\varepsilon \rightarrow 0} \|\tilde{u}_{\varepsilon,\delta} - u^{[1/\delta]}\|_{L^1(\Omega)} = 0 \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} \int_{\Omega} dQW(D\tilde{u}_{\varepsilon,\delta})(x) = \int_{\Omega} dQW(Du^{[1/\delta]})(x).$$

Since $u^{[1/\delta]} \rightarrow u$ in L^1 and

$$\int_{\Omega} dQW(Du^{[1/\delta]}) \rightarrow \int_{\Omega} dQW(Du)$$

by dominated convergence, we can choose a diagonal subsequence $u_\varepsilon := \tilde{u}_{\varepsilon,\delta(\varepsilon)}$ with $\delta(\varepsilon) \rightarrow 0$ slow enough so that

$$\lim_{\varepsilon \rightarrow 0} \|u_\varepsilon - u\|_{L^1(\Omega)} = 0 \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} \int_{\Omega} dQW(Du_\varepsilon)(x) = \int_{\Omega} dQW(Du)(x). \quad (3.11)$$

On the other hand, in view of (3.6) and (3.8), it is clear that

$$v^{\eta,\varepsilon} := v_{\text{osc}}^\eta + v_{\text{conc}}^{\eta,\varepsilon}$$

converges to v weakly* in $\mathcal{M}(\Omega; \mathbb{R}^d)$ if $\varepsilon \rightarrow 0$. Thus,

$$\begin{aligned} \overline{\mathcal{F}}(u, v, \overline{\Omega}) &\leq \liminf_{\varepsilon \rightarrow 0} \mathcal{F}(u_\varepsilon, v^{\eta,\varepsilon}, \Omega) \\ &= \liminf_{\varepsilon \rightarrow 0} \left(\int_{\Omega} dQW(Du_\varepsilon)(x) + \int_{\Omega} f_1(u_\varepsilon) f_2^{**}(v^{\eta,\varepsilon}) dx \right). \end{aligned}$$

Concerning the right-hand side, by recalling (3.11), it is sufficient to focus on the last integral of the above inequality, for which we have that

$$\liminf_{\varepsilon \rightarrow 0} \int_{\Omega} f_1(u_\varepsilon) f_2^{**}(v^{\eta,\varepsilon}) dx \leq \liminf_{\varepsilon \rightarrow 0} \left(\int_{\Omega} f_1(u_\varepsilon) f_2^{**}(v_{\text{osc}}^\eta) dx + \int_{\Omega} f_1(u_\varepsilon) (f_2^{**})^\infty(v_{\text{conc}}^{\eta,\varepsilon}) dx \right), \quad (3.12)$$

where it has been exploited that (see [19, (4.33) with $t = 1$])

$$f_2^{**}(y+z) \leq f_2^{**}(y) + (f_2^{**})^\infty(z) \quad \text{for every } y, z \in \mathbb{R}^d.$$

By construction of u_ε and $v_{\text{conc}}^{\eta, \varepsilon}$, in particular (3.9) and (3.10), we have $u_\varepsilon \equiv u_{\text{min}}^{\delta(\varepsilon)}$ on the support of $v_{\text{conc}}^{\eta, \varepsilon}$. Since $f_1(u_{\text{min}}^{\delta(\varepsilon)}) \rightarrow f_1^{\text{min}}$ as $\varepsilon \rightarrow 0$, we can replace $f_1(u_\varepsilon)$ by f_1^{min} in the second integral on the right-hand side of (3.12). This yields

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} \int_{\Omega} f_1(u_\varepsilon) f_2^{**}(v_{\text{conc}}^{\eta, \varepsilon}) dx &\leq \liminf_{\varepsilon \rightarrow 0} \left(\int_{\Omega} f_1(u_\varepsilon) f_2^{**}(v_{\text{osc}}^\eta) dx + \int_{\Omega} f_1^{\text{min}} (f_2^{**})^\infty(v_{\text{conc}}^{\eta, \varepsilon}) dx \right) \\ &\leq \int_{\Omega} f_1(u) f_2^{**}(v_{\text{osc}}^\eta) dx + \int_{\Omega} f_1^{\text{min}} (f_2^{**})^\infty(v_{\text{conc}}^\eta) dx, \end{aligned} \quad (3.13)$$

where, in the first limit, we have used the dominated convergence theorem together with bounds, and in the second we used Reshetnyak's continuity theorem (see, e.g., [3, Theorem 2.39]), exploiting the strict convergence of $v_{\text{conc}}^{\eta, \varepsilon}$ to v_{conc}^η obtained from Lemma 2.4.

Hence, together with (3.11), inequality (3.13) gives that

$$\bar{\mathcal{F}}(u, v) \leq \int_{\Omega} g(u, v) dx + \int_{\Omega} QW(\nabla u) dx + \int_{\Omega} (QW)^\infty \left(\frac{dDu^s}{d|Du^s|} \right) d|Du^s| + \eta.$$

The arbitrariness of η concludes the proof of this case.

Second step: Let $u \in BV(\Omega; \mathbb{R}^m)$ and $v \in \mathcal{M}(\bar{\Omega}; \mathbb{R}^d)$. To invoke the first step, we use the approximation $I_j v$ of v and some of its components by functions in L^1 provided in Lemma 2.5.

The first step and the lower semicontinuity of $\bar{\mathcal{F}}$ ensure that for every $u \in BV(\Omega; \mathbb{R}^m)$ and $v \in \mathcal{M}(\bar{\Omega}; \mathbb{R}^d)$,

$$\bar{\mathcal{F}}(u, v) \leq \liminf_j \int_{\Omega} g(u, I_j v) dx + \int_{\Omega} dQW(Du) dx. \quad (3.14)$$

As before for (3.7), for any $\eta > 0$, we can decompose the absolutely continuous part of v almost optimally for the infimum defining g in (1.5), i.e., $v^a = v_{\text{osc}}^{a, \eta} + v_{\text{conc}}^{a, \eta}$ so that

$$g(u(x), v^a(x)) + \eta \geq f_1(u(x)) f_2(v_{\text{osc}}^{a, \eta}(x)) + f_1^{\text{min}} f_2^\infty(v_{\text{conc}}^{a, \eta}(x)) \quad \text{for a.e. } x \in \Omega. \quad (3.15)$$

The first summand in (3.14) can be estimated as follows:

$$\begin{aligned} \liminf_j \int_{\Omega} g(u, I_j v) dx &\leq \liminf_j \int_{\Omega} f_1(u) f_2^{**}(I_j v_{\text{osc}}^{a, \eta}) + f_1^{\text{min}} (f_2^{**})^\infty(I_j [v_{\text{conc}}^{a, \eta} + v^s]) dx \\ &= \int_{\Omega} f_1(u) f_2^{**}(v_{\text{osc}}^{a, \eta}) dx + \int_{\Omega} f_1^{\text{min}} (f_2^{**})^\infty(v_{\text{conc}}^{a, \eta}) dx + f_1^{\text{min}} (f_2^{**})^\infty \left(\frac{dv^s}{d|v^s|} \right) d|v^s| \\ &= \int_{\Omega} (f_1(u) f_2^{**}(v_{\text{osc}}^{a, \eta}) + f_1^{\text{min}} (f_2^{**})^\infty(v_{\text{conc}}^{a, \eta})) dx + \int_{\bar{\Omega}} f_1^{\text{min}} (f_2^{**})^\infty \left(\frac{dv^s}{d|v^s|} \right) d|v^s| \\ &\leq \int_{\Omega} g(u, v^a) dx + \int_{\bar{\Omega}} f_1^{\text{min}} (f_2^{**})^\infty \left(\frac{dv^s}{d|v^s|} \right) d|v^s| + \eta. \end{aligned} \quad (3.16)$$

Here, for the first inequality, we used the definition of g (see (1.5)). To pass to the limit as $j \rightarrow \infty$ in the second line of (3.16), Lebesgue's dominated convergence theorem has been used for the first term (since $I_j v_1^a \rightarrow v_1^a$ in L^1 and f_1 is bounded), and Reshetnyak's continuity theorem (see, e.g., [3, Theorem 2.39]) for the other (exploiting strict convergence and positive 1-homogeneity of $(f_2^{**})^\infty$). The limit of the latter then splits because $v_{\text{conc}}^{a, \eta}$ and v^s are orthogonal measures. The last inequality in (3.16) is due to (3.15).

Finally, combining (3.14) and (3.16) gives

$$\bar{\mathcal{F}}(u, v) \leq \int_{\Omega} g(u, v^a) dx + \int_{\bar{\Omega}} f_1^{\text{min}} (f_2^{**})^\infty \left(\frac{dv^s}{d|v^s|} \right) d|v^s| + \int_{\Omega} dQW(Du)(x) + \eta.$$

As $\eta > 0$ was arbitrary, this concludes the proof.

Remark 3.1. It is easily seen that an entirely analogous proof allows us to replace the energy density W above by a function

$$f : \bar{\Omega} \times \mathbb{R}^m \times \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$$

satisfying the assumptions in [29, Theorem A], and thus leading to the following representation:

$$\begin{aligned} \bar{\mathcal{F}}(u, v) = & \int_{\bar{\Omega}} f_1^{\min}(f_2^{**})^\infty \left(\frac{dv}{d|v^s|} \right) d|v^s| + \int_{\Omega} g\left(u, \frac{dv^a}{d\mathcal{L}^N}\right) dx + \int_{\Omega} f(x, u, \nabla u) dx \\ & + \int_{\Omega} K_f(x, u^+(x), u^-(x), v_u(x)) d\mathcal{H}^{n-1} + \int_{\Omega} f^\infty\left(x, u(x), \frac{dDu^c}{d|Du^c|}\right) d|Du^c|, \end{aligned}$$

with the functions K_f and f^∞ defined by [29, p. 2] and [29, Definition 2.8], respectively. We emphasize that the same type of result could be obtained under the more stringent assumptions provided by [21].

Remark 3.2 (Properties of g). (i) According to the terminology of [6], for fixed a , $g(a, \cdot)$ denotes the *infimal convolution* of $f_1(a)f_2^{**}$ and $f_1^{\min}(f_2^{**})^\infty$.
(ii) The function $g(a, \cdot)$ is convex, as it is obtained by “projecting” a convex function defined on \mathbb{R}^{2d} to one on \mathbb{R}^d by minimizing out the other directions along a fixed linear subspace:

$$(b_1, b_2) \mapsto f_1(a)f_2^{**}(b_1) + f_1^{\min}(f_2^{**})^\infty(b_2)$$

is convex as the sum of convex functions, and $g(b)$ is its minimal value on the hypersurface $b + H$, with the fixed d -dimensional linear subspace

$$H := \{(b_1, b_2) \in \mathbb{R}^{2d} \mid b_1 + b_2 = 0\} \subset \mathbb{R}^{2d}.$$

(iii) If $f_1(a) = f_1^{\min}$, the choice $b_1 = b$ (and thus $b_2 = 0$) is optimal for the minimization in the definition of g , because the expression to be minimized in (1.5) then can be interpreted as (the limit of) a convex combination (see also [6, p. 685]):

$$\begin{aligned} f_2^{**}(b_1) + (f_2^{**})^\infty(b_2) &= \lim_{\sigma \rightarrow 0^+} (1 - \sigma)f_2^{**}(b_1) + \sigma(f_2^{**})^\infty\left(\frac{b_2}{\sigma}\right) \\ &= \lim_{\sigma \rightarrow 0^+} (1 - \sigma)f_2^{**}(b_1) + \sigma f_2^{**}\left(\frac{b_2}{\sigma}\right) \\ &\geq \lim_{\sigma \rightarrow 0^+} f_2^{**}\left((1 - \sigma)b_1 + \sigma \frac{b_2}{\sigma}\right) \\ &= f_2^{**}(b_1 + b_2). \end{aligned}$$

Conversely, if $b_1 = b$ is optimal, then

$$g(a, b) = f_1(a)f_2^{**}(b) \leq f_1^{\min}(f_2^{**})^\infty(b),$$

the latter term being the competitor for the choice $b_2 = b$. Since $(f_2^{**})^\infty(b)$ approximates $f_2^{**}(b)$ for large $|b|$, this is impossible for large $|b|$ whenever $f_1^{\min} < f_1(a)$.

Example 3.3. Consider the functional

$$\mathcal{F}(u, v) = \int_{\Omega} (W(\nabla u) + f_1(u)f_2(v)) dx,$$

subject to the constraints

$$\int_{\Omega} u dx = |\Omega|, \quad \int_{\Omega} v dx = 1,$$

with integrands given by

$$W(\nabla u) := (|\nabla u|^2 + 1)^{\frac{1}{2}}, \quad f_1(u) := 2 - \exp(-u^2), \quad f_2(v) := |v|.$$

Notice that W and f_2 are convex, $W^\infty = |\cdot|$, $f_2^\infty = |\cdot|$, $f_1^{\min} = f_1(0) = 1 \leq f_1 \leq 2$, and

$$g(a, b) = \min_{b_1 + b_2 = b} (f_1(a)|b_1| + f_1^{\min}|b_2|) = |b|. \quad (3.17)$$

In this class, the associated functional \mathcal{F} is coercive in $W^{1,1}(\Omega) \times L^1(\Omega)$. The associated relaxed functional is

$$\overline{\mathcal{F}}(u, v) = \int_{\Omega} W(\nabla u) dx + \int_{\Omega} d|D^s u| + |v|(\bar{\Omega}), \quad (u, v) \in BV(\Omega) \times \mathcal{M}(\bar{\Omega}),$$

subject to the appropriately extended constraint

$$\int_{\Omega} u dx = |\Omega|, \quad v(\bar{\Omega}) = 1. \quad (3.18)$$

Notice that u and v are fully decoupled in $\overline{\mathcal{F}}$, while \mathcal{F} clearly had coupling in the second term. In addition,

$$\inf\{\overline{\mathcal{F}}(u, v) \mid u \in BV(\Omega), v \in \mathcal{M}(\bar{\Omega}), \text{ formula (3.18) holds}\} = 1 = \overline{\mathcal{F}}(u^*, v^*),$$

where the minimizers are fully characterized by

$$u^* = 1, \quad v^* \geq 0, \quad v^*(\bar{\Omega}) = 1.$$

In particular, purely singular measures, for example $v^* = \delta_{x_0}$ with some $x_0 \in \bar{\Omega}$, appear among the minimizers.

Remark 3.4 (Concentration can be forced for minimizing sequences). In Example 3.3, for every limit pair $(u, v) \in BV \times \mathcal{M}$ with $u(x) \neq 0$ for a.e. $x \in \Omega$ (in particular, for $u = u^* = 1$), each associated recovery sequence v_k is purely concentrating in the sense that $v_k \rightarrow 0$ in measure, while simultaneously

$$\int_{\Omega} v_k(x) dx \rightarrow v(\bar{\Omega}) = 1.$$

This holds for *all* recovery sequences (u_k, v_k) associated to (u, v) with $u \neq 0$ a.e., even if v is absolutely continuous with respect to \mathcal{L}^n . In particular, we can choose $(u, v) = (u^*, v^*)$ for an arbitrary minimizer, and therefore v_k is purely concentrating for all minimizing sequences of the original functional \mathcal{F} .

To prove this concentration effect, we briefly revisit the proof of the lower bound in Theorem 1.1, where now all estimates have to hold with equality. Let

$$(u_k, v_k) \in W^{1,1}(\Omega) \times L^1(\Omega)$$

be a recovery sequence, i.e., such that $(u_k, v_k) \rightharpoonup^* (u, v)$ in $BV(\Omega) \times \mathcal{M}(\bar{\Omega})$ and $\lim_k \mathcal{F}(u_k, v_k) = \overline{\mathcal{F}}(u, v)$, and recall that we decomposed $v_k = v_k^{\text{osc}} + v_k^{\text{conc}}$ into a purely oscillating (equi-integrable) part and a purely concentrating part, $v_k^{\text{osc}} \rightharpoonup v_{\text{osc}}$ in L^1 and $v_k^{\text{conc}} \rightharpoonup^* v_{\text{conc}}$ in \mathcal{M} , respectively. As long as $a \neq 0$ and thus $f_1(a) > f_1^{\min}$, the minimum in the definition of $g(a, b)$ (see (3.17)) is attained if and only if $b_1 = 0$ and $b_2 = b$. As $u(x) \neq 0$ for a.e. $x \in \Omega$, the final estimate (3.5) thus holds with equality if and only if

$$v_{\text{osc}}(x) = 0 \quad \text{for a.e. } x.$$

In addition, the lower semicontinuity inequality of Ioffe's theorem (essentially) used in (3.3) holds with equality, namely

$$\lim_k \int_{\Omega} f_1(u) |v_k^{\text{osc}}| dx = \int_{\Omega} f_1(u) |v_{\text{osc}}| dx.$$

Since $f_1 \geq 1$, this implies that $v_k^{\text{osc}} \rightarrow 0 = v_{\text{osc}}$ strongly in L^1 . Hence, $v_k = v_k^{\text{osc}} + v_k^{\text{conc}}$ is purely concentrating as claimed.

Remark 3.5. In Example 3.3, if we replace f_1 by

$$\tilde{f}_1(x, u) := (x - x_0)^2 + 2 - \exp(-u^2), \quad \text{with a fixed } x_0 \in \bar{\Omega}$$

(adding $(x - x_0)^2$ to the original f_1 , a situation not covered by our theorems, but it is easy to see they extend to this case), then (u^*, δ_{x_0}) is still a minimizer with the same minimal energy, with u^* as defined in the example and the Dirac mass δ_{x_0} at x_0 . However, the minimizer is now unique, as it is clearly optimal to concentrate all mass of v at $x = x_0$ where $\tilde{f}_1^{\min}(x) = (x - x_0)^2 + 1$ is minimal.

Remark 3.6 (Relaxation by means of parametrized measures). Alternatively, one can extend the functional \mathcal{F} in (1.2) by continuity to a set of generalized Young measures and define the relaxation by measures in this set or use them just as a tool to derive relaxation in BV. We refer to [5] for such approach or to [24, 25] where generalized Young measures allowing for treatment of oscillations, concentrations and discontinuities were developed. This would allow us to give a different proof of Theorem 1.1. Nevertheless, the requirement that the recovery sequence $(\tilde{u}_k)_k$ can be first chosen to provide the correct limit for $(\int_{\Omega} W(\nabla \tilde{u}_k(x)) dx)_k$ as in [21], which is then carefully modified to recover also the other term in the functional, is essential and cannot be currently avoided.

4 The 1D case: Proof of Theorem 1.2

The measure representation obtained in Proposition 2.8 for $\overline{\mathcal{F}}(u, \nu, \cdot)$ in (2.12) leads us to provide an integral representation for this functional in any dimension. In the case $\Omega \subset \mathbb{R}$, the proof, achieved by showing a double inequality, will make use of the blow-up method introduced in [20], taking into account the specific structure of any Radon measure defined on subsets of \mathbb{R} .

In the one-dimensional case, it turns out that two cases have to be distinguished: the zero-dimensional contribution where the limiting measure charges individual points, and the “diffuse” rest. Here, notice that for any finite measure ν on a set I , we can split

$$\nu = \nu^0 + \nu^{\text{diff}},$$

where ν^0 is the zero-dimensional part of ν charging points, i.e.,

$$\nu^0 := \nu|_{S^0}, \quad \text{where } S^0 = S^0(\nu) := \{x \in I \mid |\nu|(\{x\}) > 0\}. \quad (4.1)$$

In particular, $\nu^0(\{x_0\}) = \nu(\{x_0\})$ for all $x_0 \in I$ (and zero for all but countably many), $|\nu^0| = |\nu|^0$ and $|\nu|^{\text{diff}} = |\nu^{\text{diff}}|$. Also recall that for a BV-function u defined on an interval I ,

$$Du = u' \mathcal{L}^1 + Du^j + Du^c = u' \mathcal{L}^1 + [u] \mathcal{H}^0 \llcorner S_u + Du^c,$$

where for \mathcal{H}^0 -a.e. $x \in S_u$ (the jump set of u) we have

$$[u](x) := u(x^+) - u(x^-), \quad \text{with } u(x^\pm) := \lim_{y \rightarrow x^\pm} u(y).$$

Accordingly, $Du^j = (Du)^0$ and $Du^{\text{diff}} = u' \mathcal{L}^1 + Du^c$.

The proof is based on the blow-up method. Let $u \in \text{BV}(\Omega; \mathbb{R}^m)$, $\nu \in \mathcal{M}(\bar{\Omega}; \mathbb{R}^d)$ and let $\overline{\mathcal{F}}_{u,\nu}$ denote the signed Radon measure on $\bar{\Omega}$ given by Lemma 2.8, uniquely extending the set function $\overline{\mathcal{F}}(u, \nu, \cdot)$ introduced in (2.11) and (2.12), i.e.,

$$\overline{\mathcal{F}}_{u,\nu}(A) = \overline{\mathcal{F}}(u, \nu, A) = \Gamma - \liminf \mathcal{F}(u, \nu, A) \quad \text{for all } A \subset \bar{\Omega} \text{ relatively open.}$$

In view of Lemma 2.8, it is enough to rephrase and prove our claim (1.7) in terms of appropriate localized statements for appropriate densities of $\overline{\mathcal{F}}_{u,\nu}$.

With the abbreviation (and the implicit convention $|Du|(\partial\Omega) = 0$)

$$\theta := \mathcal{L}^1 + |\nu| + |Du|, \quad \theta \in \mathcal{M}(\bar{\Omega}),$$

we claim that

$$\frac{d\overline{\mathcal{F}}_{u,\nu}^{\text{diff}}}{d\theta^{\text{diff}}}(x_0) = f_1(u(x_0)) \frac{df_2^{**}(\nu^{\text{diff}})}{d\theta^{\text{diff}}}(x_0) + \frac{dW^{**}((Du)^{\text{diff}})}{d\theta^{\text{diff}}}(x_0) \quad \text{for } \theta^{\text{diff}}\text{-a.e. } x_0 \in \Omega, \quad (4.2)$$

$$\overline{\mathcal{F}}_{u,\nu}^0(\{x_0\}) = f_W^0(u(x_0^+), u(x_0^-), \nu^0(\{x_0\})) \quad \text{for every } x_0 \in \Omega, \quad (4.3)$$

$$\overline{\mathcal{F}}_{u,\nu}^0(\{x_0\}) = \left(\inf_{z \in \mathbb{R}^m} f_W^0(u(x_0^+), z, \nu^0(\{x_0\})) \right) \quad \text{for } x_0 = \inf \Omega,$$

$$\overline{\mathcal{F}}_{u,\nu}^0(\{x_0\}) = \left(\inf_{z \in \mathbb{R}^m} f_W^0(z, u(x_0^-), \nu^0(\{x_0\})) \right) \quad \text{for } x_0 = \sup \Omega.$$

Here, $f_2^{**}(v^{\text{diff}})$ and $W^{**}((Du)^{\text{diff}})$ are the measures defined in Section 2.4, and $f_W^0 : \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^d \rightarrow \mathbb{R}$ is given by (1.8). As to (4.2), also notice that, for σ^{diff} -a.e. x_0 , x_0 is not a jump point of u , and therefore $u(x_0) := u(x_0^+) = u(x_0^-)$ is well defined.

Remark 4.1. It is easy to check that

$$f_W^0(u(x^+), u(x^-), 0) = (W^{**})^\infty([u](x)) \quad \text{with } [u](x) := u(x^+) - u(x^-),$$

i.e., the typical energy contribution of a jump of u at x if v does not play any role. Moreover, if $a^+ = a^- =: a$ and $b = 0$, then $f_W^0(a, a, 0) = 0$, with $u \equiv a$ and $v = 0$ as the obvious optimal choices for the infimum. For that reason, (4.3) is trivially correct for all points x_0 with $\theta^0(\{x_0\}) = 0$, and it provides nontrivial information only for points with $\theta^0(\{x_0\}) > 0$. The latter includes the jump points of u , but the most interesting case is in fact $b := v(\{x_0\}) \neq 0$. In particular, it can happen that $f_W^0(a, a, b) < f_1(a)(f_2^{**})^\infty(b)$ if reducing the energy by locally creating an artificial peak in u to reduce $f_1(u)$ at a point charged by v is cheaper than the cost for the corresponding slope of u paid in $(W^{**})^\infty$. In any such case, associated recovery sequences will not converge strictly in BV. In this aspect, the case $n = 1$ is fundamentally different to the case $n \geq 2$, where our construction for the upper bound in Section 3.2 effectively yields a strictly converging recovery sequence for u .

4.1 Lower bound

Since $f_2 \geq f_2^{**}$ and $W \geq W^{**}$, we can assume that f_2 and W are convex.

Let $(u_k, v_k) \in W^{1,1}(\Omega; \mathbb{R}^m) \times L^1(\Omega; \mathbb{R}^d)$ such that $u_k \rightarrow u$ in $L^1(\Omega; \mathbb{R}^m)$ and $v_k \xrightarrow{*} v$ in $\mathcal{M}(\bar{\Omega}; \mathbb{R}^d)$, and assume, up to a (not relabeled) subsequence, that the limit

$$\lim_{k \rightarrow +\infty} \int_{\Omega} (f_1(u_k)f_2(v_k) + W(u_k')) dx < +\infty$$

exists. For every Borel set $B \subset \Omega$, define

$$\mu_k(B) = \int_B (f_1(u_k)f_2(v_k) + W(u_k')) dx. \quad (4.4)$$

Since the functions are nonnegative and (H1)–(H3) hold, the sequences (μ_k) and $(|v_k| + |Du_k|)$ of nonnegative Radon measures, being uniformly bounded in $\mathcal{M}(\Omega)$, admit two (not relabeled) subsequences weakly* converging to two nonnegative finite Radon measures μ and λ , i.e.,

$$\mu_k \xrightarrow{*} \mu \quad \text{and} \quad |v_k| + |Du_k| \xrightarrow{*} \lambda \quad \text{in } \mathcal{M}(\bar{\Omega}). \quad (4.5)$$

We decompose μ as the sum of two mutually singular measures as described above, $\mu = \mu^0 + \mu^{\text{diff}}$, such that $|\mu^0|$ is the atomic part concentrating on points and μ^{diff} is the remaining diffuse part.

Notice that, as a consequence of the coercivity conditions in (H1)–(H3),

$$|v| + |Du| + \mathcal{L}^1 \ll \mu \quad \text{and} \quad |\mu| \ll \mathcal{L}^1 + \lambda.$$

On the other hand, Lemma 2.8, implies that a density of μ^{diff} can be computed as a Radon–Nikodým derivative with respect to $\theta^{\text{diff}} = \mathcal{L}^1 + |Du^{\text{diff}}| + |v^{\text{diff}}|$, and, if x_0 is a jump point of u (i.e., $|Du|(\{x_0\}) > 0$), or $|v|(\{x_0\}) > 0$, then $x_0 \in S^0(\mu)$, the set of atomic contributions defined in (4.1).

We now treat diffuse and atomic contributions in μ separately. Throughout, we use the intervals

$$I_{x_0, \varepsilon} := (x_0 - \varepsilon, x_0 + \varepsilon)$$

and

$$I := (-1, 1),$$

where $x_0 \in \Omega$ and $\varepsilon > 0$ is small enough so that $I_{x_0, \varepsilon} \subset \Omega$.

4.2 Diffuse contributions

Let x_0 be a point with $0 = \lambda(\{x_0\})$ such that the measures μ , $f_2(v)$ and $W(Du)$ have a finite density with respect to θ^{diff} at x_0 , i.e.,

$$\begin{aligned}\frac{d\mu}{d\theta^{\text{diff}}}(x_0) &= \lim_{\varepsilon \rightarrow 0} \frac{\mu(I_{x_0, \varepsilon})}{\theta^{\text{diff}}(I_{x_0, \varepsilon})} < +\infty, \\ \frac{df_2(v)}{d\theta^{\text{diff}}}(x_0) &= \lim_{\varepsilon \rightarrow 0} \frac{f_2(v)(I_{x_0, \varepsilon})}{\theta^{\text{diff}}(I_{x_0, \varepsilon})} < +\infty, \\ \frac{dW(Du)}{d\theta^{\text{diff}}}(x_0) &= \lim_{\varepsilon \rightarrow 0} \frac{W(Du)(I_{x_0, \varepsilon})}{\theta^{\text{diff}}(I_{x_0, \varepsilon})} < +\infty.\end{aligned}$$

In particular,

$$\begin{aligned}0 &= |\mu|(\{x_0\}) = |Du|(\{x_0\}) = |v|(\{x_0\}) = W(Du)(\{x_0\}) = f_2(v)(\{x_0\}), \\ \frac{df_2(v)}{d\theta^{\text{diff}}}(x_0) &= \frac{df_2(v)^{\text{diff}}}{d\theta^{\text{diff}}}(x_0), \\ \frac{dW(Du)}{d\theta^{\text{diff}}}(x_0) &= \frac{dW(Du)^{\text{diff}}}{d\theta^{\text{diff}}}(x_0).\end{aligned}$$

Notice that these properties hold for θ^{diff} -a.e. $x_0 \in \Omega$. In particular, $0 = \lambda(\{x_0\})$ for θ^{diff} -a.e. x_0 , because positive values are possible for at most countably many points x_0 which are negligible for the non-atomic measure θ^{diff} .

To shorten the notation when calculating densities with respect to θ^{diff} , below we will use the abbreviation

$$\vartheta_{x_0, \varepsilon} := \theta^{\text{diff}}(I_{x_0, \varepsilon}) \geq \mathcal{L}^1(I_{x_0, \varepsilon}) = 2\varepsilon.$$

Notice that $\vartheta_{x_0, \varepsilon} \rightarrow \theta^{\text{diff}}(\{x_0\}) = 0$ as $\varepsilon \rightarrow 0$, by dominated convergence.

Take a sequence $\varepsilon \rightarrow 0^+$ such that

$$\mu(\{x_0 - \varepsilon, x_0 + \varepsilon\}) = \lambda(\{x_0 - \varepsilon, x_0 + \varepsilon\}) = 0.$$

Thus $\mu(\partial I_{x_0, \varepsilon}) = 0$, and by the definition of μ and a changes of variables,

$$\begin{aligned}\frac{d|\mu|}{d\theta^{\text{diff}}}(x_0) &= \lim_{\varepsilon \rightarrow 0} \frac{\mu(I_{x_0, \varepsilon})}{\vartheta_{x_0, \varepsilon}} \\ &= \lim_{\varepsilon \rightarrow 0} \lim_{k \rightarrow +\infty} \frac{1}{\vartheta_{x_0, \varepsilon}} \int_{I_{x_0, \varepsilon}} f_1(u_k(y))f_2(v_k(y)) + W(u'_k(y)) dy \\ &= \lim_{\varepsilon \rightarrow 0} \lim_{k \rightarrow +\infty} \int_{I_{0, \varepsilon} \vartheta_{x_0, \varepsilon}^{-1}} f_1(u_k(x_0) + \vartheta_{x_0, \varepsilon} w_{k, \varepsilon}(x))f_2(\eta_{k, \varepsilon}(x)) + W(w'_{k, \varepsilon}(x)) dx, \quad (4.6)\end{aligned}$$

where

$$w_{k, \varepsilon}(x) := \frac{u_k(x_0 + \vartheta_{x_0, \varepsilon} x) - u_k(x_0)}{\vartheta_{x_0, \varepsilon}} \quad \text{and} \quad \eta_{k, \varepsilon}(x) := v_k(x_0 + \vartheta_{x_0, \varepsilon} x).$$

Choosing a sequence $\hat{\varepsilon}(k) > 0$ with $\hat{\varepsilon}(k) \rightarrow 0$ slow enough, we see that for every sequence $\varepsilon(k) \rightarrow 0^+$ with $\varepsilon(k) \geq \hat{\varepsilon}(k)$,

$$\frac{d|\mu|}{d\theta^{\text{diff}}}(x_0) = \lim_{k \rightarrow +\infty} \int_{I_{0, \varepsilon(k)} \vartheta_{x_0, \varepsilon(k)}^{-1}} f_1(u_k(x_0) + \vartheta_{x_0, \varepsilon(k)} w_{k, \varepsilon(k)}(x))f_2(\eta_{k, \varepsilon(k)}(x)) + W(w'_{k, \varepsilon(k)}(x)) dx. \quad (4.7)$$

By coercivity of W and the fact that

$$\frac{d|\mu|}{d\theta^{\text{diff}}}(x_0)$$

is finite, we infer that

$$\int_{I_{0, \varepsilon(k)} \vartheta_{x_0, \varepsilon(k)}^{-1}} |w'_{k, \varepsilon(k)}(x)| dx$$

is bounded. Since $w_{k,\varepsilon(k)}(0) = 0$, this entails that $w_{k,\varepsilon(k)}$ is bounded on

$$I_{0,\varepsilon(k)}\mathfrak{g}_{x_0,\varepsilon(k)}^{-1}$$

by a constant independent of k . Hence,

$$\mathfrak{g}_{x_0,\varepsilon(k)}w_{k,\varepsilon(k)}(x) \rightarrow 0 \quad \text{uniformly in } x \in I_{0,\varepsilon(k)}\mathfrak{g}_{x_0,\varepsilon(k)}^{-1}. \quad (4.8)$$

In addition, we claim that

$$u_k(x_0) \rightarrow u(x_0), \quad (4.9)$$

essentially because $|Du| \leq w^* - \lim|Du_k| \leq \lambda$ and $\lambda(\{x_0\}) = 0$. For a proof of (4.9), first recall that $u_k \rightarrow u$ in L^1 , and thus pointwise a.e. for a subsequence. We can therefore choose a sequence $x_j \rightarrow x_0$ with, say, $x_j \geq x_0$, such that

$$u_k(x_j) \rightarrow u(x_j) \quad \text{as } k \rightarrow \infty \text{ for each } j.$$

Since

$$\begin{aligned} |u_k(x_0) - u(x_0)| &\leq |u_k(x_j) - u(x_j)| + |u_k(x_0) - u_k(x_j)| + |u(x_0) - u(x_j)| \\ &\leq |u_k(x_j) - u(x_j)| + (|Du_k| + |Du|)([x_0, x_j]), \end{aligned}$$

we obtain (4.9) passing to the limit, first as $k \rightarrow \infty$ and then as $j \rightarrow \infty$:

$$\limsup_{k \rightarrow \infty} |u_k(x_0) - u(x_0)| \leq (\lambda + |Du|)([x_0, x_j]) \leq 2\lambda([x_0, x_j]) \xrightarrow{j \rightarrow \infty} 2\lambda(\{x_0\}) = 0.$$

Combining (4.8) and (4.9) with the fact that f_1 is uniformly continuous on bounded subsets of \mathbb{R}^m , we can replace the argument of f_1 by $u(x_0)$ in (4.7). Retracing our steps to (4.6), we conclude that

$$\frac{d|\mu|}{d\theta^{\text{diff}}}(x_0) \geq \liminf_{\varepsilon \rightarrow 0^+} \lim_{k \rightarrow +\infty} \frac{1}{\mathfrak{g}_{x_0,\varepsilon}} \int_{I_{x_0,\varepsilon}} f_1(u(x_0))f_2(v_k(y)) + W(u'_k(y)) dy. \quad (4.10)$$

On the right-hand side of (4.10), for fixed ε , we can apply standard lower semicontinuity results with respect to weak*-convergence in \mathcal{M} and BV, respectively, exploiting that f_2 and W are convex and that $|v|(\partial I_{x_0,\varepsilon}) = |Du|(\partial I_{x_0,\varepsilon}) = 0$ for all but at most countably many ε . This yields that

$$\begin{aligned} \frac{d|\mu|}{d\theta^{\text{diff}}}(x_0) &\geq \liminf_{\varepsilon \rightarrow 0^+} \frac{1}{\mathfrak{g}_{x_0,\varepsilon}} \int_{I_{x_0,\varepsilon}} (f_1(u(x_0))df_2(v)(y) + dW(Du)(y)) \\ &= f_1(u(x_0)) \frac{df_2(v^{\text{diff}})}{d\theta^{\text{diff}}}(x_0) + \frac{dW(Du^{\text{diff}})}{d\theta^{\text{diff}}}(x_0). \end{aligned} \quad (4.11)$$

As this holds for all sequences (u_k, v_k) admissible in the definition of $\overline{\mathcal{F}}(u, v, \cdot)$, inequality (4.11) implies the lower bound (“ \geq ”) in (4.2).

4.3 Contributions charging individual points in the interior

We claim that

$$\mu^0(\{x_0\}) \geq f_W^0(u(x_0^+), u(x_0^-), v(\{x_0\})) \quad \text{for all } x_0 \in \Omega \cap S^0(\mu), \quad (4.12)$$

which directly implies the lower bound (“ \geq ”) in (4.3). Here, f_W^0 is defined by (1.8), i.e.,

$$f_W^0(a^+, a^-, b) = \inf_{\substack{u \in W^{1,1}((-1,1); \mathbb{R}^m) \\ v \in L^1((-1,1); \mathbb{R}^d) \\ u(-1) = a^-, u(1) = a^+ \\ \int_{-1}^1 v dx = b}} \left\{ \int_{-1}^1 (f_1(u)(f_2^{**})^\infty(v) + (W^{**})^\infty(u')) dx \right\}. \quad (4.13)$$

Let $x_0 \in S^0(\mu)$. Since $u \in BV$ on a one-dimensional domain, $u(x_0^+)$ and $u(x_0^-)$ exist in \mathbb{R}^m (and outside of the jump set of u they are equal). In the following, let (ε) denote a sequence of positive reals converging to 0 such that

$$|v|(\{x_0 \pm \varepsilon\}) = |Du|(\{x_0 \pm \varepsilon\}) = \mu(\{x_0 \pm \varepsilon\}) = 0. \quad (4.14)$$

By virtue of dominated convergence, $\mu(\{x_0\}) = \lim_{\varepsilon \rightarrow 0^+} \mu(I_{x_0, \varepsilon})$, and by (4.4), (4.5) and a change of variables, we have that

$$\mu(\{x_0\}) \geq \lim_{\varepsilon \rightarrow 0^+} \lim_{k \rightarrow +\infty} \varepsilon \int_I (f_1(u_k(x_0 + \varepsilon y)) f_2(v_k(x_0 + \varepsilon y)) + W(u'_k(x_0 + \varepsilon y))) dy, \quad (4.15)$$

where $I = (-1, 1)$ as before. Next, let

$$v_{k, \varepsilon}(y) := v_k(x_0 + \varepsilon y).$$

Since $v_k \xrightarrow{*} v$ in $\mathcal{M}(\bar{\Omega}; \mathbb{R}^d)$ and $|v|(\{x_0 \pm \varepsilon\}) = 0$ by (4.14), for fixed ε ,

$$\int_I \varepsilon v_{k, \varepsilon}(y) dy = \int_{I_{x_0, \varepsilon}} v_k dx \xrightarrow{k \rightarrow \infty} v(I_{x_0, \varepsilon}).$$

As $\varepsilon \rightarrow 0^+$, we conclude that

$$\lim_{\varepsilon \rightarrow 0^+} \lim_{k \rightarrow +\infty} \int_I \varepsilon v_{k, \varepsilon}(y) dy = v(\{x_0\}).$$

A diagonalization argument now ensures the existence of a sequence $\bar{v}_\varepsilon := v_{k(\varepsilon), \varepsilon}$ such that

$$\lim_{\varepsilon \rightarrow 0^+} \int_I \varepsilon \bar{v}_\varepsilon(y) dy = v(\{x_0\}).$$

On the other hand, for $u_{k, \varepsilon}(y) := u_k(x_0 + \varepsilon y)$, it is easily seen that

$$\lim_{\varepsilon \rightarrow 0^+} \lim_{k \rightarrow +\infty} \|u_{k, \varepsilon} - u^\pm\|_{L^1(I)} = 0, \quad \text{where } u^\pm(y) = \begin{cases} u(x_0^+) & \text{if } y \geq 0, \\ u(x_0^-) & \text{if } y < 0. \end{cases} \quad (4.16)$$

In addition, defining

$$\bar{u}_\varepsilon := u_{k(\varepsilon), \varepsilon}, \quad \bar{v}_\varepsilon := v_{k(\varepsilon), \varepsilon},$$

with a suitable $k(\varepsilon) \rightarrow +\infty$ (fast enough), we can rewrite (4.15) as

$$\mu(\{x_0\}) \geq \lim_{\varepsilon \rightarrow 0^+} \varepsilon \left(\int_I f_1(\bar{u}_\varepsilon(y)) f_2(\bar{v}_\varepsilon(y)) + W\left(\frac{1}{\varepsilon} \bar{u}'_\varepsilon(y)\right) \right) dy. \quad (4.17)$$

Further defining

$$\hat{v}_\varepsilon(y) := \varepsilon \bar{v}_\varepsilon(y) - \frac{\varepsilon}{2} \int_I \bar{v}_\varepsilon dy + \frac{1}{2} v(\{x_0\}),$$

we observe that

$$|\hat{v}_\varepsilon(y) - \varepsilon \bar{v}_\varepsilon(y)| = \left| \int_I \frac{\varepsilon}{2} \bar{v}_\varepsilon(y) dy - \frac{1}{2} v(\{x_0\}) \right| \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \quad (4.18)$$

Together with the global Lipschitz continuity of f_2 , equation (4.18) justifies replacing \bar{v}_ε by $\frac{1}{\varepsilon} \hat{v}_\varepsilon$ in (4.17), which gives

$$\mu(\{x_0\}) \geq \lim_{\varepsilon \rightarrow 0^+} \int_I \left(f_1(\bar{u}_\varepsilon(y)) \varepsilon f_2\left(\frac{1}{\varepsilon} \hat{v}_\varepsilon(y)\right) + \varepsilon W\left(\frac{1}{\varepsilon} \bar{u}'_\varepsilon(y)\right) \right) dy.$$

Above, we may replace f_2 and W by their recession functions in (4.19), using (2.7) with $h = f_2$ and $h = W$ (both convex with linear growth) together with the fact that both \hat{v}_ε and \bar{u}'_ε are bounded in L^1 . Therefore,

$$\mu(\{x_0\}) \geq \lim_{\varepsilon \rightarrow 0^+} \int_I \left(f_1(\bar{u}_\varepsilon(y)) f_2^\infty(\hat{v}_\varepsilon(y)) + W^\infty(\bar{u}'_\varepsilon(y)) \right) dy. \quad (4.19)$$

As

$$\int_I \hat{v}_\varepsilon dx = v(\{x_0\})$$

by construction, \hat{v}_ε is admissible in the infimum defining f_W^0 in (4.13) (in place of v). Applying Lemma 2.3 to the integrand $f_1 f_2^{\text{co}} + W^{\text{co}}$, we modify \bar{u}_ε into a function \hat{u}_ε which has the same values as u^\pm on $\{\pm 1\}$; cf. (4.16). Thus, the new function \hat{u}_ε is also admissible in (4.13) (in place of u) and we obtain (4.12) from (4.19), more precisely,

$$\mu(\{x_0\}) \geq \lim_{\varepsilon \rightarrow 0^+} \int_I (f_1(\hat{u}_\varepsilon(y)) f_2^{\text{co}}(\hat{v}_\varepsilon) + W^{\text{co}}(\hat{u}'_\varepsilon(y))) dy \geq f_W^0(u(x_0^+), u(x_0^-), v(\{x_0\})).$$

4.4 Contributions charging boundary points

Let $x_0 \in \partial\Omega$, say, $x_0 = \inf \Omega$ (the other case is analogous). Passing to a subsequence if necessary, we may assume that $z := \lim u_k(x_0)$ exists. By extending $u_k(x) := u_k(x_0)$ and $v_k(x) := 0$ for all $x < x_0$, which leads to $d\mu_k(x) = f_1(u_k(x_0))f_2(0) dx$, $d\mu(x) = f_1(z)f_2(0) dx$ and $d\lambda(x) = 0$ for $x < x_0$, we can argue as for interior points, with $u(x_0^-) = z$.

4.5 Upper bound

Let $(u, v) \in \text{BV}(\Omega; \mathbb{R}^m) \times \mathcal{M}(\bar{\Omega}; \mathbb{R}^d)$. As before, without loss of generality (see Proposition 2.6), f_2 and W can be assumed to be convex, and we can use the representation (2.14) of $\bar{\mathcal{F}} = \bar{\mathcal{F}}_{**}$ obtained in Proposition 2.7. By Lemma 2.8, it is enough to prove the upper bounds for the density of $\bar{\mathcal{F}}_{u,v}$ with respect to the atomic and diffuse parts of $\theta = \theta^0 + \theta^{\text{diff}}$. We therefore have to show that

$$\frac{d\bar{\mathcal{F}}_{u,v}}{d\theta^{\text{diff}}}(x_0) \leq f_1(u(x_0)) \frac{df_2(v)}{d\theta^{\text{diff}}}(x_0) + \frac{dW(Du)}{d\theta^{\text{diff}}}(x_0) \quad \text{for } \theta^{\text{diff}}\text{-a.e. } x_0 \in \Omega, \quad (4.20)$$

$$\bar{\mathcal{F}}_{u,v}(\{x_0\}) \leq f_W^0(u^+(x_0), u^-(x_0), v(\{x_0\})) \quad \text{for every } x_0 \in \Omega, \quad (4.21)$$

$$\bar{\mathcal{F}}_{u,v}(\{x_0\}) \leq \inf_{z \in \mathbb{R}^m} f_W^0(z, u^-(x_0), v(\{x_0\})) \quad \text{for } x_0 = \sup \Omega,$$

$$\bar{\mathcal{F}}_{u,v}(\{x_0\}) \leq \inf_{z \in \mathbb{R}^m} f_W^0(u^+(x_0), z, v(\{x_0\})) \quad \text{for } x_0 = \inf \Omega.$$

Here, recall that $\theta = \mathcal{L}^1 + |v| + |Du| \in \mathcal{M}(\bar{\Omega})$ (with $|Du|(\partial\Omega) := 0$).

4.6 Diffuse contributions

As before, we set $I_{x_0, \varepsilon} := (x_0 - \varepsilon, x_0 + \varepsilon)$ and $\vartheta_{x_0, \varepsilon} := \theta^{\text{diff}}(I_{x_0, \varepsilon})$. We only consider $x_0 \in \Omega$ such that the Besicovitch derivatives

$$\frac{d\bar{\mathcal{F}}(u, v, \cdot)}{d\theta^{\text{diff}}}(x_0), \quad \frac{df_2(v)}{d\theta^{\text{diff}}}(x_0), \quad \frac{dW(Du)}{d\theta^{\text{diff}}}(x_0)$$

exist with finite values. In particular, $|v|(\{x_0\}) = |Du|(\{x_0\}) = 0$, x_0 is not a jump point of u , u has a representative which is continuous at x_0 , and $\vartheta_{x_0, \varepsilon} \rightarrow 0$ as $\varepsilon \rightarrow 0$.

For the proof of (4.20), by Proposition 2.7 (with $f = f^{**}$ and $W = QW$), it suffices to find a sequence $v_k \subset L^1(\Omega; \mathbb{R}^d)$ such that $v_k \rightharpoonup^* v$ in $\mathcal{M}(\bar{\Omega}; \mathbb{R}^d)$ and

$$\begin{aligned} & \liminf_{\varepsilon \rightarrow 0} \limsup_{k \rightarrow \infty} \frac{1}{\vartheta_{x_0, \varepsilon}} \int_{x_0 - \varepsilon}^{x_0 + \varepsilon} f_1(u) f_2(v_k) dx + dW(Du)(x) \\ & \leq \lim_{\varepsilon \rightarrow 0} \frac{1}{\vartheta_{x_0, \varepsilon}} \int_{x_0 - \varepsilon}^{x_0 + \varepsilon} f_1(u(x_0)) df_2(v)(x) + dW(Du)(x). \end{aligned} \quad (4.22)$$

As the contribution of $W(Du)$ appears on both sides, (4.22) can be reduced to

$$\liminf_{\varepsilon \rightarrow 0} \limsup_{k \rightarrow \infty} \frac{1}{\mathfrak{g}_{x_0, \varepsilon}} \int_{x_0 - \varepsilon}^{x_0 + \varepsilon} f_1(u) f_2(v_k) dx \leq f_1(u(x_0)) \lim_{\varepsilon \rightarrow 0} \frac{1}{\mathfrak{g}_{x_0, \varepsilon}} \int_{x_0 - \varepsilon}^{x_0 + \varepsilon} df_2(v). \quad (4.23)$$

We choose $(v_k) \subset L^1(\Omega; \mathbb{R}^d)$ such that $v_k \rightarrow v$ area-strictly in $\mathcal{M}(\bar{\Omega}; \mathbb{R}^d)$, for instance by using Lemma 2.5. To prove (4.23), first observe that

$$|u(x) - u(x_0)| \leq \left| \int_{x_0}^x d|Du| \right| \leq \mathfrak{g}_{x_0, \varepsilon} \quad \text{for all } x \text{ with } |x - x_0| < \varepsilon \text{ and } |Du(\{x\})| = 0. \quad (4.24)$$

Here, we excluded the case $|Du(\{x\})| > 0$ to ensure that $u(x)$ is well defined; this certainly holds for \mathcal{L}^1 -a.e. x . Since f_1 is continuous and $\mathfrak{g}_{x_0, \varepsilon} \rightarrow 0$ as $\varepsilon \rightarrow 0$, inequality (4.24) implies that

$$\|f_1(u(\cdot)) - f_1(u(x_0))\|_{L^\infty(I_{x_0, \varepsilon})} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \quad (4.25)$$

Moreover, as $v \mapsto \int df_2(v)$ is continuous with respect to area-strict convergence by Proposition 2.1,

$$\lim_{k \rightarrow \infty} \int_{x_0 - \varepsilon}^{x_0 + \varepsilon} f_2(v_k) dx = \int_{x_0 - \varepsilon}^{x_0 + \varepsilon} df_2(v) \quad \text{if } v(\{x_0 + \varepsilon\}) = v(\{x_0 - \varepsilon\}) = 0. \quad (4.26)$$

(While $v_k \rightarrow v$ area-strictly on $\bar{\Omega}$ by construction, this in general only implies area strict-convergence on the smaller set $I_{x_0, \varepsilon}$ if v does not charge its boundary.) Clearly, all but countably many ε satisfy the restriction required in (4.26), and thus

$$\liminf_{\varepsilon \rightarrow 0} \lim_{k \rightarrow \infty} \frac{1}{\mathfrak{g}_{x_0, \varepsilon}} \int_{x_0 - \varepsilon}^{x_0 + \varepsilon} f_2(v_k) dx = \lim_{\varepsilon \rightarrow 0} \frac{1}{\mathfrak{g}_{x_0, \varepsilon}} \int_{x_0 - \varepsilon}^{x_0 + \varepsilon} df_2(v). \quad (4.27)$$

Combined, (4.25) and (4.27) imply (4.23).

4.7 Contributions charging individual points in the interior

As pointed out before, we may assume without loss of generality that f_2 and W are convex. In addition, we will exploit that with

$$\tilde{u}(x) = u\left(\frac{1}{\varepsilon}(x - x_0)\right) \quad \text{and} \quad \tilde{v}(x) = \frac{1}{\varepsilon}v\left(\frac{1}{\varepsilon}(x - x_0)\right),$$

by a change of variables and the positive 1-homogeneity of the recession functions, the definition of f_W^0 in (4.13) is equivalent to

$$f_W^0(a^+, a^-, b) = \inf_{\substack{\tilde{u} \in W^{1,1}(I_{x_0, \varepsilon}; \mathbb{R}^m) \\ \tilde{v} \in L^1(I_{x_0, \varepsilon}; \mathbb{R}^d) \\ \tilde{u}(x_0 \pm \varepsilon) = a^\pm \\ \int_{I_{x_0, \varepsilon}} \tilde{v} dx = b}} \left\{ \int_{I_{x_0, \varepsilon}} (f_1(\tilde{u})(f_2^{**})^\infty(\tilde{v}) + (W^{**})^\infty(\tilde{u}')) dx \right\}, \quad (4.28)$$

independently of the choice of $\varepsilon > 0$ and $x_0 \in \mathbb{R}^n$. Here, recall that $I_{x_0, \varepsilon} = (x_0 - \varepsilon, x_0 + \varepsilon)$.

Let $u \in \text{BV}(\Omega; \mathbb{R}^m)$ and $v \in \mathcal{M}(\Omega; \mathbb{R}^d)$, and fix $x_0 \in \Omega$. We want to show that (4.21) holds, i.e., that

$$\frac{d\bar{\mathcal{F}}_{u,v}}{d\delta_{x_0}}(x_0) = \bar{\mathcal{F}}_{u,v}(\{x_0\}) \leq f_W^0(u(x_0^-), u(x_0^+), v(\{x_0\})). \quad (4.29)$$

Here, $u(x_0^-)$ and $u(x_0^+)$ denote the left- and right-hand side limits of u at x_0 . In particular,

$$Du(\{x_0\}) = u(x_0^+) - u(x_0^-).$$

The equality in (4.29) is a trivial consequence of Lebesgue's dominated convergence theorem, as $\overline{\mathcal{F}}_{u,v}$ is a finite measure. It therefore suffices to show the inequality. For each $\varepsilon > 0$, we choose

$$(U_\varepsilon, V_\varepsilon) \in W^{1,1}(I_{x_0,\varepsilon}; \mathbb{R}^m) \times L^1(I_{x_0,\varepsilon}; \mathbb{R}^m)$$

admissible and almost optimal for the infimum defining f_W^0 in (4.28), with

$$a^\pm := u(x_0^\pm) \quad \text{and} \quad b := \nu(\{x_0\}) = \lim_{\varepsilon \rightarrow 0} \nu(I_{x_0,\varepsilon}),$$

so that

$$f_W^0(u(x_0^-), u(x_0^+), b) + \varepsilon \geq \int_{x_0-\varepsilon}^{x_0+\varepsilon} f_1(U_\varepsilon) f_2^\infty(V_\varepsilon) + W^\infty(U'_\varepsilon) dx. \quad (4.30)$$

In (4.30), we may also assume without loss of generality that $f_W^0(u(x_0^-), u(x_0^+), b)$ is finite. Hence, (4.30) implies that U'_ε is bounded in L^1 by the coercivity of W^∞ inherited from W .

We define $u_\varepsilon \in \text{BV}(\Omega; \mathbb{R}^m)$ as the unique function satisfying

$$Du_\varepsilon = \begin{cases} Du & \text{on } \Omega \setminus I_{x_0,\varepsilon}, \\ DU_\varepsilon & \text{on } I_{x_0,\varepsilon}, \end{cases} \quad \text{and} \quad u_\varepsilon((x_0 - \varepsilon)^+) = a^- = u(x_0^-).$$

As $U_\varepsilon(x_0 - \varepsilon) = a^-$ (recall that U_ε is admissible in (4.13)), this entails that $u_\varepsilon = U_\varepsilon$ on $I_{x_0,\varepsilon}$. In addition,

$$u_\varepsilon(x) - u(x) = \begin{cases} u(x_0^-) - u((x_0 - \varepsilon)^+) & \text{for } x < x_0 - \varepsilon, \\ u(x_0^+) - u((x_0 + \varepsilon)^-) & \text{for } x > x_0 + \varepsilon, \end{cases}$$

whence

$$\|u_\varepsilon - u\|_{L^\infty(\Omega \setminus I_{x_0,\varepsilon}; \mathbb{R}^m)} \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (4.31)$$

Since u_ε is a bounded sequence in BV , (4.31) implies that $u_\varepsilon \rightharpoonup^* u$ in $\text{BV}(\Omega; \mathbb{R}^m)$.

For the measure ν , we define the analogous approximation

$$v_\varepsilon := \begin{cases} \nu & \text{on } \Omega \setminus I_{x_0,\varepsilon}, \\ V_\varepsilon & \text{on } I_{x_0,\varepsilon}. \end{cases}$$

Since

$$\int_{x_0-\varepsilon}^{x_0+\varepsilon} V_\varepsilon dx = b = \frac{d\nu}{d\delta_{x_0}}(x_0)$$

and V_ε is bounded in L^1 by (4.30) and the coercivity of f_2 ,

$$v_\varepsilon \rightharpoonup^* \nu|_{\Omega \setminus \{x_0\}} + b\delta_{x_0} = \nu.$$

To show the upper bound (4.29), we first consider the case $\nu \in L^1(\Omega \setminus \{x_0\}; \mathbb{R}^d)$, so that

$$(u_\varepsilon, v_\varepsilon) \in \text{BV}(\Omega; \mathbb{R}^m) \times L^1(\Omega; \mathbb{R}^d)$$

and

$$\frac{d\overline{\mathcal{F}}_{u,v}}{d\delta_{x_0}}(\{x_0\}) \leq \limsup_{r \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \int_{x_0-r}^{x_0+r} f_1(u_\varepsilon) f_2(v_\varepsilon) + W(u'_\varepsilon) dx, \quad (4.32)$$

since $(u_\varepsilon, v_\varepsilon)$ is admissible in the infimum in the representation of $\overline{\mathcal{F}} = \overline{\mathcal{F}}_{**}$ obtained in Proposition 2.7. The case of a general $\nu \in \mathcal{M}(\Omega; \mathbb{R}^d)$ is easily recovered with an additional approximation argument, mollifying $\tilde{\nu} := \nu - \nu(\{x_0\})\delta_{x_0}$.

In (4.32), we may assume without loss of generality that $r = \varepsilon$: As $u'_\varepsilon = u'$ and $v_\varepsilon = \nu$ on $\Omega \setminus I_{x_0,\varepsilon}$, and $u_\varepsilon - u \rightarrow 0$ uniformly on $\Omega \setminus I_{x_0,\varepsilon}$, the integral on $I_r \setminus I_{x_0,\varepsilon}$ converges to zero as $\varepsilon \leq r \rightarrow 0$. It therefore suffices to show that

$$\limsup_{\varepsilon \rightarrow 0} \int_{x_0-\varepsilon}^{x_0+\varepsilon} f_1(u_\varepsilon) f_2(v_\varepsilon) + W(u'_\varepsilon) dx \leq f_W^0(u(x_0^-), u(x_0^+), b). \quad (4.33)$$

Since f_2 and W are convex, we have that $f_2(v_\varepsilon) \leq f_2(0) + f_2^\infty(v_\varepsilon)$ and $W(u'_\varepsilon) \leq W(0) + W^\infty(u'_\varepsilon)$. Consequently, also exploiting that $f_1(u_\varepsilon)$ is uniformly bounded,

$$\limsup_{\varepsilon \rightarrow 0} \int_{x_0 - \varepsilon}^{x_0 + \varepsilon} f_1(u_\varepsilon) f_2(v_\varepsilon) + W(u'_\varepsilon) dx \leq \limsup_{\varepsilon \rightarrow 0} \int_{x_0 - \varepsilon}^{x_0 + \varepsilon} f_1(u_\varepsilon) f_2^\infty(v_\varepsilon) + W(u'_\varepsilon) dx. \quad (4.34)$$

As $u_\varepsilon = U_\varepsilon$ and $v_\varepsilon = V_\varepsilon$ on $I_{x_0, \varepsilon}$ by construction, (4.34) combined with (4.30) implies (4.33).

4.8 Contributions charging points on the boundary

Let $x_0 \in \partial\Omega$, say, $x_0 = \inf \Omega$ (the other case is analogous). For any $\delta > 0$, we can choose $z_0 = z_0(\delta) \in \mathbb{R}^m$ such that

$$f_W^0(u(x_0^+), z_0, \nu^0(\{x_0\})) \leq \delta + \inf_{z \in \mathbb{R}^m} f_W^0(u(x_0^+), z, \nu^0(\{x_0\})).$$

The preceding construction for interior points is easily adapted with $u(x_0^-) := z_0$, and we omit the details. It yields that

$$\bar{\mathcal{F}}(u, \nu)(\{x_0\}) \leq f_W^0(u^+(x_0), z_0, \nu(\{x_0\})) + \delta.$$

As this holds for all $\delta > 0$, we obtain the assertion.

A Proofs of auxiliary results

Here, we present the proofs of some of the auxiliary results collected in Section 2.

Proof of Proposition 2.6. Clearly, it suffices to prove

$$\bar{\mathcal{F}}(\cdot, \cdot, \cdot) \leq \bar{\mathcal{F}}_{**}(\cdot, \cdot, \cdot),$$

the opposite inequality being trivial.

In order to achieve the desired conclusion, we argue as follows. For every $A \in \mathcal{A}_r(\bar{\Omega})$, denote by $\bar{\mathcal{F}}^W$ the localized sequentially weak $W^{1,1}(\Omega \cap A; \mathbb{R}^m) \times L^1(\Omega \cap A; \mathbb{R}^d)$ lower semicontinuous envelope of F in (2.11). It was proved in [10] that for every $A \in \mathcal{A}_r(\bar{\Omega})$ (so that $A \cap \Omega$ can be any open subset of Ω),

$$\bar{\mathcal{F}}^W(u, \nu, A) = \int_{\Omega \cap A} (f_1(u) f_2^{**}(\nu) + \mathcal{Q}W(\nabla u)) dx \quad \text{for every } (u, \nu) \in W^{1,1}(\Omega \cap A; \mathbb{R}^m) \times L^1(\Omega \cap A; \mathbb{R}^d).$$

Since for every $(u, \nu) \in W^{1,1}(\Omega \cap A; \mathbb{R}^m) \times L^1(\Omega \cap A; \mathbb{R}^d)$,

$$\bar{\mathcal{F}}(u, \nu, A) \leq \bar{\mathcal{F}}^W(u, \nu, A),$$

we infer that

$$\begin{aligned} \bar{\mathcal{F}}(u, \nu, A) &= \inf \left\{ \liminf_{k \rightarrow +\infty} \bar{\mathcal{F}}(u_k, \nu_k, A) \mid \right. \\ &\quad \left. W^{1,1}(\Omega \cap A; \mathbb{R}^m) \times L^1(\Omega \cap A; \mathbb{R}^d) \ni (u_k, \nu_k) \xrightarrow{*} (u, \nu) \text{ in } \text{BV}(\Omega \cap A; \mathbb{R}^m) \times \mathcal{M}(A; \mathbb{R}^d) \right\} \\ &\leq \inf \left\{ \liminf_{k \rightarrow +\infty} \bar{\mathcal{F}}^W(u_k, \nu_k, A) \mid \right. \\ &\quad \left. W^{1,1}(\Omega \cap A; \mathbb{R}^m) \times L^1(\Omega \cap A; \mathbb{R}^d) \ni (u_k, \nu_k) \xrightarrow{*} (u, \nu) \text{ in } \text{BV}(\Omega \cap A; \mathbb{R}^m) \times \mathcal{M}(A; \mathbb{R}^d) \right\} \\ &= \inf \left\{ \liminf_{k \rightarrow +\infty} \int_A f_1(u_k) f_2^{**}(\nu_k) dx + \int_A \mathcal{Q}W(\nabla u_k) dx \mid \right. \\ &\quad \left. W^{1,1}(\Omega \cap A; \mathbb{R}^m) \times L^1(\Omega \cap A; \mathbb{R}^d) \ni (u_k, \nu_k) \xrightarrow{*} (u, \nu) \text{ in } \text{BV}(\Omega \cap A; \mathbb{R}^m) \times \mathcal{M}(A; \mathbb{R}^d) \right\} \end{aligned}$$

for every $(u, \nu) \in \text{BV}(\Omega \cap A; \mathbb{R}^m) \times \mathcal{M}(A; \mathbb{R}^d)$, which concludes the proof. \square

Proof of Proposition 2.7. It suffices to show “ \leq ”, as “ \geq ” is trivial.

For every fixed $k \in \mathbb{N}$ and $u_k \in \text{BV}(\Omega \cap A; \mathbb{R}^m)$, we can choose a sequence $(w_{k,l})_{l \in \mathbb{N}} \subset W^{1,1}(\Omega \cap A; \mathbb{R}^m)$ such that, as $l \rightarrow \infty$, we have $w_{k,l} \rightharpoonup^* u_k$ in $\text{BV}(\Omega \cap A; \mathbb{R}^m)$ and

$$\int_{\Omega \cap A} \mathcal{Q}W(\nabla w_{k,l}) \, dx = \int_{\Omega \cap A} d\mathcal{Q}W(Du_k)(x) \quad (\text{A.1})$$

(any $W^{1,1}$ recovery sequence). For instance, since $W^{1,1}$ is dense in BV with respect to area strict convergence, there exists $(w_{k,l}) \subset W^{1,1}$ such that, as $l \rightarrow \infty$, we have $w_{k,l} \rightarrow u_k$ area-strictly, which yields (A.1) by Proposition 2.1. By (H3) (coercivity and growth of W), equation (A.1) implies that

$$\limsup_l \|\nabla w_{k,l}\|_{L^1(A; \mathbb{R}^{m \times n})} \leq C(1 + |Du_k|(A))$$

with a constant $C > 0$.

In addition, given any $v_k \in L^1(\Omega \cap A; \mathbb{R}^m)$, we also have that

$$\int_{\Omega \cap A} f_1(w_{k,l}) f_2^{**}(v_k) \, dx \xrightarrow{l \rightarrow \infty} \int_{\Omega \cap A} f_1(u_k) f_2^{**}(v_k) \, dx$$

by dominated convergence, also using that $w_{k,l} \rightarrow u_k$ in L^1 , f_1 is bounded and $f_2^{**}(v_k) \in L^1(\Omega \cap A)$.

If $u_k \rightharpoonup^* u$ in BV and $v_k \rightharpoonup^* v$ in \mathcal{M} , it is therefore possible to choose a diagonal sequence $\tilde{u}_k := w_{k,l(k)}$, with $l(k) \rightarrow \infty$ fast enough as $k \rightarrow \infty$, such that for

$$\begin{aligned} \liminf_k \int_{\Omega \cap A} f_1(\tilde{u}_k) f_2^{**}(v_k) \, dx + \int_{\Omega \cap A} \mathcal{Q}W(\nabla \tilde{u}_k) \, dx \\ = \liminf_k \int_{\Omega \cap A} f_1(u_k) f_2^{**}(v_k) \, dx + \int_{\Omega \cap A} d\mathcal{Q}W(Du_k)(x), \end{aligned} \quad (\text{A.2})$$

we have

$$\limsup_k \|\nabla \tilde{u}_k\|_{L^1(\Omega \cap A; \mathbb{R}^{m \times n})} \leq C(1 + \limsup_k |Du_k|(\Omega \cap A))$$

and $\tilde{u}_k \rightarrow u$ in $L^1(\Omega \cap A; \mathbb{R}^m)$. In particular, $\tilde{u}_k \rightharpoonup^* u$ in $\text{BV}(\Omega \cap A; \mathbb{R}^m)$. This means that if

$$(u_k, v_k) \subset \text{BV}(\Omega \cap A; \mathbb{R}^m) \times L^1(\Omega \cap A; \mathbb{R}^m)$$

is an arbitrary admissible sequence for the infimum defining $\overline{\mathcal{F}}_{**}$ in (2.13), then

$$(\tilde{u}_k, v_k) \subset W^{1,1}(\Omega \cap A; \mathbb{R}^m) \times L^1(\Omega \cap A; \mathbb{R}^d)$$

is admissible, too. Hence, (A.2) yields the assertion. \square

In the following, we will discuss the measure representation for the localized relaxed functionals, i.e., for $A \in \mathcal{A}_r(\overline{\Omega})$,

$$\begin{aligned} \overline{\mathcal{F}}(u, v, A) := \inf_{k \rightarrow +\infty} \{ \liminf F(u_k, v_k, A) \mid (u_k, v_k) \in W^{1,1}(A; \mathbb{R}^m) \times L^1(A; \mathbb{R}^d), \\ (u_k, v_k) \xrightarrow{*} (u, v) \text{ in } \text{BV}(A; \mathbb{R}^m) \times \mathcal{M}(A; \mathbb{R}^d) \}. \end{aligned}$$

Here, recall that by $\mathcal{A}_r(\overline{\Omega})$ we denote the family of open subsets of $\overline{\Omega}$ in the relative topology. The following result is a close relative of [1, Lemma 2.5].

Lemma A.1. *Let $\overline{\Omega}$ be as above. Let $\lambda : \mathcal{A}_r(\overline{\Omega}) \rightarrow [0, +\infty)$ and μ be such that the following conditions hold:*

- (i) μ is a finite Radon measure on $\overline{\Omega}$.
 - (ii) $\lambda(\overline{\Omega}) \geq \mu(\overline{\Omega})$.
 - (iii) $\lambda(A) \leq \mu(A)$ for all $A \in \mathcal{A}_r(\overline{\Omega})$.
 - (iv) (Subadditivity) $\lambda(A) \leq \lambda(A \setminus \overline{U}) + \lambda(B)$ for all $A, B, U \in \mathcal{A}_r(\overline{\Omega})$ such that $U \Subset B \Subset A$.
 - (v) For all $A \in \mathcal{A}_r(\overline{\Omega})$ and $\varepsilon > 0$, there exists $U \in \mathcal{A}_r(\overline{\Omega})$ such that $U \Subset A$ and $\lambda(A \setminus \overline{U}) < \varepsilon$.
- Then $\lambda = \mu$ on $\mathcal{A}_r(\overline{\Omega})$.

Proof. The inequality $\lambda(A) \leq \mu(A)$ for every $A \in \mathcal{A}_r(\overline{\Omega})$ follows from (iii).

For what concerns the other inequality, we can observe that, by the inner regularity of μ , we can find a relatively open subset of $\bar{\Omega}$, say $A' \Subset A$, such that

$$\begin{aligned} \mu(A) &< \varepsilon + \mu(\bar{A}') \\ &= \varepsilon + \mu(\bar{\Omega}) - \mu(\bar{\Omega} \setminus \bar{A}') \\ &\leq \varepsilon + \lambda(\bar{\Omega}) - \lambda(\bar{\Omega} \setminus \bar{A}') \\ &\leq \varepsilon + \lambda(A). \end{aligned}$$

Thus, letting $\varepsilon \rightarrow 0$, we obtain the desired conclusion. \square

Proof of Lemma 2.8. In order to prove that $\bar{\mathcal{F}}(u, v, A)$ is the trace of a Radon measure, we refer to Lemma A.1, and define the increasing set function $\lambda : \mathcal{A}_r(\bar{\Omega}) \rightarrow [0, +\infty]$ by

$$\lambda(A) := \bar{\mathcal{F}}(u, v, A).$$

By the definition of $\bar{\mathcal{F}}$, we know that there exists a sequence $(u_h, v_h) \in W^{1,1}(\Omega; \mathbb{R}^m) \times L^1(\Omega; \mathbb{R}^d)$ such that $u_h \xrightarrow{*} u$ in $BV(\Omega; \mathbb{R}^m)$ and $v_h \xrightarrow{*} v$ in $\mathcal{M}(\bar{\Omega}; \mathbb{R}^d)$ such that

$$\bar{\mathcal{F}}(u, v, \bar{\Omega}) = \lim_{h \rightarrow +\infty} \int_{\Omega} (f_1(u_h) f_2(v_h) + W(\nabla u_h)) \, dx = \lambda(\bar{\Omega}).$$

Next, denoting by λ_h the measures $(f_1(u_h) f_2(v_h) + W(\nabla u_h)) \mathcal{L}^n$, we obtain that it converges weakly $*$ in the sense of measures (duality with elements in $C(\bar{\Omega})$), up to a subsequence (due to the bounds) to a measure μ . Now, due to the lower semicontinuity with respect to the weak $*$ convergence, we have

$$\mu(\bar{\Omega}) \leq \liminf_h \lambda_h(\bar{\Omega}) = \lambda(\bar{\Omega}).$$

Then, by the definition of λ , we have, for every $A \in \mathcal{A}_r(\bar{\Omega})$,

$$\lambda(A) \leq \liminf_h \int_{\Omega \cap A} (f_1(u_h) f_2(v_h) + W(\nabla u_h)) \, dx \leq \mu(\bar{A}).$$

Now by the previous lemma, we would have that $\lambda = \mu$ if we prove inner regularity and subadditivity for $\bar{\mathcal{F}}(u, v, \cdot)$.

For what concerns inner regularity (i.e., Lemma A.1 (v)), we employ the growth conditions in (H1)–(H3). It is enough to show that

$$\bar{\mathcal{F}}(u, v, A) \leq \beta(\mathcal{L}^n(A) + |Du|(A) + |v|(A)), \quad (\text{A.3})$$

with some constant $\beta > 0$, because if we substitute $A \setminus \bar{U}$ instead of A in (A.3), the inner regularity of the upper bound measures provides the asserted inner regularity for $\bar{\mathcal{F}}$. By the definition of $\bar{\mathcal{F}}$ and the growth conditions from above, we have that

$$\bar{\mathcal{F}}(u, v, A) \leq \liminf_{k \rightarrow \infty} \beta \int_A (1 + |v_k| + |\nabla u_k|) \, dx \quad (\text{A.4})$$

for any sequence $(v_k, u_k)_k \subset L^1(A; \mathbb{R}^d) \times W^{1,1}(A; \mathbb{R}^m)$ with $v_k \xrightarrow{*} v$ in \mathcal{M} and $u_k \xrightarrow{*} u$ in BV . It now suffices to choose (v_k, u_k) so that $v_k \rightarrow v$ strictly in $\mathcal{M}(A; \mathbb{R}^d)$, and $u_k \rightarrow u$ strictly in $BV(A \setminus \partial\Omega; \mathbb{R}^m)$, because then

$$\int_A (1 + |v_k| + |\nabla u_k|) \, dx \rightarrow \mathcal{L}^n(A) + |v|(A) + |Du|(A) \quad \text{as } k \rightarrow \infty.$$

By the latter, (A.4) implies (A.3). Here, recall that we use the convention $|Du|(\partial\Omega) := 0$, so that we have $|Du|(A) = |Du|(A \setminus \partial\Omega)$. By mollification similar to the classical approximation result of Meyers and Serrin, this choice of sequences is indeed possible even for arbitrary open A without boundary regularity; see, for instance, [3, Theorem 3.9] (to choose (u_k) , while the choice of (v_k) is easier and can be done similarly, or similarly to Lemma 2.5).

It remains to prove that $\overline{\mathcal{F}}(u, v, \cdot)$ is subadditive in the sense of Lemma A.1 (iv), i.e., it suffices to prove that

$$\overline{\mathcal{F}}(u, v, A) \leq \overline{\mathcal{F}}(u, v, B) + \overline{\mathcal{F}}(u, v, A \setminus \overline{U}) \quad (\text{A.5})$$

for all $A, U, B \in \mathcal{A}_r(\overline{\Omega})$ with $U \Subset B \Subset A$, $u \in \text{BV}(\Omega; \mathbb{R}^m)$ and $v \in \mathcal{M}(\overline{\Omega}; \mathbb{R}^d)$ (see, e.g., [7, Lemma 4.3.4]). Without loss of generality, in view of Proposition 2.6, we can assume f_2 convex and W quasiconvex. Fix $\eta > 0$ and find $(w_h) \subset W^{1,1}(A \setminus \overline{U}, \mathbb{R}^m)$ and $(v_h) \subset L^1(A \setminus \overline{U}, \mathbb{R}^d)$ such that $w_h \xrightarrow{*} u$ in $\text{BV}((A \setminus \overline{U}), \mathbb{R}^m)$, $v_h \xrightarrow{*} v$ in $\mathcal{M}(A \setminus \overline{U}, \mathbb{R}^d)$ and

$$\limsup_{h \rightarrow +\infty} \int_{A \setminus \overline{U}} (f_1(w_h)f_2(v_h) + W(\nabla w_h)) dx \leq \overline{\mathcal{F}}(u, v, A \setminus \overline{U}) + \eta. \quad (\text{A.6})$$

Extract a subsequence still denoted by n such that the above upper limit is a limit. Let B_0 be a relatively open subset of $\overline{\Omega}$ with Lipschitz boundary such that $U \Subset B_0 \Subset B$. Then there exist $(u_h) \subset W^{1,1}(B_0, \mathbb{R}^m)$ and $(\bar{v}_h) \subset L^1(B_0; \mathbb{R}^d)$ such that $u_h \xrightarrow{*} u$ in $\text{BV}(B_0, \mathbb{R}^m)$, $\bar{v}_h \xrightarrow{*} v$ in $\mathcal{M}(\overline{B_0}, \mathbb{R}^d)$ and

$$\overline{\mathcal{F}}(u, v, B_0) = \lim_{h \rightarrow +\infty} \int_{B_0} (f_1(u_h)f_2(\bar{v}_h) + W(\nabla u_h)) dx. \quad (\text{A.7})$$

Consider, for every $D \in \mathcal{A}_r(\overline{\Omega})$, the set function $\mathcal{G}(u, v, D) := \int_D (1 + |\nabla u|) dx + |v|(D)$. Due to (H1)–(H3), we may extract a bounded subsequence, that we will not relabel, from the sequences of measures

$$v_h := \mathcal{G}(w_h, v_h, \cdot) + \mathcal{G}(u_h, \bar{v}_h, \cdot)$$

restricted to $B_0 \setminus \overline{U}$, converging in the sense of distributions to some Radon measure ν defined on $B_0 \setminus \overline{U}$. For every $t > 0$, let

$$B_t := \{x \in B_0 \mid \text{dist}(x, \partial B_0) > t\}.$$

Define, for $0 < \delta < \eta$, the subsets

$$L_\delta := B_{\eta-2\delta} \setminus B_{\eta+\delta}.$$

Consider a smooth cut-off function $\varphi_\delta \in C^\infty(B_{\eta-\delta}, [0, 1])$ such that $\varphi_\delta = 1$ on B_η . As the thickness of the strip L_δ is of order δ , we have an upper bound of the form $\|\nabla \varphi_\delta\|_{L^\infty(B_{\eta-\delta})} \leq C/\delta$. Define

$$w'_h(x) := \varphi_\delta(x)u_h(x) + (1 - \varphi_\delta(x))w_h(x), \quad v'_h(x) := \varphi_\delta(x)\bar{v}_h(x) + (1 - \varphi_\delta(x))v_h(x).$$

Clearly, the sequences w'_h and v'_h weakly* converge to u in $\text{BV}(A, \mathbb{R}^d)$ and to ν in $\mathcal{M}(A; \mathbb{R}^m)$ as $h \rightarrow +\infty$, respectively, and

$$\nabla w'_h = \varphi_\delta \nabla u_h + (1 - \varphi_\delta) \nabla w_h + \nabla \varphi_\delta \otimes (u_h - w_h).$$

By the growth conditions (H1)–(H3), we have the estimate

$$\begin{aligned} & \int_A (f_1(w'_h)f_2(v'_h) + W(\nabla w'_h)) dx \\ & \leq \int_{B_\eta} (f_1(u_h)f_2(\bar{v}_h) + W(\nabla u_h)) dx + \int_{A \setminus \overline{B_{\eta-\delta}}} (f_1(w_h)f_2(v_h) + W(\nabla w_h)) dx \\ & \quad + C(\mathcal{G}(u_h, \bar{v}_h, L_\delta) + \mathcal{G}(w_h, v_h, L_\delta)) + \frac{1}{\delta} \int_{L_\delta} |w_h - u_h| dx \\ & \leq \int_{B_0} (f_1(u_h)f_2(\bar{v}_h) + W(\nabla u_h)) dx + \int_{A \setminus \overline{U}} (f_1(w_h)f_2(v_h) + W(\nabla w_h)) dx \\ & \quad + C(\mathcal{G}(u_h, \bar{v}_h, L_\delta) + \mathcal{G}(w_h, v_h, L_\delta)) + \frac{1}{\delta} \int_{L_\delta} |w_h - u_h| dx. \end{aligned}$$

Thus, passing to the limit as $n \rightarrow +\infty$ and making use of the lower semicontinuity of $\overline{\mathcal{F}}(\cdot, \cdot, A)$ (which is a consequence of its definition (2.12)), (A.6) and (A.7), we obtain

$$\begin{aligned} \overline{\mathcal{F}}(u, v, A) & \leq \overline{\mathcal{F}}(u, v, B_0) + \overline{\mathcal{F}}(u, v, A \setminus \overline{U}) + \eta + C\nu(\overline{L_\delta}) \\ & \leq \overline{\mathcal{F}}(u, v, B) + \overline{\mathcal{F}}(u, v, A \setminus \overline{U}) + \eta + C\nu(\overline{L_\delta}). \end{aligned}$$

Now passing to the limit as $\delta \rightarrow 0^+$, we get

$$\overline{\mathcal{F}}(u, v, A) \leq \overline{\mathcal{F}}(u, v, B) + \overline{\mathcal{F}}(u, v, A \setminus \overline{U}) + \eta + Cv(\partial B_{\eta}).$$

It suffices to choose a subsequence $\{\eta_h\}$ such that $\eta_h \rightarrow 0^+$ and $v(\partial B_{\eta_h}) = 0$, to conclude the proof of (A.5). \square

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