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Pseudoalgebraic structures and representations of the exceptional Lie superalgebra $E(5, 10)$

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Daniele Brilli
ID number 1448483

Advisor
Prof. Alessandro D'Andrea

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Author's email: brilli.daniele@gmail.com

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Lunga vita alla Zero.

Contents

Introduction

The language of Lie pseudoalgebras [BDK1], closely related to that of Lie^* algebras as in [BD], is useful in giving finite description of infinite-dimensional Lie algebras, and has proved to be a valuable tool in algebra and representation theory. One of the main applications of the theory of Lie pseudoalgebras is in the study of representations of infinite-dimensional linearly compact Lie algebras, to which they are intimately connected [BDK2, BDK3, BDK4, D].
Shortly after being introduced in [B], vertex algebras were realized as the axiomatization of the algebraic properties of local families of quantum fields in the chiral sector of a Conformal Field Theory in dimension two. These algebraic properties are encoded by the Operator Product Expansion (OPE), whose singular part can be translated into the notion of a *conformal algebra* [K1].
Lie pseudoalgebras are a generalization of conformal algebras in higher dimensions. They are defined as a Lie algebra in a certain pseudotensor category attached to a cocommutative Hopf algebra H, whence their name. One of the main features of this theory is the fact that one can associate to a Lie H-pseudoalgebra L a (usually infinite-dimensional) Lie algebra $\mathcal{A}(L)$ called the *annihilation algebra*, which, when L is finite (i.e. finitely generated as an H-module), can be endowed with a *linearly compact* topology. This connection can be used in both ways: while the structure theory of infinite-dimensional linearly compact Lie algebras led to the classification of simple finite Lie H-pseudoalgebras, the representation theory of the latter can be used to simplify and obtain new insights in the study of representations of the former.
The idea behind the research work for this thesis was to investigate the possibility of exploiting this deep connection in the Lie superalgebras setting. Basically speaking, the problem of classifying irreducible representations of a linearly compact Lie superalgebra can be reduced (in many cases) to the description of the so-called *generalized Verma modules* and in particular to the classification of *singular vectors*. In this thesis we apply pseudoalgebraic techniques to the representation theory of the exceptional Lie superalgebra $E(5,10)$. In Chapter 3 we manage to construct a *Lie superpseudoalgebra* associated to $E(5,10)$ which allows us to obtain an estimate on the degree of singular vectors. Having proved to be worthy of interest, we also classify in Chapter 4 all such pseudoalgebraic structures for $E(5,10)$. This is meant to be a first "model" for suitable other interesting cases. We will now give more details about the ideas and techniques involved in this work.

Our object of interest are representations of infinite-dimensional linearly compact Lie superalgebras, i.e. Lie superalgebras whose underlying space is a linearly compact topological vector space and the bracket is continuous. Basically, a topological vector space is linearly compact if it admits a filtration of linear subspaces of finite codimension that define

a topology in respect to which it is complete.
Let us look at the "non super" setting first. The most important example of an infinite-dimensional linearly compact Lie algebra is given by W_N, the Lie algebra of continuous derivations of $\mathcal{O}_N = \mathbb{C}[[t^1, \ldots, t^N]]$. It consists of all *formal vector fields* $D = \sum_i f_i \partial/\partial t^i$, $f_i \in \mathcal{O}_N$. Infinite-dimensional linearly compact Lie algebras have been studied, for example, in [G1, G2, GS] and the classification of simple ones goes back to Cartan [C]. The list consists of W_N and its subalgebras S_N, H_N (N even) and K_N (N odd) of divergence zero vector fields, vector fields annihilating a symplectic form and vector fields multiplying a contact form by a function, respectively. These are called *Lie algebras of Cartan type*.
Irreducible representations on discrete spaces (which are contragradient of linearly compact ones) of Lie algebras of Cartan type were classified by Rudakov and Kostrikin [R1, R2, Ko]. Given a linearly compact Lie algebra $\mathcal{L}$ with canonical filtration $\{\mathcal{L}_k\}$, Rudakov realized irreducible representations as quotients of some *induced modules* from the finite-dimensional Lie algebra $\mathcal{L}_0/\mathcal{L}_n$, where n is called the *height* of the representation. For height $n > 1$, these induced modules are all irreducible, while for height 1 a finite number of them contains proper submodules. For instance, in the case of type W and S, Rudakov found out that the degenerate (i.e. not irreducible) modules could be fitted in the De Rham complex of formal differential forms and in this case the irreducible modules were obtained as the images of the differential.
The pseudoalgebraic language provides a different strategy to the study of discrete representations of a linearly compact Lie algebra $\mathcal{L}$ in the special case where $\mathcal{L}$ is the *annihilation algebra* of a Lie pseudoalgebra. I take my chance to explain what this means. Given a cocommutative Hopf algebra H, a *Lie H-pseudoalgebra* is a left H-module L together with an H-bilinear map, called *pseudobracket*, $[\cdot * \cdot] : L \otimes L \to (H \otimes H) \otimes_H L$ satisfying the following analogues of the axioms of a Lie bracket:

Skew-commutativity $[a * b] = -(\sigma \otimes_H id)[b * a]$;

Jacobi identity $[a * [b * c]] = [[a * b] * c] + ((\sigma \otimes id) \otimes_H id)[b * [a * c]]$,

for $a, b, c \in L$, where $\sigma(h_1 \otimes h_2) = (h_2 \otimes h_1)$ and the expressions of the form $[a * [b * c]]$ have a certain interpretation in $H^{\otimes 3}$. L is finite if it is finitely generate as an H-module.
We will always reduce to the case of $H = U(\mathfrak{d})$ being the universal enveloping algebra of a finite-dimensional Lie algebra $\mathfrak{d}$. Let $X = H^*$ be the dual of H as a coalgebra. One can associate to a Lie pseudoalgebra L a Lie H-differential algebra $\mathcal{A}(L) = X \otimes_H L$ called the *annihilation algebra*, with bracket induced by the pseudobracket of L. The semidirect sum $\mathfrak{d} \ltimes \mathcal{A}(L)$ is called the *extended annihilation algebra*. When L is finite, one can use the canonical filtration of $U(\mathfrak{d})$ to define a filtration of $\mathcal{A}(L)$ that makes the latter a linearly compact Lie algebra. Similarly, one can associate to a finite L-module V a linearly compact $\mathcal{A}(L)$-module $\mathcal{A}(V)$, called *annihilation module*.
The most important example of a finite Lie pseudoalgebra is the pseudoalgebra of *pseudo vector fields*, which is the free H-module $W(\mathfrak{d}) = H \otimes \mathfrak{d}$, with pseudobracket of the form

$$[f \otimes a * g \otimes b] = (fb \otimes g) \otimes (1 \otimes a) - (f \otimes ga) \otimes (1 \otimes b) - (f \otimes g) \otimes_H (1 \otimes [a, b]).$$

Its name its due to the fact that $\mathcal{A}(W(\mathfrak{d}))$ is isomorphic, as a linearly compact Lie algebra, to W_N. The classification of finite Lie H-pseudoalgebras is achieved in [BDK1] using

Cartan's classification. The strategy, roughly speaking, is to detect the linearly compact Lie algebras associated to simple Lie pseudoalgebras (i.e. their annihilation algebras), determine all compatible H-actions and then applying the so-called *reconstruction functor*, which recovers, under mild assumptions, the pseudoalgebraic structure. The list consists of current pseudoalgebras over finite-dimensional Lie algebras and (current pseudoalgebras over) $W(\mathfrak{d})$ and its subalgebras $S(\mathfrak{d},\chi)$, $H(\mathfrak{d},\chi,\omega)$ and $K(\mathfrak{d},\theta)$, whose annihilation algebras are respectively S_N, (an extension by a one-dimensional center of) H_N and K_N. The classification depends on certain parameters χ, ω and θ due to inequivalent actions of $\mathfrak{d}$ on the annihilation algebras. The subalgebras of $W(\mathfrak{d})$ are called *primitive Lie pseudoalgebras*. The representation theory of a Lie H-pseudoalgebra L is essentially equivalent to that of $\mathcal{A}(L)^e$. In [BDK2, BDK3, BDK4] all irreducible finite representations of primitive pseudoalgebras were classified, thus obtaining again, in particular, the results of Rudakov. The role of the induced modules is played here by the so-called *tensor modules*, which are free H-modules $T(V) = H \otimes V$ where V is a $\mathfrak{d} \oplus \mathfrak{gl}(\mathfrak{d})$-(resp. $\mathfrak{d} \oplus \mathfrak{sl}(\mathfrak{d})$-)module in the type W (resp. S) case. The presence of proper submodules is detected by the existence of *singular vectors*. This occurs only in a finite number of cases, which can be grouped in exact complexes. In the case of type W and S, the singular vectors appear in degrees respectively at most 1 and 2 (where the degree of $v \in H \otimes V$ is given by the canonical grading of $H = U(\mathfrak{d})$) and the corresponding tensor modules fit in a "pseudo version" of the formal De Rham complex. It is important to stress that the pseudoalgebraic language provides a rather precise and easy to prove upper bound to the degree of possible non constant singular vectors. One would like to be able to apply the same technique in the context of discrete representations of linearly compact Lie superalgebras, and this is exactly what this thesis deals with.

Let us thus turn to the super setting. Simple infinite-dimensional linearly compact Lie superalgebras were classified by Kac in [K2] and the list is far richer then the Cartan one. It consists of ten families (of which four are $W(m,n)$, $S(m,n)$, $H(m,n)$, $K(m,n)$, "super" analogues of the algebras of Cartan type) and five exceptional ones: $E(1,6)$, $E(3,6)$, $E(3,8)$, $E(4,4)$ and $E(5,10)$. The Lie superalgebras $E(3,6)$, $E(3,8)$ and $E(5,10)$ are of particular interest because of their possible relation with particle physics (see [KR2] for more details). In [KR1, KR2, KR3] Kac and Rudakov applied the techniques developed for the Lie algebras of Cartan type to the exceptional superalgebras $E(3,6)$ and $E(3,8)$ classifying all degenerate induced modules, which in this framework are called *generalized Verma modules*.

We focus now on the exceptional Lie superalgebra $\mathfrak{L} = E(5,10)$. In [CK2] is provided a nice geometrical construction of $\mathfrak{L}$: the even part is $\mathfrak{L}_{(0)} = S_5$, the Lie algebra of zero-divergence formal vector fields in five indeterminates, while the odd one $\mathfrak{L}_{(1)} = \mathrm{d}\Omega_5^1$ consists of formal closed 2-forms. The bracket between an even and an odd element is given by the Lie derivative, while between odd elements is defined by wedge product and consequent identification $\mathrm{d}\Omega_5^3 \cong S_5$. $\mathfrak{L}$ is equipped with a $\mathbb{Z}$-grading of depth 2, consistent with the parity, for which $\mathfrak{L}_0 \cong \mathfrak{sl}_5$ and all $\mathfrak{L}_i$ are $\mathfrak{L}_0$-modules. In particular, $\mathfrak{L}_{-1} \cong \mathfrak{d} := (\mathbb{C}^5)^*$ and $\mathfrak{L}_{-1} \cong \mathfrak{s} := \bigwedge^2 \mathfrak{d}$. Minimal Verma modules (i.e. induced by an irreducible representation) for $\mathfrak{L}$ take the form $\mathfrak{T}(V) = U(\mathfrak{d} \oplus \mathfrak{s}) \otimes V$ for an irreducible finite-dimensional $\mathfrak{sl}_5 = \mathfrak{sl}(\mathfrak{d})$-module V and one can define a grading by setting $\deg \partial = 2$ for $\partial \in \mathfrak{d}$ and $\deg \xi = 1$ for $\xi \in \mathfrak{s}$. If λ is the highest weight of V, we denote $\mathfrak{T}(\lambda) = \mathfrak{T}(V)$. The study of irreducible representations of $E(5,10)$ was carried out by Cantarini, Caselli, Kac and Rudakov in [KR4, R1, CC], which produced, at the time we began our research, the following conjecture.

Conjecture 0.0.1. A minimal Verma module $\mathfrak{T}(\lambda)$ contains non constant singular vectors (i.e. of degree > 0) if and only if $\lambda = [m, n, 0, 0]$, $[0, 0, m, n]$ or $[m, 0, 0, n]$ for $m, n \in \mathbb{Z}_{\geq 0}$ and singular vectors have degree ≤ 5.

The first aim of the present thesis is to apply pseudoalgebraic constructions to determine a bound on the degree of singular vectors. Our strategy starts by constructing a pseudoalgebraic structure for $\mathfrak{L} = S_5 \oplus \mathrm{d}\Omega^1_5$, i.e. a Lie superpseudoalgebra $e(5, 10) = L_{(0)} \oplus L_{(1)}$ such that $\mathcal{A}(L_{(0)}) = S_5$ and $\mathcal{A}(L_{(1)}) = \mathrm{d}\Omega^1_5$ as linearly compact algebras and modules. The even part can be taken to be $S(\mathfrak{d}, \chi)$ (we assume for simplicity $\mathfrak{d}$ abelian and $\chi = 0$) since we know that $\mathcal{A}(S(\mathfrak{d})) \cong S_5$. Regarding the odd part, one needs a pseudo version of the formal De Rham complex, which is provided in [BDK1]: take a Lie algebra $\mathfrak{d}$ of dimension N and set $\Omega^k_H(\mathfrak{d}) = Hom(\bigwedge^k \mathfrak{d}, H) \cong H \otimes \bigwedge^k \mathfrak{d}^*$. One can construct a *pseudo differential* $\mathrm{d}_* : \Omega^k_H(\mathfrak{d}) \to \Omega^{k+1}_H(\mathfrak{d})$ which defines an exact complex of representations of $W(\mathfrak{d})$ (and in particular of $S(\mathfrak{d}, \chi)$)

$$0 \to \Omega^0_H(\mathfrak{d}) \xrightarrow{\mathrm{d}_*} \Omega^1_H(\mathfrak{d}) \xrightarrow{\mathrm{d}_*} \cdots \xrightarrow{\mathrm{d}_*} \Omega^N_H(\mathfrak{d})$$

whose cohomology is trivial but for the N one. Moreover, one can define a *pseudowedge* product $(1 \otimes \alpha) *_\wedge (1 \otimes \beta) = (1 \otimes 1) \otimes_H \alpha \wedge \beta$ that makes $\Omega_H(\mathfrak{d}) = \bigoplus_{n=0}^N \Omega^n_H(\mathfrak{d})$ an associative H-pseudoalgebra. The pseudo De Rham complex induces, via the annihilation functor, the formal one. Hence we set $e(5, 10) = S(\mathfrak{d}) \oplus \mathrm{d}_*(\Omega^1_H(5))$ and define the pseudobracket between odd elements as the pseudowedge and consequent identification of $\mathrm{d}_*\Omega^3_H(\mathfrak{d})$ with $S(\mathfrak{d})$, verifying that this is well defined and gives a morphism of $S(\mathfrak{d})$-modules. It turns out that for a first bound on the degree of singular vectors, the even part of $e(5, 10)$ is already enough. In fact, we are able to define a finite filtration of S_5-submodules of a minimal Verma module $\mathfrak{T}(V) = U(\mathfrak{d} \oplus \mathfrak{s}) = V$ by increasing odd degree (i.e. degree in $\mathfrak{s}$) and realize the successive quotients as tensor modules for $S(\mathfrak{d})$ of the form $T(\bigwedge^i \mathfrak{d}^* \otimes V) = U(\mathfrak{d}) \otimes (\bigwedge^i \mathfrak{d}^* \otimes V)$. Looking at the homogeneous components of singular vectors through this filtration tells us that the even degree must be ≤ 4 (i.e. the degree in $\mathfrak{d}$, which counts as double), which is enough to give a first estimate since the odd degree is at most 10 by construction. We refine the bound by studying when fundamental irreducible representations of $\mathfrak{sl}(\mathfrak{d})$, whose corresponding tensor modules contain non constant singular vectors, cannot occur in the decomposition of $\bigwedge^i \mathfrak{d}^* \otimes V$. Our results are stated in Theorem 3.4.1.

Theorem 0.0.1 ([Br]). *Let $\mathfrak{T}(V)$ be a minimal Verma module and let v be a singular vector. Then v has degree ≤ 12. More precisely:*

1. *if $V \ncong V([0, 0, 1, 0])$, singular vectors have degree strictly smaller than 12;*

2. *if $V \ncong V(\lambda)$ where $\lambda = [0, 0, 0, 1]$, $[0, 0, 1, 0]$, $[0, 1, 0, 0]$, $[1, 0, 0, 0]$, $[0, 1, 1, 0]$, or $[1, 0, 0, 1]$, singular vectors have degree strictly smaller than 11;*

3. *if $V \ncong V(\mu)$ where $\mu =$ $[0, 0, 0, 0]$, $[1, 0, 0, 0]$, $[0, 0, 0, 1]$, $[0, 0, 1, 0]$, $[0, 1, 0, 0]$, $[1, 1, 0, 0]$, $[0, 1, 1, 0]$, $[1, 0, 0, 1]$, $[0, 0, 1, 1]$, $[1, 0, 1, 0]$, $[0, 1, 0, 1]$, $[1, 1, 0, 1]$, $[0, 2, 0, 0]$, $[2, 0, 0, 0]$, $[1, 0, 2, 0]$, or $[3, 0, 0, 1]$, singular vectors have degree strictly smaller than 10.*

Shortly after we obtained this result, Cantarini, Caselli and Kac classified all singular vectors (and hence degenerate Verma modules) for $E(5,10)$ in [CCK1]. However, our estimate is almost optimal: in fact, they found unexpected singular vectors in degrees 7 and 11, the latter in one of the cases our argument could not rule out.
In order to apply the pseudoalgebraic language to the study of representations of a linearly compact Lie (super)algebra $\mathcal{L}$ it is essential to exhibit at least one Lie (super)pseudoalgebra inducing $\mathcal{L}$ as annihilation algebra. However, there will typically be more than one such structure, and the classification of all possible Lie (super)pseudoalgebra structures is an interesting problem in itself. In Chapter 4 we approach this problem in the special case of $E(5,10)$ and develop tools that are likely to serve for a general classification.
When we approached the task of classifying $U(\mathfrak{d})$-pseudoalgebraic structures inducing a certain linearly compact Lie (super)algebra, one of the most annoying technical steps is that of describing all possible transitive actions of $\mathfrak{d}$ on $\mathcal{L}$. This is unavoidable in the non super case, and was carried out in [BDK1, Section 12]. However, the pseudoalgebraic language is more efficient in the description of representations than as an algebra-classifying tools, and we take advantage of this fact. If $\mathcal{L} = \mathcal{L}_{(0)} \oplus \mathcal{L}_{(1)}$ is the direct sum of $\mathcal{L}$ into its even and odd part, and $\mathcal{L}_{(0)}$ admits a pseudoalgebraic description, then we employ a Grothendieck ring computation trick to provide very strict constrains on the structure of $\mathcal{L}_{(1)}$ as a module for the extended annihilation algebra corresponding to $\mathcal{L}_{(0)}$. This severely limits the possibilities for the odd part of a Lie superpseudoalgebra inducing $\mathcal{L}$. In the case of $E(5,10)$, this suffices for a classification.

Theorem 0.0.2. *For any choice of a trace form $\chi \in \mathfrak{d}^*$, there exists a unique Lie super pseudoalgebra structure of the form $L = S(\mathfrak{d},\chi) \oplus L_{(1)}$ inducing $E(5,10)$ as annihilation superalgebra. The Lie superpseudobracket identifies $L_{(1)}$ with $\mathrm{d}_\Pi(\Omega^1_H(\mathfrak{d}))$ as a $S(\mathfrak{d},\chi)$-module for a suitable choice of Π only depending on χ and the Lie superpseudobracket $L_{(1)} \otimes L_{(1)} \to (H \otimes H)_H L_{(0)}$ coincides with the pseudo de Rham wedge operation as in* (3.11).

We are confident that this is but the first step towards a classification of all finite simple Lie super pseudoalgebras.

Chapter 1

Preliminaries about linearly compact Lie (super)algebras

1.1 Linearly compact Lie algebras

Linearly compact Lie algebras arise in the context of *transitive* differential geometry. Consider the following situation (for a much more detailed discussion, see [GS, SiSt, G1, G2]).
Let M be a finite-dimensional complex manifold and let L be a Lie algebra of vector fields on M acting transitively in a neighbourhood of a point $p \in M$. One can consider a filtration $\{L_k\}_{k\geq 0}$ in L, where L_k is the subalgebra of L consisting of vector fields vanishing at p at least to order $k+1$, which satisfies the relations

$$[L_i, L_j] \subset L_{i+j} \quad \forall i, j \geq 0. \tag{1.1}$$

Consider now the topology induced by taking this filtration as a fundamental system of neighbourhoods of 0. The previous relations imply that the Lie bracket is continuous with respect to this topology, so L is a topological Lie algebra .
One can take the *formal completion* $\overline{L}$ of L in this topology. Geometrically, $\overline{L}$ is a Lie algebra of vector fields on a *formal neighbourhood* of p or, equivalently, a Lie algebra of *formal* vector fields (i.e. vector fields with coefficients in a ring of formal power series).
$\overline{L}$ is a "linearly compact" Lie algebra.

Definition 1.1.1. A *linearly compact vector space* is a topological vector space that admits a fundamental system of neighbourhoods of the origin consisting of finite-codimensional linear subspaces with respect to which is complete.

The notion of a linearly compact space is "topologically dual" to the notion of a discrete space. For instance, consider $\mathbb{C}$ as a topological field with discrete topology. Take a $\mathbb{C}$-vector space V and consider the discrete topology on it. We can consider on its topological dual $V^* = Hom^{cont}(V, \mathbb{C}) = Hom(V, \mathbb{C})$ a topology defined by taking as a fundamental system of neighbourhoods of 0 all the subspaces of the form $U^\perp = \{\phi \in V^* \mid \phi(U) = 0\}$ where U is a finite-dimensional subspace of V. The topological vector space V^* is then linearly compact.
There are several equivalent conditions to the linear compactness of a topological vector space. For a proof of next proposition, see [G1].

Proposition 1.1.1. *Let $\mathcal{L}$ be a topological vector space over the topological field $\mathbb{C}$. The following statements are equivalent:*

1. *$\mathcal{L}$ is the topological dual of a discrete space;*
2. *The topological dual $\mathcal{L}^* = Hom^{cont}(\mathcal{L}, \mathbb{C})$ of $\mathcal{L}$ is a discrete space;*
3. *$\mathcal{L}$ is the topological product of finite-dimensional discrete spaces;*
4. *$\mathcal{L}$ is the projective limit of finite-dimensional discrete spaces;*
5. *$\mathcal{L}$ has a collection of finite-codimensional open subspaces whose intersection is 0, with respect to which is complete.*

Example 1.1.1. A finite-dimensional space endowed with the discrete topology is linearly compact. In fact, in the finite-dimensional setting being linearly compact is equivalent to being a discrete space.

Definition 1.1.2. A topological Lie (or associative) algebra is *linearly compact* if its underlying topological vector space is linearly compact.

Example 1.1.2. The basic example of a linearly compact associative algebra is given by the algebra $\mathcal{O}_N = \mathbb{C}[[t^1, \ldots, t^N]]$ of formal power series in the indeterminates $t^1, \ldots, t^N$, with the usual topology for which the ideals $(t^1, \ldots, t^N)^k$ form a fundamental system of neighbourhoods of 0. It can be seen as the topological dual of the polynomial algebra $\mathbb{C}[y_1, \ldots, y_N]$ endowed with the discrete topology, where $t^i(y_j) = \delta_i^j$.

The main example of a linearly compact Lie algebra is W_N, the Lie algebra of *formal vector fields* (see [F]). It consists of all elements of the form

$$D = \sum_{i=1}^{N} f_i(t^1, \ldots, t^N) \frac{\partial}{\partial t^i}, \tag{1.2}$$

where $f_i(t^1, \ldots, t^N)$ are formal power series in $\mathcal{O}_N$. The Lie bracket of W_N is the usual one, defined by

$$\Big[\sum_i f_i \frac{\partial}{\partial t^i}, \sum_j g_j \frac{\partial}{\partial t^j}\Big] = \sum_{i,j} \Big(f_j \frac{\partial g_i}{\partial t^j} - \frac{\partial f_i}{\partial t^j} g_j\Big) \frac{\partial}{\partial t^i}. \tag{1.3}$$

W_N clearly coincides with $Der\, \mathcal{O}_N$, the Lie algebra of continuous derivations of $\mathcal{O}_N$.
The filtration of $\mathcal{O}_N$

$$\{F_p\, \mathcal{O}_N = (t^1, \ldots, t^N)^{p+1}\}_{p \geq -1} \tag{1.4}$$

induces the *canonical filtration* of W_N, $\{F_p W_n\}_{p \geq -1}$, where

$$F_p W_N = \{D \in\ W_N \mid D(F_q\, \mathcal{O}_N) \subseteq F_{p+q}\, \mathcal{O}_N)\}. \tag{1.5}$$

Explicitly, $F_p W_N$ consists of vector fields $D = \sum_{i=1}^{N} f_i \frac{\partial}{\partial t^i}$ such that all $f_i \in F_p \mathcal{O}_N$.
It is easy to see that the canonical filtration satisfies

$$[F_p W_N, F_q W_N] \subset F_{p+q} W_N, \tag{1.6}$$

thus making the Lie bracket continuous.

Let $\Omega_N = \bigoplus_{k=0}^N \Omega_N^k$ be the algebra of differential forms over $\mathcal{O}_N$.
Explicitly, $\omega \in \Omega_N^k$ will be a linear combination of elements of the form

$$f\mathrm{d}t^{i_1} \wedge \mathrm{d}t^{i_2} \wedge \cdots \wedge \mathrm{d}t^{i_k} \quad \text{for some} \quad i_1, \ldots, i_k \in \{1, \ldots, N\},\ f \in \mathbb{C}[[t^1, \ldots, t^N]].$$

The action of W_N on $\mathcal{O}_N$ extends uniquely to an action on Ω_N commuting with the differential d.
Explicitly, W_N acts on Ω_N via *Lie derivative*:

$$D \cdot \omega = \mathrm{d}(\iota_D \omega) + \iota_D(\mathrm{d}\omega) \quad \text{for} \quad D \in W_N,\ \omega \in \Omega_N \tag{1.7}$$

where d is the De Rham differential and ι_D is the contraction of ω by D.
One can define subalgebras of W_N in terms of elements annihilating certain forms.
A *volume form* is any non-zero element of Ω_N^N, i.e. a differential N-form $v = f(t_1, \ldots, t_N)\mathrm{d}t^1 \wedge \cdots \wedge \mathrm{d}t^N$ such that $f(0) \neq 0$. A *symplectic form* is a closed 2-form $s = \sum_{i,j} s_{ij}(t_1, \ldots, t_N)\mathrm{d}t^i \wedge \mathrm{d}t^j$ such that $det(s_{ij}(0)) \neq 0$ and a *contact form* is a 1-form $c = \sum_i c_i(t_1 \ldots, t_N)\mathrm{d}t^i$ such that $c \wedge (\mathrm{d}c)^{(N-1)/2}$ is a volume form.
It is a well known fact that any such form can be transformed by an automorphism of $\mathcal{O}_N$ in, respectively, the *standard* volume form $v_0 = \mathrm{d}t^1 \wedge \cdots \wedge \mathrm{d}t^N$, the standard symplectic form $s_0 = \sum_i \mathrm{d}t^i \wedge \mathrm{d}t^{n+i}$ if $N = 2n$ (otherwise a symplectic form does not exist) and the standard contact form $c_0 = \mathrm{d}t^N + \sum_i t_i\, \mathrm{d}t^{n+i}$ if $N = 2n+1$ (otherwise a contact form does not exist).
One then defines the following (closed) subalgebras of W_N:

- $S_N(v) = \{D \in W_N \mid Dv = 0\}$, $N \geq 2$;
- $H_N(s) = \{D \in W_N \mid Ds = 0\}$, N even ≥ 2;
- $K_N(c) = \{D \in W_N \mid Dc = fc \text{ for some } f \in \mathcal{O}_N\}$, N odd ≥ 3.

As special instances, denote $S_N(v_0) = S_N$, $H_N(s_0) = H_N$, $K_N(c) = K_N$, so that we know that for each type we can only consider, up to an automorphism of $\mathcal{O}_N$, the "standard" subalgebras S_N, H_N, and K_N.
Let $\mathcal{L}$ be one of these subalgebras of W_N. The canonical filtration of W_N induces a canonical filtration on $\mathcal{L}$, which equips it with a linearly compact topology.
Namely, $F_p\mathcal{L} = F_pW_N \cap \mathcal{L}$. These filtrations are all *transitive*, meaning that $\dim \mathcal{L}/F_{-1}W_N \cap \mathcal{L} = N$. Moreover, they have the following transitivity property:

$$F_k\mathcal{L} = \{D \in F_{k-1}\mathcal{L} \mid [D, \mathcal{L}] \subset F_{k-1}\mathcal{L}\}. \tag{1.8}$$

These algebras are often referred to as *Lie algebras of Cartan type*. E. Cartan in 1909 proved the following celebrated result.

Theorem 1.1.1 ([C]). *Any infinite-dimensional simple linearly compact Lie algebra is isomorphic to either W_N or one of its subalgebras S_N, H_N or K_N.*

We will mainly focus on Lie algebras of type W and S.
Consider $E = \sum_{i=1}^N t^i \frac{\partial}{\partial t^i} \in W_N$, which is called the *Euler operator*. Its adjoint action

decomposes W_N into eigenspaces $W_{N;j}$ for $j \geq -1$, thus defining a $\mathbb{Z}$-gradation which is called the *canonical gradation* of W_N. In addition, $[W_{N;p}, W_{N;q}] \subset W_{N;p+q}$.
It is a well known fact that $W_{N;0} \cong \mathfrak{gl}_N$ and $W_{N;k}$ is isomorphic, as a $\mathfrak{gl}_N$-module, to $\mathbb{C}^N \otimes (S^{k+1}\mathbb{C}^N)^*$.
Notice that $Es_0 = 2s_0$, so the canonical gradation of W_N induces the *canonical $\mathbb{Z}$-gradation* $S_{N;k}$ of S_N, for which analogously $S_{N;0} \cong \mathfrak{sl}_N$ and the $\mathfrak{sl}_N$-module $S_{N;k}$ is isomorphic to the highest component of $\mathbb{C}^N \otimes (S^{k+1}\mathbb{C}^N)^*$.

Remark 1.1.1. If we define the *divergence* map $div : W_N \to \mathcal{O}_N$ as $div(\sum_i f_i(t)\frac{\partial}{\partial t^i}) = \sum_i \frac{\partial f_i}{\partial t^i}$, then it is not hard to see that for $D \in W_N$, the condition $Dv_0 = 0$ is equivalent to $div\, D = 0$, i.e. $S_N = \{D \in W_N \mid div(D) = 0\}$.

For a satisfactory summary of the techniques that lead to Cartan's classification see [K2, Introduction].

1.2 Irreducible representations of Lie algebras of Cartan type

The study of irreducible representations of simple infinite-dimensional linearly compact Lie algebras was carried out by Rudakov ([R1, R2]) and Kostrikin ([Ko]).
To be more precise, they studied *continuous* representations of Lie algebras of Cartan type on discrete spaces of countable dimension, where we consider the linearly compact topology on the Lie algebras.
This is a reasonable class of representations to be studied: it is natural to consider continuous representations of linearly compact Lie algebras in linearly compact vector spaces. However, it turns out to be technically more convenient to work with their (topological) duals representations, which are indeed continuous representations in vector spaces with the discrete topology.
In [R1] is defined the notion of *height* of a discrete representation. It is also introduced a class of modules, called *induced* modules, which, as the name suggests, are induced by an $\mathcal{L}_0/\mathcal{L}_n$-module, where $\mathcal{L}$ is the Lie algebra of Cartan type and $\{\mathcal{L}_k\}$ its canonical filtration. Every irreducible discrete module is a quotient of an induced module and its height n determines from which $\mathcal{L}_0/\mathcal{L}_n$ induce in order to obtain this representation.
Irreducibility of induced modules in height n is related to the existence of vectors killed by the term of the canonical filtration $\mathcal{L}_n$, called *singular vectors*. When $n > 1$, it turns out that there are no singular vectors and therefore the induced modules are irreducible and they exhaust the classification of all such modules.
When $n = 1$ the picture is richer. In this case, one finds non-zero singular vectors that are killed by $\mathcal{L}_1$.
We will resume the notation and the main results regarding these representations with particular focus on those for types W and S.

Let $\mathcal{L}$ be a simple infinite-dimensional linearly compact Lie algebra of Cartan type. We will use the following properties of the topological Lie algebra $\mathcal{L}$:

1. $\mathcal{L}$ has a linear topology with respect to which it is complete. In particular, the neighbourhoods of 0 are linear subspaces of finite codimension and their intersection is 0;

2. $\mathcal{L}$ admits a decreasing filtration $\{\mathcal{L}_k\}_{k\geq -2}$, where $\mathcal{L}_{-2} = \mathcal{L}$, such that $\mathcal{L}_k \neq 0\ \forall k$, $\mathcal{L}_k = \{D \in \mathcal{L}_{k-1} \,|\, [D, \mathcal{L}_{-1}] \subset \mathcal{L}_{k-1}\}$ for $k > 0$ and the subspaces $\mathcal{L}_k$ form a fundamental system of neighbourhoods of 0.

3. Taking into account the associated graded algebra

$$\overline{L} = gr\mathcal{L} = \bigoplus_{k\geq -2} \overline{L}_i = \bigoplus_{k\geq -2} \mathcal{L}_k/\mathcal{L}_{k+1}, \tag{1.9}$$

then $\overline{L}_{-2} = [\overline{L}_{-1}, \overline{L}_{-1}]$.

Remark 1.2.1. The filtration can be taken to be the one induced by the canonical $\mathbb{Z}$-grading. For example, if $\mathcal{L} = W_N$, we can take $\mathcal{L}_k = \bigoplus_{j\leq k} W_{N;k}$. In this case, $\mathcal{L}$ and $gr\mathcal{L}$ are clearly isomorphic.

As usual, a representation of $\mathcal{L}$ (or a *$\mathcal{L}$-module*) is a vector space V together with a linear map $\rho : \mathcal{L} \to End(V)$ such that

$$\rho([a, b])(v) = \rho(a)(\rho(b)(v)) - \rho(b)(\rho(a)(v)) \quad \forall a, b \in \mathcal{L},\ v \in V. \tag{1.10}$$

We will denote a representation by (V, ρ) or just V and the action of $\mathcal{L}$ as $av = \rho(a)(v)$. The representation is *continuous* if V is a topological vector space and ρ is a continuous map. A first obvious condition on the action of $\mathcal{L}$ to be continuous on a discrete vector space is the following.

Lemma 1.2.1. *Let (V, ρ) be a representation of $\mathcal{L}$ where V is a discrete topological space. Then ρ is continuous if and only if any $v \in V$ is killed by some $\mathcal{L}_p$, i.e. exists p such that*

$$\mathcal{L}_p v = 0. \tag{1.11}$$

In this section all representations will, unless otherwise specified, be continuous in a discrete space.

Definition 1.2.1. Let V be an $\mathcal{L}$-module or an $\mathcal{L}_0$-module. The *height of the module* V is the minimal n such that $\mathcal{L}_n v = 0$ for some $0 \neq v \in V$.
The set $\{v \in V \,|\, \mathcal{L}_n v = 0\}$ is denoted by V_0.
If V is an $\mathcal{L}_0$-module and $V = V_0$, then V is said to be *homogeneous*.

Notice that since $[\mathcal{L}_0, \mathcal{L}_n] \subset \mathcal{L}_n$, V_0 is an $\mathcal{L}_0$-submodule of V. It is in particular an homogeneous $\mathcal{L}_0$-module.
Thus it is possible to see V_0 as a representation of the finite-dimensional Lie algebra $\mathcal{L}_0/\mathcal{L}_n$.
Notice also that if V is an irreducible $\mathcal{L}_0$-module, then V is homogeneous.

Remark 1.2.2. Given an $\mathcal{L}_0/\mathcal{L}_n$-module M, we can think of M as an $\mathcal{L}_0$-module by considering the trivial action of $\mathcal{L}_n$ on it. The structure of $\mathcal{L}_0$-module is said to be an *extension* of the $\mathcal{L}_0/\mathcal{L}_n$-module one.
With this terminology, an homogeneous $\mathcal{L}_0$-module of height n is an extension of an $\mathcal{L}_0/\mathcal{L}_n$-module.

A central role for the classification of irreducible representations of $\mathcal{L}$ is played by "induced modules".
Take an (arbitrary) Lie algebra $\mathfrak{g}$, a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and a representation M of $\mathfrak{h}$. A representation of $\mathfrak{h}$ is equivalently a representation of $U(\mathfrak{h})$, where $U(\mathfrak{h})$ is the universal enveloping algebra of $\mathfrak{h}$.
One can then define a $\mathfrak{g}$-module, called the *induced module* and denoted by $Ind_{\mathfrak{h}}^{\mathfrak{g}}(M)$, as

$$Ind_{\mathfrak{h}}^{\mathfrak{g}}(M) = U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} M. \tag{1.12}$$

The action of $\mathfrak{g}$ on $Ind_{\mathfrak{h}}^{\mathfrak{g}}(M)$ is given by multiplication on the left factor.

Apply this construction in our setting: let V be an $\mathcal{L}$-module and consider the induced $\mathcal{L}$-module $Ind_{\mathcal{L}_0}^{\mathcal{L}}(V_0) = U(\mathcal{L}) \otimes_{U(\mathcal{L}_0)} V_0$. One can now define a mapping

$$\begin{aligned} \varphi : U(\mathcal{L}) \otimes_{U(\mathcal{L}_0)} V_0 &\longrightarrow V \\ u \otimes v &\longmapsto uv \quad u \in U(\mathcal{L}),\, v \in V_0, \end{aligned} \tag{1.13}$$

which is clearly a morphism of $\mathcal{L}$-modules (i.e. it commutes with the action of $\mathcal{L}$).
In particular, if V is irreducible, then this morphism is surjective and V is isomorphic to a quotient of the induced module.
The problem of classifying irreducible representations is thus reduced to the study of induced modules of homogeneous $\mathcal{L}$-modules of height n.
This study is carried out in [R1] by means of *irregular submodules* and *singular vectors*. We will outline the main ideas without going too much into details.

Let M be an homogeneous $\mathcal{L}_0$-module of height $n > 0$ and consider the $\mathcal{L}$-module $I = Ind_{\mathcal{L}_0}^{\mathcal{L}}(M)$.
Using the canonical filtration of $\mathcal{L}$ one can define an increasing filtration of $\mathcal{L}_0$-submodules $\{I_p\}_{p\geq 0}$ in I where $I_0 = M$ such that

$$\mathcal{L}_k I_p \subset I_{p-k+n-1} \quad \text{for} \quad k \geq n-1. \tag{1.14}$$

Let N be an $\mathcal{L}$-submodule of I. N is called an *irregular invariant submodule* if $N \cap I_0 = 0$. One can easily see that $Ker\varphi$ is an irregular submodule. Hence, we are interested in finding out when such submodules occur.

Take an element $l \in \overline{L}_k = \mathcal{L}_k/\mathcal{L}_{k+1}$. Because of (1.14), when $k \geq n-1$, l defines a map $\overline{I}_p \longrightarrow \overline{I}_{p-k+n-1}$, where $\overline{I}_j = I_j/I_{j-1}$. We set $I_{-1} = 0$.
Notice that when $n = 1$, this defines an action of $\overline{L} = gr\mathcal{L}$ on the graded module $\overline{I} = grI = \bigoplus_{p\geq 0} \overline{I}_p$.

Definition 1.2.2. An element $v \in \overline{I}_p$, $p > 0$, is called a *singular vector* if $lv = 0$ for all $l \in \overline{L}_k$ and for all $k \geq n$.

The most useful trait of singular vectors is that they "encode" the presence of an irregular submodule.

Proposition 1.2.1. *If N is an irregular invariant submodule of the $\mathcal{L}$-module I and if j_0 is the minimal index such that $N \cap I_{j_0} \neq 0$, then all the vectors in $(N \cap I_{j_0})/I_{j_0-1} \subset \overline{I}$ are singular.*

Proof. Just notice that for $l \in \mathcal{L}_n$, $l \cdot N \cap I_j \subset N \cap I_{j-n+n-1} = N \cap I_{j-1} = 0$. Therefore, for any $l \in \overline{L}_p$ for $p \geq n$, $l \cdot (N \cap I_{j_0})/I_{j_0-1} = 0$. □

This tells us that when there are no singular vectors in grI, then $I = I(M)$ has no irregular invariant submodules. In fact, this turns out to be the case when $n > 1$.

Theorem 1.2.1. *[R1, Theorem 6.1] Let $\mathcal{L}$ be a Lie algebra of Cartan type, V an homogeneous $\mathcal{L}_0$-module of height $n > 1$. If $I = U(\mathcal{L}) \otimes_{U(\mathcal{L}_0)} V$, then the module gr I contains no singular vectors.*

Corollary 1.2.1. *If $\mathcal{L}$ is an algebra of Cartan type and M is an irreducible $\mathcal{L}_0$-module of height $n > 1$, then $I(M)$ is an irreducible $\mathcal{L}$-module.*

Proof. If M is an $\mathcal{L}$-submodule of $I(M)$, then by Proposition 1.2.1 it cannot be irregular. Hence $M \cap I_0(M) \neq 0$. But $I_0(M) \cong M$ is irreducible, so $M \supseteq I_0$. It is clear now that by allowing $\mathcal{L}$ to act on M we obtain all $I(M)$. □

Corollary 1.2.2. *If $\mathcal{L}$ is an algebra of Cartan type and if V is an irreducible $\mathcal{L}$-module of height $n > 1$, then the $\mathcal{L}_0$-module V_0 is irreducible and*

$$V \cong U(\mathcal{L}) \otimes_{U(\mathcal{L}_0)} V_0. \tag{1.15}$$

Proof. Since V_0 is homogeneous and since the height of V and V_0 is the same, by Theorem 1.2.1 there are no non-zero singular vectors, hence no irregular submodules in $U(\mathcal{L}) \otimes_{U(\mathcal{L}_0)} V_0$. This implies that the kernel of φ defined in (1.13) is 0. Therefore $V \cong U(\mathcal{L}) \otimes_{U(\mathcal{L}_0)} V_0$. Now the irreducibility of V implies that of V_0. □

Irreducible representations of height 1 were classified for W_N ([R1]), S_N and H_N ([R2]) and K_N ([Ko]).
As we mentioned before, the general picture in this case is richer. One does have non-zero singular vectors and irregular submodules in induced modules.
First of all, if V is an homogeneous $\mathcal{L}$-module of height 1, then V_0 is the extension of a module for $\mathcal{L}_0/\mathcal{L}_1$, which is isomorphic for cases W_N, S_N, H_N and K_N respectively to $\mathfrak{gl}_N$, $\mathfrak{sl}_N$, $\mathfrak{sp}_N$ and $\mathfrak{csp}_{N-1}$. So, if one is interested in understanding when an induced module is irreducible, it makes sense to study extension of irreducible $\mathcal{L}_0/\mathcal{L}_1$-modules.
Consider the induced module $I(V_0)$, where V_0 is an irreducible $\mathcal{L}_0/\mathcal{L}_1$-module. One observes that, because of (1.14), the graded associated module $\overline{I}(V_0) = gr\, I(V_0)$ has a structure of $\mathcal{L}_0$-module. One can actually verify that $I(V_0)$ and $\overline{I}(V_0)$ are isomorphic as $\mathcal{L}_0$-modules. Hence, $I(V_0)$ is a graded $\mathcal{L}_0$-module, with graded terms denoted by $I^n(V_0)$.

For W_N, it is proven that there are no non-zero singular vectors when $V_0 \ncong \bigwedge^i(\mathbb{C}^N)$ for $i = 1, \ldots, N$ where $\mathbb{C}^N$ is the standard representation of $\mathfrak{gl}_N$.
Thus in this case, following the same line of reasoning of the Corollaries of Theorem 1.2.1, one concludes that $I(V_0)$ is an irreducible $\mathcal{L}$-module.
In the remaining cases, namely when $V_0 \cong \bigwedge^i(\mathbb{C}^N)$, singular vectors only appear in degree 1 (i.e. in $I^1(V_0)$) and they altogether generate a unique irregular submodule of $I(V_0)$.
One observes that the induced module $I(\bigwedge^i(\mathbb{C}^N))$ is isomorphic to Ω^i_N, where the latter is the $\mathcal{L}$-module of differential forms of degree i. This allows to fit them in an exact sequence (contragradient of the De Rham complex) thus realizing the irreducible quotients

as $(\mathrm{d}\Omega^i)^*$.
In other words, the existence of De Rham complex provides non-trivial extensions of $\mathcal{L}_0/\mathcal{L}_1 \cong \mathfrak{gl}_N$-modules, and there are no others.
For S_N the picture is similar: induced modules $I(V_0)$ have no non-zero singular vectors when the irreducible $\mathfrak{sl}_N$-module V_0 is not isomorphic to $\bigwedge^i(\mathbb{C}^N)$ for any $i = 0, \dots, N$. In the other cases, singular vectors appear only in degree 1 for $i \neq 1$, and in degree 1 and 2 for $i = 1$ and again the correspondent modules fit in the De Rham complex.
For K_N and H_N one gets a similar picture: apart from a finite number of cases, the induced modules have no non-zero singular vectors and so are irreducible. In the exceptional cases, they have singular vectors only in degrees 1 and 2.

1.3 Linearly compact Lie superalgebras

We turn now to the "super" setting. First, a couple of standard definitions.

Definition 1.3.1. A *super vector space* is a vector space V with a $\mathbb{Z}/(2)$-grading, i.e. a decomposition as a direct sum of subspaces $V = V_{(0)} \oplus V_{(1)}$ called respectively the *even* and the *odd* part of V. Elements of $V_{(0)}$ and $V_{(1)}$ are said to be *homogeneous*.

A *superalgebra* is a super vector space endowed with a structure of an algebra that behaves coherently with the $\mathbb{Z}/(2)$-grading, i.e. such that $V_\alpha V_\beta \subset V_{\alpha+\beta}$ for $\alpha, \beta \in \mathbb{Z}/(2)$.

Example 1.3.1. Given a finite-dimensional vector space U, its *Grassman algebra* $\bigwedge(U)$ is a superalgebra where the even part is $\bigoplus_{i\,even} \bigwedge^i(V)$ and the odd part is $\bigoplus_{i\,odd} \bigwedge^i(U)$.

Definition 1.3.2. A *Lie superalgebra* $L = L_{(0)} \oplus L_{(1)}$ is a super vector space together with an bilinear map $[\cdot\,,\cdot] : L \otimes L \to L$, called *Lie superbracket*, such that $[L_\alpha, L_\beta] \subset L_{\alpha+\beta}$ for $\alpha, \beta \in \mathbb{Z}/(2)$ and that satisfies for all homogeneous $a, b, c \in L$:

Super skew-symmetry $[a, b] = -(-1)^{|a||b|}[b, a]$,

Super Jacobi identity $[a, [b, c]] = [[a, b]c] + (-1)^{|a||b|}[b, [a, c]]$,

where if $a \in V_\alpha$, $|a| = \alpha$ for $\alpha \in \mathbb{Z}/(2) = \{(0),\ (1)\}$.

Now, we are interested in (infinite-dimensional) linearly compact Lie superalgebras.
A linearly compact (Lie) superalgebra is a topological (Lie) superalgebra whose underlying topological super vector space is linearly compact.

Example 1.3.2. The $\mathbb{Z}/(2)$-grading of $\bigwedge(n) = \bigwedge(\mathbb{C}^n)$ induces a $\mathbb{Z}/(2)$-grading in $\bigwedge(m, n) = \mathcal{O}_m \otimes \bigwedge(n)$, thus a structure of superalgebra on the latter. Meanwhile, the canonical filtration of $\mathcal{O}_m$ induces a linearly compact topology on $\bigwedge(m, n)$.

Simple infinite-dimensional linearly compact Lie superalgebras were only recently classified by V. Kac in [K2]. The classification theorem is the following.

Theorem 1.3.1. *[K2] Any simple linearly compact infinite-dimensional Lie superalgebra is isomorphic to one of the following or its derived subalgebra:*

1. *eight series* $W(m, n)$, $S(m, n)$, $H(m, n)$, $K(m, n)$, $HO(n, n)$, $SHO(n, n)$, $KO(n, n+1, \beta)$, $SKO(n, n+1)$*;*

2. *two series of filtered deformations* $SHO(n,n)^{\sim}$ *(n even),* $SKO(n,n+1)^{\sim}$ *(n odd);*

3. *five exceptional Lie superalgebras:* $E(1,6)$*,* $E(3,6)$*,* $E(3,8)$*,* $E(4,4)$*,* $E(5,10)$.

The first 4 families are "super " analogous of the Cartan classification: $W(m,n)$ is the superalgebra of derivations of $\bigwedge(m,n) = \mathcal{O}_m \otimes \bigwedge(n)$ and $S(m,n)$, $H(m,n)$ and $K(m,n)$ are the subalgebras consisting respectively of "super" divergence zero vector fields, of vector fields annihilating a "super" symplectic form and of vector fields that multiply a "super" contact form by a function; in addition, in the super setting also arise the subalgebras $HO(n,n)$, $SHO(n,n)$, $KO(n,n)$, $SKO(n,n+1)$ where $HO(n,n)$ and $KO(n,n+1)$ consist of respectively vector fields annihilating an odd super symplectic form and vector fields multiplying an odd super contact form by a function, while $SHO(n,n) = HO(n,n) \cap S(n,n)$, $SKO(n,n+1,\beta) = \{D \in KO(n,n+1) \mid div^{\beta} D = 0\}$ where div^{β} is a "deformed divergence", for $\beta \in \mathbb{C}$.
The five exceptional Lie superalgebras are of particular interest, mainly because of their potential application in particle physics (see for example [KR2, Section 8] for more details) but also because they were not expected, having no "non-super" counterpart, and finally because they have explicit descriptions in terms of formal vector fields and differential forms that allow to study them and their representation theory exploiting techniques from the non-super setting.
In [CK] is provided a geometric construction of exceptional Lie superalgebras that we concisely recall now. We omit the description of $E(1,6)$.
Before doing that, we briefly recall the *twisting* of W_N-modules, which is needed to describe some of these Lie algebras.
By definition W_N acts on $\mathcal{O}_N = \mathbb{C}[[x_1,\ldots,x_N]]$. Consider the semidirect sum $\tilde{W_N} = W_N + \mathcal{O}_N$.
For any $\lambda \in \mathbb{C}$, define the the Lie algebra homomorphism $\phi_\lambda : W_N \to \tilde{W_N}$ as

$$\phi_\lambda(D) = D + \lambda div(D) \quad \text{for} \quad D \in W_N. \tag{1.16}$$

Now given a W_N-module M, if the action of W_N extends to an action of $\tilde{W_N}$, one can define a λ-*twisted* action of W_N by pulling back ϕ_λ. The module with this new action is denoted by M^λ.
W_N acts on Ω_N^k via Lie derivatives and this action naturally extends to an action of $\tilde{W_N}$. Hence we can define W_N-modules $(\Omega_N^k)^\lambda$.
Notice that when restricting to an action of S_N, by definition of the latter the twist does not modify the action.
In general, there is a one to one correspondence between vector fields and $N-1$-forms given by the map

$$\begin{aligned} \psi : W_N &\longrightarrow \Omega_N^{N-1} \\ D &\longmapsto \iota_D(v) \end{aligned} \tag{1.17}$$

where v is a fixed volume form. In general, this is not an homomorphism of W_N-modules. Anyway, a straightforward calculation shows that

Proposition 1.3.1. *ψ induces an isomorphism of W_N-modules $(\Omega_N^{N-1})^{-1} \cong W_N$ and an isomorphism of S_N-modules $\mathrm{d}\Omega_N^{N-1} \cong S_N$.*

We begin with $E(4,4)$. The even part $E(4,4)_{(0)} = W_4$ consists of all formal vector fields in 4 indeterminates, while the odd part $E(4,4)_{(1)} = (\Omega_4^1)^{-\frac{1}{2}}$.
The bracket between even elements is the Lie bracket of W_4 given by (1.3), while the bracket involving even and odd elements is defined by the Lie derivative as in (1.7) twisted by $\lambda = -\frac{1}{2}$ as described above.
The superbracket on the odd part is defined as

$$\omega_1 \otimes \omega_2 \longmapsto \mathrm{d}\omega_1 \wedge \omega_2 + \omega_1 \wedge \mathrm{d}\omega_2 \in (\Omega_4^3)^{-1} \quad \text{for} \quad \omega_1, \omega_2 \in (\Omega_4^1)^{-\frac{1}{2}} \tag{1.18}$$

where we apply Proposition 1.3.1 and identify a 3-form γ with a vector field D such that $\iota_D(v) = \gamma$ for a fixed volume form v.

Next consider $E(3,6)$.
The even part is the direct sum $E(3,6)_{(0)} = W_3 \oplus (\Omega_3^0 \otimes \mathfrak{sl}_2)$, while the odd part as an $E(3,6)_{(0)}$-module can be identified with $(\Omega_3^1)^{-\frac{1}{2}} \otimes \mathbb{C}^2$, where $\mathbb{C}^2$ is the standard representation of $\mathfrak{sl}_2$.
The brackets involving the even part are clear. The bracket between odd elements is described, for $\omega_1 \otimes v_1, \omega_2 \otimes v_2 \in (\Omega_3^1)^{-\frac{1}{2}} \otimes \mathbb{C}^2$, as follows:

$$[\omega_1 \otimes v_1, \omega_2 \otimes v_2] = (\omega_1 \wedge \omega_2) \otimes (v_1 \wedge v_2) + (\mathrm{d}\omega_1 \wedge \omega_2 + \omega_1 \wedge \mathrm{d}\omega_2) \otimes (v_1 \bullet v_2) \tag{1.19}$$

where $v_1 \wedge v_2 \in \bigwedge^2 \mathbb{C}^2$ is a complex number and $v_1 \bullet v_2$ is an element of $S^2(\mathbb{C}^2)$, the latter being identified with $\mathfrak{sl}_2$. We also identify Ω_3^0 with $(\Omega_3^3)^{-1}$ and $(\Omega_3^2)^{-1}$ with W_3.

Regarding $E(3,8)$, the even part is the semidirect sum $E(3,8)_{(0)} = W_3 + (\Omega_3^0 \otimes \mathfrak{sl}_2)$, while $E(3,8)_{(1)}$ is isomorphic to $((\Omega_3^0)^{-\frac{1}{2}} \otimes \mathbb{C}^2) + ((\Omega_3^2)^{-\frac{1}{2}} \otimes \mathbb{C}^2)$ as an $E(3,8)_{(0)}$-module.
The bracket between odd elements is defined, for $\omega_1, \omega_2 \in (\Omega_3^2)^{-\frac{1}{2}}$, $\alpha_1, \alpha_2 \in (\Omega_3^0)^{-\frac{1}{2}}$ and $v_1, v_2 \in \mathbb{C}^2$, as

$$\begin{aligned}&[\omega_1 \otimes v_1, \omega_2 \otimes v_2] = 0;\ [\alpha_1 \otimes v_1, \alpha_2 \otimes v_2] = \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \otimes v_1 \wedge v_2;\\ &[\alpha_1 \otimes v_1, \omega_1 \otimes v_2] = (\alpha_1\omega_1 \otimes v_1 \wedge v_2) + ((\alpha_1 \mathrm{d}\omega_1 - \omega_1 \mathrm{d}\alpha_1) \otimes (v_1 \bullet v_2));\end{aligned} \tag{1.20}$$

where we use the same identifications as for $E(3,6)$.

Finally, we focus on $\mathfrak{L} = E(5,10)$, for which we will give a more exhaustive description than for the previous ones.
Let $\mathfrak{d} = (\mathbb{C}^5)^*$ and let $\{\partial_1, \ldots, \partial_5\}$ and $\{x_1, \ldots, x_5\}$ be bases for respectively $\mathfrak{d}$ and $\mathfrak{d}^*$.
We can realize the even part of $\mathfrak{L}$ as zero-divergence vector fields in the indeterminates $x_1, \ldots, x_5$,

$$\mathfrak{L}_{(0)} = S_5 = \Big\{D = \sum_{i=1}^{5} f_i \partial_i \,|\, f_i \in C[[x_1, \ldots, x_5]],\ div(D) = 0\Big\},$$

and the odd part as closed 2-forms in the same indeterminates

$$\mathfrak{L}_{(1)} = \mathrm{d}\Omega_5^1 = \Big\{\omega = \sum_{i<j=1}^{5} f_{ij}\xi_{ij} \,|\, f_{ij} \in \mathbb{C}[[x_1, \ldots, x_5]],\ \xi_{ij} = \mathrm{d}x_i \wedge \mathrm{d}x_j,\\ \omega = \mathrm{d}\alpha \text{ for some } \alpha \in \Omega_5^1\Big\}.$$

The bracket between even elements is the one of the Lie algebra S_5, while the brackets between even and odds elements are given by the Lie derivative

$$[D,\omega] = \mathcal{L}_D(\omega) = \mathrm{d}(\iota_D(\omega))\,, \tag{1.21}$$

where $\iota_D(\omega)$ is the contraction of ω by D.
Finally, the bracket between odd elements is given by wedge product and, by Proposition 1.3.1, consequent identification with a zero-divergence vector field:

$$[\omega_1,\,\omega_2] = D \quad \text{where D is such that} \quad \iota_D(v) = \omega_1\wedge\omega_2, \tag{1.22}$$

v being a fixed volume form. Notice that this is $\mathbb{C}[[x_1,\dots,x_5]]$-bilinear.
In order to give an explicit formula, for $i,j,h,k\in\{1,\dots,5\}$ we set

$$(ijhk) = \begin{cases} 0 & \text{if } |\{i,j,h,k\}|<4;\\ l & \text{otherwise, where } \{i,j,h,k,l\}=\{1,\dots,5\}. \end{cases} \tag{1.23}$$

We shall adopt the following convention: whenever such an index occurs in an expression and it takes on the value 0, so does the expression.
This way the bracket between two odd elements can be defined as

$$[\xi_{ij},\xi_{hk}] = \varepsilon_{(ijhk)}\partial_{(ijhk)} \tag{1.24}$$

and then extended by $\mathbb{C}[[x_1,\dots,x_5]]$-bilinearity, where $\varepsilon_{(ijhk)}$ is the sign of the permutation $(ijhkl)$ if $(ijhk)=l\neq 0$, 0 otherwise; in a similar way $\partial_{(ijhk)}=\partial_l$ if $(ijhk)=l\neq 0$, $\partial_{(ijhk)}=0$ otherwise.

Setting $\deg x_i = -\deg\partial_i = 2$ and $\deg \xi_{hk} = -1$ provides $\mathfrak{L} = \bigoplus_{i\geq -2}\mathfrak{L}_i$ with a transitive, irreducible $\mathbb{Z}$-grading of depth 2 consistent with the superalgebra structure, which means that

$$\mathfrak{L}_{(0)} = \bigoplus_{i\,even}\mathfrak{L}_i, \quad \mathfrak{L}_{(1)} = \bigoplus_{i\,odd}\mathfrak{L}_i, \quad [\mathfrak{L}_n,\mathfrak{L}_m]\subseteq\mathfrak{L}_{n+m}. \tag{1.25}$$

We will call $\mathfrak{L}_- = \mathfrak{L}_{-2}\oplus\mathfrak{L}_{-1}$, $\mathfrak{L}_+ = \bigoplus_{i>0}\mathfrak{L}_i$ and $\mathfrak{L}_{\geq 0} = \mathfrak{L}_0\oplus\mathfrak{L}_+$.
We have an isomorphism between $\mathfrak{L}_0$ and $\mathfrak{sl}(\mathfrak{d})$ given by

$$x_i\partial_j \longmapsto -e^i_j. \tag{1.26}$$

Given the fact that $[\mathfrak{L}_0,\mathfrak{L}_n]\subseteq\mathfrak{L}_n$, we can view $\mathfrak{L}_n$ as an $\mathfrak{sl}(\mathfrak{d})$-module; in particular, $\mathfrak{L}_{-2}\cong\mathfrak{d}$ and $\mathfrak{L}_{-1}\cong\bigwedge^2\mathfrak{d}^* =: \mathfrak{s}$.
It is useful to describe also $\mathfrak{L}_1$ as an $\mathfrak{sl}(\mathfrak{d})$-module: it is the highest weight representation in $\mathfrak{d}^*\otimes\bigwedge^2\mathfrak{d}^*$, (see [CK, Section 4.3]) and it is generated by the highest weight vector $x_1\xi_{12}$.
Notice that $\mathfrak{L}_j = \mathfrak{L}_1^j$ for $j\geq 1$ and that $\mathfrak{L}_-\cong\mathfrak{d}\oplus\mathfrak{s}$ is a finite-dimensional Lie superalgebra whose superbracket is non trivial only when restricted to the odd part, where is given by (1.22).
This grading extends to the universal enveloping algebra $U(\mathfrak{L})$, and in particular to $U(\mathfrak{L}_-)$. In the latter case, as common practice, the sign of the degree is inverted in order to have a grading over $\mathbb{N}$.

We will talk in detail in Chapter 3 about the representation theory of linearly compact Lie superalgebras, in particular about $E(5,10)$. However, it is worth to mention now how the general approach for the non-super case still applies in a considerable amount of cases, which include $E(5,10)$.
One makes use of the following result, for which a proof can be found in [CK, Erratum, Lemma 1].

Lemma 1.3.1. *Let* $\mathfrak{g}$ *be a finite-dimensional Lie superalgebra and let* $\mathfrak{n}$ *be a solvable ideal of* $\mathfrak{g}$*. Let* $\mathfrak{a}$ *be an even subalgebra of* $\mathfrak{g}$ *such that* $\mathfrak{n}$ *is a completely reducible* $ad\,\mathfrak{a}$*-module with no trivial summand. Then* $\mathfrak{n}$ *acts trivially in any irreducible finite-dimensional* $\mathfrak{g}$*-module* V*.*

Now let $\mathcal{L}$ be a linearly compact Lie superalgebra with canonical filtration $\{\mathcal{L}_p\}$ and let V be an irreducible $\mathcal{L}$-module. One considers again continuous representations on discrete spaces which, in order to avoid pathological example, are assumed to be $\mathcal{K}$-locally finite for an open subalgebra $\mathcal{K} \subset \mathcal{L}$ (for which in most cases $\mathcal{L}_0$ is a good candidate), i.e. $U(\mathcal{K})v$ is finite-dimensional for any $v \in V$.
Let $\mathcal{L}_-$ be a (finite-dimensional) complement of $\mathcal{K}$ and assume it is a subalgebra of $\mathcal{L}$ (which is not always true in general). It is true, for example, for superalgebras with a consistent transitive $\mathbb{Z}$-grading of finite depth $d > 0$ $\bigoplus_{n\geq -d}\mathfrak{g}_n$, such as $E(5,10)$, by choosing $\mathcal{K} = \mathcal{L}_0 = \bigoplus_{n\geq 0}\mathfrak{g}_n$ and $\mathcal{L}_- = \bigoplus_{-d\leq n<0}\mathfrak{g}_n$.
The notion of "height" of an $\mathcal{L}$-module still makes sense, so assume the height of V is $\overline{k}$ and let $V_0 \subset V$ be the subspace of elements of V killed by $\mathcal{L}_{\overline{k}}$.
V_0 is an $\mathcal{L}_0$-submodule, hence by hypothesis it is finite-dimensional and one can apply the previous lemma for $\mathfrak{g} = \mathcal{L}/\mathcal{L}_p$, $\mathfrak{n} = \mathcal{L}_1/\mathcal{L}_p$ and $\mathfrak{a} = \mathfrak{g}_0 \subset \mathcal{L}_0/\mathcal{L}_p$, which implies that $\mathcal{L}_1$ acts trivially on V_0. In other words, $\overline{k} = 1$.
Now consider the induced module, which is most commonly known in this framework as a *generalized Verma module*, $U(\mathcal{L}) \otimes_{U(\mathcal{L}_0)} V_0 \equiv U(\mathcal{L}_-) \otimes V_0$, where the identification follows from the fact that, since the PBW theorem still holds in the superalgebra setting, fixing an ordered basis of $\mathcal{L}_-$ and $\mathcal{L}_0$, one has that $U(\mathcal{L}) = U(\mathcal{L}_-) \otimes U(\mathcal{L}_0)$.
This, by means of the natural projection $U(\mathcal{L}) \otimes_{U(\mathcal{L}_0)} V_0 \to V$, $u \otimes v_0 \mapsto uv_0$, reduces the problem of classifying irreducible representations of $\mathcal{L}$ to the study of the generalized Verma modules induced by representations of the (finite-dimensional) Lie algebra $\mathcal{L}_0/\mathcal{L}_1$.

1.4 Conformal algebras

Conformal (super)algebras were introduced by Kac ([K1]) in the context of chiral fields in conformal field theory.
They encode the "singular part" of the Operator Product Expansions (OPEs) of a family of (mutually) local fields, whereas the "regular" part is described by the normally ordered product. The whole construction's algebraic properties are described by a *vertex algebra*.

The basic (fairly simplified) idea is the following : given a vector space V, a *field* is a formal distribution in one variable $a(z) = \sum_{n\in\mathbb{Z}} a_n z^n \in (End\,V)[[z,z^{-1}]]$ such that $a(z)v$ is a Laurent series in z for all $v \in V$ (i.e. it has only finitely many negative powers of z). One is interested (because of physics) in fields satisfying the *locality principle*:

$$(z-w)^N[a(z),b(w)] = 0 \quad \text{for some} \quad N >> 0; \tag{1.27}$$

$a(z)$ and $b(z)$ are said to be *mutually local*. This locality axiom can be restated in terms of the OPE ([K1]): two fields $a(z), b(z)$ are mutually local if and only if there exist formal distributions $a_{(n)}b$ for $0 \leq n \leq N-1$ such that

$$[a(z), b(w)] = \sum_{n=0}^{N-1} \frac{1}{n!}(a_{(n)}b)(w)\partial_w^n \delta(z-w) \tag{1.28}$$

where ∂_w^n is the nth partial derivate with respect to w and $\delta(z-w)$ denotes the *delta distribution* $\partial(z,w) = \sum_{n\in\mathbb{Z}} z^n w^{-n-1}$.
The formal distributions $a_{(n)}b$ are $\mathbb{C}$-linear in a and b and are called *nth-products*.
The other operation that defines a vertex algebra is the *normally ordered product*: if one decomposes a field $a(z)$ into its *creation* part $a_+(z) = \sum_{n<0} a_n z^{-n-1}$ and *annihilation* part $a_-(z) = \sum_{n\geq 0} a_n z^{-n-1}$, then for two fields $a(z)$, and $b(z)$ one can define

$$: ab : (z) = a_+(z)b(z) + b(z)a_-(z) \tag{1.29}$$

which is well defined because of $a(z), b(z)$ being fields.
Roughly speaking, a *vertex algebra* can be thought of as a family of mutually local fields closed under derivation, normally ordered product and OPE (the latter meaning that all nth-products still belong to the family). When dealing with precise definitions, one can notice that the property of being a field is only necessary in the definition of the normally ordered product.
If one drops the latter, obtains a family of mutually local formal distributions closed under derivation and nth-products. This is called a *conformal* family and its algebraic structure is axiomatized by the following.

Definition 1.4.1. A *(Lie) conformal algebra* L is a $\mathbb{C}[\partial]$-module, where $\mathbb{C}[\partial]$ is the Hopf algebra of polynomials in the variable ∂, endowed with a $\mathbb{C}$-linear map $[\cdot_\lambda \cdot] : L \otimes L \to L[\lambda]$ called λ*-bracket*, where λ is an indeterminate.
The λ-bracket satisfies the following axioms, reminiscent of those of a Lie bracket:

(i) $[\partial a_\lambda b] = -\lambda[a_\lambda b]$; $\quad [a_\lambda \partial b] = (\partial + \lambda)[a_\lambda b]$; (Sesquilinearity)

(ii) $[b_\lambda a] = -[a_{-\lambda-\partial} b]$; (Skew-commutativity)

(iii) $[[a_\lambda [b_\eta c]] = [[a_\lambda b]_{\lambda+\eta} c] + [b_\eta [a_\lambda c]]$ (Jacobi identity)

for $a, b, c \in L$.
A conformal algebra is said to be *finite* if it is a finitely generated $\mathbb{C}[\partial]$-module.

Here one thinks of ∂ as the derivation ∂_z and the λ-bracket as the Fourier transform of (1.28).
In fact, if $[a(z), b(w)] = \sum_n \frac{1}{n!}(a_{(n)}b)(w)\partial_w^n \delta(z-w)$, then

$$[a_\lambda b] = \sum_n \frac{\lambda^n}{n!} a_{(n)} b \tag{1.30}$$

satisfies $(i) - (iii)$ for $\partial = \partial_z$.

The main examples of finite $\mathbb{C}[\partial]$-conformal algebras are the following.

Example 1.4.1 (Current conformal algebras)**.** Take a Lie algebra $\mathfrak{g}$ (over $\mathbb{C}$) and define $Cur(\mathfrak{g}) = \mathbb{C}[\partial] \otimes \mathfrak{g}$, the *current conformal algebra* associated to $\mathfrak{g}$, with λ-bracket

$$[1 \otimes a_\lambda 1 \otimes b] = 1 \otimes [a, b] \tag{1.31}$$

for $a, b \in \mathfrak{g}$ and then extended by $\mathbb{C}[\partial]$-sesquilinearity (i.e. by means of (i)).
In the language of formal distributions, it corresponds to the conformal structure of $End(\mathbb{C}[t, t^{-1}] \otimes \mathfrak{g})[[z, z^{-1}]]$, where $\mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}$ is a Lie algebra with bracket

$$[t^n \otimes a, t^m \otimes b] = t^{n+m} \otimes [a, b]. \tag{1.32}$$

Formal distributions are mutually local, since they satisfy the commutation relations

$$[g(z), h(w)] = [g, h](w)\delta(z - w) \tag{1.33}$$

where $g(z) = \sum_{n \in \mathbb{Z}} (gt^n) z^{-n-1}$.

Example 1.4.2. (Virasoro conformal algebra) Take a free $\mathbb{C}[\partial]$-module of rank 1 generated by an element l. The λ-bracket

$$[l_\lambda l] = (\partial + 2\lambda) l \tag{1.34}$$

defines a conformal algebra structure on $Vir = \mathbb{C}[\partial] l$, which is called the *Virasoro conformal algebra*.
It is the conformal structure associated to $W^{alg}(\mathbb{C}^\times)[[z, z^{-1}]]$, the conformal family of formal distributions with values in the centerless *Virasoro algebra* $W^{alg}(\mathbb{C}^\times)$, the Lie algebra of algebraic vector fields on $\mathbb{C}^\times$ (i.e. vector fields in one indeterminate with Laurent polynomial coefficients).
$W^{alg}(\mathbb{C}^\times)$ is generated by the vector fields $t^n \partial_t$ and the formal distribution $L(z) = \sum_{n \in \mathbb{Z}} (t^n \partial_t) z^{-n-1}$ satisfies the commutation relations

$$[L(z), L(w)] = \partial_w L(w) \partial(w - z) + 2L(w) \partial_w \delta(z - w). \tag{1.35}$$

The classification of simple finite conformal algebras was obtained by D'Andrea and Kac in [DK] and it actually consists of instances of the above examples.

Theorem 1.4.1 ([DK])**.** *A finite simple conformal algebra is isomorphic either to Vir or to the current conformal algebra $Cur(\mathfrak{g})$ of a simple finite-dimensional Lie algebra $\mathfrak{g}$.*

This classification uses in an essential way Cartan's classification of simple linearly compact Lie algebras (Theorem 1.1.1) via the *annihilation algebra*.
In fact, one of the motivations for the classification of infinite-dimensional linearly compact Lie superalgebra of Theorem 1.3.1 was to provide a similar tool for the classification of finite conformal superalgebras, which was obtained by Fattori and Kac in [FK].
From a physical point of view, the classification of [DK] corresponds to chiral algebras (i.e. vertex algebras) generated by a finite number of "bosonic" fields, whereas the "super" counterpart contemplates also "fermionic" fields.
One can "reverse" the construction that gives a conformal algebra from a conformal family of formal distribution: given a conformal algebra L, define the *Lie algebra of Fourier coefficients of* L, denoted by $\mathcal{A}(L)$, as the $\mathbb{C}$-linear span of symbols a_i for $a \in L$, $i \in \mathbb{Z}$

quotient the relations $(\lambda a + \eta b)_n = \lambda a_n + \eta b_n$ and $(\partial a)_n = -n a_{n-1}$.
Then $\mathcal{A}(L)$ is a Lie algebra with bracket

$$[a_m, b_n] = \sum_k \binom{m}{k} (a_{(n)}b)_{m+n-k}. \tag{1.36}$$

This is all artfully built so that the formal distributions with values in $\mathcal{A}(L)$, written as

$$a(z) = \sum_n a_n z^{-n-1}, \tag{1.37}$$

satisfy the OPE encoded in the conformal algebra.
Now, the *annihilation algebra* of L is $\mathcal{A}_-(L)$, which is the subalgebra of $\mathcal{A}(L)$ generated by elements a_n with non-negative n (notation and terminology are reminiscent of the "annihilation part" of a field).
The finiteness of L allows to define a filtration on $\mathcal{A}_-(L)$ which defines a linearly compact topology on the completion.

Example 1.4.3. For a current conformal algebra $L = Cur\mathfrak{g}$, $\mathcal{A}_-(L)$ is linearly generated by elements a_n with $a \in \mathfrak{g}$ and $n \geq 0$ where the bracket is given by $[a_n, b_m] = [a, b]_{n+m}$. It is easy to see that this is isomorphic to $\mathfrak{g}[t]$ and its completion is $\mathfrak{g}[[t]]$, with linearly compact topology induced by the standard filtration $\{t^n\mathfrak{g}[[t]]\}_{t\geq 0}$.
For instance, when $\mathfrak{g}$ is the trivial one-dimensional algebra $\mathbb{C}$, we obtain the algebra of formal power series in one indeterminate $\mathbb{C}[[t]]$ and the standard filtration is the same of Example 1.1.2.

Example 1.4.4. For the Virasoro conformal algebra $L = Vir = \mathbb{C}[\partial]l$, a linear system of generators of $\mathcal{A}_-(L)$ is given by the elements l_j, $j \geq 0$. It is easy to see that under the identification $l_j \mapsto -t^j\partial_t$, $\mathcal{A}_-(L)$ is isomorphic to $W^{alg}(\mathbb{C})$, the Lie algebra of vector fields over $\mathbb{C}$ with polynomials coefficients.
The filtration is the standard one $\{t^{n+1}\mathbb{C}[t]\partial_t\}_{n\geq -1}$ (cf. (1.5)), thus the completion is the simple linearly compact Lie algebra of Cartan type W_1.

We will not report the development of the representation theory of conformal algebras and refer to [DK, CK2] for a detailed discussion. For our purposes, it will suffice to say that, given the appropriate definitions, it is immediate to see that the natural notion of a representation V of a conformal algebra L is the same as a *conformal* representation V of the annihilation algebra $\mathcal{A}_-(L)$, where conformal here means that for every element $v \in V$ there is a term of the filtration of $\mathcal{A}_-(L)$ which acts trivially on it.
This correspondence allows to study the representation theory of linearly compact Lie algebras using conformal techniques. However, by Theorem 1.4.1 and the previous examples, this machinery only covers one Lie algebra of Cartan type, namely W_1.
Nevertheless, the techniques developed proved interesting enough to introduce a "multivariable" generalization of the notion of a conformal Lie algebra, known as a *Lie pseudoalgebra*.

Chapter 2

Hopf algebras and pseudoalgebras

2.1 Preliminaries and notation on Hopf algebras

The appropriate generalization of the notion of a conformal algebra is that of an H-*pseudoalgebra* where the base field $\mathbb{C}$ is replaced by a cocommutative Hopf algebra H. These objects where first introduced by Beilinson and Dreinfield in [BD] where they were called Lie^* algebras. They are algebras in a certain *pseudotensor category*, whence their name. In order to be able to define them, we will need to talk briefly about (cocommutative) Hopf algebras first.
In this section we review the definition of a cocommutative Hopf algebra and fix the notation, following [S] (to which we refer for details on the subject).
A *bialgebra* H is an associative algebra (over $\mathbb{C}$) with unit together with an homomorphism of algebras $\Delta : H \to H \otimes H$ called *coproduct* and a linear map $\epsilon : H \to \mathbb{C}$ called *counit* that satisfy the following axioms, for $h \in H$:

$$\textit{(Coassociativity)} \quad (\Delta \otimes id_H)\Delta(h) = (id_H \otimes \Delta)(\Delta(h)) \in (H \otimes H \otimes H); \tag{2.1}$$

$$\textit{(Counit axiom)} \quad (\epsilon \otimes id_H)\Delta(h) = (id_H \otimes \epsilon)\Delta(h) = h \in H \cong \mathbb{C} \otimes H \cong H \otimes \mathbb{C}. \tag{2.2}$$

We will use the sumless "Sweedler notation", for instance given $h \in H$:

$$\begin{aligned} \Delta(h) &= h_{(1)} \otimes h_{(2)}; \\ (\Delta \otimes id)\Delta(h) &= h_{(1)} \otimes h_{(2)} \otimes h_{(3)} = (id \otimes \Delta)\Delta(h); \\ &\text{etc.} \end{aligned}$$

Notice that $h_{(1)} \otimes h_{(2)} \otimes h_{(3)}$ is well defined because of coassociativity.
The counit axiom and the fact that Δ is an homomorphism of algebras can be rewritten, for $h, f, g \in H$, as

$$\epsilon(h_{(1)})h_{(2)} = h_{(1)}\epsilon(h_{(2)}) = h \tag{2.3}$$

$$(fg)_{(1)} \otimes (fg)_{(2)} = f_{(1)}g_{(1)} \otimes f_{(2)}g_{(2)}. \tag{2.4}$$

An *Hopf algebra* is a bialgebra endowed with a map $S : H \to H$ called the *antipode* such that

$$h_{(-1)}h_{(2)} = \epsilon(h) = h_{(1)}h_{(-2)} \quad \text{where } S(h_{(i)}) = h_{(-i)}. \tag{2.5}$$

From (2.3) and (2.5) one can deduce the useful identity in $H \otimes H$:

$$h_{(-1)}h_{(2)} \otimes h_{(3)} = h_{(1)}h_{(-2)} \otimes h_{(3)} = 1 \otimes h. \tag{2.6}$$

An Hopf algebra H is *cocommutative* if it is commutative as a coalgebra, i.e. $h_{(1)} \otimes h_{(2)} = h_{(2)} \otimes h_{(1)}$ for all $h \in H$.

Example 2.1.1 (Group algebra)**.** Given a group G , its group algebra $\mathbb{C}[G]$ has a natural structure of Hopf algebra if one defines, for $g \in G$, $\Delta(g) = g \otimes g$, $\epsilon(g) = 1$ and $S(g) = g^{-1}$. By definition, this is clearly a cocommutative Hopf algebra.
In a generic Hopf algebra H, an element $h \in H$ with coproduct $\Delta(h) = h \otimes h$ is called *group-like*. The subset of group-like elements is denoted by $G(H)$ and it is in fact a group.

Example 2.1.2 (Universal enveloping algebra)**.** Take a Lie algebra $\mathfrak{d}$. Then its universal enveloping algebra $U(\mathfrak{d})$ has a natural structure of Hopf algebra if we define for $a \in \mathfrak{d}$, $\Delta(a) = a \otimes 1 + 1 \otimes a$, $\epsilon(\mathfrak{d}) = 0$ and $S(a) = -a$. Again, by definition this is a cocommutative Hopf algebra.
Elements of a generic Hopf algebra $h \in H$ such that $\Delta(h) = h \otimes 1 + 1 \otimes h$ are called *primitive* and the subspace of primitive elements is a Lie algebra with respect to the commutator, denoted by $P(H)$.

There is a very important structure theorem on cocommutative Hopf algebras due to Konstant (for a proof and the definition of the smash product see [S]).

Theorem 2.1.1. *Let H be a cocommutative Hopf algebra over $\mathbb{C}$ (can be replaced by any algebraically closed field of characteristic 0). Then H is isomorphic to the Hopf algebra obtained by the smash product of the universal Lie algebra $U(P(H))$ and the group algebra $\mathbb{C}[G(H)]$.*

We will not explain this in details (see [BDK1]), but we mention that this result allows in many circumstances in the study of pseudoalgebras to restrict to the case of $H = U(\mathfrak{d})$ being the universal enveloping algebra of a Lie algebra $\mathfrak{d}$. For the purposes of this thesis, we will also assume $\mathfrak{d}$ to be finite-dimensional (we will highlight in some remarks throughout the work where these assumptions are needed and why).

In this case, let $\partial_1, \dots, \partial_N$ be a basis of $\mathfrak{d}$ and take as a Poincaré-Birkhoff-Witt basis of H, $\{\partial^{(I)}\}_{I \in \mathbb{Z}^N_{\geq 0}}$ where

$$\partial^{(I)} = \frac{\partial_1^{i_1}}{i_1!} \cdots \frac{\partial_N^{i_N}}{i_N!}, \quad I = (i_1, \dots, i_N) \in \mathbb{Z}^N_{\geq 0}. \tag{2.7}$$

With this choice of basis it is easy to show that

$$\Delta(\partial^{(I)}) = \sum_{J+K=I} \partial^{(J)} \otimes \partial^{(K)}. \tag{2.8}$$

Moreover, this lets us define the (canonical) increasing filtration on $U(\mathfrak{d})$

$$F^p H = span_{\mathbb{C}}\{\partial^{(I)} \mid |I| \leq p\}. \tag{2.9}$$

This filtration has the following immediate properties

$$(F^pH)(F^qH) \subset F^{p+q}H; \tag{2.10}$$

$$\Delta(F^pH) \subset \sum_{i=0}^{p} F^iH \otimes F^{p-i}H; \tag{2.11}$$

$$S(F^pH) \subset F^pH; \tag{2.12}$$

$$\bigcup_p F^pH = H; \tag{2.13}$$

$$\dim F^pH < \infty. \tag{2.14}$$

Remark 2.1.1. It is possible to define a filtration for any Hopf algebra H with similar properties but the last two, the second to last being true for any cocommutative Hopf algebra and the last being true for $H = U(\mathfrak{d})$ with finite-dimensional $\mathfrak{d}$ or its smash product with the group algebra of a finite group.

Remark 2.1.2. When $\mathfrak{d}$ is an abelian Lie algebra, $U(\mathfrak{d})$ coincides with the symmetric algebra $S(\mathfrak{d}) = \mathbb{C}[\partial_1, \ldots, \partial_N]$. In this case, instead of a filtration, one can define a grading $G^pH = span_{\mathbb{C}}\{\partial^{(I)} \,|\, |I| = p\}$, which it is easy to see that satisfies $(G^pH)(G^qH) \subset G^{p+q}H$.
This grading induces the canonical filtration by writing $F^pH = \bigoplus_{i \leq p} G^iH$.

Now let $X = H^* := Hom(H, \mathbb{C})$ be the dual of H as a coalgebra.
X can be viewed as an H-bimodule with left and right actions given respectively by

$$\langle hx, f\rangle = \langle x, S(h)f\rangle, \tag{2.15}$$

$$\langle xh, f\rangle = \langle x, fS(h)\rangle \quad \text{for } x \in X,\ h, f \in H. \tag{2.16}$$

Therefore, by associativity of H,

$$f(xg) = (fx)g \quad \text{for } f, g \in H, x \in X. \tag{2.17}$$

The left action makes X an H-*differential* algebra, i.e. an associative algebra with a left action of H such that

$$h(xy) = (h_{(1)}x)(h_{(2)}y) \quad \text{for } h \in H,\ x, y \in X. \tag{2.18}$$

Similarly, for the right action $(xy)h = (xh_{(1)})(yh_{(2)})$.
Notice also that X is commutative when H is cocommutative.

The filtration of H induces a decreasing filtration F_pX on X, where (set $F_{-1} = X$)

$$F_pX = (F^pH)^{\perp} = \{x \in X \,|\, \langle x, F^pH\rangle = 0\} \quad \text{for } p \geq -1. \tag{2.19}$$

The properties of F^pH imply the following ones:

$$(F_pX)(F_qX) \subset F_{p+q}X; \tag{2.20}$$

$$(F^pH)(F_qX) \subseteq F_{q-p}X; \tag{2.21}$$

$$\bigcap_p F_pX = 0 \quad \text{(if (2.13) holds)} \quad . \tag{2.22}$$

We can consider on X the topology induced by taking as a fundamental system of neighbourhoods of the origin the filtration $\{F_pX\}$.
By (2.20) the product in X is continuous. In addition, if we consider the discrete topology on H, by (2.21), the action of H on X is also continuous. This can be rephrased by saying that X is a topological H-differential algebra.
Moreover, when (2.14) holds, $\dim\, X/F_nX < \infty$ for all n. Hence, if also (2.22) holds, by the discussion of Section 1.1, X is linearly compact.

Remark 2.1.3. When $H = U(\mathfrak{d})$, the fact that X is a topological H-differential algebra, together with (2.18) and the definition of the coproduct in H, shows that $\mathfrak{d}$ acts on X by continuous derivations.

When $H = U(\mathfrak{d})$, $\dim\, \mathfrak{d} = N$, let $\{x_I\}$ be a dual basis of $\{\partial^{(I)}\}$, i.e. $\langle x_I, \partial^{(J)}\rangle = \delta_I^J$; in particular, we denote with $\{x^i\}$ the duals elements of the basis $\{\partial_i\}$ for $\mathfrak{d} \subset H$, which provides a basis of $\mathfrak{d}^* \subset X$.
When $H = U(\mathfrak{d})$, it follows from (2.8) that

$$x_I x_J = x_{I+J} \tag{2.23}$$

This allows us to write explicitly the basis elements x_I of X :

$$x_I = (x^1)^{i_1} \cdots (x^N)^{i_N} \quad \text{for } I = (i_1, \dots, i_N) \in \mathbb{N}^N \tag{2.24}$$

where the basis elements $x^i = x_{\varepsilon_i}$ of $\mathfrak{d}^*$ correspond to the multi index $\varepsilon_i = (0, \dots, 1, \dots, 0)$ with only a 1 in the ith position.

Example 2.1.3. When $\mathfrak{d}$ is abelian and therefore $H = U(\mathfrak{d}) = S(\mathfrak{d}) \cong \mathbb{C}[\partial_1, \dots, \partial_N]$, it is a well known fact that $X \cong \mathbb{C}[[x^1, \dots, x^N]]$. By setting $\deg x^i = 1$, X decomposes as a direct sum of spaces G^pX generated by monomials of degree p such that $G^pX \cdot G^qX \subset G^{p+q}X$ and $F_pX/F_{p+1}X \cong G^pX$.
Notice that since X is linearly compact, $\bigoplus_p G^pX$ is a dense subspace of X and any elements in X can be written as a (possibly infinite) sum of elements in G^pX. In other words, X is isomorphic to the completion of $\bigoplus_p G^pX$ in respect to the topology induced by the filtration $\{\bigoplus_{k\geq p} G^pX\}_{k\geq -1}$.

The previous example generalizes for any $\mathfrak{d}$. In fact, one can define a ring isomorphism

$$\phi : X \to \mathcal{O}_N = \mathbb{C}[[t^1, \dots, t^N]] \tag{2.25}$$

by $\phi(x_I) \mapsto t^I$, where t^I is defined similarly to (2.24).
Recall that $\mathcal{O}_N$ by Example 1.1.2 is a linearly compact associative algebra, and so is X. In fact, the filtration (2.19) corresponds via ϕ to the canonical filtration of $\mathcal{O}_N$, thus ϕ is an isomorphism of linearly compact associative algebras.
This is an important feature: it allows us to define an action of H over $\mathcal{O}_N$, in particular one of $\mathfrak{d}$, using (2.15) via ϕ.

Example 2.1.4. When $\mathfrak{d}$ is an abelian Lie algebra, one can easily compute its action on X and therefore on $\mathcal{O}_N$: for $k \in \{1, \dots, N\}$, $I = (i_1, \dots, i_N)$, $J = (j_1, \dots, j_N) \in \mathbb{N}^N$,

$$\begin{aligned}&\langle \partial_k x_I, \partial^{(J)}\rangle = -\langle x_I, \partial_k \partial^{(J)}\rangle = -\langle x_I, (j_k + 1)\partial^{(J+\varepsilon_k)}\rangle = \\ &- (j_k + 1)\delta^I_{J+\varepsilon_k} = -(j_k + 1)\delta^{I-\varepsilon_k}_J = -(j_k + 1)\langle x_{I-\varepsilon_k}, \partial^{(J)}\rangle\end{aligned}$$

where we set $x_I = 0$ if I has a negative entry.
Under the isomorphism ϕ, this translates to ∂_k acting on $\mathcal{O}_N = \mathbb{C}[[t^1, \dots, t^N]]$ as $-\frac{\partial}{\partial t^k}$.
From a geometrical point of view, if one interprets $\mathbb{C}[[t^1, \dots, t^N]]$ as the algebra of functions on a formal neighbourhood of a point p in a complex manifold M of dimension N (like at the beginning of Section 1.1), then $\mathfrak{d}$ can be viewed as a Lie algebra of vector fields on the formal neighbourhood (namely, elements of W_N) and elements of H as differential operators.

One can extend the previous argument for any Lie algebra $\mathfrak{d}$ by the following lemma.

Lemma 2.1.1. *The following formulas hold for the action of $\mathfrak{d}$ on X:*

$$x^i\partial_j = -\delta^i_j - \sum_{k<i} c^i_{kj}x^k \quad mod\, F_1X, \tag{2.26}$$

$$\partial_j x^i = -\delta^i_j + \sum_{k>i} c^i_{kj}x^k \quad mod\, F_1X \tag{2.27}$$

where c^i_{jk} are the constants defining the Lie bracket in $\mathfrak{d}$, i.e. $[\partial_j, \partial_k] = \sum_i c^i_{jk}\partial_i$.

Proof. Since we are only interested in identities $mod\, F_1X$, we only need to check the result of $\langle x^i\partial_j, h\rangle$ for $h \in F^1H$.
If $h = 1$, then $\langle x^i\partial_j, h\rangle = -\langle x^i, \partial_j\rangle = -\delta^i_j$.
If $h = \partial_k$ then $\langle x^i\partial_j, h\rangle = -\langle x^i, \partial_k\partial_j\rangle$. This is equal to 0 when $k \le j$, since in this case $\partial_k\partial_j$ is (up to a constant) an element of the basis of H. When $k > j$ we obtain $-\langle x^i, [\partial_k, \partial_j]\rangle - \langle x^i, \partial_j\partial_k\rangle$; the second term is again 0, while the first is equal to $-c^i_{kj}$
Same argument applies to $\partial_j x^i$, with a switched sign. □

We end this section by describing a very useful feature of Hopf algebras, the *left (and right) straightening*.
Take $a, b \in H$. Using (2.6) , one obtains that

$$\begin{aligned}(a \otimes b) = (a \otimes 1)(1 \otimes b) = (a \otimes 1)(b_{(-1)}b_{(2)} \otimes b_{(3)}) = \\ (ab_{(-1)} \otimes 1)(b_{(2)} \otimes b_{(3)}) = (ab_{(-1)} \otimes 1)\Delta(b_{(2)}).\end{aligned} \tag{2.28}$$

This allows us to claim that any element of $H \otimes H$ can be written as a linear combination of elements in $(H \otimes \mathbb{C})\Delta(H)$.
One can be more precise and by mean of the *Fourier transform* $\mathcal{F} : H \otimes H \to H \otimes H$ defined by the formula

$$\mathcal{F}(f \otimes g) = (f \otimes 1)(S \otimes id)\Delta(g) = fg_{(-1)} \otimes g_{(2)} \quad \text{for} \quad f, g \in H \tag{2.29}$$

prove the following lemma.

Lemma 2.1.2 ([BDK1])**.** *Every element of $H \otimes H$ can be uniquely represented in the form $\sum_i (h_i \otimes 1)\Delta(l_i)$ where $\{h_i\}$ is a fixed basis of H ((2.7) when $H = U(\mathfrak{d})$ for instance) for some $l_i \in H$. In other worlds, $H \otimes H = (H \otimes \mathbb{C})\Delta(H)$.*
In particular, for any H-module L, $(H \otimes H) \otimes_H L = (H \otimes \mathbb{C}) \otimes L = (\mathbb{C} \otimes H) \otimes_H L$.
Moreover this is compatible with filtration (2.9)*:*
$F^n(H \otimes H) = (F^nH \otimes \mathbb{C})\Delta(H) = (\mathbb{C} \otimes F^nH)\Delta(H)$ where $F^n(H \otimes H) = \sum_{i+j=n} F^iH \otimes F^jH$.

2.2 Pseudotensor categories and Lie Pseudoalgebras

The theory of pseudotensor categories, introduced by Beilinson and Drinfeld in [BD], generalizes the notion of symmetric monodial category (in fact every symmetric monodial category has a pseudotensor structure) and "fits in" between it and the concept of ordinary additive category (a pseudotensor category can be thought as an ordinary additive category with an extra structure). It also generalizes the concept of operad (an operad turns out to be, by definition, a pseudotensor category with one object).
The main benefit in working in this environment is to have the "absolute minimum" to be able to define, in terms of polylinear maps, the notions of algebras, Lie algebras, representations, etc. This way one can generalize these concepts in suitable contexts.
We are mainly only going to give an account of definitions and main results; for a more detailed discussion see [BD] or [BDK1]. In particular, our main object of interest in this framework is to introduce H-pseudoalgebras (Lie algebras in a certain pseudotensor H-category) and superalgebras (algebras in the usual $\mathcal{V}ec$ category but with a "signed" action of the symmetric group) and to define Lie super H-pseudoalgebra.

Take as an example the category $\mathcal{V}ec$ of $\mathbb{C}$-vector spaces. We can define, for any non empty finite set I and any collection of spaces $\{L_i\}_{i\in I}$, $M \in \mathcal{V}ec$, the vector space of *polylinear maps* $P_I(\{L_i\},\, M) = Hom(\otimes_{i\in I} L_i,\, M)$.
Notice that we have an action of the symmetric group on these spaces by permutation of the factors in $\otimes_{i\in I} L_i$. Moreover, for any surjective sets map $\pi : I \twoheadrightarrow J$ and for any collection of spaces $\{K_j\}_{j\in J}$, we can define (obvious) compositions of polylinear maps:

$$\begin{aligned} P_I(\{L_i\}, M) \otimes \bigotimes_{i\in I} P_{J_i}(\{K_j\}, L_i) &\longrightarrow P_J(\{K_j\}, M) \\ \phi \otimes \{\psi_i\} &\longmapsto \phi \circ \textstyle\bigotimes_{i\in I} \psi_i \equiv \phi(\{\psi_i\}) \end{aligned} \tag{2.30}$$

where we assume that $j \in J_i = \pi^{-1}(i)$.
These compositions have the following properties:

Associativity: If we have another surjective map $H \twoheadrightarrow J$, another collection $\{N_h\}_{h\in H}$ and $\chi_j \in P_{H_j}(\{N_h\}, K_j)$, then $\phi(\{\psi_i(\{\chi_j\})\}) = (\phi(\{\psi_i\}))(\{\chi_j\}) \in P_H(\{N_h\}, M)$.

Unit: For any object M there exists $id_M \in P_1(\{M\}, M)$ such that $\forall \phi \in P_I(\{L_i\}, M)$, $id_M(\phi) = \phi(\{id_{L_i}\}) = \phi$.

Equivariance: Compositions are equivariant with respect to the natural action of the symmetric group $\mathfrak{S}_I$ on$P_I(\{L_i\}, M)$.

Using this as a prototype, one defines:

Definition 2.2.1 ([BD]). A *pseudotensor category* is a class of objects $\mathcal{M}$ together with vector spaces $P_I(\{L_i\}, M)$ endowed with an action of the symmetric group $\mathfrak{S}_I$ and with composition maps as in (2.30) which satisfy the properties of Associativity, Unit and Equivariance as above.

Notice that, given a pseudotensor category $\mathcal{M}$ and two objects $L, M \in \mathcal{M}$, we can define $Hom(L, M) = P_1(\{L\}, M)$. Composition of morphisms follow from composition maps in (2.30), providing a structure of an ordinary (additive) category on $\mathcal{M}$.

In other words, we can think of pseudotensor categories as ordinary categories equipped with an additional *pseudotensor* structure given by the polylinear maps in $P_I(\{L_i\}, M)$ for $|I| \geq 2$.

Example 2.2.1. Consider the category of super vector spaces over $\mathbb{C}$, i.e. $\mathbb{Z}/(2)$-graded vector spaces over $\mathbb{C}$.
Notice that a $\mathbb{Z}/(2)$-gradation on a vector space $V = V_{(0)} \oplus V_{(1)}$ is equivalent to give a linear involution $\pi : V \to V$; in fact, V decomposes in two eigenspaces for the two eigenvalues of π, -1 and 1. We shall call the eigenvectors *homogeneous*.
We can define the *pseudotensor category of super vector spaces*, $\mathcal{SV}ec$:

Objects: pairs (V, π) where V is a vector space over $\mathbb{C}$ and $\pi \in End(V)$ such that $\pi^2 = Id_V$.

Polylinear maps:

$$P_I\big(\{(L_i, \pi_i)\}, (M, \rho)\big) =$$
$$\Big\{\phi \equiv \otimes_{i\in I}\phi_i : \otimes_{i\in I} L_i \to M \mid \phi_i \in Hom(L_i, M) \text{ and } \phi_i \circ \pi_i = \rho \circ \phi_i\Big\} \quad (2.31)$$

where tensor products are the usual ones for vector spaces and linear maps, same for compositions.

Action of $\mathfrak{S}_I$: $\mathfrak{S}_I$ acts on $P_I\big(\{(L_i, \pi_i)\}, (M, \rho)\big)$ by permutation of factors in $\otimes_{i\in I} L_i$ but following the "Koszul sign rule", which instructs to multiply by a factor $(-1)^{|x||y|}$ every time two homogeneous elements x and y swap position, where $|x| \in \mathbb{Z}/(2)$ is the degree of x (in our notation $|x| = 0$ if $\pi(x) = x$ and $|x| = 1$ if $\pi(x) = -x$).
Formally, given $\sigma \in \mathfrak{S}_I$ and $\phi \in P_I\big(\{(L_i, \pi_i)\}, (M, \rho)\big)$, then

$$(\sigma \cdot \phi)(\otimes_{i\in I} v_i) = \varepsilon(\sigma(\{v_i\}_{i\in I})) \cdot \otimes_{i\in I}\phi_{\sigma(i)}(\otimes_{i\in I} v_{\sigma(i)})$$

where v_i are homogeneous vectors of eigenvalues $\lambda_i = 1$ or -1 and ε is defined for transpositions as

$$\varepsilon((k,l)(\{v_i\}_{i\in I})) = \begin{cases} -1 & \text{if } \lambda_k = \lambda_l = -1 \\ 1 & \text{otherwise} \end{cases} \quad (2.32)$$

and then extended to $\mathfrak{S}_I$ by multiplication.

Remark 2.2.1. The construction given for $\mathcal{V}ec$ applies for any symmetric monoidal (tensor) category. Such a category is equipped with a tensor product and a map, called *braiding*, $\tau_{M,N} : M \otimes N \to N \otimes M$ such that $\tau_{N,M} \circ \tau_{M,N} = id_{M\otimes N}$ $\forall M, N \in \mathcal{M}$.
One can proceed to define the following pseudotensor structure.

- Polylinear maps: $P_I(\{L_i\}, M) = Hom(\otimes_{i\in I} L_i, M)$;
- action of $\mathfrak{S}_I$ induced by τ: for any transposition $\sigma = (j\ k) \in \mathfrak{S}_I$ and $\phi = \otimes_{i\in I}\phi_i \in P_I(\{L_i\}, M)$, $\sigma\phi = (\otimes_{i\in I}\phi_{\sigma(i)}) \circ \tau_{L_j, L_k}$;
- composition maps: same as in (2.30).

Example 2.2.2. Let H be a cocommutative bialgebra. Then the category of left H-modules $\mathcal{M}^l(H)$ is a symmetric tensor category, hence can be made into a pseudotensor category as seen above, where polylinear maps are $P_I(L_i, M) = Hom_H(\otimes_{i\in I} L_i, M)$ and the braiding is the identity.
The same can be done for the category of H-bimodules (which are often indicated as $H-H$-modules) $\mathcal{M}^b(H)$ where in this case one takes as polylinear maps morphisms of $H-H$-modules.

Example 2.2.3 (bis)**.** Given V and W super vector spaces over $\mathbb{C}$, one can define a tensor product $V \otimes W$ given by

$$(V \otimes W)_{(0)} = (V_{(0)} \otimes W_{(0)}) \oplus (V_{(1)} \otimes W_{(1)});$$
$$(V \otimes W)_{(1)} = (V_{(0)} \otimes W_{(1)}) \oplus (V_{(1)} \otimes W_{(0)}).$$

Furthermore one can define a braiding:

$$\begin{aligned} \tau_{V\otimes W} : V \otimes W &\longrightarrow W \otimes V \\ v \otimes w &\longmapsto (-1)^{|v||w|} w \otimes v \end{aligned} \tag{2.33}$$

where $v \in V_\alpha,\ w \in W_\beta$ with $\alpha, \beta \in \mathbb{Z}/(2)$ and $|v| = \alpha$.
These make the category of super vector spaces over $\mathbb{C}$ a symmetric monoidal category, thus giving another way to define the pseudotensor structure given in Example 2.2.1.
Explicitly, the polylinear maps are given by $P_I(\{L_i\}, M) = Hom^{even}(\otimes_{i\in I} L_i, M)$ (where *even* homomorphism are linear homomorphisms that preserve the $\mathbb{Z}/(2)$-grading) with action of $\mathfrak{S}_I$ given by

$$(\sigma(\otimes_{i\in I}\psi_i))(\otimes_{i\in I} v_i) = (-1)^{|\sigma(\{v_i\}_{i\in I})|} \otimes_{i\in I} \psi_{\sigma(i)}(v_{\sigma(i)}) \tag{2.34}$$

where v_i are homogeneous vectors and if $\sigma \in \mathfrak{S}_I$ is equal, as product of transpositions, as $(j_1\, l_1) \cdots (j_k\, l_k)$, then $|\sigma(\{v_i\}_{i\in I})| = |v_{j_1}||v_{l_1}| + \cdots + |v_{j_k}||v_{l_k}|$.

We hinted before that it is possible to define algebraic structures in pseudotensor categories that generalize usual ones.

Definition 2.2.2. An *associative algebra* in a pseudotensor category $\mathcal{M}$ is an object $A \in \mathcal{M}$ together with a polylinear map $\mu \in P_2(\{A, A\}, A)$ such that

$$\mu(\mu(\cdot,\cdot),\cdot) = \mu(\cdot, \mu(\cdot,\cdot)). \tag{2.35}$$

A is said to be *commutative* if, in addition,

$$\mu = \sigma_{12}\mu, \text{ where } \sigma_{12} = (1\,2) \in \mathfrak{S}_2. \tag{2.36}$$

Definition 2.2.3. A *Lie algebra* in a pseudotensor category $\mathcal{M}$ is an object $L \in \mathcal{M}$ equipped with a polylinear map $\beta \in P_2(\{L, L\}, L)$ for which the following properties apply:

Skew-commutativity $\beta = -\sigma_{12}\beta$ where $\sigma_{12} = (1\,2) \in \mathfrak{S}_2$;

Jacobi identity $\beta(\beta(\cdot,\cdot),\cdot) = \beta(\cdot,\beta(\cdot,\cdot)) - \sigma_{12}\beta(\cdot,\beta(\cdot,\cdot))$, where we are thinking σ_{12} in $\mathfrak{S}_3$.

Definition 2.2.4. A *representation* of a Lie algebra (L,β) (or L-module) is an object $V \in \mathcal{M}$ together with a polylinear map $\rho \in P_2(\{L,M\},M)$ satisfying

$$\rho(\beta(\cdot,\cdot),\cdot) = \rho(\cdot,\rho(\cdot,\cdot)) - \sigma_{12}\rho(\cdot,\rho(\cdot,\cdot)). \tag{2.37}$$

If (L,α) is an associative algebra, then ρ must instead satisfy

$$\rho(\alpha(\cdot,\cdot),\cdot) = \rho(\cdot,\rho(\cdot,\cdot)). \tag{2.38}$$

Remark 2.2.2. When the action of the symmetric group is the usual one, like in the case of $\mathcal{V}ec$, these definitions coincide with the usual ones.
In other cases we obtain new definitions. For instance, the multiplication of a Lie algebra L in the pseudotensor category of Example 2.2.1 has to satisfy $\beta(a,b) = -(-1)^{|a||b|}\beta(b,a)$ for all homogeneous elements $a,b \in L$. A similar argument applies to the Jacobi identity and commutativity; that is, we are defining superassociative (=associative) algebras, supercommutative algebras and Lie superalgebras.

Example 2.2.4. A (Lie or associative) algebra in the pseudotensor category $M^l(H)$ of Example 2.2.2 is called a (Lie or associative) H*-differential algebra*, which is a (Lie or associative) algebra that has also a structure of H-module such that the product (or the bracket) is a morphism of H-modules.

Example 2.2.5. A Lie superalgebra (resp. supercommutative algebra) is a Lie algebra (resp. commutative algebra) in the pseudotensor category of super vector spaces
Notice that a definition of an associative superalgebra would be redundant because in Definition 2.2.2 the action of $\mathfrak{S}_I$ is not involved for associative algebras (which is what determines the difference between "super" objects and usual ones).

Results involving only the polylinear maps structure over tensor categories still hold for pseudotensor categories. For instance:

Proposition 2.2.1. *Let (A,μ) be an associative algebra in a pseudotensor category $\mathcal{M}$. If we define the* commutator $\beta = \mu - \sigma_{12}\mu$, *then (A,β) is a Lie algebra in $\mathcal{M}$.*

In order to define one of the main objects of our study, Lie pseudoalgebras, we need to introduce, following [BD], the pseudotensor category $\mathcal{M}^*(H)$.
Given a cocommutative bialgebra H with coproduct Δ, $\mathcal{M}^*(H)$ is the pseudotensor category having the same objects of $\mathcal{M}^l(H)$ (i.e. left H-modules) but with a different pseudotensor structure: in this case we define the polylinear maps as

$$P_I(\{L_i\},M) = Hom_{H^{\otimes I}}(\boxtimes_{i\in I}L_i, H^{\otimes I}\otimes_H M) \tag{2.39}$$

where $\boxtimes_{i\in I}L_i$ is the $H^{\otimes I}$-module $\otimes_{i\in I}L_i$ with action given component by component, while the action of $H^{\otimes I}$ on $H^{\otimes I}\otimes_H M$ is given by left multiplication.
Compositions of polylinear maps are given accordingly to the following construction:
given $\pi : J \twoheadrightarrow I$ and polylinear maps

$$\phi = \boxtimes_{i\in I}\phi_i \in Hom_{H^{\otimes I}}(\boxtimes_{i\in I}L_i, H^{\otimes I}\otimes_H M)$$

and, for $i \in I$ and $J_i = \pi^{-1}(i)$,

$$\psi_i = \boxtimes_{j\in J_i}(\psi_i)_j \in Hom_{H^{\otimes J_i}}(\boxtimes_{j\in J_i}N_j, H^{\otimes J_i} \otimes_H L_i),$$

one can realize $\boxtimes_{i\in I}\psi_i$, consistently with π, as an element of $Hom_{H^{\otimes J}}(\boxtimes_{j\in J}N_j, H^{\otimes J} \otimes_H \boxtimes_{i\in I}L_i)$. To obtain a polylinear map $\phi(\{\psi_i\}) \in Hom_{H^{\otimes J}}(\boxtimes_{j\in J}N_j, H^{\otimes J} \otimes_H M)$ we can simply compose it with ϕ, provided, however, that since the latter is an homomorphism of $H^{\otimes I}$-modules, we give it a structure of $H^{\otimes J}$-homomorphism (consistent with π). In order to do so, one takes into account the functor $\Delta^{(\pi)} : \mathcal{M}^l(H^{\otimes I}) \to \mathcal{M}^l(H^{\otimes J})$, $M \mapsto H^{\otimes J} \otimes_{H^{\otimes I}} M$ where the right action of $H^{\otimes I}$ over $H^{\otimes J}$ is induced by the iterated coproduct according to π.
If, for example, $\pi : \{1, 2, 3, 4\} \to \{1, 2, 3\}$ where $\pi(1) = 1$, $\pi(2) = \pi(3) = 2$, $\pi(4) = 3$, then $\Delta^{(\pi)} = id\otimes\Delta\otimes id : H^{\otimes 3} \to H^{\otimes 4}$ and the action over $H^{\otimes 4}$ is given by multiplication component by component.
If, again for example, $\pi : \{1, 2, 3, 4\} \to \{1, 2\}$ with $\pi(1) = \pi(2) = 1$ e $\pi(3) = \pi(4) = 2$, then $\Delta^{(\pi)} = \Delta \otimes \Delta : H^{\otimes 2} \to H^{\otimes 4}$.
Compositions of polylinear maps is defined as

$$\phi(\{\psi_i\}_{i\in I}) = \Delta^{(\pi)}(\phi) \circ \boxtimes_{i\in I}\psi_i. \tag{2.40}$$

Explicitly, given $n_j \in N_j$, we can write

$$\psi_i(\otimes_{j\in J_i}n_j) = \sum_k h_i^k \otimes_H l_i^k$$

for some $h_i^k \in H^{\otimes J_i}$, $l_i^k \in L_i$. Similarly

$$\phi(\otimes_{i\in I}l_i^k) = \sum_t g^{kt} \otimes_H m^{kt}$$

for some $g^{kt} \in H^{\otimes I}$, $m^{rs} \in M$.
Then we define

$$(\phi(\{\psi_i\}_{i\in I}))(\otimes_{j\in J}) = \sum_{k,t}(\otimes_{i\in I}h_i^k)\Delta^{(\pi)}(g^{kt}) \otimes_H m^{kt} \tag{2.41}$$

where $\Delta^\pi : H^{\otimes I} \to H^{\otimes J}$ behaves as described above.
Finally, the symmetric group $\mathfrak{S}_I$ acts on polylinear maps permuting simultaneously factors in $\boxtimes_{i\in I}L_i$ and in $H^{\otimes I}$.

Remark 2.2.3. It is precisely the equivariance with respect to the action of the symmetric group of composition of polylinear maps that demands the cocommutativity of H. Indeed if we consider, for example, the action of $\sigma_{12} = (1\,2) \in \mathfrak{S}_2$ over $(H \otimes H) \otimes_H M$, which is given by $\sigma_{12}((f \otimes g) \otimes_H m) = (g \otimes f) \otimes_H m$, equivariance implies in particular that $\sigma_{12}\Delta = \Delta$, that is the cocommutativity of H.

One can directly define the notions of Lie and associative pseudoalgebras.

Definition 2.2.5. A *Lie (H-)pseudoalgebra* is a Lie algebra in the pseudotensor category $\mathcal{M}^*(H)$.

Definition 2.2.6. An *associative (H-)pseudoalgebra* is an associative algebra in the pseudotensor category $\mathcal{M}^*(H)$.

Anyway, we can be less cryptic by expressing the requested properties for the polylinear maps used as operations.

Definition 2.2.7. An (H-)pseudoalgebra is an H-module L together with a polylinear map $\beta \in P_2(\{L, L\}, L) = Hom_{H \otimes H}(L \otimes L, (H \otimes H) \otimes_H L)$ called the *pseudoproduct* of L. We will indicate the pseudoproduct $\beta(a, b)$ as $a * b$ for $a, b \in L$. L is said to be *finite* if it is finitely generated as an H-module.

Remark 2.2.4. By definition, β is H-bilinear, meaning that

$$fa * gb = ((f \otimes g) \otimes_H id)(a * b). \tag{2.42}$$

Explicitly, if $a * b = \sum_i (f_i \otimes g_i) \otimes_H c_i$, then $fa * gb = \sum_i (ff_i \otimes gg_i) \otimes_H c_i$.

The above definitions of associative and Lie pseudoalgebras are obtained by attaching properties to the pseudoproduct.
The definition of associativity of the pseudoproduct is the usual:

$$\textbf{[Associativity]}\ a * (b * c) = (a * b) * c \quad \forall a, b, c \in L \tag{2.43}$$

To describe it explicitly, we need to make sense of the above elements in $(H \otimes H \otimes H) \otimes_H L$ by using compositions of polylinear maps.
If we have

$$b * c = \sum_i (f_i \otimes g_i) \otimes_H v_i \tag{2.44}$$

$$a * v_i = \sum_j (h_{ij} \otimes k_{ij}) \otimes w_{ij} \tag{2.45}$$

$$\textit{then } a * (b * c) = \sum_{i,j} (h_{ij} \otimes f_i k_{ij_{(1)}} \otimes g_i k_{ij_{(2)}}) \otimes w_{ij}. \tag{2.46}$$

Similarly, if

$$a * b = \sum_i (f_i \otimes g_i) \otimes_H v_i \tag{2.47}$$

$$v_i * c = \sum_j (h_{ij} \otimes k_{ij}) \otimes w_{ij} \tag{2.48}$$

$$\textit{then } (a * b) * c = \sum_{i,j} (f_i h_{ij_{(1)}} \otimes g_i h_{ij_{(2)}} \otimes k_{ij}) \otimes w_{ij}. \tag{2.49}$$

A pseudoproduct is *commutative* when

$$\textbf{[Commutativity]}\ a * b = (\sigma \otimes_H id)(b * a) \quad \forall a, b \in L \tag{2.50}$$

where σ is the "flip" $\sigma(f \otimes g) = (g \otimes f)$.

When L is a Lie pseudoalgebra, the pseudoproduct is called *pseudobracket* and denoted by $[\cdot * \cdot]$. It has, in addition to bilinearity, the following properties:

[Skew-commutativity] $[a * b] = -(\sigma \otimes_H id)[b * a];$ (2.51)

[Jacobi identity] $[a * [b * c]] - ((\sigma \otimes id) \otimes_H id)[b * [a * c]] = [[a * b] * c],$ (2.52)

for any $a, b, c \in L$, where the identity (2.52) in $H^{\otimes 3} \otimes_H L$ is in the sense described before.

In an analogous way one can define representations of Lie and associative pseudoalgebras.

Definition 2.2.8. A representation of a Lie H-pseudoalgebra L is a left H-module V together with a polylinear map $\rho \in P_2(L, M, M)$, denoted by $\rho(a \otimes b) = a * b$ such that

$$[a * b] * v = a * (b * v) - ((\sigma \otimes id) \otimes_H id)(b * (a * v)) \quad \forall\, a, b \in L,\, v \in V. \tag{2.53}$$

V is said to be *finite* if it i finitely generated as an H-module.

We conclude this section by defining another pseudotensor category, which combines Example 2.2.1 with the pseudotensor category $\mathcal{M}^*(H)$.
We define the pseudotensor category $\mathcal{SM}^*(H)$ as follows.

Objects: $\mathbb{Z}/(2)$-graded left H-modules, or equivalently pairs (M, π) where M is a left H-module and $\pi : M \to M$ is an involution of H-modules.

Polylinear maps:

$$P_I(\{(L_i, \pi_i)\}_{i \in I}, (L, \tau)) =$$
$$\{\psi \equiv \otimes_{i \in I} \psi_i \in Hom_{H^{\otimes I}}(\boxtimes_{i \in I} L_i, H^{\otimes I} \otimes_H M) \mid \psi_i \circ \pi_i = \tau \circ \psi_i\}. \tag{2.54}$$

We are basically using the same polylinear maps as in (2.39), but we also require them to commute with the involutions, so that they behave as we expect in respect to the $\mathbb{Z}/(2)$-gradation.
Compositions of polylinear maps follows the construction given for $\mathcal{M}^*(H)$.

Action of $\mathfrak{S}_I$: The symmetric group $\mathfrak{S}_I$ acts on $P_I(\{(L_i, \pi_i)_{i \in I}\}, (L, \tau))$ by permuting simultaneously factors in $H^{\otimes I}$ and in $\boxtimes_{i \in I} L_i$ while at the same time applying the "Koszul sign rule", as given explicitly in (2.32).

Definition 2.2.9. A *Lie H-superpseudoalgebra* is a Lie algebra in the pseudotensor category $\mathcal{SM}^*(H)$.

In other words, a Lie super pseudoalgebra is a pair (L, π) (or equivalently a $\mathbb{Z}/(2)$-graded H-module $L = L_{(0)} \oplus L_{(1)}$), together with a polylinear map $\beta \in Hom_{H \otimes H}(L \otimes L, (H \otimes H) \otimes_H L)$ commuting with π (or equivalently, such that $\beta(L_{(i)} \otimes L_{(j)}) \subset (H \otimes H) \otimes_H L_{(i+j)}$ for $i, j \in \mathbb{Z}/(2)$) which we write as $\beta(a, b) = [a * b]$, with "super" versions of properties (2.51) and (2.52), i.e.

[Skew-commutativity]: $[a * b] = -(-1)^{|a||b|}(\sigma \otimes_H id)[b * a]$ (2.55)

[Jacobi identity]: $[a * [b * c]] - (-1)^{|a||b|}((\sigma \otimes id) \otimes_H id)[b * [a * c]] = [[a * b] * c]$ (2.56)

for any homogenous element $a, b \in L$ and $c \in L$.
The explicit description of these properties is the analogous "signed" version of the one in the non "super" setting.

Remark 2.2.5. It is worth to notice that if we restrict the pseudobracket to $L_{(0)} \otimes L_{(0)}$, we obtain the definition of a Lie pseudoalgebra, while if we restrict it to $L_{(0)} \otimes L_{(1)}$, Jacobi identity reads as the definition of a representation of a Lie pseudoalgebra.
In fact, one can easily see that the datum of a Lie super pseudoalgebra $L = L_{(0)} \oplus L_{(1)}$ is equivalent to that of Lie pseudoalgebra $L_{(0)}$, an $L_{(0)}$-module $L_{(1)}$ and a $L_{(0)}$-equivariant H-bilinear map : $L_{(1)} \otimes L_{(1)} \to (H \otimes H) \otimes_H L_{(0)}$ such that

$$[a * b] = (\sigma \otimes_H id)[b * a]; \tag{2.57}$$

$$[a * [b * c]] + ((\sigma \otimes id) \otimes_H id)[b * [a * c]] = [[a * b] * c] \tag{2.58}$$

$\forall\, a, b, c \in L_{(1)}$.

2.3 Pseudo linear algebra

The first example of a pseudoalgebra is given by the *current pseudoalgebra*.
Let H be a (cocommutative) Hopf algebra. Given a $\mathbb{C}$-algebra A, its *current pseudoalgebra* is the left H-module $Cur\ A = H \otimes A$ with pseudoproduct

$$(f \otimes a) * (g \otimes b) = (f \otimes g) \otimes_H ab \tag{2.59}$$

for $a, b \in L$, $f, g \in H$, where ab is the product in A. If A is associative/Lie/commutative, so is $Cur\ A$. We can also generalize this construction.

Definition 2.3.1. Let $H' \subset H$ be an Hopf subalgebra and let L be an H'-pseudoalgebra. The *current H-pseudoalgebra* of L is

$$Cur_H^{H'} L = H \otimes_{H'} L$$

where the pseudoproduct is given by

$$(f \otimes_{H'} a) * (g \otimes_{H'} b) = ((f \otimes g) \otimes_H 1)(a *_{H'} b)$$

for $f, g \in H$, $a, b \in L$, where $*_{H'}$ denotes the pseudoproduct $L \otimes L \to (H' \otimes H') \otimes_{H'} L$. The action of H is given by left multiplication on the first factor. $Cur_H^{H'} L$ is a Lie (resp. associative) pseudoalgebra when L is.

Remark 2.3.1. In other words, we are defining a pseudotensor functor $Cur_H^{H'} : \mathcal{M}^*(H') \to \mathcal{M}^*(H)$.
The previous example arises as a particular case of this construction when $H' = \mathbb{C}$. In this case we are defining a pseudotensor functor $Cur_H : \mathcal{V}ec \to \mathcal{M}^*(H)$, where the pseudotensor structure of $\mathcal{V}ec$ is the one described at the beginning of Section 2.2.

As we already mentioned, another class of examples of pseudoalgebras is provided by conformal algebras.
First of all consider the one-dimensional Lie algebra $\mathbb{C}\partial$ generated by ∂. Then $U(\mathbb{C}\partial) =$

$\mathbb{C}[\partial]$ and we can define an Hopf algebra structure on the latter.
Now, let L be a conformal algebra (so in particular an $H = \mathbb{C}[\partial]$-module) with λ-bracket given by

$$[a_\lambda b] = \sum_i p_i(\lambda) c_i \tag{2.60}$$

for some $p_i \in \mathbb{C}[\lambda]$, $c_i \in L$. Then L is a Lie H-pseudoalgebra with pseudobracket given by

$$[a * b] = \sum_i (p_i(-\partial) \otimes 1) \otimes_H c_i. \tag{2.61}$$

One can check that properties of the λ-bracket listed in Definition 1.4.1 imply H-bilinearity, skew-symmetry and Jacobi identity for (2.61).

We recover now few basic definitions and constructions about pseudoalgebras and "pseudo linear" algebra that resemble those of linear algebra.
Given an H-pseudoalgebra L, then an H-submodule $M \subset L$ is a *subalgebra* of L if

$$M * M \subset (H \otimes H) \otimes_H M. \tag{2.62}$$

A subalgebra is an *ideal* if $L * M \subset (H \otimes H) \otimes_H M$. L is *simple* if its only ideals are 0 and L. Analogous definitions are given for Lie H-pseudoalgebras.

Example 2.3.1. If $B \subset A$ are $\mathbb{C}$-algebras, then $Cur\, B$ is a subalgebra of $Cur\, A$. In particular, if A is simple, so is $Cur\, A$.

Let L_1, L_2 be H-pseudoalgebras. A morphism of pseudoalgebras is, in the language of pseudotensor categories, a polylinear map in $P_1(\{L_1\}, L_2) = Hom_H(L_1, H \otimes_H L_2) \equiv Hom_H(L_1, L_2)$ which respects the pseudoproducts .
Explicitly, it is an H-linear map $\gamma : L_1 \to L_2$ such that, for $a, b \in L_1$

$$((id \otimes id) \otimes_H \gamma)(a * b) = \gamma(a) * \gamma(b). \tag{2.63}$$

In Definition 2.2.4 we have defined representations of Lie and associative algebras in a generic pseudotensor category in terms of polylinear maps. More explicitly, for an associative (or Lie) pseudoalgebra one has the following.

Definition 2.3.2. Let L be an associative H-pseudoalgebra. A *representation* of L, or L-module, is an H-module V together with a map $\rho : L \otimes V \to (H \otimes H) \otimes_H$ written, with a slightly abuse of notation, as $a * v \rho(a, v)$ satisfying

$$a * (b * v) = (a * b) * v \quad \forall a, b \in L,\, v \in V, \tag{2.64}$$

where $a * b$ is the pseudoproduct in L.
If L is a Lie H-pseudoalgebra, the identity to be satisfied is

$$[a * b] * v = a * (b * v) - ((\sigma \otimes id) \otimes_H id)(b * (a * v)). \tag{2.65}$$

An H-submodule $U \subset V$ is called an *L-subrepresentation*, or L-submodule, if $L * U \subset (H \otimes H) \otimes_H U$.
An L-module is *irreducible* if the only submodules are $\{0\}$ and L.

In the classical setting, a representation of an associative (resp. Lie) algebra L is a morphism of associative (resp. Lie) algebras $\rho : L \longrightarrow End(V)$ (resp. $\mathfrak{gl}(V)$).
Something similar can be done in the pseudo setting. First of all, we need to define pseudolinear maps.

Definition 2.3.3. Let V, W be H-modules. An *H-pseudolinear map* from V to W is a linear map $\alpha : V \longrightarrow (H \otimes H) \otimes_H W$ such that

$$\alpha(hv) = ((1 \otimes h) \otimes_H 1)\alpha(v) \quad \forall h \in H,\, v \in V. \tag{2.66}$$

The space of pseudolinear maps from V to W is denoted by $Chom(V, W)$. This is an H-module with action defined by

$$(h\alpha)(v) = ((h \otimes 1) \otimes_H 1)\alpha(v) \quad \text{for } \alpha \in Chom(V, W),\ h \in H, v \in V. \tag{2.67}$$

When $V = W$, we denote $Cend(V) = Chom(V, V)$.

Example 2.3.2. Let L be an H-pseudoalgebra and M an L-module. Then for any $a \in L$, the map $m_a : V \longrightarrow (H \otimes H) \otimes_H V$, $v \mapsto a * v$, is an H-pseudolinear map, so $m_a \in Cend(V)$. One also has that $hm_a = m_{ha}$.

Given $\phi \in Chom(V, W)$, $\psi \in Chom(U, V)$ one can consider the "composition" $\phi \circ \psi$ in the sense of (2.40). Anyway, in order to obtain a well defined element of $Chom(U, W)$, one needs U to be finite (as an H-module).
It is shown in [BDK1, Chapter 10] that this can be done in an unique way such that the composition is "associative". When applied to the case of $V = U = W$, this gives a structure of associative pseudoalgebra for $Cend V$. Moreover, $\phi * v = \phi(v)$ defines a structure of $Cend(V)$-module on V.

Proposition 2.3.1. *If A is an H-pseudoalgebra, giving a structure of A-module on a finite H-module V is equivalent to provide an homomorphism of associative pseudoalgebras from A to $Cend(V)$.*

Proof. One associates to any $a \in A$ the pseudolinear map m_a from example 2.3.2. Then

$$(m_a * m_b) * v = m_a * (m_b * v) = a * (b * v) = (a * b) * v = m_{a*b}v,$$

for any $v \in V$, therefore $m_a * m_b = m_{a*b}$. □

By Proposition 2.2.1, given an associative pseudoalgebra one can define a Lie pseudoalgebra structure on it (same as in the non-pseudo setting). The Lie pseudoalgebra obtained by $Cend(V)$ is denoted by $\mathbf{gc}(V)$. Analogously to the previous proposition, we have the following.

Proposition 2.3.2. *If L is a Lie H-pseudoalgebra, giving a structure of L-module on a finite H-module V is equivalent to provide an homomorphism of Lie pseudoalgebras from L to $\mathbf{gc}(V)$.*

Remark 2.3.2. Let L be a Lie pseudoalgebra and V, W two finite L-modules. Then we can give to $Chom(V, W)$ a structure of an L-module where the pseudoaction of $a \in L$ on $\varphi \in Chom(V, W)$ is given by

$$(a * \varphi)(v) = a * (\varphi * v) - ((\sigma \otimes id) \otimes_H \varphi)(a * v). \tag{2.68}$$

Definition 2.3.4. Let L be an H-pseudoalgebra. A *derivation* of L is an H-pseudolinear map $\varphi \in \mathbf{gc}(L)$ satisfying, for all $a, b \in L$,

$$\varphi * (a * b) = (\varphi * a) * b + ((\sigma \otimes id) \otimes_H id)(a * (\varphi * b)) \tag{2.69}$$

if L is an associative pseudoalgebra, or

$$\varphi * [a * b] = [(\varphi * a) * b] + ((\sigma \otimes id) \otimes_H id)[a * (\varphi * b)] \tag{2.70}$$

if L is a Lie pseudoalgebra.
The set of all derivations of L is denoted by $Der\, L$.

It is easy to see from the definition that $Der\, L$ is a subalgebra of $\mathbf{gc}(L)$.

$\mathbf{gc}(V)$ is, in general, a quite big object and it is less manageable that its "linear" counterpart $\mathfrak{gl}(V)$. For instance, take a free H-module $V = H \otimes V_0$. We want V to be finite, thus V_0 is a finite-dimensional vector space.
In this case we can give an explicit description of $Cend(V)$ and $\mathbf{gc}(V)$ (see [BDK1, Proposition 10.11]).
As H-modules, they are isomorphic to $(H \otimes H) \otimes_H End(V_0)$, where H acts by left multiplication on the first factor.
The pseudoproduct in $Cend(V)$ can be written, for $f, g, h, k \in H,\ A, B \in End\, V_0$, as

$$(f \otimes g \otimes A) * (h \otimes k \otimes B) = (f \otimes hg_{(1)}) \otimes_H (1 \otimes kg_{(2)} \otimes AB). \tag{2.71}$$

The pseudobracket in $\mathbf{gc}(V)$ is given by the "commutator" of the pseudoproduct of $Cend(V)$, i.e.

$$\begin{aligned}&[(f \otimes g \otimes A) * (h \otimes k \otimes B)] = \\ &(f \otimes hg_{(1)}) \otimes_H (1 \otimes kg_{(2)} \otimes AB) - (fk_{(1)} \otimes h) \otimes_H (1 \otimes gk_{(2)} \otimes BA).\end{aligned} \tag{2.72}$$

When $V_0 = \mathbb{C}^n$ we denote $Cend_n = Cend(V)$ and $\mathbf{gc}_n = \mathbf{gc}V$.

Remark 2.3.3. Notice that if we restrict (2.71) to $H \otimes 1 \otimes End\, V_0$, we obtain

$$(f \otimes 1 \otimes A) * (h \otimes 1 \otimes B) = (f \otimes h) \otimes_H (1 \otimes 1 \otimes AB). \tag{2.73}$$

Therefore, $H \otimes 1 \otimes End(V_0)$ is a subalgebra of $Cend(V)$ and, moreover, it is isomorphic to $Cur\, End(V_0)$.
Similarly, we can embed $Cur\mathfrak{gl}(V_0) \subset \mathbf{gc}(V)$.

We focus now on the case $H = U(\mathfrak{d})$ and identify $\mathfrak{d}$ with its image in H, so that $H \otimes \mathfrak{d}$ is a subalgebra of $Cur\, H = H \otimes H$. Consider $\mathbf{gc}_1 \cong H \otimes H \otimes \mathbb{C} \equiv H \otimes H$.
If we restrict (2.72) to $H \otimes \mathfrak{d}$, we obtain (recall that for $a \in \mathfrak{d}$, $\Delta(a) = a \otimes 1 + 1 \otimes a$):

$$\begin{aligned}&[(f \otimes a) * (g \otimes b)] = \\ &(f \otimes ga_{(1)}) \otimes_H (1 \otimes ba_{(2)}) - (fb_{(1)} \otimes g) \otimes_H (1 \otimes ab_{(2)}) = \\ &(f \otimes ga) \otimes_H (1 \otimes b) + (f \otimes g) \otimes_H (1 \otimes ba) - (fb \otimes g) \otimes_H (1 \otimes a) - (f \otimes_g) \otimes_H (1 \otimes ab) = \\ &-(fb \otimes g) \otimes_H (1 \otimes a) + (f \otimes ga) \otimes_H (1 \otimes b) - (f \otimes g) \otimes_H [a, b].\end{aligned} \tag{2.74}$$

We call $W(\mathfrak{d})$ the Lie pseudoalgebra $H \otimes \mathfrak{d}$ with pseudobracket defined by this formula (with a switch sign for notational needs). $W(\mathfrak{d})$ plays a central role in the theory of finite Lie pseudoalgebras.
We give also another interesting way to realize this subalgebra of $\mathbf{gc}_1$.
By definition, elements of $\mathbf{gc}_1$ are pseudolinear maps $\phi : H \otimes \mathbb{C} \to (H \otimes H) \otimes_H (H \otimes \mathbb{C})$, that can be canonically identified with H-linear maps $H \to H \otimes H$ where the action of H on $H \otimes H$ is by multiplication on the second factor.
ϕ is therefore uniquely determined by $\phi(1)$. Assume that $\phi(1) = \sum_i f_i \otimes g_i$ for some $f_i, g_i \in H$.
Now, if we realize H trivially as $Cur\,\mathbb{C}$, then it makes sense to ask ourself when ϕ is a derivation of $Cur\,\mathbb{C} = H$. This happens when ϕ satisfies (2.69), where the pseudoproduct of H is $f * g = (f \otimes g)$.
Writing down (2.69) in this case, one obtains, under all obvious identifications, that f_i, g_i must satisfy for all $h, k \in H$

$$\sum_i f_i \otimes g_{i_{(1)}} h \otimes g_{i_{(2)}} k = \sum_i f_i \otimes g_i h \otimes k + f_i \otimes h \otimes g_i h. \tag{2.75}$$

In other words, one is asking that $\Delta(g_i) = g_i \otimes 1 + 1 \otimes g_i$, which is equivalent to say that $g_i \in \mathfrak{d}$.
This shows that $H \otimes \mathfrak{d} = W(\mathfrak{d}) = DerH$. Now, comparing this with the definition of W_N in Section 1.1 in view of (2.25), hints to a very strong connection between this Lie pseudoalgebra and the Lie algebra of Cartan type W_N. This connection will be make precise in terms of the *annihilation algebra*.

We end this section by defining other constructions we will need later, which are the *dual* and the *twisting* of finite modules.

Definition 2.3.5. Given a finite L-module V, $D(V) := Chom(V, \mathbb{C})$ is called the *dual* of V. If $\alpha : V' \to V$ is an homomorphism of L-modules, then we define $D(\alpha) = Chom(\alpha, id) : D(V) \to D(V')$, $\phi \mapsto \phi \circ \alpha$.

Definition 2.3.6. Given a finite L-module V and a finite-dimensional $\mathfrak{d}$-module Π, $T_\Pi(V) = Chom(D(V), \Pi)$ is called the *twisting* of V by Π. If $\beta : V \to V'$ is an homomorphism of L-modules, then we define the homomorphism $T_\Pi(\beta) = Chom(D(\beta), id) : T_\Pi(V) \to T_\Pi(V')$, $\phi \mapsto D(\beta) \circ \phi$.

We will only be interested in these objects for free H-modules of finite rank, which makes it possible to write them explicitly, as well as the pseudoaction of L upon them.
Let $V = H \otimes V_0$ where V_0 is a finite-dimensional vector space. Similarly to the previous case of $Cend(V)$, if W is another finite H-module it is possible to identify $Chom(V, W)$ with the free H-module $H \otimes (W \otimes V_0^*)$. Then in particular we have isomorphisms of H-modules $D(V) \cong H \otimes V_0^*$, $T_\Pi(V) \cong H \otimes (\Pi \otimes V_0)$ where again the action of H on the right hand sides is given by left multiplication on the first factor. The structure of L-modules is given explicitly by next proposition.

Proposition 2.3.3. *[BDK2] Let $V = H \otimes V_0$ be a finite L-module which is free as an H-module, V_0 a finite-dimensional vector space with basis $\{v_i\}$ and consider the dual basis $\{\psi_i\}$ of V_0^*. Fix $a \in L$ and write*

$$a * (1 \otimes v_i) = \sum_j (f_{ij} \otimes g_{ij}) \otimes_H v_j \tag{2.76}$$

for some $f_{ij},\ g_{ij} \in H$. Then the action of L on $D(V) \cong H \otimes V_0^*$ is given by

$$a * (1 \otimes \psi_k) = -\sum_j (f_{jk} g_{jk_{(-1)}} \otimes g_{jk_{(-2)}}) \otimes_H (1 \otimes \psi_j). \tag{2.77}$$

The action of L on $T_\Pi(V) \cong H \otimes (\Pi \otimes V_0)$ is given by

$$a * (1 \otimes u \otimes v_i) = \sum_j (f_{ij} \otimes g_{ij_{(1)}}) \otimes_H (1 \otimes g_{ij_{(-2)}} u \otimes v_j). \tag{2.78}$$

It is also possible to describe explicitly the homomorphisms $D(\alpha)$ and $T_\Pi(\beta)$ in Definitions 2.3.5, 2.3.6.

Proposition 2.3.4. *Let $V = H \otimes V_0$, $V' = H \otimes V_0'$ be finite L-modules which are free as H-modules. Let $\{v_i\}$, $\{v_i'\}$ be basis for respectively V_0 and V_0' and consider the dual basis $\{\psi_i\}$, $\{\psi_i'\}$ of respectively V_0^* and $(V_0')^*$. Take an homomorphism of L-modules $\beta : V \to V'$ and write*

$$\beta(1 \otimes v_i) = \sum_j f_{ij} \otimes v_j'. \tag{2.79}$$

for some $f_{ij} \in H$. Then $D(\beta) : D(V') \to D(V)$ is given by

$$D(\beta)(1 \otimes \psi_k') = \sum_j S(f_{jk}) \otimes \psi_j, \tag{2.80}$$

while $T_\Pi(\beta) : T_\Pi(V) \to T_\Pi(V')$ is given by

$$T_\Pi(\beta)(1 \otimes u \otimes v_i) = \sum_j f_{ij_{(1)}} \otimes f_{ij_{(2)}} u \otimes v_j'. \tag{2.81}$$

2.4 Annihilation algebra and primitive Lie pseudoalgebras

The main tool in the study of Lie pseudoalgebras is the *annihilation algebra*. Take as usual a cocommutative bialgebra H. Given an H-differential bialgebra Y in $\mathcal{M}^b(H)$, one can define the *annihilation functor* $\mathcal{A}_Y$ from the category $\mathcal{M}^*(H)$ to the category of H-differential modules $\mathcal{M}^l(H)$ (see [BDK1][Section 7.1]).
Remember that objects in both $\mathcal{M}^*(H)$ and $\mathcal{M}^l(H)$ are left H-modules.
This *pseudofunctor* associates to an H-module M another H-module $\mathcal{A}_Y(M) = Y \otimes_H M$ where H acts on the left factor.
Given a polylinear map $\phi \in Hom_H(\boxtimes_{i\in I} L_i, M)$ for $L_i,\ M \in \mathcal{M}^*(H)$, we associate to it a polylinear map $\mathcal{A}_Y(\phi)$ in $\mathcal{M}^l(H)$ denoted by $\mu^I \otimes_H \phi \in Hom_H(\otimes_i Y \otimes_H L_i, Y \otimes M)$ defined as the composition:

$$\begin{aligned}&\otimes_{i\in I} Y \otimes_H L_i \cong (\boxtimes_{i\in I} Y) \otimes_H (\boxtimes_{i\in I} L_i)\\ &\xrightarrow{id\otimes\phi} (\boxtimes_{i\in I} Y) \otimes_H (H^{(I)} \otimes_H M) \cong (\otimes_{i\in I} Y) \otimes_H M \xrightarrow{\mu^I \otimes id} Y \otimes_H M\end{aligned} \tag{2.82}$$

where $\mu^I = \mu(\mu \otimes id) \cdots (\mu \otimes id \otimes \cdots \otimes id)$ is the iterated multiplication in Y.
Take for example $\beta \in P_2(L_1 \otimes L_2, M)$ for some H-modules L_1, L_2, M (for instance, think

of β as the pseudobracket of a Lie H-pseudoalgebra L for $L_1 = L_2 = M = L$).
Then $\mathcal{A}_Y(\beta) : (Y \otimes_H L_1) \otimes (Y \otimes_H L_2) \to Y \otimes_H M$ is defined as:

$$A_Y(\beta)(x \otimes_H a, y \otimes_H b) = \sum_i (xf_i)(yg_i) \otimes_H e_i \tag{2.83}$$

where $\beta(a,b) = \sum_i (f_i \otimes g_i) \otimes_H e_i$, for $a, b, e_i \in L$, $f_i, g_i \in H$, $x, y \in Y$.

Definition 2.4.1. Given a Lie H-pseudoalgebra L, its *annihilation algebra* is the H-differential algebra $\mathcal{A}(L) := \mathcal{A}_X(L) = X \otimes_H L$, where as usual $X = H^*$.

This is a Lie algebra with bracket induced by the pseudobracket of L following (2.83). Notice that the left action of H on $X \otimes_H L$ satisfies:

$$h[x \otimes_H a, y \otimes_h b] = [h_{(1)}x \otimes_H a, h_{(2)} \otimes_H b] \tag{2.84}$$

which is equivalent , when $H = U(\mathfrak{d})$, to say that $\mathfrak{d} \subset H$ acts on $\mathcal{A}(L)$ by derivations.
When L is finite, the filtration of X induces one on the annihilation algebra. Take a finite-dimensional subspace $L_0 \subset L$ such that $HL_0 = L$ and define

$$F_p\mathcal{A}(L) = \{x \otimes_H a \,|\, x \in F_pX,\ a \in L_0\}. \tag{2.85}$$

This filtration satisfies

$$[F_p\mathcal{A}(L), F_q\mathcal{A}(L)] \subset F_{p+q-l}\mathcal{A}(L) \tag{2.86}$$

for some integer l depending on the choice of L_0.
The same construction can be applied to a finite L-module M. $\mathcal{A}(M) := X \otimes_H M$ is called the *annihilation module* of M and it is an $\mathcal{A}(M)$-module with action given again by applying (2.83).
If M_0 is a finite dimensional subspace that generates M as an H-module, one can again define a filtration

$$F_p\mathcal{A}(M) = \{x \otimes_H m \,|\, x \in F_pX,\ m \in M_0\}. \tag{2.87}$$

In [BDK1, Section 7.4] it is showed that the topology induced by these filtrations does not depend on L_0 and that the following holds.

Proposition 2.4.1. *Let H be a cocommutative Hopf algebra such that the finiteness property* (2.14) *holds.*
If M is a finite H-module, then $\mathcal{A}(M)$ is a linearly compact topological H-module, meaning that the action of H is continuous (where as usual we consider the discrete topology on H).
Moreover, when M is a finite Lie H-pseudoalgebra, $\mathcal{A}(M)$ is a linearly compact Lie H-differential algebra, i.e. the action of H and the Lie bracket are continuous in respect to the linearly compact topology defined by (2.85).
Similar statements hold for representations and for associative pseudoalgebras.

We will see later that one can define the so called *reconstruction functor* that allows, under some assumptions, to determine the finite H-pseudoalgebra L from the Lie H-differential algebra $\mathcal{A}_X(L)$.
From now on, $H = U(\mathfrak{d})$, where as usual $\mathfrak{d}$ is a finite-dimensional Lie algebra.

We define the *extended annihilation algebra of* L as the semidirect sum $\mathcal{A}(L)^e = \mathfrak{d} \ltimes \mathcal{A}(L)$ where for $[\partial, y \otimes_H a] = \partial y \otimes_H a$ for $\partial \in \mathfrak{d}, y \otimes_H a \in \mathcal{A}(L)$.
We can carry on the filtration (2.85) to $\mathcal{A}(L)^e$ setting $F_p\mathcal{A}(L)^e = F_p\mathcal{A}(L)$.
An $\mathcal{A}(L)^e$-module V is called *conformal* if any $v \in V$ belongs to some

$$ker_pV := \{v \in V \mid F_p\mathcal{A}(L)v = 0\}.$$

In particular $ker_{-1}V = kerV = \{v \in V \mid \mathcal{A}(L)v = 0\}$. Notice also that ker_pV is a subspace invariant under the action of $F_0\mathcal{A}(L)$.
We state now two results [BDK2, Proposition 2.1,Lemma 2.3] that are crucial in the study of representations of finite Lie pseudoalgebras.

Proposition 2.4.2. *Any module V over a Lie pseudoalgebra L has a natural structure of a conformal $\mathcal{A}(L)^e$-module, given by the action of $\mathfrak{d}$ on V and by*

$$(x \otimes_H a)v = \sum \langle x, S(f_i g_{i_{(-1)}})\rangle g_{i_{(2)}} v_i \quad \text{where } a * v = \sum_i (f_i \otimes g_i) \otimes_H v_i \tag{2.88}$$

for $a \in L$, $x \in X$, $v \in V$.
Conversely, any conformal $\mathcal{A}(L)^e$-module V has a natural structure of an L-module given by

$$a * v = \sum_{I \in \mathbb{N}^n} (S(\partial^{(I)}) \otimes 1) \otimes_H ((x_I \otimes_H a) \cdot v). \tag{2.89}$$

Moreover, V is irreducible as a module over L if and only if it is irreducible as a module over $\mathcal{A}(L)^e$.

Lemma 2.4.1. *Let L be a Lie pseudoalgebra and V an L-module. Then, if both L and V are finite, all vectors spaces $ker_pV/kerV$ are finite-dimensional. In particular, if $kerV = \{0\}$, every $v \in V$ belongs to a finite-dimensional subspace invariant under $\mathfrak{L}_0$.*

Remark 2.4.1. Notice that if one considers V as a representation of $\mathcal{A}(L) \subset \mathcal{A}(L)^e$, the condition of being conformal is equivalent to ask for V to be a continuous representation of $\mathcal{A}(L)$ if endowed with the discrete topology.

We recover the definition of the Lie pseudoalgebra $W(\mathfrak{d})$ introduced at the end of previous section.

Definition 2.4.2. The Lie pseudoalgebra $W(\mathfrak{d})$ is the free H-module $H \otimes \mathfrak{d}$ with pseudobracket given by, for $a, b \in \mathfrak{d},\ f, g \in H$,

$$[(f \otimes a) * (g \otimes b)] = (f \otimes g) \otimes_H (1 \otimes [a, b]) - (f \otimes ga) \otimes_H (1 \otimes b) + (fb \otimes g) \otimes_H (1 \otimes a). \tag{2.90}$$

We already noticed that it seems to have some similarities with the Lie algebra of Cartan type W_N. The link between the two is explained by considering the annihilation algebra of $W(\mathfrak{d})$.
Let $\mathcal{W} = \mathcal{A}(W(\mathfrak{d}))$ be the annihilation algebra of $W(\mathfrak{d})$.
Since $W(\mathfrak{d}) = H \otimes \mathfrak{d}$, $\mathcal{W} = X \otimes_H (H \otimes \mathfrak{d}) \equiv X \otimes \mathfrak{d}$. The Lie bracket of $\mathcal{W}$ is obtained from (2.90) using (2.83):

$$[x \otimes a, y \otimes b] = xy \otimes [a, b] - x(ya) \otimes b + (xb)y \otimes a \quad \text{for } a, b \in \mathfrak{d},\ x, y \in X. \tag{2.91}$$

The action of H on $\mathcal{W}$ is given by action on the first factor, as in (2.15), and, by (2.84), $\mathfrak{d}$ acts on $\mathcal{W}$ by derivations, which allows us to define the extended annihilation algebra $\mathcal{W}^e = \mathfrak{d} \ltimes \mathcal{W}$, where

$$[\partial, x \otimes a] = \partial x \otimes a \tag{2.92}$$

for $a, \partial \in \mathfrak{d}$, $x \in X$.

Since $W(\mathfrak{d})$ is a free H-module, we can choose $L_0 = \mathbb{C} \otimes \mathfrak{d}$ and obtain an induced decreasing filtration on $\mathcal{W}$:

$$\mathcal{W}_p = F_p\mathcal{W} = F_pX \otimes_H L_0 \equiv F_pX \otimes \mathfrak{d}. \tag{2.93}$$

$\mathcal{W}_{-1} = \mathcal{W}$ and it satisfies (2.86) for $l = 0$; notice also that $\mathcal{W}/\mathcal{W}_0 \cong \mathbb{C} \otimes \mathfrak{d} \equiv \mathfrak{d}$ and that $\mathcal{W}_0/\mathcal{W}_1 \cong \mathfrak{d}^* \otimes \mathfrak{d}$.
We use (2.93) to define a filtration in $\mathcal{W}^e$ as $F_p\mathcal{W}^e = \mathcal{W}_p$.
Take the usual basis $\{\partial^{(I)}\}$ and $\{x_I\}$ of H and X.
Let $e^i_j \in \mathfrak{gl}(\mathfrak{d})$ be given by $e^i_j\partial_k = \delta^i_k\partial_j$, so that e^i_j corresponds to $x^i \otimes \partial_j$ under the isomorphism $\mathfrak{gl}(\mathfrak{d}) \cong \mathfrak{d}^* \otimes \mathfrak{d}$. Lemma 2.1.1 implies the following result.

Lemma 2.4.2. *In $\mathcal{W}$ the following identities hold:*

$$[x^i \otimes \partial_j, 1 \otimes \partial_k] = -\delta^i_k 1 \otimes \partial_j \quad mod\, \mathcal{W}_0; \tag{2.94}$$

$$[x^i \otimes \partial_j, x^l \otimes \partial_k] = \delta^l_j x^i \otimes \partial_k - \delta^i_k x^l \otimes \partial_j \quad mod\, \mathcal{W}_1. \tag{2.95}$$

Proof. By (2.91)

$$[x^i \otimes \partial_j, 1 \otimes \partial_k] = x^i \otimes [\partial_j, \partial_k] - x^i(1 \cdot \partial_j) \otimes \partial_k + (x^i\partial_k) \otimes \partial_j = -\delta^i_k \otimes \partial_k \quad mod\, \mathcal{W}_0$$

where the first term equals $0\ mod\, \mathcal{W}_0$ and the rest follows from Lemma 2.1.1.
Similarly,

$$[x^i \otimes \partial_j, x^l \otimes \partial_k] = x^ix^l \otimes [\partial_j, \partial_k] - x^i(x^l\partial_j) \otimes \partial_k + (x^i\partial_k)x^l \otimes \partial_j = \delta^l_j x^i \otimes \partial_k - \delta^i_k x^l \otimes \partial_j \quad mod\, \mathcal{W}_1.$$

where again the first term is $0\ mod\, \mathcal{W}_1$. □

Notice that the right hand sight of these identities coincides to, respectively, the standard action of $\mathfrak{gl}(\mathfrak{d})$ on $\mathfrak{d}$ and the standard Lie bracket in $\mathfrak{gl}(\mathfrak{d}) \cong \mathfrak{d}^* \otimes \mathfrak{d}$. In fact, this proves:

Corollary 2.4.1. *The map $\mathcal{W}_0/\mathcal{W}_1 \to \mathfrak{gl}(\mathfrak{d})$ defined by $x_i \otimes \partial_j\, mod\, \mathcal{W}_1 \mapsto -e^i_j \in \mathfrak{gl}(\mathfrak{d})$, is an isomorphism of Lie algebras. Moreover, under this isomorphism, the adjoint action of $\mathcal{W}_0/\mathcal{W}_1$ on $\mathcal{W}/\mathcal{W}_0$ corresponds to the standard action of $\mathfrak{gl}(\mathfrak{d})$ on $\mathfrak{d}$.*

Recall that $W(\mathfrak{d})$ acts on H by derivations.
This action is given explicitly by

$$(f \otimes a) * g = -(f \otimes ga) \otimes_H 1 \tag{2.96}$$

where $f, g \in H$, $a \in \mathfrak{d}$; this induces an action of $\mathcal{W} = X \otimes_H W(\mathfrak{d})$ on $X \otimes_H H \equiv X$:

$$(x \otimes a)y = -x(ya) \quad \text{for } a \in \mathfrak{d},\ x, y \in X. \tag{2.97}$$

Remark 2.4.2. When $\mathfrak{d}$ is an abelian Lie algebra, the Lie bracket in $\mathcal{W}$ reduces, for $x, y \in X$, $a, b \in \mathfrak{d}$ to

$$[x \otimes a, y \otimes b] = (xb)y \otimes a - x(ya) \otimes b. \tag{2.98}$$

Consider the grading of X as in Example 2.1.3 and set $G^p\mathcal{W} = (G^{p+1}X) \otimes \mathfrak{d}$. From the previous equation follows easily that $[G^p\mathcal{W}, G^q\mathcal{W}] \subset G^{p+q}\mathcal{W}$, thus we are defining a $\mathbb{Z}$-grading of depth 1 on $\mathcal{W} = X \otimes \mathfrak{d}$. It is also easy to see that $G^p\mathcal{W} \cong \mathcal{W}_p/\mathcal{W}_{p+1}$.
In particular, $G^{-1}\mathcal{W} \cong \mathfrak{d}$ and $G^0\mathcal{W} \cong \mathfrak{gl}(\mathfrak{d})$. Last isomorphism provides a structure of $\mathfrak{gl}(\mathfrak{d})$-module on $G^p\mathcal{W}$ and X via respectively adjoint action and (2.97).
A direct calculation shows that, for any $p \geq 0$, $G^pX \cong S^p(\mathfrak{d}^*)$ and $G^p\mathcal{W} \cong G^pX \otimes \mathfrak{d} \cong S^p(\mathfrak{d}^*) \otimes \mathfrak{d}$ as $\mathfrak{gl}(\mathfrak{d})$-modules. $\mathcal{W}$ is the completion of $\bigoplus_p G^p\mathcal{W}$.

We can now link the dots. Recall that we have an isomorphism (2.25) $\phi : X \to \mathcal{O}_N$ compatible with corresponding filtrations and topologies.
Recall also that by Remark 2.1.3, $\mathfrak{d}$ acts on X by continuous derivations, so we can make $\mathcal{W}$ act on $\mathcal{O}_N = \mathbb{C}[[t^1, \dots t^N]]$ by continuous derivations as well. This way we are defining a Lie algebra homomorphism $\varphi : \mathcal{W} \to W_N$ such that

$$\varphi((x \otimes a)y) = \varphi(x \otimes a)\varphi(y) \tag{2.99}$$

We recall that

$$W_N = Der(\mathcal{O}_N) = \Big\{ \sum_{i=1}^{N} f_i \frac{\partial}{\partial t_i} \,|\, f_i \in \mathbb{C}[[t^1, \dots, t^N]] \Big\}$$

and that W_N has a natural filtration given by

$$F_pW_N = \Big\{ \sum_i f_i \frac{\partial}{\partial t_i} \,|\, f_i \in \mathbb{C}[[t^1, \dots, t^n]]_k \,,\, k \leq p \Big\}$$

where $\mathbb{C}[[t^1, \dots, t^N]]_k$ is the homogeneous component of degree k.
Then the following holds.

Proposition 2.4.3.

1. *$\varphi(x \otimes a) = \varphi(x)\varphi(1 \otimes a)$ $\forall x \in X,\ a \in \mathfrak{d}$;*
2. *$\varphi(1 \otimes \partial_i) = -\frac{\partial}{\partial t_i}$ mod $F_0W(n)$;*
3. *φ is an isomorphism of Lie algebras;*
4. *$\varphi(\mathcal{W}_p) = F_pW_N$ $\forall p \geq -1$.*

Proof. *1.* and *2.* follow from (2.99) and Lemma 2.1.1.
3. and *4.* follow from the previous ones and the definitions of filtrations (1.5) and (2.93). □

Since φ preserves the filtrations that define the linearly compact topologies on $\mathcal{W}$ and W_N, we can finally claim that the annihilation algebra of $W(\mathfrak{d})$ is isomorphic to the linearly compact Lie algebra of Cartan type W_N. Moreover, this isomorphism equips W_N with an action of $\mathfrak{d}$.
Due to this association, elements of $W(\mathfrak{d})$ are called *pseudo vector fields*. One can also

establish a formalism of *pseudoforms* similar to the usual one of differential forms, which we will see in next section.
Like for subalgebras of W_N of Cartan type, one can define corresponding subalgebras of $W(\mathfrak{d})$: $S(\mathfrak{d}, \chi)$, $H(\mathfrak{d}, \chi, \omega)$ and $K(\mathfrak{d}, \theta)$, whose annihilation algebras are isomorphic to S_N, P_N (which is an extension by a one-dimensional center of H_N) and K_N from Cartan classification. These, together with $W(\mathfrak{d})$, are called *primitive pseudoalgebras*.

Remark 2.4.3. One does not need the constriction of $\mathfrak{d}$ to be finite-dimensional in order to define primitive pseudoalgebras. However, it becomes necessary once we are interested in their connection with linearly compact Lie algebras, since for their annihilation algebra to be linearly compact they need to be finite H-modules and the finiteness condition (2.14) needs to hold.
On the other hand, the same finiteness condition holds also when H is the smash product of $U(\mathfrak{d})$ with the group algebra of a finite group G. Anyhow, in this case the notion of an H-pseudoalgebra is equivalent to that of an $U(\mathfrak{d})$-pseudoalgebra with an additional action of G (see [BDK1, Section 5]) and the structure theory behaves in the same way ([BDK1, Section 13.7]). So, for our purposes, there will be no loss of generalization by considering only the case of $H = U(\mathfrak{d})$ for a finite-dimensional Lie algebra $\mathfrak{d}$.

In [BDK1] the classification of simple finite Lie H-pseudoalgebras was accomplished. The classification relies on Cartan's classification (Theorem 1.1.1) and the interplay between a finite Lie pseudoalgebra and its annihilation algebra (Proposition 2.4.1 and use of the "reconstruction functor").

Theorem 2.4.1. *Let $H = U(\mathfrak{d})$ where $\mathfrak{d}$ is a finite-dimensional Lie algebra. Then any simple finite Lie H-pseudoalgebra is isomorphic to a current pseudoalgebra $Cur_H^{H'} L'$ where $H' = U(\mathfrak{d}')$ for some subalgebra $\mathfrak{d}' \subset \mathfrak{d}$ and L' is a finite-dimensional Lie algebra or a primitive H'-pseudoalgebra.*

Notice that although Cartan's classification is clean and simple, the classification in the "pseudo" setting depends on a few parameters χ, ω and θ that arise from inequivalent actions of $\mathfrak{d}$ over the annihilation algebras.
For the purposes of this thesis, we will only focus on the primitive pseudoalgebras of type W and S, so we end this section by defining more accurately $S(\mathfrak{d}, \chi)$ and recover the (main lines of) construction of its annihilation algebra from [BDK2, Section 3.4].

Define the H-linear map $div^{\chi} : W(\mathfrak{d}) \to H$ by $div^{\chi}(\sum h_i \otimes \partial_i) = \sum h_i(\partial_i + \chi(\partial_i))$, where χ is a trace form on $\mathfrak{d}$, i.e. a linear functional $\chi : \mathfrak{d} \to \mathbb{C}$ such that $\chi([\mathfrak{d}, \mathfrak{d}]) = 0$. Then

$$S(\mathfrak{d}, \chi) := \{s \in W(\mathfrak{d}) \mid div^{\chi}(s) = 0\} \tag{2.100}$$

is a subalgebra of the Lie pseudoalgebra $W(\mathfrak{d})$.
In [BDK1, Proposition 8.1] it is shown that $S(\mathfrak{d}, \chi)$ is generated as an H-module by the elements of the form:

$$s_{ab} = (a + \chi(a)) \otimes b - (b + \chi(b)) \otimes a - 1 \otimes [a, b] \quad \text{for } a, b \in \mathfrak{d}. \tag{2.101}$$

In what follows we will assume that $\dim \mathfrak{d} = n > 2$.
Let $\mathcal{S} = \mathcal{A}(S(\mathfrak{d}, \chi)) = X \otimes_H S(\mathfrak{d}, \chi)$ be the annihilation algebra of $S(\mathfrak{d}, \chi)$.

The Lie bracket is the one of $\mathcal{W}$, since the canonical injection of $S(\mathfrak{d},\chi)$ into $W(\mathfrak{d})$ induces a Lie algebra homomorphism $\iota : \mathcal{S} \hookrightarrow \mathcal{W}$.
Explicitly, if $s = \sum h_i \otimes \partial_i \in S(\mathfrak{d}) \subset W(\mathfrak{d}) = H \otimes \mathfrak{d}$,

$$\iota(x \otimes_H s) = \sum_i x h_i \otimes \partial_i \in \mathcal{W} \equiv X \otimes \mathfrak{d}$$

Choosing $L_0 = span_{\mathbb{C}}\{s_{ab} |\, a, b \in \mathfrak{d}\}$ we get a decreasing filtration of $\mathcal{S}$ as in (2.85):

$$\mathcal{S}_p = F_{p+1}\mathcal{S} = F_{p+1}X \otimes_H L_0 \quad \text{for } p \geq -2. \tag{2.102}$$

$\mathcal{S}_{-2} = \mathcal{S}$ and it satisfies (2.86) for $l = 1$.
In [BDK1, Section 8.4] it is proven that the Lie algebras $\mathcal{S}$ and S_N are isomorphic. We would also like all the filtrations and related topologies defined on these spaces to be compatible. In order to do so, one should use φ defined before for $\mathcal{W}$, which behaves well related to the filtrations.
This can be done but carefully.
First define a map $div^{\chi} : \mathcal{W} = X \otimes \mathfrak{d} \to X$ as $div^{\chi}(\sum_i y_i \otimes \partial_i) = \sum_i y_i(\partial_i + \chi(\partial_i))$.
It is not difficult to verify that

$$div^{\chi}([A, B]) = A\, div^{\chi}(B) - B\, div^{\chi}(A) \quad \forall A, B \in \mathcal{W},$$

where the action of $\mathcal{W}$ on X is given by(2.97). This implies that

$$\overline{\mathcal{S}} = \{A \in \mathcal{W} \,|\, div^{\chi}(A) = 0\}$$

is a Lie subalgebra of $\mathcal{W}$.
In [BDK2, Section 3.4] is proven first that $\iota : \mathcal{S} \xrightarrow{\sim} \overline{\mathcal{S}}$ in such a way that $\iota(\mathcal{S}_p) = \overline{\mathcal{S}} \cap \mathcal{W}_p$ $\forall p \geq -1$ [Proposition 3.5], then that ϕ maps $\overline{\mathcal{S}}$, up to a Lie algebra automorphism ψ of W_N induced by a ring automorphism of $\mathcal{O}_N$, to $S_N \subset W_N$ [Proposition 3.6].
Finally a Lie algebra isomorphism

$$\psi^{-1}\phi\iota : \mathcal{S} \xrightarrow{\sim} S_N \subset W_N \tag{2.103}$$

such that $\mathcal{S}_p$ maps onto $S_N \cap F_p W_N$ is obtained [Corollary 3.3]. In particular we have that $\mathcal{S}_{-2} = \mathcal{S}_{-1} = \mathcal{S}$.

2.5 Pseudo De Rham complex

In the study of representation theory of simple finite Lie pseudoalgebra L, one can, almost always safely, assume that L is a primitive pseudoalgebra. In fact, by Theorem 2.4.1, L is a current pseudoalgebra over either a finite-dimensional Lie algebra $\mathfrak{g}$ or a primitive pseudoalgebra. For the first case, the study of irreducible modules is easily linked to that of the representations of $\mathfrak{g}$, which in the finite-dimensional setting are well understood.
Regarding the second case, in [D] it is shown that, apart for a single class of representations of $H(\mathfrak{d},\chi,\omega)$, all irreducible representations of $L = Cur_H^{H'} L'$, where L' is a primitive pseudoalgebra, are of the form $Cur_H^{H'} V$ where V is an irreducible L'-module.

Proposition 2.4.2 sets a correspondence between representations of a Lie pseudoalgebra L and conformal modules over its extended annihilation algebra $\mathfrak{L}^e$.
This implies in particular, when L is a primitive Lie pseudoalgebra, that any L-module V is, if endowed with the discrete topology, a continuous representation of $\mathcal{L} = \mathcal{A}(L)$. One can therefore legitimately consider the height of V as in Definition 1.2.1.
Turns out that in this setting one only obtains representations of height 1. This is basically a consequence of Lemma 2.4.1: if $\mathfrak{L}_p$ acts trivially on some element of V, so that $ker_p\, V \neq 0$, the fact that $ker_p\, V/ker\, V$ (or just $ker_p\, V$ if V is irreducible) is finite-dimensional implies that $\mathfrak{L}_1$ acts trivially on some submodule of $ker_p\, V/ker\, V$ (this follows from Lemma 1.3.1). This in particular means that V has height 1.
Consider now $L = W(\mathfrak{d})$. Because of this argument, one expects to recover the classification of irreducible W_N-modules of height 1, which are induced modules together with modules obtained from the De Rham complex. It would be therefore useful to have some similar machinery for $W(\mathfrak{d})$, which is provided by the following construction in [BDK2].

First, we give a more thorough definition of the Grassmann algebra.
Given a $\mathbb{C}$-vector space V, take the tensor algebra $T(V)$ and consider the two-sided ideal I generated by the elements of the form $v \otimes v$, $v \in V$. The quotient algebra

$$\bigwedge(V) := T(V)/I$$

is called the *Grassmann algebra of* V.
The product, induced by the tensor product $\otimes$ of $T(V)$, is called the *wedge (or exterior) product* $\wedge$.
Notice that since $v \otimes v \in I$ and $v\otimes w + w \otimes v = (v+w)\otimes(v+w) - v\otimes v - w \otimes w \in I$, we have in $\bigwedge(V)$ that $v \wedge v = 0$ and $v \wedge w = -w \wedge v$ for all $v, w \in V$.
More generally, if $\sigma \in S_k$ is a permutation, for any $x_1, \dots, x_k \in V$:

$$x_1 \wedge \dots \wedge x_k = (-1)^{\sigma} v_{\sigma(1)} \wedge \dots \wedge v_{\sigma(k)}$$

where $(-1)^{\sigma}$ is the sign of σ.
This implies in particular that $x_1 \wedge \dots \wedge x_k = 0$ whenever $x_i = x_j$ for $i \neq j$.

The natural grading of $T(V)$ induces one in $\bigwedge(V)$. Notice that I is generated by elements of degree 2, hence $T^0(V) = \mathbb{C}$, $T^1(V) = V$ embed naturally in $\bigwedge(V)$.
Explicitly, for $k \geq 2$, if V is finite-dimensional and $\{v_1, \dots, v_n\}$ is a basis of V, $\bigwedge^k(V)$ is the linear span of elements of the form $v_{i_1} \wedge \dots \wedge v_{i_k}$ for any $i_1, \dots i_k \in \{1, \dots, n\}$.

Example 2.5.1. Let V be a finite-dimensional vector space and define the space of n-forms on V with constant coefficients as $\Omega^n(V) := Hom_{\mathbb{C}}(\bigwedge^n(V), \mathbb{C})$. Since $Hom_{\mathbb{C}}(\bigwedge^n(V), \mathbb{C}) \cong \bigwedge^n(V^*)$, one can use the wedge product of $\bigwedge(V^*)$ to define one on $\Omega(V) = \bigoplus_{k=0}^{\dim V} \Omega^k(V)$.
More generally the same can be done for any ring Y with $\Omega^{\bullet}_Y(V) := Hom_{\mathbb{C}}(\bigwedge^{\bullet}(V), Y) \cong Y \otimes \bigwedge^{\bullet}(V^*)$.
$\Omega_Y(V) = \bigoplus_n \Omega^n_Y(V)$, with wedge product induced by the one in $\bigwedge(V^*)$ extended by Y-bilinearity, is the Grassmann algebra of *forms with coefficients in Y* on V.

Example 2.5.2. One can define a $\mathbb{Z}/2$-grading of $\bigwedge(V)$ setting $\deg\, v_i = \bar{1}$. This grading is *consistent* with the previous one, meaning that

$$\bigwedge(V)_{\bar{0}} = \bigoplus_{i\geq 0} \bigwedge^{2i}(V), \quad \bigwedge(V)_{\bar{1}} = \bigoplus_{i \geq 0} \bigwedge^{2i+1}(V)$$

This is called a *Grassmann superalgebra*.

Remark 2.5.1. If L is a (Lie) superalgebra and $\bigwedge(V)$ is a Grassmann algebra, $L \otimes \bigwedge(V)$ is a (Lie) superalgebra, as for instance $\bigwedge(m,n)$ in Example 1.3.2.

Let as usual $H = U(\mathfrak{d})$ for a Lie algebra $\mathfrak{d}$ of dimension N and $X = H^*$ where we identify the latter with $\mathcal{O}_N$.
Consider now the cohomology complex with constant coefficients,

$$0 \to \Omega^0(\mathfrak{d}) \xrightarrow{\mathrm{d}_0} \Omega^1(\mathfrak{d}) \xrightarrow{\mathrm{d}_0} \cdots \xrightarrow{\mathrm{d}_0} \Omega^N(\mathfrak{d}) \tag{2.104}$$

where the differential d_0 is given for $\alpha \in \Omega^k(\mathfrak{d})$ and $a_1 \dots, a_{k+1} \in \mathfrak{d}$ by

$$(\mathrm{d}_0\,\alpha)(a_1\wedge\cdots\wedge a_{k+1}) = \sum_{i<j}(-1)^{i+j}\alpha([a_i,a_j]\wedge a_1\wedge\cdots\widehat{a}_i\wedge\cdots\widehat{a}_j\wedge\cdots\wedge a_{k+1}) \tag{2.105}$$

where $\widehat{a}$ means that the term a is omitted in the wedge product.

$\Omega(\mathfrak{d})$ is a $\mathfrak{d}$-module via the coadjoint action, which can be written through the Cartan formula, for $a \in \mathfrak{d}$,

$$(ad\,a)\cdot = \mathrm{d}_0\iota_a + \iota_a\mathrm{d}_0 \tag{2.106}$$

where, for $\alpha \in \Omega^k(\mathfrak{d}),\ A \in \mathfrak{gl}(\mathfrak{d}),\ a_1,\dots,a_k \in \mathfrak{d}$:

- $\iota_a : \Omega^{k+1}(\mathfrak{d}) \to \Omega^k(\mathfrak{d})$ is the *contraction* operator, defined by

$$(\iota_a\alpha)(a_1\wedge\cdots\wedge a_k) = \alpha(a\wedge a_1\wedge\cdots\wedge a_k) \quad \alpha \in \Omega^k(\mathfrak{d}),\ a_1,\cdots,a_k \in \mathfrak{d}; \tag{2.107}$$

- the action of $\mathfrak{gl}(\mathfrak{d}) \ni ad\,a$ is the natural one, given explicitly by

$$(A\cdot\alpha)(a_1\wedge\cdots\wedge a_k) = \sum_i(-1)^i\alpha(A\cdot a_i\wedge a_1\wedge\cdots\widehat{a}_i\wedge\cdots\wedge a_k). \tag{2.108}$$

Similarly, one has the *formal de Rham complex* (where we are using the identification $X \cong \mathcal{O}_N$ to write $\Omega^k_X(\mathfrak{d}) = \Omega^k_N$)

$$0 \to \Omega^0_X(\mathfrak{d}) \xrightarrow{\mathrm{d}} \Omega^1_X(\mathfrak{d}) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^N_X(\mathfrak{d}). \tag{2.109}$$

In this case the de Rham differential $\mathrm{d} : \Omega^k_X(\mathfrak{d}) \to \Omega^{k+1}_X(\mathfrak{d})$ is given explicitly by

$$\begin{aligned}(\mathrm{d}\alpha)(a_1\wedge\cdots\wedge a_{k+1}) =& \sum_i(-1)^i\alpha(a_1\wedge\cdots\wedge\widehat{a}_i\wedge\cdots\wedge a_{k+1})a_i+\\ &\sum_{i<j}(-1)^{i+j}\alpha([a_i,a_j]\wedge a_1\wedge\cdots\wedge\widehat{a}_i\wedge\cdots\wedge\widehat{a}_j\wedge\cdots\wedge a_{k+1})\end{aligned} \tag{2.110}$$

for $\alpha \in \Omega^k_x(\mathfrak{d}),\ a_1,\dots,a_{k+1} \in \mathfrak{d}$, where the right action of $\mathfrak{d}$ on X is given by (2.16).
We can define an action of $W_N = Der\,\mathcal{O}_N \cong DerX \cong X\otimes\mathfrak{d}$ on $\Omega^k_X(\mathfrak{d}) \cong X\otimes\bigwedge^k(\mathfrak{d}^*)$ via the Lie derivative, for which still holds the Cartan formula

$$\mathcal{L}_{x\otimes a}(\alpha) = \mathrm{d}\iota_{x\otimes a}(\alpha) + \iota_{x\otimes a}(\mathrm{d}\alpha), \quad x\otimes a \in X\otimes\mathfrak{d},\ \alpha \in \Omega^k_X(\mathfrak{d}), \tag{2.111}$$

where the contraction operator is defined as $\iota_{x\otimes a}(\alpha) = x\iota_a(\alpha)$.
It is a well known and easy to check fact that $\mathcal{L}_\omega \in Der(\Omega_X(\mathfrak{d}))\ \forall\omega \in W_n$.

Our next step is to define a "pseudo" version of the De Rham complex holding similar properties as the latter.
First of all, consider the H-modules $\Omega_H^k(\mathfrak{d}) = H \otimes \bigwedge^k(\mathfrak{d}^*)$ for $0 \leq k \leq N$. These are called the spaces of *k-pseudoforms*.
We want to define a pseudoaction of $W(\mathfrak{d})$ on $\Omega_H^k(\mathfrak{d})$ and a *pseudo differential* $\mathrm{d}_* : \Omega_H^k(\mathfrak{d}) \to \Omega_H^{k+1}(\mathfrak{d})$ such that:

- $\mathrm{d}_*^2 = 0$;
- the pseudoaction of $W(\mathfrak{d})$ on $\Omega_H^k(\mathfrak{d})$ is given by a formula analogous to (2.111).

We will use the isomorphisms

$$\begin{aligned} \Omega_H^i(\mathfrak{d}) = H \otimes {\textstyle\bigwedge}^i(\mathfrak{d}^*) &\cong Hom_{\mathbb{C}}({\textstyle\bigwedge}^i(\mathfrak{d}), H) \\ h \otimes \omega &\mapsto [a_1 \wedge \cdots \wedge a_i \mapsto h\omega(a_1 \wedge \cdots \wedge a_i)] \end{aligned} \tag{2.112}$$

and define the pseudo differential on $\mathbb{C} \otimes \bigwedge^k(\mathfrak{d}^*)$, for $\omega \in \bigwedge^k(\mathfrak{d}^*)$, $a_1, \ldots, a_{k+1} \in \mathfrak{d}$, as

$$\begin{aligned} &(\mathrm{d}_*(1 \otimes \omega))(a_1 \wedge \cdots \wedge a_{k+1}) = \\ &\qquad \sum_i (-1)^i \omega(a_1 \wedge \cdots \wedge \widehat{a}_i \wedge \cdots \wedge a_{k+1}) a_i + \\ &\qquad \sum_{i<j} (-1)^{i+j} \omega([a_i, a_j] \wedge a_1 \wedge \cdots \wedge \widehat{a}_i \wedge \cdots \wedge \widehat{a}_j \wedge \cdots \wedge a_{k+1}) \end{aligned} \tag{2.113}$$

and then extend it by H-linearity to $\Omega_H(\mathfrak{d})$.
By this definition, $d_*^2 = 0$ is straightforward.
The sequence

$$0 \to \Omega_H^0(\mathfrak{d}) \xrightarrow{\mathrm{d}_*} \Omega_H^1(\mathfrak{d}) \xrightarrow{\mathrm{d}_*} \cdots \xrightarrow{\mathrm{d}_*} \Omega_H^N(\mathfrak{d}) \tag{2.114}$$

is called *pseudo De Rham complex*.

Proposition 2.5.1. *[BDK1, Proposition 5.1] The n-th cohomology of the De Rham complex is trivial for $n \neq N$ and 1-dimensional for $n = N$. In particular, the sequence* (2.114) *is exact.*

Regarding the pseudoaction of $W(\mathfrak{d})$ on $\Omega_H^k(\mathfrak{d})$, first define

$$\begin{aligned} *_\iota : W(\mathfrak{d}) \otimes \Omega_H^k(\mathfrak{d}) &\longrightarrow (H \otimes H) \otimes_H \Omega_H^{k-1}(\mathfrak{d}) \\ (f \otimes a) *_\iota (g \otimes \alpha) &= (f \otimes g) \otimes_H (1 \otimes \iota_a(\alpha)). \end{aligned} \tag{2.115}$$

Then, emulating (2.111), we define the H-bilinear map

$$\begin{aligned} * : W(\mathfrak{d}) \otimes \Omega_H^k(\mathfrak{d}) &\longrightarrow (H \otimes H) \otimes_H \Omega_H^k(\mathfrak{d}) \\ w * \beta &= ((id \otimes id) \otimes_H \mathrm{d}_*)(w *_\iota \beta) + w *_\iota \mathrm{d}_* \beta. \end{aligned} \tag{2.116}$$

It is possible to write this explicitly for $w = f \otimes a \in W(\mathfrak{d})$ and $\beta = g \otimes \alpha \in \Omega^k_H(\mathfrak{d})$ (see [BDK1, Eq (8.7)]):

$$(w * \beta)(a_1 \wedge \cdots \wedge a_k) = -(f \otimes ga)\alpha(a_1 \wedge \cdots \wedge a_k) + \sum_{i=1}^{k}(-1)^i(fa_i \otimes g)\alpha(a \wedge a_1 \wedge \cdots \wedge \widehat{a}_i \wedge \cdots \wedge a_k) + \sum_{i=1}^{n}(-1)^i(f \otimes g)\alpha([a, a_i] \wedge a_1 \wedge \cdots \wedge \widehat{a}_i \wedge \cdots \wedge a_k) \quad (2.117)$$

for $k \geq 1$, where $w * \beta = -f \otimes ga$ for $\beta = g \in \Omega^0_H(\mathfrak{d}) = H$ (the latter coinciding with (2.96)).
Notice that since $\mathrm{d}_*^2 = 0$, applying $((id \otimes id) \otimes_H \mathrm{d}_*)$ to (2.116) gives $((id \otimes id) \otimes_H \mathrm{d}_*)(w * \beta) = ((id \otimes id) \otimes_H \mathrm{d}_*)(w *_\iota \mathrm{d}_*\beta)$.
On the other hand, by replacing β with $\mathrm{d}_*\beta$, it also implies $w * \mathrm{d}_*\beta = ((id \otimes id) \otimes_H \mathrm{d}_*)(w *_\iota \mathrm{d}_*\beta)$.
Therefore $((id \otimes id) \otimes_H \mathrm{d}_*)(w * \beta) = w * \mathrm{d}_*\beta$, which means that d_* is an homomorphism of $W(\mathfrak{d})$-modules.
In [BDK2, Theorem 5.1] is proven that (2.116) is a pseudoaction of $W(\mathfrak{d})$ on $\Omega^k_H(\mathfrak{d})$ $\forall\, 0 \leq k \leq n$ by noticing that it is a *tensor module*, which we will introduce in next section.
The similarities between the De Rham complex and its pseudo counterpart are not a coincidence. Indeed, by means of the annihilation functor, from the definitions one can easily derive the following.

Lemma 2.5.1. *The W_N-module $\Omega^k_X(\mathfrak{d})$ is isomorphic to $\mathcal{A}(\Omega^k_H(\mathfrak{d}))$ as $\mathcal{A}(W(\mathfrak{d})) \cong W_N$-modules for any $k = 0, \ldots, N$ and $\mathcal{A}(\mathrm{d}_*) : \mathcal{A}(\Omega^k_H(\mathfrak{d})) \equiv \Omega^k_X(\mathfrak{d}) \longrightarrow \Omega^{k+1}_X(\mathfrak{d}) \equiv \mathcal{A}(\Omega^{k+1}_H(\mathfrak{d}))$ coincides with the de Rham differential.*

Let Π be a finite-dimensional $\mathfrak{d}$-module (hence also an H-module). We can apply to the pseudo De Rham complex the "twisting" construction given in Definition 2.3.6.
This provides a complex of $W(\mathfrak{d})$-modules, called the Π-*twisted pseudo de Rham complex*,

$$0 \to T_\Pi(\Omega^0(\mathfrak{d})) \xrightarrow{\mathrm{d}_\Pi} T_\Pi(\Omega^1(\mathfrak{d})) \xrightarrow{\mathrm{d}_\Pi} \cdots \xrightarrow{\mathrm{d}_\Pi} T_\Pi(\Omega^N(\mathfrak{d})) \quad (2.118)$$

where $\mathrm{d}_\Pi = T_\Pi(\mathrm{d}_*)$ and the action of $W(\mathfrak{d})$ are given by Proposition 2.3.3.
In [BDK2, Section 5.3] is showed that this sequence is still exact.

Before moving on to describe the classification of irreducible representations of Lie pseudoalgebras of type W and S, we make a digression which looks unrelated with the subject but will come in handy later on.
Let $\mathfrak{g}$ be a semisimple finite-dimensional Lie algebra and consider the category $Rep^{ss}(\mathfrak{g})$ (possibly infinite-dimensional) $\mathfrak{g}$-modules that are obtained as a direct sum of finite-dimensional irreducible representations of $\mathfrak{g}$, where each irreducible appears only a finite number of times (up to isomorphisms). By definition, this is a semisimple category.
We can express the isomorphism class of such a representation V by a (possibly infinite) sum $\chi(V) = \sum_{U\, irr} m_U U$ for some $m_U \in \mathbb{Z}_{\geq 0}$, which we may call the *character* of V. Clearly $\chi(V \oplus W) = \chi(V) + \chi(W)$ and it is easy to see that for an exact short sequence

$0 \to V_0 \to V_1 \to V_2 \to 0$ one has $\chi(V_1) = \chi(V_0) + \chi(V_2)$.
There is no reason to expect in general that if $V, W \in Rep^{ss}(\mathfrak{g})$, so does $V \otimes W$. However, this is true when V or W is finite-dimensional and in this case $\chi(V \otimes W) = \chi(V) \cdot \chi(W)$. This shows that the characters of finite-dimensional representations form a ring, which is known as the representation ring (see [FH]) or the Grothendieck ring of $\mathfrak{g}$ (see [SeSc]), and the characters of representations in $Rep^{ss}(\mathfrak{g})$ realize a module over the said ring.

Consider now the pseudo De Rham complex and recall that $\Omega^k_H(\mathfrak{d}) = U(\mathfrak{d}) \otimes \bigwedge^k(\mathfrak{d}^*)$. Notice also that by Proposition 2.5.1 we may also consider the exact complex

$$0 \to \Omega^0_H(\mathfrak{d}) \xrightarrow{\mathrm{d}_*} \Omega^1_H(\mathfrak{d}) \xrightarrow{\mathrm{d}_*} \cdots \xrightarrow{\mathrm{d}_*} \Omega^N_H(\mathfrak{d}) \to \mathbb{C} \to 0. \tag{2.119}$$

On one hand, one can easily see that, as a $\mathfrak{gl}(\mathfrak{d})$-module, $U(\mathfrak{d}) = \bigoplus_{n\geq 0} S^n(\mathfrak{d})$ where $S^n(\mathfrak{d})$ is the n-th symmetric power of $\mathfrak{d}$; on the other, $\bigwedge^k(\mathfrak{d}^*)$ is an irreducible $\mathfrak{gl}(\mathfrak{d})$-module of finite dimension $\binom{N}{k}$. Hence, all the spaces of k-pseudoforms belong to the above-mentioned category and we can look at their characters. Because they fit in the exact sequence given above, we have the following identity:

$$\sum_{k=0}^{N}(-1)^k\Big(\sum_{n\geq 0} S^n(\mathfrak{d})\Big)\bigwedge\nolimits^k(\mathfrak{d}^*) = \mathbb{C}.$$

In other words, $\sum_{k=0}^{N}(-1)^k \bigwedge^k(\mathfrak{d}^*)$ can be viewed as the inverse of $\sum_{n\geq 0} S^n(\mathfrak{d})$ in the character ring of $Rep^{ss}(\mathfrak{gl}(\mathfrak{d}))$. One can clearly also consider the $\mathfrak{sl}(\mathfrak{d})$-action obtained by restricting the $\mathfrak{gl}(\mathfrak{d})$ one and the previous identity is still true for $Rep^{ss}(\mathfrak{sl}(\mathfrak{d}))$.
Notice that we can replace $\mathfrak{d}$ with any irreducible $\mathfrak{gl}(\mathfrak{d})$-module U since we can embed $\mathfrak{gl}(\mathfrak{d})$ in $\mathfrak{gl}(U)$.
Notice also that if we take a finite-dimensional $\mathfrak{gl}(\mathfrak{d})$-module V, the identity still holds if we tensorize both sides by V.
The identity we will use is the following:

$$\Big(\sum_{k=0}^{N}(-1)^k \bigwedge\nolimits^k(\mathfrak{d})\Big)\Big(\textstyle\sum_{n\geq 0} S^n(\mathfrak{d}^*)V\Big) = V. \tag{2.120}$$

2.6 Irreducible representations of primitive Lie pseudoalgebras of type W and S

In a series of papers by Bakalov, D'Andrea and Kac, irreducible (finite) representations of primitive pseudoalgebras were studied and classified for $W(\mathfrak{d})$ and $S(\mathfrak{d}, \chi)$ ([BDK2]), $K(\mathfrak{d}, \theta)$ ([BDK3]) and $H(\mathfrak{d}, \chi, \omega)$ ([BDK4]). As we mentioned before, we are only interested here in cases W and S. We will sketch the main ideas in [BDK2], which are similar in a way to the theory developed in [R1].

Recall that $\mathcal{W}_0/\mathcal{W}_1 \cong \mathfrak{gl}(\mathfrak{d})$. So if we take a $\mathfrak{gl}(\mathfrak{d})$-module V, we can allow $\mathcal{W}_1$ to act on it trivially, thus obtaining a $\mathcal{W}_0$-module. We can now consider the induced $\mathcal{W}$-module $U(\mathcal{W}) \otimes_{U(\mathcal{W}_0)} V$.
In order to correlate this with the Lie pseudoalgebra $W(\mathfrak{d})$, we need to take into account the action of $\mathfrak{d}$. To do so, we consider the extended annihilation algebra $\mathcal{W}^e$.
Denote by $\mathcal{N}_{\mathcal{W}}$ the normalizer of $\mathcal{W}_p$ in $\mathcal{W}^e$.

Proposition 2.6.1 ([BDK2], Section 3.3). *$\mathcal{N}_{\mathcal{W}}$ is independent of p. There is a decomposition as a direct sum of subspaces $\mathcal{W}^e = \mathfrak{d} \oplus \mathcal{N}_{\mathcal{W}}$.*
The subalgebra $\mathcal{W}_1 \subset \mathcal{N}_{\mathcal{W}}$ acts trivially on any irreducible finite-dimensional conformal $\mathcal{N}_{\mathcal{W}}$-module and $\mathcal{N}_{\mathcal{W}}/\mathcal{W}_1 \cong \mathfrak{d} \oplus \mathfrak{gl}(\mathfrak{d})$. In particular, irreducible finite-dimensional conformal $\mathcal{N}_{\mathcal{W}}$-modules (i.e modules in which every element is killed by some $\mathcal{W}_p$) are in one-to-one correspondence with irreducible finite-dimensional $\mathfrak{d} \oplus \mathfrak{gl}(\mathfrak{d})$-modules.

In [BDK2, Section 3.5] totally analogous results are proven for $\mathcal{S}$. Denote the normalizer of $\mathcal{S}_p$ in $\mathcal{S}^e = \mathfrak{d} \ltimes \mathcal{S}$ by $\mathcal{N}_{\mathcal{S}}$.

Proposition 2.6.2. *$\mathcal{N}_{\mathcal{S}}$ is independent of p. There is a decomposition as a direct sum of subspaces $\mathcal{S}^e = \mathfrak{d} \oplus \mathcal{N}_{\mathcal{S}}$.*
The subalgebra $\mathcal{S}_1 \subset \mathcal{N}_{\mathcal{S}}$ acts trivially on any irreducible finite-dimensional conformal $\mathcal{N}_{\mathcal{S}}$-module and $\mathcal{N}_{\mathcal{S}}/\mathcal{S}_1 \cong \mathfrak{d} \oplus \mathfrak{sl}(\mathfrak{d})$. In particular, irreducible finite-dimensional conformal $\mathcal{N}_{\mathcal{S}}$-modules are in one-to-one correspondence with irreducible finite-dimensional $\mathfrak{d} \oplus \mathfrak{sl}(\mathfrak{d})$-modules.

Take now a finite-dimensional $\mathfrak{d} \oplus \mathfrak{gl}(\mathfrak{d})$-module V. Letting $\mathcal{W}_1$ act trivially on it, we can define an action of $\mathcal{N}_{\mathcal{W}}$ and induce obtaining a $\mathcal{W}^e$-module $T(V) = Ind_{\mathcal{N}_{\mathcal{W}}}^{\mathcal{W}^e} V = U(\mathcal{W}^e) \otimes_{U(\mathcal{N}_{\mathcal{W}})} V$ which can be identified, as a vector space, with $H \otimes V$ since $\mathcal{W}^e = \mathfrak{d} \oplus \mathcal{N}_{\mathcal{W}}$.
We can define an action of $W(\mathfrak{d})$ on $H \otimes V$ as follows (see [BDK1, Section 4.3]):

$$\begin{aligned}(1 \otimes \partial_i) * (1 \otimes v) =&(1 \otimes 1) \otimes_H (1 \otimes (ad\, \partial_i)v) + \sum_{k=1}^{N} (\partial_k \otimes 1) \otimes_H (1 \otimes e_i^k v) \\ &-(1 \otimes \partial_i) \otimes_H (1 \otimes v) + (1 \otimes 1) \otimes_H (1 \otimes \partial_i v)\end{aligned} \tag{2.121}$$

for $v \in V$. Here, $ad\, \partial_i$ is the adjoint action of $\mathfrak{d}$ on itself and we think of it as an element of $\mathfrak{gl}(\mathfrak{d})$. So $ad\, \partial_i$ and e_k^i act on V as a $\mathfrak{gl}(\mathfrak{d})$-module, while ∂_i acts on V as a $\mathfrak{d}$-module.
$T(V)$ is called a *tensor module* for $W(\mathfrak{d})$.

Remark 2.6.1. Once we have identified W_N with $\mathcal{W} = X \otimes \mathfrak{d}$, if we drop the action of $\mathfrak{d}$ on a tensor module $T(V)$, we obtain exactly an induced module of height 1 of W_N as in Section 1.2.
One then expects to obtain a similar classification.

Remark 2.6.2. When $\mathfrak{d}$ is an abelian Lie algebra, if we take a $\mathfrak{gl}(\mathfrak{d})$-module and make it a $\mathfrak{d} \oplus \mathfrak{gl}(\mathfrak{d})$-module with trivial action of $\mathfrak{d}$, the pseudoaction of $W(\mathfrak{d})$ becomes

$$(1 \otimes \partial_i) * (1 \otimes v) = \sum_{k=1}^{N} (\partial_k \otimes 1) \otimes_H (1 \otimes e_i^k v) - (1 \otimes \partial_i) \otimes_H (1 \otimes v). \tag{2.122}$$

Definition 2.6.1. Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be Lie algebras and let V_i be $\mathfrak{g}_i$-modules for $i = 1, 2$.
We denote by $V_1 \boxtimes V_2$ the $\mathfrak{g}_1 \oplus \mathfrak{g}_2$-module $V_1 \otimes V_2$ where $\mathfrak{g}_i$ only acts on the V_i factor.

If V is of the form $\Pi \boxtimes U$, we will also denote $T(V) = T(\Pi, U)$.

Remark 2.6.3. By definitions, one can easily check that $T(\Pi, U) \cong T_\Pi(U)$, where $T_\Pi(U)$ is the Π-twisting defined in Section 2.3.

We can define on a tensor module $T(V) = H \otimes V$ the filtration

$$F^p T(V) = F^p H \otimes V \quad \text{for } p \geq -1, \tag{2.123}$$

which behaves nicely relatively to the filtration of $\mathcal{W}$.

Lemma 2.6.1. *[BDK2, Lemma 6.3] For every $p \geq 0$ we have:*

1. $\mathfrak{d} \cdot F^p T(V) \subset F^{p+1} T(V)$;
2. $\mathcal{N}_{\mathcal{W}} \cdot F^p T(V) \subset F^p T(V)$;
3. $\mathcal{W}_1 \cdot T(V) \subset F^{p-1} T(V)$.

Tensor modules play a central role in the classification of irreducible $W(\mathfrak{d})$-modules (as well as $S(\mathfrak{d})$-modules), together with the notion of *singular vectors*, in a similar fashion as induced modules do in the representation theory of Lie algebras of Cartan type.
Recall that by Proposition 2.4.2, any $W(\mathfrak{d})$-module has also a structure of a $\mathcal{W}^e$-module.

Definition 2.6.2. For a $W(\mathfrak{d})$-module V, a *singular vector* is an element $v \in V$ such that $\mathcal{W}_1 \cdot v = 0$. The space of singular vectors in V is denoted by $sing\, V$.
If we consider it as an $\mathcal{N}_{\mathcal{W}}$-module, since $\mathcal{N}_{\mathcal{W}}/\mathcal{W}_1 \cong \mathfrak{d} \oplus \mathfrak{gl}(\mathfrak{d})$ we have an action of the latter on $sing\, V$, which we denote by $\rho_{sing} : \mathfrak{d} \oplus \mathfrak{gl}(\mathfrak{d}) \to \mathfrak{gl}(sing\, V)$.

Theorem 2.6.1. *For any non-trivial finite $W(\mathfrak{d})$-module V, $sing\, V \neq \{0\}$ and $sing\, V/kerV$ is finite-dimensional.*

Proof. Since clearly $ker\, V \subset sing\, V$, we can assume without loss of generality that $ker\, V = 0$. Since V is a conformal $\mathcal{W}^e$-module, there exist a $p > 0$ such that $ker_p V \neq \{0\}$. This space, by Lemma 2.4.1, is finite-dimensional. Choose a minimal $\mathcal{N}_{\mathcal{W}}$-submodule $W \subset V$. Since it is an irreducible $\mathcal{N}_{\mathcal{W}}$-module, we know by Proposition 2.6.1 that $\mathcal{W}_1$ acts trivially on it. Hence, $W \subset sing\, V$.
The second statement follows from Lemma 2.4.1 for $p = 1$. □

One of the advantages of working in the pseudo framework is that one has a nice control over some characteristics of singular vectors. For instance, one has a direct estimate on their "degree".

Lemma 2.6.2. *Given a finite $W(\mathfrak{d})$-module V, $v \in V$ is a singular vector if and only if*

$$(1 \otimes \partial) * v \in (F^1 H \otimes \mathbb{C}) \otimes_H V \quad \forall \partial \in \mathfrak{d} \tag{2.124}$$

or equivalently

$$(1 \otimes \partial) * v \in (\mathbb{C} \otimes F^1 H) \otimes_H V \quad \forall \partial \in \mathfrak{d}. \tag{2.125}$$

Proof. If $\mathcal{W}_1 v = 0$ we have, by Proposition 2.4.2 that

$$(1 \otimes \partial) * v = (1 \otimes 1) \otimes_H ((1 \otimes \partial)v) - \sum_{k=1}^{N} (\partial_k \otimes 1) \otimes_H ((x_k \otimes \partial)v).$$

Conversely, by Lemma 2.1.2, $(1 \otimes \partial) * v$ can be written uniquely as an element of the form $\sum_{I \in \mathbb{N}^N} (\partial^{(I)} \otimes 1) \otimes_h v_I$ for some $v_I \in V$. By hypothesis, $v_I = 0$ when $|I| \geq 2$. Applying again Proposition 2.4.2, we obtain

$$(x \otimes \partial)v = \sum_{I : |I| < 2} \langle x, S(\partial^{(I)}) \rangle v_{(I)}$$

which is 0 when $(x \otimes \partial) \in \mathcal{W}_1$, since in that case $x \in F_1 X$.
The other equivalence is proven analogously. □

It is possible to write explicitly the pseudoaction of $W(\mathfrak{d})$ on a singular vector.

Lemma 2.6.3 ([BDK1]). *Let V be a $W(\mathfrak{d})$-module. Then if $v \in sing\, V$, the action of $W(\mathfrak{d})$ on v is given by*

$$(1 \otimes \partial_i) * v = \sum_{k=1}^{N} (\partial_k \otimes 1) \otimes_H \rho_{sing}(e_i^k)v - (1 \otimes 1) \otimes_H \partial_i v$$
$$+ (1 \otimes 1) \otimes_H \rho_{sing}(\partial_i + ad\, \partial_i)v. \qquad (2.126)$$

Corollary 2.6.1. *Let V be a $W(\mathfrak{d})$-module and let U be a non-trivial $\mathfrak{d} \oplus \mathfrak{gl}(\mathfrak{d})$-submodule of $sing\, V$. Then HU, the H-submodule of V generated by U, is also a $W(\mathfrak{d})$-submodule of V. In particular, if V is irreducible, $V = HU$.*

Proof. By the previous Lemma, we have that $W(\mathfrak{d}) * U \subset (H \otimes H) \otimes_H HU$. Then by H-bilinearity, $W(\mathfrak{d}) * HU \subset (H \otimes H) \otimes_H HU$. □

For $S(\mathfrak{d}, \chi)$ we follow the same construction.
Let V be an $\mathcal{S}$-module and denote by $ker_p\, V$ the space of elements $v \in V$ that are killed by $\mathcal{S}_p$; in particular, $ker\, V = ker_{-1}\, V = \{v \in V \,|\, \mathcal{S}v = 0\}$. V is said to be *conformal* if $V = \bigcup_p ker_p\, V$. Also in this case, by Proposition 2.4.2, any $S(\mathfrak{d}, \chi)$-module carries a structure of conformal $\mathcal{S}^e = \mathfrak{d} \ltimes \mathcal{S}$ and vice versa.

Definition 2.6.3. For a $S(\mathfrak{d}, \chi)$-module V, a *singular vector* is an element $v \in V$ such that $\mathcal{S}_1 \cdot v = 0$. The space of singular vectors in V is denoted by $sing\, V$.
If we consider it as a $\mathcal{N}_{\mathcal{S}}$-module, since $\mathcal{N}_{\mathcal{S}}/\mathcal{S}_1 \cong \mathfrak{d} \oplus \mathfrak{sl}(\mathfrak{d})$, we have an action of the latter on $sing\, V$, which we denote again by $\rho_{sing} : \mathfrak{d} \oplus \mathfrak{sl}(\mathfrak{d}) \to \mathfrak{gl}(sing\, V)$.

Theorem 2.6.2. *For any non-trivial finite $S(\mathfrak{d}, \chi)$-module V, $sing\, V \neq \{0\}$ and $sing\, V/kerV$ is finite-dimensional.*

Proof. The proof is the same of Theorem 2.6.1, whereas we apply Proposition 2.6.2 instead of Proposition 2.6.1. □

Recall that $S(\mathfrak{d}, \chi)$ is generated as an H-module by $s_{ij} = s_{\partial_i \partial_j}$ defined in (2.101).

Lemma 2.6.4. *Given a finite $S(\mathfrak{d}, \chi)$-module V, $v \in V$ is a singular vector if and only if*

$$(1 \otimes s_{ij}) * v \in (F^2 H \otimes \mathbb{C}) \otimes_H V \quad \forall i, j = 1, \ldots, N, \qquad (2.127)$$

or equivalently

$$(1 \otimes s_{ij}) * v \in (\mathbb{C} \otimes F^2 H) \otimes_H V \quad \forall i, j = 1, \ldots, N. \qquad (2.128)$$

Proof. Same proof as in Lemma 2.6.2 , whereas here the filtration $\{\mathcal{S}_p\}$ satisfies (2.86) for $l = 1$ instead of $l = 0$ for $\{\mathcal{W}_p\}$. □

Take now a finite-dimensional $\mathfrak{d}\oplus\mathfrak{sl}(\mathfrak{d})$-module V and let $\mathcal{S}_1$ act trivially on it so that we have an action of $\mathcal{N}_\mathcal{S}$. Consider then the $\mathcal{S}^e$-module $T_\chi(V) = Ind_{\mathcal{N}_\mathcal{S}}^{\mathcal{S}^e} V = U(\mathcal{S}^e) \otimes_{\mathcal{N}_{U(\mathcal{S})}} V$, which can again be identified, as a vector space, with $H \otimes V$.
In [BDK2, Theorem 7.3] it is proven that these modules can be obtained as the restriction of tensor modules for $W(\mathfrak{d})$, therefore we will call them again *tensor modules* for $S(\mathfrak{d}, \chi)$. H acts by left multiplication on the first factor and the action of $S(\mathfrak{d}, \chi)$ is the restriction of (2.121).
If V is of the form $\Pi \boxtimes U$, we will also the notation $T_\chi(V) = T_\chi(\Pi, U)$. In addition, we can also denote $T_\chi(V) = T_\chi(\Pi, U, c)$ when we think of U as the $\mathfrak{sl}(\mathfrak{d})$-module obtained by the restriction of an action of $\mathfrak{gl}(\mathfrak{d})$ for which $id \in \mathfrak{gl}(\mathfrak{d})$ act by scalar multiplication for $c \in \mathbb{C}$.
We can define the same filtration we defined in the $W(\mathfrak{d})$ case and have the same nice behaviour relatively to the filtration of $\mathcal{S}$.
One also has the analogue of Corollary 2.6.1.

Theorem 2.6.3. *[BDK2, Theorem 7.2] Let V be a $S(\mathfrak{d}, \chi)$-module and let U be a non-trivial $\mathfrak{d} \oplus \mathfrak{sl}(\mathfrak{d})$-submodule of $sing\, V$. Then HU, the submodule generated by U, is also a $S(\mathfrak{d}, \chi)$-submodule of V. In particular, if V is irreducible, $V = HU$.*

We summarize all the main results about tensor modules for $W(\mathfrak{d})$ and $S(\mathfrak{d}, \chi)$ and singular vectors in [BDK2, Section 7].
What happens, roughly speaking, is that the tensor modules are all irreducible but the spaces of pseudoforms. The existence of the pseudo De Rham complex in fact provides non trivial submodules via d_*.
In particular, since d_* is a morphism of $W(\mathfrak{d})$-modules and since constant vectors are always singular, one has for $\alpha \in \Omega^n(\mathfrak{d}) = \bigwedge^n \mathfrak{d}^*$, $0 \le n \le N-1$, that $\mathrm{d}_*(1 \otimes \alpha)$ is a singular vector in $F^1(T(\bigwedge^{n+1} \mathfrak{d})) = F^1 H \otimes \bigwedge^{n+1} \mathfrak{d}^*$.
We consider on $\Omega^n(\mathfrak{d}) = \bigwedge^n \mathfrak{d}^*$ the natural action of $\mathfrak{gl}(\mathfrak{d})$ (and of $\mathfrak{sl}(\mathfrak{d})$), where $\Omega^0(\mathfrak{d})$ is the trivial module $\mathbb{C}$.
Notice that we can restrict the action of $W(\mathfrak{d})$ on the twisted pseudo de Rham complex to one of $S(\mathfrak{d}, \chi)$, which provides a complex of $S(\mathfrak{d}, \chi)$-modules.

Theorem 2.6.4.

- *Every irreducible finite $W(\mathfrak{d})$-module is a quotient of a tensor module;*
- *Let Π (resp. U) be an irreducible finite-dimensional module over $\mathfrak{d}$ (resp. $\mathfrak{gl}(\mathfrak{d})$), then the $W(\mathfrak{d})$-module $T(\Pi, U)$ is irreducible if and only if U is not isomorphic to $\Omega^n(\mathfrak{d})$ for any $n \ge 1$.*
- *If $V = T(\Pi, U)$ with U not isomorphic to $\Omega^n(\mathfrak{d})$ for any $n \ge 1$, then $sing\, V = F^0 T(\Pi, U)$.*
- *If $V = T(\Pi, \Omega^n)$, $n \ne 0$, then $sing\, V \subset F^1 V$.*

Theorem 2.6.5.

- *Every irreducible finite $S(\mathfrak{d}, \chi)$-module is a quotient of a tensor module;*

- *Let* Π *(resp.* U*) be an irreducible finite-dimensional module over* $\mathfrak{d}$ *(resp.* $\mathfrak{sl}(\mathfrak{d})$*). Then the* $S(\mathfrak{d})$*-module* $T_\chi(\Pi, U)$ *is irreducible if and only if* U *is not isomorphic to* $\Omega^n(\mathfrak{d})$ *for any* $n \geq 0$*;*
- *If* $V = T_\chi(\Pi, U)$ *where* U *is not isomorphic to* $\Omega^n(\mathfrak{d})$ *for any* $n \geq 0$*, then* $sing\, V = F^0 T_\chi(\Pi, U)$*.*
- *If* $V = T_\chi(\Pi, \Omega^n)$*,* $n \neq 1$*, then* $sing\, V \subset F^1 V$*;*
- *If* $V = T_\chi(\Pi, \Omega^1)$*, then* $F^1 V \subsetneq sing\, V \subset F^2 V$*.*

Because of Remark 2.6.3, once one has this results, understanding the irreducible submodules of the tensor modules of De Rham type leads to the complete classification of irreducible finite modules.
It turns out that the images of d_Π are the unique proper $W(\mathfrak{d})$-submodules of $T(\Pi, \Omega^n(\mathfrak{d}))$ for any $1 \leq n \leq N-1$ and it is still minimal (hence irreducible) for $n = N$.
The same goes for the restriction $T_\chi(\Pi, \Omega^n(\mathfrak{d}))$ of $T(\Pi, \Omega^n(\mathfrak{d}))$ to the action of $S(\mathfrak{d}, \chi)$, except for $n = 1$, in which case $\mathrm{d}_\Pi T_\chi(\Pi, \Omega^0)$ has a proper submodule isomorphic to $\mathrm{d}_{\Pi'} T_\chi(\Pi', \Omega^{N-1})$ where $\Pi' = \Pi \otimes \mathbb{C}_{tr\, ad - \chi}$. Here $\mathbb{C}_\sigma$, $\sigma \in \mathfrak{d}^*$, is the 1-dimensional $\mathfrak{d}$-module where $a \in \mathfrak{d}$ acts as multiplication by the scalar $\sigma(a)$. We can now finally state the classification theorems.

Theorem 2.6.6. *Any irreducible finite* $W(\mathfrak{d})$*-module is isomorphic to one of the following:*

- *Tensor modules* $T(\Pi, U)$ *where* Π *is an irreducible finite-dimensional* $\mathfrak{d}$*-module and* U *is a finite-dimensional irreducible* $\mathfrak{gl}(\mathfrak{d})$*-module not isomorphic to* $\Omega^k(\mathfrak{d})$ *for any* $k \geq 1$*. In this case,* $sing\, T(\Pi, U) \cong \Pi \boxtimes (U \otimes \mathbb{C}_{tr})$ *as* $\mathfrak{d} \oplus \mathfrak{gl}(\mathfrak{d})$*-modules.*
- *Images* $\mathrm{d}_\Pi T(\Pi, \Omega^k(\mathfrak{d}))$ *where* Π *is an irreducible finite-dimensional* $\mathfrak{d}$*-module and* $1 \leq k \leq N-1$*. In this case,* $sing\, \mathrm{d}_\Pi T(\Pi, \Omega^k(\mathfrak{d})) \cong \Pi \boxtimes (\bigwedge^k(\mathfrak{d}^*) \otimes \mathbb{C}_{tr})$ *as* $\mathfrak{d} \oplus \mathfrak{gl}(\mathfrak{d})$*-modules.*

Theorem 2.6.7. *Any irreducible finite* $S(\mathfrak{d}, \chi)$*-module is isomorphic to one of the following:*

- *Tensor modules* $T_\chi(\Pi, U, 0)$ *where* Π *is an irreducible finite-dimensional* $\mathfrak{d}$*-module and* U *is a finite-dimensional irreducible* $\mathfrak{sl}(\mathfrak{d})$*-module not isomorphic to* $\Omega^k(\mathfrak{d})$ *for any* $k \geq 0$*. In this case,* $sing\, T_\chi(\Pi, U, 0) \cong (\Pi \otimes \mathbb{C}_{-\chi/N}) \boxtimes U$ *as* $\mathfrak{d} \oplus \mathfrak{sl}(\mathfrak{d})$*-modules.*
- *Images* $\mathrm{d}_\Pi T_\chi(\Pi, \Omega^k(\mathfrak{d}))$ *where* Π *is an irreducible finite-dimensional* $\mathfrak{d}$*-module and* $1 \leq k \leq N-1$*. In this case,* $sing\, \mathrm{d}_\Pi(T_\chi(\Pi, \Omega^k(\mathfrak{d}))) \cong (\Pi \otimes \mathbb{C}_{-\chi + k(\chi - tr\, ad)/N}) \boxtimes \bigwedge^k(\mathfrak{d}^*)$ *as* $\mathfrak{d} \oplus \mathfrak{sl}(\mathfrak{d})$*-modules.*

In both cases we will refer to irreducible modules obtained as images of d_Π as irreducible modules *of De Rham type*.

Chapter 3

Pseudoalgebraic approach to the representation theory of $E(5, 10)$

3.1 Generalized Verma modules for E(5,10)

As was explained at the end of Section 1.3, the representation theory of (among other Lie superalgebras) $E(5, 10)$ can be reduced to the study of generalized Verma modules. We briefly review now the main ideas applied to $E(5, 10)$ following [R2](see also [KR4, CC]). Fix $\mathfrak{d} = (\mathbb{C}^5)^*$, $\{\partial_1, \dots, \partial_5\}$ and $\{x_1, \dots, x_5\}$ dual bases of $\mathfrak{d}$ and $\mathfrak{d}^*$, and set as usual $H = U(\mathfrak{d})$ and $X = H^*$.
Recall that $E(5, 10)_{(0)} = S_5$, $E(5, 10)_{(1)} = \mathrm{d}\Omega^1(5)$ and that $\mathfrak{L} = E(5, 10) = \bigoplus_{i \geq -2} \mathfrak{L}_i$ has a transitive, irreducible, consistent $\mathbb{Z}$-grading of depth 2 such that $\mathfrak{L}_0 \cong \mathfrak{sl}(\mathfrak{d})$.
Recall also that we set the notation $\mathfrak{L}_{\geq 0} = \bigoplus_{i \geq 0} \mathfrak{L}_i$ and $\mathfrak{L}_+ = \bigoplus_{i > 0} \mathfrak{L}_i$.
Given an $\mathfrak{sl}(\mathfrak{d}) \cong \mathfrak{L}_0$-module V, we can extend it to a $\mathfrak{L}_{\geq 0}$-module by letting $\mathfrak{L}_+$ act trivially on it; we can then consider the induced $\mathfrak{L}$-module

$$\mathfrak{T}(V) = U(\mathfrak{L}) \otimes_{U(\mathfrak{L}_{\geq 0})} V \tag{3.1}$$

where the action of $\mathfrak{L}$ is given by left multiplication.

Definition 3.1.1. Let V be an $\mathfrak{sl}(\mathfrak{d})$-module. The $\mathfrak{L}$-module $\mathfrak{T}(V)$ is called *generalized Verma module*.
If V is a finite-dimensional irreducible $\mathfrak{sl}(\mathfrak{d})$-module, we call $\mathfrak{T}(V)$ *minimal*.
A minimal Verma module is called *non-degenerate* if it is irreducible, *degenerate* otherwise.

Remark 3.1.1. Let us notice that, as vector spaces, $\mathfrak{T}(V) \cong U(\mathfrak{L}_-) \otimes V$. We will often use this isomorphism omitting the subscript $U(\mathfrak{L}_{\geq 0})$ on the tensor product.

When $V = V(\lambda)$ is an irreducible $\mathfrak{sl}(\mathfrak{d})$-module of highest weight λ, we may use the notation $\mathfrak{T}(V) = \mathfrak{T}(\lambda)$. A dominant weight λ for $\mathfrak{sl}(\mathfrak{d})$ will be expressed in terms of a quadruple $[a_1, a_2, a_3, a_4] \in \mathbb{Z}^4_{\geq 0}$ where $\lambda = a_1\omega_1 + \cdots + a_4\omega_4$ and $\omega_1, \dots, \omega_4$ are the fundamental weights of $\mathfrak{sl}(\mathfrak{d})$.
One should pay special attention because, since we set $\mathfrak{d} = (\mathbb{C}^5)^*$, all the highest weights modules are the duals of the $\mathfrak{sl}_5$ usual ones. For example, in our notation the $\mathfrak{sl}_5$ standard representation, which has highest weight $[1, 0, 0, 0]$, will be $\mathfrak{d}^*$.

The grading of $U(\mathfrak{L}_-)$ induces one on $\mathfrak{T}(V)$.
Poincaré-Birkhoff-Witt Theorem is still true in the superalgebra setting (see for example [M, 6.1]), so fixed an ordered basis $\{\partial_1 \dots, \partial_5, \xi_{12}, \dots, \xi_{45}\}$ of $\mathfrak{L}_-$, we can choose as a PBW-basis for $U(\mathfrak{L}_-)$ the monomials $\partial^{(I)}\xi^K$ where,

$$\partial^{(I)} = \frac{\partial_1^{i_1}}{i_1!} \cdots \frac{\partial_5^{i_5}}{i_5!}, \quad I = (i_1, \dots, i_5) \in \mathbb{N}^5 \tag{3.2}$$

$$\xi^K = \xi_{12}^{k_{12}} \cdots \xi_{45}^{k_{45}}, \quad K = (k_{12}, \dots, k_{45}) \in \{0,1\}^{10}. \tag{3.3}$$

The basis elements are, by definition of the grading, homogeneous of degree $p = 2|I|+|K|$, where $|I| = i_1 + \cdots + i_5$ and $|K| = k_{12} + \cdots + k_{45}$; they generate the homogeneous subspaces $U^p(\mathfrak{L}_-)$. We can thus equip $\mathfrak{T}(V)$ with a grading $\mathfrak{T}^p(V) = U^p(\mathfrak{L}_-) \otimes V$.
We should notice that this grading and the grading of $\mathfrak{L}$ are compatible, by which we mean that

$$\mathfrak{L}_n \mathfrak{T}^p(V) \subseteq \mathfrak{T}^{p-n}(V). \tag{3.4}$$

We will call the elements of $\mathfrak{T}^p(V)$ *homogeneous vectors of degree* p; in particular, we will call the degree 0 ones *constant*.
For instance, $\mathfrak{T}^0(V) = \mathbb{C} \otimes V$, $\mathfrak{T}^1(V) = \mathfrak{s} \otimes V$, $\mathfrak{T}^2(V) = \mathfrak{d} \otimes V + \bigwedge^2(\mathfrak{s}) \otimes V$, $\mathfrak{T}^3(V) = \mathfrak{d}\mathfrak{s} \otimes V + \bigwedge^3(\mathfrak{s}) \otimes V$, etc.
If $v \in S^n(\mathfrak{d}) \bigwedge^m(\mathfrak{s}) \otimes V$, to keep track of the even and odd degrees, we will say that v has degree $(n|m)$.

Remark 3.1.2. Take a generalized Verma module $\mathfrak{T}(V) = U(\mathfrak{L}_-) \otimes V = U(\mathfrak{d} + \mathfrak{s}) \otimes V$. It is in particular an S_5-module and, in view of (2.103), also an $\mathcal{S}$-module. Furthermore, considering the action of $\mathfrak{d}$ as left multiplication in $U(\mathfrak{d} + \mathfrak{s})$, we can view $\mathfrak{T}(V)$ as a $\mathcal{S}^e$-module. Since all these identifications are compatible with the filtrations, (3.4) implies that $\mathfrak{T}(V)$ is a conformal $\mathcal{S}^e$-module. By Proposition 2.4.2, it has a natural structure of an $S(\mathfrak{d}, \chi)$-module.

Just like the presence of irregular submodules in induced modules for Lie algebras of Cartan type depended on the existence of singular vectors, degeneracy of minimal Verma modules can be reformulated in terms of *singular vectors*.

Definition 3.1.2. Let $\mathfrak{T}(V)$ be a Verma module. $v \in \mathfrak{T}(V)$ is called a *singular vector* if $\mathfrak{L}_1 v = 0$.
The space of singular vectors will be denoted by $sing\,\mathfrak{T}(V)$.

Example 3.1.1. Any constant vector $v \in \mathfrak{T}^0(V)$ is singular, since in that case $\mathfrak{L}_1 v \in \mathfrak{T}^{-1}(V) = 0$.

Remark 3.1.3. Take $v \in sing\,\mathfrak{T}(V)$ and $z \in \mathfrak{L}_0$. Then, for any $y \in \mathfrak{L}_1$, $yv = 0$ and so

$$y(zv) = [y,z]v + z(yv) = [y,z]v = 0$$

where the last identity follows from the singularity of v and the fact that $[\mathfrak{L}_1, \mathfrak{L}_0] \subseteq \mathfrak{L}_1$.
In other terms, $sing\,\mathfrak{T}(V)$ is a $\mathfrak{L}_0$-submodule of $\mathfrak{T}(V)$. In particular, since $\mathfrak{T}(V) = \bigoplus_p \mathfrak{T}^p(V)$ as $\mathfrak{L}_0$-modules, homogeneous components of a singular vector are singular. We will always assume that a singular vector is homogeneous.
The same holds for the weight components of a singular vector so we will assume, whenever possible, that a singular vector is also a weight vector.

Proposition 3.1.1. *A minimal Verma module $\mathfrak{T}(V)$ is degenerate if and only if it contains non constant singular vectors.*

Proof. Assume $0 \neq v \in sing\mathfrak{T}^p(V)$ for some $p > 0$. Since $\mathfrak{L}_-$ and $\mathfrak{L}_0$ can only respectively rise the degree of v or preserve it and since $\mathfrak{L}_+.v = 0$ by assumption, $\mathfrak{L}v$ is an $\mathfrak{L}$-submodule of $\mathfrak{T}(V)$, which is proper because it only contains vectors of degree $\geq p$.
Vice versa, let $W \subset \mathfrak{T}(V)$ be a non trivial proper $\mathfrak{L}$-submodule and take $0 \neq w \in W$. Since the action of $\mathfrak{L}_1$ lowers strictly the degree of homogeneous components of w, we know that eventually $(\mathfrak{L}_1)^n w = 0$ for some finite $n \geq 1$; thus we can assume without loss of generalization that w is singular. Now, if w was constant, by irreducibility of V we would have $\mathfrak{L}_0 w = \mathbb{C} \otimes V$ and therefore, by iterated action of $\mathfrak{L}_-$, we would obtain $\mathfrak{L}w = \mathfrak{T}(V)$. But $\mathfrak{T}(V) \supsetneq W \supseteq \mathfrak{L}w = \mathfrak{T}(V)$, a contradiction. Hence w is a non constant singular vector. □

The proof of the proposition shows vividly how singular vectors "detect" degeneracy of minimal Verma modules.

Example 3.1.2. Let $V = V([0,0,0,1]) \cong \mathfrak{d}$ and let $v = \sum_i \xi_{1i} \otimes \partial_i \in \mathfrak{T}(\mathfrak{d})$.
A generic element of $\mathfrak{L}_1$ is of the form $y = x_h\xi_{kl} + x_k\xi_{hl}$ for some $h, k, l \in \{1, \ldots, 5\}$. To check if v is singular, we can carry out the computation:

$$\begin{aligned} y \cdot v = &\sum_i x_h[\xi_{kl}, \xi_{1i}] \otimes \partial_i + x_k[\xi_{hl}, \xi_{1i}] \otimes \partial_i = \\ &\sum_i \varepsilon_{(kl1i)} \otimes (x_h\partial_{(kl1i)})\partial_i + \varepsilon_{(hl1i)} \otimes (x_k\partial_{(hl1i)})\partial_i = \\ &\sum_i -\varepsilon_{(kl1i)} \otimes e^h_{(kl1i)}\partial_i - \varepsilon_{(hl1i)} \otimes e^k_{(hl1i)}\partial_i = \\ &\sum_i -\varepsilon_{(kl1i)} \otimes \delta^h_i\partial_{(kl1i)} - \varepsilon_{(hl1i)} \otimes \delta^k_i\partial_{(hl1i)} = \\ &- \varepsilon_{(kl1h)} \otimes \partial_{(kl1h)} - \varepsilon_{(hl1k)} \otimes \partial_{(hl1k)} = \\ &- \varepsilon_{(kl1h)} \otimes \partial_{(kl1h)} + \varepsilon_{(kl1h)} \otimes \partial_{(kl1h)} = 0. \end{aligned}$$

So $v \in sing\,\mathfrak{T}^1(V)$.
Alternatively, one could notice that v is an highest weight vector, therefore realize that it is sufficient to check that the lowest weight vector of $\mathfrak{L}_1$, $x_5\xi_{45}$, acts trivially on v, which is clearly easier (or at least shorter) (see [CC, Ch.3]).

In [KR4] Kac and Rudakov built complexes of modules for $E(5,10)$ which provided families of degenerate Verma modules and they conjectured there were no others.

Conjecture 3.1.1. The complete list of degenerate minimal Verma modules for $E(5,10)$ is given, for $m, n \in \mathbb{Z}_{\geq 0}$, by $\mathfrak{T}(\lambda)$ for $\lambda = [m,n,0,0]$, $[0,0,m,n]$ and $[m,0,0,n]$.

In [R2] this conjecture was refined in terms of singular vectors and *morphism of positive degree*.
Given a morphism of $\mathfrak{L}$-modules $\varphi : \mathfrak{T}(V) \to \mathfrak{T}(W)$, one can associate to it $\phi \in U(\mathfrak{L}_-) \otimes Hom(V,W)$ such that $\varphi(1 \otimes v) = \phi(v)$ for any $v \in V$.

One says that φ has degree k when $\phi = \sum_i u_i \otimes A_i$ for some $u_i \in U(\mathfrak{L}_-)$, $A_i \in Hom(V,W)$ and all $u_i \in U^k(\mathfrak{L}_-)$.
One can characterize morphisms of Verma modules by the following proposition.

Proposition 3.1.2 ([R2])**.** *Consider a linear map $\varphi : \mathfrak{T}(V) \to \mathfrak{T}(W)$ and consider the associated element $\phi \in U(\mathfrak{L}_-) \otimes Hom(V,W)$. φ is a morphism of $\mathfrak{L}$-modules if and only if $\mathfrak{L}_0\phi = 0$ and $y\phi(v) = 0$ for any $v \in V, y \in \mathfrak{L}_1$.*

Now one can restate the degeneracy of a minimal Verma module with the existence of positive degree morphisms.

Proposition 3.1.3. *Let $\mathfrak{T}(V)$ be a minimal Verma module. Then the following conditions are equivalent.*

1. *$\mathfrak{T}(V)$ is degenerate.*
2. *$\mathfrak{T}(V)$ contains non constant singular vectors.*
3. *there exists a minimal Verma module $\mathfrak{T}(U)$ and a morphism $\varphi : \mathfrak{T}(U) \to \mathfrak{T}(V)$ of degree $k > 0$.*

Proof. We already know that the first two conditions are equivalent. If $\varphi : \mathfrak{T}(U) \to \mathfrak{T}(V)$ is a morphism of degree $k > 0$, then, for any $u \in U$, $\varphi(1 \otimes u)$ is a singular vector in $\mathfrak{T}(V)$ of degree k.
On the other hand, if $v \in sing\,\mathfrak{T}(V)$ has degree $k > 0$, one can define $\varphi : \mathfrak{T}(U) \to \mathfrak{T}(V)$ as the unique morphism of $\mathfrak{L}$-modules such that $\varphi(1 \otimes u) = v$ where u is a highest weight vector in U. φ is clearly of degree k. □

The maps in the complexes built in [KR4] were $\mathfrak{L}$-morphisms of degree 1. Rudakov proved in [R2] that there were no others and obtained morphisms of degree 2 and 3 combining, when possible, morphisms of degree 1. He also found morphisms of degree 4 and 5, all involving only minimal Verma modules in the list of Conjecture 3.1.1, and conjectured there were no others.
In [CC] Caselli and Cantarini developed some combinatorial aspects of morphisms between $\mathfrak{L}$-modules that allowed them, in particular, to confirm Rudakov's conjecture up to degree 3.
Kac-Rudakov conjectures can be restated in terms of singular vectors.

Conjecture 3.1.2. A minimal Verma module $\mathfrak{T}(\lambda)$ contains non constant singular vectors if and only if $\lambda = [m,n,0,0]$, $[0,0,m,n]$ or $[m,0,0,n]$ and singular vectors have degree ≤ 5.

Despite a visible fair amount of understanding of these objects, an explicit bound on the degree of singular vectors (or equivalently morphisms) was only implicitly conjectured (at the beginning of the research work for this thesis).
We have seen in Section 2.6 that in the pseudo setting one has an explicit technique which allows to bound the degree of singular vectors from above.
This suggested us the following strategy: verify if it was possible to present a *Lie superpseudoalgebra* which induces, via the annihilation functor, $E(5,10)$ and check if, exploiting the "pseudo" language, this could provide useful new informations.
The idea was to utilize this case as a model to develop arguments that could be generalized to be applied to suitable other linearly compact Lie superalgebras.

3.2 A pseudoalgebraic structure for $E(5,10)$

Let $H = U(\mathfrak{d})$ be the universal enveloping algebra of a Lie algebra $\mathfrak{d}$ of dimension 5. In this section we are going to define a finite Lie H-superpseudoalgebra $L = e(5,10)$ such that $\mathcal{A}(L_{(0)}) \cong E(5,10)_{(0)}$, $\mathcal{A}(L_{(1)}) \cong E(5,10)_{(1)}$ and the Lie bracket of $E(5,10)$ is induced via the annihilation functor by the pseudobracket of L.
Since the even and odd parts of $\mathfrak{L} = E(5,10)$ are explicitly known in terms of formal vector fields and differential forms, the strategy is to look at the analogues "pseudo" concepts.

Recall that $X = H^*$ can be identified with $\mathbb{C}[[x_1,\dots,x_5]]$. We will denote $\xi_{ij} = x_i \wedge x_j \in \bigwedge^2(\mathfrak{d}^*)$.
Consider the even part $\mathfrak{L}_{(0)} = S_5$.
We already know that it is isomorphic to the annihilation algebra of $S(\mathfrak{d},\chi)$.
Since $\mathcal{A}(S(\mathfrak{d},\chi))$ is isomorphic to S_5 regardless of $\mathfrak{d}$ and χ, we will assume for simplicity that $\mathfrak{d}$ is abelian and that $\chi = 0$ and we will denote $S(\mathfrak{d}) = S(\mathfrak{d},0)$.

We set $e(5,10)_{(0)} = S(\mathfrak{d})$ with the usual pseudobracket inherited from $W(\mathfrak{d})$:

$$[(f\otimes a) * (g\otimes b)] = ((fa)\otimes g)\otimes_H (1\otimes b) - (f\otimes (gb))\otimes_H (1\otimes a) \tag{3.5}$$

for $f\otimes a,\ g\otimes b \in S(\mathfrak{d}) \subset H\otimes \mathfrak{d}$.

The odd part of $E(5,10)$ is $\mathrm{d}\Omega^1_5$, the space of differential closed 2-forms in the indeterminates $x_1,\dots,x_5$.
Under the identification of X with $\mathbb{C}[[x_1,\dots,x_5]]$, the space of differential forms Ω_5 can be identified with $\Omega_X(\mathfrak{d}) = X \otimes \bigwedge(\mathfrak{d}^*)$.
Recall that in Section 2.5 we have defined the pseudo De Rham complex with the pseudo differential d_* and a pseudoaction of $W(\mathfrak{d})$, with the properties that $\mathcal{A}(\Omega^k_H(\mathfrak{d})) \cong \Omega^k_X(\mathfrak{d})$ and that $\mathcal{A}(\mathrm{d}_*) = \mathrm{d}$.
In particular, we know that $\mathrm{d}\Omega^1_X(\mathfrak{d}) \cong \mathcal{A}(\mathrm{d}_*\Omega^1_H(\mathfrak{d}))$ and that the action of W_N via Lie derivative coincides with the one induced by $W(\mathfrak{d})$ on $\mathrm{d}_*\Omega^1_H(\mathfrak{d})$.
If we consider the action of S_N and $S(\mathfrak{d})$ obtained by restriction of the previous ones, we can effectively equip $\mathrm{d}\Omega^1_X(\mathfrak{d})$ with a structure of $\mathcal{A}(S(\mathfrak{d}))$-module induced by the pseudoaction of $S(\mathfrak{d})$ on $\mathrm{d}_*\Omega^1_H(\mathfrak{d})$.
We set $e(5,10)_{(1)} = \mathrm{d}_*\Omega^1_H(\mathfrak{d})$ and define the pseudobracket between the even and the odd part as the restriction of the pseudoaction of $W(\mathfrak{d})$ on $\Omega^2_H(\mathfrak{d})$ given in (2.116) (where the second term is missing because $\mathrm{d}_*^2 = 0$):

$$[(f\otimes a) * (g\otimes \alpha)] = ((1\otimes 1)\otimes_H \mathrm{d}_*)((f\otimes a)) *_\iota (g\otimes\alpha) \tag{3.6}$$

for $f\otimes a \in S(\mathfrak{d}) \subset H\otimes\mathfrak{d}$ and $g\otimes\alpha \in \mathrm{d}_*\Omega^1_H(\mathfrak{d}) \subset H\otimes \bigwedge^2(\mathfrak{d}^*)$.

The superbracket in $E(5,10)$ restricted to the odd part is given by (1.22) and explicitly by (1.24).
In order to define the super pseudobracket on $\mathrm{d}_*\Omega^1_H(\mathfrak{d})$, we first define the map

$$\begin{aligned}\phi : \Omega^2_H(\mathfrak{d})\otimes\Omega^2_H(\mathfrak{d}) &\longrightarrow (H\otimes H)\otimes_H W(\mathfrak{d})\\ (1\otimes\xi_{ij})\otimes(1\otimes\xi_{hk}) &\longmapsto (1\otimes 1)\otimes_H (\varepsilon_{(ijhk)}\otimes\partial_{(ijhk)})\end{aligned} \tag{3.7}$$

and extend it by H-bilinearity.
The pseudobracket will be then the restriction of ϕ to $\mathrm{d}_*\Omega^1_H(\mathfrak{d})\otimes\mathrm{d}_*\Omega^1_H(\mathfrak{d})$, so for $f\otimes\xi_{ij}$, $g\otimes\xi_{hk}\in\mathrm{d}_*\Omega^1_H(\mathfrak{d})\subset H\otimes\bigwedge^2(\mathfrak{d}^*)$ we have

$$[f\otimes\xi_{ij}*g\otimes\xi_{hk}]=\varepsilon_{(ijhk)}(f\otimes g)\otimes_H(1\otimes\partial_{(ijhk)}). \tag{3.8}$$

First of all, we need to be sure that this is well defined.

Lemma 3.2.1. $[\mathrm{d}_*\Omega^1_H(\mathfrak{d})*\mathrm{d}_*\Omega^1_H(\mathfrak{d})]\subseteq(H\otimes H)_HS(\mathfrak{d})$.

Proof. A generic element of $\Omega^1_H(\mathfrak{d})=H\otimes\mathfrak{d}^*$ can be written as $\sum_{i=1}^{5}f_i\otimes x_i$ for some $f_i\in H$.
Applying d_* we obtain elements of the form $\sum_{i,j}f_i\partial_j\otimes\xi_{ij}$.
$[\sum_{i,j}f_i\partial_j\otimes\xi_{ij}*\sum_{h,k}g_h\partial_k\otimes\xi_{hk}]=\sum_{i,j,h,k}\varepsilon_{(ijhk)}(f_i\partial_j\otimes g_h\partial_k)\otimes_H(1\otimes\partial_{(ijhk)})$.
Applying $(1\otimes1)\otimes_H\,div$ to the above expression, we expect to obtain 0.

$$\begin{aligned}&\sum_{i,j,h,k}\varepsilon_{(ijhk)}(f_i\partial_j\otimes g_h\partial_k)\otimes_H(\partial_{(ijhk)})\equiv\\&\sum_{i,j,h,k}\varepsilon_{(ijhk)}((f_i\partial_j\partial_{(ijhk)}\otimes g_h\partial_k)+(f_i\partial_j\otimes g_h\partial_k\partial_{(ijhk)}))=\\&\sum_{\{i\neq j\neq h\neq k\}}\varepsilon_{(ijhk)}((f_i\partial_j\partial_{(ijhk)}\otimes g_h\partial_k)+(f_i\partial_j\otimes g_h\partial_k\partial_{(ijhk)}))\end{aligned}$$

Take the terms of the form $\varepsilon_{(ijhk)}(f_i\partial_j\partial_{(ijhk)}\otimes g_h\partial_k)$. Notice that they are different from 0 if and only if $\{i,j,h,k,(ijhk)\}=\{1,\dots,5\}$. If we fix h and k, we obtain exactly $3!=6$ terms of the form $f_i\partial_j\partial_{(ijhk)}\otimes g_h\partial_k$ different from 0 (one for any way we order the remaining 3 indices in $f_i\partial_j\partial_{(ijhk)}$). If we also fix i, we get 2 choices for j and then an obliged choice for $(ijhk)$. The terms obtained from the two choices for j are opposite (because of the definition of ε) and with ∂_j and $\partial_{(ijhk)}$ swapped. Explicitly, we get $\pm f_i\partial_j\partial_{(ijhk)}\otimes g_h\partial_k\mp f_i\partial_{(ijhk)}\partial_j\otimes g_h\partial_k$. But we assumed $\mathfrak{d}$ to abelian, so we can switch ∂_j and $\partial_{(ijhk)}$ and obtain 0. This happens for any choice of h, k and i. A totally analogous argument applies for the terms of the form $\varepsilon_{(ijhk)}(f_i\partial_j\otimes g_h\partial_k\partial_{(ijhk)})$, thus exhausting the sum. □

From this definition and by $\mathbb{C}[[x_1,\dots,x_5]]$-bilinearity of the bracket of $E(5,10)$, it is immediate to verify that the induced map $[\,,\,]:\mathcal{A}(\mathrm{d}_*\Omega^1_H(\mathfrak{d}))\otimes\mathcal{A}(\mathrm{d}_*\Omega^1_H(\mathfrak{d}))$ is equal to (1.22):

$$\begin{aligned}[x\otimes\xi_{ij},y\otimes\xi_{hk}]&\equiv[x\otimes_H(1\otimes\xi_{ij}),y\otimes_H(1\otimes\xi_{hk})]=\\&\varepsilon_{(ijhk)}xy\otimes_H(1\otimes\partial_{(ijhk)})\equiv\varepsilon_{(ijhk)}xy\otimes\partial_{(ijhk)}.\end{aligned}\tag{3.9}$$

However, since the bracket in $E(5,10)$ makes use of the wedge product in Ω_5, we want to verify if we can define a *pseudowedge* in $\Omega_H(\mathfrak{d})$ and obtain an analogous description of the pseudobracket.
We can define on $\Omega_H(\mathfrak{d})=\bigoplus_{k=0}^n\Omega^k_H(\mathfrak{d})=H\otimes\bigwedge(\mathfrak{d}^*)$ a structure of an associative

H-pseudoalgebra by considering it as the current algebra (cf. Definition 2.3.1) over the Grassman algebra $\bigwedge(\mathfrak{d}^*)$ (cf. Example 2.5.1).:

$$Cur_H \bigwedge(\mathfrak{d}^*) = H \otimes \bigwedge \mathfrak{d}^*. \tag{3.10}$$

The pseudoproduct, which we call *pseudowedge* and indicate by $*_\wedge$, is induced by the wedge product in $\bigwedge(\mathfrak{d}^*)$:

$$(f \otimes \alpha) *_\wedge (g \otimes \beta) = (f \otimes g) \otimes_H (1 \otimes \alpha \wedge \beta) \tag{3.11}$$

for $f, g \in H,\ \alpha, \beta \in \bigwedge(\mathfrak{d}^*)$.

Remark 3.2.1. It is easy to check that for $w \in W(\mathfrak{d})$ and $\alpha \in \bigwedge^p(\mathfrak{d}), \beta \in \bigwedge^q(\mathfrak{d})$

$$\begin{aligned}
&w * ((1 \otimes \alpha) *_\wedge (1 \otimes \beta)) = \\
&(w * (1 \otimes \alpha)) *_\wedge (1 \otimes \beta) + ((\sigma \otimes 1) \otimes_H 1)((1 \otimes \alpha) *_\wedge (w * (1 \otimes \beta))), && (3.12)\\
&w *_\iota ((1 \otimes \alpha) *_\wedge (1 \otimes \beta)) = \\
&(w *_\iota (1 \otimes \alpha)) *_\wedge \beta + (-1)^p ((\sigma \otimes 1) \otimes_H 1)((1 \otimes \alpha) *_\wedge (w *_\iota (1 \otimes \beta))) && (3.13)
\end{aligned}$$

and therefore it is true, by H-bilinearity, in $\Omega_H(\mathfrak{d})$.
In other words, $W(\mathfrak{d})$ (and therefore $S(\mathfrak{d}, \chi)$) acts on $\Omega_H(\mathfrak{d})$ by *superderivations*.

If we restrict now the pseudowedge to $\Omega^2_H(\mathfrak{d})$ we obtain an H-bilinear map

$$\begin{aligned}
\Omega^2_H(\mathfrak{d}) \otimes \Omega^2_H(\mathfrak{d}) &\longrightarrow (H \otimes H) \otimes_H \Omega^4_H(\mathfrak{d}) \\
\alpha \otimes \beta &\longmapsto \alpha *_\wedge \beta.
\end{aligned} \tag{3.14}$$

In particular, if $\alpha = (f \otimes \omega_1)$ and $\beta = (g \otimes \omega_2)$, then

$$\alpha *_\wedge \beta = (f \otimes g) \otimes_H (1 \otimes \omega_1 \wedge \omega_2), \tag{3.15}$$

for $f, g \in H, \omega_1, \omega_2 \in \Lambda^2(\mathfrak{d}^*)$.

Define now a map

$$\begin{aligned}
\phi : \Omega^4_H(\mathfrak{d}) &\longrightarrow W(\mathfrak{d}) \\
(1 \otimes \gamma) &\longmapsto D
\end{aligned} \tag{3.16}$$

where D is such that $D *_\iota (1 \otimes v) = (1 \otimes 1) \otimes_H \gamma$ where $v = x_1 \wedge \cdots \wedge x_5 \in \bigwedge^5(\mathfrak{d}^*)$.
By (2.115), $\phi(1 \otimes \gamma) = 1 \otimes a$ for some $a \in \mathfrak{d}$. We can therefore extend ϕ by H-linearity.
The following is an equivalent statement of Lemma 3.2.1.

Lemma 3.2.2. *If* $\alpha, \beta \in \mathrm{d}_*\Omega^1_H(\mathfrak{d}) \subset \Omega^2_H(\mathfrak{d})$, *then* $\alpha *_\wedge \beta \in (H \otimes H) \otimes_H \mathrm{d}_*\Omega^3_H(\mathfrak{d})$.

Proof. In the same notation of Lemma 3.2.1, a generic element of $\mathrm{d}_*\Omega^1_H(\mathfrak{d})$ is of the form $\sum_{i,j} f_i\partial_j \otimes \xi_{ij}$. So given $\alpha = \sum_{i,j} f_i\partial_j \otimes \xi_{ij}$ and $\beta = \sum_{h,k} g_h\partial_k \otimes \xi_{hk}$, then

$$\alpha *_\wedge \beta = \sum_{i,j,h,k} (f_i\partial_j \otimes g_h\partial_k) \otimes_H (1 \otimes \xi_{ij} \wedge \xi_{hk}).$$

By Proposition 2.5.1, we only need to check that applying $(id\otimes id)\otimes_H \mathrm{d}_*$ to the expression above gives 0.
Notice that $\xi_{ij}\wedge\xi_{hk}\neq 0$ only when i,j,h,k are all different from each other. Notice also that $\mathrm{d}_*(1\otimes\xi_{ij}\wedge\xi_{hk})\in\Omega^5_H(\mathfrak{d})\cong H\otimes\bigwedge^5(\mathfrak{d}^*)$ where $\bigwedge^5(\mathfrak{d}^*)=\mathbb{C}v$ where $v=x_1\wedge\cdots\wedge x_5$.
It is easy then to see that $\mathrm{d}_*(1\otimes\xi_{ij}\wedge\xi_{hk})=\varepsilon_{(ijhk)}\partial_{(ijhk)}\otimes x_1\wedge\cdots\wedge x_5$, where the sign ε comes by rearranging the indexes $i,j,h,k,(i,j,h,k)$.
The claim now follows from the same calculation carried out in the proof of Lemma 3.2.1.
□

Lemma 3.2.3. *The restriction of ϕ to $\mathrm{d}_*\Omega^3(\mathfrak{d})$ is an isomorphism of $S(\mathfrak{d})$-modules between $\mathrm{d}_*\Omega^3(\mathfrak{d})$ and $S(\mathfrak{d})$.*

Proof. A generic element of $\omega\in\Omega^4_H(\mathfrak{d})$ is of the form $\omega=\sum_{j<h<k<l} f_{jhkl}\otimes x_j\wedge x_h\wedge x_k\wedge x_l$ for some $f_{jhkl}\in H$. Denoting $\hat{x_i}=x_1\wedge\cdots\wedge\hat{x_i}\wedge\cdots\wedge x_5$ we can rewrite it as $\sum_{i=1}^5 f_i\otimes\hat{x_i}$. Then $\phi(\sum_i f_i\otimes\hat{x_i})=D$ where $D\in W(\mathfrak{d})$ such that $D*_\iota(1\otimes v)=\sum_i(f_i\otimes 1)\otimes_H\hat{x_i}$. One is easily convinced of the fact that $D=\sum_i(-1)^i f_i\otimes\partial_i$.
Assume that $div\,D=0$, which means that $\sum_i(-1)^i f_i\partial_i=0$. Applying now d_* to ω, we obtain an element in $\Omega^5_H(\mathfrak{d})=Hom(\bigwedge^5(\mathfrak{d}),H)$ which can be written explicitly using (2.113) (remember that we are assuming for simplicity that $\mathfrak{d}$ is abelian) :

$$(\mathrm{d}_*(\sum_i f_i\otimes\hat{x_i}))(\partial_1\wedge\cdots\wedge\partial_5)=$$
$$\sum_{i,j}(-1)^j f_i\partial_j(x_1\wedge\cdots\wedge\hat{x_i}\wedge\cdots\wedge x_5)(\partial_1\wedge\cdots\wedge\hat{\partial_j}\wedge\cdots\wedge\partial_5).$$

The generic term of the sum is different from 0 if and only if $i=j$. Thus we obtain $\sum_i(-1)^i(f_i\partial_i)$ which is 0 by assumption.
On the other hand, a generic element $\gamma\in\Omega^3_H(\mathfrak{d})$ is of the form $\sum_{h<k<l} f_{hkl}\otimes x_h\wedge x_k\wedge x_l$, so $\mathrm{d}_*\gamma=\sum_{j,h<k<l} f_{hkl}\partial_j\otimes x_h\wedge x_j\wedge x_l\wedge x_j$ which can be rearranged as $\sum_{j<h<k<l}(-f_{hkl}\partial_j+f_{jkl}\partial_h-f_{jkl}\partial_k+f_{jhk}\partial_l)\otimes x_j\wedge x_h\wedge x_k\wedge x_l$.
If we again denote it as $\sum_i f_i\otimes\hat{x_i}$ where i is the missing index such that $\{i\}\cup\{j,h,k,l\}=\{1,2,3,4,5\}$, then $\phi(\mathrm{d}_*)=\sum_i(-1)^i f_i\otimes\partial_i$.
Now, computing the divergence leads to the expression $\sum_{j<h<k<l}(-1)^i(-f_{hkl}\partial_j\partial_i+f_{jkl}\partial_h\partial_i-f_{jhl}\partial_k\partial_i+f_{jhk}\partial_l\partial_i)$.
A straightforward combinatorial calculation similar to the one used to prove Lemma 3.2.1 shows that this equals 0.
In order to prove that ϕ is also a morphism of $S(\mathfrak{d})$-modules, it is sufficient to verify it when the action is carried out by the elements $s_{ij}=s_{\partial_i\partial_j}$ defined in (2.101).
A direct calculation shows indeed that for any $\sum_h f_h\otimes\hat{x_h}\in\mathrm{d}_*\Omega^3_H(\mathfrak{d})$,
$((id\otimes id)\otimes_H\phi)(s_{ij}*\sum_h f_h\otimes\hat{x_h})=s_{ij}*(\phi(\sum_h f_h\otimes\hat{x_h})$ □

We only need to notice now that $\phi((1\otimes\xi_{ij})*_\wedge(1\otimes\xi_{hk}))=\phi(1\otimes\xi_{ij}\wedge\xi_{hk})=\varepsilon_{(ijhk)}1\otimes\partial_{(ijhk)}$.
In other words, we can define the pseudobracket between odd elements as the composition of $*_\wedge$ and ϕ.
Now we are ready to prove the following:

Proposition 3.2.1. $e(5,10)$ *is a Lie superpseudoalgebra and* $\mathcal{A}(e(5,10))\cong E(5,10)$.

Proof. We only need to show that $[\cdot * \cdot]$ is in fact a super pseudobracket.
H-bilinearity and super skew symmetry follow directly by the definitions (3.5), (3.6) and (3.8).
Regarding the Jacobi identity, it is satisfied when one even element and two odd ones are involved because of Lemma 3.2.3, and Remark 3.2.1. When only odd elements are involved, a direct but tedious computation shows that the Jacobi identity is still satisfied. □

Remark 3.2.2. For different choices of $\mathfrak{d}$ and χ, it is still possible to define a pseudoalgebraic structure by taking a suitable 1-dimensional $\mathfrak{d}$-module $\Pi = \mathbb{C}_\varepsilon$, where $\varepsilon \in \mathfrak{d}^*$ and consider the Π-twisting of the De Rham complex (ε will be an adequate linear combination of $tr\, ad$ and χ).

Remark 3.2.3. It is worth to mention here that in [CCK2], in a similar fashion, $E(5,10)$ is realized as the annihilation algebra of a Lie conformal superalgebra.

3.3 Bound on degree of singular vectors

We will apply now the pseudoalgebraic structure $e(5,10)$ to the representation theory of $E(5,10)$. As we have seen in Section 3.1, the classification of irreducible $E(5,10)$-modules can be translated in terms of classification of singular vectors in generalized Verma modules.
Direct computation of singular vectors is not immediate, especially in high degrees. One thing one can do is trying, as a start, to rule out as many options as possible. This can be done, for example, by looking for restricting conditions on the degree of singular vectors, which is what we are about to do.
Recall that by Theorem (2.6.5), we have a very important result that points in this direction in the "pseudo" setting. Having a pseudoalgebraic structure for $E(5,10)$ might therefore be useful. Turns out, in order to get a first bound on the degree of singular vectors, it is enough to take into account just the even structure of $E(5,10)$, exploiting the pseudoalgebra techniques already available in the non super setting.
To do so, we first define some subspaces of $\mathfrak{T}(V)$.
For $i = 0, \dots, 10$, let

$$\Gamma_i(V) := \Big\{ v = \sum_{I,K} \partial^{(I)} \xi^K \otimes v_{IK} \in \mathfrak{T}(V) \,\big|\, |K| \leq i \text{ se } v_{IK} \neq 0 \Big\}. \tag{3.17}$$

In other words, $\Gamma_i(V)$ consists of vectors with "odd degree" at most i. We may also describe $\Gamma_i(V)$ as the space generated by the PBW monomials of degree $(n \,|\, j)$ with $j \leq i$.
It is easy to check the following properties:

- $\Gamma_i(V) \subseteq \Gamma_{i+1}(V)$;
- $\Gamma_0(V) = U(\mathfrak{d}) \otimes V$;
- $\Gamma_{10}(V) = \mathfrak{T}(V)$;
- $\Gamma_i(V)$ is a $\mathfrak{L}_{(0)} \cong S_5$-submodule of $\mathfrak{T}(V)$.

Basically, we have built a finite filtration of S_5-modules of $\mathfrak{T}(V)$.
The last property follows from the fact that the action of an even element in $\mathfrak{L}$ can only lower the odd degree in $\mathfrak{T}(V)$. Take for example $y \in \mathfrak{L}_2$, $v = \partial^{(I)}\xi^K \otimes v_I^K \in \mathfrak{T}(V)$.
We have

$$y \cdot v = y \cdot (\partial^{(I)}\xi^K \otimes v_I^K) = [y, \partial^{(I)}]\xi^K \otimes v_I^K + \partial^{(I)}[y, \xi^K] \otimes v_I^K + \partial^{(I)}\xi^K \otimes y \cdot v_I^K.$$

Here the third term is 0 because $\mathfrak{L}_+$ acts trivially on V; the first term consists of elements of degree $(\,|I| - 1 \mid |K|\,)$; lastly, the second term, once expanded the bracket and sorted everything, can only contribute with elements of degree $(|I| \mid |K|-2)$ or $(|I|+1 \mid |K|-4)$. In any case the odd degree cannot increase.

These properties allow us to talk about quotients.
Let us consider the quotients of S_5-modules $\Gamma_i(V)/\Gamma_{i-1}(V)$ for $i = 0, \dots, 10$ (where we impose $\Gamma_{-1}(V) = 0$).
As $\mathfrak{sl}(\mathfrak{d})$-modules, they are isomorphic to $U(\mathfrak{d}) \otimes (\bigwedge^i(\mathfrak{s}) \otimes V)$ (follows from [M, Corollary 6.4.5]). The latter look a lot like tensor modules $T(\bigwedge^i(\mathfrak{s}) \otimes V)$ for $S(\mathfrak{d})$: if the action of S_5 can be interpreted as the action of the annihilator algebra associated to pseudoaction of $S(\mathfrak{d})$, we can put to use Proposition 2.4.2. This is, in fact, possible in the following way.

The action of $y \in \mathfrak{L}_{2j}$ on a class $\overline{u \cdot \xi^K \otimes v} \in \Gamma_i(V)/\Gamma_{i-1}(V)$, where $u \in U(\mathfrak{d})$, $|K| = i$ and $v \in V$, behaves, depending on j, like:

for j $= -1$ $y \cdot \overline{u\xi^K \otimes v} = \overline{(yu)\xi^K \otimes v}$, since in this case $y \in \mathfrak{d} \subseteq U(\mathfrak{d})$;

for j $= 0$ $y \cdot \overline{u\xi^K \otimes v} = \overline{[y, u]\xi^K \otimes v + u[y, \xi^K] \otimes v + u\xi^K \otimes (y \cdot v)}$;

for j > 0
$$\begin{aligned} y \cdot \overline{u\xi^K \otimes v} &= \overline{[y, u]\xi^K \otimes v + u[y, \xi^K] \otimes v + u\xi^K \otimes (y \cdot v)} \\ &= \overline{[y, u]\xi^K \otimes v} \end{aligned}$$

where the last equality is due to the fact that y lowers the odd degree of at least 1, sending the second term to 0 in the quotient, and acts trivially on V.
Notice that in the case $j = 0$, the action of $y \in \mathfrak{L}_0 \cong \mathfrak{sl}(\mathfrak{d})$ on ξ^K is actually the same as the one on the $\mathfrak{sl}(\mathfrak{d})$-module $\bigwedge^i(\mathfrak{s})$ (since the other terms that usually appear in the bracket $[y, \xi^K]$ once sorted are 0 in the quotient).

Summing up, we have that $\mathfrak{L}_{(0)} = S_5$ acts on $\Gamma_i(V)/\Gamma_{i-1}(V)$

- by left multiplication on the $H = U(\mathfrak{d})$ factor with the negative degree part;
- by the natural action of $\mathfrak{sl}(\mathfrak{d})$ on $U(\mathfrak{d}) \otimes (\bigwedge^i(\mathfrak{s}) \otimes V)$ with the degree 0 part;
- trivially on $\bigwedge^i(\mathfrak{s}) \otimes V$ with the positive degree part.

Since this is exactly the action of $\mathcal{S}$ on the tensor module $T_\chi(\bigwedge^i(\mathfrak{s}) \otimes V)$ (with $\chi = 0$, so it can be omitted) via the isomorphism in (2.103), we can state:

Proposition 3.3.1. *Let V be a finite-dimensional irreducible $\mathfrak{sl}(\mathfrak{d})$-module. Then we have an isomorphism of $S_5 \cong \mathcal{A}(S(\mathfrak{d}))$-modules*

$$\Gamma_i(V)/\Gamma_{i-1}(V) \cong T(\bigwedge^i(\mathfrak{s}) \otimes V). \tag{3.18}$$

Take now $v \in sing\,\mathfrak{T}^p(V)$. It is, in particular, a singular vector for $\mathfrak{L}_{(0)} = S_5$ of degree p. We denote the space of such vectors with $sing_{S_5}\,\mathfrak{T}(V)$.
If we consider $\overline{v} \in \Gamma_i(V)/\Gamma_{i-1}(V)$ for a suitable $i = 0, \dots, 10$, it will still be a singular vector in what we now know is a tensor module. Therefore, by Theorem 2.6.5, the even degree of $\overline{v}$ must be ≤ 2. Since the odd degree of a vector cannot be larger than 10, these ideas, formalized, prevent vectors with a sufficiently high enough degree from being singular.

Theorem 3.3.1. *Let $\mathfrak{T}(V)$ be a minimal Verma module and let $v \in sing\,\mathfrak{T}(V)$. Then v has degree at most* 14.

Proof. We can assume that v is homogeneous of degree p.
We have, if p is either even or odd:

p=2n $\mathfrak{T}^p(V) = S^n(\mathfrak{d}) \otimes V + S^{n-1}(\mathfrak{d}) \bigwedge^2(\mathfrak{s}) \otimes V + \cdots + S^{n-5}(\mathfrak{d}) \bigwedge^{10}(\mathfrak{s}) \otimes V;$

p=2n+1 $\mathfrak{T}^p(V) = S^n(\mathfrak{d})\, \mathfrak{s} \otimes V + S^{n-1}(\mathfrak{d}) \bigwedge^3(\mathfrak{s}) \otimes V + \cdots + S^{n-4}(\mathfrak{d}) \bigwedge^9(\mathfrak{s}) \otimes V.$

We study the case $p = 2n$.
Let $0 \leq m_0 \leq 10$ be the greatest index such that the term of v in degree $(\frac{p-m_0}{2}|m_0)$ is not 0 (i.e. the term of v in $S^{(p-m_0)/2}(\mathfrak{d}) \bigwedge^{m_0}(\mathfrak{s}) \otimes V$).
Therefore $v \in \Gamma_{m_0}(V)$ and it is a combination of terms in degrees

$$(\tfrac{p-m_0}{2}|m_0),\ (\tfrac{p-m_0}{2}+1|m_0-2), \dots,\ (\tfrac{p}{2}|0).$$

We can then consider

$$\overline{v} \in \Gamma_{m_0}(V)/\Gamma_{m_0-1}(V) \cong U(\mathfrak{d}) \otimes (\bigwedge^{m_0}(\mathfrak{s}) \otimes V).$$

Notice that given $y \in \mathfrak{L}_2$, $y \cdot v \in \mathfrak{T}^{p-1}(V)$ and the term of degree $(\frac{p-m_0-1}{2}|m_0)$ can be obtained only acting with y on the term of v of degree $(\frac{p-m_0}{2}|m_0)$. Hence, if v is singular for S_5, so must be $\overline{v}$.
To recap: if $v \in sing\,\mathfrak{T}^p(V)$, $\overline{v} \in U(\mathfrak{d}) \otimes (\bigwedge^{m_0}(\mathfrak{s}) \otimes V) = T(\bigwedge^{m_0}(\mathfrak{s}) \otimes V)$ is a singular vector for S_5 of (even) degree $(p - m_0)/2$ in a tensor module. By Theorem 2.6.5, the even degree of $\overline{v}$ must be less than or equal to 2, which means that $(p - m_0)/2 \leq 2$, that is $p \leq m_0 + 4 \leq 14$. When p is odd the same argument holds. □

The proof actually tells us more than the statement of the theorem: we can not only estimate the singular vectors' degree, but we can also rule out straightforwardly most of the irreducible $\mathfrak{sl}(\mathfrak{d})$-modules whose induced modules we expect to possibly contain singular vectors of a certain degree. We recall that in our notation $\Omega^1(\mathfrak{d}) = \mathfrak{d}^* \cong V([1, 0, 0, 0])$. Similarly $\Omega^2(\mathfrak{d}) \cong V([0, 1, 0, 0])$, $\Omega^3(\mathfrak{d}) \cong V([0, 0, 1, 0])$ and $\Omega^4(\mathfrak{d}) \cong V([0, 0, 0, 1])$.

Now apply, for example, the proof's arguments on degree 14:
let v be a singular vector of degree 14 in a minimal Verma module $\mathfrak{T}(V)$ and consider

$\overline{v} \in \Gamma_{10}(V)/\Gamma_9(V) \cong U(\mathfrak{d}) \otimes (\bigwedge^{10}(\mathfrak{s}) \otimes V) \cong U(\mathfrak{d}) \otimes V$. If $\overline{v} \neq 0$, this means that the term of v of degree $(2|10)$ is not 0.
We know by Theorem 2.6.5 that we can find singular vectors of (even) degree 2 in a tensor module $T(V)$ where V is irreducible if and only if $V \cong \Omega^1(\mathfrak{d}) \cong V([1,0,0,0])$.
Therefore, if we assume that $V \ncong \mathfrak{d}^*$, $\overline{v}$ will necessarily be 0. This implies that $v \in \Gamma_8(V)$ and we can consider $\overline{v} \in U(\mathfrak{d}) \otimes (\bigwedge^8(\mathfrak{s}) \otimes V)$, thus obtaining a singular vector in $T(\bigwedge^8(\mathfrak{s}) \otimes V)$ of (even) degree $(14-8)/2 = 3$. This cannot happen, so that the only possible solution is $\overline{v} = 0$. Iterating, we discover that v must be 0.
We proved:

Lemma 3.3.1. *If $V \ncong \mathfrak{d}^* \cong V([1,0,0,0])$, $sing\,\mathfrak{T}^{14}(V) = \{0\}$.*

3.4 Bound refining

A simple lemma will be extremely useful:

Lemma 3.4.1. *Let $\mathfrak{T}(V)$ be a Verma module for $\mathfrak{L}$. If $v \in sing\,\mathfrak{T}(V)$ and $\xi \in \mathfrak{L}_{-1} \cong \mathfrak{s}$, then $\xi v \in sing_{S_5}\,\mathfrak{T}(V)$.*

Proof. Take $y \in \mathfrak{L}_2$. Then

$$y \cdot (\xi v) = [y, \xi] \cdot v + \xi(y \cdot v) = 0,$$

where the second term is 0 because v is singular. The same goes for the first term since $[y, \xi] \in \mathfrak{L}_1$. □

We apply this new piece of information to the case $V \cong \mathfrak{d}^*$.
Let $v \in sing\,\mathfrak{T}^{14}(\mathfrak{d}^*)$. By the lemma, given any $\xi \in \mathfrak{s}$, ξv (which has now degree 15), is still singular for the action of S_5. Consider the term of degree $(3|9)$ and the corresponding $\overline{\xi v} \in \Gamma_9(V)/\Gamma_8(V) \cong T(\bigwedge^9(\mathfrak{s}) \otimes V)$. Like before, it is still singular and has even degree 3, which implies that $\overline{\xi v} = 0$. We can then consider $\overline{\xi v} \in \Gamma_7(V)/\Gamma_6(V)$ and, iterating the argument, obtain that ξv must be 0 $\forall \xi \in \mathfrak{s}$.
We remark that $\mathfrak{L}_{-2} = [\mathfrak{L}_{-1}, \mathfrak{L}_{-1}]$ which means that, for any $\partial \in \mathfrak{d} \cong \mathfrak{L}_{-2}$, we can find $\xi_1, \xi_2 \in \mathfrak{s} \cong \mathfrak{L}_{-1}$ such that $\partial = [\xi_1, \xi_2]$. This in particular implies that $\partial v = 0\ \forall \partial \in \mathfrak{L}_{-2}$. Since the action of $\mathfrak{L}_{-2}$ on a tensor module is simply given by left multiplication, it can only mean that $v = 0$.
In conclusion, we showed that even if $V \cong \mathfrak{d}^*$, $\mathfrak{T}(V)$ cannot have singular vectors of degree 14.
These ideas can be applied systematically to perform a refining of the bound in Theorem 3.3.1.

Theorem 3.4.1. *Let $\mathfrak{T}(V)$ be a minimal Verma module and let $v \in sing\,\mathfrak{T}(V)$. Then v has degree ≤ 12. More precisely:*

1. *if $V \ncong V([0,0,1,0])$, singular vectors have degree strictly smaller than 12;*

2. *if $V \ncong V(\lambda)$ where $\lambda = [0,0,0,1]$, $[0,0,1,0]$, $[0,1,0,0]$, $[1,0,0,0]$, $[0,1,1,0]$, or $[1,0,0,1]$, singular vectors have degree strictly smaller than 11;*

3. $ifV \ncong V(\mu)$ *where* $\mu =[0,0,0,0],\ [1,0,0,0],\ [0,0,0,1],\ [0,0,1,0],\ [0,1,0,0],$
$[1,1,0,0],\ [0,1,1,0],\ [1,0,0,1],\ [0,0,1,1],\ [1,0,1,0],$
$[0,1,0,1],\ [1,1,0,1],\ [0,2,0,0],\ [2,0,0,0],\ [1,0,2,0],$
or $[3,0,0,1]$, *singular vectors have degree strictly smaller than* 10.

The proof revolves around arguments similar to the previous ones. To that end, because of Proposition 3.3.1 and Theorem 2.6.5, we will need to be able to determine when, given an irreducible $\mathfrak{sl}(\mathfrak{d})$-module V, we can (or rather cannot) find a copy of $V(\omega_i)$ in $\bigwedge^j(\mathfrak{s}) \otimes V$. Recall that every irreducible $\mathfrak{sl}(\mathfrak{d})$-module V has a highest weight vector and that V is uniquely determined by the highest weight. Here, as before, $\omega_1 = [1,0,0,0]$, $\omega_2 = [0,1,0,0]$, $\omega_3 = [0,0,1,0]$, $\omega_4 = [0,0,0,1]$ and $\omega_0 = [0,0,0,0]$.
By Frobenius duality (keeping in mind that these are all finite-dimensional modules), $V(\omega_i) \subseteq \bigwedge^j(\mathfrak{s}) \otimes V$ if and only if $V \subseteq \bigwedge^j(\mathfrak{s}^*) \otimes V(\omega_i)$.
In the following table we list the highest weights of the irreducible representations that appear in the decomposition of those tensor products. It was obtained using computer software "LiE" (see [LiE] for further informations).

	$\bigwedge^9(\mathfrak{s}^*) \otimes V(\omega_i)$	$\bigwedge^8(\mathfrak{s}^*) \otimes V(\omega_i)$	$\bigwedge^7(\mathfrak{s}^*) \otimes V(\omega_i)$	$\bigwedge^6(\mathfrak{s}^*) \otimes V(\omega_i)$
i=0	[0,1,0,0]	[1,0,1,0]	[0,0,2,0], [2,0,0,1]	[1,0,1,1], [3,0,0,0]
i=1	[0,0,1,0], [1,1,0,0]	[0,1,1,0], [1,0,0,1], [2,0,1,0]	[0,0,1,1], [1,0,2,0], [1,1,0,1], [2,0,0,0], [3,0,0,1]	[0,1,1,1], [1,0,0,2], [1,0,1,0], [2,0,1,1], [2,1,0,0], [4,0,0,0]
i=2	[0,0,0,1], [0,2,0,0], [1,0,1,0]	[0,0,2,0], [0,1,0,1], [1,0,0,0], [1,1,1,0], [2,0,0,1]	[0,0,1,0], [0,1,2,0], [1,0,1,1], [1,1,0,0], [2,1,0,1], [3,0,0,0]	[0,0,2,1], [0,1,0,2], [0,1,1,0], [1,0,0,1], [1,1,1,1], [2,0,0,2], [2,0,1,0], [3,1,0,0]
i=3	[0,0,0,0], [0,1,1,0], [1,0,0,1]	[0,0,1,1], [0,1,0,0], [1,0,2,0], [1,1,0,1], [2,0,0,0]	[0,0,3,0], [0,1,1,1], [1,0,0,2], [1,0,1,0], [2,0,1,1], [2,1,0,0]	[0,0,1,2], [0,0,2,0], [0,1,0,1], [1,0,2,1], [1,1,0,2], [1,1,1,0], [2,0,0,1], [3,0,1,0]
i=4	[0,1,0,1], [1,0,0,0]	[0,0,1,0], [1,0,1,1], [1,1,0,0]	[0,0,2,1], [0,1,1,0], [1,0,0,1], [2,0,0,2], [2,0,1,0]	[0,0,1,1], [1,0,1,2], [1,0,2,0], [1,1,0,1], [2,0,0,0], [3,0,0,1]

Proof of Theorem 3.4.1. We will outline the various steps in a schematic way. The ideas are the same we have already discussed.
Degree 13:
Let $v \in sing\,\mathfrak{T}^{13}(V)$. We first consider the term of v in degree $(2|9)$; it can be different from zero only if V appears in the decomposition of $\bigwedge^9(\mathfrak{s}^*) \otimes V(\omega_1) \cong V([0,0,1,0]) \oplus V([1,1,0,0])$. So if V is not isomorphic to one of these two representations, the term in $(2|9)$ of v must be equal to 0, we look at next term, which is also 0 because has degree $(3|7)$. Iterating, we deduce that $v = 0$.
If $V \cong V([0,0,1,0])$ or $V([1,1,0,0])$, take $\xi \in \mathfrak{s}$ and consider the term in degree $(2|10)$ of ξv. It can be non zero only if V does not appear in $\bigwedge^{10}(\mathfrak{s}^*) \otimes V(\omega_1) \cong V(\omega_1)$. It follows that it must be 0 and, iterating, so does $\xi v\ \forall \xi \in \mathfrak{s}$ which in turn implies, as we already saw, that v must be 0. In conclusion, $sing\,\mathfrak{T}^{13}(V)$ is always 0.
Degree 12:

Let $v \in sing\,\mathfrak{T}^{12}(V)$. We consider the term of degree $(1|10)$; it can be non zero only if V appears in $\bigwedge^{10}(\mathfrak{s}^*) \otimes V(\omega_i) \cong \omega_i$ for $i = 0, \dots, 4$. Therefore if V is not of the form $V(\omega_i)$, we check the term of degree $(2|8)$ which must be 0 unless a copy of V appears in $\bigwedge^8(\mathfrak{s}^*) \otimes V(\omega_1) = V([0,1,1,0]) \oplus V([1,0,0,1]) \oplus V([2,0,1,0])$. In the remaining cases $v = 0$.
Now assume $V \cong V([0,1,1,0])$, $V([1,0,0,1])$, $V([2,0,1,0])$ or $V(\omega_i)$ with $i = 0, \dots, 4$. Take $\xi \in \mathfrak{s}$ and consider the term of ξv of degree $(2|9)$. It cannot be non zero if V does not appear in $\bigwedge^9(\mathfrak{s}^*) = V([0,0,1,0]) \oplus V([1,1,0,0])$. So if $V \ncong V(\omega_3)$ $\xi v = 0$ $\forall \xi \in \mathfrak{s}$ and again it implies that $v = 0$. Therefore the only case in which we cannot rule out the presence of singular vectors of degree 12 is for $V \cong V(\omega_3)$.
Degree 11: Let $v \in sing\,\mathfrak{T}^{11}(V)$. We consider the term of degree $(1|9)$; it can be different from zero when V appears in $\bigwedge^9(\mathfrak{s}^*) \otimes V(\omega_i)$ for $i = 0, \dots, 4$. It happens when V has as highest weight one belonging to the first column of the table.
If we assume V is not isomorphic to any of them, we can move to the next term which has degree $(2|7)$. According to the table, if V is also not isomorphic to $V([0,0,1,1])$, $V([1,0,2,0])$, $V([1,1,0,1])$, $V([2,0,0,0])$ and $V([3,0,0,1])$, $v = 0$.
Assume V is isomorphic to one of these representations, a fundamental representation or the trivial one; take $\xi \in \mathfrak{s}$ and check the term of degree $(1|10)$; if V is not a fundamental representation or the trivial one, we can move to the term of degree $(2|8)$ which will be 0 unless a copy of V appears in $\bigwedge^8(\mathfrak{s}^*) \otimes V(\omega_1) \cong V([0,1,1,0]) \oplus V([1,0,0,1]) \oplus V([2,0,1,0])$. Therefore if V is not isomorphic to $V([0,0,0,1])$, $V([0,0,1,0])$, $V([0,1,0,0])$, $V([1,0,0,0])$, $V([0,1,1,0])$ or $V([1,0,0,1])$, $sing\,\mathfrak{T}^{11}(V) = 0$.
Degree 10:
Let $v \in sing\,\mathfrak{T}^{10}(V)$. The term of v with the greatest odd degree is the one of degree $(0|10)$. In this case we cannot deduce anything, since this term is constant from the point of view of S_5, therefore singular.
We relay again on Lemma 3.4.1: take $\xi \in \mathfrak{s}$ and consider ξv, which has now degree 11. This time we can still look at $\overline{\xi v} \in \Gamma_9(V)/\Gamma_8(V) \cong T(\bigwedge^9(\mathfrak{s}) \otimes V))$ which has even degree 1. Again, we know that it is 0 unless V appears as an irreducible component of $\bigwedge^9(\mathfrak{s}^*) \otimes V(\omega_i)$, $i = 0\dots,5$. So if the highest weight of V does not appear in the first column of the table, $\overline{\xi v} = 0$. We can then move to the next term, which has degree $(2|7)$. Here we look at the irreducible modules in $\bigwedge^7(\mathfrak{s}^*) \otimes V(\omega_1) \cong V([0,0,1,1]) \oplus V([1,0,2,0]) \oplus V([1,1,0,1]) \oplus V([2,0,0,0]) \oplus V([3,0,0,1])$. So if in addition we ask that V is not isomorphic to these modules, $\xi v = 0$ $\forall \xi \in \mathfrak{s}$ and again we cannot have singular vectors of degree 10 in $\mathfrak{T}(V)$. □

Almost at the same time as we obtained these results, Cantarini, Caselli and Kac presented in [CCK1] the complete classification of singular vectors and hence of all degenerate Verma modules for $E(5,10)$, which confirmed Conjecture 3.1.1. However, the statement about the degrees of singular vectors of Conjecture 3.1.2 turned out to be false: in addition to the singular vectors already found by Rudakov, they found a singular vector of degree 11 and one of degree 7 in $\mathfrak{T}(V)$, where V is isomorphic respectively to $V([0,0,0,1])$ and $V([0,0,0,2])$.
Of course, our results are compatible with the classification, since $V([0,0,0,1])$ appears in our list of exceptions for degree 11. Remarkably enough, the existence of a singular vector of degree 11 means that, except for the single case in degree 12, the estimate could not have directly lowered any further (except by studying each case individually).

The still satisfactory preciseness of the first rough estimate of Lemma 3.3.1 confirms a promising "trend" which already held in the non "super" setting. Namely, the estimate for the degree of singular vectors for $W(\mathfrak{d})$ and $S(\mathfrak{d}, \chi)$ provided respectively in Lemma 2.6.2 (degree less or equal to 1) and Lemma 2.6.4 (degree less or equal to 2) turned out to be optimal, which is also true for the analogous situations of type K (degree less or equal to 2, [BDK3]) and H (degree less or equal to 2, [BDK4]). In particular, having an estimate from above and a finite number of tensor modules that can possess non-trivial singular vectors, makes the problem "finite" and possibly approachable with the help of a computer. Our hope is that a similar strategy could be applied for most linearly compact Lie superalgebras, provided that one can exhibit a pseudoalgebraic structure.

Chapter 4

Classification of pseudoalgebraic structures for $E(5,10)$

4.1 The reconstruction functor

In order to investigate the possible pseudoalgebraic structures for $E(5,10)$, a very important tool is the *reconstruction functor*, which we will introduce in this section following [BDK1, Section 11].
Let H be a cocommutative Hopf algebra and let $\mathcal{L}$ be a topological left H-module.
The reconstruction functor $\mathcal{C}$ associates to $\mathcal{L}$ the left H-module

$$\mathcal{C}(\mathcal{L}) = Hom_H^{cont}(X, \mathcal{L}), \tag{4.1}$$

the space of continuous H-homomorphisms, where as usual H is endowed with the discrete topology and $X = H^*$.
The action of H is given by

$$(h\alpha)(x) = \alpha(xh) \quad \text{for } h \in H, x \in X, \alpha \in \mathcal{C}(\mathcal{L}). \tag{4.2}$$

Then $\mathcal{C}$ is a covariant functor from the category of topological H-modules to the category of H-modules.
The reconstruction functor was introduced, as the name suggests, to "reconstruct" a finite H-pseudoalgebra L from its annihilation algebra, which we recall is a linearly compact H-differential algebra.
We want to use it in a similar fashion, applying it also to annihilation modules of finite L-modules, therefore we will always assume that $\mathcal{L}$ is a linearly compact H-module.

Given a finite H-module M and its annihilation module $\mathcal{A}(M) = X \otimes_H M$, we denote its reconstruction by $\widehat{M} = \mathcal{C}(\mathcal{A}(M)) = Hom_H^{cont}(X, X \otimes_H M)$. There is a natural map $\Phi : M \longrightarrow \widehat{M}$ defined by

$$\Phi(m) = \phi_m \text{ where } \phi_m(x) = x \otimes_H m \text{ for } m \in M, x \in X. \tag{4.3}$$

By the definitions, we have

$$(h\Phi(m))(x) = \phi_m(xh) = xh \otimes_H m = x \otimes_H hm = \phi_{hm}(x) = (\Phi(hm))(x),$$

thus Φ is an homomorphism of H-modules.
In [BDK1] is showed that if L is a finite Lie H-pseudoalgebra, it is possible to define a pseudobracket on $\widehat{L}$ and $\Phi : L \to \widehat{L}$ is an homomorphism of Lie H-pseudoalgebras.
Similarly, if M is a finite L-module, one can define a pseudoaction of L on $\widehat{M}$ and $\Phi : M \to \widehat{M}$ is a morphism of L-modules.
It is also proven the following result, which gives us useful properties of $\mathcal{C}$.

Lemma 4.1.1. *1. $\mathcal{C}$ is left exact: if i is injective, so is $\mathcal{C}(i)$;*

2. $\mathcal{C}$ preserves direct sums: $\mathcal{C}(\mathfrak{L} \oplus \mathcal{M}) = \mathcal{C}(\mathfrak{L}) \oplus \mathcal{C}(\mathcal{M})$;

3. If $H = U(\mathfrak{d})$, then $\mathcal{C}(\mathcal{L})$ is a torsion-free H-module.

When M is a free H-module, Φ behaves very nicely.

Proposition 4.1.1. *[BDK1, Proposition 11.3] Let M be a free H-module. Then $\Phi : M \longrightarrow \widehat{M}$ is an isomorphism of H-modules.*

In particular, if L is a Lie H-pseudoalgebra free as an H-module, $\Phi : L \to \widehat{L}$ is an isomorphism of Lie H-pseudoalgebras. Similarly, if M is an L-module free as H-module, $\Phi : M \to \widehat{M}$ is an isomorphism of L-modules.

Example 4.1.1. If M is a tensor module for $W(\mathfrak{d})$ or $S(\mathfrak{d}, \chi)$, since M is a free H-module, we have $\widehat{M} = \mathcal{C}(\mathcal{A}(M)) \cong M$.

Remark 4.1.1. Let L be a Lie H-pseudoalgebra, $M = H \otimes M_0$ be an L-module which is free as an H-module and let $N \subset M$ be an L-submodule. Then, by definition, $\mathcal{A}(N) \subset \mathcal{A}(M)$, which implies, since $\mathcal{C}$ is left exact, that $\widehat{N} = \mathcal{C}(\mathcal{A}(N)) \subseteq \mathcal{C}(\mathcal{A}(M)) = \widehat{M} \cong M$ where last isomorphism follows from the fact that M is a free H-module. In other words, $\widehat{N}$ is isomorphic to a submodule of M. This is enough to determine the reconstruction of irreducible modules of De Rham type for $W(\mathfrak{d})$ (and $S(\mathfrak{d}, \chi)$) because there is only one proper submodule in the corresponding tensor module. However, one should be able to verify that in this case N and M also induce the same annihilation algebra. This can be useful for the cases of type H and K, for which not irreducible tensor modules can have up to 3 submodules.

Lemma 4.1.2. *If M is a finite irreducible representation of $W(\mathfrak{d})$ or $S(\mathfrak{d}, \chi)$, then $\widehat{M} = \mathcal{C}(\mathcal{A}(M)) \cong M$.*

Proof. We already know that the claim is true if M is a tensor module by Proposition 4.1.1.
If $M = \mathrm{d}_*(\Omega^k_H(\mathfrak{d}))$ is a module of De Rham type, it is in particular a submodule of the tensor module $T(\bigwedge^{k+1} \mathfrak{d}^*)$. By the previous Remark, $\widehat{M}$ is isomorphic to a submodule of $T(\bigwedge^{k+1} \mathfrak{d}^*)$ because the latter is a free H-module.
We know that M is the only proper submodule of $T(\bigwedge^{k} \mathfrak{d}^*)$, hence $\widehat{M} \cong M$. □

In the non-free case, Φ is in general neither injective nor surjective. However, when $H = U(\mathfrak{d})$, its injectivity can still be controlled in terms of H-torsion. In fact, one has (see [BDK1, Lemma 7.4, Proposition 11.5]):

Proposition 4.1.2. *1. if H is a Noetherian domain which has a skew-fields of fractions K and L is a finite H-module, then there is an homomorphism $i : L \to F$, where F is a free H-module, whose kernel is $Tor\ L$.*

2. *If L is a finite $U(\mathfrak{d})$-module and $a \in L$ is H-torsion, i.e. $ha = 0$ for some $0 \neq h \in U(\mathfrak{d})$, then $X \otimes_H a = 0$.*

Corollary 4.1.1. $ker\Phi = Tor\, L$.

Proof. It is a well known fact that $H = U(\mathfrak{d})$ is a Noetherian domain and that any two non-zero elements in H have a non-zero left (and right) common multiple, which in particular implies that H has a skew-field of fractions.
If $a \in Tor\, L$, then $\Phi(a)(x) = x \otimes_H a \equiv 0$, so $a \in ker\Phi$.
On the other hand, if a is not $U(\mathfrak{d})$-torsion, $i(a) \neq 0$. The induced map $\mathcal{A}(i)$ maps $x \otimes_H a$ to $x \otimes_H i(a) \in X \otimes_H F$. Proposition 4.1.1 implies that there exists $x \in X$ such that $x \otimes_H i(a) \neq 0$, hence $x \otimes_H a \neq 0$. □

We very briefly review now the reconstruction of primitive Lie pseudoalgebras of type W and S. For more details, see [BDK1, Section 12].

Let $\mathcal{L}$ be a subalgebra of W_N with its canonical filtration $\{\mathcal{L}_k\}$. The latter induces a filtration of the Lie algebra $Der\,\mathcal{L}$, $(Der\,\mathcal{L})_j = \{d \in Der\mathcal{L} \mid d(\mathcal{L}_i) \subset \mathcal{L}_{j+i}\}$.
Recall that $X = H^*$ can be identified with $\mathcal{O}_N$ and that under this identification W_N is isomorphic to $\mathcal{A}(W(\mathfrak{d}))$ as linearly compact Lie algebras. By (2.84), this lets $\mathfrak{d}$ act on W_N by derivations. The action of $\mathfrak{d}$ on $\mathcal{L}$ is called *transitive* if the composition of the map $\mathfrak{d} \to Der\,\mathcal{L}$ with the projection $Der\,\mathcal{L} \to Der\,\mathcal{L}/(Der\,\mathcal{L})_0$ is a linear isomorphism.
In [BDK1, Lemma 12.1] is shown that if L is a simple Lie H-pseudoalgebra (or a current pseudoalgebra over one), then the canonical action of $\mathfrak{d}$ (2.84) on $\mathcal{A}(L)$ is transitive.

Recall also that under the identification $X \cong \mathcal{O}_N$, $\mathfrak{d}$ acts on $\mathcal{O}_N$ by linear differential operators, which is equivalent to say that there is an embedding $\mathfrak{d} \hookrightarrow Der\mathcal{O}_N = W_N$, which is called the *canonical embedding*. This embedding is transitive, i.e. $\mathfrak{d} \subset W_N$ is a complementary of $F_0 W_N$.

A structure of an H-differential algebra on W_N is equivalent to a transitive action of $\mathfrak{d}$ on W_N by derivations, which, since $Der\, W_N = W_N$, is equivalent to a transitive embedding of $\mathfrak{d}$ in W_N.
In [BDK1] is proved, applying a theorem of Guillemin and Sternberg [GS] ([BDK1, Proposition 6.9]), that any two such embeddings are conjugated by an automorphism of W_N.
Therefore, up to an automorphism of W_N, we can assume that the transitive embedding is the canonical one, which corresponds to the canonical action of $\mathfrak{d}$. With the corresponding action of H, W_N is isomorphic to $\mathcal{A}(W(\mathfrak{d}))$.
Since $W(\mathfrak{d})$ is a free H-module, by Proposition 4.1.1 $\mathcal{C}(W_N) = W(\mathfrak{d})$.
In [BDK1, Section 12.6] it is proven that, even if it is not a free H-module, the same holds for $S(\mathfrak{d}, \chi)$.

Lemma 4.1.3. *The reconstruction of the Lie algebra S_N, provided with a transitive action of $\mathfrak{d}$, is $S(\mathfrak{d}, \chi)$ where χ is a trace form on $\mathfrak{d}$.*

Remark 4.1.2. We should stress the fact that $\mathcal{A}(W(\mathfrak{d}))$ is always isomorphic, as a linearly compact Lie algebra, to W_N regardless of the Lie bracket of $\mathfrak{d}$ (apart for its dimension, which has to be N). In order to apply the reconstruction functor then one needs an H-differential algebra structure, which clearly depends on $\mathfrak{d}$ and its action on W_N; the same

is true for S_N and $S(\mathfrak{d},\chi)$ if one takes different $\mathfrak{d}$ or χ.
Similarly, if $V = H \otimes V_0$ is a tensor module for $W(\mathfrak{d})$ (or $S(\mathfrak{d},\chi)$), then $\mathcal{A}(V)$ is always isomorphic, as linearly compact representations of W_N (or S_N), to $X \otimes V_0$, while it can have a different structure as an H-module. The same holds for modules of De Rham type.

One final fact that will be useful later is the following.

Proposition 4.1.3. *[BDK1, Proposition 11.13] If L is a Lie H-pseudoalgebra which is finite and torsionless as an H module and if $\mathcal{A}(L)$ has a finite-dimensional center, then $\widehat{L}$ is a Lie H-pseudoalgebra containing L as an ideal.*

4.2 Reconstruction of irreducible representations of simple Lie pseudoalgebras of type W and S

The reconstruction functor is used in [BDK1] to achieve the classification of finite simple Lie pseudoalgebras. Roughly speaking, the idea is the following: one starts by understanding the structure of the linearly compact Lie algebra associated to a simple Lie pseudoalgebras (i.e. its annihilation algebra), then one checks for all possible compatible H-module structures and finally applies the reconstruction functor.
The second step is far from being trivial and one expects it to be even harder in the "super" setting, so while exploiting it for the even part of a Lie superpseudoalgebra, we will try to avoid it for the odd one.
In this section, we develop a technique that works around this issue and that can be applied to the odd part of $e(5,10)$ and hopefully to other cases.

Let L be a Lie H-pseudoalgebra isomorphic to either $W(\mathfrak{d})$ or $S(\mathfrak{d},\chi)$ and let M be a finite L-module.
Generally speaking, one should be able to reconstruct M from its annihilation module $\mathcal{M} = \mathcal{A}(M)$ if a structure of an $\mathcal{L}^e = \mathcal{A}(L)^e$-module for the latter is known, which is equivalent to know the action of both $\mathcal{L} = \mathcal{A}(L)$ and H. Our aim is to understand if, under suitable conditions, the action of $\mathcal{L}$ is actually enough. This turns out to be true, and this section will be devoted to prove the following result.

Proposition 4.2.1. *Let M be an irreducible L-module. If M' is another L-module such that $\mathcal{A}(M') \cong \mathcal{A}(M)$ as $\mathcal{L}$-modules, then $M' \cong M$.*

Remark 4.2.1. We will carry out the calculations for $W(\mathfrak{d})$ and $S(\mathfrak{d},\chi)$. However, the technique should work, with adequate technical adjustments, for any simple finite Lie pseudoalgebra.

As explained in Remark 2.4.3, in our situation is not restricting to assume that $H = U(\mathfrak{d})$. We will also assume all H-modules to be torsionless because we are interested in their connection with the associated linearly compact objects via the annihilation functor which, by Proposition 4.1.2, kills all torsion elements.
Recall that $\mathcal{L}$ has a canonical filtration such that $F_0\mathcal{L}/F_1\mathcal{L} \cong \mathfrak{g}_L$ where $\mathfrak{g}_L = \mathfrak{gl}(\mathfrak{d})$ if $L = W(\mathfrak{d})$ and $\mathfrak{g}_L = \mathfrak{sl}(\mathfrak{d})$ if $L = S(\mathfrak{d},\chi)$.

Recall that $sing\,M$ is a $\mathfrak{d} \oplus \mathfrak{g}_L$-module with action denoted by ρ_{sing} and that by Theorems 2.6.6 and 2.6.7 it uniquely determines the irreducible L-module M.
Recall also that by Corollary 2.6.1 and Theorem 2.6.3 it generates M as an H-module and by Theorems 2.6.1 and 2.6.2 it is finite-dimensional.
This allows us to define the filtration (2.87) on $\mathcal{M} = X \otimes_H M$. Namely,

$$F_p\mathcal{M} = F_pX \otimes_H sing\,M. \tag{4.4}$$

Consider the associated graded space of $\mathcal{M}$: $gr\,\mathcal{M} = \bigoplus_p gr^p\mathcal{M}$ where

$$gr^p\mathcal{M} = F_p\mathcal{M}/F_{p+1}\mathcal{M} = F_pX \otimes_H sing\,M/F_{p+1}X \otimes_H sing\,M.$$

It is easy to see that $gr^p\mathcal{M}$ is isomorphic, as a vector space, to $S^p(\mathfrak{d}^*) \otimes sing\,M$.

Lemma 4.2.1. *If $L = W(\mathfrak{d})$, then $gr^p\mathcal{M}$ is an $F_0\mathcal{W}/F_1\mathcal{W} \cong \mathfrak{gl}(\mathfrak{d})$-module isomorphic to $S^p(\mathfrak{d}^*) \otimes sing\,M \otimes \mathbb{C}_{tr}$*

Proof. The pseudoaction of $W(\mathfrak{d})$ on $v \in sing\,M$ is given explicitly by (2.126):

$$\begin{aligned}(1 \otimes \partial_i) * v = \sum_{k=1}^{N} (\partial_k \otimes 1) \otimes_H \rho_{sing}(e_i^k)v - (\partial_i \otimes 1) \otimes_H v \\ -(1 \otimes \partial_i) \otimes_H v + (1 \otimes 1) \otimes_H \rho_{sing}(\partial_i + ad\,\partial_i)v.\end{aligned}$$

Therefore, the action of $\mathcal{W}$ on $F_p\mathcal{M}$ can be written, for $x \in X, y \in F_pX, v \in sing\,M$ as:

$$\begin{aligned}(x \otimes \partial_i)(y \otimes_H v) = \sum_{k=1}^{N} (x\partial_k)y \otimes_H \rho_{sing}(e_i^k)v - ((x\partial_i)y + x(y\partial_i)) \otimes_H v \\ + xy \otimes_H \rho_{sing}(\partial_i + ad\,\partial_i)v.\end{aligned} \tag{4.5}$$

Notice that if $x \in F_qX$ (cf. (2.93)), then the right-hand side is in $F_{p+q}X \otimes_H sing\,M$. In other words,

$$F_q\mathcal{W} \cdot F_p\mathcal{M} \subset F_{q+p}\mathcal{M}. \tag{4.6}$$

The previous relation implies in particular that each $gr^p\mathcal{M}$ is a $F_0\mathcal{W}/F_1\mathcal{W} \cong \mathfrak{gl}(\mathfrak{d})$-module.
Writing out this action from equation (4.5) for $x = x_j$ under the isomorphism $x_j \otimes \partial_i \mapsto -e_i^j$, we obtain

$$e_i^j(y \otimes_H v) = y \otimes_H e_i^j v - \partial_j^i y \otimes_H v + (e_j^i y) \otimes_H v,$$

where we used (2.97).
y can be identified with an element $x_I \in F_pX/F_{p+1}X \cong S^p(\mathfrak{d}^*)$ (as vector spaces) where $x_I = x_1^{i_1} \cdots x_N^{i_N}$ for $I = (i_1, \ldots, i_N)$ and $|I| = p$. Using Lemma 2.1.1, a straightforward computation shows that $(x_j \otimes \partial_i)x_I = x_j(x_I\partial_i)$ coincides with the action of $-e_i^j$ on $x_I \in S^p(\mathfrak{d}^*)$, where $e_i^j x_k = -\delta_i^k x_j$.
Finally, notice that δ_j^i can be interpreted as the action of e_i^j on $\mathbb{C}_{tr}$, hence the claim is true. □

A similar statement holds for $S(\mathfrak{d}, \chi)$.

Lemma 4.2.2. *If $L = S(\mathfrak{d},\chi)$, then $gr^p\mathcal{M}$ is an $F_0\mathcal{S}/F_1\mathcal{S} \cong \mathfrak{sl}(\mathfrak{d})$-module isomorphic to $S^p(\mathfrak{d}^*) \otimes sing\, M$*

Proof. For simplicity, assume that $\chi = 0$ and $\mathfrak{d}$ is an abelian Lie algebra. This is motivated by the fact that the terms appearing in the action of $S(\mathfrak{d},\chi)$ on singular vectors and involving χ and the adjoint action of $\mathfrak{d}$ are killed by the quotient in $gr^p\mathcal{M}$ when one consider the induced annihilation action of $\mathcal{S}$.
Since all irreducible $S(\mathfrak{d})$-modules are tensor modules (or submodules of tensor modules) and since all tensor modules for $S(\mathfrak{d})$ are restrictions of tensor modules for $W(\mathfrak{d})$, we can use formula (2.126) to derive an analogous one for $S(\mathfrak{d})$. Recall that $S(\mathfrak{d})$ is generated, in our assumptions, as an H-module by elements $s_{ij} = s_{\partial_i\partial_j} = \partial_i \otimes \partial_j - \partial_j \otimes \partial_i$. Hence, for $v \in sing\, M$, using H-bilinearity of the $W(\mathfrak{d})$-pseudoaction, we obtain:

$$\begin{aligned} s_{ij} * v = &\sum_{k=1}^{N} (\partial_k\partial_i \otimes 1) \otimes_H \rho_{sing}(e_j^k)v - (\partial_i \otimes \partial_j) \otimes_H v + (\partial_i \otimes 1) \otimes_H \rho_{sing}(\partial_j)v \\ &- \sum_{k=1}^{N} (\partial_k\partial_j \otimes 1) \otimes_H \rho_{sing}(e_i^k)v + (\partial_j \otimes \partial_i) \otimes_H v - (\partial_j \otimes 1) \otimes_H \rho_{sing}(\partial_i)v. \end{aligned}$$

This let us write down the action of $\mathcal{S}$ on $F_p\mathcal{M}$. For $x \in X, y \in F_pX, v \in sing\, M$,

$$\begin{aligned} (x \otimes_H s_{ij})(y \otimes_H v) = &\sum_{k=1}^{N} (x\partial_k)(y\partial_i) \otimes_H \rho_{sing}(e_j^k)v - (x\partial_k)(y\partial_j) \otimes_H \rho_{sing}(e_i^k)v \\ &+((x\partial_j)(y\partial_i) - (x\partial_i)(y\partial_j)) \otimes_H v + (x\partial_i)y \otimes_H \rho_{sing}(\partial_j)v - (x\partial_j)y \otimes_H \rho_{sing}(\partial_i)v. \end{aligned}$$

Now, if $x \in F_{q+1}X$ (cf. (2.102)), $(x \otimes_H s_{ij})(y \otimes_H v) \in F_{p+q}\mathcal{M}$, so again

$$F_q\mathcal{S} \cdot F_p\mathcal{M} \subset F_{q+p}\mathcal{M}.$$

Now $gr^p\mathcal{M}$ is a $F_0\mathcal{S}/F_1\mathcal{S} \cong \mathfrak{sl}(\mathfrak{d})$. A tedious but direct calculation shows that the action coincides with the $\mathfrak{sl}(\mathfrak{d})$-one expected. □

Remark 4.2.2. Notice that all the extra structure regarding the $\mathfrak{d}$-action and the trace form χ do not appear, as we expected since we are only considering the action of the linearly compact Lie algebra $\mathcal{L}$ on the $\mathcal{M}$.
In this regard, notice in particular that the $\mathfrak{d}$-module structure of $sing\, M$ does not affect this construction.
This means that if M is a tensor module $T(\Pi, V)$ or a submodule of a Π-twisted De Rham module $\mathrm{d}_\Pi(\Omega_H^k(\mathfrak{d}))$ for an irreducible $\mathfrak{d}$-module Π, one will simply find $\dim \Pi$ copies of V in the $\mathfrak{g}_L$-module $gr\,\mathcal{M}$.

To adjust the notation, let V_M be the irreducible $\mathfrak{g}_L$-module correspondent to M, i.e. $V_M = V$ if $M = T(\Pi, V)$ or $V_M = \bigwedge^k \mathfrak{d}^*$ if $M = \mathrm{d}_\Pi(\Omega^k(\mathfrak{d}))$.
Comparing the above lemmas with Theorems 2.6.6 and 2.6.7 we can summarize it as follows.

Proposition 4.2.2. *If M is an irreducible L-module, then $gr^p\mathcal{M}$ is isomorphic, as a $\mathfrak{g}_L$-module, to $S^p(\mathfrak{d}^*) \otimes V_M$.*

Building on last remark, if we put ourself in the assumptions that $\mathfrak{d}$ is abelian, it acts trivially on $sing\,M$ and $\chi = 0$, we obtain the same construction, whereas we can directly define a grading on $\mathcal{M}$ (cf. Example 2.1.3 and Remark 2.4.2):

$$G^p\mathcal{M} = G^pX \otimes_H sing\,M \tag{4.7}$$

such that $G^p\mathcal{M} \cong F_p\mathcal{M}/F_{p+1}\mathcal{M}$.
To recap, we have associated to an irreducible L-module M a dense $\mathfrak{g}_L$-submodule of $\mathcal{M}$ which decomposes as an (infinite) direct sum of finite-dimensional $\mathfrak{g}_L$-modules.

Remark 4.2.3. The assumptions of $\chi = 0$, $\mathfrak{d}$ being abelian and acting trivially on $sing\,M$ are actually not necessary. In fact, by Proposition 1.1.1 follows that $\mathcal{M}$ is isomorphic to the completion of the graded module associated to the filtration defining the linearly compact topology. Moreover, it is showed in [BDK1] that the topology induced on $\mathcal{M}$ does not depend on the choice of the finite-dimensional subspace used to define the filtration, so this will not be an issue in the what follows.

We intend to compare two irreducible L-modules with isomorphic linearly compact annihilation modules via these associated semisimple $\mathfrak{g}_L$-submodules.
In order to do so, we need first to verify that there are no other finite-dimensional $\mathfrak{g}_L$-submodules in $\mathcal{M}$.

Lemma 4.2.3. *All finite-dimensional irreducible $\mathfrak{g}_L$-submodules of $\mathcal{M}$ appear in $\bigoplus_p G^p\mathcal{M}$.*

Proof. Let V be an irreducible finite-dimensional $\mathfrak{g}_L$-submodule of $\mathcal{M}$.
Since every element in $\mathcal{M}$ is a (possibly infinite) direct sum of elements in $G^p\mathcal{M}$, we can consider the projections $\pi_p : \mathcal{M} \to G^p\mathcal{M}$.
π_p is a $\mathfrak{g}_L$-homomorphism, so when restricted to V, which is irreducible and finite-dimensional, it is either 0 or an isomorphism into an irreducible in $G^p\mathcal{M} \cong S^p(\mathfrak{d}^*) \otimes V_M$.
Now, let λ be the highest weight of V. The highest weights of the irreducible components of $S^p(\mathfrak{d}^*) \otimes V_M$ diverge as p goes to infinity (the highest weight of $S^p(\mathfrak{d}^*)$ is $[0, \dots, 0, p]$). Alternatively, one can notice that by Frobenius reciprocity, since all the representations involved are irreducible, $V \subset S^k(\mathfrak{d}^*) \otimes V_M$ if and only if $S^k(\mathfrak{d}^*) \subset V \otimes V_M^*$. Either way, it is clear that there exists an index $\overline{p}$ such that $\pi_p = 0$ for all $p > \overline{p}$. This implies that the isotypic component of V in $\mathcal{M}$ is contained in $\bigoplus_{p \leq \overline{p}} G^p\mathcal{M} \subset \bigoplus_p G^p\mathcal{M}$. □

The final ingredient we need is provided by the discussion at the end of Section 2.5. In the proof of the previous lemma, we implicitly showed also that the following holds.

Corollary 4.2.1. $\bigoplus_p S^p(\mathfrak{d}^*) \otimes V_M \in Rep^{ss}(\mathfrak{g}_L)$.

Proof. We proved that the isotypic component of any irreducible finite-dimensional $\mathfrak{gl}(\mathfrak{d})$-submodule of $\mathcal{M}$ is contained in a finite-dimensional submodule of $\bigoplus_p S^p(\mathfrak{d}^*) \otimes V_M$, hence its multiplicity is finite. □

This result allows us to apply identity (2.120) and obtain

$$\Big(\sum_{k=0}^{N} (-1)^k \textstyle\bigwedge^k(\mathfrak{d})\Big)\Big(\sum_{n\geq 0} S^n(\mathfrak{d}^*)V_M\Big) = V_M. \tag{4.8}$$

We are finally ready to prove Proposition 4.2.1.

Proof. Let $N \subseteq M'$ be an irreducible L-submodule.
First, we claim that if $\mathcal{A}(N) = \mathcal{A}(M')$, then $N = M'$. In fact, applying the reconstruction functor, we obtain that $\widehat{M'} = \mathcal{C}(\mathcal{A}(M')) = \mathcal{C}(\mathcal{A}(N)) \cong N$, where the last isomorphism follows from Lemma 4.1.2.
Thus we have an homomorphism of L-modules $\Phi : M' \longrightarrow \widehat{M'} \cong N$ given by (4.3), which is injective because we assumed all modules to be H-torsionless. Hence, $N = M'$.
Now, we can apply the construction explained in this section and associate to N the $\mathfrak{g}_L$-module $\bigoplus_p S^p(\mathfrak{d}^*) \otimes V_N$, which is a dense $\mathfrak{g}_L$-submodule of $\mathcal{N}$. By assumption, $\mathcal{N}$ is, up to isomorphism, a submodule of $\mathcal{M}$.
By Lemma (4.2.3), every $S^p(\mathfrak{d}^*) \otimes V_N$ is contained in $\bigoplus_p S^p(\mathfrak{d}^*) \otimes V_M$ and by the same Lemma the opposite also holds.
In other words, $\bigoplus_p S^p(\mathfrak{d}^*) \otimes V_M \cong \bigoplus_p S^p(\mathfrak{d}^*) \otimes V_N$. Applying the identity (4.8), we get

$$V_M \cong V_N.$$

By definition, this means that $M \cong N$.
Now the claim at the beginning of this proof implies that $N = M' \cong M$. □

Remark 4.2.4. Since the odd part of $e(5,10)$ presented in Section 3.2 is an irreducible $S(\mathfrak{d})$-module, this is enough for our purposes. However, this argument should be easily generalized to determine the uniqueness, up to isomorphism, of a completely reducible L-module.

4.3 Classification of $U(\mathfrak{d})$-pseudoalgebraic structures for $E(5,10)$

We are interested in developing techniques that will allow us to characterize all possible pseudoalgebraic structures of a given linearly compact Lie superalgebra.
As a "proof of concept", we present here the application for the Lie superalgebra $E(5,10)$.

By an *H-pseudoalgebraic structure* for a linearly compact Lie superalgebra $\mathcal{L} = \mathcal{L}_{(0)} \oplus \mathcal{L}_{(1)}$ we mean a Lie H-superpseudoalgebra $L = L_{(0)} \oplus L_{(1)}$ (where the decomposition is also of H-modules) such that:

1. $\mathcal{A}(L_{(0)}) \cong \mathcal{L}_{(0)}$ as linearly compact Lie algebras;
2. $\mathcal{A}(L_{(1)}) \cong \mathcal{L}_{(1)}$ as linearly compact representations of the linearly compact Lie algebra $\mathcal{L}_{(0)} \cong \mathcal{A}(L_{(0)})$;
3. the restriction of the pseudobracket of L to $L_{(1)}$ induces via the annihilation functor the restriction of the Lie bracket of $\mathcal{L}$ to $\mathcal{L}_{(1)}$ up to an automorphism of $\mathcal{L}_{(1)}$ as an $\mathcal{L}_{(0)}$-module.

Example 4.3.1. The Lie superpseudoalgebra $e(5,10)$ constructed in Section 3.2 is a $U(\mathfrak{d})$-pseudoalgebraic structure for $E(5,10)$ with $\mathfrak{d}$ being an abelian Lie algebra.

In this section H will be the universal enveloping algebra of a Lie algebra of dimension 5. Because of Proposition 4.1.2, we will assume all H-modules to be H-torsionless and as usual finite.

If $L_{(0)}$ is a Lie H-pseudoalgebra such that $\mathcal{A}(L_{(0)}) = E(5,10)_{(0)} = S_5$, we know that the canonical action of $\mathfrak{d}$ on $\mathcal{A}(L_{(0)})$ is transitive, hence, by Lemma 4.1.3, $\mathcal{C}(\mathcal{A}(L_{(0)})) \cong S(\mathfrak{d},\chi)$. The assumption of no H-torsion, Proposition 4.1.3 and simplicity of $S(\mathfrak{d},\chi)$ imply that $\Phi : L_{(0)} \longrightarrow \widehat{L_{(0)}} \cong S(\mathfrak{d},\chi)$ (4.3) is an isomorphism of Lie H-pseudoalgebras.

Our main result is the classification of $U(\mathfrak{d})$-pseudoalgebraic structures for $E(5,10)$.

Theorem 4.3.1. *For any choice of a trace form $\chi \in \mathfrak{d}^*$, there exists a unique Lie super pseudoalgebra structure of the form $L = S(\mathfrak{d},\chi) \oplus L_{(1)}$ inducing $E(5,10)$ as annihilation superalgebra. The Lie superpseudobracket identifies $L_{(1)}$ with $\mathrm{d}_\Pi(\Omega^1_H(\mathfrak{d}))$ as a $S(\mathfrak{d},\chi)$-module for a suitable choice of Π only depending on χ and the Lie superpseudobracket $L_{(1)} \otimes L_{(1)} \to (H \otimes H)_H L_{(0)}$ coincides with the pseudo de Rham wedge operation as in* (3.11).

We will provide here a proof for the case of $\mathfrak{d}$ being abelian and $\chi = 0$ to keep the exposition clear and simple.
First we show how the result of the previous section takes care of the odd part of the pseudoalgebraic structure $L_{(1)}$.
If $S(\mathfrak{d}) \oplus L'_{(1)}$ is another potential pseudoalgebraic structure for $E(5,10)$, this means in particular that the annihilation module $\mathcal{A}(L'_{(1)})$ is isomorphic to the S_N-module $\mathcal{A}(\mathrm{d}_*(\Omega^1_H(\mathfrak{d})))$.
Since $\mathrm{d}_*(\Omega^1_H(\mathfrak{d}))$ is an irreducible $S(\mathfrak{d})$-module, Proposition 4.2.1 straightforwardly implies that $L'_{(1)} \cong \mathrm{d}_*(\Omega^1_H(\mathfrak{d}))$.

Now that both the even and the odd parts are, up to isomorphisms, fixed, we just need to check that the same happens for the pseudobracket restricted to $L_{(1)} = \mathrm{d}_*(\Omega^1_H(\mathfrak{d}))$.
Let $\rho, \tau : L_{(1)} \otimes L_{(1)} \longrightarrow S(\mathfrak{d})$ be H-bilinear maps such that they induce via the annihilation functor the same map $\mathcal{A}(L_{(1)}) \otimes \mathcal{A}(L_{(1)}) \longrightarrow S_N$ up to an automorphism ψ of $\mathcal{A}(L_{(1)})$ as an S_N-module, i.e.

$$\mathcal{A}(\rho) = \psi \circ \mathcal{A}(\tau). \tag{4.9}$$

To understand what such an automorphism looks like, we can relay on the pseudoalgebraic structure. In fact, we can consider the morphism of $S(\mathfrak{d})$-modules

$$\begin{aligned} \mathcal{C}(\psi) : \mathcal{C}(\mathcal{A}(L_{(1)})) = Hom^{cont}_H(X, \mathcal{A}(L_{(1)})) &\longrightarrow \mathcal{C}(\mathcal{A}(L_{(1)})) = Hom^{cont}_H(X, \mathcal{A}(L_{(1)})) \\ \phi &\mapsto \psi \circ \phi. \end{aligned} \tag{4.10}$$

Moreover, we know that $L_{(1)} \cong \mathcal{C}(\mathcal{A}(L_{(1)}))$ via (4.3). Explicitly, this isomorphism is given by $v \mapsto \phi_v$, where $\phi_v(x) = x \otimes_H v$ for $v \in L_{(1)}$, which tells us that

$$(\mathcal{C}(\psi)\phi_v)(x) = \psi \circ \phi_v(x) = \psi(x \otimes_H v). \tag{4.11}$$

Recall that by Proposition 2.4.2 a $W(\mathfrak{d})$-module naturally carries a structure of conformal $\mathcal{A}(W(\mathfrak{d}))^e$-module. It is straightforward by definitions that a morphism of $W(\mathfrak{d})$-modules commutes with the action of $\mathcal{A}(W(\mathfrak{d}))^e$.
This implies in particular that $\mathcal{C}(\phi)$ sends singular vectors to singular vectors and that it commutes with the action of $\mathfrak{sl}(\mathfrak{d}) \cong F_0\mathcal{S}/F_1\mathcal{S}$.
By Schur lemma, $\mathcal{C}(\psi) \equiv \lambda Id$ for some scalar $\lambda \in \mathbb{C}$. However, recall that $L_{(1)}$ is generated as an H-module by $sing\, L_{(1)}$. Since a morphism of $W(\mathfrak{d})$-modules is in particular a morphism of H-modules, we have that for any $\alpha \in L_{(1)}$ there exist $h \in H$ and

$v \in sing\, L_{(1)}$ such that $\alpha = hv$. Therefore,

$$\mathcal{C}(\psi)(\alpha) = \mathcal{C}(\psi)(hv) = h\mathcal{C}(\psi)(v) = \lambda hv = \lambda\alpha.$$

Combining the previous identity with (4.11) gives us, for $x \in X, v \in M_i$,

$$\psi(x \otimes_H v) = (\mathcal{C}(\psi)\phi_v)(x) = (\phi_{\lambda v}(x)) = \lambda x \otimes_H v. \tag{4.12}$$

Now, applying $\mathcal{C}$ to the relation (4.9) tells us that $\rho = \lambda\tau$, which implies in particular that any two superpseudobrackets inducing the Lie supebracket of $E(5,10)$ define the same Lie superpseudoalgebra structure up to a constant.
This concludes the proof.

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