The demographic risk assessment: from the traditional Local GAAP framework to the Solvency II market-consistent one

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Abstract

The purpose of this thesis is to provide a stochastic model consistent with the accounting principles envisaged by Directive 138/2009/EC aimed at quantifying demographic risk. This model, already developed in a local accounting context, is then adapted to market-consistent valuation, highlighting the bridge between the two frameworks, the individual risks that are quantified and the differences in terms of numerical results. The application of the model leads to the identification of some sub-risks inherent to the demographic one: the idiosyncratic, the trend one and a risk linked to risk-free rate changes. For the first risk, using an approach based on the concept of *cohorts*, closed formula results are obtained about expected value, standard deviation and skewness. Moreover, it is possible to numerically quantify the capital requirement and, at the same time, approximate it according to the characteristics of the profit (loss) random variable. Regarding the second risk, the model pursues the aim of quantifying the Solvency Capital Requirement by simulating the possible trajectories of mortality and obtaining the simulated distribution of the company's profit (loss) linked to the possible variation in demographic expectations. Thus, it was possible to numerically quantify the capital requirement coherently with in force regulatory principles.
Keywords: `Life insurance; Solvency Capital Requirement; Demographic Profit; Risk Theory; Cohort approach.`

To my mentors,

professor Clemente and professor Savelli

Contents

Introduction

The introduction of Directive 2009/138/EC, commonly called "Solvency II", has deeply changed the structural aspects of the insurance world. In particular, both the risk assessment and the capital requirement quantification shifted from factor-based to a more realistic, a more relevant methodology that went into the direction of the undertaking risk profile assessment. Similarly, the valuation of the liabilities, i.e. the technical provisions, moved towards fair value principle defined as the theoretical value at which they could be transferred between system operators. In this context, this thesis proposes a new methodological framework, coherent with the accounting principles of Solvency II, based on the cohort approach and aimed both at identifying the different sources of demographic risk and at quantifying the Solvency Capital Requirements (SCRs). With this purpose, it separately models, within demographic risk, an idiosyncratic (or unsystematic) component and a trend (or systematic) component. A unified model, focused on quantifying the SCR in a closed form, is provided for policies both with survival-linked benefits and for death-linked benefits.
In literature, many works deal with this topic. Savelli & Clemente (see [34]) proposed a stochastic model, based on the cohort approach. Even if this model is developed in a Local Generaly Acceptec Accounting Principles (Local GAAP) context, it presents fundamental relationships that this thesis demonstrates to be underlying also in the market consistent framework. In this regard, Section 3 of this thesis is based on a joint work with Savelli & Clemente (see [13]). In particular, in that work it is proved how it is possible to adapt the cohort model presented in [34] to the market consistent

context envisaged by Solvency II. This gap is bridged by abandoning the valuation of mathematical reserves with locked and prudential technical bases (used in pricing phase) and adopting realistic bases that can be updated over time. This thesis extends [13] through the identification of a systematic component (trend risk) and an unsystemic component (idiosyncratic risk) of the demographic risk. With reference to the latter, by using the cohort approach, it is possible to accurately quantify the moments of distribution; with reference to the trend component, a bootstrap model will be applied in order to estimate the volatility of the mortality rates and, consequently, the Solvency Capital Requirement. Both risk identification and SCR assessment (which concern Sections 4 and 5 of the thesis) are subject to publication.[1].In this field, Pitacco & Olivieri (see [27]) analyse longevity risk by referring to a portfolio of annuities. In particular, through risk-neutral approaches, the authors reconcile the traditional methodology with the market-consistent one. Jarner & Møller (see [24]) propose a partial internal model for the longevity risk component, which incorporates an unsystematic element linked to the size of the portfolio. Similarly to [24], this thesis overcomes the methodology provided by the Standard Formula based on a longevity shock. However, this approach is different since it includes also mortality risk in the evaluation and consider the volatility of the sums insured within the portfolio, usually neglected in literature.

This thesis therefore takes up the stochastic model based on the cohort ap-

[1]In April 2022, the paper titled "A stochastic model for capital requirement assessment for mortality and longevity risk, focusing on idiosyncratic and trend components" is awaiting second reviewer assignment

proach aimed at quantifying the capital requirement in the Local GAAP context, and analytically quantifies the bridge towards market consistent valuation. This first step makes it possible to highlight how the fair value measurement of liabilities structurally modifies the quantification of the expected future demographic profit and its volatility. The second step involves the identification of the two components of demographic profit, the idiosyncratic (or unsystemic) and the trend (or systemic) risks. Regarding idiosyncratic one, thanks to the cohort approach, it is possible to quantify in a closed formula the main characteristics of the distribution and, consequently, to assume a proxy to estimate the SCR consistent with the principles of regulatory system. Regarding the trend component, the proposed framework is not bound to the use of a specific model for the projection of mortality rates: from a practical point of view, therefore, the model of this thesis can be extended to any model present in the literature focused on longevity/mortality assessment. For example, citing the main ones, in [30] a mortality model is proposed for forecasting trends. In [9], the shock given by Solvency II is compared with the results of the forward models proposed by Bauer et al. (see [5] and [6]). In [10], the authors propose an ad-hoc mortality model that considers both longevity and mortality and the dependency structure between the different cohorts. Moreover, Gylys and Šiaulys (see [22]) compare the run-off and the one-year approach fitting stochastic mortality data on different years than those used for Solvency II calibration by EIOPA. Zhou et al. (see [39]) model a multi-population mortality model overcoming the common assumptions of dominant population.

This thesis is organized as follows: Section 1 presents the regulatory context:

in particular, an excursus of the capital requirement for life insurance is presented. This section also deals with the issue of the fair value valuation of non-hedgeable technical liabilities, i.e. those liabilities for which it is necessary to calculate the Best Estimate and Risk Margin separately.
Section 2 takes up the model presented by Savelli and Clemente (see [34]) reproducing the numerical results using the most up-to-date technical bases and neglecting the effects of lapses and expenses: the aim is to obtain results related to the Local GAAP framework that are comparable with those of the following sections, developed in a market-consistent context.
Section 3 presents the innovative mathematics underlying the stochastic model of the market-consistent framework: since there is a misalignment between the first-order and second-order demographic base, it is necessary to revisit the Fouret equation to guarantee the recursion property. In the same section, the topic of identifying the sources of risk linked to the insurance business is introduced, by re-adapting the Homans breakdown.
Section 4, with reference to demographic risk, focuses attention on the two components of greatest importance: the idiosyncratic and the trend one. As regards the first, the use of the cohort approach allows to reach results in a closed formula with reference to the characteristics of the distribution. In particular, closed formulas are obtained for expected value, standard deviation and skewness: through these statistics it is possible to present a compact formulation (similar to QIS n.2, cfr. [14]) for the calculation of the SCR. With reference to the trend component, on the other hand, the Poisson log-bilinear model is presented to estimate the volatility around the company's expectations (see [11]).

In conclusion, Section 5 shows the results of the stochastic model, comparing them with the analogues of the Local GAAP context, highlighting how the market-consistent valuation allows to identify new sources of risk (first and foremost structural longevity and mortality risk) and to carry out a risk-based assessment of the company risk profile.

1. The legislative framework evolution for Capital Requirement in insurance

1.1. Introduction: Solvency in life-insurance

[2] Solvency, understood as a situation of general well-being of the insurance company both from the point of view of the supervisory authority and from the point of view of management, is a constant frontier issue for actuarial disciplines.

The two concepts aforementioned do not coincide perfectly: solvency in the strict sense refers to the ability of the insurance undertaking to meet the regulatory requirements in terms of capitalization in order for the supervisory authority to deem the requirement in question satisfied. On the other hand, the expression "financial strength" refers to the ability of the insurance company to cope with adverse scenarios and, consequently, the ability to operate in the long term. In the study of insurance solvency, the ultimate aim is to guarantee policyholders that the company is able, even in extreme scenarios, to meet its contractual commitments: obviously this purpose is not in an absolute sense, but a threshold is assumed, in terms of probability, within which the company must be solvent.

This ability to meet contractual commitments can be pursued with three different types of approaches. In the wind-up approach, a completely static situation is considered: in this context, a company must be solvent with

[2] This subsection is taken from [33], a text of particular inspiration for the whole of this thesis

exclusive reference to the business hired up to the moment of evaluation. In the run-off approach it is assumed that the acquisition of the business does not cease immediately, but that it continues involving immediately predictable contracts (generally within 2 years): in this case, therefore, reference is made to a dynamic solvency criterion applied to a portfolio semi-closed. The latter approach, called going-on, assumes that the insurance company will continue building business without any future disruption. This type of approach, which concerns the concept of dynamic solvency for an open portfolio, implies the correct evaluation of the insurance liabilities (contracts not commonly transferred between insurers) so that the transfer of risks by the company in a state of emergency to solid companies is conceivable. Finally, it is specified that this last approach, necessary from a perspective point of view, is typically the same as adopted by the management since the going concern is a prerequisite for the exercise of the industrial activity itself.

1.2. From Solvency 0 to Solvency II

Before the promulgation of Law 742 of 1986, the exercise of insurance activity by Italian insurance undertakings, from the point of view of the capital requirement, was subject to the possession of a minimum share capital, calculated according to the Line of Business (LoBs) exercised.
With the entry into force of Directive 79/267/EEC (see [18]) also known as Solvency 0, the assumption and exercise of the autonomous direct insurance business carried out by undertakings established in a Member State is subject to the holding of capital (see art. 18) at least equal to a specific Minimum Solvency Margin (see art. 19). The proposed formula, which was identified

for the different Lines of Business and policies, was defined as:[3]

$$MSM = 4\% \cdot VB \cdot r_1 + 3‰ \cdot (C - VB)^+ \cdot r_2 \tag{1}$$

Where:

- VB means the mathematical reserves gross of reinsurance,
- The ratio r_1 was calculated as $r_1 = \max\left(\frac{VB^{net}}{VB^{gross}}; 85\%\right)$ where VB were the mathematical reserve, respectively net and gross of reinsurance; it is therefore possible to deduce that the reduction in the Minimum Solvency Margin thanks to reinsurance was equal to a maximum of 6‰ of the mathematical reserves,
- $(C - VB)^+$ are the so-called sums at risk, i.e. the difference between the insured sums C and aforementioned reserves VB. We highlights the fact that the legislator considered these values only if positive: as we will see in the continuation of this thesis, this formulation only concerns policies whose benefits are mainly linked to the death of the insured (Term Insurance and Endowment policies),
- The ratio r_2 was defined as $r_2 = max\left(\frac{(C - VB)^{+,net}}{(C - VB)^{+,gross}}; 50\%\right)$. Also in this case, it can be deduced that the maximum reduction of the MSM due to reinsurance was equal to 1.5‰ of the sums at risk when positive.

An interesting aspect lies in the fact that the aforementioned 4% of the mathematical reserves can be understood as the sum of two capital requirements:

[3]The presence of additional reserves for specific LoBs is reminded

3% of the mathematical reserves is linked to the investment risk (a preliminary form of market risk) while the remaining 1% is linked to the expense risk. Therefore, the second component of formula (1) is to be understood as the first formulation of a capital requirement linked to demographic risk: however, since it refers only to policies with positive Sum at Risk, $3‰ \cdot (C - VB)^+$ should be treated as a capital requirement linked to mortality risk.[4]

In regard to Solvency 0 we underline that the mathematical reserves mentioned above are equal to the sum of the pure mathematical reserve and the mathematical reserve for expenses. The pure one is defined as the expected present value of the benefits net of the expected present value of the premiums: both the aforementioned quantities are calculated on first-order (financial and demographic) bases, i.e. fixed and prudential technical bases. Neglecting here the presence of guarantees and options, which will be discussed when the Solvency II framework will be presented, right now it is emphasized that at the time of subscription it is possible to define the value of the mathematical reserve rate (defined as the ratio between the mathematical reserve and sum insured) over the entire contractual time span. Its reason relies on the fact that the technical rate used is deterministic and constant and that the survival probabilities used are not subject to revision.

Directive 2002/83/EC, with the aim of increasing the clarity which disappeared after the introduction of the Second and Third Directives (in the Life insurance they were respectively 90/619 and 92/96 which, however, do not

[4]In the event that reference was made to Term Insurance with a duration lower than 3 years, the percentage shifted from 0.3% to 0.10%; for TIs with a duration of 3 to 5 years, it was 0.15%.

modify the Capital Requirement) recast the regulation of life insurance undertakings and is a prelude to the introduction of a new system that fully regulates the insurance sector: Solvency I.
From this brief summary of the legislative landscape prior to the introduction of Solvency II, it is clear that despite the innovativeness of the juridical law aimed at guaranteeing the solvency of insurance undertakings, the factor-based approach does not allow specific identification of risks and precise quantification of the same.In particular, it is remarked that:

- Some risks with a particularly pronounced impact are completely neglected: in particular the longevity risk, the catastrophe risk and the risk linked to lapses,
- The benefit of diversification is not considered in any way: by neglecting the dependencies between risks, it is not possible to exploit natural hedging in order to reduce the risk profile of the undertaking,
- Reinsurance is treated without differentiating according to the type of contract: whatever reinsurance was underwritten, it was considered only through the factors r_1 and r_2 mentioned in formula (1),
- The valuation of technical liabilities with locked demographic and financial bases does not allow for the acknowledgement of any exogenous changes/shocks: in particular, neither a change in demographic assumptions nor a structural change in the financial context affect the valuation of reserves.

The aforementioned limits, together with a profoundly changed financial framework (i.e. the evolution of the risk-free rate curve) has led to a radical change in prudential supervision, setting the goal of creating a system oriented to the quantification of economic capital, intended as an indicator of the effective risk profile of the individual undertaking.

1.3. Solvency II: introduction and structure

The process of conception and adoption of Solvency II was as long as it was troubled: the turning point coincides with the approval by the European Parliament of Directive 2009/138/EC on 22^{nd} April 2009; subsequently it was modified until reaching the current version of 30^{th} June 2021.
Its regulatory structure follows the *Lamfalussy* process: it is built on four levels, briefly summarized below:

- The first level concerns the Directive itself, stating the general principles and introducing the 3-pillar system,
- The second level concerns implementing measures: technical implementing measures adopted by the European Commission in the form of Delegated Regulation (see [19]). This second level therefore contains both the details of the Standard Formula and the methodologies for implementing the principles of the Directive,
- The Omnibus II Directive (see [20]) introduces an intermediate light concerning the technical measures proposed by the European Insurance and Occupational Pensions Authority (EIOPA) and adopted by the European Commission. These technical standards are oriented towards

internationalization (regulatory technical standards) and the standardization of the European regulatory landscape (implementing technical standards),

- The third level consists of the non-binding guidelines addressed to the individual Supervisory Authorities: these are neither mandatory nor coercive measures, but if they are rejected it is necessary to explain the reason.

- The fourth level concerns the verification and application of the directive by individual states and provides for sanctions against those who have not complied with Community legislation.

This rigid structure, aimed at limiting the freedom of manoeuvrer of national legislators, makes it possible to standardize insurance and insurance activities within the Member States. The risk-based economic approach takes the form of a three-pillar structure, where each pillar concerns a specific macro-area. Pillar I concerns the valuation of assets and liabilities, the definition and quantification of both the SCR and the Minimum Capital Requirement and, in conclusion, the identification and classification (tiering) criteria of own funds. Pillar II, on the other hand, concerns the principles of corporate governance, management and management rules: the proposal of a qualitative management aims to reduce all those risks not identified by the first Pillar, but whose economic effects would have repercussions on the prospective trend of the undertaking: an example is reputational risk. Pillar II also concerns the *Own Risk and Solvency Assessment* (known by its acronym, ORSA), that is the procedure according to which each insurance undertak-

ing is required to assess both its own risk profile and the consistency of the calculation criteria, of the capital requirement with its own profile. Pillar II ends with the supervisory review process: the supervisory authority has the task of evaluating the ORSA by requesting any capital add-ons if necessary to improve the solvency situation of the undertaking.
In conclusion, the third pillar regulates the information that each undertaking must provide on the one hand to the Supervisory Authority and, on the other hand, to the market: this information is both periodic (Report to Supervisors and Quantitative Report Templates) and extraordinary. The introduction of a high degree of transparency and market discipline encourages undertakings to better manage risks, precisely in order not to suffer penalties from the market.

1.4. The valuation of assets and liabilities in the Economic Balance Sheet

The valuation of assets and liabilities in the Economic Balance Sheet is a key element both from a Solvency II perspective and for the implementation of stochastic models aimed at quantifying the SCR. The starting point for the assessment in question is Article 75 of the Directive: it, called the fair value principle, states that "*assets shall be valued at the amount for which they could be exchanged between knowledgeable willing parties in an arm's length transaction*"while the liabilities "*shall be valued at the amount for which they could be transferred, or settled, between knowledgeable willing parties in an arm's length transaction*". Furthermore, where there is consistency in terms of fair value with the Directive, undertakings must also consider Regulation

(EC) N°. 1606/2002 containing provisions on the use of International Accounting Standards in order to ensure homogeneity and transparency in the valuations.
From an application point of view, Article 10 of the Delegated Acts defines 3 criteria (in preferential order) to be used:

1. If the asset or liability is regularly traded on an active market, its value coincides with "*quoted market prices in active markets*". In this context, the "replicating portfolio method" is perfectly allowed: it allows you to calculate the value (price) of a certain financial instrument with certain cash flows, such as the sum of the market prices of the instruments which, if considered within a portfolio, exactly replicate the cash flows of the instrument being valued,

2. If the quoted price is not available, it is possible to use the quoted price of similar assets / liabilities, applying corrections to take into account the difference element,

3. If it is in no way possible to exploit neither market prices nor other inputs directly observable on the market, the Delegated Acts allow for the undertakings to use techniques consistent with at least one of the following methods: market approach (e.g. matrix pricing), income approach (based on the conversion of future cash flows into current values, e.g. option pricing) and cost approach (linked to the cost of replacing the financial instrument).

1.5. The valuation of technical provisions in the Economic Balance Sheet

In this subsection the criteria linked to the evaluation of Technical Provisions (TPs) are investigated, paying particular attention to the *Best Estimate* component: as will be seen in the following, the Risk Margin is not considered in quantifying the SCR, therefore a correct use of the first component, will allow the drafting of stochastic models consistent with the legislation.
First of all, it is specified that, consistently with Art. 10 investigated previously, if the valuation of liabilities is possible by observing the quoted prices of active markets, this solution remains the best-choice: i.e. policies without demographic risk and with a single financial component (particular types of Index-Liked and specific Units -Linked). The situation is much more thorny when considering traditional policies or, more simply, policies with a demographic component: taking a simple Term Insurance as an example, it is intuitive to grasp that neither financial products nor replicating portfolios can produce the same outflows of the policy. Such liabilities, called *non-hedgeable*, must be valued in accordance with Art. 77 of the Directive. The purpose is to calculate the so-called *current exit value*, which is the value that the undertaking would have to pay if it were to transfer the obligation to another company.
In this regard, the Directive specifies that the value of the technical provisions must be calculated as the sum of two components: Best Estimate and Risk Margin.

1.5.1. The Best Estimate

The first component is defined as "*the probability-weighted average of future cash-flows, taking account of the time value of money (expected present value of future cash-flows), using the relevant risk-free interest rate term structure*";[5] furthermore, the Directive specifies that it must be calculated gross of reinsurance, based on credible, realistic and up-to-date information.

The first element to consider is the the time span within which the cash flows must be considered: this topic, defined as *boundary of the contract*, refers to the fact that the boundary coincides with the last moment when the undertaking has the unilateral right to terminate the contract, to refuse the award or has the unconditional ability to modify the awards and future benefits.

The second topic regards the breadth of the spectrum of cash flows to be considered within the Best Estimate considering the closed portfolio with respect to the future new business. Art. 28 of the Delegated Acts provides a list of cash flows to be considered in calculating the Best Estimate. Starting from the cash outflows, on the first hand, they concern the benefits to be paid to policyholders (maturity, death, surrender, annuities, disability benefits and medical expenses). On the second hand, they also concern all forms of expenses (direct and indirect administrative, investment management and claims management), any fees (both management and structural) and taxation payments which charged to policyholders / are required to settle the insurance obligations. Cash inflows regard every premium that will be collected by policyholders, both discretionary and those expected due to the

[5] Article 77, paragraph 2, Directive 2009/138/EC

contracts.
We should observe that the undertaking, based on one's specific past experience, produces a large number of realistic bases (so-called second order).
Two other elements of fundamental importance in the evaluation of the best estimates related to the Life business concern *future discretionary benefits* and *financial guarantees / contractual options*.
Future management actions are those actions of management that affect the benefit due to the policyholder. Since both future discretionary benefits and the capital gains management (e.g. in case of segregated funds) make the valuation of technical provisions discretionary, Art. 23 of the Delegated fully regulates future management measures. Briefly, they must be declared in advance, they must be consistent both with each other and with the undertaking business strategies.
With reference to financial guarantees, Art. 79 of the Delegated Acts specifies that the undertaking must consider *the value of financial guarantees and any contractual options included in insurance and reinsurance policies.* Further information derives from the Technical Specifications (TP.2.120), where it is specified that in the absence of quoted prices in the ADLT (active, deep, liquid and transparent) markets, it is necessary to use a mark-to-model approach. This result must be:

- consistent with the ADLT markets
- consistent with the assumption of absence of arbitrage
- consistent with the risk-free rate structure

This means that, indicating with V_t the mark-to-model value of the insurance obligation and with Z_t the known risk factors at time t:

$$V_t = f(t, Z_t) = \mathbb{E}^{\mathbb{Q}}[\text{FDCF}|\mathcal{F}_t] \tag{2}$$

From a mathematical point of view, we choose a sufficiently rich probability space $(\Omega, \mathcal{F}, P)$ and an increasing sequence of σ-fields $\mathbb{F} = (\mathcal{F}_t)_{t=0,\ldots,n}$ on $(\Omega, \mathcal{F}, P)$. Therefore, $(\Omega, \mathcal{F}, P, \mathbb{F})$ is a filtered probability space with filtration $\mathbb{F}$, while the σ-field $\mathcal{F}_t$ represents the information available at time t.
We assume that every component of the $\mathbb{F}$-adapted cash flow Z on $(\Omega, \mathcal{F}, P, \mathbb{F})$ is square integrable. Considering formula (2), f is a function that assigns a monetary value at time t to the cash flow Z_t, i.e. it attaches an $\mathcal{F}_t$-measurable price to the cash flow Z_t. $FDCF$ stands for future discounted cash flows (discounted with specific financial rates, i.e., risk-free rates) while $\mathbb{Q}$ is the "risk-neutral" probability measure. It is evident that the evaluation of the expected value defined in formula (2) requires the use of stochastic models where the solution is not known (considering exceptions, the evaluation of a European or American call: the Black-Scholes model and Longstaff & Schwartz'one propose solutions obtainable without the use of nested simulations).
The last aspect to be clarified in relation to the Best Estimate concerns the discounting factor: as can be seen from the definition, the interest rate used on the individual maturity must be the risk-free one. The rationale coincides with the fact that the risk adjustment (where necessary, for example in the valuation of hedgeable liabilities the value of the replicating portfolio already contains the compensation in question) is considered with the addition of the Risk Margin. The risk-free interest rate chosen is the interest rate swaps one,

adjusted for credit risk: this deduction must be between 10 and 35 bps. The calculation of the risk-free rate curve is carried out monthly by EIOPA for each individual currency: the risk-free rates are considered up to a maturity called Last Liquidity Point and subsequently extrapolated so that the long-term forward rate (so-called Ultimate Forward Rate) coincides with the predetermined forward rate.
Ultimately, the presence of alternative adjustments to the interest rate is allowed:

- The Matching Adjustment, the use of which is subject to the approval of the Supervisory Authority, which is applicable only if there is an intention to hold the assets to cover the technical reserves until maturity (the liabilities must therefore not be liquid: for example there must be no options for the policyholder to surrender early),
- The Volatility Adjustment, provided by EIOPA and applicable up to the Last Liquidity Point, which aims to reduce the impact of market volatility in the short term in times of market stress,
- Finally, it is specified that the quantification of aforementioned instruments is subject to improvement/modification with the in progress Solvency review .

1.5.2. The Risk Margin

As previously mentioned, the Risk Margin is that quantity which, added to the Best Estimate, allows to calculate the current exit value of liabilities; moreover, the Risk Margin satisfies the need to consider the risk aversion of financial operators.

It is possible to write the Risk Margin formula as:[6]

$$RM(P) = CoC \cdot \sum_{t \geq 0} \frac{SCR_t^{RM}}{(1 + i(0, t+1))^{t+1}} \tag{3}$$

where P is the whole undertaking's portfolio, SCR_t^{RM} is the SCR specific to the calculation of the Risk Margin (ratios and differences will be explained in the next lines), CoC is the Cost of Capital and $i(0, t+1)$ is the risk-free rates between 0 and $t+1$.

The first interesting topic concerns the level of aggregation: as can be seen from formula (3), the Risk Margin is calculated at the level of the entire undertaking portfolio. As will be shown shortly, unlike the Best Estimate, the Risk Margin is a sub-additive function and an assessment at the level of the entire portfolio allows to exploit the maximum benefit deriving from diversification: in this way, the undertaking can benefit from both the diversification between the LoBs and within the LoBs. However, it should be added that the Technical Specifications, with reference to the calculation of the technical provisions, also require to calculate them for homogeneous groups (which bear the same risks), at most by dividing into LoBs: it follows that it is a duty of the undertakings to use methodologies to allocate the portions of Risk Margin to the portfolio segments. In order to put the description of the Risk Margin before the topic of its allocation, we now proceed to the description of the components and finally the discussion will be concluded. CoC stands for Cost of Capital: it is a coefficient equal to 6%

[6]The Solvency II Review 2020 brings a slight change regarding the discounting of the most distant SCRs, in particular those over 20 years

and, citing Article 75 paragraph 5 of the Directive, it "*shall be equal to the additional rate, above the relevant risk-free interest rate, that an insurance [...] undertaking would incur holding an amount of eligible own funds [...] equal to the Solvency Capital Requirement necessary to support insurance and reinsurance obligations over the lifetime of those obligations*".

It is therefore noted that the CoC rate is multiplied by the sum of all future SCRs, discounted to the current date. It is emphasized that the difference in the subscripts between the SCRs and the qualification factors is linked to the need to hold the SCR for the entire following year. In conclusion, it is specified that the SCRs of the various years that fall within the calculation of the Risk Margin only concern the following risks:

- The underwriting risk linked to the existing business
- The market risk different from the interest rate risk (only if relevant)
- Credit risk linked to insurance contracts, vehicle companies, intermediaries and contractors
- The counterparty risk with respect to passive reinsurance contracts

In conclusion, a methodology proposed in the Technical Specifications is presented to distribute the Risk Margin, calculated on the entire portfolio benefiting from the effect of diversification, to the individual LoBs:

$$RM(LoB) = \frac{SCR_{RU,LoB}(0)}{\sum_{LoB} SCR_{RU,LoB}(0)} \cdot RM(P) \tag{4}$$

It is specified that this proxy allocates an aggregate value as a function of values that do not consider the benefit of diversification.

1.6. SCR and Life Underwriting Risk

1.6.1. The Solvency Capital Requirement: definition and operational aspects

According to Section 4 of the Directive, each undertaking must possess sufficient eligible own funds to cover the SCR: this requirement must be calculated on the risk profile of the individual company at least once a year, must be communicated to the Supervisory Authority and must be based on the assumption of business continuity.
Quoting Art. 100 of the Directive, paragraph 4, it *It shall correspond to the Value-at-Risk of the basic own funds of an insurance or reinsurance undertaking subject to a confidence level of* 99.5% *over a one-year period.* Hence, indicating with $\tilde{E}_{t+1}$[7] the random expenses amount and with $i(0,1)$ the risk-free spot rate between 0 and 1, the company's eligible own funds SCR at time $t+1$, are defined as the x that satisfies the following relation:

$$\begin{aligned} SCR &= inf\left[x : \mathbb{P}[\tilde{E}_{t+1} + x \cdot (1 + i(t,t+1)) \geq 0] = 1 - \alpha\right] \\ E_t + SCR &= VaR_{1-\alpha}[\Delta\tilde{E}_{t+1}] \end{aligned} \tag{5}$$

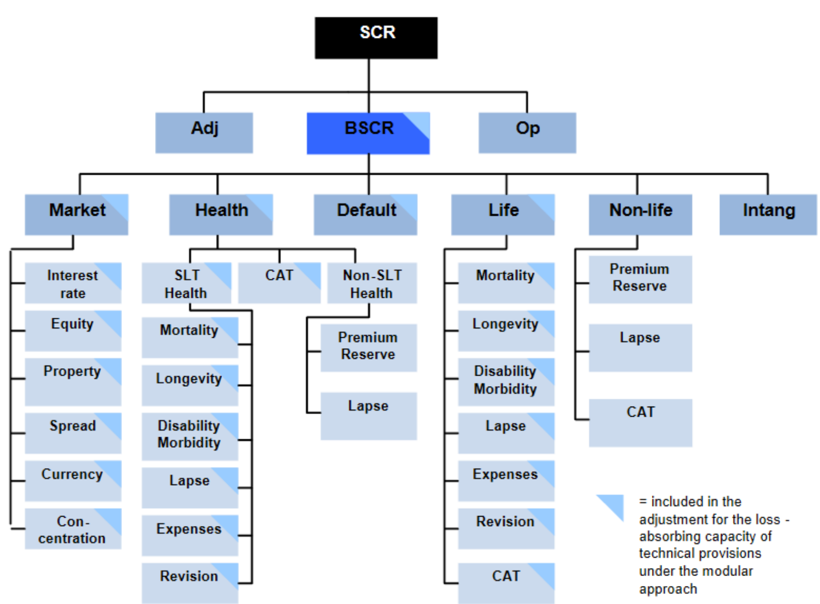
Qualitatively, therefore, the SCR coincides with the $1-\alpha$ percentile of the distribution of the variation of the eligible own funds. An extremely important aspect that will have significant consequences in the stochastic models presented in the next sections, concerns a specification of the Technical Provisions: in the calculation of the SCR, in order to avoid circularity, any reference to the Technical Provisions is to be understood net of the Risk

[7] All the random variables of this Thesis are indicated with the tilde

Margin. In other words, therefore, the Risk Margin is not to be considered in the models designed to quantify the SCR.
The SCR must be calculated with one of the following three methods, listed from the simplest to the most complex:

- *Standard Formula*: It presents the essential risks that any undertaking must consider (these macro modules are: Market risk, Health underwriting risk, Default Risk, Life underwriting risk, Non-Life underwriting risk and Intangible), proposes correlation matrices both to aggregate the SCRs of the aforementioned macro modules , both to aggregate the risks of the sub-modules and, finally, proposes calculation methods to calculate the individual SCRs. The structure of the Standard Formula is shown in Figure 1. These methodologies are the "factor based"

Figure 1: The structure of the Standard Formula



and "scenario based" approaches.

In the first, the coefficients (sigma factors) set by the legislation are applied to the typical undertaking quantities. It is noted that the MSM proposed for demographic risk by Solvency I pertained to this category. The "scenario based" approach, the SCR is calculated by observing the difference between two scenarios: that of normality and the "stressed" one, where some grades are precisely stressed according to the provisions of the Delegated Acts. It is emphasized that, with the scenario-based method, in addition to the Risk Margin which must not be changed in any way, the stressed scenario does not even concern the values of deferred tax assets and liabilities, the value of future profit sharing and, above all, the value of the company does not adopt extraordinary measures.

- *Undertaking Specific Parameters* (USP): The USP approach allows the undertaking to independently calibrate parameters set by the Delegated Acts in a Standard Formula context. This methodology allows you to grasp the actual risk profile of the company with more precision: in particular, the USP approach allows you to modify the sigma factors of the tariff and reserve risk for a Non-Life partner and the increases in the amount of benefits of annuities in the "revision" submodule of the Life & Health Underwriting risk macro modules.

- *Internal Model (Partial or Full)*: these methodologies, subject to a strong review and approval by the Supervisory Authority, allow the undertaking to autonomously model one or more macro risk modules

1.7. Life Underwriting Risk in the Standard Formula

As anticipated when the Standard Formula was discussed and as evident from Figure 1, the Life Underwriting Risk is one of the macro-modules, it concerns one of the specific and technical risks of the insurance business and is the result of the following sub-modules:

- Mortality,
- Longevity,
- Disability,
- Life expense,
- Revision,
- Lapse,
- Life catastrophe.

The aggregation of the sub modules takes place using the following formula and using the linear correlation coefficients shown in Table 1:

$$SCR_{LUR} = \sqrt{\sum_{i,j} CorrNL_{i,j} \cdot SCR_i \cdot SCR_j} \tag{6}$$

where $CorrNL_{i,j}$ denotes the correlation parameter for life underwriting risk for sub-modules i and j. An interesting and singular aspect is the only negative correlation coefficient predicted by the Standard Formula, that is the coefficient between Longevity and Mortality, exactly the risks I will focus on. The two risks, as you can intuitively think, are "opposite" but not completely: although it is logical that the increase in survival leads to a reduction

Table 1: Correlation matrix of Life Underwriting Risk

i,j	**Mortality**	**Longevity**	**Disability**	**Life Expense**	**Revision**	**Lapse**	**CAT**
Mortality	1						
Longevity	-0.25	1					
Disability	0.25	0.25	1				
Life Expense	0.25	0.25	0.5	1			
Revision	0	0.25	0	0.5	1		
Lapse	0	0.25	0	0.5	0	1	
CAT	0.25	0	0.25	0.25	0	0.25	1

in mortality (so-called natural hedging), usually the two risks act on insurance portfolios profoundly different in terms of policyholders. The mortality risk, generated mainly by Term Insurance policies, derives from policyholders exposed to the risk of death, i.e. policyholders who tend to be elderly. The longevity risk, on the other hand, concerns policyholders with a high life expectancy, therefore generally young and in good health. The phenomenon whereby the intrinsic characteristics of a policy attract certain types of policyholders relates to the issues, known in the literature, of self-selection and adverse selection.

1.7.1. Mortality and longevity risks: the Standard Formula approach

Mortality risk is defined as "*the risk of loss, or of adverse change in the value of insurance liabilities, resulting from changes in the level, trend, or volatility of mortality rates, where an increase in the mortality rate leads to an increase in the value of insurance liabilities*".

This risk affects those policies that provide for the payment of a benefit, subject to the death of the insured, which is greater than the technical provisions: this definition, attributable to that of positive Sum at Risk, is linked to Term Insurance and Endowment policies. For the latter, the EIOPA Technical Specifications exclude unbundling, i.e. allow the "natural" compensation between the component of the benefit in the event of death and the component of the benefit in the event of survival. The $SCR_{LUR,mortality}$ is calculated as a variation of the Basic Own Funds (BOF):

$$SCR_{LUR,mort} = \Delta BOF|(shock_{mort}) \tag{7}$$

From a qualitative point of view, the SCR is therefore calculated as the difference between the Basic Own Funds of the undertaking in the ordinary situation and the Basic Own Funds in the scenario with mortality rates increased instantly and permanently by 15% ($shock_{mort}$). A simplified methodology is proposed in the Technical Specifications (SCR.7.19.) and it can be applied subject to the so-called "principle of proportionality", i.e. if the calculation with the Standard Formula is disproportionate and excessive compared to the size of the portfolio.

$$SCR_{LUR,mort} = 0.15 \cdot SaR \cdot q \cdot \sum_{k=1}^{n} \left(\frac{1-q}{1+i(0,k)} \right)^{k-0.5} \tag{8}$$

where:

- SaR stands for Sum At Risk,
- q is the weighted average of the mortality rates, where the weights coincide with the insured sums,
- n is the modified duration expressed in years relating to the possible

payments in the event of death included in the calculation of the Best estimate,

- $i(0,k)$ is the risk-free spot rate relating to maturity k.

Longevity risk, on the other hand, is defined as *the risk of loss, or of adverse change in the value of insurance liabilities, resulting from changes in the level, trend, or volatility of mortality rates, where a decrease in the mortality rate leads to an increase in the value of insurance liabilities*".

This risk therefore affects policies that provide for the payment of a benefit subject to the survival of the policyholder at a certain date: in this case, therefore, a reduction in mortality rates implies an increase in the expected present value of the benefits and a consequent increase in the Best Estimate. The SCR linked to the longevity risk proposed by the Standard Formula, also calculated with a scenario based approach, is equal to:

$$SCR_{LUR,long} = \Delta BOF|(shock_{long}) \qquad (9)$$

where $shock_{long}$ represents a decrease of mortality rates instantly and permanently by 20%. As for the mortality risk, the Regulation proposes a simplified formulation in order to respect the principle of proportionality, i.e. when the calculation with the approach envisaged by the Standard Formula is disproportionate to the undertaking business.

$$SCR_{LUR,long} = 0.2 \cdot q \cdot n \cdot 1.1^{(n-1)/2} \cdot BE_{long} \qquad (10)$$

Where:

- q is the average mortality rate expected by policyholders over the next 12 months; the weights of the weighting coincide with the sums insured,

- n is the duration in years of the benefits to be provided to policyholders included in the calculation of the Best Estimate,

- BE_{long} is the Best estimate of liabilities exposed to longevity risk.

2. The stochastic model for demographic profit in the Local GAAP framework

2.1. Introduction

The previous Section 1 showed the regulatory evolution of the capital requirement linked to demographic risk. As stated in the Introduction, with the aim of presenting a stochastic model that is consistent with the market consistent valuation of Solvency II, this section presents a model known in literature, developed in a Local Generally Accepted Accounting Principles (Local GAAP) context (see [33], [34]).

The purpose is to show what the underlying assumptions are, what types of risks the model identifies and what (numerical) results it brings: it is observed that the fundamental quantities of the model (insured sums of occurred deaths and insured sums of occurred lapses) are the same of a market consistent model, with the difference that in the latter case the assessment criteria of assets and liabilities changes radically.

In conclusion, it is anticipated that the case study reproduced will be applied to two traditional without-profit policies (policies whose benefit is deterministic and is not linked to any financial instrument): the aim is to isolate the demographic risk and prevent the results from being linked to market risk.

2.2. From Technical to Demographic Profit

The starting point coincides with the presentation of a "corporate" equation aimed at quantifying the Risk Reserve. It, also called surplus or asset margin, indicates the difference between assets and liabilities and, therefore, can be considered as free own funds.

Before presenting the aforementioned equation, two particularly important elements are highlighted for the understanding of the theoretical actuarial aspects of this thesis:

- $\mathcal{F}_t$ indicates natural filtration containing the information available at time t. Each random variable, indicated with the tilde symbol, refers to the respective stochastic process and is conditioned by the natural filtration of the evaluation instant. Except in cases where it will be explicitly indicated, all the stochastic processes are adapted to $\mathcal{F}_t$: therefore the values assumed at the time of evaluation are deterministic.
- The evaluation instant is always $t = 0$. This implies that, technically, even the random variables evaluated at time t would be random. Since the objective of each stochastic model presented in this article is to evaluate the capital requirement between t and $t + 1$, it is assumed that we position at instant t where all the stochastic processes have assumed their average value so as to be able to evaluate the volatility in the time span alone $[t, t + 1)$.

Resuming the definition of the risk reserve $\tilde{U}_{t+1}$, it is now possible to present the formulation:

$$\tilde{U}_{t+1} = U_t + UN_{t+1} + [(\tilde{YL}_{t+1} + \tilde{CH}_{t+1}) - \tilde{TX}_{t+1} - \tilde{DV}_{t+1}] \qquad (11)$$

Where:

- UN_{t+1} is the monetary amount of the (possible) payment of further funds by the shareholders: it is noted that this value is not random as it is assumed that the planning of capital payments takes place at the beginning of the year,
- $\tilde{YL}_{t+1}$ is the gross ordinary profit for the year, not influenced by extraordinary items and before taxes,
- $\tilde{CH}_{t+1}$ is the random variable of the gains (losses) linked to changing in value of the assets market values,
- $\tilde{TX}_{t+1}$ is the random variable for taxes pertaining to the year,
- $\tilde{DV}_{t+1}$ is the random variable dividends to be distributed to shareholders.

Considering the case of without-profit policies[8], gross ordinary profit $\tilde{YL}_{t+1}$ can be defined as:

$$\tilde{YL}_{t+1} = [\tilde{Y}_{t+1} + (\tilde{JT}_{t+1} - \tilde{J}_{t+1})] \tag{12}$$

The gross ordinary profit $\tilde{YL}_{t+1}$ is therefore defined as the sum of the insurance profit for the year $\tilde{Y}_{t+1}$ and the financial return due to the difference between the financial interests realized by all the activities of the company

[8] In the Local GAAP context, the addition of revaluations due to Segregated Funds would not involve particular changes in the model as the insured sums are valued at the valuation date and any financial guarantees do not change their value

$\tilde{JT}_{t+1}$ net of the investments realized only on the insurance resources $\tilde{J}_{t+1}$, hence premiums and reserves. it is now possible to define the insurance profit for the year, it is equal to:

$$\tilde{Y}_{t+1} = [VB_t + B_{t+1} + \tilde{J}_{t+1}] - [\tilde{E}_{t+1} + \tilde{S}_{t+1} + \tilde{X}_{t+1} + \tilde{VB}_{t+1}] \qquad (13)$$

Where:

- VB_t represents the complete mathematical reserve calculated at time t. It therefore coincides with the sum of the pure mathematical reserve with the reserve for expenses: the first is equal to the expected present value of the benefits net of the expected present value of the premiums, while the second is equal to the expected present value of the expenses net of the value current expected charges for expenses. It is specified that both the pure mathematical reserve and the reserve for expenses are calculated (consistently with the Local GAAP framework) on a first-rate technical basis: in this case, the expected present values are calculated using the same prudential demographic base and the same rate technical used in the pricing phase.

- B_{t+1} represents the amount of gross premiums collected by the undertaking therefore they, called also tariff premiums, include both the pure premium (calculated on the technical basis of the first order) and the expenses loadings. It is assumed that they are collected at the beginning of the year: therefore late payments by policyholders are excluded and, for these reasons, they are deterministic.

- $\tilde{E}_{t+1}$ indicates the stochastic amount of expenses incurred by the undertaking. They are also assumed to be paid at the beginning of the year.

- $\tilde{S}_{t+1}$ indicates the total amount of lapses: they are also assumed at the beginning of the time span $[t, t+1)$.

- $\tilde{X}_{t+1}$ represents the amount of claims occurred in the year and paid at the end of the same year. From an exclusively qualitative point of view, $\tilde{X}_{t+1}$ is equal to 0 in the case of Pure Endowment, is equal to the insured sum in the case of Term Insurance and Endowment and equal to the instalment in the case of annuity.

The starting point of this thesis coincides with the presentation of a stochastic model based on the theory of risk oriented to the quantification of demographic risk in a Local GAAP context (see [33], [34]).
First, we consider the formulation of the technical profit at a generic instant $t+1$:

$$\tilde{Y}_{t+1}^{LG} = [VB_t + B_{t+1} + \tilde{J}_{t+1}] - [\tilde{E}_{t+1} + \tilde{S}_{t+1} + \tilde{X}_{t+1} + V\tilde{B}_{t+1}] \tag{14}$$

Before explaining the meaning of the quantities present in formula (14), we specify that each random variable is indicated with the letter tilde: since the objective of this thesis is to evaluate the SCR over an annual time horizon (see Art. 101 of [21]), we assume to be at time t and to make an assessment considering the time span $[t, t+1)$. From a probabilistic point of view then, we are assuming that all of stochastic processes are adapted to filtrations $(\mathcal{F}_t)_{t \epsilon I}$, therefore measurable with respect to the natural filtrations $\mathcal{F}_t$[9].

[9] I is a index set with a total order $\leq$

Returning to formula (14) we observe that it is calculated as the difference between two sums. The first one concerns the complete mathematical provisions VB_t calculated with Local Generally Accepted Accounting Principles (Local GAAP)[10], the volume of premiums collected B_{t+1} and the financial returns obtained from the investment of business assets $\tilde{J}_{t+1}$ in the time span $[t, t+1)$. The minuend is made up of the sum of expenses $\tilde{E}_{t+1}$, lapses $\tilde{S}_{t+1}$, claims occurred during the year $\tilde{X}_{t+1}$ and settled at instant $t+1$ and, finally, the new complete mathematical reserves $\tilde{VB}_{t+1}$. As demonstrated in [33], it is possible to break down the technical profit described in formula (14) into 5 profit components: demographic, financial, expenses loading, lapses and residual. Since our aim is to analyse the demographic profit and quantify a SCR that is consistent with the market consistent valuation introduced by Solvency II, we now report the first two components previously mentioned, defining the quantities that constitute them.

Before presenting the two aforementioned components, we introduce the rate-based notation: in the continuation we will use lowercase letters to indicate the "rates", that is, quantities linked to unitary insured sums precisely in order to highlight the main drivers of the effects on the various components of profit.

$$\begin{aligned} \tilde{y}_{t+1}^{LG,Dem} =& [v_t^b + b_{t+1} \cdot (1 - \alpha^* - \beta^*) - \gamma^*] \cdot (w_t - \tilde{s}_{t+1}) \cdot (1 + j^*) \\ & - (\tilde{x}_{t+1} + \tilde{w}_{t+1} \cdot v_{t+1}^b) \end{aligned} \tag{15}$$

[10]The complete mathematical reserve is therefore the sum of the expenses reserve and the pure mathematical reserve. Both are calculated with locked and prudential technical bases: that is, they are calculated with the technical rate and with the mortality table used in the pricing phase.

The random variable "demographic profit in LG context at time $t + 1$" $\tilde{y}_{t+1}^{LG,Dem}$ is defined as the difference of two terms. The first one concerns the sum of the complete technical provisions v_t^b and the pure premiums collected in $t+1$ (defined as the tariff premiums b_{t+1} net of loading for acquisition α^*, collection β^* and management costs γ^*). This first term is multiplied by the monetary amounts, defined as the insured sums calculated at time t w_t net of the sums eliminated because of lapses $\tilde{s}_{t+1}$, and subsequently capitalized for one year at the technical rate j^*, technical financial basis (i.e. technical rate).[11]

The second term, on the other hand, is composed of the sum of the amount of the claims occurred in year $\tilde{x}_{t+1}$ and the new complete technical provisions established in year $t+1$, v_{t+1}^b. We note that the definition of $\tilde{x}_{t+1}$, typical in the non-life insurance context, indicates the payments of the insured sums in the face of the deaths of policyholders: this implies that, as will be detailed below, this random variable will be considered only in the case of Endowments and Terms Insurance, while it will take on the value 0 in the case of Pure Endowments and annuities except in the distribution phase. Standing

[11] At this stage we introduce the notation linked to the technical (demographic and financial) bases: those of the first order, prudential and used in the pricing phase, will be indicated with an asterisk $*$, while those of the second order considered "realistic" by the undertaking, will be referred to as j and q

$\tilde{z}_{t+1}$ the insured sums of occurred deaths:

$$\tilde{x}_{t+1} = \begin{cases} \tilde{z}_{t+1}, & \text{for term insurance and endowment policies;} \\ 0, & \text{for pure endowment policies and annuities in the accumulation period;} \\ \tilde{w}_{t+1}, & \text{for annuities in the benefit period.} \end{cases} \tag{16}$$

Although this is not exactly the point of this thesis, we now present the formulation of financial profit: it is intuitive to think that in a market consistent context, where non-hedgeable liabilities are valued at risk-free rates, it is impossible to analyse the demographic profit without considering the iterations with financial profit.

$$\begin{aligned} \tilde{y}_{t+1}^{LG,Fin} =& (\tilde{j}_{t+1} - j^*) \cdot \Big[v_t^b \cdot w_t + b_{t+1} \cdot (1 - \alpha^* - \beta^*) \cdot (w_t - \tilde{s}_{t+1}) \\ & - \gamma^* \cdot w_t - g_t \cdot v_t^z \cdot \tilde{s}_{t+1} \Big] \end{aligned} \tag{17}$$

where g_t is a specific penalization coefficient applied in case of surrender and v_t^z is the Zillmer reserve. The most interesting aspect to underline is that the financial profit depends on the difference between the yield actually achieved by the undertaking and that assumed during the pricing phase: the financial profit will therefore be as much greater as the yield. obtained from insurance investments exceeds the return assigned to policyholders.
Taking up the formulation of demographic profit in the Local GAAP context defined in formula (15), it is possible (see [34]) to define a "more compact" version, which highlights the model drivers:

$$\tilde{y}_{t+1}^{LG,Dem} = D_{t+1}^{LG,compl} \cdot [q_{x+t}^* \cdot (w_t - \tilde{s}_{t+1}) - \tilde{z}_{t+1}] \tag{18}$$

Therefore follows that the demographic profit assessed in a Local GAAP context depends on the probabilities of death used in the pricing phase q_{x+t}^*,

the insured capital w_t net of the insured sums of occurred lapses and the insured sums of occurred deaths.
The first term, $D_{t+1}^{LG,compl}$, indicates the complete (of expenses) Sum-at-Risk (SaR) rate, defined as:

$$D_{t+1}^{LG,compl} = \begin{cases} (1 - v_{t+1}^b), & \text{for term insurance and endowment policies;} \\ -v_{t+1}^b, & \text{for pure endowment policies;} \\ -v_{t+1}^b, & \text{for annuity in accumulation period (t} \leq \text{m) ;} \\ -(1 + v_{t+1}^b), & \text{for annuity in the payment period (t} \geq \text{m).} \end{cases} \tag{19}$$

2.3. The cohort approach and the exact individual approach

In this section we present the general aspects of the model, in particular we follow the so-called cohort approach. Let's assume that the portfolio is divided into sub-portfolios of homogeneous risks. In this case, each policyholder within the same cohort has the same age, the same gender, the same survival probability and so on: the only element of differentiation between policyholders within the same cohort concerns the sums insured, denoted with C_i for the policyholder i. Particularly notable is that policyholders within the same cohort are assumed to be independent and identically distributed.
The use of this assumption has two important consequences. On the one hand it implies that the aggregation and dependencies between the different cohorts must be specifically modelled. On the other hand it is possible

to describe the survival of each policyholder in a given time span with a dichotomous random variable as a Bernoulli distribution. Additionally, the cohort approach is consistent with the framework defined by Solvency II and IFRS17.

We denote with $w_0 = \sum_{i=0}^{l_0} C_i$ the total sums insured of a cohort with l_0 policyholders at the inception of the contract $t = 0$. We assume that the sums insured of a cohort follows the following rule over time:

$$\tilde{w}_t = w_{t-1} - \tilde{s}_t - \tilde{z}_t \tag{20}$$

where $\tilde{z}_t$ are the sums insured of occurred deaths between $t-1$ and t.

In this context, in order to quantify the characteristics of the demographic profit random variable and estimate its confidence intervals, it is necessary to model the insured sums, the insured sums of occurred deaths and, if you want to quantify also the lapse risk, the insured sums of occurred lapse. Two different approaches to the problem were considered.

Considering the exact individual approach, the algorithm provides that:

- We assume that the vector of the exact monetary amounts of the insured sums of each policyholder belonging to the cohort is known. This assumption is consistent with reality, where the vector of insured sums of each policyholder is contained in undertaking's data warehouse. This approach, coherently with the following points, provides for the simulation of the survival/death of each individual policyholder. This is the main difference with the adjusted individual approach presented below, where instead of working with a vector of insured sums, it is considered a generic probability distribution, calibrated on available data, from which to simulate the single insured sum. The simple moment of n order is indicated with $\bar{C}^n_{t+1}$; hence, assuming a cohort composed of l_t policyholders at time t,

$$\bar{C}_{t+1} = \frac{\sum_{i=1}^{l_t} C_i}{l_t} \tag{21}$$

$$\bar{C}^2_{t+1} = \frac{\sum_{i=1}^{l_t} C_i^2}{l_t} \tag{22}$$

$$\bar{C}^3_{t+1} = \frac{\sum_{i=1}^{l_t} C_i^3}{l_t} \tag{23}$$

It is therefore anticipated that in an exact individual approach each policyholder has his specific insured sum: this phenomenon leads to volatility within the insured sums. This volatility, as explained in the next point, is quantified by individually simulating the survival of the individual insured and, therefore, the fact that her/his specific insured sum remains or not within the cohort insured sums. This approach obviously has longer computational times than an approach where each single sum insured is treated as the realization of an appropriately parameterized random variable, but allows to exactly quantify the effect of death (and therefore of the elimination from the portfolio of the respective insured sum) of the individual policyholder.

- Setting the number of cohort policyholders equal to l_0, its permanence in the portfolio is simulated assuming a lapse rate equal to δ and a probability of death equal to q_x. This simulation takes place assuming that each policyholder is described by a Bernoulli r.v.[12] and that people

[12]The support $[0, 1]$ is therefore associated with the states "survived" and "deceased"

are independent from each other and identically distributed;

- The sums insured of occurred deaths $\tilde{z}_{t+1}$ and the insured sums of occurred lapse $\tilde{s}_{t+1}$ are calculated. The new sums insured amounts are updated using formula (20);

- The procedure is repeated N times.

The adjusted individual approach (not used in this thesis), is presented with the purpose of showing a proxy:

- The insured sums are described by a known probability distribution (for example a Log-normal) which parameters are determined according to the moments of the insured sums of the portfolio relative to the specific generation examined. It is also assumed that this distribution remains constant over time;

- Through suitable distributions (for example Binomial) the number $\tilde{r}$ of lapses is simulated

- The sums insured of a number of policyholders equal to the number extracted in point 2 are extracted from the distribution mentioned in the first point;

- The procedure used to simulate the insured sums of occurred lapse is replicated to simulate the insured sums of occurred deaths;

- The procedure is repeated N times.

Using the exact individual approach, both the characteristics of the main random variables inherent to the model and the characteristics of the demographic profit will now be presented, with the aim of identifying a SCR consistent with the legislation. A theme that remains open, possible for in-depth research, is the aggregation (and therefore of dependencies) of the results relating to different cohorts.

2.4. The characteristics of the main variables

Now, we start considering the individual approach and a one-year time horizon. The lapse of each individual policyholder is described as a Bernoulli distribution with parameter δ_t, that represents the expected annual lapse rate at time t. Lapses are here assumed independent and identically distributed random variables. Therefore, considering a cohort of l_0 policyholders it is possible to obtain the following relations (representing the mean, the variance and the skewness, respectively) related to the r.v. number of lapses $\tilde{r}_{t+1}$ during the period $(t, t+1]$ in the cohort:

$$\begin{aligned} \mathbb{E}\left[\tilde{r}_{t+1}\right] &= l_t \cdot \delta_t \\ \sigma^2[\tilde{r}_{t+1}] &= l_t \cdot \delta_t \cdot (1-\delta_t) \\ \gamma[\tilde{r}_{t+1}] &= \frac{(1-2\cdot\delta_t)}{\sqrt{l_t \cdot \delta_t \cdot (1-\delta_t)}} \end{aligned} \tag{24}$$

In a analogous way, we define the random variable $\tilde{d}_{t+1}$ number of deaths during the period $(t, t+1]$ as the sum of Bernoulli random variables, each one with parameter $q_t \cdot (1-\delta_t)$, we obtain:

$$\begin{aligned} \mathbb{E}\left[\tilde{d}_{t+1}\right] &= l_t \cdot q_t \cdot (1-\delta_t) \\ \sigma^2[\tilde{d}_{t+1}] &= l_t \cdot q_t \cdot (1-\delta_t) \cdot [1 - q_t \cdot (1-\delta_t)] \\ \gamma[\tilde{d}_{t+1}] &= \frac{[1 - 2 \cdot q_t \cdot (1-\delta_t)]}{\sqrt{l_t \cdot q_t \cdot (1-\delta_t) \cdot [1 - q_t \cdot (1-\delta_t)]}} \end{aligned} \tag{25}$$

First of all, we observe that by keeping the number of policyholders l_t and the actual probabilities of lapse fixed, the average of $\tilde{d}_{t+1}$ grows, very intuitively, as the actual probabilities of death increase. With reference to the variability instead, it has a paraboloidal trend that increases as q_t tends to 0.5, then decreases towards 0. An interesting aspect is that, by calculating the Coefficient of Volatility (CoV), we obtain:

$$CoV(\tilde{d}_{t+1}) = \sqrt{\frac{\sigma^2[\tilde{d}_{t+1}]}{\mathbb{E}\left[\tilde{d}_{t+1}\right]^2}} = \sqrt{\frac{[1 - q_t \cdot (1-\delta_t)]}{l_t \cdot q_t \cdot (1-\delta_t)}} \tag{26}$$

The CoV is therefore a hyperbolic function that tends to $+\infty$ as q tends to 0 and tends to 0 as q tends to 1: this trend highlights the relative riskiness of policies that pay benefits in the event of death, for example Term Insurance. Considering a generic policyholder i and a specific time period (e.g. $(t, t+1]$), we define the moment generating function $M_{\tilde{s}_i}(s)$ of the r.v. $\tilde{s}_i$ that denotes the sum insured eliminated due to lapses of the i-th policyholder: in this context the insured sum is not unitary, but equal to C_i. Notice that for the sake of simplicity, we neglected the notation related to the time period.
Hence, the moment generating function of $\tilde{s}_i$ is defined as:

$$M_{\tilde{s}_i}(s) = 1 - \delta + e^{s \cdot C_i} \cdot \delta \tag{27}$$

Considering now the whole cohort, under the assumption of dd. of the policyholders, we obtain the cumulant generating function of the r.v. s:

$$\Psi_{\tilde{s}}(s) = \sum_{i=1}^{l} ln(1 - \delta + e^{s \cdot C_i} \cdot \delta) \tag{28}$$

The characteristics of the distribution of $\tilde{s}_{t+1}$ are easily obtained:

$$\begin{aligned}
&\mathbb{E}\left[\tilde{s}_{t+1}\right] = l_t \cdot \delta_t \cdot \bar{C}_{t+1} = \mathbb{E}\left[\tilde{r}_{t+1}\right] \cdot \bar{C}_{t+1} \\
&\sigma^2[\tilde{s}_{t+1}] = l_t \cdot \delta_t \cdot (1 - \delta_t) \cdot \bar{C}^2_{t+1} = \sigma^2[\tilde{r}_{t+1}] \cdot (\bar{C}_{t+1})^2 \cdot r_{2,C_{t+1}} \\
&\gamma[\tilde{s}_{t+1}] = \frac{(1 - 2 \cdot \delta_t)}{\sqrt{l_t \cdot \delta_t \cdot (1 - \delta_t)}} \cdot \frac{\bar{C}^3_{t+1}}{(\bar{C}^2_{t+1})^{3/2}} = \gamma[\tilde{r}_{t+1}] \cdot \frac{r_{3,C_{t+1}}}{(r_{2,C_{t+1}})^{3/2}}
\end{aligned} \tag{29}$$

where risk indices of order n are defined as $r_{n,C_{t+1}} = \frac{\bar{C}^n_{t+1}}{(\bar{C}_{t+1})^n}$. Similarly, for the r.v. sums insured in case of death, we define the cumulant generating function of the random variable $\tilde{z}$. Notice that for the sake of simplicity, we neglected the notation related to the time period.

$$\Psi_{\tilde{z}}(s) = \sum_{i=1}^{l} ln\Bigg(1 - ((1 - q) \cdot \delta) + e^{s \cdot C_i} \cdot q \cdot (1 - \delta)\Bigg) \tag{30}$$

As for the r.v. sums insured of occurred lapse, the following cumulants are obtained for $\tilde{z}_{t+1}$:

$$\begin{aligned}
&\mathbb{E}\left[\tilde{z}_{t+1}\right] = \mathbb{E}\left[\tilde{d}_{t+1}\right] \cdot \bar{C}_{t+1} \\
&\sigma^2[\tilde{z}_{t+1}] = \sigma^2[\tilde{d}_{t+1}] \cdot \bar{C}^2_{t+1} \cdot r_{2,C_{t+1}} \\
&\gamma[\tilde{z}_{t+1}] = \gamma[\tilde{d}_{t+1}] \cdot \frac{r_{3,C_{t+1}}}{r^{3/2}_{2,C_{t+1}}}
\end{aligned} \tag{31}$$

We observe that the expected value of the sums insured eliminated due to deaths increases on average with both the number of deaths and the amounts of sums insured. Volatility increases both as a function of the variance of the number of deaths and as the relative volatility of the sums insured increases. The sign of the skewness depends exclusively on the sign of the skewness of the number of deaths.

2.5. The characteristics of demographic profit in a Local GAAP context

In this subsection, we highlight the analytical results related to the characteristics of the r.v. "demographic profit in a Local GAAP context" neglecting the effects of lapses: if they were considered, they would allow us to grasp the joint effect with mortality, but would be inconsistent with the idea of modelling only the demographic risk. As with the aggregation of several cohorts, we believe that the diversification between mortality risk and surrender must be treated separately.
Starting from formula (18), the expected profit is equal to:

$$\mathbb{E}\left[\tilde{y}_{t+1}^{LG,Dem}\right] = D_{t+1}^{LG,compl} \cdot \lambda_{x+t} \cdot \mathbb{E}\left[\tilde{d}_{t+1}\right] \cdot \bar{C}_{t+1} \tag{32}$$

The first interesting aspect is that the expected gain depends on the SaR rate $D_{t+1}^{LG,compl}$ calculated using first order basis (i.e. the technical pricing bases): in what follows the SaR rate trends for different policies will be shown; thus we highlight that these values are very different in the various policies. For example, in Pure Endowment and Endowment the SaR rates start from 0 at the inception and are worth 1 at maturity, while in Term Insurance they are worth 0 both at the beginning and at maturity, while during the contractual duration they are very close to 0.
λ_{x+t} is a measure of the implicit safety loading: it is calculated as $\lambda_{x+t} =$

$\dfrac{q^*_{x+t} - q_{x+t}}{q_{x+t}}$ and therefore depends on the difference between the demographic base used in the pricing phase q^* and the realistic one, named q. One aspect that should be pointed out is that, as can be observed, the sign of the expected gain (profit) depends on the combined effect of $D^{LG,compl}_{t+1}$ and λ_{x+t}: in policies with a positive SaR rate (Term Insurance and Endowment), in order to obtain a positive expected profit, it is necessary to use a demographic base with a greater probability of death than those believed to be true during the pricing phase, i.e. it is necessary to use an older demographic base. If, on the other hand, the policy provides for a negative SaR rate (Pure Endowment and annuity), the expected profit is reached only with a negative safety loading, hence the pricing must be conducted with a "future" demographic base compared to the realistic one: this necessity implies the need to create selected and, above all, projected mortality tables. Considering formula (18), the standard deviation can be defined as follows:

$$\begin{aligned}\sigma[\tilde{y}^{LG,Dem}_{t+1}] &= \sqrt{(D^{LG,compl}_{t+1})^2 \cdot \sigma^2[\tilde{z}_{t+1}]} \\ &= |D^{LG,compl}_{t+1} \cdot \bar{C}_{t+1} \cdot l_{x+t}| \cdot \sqrt{\frac{q_{x+t} \cdot (1 - q_{x+t})}{l_{x+t}} \cdot r_{2,C}}\end{aligned} \tag{33}$$

On the first hand, we observe that as for the expected profit, the SaR rate value is one of the main drivers of standard deviation.The second factor of considerable interest is $\dfrac{q_{x+t} \cdot (1 - q_{x+t})}{l_{x+t}}$: it grows as q grows from 0 to 0.5, then decreases towards 0. The interesting aspect is that except for extreme ages (i.e. $\omega > 95$), this factor increases when the policy approaches maturity: this phenomenon is therefore common to the policies on the market. The last aspect concerns the presence of $r_{2,C}$: i.e. a volatility index of the insured sums.

In conclusion, we show the results related to skewness:

$$\gamma[\tilde{y}_{t+1}^{LG,Dem}] = -\frac{(D_{t+1}^{LG,compl})^3}{|(D_{t+1}^{LG,compl})|^3} \cdot \frac{(1 - 2 \cdot q_{x+t})}{\sqrt{l_{x+t} \cdot q_{x+t} \cdot (1 - q_{x+t})}} \cdot \frac{r_{3,C}}{(r_{2,C})^{3/2}} \tag{34}$$

The skewness index depends on three factors: the first ratio exclusively determines its sign (except in cases where the cohort has probability of death greater than 50%). In particular, the sign of skewness is equal to opposite of the sign of the SaR rate. This means that the distribution of demographic profit in a local GAAP context has positive skewness in the case of Pure Endowment and annuities and negative skewness in the case of Term Insurance and Endowment. The second term, as previously mentioned, is positive only if $q_{x+t} < 0.5$; moreover, it highlights the benefit of diversification, tending to 0 as l_{x+t} grows. The last term, which depends on the volatility of the insured sums, affects only the order of magnitude of the skewness index.

2.6. Determination of the Capital Requirement and QIS n.2

The purpose of the model presented is to quantify the SCR linked to the demographic risk, such that it is consistent with the legislation, indeed it is calculated with the Value at Risk measure, the time horizon is annual and the confidence level is equal to 99.5%. Consistently with this definition, Risk Based Capital is defined as:

$$SCR_{t+1}^{LG,Dem} = -VaR(\tilde{y}_{t+1}^{LG,Dem}) \tag{35}$$

An aspect that should be emphasized concerns the possibility of quantifying the capital requirement with a proxy, as occurs in the Non-Life insurance context[13].

Since the value of both the volatility and the Potential Releases can be easily deduced from the model, the goal becomes that of identifying a multiplier that approximates formula (35). Hence, we propose to compute the capital requirement as:

$$SCR_{t+1}^{LG,Dem} = k[\gamma(\tilde{y}_{t+1}^{LG,Dem})] \cdot \sqrt{\frac{q_t \cdot (1-q_t)}{l_t} \cdot r_{2,C_{t+1}}} \cdot \left(|D_{t+1}^{LG,compl}| \cdot \bar{C}_{t+1} \cdot l_t\right) \tag{36}$$

where $k[\gamma(\tilde{y}_{t+1}^{LG,Dem})]$ is a deterministic multiplier of the standard deviation, obviously depending on skewness of the demographic profit $\gamma(\tilde{y}_{t+1}^{LG,Dem})$. In this regard, let us go back to the formulation proposed by the Quantitative Impact Study n.2 (see [14]) proposed by CEIOPS[14] prior to the drafting of Solvency II. It provided that the demographic risk was broken down into two macro-components: the risk of mortality and the risk of longevity. The first was in turn divided into "volatility", "trend" and "CAT". The former only in "volatility"and "trend". In formulas:

$$\begin{aligned} SCR_{mort} &= SCR_{mort,vol} + SCR_{mort,trend} + SCR_{CAT} \\ SCR_{long} &= SCR_{long,vol} + SCR_{long,trend} \end{aligned} \tag{37}$$

[13]According to the Standard Formula, it is quantified as $3 \cdot \sigma_{NonLife} \cdot V_{NonLife}$, where $\sigma_{NonLife}$ is the sigma factor linked to the Non Life Underwriting Risk, $V_{NonLife}$ is the sum of the volumes of premiums and best estimates (technical provisions) net of reinsurance.

[14]CEIOPS (Committee of European Insurance and Occupational Pensions Supervisors), successively transformed into European Institution EIOPA in 2011

We now report the definition of the two risks linked to volatility:

$$
\begin{aligned}
SCR_{mort,vol} &= 2.58 \cdot \sqrt{\frac{q_x \cdot (1-q_x)}{N}} \cdot SumAtRisk \\
SCR_{long,vol} &= 2.58 \cdot \sqrt{\frac{q_x \cdot (1-q_x)}{N}} \cdot PotentialRelease
\end{aligned}
\tag{38}
$$

where N indicates the number of policyholders within the cohort (indicated in our model with l_t). Comparing the two requirements of formula (38), it is observed that in the case of mortality risk, the Sums-at-Risk are considered and in the case of longevity risk, the Potential Releases (PRs) are considered. PRs are defined as the mathematical reserve, i.e., the opposite of the Sum-at-Risk.[15] It is highlighted that in the stochastic model discussed in [34], through the notation $\left(|D_{t+1}^{LG,compl}| \cdot \bar{C}_{t+1} \cdot l_t\right)$ it is possible to indicate both "SumAtRisk" and "PotentialRelease" at the same time thanks to the absolute value of the Sum-at-Risk, as reported in formula (36). Therefore, the quantity $\left(|D_{t+1}^{LG,compl}| \cdot \bar{C}_{t+1} \cdot l_t\right)$ indicates the Sum-at-Risk in case of death-linked policies and the PRs in case of survival-linked policies..

Comparing formula (36) with formulas (38), we observe the following main differences:

- The multiplier, indicated in formula (36) with $k[\gamma(\tilde{y}_{t+1}^{LG,Dem})]$, chosen in QIS n.2 is the 99.5% percentile of a standard normal (2.58) therefore, in other words, it is assumed that the distribution of the demographic profit in a Local GAAP context is Gaussian.

[15]It is specified that this is correct in the case of Pure Endowments and Annuities in the accumulation phase, for the Annuities in the disbursement phase it is also necessary to consider the installment if advanced

- Components linked to volatility are equal if in formula (36) we set $r_{2,C_{t+1}}$ equal to 1; in other words, the two formulas are the same if the volatility within the vector of the sums insured is neglected.

In conclusion, the model presented in [34] allows to quantify the risk linked to the volatility of mortality, the so-called idiosyncratic risk. In order to adapt this stochastic model to Solvency II, it is first necessary to switch from the locked and prudential valuation of the liabilities to the market consistent valuation. Secondly, it is also necessary to introduce volatility due to structural fluctuations in the mortality curve, i.e. to introduce trend risk. For the reasons set out above, formula (36) can be understood as an improvement of the SCR where the only source of uncertainty is one linked to the idiosyncratic volatility of the deaths of the cohort.

2.7. Numerical results

This subsection presents the results of applying the model described consistent with the Local GAAP context.

Table 2 presents all the input data: from a purely qualitative point of view, it is specified that a cohort of $15,000$ i.i.d. policyholders entering the portfolio in $t = 0$, i.e. in 2018, where each policyholder has a different sum insured.

It is specified that the individual insured sums were generated *ex-nihilo* from a Log-normal with an expected value of $100,000$ and a CoV equal to 2; subsequently they have been saved and will be used for all the case studies of this thesis: it is thus possible to compare the results with the same input data.

The model was applied to two distinct contractual forms: a Pure Endowment

Table 2: Model parameters

Policyholders date of birth	31st December 1978
Policyholders gender	Males
Policies moment of issue	31st December 2018 ($t = 0$)
Policy duration	20 y
Premium type	Annual premiums (20 y)
Initial number of policyholders (l_0)	15,000
Expected value of the single insured sum (CU)[a]	100,000
CoV of the insured sums	2
First order financial rate j^*	1%

[a]CU stands for Currency Unit

and a Term Insurance. This choice arises from the need to seize the fevers linked to two structurally opposite policies[16] and that, above all, if combined they generate all the most common policies on the insured market: for example an Endowment coincides with the sum of a Pure Endowment and a Term Insurance that have the same maturity and the same technical bases, while an Annuity is a sum of Pure Endowments with different maturities and the same technical bases.

[16]In this case, Pure Endowment gives the right to a benefit that expires only when the insured survives to maturity, while Term Insurance pays a benefit to the beneficiary at the end of the year if the policyholder dies during the year

2.7.1. Pure Endowment - Local GAAP context

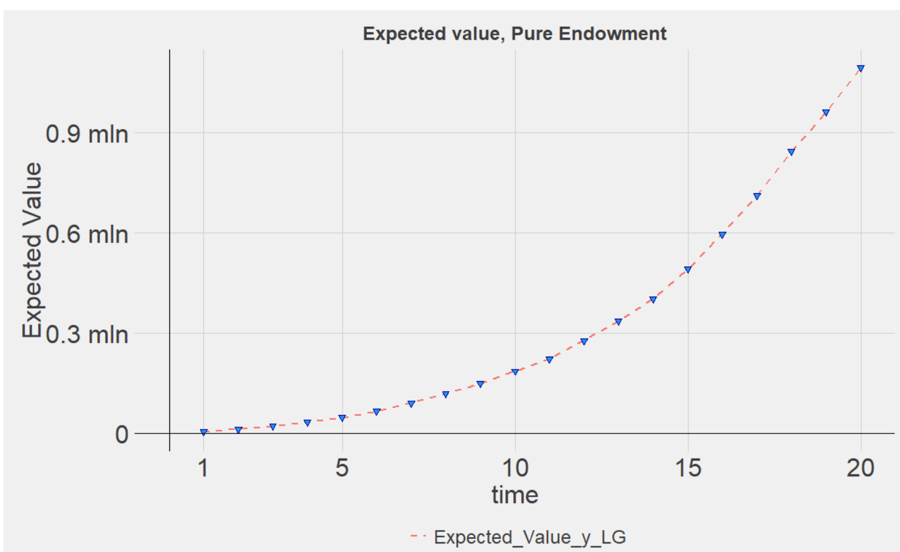
With reference to the application of the stochastic model to the cohort of 15,000 policyholders born in 1978 holders of Pure Endowments, the technical bases used are now specified.

The second order demographic base q_t, i.e. the one considered the best estimate of the mortality of the cohort, was calculated by applying the Lee Carter model (details about calibration are provided later) applied to the data relating to the Italian population from 1872 to 2018 contained in the Human Mortality Database. The first order demographic basis, therefore the one used in the pricing phase and containing an implicit safety loading, is calculated by reducing the mortality rates of the second order demographic base by 20%. Hence, with reference to formula 32 of the profit expected value in a Local GAAP context, $\lambda_{x+t} = \dfrac{q^*_{x+t} - q_{x+t}}{q_{x+t}} = -20\%$.

To facilitate the presentation of the results, the following Figure 2 shows the trend of the SaR calculated with the first-order technical bases previously declared. The strong variation of $(D_{t+1}^{LG,compl})$ over the 20-year time span is observed: it decreases up to -100% at maturity, with a concavity proportional to both the technical rate and the probability of death. Therefore, considering the expected value of the demographic profit calculated in a Local GAAP context and the standard deviation, it is possible to anticipate that the magnitude of the SaR rate will deeply affect them. Consistently with what was previously stated, it is noted that for the same λ_{x+t}, the expected value of the demographic profit in the Local GAAP context varies according to two drivers: $(D_{t+1}^{LG,compl})$ and $\mathbb{E}\left[\tilde{d}_{t+1}\right]$. The first of them, as presented in Figure 2, is strictly increasing in absolute value with consider-

Figure 2: Behaviour of SaR rate - Pure Endowment Case

Figure 3: Behaviour of Expected Value - Pure Endowment Case



able variations, the second undergoes two effects: the increasing probability of death over time one and that of the increasingly reduced cohort. It is specified that although $\mathbb{E}\left[\tilde{d}_{t+1}\right]$ is slightly increasing (the relative value from

0.04% to 0.36%) the main driver of the expected value is the SaR rate.
Figure 4, on the other hand, shows the trend of the standard deviation: also in this case it is possible to observe that the trend is mainly dictated by that of the SaR rate. In this context, however, it is observed that the volatility of the insured sums of procured deaths $\tilde{z}_{t+1}$ increases as t grows up: in fact, observing the ratio $q_t \cdot (1-q_t)$, it is noted how it grows as q_t tends to values close to 0.5 (see formula (33)).
An interesting aspect concerns the analysis of the CoV, understood as the ratio between the standard deviation and the expected value. It is observed that as the valuation instant t increases, this ratio increases, passing from 22% to about 67%: the relative volatility therefore increases where the effect of term $q_t \cdot (1-q_t)$ becomes more relevant.

Figure 4: Behaviour of Standard Deviation - Pure Endowment Case

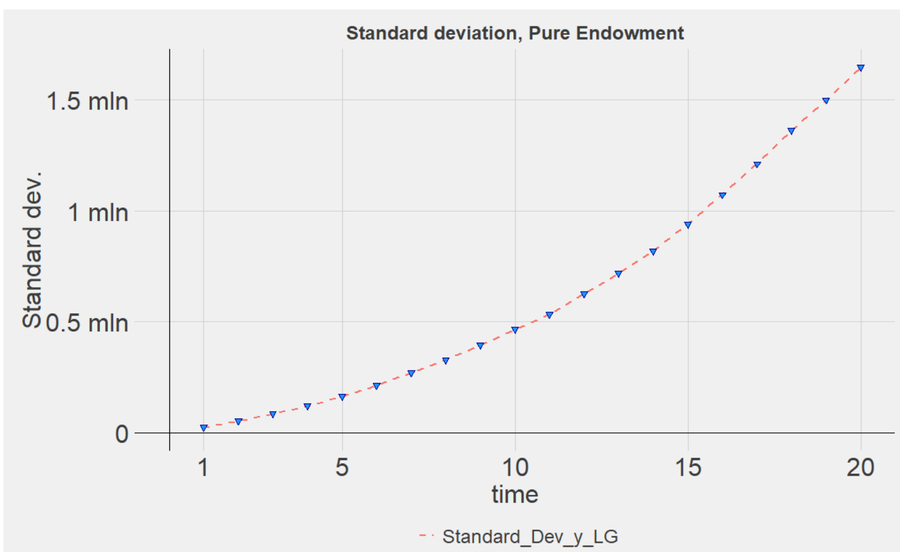
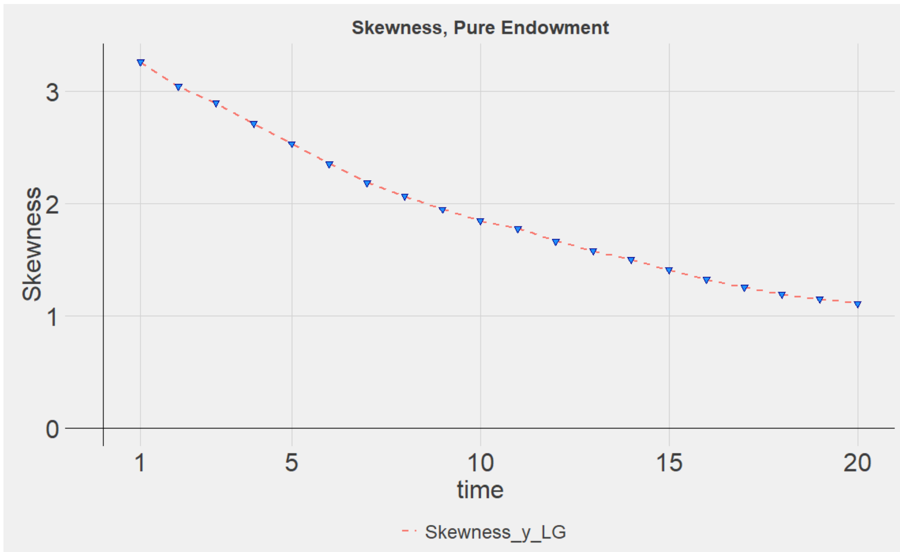


Figure 5 shows the skewness trend: with reference to formula (34), it is observed that since the SaR rate is negative, the skewness of the distribution

Figure 5: Behaviour of Skewness - Pure Endowment Case



is always positive if $q_t < 0.5$. This phenomenon occurs by virtue of the non-extreme age of the policyholders: however, as t increases, the nominator of the above mentioned formula increases, reducing the value and showing distributions more and more similar to the Normal one. It is specified that the effect is not linear because the denominator grows quadratically for the reasons explained when the trend of the standard deviation was shown.

Table 3 shows the results relating to the simulations of the random variable $\tilde{y}_{t+1}^{LG,Dem}$ in three different years: the first year of the contract ($t = 1$), in tenth year ($t = 10$) and in last year ($t = 20$). Each time span was simulated with 10 millions of scenarios: this choice derives from the fact that the possibility of converging towards the exact moments of the distribution requires only 17 minutes, hence it's not time consuming[17]. On the one hand it is observed

[17]Working in parallel with a Intel i7 8700K processor (6 Cores, 12 Threads)

Table 3: Simulated and theoretical characteristics of idiosyncratic profit and loss distribution for a Pure Endowment contract for three different time periods in a Local GAAP context

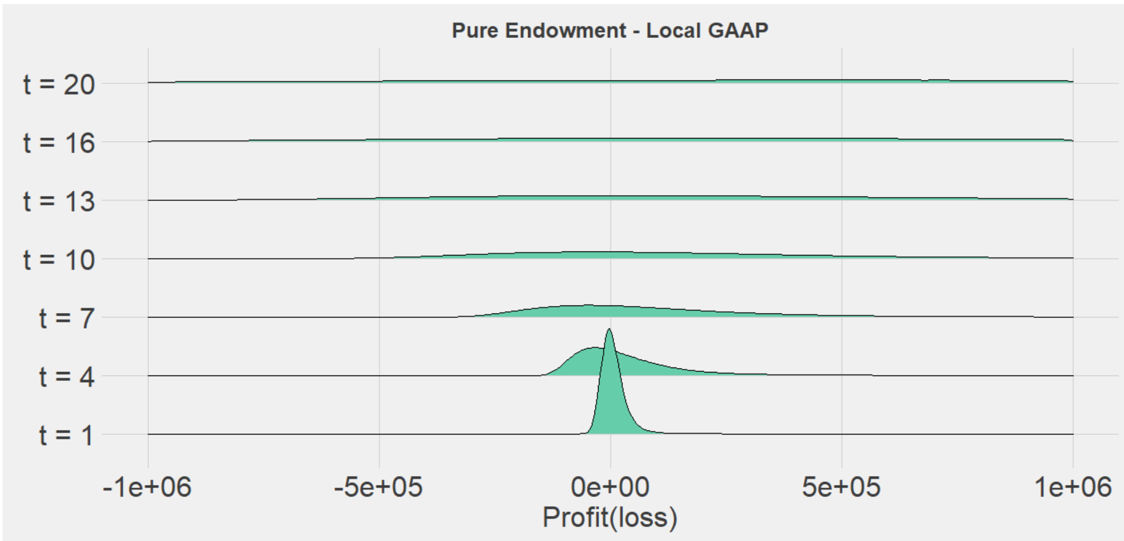
Pure Endowment	**t=1**	**t=10**	**t=20**
Theoretical expected value	5,706	186,600	1,095,008
Simulated mean	5,749	187,882	1,101,972
Theoretical standard deviation	25,295	468,605	1,657,030
Simulated standard deviation	25,306	467,042	1,649,935
Theoretical skewness	3.26	1.84	1.10
Simulated skewness	3.25	1.83	1.09
Solvency Capital Requirement	22,178	522,639	1,880,416
SCR/Sums insured	0.001%	0.035%	0.129%

that the high number of simulations allows to obtain simulated results very close to theoretical results: obviously these results are a good approximation to decreasing volatility. On the other hand it is observed that through Monte Carlo simulations it is possible to identify the percentiles of the $\tilde{y}_{t+1}^{LG,Dem}$ distribution, in particular the order 0.5% one: it is therefore an estimate of the SCR in a Local GAAP context consistent with formula (5).

In conclusion, Figure 6 shows the simulated distributions over different time horizons t: it is interesting to observe how all the effects described so far can be seen jointly. As t increases, both the expected value (shifting the distributions to the right) and the standard deviation increase (flattening the distributions), while the skewness reduces making the distributions more asymmetric.

In this regard, the relationship between the SCR and the standard deviation

Figure 6: Simulated profit distributions - Pure Endowment Case



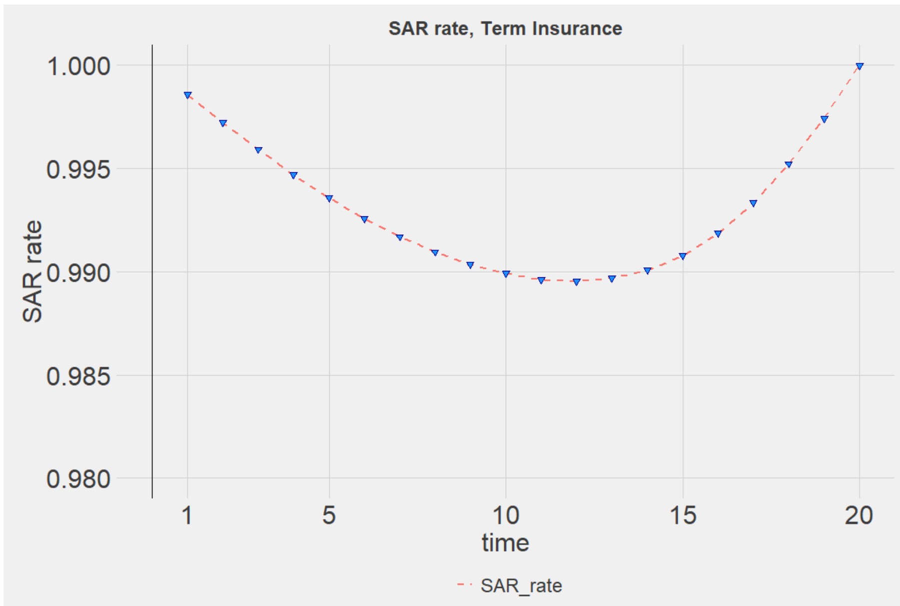
is observed: net of the effect of the SaRs, it is observed that this value increases from 0.88 to 1.14. This phenomenon is mainly linked to the decrease in skewness over time: since the distribution of the demographic profit tends (without reaching it, see at the value equal to 1.10 in $t = 20$) to a Gaussian distribution, the limit is that of the 2.58 identified from the QIS n.2 (see formula (38)).

2.7.2. Term Insurance - Local GAAP context

Also in this case study the data shown in Table 2 is used: the only substantial difference concerns the first-order demographic technical basis.
IAlso in this case, the demographic basis of the second order is still calculated by applying the Lee-Carter model to the data from 1972 to 2018 of the Italian population contained in the HMD. The first order demographic base is calculated by increasing the mortality rates of the second order basis by

20%. It follows that $\lambda_{x+t} = 20\%$.

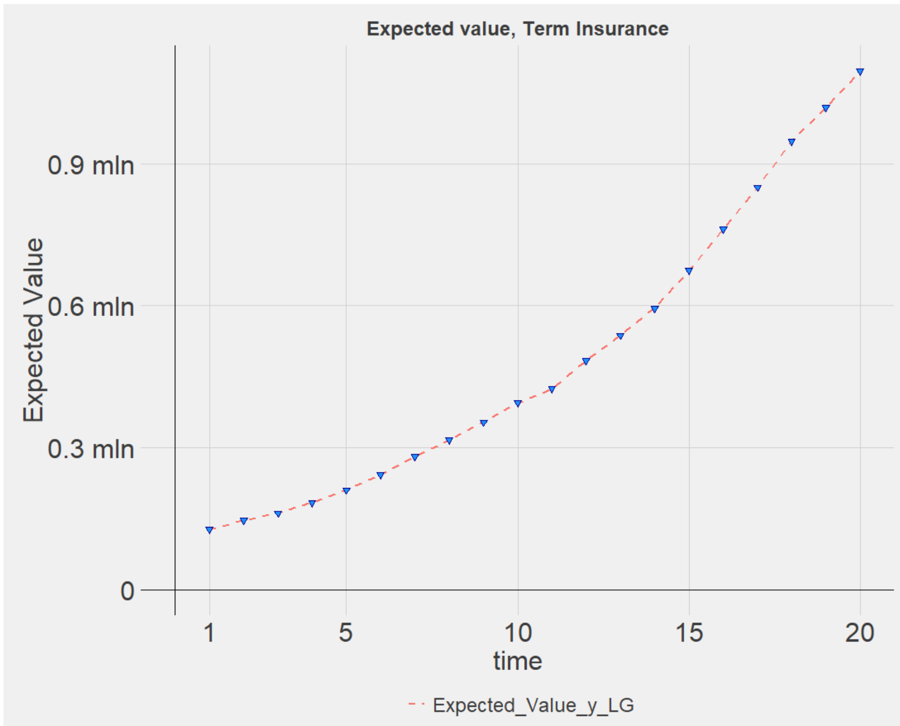
Figure 7: Behaviour of Skewness - Pure Endowment Case



In Figure 7 we show the behaviour of the SaR rate which, consistent with formula (19), is calculated as the complement to 1 of the mathematical reserve rate computed with the prudential and locked bases of the first order. It is observed that this behaviour is diametrically opposite to that of a Pure Endowment: on the first hand its shape is convex (therefore decreasing in the first development years and subsequently increasing), secondly its range of variation is extremely limited, in fact it is always very close to the value of 1.

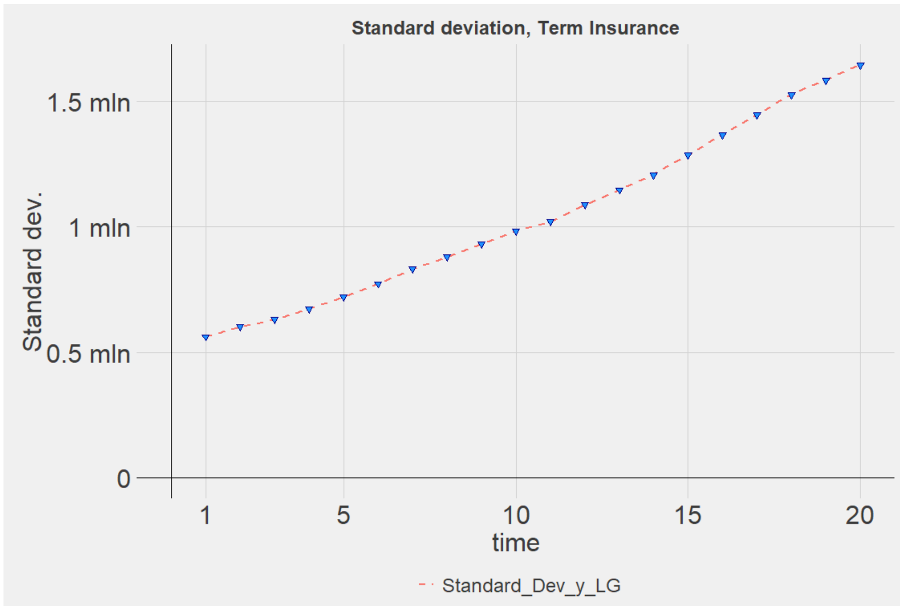
The first intuition is that in this context, the structural characteristics of the policy will not influence so much the determinations of the expected value and the standard deviation, while the fact that the SaR rate is always positive will profoundly modify the skewness. Figure 8 shows the trend of the

Figure 8: Behaviour of Standard Deviation - Pure Endowment Case



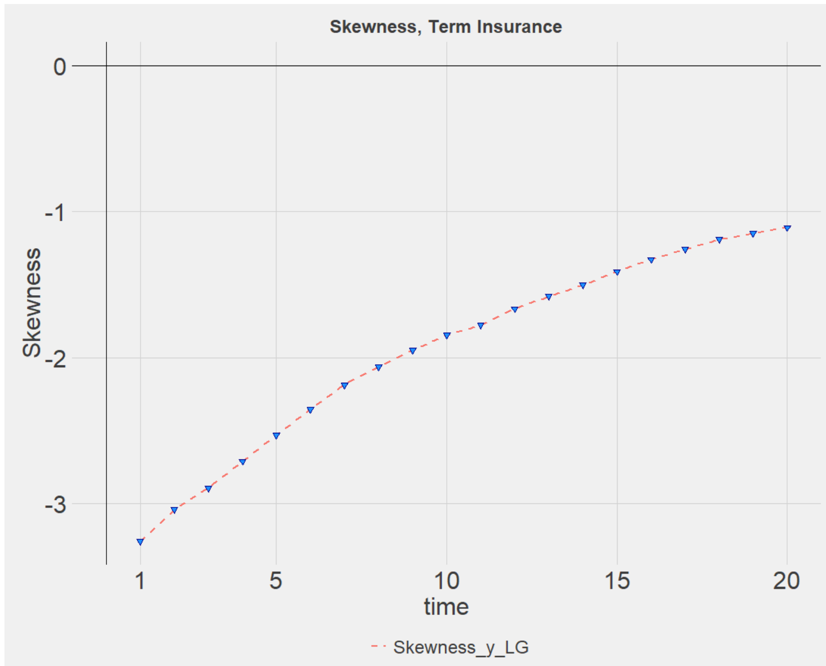
expected value of the demographic profit: it is observed that since the absolute value of λ_{x+t} and $\mathbb{E}\left[\tilde{z}_{t+1}\right]$ are unchanged, the only difference between the case of Term insurance and that of Pure Endowment concerns the trend of the SaR rate. Since both SaR rates converge to the value of 1 at maturity, the most evident differences can be seen in the first part of the behaviour: in this regard, it should be remembered that the expected value of the demographic profit calculated in a Local GAAP context in the case of Pure Endowment was just over 5,000 euros: in this case $\mathbb{E}\left[\tilde{y}_1^{LG,Dem}\right] \simeq 128,000$. With reference to the standard deviation, it is observed that it undergoes the same effects as the expected value: in particular, it is observed that both the standard deviation of Pure Endowment and that of Term Insurance grow

Figure 9: Behaviour of Skewness - Pure Endowment Case



in the same way with reference only to the $\sqrt{\dfrac{q_{x+t} \cdot (1-q_{x+t})}{l_{x+t}}} \cdot r_{2,C}$ factor. The difference between the two standard deviations concerns exclusively the SaR rate: in particular, at maturity (where the absolute values of both SaR rates are equal to 1) the two standard deviations coincide; in the first times the volatility of Term Insurance is much higher. With reference to skewness, a singular phenomenon is observed: since the intrinsic characteristics of the policies are reflected only on the SaR rate and this rate affects only the skewness sign, the Pure Endowment skewness module is equal to the skewness module of Term Insurance. Obviously, since Term Insurance's SaR rate is always positive in a Local GAAP context (unlike that of Pure Endowment always negative), the Skewness sign of Term Insurance is opposite to that of Pure Endowment.Therefore, as shown in Figure (10), in the first instants

Figure 10: Behaviour of Skewness - Pure Endowment Case



it assumes strongly negative values close to -3 and subsequently increases reaching values close to -1.

Table 4 shows the simulated and analytically results related to the characteristics of the demographic profit distribution calculated in a Local GAAP context of a cohort of 15,000 policyholders holding Term Insurances.

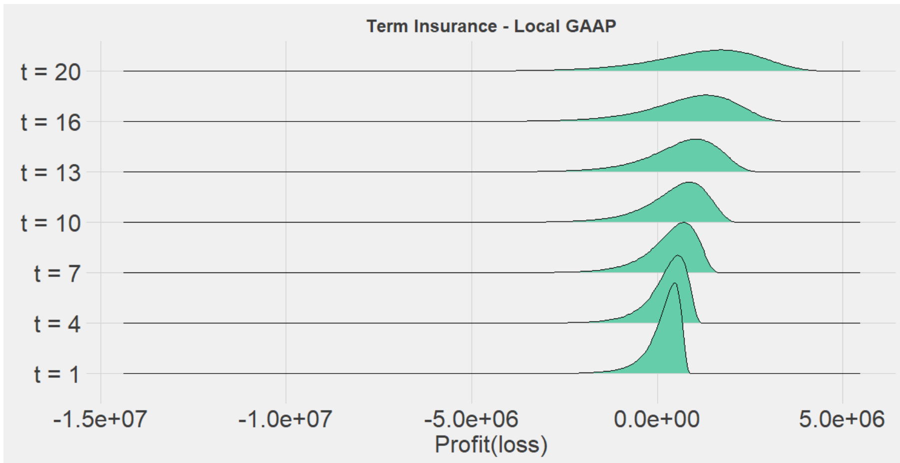
The first interesting aspect is that when the SaR rate of the policy is equal to 1 (at maturity), the exact moments coincide with those of the Pure Endowment case. Since also in this case 10 million scenarios have been simulated for each distribution, the simulated results are very close to the theoretical results and this element allows to calculate the Capital-at-Risk with a high precision. In this regard, the SCR grows as the standard deviation increases, but the ratio between the simulated value and the standard deviation itself

Table 4: Simulated and theoretical characteristics of idiosyncratic profit and loss distribution for a Term Insurance contract for three different time periods in a Local GAAP context

Term Insurance	**t=1**	**t=10**	**t=20**
Theoretical expected value	127,841	396,959	1,095,008
Simulated mean	128,426	394,563	1,102,456
Theoretical standard deviation	566,728	990,865	1,657,030
Simulated standard deviation	567,359	988,636	1,652,745
Theoretical skewness	-3.26	-1.84	-1.10
Simulated skewness	-3.25	-1.83	-1.09
Solvency Capital Requirement	2,723,310	4,027,834	4,954,808
SCR/Sums insured	0.182%	0.271%	0.341%

decreases, passing from values about to 4.83 to 3.This effect is due to the shape of the distributions: when the time horizon t is close to the signing of the contract, the distributions are particularly asymmetric (with negative skewness) and the percentile is in the far left tail. When t grows, the distribution is more and more similar to a Normal which would have a multiplier equal to 2.58 identified by QIS n.2 (see formula (37)). In conclusion, as in the case of the Pure Endowment, the distributions of profit over different time spans are shown in Figure 11: it should be noted that the distributions are sharper than the analogue relating to PE; they also are more volatile and less concentrated around the average value when t is far to 0. From the figure it is also possible to see that the shape of the distributions is asymptotically convergent to a Normal: it is therefore evident that the Capital Requirement can be approximated by $2.58 \cdot \sigma(\tilde{z}_{t+1})$ only when it is calculated for cohorts

Figure 11: Simulated profit distributions - Term Insurance Case



with annual probability of death close to 50% and without expected profit.

3. The market consistent valuation: the bridge between the Local GAAP framework and the market consistent one

3.1. Introduction

Up to now, a model based on risk theory has been adopted which, through the cohort approach, allows to model both the demographic profit quantified in a Local GAAP context, and the main variables that characterize it.

This section is not exclusively oriented to present an innovative stochastic model which aims to the calculation of the SCR, but rather to present a bridge that highlights the transition from the Local GAAP context to the market-consistent one introduced by Solvency II.

The first purpose is to present a decomposition of the company's technical profit, as presented in formula (14), considering however that the Technical Provisions are no longer calculated as the expected present value of cash flows with locked and prudential bases, but using the Solvency II principles presented in Section 1. In this context, attention is paid to the demographic profit which, intuitively, is not only affected by the idiosyncratic volatility deriving from stochastic deaths, but also by the risks associated with the possibility of changing the best estimates of mortality rates and the risk associated with changes risk-free rates. In conclusion, a fundamental recursive equation is presented in the market consistent context that will allow us to quantify the exact moments of the "idiosyncratic demographic useful" distribution in the next section.

3.2. Homans' revised decomposition

The first goal is to convert formula (13) so that it is consistent with the regulations relating to Solvency II, (see, e.g. [21] and [19]). The problem therefore concerns the calculation of the technical provisions calculated in the Local GAAP context as the expected present value of cash flows, using the first-order locked and prudential bases as technical bases.

It is noted that the liabilities linked to life-time insurance policies are non-hedgeable: in particular it is not possible to think of any financial instrument commonly traded on the markets (or a portfolio of instruments) whose cash flows can replicate those of the policy in object. For these reasons, consistently with article 77 of the Delegated Acts, they must be calculated as the sum of the Best Estimate and Risk Margin.

As detailed in Section 1, the replacement of the technical provisions with the sum of the Best Estimate and Risk Margin would be consistent in models oriented to liabilities valuation or to undertaking profit assessment, however not the case of SCR calculation. In the case relating to the SCR, the legislation clearly specifies (see, e.g. Art. 83 Technical Specifications) that changes in the Risk Margin must not be included in the calculation of the capital requirement. Hence, in the continuation of the report, we will speak of random variable "profit" not in the business sense, but only with reference to the calculation of the capital requirement indeed the technical provisions will be represented only by the Best Estimate component.

The market-consistent (MC) insurance profit is divided into 5 components, so that the effects of the main drivers are separated, as far as possible.

$$\tilde{Y}_{t+1}^{MC} = \tilde{y}_{t+1}^{MC,Dem} + \tilde{y}_{t+1}^{MC,Fin} + \tilde{y}_{t+1}^{MC,Lapse} + \tilde{y}_{t+1}^{MC,Exp} + \tilde{y}_{t+1}^{MC,Res} \tag{39}$$

It is specified that the proof relating to the decomposition into the five components follows simple algebra. With reference to formula (39), it is noted that it is possible to break down the technical-insurance profit of the undertaking into 5 components, i.e according to the drivers underlying the model. In particular, $\tilde{y}_{t+1}^{MC,Dem}$ is the profit linked to fluctuations in mortality rates, whatever their nature (therefore both idiosyncratic variations and variations in the mortality trend are considered). $\tilde{y}_{t+1}^{MC,Fin}$ is the profit linked to financial aspects: it arises from the difference in various financial technical bases used: the premiums are calculated with the locked and prudential technical rate j^*, the Best Estimate is discounted with the risk-free rates and the undertaking, through the investment of premiums and reserves, obtains a specific (stochastic) financial return. The third component, $\tilde{y}_{t+1}^{MC,Lapse}$, is the lapse margin and is linked to the difference between the assumptions in terms of surrenders and the actual realizations of the homonymous random variable. The forth component, $\tilde{y}_{t+1}^{MC,Exp}$, is the so-called expense margin and arises from the difference between the expense loadings collected by the undertaking and the expenses actually incurred, i.e. those of acquisition, collection and management. The last component, $\tilde{y}_{t+1}^{MC,Res}$, which the relative effect on the insurance profit is almost negligible, is the residual profit and arises from the interaction between expense loadings and financial returns.

Now we introduce, from a quantitative point of view, the demographic profit (technical profit first component):

$$\begin{aligned}\tilde{y}_{t+1}^{MC,Dem} = {} & [be_t^{Rf(t),q(t)} + b_{t+1}(1-\alpha^*-\beta^*)-\gamma^*](w_t-\tilde{s}_{t+1})(1+j^*)+ \\ & -(\tilde{x}_{t+1}+\tilde{w}_{t+1}\cdot \tilde{be}_{t+1}^{\tilde{R}f(t+1),\tilde{q}(t+1)})\end{aligned} \tag{40}$$

It is possible to observe that, although this component is oriented to quantify the effects of changes in both expectations and mortality realizations, other random variables come into play: in particular the volatility linked to lapses and that linked to the new interest risk-free curve $\tilde{Rf}(t+1)$, available in $t+1$. Considering again formula (40), it is possible to observe that it is very similar to formula (15): both calculate the demographic profit as the difference between the sum of reserves and premiums net of the sum of claims paid and new reserves, but in the market-consistent context the reserves VB_{t+1} are replaced by the Best Estimate rate $\tilde{be}_{t+1}^{\tilde{Rf}(t+1),\tilde{q}(t+1)}$.
Due to the presence of risk-free rates in the formulation of the demographic profit, it is difficult to treat it as a Leibnizian monad. For this reason, the formulation of the financial component is now presented specifying the importance to give an insight into the double effect of risk-free rates (and, more generally, of the financial basis), both on the first and on the second component of the technical profit.

$$\begin{aligned} \tilde{y}_{t+1}^{MC,Fin} =& (\tilde{j}_{t+1} - j^*) \cdot (be_t^{Rf(t),q(t)} \cdot w_t + b_{t+1}(1-\alpha^* - \beta^*)(w_t - \tilde{s}_{t+1}) + \\ & - (\gamma^* w_t) - (g_t^* \cdot be_t^{Rf(t),q(t)} \cdot \tilde{s}_{t+1}) \end{aligned} \tag{41}$$

where g_t^* is a specific penalization coefficient applied in case of surrender. It is noted that net of the part relating to the penalty of lapses, the financial profit mainly depends on the difference between $\tilde{j}_{t+1}$ and j^*. In this case, it is equal to the sum of the technical provisions calculated in a market-consistent context and the pure premiums multiplied by the spread between the actual returns obtained from the company's investments (of reserves and premiums) and the technical rate (first-order financial basis) guaranteed to policyholders. It is noted that in this context the technical rate, which works

as a financial guarantee returned to the policyholders, represents a share of the financial profit obtained by the company: the profit is positive only if what is obtained as an investment is sufficient to cover what is guaranteed to the policyholders.
The third component, the lapses linked one, is defined as:

$$\tilde{y}_{t+1}^{MC,Lapse} = \left(be_t^{Rf(t),q(t)} - \gamma^* - g_t^* \cdot be_t^{Rf(t),q(t)}\right) \cdot (1+j^*) \cdot \tilde{s}_{t+1} \tag{42}$$

From formula (42) it is understood that the profit in the event of lapses derives from two conditions:

- Surrenders occurs: in this context, policyholders are treated as independent and identically distributed random variables with the same lapse rate
- In the event of surrender, the policyholder receives an amount lower than the amount allocated to the reserve. In this context, the previously mentioned coefficient g_t^* is precisely aimed to penalize the policyholders by guaranteeing a profit that can replace the demographic and/or financial one.

$$g_t^* = \begin{cases} 0 & if\ t < \tau \\ (1+j_s)^{-(m-t)} & if\ t >= \tau \end{cases} \tag{43}$$

where $\tau > 0$ and $j_s^* < j^*$ are fixed by the undertaking. I add a proposal relating to g_t^* with a typical increasing and convex trend. It is observed that for the first moments the lapse is not even allowed, subsequently it assumes values typically calibrated above 80% and then converges to 1 at maturity.

We define the expenses linked component, the fourth one, as:

$$\tilde{y}_{t+1}^{MC,Exp} = (1 + j^*)[(\Delta\alpha_{t+1}^* + \Delta\beta_{t+1}^*) \cdot b_{t+1} \cdot (w_t - \tilde{s}_{t+1}) + \Delta\gamma_{t+1}^* \cdot w_t] \quad (44)$$

where $\Delta\alpha_{t+1}^*$, $\Delta\beta_{t+1}^*$ and $\Delta\gamma_{t+1}^*$ depend on the differences between the first order expense assumptions and the realistic ones.
The last component, is the residual profit:

$$\begin{aligned} \tilde{y}_{t+1}^{MC,Res} = (\tilde{j}_{t+1} - j^*)[(\Delta\alpha_{t+1}^* + \Delta\beta_{t+1}^*) \cdot b_{t+1} \cdot (w_t - \tilde{s}_{t+1}) + \\ + \Delta\gamma_{t+1}^* \cdot w_t] \end{aligned} \quad (45)$$

3.3. The bridge from Local GAAP to MCV framework

The purposes of this and next subsections are to present two fundamental aspects of the model:

1. how to break down the demographic profit of formula (40) into separate submodules;

2. what is the relationship between the demographic profit of formula (15) (Local GAAP) and the market consistent formula (40)

With reference to the breakdown of the market consistent demographic profit, we consider adding and subtracting the following three terms:

- $\tilde{w}_{t+1} \cdot \tilde{be}_{t+1}^{Rf(t+1),\tilde{q}(t+1)}$: representing the product between the insured sums (stochastic) at time $t+1$ and the best estimate rate calculated with a deterministic curve of risk-free rates and stochastic assumptions on mortality $\tilde{q}(t+1)$. In particular, it is specified that the risk-free rate curve used in $t+1$, $Rf(t+1)$ coincides with the forward rates inferable from the spot rates of the curve available at time t,

- $\tilde{w}_{t+1} \cdot be_{t+1}^{Rf(t+1),q(t+1)}$: representing the product between the insured sums at time $t+1$, $\tilde{w}_{t+1}$ and the best estimate of the reserves using the risk-free rate curve achievable from the curve available at time t (i.e. the forward rates) $Rf(t+1)$ and the same demographic assumptions of the previous instant $q(t+1) = q(t) = q$, as if the company confirmed them.

- $i_t(t, t+1) \cdot (w_t - \tilde{s}_{t+1}) \cdot [be_t^{Rf(t),q(t)} + \pi]$: representing the product between the sum of the best estimate at the valuation instant t and the pure premiums amounts and the risk-free rate $i_t(t, t+1)$

The result is the following:

$$\tilde{y}_{t+1}^{MC,Dem} = \tilde{y}_{t+1}^{Idios} + \tilde{y}_{t+1}^{Trend} + \tilde{y}_{t+1}^{RiskFree} + \tilde{y}_{t+1}^{NDM} \tag{46}$$

where:

1. The idiosyncratic risk:

$$\begin{aligned} \tilde{y}_{t+1}^{Idios} = {} & [be_t^{Rf(t),q(t)} + b_{t+1}(1 - \alpha^* - \beta^*) - \gamma^*](w_t - \tilde{s}_{t+1})(1 + i_t(t, t+1)) + \\ & - (\tilde{x}_{t+1} + \tilde{w}_{t+1} \cdot be_{t+1}^{Rf(t+1),q(t+1)}) \end{aligned} \tag{47}$$

   It quantifies the variability linked to the stochastic process of deaths in the strict sense, i.e. to the fact that we are working with random variables. It should be noted that in this context, the best estimate rate is also deterministic for $t+1$: it is calculated on the same technical basis used at the beginning of the year, therefore the only source of variability is linked to the deaths of the cohort and, consequently, with formula (20), to the sums insured,

2. The trend risk:

$$\tilde{y}_{t+1}^{Trend} = -\tilde{w}_{t+1} \cdot (\tilde{be}_{t+1}^{Rf(t+1),\tilde{q}(t+1)} - be_{t+1}^{Rf(t+1),q(t+1)}) \tag{48}$$

This second component mainly concerns the difference between the best estimate calculated at time $t+1$ with the same demographic assumptions used at time t and the new assumptions deriving from the new information contained in the filtration $\mathcal{F}_t$. It is specified that this component also depends on the volatility of the insured sums since the amounts of the best estimates are stochastic.

3. The risk-free linked risk:

$$\tilde{y}_{t+1}^{RiskFree} = \tilde{w}_{t+1} \cdot (\tilde{be}_{t+1}^{\tilde{Rf}(t+1),\tilde{q}(t+1)} - \tilde{be}_{t+1}^{Rf(t+1),\tilde{q}(t+1)}) \tag{49}$$

This third component, net of the amounts $\tilde{w}_{t+1}$, is equal to the difference between the effective best estimate computed on the basis on rates published by EIOPA in $t+1$ and that calculated in $t+1$ using the risk-free interest rate curve available at time t . A thorny aspect is linked to the fact that the demographic base used is the effective second order one used in $t+1$, therefore unknown at time t. However, it is considered that within the round brackets there is the difference between two best estimates where they have both the same number of terms, the same amounts and the same stochastic survival/death probabilities.

It is therefore possible to observe that if the risk-free curve available in $t+1$ is equal to what is predicted in t, this component disappears. Depending on the variation of the rates (only and only if the risk-free rates are specified) there will be a profit/loss whose magnitude depends on the best estimate of mortality at $t+1$. Since the sign of the component depends only on the risk-free rates, it is believed that this third component belongs to the Financial Profit, second component, of the Demographic risk, as presented in formula (39). The joint study of mortality and risk-free rates can be a further starting point for research either to quantify the dependence between the two risks and to grasp the overall effect of the two sources on the company's SCR.

4. The non mortality-depending risk:

$$\tilde{y}_{t+1}^{NDM} = (j^* - i_t(t, t+1)) \cdot (w_t - \tilde{s}_{t+1}) \cdot [be_t^{Rf(t),q(t)} + \pi] \tag{50}$$

   The last component does not depend on mortality. Therefore, if the effect of lapses were ignored, a purely deterministic amount would be obtained whose main driver is the spread between the risk-free rate and the guaranteed technical rate. The analysis of this risk free component is not dealt in this thesis because it does not modify the volatility of the demographic profit; it should be considered when evaluating the overall undertaking's profit.

3.4. Model algebra and underlying recursive formula

Having presented the four components of demographic risk (idiosyncratic, trend, risk-free and not mortality-depending), we now demonstrate the relationship with the demographic risk of the Local GAAP context reported in formula (15).

By definition, the demographic risk of the Local GAAP context measures the risk of suffering losses linked to adverse fluctuations in mortality rates: it is noted that this definition coincides with that of idiosyncratic risk. it is therefore natural to start from formula (47) to try to rewrite it as a function of the compact version of formula (18). In order to clearly show all the drivers of the model, the component linked to the volatility of risk-free rates is also considered: it has been stated that $\tilde{y}_{t+1}^{RiskFree}$ should be allocated to the $\tilde{y}_{t+1}^{MC,Fin}$ component and the reason will be clear from the following demonstration. To this end, a new quantity is introduced, indicated with $epv_t^{j^*,q}$, which coincides with the expected present value of the cash-flows net of the expected present values of the premiums. The technical bases used are a hybrid between the locked and prudentials of the technical provisions of the Local GAAP context and the dynamic and updated ones of the Solvency II framework. The technical rate coincides with the first-order financial base j^*, while the demographic base coincides with the second-order table q.

$$\tilde{y}_{t+1}^{Idios} + \tilde{y}_{t+1}^{RiskFree} + \tilde{y}_{t+1}^{NMD} = \tilde{y}_{t+1}^{LG,Dem} + \tilde{y}_{t+1}^{MCVRf-j*} +_1 \tilde{y}_{t+1}^{MCVq-q*} \tag{51}$$

The proof of the above derives from simple algebra and from the use of the recursive formula (20).

In this context,

- $\tilde{y}_{t+1}^{LG,Dem}$ is defined exactly as in formula (15),
- $\tilde{y}_{t+1}^{MCVRf-j*}$ is defined as:

$$\begin{aligned}\tilde{y}_{t+1}^{MCVRf-j*} =& (w_t - \tilde{s}_{t+1}) \cdot [(be_t^{Rf(t),q} - epv_t^{j^*,q})(1+j^*) - (\tilde{be}_{t+1}^{Rf(\tilde{t}+1),q} - epv_{t+1}^{j^*,q})] \\ & + (\tilde{be}_{t+1}^{Rf(\tilde{t}+1),q} - epv_{t+1}^{j^*,q}) \cdot \tilde{z}_{t+1}\end{aligned} \tag{52}$$

Net of the component linked to the insured sums of occurred deaths $\tilde{z}_{t+1}$, this is greater than 0 when the jump between the best estimate and the expected present value calculated on the same demographic basis, but with a financial basis equal to the first-order technical rate, is greater than the analogous jump in $t+1$, discounted for one year.

- $$\begin{aligned}\tilde{y}_{t+1}^{MCVq-q*} =&(w_t - \tilde{s}_{t+1}) \cdot [(epv_t^{j^*,q} - v_t^b)(1+j^*) - (epv_{t+1}^{j^*,q} - v_{t+1}^b)] + \\ &(epv_{t+1}^{j^*,q} - v_{t+1}^b) \cdot \tilde{z}_{t+1}\end{aligned} \tag{53}$$

  Net of the component linked to the insured sums of occurred deaths $\tilde{z}_{t+1}$, this is greater than 0 when the jump between the expected present value calculated and the Local GAAP mathematical reserve calculated on the same financial basis, but with a demographic basis equal to the first-order mortality rate, is greater than the analogous jump in $t+1$, discounted for one year.

Remark. *The following proof, aimed at proving that* $\mathbb{E}\left[\tilde{y}_{t+1}^{LG,Dem}\right] = -\mathbb{E}\left[\tilde{y}_{t+1}^{MCVq-q*}\right]$, *allows to highlight two fundamental aspects of the model represented by formula (46):*

1. *Since various sources of uncertainty come into play in the definition of market-consistent demographic profit, it is necessary to divide them according to the source of risk itself (idiosincratyc, trend, iteration between risk-free and mortality and lapse risk). For this reason, the present proof will give an important indication that leads to the specific decomposition of formula 46*

2. *The definition of the 4 components and of the idiosyncratic one in particular, arises from a particular recursive equation (which will be demonstrated in the Theorem) whose definition, however, arises from the demonstration that* $\mathbb{E}\left[\tilde{y}_{t+1}^{LG,Dem}\right] = -\mathbb{E}\left[\tilde{y}_{t+1}^{MCVq-q*}\right]$

Considering formula (20), we can rewrite the right member of formula (51) as:

$$\begin{aligned}
{}_1\tilde{y}_{t+1}^{LG} +_1 \tilde{y}_{t+1}^{MCVq-q^*} &= \left(\pi + epv_t^{j^*,q}\right) \cdot (w_t - \tilde{s}_{t+1}) \cdot (1 + j^*) \\
&\quad - \tilde{x}_{t+1} - epv_{t+1}^{j^*,q} \cdot (w_t - \tilde{s}_{t+1} - \tilde{z}_{t+1}) \\
&= (w_t - \tilde{s}_{t+1}) \cdot \left[\left(\pi + epv_t^{j^*,q}\right) \cdot (1 + j^*) - epv_{t+1}^{j^*,q}\right] \\
&\quad + epv_{t+1}^{j^*,q} \cdot \tilde{z}_{t+1} - \tilde{x}_{t+1}
\end{aligned} \tag{54}$$

The next step is to calculate the expected value. Since the equality concerns "best estimates" and "expected present values" it is necessary to define these quantities from a practical point of view. To this end, the most generic case possible is considered, namely that of an Endowment with regular premiums. Obviously, the case of regular premiums is the general case of single premiums since in the latter case it is sufficient to set all the terms of the premium carrier to 0, except for the first. It is also specified that an Endowment is a policy that guarantees a specific sum insured in the event of the death of the insured at each possible anniversary of the policy in the event of death or sum insured to maturity in the event of survival. For this reason, by incorporating both benefits in the event of death and benefits in the event of survival, any demonstration involving it can be extended to both policies with benefits only in the event of death (Term Insurance) and policies with benefits only in the event of survival. (Pure Endowments or Annuities).

Therefore formula (54) is considered and its expected value is calculated:

$$\mathbb{E}\left[{}_1\tilde{y}^{LG}_{t+1} +_1 \tilde{y}^{MCV q-q^*}_{t+1}\right] =$$
$$\mathbb{E}\left[w_t - \tilde{s}_t\right]\left[\left(\pi + epv^{j^*,q}_t\right) \cdot (1+j^*) - epv^{j^*,q}_{t+1}\right] + epv^{j^*,q}_{t+1} \cdot \mathbb{E}\left[\tilde{z}_{t+1}\right] - \mathbb{E}\left[\tilde{x}_{t+1}\right]$$

Since $\tilde{x}_{t+1} = \tilde{z}_{t+1}$ for an endowment contract and $\mathbb{E}\left[\tilde{z}_{t+1}\right] = q_{x+t} \cdot \mathbb{E}\left[w_t - \tilde{s}_t\right]$, we have:

$$\mathbb{E}\left[{}_1\tilde{y}^{LG}_{t+1} +_1 \tilde{y}^{MCV q-q^*}_{t+1}\right] =$$
$$\mathbb{E}\left[w_t - \tilde{s}_t\right]\left(\left(\pi + epv^{j^*,q}_t\right) \cdot (1+j^*) - epv^{j^*,q}_{t+1} \cdot p_{x+t} - q_{x+t}\right)$$

Since empirical analysis shows that the expected value of the two components is equal to 0 when $t > 0$, we proceed to the analysis of the recursive formula highlighted in the round brackets.

$$\left(\pi + epv^{j^*,q}_t\right) \cdot (1+j^*) = epv^{j^*,q}_{t+1} \cdot p_{x+t} + q_{x+t} \tag{55}$$

In the case of a endowment contract with unitary sum insured, we have:

$$\begin{aligned}
epv^{j^*,q}_t =& {}_{n-t}p_{x+t} \cdot (1+j^*)^{-(n-t)} + \sum_{h=0}^{n-t-1} {}_{h/1}q_{x+t} \cdot (1+j^*)^{-(h+1)} + \\
&- \pi \sum_{h=0}^{n-t-1} {}_h p_{x+t} \cdot (1+j^*)^{-h} = \\
&{}_{n-t}p_{x+t} \cdot (1+j^*)^{-(n-t)} + \\
&+ (q_{x+t} \cdot (1+j^*)^{-1} + \sum_{h=1}^{n-t-1} {}_{h/1}q_{x+t} \cdot (1+j^*)^{-(h+1)}) + \\
&- \pi \cdot (1 + \sum_{h=1}^{n-t-1} {}_h p_{x+t} \cdot (1+j^*)^{-h})
\end{aligned}$$

That, for $s = h - 1$, could be rewritten as:

$$\begin{aligned} epv_t^{j^*,q} =& {}_{n-t}p_{x+t} \cdot (1+j^*)^{-(n-t)} + \\ & + (q_{x+t} \cdot (1+j^*)^{-1} + \sum_{s=0}^{n-t-2} {}_{(s+1)/1}q_{x+t} \cdot (1+j^*)^{-(s+2)}) + \\ & - \pi \cdot (1 + \sum_{s=0}^{n-t-2} {}_{s}p_{x+t} \cdot (1+j^*)^{-(s+1)}) \end{aligned}$$

Considering now:

$$\begin{aligned} \frac{epv_t^{j^*,q}}{{}_1E_{x+t}} =& {}_{n-t-1}p_{x+t+1} \cdot (1+j^*)^{-(n-t-1)} + \\ & \left(\frac{q_{x+t}}{{}_1p_{x+t}} + \sum_{s=0}^{n-t-2} \frac{{}_{(s+1)/1}q_{x+t}}{{}_1p_{x+t}} \cdot (1+j^*)^{-(s+1)} \right) + \\ & - \frac{\pi}{{}_1E_{x+t}} - \pi \cdot \sum_{s=0}^{n-t-2} {}_{s}p_{x+t+1} \cdot (1+j^*)^{-s} \end{aligned}$$

we have:

$$\frac{epv_t^{j^*,q}}{{}_1E_{x+t}} = epv_{t+1}^{j^*,q} - \frac{\pi}{{}_1E_{x+t}} + \frac{q_{x+t}}{{}_1p_{x+t}}$$

and, with simple algebra, easily follows equation (55). Therefore, it has been shown that formula (55) is a recursive equation: it is similar to Fouret one, with the difference that $epv_{t+1}^{j^*,q}$ and π are calculated on a different demographic basis (respectively q and q^*).

It follows that, if all the components are well defined (i.e. when none assumes a value of 0 by construction, e.g. when $t = 0$), $\mathbb{E}\left[\tilde{y}_{t+1}^{LG,Dem}\right] = -\mathbb{E}\left[\tilde{y}_{t+1}^{MCVq-q*}\right]$.

The most interesting aspect of this proof concerns two insights into formula (40):

- On the one hand, it is necessary to isolate the effects of $(1 + j^*)$: in particular, the sum of the best estimate rate and the pure premium is not consistent with the capitalization rate. To this end, through the Theorem proposed below it will be shown that the coherent rate is the risk free one between t and $t+1$, $i_t(t, t+1)$: for these reasons the $\tilde{y}_{t+1}^{NDM}$ component is built in formula (46).

- On the second hand, it is necessary to consider separately the differences between the actual realizations of the risk free and mortality random variables from the respective average values. For these reasons, these effects are specified in the Trend Risk and Risk Free Risk components of formula (46), called respectively $\tilde{y}_{t+1}^{Trend}$ and $\tilde{y}_{t+1}^{RiskFree}$.

Theorem 1. [18] *Considering a without-profit endowment insurance contract that pays a lump sum equal to 1 either in case of death or in case of survival at the end of the contract and without benefits in case of lapses, if second-order technical bases at time t and time t + 1 are the same, the following recursive equation holds:*

$$(be_t + \pi)(1 + i_t(0,1)) =_{/1} q_{x+t} + be_{t+1} \cdot_1 p_{x+t} \tag{56}$$

where π *is the regular premium rate,* $i_t(0,1)$ *is the one-year risk-free spot rate in force at time t and* be_t *and* be_{t+1} *are the pure best estimate rates computed using realistic demographic and financial assumptions in force at time t and neglecting expenses and expenses loadings.*

Proof. We recall here the definition of the best estimate rate of an endowment policy computed using realistic demographic assumption q and the risk-free

[18] The original version of this Theorem can be found in [13]

rate curve it in force at time t.

$$\begin{aligned} be_t =_{n-t}p_{x+t} \cdot \left[\prod_{h=0}^{n-t-1}(1+i_t(0,h,h+1))\right]^{-1} + \\ \sum_{k=0}^{n-t-1} {}_{k/1}q_{x+t}\left[\prod_{h=0}^{k}(1+i_t(0,h,h+1))\right]^{-1} - \pi \cdot \ddot{a}_{(x+t):(n-t)} \end{aligned} \tag{57}$$

where $i_t(0,h,h+1)$ is a risk-free forward rates.

Previous formula is also equal to:

$$\begin{aligned} be_t =_{n-t}p_{x+t}\left[\prod_{h=0}^{n-t-1}(1+i_t(0,h,h+1))\right]^{-1} +_{/1} q_{x+t} \cdot (1+i_t(0,1))^{-1} + \\ \sum_{k=1}^{n-t-1} {}_{k/1}q_{x+t}\left[\prod_{h=0}^{k}(1+i_t(0,h,h+1))\right]^{-1} - \pi \cdot \sum_{h=1}^{n-t-1} {}_hE_{x+t} - \pi \end{aligned} \tag{58}$$

From formula (58), we have that the following relation holds:

$$\begin{aligned} (be_t+\pi)\cdot(1+i_t(0,1)) -_{/1} q_{x+t} =_{n-t} p_{x+t}\left[\prod_{h=1}^{n-t-1}(1+i_t(0,h,h+1))\right]^{-1} \\ + \sum_{s=0}^{n-t-2}\left({}_{(s+1)/1}q_{x+t}\left[\prod_{h=1}^{s+1}(1+i_t(0,h,h+1))\right]^{-1}\right) \\ - \pi \cdot \sum_{h=1}^{n-t-1}\left({}_hp_{x+t}\left[\prod_{j=1}^{h}(1+i_t(0,j,j+1))\right]^{-1}\right) \end{aligned} \tag{59}$$

Since, the estimation of the best estimate at time $t+1$ under the assumption in force at time t is equal to:

$$\begin{aligned} be_{t+1} =_{n-t-1}p_{x+t+1}\left[\prod_{h=1}^{n-t-1}(1+i_t(0,h,h+1))\right]^{-1} + \\ + \sum_{k=0}^{n-t-2}\left({}_{k/1}q_{x+t+1}\cdot\left[\prod_{h=1}^{k+1}(1+i_t(0,h,h+1))\right]^{-1}\right) + \\ - \pi \cdot \ddot{a}_{(x+t+1):(n-t-1)} \end{aligned} \tag{60}$$

it is noticeable that the right-hand side of formula (59) is equal to $be_{t+1} \cdot_1 p_{x+t}$. Hence, we have:

$$(be_t + \pi)(1 + i_t(0,1)) =_{/1} q_{x+t} + be_{t+1} \cdot_1 p_{x+t} \tag{61}$$

It is worth pointing out that the proof can be easily adapted to the cases of single premiums, pure endowment or term insurance contracts and flat rates, that have been also analysed in the thesis. All of these combinations can be considered as special cases of the one that has been proved. □

3.5. Conclusions

Section 3, full of technical demonstrations and analyses, is necessary in the first place because it shows the need to divide the effect of compounding with the technical rate $(1+j^*)$ by formula (46). On the other hand, by defining of formula (61) as a recursive formula, highlights that this component (which coincides with the idiosyncratic profit, see formula (47), must be treated separately. For these reasons it is necessary that formula (46)) divides the effects of changes in expectations, whether demographic ($\tilde{y}_{t+1}^{Trend}$) or financial ($\tilde{y}_{t+1}^{RiskFree}$), from idiosyncratic risk itself.

4. Idiosyncratic and trend risks

4.1. Introduction

In this section the concepts of idiosyncratic risk and trend risk are taken up, their characteristics are studied by reaching closed formulas, the model is applied to a case study and, by comparing the results with those of the Local GAAP context, a methodology is proposed for the calculation of the SCR partially similar to that used by Solvency II to quantify the Non-Life Underwriting Risk.

In order to facilitate the reader's understanding, the component of the demographic profit (see formula 46) relating to idiosyncratic profit is taken up:

$$\begin{aligned}\tilde{y}_{t+1}^{Idios} = {} & [be_t^{Rf(t),q(t)} + b_{t+1}(1-\alpha^* - \beta^*) - \gamma^*](w_t - \tilde{s}_{t+1})(1 + i_t(t, t+1)) + \\ & - (\tilde{x}_{t+1} + \tilde{w}_{t+1} \cdot be_{t+1}^{Rf(t+1),q(t+1)})\end{aligned}$$

In this context, it is noted that the only source of volatility relates to deaths (hence the claims $\tilde{x}_{t+1}$) and, consequently, the sums insured $\tilde{w}_{t+1}$ at the end of the valuation period. The most interesting issue is that the best estimate rate, improperly defined, is calculated on the same realistic (and therefore second-order) basis used in t, i.e. in the previous instant.

Conversely, considering the trend risk component shown below, the volatility of interest concerns the mortality estimates that the company makes at the time of valuation, generally different from those used in the previous instant.

$$\tilde{y}_{t+1}^{Trend} = \tilde{w}_{t+1} \cdot (\tilde{be}_{t+1}^{Rf(t+1),\tilde{q}(t+1)} - be_{t+1}^{Rf(t+1),q(t+1)})$$

4.2. Idiosyncratic Risk

Now take formula (47) and calculate the expected value; the second step will be to rewrite it with a more compact notation, similar to formula (18), which allows you to calculate standard deviation and skewness.
With reference to the expected value, it should be remembered that, where the formula is well defined, it is of the recursive type: it is therefore reasonable to expect that this expected value is equal to 0 when $t > 1$. The main problem concerns the instant $t = 0$ where the rate of best estimate, not yet set aside, is equal to 0.

4.2.1. The expected value

Lemma 2. *Considering a generic risk-free rate curve $Rf(t)$ and a generic second order demographic assumption $q(t)$, regardless of the first order pricing bases (demographic assumption q^* and technical rate j^*), when $t = 1$ it is possible to define:*

$$\mathbb{E}\left[\tilde{y}_1^{Idios}\right] = -be_{0+}^{Rf(0),q(0)} \cdot \Bigg(1 + i_0(0,1)\Bigg) \cdot \left(w_0 - \mathbb{E}\left[\tilde{s}_1\right]\right) \tag{62}$$

where $be_{0+}^{Rf(0),q(0)}$ is the expected present value of the benefits net of the expected present value of the premiums, calculated in $t = 1$ using as demographic base $q(0)$ and $Rf(0)$ as risk-free discount curve.
Whereas when $t > 0$:

$$\mathbb{E}\left[\tilde{y}_{t+1}^{Idios}\right] = (be_t^{Rf(t),q(t)} + \pi) \cdot (1 + i_t(t, t+1)) - p_{x+t} \cdot be_{t+1}^{Rf(t),q(t)} - q_{x+t} = 0 \tag{63}$$

For $t > 0$, formula (63) is a recursive equation, therefore if there are no changes in second-order demographic bases (i.e. the company does not change its expectations on future mortality), $\mathbb{E}\left[\tilde{y}_{t+1}^{Idios}\right]$ is equal to 0.

When $t = 1$, formula (62) of Lemma 1, is proved with following simple algebra:

$$\begin{aligned}\mathbb{E}\left[\tilde{y}_1^{Idios}\right] =&(w_0 - \mathbb{E}\left[\tilde{s}_1\right]) \cdot (0 + \pi) \cdot \Big(1 + i_0(0,1)\Big) + \\ &- \Big[\mathbb{E}\left[\tilde{w}_1\right] \cdot be_1^{Rf(1),q(0)} + \mathbb{E}\left[\tilde{x}_1\right]\Big].\end{aligned} \tag{64}$$

because $be_0^{Rf(0),q(0)} = 0$ in $t = 0$.

Since for an Endowment $\tilde{x}_{t+1} = \tilde{z}_{t+1}$[19] and $\mathbb{E}\left[\tilde{z}_{t+1}\right] = q_{x+t} \cdot \mathbb{E}\left[\tilde{w}_t - \tilde{s}_t\right]$, we have:

$$\begin{aligned}\mathbb{E}\left[\tilde{y}_1^{Idios}\right] =&(w_0 - \mathbb{E}\left[\tilde{s}_1\right]) \cdot \\ &\Big[(0 + \pi) \cdot \Big(1 + i_0(0,1)\Big) - \Big(p_x \cdot be_1^{Rf(1),q(0)} + q_x\Big)\Big]\end{aligned} \tag{65}$$

For an endowment contract without lapses, we can rewrite (65) in the following way:

$$\begin{aligned}\mathbb{E}\left[\tilde{y}_1^{Idios}\right] =&(w_0 - \mathbb{E}\left[\tilde{s}_1\right]) \cdot \Bigg(\pi \cdot \Big(1 + i_0(0,1)\Big) - \Bigg[p_x \cdot \\ &\Bigg({}_{n-1}p_{x+1} \left[\prod_{h=1}^{n-1} (1 + i_1(0,h,h+1))\right]^{-1} + \\ &+ \sum_{k=0}^{n-2} \left({}_{k/1}q_{x+1} \cdot \left[\prod_{h=1}^{k+1} (1 + i_1(0,h,h+1))\right]^{-1} \right) + \\ &- \pi \cdot \ddot{a}_{(x+1):(n-1)} \Bigg) + q_x \Bigg]\Bigg)\end{aligned} \tag{66}$$

[19] Using an Endowment policy is a sufficient condition to verify the validity of any other policy: if we had considered a Pure Endowment (or an annuity) $\tilde{x}_{t+1} = 0$, if instead we had considered a Term Insurance $\tilde{x}_{t+1} = \tilde{z}_{t+1}$

The best estimate at $t = 0^+$ can be defined as follows:

$$
\begin{aligned}
be_{0^+}^{Rf(0),q(0)} =&{}_n p_x \left[\prod_{h=0}^{n-1} (1 + i_0(0,h,h+1)) \right]^{-1} +_{/1} q_x \cdot (1 + i_0(0,0,1))^{-1} + \\
&\sum_{k=1}^{n-1} {}_{k/1}q_x \left[\prod_{h=0}^{k} (1 + i_0(0,h,h+1)) \right]^{-1} - \pi \cdot \sum_{h=1}^{n-1} {}_hE_x - \pi
\end{aligned}
\tag{67}
$$

Therefore, from (66) and (67), we have formula (62).

It is noteworthy that formula (62) defines the profit or loss that is released at the inception of the contract. Demographic profit or loss is indeed generated by the differences between the life table used for computing premiums and the assumptions used for the computation of the best estimate ($be_{0^+}^{Rf(0),q(0)}$ and $be_1^{Rf(1),q(0)}$). Obviously in case no differences are observed, also the expected value in formula (62) is zero since $be_{0^+}^{Rf(0),q(0)}$ is zero.

4.2.2. The more compact version of the idiosyncratic risk

We focus now on the other characteristics of the distribution of the r.v. $\tilde{y}_{t+1}^{Id}$, i.e. standard deviation and skewness; for this purpose a more compact version of formula (47) is presented. We break down the best estimate rate into a pure component be_t^P (expected present value of benefits net of expected present value of premiums) and a component relating to expenses be_t^E (expected present value of expenses net of expected present value of loadings per expense). Since both the financial basis used, i.e. the risk free curve available at time t, $Rf(t)$, and the demographic basis, best estimate of mortality rates at time t, $q(t)$, are the same both in the calculation of $be_t^{Rf(t),q(t)}$ and in the calculation of $be_{t+1}^{Rf(t+1),q(t+1)}$, to facilitate understanding of the reader, the information on the technical basis is neglected.

$$\begin{aligned}\tilde{y}_{t+1}^{Idios} =& (w_t - \tilde{s}_{t+1}) \cdot \left[be_t^P + be_t^E + \pi \right] \cdot (1 + i_t(t, t+1)) + \\ & - \left[\tilde{w}_{t+1} \cdot (be_{t+1}^P + be_{t+1}^E) + \tilde{x}_{t+1} \right]\end{aligned} \tag{68}$$

Considering the recursive formula (20) relating to the insured sums, we obtain:

$$\begin{aligned}\tilde{y}_{t+1}^{Idios} =& (w_t - \tilde{s}_{t+1}) \cdot \left[be_t^P + be_t^E + \pi \right] \cdot (1 + i_t(t, t+1)) + \\ & - \left[(w_t - \tilde{s}_{t+1} - \tilde{z}_{t+1}) \cdot (be_{t+1}^P + be_{t+1}^E) + \tilde{x}_{t+1} \right]\end{aligned} \tag{69}$$

By simple steps, we have:

$$\begin{aligned}\tilde{y}_{t+1}^{Idios} =& \left[(be_t^P + \pi)(1 + i_t(t, t+1)) - be_{t+1}^P \right] (w_t - \tilde{s}_{t+1}) - \tilde{x}_{t+1} + be_{t+1}^P \tilde{z}_{t+1} \\ & - be_t^E \left[w_t - \tilde{s}_{t+1}(1 + i_t(t, t+1)) \right] - be_{t+1}^E (w_t - \tilde{s}_{t+1}) + be_{t+1}^E \tilde{z}_{t+1}\end{aligned} \tag{70}$$

We consider now a further break down of formula (70): the pure premium π is discomposed into the risk premium rate π^r and the rate of savings premium π^s. We could do it because formula (63) is a recursive equation (see Theorem 1).

The two components of the π premium are defined as:

$$\begin{aligned}\pi^r &= q_t(\mathbb{1}_k - be_{t+1}^P)(1 + i_t(t, t+1))^{-1} \\ \pi^s &= be_{t+1}^P (1 + i_t(t, t+1))^{-1} - be_t^P\end{aligned} \tag{71}$$

where

$$\mathbb{1}_k = \begin{cases} 0 & \text{in case of Pure Endowments \& Annuities} \\ 1 & \text{in case of Term Insurances \& Endowments} \end{cases} \tag{72}$$

It follows that the π^r risk premium rate can be written as:

$$\pi^r = (\pi + be_t) - be_{t+1}(1 + i_t(t, t+1))^{-1} \tag{73}$$

In a completely analogous way, it is possible to define for the component relating to expenses:

$$\pi^{er} = (\pi^e + be_t^E) - be_{t+1}^E(1 + i_t(t, t+1))^{-1} \tag{74}$$

By formula (70), we can write:

$$\begin{aligned}\tilde{y}_{t+1}^{Idios} =& \Big[\pi^r(1 + i_t(t, t+1))(w_t - \tilde{s}_{t+1})\Big] - \tilde{x}_{t+1} + be_{t+1}^P + \\ & - \pi_{t+1}^{er}(1 + i_t(t, t+1))(w_t - \tilde{s}_{t+1}) + be_{t+1}^E \tilde{z}_{t+1}\end{aligned} \tag{75}$$

and

$$\begin{aligned}\tilde{y}_{t+1}^{Idios} =& q_t(I - be_{t+1}^P)(w_t - \tilde{s}_{t+1}) - \tilde{x}_{t+1} + be_{t+1}^P \tilde{z}_{t+1} \\ & - be_{t+1}^E(q_t(w_t - \tilde{s}_{t+1}) - \tilde{z}_{t+1})\end{aligned} \tag{76}$$

On the one hand, we now consider the case of Term Insurance and Endowment policies, where $\tilde{x}_t = \tilde{z}_t$ and $\mathbb{1}_k = 1$, and on the other hand the case of Pure Endowment and Annuities where $\tilde{x}_t = 0$ and $\mathbb{1}_k = 0$. In both cases it is possible to rewrite the demographic profit linked to the idiosyncratic risk as:

$$\tilde{y}_{t+1}^{Idios} = D_{t+1}^{Rf(t),q(t)}\Big[q_t(w_t - \tilde{s}_{t+1}) - \tilde{z}_{t+1}\Big] \tag{77}$$

4.2.3. Standard deviation and skewness of the idiosyncratic risk

Having defined a more compact version of formula (47), it is easy to calculate both the standard deviation and the skewness of the idiosyncratic component of demographic risk in a market-consistent context. To this end, the effect of lapses is neglected. As for the case study developed in the Local GAAP framework, the dynamics of lapses impact the amounts of the insured sums which, in turn, impact the idiosyncratic component. However, since the aim is to calculate the SCR linked only to the components that depend on the risk of mortality, the absence of surrenders is also neglected here. In this way, it is also possible to compare the results with those of Section 2.

$$\begin{aligned}
\sigma(\tilde{y}_{t+1}^{Idios}) &\approx D_{t+1}^{Rf(t),q(t)} \cdot \sigma(\tilde{z}_{t+1}) \\
&= |D_{t+1}^{Rf(t),q(t)}| \cdot \sqrt{\sigma^2(\tilde{d}_{t+1}) \cdot \bar{C}_{t+1}^2 \cdot r_{2,C_{t+1}}} \\
&= \left(|D_{t+1}^{Rf(t),q(t)}| \cdot \bar{C}_{t+1} \cdot l_t\right) \cdot \sqrt{\frac{q_t \cdot (1-q_t)}{l_t} \cdot r_{2,C_{t+1}}}
\end{aligned} \tag{78}$$

$$\begin{aligned}
\gamma(\tilde{y}_{t+1}^{Idios}) &\approx -\frac{(D_{t+1}^{Rf(t),q(t)})^3}{|D_{t+1}^{Rf(t),q(t)}|^3} \cdot \gamma(\tilde{z}_{t+1}) \\
&= -\frac{(D_{t+1}^{Rf(t),q(t)})^3}{|D_{t+1}^{Rf(t),q(t)}|^3} \cdot \gamma(\tilde{d}_{t+1}) \cdot \frac{r_{3,C_{t+1}}}{(r_{2,C_{t+1}})^{3/2}} \\
&= -\frac{(D_{t+1}^{Rf(t),q(t)})^3}{|D_{t+1}^{Rf(t),q(t)}|^3} \cdot \frac{(1-2q_t)}{\sqrt{l_t \cdot q_t \cdot (1-q_t)}} \cdot \frac{r_{3,C_{t+1}}}{(r_{2,C_{t+1}})^{3/2}}
\end{aligned} \tag{79}$$

The results obtained are now compared with the results of the model developed in the Local GAAP context. Following aspects are noticeable:

- By construction, the demographic risk constructed in the Local GAAP context (see formula (18)) is attributable only to the idiosyncratic component. Actually, a change in demographic expectations (or in risk-free rates) does not affect the results of the Local GAAP model: in the same way, the idiosyncratic risk component is sensitive only and exclusively to actual changes in mortality rates with respect to their predicted value.

- Comparing formula (18) with formula (77) we observe that the only differences concern $D_{t+1}^{Rf(t),q(t)}$ and q_t: the first is a multiplicative component of the only random variable of the formula, $\tilde{z}_{t+1}$, while the second is an additive component. The properties of variance are well known, primarily that of translational invariance, the variability of the Local GAAP context depends on the SaR rate calculated on the basis of the locked and prudential first order, the variability of idiosyncratic risk in the market consistent context depends on from the same quantity, but calculated on the updated second order bases $Rf(t)$ and $q(t)$.

4.3. Trend risk

In this subsection we will deal with studying the trend risk component of the demographic profit which, for simplicity, is reported below:

$$\tilde{y}_{t+1}^{Trend} = -\tilde{w}_{t+1} \cdot (\tilde{be}_{t+1}^{Rf(t+1),\tilde{q}(t+1)} - be_{t+1}^{Rf(t+1),q(t+1)})$$

With reference to formula (48), above for convenience, it is observed that there are two sources of volatility: the first one is related to insured sums $\tilde{w}_{t+1}$ and the second one is linked to expectations (second order) on the mortality rates which the company will adopt in $t+1$, hence stochastic rates.
Insured sums, stochastic, are always a positive value while the driver of the trend risk is the difference between the (deterministic) demographic assumptions $q(t+1)$ and the stochastic ones $\tilde{q}(t+1)$. Therefore, the random variable $\tilde{z}_{t+1}$ affects the amount of profit (if the spread is positive) or loss (in case of negative spread), but not the sign.
For this reason, the main interest of this section will be oriented to the

analysis of the distribution of $\tilde{be}_{t+1}^{Rf(t+1),\tilde{q}(t+1)}$ and, consequently, of the aforementioned spread. Even in terms of quantifying the SCR, the volatility of $\tilde{w}_{t+1}$ will be neglected: the joint assessment of $\tilde{w}_{t+1}$ and $\tilde{q}(t+1)$ is instead necessary for quantifying the dependence between the two random variables and, therefore, for model the aggregation between SCR_{t+1}^{Trend} and SCR_{t+1}^{Idios}. Therefore, the target on $\tilde{be}_{t+1}^{Rf(t+1),\tilde{q}(t+1)}$ doesn't regard the estimation of the evolution of $\tilde{q}(t+1)$, but rather the volatility around the aforementioned estimation assuming that the procedure is carried out in t as $q(t)$ represents the best estimate of the future mortality of the cohort. In this way it is therefore possible to construct a confidence interval and, above all, to quantify the percentile of order ϵ of the distribution: if this evaluation is made on an annual time horizon and $\epsilon = 0.5\%$, then this methodology is completely consistent with Solvency II.
In this thesis, the choice fell on *Bootstrapping the Poisson log-bilinear model for mortality forecasting* (see [11]), a model that will be presented later, as it manages to capture both the volatility linked to the estimate of the parameters, and the volatility of deaths.

4.3.1. The Poisson log-bilinear model

In this paragraph we recall the structure of the Poisson log-bilinear model in order to explain the results of the numerical section and, above all, to highlight the strengths and weaknesses of the model.

- The starting point coincides with the definition of the random variable $\tilde{T}_x$ representing the residual life span of an individual of age x. It follows that, for example, the probability that an individual of age x survives for h years is equal to:

$${}_h p_x = \mathbb{P}[\tilde{T}_x > h]$$

μ_x is defined as the force of mortality (also called mortality intensity) as the ratio between the instantaneous probability of death and a time span of infinitesimal amplitude. In formulas:

$$\begin{aligned} \mu_x(t) &= \lim_{t \to +0^+} \frac{\mathbb{P}[\tilde{T}_x \leq t]}{t} \\ &= \lim_{t \to +0^+} \frac{{}_t q_x}{t} \end{aligned} \tag{80}$$

We then introduce the survival function $S_x(t)$ defined as:

$$S_x(t) = \mathbb{P}[\tilde{T}_x > t]$$

The definition of $S_x(t)$ is necessary to define the central mortality rate $m_x(t)$. Defined an age x, it is equal to the weighted average of the mortality forces, with weights equal to the values of the survival function. In formulas:

$$m_x(t) = \frac{\int_0^1 S_{x+u}(t) \cdot \mu_{x+u}(t) du}{\int_0^1 S_{x+u}(t) du} \tag{81}$$

in this context it is assumed that the mortality force is piecewise constant therefore, when $0 \leq t \leq 1$:

- Constant and continuous function on its right. When $0 \leq k < 1$:

$$\mu_{x+k}(t) = \mu_x(t) \tag{82}$$

- Coincidence between central mortality rate and force of mortality:

$$\mu_x(t) = m_x(t) \tag{83}$$

– Exponential formula of the probability of survival:

$$ {}_tp_x = e^{-t \cdot \mu_x(t)} \tag{84} $$

- We assume that the number of deaths follows a Poisson distribution based on the following assumptions:

$$ \begin{aligned} &D_{x,t} \sim Poisson\Big(E_{x,t}\mu_x(t)\Big) \\ &\mu_x(t) = exp(\alpha_x + \beta_x \kappa_t) \end{aligned} \tag{85} $$

where $E_{x,t}$ is used to indicate the exposure-to-risk at age x during calendar year t, i.e. the total time lived by people aged x in year t and where $\mu_x(t)$ is described by the Lee-Carter model (see [25]). In particular, it is assumed that the logarithm of the mortality force is described by a linear combination of three vectors of parameters $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\kappa}$ and by a disturbance factor.
α_x describes the general shape of mortality according to different ages, κ_t reproduces the underlying time trend, while the term β_x is considered in order to take into account the different effect of time t at each age.

- Instead of resorting to Singular Value Decomposition, we estimate the model parameters by maximizing the log-likelihood function defined as:

$$ L(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\kappa}) = \sum_t \sum_x \Big(D_{x,t}(\alpha_x + \beta_x \kappa_t) - E_{x,t} e^{(\alpha_x + \beta_x \kappa_t)} \Big) + constant \tag{86} $$

Vectors of parameters $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\kappa}$ are obtained by an iterative algorithm and under properly constraints to assure a unique solution: $\sum_t k_t = 0$ and $\sum_x \beta_x = 1$.

- Finally, Box-Jenkins methodology is used to generate the appropriate ARIMA time series model for forecasting of k_t. In particular, the parameter κ_t is considered as a discrete stochastic process of the ARIMA type (ARIMA (0,1,0) for men and ARIMA (0,1,1) for women, respectively). The methodology allows to obtain an estimate $\hat{\mu}_x(t)$ for each age x and time t.

- The procedure is iterated n times, following the parametric bootstrap proposed in [11]. Bootstrap samples have been derived by simulating $D^n_{x,t}$ from a Poisson:
$$D^n_{x,t} = E_{x,t} \cdot \mu_x(t) \tag{87}$$
Therefore, it is added a heteroskedatic noise that will consider greater volatility for the most extreme ages. Given the new set of deaths $D^n_{x,t}$ obtained in each iteration, the Lee-Carter methodology is applied in order to estimate $\hat{\mu}^n_x(t)$. The procedure involves the estimation of n Poisson Lee-Carter models and as many forecasts. It is time-consuming but allows to quantify the two volatilities previously mentioned in the calculation of future best estimates rates.

5. Numerical results

5.1. Introduction

The purpose of this concluding section is to show the application of the model presented in the previous sections. In particular, the purpose is dual: on the one hand we want to show the ability of the model to intercept the different sources of uncertainty while remaining consistent with the accounting principles of Solvency II, on the other hand we want to show how the new framework is distant, in terms of capital requirement, from the Local GAAP one.

In order to cover all aspects of a comparative nature between the two frameworks, the characteristics of the cohorts and those of the market will be kept unchanged, so that the only drivers of the differences between the two models will be only the evaluation criteria linked to the technical liabilities of the company.

We then proceed to the presentation of the results linked to the two components of the demographic risk linked exclusively to the achievements / expectations relating to mortality, i.e. the idiosyncratic and trend components. Also in this section the two components will be treated separately and evaluated in relation to the two elementary policies through which it is possible to build each policy on the insurance market: Pure Endowment and Term Insurance.

5.2. Idiosyncratic risk results

In this section we will present the results of the model developed in Section 4 applied to both a Pure Endowment and a Term Insurance. As anticipated, a particularly interesting aspect will be the comparison of the market consistent results with those relating to the Local GAAP context: for this reason, it is specified that the model parameters will be those proposed in Table 2. This table will be shown below with an addition relating to risk-free rates: from what has been learned, it is necessary to specify the risk-free rate curve available at the time of valuation. Although the exact moments of the idiosyncratic demographic profit do not depend on the shape of the curve (generally shared by EIOPA), it is observed that the formulation of the expected value of the market-consistent demographic profit when $t = 1$ (see formula (62)) depends from the ratio between the first-order financial basis used in the pricing phase j^* and the risk-free rates of the curve. Because of this reason, the assumption that the risk-free rates are constant and equal to the value of the technical rate j^* is added in the table below.
Therefore, in conclusion, this simplifying hypothesis is not strictly necessary for the application of the model, but allows to neutralize the effect of the spread between the two aforementioned financial rates on the expected value. In this way it is possible to compare the proposed results with those of Section 2 without having an extra distorting component.

5.2.1. Idiosyncratic risk - Pure Endowment

The starting point to analyse the characteristics of the idiosyncratic profit is the SaR rate which, in the case of the Pure Endowment, is calculated as

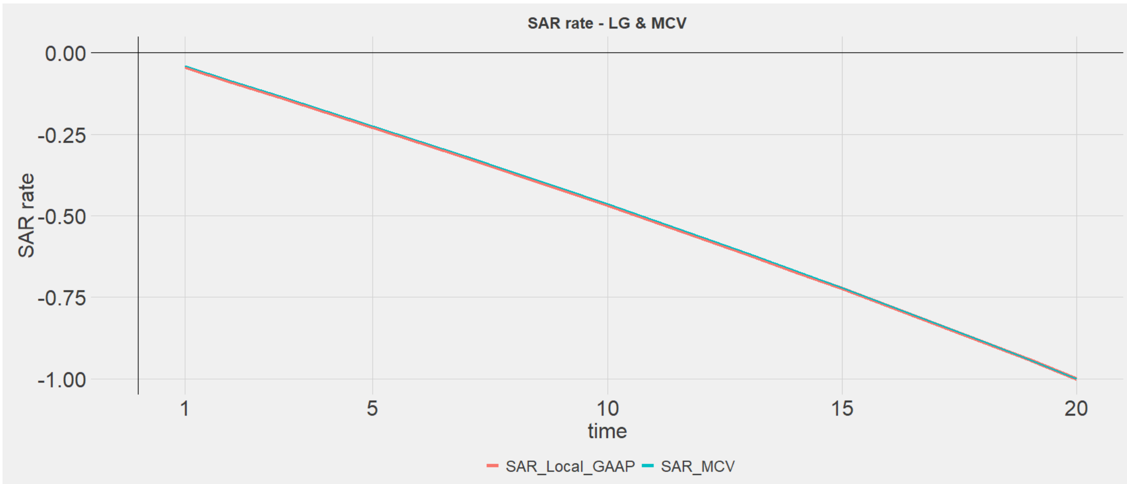
Table 5: Model parameters

Policyholders date of birth	31st December 1978
Policyholders gender	Males
Policies moment of issue	31st December 2018 ($t = 0$)
Policy duration	20 y
Premium type	Annual premiums (20 y)
Number of policyholders (l_0)	15,000
Expected value of the single insured sum (CU)	100,000
CoV of the insured sums	2
Risk-free rates	1% (flat)
First order financial rate j^*	1%

the opposite of the mathematical reserve: for this purpose it is presented in Figure 12 the trend of the two SaRs, i.e. the market consistent one and the one consistent with the Local GAAP framework. First of all it is specified that although the two curves are very similar, both of them are calculated on a unitary insured sum: a variation (even slight) is then multiplied by the insured sums of the entire portfolio, so the overall effect is amplified.

Secondly, it is observed that for each value of t, the SaR rate Local GAAP is strictly lower than the market consistent one: to understand the reason it is sufficient to remember that, with the same technical-financial rate, the mathematical reserve curve of the Local context GAAP is less convex than the best estimate one since the first order death probabilities are lower than the second order ones, hence it should be recalled that $\lambda_{x+t} = \dfrac{q^*_{x+t} - q_{x+t}}{q_{x+t}} = -20\%$.

Figure 12: SaR rates comparison: the market consistent one and the Local GAAP consistent one



With reference to Table 6, the first element on which to focus is the expected value. On the first hand, it is observed that, consistently with the recursive formula (63), this expected value is always equal to 0 when $t > 1$. The most interesting aspect is that the expected value, which, when $t = 1$, depends mostly on $be_{0+}^{Rf(0),q(0)}$, i.e. the expected present value of the benefits net of the expected present value of the premiums. Since the cash flows are discounted with the same technical rate of the first order (financial bases) with which the pricing was done and since the demographic base of the first order is prudential, the so-called $be_{0+}^{Rf(0),q(0)}$ is negative and the expected profit in $t = 1$ is positive.

The fact that the expected profit is fully expected at the end of the first time span, i.e. when $t = 1$, has two notable implications:

- Unlike the Local GAAP context (see Table 3), where the expected profit depends on the differnce between what is “expected” in the accounting

Table 6: Simulated and theoretical characteristics of idiosyncratic profit and loss distribution for a Pure Endowment contract for three different time periods in the market-consistent context. Last two rows summarize SCR and SCR ratio with respect to sums insured.

Pure Endowment	**t=1**	**t=10**	**t=20**
Theoretical expected value	5,807,038	0	0
Simulated mean	5,807,024	-122	124
Theoretical standard deviation	23,115	464,869	1,657,030
Simulated standard deviation	23,091	463,478	1,632,732
Theoretical skewness	3.26	1.84	1.10
Simulated skewness	3.25	1.84	1.09
Solvency Capital Requirement	-5,784,999	704,780	2,985,434
SCR/Sums insured	-0.38%	0.05%	0.21%

year and what actually happens, in the market-consistent context the profit is expected when a "new" difference is observed between the technical bases of the first order and those of the second order. Since the financial basis in this case study is the same, all expected profit is entirely demographic.

- Having all the expected profit accounted for in a single payment is certainly a phenomenon to be managed for the company. This thesis does not aim to give a practical solution, even if all the actuarial practices known in the literature on the subject are feasible. An example could be to split the expected profit in the various time-periods according to the volume of premiums collected compared to the total.

It is observed that the volatility of the demographic profit grows both as a function of the increase in SaRs, and as the (realistic) probability of death of the policyholders increases. These probabilities, at the same time, are strictly lower than 0.5, hence the skewness decreases approaching its limit of 0. Comparing Table 3 (demographic profit distribution in the Local GAAP context) with Table 6 (demographic profit distribution in the market consistent context) and analysing the theoretical standard deviation, the absolute value of the SaR rate is higher in the Local GAAP context, while the $\sqrt{\frac{q_t \cdot (1-q_t)}{l_t}}$ factor is the same in the two frameworks. The result is that the theoretical standard deviation is greater in the Local GAAP context when the difference between the two SaR rates is greater: actually it occurs in the first time spans then it's decreasing until the expire date where the SaR rates are the same, hence the volatilities are identical.
Finally, it is noted that the company's SCR strictly depends on the volatility of the demographic profit, therefore the idea behind the QIS n.2 is partially confirmed, with the following specifications:

1. With reference to formula (36), it is possible to calculate it as the product of a coefficient and the standard deviation of the demographic profit,

2. The standard deviation must consider both the volatility of the deaths of the cohort and the volatility of the sums insured. Both must be calculated on realistic and non-prudential demographic bases used in the pricing phase,

3. The multiplier of the standard deviation, indicated with $k[\gamma(\tilde{y}_{t+1}^{MC,Dem})]$,

cannot be set equal to 2.58 because the shape of the distribution is not that of a Gaussian and the skewness is strictly different from 0,

4. From an Internal Model perspective, the company must first calculate the expected value of the demographic profit deriving from the use, at the pricing stage, of prudential demographic bases, and then it must spread it over the various time points with a methodology consistent with the business of the company.

Figure 13: Simulated distributions of idiosyncratic profit and loss for a pure endowment

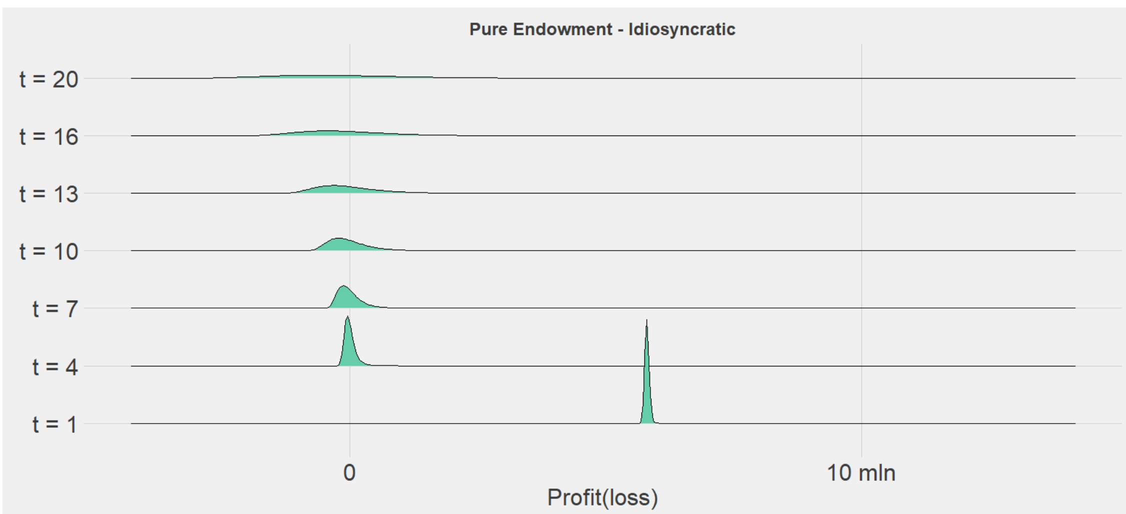
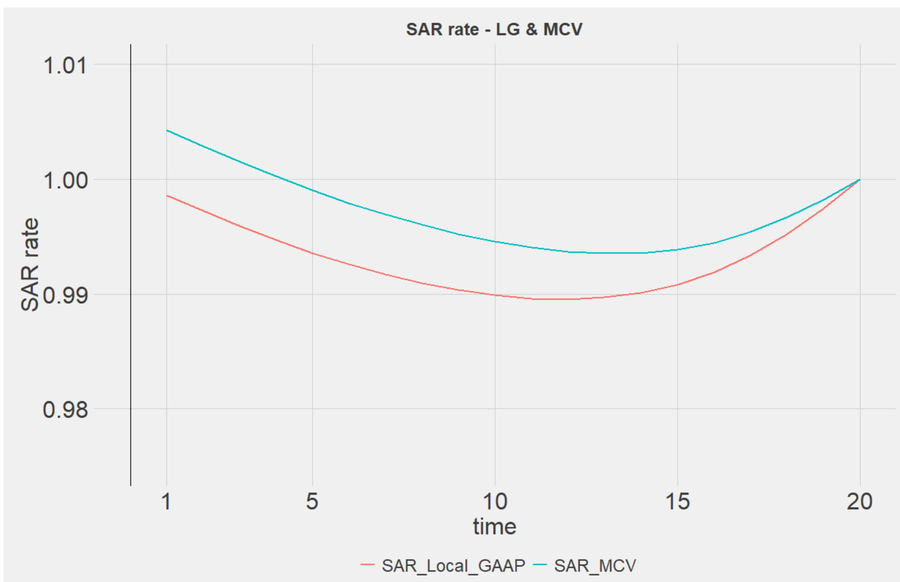


Figure 13 show the simulated distributions of the demographic profit random variable over different time instants. Comparing it with Figure 11 it is observed that the shapes of the distributions are very similar, with the difference that the first is shifted to the right due to the profit deriving from the use of prudential technical bases and the other distributions are centred on 0.

5.2.2. Idiosyncratic risk - Term Insurance

Also in this section, the starting point between the results of the application of the stochastic model developed in the Local GAAP context and that developed in the model consistent with Solvency II, coincides with the analysis of the SaR rate. Figure 14, shows two characteristics of the SaR

Figure 14: SaR rates comparison: the market consistent one and the Local GAAP consistent one



rate on which to focus: on the one hand the first values of the SaR rate of the market-consistent scenario are greater than unity since, also in this context, the demographic base of the first order is obtained with the goal of having a constant absolute lambda value equal to 20%[20]. On the other hand, net of the graphical effect of the figure, it is observed that since only

[20]Instead of reducing them, in the case of Term Insurance, first-order mortality rates are calculated by increasing second-order mortality rates by 20%

Table 7: Simulated and theoretical characteristics of idiosyncratic profit and loss distribution for a Term Insurance contract for three different time periods in the market-consistent context. Last two rows summarize SCR and SCR ratio with respect to sums insured.

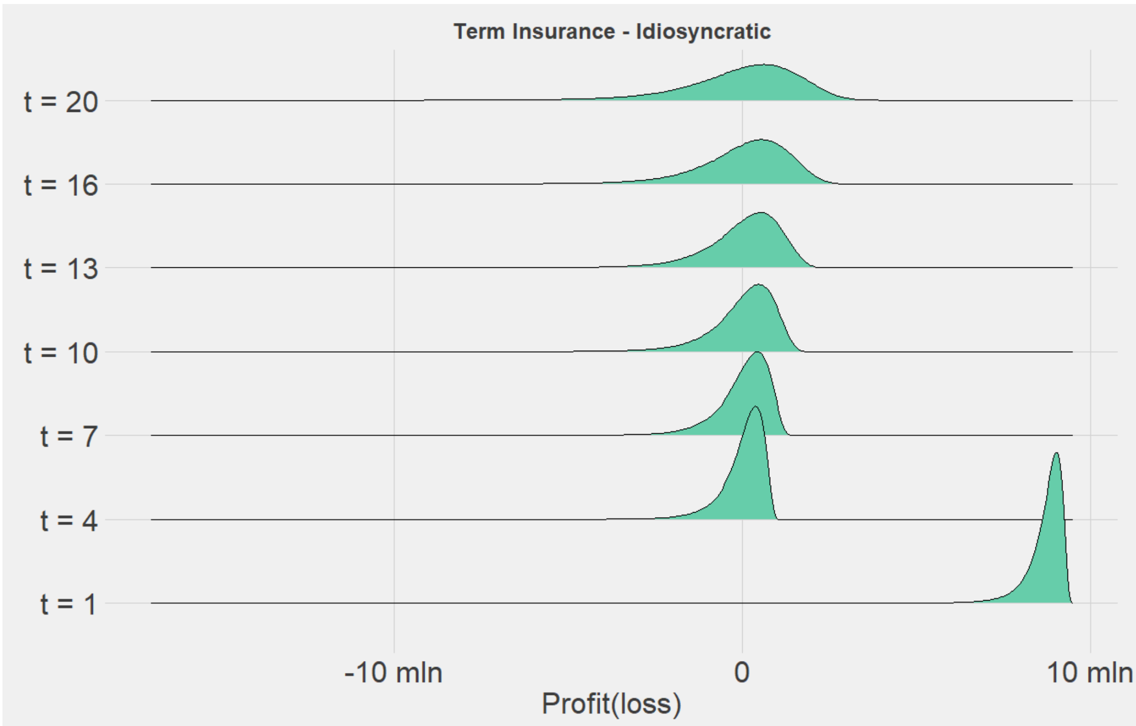
Term Insurance	**t=1**	**t=10**	**t=20**
Theoretical expected value	8,696,520	0	0
Simulated mean	8,696,373	-49	331
Theoretical standard deviation	569,948	995,523	1,657,030
Simulated standard deviation	570,607	991,701	1,632,169
Theoretical skewness	-3.26	-1.84	-1.10
Simulated skewness	-3.25	-1.83	-1.09
Solvency Capital Requirement	-5,826,888	4,444,835	6,057,733
SCR/Sums insured	-0.38%	0.29%	0.41%

the demographic bases vary between the two assessments (Local GAAP and market-consistent), the two SaR rate curves are very close, but with the market-consistent one always above. The effect deriving from the SaR rate will therefore be to further amplify the volatility of the idiosyncratic demographic profit. With reference to the results of the simulation model reported in Table (7), first of all it is observed that also in this market-consistent case all the expected profit is recognized during the first time span, while subsequently it is null. An interesting aspect, comparing the expected profit in $t = 1$ with the analogous value of the Pure Endowment is that the Term Insurance one is higher: the reason lies in the fact that in relative terms, the safety loading in the demographic base of the first order compared to second order, is greater. Despite in both cases $|\lambda_{x+t}| = 20\%$, it is specified that in the case of the Pure Endowment, the benefit is paid in the event of survival,

while in the Term Insurance in the event of death: from this point of view it is clear that the relationship between the demographic base of the first order and the second order one is greater in the case $\frac{q^*_{x+t}}{q_{x+t}}$ of the Term Insurance.
With reference to the trend of the standard deviation, the main driver are the probabilities of death which (see formula (78)) increase over time, approaching the absolute maximum of 0.5. Since the first order death probabilities are greater than those of the second order (by construction), the SaR rate of the market consistent framework is higher than the Local GAAP framework one; therefore, except when $t = 20$, the standard deviation of the market consistent demographic profit is higher than the analogous value in the Local GAAP framework (see Table 4 and Table 7).
In this context, as in the analogous Local GAAP one, the SaR rate is not a fundamental driver because, as per Figure 14, its value is always very close to 1: it is interesting, however, to observe that the difference between the market-consistent SaR and the Local GAAP SaR justify the difference between the theoretical standard deviations of the two models.
The SCR, on the other hand, defined as the opposite of the 0.5% order percentile of the demographic profit distribution is particularly strong since the distributions of $\tilde{y}^{Idios}_{t+1}$, shown in Figure 15, has a particularly important volatility and a skewness that is always negative[21], in absolute value above 3 in the first time spans.

[21] Remember that the skewness, as demonstrated in formula (79), depends on the opposite of the sign of the SaR

Figure 15: Simulated distributions of idiosyncratic profit and loss for a term insurance

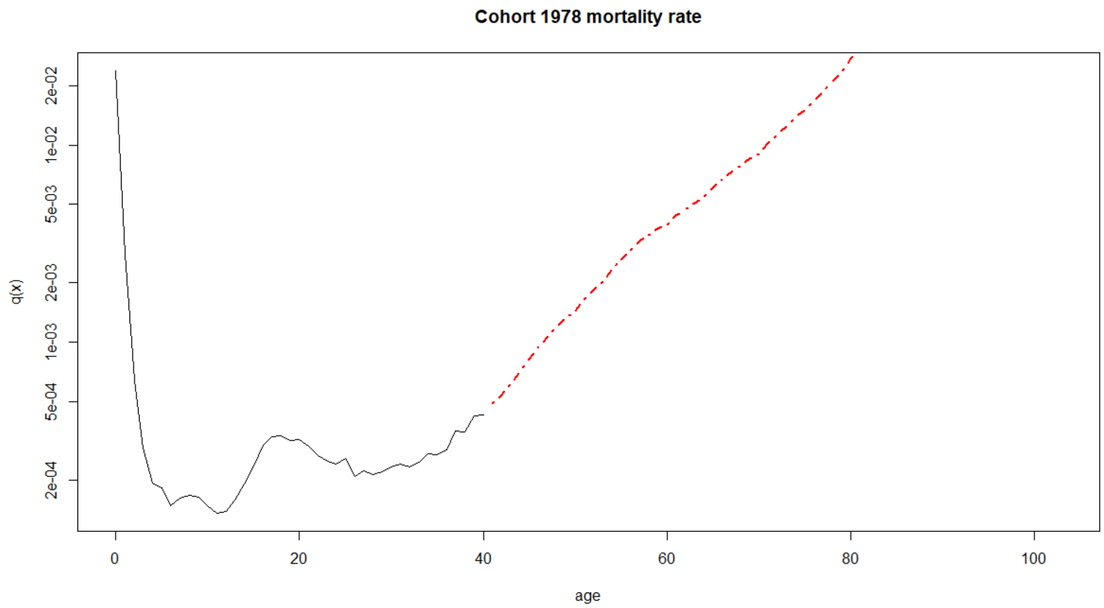


5.3. Trend risk results

This subsection presents the numerical results of the application linked to the log-bilinear Poisson bootstrap model in order to obtain an estimate of the variability around mortality expectations.

First of all, it is specified that the data used are those from 1872 to 2018 relating to the Italian population present in the Human Mortality Database: the choice not to exclude the years relating to the World Wars and the Spanish Fever derives from the fact that, as in the case of the 2019 pandemic, we do not want to arbitrarily reduce volatility by excluding future semi-catastrophic scenarios. Figure 16 shows the estimated mortality rates and

Figure 16: Estimated and forecasted mortality rates - cohort 1978

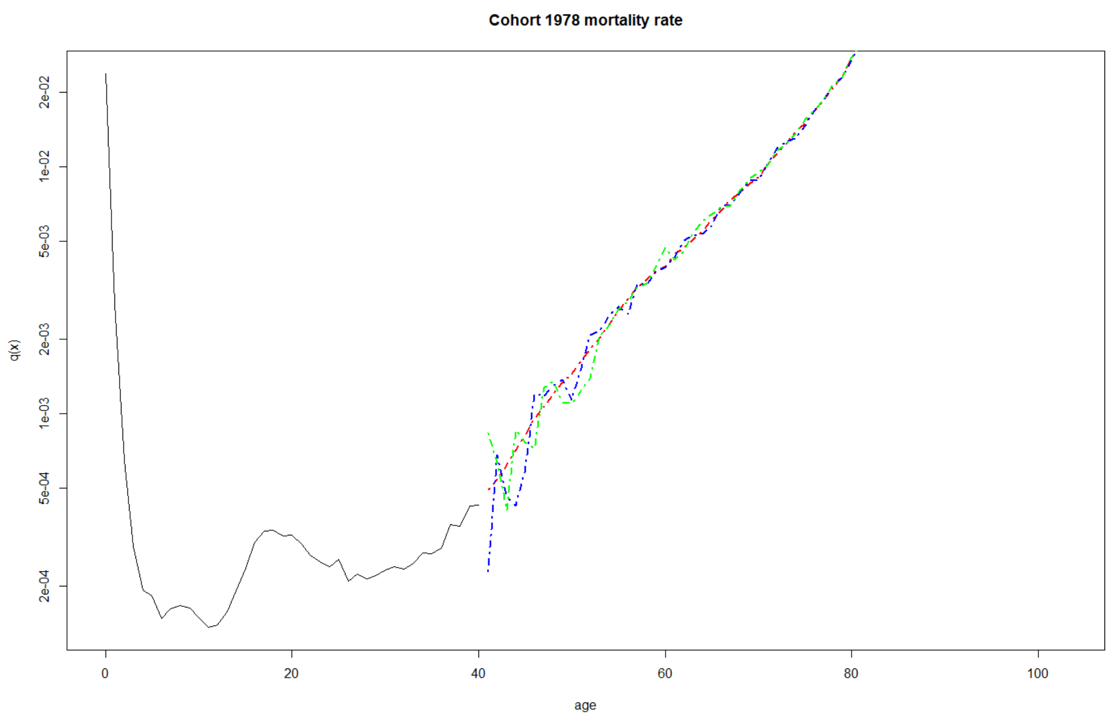


their projection for the generation of policyholders in 1978, which in 2018 (age of underwriting of twenty-year policies) reached 40 years. This projection assumes particular importance for two main reasons: on the first hand it coincides with the second order demographic table used by the company. On the second hand, in order to calculate the SCR over a time span of one year, for $t > 0$ the deaths of the cohort are equal to those expected up to the age of $x + t - 1$ and , subsequently evolve randomly. This last aspect is of particular importance because it allows to incorporate in the model not all the volatility between 2018 and the valuation year, but only the annual one useful for the purposes of the SCR.

Taking up the methodological steps shown in Subsection 4.3, a bootstrap model is then applied for each t to describe the probability distribution of the death expectations and, consequently, the best estimate of the company's liabilities $\tilde{be}_{t+1}^{Rf(t),\tilde{q}_{t+1}}$.

As shown in Figures 17 it is not possible to identify the expected death

Figure 17: Estimated and forecasted mortality rates - cohort 1978



curve that implies the worst case scenario of the company, transcending the type of policy. The use of the bootstrap model makes it possible to simulate different trajectories of the probability of death of the cohort which, however, intersect with each other: this means that it is necessary to directly associate the corresponding value of the best estimate to each curve which, in turn, depends on the structure of the policy and from different cash-flows at different time points.

The last aspect, demonstrable empirically by the execution of the bootstrap log-bilinear Poisson model is that for any age of the cohort, the distribution of the probability of death has a positive skewness: consider, intuitively, that it is more probable that the whole community survive rather than die entirely. The mathematical reason lies in the use of the Poisson random variable to describe the number of deaths: it is known that Poisson has a positive skewness that tends to 0 as the parameter increases.

Table 8: Simulated and theoretical characteristics of trend profit and loss distribution for a Pure Endowment contract for three different time periods. Last two rows summarize SCR and SCR ratio with respect to sums insured.

Pure Endowment	**t=1**	**t=10**
Be rate	0.04	0.46
Simulated mean	0.04	0.46
Simulated standard deviation	0.25%	0.14%
Simulated skewness	-0.43	-0.29
Solvency Capital Requirement	7,843,214	5,064,797
SCR/Sums Insured	0.52%	0.33%

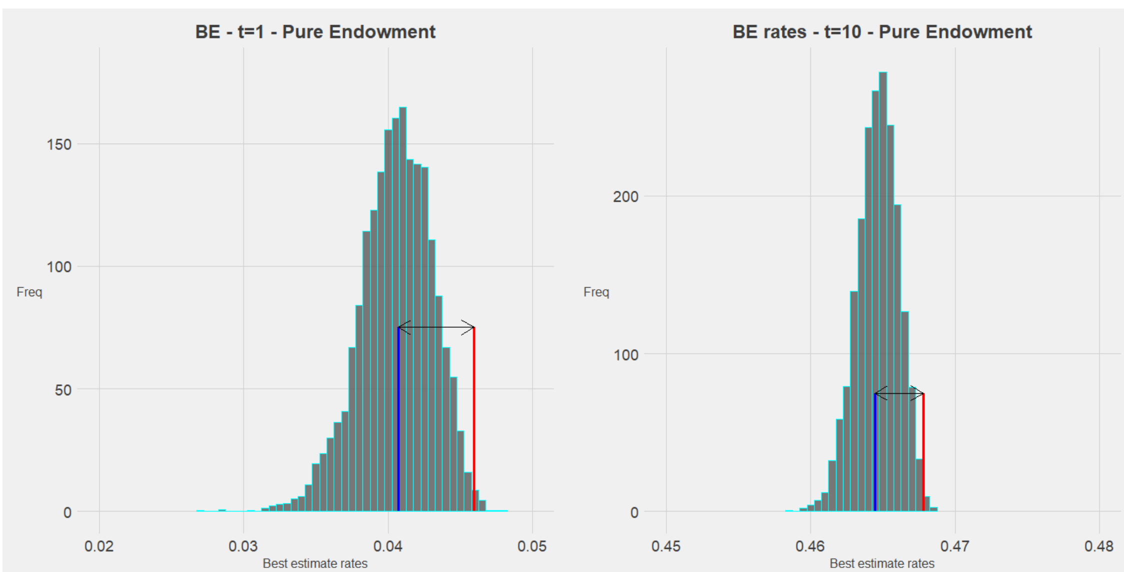
5.3.1. Trend risk - Pure Endowment

The results relating to the application of the Poisson log-bilinear model to the cohort of policyholders holding Pure Endowments are presented in this subsection.

It is specified that the application of the bootstrap model is by its nature extremely time consuming, in fact for each bootstrapped scenario it is necessary to re-calibrate the parameters of the Lee Carter model: the consequence is that in this context, five thousand simulations are presented for each scenario.

It is observed that despite the limited number of simulations, the estimate of the best estimate at instant $t+1$ is extremely close to its average (see Table 8). With reference to the standard deviation, on the other hand, a decreasing trend is identified: the reason lies in the fact that, as the time horizon increases, a revision of the mortality expectations of the cohort impacts on a number of smaller future time spans (intuitively, consider that a revision of expectations in t before expiration, only affects the benefit at maturity). Since the asymmetry of the death distribution is distributed as a Poisson, with positive asymmetry, the distribution of the best estimate has negative skewness: the highest value of $\tilde{be}_{t+1}^{Rf(t),\tilde{q}_{t+1}}$ correspond to cases where a large increase in the probability of survival is expected.

Figure 18: Distribution of BE rates at time 1 and time 10 for a Pure Endowment. Blue lines represent the expected values of the distributions, therefore $be_1^{Rf(1),q(0)}$ for the left figure and $be_{10}^{Rf(10),q(9)}$ for the right figure. The red lines indicate the 99.5% percentile of the distributions; then the black lines indicate the spreads between the expected values and the stressed values indicate the SCRs.



Hence, Figure 18 shows the simulated distributions of $\tilde{be}_1^{Rf(0),\tilde{q}_1}$ and $\tilde{be}_{10}^{Rf(9),\tilde{q}_{10}}$ highlighting, in addition to the characteristics described above, how the SCR linked to the Trend risk decreases as the time horizon increases.

A particularly important aspect is therefore the SCR linked to demographic risk for the Pure Endowment cohort: it is observed that on the one hand the capital for idiosyncratic risk increases as the time horizon increases because the volatility linked to deaths increases and increases the impact due to the absolute value of the SaRs, on the other hand the requirement linked to the trend risk decreases because, even in the event of a revision of the second-order demographic base, the impact on the best estimates is increasingly limited . In spite of this, however, the impact of a possible change in mortality has a weight, in terms of SCR, that is much more relevant than that deriving from fluctuations in mortality: in this context, therefore, the choice of the model used to estimate the evolution and, above all, the volatility linked to the company's future estimates.

Table 9: Simulated and theoretical characteristics of trend profit and loss distribution for a Term Insurance contract for three different time periods. Last two rows summarize SCR and SCR ratio with respect to sums insured.

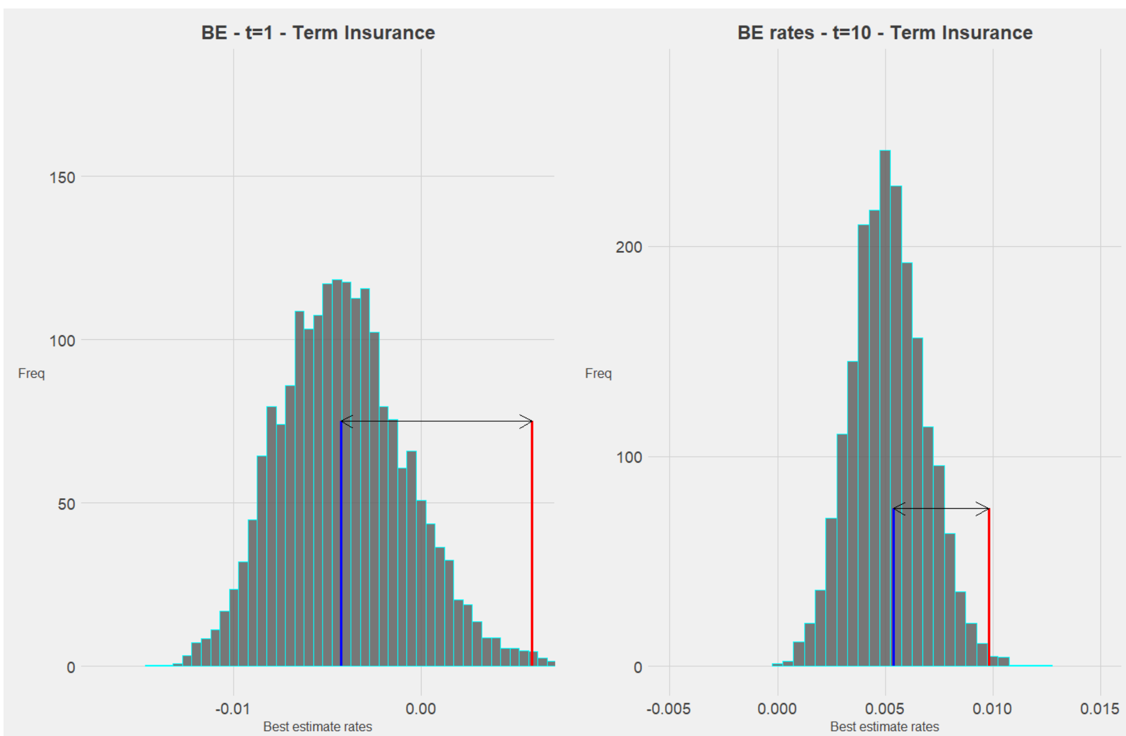
Term Insurance	**t=1**	**t=10**
Be rate	-0.4%	0.5%
Simulated mean	-0.4%	0.5%
Simulated standard deviation	0.34%	0.17%
Simulated skewness	0.33	0.19
Solvency Capital Requirement	15,382,318	6,618,003
SCR/Sums Insured	1.02%	0.44%

5.3.2. Trend risk - Term Insurance

As for the cohort of policyholders holding Pure Endowments, the application of the Poisson log-bilinear bootstrap model to the cohort of policies holding Term Insurances is particularly time consuming. Even in this case, however, the five thousand simulations are sufficient to ensure that the estimated average of the best estimate at the next instant coincides with the theoretical one: obviously these results are consistent and robust from a theoretical point of view and present a challenging theme in term application in relation to the estimate of the extreme quantiles of the distribution. As in the case of the Pure Endowment, it is observed that the volatility of the best estimate decreases as the time horizon increases because, also in this context, any new demographic bases influence an increasingly limited number of time points that provide for the hypothetical benefit to the policyholder.

A completely different element from what we have seen so far concerns the skewness analysis: as mentioned, the distribution of policyholder deaths follows a Poisson distribution with positive asymmetry. In this context, therefore, the extreme cases on the right tail predict a high number of deaths: for this reason, even the skewness of the trend risk distribution has a positive skewness that strongly influences the SCR values. In conclusion, Figure 19

Figure 19: Distribution of BE rates at time 1 and time 10 for a Term Insurance. Blue lines represent the expected values of the distributions, therefore $be_1^{Rf(1),q(0)}$ for the left figure and $be_{10}^{Rf(10),q(9)}$ for the right figure. The red lines indicate the 99.5% percentile of the distributions; then the black lines indicate the spreads between the expected values and the stressed values indicate the SCRs.



shows the representation of the difference between the best estimate in the strict sense and the percentile of order of 99.5% of the same, driver of the

SCR. It is noted that, although the SCR decreases as the valuation time increases, the amounts are multiples of the analogous value referred to the idiosyncratic risk. For a company that operates with policies whose benefits are linked to mortality, the keystone of the business model will not be so much the intrinsic volatility of the business, but rather the structural fluctuations in mortality rates.

6. Conclusions

This thesis aims to resume the stochastic model, based on the cohort approach and built up in a Local GAAP context, in order to extend it to the MCV framework introduced by Solvency II.
The process starts with the presentation of demographic profit in a Local GAAP context, where the expected profit appears over time as the well-known function of the difference between the first-order demographic bases (used in the pricing phase) and the second-order demographic bases considered realistic at the evaluation date. Similarly, volatility and skewness of the r.v. demographic profit depend on the structural characteristics of the contract (i.e. the events involving the payment of benefits, whether related to survival or death) and to the features of the policyholders cohort.
Section 3 presents the mathematical aspects necessary to shift from the Local GAAP framework to the MCV one: the most interesting theme is that using the definition of Best Estimate as it is under Solvency II legislation (see Section 1), we arrive at the definition of market-consistent demographic profit with all the desired properties. In particular, we obtain a random variable that incorporates demographic, risk-free rate, lapse and expense risks where the expected profit appears entirely when the company observes differences between the demographic basis. It occurs certainly upon underwriting, in case of prudential bases, and it could be observed also during the contract lifespan if the expectations are revised.
It is worth pointing out that we provide a bridge between the demographic profit of the Local GAAP context (which recognizes only the idiosyncratic component) and the MCV demographic profit which, by its nature, is com-

posed by an idiosyncratic component, by a trend component and, moreover, takes into account the effects of changes in the risk-free rate curve. The cohort approach was then applied to the components of the market-consistent demographic profit. By this way, expected value, standard deviation and skewness of the idiosyncratic component were calculated in closed forms. These results appear to be of particular relevance because they allow to provide formulations of the SCR similar to those proposed in the Life Underwriting Risk - idiosyncratic risk of QIS n.2.
Therefore, it is possible to quantify the capital requirement taking into account the following aspects:

(a) the expected profit estimated by the company: it generally derives from the use of prudential demographic (and financial) basis;

(b) the volatility of the idiosyncratic risk, which depends on the cohort characteristics and on underwritten contracts types;

(c) the skewness of the distribution: known that Solvency II uses the Value at Risk as risk measure, the knowledge of the skewness is fundamental to understand the shape of the distribution and, consequently, the extreme quantile.

As regards the trend component of the demographic profit, it is not possible to obtain the main characteristics of the distribution in closed form because they depend on the variation of the technical provisions as the company expectations on cohort mortality vary. In this context, therefore, the application of a stochastic model was proposed to quantify the volatility around

the expectations of death rates to arrive at the computation of a SCR consistent with Solvency II legislation.
On the one hand, the proposed model, after identifying the underlying risks, bases all its formulations on the cohort approach. From an operational point of view, it is certainly complex and nearly unreal to break up the undertaking's portfolio into hundreds of cohorts (sometimes thousands, depending on the size of the portfolio). On the other hand, we specify two fundamental aspects:

(a) the first aspect regards that the model is perfectly adaptable to the operational reality. It is possible to divide the undertaking portfolio in order to build clusters capable of identifying model points with a certain degree of approximation. They are defined as policies representative of the single clusters and the model is suitable for assessing their risk. Using this approach, it is therefore possible to assess the risk of the overall portfolio as a function of the model points risk;

(b) the second important aspect concerns the extraordinary computational capacity, from an IT point of view, of the undertakings: it allows the possibility of carrying out analysis at a single contract level. Both in the case of individual policy analysis and in the case of cohort evaluation, it is possible to achieve an aggregate result by modelling the dependencies. In this regard, it is necessary to use methodologies that contemplate non-linear dependencies and a typical solution concerns the use of copula functions.

It should also be noted that each undertaking is required to carry out the

Own Risk and Solvency Assessment (ORSA) in accordance with Solvency II legislation. The purpose of ORSA is to evaluate the risk profile of the company and, in case of Standard Formula use, to verify that it does not underestimate the capital requirement. In this context, the present model may be adopted as a tool both for identifying the risk sources and their quantification.

This use related to demographic risk quantification is of strategic relevance since it identifies the volatility of the pure demographic component, without keeping the results obscured by the financial part. The most suitable case concerns undertaking with significant Term Insurance portfolios. It is known that these policies involve the premiums payment and the reserves allocation, which amounts are very low. However, these policies certainly have a not negligible weight on the company economic result; hence, their assessment plays a strategic role on the undertaking business and in planning future portfolio value.

Therefore, if the purely demographic risk component is the main driver of the overall undertaking risk profile (in addition to the case of Term Insurance portfolio it is also necessary to mention the Endowments case), the model may be suitable as an easily adaptable (to the characteristics of the cohorts) proxy for the quantification of the capital requirement. In conclusion, it is emphasized that this use as a SCR proxy is consistent with the Undertaking Specific Parameters approach envisaged by Solvency II. Furthermore, and in conclusion, the possibility of using a model with closed formulas allows the possibility of knowing deeply the demographic risk and, at the same time, of optimizing any reinsurance operations to contain the volatility of the profit.

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8. Acknowledgements

I confess: this time too I followed the questionable idea of writing these lines the evening before delivery, but I am convinced that, after a hundred pages read and reread, that's okay.

This is one of those moments when you stop and think (just as when I first saw formula (47)) and I can only turn my first thought to my parents. You chose to give me everything and you did it, I promised you that I would make you proud of me and I swear I repeat it every morning as soon as I wake up (maybe too much!). Probably these lines will not be enough, just as this Thesis will not be enough and how nothing would be enough to reciprocate your efforts, but the concept of love that you have taught and transmitted to me is the same that I address to you in the hope of giving you what for me is the greatest gift of life, happiness.

Lowering my head and expressing my purest thanks, I turn to Camilla, the girl I love and who has supported me for more than 9 years. You are the life partner who makes me have fun, grow and hope. This thesis is our goal and I hope it can contribute to happiness in our future.

Mum and dad didn't give me brothers, but life gave me Nicolò and Federico, we're okay.

Everyone in my family knows who Professor Clemente, Gian Paolo, and Professor Savelli are. I cherish the hope that one day they too will know them, in the meantime I thank them for having transmitted to me passion and friendship. I am proud when I talk about moments spent with you and I am even more proud to see you not only as my professors, but as part of my world.

There are other people to whom I owe my thanks, people who have made my life something truly beautiful. They are all really smart people that I'm sure they find themselves in these lines, I love you. *deltaZ*

Last but not least, thanks Max.