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A multilinear approach to the theory of decision-making based on disaggregate and aggregate measures

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Dedicated to Rosa

Abstract

This research work studies the criteria of rational choices being made by the decision-maker under conditions of certainty or uncertainty and riskiness. With regard to these choices, a same logical framework is shown. Indeed, the incompleteness of the state of information and knowledge associated with a given decision-maker underlies it. The criteria of rational choices being made by the decision-maker under claimed conditions of certainty focus on non-negative and finitely additive masses, where each non-negative mass is associated with a possible alternative whose nature is objective, and utility. This is because actual situations, such as the total amount of money the decision-maker has to spend, are uncertain at the time of choice, so possible alternatives are handled as a consequence. The criteria of rational choices being made by the decision-maker under conditions of uncertainty and riskiness focus on probability and utility. This research work is accordingly connected with the international literature on the subject of probability viewed to be as a mass, where it is moved in whatever coherent way the decision-maker likes, and on the one of preference. We study choices subjected to budget constraint being made by the decision-maker who is modeled as being a consumer. She chooses bundles of two marginal goods, where each bundle operationally coincides with a bilinear measure of a metric nature. This measure is obtained from a summarized nonparametric distribution of mass. It is a joint distribution of mass. A bilinear measure is always decomposed into two linear measures obtained from two summarized nonparametric marginal distributions of mass. A nonparametric joint distribution of mass has always to reflect the knowledge hypothesis underlying each evaluation concerning all joint masses characterizing it. This hypothesis is made clear by the decision-maker from time to time. Our goal is to extend rational choice behaviors. Our goal is to study multiple choices. They are associated with multiple goods. Each multiple choice is based on different summarized joint distributions of mass. Each multiple choice is rational if and only if all these summaries of joint distributions of mass are coherent.

In Chapter 1, we define the notion of random good as well as the one of prevision bundle. We prove a theorem showing that there exists a full analogy between properties concerning average quantities of consumption of random goods and well-behaved preferences. We focus on axioms of revealed preference theory applied to average quantities of consumption of goods. Revealed preference theory gives empirical meaning to the neoclassical economic hypothesis according to which the best rational choice being made by the decision-maker inside of her budget set has to be the one maximizing her utility. We show that the best rational choice being made by the decision-maker inside of her budget set deals with average quantities of consumption of goods. After decomposing the object of decision-maker choice under conditions of uncertainty and riskiness inside of a subset of a two-dimensional linear space over $\mathbb{R}$, we define the decision-maker's demand functions that give the average consumption amounts associated with each random good under consideration. We show that it is possible to unify the empirical content of specific theories referred to coherent previsions of random goods in specific economic environments.

In Chapter 2, we prove a theorem showing how to transfer all the n states of the world of a contingent consumption plan on a one-dimensional straight line on which an origin, a unit of length, and an orientation are chosen. All the n states of the world of a contingent consumption plan are possible alternatives. They are not studied inside of E^n only, where E^n is an n-dimensional linear space over $\mathbb{R}$ having a Euclidean structure. This is because they

are also transferred on a one-dimensional straight line on which an origin, a unit of length, and an orientation are established. We do not consider an n-dimensional point referred to a random good, where a random good identifies a contingent consumption plan, but we study a finite set of n one-dimensional points. We do not deal with n masses associated with n possible states of the world of a contingent consumption plan yet. We focus on the two-good assumption, so X_1 and X_2 are two marginal random goods. Each of them has n possible consumption levels. The n possible values for each good under consideration are transferred on two one-dimensional straight lines on which an origin, a same unit of length, and an orientation are established. Such lines are the two axes of a two-dimensional Cartesian coordinate system. The space where the decision-maker chooses is her budget set. If we take her budget set into account then all masses associated with all possible consumption levels come into play. Her budget set is an uncountable subset of a two-dimensional linear space over $\mathbb{R}$. Her budget set contains points whose number is infinite. It is a right triangle belonging to the first quadrant of a two-dimensional Cartesian coordinate system. The point given by $(0,0)$ identifies its right angle, whereas the budget line whose slope is negative identifies its hypotenuse. Her budget set contains infinite coherent bilinear previsions associated with a joint random good denoted by $X_1 X_2$ and infinite coherent linear previsions associated with two marginal random goods denoted by X_1 and X_2. Two marginal random goods always identify a joint random good. Each bilinear prevision is denoted by $\mathbf{P}(X_1 X_2)$, where $\mathbf{P}(X_1 X_2)$ is always decomposed into two linear previsions denoted by $\mathbf{P}(X_1)$ and $\mathbf{P}(X_2)$ respectively. The decision-maker chooses one bilinear prevision denoted by $\mathbf{P}(X_1 X_2)$ among infinite coherent bilinear previsions. She chooses a bundle of two random goods operationally identified with $\mathbf{P}(X_1 X_2)$. Since $\mathbf{P}(X_1 X_2)$ belongs to a two-dimensional convex set, we express it in the form given by $(\mathbf{P}(X_1), \mathbf{P}(X_2))$. Accordingly, she also chooses $\mathbf{P}(X_1)$ and $\mathbf{P}(X_2)$ because $\mathbf{P}(X_1 X_2)$ is always decomposed into $\mathbf{P}(X_1)$ and $\mathbf{P}(X_2)$ respectively. We pass from $\mathbf{P}(X_1 X_2)$, where $\mathbf{P}(X_1 X_2)$ is found inside of a subset of a two-dimensional linear space over $\mathbb{R}$, to $\mathbf{P}(X_1)$ and $\mathbf{P}(X_2)$, where $\mathbf{P}(X_1)$ and $\mathbf{P}(X_2)$ are found on two different and mutually orthogonal one-dimensional straight lines. A nonparametric joint distribution of mass gives rise to a continuous subset of $\mathbb{R} \times \mathbb{R}$. This is because all coherent previsions of a joint random good are considered. They are obtained by taking all values between 0 and 1, end points included, into account for each mass associated with a possible value for two random goods which are jointly considered. The number of these values is infinite. Two nonparametric marginal distributions of mass give rise to two continuous subsets of $\mathbb{R}$, where each of them identifies a line segment belonging to one of the two axes of a two-dimensional Cartesian coordinate system. This is because all coherent previsions of marginal random goods are considered. All coherent previsions of two marginal random goods identify the two catheti of the right triangle under consideration. Such previsions are obtained by taking all values between 0 and 1, end points included, into account for each mass associated with a possible consumption level concerning a random good. The number of these values is infinite. We show that the continuous subset of $\mathbb{R} \times \mathbb{R}$ is a subset of the direct product of $\mathbb{R}$ and $\mathbb{R}$, where the latter is a two-dimensional linear space over $\mathbb{R}$.

In Chapter 3, we define multiple goods of order 2 whose possible values are not necessarily of a monetary nature. We show a numerical example referred to a multiple physical good of order 2. Given the two-good assumption, the objects of decision-maker choice are studied by using bilinear measures of a metric nature. Such measures are firstly decomposed into two linear measures inside of the budget set of the decision-maker. We secondly establish aggregate measures which are strictly connected with multiple goods. Aggregate measures are based on what the decision-maker chooses inside of her budget set. They are studied outside of her budget set. The Cartesian product of two finite sets of possible quantities of consumption associated with two goods which are separately considered can be released from the notion of ordered pair of possible quantities of consumption connected with each good under consideration. This implies that an extension of the notion of bundle of goods is caught. Accordingly, we define the notion of consumption matrix. For the purpose, disaggregate and aggregate measures of a metric nature are considered. We calculate the average consumption as well as the variability of it associated with a multiple good of order 2. The variability of consumption is expressed by using the Bravais-Pearson correlation coefficient. We use the Bravais-Pearson correlation coefficient because the variability of a nonparametric joint distribution of mass is expressed by its numerator. This variability always depends on how the decision-maker estimates all the joint masses under consideration. She estimates them according to her variable state of information and knowledge. Accordingly, mean quadratic differences connected with multiple goods of order 2 are shown. The Bravais-Pearson correlation coefficient associated with each bundle of two goods being chosen by the decision-maker inside of her budget set is used in order to check the weak axiom of revealed preference. We refer ourselves to this axiom because it is the basic axiom of the theory of decision-making whenever the decision-maker is modeled as being a consumer whose choices are subjected to budget constraint. We realize that a marginal random good can always be studied by using a particular joint distribution of mass. Consumption data are dealt with by using metric measures. Disaggregate measures are obtained by using a linear and quadratic metric. Aggregate measures are obtained by using a multilinear and quadratic metric.

In Chapter 4, we define a multiple random good of order 2 denoted by X_{12} whose possible values are of a monetary nature. A two-risky asset portfolio is a multiple random good of order 2. It is firstly possible to establish its expected return by using a linear metric. Given ${}_1X$ and ${}_2X$, where ${}_1X$ and ${}_2X$ are the components of $X_{12} = \{{}_1X, {}_2X\}$, whenever we use a linear metric in order to establish the expected return on a two-risky asset portfolio, we focus on the components of X_{12} only. We secondly establish the expected return on X_{12} denoted by $\mathbf{P}(X_{12})$ by using a multilinear metric. Whenever we use a multilinear metric in order to establish the expected return on a two-risky asset portfolio, we focus on X_{12}. It is viewed to be as a stand-alone good. Whenever we use a multilinear metric, we are not interested in studying separately the components of X_{12} denoted by ${}_1X$ and ${}_2X$. If the decision-maker is risk neutral then $\mathbf{P}(X_{12})$ is a subjective price coinciding with the certainty which is judged to be equivalent to X_{12} by her. An extension of the notion of mathematical expectation of X_{12} denoted by $\mathbf{P}(X_{12})$ is carried out by using the notion of α-norm of an antisymmetric tensor of order 2. We prove a theorem about this. An extension of the notion of variance of X_{12} denoted by $\mathrm{Var}(X_{12})$ is shown by using the notion of α-norm of an antisymmetric tensor of order 2 based on changes of origin. We prove a theorem about this. An extension of the notion of expected utility connected with X_{12} is considered. An extension of Jensen's inequality is shown as well. Whenever the decision-maker maximizes the expected utility of X_{12}, she maximizes the utility of average quantities of consumption. We focus on how the decision-maker maximizes the expected utility connected with multiple random goods of order 2 being chosen by her under conditions of uncertainty and riskiness. What she actually chooses inside of her budget set underlies this.

In Chapter 5, we study m risky assets identifying a multiple random good of order m whose possible values are of a monetary nature. Any two risky assets of m risky assets are always studied inside of the budget set of the decision-maker. Two or more than two risky assets are also studied outside of her budget set. Whenever changes of origin are considered, we go away from her budget set. Given m risky assets subjected to m changes of origin, we study an m-dimensional linear manifold embedded in E^n. It is spanned by m basic risky assets, where each of them is subjected to a change of origin. Each of them has n possible values. Each linear combination of m basic risky assets identifies an n-dimensional vector belonging to an m-dimensional linear manifold embedded in E^n, where this n-dimensional vector is a risky asset. This n-dimensional vector identifies a nonparametric marginal distribution of mass. The number of all linear combinations of m basic risky assets is infinite. All risky assets belonging to an m-dimensional linear manifold embedded in E^n are dealt with. We are also interested in knowing the starting possible values for each risky asset under consideration as well as all marginal masses associated with them. We show that all risky assets contained in an m-dimensional linear manifold embedded in E^n are intrinsically related. In particular, we realize that any two risky assets of them are α-orthogonal, so their covariance is equal to 0. We define the notion of α-metric tensor. It is used to study how all risky assets contained in an m-dimensional linear manifold embedded in E^n are intrinsically related. On the other hand, eigenvalues, eigenvectors, eigenequation, and eigenspaces derive from the notion of α-metric tensor. We show that all principal components coincide with basic risky assets. Constants of riskiness explain the variance of all risky assets belonging to an m-dimensional linear manifold embedded in E^n. We show that all risky assets belonging to a specific m-dimensional linear manifold embedded in E^n are proportional. Non-classical inferential results are obtained. We realize that the price of risk is based on multilinear indices. This price measures how risk and return can be traded off in making portfolio choices.

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Contents

Introduction

This research work is mostly placed in the field of studies connected with revealed preference theory. The international literature on this subject is abundant (see also [43]). Revealed preference theory is important because information associated with the choice being made by the decision-maker can be used to derive information about her preferences. To discover the preferences of those who decide it is necessary to observe their actual behavior. In general, information about preferences can be essential in making policy decisions. Most economic policy involves trading off one good for another. For example, if we put a tax on washing machines and subsidize televisions then we will probably end up having more televisions and fewer washing machines. In order to evaluate the desirability of this economic policy, it is important to have some idea of what the decision-maker's preferences between televisions and washing machines look like. It is possible to use revealed preference and related techniques in order to derive such information (see also [42]). Revealed preference theory is meant as a branch of the theory of decision-making. The decision-maker is modeled as being a consumer whose choices are subjected to budget constraint (see also [71]). She can also choose prevision bundles consisting of random goods such as risky assets. We realize that the logical framework used to study bound choices being made by the decision-maker under claimed conditions of certainty is the same as the one used to study bound choices being made by her under conditions of uncertainty and riskiness.

We use a quadratic metric, so it is not practically possible to consider more than two goods at a time from a metric point of view (see also [64]). The two-good assumption is not a restriction from a mathematical point of view. Such an assumption is not a restriction from a conceptual point of view either. This assumption is more general than one might think at first. This is because we can interpret one of the two goods under consideration as representing everything else the decision-maker might want to choose.

The quantity of consumption actually demanded for each good into account by the decision-maker under conditions of certainty or uncertainty and riskiness is an average quantity. This quantity always depends on objective and subjective elements (see also [25]). Such a quantity does not depend on objective elements only. Since what she actually chooses is an average consumption, she uses masses expressing her expectations and subjective sensations in order to estimate average quantities of consumption. Accordingly, an axiomatic approach to the theory of decision-making is not alone sufficient to explain choices being made by the decision-maker inside of her budget set. This approach does not sufficiently explain all intrinsic elements of bound choices. The insufficiency of explanation is more serious because the rational criterion of decision-making depends on two subjective notions. Since the notion of average quantity connected with bound choices uses masses that are subjectively chosen, it is possible to study it, together with utility, inside of linear spaces and subspaces provided with a Euclidean metric. Bound choices can be studied inside of linear structures over $\mathbb{R}$ having different dimensions.

We put forward a unified approach to an integrated formulation of decision theory in its two subjective components. The inability to imagine something extraneous to our mental habits could sometimes appear to us as the criterion of a particular truth. It is a priori truth. This truth is the stable foundation of a predicate science. It is a science that takes place on a basis that is guaranteed once and for all. Such a basis is almost always of an axiomatic nature. Nevertheless, economics is not a predicate science, but it is a dialectical science. Accordingly, it is a science that, at every moment of its historical development, appears to satisfy a set of intellectual and applicative needs. With regard to its evolution, economics viewed to be as a dialectical science can even contain the elements of future crises. The same is true with regard to mathematics and statistics on which economics is based.

Since there exist different extraneous factors that play a part in decision-making, if we generically identify decisions being made by the decision-maker and her preferences then we ignore them. Accordingly, in general it is not appropriate to identify decisions and preferences. Indeed, nobody accepts all the opportunities that she judges favorable as well as perhaps we all sometimes enter into situations that we judge unfavorable. To reduce the influence of these extraneous factors, it is possible to observe the phenomena under consideration in such a way that they are isolated in their most simple forms (see also [46]). Accordingly, given a random good denoted by X whose possible values belonging to $I(X) = \{x^1, x^2, \ldots, x^n\}$ are explicitly of a monetary nature, where it turns out to be $x^1 < x^2 < \ldots < x^n$, we have to force the decision-maker to make her rational choices releasing her from passivity and preserving her from fancy (see also [70]). This is possible whenever she has to bet on single alternatives or events, where all these single alternatives constitute a finite partition of mutually exclusive events. It follows that to bet on them means to bet on the certain event denoted by Ω (see also [20]). Since the certain gain denoted by $p = \mathbf{P}(\Omega)$ and judged by the decision-maker to be equivalent to a unit gain conditional on the occurrence of Ω is coherently equal to 1, the sum of all masses associated with all the single alternatives into account has coherently to be equal to 1. Accordingly, a nonparametric distribution of mass associated with X arises. Strictly speaking, we especially take $p_1 S_1$ equivalent to S_1 conditional on E_1, where E_1 is a single event coinciding with the possible consumption level for X given by x^1. We arbitrarily choose a unit amount of money, so it turns out to be $S_1 = 1$. We especially take $p_2 S_2$ equivalent to S_2 conditional on E_2, where E_2 is a single event coinciding with the possible consumption level for X given by x^2. We arbitrarily choose a unit amount of money, so it turns out to be $S_2 = 1$. Finally, we especially take $p_n S_n$ equivalent to S_n conditional on E_n, where E_n is a single event coinciding with the possible consumption level for X given by x^n. We arbitrarily choose a unit amount of money, so it turns out to be $S_n = 1$. Thus, it is possible to write

$$p_1 + p_2 + \ldots + p_n = 1.$$

We say that the decision-maker actually chooses $\mathbf{P}(X)$, where

$$\mathbf{P}(X) = x^1 p_1 + x^2 p_2 + \ldots + x^n p_n$$

is the prevision of X. It is the mathematical expectation of X or its mean value [1]. If $x^1 = 0$ and $p_1 = 0$, where it turns out to be $p_2 + \ldots + p_n = 1$ as well as $p_1 + p_2 + \ldots + p_n = 1$, then

[1] She chooses two mean values connected with two marginal goods studied inside of her budget set. Her utility associated with these two goods has to be the highest.

nothing changes. This also holds if X is a random good whose possible values belonging to $I(X) = \{x^1, x^2, \ldots, x^n\}$ are not of a monetary nature. This also holds if the decision-maker chooses an ordinary good under claimed conditions of certainty. She always chooses a weighted average inside of her budget set. In particular, given the two-good assumption, she chooses two weighted averages inside of her budget set, where every weighted average is associated with one of the two ordinary goods under consideration.

There also exist two extreme situations connected with the calmness and accuracy and thus the reliability of the evaluations of probability connected with random goods being made by the decision-maker. Such evaluations underlie every decision being made by her inside of her budget set. On the one hand, the direct interest in decisions being made by the decision-maker can both encourage and block the reliability in terms of coherence of her evaluations of probability. On the other, the lack of a direct interest in decisions being made by the decision-maker can do the same effect. It is also possible to consider an intermediate case which is the one of a decision-maker who is consulted about a decision in which others are actually interested. This case could be of interest because it leads to responsibility in the judgment without influencing the calmness of the individual who decides. Moreover, the accuracy of the evaluations of probability in terms of coherence could also be connected with the self-respect of the decision-maker in some competitive situation characterized by prizes which are materially meaningless, but they are intrinsically associated with the significance of the competition. Probability evaluations being made by the decision-maker choosing random goods are well-founded, reasonable, serious if and only if they are coherent (see also [11]). Although they are of a subjective nature, they are not improvised. They are always carried out with all the attention of those who consider them as objective. If necessary, they are carried out with a greater sense of responsibility deriving from not having illusions regarding their false objective nature. Probability evaluations can be based on symmetric probabilities implying a judgment of equal probability being made by the decision-maker who deals with all the single alternatives into account. They can be based on frequencies. Nevertheless, they always exist according to her subjective judgment. If the decision-maker chooses ordinary goods inside of her budget set then the evaluations of mass being made by her underlie every choice. This is because she chooses weighted averages. The evaluations of mass underlying average quantities of consumption associated with ordinary goods being chosen by the decision-maker under claimed conditions of certainty have to be coherent as well (see also [12]).

Given two random goods studied inside of the budget set of the decision-maker, one of strengths of this research work is to establish which is the price that she is willing to pay to purchase the right to participate in a bet that places her in the uncertain situation denoted by X_{12}, where X_{12} is a multiple random good of order 2 whose possible values are explicitly of a monetary nature. Accordingly, we handle $X_{12} = \{{}_1X, {}_2X\}$. We are able to determine its mathematical expectation denoted by $\mathbf{P}(X_{12})$. We have to consider the cardinal utility function associated with X_{12}. It depends on her subjective attitude towards risk. We have to go away from the budget set of the decision-maker after observing what she chooses inside of it with respect to the two components of X_{12}. The two components of X_{12} are the two marginal random goods studied inside of her budget set. If we go away from the budget set of the decision-maker then the criteria of rational decision-making consist of the choice of any coherent evaluation of the probabilities being made by her and any utility function having the necessary mathematical properties, where such properties have to comply with her subjective attitude towards risk. She also fixes as her goal the maximization of the prevision or mathematical expectation of her cardinal utility. Accordingly, we are able to extend to multiple random goods the notion of moral expectation which has been put forward by Daniel Bernoulli and developed by John von Neumann and Oskar Morgenstern at a later time. On the other hand, it is also possible to behave rationally with respect to decisions being made by the decision-maker and her preferences without knowing anything about probability and utility. However, in this case the decision-maker has to behave as if she is acting in the above manner. In other words, she must behave as if obeying a coherent evaluation of probability and a scale of utility underlying her way of thinking and acting, although without realizing this thing. It could also be possible to individuate the two subjective components on which the correct criterion of decision-making depends. For the purpose, one should succeed in exploring in an appropriate way how the decision-maker thinks and acts. Anyway, only the completely agnostic point of view is unacceptable. It is not correct to think that every subjective attitude is a fact on which it is not possible to express an objective judgment based on rational conditions made clear by economists, mathematicians, and statisticians. It is true that everyone has her subjective attitude towards risk caught by a specific utility function. It is true that everyone is able to evaluate all probabilities under consideration according to one's own subjective judgment. It is false that everyone is able to choose which rational rules have to be complied with. Such rules cannot arbitrarily be chosen by anyone. They are always the same. They are logical rules. In this research work, we extend them.

We do not consider an infinite number of possible alternatives whose unique and well-determined summary, if there exists, belongs to a set containing a finite number of elements. We study a finite number of possible outcomes whose nature is objective. All their coherent summaries belong to a set containing an infinite number of elements. It is the budget set of the decision-maker. It follows that in order to study choices inside of her budget set it is possible to compare a concrete distribution of mass with a model which is not a continuous function such as the density function of a continuous random variable, but it is itself a distribution of mass. We do not embed what is actually observed inside of a framework characterized by a continuous function such as the density function of a continuous random variable. On the other hand, what is actually observed is always characterized by a finite number of elements. Only a finite number of possible alternatives it is natural to admit into the formulation of any choice problem which is studied inside of the budget set of the decision-maker and into the arguments required for its solution. These arguments are basically connected with Bayes' rule (see also [15]). It is not necessary to deal with a σ-algebra defined on a nonempty set of outcomes. It is conversely necessary that each measure obtained by summarizing a nonparametric distribution of mass complies with a logical criterion.

A summary of a nonparametric joint distribution of mass coincides with each point of the budget set of the decision-maker (see also [9]). One of strengths of this research work is to realize what it implies that even this summary is subjected to a choice.

Prevision is not something which in itself can be known with certainty. If prevision can be known with certainty then the decision-maker would be able to guess which alternative will occur among the possible ones. In this case, prevision would be a prediction or prophecy and the decision-maker developing it would be a magician. Prevision is not either something which in itself can be not known. Prevision takes place in that it serves to express, for each decision-maker inside of her budget set, her choice in her given state of ignorance. The same is true by considering weighted averages of possible quantities of consumption of ordinary goods being chosen by the decision-maker under claimed conditions of certainty. Each point of the budget set of the decision-maker is a weighted average of possible quantities of consumption of two ordinary goods which are jointly considered. Such a weighted average is always decomposed into two weighted averages, where each of them is of a linear nature (see also [28]).

If we extend the preeminent role of linearity in the theory of decision-making then we realize that average quantities obtained by using non-negative masses subjectively chosen and utility are two instruments which conform themselves strictly to the exigencies of the field where they turn out to be applied. Whenever the decision-maker chooses by summarizing her information and knowledge, she will exercise great attention in not going far beyond the consideration of alternatives immediately at hand and directly interesting (see also [39]). It is not appropriate to impose axioms not required for essential reasons. It is not absolutely correct to impose axioms which could even be in conflict with them. However, we do not put in doubt anything. We do not put in doubt axioms that are widely and authoritatively accepted. We simply limit ourselves to what serves the purpose and is of interest according to a substantial approach having an operational nature.

Chapter 1

Optimal choices based on bilinear and disaggregate measures

1.1 Introduction

1.1.1 The real nature of the objects of decision-maker choice under claimed conditions of certainty

The decision-maker is modeled as being a consumer. Bound choices being made by her under conditions of certainty are intrinsically characterized by the incompleteness of her state of information and knowledge. The conditions of certainty are not real, but they are ideal. They are a simplification. Indeed, in almost all circumstances, and at all times, we all find ourselves in a state of ignorance. Hence, average quantities of consumption are treated. They are treated by the logic of prevision. Given the two-good assumption, the objects of decision-maker choice are of a bilinear nature. Given two ordinary goods having downward-sloping demand curves, $({}_1y, {}_2y)$ represents what is actually demanded for each of them by her inside of her budget set. We establish the following

Definition 1. *The quantity of consumption associated with two ordinary goods actually demanded by the decision-maker under claimed conditions of certainty is an average quantity. We write*

$$ {}_1y = y_1^1\, p_1^1 + \ldots + y_1^n\, p_1^n \tag{1.1}$$

and

$$ {}_2y = y_2^1\, p_2^1 + \ldots + y_2^n\, p_2^n, \tag{1.2}$$

where $\{p_1^i\}$ *and* $\{p_2^j\}$ *are two sets of n masses, with* $0 \le p_i^j \le 1$, $j = 1, \ldots, n$, $i = 1, 2$, *whose sum is always equal to* 1 *with regard to each of them. The possible quantities of consumption for good* 1 *are expressed by* $\{y_1^1, \ldots, y_1^n\}$, *whereas the possible quantities of consumption for good* 2 *are given by* $\{y_2^1, \ldots, y_2^n\}$.

We are found inside of the budget set of the decision-maker. It is a right triangle belonging to the first quadrant of a two-dimensional Cartesian coordinate system. The point given by $(0,0)$ identifies its right angle. The budget line whose slope is negative coincides with its hypotenuse. We also deal with the weighted average of n^2 possible quantities of consumption for good 1 and good 2 that are jointly considered. They derive from $\{y_1^1, \ldots, y_1^n\} \times \{y_2^1, \ldots, y_2^n\}$, where n^2 non-negative masses are associated with each pair of this Cartesian product. Given $({}_1y, {}_2y)$, the weighted average of n^2 possible quantities of consumption for good 1 and good 2 is a synthesized element of the Fréchet class. We establish the following

Definition 2. *The set of all weighted averages of n^2 possible quantities of consumption for good* 1 *and good* 2 *that are jointly considered, with the same given marginal weighted averages of n possible quantities of consumption for good* 1 *and n possible quantities of consumption for good* 2*, constitutes the Fréchet class.*

It is clear that ${}_1y$ is the given marginal weighted average of n possible quantities of consumption for good 1, whereas ${}_2y$ is the given marginal weighted average of n possible quantities of consumption for good 2 [1]. With regard to the two goods that are separately considered, it is evident that every weighted average of n possible quantities of consumption for each of them is always found between the lowest possible quantity of consumption and the highest possible one for each good under consideration. The same is true with regard to n^2 possible quantities of consumption for the two goods into account that are jointly considered, where each pair of n^2 pairs is handled by taking the arithmetic product of its corresponding elements into account. Notice that $({}_1y, {}_2y)$ is a bilinear and disaggregate measure belonging to a subset of $\mathbb{R}\times\mathbb{R}$, where $\mathbb{R}\times\mathbb{R}$ is the direct product of $\mathbb{R}$ and $\mathbb{R}$ (see Chapter 2). It is decomposed into two linear measures, where each of them belongs to a subset of $\mathbb{R}$. We observe reductions of dimension by passing from n^2 to 2 (being equal to 2 the dimension of the plane) as well as from n to 1 (being equal to 1 the dimension of the straight line). All coherent weighted averages of n^2 possible quantities of consumption for the two goods into account that are jointly considered identify a two-dimensional convex set coinciding with a subset of $\mathbb{R}\times\mathbb{R}$. They are obtained by taking all values lying between 0 and 1, end points included, into account with regard to each mass of n^2 masses. The number of these values is infinite. Moreover, all these averages also identify two one-dimensional convex sets coinciding with two line segments belonging to the two mutually orthogonal axes of a two-dimensional Cartesian coordinate system. They are obtained by taking all values lying between 0 and 1, end points included, into account with regard to each mass of n masses. The number of these values is infinite. Strictly speaking, we refer ourselves to two half-lines, where each of them extends indefinitely in a positive direction from zero before being restricted. Boundary points that are found on each restricted half-line identify degenerate averages. The budget line identifying the budget set of the decision-maker is nothing but a hyperplane embedded in a two-dimensional Cartesian coordinate system. Its slope depends on the known and objective prices of the two goods under consideration. The budget constraint of the decision-maker is written in the form

$$b_1\,({}_1y) + b_2\,({}_2y) \leq b,$$

[1] Given $({}_1y, {}_2y)$, we firstly handle a closed neighborhood of ${}_1y$ denoted by $[{}_1y-\varepsilon\,;\,{}_1y+\varepsilon']$ on the horizontal axis as well as a closed neighborhood of ${}_2y$ denoted by $[{}_2y-\varepsilon\,;\,{}_2y+\varepsilon']$ on the vertical one, where both ε and ε' are two small positive quantities. Since the state of information and knowledge associated with a given decision-maker is assumed to be incomplete at the time of choice, n possible quantities of consumption for good 1 belong to $[{}_1y-\varepsilon\,;\,{}_1y+\varepsilon']$ and n possible quantities of consumption for good 2 belong to $[{}_2y-\varepsilon\,;\,{}_2y+\varepsilon']$. It is not necessary that one of n possible alternatives coincides with ${}_1y$. The same is true with regard to ${}_2y$. It follows that n^2 possible quantities of consumption for good 1 and good 2 jointly considered are handled. After determining $\{y_1^1,\ldots,y_1^n\}$, $\{y_2^1,\ldots,y_2^n\}$, and $\{y_1^1,\ldots,y_1^n\}\times\{y_2^1,\ldots,y_2^n\}$, two nonparametric marginal distributions of mass together with a nonparametric joint distribution of mass are estimated in such a way that $({}_1y, {}_2y)$ is their chosen synthesis.

where the prices of good 1 and good 2 are (b_1, b_2), whereas the amount of money she has to spend is given by b. Notice that b is assumed to be an uncertain or possible element at the time of choice [2] (see Chapter 3). Given (b_1, b_2) and $({}_1y, {}_2y)$, these two pairs are consumption data. They are observed inside of the budget set of the decision-maker. The decision-maker always chooses a point belonging to her starting budget set. If good 1 becomes more expensive then the budget line changes its negative slope. It becomes steeper. This means that the budget set of the decision-maker changes. It becomes a right triangle which is different from the starting one. If her starting budget set changes then she chooses a point belonging to her changed budget set. Each point representing a choice being made by the decision-maker is associated with a given budget set. Two points representing two choices being made by the decision-maker are associated with two different budget sets. We could also understand one of the two goods into account as representing everything else she might want to choose.

We note the following

Remark 1. The prices of good 1 and good 2 are the two real coefficients identifying the slope of a hyperplane embedded in a two-dimensional Cartesian coordinate system. Since $({}_1y, {}_2y)$ is a point belonging to a two-dimensional convex set, the budget line does not separate $({}_1y, {}_2y)$ from the set of possible points denoted by $\{y_1^1, \ldots, y_1^n\} \times \{y_2^1, \ldots, y_2^n\}$. It does not separate ${}_1y$ from $\{y_1^1, \ldots, y_1^n\}$, nor ${}_2y$ from $\{y_2^1, \ldots, y_2^n\}$. By definition, the budget line is a hyperplane. Since there exists a hyperplane, possible alternatives rightly come into play. It is appropriate to study possible consumption levels inside of the budget set of the decision-maker for this reason. She summarizes them by means of masses subjectively chosen. □

1.1.2 Logically independent random goods

All possible quantitative states of the world of a contingent consumption plan coincide with an n-dimensional consumption vector of E^n denoted by

$$\mathbf{x} = (x_1, x_2, \ldots, x_n), \tag{1.3}$$

where E^n is an n-dimensional linear space over $\mathbb{R}$ provided with a quadratic metric. E^n has a Euclidean structure. A located vector at the origin of E^n is completely established by its end point. An ordered n-tuple of real numbers can be either a point of $\mathscr{E}^n$ (affine space) or a vector of E^n, where $\mathscr{E}^n$ and E^n are isomorphic. Given an orthonormal basis of E^n, (1.3) identifies the components of $\mathbf{x}$ with respect to it. They are uniquely established with respect to it. They are the possible values for X. They are possible consumption levels. The true value of X is unique, but it is unknown to the decision-maker at the time of choice. She is in doubt between n possible values for it. The set of all possible values for X at a given time is denoted by

$$I(X) = \{x_1, x_2, \ldots, x_n\}, \tag{1.4}$$

[2] Bound choice being made by the decision-maker is always relative to her state of information and knowledge. It is assumed to be incomplete. If there is no ignorance any more because further information is later acquired then it is possible to observe a parallel shift outward of the budget line. Its slope is accordingly unchanged. On the other hand, it is also possible to observe its parallel shift inward. Moreover, it is possible that the budget line does not shift.

where it turns out to be $x_1 < x_2 < \ldots < x_n$. We have $\inf I(X) = x_1$ and $\sup I(X) = x_n$. We deal with a finite partition of mutually exclusive states of the world of a contingent consumption plan. Each element of (1.4) is an elementary event meant as a proposition susceptible of being true or false at the right time. A probability is associated with each element of (1.4). With regard to each probability of n probabilities connected with an element belonging to $I(X)$, it is possible to take all values lying between 0 and 1, end points included, into account. The number of these values is infinite. It is possible to assign to each element belonging to $I(X)$ a probability lying between 0 and 1, end points included, because only absolutely inadmissible probabilistic evaluations must be excluded. We obtain all coherent previsions of X denoted by $\mathbf{P}(X)$ in this way. Each possible value for X belonging to $I(X)$ can coincide with a coherent prevision of X.

We establish the following

Definition 3. *X is a random good such that the possible quantities of consumption associated with it that can be actually chosen under conditions of uncertainty and riskiness by the decision-maker inside of her budget set coincide with all coherent previsions of X denoted by $\mathbf{P}(X)$. They are average quantities of consumption. X has an elastic demand. If the price of X (coinciding with the one of $\mathbf{P}(X)$) changes then X has an elastic demand with respect to the new price.*

If an n-dimensional consumption vector of E^n identifies the states of the world of a contingent consumption plan then two n-dimensional consumption vectors of E^n that are separately considered identify the states of the world of two contingent consumption plans that are separately considered. It follows that a joint contingent consumption plan is established by two n-dimensional consumption vectors of E^n that are separately considered besides an affine tensor of order 2 belonging to $E^n \otimes E^n$. It is uniquely established whenever a joint contingent consumption plan is studied (see Chapter 4). The number of the components of this tensor is overall equal to n^2. Indeed, we write $\dim(E^n \otimes E^n) = n^2$. Each component of it is a mass of a joint distribution of mass. Each component of this tensor is associated with an ordered pair of components of two n-dimensional consumption vectors of E^n separately considered. Since the arithmetic product of two quantitative states of the world of two single (marginal) contingent consumption plans is dealt with, it is not necessary that n^2 states of the world of a joint contingent consumption plan are all different. It is conversely necessary that the sum of n^2 joint masses is equal to 1. A joint contingent consumption plan is always dealt with whenever we are found inside of the budget set of the decision-maker. Given X_1 and X_2, let $I(X_1) = \{x_{11}, \ldots, x_{1n}\}$ and $I(X_2) = \{x_{21}, \ldots, x_{2n}\}$ be the sets of their possible values. Since X_1 and X_2 are jointly considered inside of the budget set of the decision-maker, they are logically independent if and only if there are n^2 possible values for them. They belong to the set denoted by $I(X_1) \times I(X_2)$.

1.1.3 Probability viewed to be as a mass

The probability of a state of the world of a contingent consumption plan is viewed to be as a mass (see also [7]). It is always a non-negative and additive function whose value on the whole space of states of the world of a single or joint contingent consumption plan is equal to 1. Axiomatic probability theory is satisfied (see also [83]). Nevertheless, the notion of probability associated with a state of the world of a contingent consumption plan is not undefined. It is the degree of belief in the occurrence of it attributed by a given decision-maker at a given instant and with a given set of information and knowledge. The unit mass of probability can freely be distributed without altering its geometric support and the measure that appears more natural to consider inside of a subset of a linear space over $\mathbb{R}$. Symmetric probabilities as well as evaluations based on statistical frequencies can be considered. Nevertheless, they never exist outside of the decision-maker's judgment whose nature is always subjective.

1.1.4 Reductions of dimension connected with random goods which are jointly considered

Let X_1 and X_2 be two random goods (see also [84]). Given an orthonormal basis of E^n denoted by $\mathscr{B}_n^{\perp} = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$, we consider two linearly independent vectors of E^n denoted by $\mathbf{x}_1$ and $\mathbf{x}_2$. Since they are uniquely obtained by means of two linear combinations of n basis vectors, we write

$$\mathbf{x}_1 = x_{11}\,\mathbf{e}_1 + \ldots + x_{1n}\,\mathbf{e}_n \tag{1.5}$$

and

$$\mathbf{x}_2 = x_{21}\,\mathbf{e}_1 + \ldots + x_{2n}\,\mathbf{e}_n. \tag{1.6}$$

All real coefficients of $\mathbf{x}_1$ coincide with $I(X_1) = \{x_{11}, \ldots, x_{1n}\}$, whereas all real coefficients of $\mathbf{x}_2$ coincide with $I(X_2) = \{x_{21}, \ldots, x_{2n}\}$. These coefficients are respectively the components of $\mathbf{x}_1$ and $\mathbf{x}_2$ with respect to $\mathscr{B}_n^{\perp}$.

Two marginal random goods, where each of them is characterized by n possible and distinct values, always give rise to a joint random good denoted by $X_1 X_2$ inside of the budget set of the decision-maker. We establish the following

Definition 4. *The possible values for two logically independent random goods which are jointly considered inside of the budget set of the decision-maker identify all possible quantitative states of the world for a joint contingent consumption plan. A joint random good denoted by $X_1 X_2$ is a function written in the form*

$$X_1 X_2 \colon I(X_1) \times I(X_2) \to \mathbb{R}.$$

Each element of its codomain coincides with the arithmetic product of two values belonging to two sets denoted by $I(X_1)$ and $I(X_2)$ respectively.

We note the following

Remark 2. All the n possible values for random good 1 identified with $\mathbf{x}_1 \in E^n$ can be transferred on a one-dimensional straight line on which an origin, a unit of length, and an orientation are chosen. We observe a reduction of dimension in this way. We pass from n to 1. All the n possible values for random good 2 identified with $\mathbf{x}_2 \in E^n$ can be transferred on another one-dimensional straight line on which an origin, a same unit of length, and an orientation are established. We observe a reduction of dimension in this way. We pass from n to 1. Since $\mathbf{x}_1$ and $\mathbf{x}_2$ are assumed to be linearly independent, it is possible to admit that these two one-dimensional straight lines are mutually orthogonal. The budget set of the decision-maker deals with two mutually orthogonal straight lines. All the n^2 possible values for two random goods which are jointly considered identify a finite subset of a two-dimensional Cartesian coordinate system. We observe n^2 two-dimensional points inside of it. Each point of them has two Cartesian coordinates telling us that we passed from n to 1 with respect to every coordinate associated with a given axis. Accordingly, we pass from n^2 to 2. □

This remark is mathematically made clear by Theorem 2 (see Chapter 2).

1.2 Single random goods and bundles of ordinary goods: their relationships with the notion of prevision and utility

Let X_1 and X_2 be two random goods which are separately considered outside of the budget set of the decision-maker. We denote by $I(X_1) = \{x_{11}, \ldots, x_{1n}\}$ the set of the possible values for X_1, where it turns out to be $x_{11} < \ldots < x_{1n}$. We denote by $I(X_2) = \{x_{21}, \ldots, x_{2n}\}$ the set of the possible values for X_2, where it turns out to be $x_{21} < \ldots < x_{2n}$. In this section, X_1 and X_2 are two random goods such that their possible values are explicitly of a monetary nature. Thus, x_{11} is the return on X_1 if E_{11} occurs with probability denoted by p_{11}, …, x_{1n} is the return on X_1 if E_{1n} occurs with probability denoted by p_{1n}. It is evident that x_{11} is the wealth that X_1 yields and that can be spent by the decision-maker if E_{11} occurs with probability denoted by p_{11}, …, x_{1n} is the wealth that X_1 yields and that can be spent by the decision-maker if E_{1n} occurs with probability denoted by p_{1n}. The same is true with respect to the possible values for X_2 identifying $I(X_2)$ (see also [45]). We are faced with two different partitions of incompatible and exhaustive states of the world of two single contingent consumption plans, where each of them is characterized by n states of the world generically denoted by E_{ij}, with $i = 1, 2$, $j = 1, \ldots, n$. It turns out to be $|E_{ij}| = 1$ or $|E_{ij}| = 0$ whenever uncertainty ceases.

Uncertainty about a state of the world of a contingent consumption plan depends on a lack of information and knowledge by the decision-maker under consideration (see also [13]). Uncertainty about a state of the world of a contingent consumption plan ceases, for any given decision-maker, only when she receives certain information about it (see also [62]).

It is possible that the decision-maker is only interested in the mathematical expectation of her monetary wealth. Accordingly, she prefers X_1 to X_2 if and only if it turns out to be $\mathbf{P}(X_1) > \mathbf{P}(X_2)$, where $\mathbf{P}$ denotes the prevision or mathematical expectation of a random good. $\mathbf{P}$ is a summary index. In this section, it also represents the price of a random good whose possible values are explicitly of a monetary nature. This price is subjectively established by the decision-maker. We accordingly write

$$\mathbf{P}(X_1) = x_{11}\, p_{11} + \ldots + x_{1n}\, p_{1n} \tag{1.7}$$

as well as

$$\mathbf{P}(X_2) = x_{21}\, p_{21} + \ldots + x_{2n}\, p_{2n}, \tag{1.8}$$

with

$$p_{11} + \ldots + p_{1n} = 1 \tag{1.9}$$

and

$$p_{21} + \ldots + p_{2n} = 1, \tag{1.10}$$

where we have $0 \leq p_{1i} \leq 1$, $i = 1, \ldots, n$, $0 \leq p_{2j} \leq 1$, $j = 1, \ldots, n$. Among those decisions leading to different random goods which are separately considered, the decision-maker chooses that random good with the highest prevision or price (see also [66]). The decision-maker's behavior is directly observable. It is based on the notion of prevision of a random good. We note the following

Remark 3. Let X be a random good whose possible values belonging to $I(X) = \{x_1, \ldots, x_n\}$ are explicitly of a monetary nature. The certain gain that the decision-maker subjectively judges to be equivalent to X is expressed by $\mathbf{P}(X)$. It is the price of X for her whenever her cardinal utility function is the 45-degree line. $\mathbf{P}(X)$ coincides with the prevision of X being made by her. Accordingly, the certain gain that the decision-maker subjectively judges to be equivalent to a unit gain conditional on the occurrence of E_j is denoted by $\mathbf{P}(E_j) = p_j$, with $j = 1, \ldots, n$. It is the probability of E_j for her. □

Given two bundles of ordinary goods denoted by A and B, if the decision-maker always chooses A when B is available then she strictly prefers A to B inside of her budget set. The decision-maker's behavior is directly observable. It is based on her preference (see also [85]). Choices being made by the decision-maker are rational if and only if the underlying preference relations work in a coherent way. Some basic assumptions about coherence of the decision-maker's preferences coincide with completeness, reflexivity, and transitivity of preferences. A preference ordering is summarized by using a utility function. If A is preferred to B then in the scale of decision-maker's preference the utility of A denoted by $u(A)$ is greater than the utility of B denoted by $u(B)$. We write $u(A) > u(B)$, where u is a summary index expressing an ordinal utility within this context.

We note the following

Remark 4. Random goods and bundles of ordinary goods are ranked by the decision-maker through two subjective measures, prevision (probability) and utility. They are the two notions on which the rational criterion of decision-maker choice depends. □

1.3 Bound choices being made by the decision-maker under conditions of uncertainty and riskiness

1.3.1 Coherence properties of the notion of average consumption of a random good

Given the two-good assumption, necessary and sufficient conditions for coherence of $\mathbf{P}$ [3] allow to deal with two half-lines, where each of them extends indefinitely in a positive direction from zero before being restricted. Let X_1 be a random good whose possible values are found on the horizontal axis of a two-dimensional Cartesian coordinate system and let X_2 be another random good whose possible values are found on the vertical one. Since $\mathbf{P}$ is linear, we write

$$\mathbf{P}(aX_1) = a\mathbf{P}(X_1) \tag{1.11}$$

as well as

$$\mathbf{P}(aX_2) = a\mathbf{P}(X_2) \tag{1.12}$$

for every real number denoted by a. More generally, we write

$$\mathbf{P}(aX_1' + bX_1'' + cX_1''' + \ldots) = a\mathbf{P}(X_1') + b\mathbf{P}(X_1'') + c\mathbf{P}(X_1''') + \ldots \tag{1.13}$$

[3] Notice that **P** is both bilinear and linear. Whenever **P** is bilinear, we decompose it into two linear measures. Each of them is still denoted by **P**.

for any finite number of random goods X_1', X_1'', X_1''', ... that are considered on the horizontal axis and

$$\mathbf{P}(aX_2' + bX_2'' + cX_2''' + \ldots) = a\mathbf{P}(X_2') + b\mathbf{P}(X_2'') + c\mathbf{P}(X_2''') + \ldots \tag{1.14}$$

for any finite number of summands X_2', X_2'', X_2''', ... that are considered on the vertical one, with a, b, c, ... any real numbers. With respect to (1.11), if a is a real number then a has to lie between a' and a''. This implies that aX_1 has to lie between $a'X_1$ and $a''X_1$. Nevertheless, this is true when and only when all possible values for X_1 are non-negative. If all possible values for X_1 are not non-negative then this is false. However, we can always write $X_1 = Y_1 - Z_1$, where we have $Y_1 = X_1\,(X_1 \geq 0)$ and $Z_1 = -X_1\,(X_1 \leq 0)$. The possible values for Y_1 and Z_1 are always non-negative because it turns out to be $Y_1 = X_1$, if $X_1 > 0$ and zero otherwise, as well as $Z_1 = -X_1$, if $X_1 < 0$ and zero otherwise. The conclusion is therefore valid for Y_1 and Z_1. It is consequently valid for $X_1 = Y_1 - Z_1$. Thus, all possible values for X_1 are non-negative. The same is true with respect to (1.12). It follows that it is sufficient to consider two half-lines, where the point at which they meet each other is given by $(0,0)$. We write

$$\mathbf{P}(Y_1 - Z_1) = \mathbf{P}(Y_1) - \mathbf{P}(Z_1) \tag{1.15}$$

by virtue of additivity property of $\mathbf{P}$. This means that $\mathbf{P}(Y_1 - Z_1)$, $\mathbf{P}(Y_1)$, and $\mathbf{P}(Z_1)$ have the same masses. Differently, (1.15) does not work. The number of the possible values for Y_1 is the same as the one for Z_1. We suppose that zero (whose probability can be equal to 0 because it is a possible value for a random good, so we write $0 \cdot 0 = 0$) always separates them. Moreover, it turns out to be

$$\inf I(Y_1 - Z_1) \leq \mathbf{P}(Y_1 - Z_1) \leq \sup I(Y_1 - Z_1), \tag{1.16}$$

with $Y_1 - Z_1 = X_1$, because $\mathbf{P}$ is convex. We similarly write

$$\mathbf{P}(Y_2 - Z_2) = \mathbf{P}(Y_2) - \mathbf{P}(Z_2) \tag{1.17}$$

as well as

$$\inf I(Y_2 - Z_2) \leq \mathbf{P}(Y_2 - Z_2) \leq \sup I(Y_2 - Z_2). \tag{1.18}$$

Any transgression of such properties of $\mathbf{P}$ leads to choices which are not rational (see also [30]).

If we consider all coherent previsions of Y_1 then the evaluation of the probability of each state of the world of the contingent consumption plan under consideration permits the choice of any value in the interval from 0 to 1, end points included. The number of these values is infinite. Hence, we observe that $\mathbf{P}(Y_1)$ geometrically identifies a line segment on the horizontal axis because $\mathbf{P}$ is convex. One of the two end points of it coincides with zero. If we consider all coherent previsions of $(Y_1 - Z_1)$ then we observe that $\mathbf{P}(Y_1 - Z_1)$ geometrically identifies a more extended line segment on the horizontal axis because the absolute value of each element of the set $I(Y_1 - Z_1)$ is not lower than the one of each element of $I(Y_1)$ (see also [29]). The same is true whenever we consider all coherent previsions of Y_2 and $(Y_2 - Z_2)$ on the vertical axis.

It is always possible to extend and, symmetrically, to restrict the absolute value of each element belonging to the set of possible values for a random good. In particular, it is always possible to extend and, symmetrically, to restrict the absolute value of the highest value belonging to the set of possible values for a random good. This set depends on the state of information and knowledge associated with a given decision-maker at a certain instant, so it is always of a relative nature. It depends on what she knows, or does not know, at a certain instant, so it is of an objective nature. It does not depend on her opinion on what is uncertain or possible for her, so it is never of a subjective nature.

1.3.2 A projection of a bilinear and disaggregate measure identifying the object of decision-maker choice onto two mutually orthogonal axes

Two marginal random goods denoted by X_1 and X_2 are jointly considered by us. Hence, they give rise to a joint random good denoted by X_1X_2. Its coherent prevision is of a bilinear nature (see also [87]). We denote it by $\mathbf{P}(X_1X_2)$. By definition, it is linear with regard to each marginal random good that is separately considered (see also [22]). We denote by $\mathscr{P}$ the set of all coherent previsions denoted by $\mathbf{P}$ connected with two logically independent random goods which are jointly considered. We summarize their intrinsic joint distribution by means of $\mathbf{P}$ (see also [19]). We note that $\mathscr{P}$ is an uncountable subset of a two-dimensional linear space over $\mathbb{R}$. All pairs of real numbers denoted by $(\mathbf{P}(X_1),\mathbf{P}(X_2))$ are the Cartesian coordinates of all points of this subset. We always project $(\mathbf{P}(X_1),\mathbf{P}(X_2))$ onto the two mutually orthogonal axes of a two-dimensional Cartesian coordinate system whose intersection is given by the point $(0,0)$. We always decompose $(\mathbf{P}(X_1),\mathbf{P}(X_2))$ into two linear measures denoted by $\mathbf{P}(X_1)$ and $\mathbf{P}(X_2)$ respectively. We are also interested in knowing all coherent previsions of each marginal random good (see also [4]). The two-dimensional set $\mathscr{P}$ of all coherent previsions $\mathbf{P}$ connected with two logically independent random goods which are jointly considered is a convex set. It is the closed convex hull of the set $\mathscr{Q}=I(X_1)\times I(X_2)$ of all possible values for X_1 and X_2 respectively.

We analytically consider a linear inequality given by

$$c_1X_1+c_2X_2\leq c, \tag{1.19}$$

where c_1, c_2, c are strictly positive real numbers. It must also be satisfied by the corresponding marginal previsions denoted by $\mathbf{P}(X_1)$ and $\mathbf{P}(X_2)$, so we have

$$c_1\mathbf{P}(X_1)+c_2\mathbf{P}(X_2)\leq c. \tag{1.20}$$

We are found inside of the budget set of the decision-maker. It obeys the rules of the logic of prevision. Finite sets of possible consumption levels remain finite sets whenever they are studied by ordinary logic characterized by the principle of bivalence [4]. Finite sets of possible consumption levels do not remain finite sets whenever they are studied by the logic of prevision. It follows that we deal with uncountable sets of coherent previsions identifying continuous goods whose average quantities of consumption are denoted by $\mathbf{P}(X_1)$ and $\mathbf{P}(X_2)$. The expression given by

$$c_1\mathbf{P}(X_1)+c_2\mathbf{P}(X_2)=c \tag{1.21}$$

is an equation of a linear function expressed in an implicit form. It represents a line whose slope is given by $-\frac{c_1}{c_2}$. Its horizontal intercept is given by $\frac{c}{c_1}$, whereas its vertical intercept is given by $\frac{c}{c_2}$. Such a line is a hyperplane embedded in a two-dimensional Cartesian coordinate system. By definition, the line given by (1.21) does not separate any point $\mathbf{P}$ of $\mathscr{P}$ from the

[4] If the state of information and knowledge associated with a given decision-maker is assumed to be complete at the time of choice then ordinary logic takes place.

set $\mathscr{Q}$ of all possible points for X_1 and X_2. Also, it does not separate any point $\mathbf{P}$ of $\mathscr{P}$ from $I(X_1)$, nor from $I(X_2)$. The point denoted by

$$\big(\sup I(Y_1), \sup I(Y_2)\big) \tag{1.22}$$

always belongs to it, where we have $Y_1 - Z_1 = X_1$ as well as $Y_2 - Z_2 = X_2$. The line given by (1.21) always passes through the point whose coordinates are expressed by (1.22) whenever it changes its negative slope as the state of information and knowledge associated with a given decision-maker changes. For instance, her state of information and knowledge changes if one of the two objective prices expressed by c_1 and c_2 changes. This modifies her prevision denoted by $\mathbf{P}$ conditional on her changed state of information and knowledge according to Bayes' theorem. In general, it is possible to say that rational choices being made by the decision-maker inside of her budget set depend on her variable state of information and knowledge according to Bayes' rule.

1.3.3 A full analogy between properties concerning average consumption of random goods and well-behaved preferences

There exists a dichotomy between $I(X_1) = \{x_{11}, \ldots, x_{1n}\}$ and $\mathbf{P}(X_1)$. There exists a dichotomy between $I(X_2) = \{x_{21}, \ldots, x_{2n}\}$ and $\mathbf{P}(X_2)$ as well as between $\mathscr{Q}$ and $\mathbf{P}(X_1 X_2)$, where $\mathbf{P}(X_1 X_2) \in \mathscr{P}$ is also a point whose coordinates are given by $(\mathbf{P}(X_1), \mathbf{P}(X_2))$. Indeed, $I(X_1)$, $I(X_2)$, $\mathscr{Q}$ contain a finite number of possible points unlike $\mathbf{P}(X_1)$, $\mathbf{P}(X_2)$, $\mathbf{P}(X_1 X_2)$. Within this context, we focus on $\mathbf{P}(X_1 X_2)$ as being a two-dimensional point. $\mathscr{P}$ is a right triangle. Only one point of the ones of $\mathscr{P}$, where $\mathscr{P}$ depends on (c_1, c_2, c), is the one chosen by the decision-maker. Every point denoted by $(\mathbf{P}(X_1), \mathbf{P}(X_2))$ is a prevision bundle viewed to be as a consumption bundle, with $\mathbf{P}(X_1)$ and $\mathbf{P}(X_2)$ that tell us how much the decision-maker modeled as being a consumer is choosing to demand for X_1 and how much she is choosing to demand for X_2. We are talking about average quantities of consumption. The prices of $\mathbf{P}(X_1)$ and $\mathbf{P}(X_2)$ are respectively c_1 and c_2, whereas the amount of money she has to spend is equal to c. The inequality given by (1.20) represents the budget constraint of the decision-maker with respect to X_1 and X_2. Indeed, the amount of money spent on $\mathbf{P}(X_1)$ and on $\mathbf{P}(X_2)$ must be no more than the total amount she has to spend. All affordable prevision bundles are those that do not cost any more than c. The set of all affordable prevision bundles at prices (c_1, c_2) and income c is her budget set with respect to X_1 and X_2. The expression coinciding with (1.21) represents the budget line concerning X_1 and X_2. It is the set of prevision bundles that cost exactly c.

We prove the following

Theorem 1. Let X_1 and X_2 be two logically independent random goods which are jointly considered inside of the budget set of the decision-maker. Their possible values are expressed by $I(X_1) = \{x_{11}, \ldots, x_{1n}\}$ and $I(X_2) = \{x_{21}, \ldots, x_{2n}\}$, where it turns out to be $x_{11} < \ldots < x_{1n}$ and $x_{21} < \ldots < x_{2n}$. All coherent previsions of $X_1 X_2$ being established by the decision-maker and denoted by $\mathbf{P}(X_1 X_2)$ are of a bilinear nature. Each prevision of $X_1 X_2$ is decomposed into two linear previsions inside of her budget set. Their properties coincide with the ones of well-behaved preferences, where the latter are monotonic and convex preferences.

Proof. The decision-maker ranks inside of her budget set various prevision possibilities for which usual assumptions of completeness, reflexivity, and transitivity are valid. They are all formally admissible in terms of coherence. Additivity and convexity of $\mathbf{P}$ with respect to two logically independent random goods which are jointly considered correspond to monotonicity and convexity of well-behaved preferences considered inside of the budget set of the decision-maker. Well-behaved preferences are monotonic because more of both goods is better. They are also convex because averages are weakly preferred to extremes. We are talking about goods, not bads. We imagine indifference curves which are parallel lines restricted to the first quadrant of a two-dimensional Cartesian coordinate system. The budget set of the decision-maker contains all indifference curves we graphically imagine. They have the same slope as the one of the budget line concerning X_1 and X_2 given by (1.21). We think of indifference curves representing perfect substitutes, so the weighted average of two indifferent and extreme prevision bundles is not preferred to the two extreme prevision bundles, but it is as good as the two extreme prevision bundles. Every prevision bundle is getting a utility level and those prevision bundles on higher indifference curves are getting larger utility levels. This kind of utility is referred to as ordinal utility. The direction of increasing preference is up and to the right. It is towards the direction of increased random good 1 average consumption and increased random good 2 average consumption. With regard to (1.21), we write

$$\frac{\Delta\mathbf{P}(X_2)}{\Delta\mathbf{P}(X_1)} = -\frac{c_1}{c_2} \tag{1.23}$$

because if the decision-maker increases $\mathbf{P}(X_1)$ then she must decrease $\mathbf{P}(X_2)$ and vice versa in order to move along it. If she is choosing $(\mathbf{P}(X_1), \mathbf{P}(X_2))$ inside of her budget set then we write

$$MU_1 = \frac{\Delta U}{\Delta\mathbf{P}(X_1)} = \frac{u\left(\mathbf{P}(X_1) + \Delta\mathbf{P}(X_1), \mathbf{P}(X_2)\right) - u\left(\mathbf{P}(X_1), \mathbf{P}(X_2)\right)}{\Delta\mathbf{P}(X_1)}, \tag{1.24}$$

where MU_1 measures the rate of change in utility, denoted by ΔU, associated with a small change in the amount of random good 1 expressed by $\Delta\mathbf{P}(X_1)$. MU_1 is the marginal utility with respect to random good 1. The amount of random good 2 is held fixed. We can multiply the change in average consumption of random good 1 by the marginal utility with respect to random good 1. This allows to calculate the change in utility associated with a small change in average consumption of random good 1. We therefore write

$$\Delta U = MU_1\, \Delta\mathbf{P}(X_1). \tag{1.25}$$

On the other hand, the marginal utility with respect to random good 2 is defined in the following form

$$MU_2 = \frac{\Delta U}{\Delta\mathbf{P}(X_2)} = \frac{u\left(\mathbf{P}(X_1), \mathbf{P}(X_2) + \Delta\mathbf{P}(X_2)\right) - u\left(\mathbf{P}(X_1), \mathbf{P}(X_2)\right)}{\Delta\mathbf{P}(X_2)}. \tag{1.26}$$

If we calculate the marginal utility with respect to random good 2 then we keep the amount of random good 1 constant. We can evidently write

$$\Delta U = MU_2\, \Delta\mathbf{P}(X_2). \tag{1.27}$$

Marginal utility is used to calculate the marginal rate of substitution (MRS). It is the rate at which the decision-maker is willing to substitute a small amount of random good 2 for random good 1. We focus on that indifference curve imagined inside of the budget set of the decision-maker whose utility level is larger. It coincides with the budget line. We consider a change in the average consumption of each random good such that it keeps utility constant. It is denoted by $(\Delta\mathbf{P}(X_1), \Delta\mathbf{P}(X_2))$. This change moves her along that indifference curve imagined inside of her budget set whose utility level is larger. We write

$$MU_1\,\Delta\mathbf{P}(X_1) + MU_2\,\Delta\mathbf{P}(X_2) = \Delta U = 0. \tag{1.28}$$

If we solve for the slope of the indifference curve then we obtain

$$\text{MRS} = \frac{\Delta\mathbf{P}(X_2)}{\Delta\mathbf{P}(X_1)} = -\frac{MU_1}{MU_2}. \tag{1.29}$$

Since the MRS measures the slope of the indifference curve under consideration, if we consider its algebraic sign then we should write

$$\text{MRS} = -\frac{\Delta\mathbf{P}(X_2)}{\Delta\mathbf{P}(X_1)}$$

because it is negative. Nevertheless, (1.29) tells us that we consider the absolute value of the MRS by means of the ratio of marginal utilities. The ratio of marginal utilities is independent of the particular way being chosen by the decision-maker to represent her preferences. Let $(\mathbf{P}(X_1), \mathbf{P}(X_2))$ be a point belonging to the indifference curve whose utility level is larger. After projecting $(\mathbf{P}(X_1), \mathbf{P}(X_2))$ onto the two mutually orthogonal axes of a two-dimensional Cartesian coordinate system, we note that additivity and convexity of $\mathbf{P}$ with respect to marginal previsions of X_1 and X_2 correspond to monotonicity and convexity of well-behaved preferences referred to each axis of a two-dimensional Cartesian coordinate system. Hence, whenever we say that more is better, we mean that a line segment is increasingly large on the horizontal axis as well as a line segment is increasingly large on the vertical one. Also, any line segment on the horizontal axis is a one-dimensional convex set in the same way as any line segment on the vertical one. Since preferences for perfect substitutes are expressed by a utility function whose form is additive, we observe that

$$u\,(\mathbf{P}(X_1), \mathbf{P}(X_2)) = \mathbf{P}(X_1) + \mathbf{P}(X_2) \tag{1.30}$$

is constant along all indifference curves we graphically imagine. In particular, it is constant along the budget line. □

We note the following

Remark 5. The budget line contains those prevision bundles whose utility level is larger. An optimal choice for the decision-maker modeled as being a consumer is her best rational choice. Such a choice is where the indifference curve is tangent to the budget line. The indifference curve is rounded. Its slope is negative. Since we are not interested in proving that prevision (probability) and utility are the two sides of the same coin, it is not necessary to think of indifference curves representing perfect substitutes. □

In general, an optimal choice for the decision-maker is always of a relative nature. It depends on her objective state of information and knowledge. Since we are handling random goods, it is intrinsically incomplete at the time of choice.

We establish the following

Definition 5. *After decomposing* $\mathbf{P}(X_1 X_2)$ *inside of a subset of a two-dimensional linear space over* $\mathbb{R}$*, the decision-maker's demand functions that give the average consumption amounts of each of the two random goods under consideration are expressed by*

$$\mathbf{P}(X_1) = \{\mathbf{P}(X_1)[(c_1, c_2, c)]\} \tag{1.31}$$

and

$$\mathbf{P}(X_2) = \{\mathbf{P}(X_2)[(c_1, c_2, c)]\}, \tag{1.32}$$

where $\mathbf{P}$ *is additive and convex as a consequence of its coherence.*

Since $\mathbf{P}(X_1 X_2)$, $\mathbf{P}(X_1)$, $\mathbf{P}(X_2)$ belong to convex sets, it is evident that the quantities demanded denoted by $\mathbf{P}(X_1)$ and $\mathbf{P}(X_2)$ depend on the three objective elements identifying a two-dimensional convex set inside of which $\mathbf{P}(X_1 X_2)$ is coherently decomposed into $\mathbf{P}(X_1)$ and $\mathbf{P}(X_2)$. It follows that $\mathbf{P}(X_1)$ and $\mathbf{P}(X_2)$ depend on objective and subjective elements, where the latter are given by all masses being subjectively chosen by the decision-maker. They never depend on objective elements only. Accordingly, an axiomatic approach to the theory of decision-making is not alone sufficient to explain all choices being made by the decision-maker inside of her budget set.

We note the following

Remark 6. The decision-maker estimates both marginal probabilities associated with X_1 and X_2 and the joint ones associated with $X_1 X_2$. She is subjected to $2n-1$ constraints only in order to estimate all joint probabilities associated with $X_1 X_2$. Such constraints coincide with $2n-1$ marginal masses. Marginal probabilities associated with X_1 and X_2 give rise to $\mathbf{P}(X_1)$ and $\mathbf{P}(X_2)$, where $\mathbf{P}(X_1)$ and $\mathbf{P}(X_2)$ represent what is actually demanded for X_1 and X_2 by her inside of her budget set. Given $(\mathbf{P}(X_1), \mathbf{P}(X_2))$, a bilinear and disaggregate measure coinciding with $\mathbf{P}(X_1 X_2)$, where $\mathbf{P}(X_1 X_2)$ is decomposed into $\mathbf{P}(X_1)$ and $\mathbf{P}(X_2)$, is a synthesized element of the Fréchet class. Given $(\mathbf{P}(X_1), \mathbf{P}(X_2))$, the decision-maker also chooses a synthesized element of the Fréchet class such that $\mathbf{P}(X_1)$ and $\mathbf{P}(X_2)$ never change. A synthesized element of the Fréchet class established by the marginal distributions of X_1 and X_2 such that $\mathbf{P}(X_1)$ and $\mathbf{P}(X_2)$ do not change corresponds to each point of the budget set of the decision-maker. In particular, she can choose a coherent summary of $X_1 X_2$ denoted by $\mathbf{P}(X_1 X_2)$ and decomposed into $\mathbf{P}(X_1)$ and $\mathbf{P}(X_2)$ such that there is no linear correlation between random good 1 and random good 2. She could also choose a coherent summary of $X_1 X_2$ denoted by $\mathbf{P}(X_1 X_2)$ and decomposed into $\mathbf{P}(X_1)$ and $\mathbf{P}(X_2)$ such that there is an inverse or direct linear relationship between random good 1 and random good 2. □

Remark 7. Given c_1, X_1 is a random good such that the possible quantities of consumption associated with it that can be demanded by the decision-maker inside of her budget set coincide with all coherent previsions of X_1 denoted by $\mathbf{P}(X_1)$. They are average quantities of consumption. X_1 has an elastic demand. Given c_2, X_2 is a random good such that the possible quantities of consumption associated with it that can be demanded by her inside of her budget set coincide with all coherent previsions of X_2 denoted by $\mathbf{P}(X_2)$. They are average quantities of consumption. X_2 has an elastic demand. If the prices of X_1 (coinciding with the one of $\mathbf{P}(X_1)$) and X_2 (coinciding with the one of $\mathbf{P}(X_2)$) change then X_1 and X_2 have an elastic demand with respect to the new prices identifying a different budget set of the decision-maker. □

From now until the end of this chapter, we focus on axioms of revealed preference theory applied to average quantities of consumption. It is clear that average quantities of consumption can be referred to random or ordinary goods whenever the state of information and knowledge associated with a given decision-maker is assumed to be incomplete at the time of choice, in one way or another. Revealed preference theory viewed to be as a branch of the theory of decision-making gives empirical meaning to the neoclassical economic hypothesis according to which the best rational choice being made by the decision-maker inside of her budget set has to be the one maximizing her utility. We realize that the best rational choice being made by the decision-maker inside of her budget set deals with average quantities of consumption. We show that it is possible to unify the empirical content of specific theories referred to coherent previsions of random goods in specific economic environments (see also [69]). This is possible because a coherent prevision of a joint random good is always decomposed into two previsions of two marginal random goods inside of the budget set of the decision-maker.

The idea of revealed preference has an operational nature. The same is true with regard to the notion of prevision (probability) and utility.

1.4 Metric aspects of the neoclassical decision-maker choice theory applied to average quantities of consumption

We consider a two-dimensional linear space over $\mathbb{R}$ denoted by E^2 [5]. It has a Euclidean structure. The set of all $\mathbf{x} \in E^2$, with $x_1 = \mathbf{P}(X_1) \geq 0$ and $x_2 = \mathbf{P}(X_2) \geq 0$, is denoted by E^2_+. The set of all $\mathbf{x} \in E^2$, with $x_1 = \mathbf{P}(X_1) > 0$ and $x_2 = \mathbf{P}(X_2) > 0$, is denoted by E^2_{++}. Each decision-maker's prevision concerning a joint random good is chosen inside of her budget set. In general, it is possible to observe a different budget set whenever the budget line changes its negative slope. Each choice being made by the decision-maker is associated with a budget set characterized by a budget line. All decision-maker's previsions concerning joint random goods that are chosen whenever the budget line changes its negative slope identify a finite sequence belonging to E^2 and denoted by

$$\{\mathbf{x}^k | k = 1, \ldots, K\}. \tag{1.33}$$

For each k, it is possible to consider a pair of real numbers written in the form (x_1^k, x_2^k). If we suppose, to fix ideas, that it turns out to be $K = 2$ then we are faced with a balanced sequence of pairs given by

$$(x_1^1, x_2^1), (x_1^2, x_2^2). \tag{1.34}$$

The space where the decision-maker chooses is denoted by E^2_+. It coincides with the first quadrant of a two-dimensional Cartesian coordinate system. We consider a collection denoted by $\mathscr{U}$ of utility functions written in the form

$$U : E^2_+ \to \mathbb{R}. \tag{1.35}$$

[5] The space of alternatives is an n-dimensional linear space over $\mathbb{R}$ denoted by E^n. Since we study two marginal random goods at a time, two one-dimensional linear subspaces of E^n are considered. Each of them is transferred on a one-dimensional straight line on which an origin, a same unit of length, and an orientation are chosen. We do not handle two one-dimensional straight lines, but we deal with two half-lines. This is because $\mathbf{P}$ is involved together with its coherence properties. We accordingly use n non-negative and finitely additive masses with regard to each marginal random good, so E^2 coincides with $\mathbb{R} \times \mathbb{R}$. On the other hand, it turns out to be $\dim E^2 = \dim(\mathbb{R} \times \mathbb{R}) = 2$.

The elements of the pair expressed by

$$(E^2_+, U), \tag{1.36}$$

with $U \in \mathscr{U}$, are said to be primitive members according to the neoclassical decision-maker choice theory.

Each decision-maker's prevision concerning a joint random good consists of a two-dimensional vector $\mathbf{x} \in E^2_+$ obtained from her budget set denoted by

$$B(\mathbf{c}, c) = \{\mathbf{x} \in E^2_+ | \, \mathbf{c} \cdot \mathbf{x} \leq c\}, \tag{1.37}$$

where $\mathbf{c} = (c_1, c_2)$ is a price vector, whereas c is the amount of money she has to spend. We observe that $\mathbf{c} \cdot \mathbf{x}$ is a scalar or inner product characterizing E^2 from a metric point of view. Notice that (1.37) coincides with (1.20).

In general, if the line expressed by (1.21) changes its negative slope then it is possible to observe both those quantities being actually chosen by the decision-maker and their prices. The former coincide with coherent previsions of a joint random good which are decomposed into two previsions of two marginal random goods, whereas the latter geometrically coincide with the real coefficients identifying the negative slope of a hyperplane embedded in E^2. A generic pair denoted by

$$(\mathbf{x}, \mathbf{c}) \in E^2_+ \times E^2_{++} \tag{1.38}$$

represents all we need to know about a coherent prevision of a random good being made by the decision-maker and about her budget. It follows that a finite collection of pairs written in the form

$$\{(\mathbf{x}^1, \mathbf{c}^1), \ldots, (\mathbf{x}^K, \mathbf{c}^K)\} \tag{1.39}$$

expresses a dataset [6] (see also [33]). The consumption data can pragmatically be obtained from consumption surveys drawn from the field, from a laboratory experiment or from a hybrid design.

All datasets that are coherent with revealed preference theory represent its empirical content (see also [18]). Given a collection $\mathscr{U}$ of utility functions, it is possible to say that a dataset expressed by (1.39) is $\mathscr{U}$-rational if there exists $U \in \mathscr{U}$ such that we write, for each k,

$$\mathbf{x}^k \in \text{argmax}\,\{U(\mathbf{x}) | \, \mathbf{x} \in B(\mathbf{c}^k, \mathbf{c}^k \cdot \mathbf{x}^k)\}. \tag{1.40}$$

[6] Given n possible alternatives for random good 1, we obtain $\mathbf{P}(X_1)$ by using n non-negative masses. A nonparametric marginal distribution of mass is summarized only. The same is true with regard to random good 2. Conversely, if $\mathbf{P}(X_1)$ is an element contained in the dataset under consideration then it is possible to identify n possible alternatives belonging to a closed neighborhood of $\mathbf{P}(X_1)$ denoted by $[\mathbf{P}(X_1) - \varepsilon\,; \mathbf{P}(X_1) + \varepsilon']$ on the horizontal axis, where both ε and ε' are two small positive quantities. The same is true with regard to a closed neighborhood of $\mathbf{P}(X_2)$ denoted by $[\mathbf{P}(X_2) - \varepsilon\,; \mathbf{P}(X_2) + \varepsilon']$ on the vertical one. It follows that n possible alternatives are summarized in such a way that $\mathbf{P}(X_1)$ is actually chosen. A nonparametric marginal distribution of mass is accordingly estimated. Solutions of a linear equation, whose unknowns are probabilities, can be established. The same is true with regard to n possible alternatives associated with random good 2. It is not necessary that one of n possible alternatives coincides with $\mathbf{P}(X_1)$. The same is true with regard to $\mathbf{P}(X_2)$. A nonparametric joint distribution of mass derives from two marginal distributions of mass. They are all estimated before being summarized.

1.4.1 General utilities whose arguments are average quantities of consumption

Let $(\mathbf{x}^k, \mathbf{c}^k)_{k=1}^K$ be a dataset.

It is possible to define two binary relations on E_+^2. We firstly establish the following

Definition 6. *It is possible to say that* $\mathbf{x}$ *is revealed preferred to* $\mathbf{y}$*, that is* $\mathbf{x} R^P \mathbf{y}$*, if there exists k such that it turns out to be* $\mathbf{x} = \mathbf{x}^k$ *as well as* $\mathbf{c}^k \cdot \mathbf{y} \leq \mathbf{c}^k \cdot \mathbf{x}^k$.

We secondly establish the following

Definition 7. *It is possible to say that* $\mathbf{x}$ *is strictly revealed preferred to* $\mathbf{y}$*, that is* $\mathbf{x} P^P \mathbf{y}$*, if there exists k such that it turns out to be* $\mathbf{x} = \mathbf{x}^k$ *as well as* $\mathbf{c}^k \cdot \mathbf{y} < \mathbf{c}^k \cdot \mathbf{x}^k$.

We observe that a dataset expressed by $(\mathbf{x}^k, \mathbf{c}^k)_{k=1}^K$ satisfies the Weak Axiom of Revealed Preference (WARP) if there is no pair of observations k and k' such that it is possible to observe $\mathbf{x}^k R^P \mathbf{x}^{k'}$, whereas it also turns out to be $\mathbf{x}^{k'} P^P \mathbf{x}^k$. If $(\mathbf{x}^k, \mathbf{c}^k)_{k=1}^K$ does not satisfy the WARP then it cannot be $\mathscr{U}_{LNS}$-rational, where $\mathscr{U}_{LNS}$ denotes the set of locally non-satiated utility functions.

We note that a dataset expressed by $(\mathbf{x}^k, \mathbf{c}^k)_{k=1}^K$ satisfies the Generalized Axiom of Revealed Preference (GARP) when, for any finite sequence $(k_i)_{i=1}^M$ in $\{1, \ldots, K\}$, if we observe $\mathbf{x}^{k_i} R^P \mathbf{x}^{k_{i+1}}$, with $i = 1, \ldots, M-1$, then it is false to observe $\mathbf{x}^{k_M} P^P \mathbf{x}^{k_1}$.

Hence, we observe that the empirical content of the rational behavior of a locally non-satiated decision-maker maximizing her utility associated with a bundle of two goods is the same as the one of a decision-maker with a strictly increasing and concave utility function. The decision-maker is modeled as being a consumer. We note that the set of strictly increasing and concave utility functions is denoted by $\mathscr{U}_{MC}$. Also, those datasets that are coherent with revealed preference theory are those satisfying the GARP (see also [40]).

1.4.2 Additive separability of utility of prevision bundles

Let $\mathscr{U}_{AS}$ be the set of utility functions denoted by $U \colon E_+^2 \to \mathbb{R}$ for which there exists a concave and strictly increasing utility function denoted by $u \colon \mathbb{R}_+ \to \mathbb{R}$. Hence, it is possible to write $U(\mathbf{r}) \geq U(\mathbf{s})$ if and only if we observe

$$u(r_1) + u(r_2) \geq u(s_1) + u(s_2), \tag{1.41}$$

where it turns out to be $r_1 = \mathbf{P}(X_1)$, $r_2 = \mathbf{P}(X_2)$ as well as $s_1 = \mathbf{P}(X_1)$, $s_2 = \mathbf{P}(X_2)$, with $r_1 \neq s_1$ and $r_2 \neq s_2$. The decision-maker is consequently faced with the problem expressed by

$$\max\Big\{u(r_1) + u(r_2) \,|\, \mathbf{r} \in B(\mathbf{c}, c)\Big\}. \tag{1.42}$$

If u is smooth then the first-order conditions of the maximization problem, where an interior solution is assumed to exist, require that it turns out to be

$$\frac{u'(r_1)}{u'(r_2)} = \frac{c_1}{c_2}. \tag{1.43}$$

The first-order conditions, together with the concavity of u, involve that whenever it turns out to be $r_1 > r_2$, it has then to be the case that $\frac{c_1}{c_2} \leq 1$. This means that $\mathbf{P}(X_1\, X_2)$ is decomposed into $\mathbf{P}(X_1)$ and $\mathbf{P}(X_2)$ inside of a subset of a linear space over $\mathbb{R}$ in order that a larger consumption of the random good X_1 expressed by $\mathbf{P}(X_1)$ is only possible when it is cheaper than $\mathbf{P}(X_2)$. In other words, demand slopes down (see also [24]). We are talking about average quantities of consumption.

Suppose that we are faced with a budget set where $\mathbf{P}(X_1)$ is cheaper than $\mathbf{P}(X_2)$. This means that the budget set below the 45-degree line is larger than the one above. If a dataset includes a coherent prevision of $X_1 X_2$ being made by the decision-maker that is found on the budget line above the 45-degree line then it violates downward-sloping demand property (see also [44]). It is possible to say that it is not $\mathscr{U}_{AS}$-rational. If a dataset conversely includes a coherent prevision of $X_1 X_2$ being made by the decision-maker that is found on the budget line below the 45-degree line then it does not violate downward-sloping demand property. Such a prevision is then compatible with a $\mathscr{U}_{AS}$-rational decision-maker (see also [55]).

To fix ideas, let $(\mathbf{r}^k, \mathbf{c}^k)_{k=1}^3$ be a dataset with three observations. If they violate the WARP then such a dataset cannot be rationalized by any utility function. It is then evident that it cannot even be rationalized by an element belonging to $\mathscr{U}_{AS}$. Given a balanced sequence of pairs denoted by

$$(r_1^1, r_2^1), (r_1^2, r_2^2), (r_1^3, r_2^3), \tag{1.44}$$

it is possible to say that (1.44) has the downward-sloping demand property if

$$r_1^1 > r_2^1,\ r_1^2 > r_2^2, \text{ and } r_1^3 > r_2^3 \quad \text{imply that } \frac{c_1^1}{c_2^1} \cdot \frac{c_1^2}{c_2^2} \cdot \frac{c_1^3}{c_2^3} \leq 1. \tag{1.45}$$

If any balanced sequence of pairs has the downward-sloping demand property then the Strong Axiom of Revealed Additively Separable Utility is satisfied. Such an axiom is a test for whether a dataset is $\mathscr{U}_{AS}$-rational (see also [26]).

We note the following

Remark 8. Let X_1 and X_2 be two marginal random goods studied inside of the budget set of the decision-maker. Each point of it is a coherent summary of a joint distribution of X_1 and X_2. A joint distribution of X_1 and X_2 is coherently summarized by using a bilinear index denoted by $\mathbf{P}(X_1 X_2)$. It is decomposed into two linear indices denoted by $\mathbf{P}(X_1)$ and $\mathbf{P}(X_2)$ respectively. Since we are faced with all coherent previsions of $X_1 X_2$, each possible value for $X_1 X_2$, X_1, and X_2 belonging to $I(X_1) \times I(X_2)$, $I(X_1)$, and $I(X_2)$ enables the choice of any probabilistic value in the interval from 0 to 1, end points included. The number of these probabilistic values is infinite. It follows that it is not necessary that all possible values for $X_1 X_2$, X_1, and X_2 have only to be equally probable. Additive separability of utility of prevision bundles is a decomposition of utility which is based on the decomposition of a coherent prevision of a joint random good. □

1.5 Choices under risk: the case of random goods identifying average quantities of consumption

Let $\Delta_{++} = \{\boldsymbol{\mu}^* \in E_{++}^2 \,|\, \sum_{s=1}^2 \mu_s^* = 1\}$ be the set of strictly positive weights measuring the relative importance of the two marginal previsions concerning the two marginal random goods into account. Risk is common in experimental designs. It is possible to assume that there exists a given, known, and objective measure associated with marginal previsions identifying average quantities of consumption. On the other hand, data drawn from laboratory experiments can be considered by using known and objective measures denoted by $\boldsymbol{\mu}^*$

associated with coherent previsions of single random goods identifying average quantities of consumption. Let $\mathscr{U}_{EU}$ be the class of utility functions denoted by $U: E_+^2 \to \mathbb{R}$ for which there exists a concave and strictly increasing utility function denoted by $u: \mathbb{R}_+ \to \mathbb{R}$. Hence, it is possible to write $U(\mathbf{r}) \geq U(\mathbf{s})$ if and only if we observe

$$\mu_1^* u(r_1) + \mu_2^* u(r_2) \geq \mu_1^* u(s_1) + \mu_2^* u(s_2), \tag{1.46}$$

where it turns out to be $r_1 = \mathbf{P}(X_1)$, $r_2 = \mathbf{P}(X_2)$ as well as $s_1 = \mathbf{P}(X_1)$, $s_2 = \mathbf{P}(X_2)$, with $r_1 \neq s_1$ and $r_2 \neq s_2$. We are faced with a problem concerning the expected utility of the decision-maker (see also [63]).

We note that the first-order conditions concerning the decision-maker's optimization problem are expressed by

$$\frac{u'(r_1)}{u'(r_2)} = \frac{\rho_1}{\rho_2}, \tag{1.47}$$

where risk-neutral prices appearing on the right-hand side of (1.47) are defined as

$$\rho_s^k = \frac{c_s^k}{\mu_s^*}, \tag{1.48}$$

with $k \in K$ and $s \in I_2 = \{1, 2\}$.

It is then possible to say that a dataset is $\mathscr{U}_{EU}$-rational if and only if it satisfies the Strong Axiom of Revealed Objective Expected Utility (SAROEU). Given any balanced sequence of pairs denoted by $(r_1^1, r_2^1), \ldots, (r_1^K, r_2^K)$, if it has the downward-sloping demand property then SAROEU is satisfied (see also [54]).

1.6 Choices under uncertainty: the case of random goods identifying average quantities of consumption

Faced with uncertainty meant in a personalistic sense, the decision-maker feels a more or less strong propensity to expect that a particular state of the world of a contingent consumption plan rather than others will turn out to be true at the right time. She attributes to the various possible quantitative states of the world of a contingent consumption plan a greater or lesser degree of a subjective and psychological factor expressing such an attitude. In other words, she distributes among all possible values for $X_1 X_2$ her sensations of probability after distributing them among all possible values for X_1 and X_2. Hence, a nonparametric joint distribution associated with $X_1 X_2$ arises after two nonparametric marginal distributions associated with X_1 and X_2 arise. We do not assume a probability distribution as already attached to a random good. Each random good can coherently be assigned a distribution of mass as an expression of the attitude connected with the decision-maker under consideration. We say that a distribution of mass can vary from individual to individual as well as it can vary in accordance with the state of information and knowledge associated with a given decision-maker (see also [74]). A distribution of mass has to be summarized, where its first row moment and its second central moment always take place within this context. It has to be summarized in a coherent way (see also [56]). We note the following

Remark 9. The decision-maker maximizes her utility associated with prevision bundles. Her behavior is rational if and only if it obeys the underlying logical rules of coherence, so each mass under consideration is non-negative and their sum is finitely equal to 1. Such rules have specific needs that must be satisfied. Specific needs of logical rules of coherence are satisfied whenever a coherent prevision of a joint random good expressed by $\mathbf{P}(X_1 X_2)$ is decomposed into two coherent previsions of two marginal random goods inside of the budget set of the decision-maker. □

Let $\mathscr{U}_{SEU}$ be the class of utility functions denoted by $U\colon E_+^2 \to \mathbb{R}$ for which there exists a subjective measure denoted by $\boldsymbol{\mu} \in \Delta_{++}$ that goes into constructing the expected utility of the decision-maker and a concave and strictly increasing utility function denoted by $u\colon \mathbb{R}_+ \to \mathbb{R}$. Hence, it is possible to write $U(\mathbf{r}) \geq U(\mathbf{s})$ if and only if we observe

$$\mu_1\, u(r_1) + \mu_2\, u(r_2) \geq \mu_1\, u(s_1) + \mu_2\, u(s_2), \tag{1.49}$$

where it turns out to be $r_1 = \mathbf{P}(X_1)$, $r_2 = \mathbf{P}(X_2)$ as well as $s_1 = \mathbf{P}(X_1)$, $s_2 = \mathbf{P}(X_2)$, with $r_1 \neq s_1$ and $r_2 \neq s_2$. It is possible to decompose both prevision of a joint random good and utility of a two-dimensional vector.

The first-order condition of the maximization problem is expressed by

$$\frac{\mu_1\, u'(r_1)}{\mu_2\, u'(r_2)} = \frac{c_1}{c_2}. \tag{1.50}$$

Suppose that we are faced with a budget set where $\mathbf{P}(X_1)$ is cheaper than $\mathbf{P}(X_2)$. If a dataset includes a coherent prevision being made by the decision-maker that is found on the budget line above the 45-degree line then it is $\mathscr{U}_{SEU}$-rational. This is because she could choose of consuming more in the more expensive situation. It is then characterized by a greater subjective weight than the other situation.

Suppose that we are faced with a dataset whose observations are two. We then consider two budget lines. Each negative slope of one of them is different from the one of the other. We note that there is no violation of subjective expected utility if and only if choices being made by the decision-maker are found on the two budget lines where the 45-degree line exactly crosses both of them. This is because it turns out to be $r_1 = r_2$ anywhere on the 45-degree line, so the tangent expressing the first-order conditions of the maximization problem has slope given by

$$\frac{\mu_1\, u'(r_1)}{\mu_2\, u'(r_2)} = \frac{\mu_1}{\mu_2}. \tag{1.51}$$

The slope of the tangent is the same anywhere on the 45-degree line. We then write

$$\frac{\mu_1}{\mu_2} = \frac{c_1}{c_2}, \tag{1.52}$$

so we are also able to observe the real nature of $\boldsymbol{\mu} \in \Delta_{++}$.

Given a doubly balanced sequence of pairs denoted by $(r_1^1, r_1^2), (r_2^1, r_2^2)$, it has then the downward-sloping demand property. In general, the Strong Axiom of Revealed Subjective Expected Utility tells us that any doubly balanced sequence of pairs has the downward-sloping demand property. It follows that a dataset is $\mathscr{U}_{SEU}$-rational if and only if it satisfies such an axiom.

On the other hand, all of this allows us to consider the theory of maxmin expected utility as well (see also [50]). Let $\mathscr{U}_{MEU}$ be the class of utility functions denoted by $U\colon E_+^2 \to \mathbb{R}$ for which there exists a set denoted by $\mathbf{M} \subseteq \Delta_{++}$ as well as a concave and strictly increasing utility function denoted by $u\colon \mathbb{R}_+ \to \mathbb{R}$. Hence, it is possible to write $U(\mathbf{r}) \geq U(\mathbf{s})$ if and only if we observe

$$\min\Big\{\mu_1\, u(r_1) + \mu_2\, u(r_2) | \,\boldsymbol{\mu} \in \mathbf{M}\Big\} \geq \min\Big\{\mu_1\, u(s_1) + \mu_2\, u(s_2) | \,\boldsymbol{\mu} \in \mathbf{M}\Big\}, \tag{1.53}$$

where it turns out to be $r_1 = \mathbf{P}(X_1)$, $r_2 = \mathbf{P}(X_2)$ as well as $s_1 = \mathbf{P}(X_1)$, $s_2 = \mathbf{P}(X_2)$, with $r_1 \neq s_1$ and $r_2 \neq s_2$.

The theory of maxmin expected utility tells us that it is absolutely natural to choose different weights for different observations in order to rationalize a dataset. Thus, if it is $r_1^k \leq r_2^k$, where we have $k = 1, \ldots, K$, then we have to observe a pessimistic weight about a marginal prevision of X_1 identifying an average consumption established by the decision-maker. If it is conversely $r_2^k \leq r_1^k$ then we have to observe a pessimistic weight about a marginal prevision of X_2 identifying an average consumption established by her.

We note that many laboratory experiments operate with two outcomes only. To decompose $\mathbf{P}(X_1\, X_2)$ inside of the budget set of the decision-maker is more general than one could think at first. On the other hand, it is not absolutely manageable to consider more than two random goods at a time from a metric point of view. This is because we use a quadratic metric.

A dataset is $\mathscr{U}_{MEU}$-rational if and only if it satisfies the Strong Axiom of Revealed Maxmin Expected Utility according to which any balanced sequence of pairs identifying average quantities of consumption established by the decision-maker has the downward-sloping demand property.

1.7 Intertemporal choices without exponential discounting and risk aversion: the case of random goods identifying average quantities of consumption

In this section, we accept as pragmatically valid the hypothesis of rigidity in the face of risk of the decision-maker. We accept the hypothesis according to which there exists the identity of monetary value and utility within the limits of everyday affairs. Such an assumption is realistic to an adequate degree enough whenever we consider all those transactions being made by the decision-maker whose outcome has no significant effect on her fortune identified with the monetary value of her assets. It is therefore possible to say that all outcomes we consider do not give rise to remarkable improvements in her situation, nor to heavy losses. On the other hand, since each point of the two-dimensional budget set of the decision-maker is a synthesized element of the Fréchet class, she can choose a coherent summary of a joint distribution of mass identifying a joint random good such that there is no linear correlation between random good 1 and random good 2. They are stochastically independent.

We consider those prevision bundles being chosen by the decision-maker at two different time periods (see also [2]). We want to compare how average consumption has changed from one time period to the other by using an appropriate summary index. Let b be the base time period. Let t be the other time period. We suppose that prices are (c_1^t, c_2^t) at the time period denoted by t. The decision-maker chooses (r_1^t, r_2^t), where r_1^t and r_2^t are two marginal previsions of the two random goods into account. We observe that prices are (c_1^b, c_2^b) in the base time period denoted by b. She chooses (r_1^b, r_2^b), where r_1^b and r_2^b are two marginal previsions of the two random goods under consideration. We wonder how the consumption has changed from one time period to the other (see also [5]). We have to summarize it in order to answer such a question. We use two strictly positive weights such that their sum is equal to 1. Also, we have to take her utility function into account as well. The decision-maker is risk neutral. Her utility function is linear. It represents the identity of monetary value and utility, so it coincides with the 45-degree line restricted to the first quadrant of a two-dimensional Cartesian coordinate system.

Let w_1 and w_2 be two strictly positive weights. The following index given by

$$I_q = \frac{w_1\, r_1^t + w_2\, r_2^t}{w_1\, r_1^b + w_2\, r_2^b} \tag{1.54}$$

expresses the ratio of an average consumption to another one. If I_q is greater than 1 then average consumption has gone up when the decision-maker passes from b to t. If it is less than 1 then average consumption has gone down when she passes from b to t.

If it turns out to be

$$U(r_1^t, r_2^t) > U(r_1^b, r_2^b) \tag{1.55}$$

then it is possible to write

$$w_1\, r_1^t + w_2\, r_2^t > w_1\, r_1^b + w_2\, r_2^b \tag{1.56}$$

because we use a utility function coinciding with the 45-degree line. We write $u(x) = x$ whenever $u(x)$ represents the value of a function of x with respect to a two-dimensional Cartesian coordinate system. With regard to a problem of exponential discounting, we note that in addition to weights we also consider a discount function equal to 1 within this context. We evidently number a constant among functions although it is in general possible not to call a constant a function if there is no good reason (see also [21]). We note that I_q is therefore greater than 1. We are able to say that average consumption representing the expected utility of the decision-maker who is a risk-neutral expected-utility agent has gone up in the movement from b to t.

After transforming the t time period prices, we choose them as weights. Hence, if we substitute them for the weights contained in (1.56) then we write

$$\frac{c_1^t}{c_1^t + c_2^t}\, r_1^t + \frac{c_2^t}{c_1^t + c_2^t}\, r_2^t > \frac{c_1^t}{c_1^t + c_2^t}\, r_1^b + \frac{c_2^t}{c_1^t + c_2^t}\, r_2^b. \tag{1.57}$$

The budget line does not change its slope. This is because changes of unit of measurement concerning the decision-maker's budget are inessential. It follows that all of this corroborates that she is better off at the time period denoted by t than at the time period denoted by b. We then say that (r_1^t, r_2^t) is revealed preferred to (r_1^b, r_2^b). The same is true if we choose the b time period prices as weights contained in (1.56). We then write

$$\frac{c_1^b}{c_1^b + c_2^b}\, r_1^t + \frac{c_2^b}{c_1^b + c_2^b}\, r_2^t > \frac{c_1^b}{c_1^b + c_2^b}\, r_1^b + \frac{c_2^b}{c_1^b + c_2^b}\, r_2^b. \tag{1.58}$$

On the other hand, if I_q is less than 1 then it turns out to be

$$U(r_1^t, r_2^t) < U(r_1^b, r_2^b). \tag{1.59}$$

We consequently write

$$w_1\, r_1^t + w_2\, r_2^t < w_1\, r_1^b + w_2\, r_2^b, \tag{1.60}$$

so (r_1^b, r_2^b) is revealed preferred to (r_1^t, r_2^t).

We note the following

Remark 10. The decision-maker is better off at the time period denoted by t than at the time period denoted by b if and only if average consumption at the time period denoted by t is greater than the one at the time period denoted by b. Conversely, she is better off at the time period denoted by b than at the time period denoted by t if and only if average consumption at the time period denoted by b is greater than the one at the time period denoted by t. □

Remark 11. A balanced sequence of pairs denoted by $(r_1^b, r_2^b), (r_1^t, r_2^t)$ has the downward-sloping demand property. □

Chapter 2

Distributions of mass transferred on straight lines: reductions of dimension

2.1 Introduction

The n states of the world of a contingent consumption plan are points (real numbers) in the space of random goods (random quantities), where the latter is an n-dimensional linear space over $\mathbb{R}$ denoted by E^n. In the case of states of the world (elementary events) of a contingent consumption plan, the most useful arguments are not available if one thinks in terms of the set of them without reference to the linear space (space of alternatives) in which this set is naturally embedded and in which it is necessary to see it embedded. In this chapter, two n-dimensional linear spaces are superposed by using an orthonormal basis of E^n. All possible consumption levels associated with a random good are obtained by using homogeneous linear combinations of a finite number of states of the world of a contingent consumption plan.

If the two-good assumption holds then we deal with two marginal random goods denoted by X_1 and X_2, where each of them has n possible values corresponding to n states of the world of a contingent consumption plan that is separately considered. Each of them has n possible alternatives. The n possible values for each good under consideration are transferred on two one-dimensional straight lines on which an origin, a same unit of length, and an orientation are established. Such lines are the two axes of a two-dimensional Cartesian coordinate system. The space where the decision-maker chooses is an uncountable subset of a two-dimensional linear space over $\mathbb{R}$. It is her budget set. Her budget set contains points whose number is infinite. It contains infinite coherent previsions, where each mass is associated with a possible consumption level whose nature is objective. We say that the budget set of the decision-maker contains infinite bilinear previsions associated with a joint random good denoted by $X_1 X_2$ and infinite linear previsions associated with two marginal random goods denoted by X_1 and X_2. We say that X_1 and X_2 always give rise to $X_1 X_2$. Each bilinear prevision is denoted by $\mathbf{P}(X_1 X_2)$, where $\mathbf{P}(X_1 X_2)$ is steadily decomposed into two linear previsions denoted by $\mathbf{P}(X_1)$ and $\mathbf{P}(X_2)$ respectively. After transferring the n possible values for each good under consideration on two one-dimensional straight lines, we observe a reduction of dimension inside of the budget set of the decision-maker as well. We pass from $\mathbf{P}(X_1 X_2)$ to $\mathbf{P}(X_1)$ and $\mathbf{P}(X_2)$ respectively. It is clear that $\mathbf{P}(X_1 X_2)$ is a point of a two-dimensional convex set obtained by summarizing n^2 two-dimensional points, whereas $\mathbf{P}(X_1)$ is a point of a one-dimensional convex set obtained by summarizing n one-dimensional points. The same is true with respect to $\mathbf{P}(X_2)$. Whenever we say that $\mathbf{P}(X_1 X_2)$ is a point obtained by summarizing n^2 two-dimensional points by means of n^2 non-negative and finitely additive masses, we express it in the form given by $(\mathbf{P}(X_1), \mathbf{P}(X_2))$.

2.2 A random good and its linear nature

Let $\mathscr{B}_n^{\perp} = \{\mathbf{e}_i\}$, $i = 1, \ldots, n$, be an orthonormal basis of E^n, where E^n is an n-dimensional linear space over $\mathbb{R}$. E^n has a Euclidean structure. If $\mathbf{x} \in E^n$ then it is possible to write

$$\mathbf{x} = x^i \mathbf{e}_i \tag{2.1}$$

by using the Einstein summation convention. We note that $\{x^i\}$ is the set of all contravariant components of $\mathbf{x}$. They are uniquely determined with respect to $\mathscr{B}_n^{\perp}$. Given $\mathscr{B}_n^{\perp}$, they uniquely identify $\mathbf{x}$. If we write

$$\mathbf{x} = (x^1, x^2, \ldots, x^n) \tag{2.2}$$

then (2.2) is an n-dimensional consumption vector belonging to E^n (where E^n is the space of alternatives) whose contravariant components represent different and possible consumption levels (see also [80]). A possible consumption level coincides with a state of the world of a contingent consumption plan. Each contravariant component of $\mathbf{x}$ is a possible value for X, where X is a random good. The set of all possible values for X is denoted by $I(X) = \{x^1, \ldots, x^n\}$, where we have $x^1 < \ldots < x^n$ without loss of generality because a finite partition of mutually exclusive states of the world of a contingent consumption plan is considered. Let E_i, $i = 1, \ldots, n$, be the generic state of the world of a contingent consumption plan. Since it is an unequivocally individuated proposition identified with a real number and susceptible of being either true or false at the right time, we write

$$X = x^1|E_1| + x^2|E_2| + \ldots + x^n|E_n|, \tag{2.3}$$

where we have

$$|E_i| = \begin{cases} 1, & \text{if } E_i \text{ is true} \\ 0, & \text{if } E_i \text{ is false} \end{cases} \tag{2.4}$$

for every $i = 1, \ldots, n$. It is clear that X is linearly dependent on n states of the world, where linear dependence is a special case of logical dependence. Indeed, linear dependence is more restrictive than logical dependence. Logical dependence of one random entity on others has the same meaning that it has in mathematical analysis with respect to a one-valued function of several variables. Accordingly, X is also "a posteriori" logically dependent on $|E_i|$, $i = 1, \ldots, n$. All possible linear combinations expressed by (2.3) give rise to different random goods identifying a linear space denoted by E^{n*}. It is dual to E^n. Such dual spaces are superposed by means of a quadratic metric introduced by considering an orthonormal basis of E^n. This is because we use the notion of scalar or inner product in order to say that an orthonormal basis of E^n consists of unit vectors and orthogonal to each other. All possible consumption levels of a contingent consumption plan are also "a priori" logically independent because they are all uncertain or possible.

If uncertainty about a state of the world of a contingent consumption plan does not cease then it makes sense that the evaluation of probability associated with all possible consumption levels is made by the decision-maker. Accordingly, a nonparametric distribution of mass is associated with a contingent consumption plan. It is viewed to be as an expression of the attitude characterizing a given decision-maker with regard to all uncertain consumption levels belonging to $I(X)$. We do not think of a distribution of mass as already attached to X. Such a distribution can vary from decision-maker to decision-maker (see also [61]). It can also vary with respect to her state of information and knowledge. We write it in the form of a finite sequence expressed by

$$(x^1, p_1), (x^2, p_2), \ldots, (x^n, p_n).$$

X is a random good such that x^1 is the possible consumption level associated with it if E_1 occurs with probability denoted by p_1, ..., x^n is the possible consumption level associated with it if E_n occurs with probability denoted by p_n. Since E_1 occurs or does not occur only when uncertainty ceases, ..., E_n occurs or does not occur only when uncertainty ceases, further consumption levels have to be taken into account in addition to the starting ones. They are average consumption levels.

2.3 Two n-dimensional linear spaces which are superposed from a metric point of view

Let Φ^j be a linear function such that it is possible to write

$$\Phi^j \colon E^n \to \mathbb{R}. \tag{2.5}$$

If $\mathbf{t} \in E^n$ then we write

$$\mathbf{t} = t^j \mathbf{e}_j. \tag{2.6}$$

Hence, it turns out to be

$$\Phi^j(\mathbf{t}) = t^j, \quad j = 1, \ldots, n. \tag{2.7}$$

It follows that we obtain

$$\Phi^j(\mathbf{e}_i) = \delta_i^j, \tag{2.8}$$

where δ_i^j is the Kronecker delta, so we have $\delta_i^j = 1$ if $i = j$ and $\delta_i^j = 0$ if $i \neq j$. Thus, it is possible to show that $\{\Phi^j\}$, $j = 1, \ldots, n$, is a basis of E^{n*}, where E^{n*} is the dual space of E^n. This means that we write

$$\Phi = u_1 \Phi^1 + u_2 \Phi^2 + \ldots + u_n \Phi^n, \tag{2.9}$$

with $\Phi \in E^{n*}$. Let F be a linear function such that we write

$$F \colon E^n \to \mathbb{R}. \tag{2.10}$$

We consequently obtain

$$F(\mathbf{x}) = F(x^j \mathbf{e}_j) = x^j F(\mathbf{e}_j), \quad \mathbf{x} \in E^n. \tag{2.11}$$

Given (2.7), it turns out to be

$$F(\mathbf{x}) = F(\mathbf{e}_j)\Phi^j(\mathbf{x}) = \left[F(\mathbf{e}_j)\Phi^j\right](\mathbf{x}), \quad \mathbf{x} \in E^n, \tag{2.12}$$

where (2.12) is valid for every $\mathbf{x} \in E^n$. It follows that F and $F(\mathbf{e}_j)\Phi^j$ are linear functions that coincide, so we write

$$F = F(\mathbf{e}_j)\Phi^j, \tag{2.13}$$

where it turns out to be $F(\mathbf{e}_j) = u_j \in \mathbb{R}$. We therefore say that $\{\Phi^j\}$, $j = 1, \ldots, n$, is a basis of E^{n*}. Its elements span E^{n*}. Indeed, we write

$$\Phi(\mathbf{x}) = u_1\Phi^1(\mathbf{x}) + u_2\Phi^2(\mathbf{x}) + \ldots + u_n\Phi^n(\mathbf{x}) = u_1 x^1 + u_2 x^2 + \ldots + u_n x^n, \quad \mathbf{x} \in E^n, \tag{2.14}$$

with $\Phi \in E^{n*}$, where (2.14) denotes the scalar or inner product of two n-dimensional vectors. The former is denoted by $\mathbf{x} = (x^1, x^2, \ldots, x^n) \in E^n$, whereas the latter is denoted by $\mathbf{u} = (u_1, u_2, \ldots, u_n) \in E^{n*}$.

We have to show that $\Phi^1, \Phi^2, \ldots, \Phi^n$ are linearly independent. We suppose that it is possible to write

$$\alpha_j\Phi^j = 0, \tag{2.15}$$

where it turns out to be $\alpha_j \in \mathbb{R}$. From (2.15) it follows that we can write

$$\alpha_j\Phi^j(\mathbf{x}) = 0, \quad \forall\mathbf{x} \in E^n. \tag{2.16}$$

In particular, if we choose $\mathbf{x} = \mathbf{e}_k$ then we write

$$\alpha_j\Phi^j(\mathbf{e}_k) = \alpha_k = 0 \tag{2.17}$$

because we refer ourselves to the expression given by (2.8). Since (2.16) is valid for every $\mathbf{x} \in E^n$, we observe that (2.17) is true for every $\mathbf{e}_j \in \mathscr{B}_n^{\perp}$, $j = 1, \ldots, n$. This means that we obtain

$$\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0, \tag{2.18}$$

so we prove that all linear functions of the set $\{\Phi^j\}$ are linearly independent. They represent a basis of E^{n*}. We then write

$$\dim E^n = \dim E^{n*} = n. \tag{2.19}$$

2.4 How to obtain all possible consumption levels

Let $\mathscr{B}_n^{\perp}$ be an orthonormal basis of E^n. In general, it is possible to study n quantitative states of the world of a contingent consumption plan inside of E^n because they coincide with n real numbers. These n real numbers are the contravariant components of an n-dimensional vector of E^n. If there is no uncertainty about n states of the world of a contingent consumption plan then the real coefficients of each linear combination of n basis vectors belonging to $\mathscr{B}_n^{\perp}$ coincide with 0 or 1 only (see also [32]). Let $\mathscr{A} \subset E^n$ be the set of elements denoted by $\mathbf{a}$. They are n-dimensional vectors having their contravariant components all equal to 0 or 1 only. We write

$$\mathbf{a} = a^j\mathbf{e}_j, \tag{2.20}$$

where if $a^j = 1$ and $a^i = 0$ ($\forall i \neq j$) then E_j corresponding to $\mathbf{e}_j$ is true, whereas all others are false. Given n states of the world of a contingent consumption plan generically denoted by $E_1, E_2, \ldots, E_n$, we note that $\mathscr{A} \subset E^n$ contains all the constituents of $E_1, E_2, \ldots, E_n$ whose number is at most equal to $k = 2^n$. In general, each constituent of $E_1, E_2, \ldots, E_n$ is nothing but a state of the world obtained through the logical product concerning E_1 or its negation denoted by $\bar{E}_1$, E_2 or its negation denoted by $\bar{E}_2$, ..., E_n or its negation denoted by $\bar{E}_n$. This product having n factors coincides with the arithmetic one. For instance,

$$E_1 E_2 \ldots E_n,$$

$$\bar{E}_1 \bar{E}_2 \ldots \bar{E}_n,$$

and

$$\bar{E}_1 E_2 \ldots E_n$$

are products identifying some constituents of $E_1, E_2, \ldots, E_n$. In general, all possible constituents of $E_1, E_2, \ldots, E_n$, whose number is equal to $k \leq 2^n$, give rise to a finite partition of incompatible and exhaustive states of the world of a contingent consumption plan. In particular, if $E_1, E_2, \ldots, E_n$ are incompatible and exhaustive consumption levels associated with a random good denoted by X then it turns out to be $k = n$. Hence, with respect to the elements of $\mathscr{A} \subset E^n$, it is possible to define the following linear function

$$F(\mathbf{a}) = u_j \Phi^j(\mathbf{a}) = u_1 a^1 + u_2 a^2 + \ldots + u_n a^n, \tag{2.21}$$

where it turns out to be $F(\mathbf{a}) \in E^{n*}$. It represents the possible value for X which is associated with $\mathbf{a}$. In particular, we denote by u_j the possible value for X which is associated with $\mathbf{e}_j$ to which $\mathbf{a}$ reduces when and only when E_j is true. In general, (2.21) tells us that consumption levels which are associated with the occurrence of different states of the world of a contingent consumption plan are additive. From (2.21) and (2.8) it follows that it is possible to write

$$F(\mathbf{e}_k) = u_j \Phi^j(\mathbf{e}_k) = u_j \delta_k^j = u_k, \tag{2.22}$$

with $u_k \in \mathbb{R}$. On the other hand, F is a linear map, so it is determined whenever we know its value on basis elements. Hence, u_j is a real number being determined by the decision-maker. It objectively depends on her state of information and knowledge. Since $\mathbf{a} \in \mathscr{A}$ is a random vector, $F(\mathbf{a})$ is a scalar or inner product representing all possible values for X as the contravariant components of $\mathbf{a} \in \mathscr{A}$ vary by taking 0 and 1 only. The two dual spaces denoted by E^n and E^{n*} are superposed, so $F(\mathbf{a})$ identifies homogeneous linear combinations. The number of the possible and different values for $F(\mathbf{a})$, $\mathbf{a} \in \mathscr{A}$, is overall equal to n. Thus, it turns out to be

$$u_1 \neq u_2 \neq \ldots \neq u_n. \tag{2.23}$$

We denote them by $b_1, b_2, \ldots, b_n$, so we write

$$(u_1 = b_1) \neq (u_2 = b_2) \neq \ldots \neq (u_n = b_n). \tag{2.24}$$

On the other hand, they exactly correspond to $x^1, x^2, \ldots, x^n$, so we observe $b_i = x^i$, $i = 1, \ldots, n$. This is because if an orthonormal basis of E^n is considered then the contravariant and covariant components of a same vector of E^n coincide. Thus, we write

$$\mathscr{S}_r = \{\mathbf{a} \in \mathscr{A} \,|\, F(\mathbf{a}) = b_r\}, \tag{2.25}$$

so it turns out to be

$$\mathscr{A} = \bigcup_{r=1}^{n} \mathscr{S}_r \tag{2.26}$$

as well as

$$\mathscr{S}_r \cap \mathscr{S}_t = \emptyset, \quad r \neq t. \tag{2.27}$$

If we extend $\mathscr{A}$ to E^n then (2.21) is an expression of a hyperplane embedded in E^n. We rewrite it in the following form

$$u_1 a^1 + u_2 a^2 + \ldots + u_n a^n,$$

so different n-dimensional vectors denoted by $\mathbf{a} \in E^n$ give rise to different values for it denoted by $b_1, b_2, \ldots, b_n$. Each b_i, $i = 1, \ldots, n$, represents a hyperplane of consumption.

2.5 Convex combinations of possible consumption levels

The set of n possible values for X is embedded in an n-dimensional linear space over $\mathbb{R}$ provided with a quadratic metric (see also [8]). All its possible values can be expressed by means of two different and possible values for it in the form of a convex combination after transferring them on a one-dimensional straight line on which an origin, a unit of length, and an orientation are chosen. Given two points denoted by A and B belonging to a one-dimensional straight line and identifying two different states of the world of a contingent consumption plan, every point P belonging to the same one-dimensional straight line and identifying another state of the world of the same contingent consumption plan is expressed by

$$P = tA + (1-t)B, \tag{2.28}$$

where it turns out to be $0 \leq t \leq 1$. We deal with a linear combination of two different points associated with two different states of the world of the same contingent consumption plan. It is a convex combination, so we note

$$t + (1-t) = 1. \tag{2.29}$$

How to transfer all the n possible values for X on a one-dimensional straight line on which an origin, a unit of length, and an orientation are chosen is proved by the following

Theorem 2. Let $\mathscr{B}_n^{\perp} = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be an orthonormal basis of E^n and let $b_1, b_2, \ldots, b_n$ be the possible values for a random good denoted by X. If each possible value for it is obtained by means of a homogeneous linear combination of n states of the world of a contingent consumption plan then every possible value for X is expressed, with respect to a one-dimensional straight line, as a convex combination of two different and possible values for X.

Proof. We firstly note that an n-dimensional located vector at the origin of E^n is entirely determined by its end point. It is then possible to call an ordered n-tuple of real numbers either a vector of E^n or a point of $\mathscr{E}^n$ (affine space). It follows that E^n and $\mathscr{E}^n$ are isomorphic. This means that there exists a one-to-one correspondence between the vectors of E^n and the points of $\mathscr{E}^n$. If we write

$$F(\mathbf{x}) = u_j \Phi^j(\mathbf{x}) = b_r, \quad \mathbf{x} \in E^n, \tag{2.30}$$

then it turns out to be $F(\mathbf{x}) \in E^{n*}$. This means that (2.30) is a hyperplane embedded in E^n. It is also a hyperplane embedded in $\mathscr{E}^n$. On the other hand, if we write

$$F(\mathbf{a}) = u_j \Phi^j(\mathbf{a}) = b_r, \quad \mathbf{a} \in \mathscr{A}, \tag{2.31}$$

then it turns out to be $F(\mathbf{a}) \in E^{n*}$. This means that (2.31) is a hyperplane embedded in E^n. It is also a hyperplane embedded in $\mathscr{E}^n$. We observe that (2.30) and (2.31) are characterized by the same possible value for X. Let

$$\mathbf{u} = u^j \mathbf{e}_j \in E^n \tag{2.32}$$

be an n-dimensional vector and let $\rho_0 \in \mathscr{E}^n$ be that straight line whose geometric nature is given by

$$\{\lambda\, \mathbf{u} \,|\, \forall \lambda \in \mathbb{R}\}. \tag{2.33}$$

The straight line given by (2.33) is orthogonal to all hyperplanes established by (2.30). They are obtained as b_r varies, $r = 1, \ldots, n$. In particular, (2.33) is orthogonal to the hyperplane given by $F(\mathbf{x}) = 0$. It passes through the point of $\mathscr{E}^n$ whose coordinates are all equal to 0. It follows that (2.30) identifies a sheaf of parallel hyperplanes. Given two vectors of E^n expressed by

$$\mathbf{x}' = \lambda'\, \mathbf{u} + \mathbf{x}'_0 \tag{2.34}$$

and

$$\mathbf{x}'' = \lambda''\, \mathbf{u} + \mathbf{x}''_0, \tag{2.35}$$

we note that the parallel components to the vector $\mathbf{u} \in E^n$ of the vectors $\mathbf{x}'$ and $\mathbf{x}''$ are respectively $\lambda'\, \mathbf{u}$ and $\lambda''\, \mathbf{u}$, whereas $\mathbf{x}'_0$ and $\mathbf{x}''_0$ are the orthogonal components to $\mathbf{u} \in E^n$ of $\mathbf{x}'$ and $\mathbf{x}''$. We observe that (2.34) and (2.35) can be viewed as two n-dimensional located vectors at the origin of E^n whose end points belong to two hyperplanes, where each of them is expressed by (2.30). Their value is respectively given by $b_{r'}$ and $b_{r''}$, so it turns out to be

$$F(\mathbf{x}') = \left\langle \mathbf{u}, \mathbf{x}' \right\rangle = b_{r'} \tag{2.36}$$

as well as

$$F(\mathbf{x}'') = \left\langle \mathbf{u}, \mathbf{x}'' \right\rangle = b_{r''}. \tag{2.37}$$

We have evidently

$$F(\mathbf{x}') = \left\langle \mathbf{u}, \mathbf{x}' \right\rangle = \left\langle \mathbf{u}, \lambda'\, \mathbf{u} + \mathbf{x}'_0 \right\rangle = \lambda'\, \|\mathbf{u}\|^2 + \left\langle \mathbf{u}, \mathbf{x}'_0 \right\rangle = b_{r'} \tag{2.38}$$

as well as

$$F(\mathbf{x}'') = \left\langle \mathbf{u}, \mathbf{x}'' \right\rangle = \left\langle \mathbf{u}, \lambda''\, \mathbf{u} + \mathbf{x}''_0 \right\rangle = \lambda''\, \|\mathbf{u}\|^2 + \left\langle \mathbf{u}, \mathbf{x}''_0 \right\rangle = b_{r''}. \tag{2.39}$$

Since it turns out to be $\langle \mathbf{u}, \mathbf{x}'_0 \rangle = 0$ as well as $\langle \mathbf{u}, \mathbf{x}''_0 \rangle = 0$, we note that it is possible to write

$$\lambda'\, \|\mathbf{u}\|^2 = b_{r'} \tag{2.40}$$

and

$$\lambda''\, \|\mathbf{u}\|^2 = b_{r''}. \tag{2.41}$$

In general, all points of a hyperplane characterized by the same value denoted by b_r can be summarized by using the intersection of it and the straight line denoted by $\rho_0 \in \mathscr{E}^n$. Such an intersection coincides with the real number given by

$$\lambda = \frac{b_r}{||\mathbf{u}||^2}. \tag{2.42}$$

This means that the orthogonal component of the vectors $\mathbf{x} \in E^n$ and $\mathbf{a} \in \mathscr{A}$ is insignificant with respect to the straight line denoted by $\rho_0 \in \mathscr{E}^n$. Hence, we refer ourselves to such a line instead of different hyperplanes. It is consequently evident that every point belonging to $\rho_0 \in \mathscr{E}^n$ can be expressed as a convex combination of two different points belonging to it. Given three different possible values for X denoted by $b_{r'}$, $b_{r''}$, and $b_{r'''}$, we write

$$\mathbf{x}_1' = \lambda' \mathbf{u}, \tag{2.43}$$

$$\mathbf{x}_1'' = \lambda'' \mathbf{u}, \tag{2.44}$$

and

$$\mathbf{x}_1''' = \lambda''' \mathbf{u}. \tag{2.45}$$

It is then clear that it turns out to be $F(\mathbf{x}_1') = b_{r'}$, $F(\mathbf{x}_1'') = b_{r''}$, $F(\mathbf{x}_1''') = b_{r'''}$. We obtain the following expression

$$\lambda'' \mathbf{u} = t\,\lambda' \mathbf{u} + (1-t)\,\lambda''' \mathbf{u}, \tag{2.46}$$

with $0 \leq t \leq 1$. We take (2.42) into account, so we write

$$b_{r''}\,\mathbf{u} = t\,b_{r'}\,\mathbf{u} + (1-t)\,b_{r'''}\,\mathbf{u} \tag{2.47}$$

after multiplying by $||\mathbf{u}||^2$ both sides of (2.46). After dividing by $\mathbf{u}$ both sides of (2.47), we write

$$b_{r''} = t\,b_{r'} + (1-t)\,b_{r'''}, \tag{2.48}$$

so it turns out to be

$$t = \frac{b_{r''} - b_{r'''}}{b_{r'} - b_{r'''}}. \tag{2.49}$$

We note that (2.48) shows the stated property concerning the one-dimensional points identifying the possible values for X. □

We note the following

Remark 12. There exists a one-to-one correspondence between the elements of a sheaf of parallel hyperplanes and the points of intersection of them and a straight line denoted by $\rho_0 \in \mathscr{E}^n$. Given X, since the decision-maker does not know which possible value for it belonging to $I(X)$ will be true at the right time, she focuses on one of n axes of an n-dimensional Cartesian coordinate system. They are pairwise orthogonal. She considers all collinear vectors with respect to one of n basis vectors. Such collinear vectors give rise to $\rho_0 \in \mathscr{E}^n$, where ρ_0 is orthogonal to all parallel hyperplanes under consideration. The points of intersection of all these parallel hyperplanes and a straight line denoted by $\rho_0 \in \mathscr{E}^n$ are real numbers transferred on a one-dimensional straight line on which an origin, a unit of length, and an orientation are established. They coincide with all possible values for X belonging to $I(X)$. After focusing on another axis of n axes of an n-dimensional Cartesian coordinate system, if the decision-maker considers all collinear vectors with respect to another vector of n basis vectors then the same possible values for X belonging to $I(X)$ are obtained. □

Remark 13. The dual space of E^n denoted by E^{n*} contains all those random goods obtained by considering all possible homogeneous linear combinations of n states of the world generically denoted by $E_1, E_2, \ldots, E_n$ of a contingent consumption plan. If we write

$$X = b_1|E_1| + b_2|E_2| + \ldots + b_n|E_n|,$$

where it turns out to be

$$|E_i| = \begin{cases} 1, & \text{if } E_i \text{ is true} \\ 0, & \text{if } E_i \text{ is false} \end{cases}$$

for every $i = 1, \ldots, n$, then X is that random good such that its possible and different values coincide with $I(X) = \{b_1, b_2, \ldots, b_n\}$. They are found on distinct hyperplanes expressed by

$$b_i\, a^i = \text{constant},$$

where $b_1, b_2, \ldots, b_n$ are coordinates of points of E^{n*}, whereas $a^1, a^2, \ldots, a^n$ are components of vectors of E^n, with the values identifying each a^i, $i = 1, \ldots, n$, that coincide with 0 or 1 only. □

Remark 14. Since E^n is a linear space over $\mathbb{R}$ having a Euclidean structure, E^n and E^{n*} coincide. The contravariant and covariant components of a same n-dimensional vector coincide because an orthonormal basis of E^n is considered. Points of E^{n*} and vectors of E^n can be identified because the origin given by the n-tuple denoted by $(0, 0, \ldots, 0)$ has meaning in both of them. After noting that it turns out to be $u_i = b_i$, $i = 1, \ldots, n$, we write $b_i = x^i$, $i = 1, \ldots, n$. This means that all possible values for X are given by $I(X) = \{b_1, \ldots, b_n\} = \{x^1, \ldots, x^n\}$. □

2.6 A reduction of dimension characterizing the budget set of the decision-maker

2.6.1 Contravariant and covariant components of vectors and tensors

If X_1 and X_2 are two random goods, where the number of the possible values for each of them is equal to n, then they identify two contingent consumption plans. Each contingent consumption plan is uniquely individuated by an n-dimensional vector of E^n, where we have $\dim E^n = n$. Two contingent consumption plans identify a joint contingent consumption plan as well. A joint contingent consumption plan is uniquely individuated by an affine tensor of order 2 belonging to $E^n \otimes E^n$, where we have $\dim(E^n \otimes E^n) = n^2$. It is clear that it turns out to be

$$\dim E^n \neq \dim(E^n \otimes E^n),$$

where $n \geq 2$ is an integer. Given an orthonormal basis of E^n, the contravariant components of ${}_{(1)}\mathbf{x}$ expressed by

$${}_{(1)}\mathbf{x} = {}_{(1)}x^i\, \mathbf{e}_i$$

uniquely represent the possible values for X_1. We use the Einstein notation. There exists one and only one set of contravariant components of ${}_{(1)}\mathbf{x}$ with regard to an orthonormal basis of E^n. Even though we use a contravariant notation, the contravariant and covariant components of a same vector of E^n coincide whenever an orthonormal basis of E^n is considered. The same is true with regard to the contravariant components of ${}_{(2)}\mathbf{x}$ expressed by

$$ {}_{(2)}\mathbf{x} = {}_{(2)}x^j \, \mathbf{e}_j $$

whose number is overall equal to n. Given

$$ {}_{(1)}\mathbf{p} = {}_{(1)}p_i \, \mathbf{e}^i, $$

where the covariant components of ${}_{(1)}\mathbf{p}$ identify all the masses associated with all possible values for X_1, all coherent summaries of the possible values for X_1 denoted by $\mathbf{P}(X_1)$ are expressed by

$$ \mathbf{P}(X_1) = {}_{(1)}x^i \; {}_{(1)}p_i \, . \tag{2.50} $$

Each mass associated with a possible value for X_1 can take all values between 0 and 1, end points included, into account. The number of these values is infinite. Even though we use a covariant notation, the contravariant and covariant components of a same vector of E^n coincide whenever an orthonormal basis of E^n is considered. Given

$$ {}_{(2)}\mathbf{p} = {}_{(2)}p_j \, \mathbf{e}^j, $$

where the covariant components of ${}_{(2)}\mathbf{p}$ identify all the masses associated with all possible values for X_2, all coherent summaries of the possible values for X_2 denoted by $\mathbf{P}(X_2)$ are given by

$$ \mathbf{P}(X_2) = {}_{(2)}x^i \; {}_{(2)}p_i \, . \tag{2.51} $$

Each mass associated with a possible value for X_2 can take all values between 0 and 1, end points included, into account. The number of these values is infinite. We note that $\mathbf{P}(X_1)$ and $\mathbf{P}(X_2)$ are two scalar or inner products of two n-dimensional vectors belonging to the same linear space over $\mathbb{R}$. This is because E^n and E^{n*} coincide. Given the two affine tensors of order 2 expressed by

$$ T = {}_{(1)}x^i \, {}_{(2)}x^j \, \mathbf{e}_i \otimes \mathbf{e}_j $$

and

$$ P = p_{ij} \, \mathbf{e}^i \otimes \mathbf{e}^j, $$

where each of them has n^2 components, all coherent summaries of the possible values for $X_1 X_2$ denoted by $\mathbf{P}(X_1 X_2)$ are expressed by

$$ \mathbf{P}(X_1 X_2) = {}_{(1)}x^i \, {}_{(2)}x^j \; p_{ij}. \tag{2.52} $$

Each mass associated with a possible value for $X_1 X_2$ belonging to $I(X_1) \times I(X_2)$ can take all values between 0 and 1, end points included, into account. The number of these values is infinite. Given an orthonormal basis of E^n, the contravariant and covariant components of a same affine tensor of order 2 coincide whenever a basis of $E^n \otimes E^n$ is considered. Accordingly, if we use a covariant notation then the components of T are expressed by the same numbers. If we use a contravariant notation then the components of P are expressed by the same numbers.

2.6.2 A metric notion: α-product

Two marginal random goods denoted by X_1 and X_2 always give rise to a joint random good denoted by $X_1 X_2$. All its possible values are obtained by considering the Cartesian product of the possible values for X_1 and X_2. They belong to $I(X_1) = \{{}_{(1)}x^1, \ldots, {}_{(1)}x^n\}$ and $I(X_2) = \{{}_{(2)}x^1, \ldots, {}_{(2)}x^n\}$ respectively, so their Cartesian product is given by $I(X_1) \times I(X_2)$. The values of $I(X_1)$ and $I(X_2)$ coincide with the contravariant components of two n-dimensional vectors. The covariant components of an affine tensor of order 2 represent the joint probabilities of the joint distribution of X_1 and X_2. Their number is overall equal to n^2. We pass from n^2 ordered pairs of numbers to 1 ordered pair of numbers by using the notion of α-product between ${}_{(1)}\mathbf{x}$ and ${}_{(2)}\mathbf{x}$. It is a scalar or inner product obtained by using the joint probabilities denoted by p_{ij} of the joint distribution of X_1 and X_2 together with the contravariant components of ${}_{(1)}\mathbf{x}$ and ${}_{(2)}\mathbf{x}$. For instance, from the table

random good 2 / random good 1	0	4	5	Sum
0	0	0	0	0
2	0	0.1	0.2	0.3
3	0	0.5	0.2	0.7
Sum	0	0.6	0.4	1

it follows that it turns out to be $\mathbf{P}(X_1 X_2) = 11.8$. Given the contravariant components of ${}_{(2)}\mathbf{x}$ identifying the following column vector

$$\begin{pmatrix} 0 \\ 4 \\ 5 \end{pmatrix},$$

its covariant components are expressed by

$$0 \cdot 0 + 4 \cdot 0 + 5 \cdot 0 = 0,$$

$$0 \cdot 0 + 4 \cdot 0.1 + 5 \cdot 0.2 = 1.4,$$

$$0 \cdot 0 + 4 \cdot 0.5 + 5 \cdot 0.2 = 3,$$

so it is possible to write the following result

$$\left\langle \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1.4 \\ 3 \end{pmatrix} \right\rangle = \langle {}_{(1)}\mathbf{x}, {}_{(2)}\mathbf{x} \rangle_\alpha = \mathbf{P}(X_1 X_2) = 11.8.$$

On the other hand, after calculating the covariant components of ${}_{(1)}\mathbf{x}$ in a similar way, we write

$$\left\langle \begin{pmatrix} 0 \\ 1.7 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 5 \end{pmatrix} \right\rangle = \langle {}_{(1)}\mathbf{x}, {}_{(2)}\mathbf{x} \rangle_\alpha = \mathbf{P}(X_1 X_2) = 11.8.$$

It is clear that $\mathbf{P}(X_1 X_2)$ is a notion of a metric nature. Since we pass from n^2 ordered pairs to 1 ordered pair after using n^2 non-negative masses, we write

$$(\mathbf{P}(X_1), \mathbf{P}(X_2))$$

in order to identify $\mathbf{P}(X_1 X_2)$. This is because $\mathbf{P}(X_1 X_2)$ is always decomposed into two linear previsions. The notion of α-norm is a particular case of the one of α-product. From the table

random good 1 \ random good 1	0	2	3	Sum
0	0	0	0	0
2	0	0.3	0	0.3
3	0	0	0.7	0.7
Sum	0	0.3	0.7	1

it follows that it turns out to be $\|_{(1)}\mathbf{x}\|_{\alpha}^{2} = \mathbf{P}(X_1 X_1) = 7.5$, whereas from the table

random good 2 \ random good 2	0	4	5	Sum
0	0	0	0	0
4	0	0.6	0	0.6
5	0	0	0.4	0.4
Sum	0	0.6	0.4	1

it follows that it is possible to write $\|_{(2)}\mathbf{x}\|_{\alpha}^{2} = \mathbf{P}(X_2 X_2) = 19.6$.

2.6.3 Coherent summaries of nonparametric distributions of mass transferred on straight lines

All the n states of the world of a contingent consumption plan are transferred on a one-dimensional straight line on which an origin, a unit of length, and an orientation are chosen. We observe a reduction of dimension because we pass from n to 1. We do not consider an n-dimensional point referred to a random good, but we study a finite set of n one-dimensional points. We do not deal with n masses associated with n possible states of the world of a contingent consumption plan yet. We focus on the two-good assumption, so X_1 and X_2 are two marginal random goods. Each of them has n possible consumption levels. The n possible values for each good under consideration are transferred on two one-dimensional straight lines on which an origin, a same unit of length, and an orientation are established. Such lines are the two axes of a two-dimensional Cartesian coordinate system.

The space where the decision-maker chooses is her budget set. If we take her budget set into account then all masses associated with all possible consumption levels come into play. Her budget set is an uncountable subset of a two-dimensional linear space over $\mathbb{R}$. Her budget set contains points whose number is infinite. It is a right triangle belonging to the first quadrant of a two-dimensional Cartesian coordinate system. The point given by $(0,0)$ identifies its right angle, whereas the budget line whose slope is negative identifies its hypotenuse. Her budget set contains infinite coherent bilinear previsions associated with a joint random good denoted by $X_1 X_2$ and infinite coherent linear previsions associated with two marginal random goods denoted by X_1 and X_2. Each bilinear prevision is denoted by $\mathbf{P}(X_1 X_2)$, where $\mathbf{P}(X_1 X_2)$ is always decomposed into two linear previsions denoted by $\mathbf{P}(X_1)$ and $\mathbf{P}(X_2)$ respectively. The decision-maker chooses one bilinear prevision denoted by $\mathbf{P}(X_1 X_2)$ among infinite coherent bilinear previsions. This is her rational choice. Indeed, she chooses a bundle of two random goods operationally identified with $\mathbf{P}(X_1 X_2)$.

Since $\mathbf{P}(X_1 X_2)$ belongs to a two-dimensional convex set, we express it in the form given by $(\mathbf{P}(X_1), \mathbf{P}(X_2))$. Accordingly, the decision-maker also chooses $\mathbf{P}(X_1)$ and $\mathbf{P}(X_2)$ because $\mathbf{P}(X_1 X_2)$ is always decomposed into $\mathbf{P}(X_1)$ and $\mathbf{P}(X_2)$ respectively. Hence, we secondly observe a reduction of dimension because we pass from 2 to 1. Indeed, we pass from $\mathbf{P}(X_1 X_2)$, where $\mathbf{P}(X_1 X_2)$ is found inside of an uncountable subset of a two-dimensional linear space over $\mathbb{R}$, to $\mathbf{P}(X_1)$ and $\mathbf{P}(X_2)$, where $\mathbf{P}(X_1)$ and $\mathbf{P}(X_2)$ are found on two different and mutually orthogonal one-dimensional straight lines. A joint distribution of mass gives rise to a continuous subset of $\mathbb{R} \times \mathbb{R}$. This is because all coherent previsions of a joint random good are considered. They are obtained by taking all values between 0 and 1, end points included, into account for each mass associated with a possible value for two random goods which are jointly considered. The number of these values is infinite. Two nonparametric marginal distributions of mass give rise to two continuous subsets of $\mathbb{R}$, where each of them identifies a line segment belonging to one of the two axes of a two-dimensional Cartesian coordinate system. This is because all coherent previsions of marginal random goods are considered. All coherent previsions of two marginal random goods identify the two catheti of the right triangle under consideration. Such previsions are obtained by taking all values between 0 and 1, end points included, into account for each mass associated with a possible consumption level concerning a random good. The number of these values is infinite.

Each point of the budget set of the decision-maker is a bilinear and disaggregate measure coinciding with a synthesized element of the Fréchet class.

2.6.4 The direct product of $\mathbb{R}$ and $\mathbb{R}$

We note that $\mathbb{R} \times \mathbb{R}$ is the direct product of $\mathbb{R}$ and $\mathbb{R}$, so $\mathbb{R} \times \mathbb{R}$ is a two-dimensional linear space over $\mathbb{R}$. The two-dimensional budget set of the decision-maker is an uncountable subset of $\mathbb{R} \times \mathbb{R}$ consisting of points whose number is infinite. Two half-lines are firstly considered instead of two one-dimensional straight lines. Each of them extends indefinitely in a positive direction from zero before being restricted. Two line segments belonging to these two half-lines are obtained whenever all coherent previsions of two marginal random goods are taken into account.

Formally, we observe the set of all pairs denoted by $(\mathbf{P}_0^+(X_1), \mathbf{P}_0^+(X_2))$ whose first component is an element $\mathbf{P}_0^+(X_1) \in \mathbb{R}_0^+$ and whose second component is an element $\mathbf{P}_0^+(X_2) \in \mathbb{R}_0^+$, where it turns out to be $\mathbf{P}_0^+(X_1) \geq 0$ as well as $\mathbf{P}_0^+(X_2) \geq 0$. The addition of such pairs works componentwise. Accordingly, if $(\mathbf{P}_0^+(X_1'), \mathbf{P}_0^+(X_2')) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+$ and $(\mathbf{P}_0^+(X_1''), \mathbf{P}_0^+(X_2'')) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+$ then it is possible to write

$$(\mathbf{P}_0^+(X_1'), \mathbf{P}_0^+(X_2')) + (\mathbf{P}_0^+(X_1''), \mathbf{P}_0^+(X_2'')) = (\mathbf{P}_0^+(X_1') + \mathbf{P}_0^+(X_1''), \mathbf{P}_0^+(X_2') + \mathbf{P}_0^+(X_2'')). \tag{2.53}$$

If $k \in \mathbb{R}$, the product given by $k\,(\mathbf{P}_0^+(X_1'), \mathbf{P}_0^+(X_2'))$ is written in the following form expressed by

$$k\,(\mathbf{P}_0^+(X_1'), \mathbf{P}_0^+(X_2')) = (k\,\mathbf{P}_0^+(X_1'), k\,\mathbf{P}_0^+(X_2')). \tag{2.54}$$

Chapter 3

Disaggregate and aggregate measures identifying multiple goods

3.1 Introduction

3.1.1 Choices based on disaggregate and aggregate measures

Given the two-good assumption, the objects of decision-maker choice are of a bilinear nature. They are studied by using bilinear measures. Such measures are firstly decomposed into two linear measures inside of an uncountable subset of a two-dimensional linear space over $\mathbb{R}$. They are disaggregate measures. We are not interested in knowing their objective elements only. We are interested in knowing their subjective elements as well. The decision-maker chooses ordinary or random goods inside of her budget set, so it is also possible to establish aggregate measures in order to study choices connected with multiple goods under conditions of certainty or uncertainty and riskiness. Aggregate measures are based on what the decision-maker chooses inside of her budget set. Their nature is bilinear in the case of multiple goods of order 2. They are studied outside of her budget set.

3.1.2 A finite partition of mutually exclusive states of the world of a contingent consumption plan

Given a family of n states of the world of a contingent consumption plan for which it is certain that one and only one of them will be true at the right time, a random good denoted by X and written in the form expressed by

$$X = x^1 |E_1| + x^2 |E_2| + \ldots + x^n |E_n|,$$

where $|E_i|$, $i = 1, \ldots, n$, coincides with 0 or 1, expresses a finite partition of mutually exclusive states of the world of a contingent consumption plan (see also [51]). If all possible and different numbers belonging to $I(X) = \{x^1, x^2, \ldots, x^n\}$, where we have $x^1 < x^2 < \ldots < x^n$, are subjected to a change of origin obtained by using a same real constant then X continues to be the same object from a randomness point of view.

A state of the world of a contingent consumption plan is not a measurable set, but it is a well-determined proposition coinciding with a real number. It is specified in such a way that a possible bet based upon it can be decided without question. The probability of a state of the world of a contingent consumption plan is not a first principle, but it is a practical notion of a relative and subjective nature. Indeed, it is firstly necessary to take all the circumstances which are known to be relevant at the time into account. These circumstances are secondly evaluated by the decision-maker considering them. The probability of a state of the world of a contingent consumption plan can be moved in whatever coherent way the decision-maker likes whenever she distributes a unit mass of probability among the different elements of a finite partition of them. Each random good can be assigned a nonparametric probability distribution expressing the attitude shown by a given decision-maker with regard to every uncertain or possible state of the world (elementary event) of a contingent consumption plan. We say that uncertainty about a state of the world of a contingent consumption plan stands for ignorance by the decision-maker (see also [37]). It depends on her incomplete state of information and knowledge, so it is of a personalistic nature. For any given decision-maker, uncertainty about a state of the world of a contingent consumption plan ceases only when she receives sure information about it. We consider concrete probability distributions measuring uncertainty (see also [60]). They are distributions of mass. They are discrete distributions referred to objects whose variability has an origin that is not random, but it depends on the variable state of information and knowledge connected with a given decision-maker.

3.2 Goods demanded by the decision-maker under different conditions

3.2.1 Random goods demanded under conditions of uncertainty and riskiness

Let ${}_1X$ and ${}_2X$ be two marginal random goods, where each of them has n (with $n > 2$ which is an integer) possible values denoted by $I({}_1X) = \{{}_{(1)}x^1, \ldots, {}_{(1)}x^n\}$ and $I({}_2X) = \{{}_{(2)}x^1, \ldots, {}_{(2)}x^n\}$. Each of them has n possible alternatives whose nature is objective. Indeed, such a nature is based on the state of information and knowledge associated with a given decision-maker. The possible values for a marginal random good are summarized by means of n non-negative masses. It turns out to be ${}_{(1)}x^1 < \ldots < {}_{(1)}x^n$ as well as ${}_{(2)}x^1 < \ldots < {}_{(2)}x^n$, so it is clear that we deal with two finite partitions of incompatible and exhaustive states of the world of two contingent consumption plans, where each state of the world of one of them is expressed by a real number. We write $\inf I({}_1X) = {}_{(1)}x^1$ and $\sup I({}_1X) = {}_{(1)}x^n$. We observe $\inf I({}_2X) = {}_{(2)}x^1$ and $\sup I({}_2X) = {}_{(2)}x^n$.

What is actually demanded by the decision-maker for ${}_1X$ and ${}_2X$ coincides with their mathematical expectation or prevision denoted by

$$\mathbf{P}({}_1X) = {}_{(1)}x^1\,{}_{(1)}p_1 + \ldots + {}_{(1)}x^n\,{}_{(1)}p_n$$

and

$$\mathbf{P}({}_2X) = {}_{(2)}x^1\,{}_{(2)}p_1 + \ldots + {}_{(2)}x^n\,{}_{(2)}p_n,$$

where it turns out to be

$${}_{(1)}p_1 + \ldots + {}_{(1)}p_n = 1,$$

with $0 \leq {}_{(1)}p_i \leq 1$, $i = 1, \ldots, n$, as well as

$${}_{(2)}p_1 + \ldots + {}_{(2)}p_n = 1,$$

with $0 \leq {}_{(2)}p_j \leq 1$, $j = 1, \ldots, n$.

3.2.2 One-dimensional and two-dimensional convex sets

The possible values for ${}_1X$ and ${}_2X$ are found on two mutually orthogonal axes of a two-dimensional Cartesian coordinate system. In particular, they are found on two half-lines. Two separately considered random goods are denoted by ${}_1X$ and ${}_2X$. They are jointly considered as well, so they give rise to a joint random good denoted by ${}_1X\,{}_2X$. Two random goods are always jointly considered inside of the budget set of the decision-maker. A coherent prevision of ${}_1X\,{}_2X$ is denoted by $\mathbf{P}({}_1X\,{}_2X)$. We denote by $\mathscr{P}$ the set of all coherent previsions denoted by $\mathbf{P}$ connected with two random goods which are jointly considered. This set is a two-dimensional convex set. Possible pairs of real numbers denoted by $(\mathbf{P}({}_1X), \mathbf{P}({}_2X))$ are the Cartesian coordinates of possible points of $\mathscr{P}$. We always project the point denoted by $(\mathbf{P}({}_1X), \mathbf{P}({}_2X))$ onto the two mutually orthogonal axes of a two-dimensional Cartesian coordinate system whose intersection is given by the point $(0,0)$. This is because we are also interested in knowing all coherent previsions of each marginal random good. All coherent previsions of each marginal random good identify two one-dimensional convex sets. The two-dimensional budget set of the decision-maker is a right triangle belonging to the first quadrant of a two-dimensional Cartesian coordinate system. The point given by $(0,0)$ is the vertex of its right angle. The budget line whose slope is negative coincides with its hypotenuse. Its catheti are the two one-dimensional convex sets under consideration.

3.2.3 Ordinary goods demanded under claimed conditions of certainty

Given two ordinary goods having downward-sloping demand curves, the quantity of consumption actually demanded for each of them by the decision-maker under claimed conditions of certainty is an average quantity. It is denoted by (x_1, x_2), where the two numbers of the list under consideration could also be equal. We write

$$x_1 = x_1^1\, p_1^1 + \ldots + x_1^n\, p_1^n \tag{3.1}$$

and

$$x_2 = x_2^1\, p_2^1 + \ldots + x_2^n\, p_2^n, \tag{3.2}$$

where $\{p_1^i\}$ and $\{p_2^j\}$ are two sets of n non-negative masses whose sum is always equal to 1 with regard to each of them. It is clear that n non-negative masses of each set under consideration are finitely additive. The possible quantities of consumption for good 1 are expressed by $\{x_1^1, \ldots, x_1^n\}$, whereas the possible quantities of consumption for good 2 are given by $\{x_2^1, \ldots, x_2^n\}$. Actual situations are uncertain. The state of information and knowledge associated with a given decision-maker is assumed to be incomplete at the time of choice, so she finds herself in a condition of real ignorance. Faced with her current state of information and knowledge, a fictitious certainty is handled. It follows that the logic of prevision is actually involved [1]. (see also [33]).

The decision-maker chooses a bilinear and disaggregate measure inside of her budget set such that x_1 and x_2 are two linear measures coinciding with (3.1) and (3.2). Even though all the masses into account are subjectively chosen, every evaluation being made by her is in general carried out with all the attention of those who consider them as objective and, if

[1] See Chapter 1 with regard to n possible quantities of consumption for good 1 belonging to a closed neighborhood of x_1 and n possible quantities of consumption for good 2 belonging to a closed neighborhood of x_2. Two nonparametric marginal distributions of mass are accordingly estimated together with a nonparametric joint distribution of mass before being summarized in such a way that (x_1, x_2) is actually chosen.

necessary, with a greater sense of responsibility deriving from not having illusions regarding their false objective nature.

Two ordinary goods are always jointly considered inside of the budget set of the decision-maker. Given (x_1, x_2), the weighted average of n^2 possible quantities of consumption for good 1 and good 2 that are jointly considered is a synthesized element of the Fréchet class. The decision-maker also chooses this synthesized element in addition to (x_1, x_2). She is subjected to $2n-1$ constraints in order to synthesize this element. They coincide with $2n-1$ marginal masses. Each point of the budget set of the decision-maker is a weighted average of n^2 possible quantities of consumption for good 1 and good 2 that are jointly considered. Degenerate averages can be obtained with respect to boundary points.

We note the following

Remark 15. Given two ordinary goods having downward-sloping demand curves, the quantity of consumption actually demanded for each of them by the decision-maker under claimed conditions of certainty is an average quantity. It always depends on objective and subjective elements. Prices and income are objective elements, whereas the masses which are estimated by the decision-maker in order to summarize her incomplete information and knowledge are subjective elements. The quantity of consumption actually demanded for each good under consideration coincides with a linear index. It is obtained by decomposing a bilinear measure into two linear measures inside of her budget set. □

Remark 16. The possible quantities of consumption concerning each good under consideration together with their masses identify nonparametric distributions of mass which are estimated and summarized by the decision-maker inside of her budget set. Starting from a decision-maker choice which is observed inside of the budget set of the decision-maker, we can identify the possible quantities of consumption for good 1 and good 2 which have had an effect on it. Whenever the possible quantities of consumption for each good are summarized by means of n non-negative masses, we obtain what the decision-maker actually chooses with respect to it inside of her budget set. □

It is clear that the objects of decision-maker choice studied inside of her budget set have to maximize her utility, where the latter is of a subjective nature (see also [49]). These objects of decision-maker choice identify her optimal choices (see also [1]). They are always relative to a specific state of information and knowledge associated with a given decision-maker. It is assumed to be incomplete at the time of choice.

3.3 Unit of measurement

Since $\mathbf{P}$ is bilinear, it is linear with respect to each of its arguments. We denote them by ${}_1X$ and ${}_2X$. After decomposing $\mathbf{P}$ into two linear measures, additivity and convexity of $\mathbf{P}$ involve that $\mathbf{P}$ is linear. In particular, let ${}_1X$ be a random good whose possible values are of a monetary nature. It is possible to observe

$$\mathbf{P}(a\,{}_1X) = a\mathbf{P}({}_1X)$$

for every real number denoted by a, so $\mathbf{P}$ is a weighted average whose nature is linear. Given this character of $\mathbf{P}$, it is possible to extend the definition of $\mathbf{P}({}_1X)$ to the case in which ${}_1X$ is a random good whose possible values are not of a monetary nature. Indeed, it is possible to choose a coefficient denoted by a such that $a\,{}_1X$ is a random good whose possible values are of a monetary nature. For instance, in the case of weight expressed by means of kilogram, we could take $a = \text{dollar}/kg = \$/kg$ into account. Since it turns out to be

$$\mathbf{P}({}_1X) = (1/a)\mathbf{P}(a\,{}_1X),$$

it is clear that $\mathbf{P}({}_1X)$ is well-defined. Indeed, $\mathbf{P}({}_1X)$ is invariant with respect to the choice of a.

On the other hand, if ${}_1X$ is a random good whose possible values are pure numbers then it is possible to choose a coefficient denoted by a such that the possible values for $a\,{}_1X$ are of a monetary nature. For instance, we could take $a = \text{dollar} = \$$ into account. The same is true with regard to ${}_2X$. It is therefore possible to extend the definition of $\mathbf{P}({}_2X)$ to the case in which ${}_2X$ is a random good whose possible values are not of a monetary nature. It is also possible to extend the definition of $\mathbf{P}({}_2X)$ to the case in which ${}_2X$ is a random good whose possible values are pure numbers.

With regard to a bundle of two ordinary goods being chosen by the decision-maker inside of her budget set, any weighted average of n^2 possible consumption levels is always decomposed into two weighted averages of n possible consumption levels. The non-negative weights associated with each of them have their sum being equal to 1. Possible consumption levels for good 1 and good 2 can always be viewed as possible quantities whose nature is monetary.

Since each point of the budget set of the decision-maker is a synthesized element of the Fréchet class, it is possible to handle ordinal utility functions associated with bundles of two random or ordinary goods whose form is concave or convex or linear. Accordingly, two different scales can be considered. They are the monetary scale and the one of utility.

3.4 An extension of the notion of bundle of goods: a consumption matrix

Given (x_1, x_2), (3.1) and (3.2) are obtained by decomposing inside of the budget set of the decision-maker the following bilinear measure expressed by

$$x_1 x_2 = x_1^1 x_2^1 p_{11} + \ldots + x_1^n x_2^n p_{nn}, \tag{3.3}$$

where it turns out to be

$$p_{11} + p_{12} + \ldots + p_{1n} + \ldots + p_{nn} = 1, \tag{3.4}$$

with $0 \leq p_{ij} \leq 1$, $i, j = 1, \ldots, n$. We deal with n^2 joint masses characterizing (3.3). We can think of putting them into a two-way table having the same number of rows and columns. We consider a two-way table having n rows and n columns. Such masses are mathematically the covariant components of an affine tensor of order 2. They identify, together with $\{x_1^1, \ldots, x_1^n\} \times \{x_2^1, \ldots, x_2^n\}$, a point belonging to a two-dimensional convex set. It is the budget set of the decision-maker. Any rational choice being made by her is found inside of this convex set. All the n^2 joint masses are subjectively estimated in such a way that the marginal masses identifying the sets $\{p_1^i\}$ and $\{p_2^j\}$ remain unchanged. It being understood that the marginal masses always remain unchanged whenever (x_1, x_2) is chosen, we also consider

$$x_1 x_1 = x_1^1 x_1^1 p_{11} + \ldots + x_1^n x_1^n p_{nn}, \tag{3.5}$$

where all off-diagonal masses are equal to 0,

$$x_2 x_2 = x_2^1 x_2^1 p_{11} + \ldots + x_2^n x_2^n p_{nn}, \tag{3.6}$$

where all off-diagonal masses are equal to 0, and

$$x_2 x_1 = x_2^1 x_1^1 p_{11} + \ldots + x_2^n x_1^n p_{nn}. \tag{3.7}$$

It follows that we write a symmetric matrix of order 2 denoted by

$$C = \begin{bmatrix} x_1 x_1 & x_1 x_2 \\ x_2 x_1 & x_2 x_2 \end{bmatrix}. \tag{3.8}$$

We call it a consumption matrix. Whenever we deal with $x_1 x_1$ and $x_2 x_2$, the slope of the corresponding budget line is equal to -1. Indeed, the two catheti of the right triangle under consideration are equal.

We establish the following

Definition 8. *Given two marginal goods, a consumption matrix is a square matrix of order* 2 *containing four bilinear measures, where each of them can be decomposed into two linear measures inside of the budget set of the decision-maker.*

We actually decompose $x_1 x_2$ into two linear measures inside of the budget set of the decision-maker. Since we want to release the notion of bundle of two goods from the one of ordered pair of quantities of consumption being demanded by the decision-maker, we obtain four measures coinciding with all elements of the square matrix of order 2 denoted by C. An aggregate measure of a bilinear nature is given by

$$x_{12} = \begin{vmatrix} x_1 x_1 & x_1 x_2 \\ x_2 x_1 & x_2 x_2 \end{vmatrix} = x_1 x_1 x_2 x_2 - x_1 x_2 x_2 x_1. \tag{3.9}$$

It represents the average quantity of consumption of a double and stand-alone good consisting of good 1 and good 2, where good 1 and good 2 are expressed by using the same unit of measurement. A double good is nothing but a multiple good of order 2, where good 1 and good 2 are its components. By definition, a multiple good of order 2 is such that it has only two marginal goods as its components. There accordingly exist multilinear relationships between good 1 and good 2. We study them.

3.4.1 Another consumption matrix: changes of origin

Given x_1, it is possible to consider a change of origin expressed by

$$d_1 = (x_1^1 - x_1)\, p_1^1 + \ldots + (x_1^n - x_1)\, p_1^n, \tag{3.10}$$

where it turns out to be $(x_1^i - x_1) = d_1^i$, $i = 1, \ldots, n$. All deviations from x_1 of the possible quantities of consumption associated with good 1 are considered in this way. Given x_2, it is similarly possible to consider another change of origin given by

$$d_2 = (x_2^1 - x_2)\, p_2^1 + \ldots + (x_2^n - x_2)\, p_2^n, \tag{3.11}$$

where it turns out to be $(x_2^j - x_2) = d_2^j$, $j = 1, \ldots, n$. All deviations from x_2 of the possible quantities of consumption associated with good 2 are considered in this way. It follows that it is possible to obtain

$$d_1\, d_2 = d_1^1\, d_2^1\, p_{11} + \ldots + d_1^n\, d_2^n\, p_{nn}, \tag{3.12}$$

where we have

$$p_{11} + p_{12} + \ldots + p_{1n} + \ldots + p_{nn} = 1, \tag{3.13}$$

with $0 \leq p_{ij} \leq 1$, $i, j = 1, \ldots, n$. We also consider

$$d_1\, d_1 = d_1^1\, d_1^1\, p_{11} + \ldots + d_1^n\, d_1^n\, p_{nn}, \tag{3.14}$$

$$d_2\, d_2 = d_2^1\, d_2^1\, p_{11} + \ldots + d_2^n\, d_2^n\, p_{nn}, \tag{3.15}$$

$$d_2\, d_1 = d_2^1\, d_1^1\, p_{11} + \ldots + d_2^n\, d_1^n\, p_{nn}. \tag{3.16}$$

It is clear that all marginal and joint masses do not change. They are the same masses that have been established by the decision-maker inside of her budget set with regard to the possible quantities of consumption connected with $x_1, x_2, x_1\, x_2$. We write another symmetric matrix of order 2 denoted by

$$C' = \begin{bmatrix} d_1\, d_1 & d_1\, d_2 \\ d_2\, d_1 & d_2\, d_2 \end{bmatrix}. \tag{3.17}$$

It contains four bilinear measures. They are considered outside of the budget set of the decision-maker. It follows that another aggregate measure of a bilinear nature expressing the variability of consumption of a multiple good of order 2 is given by

$$d_{12} = \begin{vmatrix} d_1\, d_1 & d_1\, d_2 \\ d_2\, d_1 & d_2\, d_2 \end{vmatrix} = d_1\, d_1\, d_2\, d_2 - d_1\, d_2\, d_2\, d_1. \tag{3.18}$$

It is based on two changes of origin. A change of origin with regard to the possible quantities of consumption associated with good 1 is considered. A change of origin with regard to the possible quantities of consumption associated with good 2 is considered.

3.5 How to check the weak axiom of revealed preference by using aggregate measures

3.5.1 The Bravais-Pearson correlation coefficient associated with each bundle of two goods being chosen by the decision-maker inside of her budget set

Given (x_1, x_2), we consider two aggregate measures based on changes of origin. The former is expressed by (3.18), whereas the latter coincides with

$$\hat{d}_{12} = \begin{vmatrix} d_1\, d_1 & 0 \\ 0 & d_2\, d_2 \end{vmatrix}. \tag{3.19}$$

After some mathematical steps, we write

$$-1 \leq \left(1 - \frac{\begin{vmatrix} d_1\, d_1 & d_1\, d_2 \\ d_2\, d_1 & d_2\, d_2 \end{vmatrix}}{\begin{vmatrix} d_1\, d_1 & 0 \\ 0 & d_2\, d_2 \end{vmatrix}} \right)^{1/2} \leq 1, \tag{3.20}$$

where it is possible to realize that the expression within the parentheses coincides with the Bravais-Pearson correlation coefficient intrinsically referred to a double and stand-alone good consisting of good 1 and good 2. We write it in the following form given by

$$r_{12} = \frac{d_1\, d_2}{\sqrt{d_1\, d_1}\,\sqrt{d_2\, d_2}}. \tag{3.21}$$

It is a measure of a linear correlation between two sets of possible quantities of consumption for good 1 and good 2. It geometrically measures the angle between two n-dimensional vectors of E^n, where the components of each of them represent all deviations from a mean value. Since the budget set of the decision-maker does not change whenever we multiply all prices and income by $t > 0$, her optimal choice from her budget set cannot change either. Moreover, if we multiply all possible quantities of consumption for good 1 and good 2 connected with x_1 and x_2 by $t > 0$ then the Bravais-Pearson correlation coefficient does not change either. It is possible to consider metric measures inside of linear spaces and subspaces over $\mathbb{R}$ furnished with a measure in order to describe choices being made by the decision-maker. Choices being made by her are studied by using disaggregate and aggregate measures inside of linear spaces and subspaces over $\mathbb{R}$. Such structures have different dimensions. On the other hand, everything can vectorially be studied from a statistical and economic point of view provided one takes a sufficient number of dimensions.

We note the following

Remark 17. Whenever we consider a bilinear measure that is decomposed into two linear measures, we refer ourselves to a continuous subset of a two-dimensional linear space over $\mathbb{R}$ coinciding with the budget set of the decision-maker. Such a bilinear measure is decomposed into two linear measures inside of a convex set coinciding with her budget set. Her budget set is generated by two one-dimensional straight lines on which an origin, a same unit of length, and an orientation are established. They identify two mutually orthogonal axes of a two-dimensional Cartesian coordinate system. With regard to two one-dimensional straight lines, we consider two "reductions of dimension". We firstly pass from 2 to 1. Indeed, a point of a two-dimensional convex set is always decomposed into two points of two one-dimensional convex sets. We secondly pass from n to 1. Indeed, if the decision-maker synthesizes n possible quantities of consumption corresponding to n one-dimensional points by using n non-negative masses subjectively chosen then she obtains a real number with respect to each axis. Strictly speaking, she obtains a real number with respect to each line segment belonging to each half-line. □

Remark 18. Whenever we consider an aggregate measure referred to a multiple good of order 2, we take four bilinear measures into account. Each of them derives from an element belonging to a linear space that is a subset of an n^2-dimensional linear space over $\mathbb{R}$ denoted by $E^n \otimes E^n$. Each of them is obtained by considering the covariant components of an affine tensor of order 2. Such components coincide with the n^2 joint masses that are estimated by the decision-maker. After establishing the marginal masses, she has to take them into account in order to estimate n^2 joint masses. The aggregate measure under consideration coincides with the determinant of a square matrix of order 2. It is a real number obtained by considering a bilinear function. The two columns of the matrix under consideration are two column vectors, where each of them has two components expressed by two real numbers. Whenever we consider an aggregate measure, we go away from the budget set of the decision-maker. □

Remark 19. The Bravais-Pearson correlation coefficient is intrinsically based on the bilinear object being chosen by the decision-maker inside of her budget set. Whenever we use the Bravais-Pearson correlation coefficient, we go away from the budget set of the decision-maker. □

3.5.2 A violation of the weak axiom of revealed preference

We say that (x_1, x_2) is demanded by the decision-maker at prices (b_1, b_2). She is modeled as being a consumer. We denote by r_{12} the Bravais-Pearson correlation coefficient associated with (x_1, x_2). Such a coefficient is expressed by (3.21). To fix ideas, we say that the quantity of consumption that is demanded by her for good 1 is found on the horizontal axis, whereas the quantity of consumption that is demanded for good 2 is found on the vertical one. If good 1 becomes more expensive and good 2 becomes less expensive then the budget line changes its negative slope. We also say that the budget line changes its negative slope because the state of information and knowledge associated with a given decision-maker changes. The budget line becomes steeper. It follows that (y_1, y_2) is the bundle of goods being demanded by her at prices (q_1, q_2), where it is evident that it turns out to be $q_1 > b_1$ and $q_2 < b_2$. The Bravais-Pearson correlation coefficient associated with (y_1, y_2), where we have $(y_1, y_2) \neq (x_1, x_2)$, is denoted by

$$r'_{12} = \frac{d'_1\, d'_2}{\sqrt{d'_1\, d'_1}\,\sqrt{d'_2\, d'_2}}. \tag{3.22}$$

We have

$$d'_1 = (y_1^1 - y_1)\, p'^1_1 + \ldots + (y_1^n - y_1)\, p'^n_1 \tag{3.23}$$

as well as

$$d'_2 = (y_2^1 - y_2)\, p'^1_2 + \ldots + (y_2^n - y_2)\, p'^n_2, \tag{3.24}$$

where the possible quantities of consumption for good 1 at price q_1 and connected with y_1 are expressed by $\{y_1^1, \ldots, y_1^n\}$, whereas the possible quantities of consumption for good 2 at price q_2 and connected with y_2 are given by $\{y_2^1, \ldots, y_2^n\}$. It is clear that $\{p'^i_1\}$ and $\{p'^j_2\}$ are two sets of n non-negative masses whose sum is always equal to 1 with regard to each of them [2].

If a violation of the weak axiom of revealed preference is observed then the quantity of consumption that is demanded for good 1 and denoted by y_1 does not decrease, but it increases. Moreover, the quantity of consumption that is demanded for good 2 and denoted by y_2 does not increase, but it decreases. We observe a violation of the principle according to which the demand curve for each of the two goods under consideration slopes downwards. This implies that if a negative number is used to show r_{12} because there exists an inverse relationship between quantity of consumption associated with a good and its price then a positive number has to be used to show r'_{12}. This is because we have to consider a change of sign. On the other hand, each point of the budget set of the decision-maker is a synthesized element of the Fréchet class. Accordingly, she can estimate all the joint probabilities expressing the variability of consumption of the two marginal goods under consideration based on her variable state of information and knowledge.

[2] See Chapter 1 with regard to n possible quantities of consumption for good 1 belonging to a closed neighborhood of y_1 and n possible quantities of consumption for good 2 belonging to a closed neighborhood of y_2. Two nonparametric marginal distributions of mass are consequently estimated together with a nonparametric joint distribution of mass before being summarized in such a way that (y_1, y_2) is actually chosen.

3.5.3 Decision-maker choices that satisfy the weak axiom of revealed preference

We say that (x_1, x_2) is demanded by the decision-maker at prices (b_1, b_2). We denote by r_{12} the Bravais-Pearson correlation coefficient associated with it. Such a coefficient is expressed by (3.21). If good 1 becomes less expensive and good 2 becomes more expensive then the budget line changes its negative slope. It becomes flatter. Let (y_1, y_2) be the bundle of goods being demanded by her at prices (q_1, q_2), where it is clear that it turns out to be $q_1 < b_1$ and $q_2 > b_2$. The Bravais-Pearson correlation coefficient associated with (y_1, y_2), where we have $(y_1, y_2) \neq (x_1, x_2)$, is denoted by (3.22).

If the weak axiom of revealed preference is satisfied then the quantity of consumption that is demanded for good 1 and denoted by y_1 does not decrease, but it increases. Moreover, the quantity of consumption that is demanded for good 2 and denoted by y_2 does not increase, but it decreases. Accordingly, we do not observe a violation of the principle according to which the demand curve for each of the two goods under consideration slopes downwards. This implies that if a negative number is used to show r_{12} then a negative number has to be used to show r'_{12}. This is because we do not need to consider a change of sign.

3.5.4 A summary of consumption data based on subjective elements as well

After observing several choices of bundles of goods at different prices, we obtain different measures in order to check the weak axiom of revealed preference (see also [27]). We are indirectly interested in knowing how much it costs the decision-maker to purchase each bundle of two goods at each corresponding set of prices. This is because we want to consider subjective elements as well. They are always connected with choices of bundles of two goods being made by her at different prices. Thus, we are directly interested in knowing the sign of each correlation coefficient associated with each observation characterized by a given set of prices and quantities of consumption demanded by the decision-maker. Each observation identifies a choice being made by her inside of her budget set.

In general, the chain of direct comparisons can be of any finite length. Assumptions about how the decision-maker's preferences work tell us that any two bundles of goods can directly be compared. Hence, to say that any two bundles of goods can directly be compared means that the decision-maker can choose between any two given bundles of goods. Any two bundles of goods can directly be compared for a reason of a metric nature. For instance, let (x_1, x_2) be the consumption bundle which is demanded when prices are (c_1, c_2). Let (y_1, y_2) be another consumption bundle such that we write the following expression

$$c_1 x_1 + c_2 x_2 \geq c_1 y_1 + c_2 y_2$$

having a metric nature. If the decision-maker is choosing the most preferred bundle she can afford then (x_1, x_2) is strictly preferred to (y_1, y_2). It is strictly preferred to (y_1, y_2) for a reason of a metric nature.

In particular, we observe the same sign expressed by the minus symbol referred to each correlation coefficient whenever the decision-maker chooses the best things she can afford. We conversely observe different signs referred to all correlation coefficients that have been calculated whenever she does not choose the best things she can afford. All choices being made by her are not coherent with revealed preference theory (see also [3]).

3.6 Other variability measures: mean quadratic differences

A marginal distribution of ${}_1X$, where ${}_1X$ is a random good, can be interpreted as a joint distribution of ${}_1X$ and ${}_2X = \phi$. For instance, from the table

${}_2X = \phi$ / ${}_1X$	1	1	1	Sum
0	0	0	0	0
2	0	0.3	0	0.3
3	0	0	0.7	0.7
Sum	0	0.3	0.7	1

it follows that it turns out to be $\mathbf{P}({}_1X) = \mathbf{P}({}_1X\,{}_2X) = \mathbf{P}({}_2X\,{}_1X) = 2.7$. Since we have $\mathbf{P}({}_1X\,{}_1X) = 7.5$ and $\mathbf{P}({}_2X\,{}_2X) = 1$, the variability of consumption of ${}_1X$ can be expressed by

$$\sigma^2_{{}_1X} = \begin{vmatrix} \mathbf{P}({}_1X\,{}_1X) = 7.5 & \mathbf{P}({}_1X\,{}_2X) = 2.7 \\ \mathbf{P}({}_2X\,{}_1X) = 2.7 & \mathbf{P}({}_2X\,{}_2X) = 1 \end{vmatrix} = 0.21.$$

The variability of consumption of ${}_1X$ is processed outside of the budget set of the decision-maker, whereas $\mathbf{P}({}_1X)$ is actually chosen by her inside of her budget set. If $\mathbf{P}({}_1X)$ is chosen by her inside of her budget set then ${}_1X$ is associated with another random good whose possible values are all different. This is because they identify a finite partition of states of the world of a contingent consumption plan. ${}_1X$ can also be associated with ${}_1X$ itself. The variability of consumption of ${}_1X$ is processed on the basis of what is actually chosen by her inside of her budget set. A nonparametric distribution of mass characterizing a marginal random good denoted by ${}_1X$ is summarized by using the notion of α-norm of an antisymmetric tensor of order 2 denoted by ${}_{(1)}f$ (see Chapter 4). A measure of variability of consumption of ${}_1X$ is obtained by calculating the α-norm of ${}_{(1)}f$ denoted by $\|{}_{(1)}f\|^2_\alpha$. The relation between the mean quadratic difference of ${}_1X$ given by

$${}^2\Delta^2({}_1X) = \|{}_{(1)}f\|^2_\alpha = \frac{1}{2}\begin{vmatrix} 2\,\|{}_{(1)}\mathbf{x}\|^2_\alpha & 2\,{}_{(1)}\bar{\mathbf{x}} \\ 2\,{}_{(1)}\bar{\mathbf{x}} & 2 \end{vmatrix},$$

where ${}_{(1)}\bar{\mathbf{x}}$ has all equal its contravariant components because it vectorially denotes the expected value of ${}_1X$, and its standard deviation has been established by Corrado Gini. We consider the square of it, so we write

$${}^2\Delta^2({}_1X) = 2\,\sigma^2_{{}_1X}.$$

The linear mean quadratic difference of X_{12} given by

$${}^2_L\Delta^2(X_{12}) = \|{}_{(1)}\mathbf{d} - {}_{(2)}\mathbf{d}\|^2_\alpha = \|{}_{(1)}\mathbf{d}\|^2_\alpha + \|{}_{(2)}\mathbf{d}\|^2_\alpha - 2\langle{}_{(1)}\mathbf{d},\,{}_{(2)}\mathbf{d}\rangle_\alpha$$

is obtained by using a linear and quadratic metric (see also [14]). It is clear that X_{12} is a multiple random good of order 2. The non-linear (multilinear) mean quadratic difference of X_{12} expressed by

$${}^2_{NL}\Delta^2(X_{12}) = \frac{3^2}{3!}\left[\|{}_{(1)}\mathbf{d}\|^2_\alpha\,\|{}_{(2)}\mathbf{d}\|^2_\alpha - \left(\langle{}_{(1)}\mathbf{d},\,{}_{(2)}\mathbf{d}\rangle_\alpha\right)^2\right]$$

is obtained by using a non-linear and quadratic metric (see also [48]). All possible values identifying ${}_{(1)}\mathbf{d}$ represent deviations from a mean value. All possible values identifying ${}_{(2)}\mathbf{d}$ represent deviations from a mean value. After observing that

$$r_{12} = \frac{\langle {}_{(1)}\mathbf{d}, {}_{(2)}\mathbf{d} \rangle_\alpha}{\| {}_{(1)}\mathbf{d} \|_\alpha \, \| {}_{(2)}\mathbf{d} \|_\alpha}$$

is the measure of correlation with respect to X_{12} whose possible values for its components denoted by ${}_1X$ and ${}_2X$ are subjected to changes of origin, it turns out to be

$${}^{2}_{NL}\Delta^2(X_{12}) = \frac{3}{2} \| {}_{(1)}\mathbf{d} \|^2_\alpha \, \| {}_{(2)}\mathbf{d} \|^2_\alpha \left(1 - r^2_{12}\right).$$

3.7 Multiple physical goods of order 2: a numerical example dealing with real data

For convenience, we consider a simplified dataset whose real observations are only two. The state of information and knowledge associated with a given decision-maker is assumed to be incomplete at the time of choice. From the following table

Observation	c_1	c_2	x_1	x_2
1	4	5	2	3
2	6	3	1.5	4

we note that the prices of the two single physical goods under consideration are denoted by c_1 and c_2, whereas the quantities actually chosen by the decision-maker inside of her budget set are denoted by x_1 and x_2. These quantities are associated with two different kinds of cheese. These two different kinds of cheese are respectively good 1 and good 2. With regard to the first observation, a joint distribution of mass is estimated. We estimate it by taking into account that our goal is also to check the weak axiom of revealed preference. We estimate it by taking two logical criteria into account. They obey the rules of the logic of prevision as well as the ones of ordinary logic. The logic of prevision is involved whenever the state of information and knowledge associated with a given decision-maker is incomplete. Given a finite number of possible alternatives, such a logic admits an infinite number of values connected with each non-negative mass. Conversely, ordinary logic admits only two values associated with each non-negative mass, either true $= 1$ or false $= 0$, whenever the state of information and knowledge associated with a given decision-maker is assumed to be complete. If ignorance ceases then a consumption level is not uncertain or possible any more, but it is either true or false [3]. From the following table

[3] Two types of logical reasoning can separately be handled. Given n possible consumption levels belonging to a set whose nature is objective, if we want to obtain their coherent synthesis then a deductive reasoning takes place. On the other hand, given an observed quantity contained in the dataset under consideration, if we pass from n possible consumption levels (belonging to a closed neighborhood of the observed quantity into account) to their coherent synthesis coinciding with this observed quantity then a deductive reasoning still takes place. In general, a deductive reasoning always uses n non-negative and finitely additive masses, where each of them can take infinite values between 0 and 1, end points included, into account. It is not necessary that one of n possible alternatives coincides with their synthesis. Real situations where the state of information and knowledge associated with a given decision-maker is incomplete at the time of choice are therefore dealt with by the logic of prevision. Conversely, if we pass from an observed value contained in the dataset under consideration to n

good 1 \ good 2	0	2	3	4	Sum
0	0	0	0	0	0
1	0	0	0	1/3	1/3
2	0	0	1/3	0	1/3
3	0	1/3	0	0	1/3
Sum	0	1/3	1/3	1/3	1

it turns out to be $x_1 x_2 = x_2 x_1 = 5.33$, $x_1 = 2$, $x_2 = 3$. From the following table

good 1 \ good 1	0	1	2	3	Sum
0	0	0	0	0	0
1	0	1/3	0	0	1/3
2	0	0	1/3	0	1/3
3	0	0	0	1/3	1/3
Sum	0	1/3	1/3	1/3	1

it is possible to write $x_1 x_1 = 4.67$. All off-diagonal elements have to be equal to 0. The slope of the corresponding budget line is equal to -1. With regard to the budget set of the decision-maker containing points whose number is infinite, each joint probability can take all values from 0 to 1, end points included, into account. Conversely, how to estimate $x_1 x_1$ is constrained. From the following table

good 2 \ good 2	0	2	3	4	Sum
0	0	0	0	0	0
2	0	1/3	0	0	1/3
3	0	0	1/3	0	1/3
4	0	0	0	1/3	1/3
Sum	0	1/3	1/3	1/3	1

determined consumption levels (belonging to a closed neighborhood of the observed quantity into account) then an inductive reasoning takes place. This reasoning does not characterize real situations, but it characterizes ideal situations where the state of information and knowledge associated with a given decision-maker is assumed to be complete at the time of choice. Thus, it always uses n masses such that $n-1$ masses are coherently equal to 0, whereas only one mass of n masses is coherently equal to 1. Ordinary logic is consequently involved, so degenerate distributions of mass to be summarized always appear. Since linear spaces and subspaces over $\mathbb{R}$ are taken into account to study bound choices, infinite ordered n-tuples of real numbers (belonging to a closed neighborhood of the observed quantity under consideration) can be determined by the decision-maker before transferring them on a one-dimensional straight line on which an origin, a unit of length, and an orientation are chosen. Each of them identifies n outcomes such that only one alternative expressed by a real number is true. All others are false. It is true because it is objectively observed inside of her budget set under ideal conditions of certainty. This number is therefore that value appearing in the dataset under consideration. Hence, it is absolutely necessary that one of n alternatives coincides with the observed value contained in the dataset into account. In this section, if the incompleteness of the state of information and knowledge associated with a given decision-maker ceases then her bound choice does not change. This means that new piece of information later acquired in such a way that there is no ignorance any more is unimportant with regard to her bound choice.

it is possible to obtain $x_2 x_2 = 9.67$, so we write

$$x_{12} = \begin{vmatrix} 4.67 & 5.33 \\ 5.33 & 9.67 \end{vmatrix} = 16.75.$$

Our simplified dataset contains pure numbers, so 16.75 represents the average quantity of consumption referred to a multiple physical good of order 2. The two single physical goods which are the components of this multiple physical good of order 2 are not fused together from a physical point of view. If this happens then we obtain another single physical good. Conversely, we want to obtain a multiple physical good of order 2 whose components are two single physical goods. A multiple physical good of order 2 is a portfolio containing two single physical goods which give rise to an aggregate good. It is viewed to be as a stand-alone good from a conceptual point of view. It does not live from a material point of view. Conversely, its components live from a material point of view.

With regard to the second observation, a joint distribution of mass is estimated. We estimate it by taking two logical criteria into account. We estimate it by taking into account that our goal is also to check the weak axiom of revealed preference. From the following table

good 2 / good 1	0	3	4	5	Sum
0	0	0	0	0	0
0.5	0	0	0	1/3	1/3
1.5	0	0	1/3	0	1/3
2.5	0	1/3	0	0	1/3
Sum	0	1/3	1/3	1/3	1

it turns out to be $x_1 x_2 = x_2 x_1 = 5.33$, $x_1 = 1.5$, $x_2 = 4$. From the following table

good 1 / good 1	0	0.5	1.5	2.5	Sum
0	0	0	0	0	0
0.5	0	1/3	0	0	1/3
1.5	0	0	1/3	0	1/3
2.5	0	0	0	1/3	1/3
Sum	0	1/3	1/3	1/3	1

it is possible to write $x_1 x_1 = 2.917$. All off-diagonal elements have to be equal to 0. From the following table

good 2 / good 2	0	3	4	5	Sum
0	0	0	0	0	0
3	0	1/3	0	0	1/3
4	0	0	1/3	0	1/3
5	0	0	0	1/3	1/3
Sum	0	1/3	1/3	1/3	1

it is possible to obtain $x_2 x_2 = 16.67$, so we write

$$x_{12} = \begin{vmatrix} 2.917 & 5.33 \\ 5.33 & 16.67 \end{vmatrix} = 20.21749.$$

It is clear that 20.21749 represents the average quantity of consumption referred to a multiple physical good of order 2. This quantity has been obtained by considering the quantities actually chosen by the decision-maker inside of her budget set characterized by a specific pair of prices. If we consider changes of origin then it is possible to study the variability of consumption referred to a multiple physical good of order 2. It is also possible to check the weak axiom of revealed preference. From the following table

Observation	c_1	c_2	x_1	x_2	x_{12}	d_{12}	r_{12}
1	4	5	2	3	16.75	0	- 1
2	6	3	1.5	4	20.21749	0	- 1

it follows that it is possible to consider aggregate measures as well. We calculate some aggregate measures. They are x_{12}, d_{12}, and r_{12}. The weak axiom of revealed preference is satisfied. Choices connected with two single physical goods are rational. They are optimal choices of a relative nature [4]. Accordingly, multiple choices connected with a multiple physical good of order 2 are rational as well. They are optimal choices of a relative nature as well.

[4] Optimal choices are always referred to the variable state of information and knowledge associated with a given decision-maker. Their nature is therefore relative. Since actual situations are uncertain, a less arbitrary origin is considered. By studying average quantities, the possibility that bound choices are relative to a specific state of information and knowledge associated with a given decision-maker is handled. Indeed, variations in the total amount of money the decision-maker has to spend could happen. On the other hand, risks of external origin determining variations in her income could occur as well. Hence, optimal choices coincide with average quantities. They intrinsically depend on the notion of prevision consisting in distributing among all the possible alternatives into account subjective expectations and sensations identified with non-negative and finitely additive masses. Every marginal prevision treated inside of the budget set of the decision-maker is not less than the lower bound of the set of possible values under consideration, nor greater than the upper bound. Every marginal choice studied inside of her budget set is intrinsically a value that is found between these two bounds.

Chapter 4

Certainties equivalent to a multiple random good

4.1 Introduction

4.1.1 A contingent consumption plan

In this chapter, the investor is modeled as being a consumer. It is not money alone that matters, but it is the average consumption that money can buy that is the ultimate good being chosen by her. A state of the world of a contingent consumption plan is a proposition identified with a real number such that, by betting on it, it is possible to establish whether it is true or false (see also [35]).

Let X be a random good. Let $I(X) = \{x^1, x^2, \ldots, x^m\}$ be the set of the possible values for X, where we have $x^1 < x^2 < \ldots < x^m$ because $I(X)$ identifies a finite partition of m mutually exclusive states of the world of a contingent consumption plan. We write $\inf I(X) = x^1$ and $\sup I(X) = x^m$. The elements of $I(X)$ are of a monetary nature. They give rise to an m-dimensional consumption vector denoted by

$$(x^1, x^2, \ldots, x^m).$$

It expresses all possible quantitative states of the world of a contingent consumption plan. It expresses m possible alternatives. It is possible to verify that it is contained in a closed structure coinciding with an m-dimensional linear space over $\mathbb{R}$ (see also [86]). It is a space (space of alternatives) furnished with a measure. We denote it by E^m. It has a Euclidean structure. A located vector at the origin of E^m is entirely determined by its end point. Accordingly, an ordered m-tuple of real numbers can be called either a point of $\mathscr{E}^m$ or a vector of E^m, where $\mathscr{E}^m$ and E^m are isomorphic. It is clear that $\mathscr{E}^m$ is an affine space whose elements are m-dimensional points.

Uncertainty about a state of the world of a contingent consumption plan is of a personalistic nature in the sense that it ceases only when the investor receives sure information about it (see also [23]).

It is clear that m possible alternatives whose nature is objective have to be summarized by using m non-negative masses. The probability associated with a state of the world of a contingent consumption plan is the degree of belief in the occurrence of it attributed by a given investor at a given instant and with a given set of information and knowledge. We think of probability as being a mass. It does not exist independently of the evaluations the investor makes of it mentally or instinctively. Such evaluations can be based on objective elements such as a judgment of equal probability expressing symmetry or a judgment based on statistical frequencies (see also [36]). Nevertheless, they do not exist outside of the investor's judgment whose nature is always subjective.

A function defined on the set of all possible quantitative states of the world of a contingent consumption plan coincides with X. Its domain expressed by $I(X)$ is a finite collection of possible and elementary events, where each of them is generically denoted by E_i, $i = 1, \dots, m$. We write

$$X = x^1|E_1| + x^2|E_2| + \dots + x^m|E_m|,$$

where we have

$$|E_i| = \begin{cases} 1, & \text{if } E_i \text{ is true} \\ 0, & \text{if } E_i \text{ is false} \end{cases}$$

for every $i = 1, \dots, m$.

One and only one of all possible quantitative states of the world of a contingent consumption plan belonging to $I(X)$ will be true at the right time (see also [34]). We establish the following

Definition 9. *Let* $id_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ *be the identity function on* $\mathbb{R}$*, where* $\mathbb{R}$ *is a linear space over itself. Given m incompatible and exhaustive states of the world of a contingent consumption plan, a random good denoted by X is the restriction of* $id_{\mathbb{R}}$ *to* $I(X) = \{x^1, x^2, \dots, x^m\} \subset \mathbb{R}$ *such that it turns out to be* $id_{\mathbb{R}|I(X)} : I(X) \to \mathbb{R}$.

We consider the finest possible partition into elementary events. They are not further subdivisible for the purposes of the problem under consideration. We do not consider other events. That alternative which will turn out to be verified "a posteriori" is nothing but a random point contained in $I(X)$. It expresses everything there is to be said whenever uncertainty ceases (see also [58]).

We say that a nonparametric probability distribution can vary from investor to investor. It can vary in accordance with the state of information and knowledge associated with each investor. Each investor is faced with m masses denoted by $p_1, p_2, \dots, p_m$ such that it is possible to write $p_1 + p_2 + \dots + p_m = 1$, where we have $0 \le p_j \le 1$, $j = 1, \dots, m$. They are located on m real numbers denoted by $x^1, x^2, \dots, x^m$.

Each single state of the world of a contingent consumption plan could uniquely be expressed by infinite real numbers, so we could also write

$$\{x^1 + a, x^2 + a, \dots, x^m + a\},$$

where $a \in \mathbb{R}$ is an arbitrary constant. We consider infinite changes of origin in this way. It is possible to consider different quantities from a geometric point of view. They are nevertheless the same quantity from a randomness point of view because states of the world and probabilities associated with them do not change.

4.1.2 Contravariant and covariant indices associated with a contingent consumption plan

Let E_i, $i = 1, \dots, m$, be a generic state of the world of a contingent consumption plan. We establish the following

Definition 10. *Let X be a random good whose possible values are of a monetary nature. The investor is in doubt between m values for X, so x^1 is the return on X if E_1 occurs with probability denoted by p_1, ..., x^m is the return on X if E_m occurs with probability denoted by p_m, that is, x^1 is the wealth that X yields and that can be spent by her if E_1 occurs with probability denoted by p_1, ..., x^m is the wealth that X yields and that can be spent by her if E_m occurs with probability denoted by p_m.*

We write

$$(x^1, p_1), (x^2, p_2), \ldots, (x^m, p_m)$$

in order to identify a nonparametric probability distribution associated with the possible values for X. We use covariant indices together with contravariant ones. We wish to distinguish possibility from probability in this way. We use contravariant indices to identify the possible values for X whose nature is objective. We use covariant indices to denote the corresponding probabilities that are subjectively assigned to them.

The conditions of coherence impose no limits on the probabilities that the investor may assign, except that the sum of all non-negative masses under consideration has to be equal to 1 (see also [10]).

4.2 Logical and probabilistic aspects concerning an ordered pair of contingent consumption plans

Let $\mathscr{B}_m^{\perp} = \{\mathbf{e}_i \,|\, i \in I_m = \{1, \ldots, m\}\}$ be an orthonormal basis of E^m. Two random goods denoted by ${}_1X$ and ${}_2X$ always give rise to a joint random good denoted by ${}_1X\,{}_2X$. All its possible monetary values are obtained by considering the Cartesian product of the possible values for ${}_1X$ and ${}_2X$ belonging to $I({}_1X)$ and $I({}_2X)$ respectively. Two random goods are logically independent if and only if there are m^2 possible values for ${}_1X\,{}_2X$. Let $({}_1X, {}_2X)$ be an ordered pair of random goods (see also [67]). We are faced with two different partitions, where each of them is characterized by m incompatible and exhaustive events. After considering $I({}_1X) = \{{}_{(1)}x^1, \ldots, {}_{(1)}x^m\}$ and $I({}_2X) = \{{}_{(2)}x^1, \ldots, {}_{(2)}x^m\}$, where it is possible to put ${}_{(1)}x^1 = {}_{(2)}x^1 = 0$, we establish the following

Definition 11. *All states of the world of an ordered pair of contingent consumption plans are obtained by considering the Cartesian product of the possible values for two logically independent random goods denoted by ${}_1X$ and ${}_2X$. Such random goods give rise to a joint random good denoted by ${}_1X\,{}_2X$. It is a function written in the form ${}_1X\,{}_2X : I({}_1X) \times I({}_2X) \to \mathbb{R}$, where it turns out to be ${}_1X\,{}_2X({}_{(1)}x^i, {}_{(2)}x^j) = {}_{(1)}x^i\,{}_{(2)}x^j$, with $i, j = 1, \ldots, m$.*

We are evidently faced with

$$ {}_1X\,{}_2X = {}_{(1)}x^1\,{}_{(2)}x^1 |{}_{(1)}E_1||{}_{(2)}E_1| + \ldots + {}_{(1)}x^m\,{}_{(2)}x^m |{}_{(1)}E_m||{}_{(2)}E_m|, \tag{4.1}$$

where it is possible to write

$$ |{}_{(1)}E_i||{}_{(2)}E_j| = \begin{cases} 1, & \text{if } {}_{(1)}E_i \text{ and } {}_{(2)}E_j \text{ are both true} \\ 0, & \text{otherwise} \end{cases} \tag{4.2}$$

for every $i, j = 1, \ldots, m$.

We geometrically consider ${}_{(1)}\mathbf{x} \in E^m$ as well as ${}_{(2)}\mathbf{x} \in E^m$. We write

$${}_{(1)}\mathbf{x} = {}_{(1)}x^i \mathbf{e}_i$$

and

$${}_{(2)}\mathbf{x} = {}_{(2)}x^i \mathbf{e}_i,$$

where we use the Einstein summation convention. We note that ${}_{(1)}\mathbf{x}$ and ${}_{(2)}\mathbf{x}$ are uniquely represented with respect to $\mathscr{B}_m^{\perp}$. There exists one and only one m-tuple of real numbers coinciding with the set $\{{}_{(1)}x^i\}$ and satisfying the first linear combination appearing. There also exists one and only one m-tuple of real numbers coinciding with the set $\{{}_{(2)}x^i\}$ and satisfying the second linear combination appearing. We associate the contravariant components of ${}_{(1)}\mathbf{x}$ and ${}_{(2)}\mathbf{x}$ with the possible values for ${}_1X\,{}_2X$ expressed in the same unit of measurement (see also [75]). We note that ${}_1X$ and ${}_2X$ are two marginal random goods with regard to ${}_1X\,{}_2X$.

The covariant components of an affine tensor of order 2 represent the joint probabilities of the joint distribution of ${}_1X$ and ${}_2X$. We associate in an orderly manner the covariant components of an affine tensor of order 2 with the joint probabilities of the joint distribution of ${}_1X$ and ${}_2X$. Their number is overall equal to m^2. We write

$$p = p_{ij}, \tag{4.3}$$

with $p \in E^m \otimes E^m$. Since it turns out to be

$$\sum_{i=1}^{m} \sum_{j=1}^{m} p_{ij} = 1, \tag{4.4}$$

all probabilistic evaluations being made by the investor are coherent. Conditions of coherence pertain to the meaning of probability. They do not pertain to motives of a mathematical nature (see also [72]).

We note the following

Remark 20. Given an orthonormal basis of E^m, the contravariant and covariant components of a same vector of E^m coincide. They represent the same numbers. Accordingly, we could use lower indices instead of upper ones and vice versa. □

4.2.1 Metric aspects concerning an ordered pair of contingent consumption plans

We say that an ordered pair of random goods denoted by $({}_1X, {}_2X)$ is represented by an ordered triple of geometric entities denoted by

$$\left({}_{(1)}\mathbf{x}, {}_{(2)}\mathbf{x}, p_{ij}\right), \tag{4.5}$$

with $(i, j) \in I_m \times I_m$.

We consider the notion of α-product between ${}_{(1)}\mathbf{x}$ and ${}_{(2)}\mathbf{x}$ in order to establish a quadratic metric on E^m. It is a scalar or inner product obtained by using the joint probabilities of the joint distribution of ${}_1X$ and ${}_2X$ together with the contravariant components of ${}_{(1)}\mathbf{x}$ and ${}_{(2)}\mathbf{x}$. We write

$$\langle {}_{(1)}\mathbf{x}, {}_{(2)}\mathbf{x} \rangle_\alpha = {}_{(1)}x^i\, {}_{(2)}x^j p_{ij} = {}_{(1)}x^i\, {}_{(2)}x_i, \tag{4.6}$$

where

$$ {}_{(2)}x^{j}p_{ij} = {}_{(2)}x_{i} \tag{4.7} $$

is a vector homography by means of which we pass from ${}_{(2)}x^{j}$ to ${}_{(2)}x_{i}$ by using p_{ij}. For instance, from the table

random good 2 / random good 1	0	10	11	Sum
0	0	0	0	0
6	0	0.3	0.1	0.4
7	0	0.1	0.5	0.6
Sum	0	0.4	0.6	1

it follows that it turns out to be $\mathbf{P}({}_{1}X\,{}_{2}X) = 70.1$. Given the contravariant components of ${}_{(2)}\mathbf{x}$ identifying the following column vector

$$ \begin{pmatrix} 0 \\ 10 \\ 11 \end{pmatrix}, $$

its covariant components are expressed by

$$ 0 \cdot 0 + 10 \cdot 0 + 11 \cdot 0 = 0, $$

$$ 0 \cdot 0 + 10 \cdot 0.3 + 11 \cdot 0.1 = 4.1, $$

$$ 0 \cdot 0 + 10 \cdot 0.1 + 11 \cdot 0.5 = 6.5, $$

so it is possible to write the following result

$$ \left\langle \begin{pmatrix} 0 \\ 6 \\ 7 \end{pmatrix}, \begin{pmatrix} 0 \\ 4.1 \\ 6.5 \end{pmatrix} \right\rangle = \langle {}_{(1)}\mathbf{x}, {}_{(2)}\mathbf{x} \rangle_{\alpha} = \mathbf{P}({}_{1}X\,{}_{2}X) = 70.1. $$

On the other hand, after calculating the covariant components of ${}_{(1)}\mathbf{x}$ in a similar way, we write

$$ \left\langle \begin{pmatrix} 0 \\ 2.5 \\ 4.1 \end{pmatrix}, \begin{pmatrix} 0 \\ 10 \\ 11 \end{pmatrix} \right\rangle = \langle {}_{(1)}\mathbf{x}, {}_{(2)}\mathbf{x} \rangle_{\alpha} = \mathbf{P}({}_{1}X\,{}_{2}X) = 70.1. $$

From the notion of α-product it follows the one of α-norm of an m-dimensional vector. We write

$$ \|{}_{(1)}\mathbf{x}\|^{2}_{\alpha} = \langle {}_{(1)}\mathbf{x}, {}_{(1)}\mathbf{x} \rangle_{\alpha} = {}_{(1)}x^{i}\,{}_{(1)}x^{i}p_{ii} = {}_{(1)}x^{i}\,{}_{(1)}x_{i} \tag{4.8} $$

as well as

$$ \|{}_{(2)}\mathbf{x}\|^{2}_{\alpha} = \langle {}_{(2)}\mathbf{x}, {}_{(2)}\mathbf{x} \rangle_{\alpha} = {}_{(2)}x^{i}\,{}_{(2)}x^{i}p_{ii} = {}_{(2)}x^{i}\,{}_{(2)}x_{i} \tag{4.9} $$

because the joint probabilities of the particular joint distributions under consideration whose covariant indices are not equal coincide with 0. For instance, from the table

random good 1 \ random good 1	0	6	7	Sum
0	0	0	0	0
6	0	0.4	0	0.4
7	0	0	0.6	0.6
Sum	0	0.4	0.6	1

it follows that it turns out to be

$$\left\langle \begin{pmatrix} 0 \\ 6 \\ 7 \end{pmatrix}, \begin{pmatrix} 0 \\ 2.4 \\ 4.2 \end{pmatrix} \right\rangle = \langle {}_{(1)}\mathbf{x}, {}_{(1)}\mathbf{x} \rangle_\alpha = \| {}_{(1)}\mathbf{x} \|^2_\alpha = 43.8.$$

Also, it is possible to show two metric inequalities. The Schwarz's α-generalized inequality is given by

$$\left| \langle {}_{(1)}\mathbf{x}, {}_{(2)}\mathbf{x} \rangle_\alpha \right| \leq \| {}_{(1)}\mathbf{x} \|_\alpha \| {}_{(2)}\mathbf{x} \|_\alpha, \tag{4.10}$$

whereas the α-triangle inequality is expressed by

$$\| {}_{(1)}\mathbf{x} + {}_{(2)}\mathbf{x} \|_\alpha \leq \| {}_{(1)}\mathbf{x} \|_\alpha + \| {}_{(2)}\mathbf{x} \|_\alpha. \tag{4.11}$$

From (4.10) it follows the notion of α-cosine, so it is possible to write

$$\cos({}_{(1)}\mathbf{x}, {}_{(2)}\mathbf{x})_\alpha = \frac{\langle {}_{(1)}\mathbf{x}, {}_{(2)}\mathbf{x} \rangle_\alpha}{\| {}_{(1)}\mathbf{x} \|_\alpha \| {}_{(2)}\mathbf{x} \|_\alpha}. \tag{4.12}$$

4.2.2 The relative and subjective nature of the joint probabilities associated with an ordered pair of contingent consumption plans

The covariant components of an affine tensor of order 2 belonging to $E^m \otimes E^m$ are joint probabilities whose nature is relative (see also [17]). They depend on the variable group of circumstances supposed to be of interest to the occurrence of a specific state of the world characterizing ${}_1X\,{}_2X$. Such circumstances are known at the time. They generally vary from instant to instant. It follows that probabilities vary according to the state of information and knowledge associated with a given investor which can be enriched by the flow of information and results that are learned or observed with respect to more or less similar cases. We note that each new piece of information is able to modify the evaluations of probability being made by the investor according to Bayes' rule. The nature of the joint probabilities is also subjective in the sense that a probability concerning a state of the world of a contingent consumption plan and depending on the state of information and knowledge associated with a given investor is intrinsically personalized. It follows that different investors having the same state of information and knowledge could give a greater attention to certain circumstances than to others. The state of information and knowledge associated with a given investor can also modify the set of all possible quantitative states of the world of a contingent consumption plan, where each state of the world of it is a real number. Accordingly, the absolute value of each real number can change.

Since it turns out to be

$$\dim (E^m \otimes E^m) = m^2,$$

there exists an isomorphism between $E^m \otimes E^m$ and E^{m^2}. We can think of locating m^2 non-negative masses on m^2 points, where each point of them expressed by $\big({}_{(1)}x^i, {}_{(2)}x^j\big)$ is a real number denoted by ${}_{(1)}x^i{}_{(2)}x^j$, $i, j = 1, \ldots, m$. We write

$$(\mathbf{x}, \mathbf{p}) \subset E^{m^2},$$

where $\mathbf{x}$ and $\mathbf{p}$ are two m^2-dimensional vectors.

If we write

$$\langle {}_{(1)}\mathbf{x}, {}_{(2)}\mathbf{x} \rangle_\alpha = {}_{(1)}x^i{}_{(2)}x^j p_{ij} = \mathbf{P}({}_1X\,{}_2X)$$

then we observe a "reduction of dimension" because we pass from m^2 points to 1 point, where the latter is always studied together with its Cartesian coordinates. After transferring the possible values for ${}_1X$ and ${}_2X$ on two one-dimensional straight lines, each coherent prevision of ${}_1X\,{}_2X$ denoted by $\mathbf{P}({}_1X\,{}_2X)$ is a point of a two-dimensional convex set coinciding with the budget set of the investor (see also [16]). This point is denoted by $(\mathbf{P}({}_1X), \mathbf{P}({}_2X))$. The budget set of the investor is a continuous subset of $\mathbb{R} \times \mathbb{R}$. All coherent previsions of ${}_1X\,{}_2X$ are obtained by taking all the values between 0 and 1, end points included, into account for each mass of m^2 masses. The number of such values is infinite. $\mathbf{P}({}_1X\,{}_2X)$ is always decomposed into two linear measures, $\mathbf{P}({}_1X)$ and $\mathbf{P}({}_2X)$ respectively. Each of them shows a "reduction of dimension" because we pass from m one-dimensional points which are found on a one-dimensional straight line to 1 one-dimensional point which is found on the same line. All coherent previsions of ${}_1X$ and ${}_2X$ are obtained by taking all the values between 0 and 1, end points included, into account for each mass of m masses. The number of such values is infinite.

4.3 Two contingent consumption plans jointly considered that are independent of the notion of ordered pair

We note the following

Remark 21. Let ${}_1X$ and ${}_2X$ be two marginal random goods, where each of them is characterized by m possible values whose nature is objective. The two m-dimensional vectors, whose contravariant components represent the possible values for two random goods which are separately considered, are assumed to be linearly independent. The possible values for two logically independent random goods which are jointly considered have to be represented by the contravariant components of a tensor of order 2. It is an antisymmetric tensor of order 2 whenever we are interested in handling a multiple random good of order 2 denoted by X_{12}, where its components are expressed by ${}_1X$ and ${}_2X$ respectively. □

We pass from an ordered pair of contingent consumption plans to two contingent consumption plans which are jointly considered regardless of the notion of ordered pair. We have to consider a multiple random good of order 2 (double random good) denoted by

$$X_{12} = \{{}_1X, {}_2X\} \tag{4.13}$$

whose possible values coincide with the contravariant components of an antisymmetric tensor of order 2. Given the marginal probabilities of ${}_1X$ and ${}_2X$, after choosing m^2 joint probabilities connected with ${}_1X\,{}_2X$, it is necessary to consider four joint distributions characterizing ${}_1X\,{}_1X$, ${}_1X\,{}_2X$, ${}_2X\,{}_1X$, ${}_2X\,{}_2X$, with

$$ {}_1X\,{}_1X : I({}_1X) \times I({}_1X) \to \mathbb{R}, \tag{4.14} $$

$$ {}_2X\,{}_2X : I({}_2X) \times I({}_2X) \to \mathbb{R}, \tag{4.15} $$

$$ {}_2X\,{}_1X : I({}_2X) \times I({}_1X) \to \mathbb{R}, \tag{4.16} $$

in order to release X_{12} from the notion of ordered pair of contingent consumption plans. We note that ${}_1X$ and ${}_2X$ are not put near unlike what happens when we jointly consider ${}_iX$ and ${}_jX$, where we have $i, j = 1, 2$. We can think of putting the m^2 joint probabilities into a two-way table having m rows and m columns. Each nonparametric probability distribution of a marginal random good is viewed to be as a particular joint distribution. This is because all off-diagonal joint probabilities of the two-way table under consideration coincide with 0. It is possible to show that the mathematical expectation of ${}_iX\,{}_jX$, with $i, j = 1, 2$, is always bilinear. This is because it is separately linear in each marginal random good. We prove the following

Theorem 3. The mathematical expectation of $X_{12} = \{{}_1X, {}_2X\}$ denoted by $\mathbf{P}(X_{12})$ coincides with the determinant of a square matrix of order 2. Each element of such a determinant is a real number coinciding with the mathematical expectation of ${}_iX\,{}_jX$, where we have $i, j = 1, 2$.

Proof. An affine tensor of order 2 representing the possible values for ${}_1X\,{}_2X$, where ${}_1X\,{}_2X$ corresponds to $({}_1X, {}_2X)$, is written in the form

$$ T = {}_{(1)}\mathbf{x} \otimes {}_{(2)}\mathbf{x} = {}_{(1)}x^i\,{}_{(2)}x^j\,\mathbf{e}_i \otimes \mathbf{e}_j. \tag{4.17} $$

An affine tensor of order 2 representing the possible values for ${}_2X\,{}_1X$, where ${}_2X\,{}_1X$ corresponds to $({}_2X, {}_1X)$, is conversely written in the form

$$ T = {}_{(2)}\mathbf{x} \otimes {}_{(1)}\mathbf{x} = {}_{(2)}x^j\,{}_{(1)}x^i\,\mathbf{e}_j \otimes \mathbf{e}_i. \tag{4.18} $$

We wrote a same affine tensor of order 2 denoted by T whose m^2 contravariant components are not the same. If we pass from (4.17) to (4.18) then we note that the contravariant components whose upper indices are equal do not change. If we pass from (4.17) to (4.18) then we note that the contravariant components whose upper indices are not equal change. It follows that we write an antisymmetric tensor of order 2 in the form

$$ T = \sum_{i<j} \left({}_{(1)}x^i\,{}_{(2)}x^j - {}_{(1)}x^j\,{}_{(2)}x^i \right) \mathbf{e}_i \otimes \mathbf{e}_j \tag{4.19} $$

because we have to consider (4.17) and (4.18) together. We wrote $i < j$ under the summation symbol because if it turns out to be $i = j$ then every contravariant component inside parentheses is equal to 0. Hence, we denote by ${}_{12}x$ an antisymmetric tensor of order 2 identifying X_{12}. We write

$$ {}_{12}x^{(ij)} = \begin{vmatrix} {}_{(1)}x^i & {}_{(1)}x^j \\ {}_{(2)}x^i & {}_{(2)}x^j \end{vmatrix} = {}_{(1)}x^i\,{}_{(2)}x^j - {}_{(1)}x^j\,{}_{(2)}x^i \tag{4.20} $$

in order to identify the strict contravariant components of it. We have $i < j$. The number of such components is overall equal to

$$\binom{m}{2}.$$

The corresponding strict covariant components of ${}_{12}x$ are given by

$$ {}_{12}x_{(ij)} = \begin{vmatrix} {}_{(1)}x_i & {}_{(1)}x_j \\ {}_{(2)}x_i & {}_{(2)}x_j \end{vmatrix} = \begin{vmatrix} {}_{(1)}x^j p_{ji} & {}_{(1)}x^i p_{ij} \\ {}_{(2)}x^j p_{ji} & {}_{(2)}x^i p_{ij} \end{vmatrix}, \tag{4.21}$$

where we have $i < j$. We do not compute the scalar value of (4.21). The number of the strict contravariant and covariant components of ${}_{12}x$ is absolutely unimportant. We always obtain the same outcome independently of such a number. We put together (4.20) and (4.21), where (4.20) and (4.21) contain all strict contravariant and covariant components of ${}_{12}x$ at the same time. We always put together (4.20) and (4.21) in the same way. We always associate ${}_{(1)}x^i$ with ${}_{(1)}x_i$, ${}_{(1)}x^j$ with ${}_{(2)}x_j$, ${}_{(2)}x^i$ with ${}_{(1)}x_i$, ${}_{(2)}x^j$ with ${}_{(2)}x_j$. After putting together (4.20) and (4.21), whose structure is evidently the one of two determinants because we are considering multilinear matters, we obtain different single terms (monomials). It follows that a variable index appearing twice in a monomial implies summation of it over all values of the index (hence, every time it is possible to obtain a polynomial by using the Einstein notation). On the other hand, all strict contravariant and covariant components of ${}_{12}x$ are simultaneously identified with two determinants because, in general, the determinant of a square matrix is the most exemplary multilinear relationship as well as a linear combination of basis vectors is the most exemplary linear relationship. We obtain the mathematical expectation of X_{12} given by

$$\|{}_{12}x\|_\alpha^2 = \begin{vmatrix} \|{}_{(1)}\mathbf{x}\|_\alpha^2 & \langle {}_{(1)}\mathbf{x}, {}_{(2)}\mathbf{x}\rangle_\alpha \\ \langle {}_{(2)}\mathbf{x}, {}_{(1)}\mathbf{x}\rangle_\alpha & \|{}_{(2)}\mathbf{x}\|_\alpha^2 \end{vmatrix} = \|{}_{(1)}\mathbf{x}\|_\alpha^2 \|{}_{(2)}\mathbf{x}\|_\alpha^2 - \left(\langle {}_{(1)}\mathbf{x}, {}_{(2)}\mathbf{x}\rangle_\alpha\right)^2, \tag{4.22}$$

where we observe

$$\langle {}_{(1)}\mathbf{x}, {}_{(2)}\mathbf{x}\rangle_\alpha = \langle {}_{(2)}\mathbf{x}, {}_{(1)}\mathbf{x}\rangle_\alpha. \tag{4.23}$$

By putting together (4.20) and (4.21), we are always faced with four joint distributions characterizing ${}_1X\,{}_1X$, ${}_1X\,{}_2X$, ${}_2X\,{}_1X$, and ${}_2X\,{}_2X$ that are all summarized. We write

$$\|{}_{12}x\|_\alpha^2 = \mathbf{P}(X_{12}) > 0, \tag{4.24}$$

where it turns out to be

$$\begin{aligned} \mathbf{P}(X_{12}) &= \begin{vmatrix} \|{}_{(1)}\mathbf{x}\|_\alpha^2 & \langle {}_{(1)}\mathbf{x}, {}_{(2)}\mathbf{x}\rangle_\alpha \\ \langle {}_{(2)}\mathbf{x}, {}_{(1)}\mathbf{x}\rangle_\alpha & \|{}_{(2)}\mathbf{x}\|_\alpha^2 \end{vmatrix} \\ &= \begin{vmatrix} {}_{(1)}x^i\, {}_{(1)}x^i p_{ii}^{(11)} = {}_{(1)}x^i\, {}_{(1)}x_i & {}_{(1)}x^j\, {}_{(2)}x^i p_{ij}^{(12)} = {}_{(1)}x^j\, {}_{(2)}x_j \\ {}_{(2)}x^i\, {}_{(1)}x^j p_{ji}^{(21)} = {}_{(2)}x^i\, {}_{(1)}x_i & {}_{(2)}x^j\, {}_{(2)}x^j p_{jj}^{(22)} = {}_{(2)}x^j\, {}_{(2)}x_j \end{vmatrix}. \end{aligned} \tag{4.25}$$

We note that $p^{(11)}$ is the tensor of all joint probabilities associated with $({}_{(1)}\mathbf{x}, {}_{(1)}\mathbf{x})$. The same is true for all others contained in (4.25). It is possible to observe that in general it turns out to be

$$\mathbf{P}({}_1X\,{}_2X) \neq \mathbf{P}(X_{12}). \tag{4.26}$$

We finally write

$$\mathbf{P}(X_{12}) = \begin{vmatrix} \mathbf{P}({}_1X\,{}_1X) & \mathbf{P}({}_1X\,{}_2X) \\ \mathbf{P}({}_2X\,{}_1X) & \mathbf{P}({}_2X\,{}_2X) \end{vmatrix}, \tag{4.27}$$

where the determinant of the square matrix of order 2 under consideration is a bilinear function of the columns of it. □

Given ${}_1X$ and ${}_2X$ and their coherent previsions denoted by

$$\mathbf{P}({}_1X) = {}_{(1)}x^1\,{}_{(1)}p_1 + \ldots + {}_{(1)}x^m\,{}_{(1)}p_m$$

and

$$\mathbf{P}({}_2X) = {}_{(2)}x^1\,{}_{(2)}p_1 + \ldots + {}_{(2)}x^m\,{}_{(2)}p_m,$$

where it turns out to be

$${}_{(1)}p_1 + \ldots + {}_{(1)}p_m = 1$$

as well as

$${}_{(2)}p_1 + \ldots + {}_{(2)}p_m = 1,$$

with $0 \le {}_{(1)}p_i \le 1$, $0 \le {}_{(2)}p_j \le 1$, $i, j = 1, \ldots, m$, it is possible to consider all deviations from $\mathbf{P}({}_1X)$ and $\mathbf{P}({}_2X)$ of the possible values for ${}_1X$ and ${}_2X$. We are evidently faced with the marginal distributions of the joint distribution of ${}_1X$ and ${}_2X$ (see also [78]). We prove the following

Theorem 4. The variance of $X_{12} = \{{}_1X, {}_2X\}$ denoted by $\mathrm{Var}(X_{12})$ coincides with the determinant of a square matrix of order 2. Each element of such a determinant is a real number coinciding with the variance of ${}_1X$ and ${}_2X$, and with their covariance.

Proof. All deviations from $\mathbf{P}({}_1X)$ and $\mathbf{P}({}_2X)$ of the possible values for ${}_1X$ and ${}_2X$ are translations. They are changes of origin. It is possible to write

$$\|{}_{12}d\|_\alpha^2 = \begin{vmatrix} \|{}_{(1)}\mathbf{d}\|_\alpha^2 & \langle{}_{(1)}\mathbf{d},\,{}_{(2)}\mathbf{d}\rangle_\alpha \\ \langle{}_{(2)}\mathbf{d},\,{}_{(1)}\mathbf{d}\rangle_\alpha & \|{}_{(2)}\mathbf{d}\|_\alpha^2 \end{vmatrix} = \|{}_{(1)}\mathbf{d}\|_\alpha^2\|{}_{(2)}\mathbf{d}\|_\alpha^2 - \left(\langle{}_{(1)}\mathbf{d},\,{}_{(2)}\mathbf{d}\rangle_\alpha\right)^2, \tag{4.28}$$

where ${}_{12}d$ is an antisymmetric tensor of order 2 representing X_{12} from a logical point of view. We are faced with changes of origin of the possible values for ${}_1X$ and ${}_2X$. We write

$$\|{}_{12}d\|_\alpha^2 = \mathrm{Var}(X_{12}) = \sigma_{X_{12}}^2. \tag{4.29}$$

We note that it turns out to be

$$\langle{}_{(1)}\mathbf{d},\,{}_{(2)}\mathbf{d}\rangle_\alpha = \langle{}_{(2)}\mathbf{d},\,{}_{(1)}\mathbf{d}\rangle_\alpha = \mathrm{Cov}({}_1X,\,{}_2X) = \mathrm{Cov}({}_2X,\,{}_1X), \tag{4.30}$$

so it is possible to write

$$\mathrm{Var}(X_{12}) = \begin{vmatrix} \mathrm{Var}({}_1X) & \mathrm{Cov}({}_1X,\,{}_2X) \\ \mathrm{Cov}({}_2X,\,{}_1X) & \mathrm{Var}({}_2X) \end{vmatrix}. \tag{4.31}$$

If we are faced with the variance of X_{12} then ${}_1X$ and ${}_2X$ are fused together. In general, if we compute only the covariance of ${}_1X$ and ${}_2X$ (in addition to the variance of each of them) then they are simply put near. □

We note the following

Remark 22. Given X_{12}, $\mathbf{P}(X_{12})$ is coherent in the same way as $\mathbf{P}({}_1X)$, $\mathbf{P}({}_2X)$, $\mathbf{P}({}_1X\,{}_2X) = \mathbf{P}({}_2X\,{}_1X)$, where $\mathbf{P}(X_{12})$ and $\mathbf{P}({}_1X\,{}_2X) = \mathbf{P}({}_2X\,{}_1X)$ are both of them bilinear indices. $\mathbf{P}(X_{12})$ is an aggregate index, whereas $\mathbf{P}({}_1X\,{}_2X) = \mathbf{P}({}_2X\,{}_1X)$ is a disaggregate index. □

Remark 23. The origin of the variability of X_{12} is not standardized, but it depends on the variable state of information and knowledge associated with a given investor. All deviations from $\mathbf{P}({}_1X)$ and $\mathbf{P}({}_2X)$ of the possible values for ${}_1X$ and ${}_2X$ depend on her variable state of information and knowledge. □

4.4 The budget set of the investor

Given the two-good assumption, the objects of investor choice are of a bilinear nature. We consider two mutually orthogonal axes of a two-dimensional Cartesian coordinate system on which an origin, a same unit of length, and an orientation are established. All the m^2 possible states of the world of two contingent consumption plans which are jointly considered belong to a finite subset of a two-dimensional Cartesian coordinate system, where each axis of it contains m possible states of the world of a contingent consumption plan. It is possible to consider two half-lines, where each of them extends indefinitely in a positive direction from zero before being restricted. Only a joint distribution at a time is considered inside of the budget set of the investor. We obtain a bilinear measure whenever we summarize it. It is a synthesized element of the Fréchet class. We consider all coherent previsions of a joint random good. All coherent previsions of a joint random good denoted by ${}_1X\,{}_2X$ are expressed by $\mathbf{P}({}_1X\,{}_2X)$. They are disaggregate and bilinear measures, so $\mathbf{P}({}_1X\,{}_2X)$ is always decomposed into two coherent previsions of two marginal random goods denoted by $\mathbf{P}({}_1X)$ and $\mathbf{P}({}_2X)$ respectively. All coherent previsions of a joint random good identify a two-dimensional convex set denoted by $\mathscr{P} \subset \mathbb{R} \times \mathbb{R}$. It is a right triangle belong to the first quadrant of a two-dimensional Cartesian coordinate system. Its catheti meet at the point denoted by $(0,0)$. Its hypotenuse is the budget line identifying the budget set of the investor. It is a hyperplane embedded in a two-dimensional Cartesian coordinate system. It does not separate any point $\mathbf{P}$ of $\mathscr{P}$ from the set $\mathscr{Q} = I({}_1X) \times I({}_2X)$ of all possible points for ${}_1X$ and ${}_2X$ belonging to $I({}_1X)$ and $I({}_2X)$ respectively. Also, it does not separate $\mathbf{P}({}_1X)$ from $I({}_1X)$, nor $\mathbf{P}({}_2X)$ from $I({}_2X)$. Given the marginal previsions of ${}_1X$ and ${}_2X$ denoted by $\mathbf{P}({}_1X)$ and $\mathbf{P}({}_2X)$, the budget constraint of the investor is an inequality expressed by

$$c_1\,\mathbf{P}({}_1X) + c_2\,\mathbf{P}({}_2X) \leq c.$$

It is characterized by three strictly positive real numbers. They are the two objective prices, c_1 and c_2, of the two goods under consideration besides the amount of money she has to spend. The latter is denoted by c. The two objective prices of the two goods under consideration identify the negative slope of the budget line. The two objective prices identify the two real coefficients expressing the slope of a hyperplane embedded in a two-dimensional Cartesian coordinate system. This means it makes sense to consider the possible values for ${}_1X$, ${}_2X$, and ${}_1X\,{}_2X$. Their nature is objective. We deal with two continuous goods because what the investor actually chooses inside of her budget set is an average quantity of consumption associated with each of them. Two one-dimensional convex sets are identified because the two-dimensional convex set coinciding with the budget set of the investor contains infinite coherent previsions of a bilinear nature, where each of them is always decomposed into two previsions of a linear nature. Two one-dimensional convex sets coincide with two line segments belonging to the two axes of a two-dimensional Cartesian coordinate system. Each average quantity of consumption associated with random good 1 and random good 2 does not depend on objective elements only, but it depends on subjective elements as well.

We multiply c_1, c_2, c by a positive number. The investor divides her relative monetary wealth given by

$$\frac{c_1}{c_1+c_2} \tag{4.32}$$

and

$$\frac{c_2}{c_1+c_2} \tag{4.33}$$

between the two random goods denoted by ${}_1X$ and ${}_2X$. It follows that it turns out to be

$$\frac{c_1}{c_1+c_2}+\frac{c_2}{c_1+c_2}=1. \tag{4.34}$$

The budget set of the investor does not change. She always chooses one and only one of the points of $\mathscr{P}$ from her budget set. All points of $\mathscr{P}$ are admissible in terms of coherence of $\mathbf{P}$. We write

$$\frac{c_1}{c_1+c_2}\mathbf{P}({}_1X)+\frac{c_2}{c_1+c_2}\mathbf{P}({}_2X)\leq\frac{c}{c_1+c_2} \tag{4.35}$$

whenever we deal with a bilinear measure that is decomposed into two linear measures. The left-hand side of (4.35) is a weighted average of the two expected returns on ${}_1X$ and ${}_2X$.

4.4.1 To go away from the budget set of the investor: changes of origin

Let ${}_1X$ and ${}_2X$ be two random goods coinciding with two risky assets. Given

$$\mathbf{y}=\mu_1\,{}_{(1)}\mathbf{d}+\mu_2\,{}_{(2)}\mathbf{d}, \tag{4.36}$$

with $\mu_1=\frac{c_1}{c_1+c_2}\in\mathbb{R}$ and $\mu_2=\frac{c_2}{c_1+c_2}\in\mathbb{R}$, it is possible to obtain, outside of the budget set of the investor, the following expression given by

$$\|\mathbf{y}\|_\alpha^2=\|\mu_1\,{}_{(1)}\mathbf{d}+\mu_2\,{}_{(2)}\mathbf{d}\|_\alpha^2=(\mu_1)^2\,\|{}_{(1)}\mathbf{d}\|_\alpha^2+2\mu_1\,\mu_2\langle{}_{(1)}\mathbf{d},\,{}_{(2)}\mathbf{d}\rangle_\alpha+(\mu_2)^2\,\|{}_{(2)}\mathbf{d}\|_\alpha^2. \tag{4.37}$$

We focus on the riskiness of the components of X_{12} only, where ${}_1X$ and ${}_2X$ are the components of X_{12}. This is because we use a linear metric.

We establish the following

Definition 12. *We call linear metric the expression given by* (4.37)*. Since it is also possible to write* $\|{}_{(1)}\mathbf{d}-{}_{(2)}\mathbf{d}\|_\alpha^2=\|{}_{(1)}\mathbf{d}\|_\alpha^2+\|{}_{(2)}\mathbf{d}\|_\alpha^2-2\langle{}_{(1)}\mathbf{d},\,{}_{(2)}\mathbf{d}\rangle_\alpha$*, it derives from the notion of* α*-distance between the two components of* X_{12} *denoted by* ${}_1X$ *and* ${}_2X$ *whose possible values are subjected to two changes of origin.*

We note the following

Remark 24. Whenever we consider a linear and quadratic metric, we are faced with a joint distribution only. Such a distribution depends on the notion of ordered pair of random goods. □

We establish the following

Definition 13. *We call non-linear (multilinear) metric the expression given by* (4.28). *It is the area of a* 2*-parallelepiped whose edges are two marginal random goods having their possible values that are subjected to two changes of origin. The strict components of* ${}_{12}d$ *are the coordinates of such edges denoted by* ${}_{(1)}\mathbf{d}$ *and* ${}_{(2)}\mathbf{d}$.

By using (4.28), where (4.28) is an aggregate measure of a bilinear nature, it is possible to obtain the Bravais-Pearson correlation coefficient. We firstly write

$$\|{}_{12}\hat{d}\|^2_{\alpha} = \begin{vmatrix} \|{}_{(1)}\mathbf{d}\|^2_{\alpha} & 0 \\ 0 & \|{}_{(2)}\mathbf{d}\|^2_{\alpha} \end{vmatrix}. \tag{4.38}$$

After some mathematical steps, we obtain

$$-1 \leq \left(1 - \frac{\|{}_{12}d\|^2_{\alpha}}{\|{}_{12}\hat{d}\|^2_{\alpha}}\right)^{1/2} \leq 1, \tag{4.39}$$

where it is possible to realize that the expression within the parentheses coincides with the Bravais-Pearson correlation coefficient intrinsically referred to X_{12}. We write it in the following form expressed by

$$r_{12} = \frac{\langle {}_{(1)}\mathbf{d}, {}_{(2)}\mathbf{d} \rangle_{\alpha}}{\|{}_{(1)}\mathbf{d}\|_{\alpha}\, \|{}_{(2)}\mathbf{d}\|_{\alpha}}. \tag{4.40}$$

4.5 Uncertainty and riskiness: probability and cardinal utility connected with multiple random goods of order 2

Since $E^m \otimes E^m$ is isomorphic to E^{m^2}, it is possible to transfer all the m^2 possible states of the world of two contingent consumption plans jointly considered identifying a joint random good on a one-dimensional straight line on which an origin, a unit of length, and an orientation are established. We deal with four joint random goods. We deal with four joint distributions, where each of them can be considered inside of the budget set of the investor. We go away from the budget set of the investor. We transfer four joint distributions on a one-dimensional straight line on which an origin, a unit of length, and an orientation are chosen.

Any distribution of mass is completely characterized by its mathematical expectation and variance, where the latter is a measure of the riskiness of the wealth distribution under consideration (see also [41]). Both mathematical expectation and variance of X_{12} have been obtained by means of the notion of α-norm of an antisymmetric tensor of order 2. Accordingly, in general, they are both of them greater than zero. If the investor estimates all joint probabilities of ${}_1X\,{}_2X$ in such a way that there exists an inverse linear relationship between random good 1 and random good 2 then a higher mathematical expectation of X_{12} is good in her opinion, other things being equal, and a higher variance or standard deviation is bad. She is averse to risk. Her continuous utility function denoted by $u(x)$ is a strictly increasing and concave function, where its slope gets flatter as wealth increases (see also [38]). The form and extent of the aversion to risk which is caught by the utility function under consideration will depend on her temperament, her current mood, and some other circumstance. This function is graphically represented outside of the budget set of the investor by using two mutually orthogonal axes of a two-dimensional Cartesian coordinate system on which an origin, a same unit of length, and an orientation are established (see also [88]). It follows that we have ${}_{(1)}x^i{}_{(2)}x^j$, $i, j = 1, \dots, m$, together with their masses on the horizontal axis. We have consequently $u({}_{(1)}x^i{}_{(2)}x^j)$ together with their masses on the vertical one. We are evidently faced with m^2 masses located on $u(x)$ as well as on two mutually orthogonal axes. We have also to consider ${}_{(1)}x^i{}_{(1)}x^i$ and $u({}_{(1)}x^i{}_{(1)}x^i)$ together with m^2 masses as well as ${}_{(2)}x^i{}_{(2)}x^i$ and $u({}_{(2)}x^i{}_{(2)}x^i)$ together with m^2 masses. It is clear that ${}_{(2)}x^j{}_{(1)}x^i$, $j, i = 1, \dots, m$, together with their masses on the horizontal axis as well as $u({}_{(2)}x^j{}_{(1)}x^i)$ together with their masses on the vertical one give rise to the same values as ${}_{(1)}x^i{}_{(2)}x^j$ and $u({}_{(1)}x^i{}_{(2)}x^j)$. Each possible value that is considered on the horizontal axis of a two-dimensional Cartesian coordinate system is expressed by using the arithmetic product of two values associated with two contingent consumption plans which are separately considered. Accordingly, all coherent arithmetic means are considered. They transfer on a one-dimensional straight line all coherent α-products.

We note the following

Remark 25. Let $u(x)$ be the utility function identifying a risk-averse investor. It is considered outside of the budget set of the investor. This function lives inside of a two-dimensional Cartesian coordinate system. All masses characterizing each joint distribution which is considered in order to release X_{12} from the notion of ordered pair of contingent consumption plans are located on some points of its diagram. They identify different one-dimensional convex sets as joint probabilities of every joint distribution vary in the interval from 0 to 1 by taking all the values between 0 and 1, end points included, into account. The number of these values is infinite. There are different one-dimensional convex sets on the horizontal axis as well as different one-dimensional convex sets on the vertical one. All marginal probabilities of every marginal distribution under consideration vary in the interval from 0 to 1 by taking all the values between 0 and 1, end points included, into account. The number of these values is infinite. We note that

$$(\mathbf{P}(X_{12}), \mathbf{P}[u(X_{12})])$$

is a point of a two-dimensional Cartesian coordinate system belonging to the union of different one-dimensional convex sets. Such one-dimensional convex sets are found on the horizontal axis to which $\mathbf{P}(X_{12})$ belongs as well as on the vertical one to which $\mathbf{P}[u(X_{12})]$ belongs. If $u(x)$ identifies a risk-loving investor or a risk-neutral decision-maker then all of this continues to be valid. If X is a random good having m possible values then $(\mathbf{P}(X), \mathbf{P}[u(X)])$ is a two-dimensional point expressing two barycenters of two nonparametric distributions of mass, where it turns out to be $\mathbf{P}(X) = x^k p_k$ and $\mathbf{P}[u(X)] = u(x^k)\, p_k$ respectively ($k \in I_m = \{1, 2, \dots, m\}$). □

We observe that X_{12} has been constructed in such a way that the marginal distributions of ${}_1X$ and ${}_2X$ never change with respect to the starting ones. The marginal distributions of the joint distribution connected with ${}_1X\,{}_1X$ coincide with the probability distribution of ${}_1X$. The marginal distributions of the joint distribution connected with ${}_2X\,{}_2X$ coincide with the probability distribution of ${}_2X$. The marginal distributions of the joint distribution connected with ${}_1X\,{}_2X$ coincide with the probability distribution of ${}_1X$ and ${}_2X$ respectively. The marginal distributions of the joint distribution connected with ${}_2X\,{}_1X$ coincide with the probability distribution of ${}_2X$ and ${}_1X$ respectively.

The investor estimates all joint probabilities of ${}_1X\,{}_2X$ inside of her budget set in such a way that there exists an inverse linear relationship between ${}_1X$ and ${}_2X$. She is risk averse. For a risk-averse investor, the utility of the mathematical expectation of X_{12} is greater than the expected utility of X_{12} given by

$$\begin{aligned}\mathbf{P}[u(X_{12})] &= \begin{vmatrix} u({}_{(1)}x^{i}\,{}_{(1)}x^{i})p_{ii} & u({}_{(1)}x^{i}\,{}_{(2)}x^{j})p_{ij} \\ u({}_{(2)}x^{j}\,{}_{(1)}x^{i})p_{ji} & u({}_{(2)}x^{i}\,{}_{(2)}x^{i})p_{ii} \end{vmatrix} \\ &= \left[u({}_{(1)}x^{i}\,{}_{(1)}x^{i})p_{ii}\,u({}_{(2)}x^{i}\,{}_{(2)}x^{i})p_{ii} - u({}_{(1)}x^{i}\,{}_{(2)}x^{j})p_{ij}\,u({}_{(2)}x^{j}\,{}_{(1)}x^{i})p_{ji}\right] > 0,\end{aligned} \tag{4.41}$$

where (4.4) holds with regard to each factor characterizing the minuend and the subtrahend of (4.41). We consider an extension of Jensen's inequality connected with a discrete probability distribution. We denote by

$$x_{12} = \mathbf{P}_u(X_{12}) \tag{4.42}$$

the certainty equivalent to X_{12} given by

$$\mathbf{P}_u(X_{12}) = u^{-1}\{\mathbf{P}[u(X_{12})]\}. \tag{4.43}$$

We note that (4.43) represents an associative mean. It is an increasing transform of the arithmetic mean considered by means of u and obtained by using a bilinear function of the columns of a square matrix of order 2 (see Theorem 3). Since it turns out to be $x_{12} < \mathbf{P}(X_{12})$ on the horizontal axis, it is possible to say that X_{12} is not preferred to x_{12} in opinion of a risk-averse investor. In all cases she will prefer the certain alternative to the uncertain one. She would content herself with receiving with certainty x_{12} which is less than $\mathbf{P}(X_{12})$ in exchange for the hypothetical gain given by $2\mathbf{P}(X_{12})$ whose probability is judged to be equal to $1/2$ by her. In the scale of utility in which her judgments of indifference are based, it is possible to observe equal levels on the vertical axis in passing from 0 to x_{12} and from x_{12} to $2\mathbf{P}(X_{12})$ on the horizontal axis, where 0 and $2\mathbf{P}(X_{12})$ express two equiprobable events of a partition of two incompatible and exhaustive events. The possibility of inserting the degree of preferability of X_{12} into the scale of the certain amounts is a necessary condition of all rational decision-making criteria that can be followed (see also [82]).

The investor estimates all joint probabilities of ${}_1X\,{}_2X$ inside of her budget set in such a way that there exists a direct linear relationship between ${}_1X$ and ${}_2X$. She is risk lover. For a risk-loving investor, the expected utility of X_{12} is greater than the utility of the mathematical expectation of X_{12}. Her continuous utility function is a strictly increasing and convex function, where its slope gets steeper as wealth increases. The form and extent of this attitude towards risk which is caught by the utility function under consideration will depend on her temperament, her current mood, and some other circumstance.

The investor estimates all joint probabilities of ${}_1X\,{}_2X$ inside of her budget set in such a way that ${}_1X$ and ${}_2X$ are stochastically independent. She is risk neutral. Accordingly, it is possible to observe that among those decisions leading to different joint contingent consumption plans her best choice under conditions of uncertainty and riskiness must be the one leading to the plan with the highest mathematical expectation denoted by $\mathbf{P}(X_{12})$. Her continuous utility function is an increasing linear function. It is the 45-degree line. Its graphical form is always the same unlike the previous cases.

4.5.1 The criteria of rational choices being made by the investor: multiple random goods of order 2

We establish the following

Definition 14. *The certain amount that the investor subjectively judges to be equivalent to a double random good denoted by* X_{12} *is expressed by* $\mathbf{P}(X_{12})$ *whenever she is only interested in the mathematical expectation of* X_{12}*. It is the price of* X_{12} *for her whenever her utility function coincides with the* 45*-degree line. It coincides with its coherent prevision given by*

$$\mathbf{P}(X_{12}) = \begin{vmatrix} \mathbf{P}({}_1X\,{}_1X) & \mathbf{P}({}_1X\,{}_2X) \\ \mathbf{P}({}_2X\,{}_1X) & \mathbf{P}({}_2X\,{}_2X) \end{vmatrix} = \mathbf{P}({}_1X\,{}_1X)\,\mathbf{P}({}_2X\,{}_2X) - \mathbf{P}({}_1X\,{}_2X)\,\mathbf{P}({}_2X\,{}_1X) > 0. \quad (4.44)$$

It represents the price that the investor is willing to pay in order to purchase the right to participate in a bet that places her in the uncertain situation denoted by $X_{12} = \{{}_1X,\, {}_2X\}$*.*

The slope of the budget line is equal to -1 whenever two random goods are the same. We note the following

Remark 26. A choice being made by the investor under conditions of uncertainty and riskiness is rational if and only if she chooses any coherent evaluation of the marginal probabilities together with the joint ones characterizing m^2 possible quantitative states of the world of two contingent consumption plans that are jointly considered. She chooses a continuous and strictly increasing utility function in accordance with her subjective attitude towards risk. She fixes as her goal the maximization of the expected value of her cardinal utility, where the nature of such an expected value is firstly bilinear. □

Given a concave utility function denoted by $u(x)$, where x coincides with the monetary wealth of a risk-averse investor, it is possible to say that X_{12} is preferred to another double random good denoted by X_{34} if and only if it turns out to be

$$\mathbf{P}[u(X_{12})] > \mathbf{P}[u(X_{34})] \quad (4.45)$$

on the vertical axis of a two-dimensional Cartesian coordinate system. It follows that it turns out to be

$$\mathbf{P}_u(X_{12}) > \mathbf{P}_u(X_{34}) \quad (4.46)$$

on the horizontal axis, where $\mathbf{P}_u(X_{12})$ is less than $\mathbf{P}(X_{12})$, whereas $\mathbf{P}_u(X_{34})$ is less than $\mathbf{P}(X_{34})$. If a risk-averse investor is firstly faced with X_{34} then to pass from X_{34} to X_{12} is an advantageous transaction to her because $\mathbf{P}_u$ increases. Given a convex utility function denoted by $u(x)$, it is possible to say that X_{12} is preferred to X_{34} by a risk-loving investor if and only if (4.45) and (4.46) hold. We observe that $\mathbf{P}_u(X_{12})$ is greater than $\mathbf{P}(X_{12})$ as well as $\mathbf{P}_u(X_{34})$ is greater than $\mathbf{P}(X_{34})$ on the horizontal axis. Given the 45-degree line denoted by $u(x)$ identifying the identity of monetary value and utility, it is possible to say that X_{12} is preferred to X_{34} by a risk-neutral investor if and only if (4.45) and (4.46) continue to be valid. We note that it turns out to be $\mathbf{P}_u(X_{12}) = \mathbf{P}(X_{12})$ as well as $\mathbf{P}_u(X_{34}) = \mathbf{P}(X_{34})$ on the horizontal axis.

It is also possible to compare more than two joint contingent consumption plans. If the investor prefers X_{12} to X_{34} then it turns out to be $\mathbf{P}[u(X_{12})] > \mathbf{P}[u(X_{34})]$. If she prefers X_{34}

to X_{56}, where X_{56} is different from X_{12} and X_{34}, then we observe $\mathbf{P}[u(X_{34})] > \mathbf{P}[u(X_{56})]$. It follows that she rationally prefers X_{12} to X_{56}, so we note $\mathbf{P}[u(X_{12})] > \mathbf{P}[u(X_{56})]$.

A choice is optimal if and only if there exists a utility function whose maximum expected value is firstly bilinear (see also [81]). An extension of Daniel Bernoulli's approach to the notion of expected utility is carried out. All rational choices under conditions of uncertainty and riskiness can be ranked by the investor inside of a linear space over $\mathbb{R}$ provided with a quadratic metric. Accordingly, she is able to establish which is her best choice (see also [65]).

Even if we jointly study three or more than three contingent consumption plans, it is never practically possible to consider more than two contingent consumption plans at a time from a metric point of view (see also [47]). This is because we use a quadratic metric. It can be a linear or multilinear metric.

Chapter 5

Principal components, eigenequation, and eigenspaces in the theory of decision-making

5.1 Introduction

5.1.1 The investor modeled as being a consumer

Given the two-good assumption, let X_1 and X_2 be two random goods coinciding with two risky assets. We denote by $(\mathbf{P}(X_1), \mathbf{P}(X_2))$ the list of two numbers representing the average consumption of X_1 and X_2. Such a list is analogous to the one characterizing the consumer's consumption bundle whose goods are two. The two numbers of this list could also be equal.

In general, the objects of investor choice are of a multilinear nature. In particular, they are of a bilinear nature whenever we consider the two-good assumption. It is more general than one could think at first, since it is always possible to interpret one of the two goods under consideration as identifying everything else the investor might want to choose. The objects of investor choice are studied by using bilinear and multilinear indices of a metric nature. They are firstly of a bilinear nature whenever they are determined inside of a subset of a two-dimensional linear space over $\mathbb{R}$ coinciding with the investor's choice space. Such a space is furnished with a quadratic metric.

The investor is modeled as being a consumer. The ultimate good being chosen by her is the average consumption that money can purchase. The set of all possible alternatives for each random good under consideration consists of a finite number of single points, where every single point is a real number identifying a well-determined proposition. The latter will be true or false at the right time. We are not interested in combining such points to form new events. On the other hand, since we deal with finite partitions of mutually exclusive events, this has no meaning for our purposes. We are interested in attributing to all single points of the set of all possible alternatives for each random good into account all those probabilities generating a coherent prevision of it. It follows that we summarize discrete probability distributions inside of a subset of a linear space over $\mathbb{R}$. What the investor actually chooses inside of her budget set is a coherent summary of a nonparametric distribution of mass. In this chapter, we focus on nonparametric distributions of mass as well. We do not focus on coherent summaries of them only.

5.1.2 The basic thing that is being chosen by the investor under conditions of uncertainty and riskiness

The basic thing that is being chosen by the investor under conditions of uncertainty and riskiness is a nonparametric distribution of mass connected with a random good. A risky asset is a random good whose possible values are of a monetary nature. It is studied inside of the budget set of the investor. Since the investor does not know its true value, she is in doubt between at least two possible values. She is in doubt between n possible values whenever a risky asset is characterized by n possible and distinct values. We consider a finite partition of n incompatible and exhaustive elementary events connected with n possible values for a risky asset. They are points in the space of random goods, where the latter is a linear space over $\mathbb{R}$ provided with a quadratic metric. By considering m risky assets, we suppose that it always turns out to be $n > m$. We say that m risky assets are logically independent, so there are n^m possible values for a multiple risky asset of order m. They coincide with the Cartesian product of the sets of possible values for every risky asset which is the component of a multiple risky asset of order m. A multiple risky asset of order m is a multiple random good of order m.

Let X_i be a risky asset. We write $I_m = \{1, \ldots, m\}$, so it turns out to be $i \in I_m$. The generic and possible value for X_i is denoted by X_{β_i}, where we have $\beta_i \in I_n = \{1, \ldots, n\}$. It follows that if $E_{\beta_1} \ldots E_{\beta_m}$ is an event of a finite partition of n^m elementary events then we consider an ordered m-tuple of corresponding values denoted by $(X_{\beta_1}, X_{\beta_2}, \ldots, X_{\beta_m})$. A multiple risky asset of order m is denoted by

$$X_{\{m\}} = \{(X_{\beta_1}, X_{\beta_2}, \ldots, X_{\beta_m}); p_{\beta_1 \ldots \beta_m} \,|\, (\beta_1, \beta_2, \ldots, \beta_m) \in I_n^{(m)}\}, \tag{5.1}$$

where we have coherently

$$\sum_{\beta_1 \ldots \beta_m = 1}^{n} p_{\beta_1 \ldots \beta_m} = 1. \tag{5.2}$$

In particular, if we deal with a joint risky asset then we write

$$\sum_{\beta_r \beta_s = 1}^{n} p_{\beta_r \beta_s} = 1, \quad \forall (r, s) \in I_m^{(2)}. \tag{5.3}$$

If the generic and possible value for a risky asset is denoted by ${}_rX_\beta$ then it is also possible to write

$$\sum_{\beta\gamma = 1}^{n} {}_{rs}p_{\beta\gamma} = 1, \quad \forall (r, s) \in I_m^{(2)} \tag{5.4}$$

instead of (5.3). In this chapter, we focus on mathematical, statistical, and economic aspects characterizing a set of m risky assets.

5.2 Risky assets viewed to be as elements of a linear manifold whose possible values are subjected to changes of origin

Let E^n be a linear space over $\mathbb{R}$ and let ${}_nB_{\mathbf{e}}^{\perp} = \{\mathbf{e}_\beta; \beta \in I_n\}$ be an orthonormal basis of it. We say that E^n is provided with a Euclidean metric. This is because we are able to consider a metric tensor with respect to ${}_nB_{\mathbf{e}}^{\perp}$. It belongs to $E^n \otimes E^n$. We write

$${}_{\mathbf{e}}g_{\beta\gamma} = \langle \mathbf{e}_\beta, \mathbf{e}_\gamma \rangle = \delta_{\beta\gamma}, \tag{5.5}$$

where $\delta_{\beta\gamma}$ is the Kronecker delta. It is clear that (5.5) represents the generic component of a tensor of order 2. All components of it are scalars. They give origin to an $n \times n$ identity matrix expressed by

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}. \tag{5.6}$$

The possible values for a risky asset are uniquely expressed by the contravariant components of an n-dimensional vector of E^n with respect to ${}_nB_{\mathbf{e}}^{\perp}$. We write

$$\mathbf{x}_i = x_i^{\beta} \mathbf{e}_{\beta} \in E^n \tag{5.7}$$

by using the Einstein notation. We consider m risky assets, so we write

$$\mathbf{x}_i = x_i^{\beta} \mathbf{e}_{\beta}, \quad \forall i \in I_m.$$

We suppose that all vectors expressed by (5.7) are linearly independent without loss of generality.

We observe that

$$I_m^{[\mathbf{x}]} = \{\mathbf{x}_i; i \in I_m\} \tag{5.8}$$

represents a basis of an m-dimensional linear manifold embedded in E^n and denoted by ${}_{\mathbf{x}}\mathscr{M}^m$. All linear combinations of the basis vectors contained in (5.8) give origin to the elements of ${}_{\mathbf{x}}\mathscr{M}^m$. We consider all their translations with respect to an n-dimensional vector of E^n coinciding with the one whose elements are all equal to 0. We write

$$\mathbf{x} = x^i \mathbf{x}_i \in {}_{\mathbf{x}}\mathscr{M}^m, \quad \forall x^i \in \mathbb{R}, \tag{5.9}$$

where we have $i = 1, \dots, m$. We note the following

Remark 27. Let X_i be a risky asset whose possible monetary values are denoted by $I(X_i) = \{x_i^1, x_i^2, \dots, x_i^n\}$, $i = 1, \dots, m$. It follows that x_i^1 is the return on X_i if E_{i1} occurs with probability denoted by p_{i1}, x_i^2 is the return on X_i if E_{i2} occurs with probability denoted by p_{i2}, ..., x_i^n is the return on X_i if E_{in} occurs with probability denoted by p_{in}. It is possible to note that E_{i1}, ..., E_{in} are elementary events of a finite partition of events. It is possible to say that x_i^1 is the wealth that X_i yields and that can be spent by the investor if E_{i1} occurs with probability denoted by p_{i1}, x_i^2 is the wealth that X_i yields and that can be spent by her if E_{i2} occurs with probability denoted by p_{i2}, ..., x_i^n is the wealth that X_i yields and that can be spent by her if E_{in} occurs with probability denoted by p_{in}. □

We observe that $\overline{\mathbf{x}}_i$, $i = 1, \dots, m$, represents a vector of E^n having all its contravariant components equal to the expected return on X_i denoted by $\mathbf{P}(X_i)$, $i = 1, \dots, m$. If the sum of all probabilities connected with the possible values for X_i, $i = 1, \dots, m$, is equal to 1 then $\mathbf{P}(X_i)$ is coherent. We write

$$\overline{\mathbf{x}}_i = \begin{pmatrix} \overline{x}_i^1 = \mathbf{P}(X_i) \\ \overline{x}_i^2 = \mathbf{P}(X_i) \\ \vdots \\ \overline{x}_i^n = \mathbf{P}(X_i) \end{pmatrix}, \tag{5.10}$$

where we have $i = 1, \ldots, m$. We write

$$\bar{\mathbf{x}}_i = \bar{x}_i^{\beta} \mathbf{e}_{\beta}, \quad \forall i \in I_m.$$

We observe that every $\bar{\mathbf{x}}_i$, $i = 1, \ldots, m$, represents an n-dimensional vector of E^n with respect to which it is possible to consider an m-dimensional linear manifold denoted by ${}_{\bar{\mathbf{x}}}\mathcal{M}^m$. The m-dimensional linear manifold denoted by ${}_{\mathbf{d}}\mathcal{M}^m_{(O)}$ is expressed by

$${}_{\mathbf{d}}\mathcal{M}^m_{(O)} = {}_{\mathbf{x}}\mathcal{M}^m \ominus {}_{\bar{\mathbf{x}}}\mathcal{M}^m. \tag{5.11}$$

It is obtained by considering the direct difference of two m-dimensional linear manifolds. Every vector denoted by $\mathbf{d}_i \in {}_{\mathbf{d}}\mathcal{M}^m_{(O)}$ represents all deviations of the possible and distinct values for X_i from its expected return vectorially expressed by $\bar{\mathbf{x}}_i$, $i = 1, \ldots, m$ (see also [79]). Every vector denoted by $\mathbf{d}_i \in {}_{\mathbf{d}}\mathcal{M}^m_{(O)}$ represents a basic risky asset whose possible values are subjected to a change of origin. We consequently write

$${}_{(O)}I_m^{[\mathbf{d}]} = \{\mathbf{x}_i - \bar{\mathbf{x}}_i; i \in I_m\} = \{\mathbf{d}_i; i \in I_m\}. \tag{5.12}$$

We observe that (5.12) represents a basis of ${}_{\mathbf{d}}\mathcal{M}^m_{(O)}$. We denote it by ${}_mB_{\mathbf{d}}$. All linear combinations of the basic risky assets contained in (5.12) span ${}_{\mathbf{d}}\mathcal{M}^m_{(O)}$, so we write

$$\mathbf{d} = d^i \mathbf{d}_i \in {}_{\mathbf{d}}\mathcal{M}^m_{(O)}, \quad \forall d^i \in \mathbb{R}, \tag{5.13}$$

where it turns out to be $i = 1, \ldots, m$. The number of all these linear combinations is infinite. We denote by $D_{\{m\}}$ a multiple risky asset of order m whose components are m risky assets. The first component of $D_{\{m\}}$ is a risky asset whose possible and distinct values coincide with all deviations of the possible values for X_1 from $\mathbf{P}(X_1)$, ..., the m-th component of $D_{\{m\}}$ is a risky asset whose possible and distinct values coincide with all deviations of the possible values for X_m from $\mathbf{P}(X_m)$ (see also [53]). We say that $D_{\{m\}}$ is defined with respect to $X_{\{m\}}$.

5.3 A multiple risky asset of order m and its probabilities

Let $\mathbf{y}_h$ be a vector representing a risky asset (see also [52]). It is obtained by considering a linear combination of m basic risky assets, where the possible values for each of them are subjected to a change of origin. We write

$$\mathbf{y}_h = y_h^i \mathbf{d}_i \in {}_{\mathbf{d}}\mathcal{M}^m_{(O)}. \tag{5.14}$$

We say that $\{y_h^i\}$ is the set of m contravariant components of $\mathbf{y}_h$ with respect to ${}_mB_{\mathbf{d}}$. If we want to obtain one of the vectors of ${}_mB_{\mathbf{d}}$ then only one contravariant component of $\mathbf{y}_h$ has to be equal to 1. All other contravariant components of $\mathbf{y}_h$ have to be equal to 0.

Since ${}_{\mathbf{d}}\mathcal{M}^m_{(O)}$ is embedded in E^n, it is possible to write the same vector with respect to ${}_nB_{\mathbf{e}}^{\perp}$. We obtain

$$\mathbf{y}_h = y_h^i x_i^{\beta} \mathbf{e}_{\beta} \in E^n, \tag{5.15}$$

where $\{y_h^i x_i^{\beta}\}$ is the set of n contravariant components of $\mathbf{y}_h$ with respect to ${}_nB_{\mathbf{e}}^{\perp}$.

We deal with a multiple risky asset of order m, so we have to take association probabilities into account (see also [31]). A multiple risky asset of order m is then characterized by an affine tensor of order m whose components identify association probabilities. A basis of

$$\overbrace{E^n \otimes E^n \otimes \ldots \otimes E^n}^{m\ times}$$

is denoted by $B_{n^m} = \{\mathbf{e}_{\beta_1} \otimes \ldots \otimes \mathbf{e}_{\beta_m}\}$, so we write

$$p_{1\ldots m} = {}_{1\ldots m}p^{\beta_1\ldots\beta_m}\mathbf{e}_{\beta_1} \otimes \ldots \otimes \mathbf{e}_{\beta_m}, \tag{5.16}$$

where (5.16) is the generic component of an affine tensor of order m representing association probabilities whose sum has coherently to be equal to 1 on the whole partition of n^m events. We note that $\mathbf{y}_h \in E^n$ is nothing but a tensor of order 1. We suppose that all contravariant components of an n-dimensional vector of E^n denoted by ϕ are equal to 1 with respect to ${}_nB_{\mathbf{e}}^{\perp}$, so it is possible to construct an affine tensor of order $m-1$ expressed by

$$\phi_{\beta_i}^{\beta_1\ldots\beta_m} = \prod_{\substack{j=1 \\ j\neq i}}^{m} \phi^{\beta_j}. \tag{5.17}$$

We construct (5.17) with respect to ϕ. The following pair of expressions

$$\begin{cases} Y_h^{\beta_1\ldots\beta_m} = y_h^i x_i^{\beta_i} \phi_{\beta_i}^{\beta_1\ldots\beta_m} \\ {}_{1\ldots m}p_{\beta_1\ldots\beta_m} \end{cases} \tag{5.18}$$

allows us to represent all deviations concerning all single risky assets of a multiple risky asset of order m together with their association probabilities. We note that the tensor product between a tensor whose order is equal to 1 and a tensor whose order is equal to $m-1$ coincides with a tensor whose order is equal to $1+m-1=m$. Moreover, by using an orthonormal basis of E^n, we could represent the generic component of an affine tensor of order m identifying association probabilities by means of contravariant or covariant indices. We choose a covariant notation with regard to them.

5.3.1 A symmetric tensor obtained by using joint probabilities

If we deal with two risky assets which are jointly considered then we need to take the tensor product of two n-dimensional linear spaces over $\mathbb{R}$ into account (see also [6]). We prove the following

Proposition 1. Let $E^n \otimes E^n$ be a linear space over $\mathbb{R}$ containing affine tensors of order 2 and let $B_{n^2} = \{\mathbf{e}_\beta \otimes \mathbf{e}_\gamma\}$ be a basis of it. Let $(\mathbf{d}_i, \mathbf{d}_j)$ be an ordered pair of basic risky assets, where every risky asset of it belongs to ${}_{\mathbf{d}}\mathscr{M}_{(O)}^m$. All possible ordered pairs of basic risky assets of ${}_{\mathbf{d}}\mathscr{M}_{(O)}^m$ give rise to an α-metric tensor whose components are symmetric. They give origin to an $m \times m$ symmetric matrix.

Proof. If we consider an ordered pair of basic risky assets then we have to take their nonparametric joint distribution into account. Association probabilities are now joint probabilities. The affine tensor of order 2 whose components represent all joint probabilities under consideration is expressed by

$$p_{ij} = {}_{ij}p^{\beta\gamma}\mathbf{e}_\beta \otimes \mathbf{e}_\gamma, \tag{5.19}$$

where ${}_{ij}p^{\beta\gamma}$ is the generic component of it. It is a joint probability. We write

$$_{\mathbf{d};\alpha}g_{ij} = {}_{\mathbf{d}}g_{ij} = \langle \mathbf{d}_i, \mathbf{d}_j \rangle_\alpha = d_i^\beta d_j^\gamma \; {}_{ij}p_{\beta\gamma} \tag{5.20}$$

with respect to ${}_mB_\mathbf{d}$, where we use covariant indices with regard to a generic joint probability. We observe that (5.20) represents a tensor of order 2 called α-metric tensor with respect to an m-dimensional linear manifold denoted by ${}_\mathbf{d}\mathscr{M}^m_{(O)}$. It is a symmetric tensor. Indeed, its components are symmetric. If it turns out to be $m > 2$ then different comparisons between two risky assets of ${}_mB_\mathbf{d}$ are necessary. Only pairwise comparisons are possible. This is because we use a quadratic metric. An $m \times m$ symmetric matrix is then generated by all possible pairwise comparisons. The number of its distinct elements is given by

$$C^r_{m,2} = \frac{1}{2}m(m+1). \tag{5.21}$$

From (5.20), it is possible to derive the notion of α-norm. Accordingly, we write

$$_\mathbf{d}g_{ii} = \|\mathbf{d}_i\|^2_\alpha = d_i^\beta d_i^\beta \; {}_ip_\beta. \tag{5.22}$$

We therefore obtain the following $m \times m$ matrix

$$\begin{bmatrix} {}_\mathbf{d}g_{11} & {}_\mathbf{d}g_{12} & \cdots & {}_\mathbf{d}g_{1m} \\ {}_\mathbf{d}g_{21} & {}_\mathbf{d}g_{22} & \cdots & {}_\mathbf{d}g_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ {}_\mathbf{d}g_{m1} & {}_\mathbf{d}g_{m2} & \cdots & {}_\mathbf{d}g_{mm} \end{bmatrix} \tag{5.23}$$

whose structure is evidently symmetric. □

The Schwarz's α-generalized inequality is considered in order to complete the α-metric structure of ${}_\mathbf{d}\mathscr{M}^m_{(O)}$. It is given by

$$\left|{}_\mathbf{d}g_{ij}\right| \leq \sqrt{{}_\mathbf{d}g_{ii}}\,\sqrt{{}_\mathbf{d}g_{jj}}. \tag{5.24}$$

We observe that (5.5) and (5.20) are two tensors of order 2. The former is defined with respect to an orthonormal basis of E^n without considering joint probabilities, whereas the latter is defined with respect to an ordered pair of basic risky assets of ${}_\mathbf{d}\mathscr{M}^m_{(O)}$. We note that an affine tensor of order 2 whose components represent all joint probabilities under consideration always corresponds to this pair of basic risky assets of ${}_\mathbf{d}\mathscr{M}^m_{(O)}$. It must be used to obtain (5.20). It follows that (5.5) and (5.20) are conceptually different.

We note the following

Remark 28. If $m = 2$ then we obtain the following 2×2 matrix

$$\begin{bmatrix} {}_\mathbf{d}g_{11} & {}_\mathbf{d}g_{12} \\ {}_\mathbf{d}g_{21} & {}_\mathbf{d}g_{22} \end{bmatrix}$$

whose structure is symmetric. The determinant of this matrix allows us to obtain an aggregate measure based on changes of origin. We obtain it outside of the budget set of the investor. Such a measure is itself based on what the investor chooses inside of her budget set. What the investor chooses inside of her budget set is a bilinear measure which is always decomposed into two linear measures, $\mathbf{P}(X_1)$ and $\mathbf{P}(X_2)$. It is evidently a disaggregate measure. □

5.4 Eigenvectors connected with a symmetric tensor obtained by using joint probabilities: their representation

We prove in the appendix the following

Proposition 2. Let ${}_{\mathbf{d}}g_{ij}$ be an α-metric tensor connected with ${}_{\mathbf{d}}\mathscr{M}^m_{(O)}$ whose components identify an $m \times m$ symmetric matrix. Suppose that all entries of it outside of its main diagonal are equal to zero. Suppose that all its main diagonal entries are different. Since ${}_{\mathbf{d}}g_{ij}$ identifies an eigenequation and all eigenvectors associated with ${}_{\mathbf{d}}g_{ij}$ are pairwise α-orthogonal, such eigenvectors can be represented with regard to two distinct orthonormal bases of ${}_{\mathbf{d}}\mathscr{M}^m_{(O)}$.

We consider the following eigenequation

$$\left({}_{\mathbf{d}}g_{kh} - {}_{\mathbf{d}}\lambda_{(k)}\,\delta_{kh}\right)v^k_{(k)} = 0.$$

We deal with m different values characterizing the main diagonal entries of an $m \times m$ symmetric matrix. It is an $m \times m$ diagonal matrix. Given a basis of ${}_{\mathbf{d}}\mathscr{M}^m_{(O)}$ consisting of m basic risky assets identifying m nonparametric marginal distributions of mass, whenever we jointly consider two different basic risky assets of it, we observe that the property of α-orthogonality is satisfied. This means that the covariance of these two risky assets is equal to 0, so a pairwise non-correlation takes place. All of this is possible because each point of the budget set of the investor is a synthesized element of the Fréchet class. Since we consider changes of origin outside of her budget set, marginal and joint masses do not change. The state of information and knowledge associated with a given investor is taken into account. Accordingly, a pairwise non-correlation has never an absolute meaning. Its meaning is always of a relative and subjective nature.

5.5 The projection of a linear manifold onto another one: its reason

Any evaluation of probability referred to an event always depends on the variable group of circumstances assumed to be relevant to its occurrence. Such circumstances are known at the time. In general, they vary from moment to moment. This means that any evaluation of probability referred to an event can vary according to the state of information and knowledge associated with a given investor. Her state of information and knowledge even affects the set of all possible alternatives concerning every risky asset under consideration.

Accordingly, it is possible to consider ${}_{\mathbf{d}'}\mathscr{M}^m_{(O)}$ for all above reason. It is an m-dimensional linear manifold embedded in E^n. It is therefore a translation of an m-dimensional linear subspace of E^n with respect to the zero vector of E^n. Such a linear manifold is spanned by m linearly independent vectors, where each of them represents ordered deviations from a mean value which are subjectively and coherently determined by a given investor. They identify m basic risky assets which are the components of a multiple risky asset of order m denoted by $D'_{\{m\}}$. They identify m nonparametric marginal distributions of mass. All linear combinations of these m basic risky assets span risky assets which are intrinsically related. They belong to ${}_{\mathbf{d}'}\mathscr{M}^m_{(O)}$. If we consider two risky assets of them together with their joint probabilities then we observe that they are α-orthogonal, so their covariance is equal to 0. The marginal masses of two single risky assets are always the same, so the investor estimates their joint probabilities in such a way that their covariance is equal to 0. All of this is possible because each point of the budget set of the investor where two risky assets are studied is a synthesized element of the Fréchet class.

Given $D_{\{m\}}$, we suppose that every risky asset belonging to $D_{\{m\}}$ is associated with every risky asset belonging to $D'_{\{m\}}$. This means that we consider two multiple risky assets of order m denoted by $D_{\{m\}}$ and $D'_{\{m\}}$ as well as a set of m^2 joint risky assets whose generic element is denoted by (D_i, D'_j). We also consider the α-orthogonal projection of ${}_{\mathbf{d}'}\mathscr{M}^m_{(O)}$ onto ${}_{\mathbf{d}}\mathscr{M}^m_{(O)}$. It is denoted by ${}_{\hat{\mathbf{d}}}\mathscr{M}^m_{(O)}$. All vectors belonging to ${}_{\mathbf{d}}\mathscr{M}^m_{(O)}$ represent the starting risky assets, where the possible values for each of them are subjected to changes of origin. All vectors belonging to ${}_{\hat{\mathbf{d}}}\mathscr{M}^m_{(O)}$ express a logical and formal hypothesis with respect to a given structure of the nonparametric distributions of mass identifying all risky assets under consideration. The condition of invariance of the covariance of any two risky assets expresses this knowledge hypothesis.

We prove the following

Proposition 3. Let $\mathbf{d}'_j$ be an element of ${}_{\mathbf{d}'}\mathscr{M}^m_{(O)}$. Let $\hat{\mathbf{d}}_j$ be the corresponding element of ${}_{\hat{\mathbf{d}}}\mathscr{M}^m_{(O)}$, where we have $j = 1, \ldots, m$. Hence, the covariant components of $\hat{\mathbf{d}}_j$ with respect to ${}_mB_{\mathbf{d}}$ are expressed by $d_{ji} = \langle \hat{\mathbf{d}}_j, \mathbf{d}_i \rangle_\alpha$. Its contravariant components are given by $d^k_j = d_{ji}\, {}_{\mathbf{d}}g^{ki}$.

Proof. It is possible to write

$$\mathbf{d}^*_j = \mathbf{d}'_j - \hat{\mathbf{d}}_j, \quad \forall j \in I_m. \tag{5.25}$$

We construct ${}_{\hat{\mathbf{d}}}\mathscr{M}^m_{(O)}$ by solving the following system of m linear equations for every value of $j \in I_m$. It is given by

$$\langle \mathbf{d}^*_j, \mathbf{d}_i \rangle_\alpha = 0, \quad \forall i \in I_m. \tag{5.26}$$

We consider m systems expressed by (5.26). It is evident that we can write

$$\langle \mathbf{d}'_j, \mathbf{d}_i \rangle_\alpha - \langle \hat{\mathbf{d}}_j, \mathbf{d}_i \rangle_\alpha = 0, \quad \forall i \in I_m. \tag{5.27}$$

We note that (5.27) tells us that the covariance of two risky assets is invariant. We observe that it is possible to write

$$\hat{\mathbf{d}}_j = d^h_j \mathbf{d}_h, \tag{5.28}$$

where we have $\hat{\mathbf{d}}_j \in {}_{\hat{\mathbf{d}}}\mathscr{M}^m_{(O)}$, $j = 1, \ldots, m$. We note that it turns out to be $\mathbf{d}_h \in {}_mB_{\mathbf{d}}$. We put (5.28) into (5.27) as well as we remind how the α-metric tensor concerning ${}_{\mathbf{d}}\mathscr{M}^m_{(O)}$ has been defined. Hence, we write

$$\langle \mathbf{d}'_j, \mathbf{d}_i \rangle_\alpha - d^h_j\, {}_{\mathbf{d}}g_{hi} = \langle \mathbf{d}'_j, \mathbf{d}_i \rangle_\alpha - d_{ji} = 0. \tag{5.29}$$

It follows that we obtain

$$d_{ji} = \langle \mathbf{d}'_j, \mathbf{d}_i \rangle_\alpha. \tag{5.30}$$

We note that the covariant components of $\hat{\mathbf{d}}_j$ with respect to ${}_mB_{\mathbf{d}}$ are expressed by (5.30). Since the subtrahend of (5.27) is given by

$$\langle \hat{\mathbf{d}}_j, \mathbf{d}_i \rangle_\alpha = d^h_j \langle \mathbf{d}_h, \mathbf{d}_i \rangle_\alpha = d^h_j\, {}_{\mathbf{d}}g_{hi} = d_{ji}, \tag{5.31}$$

we note that the covariant components of $\mathbf{d}'_j$ and $\hat{\mathbf{d}}_j$ with respect to ${}_mB_{\mathbf{d}}$ are obtained in the same way. We have to establish the contravariant components of $\hat{\mathbf{d}}_j$ in order to complete the α-orthogonal projection of ${}_{\mathbf{d}'}\mathscr{M}^m_{(O)}$ onto ${}_{\mathbf{d}}\mathscr{M}^m_{(O)}$. We therefore write

$$d^h_j\, {}_{\mathbf{d}}g_{hi}\, {}_{\mathbf{d}}g^{ki} = d_{ji}\, {}_{\mathbf{d}}g^{ki}, \tag{5.32}$$

so we obtain

$$d^k_j = d_{ji}\, {}_{\mathbf{d}}g^{ki}, \tag{5.33}$$

where the contravariant components of the α-metric tensor under consideration denoted by ${}_{\mathbf{d}}g^{ki}$ are given by

$$\begin{bmatrix} \frac{1}{{}_{\mathbf{d}}g_{11}} & 0 & \dots & 0 \\ 0 & \frac{1}{{}_{\mathbf{d}}g_{22}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{{}_{\mathbf{d}}g_{mm}} \end{bmatrix}. \tag{5.34}$$

They form a square matrix of order m. It is a diagonal matrix. □

We note that m linear combinations of the basic risky assets of ${}_{\mathbf{d}}\mathscr{M}^m_{(O)}$ give origin to m risky assets of ${}_{\hat{\mathbf{d}}}\mathscr{M}^m_{(O)}$ represented by m vectors belonging to ${}_{\hat{\mathbf{d}}}\mathscr{M}^m_{(O)}$ according to (5.28). The contravariant components of each $\hat{\mathbf{d}}_j$, $j = 1,\dots,m$, coincide with the coefficients of each linear combination of m linear combinations of the basic risky assets of ${}_{\mathbf{d}}\mathscr{M}^m_{(O)}$. Given such components, it is possible to obtain the covariant components of the risky assets of ${}_{\hat{\mathbf{d}}}\mathscr{M}^m_{(O)}$ by using the covariant components of ${}_{\mathbf{d}}g_{ij}$ expressed by (5.23). Given the covariant components of the risky assets belonging to ${}_{\hat{\mathbf{d}}}\mathscr{M}^m_{(O)}$, it is conversely possible to obtain their contravariant components by using the contravariant components of ${}_{\mathbf{d}}g^{ij}$ expressed by (5.34). It is clear that all of this is possible by using the covariant or contravariant components of the α-metric tensor identifying an $m \times m$ symmetric matrix.

5.6 An appropriate basis of a linear manifold: a definition of principal components

Let $\{{}_{\mathbf{d}}\lambda_{(k)};\, (k) \in I_m\}$ be the set containing all eigenvalues of the α-metric tensor which has been constructed with respect to ${}_{\mathbf{d}}\mathscr{M}^m_{(O)}$. We suppose that they are all distinct. Let $\{\mathbf{v}_{(k)};\, (k) \in I_m\}$ be the corresponding set containing all normalized eigenvectors (see also [59]). They are pairwise α-orthogonal. We establish the following

Definition 15. *Given $D_{\{m\}}$, the principal components with respect to $D_{\{m\}}$ and denoted by $\mathbf{w}_{(h)}$, $(h) = 1,\dots,m$, are all linear combinations of vectors, where each of them represents a basic risky asset of $D_{\{m\}}$, whose coefficients are the components of a normalized eigenvector. We write*

$$\mathbf{w}_{(h)} = v^i_{(h)}\mathbf{d}_i, \quad \forall (h) \in I_m. \tag{5.35}$$

All principal components represent a basis of ${}_{\mathbf{d}}\mathscr{M}^m_{(O)}$. All principal components represent basic risky assets. Such a basis is denoted by ${}_mB_{\mathbf{w}_{\mathbf{d}}}$.

We prove in the appendix the following

Proposition 4. All principal components representing a basis of ${}_{\mathbf{d}}\mathscr{M}^m_{(O)}$ denoted by ${}_mB_{\mathbf{w_d}}$ identify an α-metric tensor which is a diagonal tensor.

By putting together (.85) and (.88) we write

$$ {}_{\mathbf{w}}g_{(k)(h)}\,{}_{\mathbf{w}}g^{(k)(h)} = \lambda_{(k)}\,\delta_{(k)(h)}\,\lambda^{(k)}\,\delta^{(k)(h)} = \lambda_{(k)}\,\lambda^{(j)}\,\delta^{(j)}_{(k)}. \tag{5.36} $$

We say that (5.36) identifies a mixed and α-metric tensor whose generic component can be denoted by ${}_{\mathbf{w}}g^{(j)}_{(k)}$. It follows that it turns out to be ${}_{\mathbf{w}}g^{(k)}_{(k)} = m$. This is because we consider a product of matrices by means of which we obtain an $m \times m$ identity matrix of which we compute its trace.

5.6.1 The projection of a linear manifold onto another one obtained by choosing a particular basis of it

If we project ${}_{\mathbf{d}'}\mathscr{M}^m_{(O)}$ onto ${}_{\mathbf{d}}\mathscr{M}^m_{(O)}$ then we obtain an m-dimensional linear manifold embedded in E^n denoted by ${}_{\hat{\mathbf{d}}}\mathscr{M}^m_{(O)}$. We observe that ${}_mB_{\mathbf{w_d}}$ is a basis of it. We therefore prove in the appendix the following

Proposition 5. If it is possible to write $\hat{\mathbf{d}}_j = d_j^{(i)}\mathbf{w}_{(i)}$, $\forall j \in I_m$, then the covariant components of $\hat{\mathbf{d}}_j$ are given by $d_{j(k)} = \langle \hat{\mathbf{d}}_j, \mathbf{w}_{(k)} \rangle_\alpha$. The contravariant components of $\hat{\mathbf{d}}_j$ are given by $d_j^{(h)} = \frac{\langle \mathbf{d}'_j, \mathbf{w}_{(h)} \rangle_\alpha}{\lambda_{(h)}}$.

5.7 A proportionality existing between risky assets: a particular case

Every risky asset whose possible values are subjected to a change of origin coincides with an n-dimensional vector of E^n identifying a nonparametric marginal distribution of mass. An n-dimensional vector of E^n is isomorphic to a point expressed by an ordered n-tuple of real numbers. This is because a located vector at the origin of E^n is entirely determined by its end point. Accordingly, an ordered n-tuple of real numbers can be called either a point of an affine space containing points or a vector of a linear space containing vectors. If we consider m risky assets then we deal with m ordered sets of real numbers. We say that two ordered sets of two nonzero real numbers denoted by $\{d_A^1, d_A^2\}$ and $\{d_B^1, d_B^2\}$ are proportional if it is possible to write

$$ d_A^1 : d_B^1 = d_A^2 : d_B^2. \tag{5.37} $$

This means that there exists a constant of proportionality denoted by h such that we have

$$ \frac{d_A^1}{d_B^1} = \frac{d_A^2}{d_B^2} = h. \tag{5.38} $$

In general, given two ordered sets of n real numbers denoted by $\{d_A^1, d_A^2, \ldots, d_A^n\}$ and $\{d_B^1, d_B^2, \ldots, d_B^n\}$, we say that they are proportional if it is possible to write

$$ \begin{pmatrix} d_A^1 = h\,d_B^1 \\ d_A^2 = h\,d_B^2 \\ \vdots \\ d_A^n = h\,d_B^n \end{pmatrix}. \tag{5.39} $$

It follows that we can write

$$\begin{pmatrix} d_A^1 - d_B^1 = (h-1)d_B^1 \\ d_A^2 - d_B^2 = (h-1)d_B^2 \\ \vdots \\ d_A^n - d_B^n = (h-1)d_B^n \end{pmatrix}. \tag{5.40}$$

Given the direct difference between $\{d_A^1, d_A^2, \ldots, d_A^n\}$ and a homothetic transformation of $\{d_B^1, d_B^2, \ldots, d_B^n\}$, if we say that such a difference is proportional to a third set of n real numbers denoted by $\{d_C^1, d_C^2, \ldots, d_C^n\}$ then we write

$$\begin{pmatrix} d_A^1 - x\,d_B^1 = y\,d_C^1 \\ d_A^2 - x\,d_B^2 = y\,d_C^2 \\ \vdots \\ d_A^n - x\,d_B^n = y\,d_C^n \end{pmatrix}. \tag{5.41}$$

We note that y is an average constant of proportionality. We suppose that all equalities expressed by (5.41) do not hold. We consequently establish a criterion by means of which it is possible to construct an ordered set of n real numbers whose elements are given by $\{d_{C'}^1, d_{C'}^2, \ldots, d_{C'}^n\}$. We note that $\{d_{C'}^1, d_{C'}^2, \ldots, d_{C'}^n\}$ must have pre-established characteristics with respect to $\{d_C^1, d_C^2, \ldots, d_C^n\}$. The following equalities

$$\begin{pmatrix} d_A^1 - x\,d_B^1 = y\,d_{C'}^1 \\ d_A^2 - x\,d_B^2 = y\,d_{C'}^2 \\ \vdots \\ d_A^n - x\,d_B^n = y\,d_{C'}^n \end{pmatrix} \tag{5.42}$$

must then be satisfied.

5.7.1 The condition of invariance of the covariance of two risky assets

In this subsection, we suppose that it turns out to be $m = 2$. All risky assets belonging to $_{\hat{\mathbf{d}}}\mathscr{M}_{(O)}^2$ are nothing but units of measurement with respect to which a given investor can measure and characterize all risky assets belonging to $_{\mathbf{d}}\mathscr{M}_{(O)}^2$. We write

$$\mathbf{d}_1^* = \mathbf{d}_1' - \hat{\mathbf{d}}_1 \tag{5.43}$$

and

$$\mathbf{d}_2^* = \mathbf{d}_2' - \hat{\mathbf{d}}_2. \tag{5.44}$$

It follows that we have to consider two systems of two linear equations in order to construct $_{\hat{\mathbf{d}}}\mathscr{M}_{(O)}^2$. We have

$$\begin{cases} \langle \mathbf{d}_1^*, \mathbf{d}_1 \rangle_\alpha = 0 \\ \\ \langle \mathbf{d}_1^*, \mathbf{d}_2 \rangle_\alpha = 0 \end{cases} \tag{5.45}$$

as well as

$$\begin{cases} \langle \mathbf{d}_2^*, \mathbf{d}_1 \rangle_\alpha = 0 \\ \\ \langle \mathbf{d}_2^*, \mathbf{d}_2 \rangle_\alpha = 0 \end{cases}. \tag{5.46}$$

If we consider (5.43) and (5.44) then we write

$$\begin{cases} \langle \mathbf{d}'_1, \mathbf{d}_1 \rangle_\alpha = \left\langle \hat{\mathbf{d}}_1, \mathbf{d}_1 \right\rangle_\alpha \\ \\ \langle \mathbf{d}'_1, \mathbf{d}_2 \rangle_\alpha = \left\langle \hat{\mathbf{d}}_1, \mathbf{d}_2 \right\rangle_\alpha \end{cases} \tag{5.47}$$

as well as

$$\begin{cases} \langle \mathbf{d}'_2, \mathbf{d}_1 \rangle_\alpha = \left\langle \hat{\mathbf{d}}_2, \mathbf{d}_1 \right\rangle_\alpha \\ \\ \langle \mathbf{d}'_2, \mathbf{d}_2 \rangle_\alpha = \left\langle \hat{\mathbf{d}}_2, \mathbf{d}_2 \right\rangle_\alpha \end{cases}. \tag{5.48}$$

Given two basic risky assets of ${}_{\mathbf{d}}\mathscr{M}^2_{(O)}$ denoted by $\mathbf{d}_1$ and $\mathbf{d}_2$, it is possible to consider

$$\mathbf{d}_1 - x\mathbf{d}_2 = y\hat{\mathbf{d}}_1 \tag{5.49}$$

and

$$\mathbf{d}_1 - x'\mathbf{d}_2 = y'\hat{\mathbf{d}}_2, \tag{5.50}$$

where $\hat{\mathbf{d}}_1$ and $\hat{\mathbf{d}}_2$ are two basic risky assets of ${}_{\hat{\mathbf{d}}}\mathscr{M}^2_{(O)}$. We note that y and y' are average constants of proportionality because they are referred themselves to distributions of mass, whereas x and x' are coefficients of adjustment. They adjust the difference of $\mathbf{d}_1$ and $\mathbf{d}_2$ to distributions that should exist with respect to a probabilistic and economic hypothesis identifying the invariance of the covariance of two risky assets.

We note the following

Remark 29. We deal with a Bayesian adjustment because $\mathbf{d}_1$ and $\mathbf{d}_2$ can be viewed as two prior distributions of mass, whereas $\hat{\mathbf{d}}_1$ and $\hat{\mathbf{d}}_2$ can be viewed as two posterior distributions of mass characterizing a specific hypothesis. A distance between two prior distributions of mass is proportional to a posterior distribution of mass characterizing the right-hand side of (5.49). A distance between two prior distributions of mass is similarly proportional to a posterior distribution of mass appearing on the right-hand side of (5.50). □

We note that $\hat{\mathbf{d}}_1$ in (5.49) and $\hat{\mathbf{d}}_2$ in (5.50) are obtained by means of linear combinations of $\mathbf{d}_1$ and $\mathbf{d}_2$. We can refer ourselves to (5.28). The condition of invariance of the covariance of two risky assets expressed by (5.47) and (5.48) is equal to the condition according to which $\mathbf{d}_1^* = \mathbf{d}'_1 - \hat{\mathbf{d}}_1$ and $\mathbf{d}_2^* = \mathbf{d}'_2 - \hat{\mathbf{d}}_2$ are orthogonal to a hyperplane embedded in E^n. It is described by using a linear equation satisfied by $\mathbf{d}_1$ and $\mathbf{d}_2$. It is possible to show that two risky assets identify a parallelogram (2-parallelepiped) with two pairs of parallel sides, where every risky asset is a side of it. Such a parallelogram (2-parallelepiped) recognizes a multiple risky asset of order 2. We observe that $\hat{\mathbf{d}}_1$ coincides with the orthogonal projection of $\hat{\mathbf{d}}_1$ onto $\mathbf{d}'_1$ given by

$$\operatorname{proj}_{\mathbf{d}'_1}(\hat{\mathbf{d}}_1) = \frac{\mathbf{d}'_1 \cdot \hat{\mathbf{d}}_1}{\|\mathbf{d}'_1\|^2}\mathbf{d}'_1 \tag{5.51}$$

as well as $\hat{\mathbf{d}}_2$ coincides with the orthogonal projection of $\hat{\mathbf{d}}_2$ onto $\mathbf{d}'_2$ expressed by

$$\text{proj}_{\mathbf{d}'_2}(\hat{\mathbf{d}}_2) = \frac{\mathbf{d}'_2 \cdot \hat{\mathbf{d}}_2}{\|\mathbf{d}'_2\|^2}\mathbf{d}'_2. \tag{5.52}$$

On the other hand, $\mathbf{d}'_1$ coincides with the orthogonal projection of $\mathbf{d}'_1$ onto $\hat{\mathbf{d}}_1$ as well as $\mathbf{d}'_2$ coincides with the orthogonal projection of $\mathbf{d}'_2$ onto $\hat{\mathbf{d}}_2$.

5.8 Non-classical inferential results

We say that (5.43) and (5.44) are orthogonal vectors with respect to the hyperplane embedded in E^n described by $\mathbf{d}_1$ and $\mathbf{d}_2$. It follows that it turns out to be

$$\begin{cases} d_1^1 f_1 + d_1^2 f_2 + \ldots + d_1^n f_n &= 0 \\ d_2^1 f_1 + d_2^2 f_2 + \ldots + d_2^n f_n &= 0 \end{cases} \tag{5.53}$$

as $\mathbf{f} = (f_1, f_2, \ldots, f_n)$ varies over $\mathbb{R}$. We note that the possible values for $\mathbf{d}_1$ and $\mathbf{d}_2$ are given by

$$\{d_1^1, d_1^2, \ldots, d_1^n\} \tag{5.54}$$

and

$$\{d_2^1, d_2^2, \ldots, d_2^n\}. \tag{5.55}$$

In particular, one of the ordered n-tuples denoted by $\mathbf{f} = (f_1, f_2, \ldots, f_n)$ has to be written in such a way that the following expression

$$f_1 + f_2 + \ldots + f_n = 1 \tag{5.56}$$

holds, with $0 \leq f_i \leq 1$, $i = 1, \ldots, n$. This means that (5.56) tells us that the sum of n non-negative masses must coherently be equal to 1 whenever they are associated with n incompatible and exhaustive events of a finite partition of elementary events (see also [77]). We write

$$\begin{cases} \langle \mathbf{d}'_1 - \hat{\mathbf{d}}_1, \mathbf{f} \rangle &= 0 \\ \langle \mathbf{d}'_2 - \hat{\mathbf{d}}_2, \mathbf{f} \rangle &= 0 \end{cases} \tag{5.57}$$

to denote the orthogonality of $\mathbf{d}_1^*$ and $\mathbf{d}_2^*$.

We note the following

Remark 30. We suppose that it turns out to be $m = 2$. Given all deviations from $\mathbf{P}(X_1)$ and $\mathbf{P}(X_2)$ of the starting possible values for two basic risky assets of ${}_{\mathbf{d}}\mathscr{M}^2_{(O)}$, we denote by $\mathbf{d}_1$ and $\mathbf{d}_2$ the corresponding n-dimensional vectors. We go back to the possible values for X_1 and X_2 by using the marginal masses of the nonparametric marginal distributions of mass of X_1 and X_2. If we deal with $\hat{\mathbf{d}}_1$ and $\hat{\mathbf{d}}_2$ then they are obtained by considering two different linear combinations of $\mathbf{d}_1$ and $\mathbf{d}_2$. We observe that it is possible to go back to the starting values of which $\hat{\mathbf{d}}_1$ and $\hat{\mathbf{d}}_2$ represent deviations by using that n-dimensional vector denoted by $\mathbf{f}$ such that it turns out to be equal to 1 the sum of all non-negative masses expressed by f_i, $i = 1, \ldots, n$. We refer ourselves to (5.56). □

Remark 31. All probabilities associated with the possible values for the risky assets belonging to ${}_{\hat{\mathbf{d}}}\mathscr{M}^m_{(O)}$ and obtained by considering linear combinations of basic risky assets constituting ${}_mB_{\mathbf{d}}$ can be determined by using $\mathbf{f}$. The same is true with regard to all probabilities associated with the possible values for the risky assets belonging to ${}_{\mathbf{d}}\mathscr{M}^m_{(O)}$ and obtained by considering linear combinations of basic risky assets constituting ${}_mB_{\mathbf{d}}$. □

Similar results can be obtained by considering the α-orthogonal projection of ${}_{\mathbf{d}}\mathscr{M}^m_{(O)}$ onto ${}_{\mathbf{d}'}\mathscr{M}^m_{(O)}$.

5.8.1 A two-dimensional linear manifold expressed as a direct sum of two eigenspaces

We prove in the appendix the following

Theorem 5. Given (5.28), suppose that it turns out to be $I_2 = \{1,2\}$. If the condition expressed by $\langle \hat{\mathbf{d}}_1, \hat{\mathbf{d}}_2 \rangle_\alpha = 0$ is satisfied then $\hat{\mathbf{d}}_1$ and $\hat{\mathbf{d}}_2$ identify the principal components connected with $D_{\{2\}}$.

We note the following

Remark 32. The above theorem can be extended by considering an m-dimensional linear manifold. It is decomposed into two complementary linear manifolds. The former is a one-dimensional linear manifold, whereas the latter is an $(m-1)$-dimensional linear manifold. It is possible to consider the following conditions given by

$$\langle \hat{\mathbf{d}}_i, \hat{\mathbf{d}}_j \rangle_\alpha = 0, \quad \forall i < j \in I_m = \{1,2,\ldots,m\}, \tag{5.58}$$

whose number is overall equal to

$$\binom{m}{2}.$$

We write

$${}_{\mathbf{d}}\mathscr{M}^m_{(O)} = Z_i \oplus Z_i^*, \quad \forall i < j \in I_m = \{1,2,\ldots,m\}, \tag{5.59}$$

to denote one of the possible direct sums. The number of the principal components connected with $D_{\{m\}}$ is overall equal to m. □

Remark 33. If we refer ourselves to (.103) then $\hat{\mathbf{d}}_1$ and $\hat{\mathbf{d}}_2$ can be led back to $\mathbf{d}_1$ and $\mathbf{d}_2$. The variance of each risky asset under consideration is characterized by a constant of riskiness. All different eigenvalues do not change. They are constants of riskiness. The possible values for each risky asset change. We use specific probabilities in order to lead back them to the starting possible values expressed without considering deviations. The same is true if we refer ourselves to (5.59). We therefore write

$$G(\mathbf{v}) = \lambda \mathbf{v}, \tag{5.60}$$

where G is the square matrix of order m whose elements coincide with the covariant components of the α-metric tensor defined with regard to ${}_{\mathbf{d}}\mathscr{M}^m_{(O)}$. Such a matrix is identified with (5.23). We note that $\lambda \in \mathbb{R}$ is an eigenvalue of G, whereas $\mathbf{v}$ is an eigenvector of G that uniquely identifies an element of ${}_{\mathbf{d}}\mathscr{M}^m_{(O)}$. All eigenvectors of G are nonzero and column vectors spanning a linear subspace of E^m. It is the eigenspace corresponding to $\lambda \in \mathbb{R}$. □

Remark 34. We refer ourselves to (.103). We write $\mu_1\,\hat{\mathbf{d}}_1$, $\mu_1 \in \mathbb{R}$, and $\mu_2\,\hat{\mathbf{d}}_2$, $\mu_2 \in \mathbb{R}$. We calculate $\|\mu_1\,\hat{\mathbf{d}}_1\|_\alpha^2$ as well as $\|\mu_2\,\hat{\mathbf{d}}_2\|_\alpha^2$ by using $\mathbf{f} = (f_1, f_2, \ldots, f_n)$ such that it turns out to be $f_1 + f_2 + \ldots + f_n = 1$, with $0 \leq f_i \leq 1$, $i = 1, \ldots, n$. We have firstly to solve for the value of x, where x is an unknown, that satisfies the following expression written in the form

$$x \begin{pmatrix} \lambda_1 \\ 0 \end{pmatrix} = \begin{pmatrix} \|\mu_1\,\hat{\mathbf{d}}_1\|_\alpha^2 \\ 0 \end{pmatrix}. \tag{5.61}$$

Two column vectors meant as eigenvectors characterize both sides of (5.61). We observe that $\lambda_1 \in \mathbb{R}$ is that eigenvalue obtained by using $\mathbf{d}_1$ together with its starting marginal probabilities. It follows that it is possible to write

$$\frac{\|\mu_1\,\hat{\mathbf{d}}_1\|_\alpha^2}{\lambda_1} \begin{pmatrix} \lambda_1 \\ 0 \end{pmatrix}, \tag{5.62}$$

so we calculate $\left\| \sqrt{\frac{\|\mu_1\,\hat{\mathbf{d}}_1\|_\alpha^2}{\lambda_1}}\,\mathbf{d}_1 \right\|_\alpha^2$ by using the starting marginal probabilities associated with $\mathbf{d}_1$. In particular, it turns out to be

$$\left\| \sqrt{\frac{\|\mu_1\,\hat{\mathbf{d}}_1\|_\alpha^2}{\lambda_1}}\,\mathbf{d}_1 \right\|_\alpha^2 = \|\mu_1\,\hat{\mathbf{d}}_1\|_\alpha^2 \tag{5.63}$$

whenever all probabilities associated with $\mathbf{d}_1$ and $\hat{\mathbf{d}}_1$ coincide. In all cases, the left-hand side of (5.63) is proportional to a constant of riskiness denoted by λ_1. Given $\mu_1\,\hat{\mathbf{d}}_1$, $\mu_1 \in \mathbb{R}$, we go back to the starting and basic risky asset denoted by $\mathbf{d}_1$. Since a real constant is considered together with $\mathbf{d}_1$, we write $\sqrt{\frac{\|\mu_1\,\hat{\mathbf{d}}_1\|_\alpha^2}{\lambda_1}}\,\mathbf{d}_1$. It is clear that it is also possible to solve for the value of x that satisfies the following expression written in the form

$$x \begin{pmatrix} 0 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \|\mu_2\,\hat{\mathbf{d}}_2\|_\alpha^2 \end{pmatrix}. \tag{5.64}$$

The same is true if we refer ourselves to (5.59). □

Remark 35. The possible values for each risky asset under consideration can be subjected to infinite translations. The finite partition of mutually exclusive events to which the possible values for each risky asset under consideration correspond is the same from a randomness point of view. All probabilities associated with them do not change. All probabilities into account can also be based on a judgment of equal probability. Anyway, such a judgment is always of a subjective nature. All probabilities into account can also be based on an opinion shared by all reasonable people. However, such an opinion always remains of a subjective nature. All probabilities into account can even be based on statistical frequencies. Nevertheless, the investor has subjectively to specify the meaning and all conditions associated with relating probability back to frequency. Differently, to relate probability back to frequency has no meaning. □

5.9 Mean-variance utility

Given a portfolio consisting of two different types of assets, we suppose that one of them is not a risky asset, but it is a risk-free asset. It always pays an amount of money denoted by r_f regardless of what happens. Its return is a positive constant, whereas its standard deviation is equal to 0 because there is no riskiness.

The other asset is a set of m risky assets. They give origin to a multiple risky asset of order m denoted by $X_{\{m\}} = X_{12\ldots m}$. A set of m risky assets is a multiple random good of order m. The mean-variance model assumes that the subjective utility of a nonparametric distribution of mass referred to $X_{12\ldots m}$ can be expressed as a function of the mean and variance of it. It is appropriate to make the natural assumption that a higher expected return on $X_{12\ldots m}$ is good when all other things do not change. A higher variance is conversely bad. This evidently means that the natural assumption of aversion to risk holds (see also [68]). It is possible to assume that an investor's preferences depend only on the mean and variance of the nonparametric distribution of mass referred to $X_{12\ldots m}$. It is possible to consider indifference curves illustrating an investor's preferences for return and risk. If she is overall a risk-averse investor then a higher expected return on $X_{12\ldots m}$ makes her better off as well as a higher standard deviation makes her worse off. Riskiness identified with the variance of the nonparametric distribution of mass referred to $X_{12\ldots m}$ is bad, so the indifference curves characterizing her subjective utility function must have a positive slope (see also [89]).

We describe the nonparametric distribution of mass of a multiple risky asset of order m by using a few parameters. We are interested in summarizing the nonparametric distribution of mass of a multiple risky asset of order m. This is because the subjective utility function characterizing the mean-variance model must be defined over those parameters concerning such a distribution. An investor's preferences can be described by considering just a few summary statistics about a nonparametric distribution of mass of a multiple risky asset of order m. We decompose it inside of a linear space over $\mathbb{R}$ provided with a quadratic metric. We write

$$\mathbf{P}(X_{12\ldots m}) = \begin{vmatrix} \mathbf{P}(X_1 X_1) & \mathbf{P}(X_1 X_2) & \ldots & \mathbf{P}(X_1 X_m) \\ \mathbf{P}(X_2 X_1) & \mathbf{P}(X_2 X_2) & \ldots & \mathbf{P}(X_2 X_m) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{P}(X_m X_1) & \mathbf{P}(X_m X_2) & \ldots & \mathbf{P}(X_m X_m) \end{vmatrix}, \tag{5.65}$$

where $\mathbf{P}$ denotes the expected return on m risky assets identifying a multiple risky asset of order m. It coincides with the coherent prevision or mathematical expectation of a multiple random good of order m. We also write

$$\mathrm{Var}(X_{12\ldots m}) = \begin{vmatrix} \mathrm{Var}(X_1) & \mathrm{Cov}(X_1,X_2) & \ldots & \mathrm{Cov}(X_1,X_m) \\ \mathrm{Cov}(X_2,X_1) & \mathrm{Var}(X_2) & \ldots & \mathrm{Cov}(X_2,X_m) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{Cov}(X_m,X_1) & \mathrm{Cov}(X_m,X_2) & \ldots & \mathrm{Var}(X_m) \end{vmatrix} = \begin{vmatrix} \|\mathbf{d}_1\|_\alpha^2 & \langle \mathbf{d}_1,\mathbf{d}_2\rangle_\alpha & \ldots & \langle \mathbf{d}_1,\mathbf{d}_m\rangle_\alpha \\ \langle \mathbf{d}_2,\mathbf{d}_1\rangle_\alpha & \|\mathbf{d}_2\|_\alpha^2 & \ldots & \langle \mathbf{d}_2,\mathbf{d}_m\rangle_\alpha \\ \vdots & \vdots & \ddots & \vdots \\ \langle \mathbf{d}_m,\mathbf{d}_1\rangle_\alpha & \langle \mathbf{d}_m,\mathbf{d}_2\rangle_\alpha & \ldots & \|\mathbf{d}_m\|_\alpha^2 \end{vmatrix} \tag{5.66}$$

in order to obtain the variance of a multiple risky asset of order m. We note that the variability of $X_{12\ldots m}$ is not standardized, but it depends on the state of information and knowledge associated with a given investor. This means that the origin of the variability of $X_{12\ldots m}$ is not random. Moreover, any two risky assets of m risky assets are supposed to be stochastically independent. This thing is possible because each point of the budget set of the investor where two risky assets at a time are studied is a synthesized element of the Fréchet class. Since we consider changes of origin outside of her budget set, marginal and joint masses do not change. She is able to estimate all joint probabilities under consideration in such a way that the covariance of any two risky assets, where the possible values for each of them are subjected to two changes of origin, is equal to 0. She is subjected to $2n-1$ constraints only. They coincide with $2n-1$ marginal masses whenever she estimates all joint probabilities into account.

At the optimal choice of mean return and standard deviation of return we observe that the slope of the indifference curve must be equal to the slope of the budget line. Such a line measures the cost of obtaining a larger expected return in terms of the increased standard deviation of the return. It describes the market trade-off between return and risk. Its vertical intercept coincides with the return associated with the risk-free asset under consideration denoted by r_f. It follows that the price of risk, whose nature is always characterized by objective and subjective elements, is given by

$$p = \frac{\mathbf{P}(X_{12\ldots m}) - r_f}{\sqrt{\mathrm{Var}(X_{12\ldots m})}}, \tag{5.67}$$

where $\mathbf{P}(X_{12\ldots m})$ and $\sqrt{\mathrm{Var}(X_{12\ldots m})}$ are two multilinear indices. They are two determinants of two square matrices of order m. Such indices deal with two tensors identifying the same multiple random good of order m. In this chapter, we especially focus on the α-metric tensor whose components coincide with the elements of the determinant given by (5.66).

Conclusions

A multilinear approach to the theory of decision-making consists in establishing disaggregate and aggregate measures based on what the decision-maker actually chooses inside of her budget set. Aggregate measures obtained by using a multilinear metric allow to identify multiple choices connected with multiple goods. A multilinear approach to the theory of decision-making shows that an evolution towards a synthesis is intentionally tried. This evolution is necessary. A conceptually unsatisfactory attitude consists in breaking knowledge up into self-contained compartments. For instance, the justification of any distinction of principle concerning ignorance, risk, and uncertainty as supported by Frank Knight is based on the possibility, or not, of bringing a probability evaluation back to the special classical or statistical definitions. If this conceptually unsatisfactory distinction is got over then there is a gain in the meaning of the conclusions as well as in the possibility of their extension. On the other hand, everything can vectorially be studied from a statistical and economic point of view provided one takes a sufficient number of dimensions. An adequate number of dimensions is necessary in order to study multiple goods. Mistaken ideas are avoided such as the attempts to study decisions under conditions of uncertainty forcing everything to intervene except the evaluation of probability being made by the decision-maker. It is a fundamental element. It is a basic and unavoidable result associated with specific conditions identifying decisions where sure elements are absent. It is not helpful to disconnect the theory of decision-making from properties of the notion of average quantity obtained by using non-negative masses subjectively chosen. These properties are the same as the ones characterizing the intuitive notion of probability whose nature is intrinsically subjective. Bound choices being made by the decision-maker under conditions of uncertainty and riskiness have to be studied by considering all elements characterizing them. They are not of an objective nature only, but they are also of a subjective nature. Accordingly, an evolution towards a unitary vision in which it appears that a place and a link are found for theories previously viewed as unconnected parts must intentionally be tried. It is possible to recompose these parts in an organized way by using the properties of the notion of average quantity obtained by using non-negative masses subjectively chosen. The budget set of the decision-maker coincides with infinite coherent average quantities obeying the rules of the logic of prevision.

The variability of a nonparametric joint distribution of mass depends on how the decision-maker estimates all the joint masses under consideration. These masses are estimated by her according to her variable state of information and knowledge at the time of choice. It follows that the origin of the variability of any nonparametric distribution of mass is not random. It is not standardized because the decision-maker makes explicit, from time to time, the knowledge hypothesis underlying it. The origin of the variability of any nonparametric distribution of mass is not connected with the theory of measurement errors, where such errors are of a random nature. Different measures based on this origin can be used outside of the budget set of the decision-maker in order to process real data.

The decision-maker also expresses, from time to time, the knowledge hypothesis underlying the principal components. It is always up to her to establish which is the most appropriate instrument in connection with the hypotheses and the knowledge purposes. Accordingly, it is a question of investigating a system of knowledge hypotheses leading to many solutions and identifying the one among them that allows to find the framework and all instruments associated with principal component analysis.

All alternatives referred to ordinary or random goods are firstly well-determined propositions coinciding with real numbers. Hence, they have to be considered within the ambit of ordinary logic where only two values, either true or false, are involved whenever uncertain elements are ultimately absent. The variable state of information and knowledge associated with a given decision-maker permits her to exclude an enormous number of outcomes as impossible. All the others remain possible for her at the time of choice. They are possible alternatives whose number is finite. Since each point of the budget set of the decision-maker obeys the rules of the logic of prevision, the decision-maker rationally behaves if she obeys them. They are characterized by logical needs that must be satisfied. Logical needs characterizing rules of the logic of prevision are satisfied whenever an average quantity of consumption associated with a joint good is decomposed into two coherent average quantities of consumption associated with two marginal goods. With regard to the necessary evolution towards a synthesis of the theory of decision-making, logical aspects of decision-making could also be fused together by means of a many-valued logic such as fuzzy logic. On the other hand, aggregate measures can be used outside of decision theory in order to study multilinear relationships between variables. It is possible to consider such measures together with parametric probability distributions such as normal distributions to solve specific inference problems.

Appendix

.1 Proof of Proposition 2.

Let ${}_{m}B^{\perp}_{\mathbf{e}} = \{\mathbf{e}_i;\, i \in I_m\}$ and ${}_{m}B^{\perp}_{\mathbf{e}(\,)} = \{\mathbf{e}_{(j)};\, (j) \in I_m\}$ be two distinct orthonormal bases of ${}_{\mathbf{d}}\mathscr{M}^{m}_{(O)}$. We therefore write

$$\left\langle \mathbf{e}_i, \mathbf{e}_j \right\rangle = \delta_{ij} \tag{.68}$$

as well as

$$\left\langle \mathbf{e}_{(i)}, \mathbf{e}_{(j)} \right\rangle = \delta_{(i)\,(j)}. \tag{.69}$$

If the set of the contravariant components of $\mathbf{e}_i$ with respect to ${}_{m}B^{\perp}_{\mathbf{e}(\,)}$ is denoted by $\{A_i^{(j)};\, (j) \in I_m\}$ then we write

$$\mathbf{e}_i = A_i^{(j)} \mathbf{e}_{(j)}, \tag{.70}$$

where we have $i = 1, \ldots, m$. We evidently consider m linear combinations identifying a nonsingular matrix denoted by $A = \{A_i^{(j)};\, (j) \in I_m,\, i \in I_m\}$. We can also write

$$\mathbf{e}_{(k)} = A^{i}_{(k)} \mathbf{e}_i, \tag{.71}$$

where we have $(k) = 1, \ldots, m$. We obtain $A^{-1} = \{A^{i}_{(k)};\, i \in I_m,\, (k) \in I_m\}$. This means that it turns out to be

$$A_i^{(j)} A^{i}_{(k)} = \delta^{(j)}_{(k)} \tag{.72}$$

as well as

$$A_i^{(j)} A^{h}_{(j)} = \delta^{h}_{i}. \tag{.73}$$

Let $\mathbf{v}_{(k)}$ be an eigenvector of the α-metric tensor whose components identify an $m \times m$ symmetric matrix. It is then associated with the eigenvalue denoted by ${}_{\mathbf{d}}\lambda_{(k)}$ (see also [73]). Hence, $\mathbf{v}_{(k)}$ is expressed by

$$\mathbf{v}_{(k)} = v^{i}_{(k)} \mathbf{e}_i \tag{.74}$$

with respect to ${}_{m}B^{\perp}_{\mathbf{e}}$. Let $\mathbf{v}_k$ be an eigenvector of the α-metric tensor whose components identify an $m \times m$ symmetric matrix. It is then associated with the eigenvalue denoted by ${}_{\mathbf{d}}\lambda_k$. Hence, $\mathbf{v}_k$ is expressed by

$$\mathbf{v}_k = v_k^{(j)} \mathbf{e}_{(j)} \tag{.75}$$

with respect to ${}_{m}B^{\perp}_{\mathbf{e}(\,)}$. We observe that $\{{}_{\mathbf{d}}\lambda_{(k)};\, (k) \in I_m\}$ as well as $\{{}_{\mathbf{d}}\lambda_k;\, k \in I_m\}$ are two different enumerations identifying the same eigenvalues. The same eigenvalues are contained in both sets. We cannot pass from a set to another one because we deal with different enumerations. If $\mathbf{v}_{(k)}$ and $\mathbf{v}_k$ are normalized, with $(k) = 1, \ldots, m$ as well as $k = 1, \ldots, m$, then it turns out to be

$$\left\langle \mathbf{v}_{(k)}, \mathbf{v}_{(h)} \right\rangle = \langle v^i_{(k)}\, \mathbf{e}_i,\, v^j_{(h)}\, \mathbf{e}_j \rangle = v^i_{(k)}\, v^j_{(h)} \langle \mathbf{e}_i,\, \mathbf{e}_j \rangle = v^i_{(k)}\, v^j_{(h)}\, \delta_{ij} = \delta_{(k)\,(h)} \tag{.76}$$

and

$$\langle \mathbf{v}_k, \mathbf{v}_h \rangle = \delta_{kh}. \tag{.77}$$

All these eigenvectors are orthonormal. They identify an orthogonal matrix (see also [57]). All eigenvectors associated with ${}_{\mathbf{d}}g_{ij}$ are pairwise α-orthogonal. Since they are associated with ${}_{\mathbf{d}}g_{ij}$, we refer ourselves to the property of α-orthogonality. □

.2 Proof of Proposition 4.

Every risky asset belonging to ${}_{\mathbf{d}}\mathscr{M}^m_{(O)}$ can be expressed as a linear combination of the basic risky assets belonging to ${}_mB_{\mathbf{w}_\mathbf{d}}$. In particular, we can write

$$\mathbf{d}_i = v_i^{(h)} \mathbf{w}_{(h)}, \quad \forall i \in I_m. \tag{.78}$$

We consequently consider the following eigenequation

$$\left({}_{\mathbf{d}}g_{kh} - \lambda_{(k)}\, \delta_{kh}\right) v^k_{(k)} = 0. \tag{.79}$$

From (.79), it follows

$${}_{\mathbf{d}}g_{kh}\, v^k_{(k)} = \lambda_{(k)}\, \delta_{kh}\, v^k_{(k)}. \tag{.80}$$

If we use the contravariant components of $\mathbf{v}_h$ in both sides of (.80) then we obtain

$$v^k_{(k)}\, v^h_{(h)}\, {}_{\mathbf{d}}g_{kh} = \lambda_{(k)}\, v^k_{(k)}\, v^h_{(h)}\, \delta_{kh}. \tag{.81}$$

On the other hand, it is also possible to write

$${}_{\mathbf{d}}g_{kh} = v_k^{(j)}\, v_h^{(i)} \left\langle \mathbf{w}_{(j)}, \mathbf{w}_{(i)} \right\rangle \alpha \tag{.82}$$

after considering (.78). We observe that it turns out to be

$$v^k_{(k)}\, v^h_{(h)}\, \delta_{kh} = \delta_{(k)(h)}, \tag{.83}$$

so we obtain

$$\left\langle \mathbf{w}_{(k)}, \mathbf{w}_{(h)} \right\rangle \alpha = \lambda_{(k)}\, \delta_{(k)(h)}. \tag{.84}$$

We obtain (.84) after putting (.82) into (.81). If (k) and (h) vary over I_m then we note that (.84) identifies an α-metric tensor with respect to a basis of ${}_{\mathbf{d}}\mathscr{M}^m_{(O)}$ whose elements are the principal components concerning $D_{\{m\}}$. We therefore write

$${}_{\mathbf{w}}g_{(k)(h)} = \left\langle \mathbf{w}_{(k)}, \mathbf{w}_{(h)} \right\rangle \alpha = \lambda_{(k)}\, \delta_{(k)(h)}. \tag{.85}$$

The generic covariant component of a diagonal tensor is expressed by (.85). We define to be ${}_mB_{\mathbf{w}_\mathbf{d}}$ an α-orthogonal basis of ${}_{\mathbf{d}}\mathscr{M}^m_{(O)}$, so its elements are pairwise α-orthogonal. Also, every eigenvalue corresponding to the normalized eigenvector whose components are contained in (5.35) coincides with the α-norm of a principal component belonging to ${}_mB_{\mathbf{w_d}}$ (see also [76]). Since we write

$$\begin{bmatrix} {}_{\mathbf{w}}g_{(1)(1)} & {}_{\mathbf{w}}g_{(1)(2)} & \cdots & {}_{\mathbf{w}}g_{(1)(m)} \\ {}_{\mathbf{w}}g_{(2)(1)} & {}_{\mathbf{w}}g_{(2)(2)} & \cdots & {}_{\mathbf{w}}g_{(2)(m)} \\ \vdots & \vdots & \ddots & \vdots \\ {}_{\mathbf{w}}g_{(m)(1)} & {}_{\mathbf{w}}g_{(m)(2)} & \cdots & {}_{\mathbf{w}}g_{(m)(m)} \end{bmatrix}, \tag{.86}$$

we denote by ${}_{\mathbf{w}}a^{(k)(h)}$ the cofactor of ${}_{\mathbf{w}}a_{(k)(h)}$, where both ${}_{\mathbf{w}}a^{(k)(h)}$ and ${}_{\mathbf{w}}a_{(k)(h)}$ are contained in (.86). Moreover, we denote by ${}_{\mathbf{w}}g$ the determinant of (.86). The generic contravariant component of the α-metric tensor under consideration is therefore given by

$${}_{\mathbf{w}}g^{(k)(h)} = \frac{{}_{\mathbf{w}}a^{(k)(h)}}{{}_{\mathbf{w}}g}. \tag{.87}$$

If we write

$$\frac{1}{\lambda_{(k)}} = \lambda^{(k)}$$

then it is possible to obtain

$${}_{\mathbf{w}}g^{(k)(h)} = \lambda^{(k)}\,\delta^{(k)(h)}, \tag{.88}$$

where we write $\delta^{(k)(h)} = 1$ if and only if it turns out to be $(k) = (h)$. □

.3 Proof of Proposition 5.

We remind (5.29). The following condition

$$\left\langle \mathbf{d}'_j, \mathbf{w}_{(k)} \right\rangle_\alpha - d_j^{(h)}\,{}_{\mathbf{w}}g_{(h)(k)} = \left\langle \mathbf{d}'_j, \mathbf{w}_{(k)} \right\rangle_\alpha - d_{j(k)} = 0 \tag{.89}$$

allows us to compute the covariant components of $\hat{\mathbf{d}}_j \in {}_{\hat{\mathbf{d}}}\mathscr{M}^m_{(O)}$ with respect to ${}_mB_{\mathbf{w_d}}$. Given (.89), it turns out to be

$$d_{j(k)} = \left\langle \mathbf{d}'_j, \mathbf{w}_{(k)} \right\rangle_\alpha. \tag{.90}$$

We take (.88) into account, so we write

$$d_{j(k)}\,{}_{\mathbf{w}}g^{(h)(k)} = d_{j(k)}\,\lambda^{(h)}\,\delta^{(h)(k)} = d_{j(h)}\,\lambda^{(h)}. \tag{.91}$$

We observe that the h index in the third side of (.91) is a free index unlike the k index in the second side of it. The contravariant components of $\hat{\mathbf{d}}_j \in {}_{\hat{\mathbf{d}}}\mathscr{M}^m_{(O)}$ with respect to ${}_mB_{\mathbf{w_d}}$ are then expressed by

$$d_{j(h)}\,\lambda^{(h)} = \frac{\langle \mathbf{d}'_j, \mathbf{w}_{(h)} \rangle_\alpha}{\lambda_{(h)}} = d_j^{(h)}. \tag{.92}$$

We lastly observe that the covariant components of $\hat{\mathbf{d}}_j$ and $\mathbf{d}'_j$ with respect to ${}_mB_{\mathbf{w_d}}$ are obtained in the same way. This means that it is possible to write

$$d_{j(k)} = \left\langle \hat{\mathbf{d}}_j, \mathbf{w}_{(k)} \right\rangle_\alpha \tag{.93}$$

in order to obtain them. □

.4 Proof of Theorem 5.

Firstly, we establish a bilinear relationship between two α-metric tensors of order 2 defined with regard to ${}_2B_{\hat{\mathbf{d}}}$ and ${}_2B_{\mathbf{d}}$. Such a relationship is studied with respect to ${}_{\mathbf{d}}\mathscr{M}^2_{(O)}$. It follows that, after writing

$$ {}_{\hat{\mathbf{d}}}g_{12} = \left\langle \hat{\mathbf{d}}_1, \hat{\mathbf{d}}_2 \right\rangle_\alpha, \tag{.94}$$

it is possible to obtain

$$ {}_{\hat{\mathbf{d}}}g_{12} = \left\langle \hat{\mathbf{d}}_1, \hat{\mathbf{d}}_2 \right\rangle_\alpha = d_1^k d_2^h \, {}_{\mathbf{d}}g_{kh}, \tag{.95}$$

where we note that (5.28) has taken the place of the two vectors considered into (.94). This means that we refer ourselves to the definition of α-metric tensor with respect to ${}_2B_{\mathbf{d}}$ in order to obtain (.95). Hence, the condition expressed by $\langle \hat{\mathbf{d}}_1, \hat{\mathbf{d}}_2 \rangle_\alpha = 0$ becomes

$$ {}_{\hat{\mathbf{d}}}g_{12} = d_1^k d_2^h \, {}_{\mathbf{d}}g_{kh} = 0. \tag{.96}$$

We are only interested in it. Secondly, we refer ourselves to the eigenequation given by (.79). We consequently write

$$ {}_{\mathbf{d}}\mathscr{M}^m_{(O)} = \bigoplus_{(k)=1}^{m} N_{(k)}, \tag{.97}$$

where we denote by $N_{(k)}$ the eigenspace corresponding to the eigenvalue expressed by $\lambda_{(k)}$. Such an eigenspace contains all the eigenvectors associated with $\lambda_{(k)}$. In particular, since it turns out to be $m = 2$, we write

$$ {}_{\mathbf{d}}\mathscr{M}^2_{(O)} = \bigoplus_{(k)=1}^{2} N_{(k)}, \tag{.98}$$

where $N_{(1)}$ and $N_{(2)}$ are α-orthogonal because the eigenvalues under consideration are supposed to be all different. Accordingly, each element of ${}_{\mathbf{d}}\mathscr{M}^2_{(O)}$ can uniquely be expressed as a direct sum of elements, where each of them belongs to an eigenspace only. Thirdly, we define two vectors written in the form

$$ \mathbf{z}_1 = \mu_1 \hat{\mathbf{d}}_1 \tag{.99}$$

and

$$ \mathbf{z}_2 = \mu_2 \hat{\mathbf{d}}_2. \tag{.100}$$

Two one-dimensional and complementary linear manifolds are therefore spanned as μ_1 and μ_2 vary over $\mathbb{R}$, so we denote them by

$$ Z_1 : \mathbf{z}_1 = \mu_1 \hat{\mathbf{d}}_1, \quad \mu_1 \in \mathbb{R}, \tag{.101}$$

as well as

$$ Z_1^* : \mathbf{z}_2 = \mu_2 \hat{\mathbf{d}}_2, \quad \mu_2 \in \mathbb{R}. \tag{.102}$$

Since Z_1 and Z_1^* are α-orthogonal, it turns out to be

$$ {}_{\mathbf{d}}\mathscr{M}^2_{(O)} = Z_1 \oplus Z_1^*. \tag{.103}$$

If we write $\hat{\mathbf{d}}_1 \in Z_1$ and $\mu_2 \hat{\mathbf{d}}_2 \in Z_1^*$, where $\mu_2 \in \mathbb{R}$ is arbitrarily chosen, then (.96) can be rewritten in the form

$$d_1^k \mu_2 d_2^h \,{}_{\mathbf{d}}g_{kh} = 0, \tag{.104}$$

where $\{d_1^k\}$ is the set of the contravariant components of $\hat{\mathbf{d}}_1 \in Z_1$ with regard to ${}_2B_{\mathbf{d}}$, whereas $\{\mu_2 d_2^h\}$ is the set of the contravariant components of $\mu_2 \hat{\mathbf{d}}_2 \in Z_1^*$ with regard to the same basis denoted by ${}_2B_{\mathbf{d}}$. On the other hand, the set of the covariant components of $\hat{\mathbf{d}}_1 \in Z_1$ with regard to ${}_2B_{\mathbf{d}}$ is given by $\{d_1^k \,{}_{\mathbf{d}}g_{kh}\}$, whereas the covariant components of $\mathbf{z}_1$ are expressed by $z_{1h} = \mu_1 d_{1h}$, with $h \in I_2$. It follows that the vectors whose covariant components are given by $d_1^k \,{}_{\mathbf{d}}g_{kh}$ and $\mu_1 d_{1h}$ belong to the same eigenspace denoted by Z_1, so there exists one and only one scalar denoted by $\tau_1 \in \mathbb{R}$ such that it turns out to be

$$d_1^k \,{}_{\mathbf{d}}g_{kh} = \tau_1 \mu_1 d_{1h}. \tag{.105}$$

From (.105) it is therefore possible to derive the following eigenequation

$$\left({}_{\mathbf{d}}g_{kh} - \tau_i \mu_i \delta_{kh}\right) d_i^k = 0. \tag{.106}$$

We note that it turns out to be $\delta_{kh} d_i^k = d_{ih}$, so d_{1h} appearing on the right-hand side of (.105) is intrinsically of a contravariant nature. Fourthly, we compare (.79) with (.106). We consequently note that ${}_{\mathbf{d}}g_{kh}$ is the same, so (.79) and (.106) admit the same eigenvalues. Moreover, (.79) and (.106) admit the same normalized eigenvectors. This means that $\lambda_i = \tau_i \mu_i$ are different from the ones contained in (.79) with regard to their enumeration only. The same is true with regard to the normalized eigenvectors denoted by $\mathbf{v}_{(k)}$ and $\mathbf{d}_i$. We lastly say that the condition given by $\langle \hat{\mathbf{d}}_1, \hat{\mathbf{d}}_2 \rangle_\alpha = 0$ characterizes $\hat{\mathbf{d}}_1$ and $\hat{\mathbf{d}}_2$ to be the principal components connected with $D_{\{2\}}$. Both of them are obtained by means of a linear combination. Such principal components are α-orthogonal and the α-norm of each of them coincides with an eigenvalue identifying a specific eigenspace. □

Bibliography

[1] M. Abdellaoui, H. Bleichrodt, O. l'Haridon, and C. Paraschiv. Is there one unifying concept of utility? an experimental comparison of utility under risk and utility over time. *Management Science*, 59(9):2153–2169, 2013.

[2] A. Adams, L. Cherchye, B. De Rock, and E. Verriest. Consume now or later? time inconsistency, collective choice, and revealed preference. *American Economic Review*, 104(12):4147–4183, 2014.

[3] S. N. Afriat. The construction of utility functions from expenditure data. *International Economic Review*, 8(1):67–77, 1967.

[4] D. Ahn, S. Choi, D. Gale, and S. Kariv. Estimating ambiguity aversion in a portfolio choice experiment. *Quantitative Economics*, 5(2):195–223, 2014.

[5] J. Andreoni and C. Sprenger. Estimating time preferences from convex budgets. *American Economic Review*, 102(7):3333–3356, 2012.

[6] P. Angelini. A reinterpretation of principal component analysis connected with linear manifolds identifying risky assets of a portfolio. *International Journal of Applied Mathematics*, 33(4), 2020.

[7] P. Angelini and F. Maturo. Non-parametric probability distributions embedded inside of a linear space provided with a quadratic metric. *Mathematics*, 8(11), 2020.

[8] P. Angelini and F. Maturo. The consumer's demand functions defined to study contingent consumption plans. *Quality & Quantity*, https://doi.org/10.1007/s11135-021-01170-2, 2021.

[9] P. Angelini and F. Maturo. Summarized distributions of mass: a statistical approach to consumers' consumption spaces. *Journal of Intelligent & Fuzzy Systems*, 41(2):3093–3105, 2021.

[10] F. J. Anscombe and R. J. Aumann. A definition of subjective probability. *The Annals of Mathematical Statistics*, 34(1):199–205, 1963.

[11] J. Baron. Second-order probabilities and belief functions. *Theory and Decision*, 23(1):25–36, 1987.

[12] J. Baron and I. Ritov. The role of probability of detection in judgments of punishment. *Journal of Legal Analysis*, 1(2):553–590, 2009.

[13] P. Battigalli and M. Siniscalchi. Rationalization and incomplete information. *The B. E. Journal of Theoretical Economics*, 3(1), 2003.

[14] M. Berkhouch, G. Lakhnati, and M. Brutti Righi. Extended Gini-type measures of risk and variability. *Applied Mathematical Finance*, 25(3):295–314, 2018.

[15] P. Berti, E. Dreassi, and P. Rigo. A notion of conditional probability and some of its consequences. *Decisions in Economics and Finance*, 43(1):3–15, 2020.

[16] P. Berti, E. Regazzini, and P. Rigo. Strong previsions of random elements. *Statistical Methods and Applications (Journal of the Italian Statistical Society)*, 10(1):11–28, 2001.

[17] P. Berti and P. Rigo. On coherent conditional probabilities and disintegrations. *Annals of Mathematics and Artificial Intelligence*, 35(1):71–82, 2002.

[18] R. W. Blundell, M. Browning, and I. A. Crawford. Nonparametric Engel curves and revealed preference. *Econometrica*, 71(1):205–240, 2003.

[19] P. Bossaerts, P. Ghirardato, S. Guarnaschelli, and W. R. Zame. Ambiguity in asset markets: theory and experiment. *Review of Financial Studies*, 23(4):1325–1359, 2010.

[20] J. Bröcker. Reliability, sufficiency, and the decomposition of proper scores. *Quarterly Journal of the Royal Meteorological Society*, 135(643):1512–1519, 2009.

[21] M. Browning. A nonparametric test of the life-cycle rational expectations hypothesis. *International Economic Review*, 30(4):979–992, 1989.

[22] C. Camerer and M. Weber. Recent developments in modeling preferences: uncertainty and ambiguity. *Journal of Risk and Uncertainty*, 5(4):325–370, 1992.

[23] A. Capotorti, G. Coletti, and B. Vantaggi. Standard and nonstandard representability of positive uncertainty orderings. *Kybernetika*, 50(2):189–215, 2014.

[24] A. Carvajal, I. Ray, and S. Snyder. Equilibrium behavior in markets and games: testable restrictions and identification. *Journal of Mathematical Economics. Part of special issue: aggregation, equilibrium and observability in honor of Werner Hildenbrand*, 40(1–2):1–40, 2004.

[25] G. Cassese, P. Rigo, and B. Vantaggi. A special issue on the mathematics of subjective probability. *Decisions in Economics and Finance*, 43(1):1–2, 2020.

[26] C. P. Chambers and F. Echenique. Supermodularity and preferences. *Journal of Economic Theory*, 144(3):1004–1014, 2009.

[27] C. P. Chambers, F. Echenique, and E. Shmaya. The axiomatic structure of empirical content. *American Economic Review*, 104(8):2303–2319, 2014.

[28] C. P. Chambers, F. Echenique, and E. Shmaya. General revealed preference theory. *Theoretical Economics*, 12(2):493–511, 2017.

[29] L. Cherchye, T. Demuynck, and B. De Rock. Normality of demand in a two-goods setting. *Journal of Economic Theory*, 173:361–382, 2018.

[30] S. Choi, S. Kariv, W. Müller, and D. Silverman. Who is (more) rational? *American Economic Review*, 104(6):1518–1550, 2014.

[31] T. Chudjakow and F. Riedel. The best choice problem under ambiguity. *Economic Theory*, 54(1):77–97, 2013.

[32] G. Coletti, D. Petturiti, and B. Vantaggi. When upper conditional probabilities are conditional possibility measures. *Fuzzy Sets and Systems*, 304:45–64, 2016.

[33] I. Crawford and B. De Rock. Empirical revealed preference. *Annual Review of Economics*, 6:503–524, 2014.

[34] B. de Finetti. The role of "Dutch Books" and of "proper scoring rules". *The British Journal of Psychology of Sciences*, 32:55–56, 1981.

[35] B. de Finetti. Probability: the different views and terminologies in a critical analysis. In L. J. Cohen, J. Łoś, H. Pfeiffer, and K.-P. Podewski, editors, *Logic, Methodology and Philosophy of Science VI*, pages 391–394. North-Holland Publishing Company, Amsterdam, 1982.

[36] B. de Finetti. The proper approach to probability. In G. Koch and F. Spizzichino, editors, *Exchangeability in Probability and Statistics*, pages 1–6. North-Holland Publishing Company, Amsterdam, 1982.

[37] B. de Finetti. Probabilism: a critical essay on the theory of probability and on the value of science. *Erkenntnis*, 31(2-3):169–223, 1989.

[38] G. Debreu. Stochastic choice and cardinal utility. *Econometrica*, 26(3):440–444, 1958.

[39] T. Denti, M. Marinacci, and L. Montrucchio. A note on rational inattention and rate distortion theory. *Decisions in Economics and Finance*, 43(1):75–89, 2020.

[40] W. E. Diewert. Afriat and revealed preference theory. *Review of Economic Studies*, 40(3):419–425, 1973.

[41] S. Drapeau and M. Kupper. Risk preferences and their robust representation. *Mathematics of Operations Research*, 38(1):28–62, 2013.

[42] P. Dziewulski. Revealed time preference. *Games and Economic Behavior*, 112:67–77, 2018.

[43] F. Echenique. New developments in revealed preference theory: decisions under risk, uncertainty, and intertemporal choice. *Annual Review of Economics*, 12(1):299–316, 2020.

[44] F. Echenique and K. Saito. Savage in the market. *Econometrica*, 83(4):1467–1495, 2015.

[45] L. G. Epstein. Are probabilities used in markets? *Journal of Economic Theory*, 91(1):86–90, 2000.

[46] C. A. T. Ferro and T. E. Fricker. A bias-corrected decomposition of the Brier score. *Quarterly Journal of the Royal Meteorological Society*, 138(668):1954–1960, 2012.

[47] M. Friedman and L. J. Savage. The expected-utility hypothesis and the measurability of utility. *Journal of Political Economy*, 60(6):463–474, 1952.

[48] C. Gerstenberger and D. Vogel. On the efficiency of Gini's mean difference. *Statistical Methods & Applications*, 24(4):569–596, 2015.

[49] P. Ghirardato, F. Maccheroni, and M. Marinacci. Certainty independence and the separation of utility and beliefs. *Journal of Economic Theory*, 120(1):129–136, 2005.

[50] I. Gilboa and D. Schmeidler. Maxmin expected utility with a non-unique prior. *Journal of Mathematical Economics*, 18(2):141–153, 1989.

[51] A. Gilio and G. Sanfilippo. Conditional random quantities and compounds of conditionals. *Studia logica*, 102(4):709–729, 2014.

[52] B. Grechuk, A. Molyboha, and M. Zabarankin. Mean-deviation analysis in the theory of choice. *Risk Analysis: An International Journal*, 32(8):1277–1292, 2012.

[53] B. Grechuk and M. Zabarankin. Inverse portfolio problem with mean-deviation model. *European Journal of Operational Research*, 234(2):481–490, 2014.

[54] R. C. Green and S. Srivastava. Expected utility maximization and demand behavior. *Journal of Economic Theory*, 38(2):313–323, 1986.

[55] Y. Halevy, D. Persitz, and L. Zrill. Parametric recoverability of preferences. *Journal of Political Economy*, 126(4):1558–1593, 2018.

[56] J. D. Hey and N. Pace. The explanatory and predictive power of non two-stage-probability theories of decision making under ambiguity. *Journal of Risk and Uncertainty*, 49(1):1–29, 2014.

[57] H. Hotelling. Relations between two sets of variates. *Biometrika*, 28(3-4):321–377, 1936.

[58] E. J. Johnson and J. W. Payne. Effort and accuracy in choice. *Management Science*, 31(4):395–414, 1985.

[59] I. T. Jolliffe and J. Cadima. Principal component analysis: a review and recent developments. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 374(2065), 2016.

[60] K. Jurado, S. C. Ludvigson, and S. Ng. Measuring uncertainty. *American Economic Review*, 105(3):1177–1216, 2015.

[61] W. Kip Viscusi and W. N. Evans. Behavioral probabilities. *Journal of Risk and Uncertainty*, 32(1):5–15, 2006.

[62] S. Krasa and N. C. Yannelis. The value allocation of an economy with differential information. *Econometrica*, 62(4):881–900, 1994.

[63] F. Kubler, L. Selden, and X. Wei. Asset demand based tests of expected utility maximization. *American Economic Review*, 104(11):3459–3480, 2014.

[64] R. La Haye and P. Zizler. The Gini mean difference and variance. *Metron*, 77(1):43–52, 2019.

[65] M. J. Machina. Choice under uncertainty: problems solved and unsolved. *Journal of Economic Perspectives*, 1(1):121–154, 1987.

[66] M. J. Machina and D. Schmeidler. A more robust definition of subjective probability. *Econometrica*, 60(4):745–780, 1992.

[67] H. Markowitz. The utility of wealth. *Journal of Political Economy*, 60(2):151–158, 1952.

[68] H. M. Markowitz. The optimization of a quadratic function subject to linear constraints. *Naval Research Logistics Quarterly*, 3(1-2):111–133, 1956.

[69] R. L. Matzkin and M. K. Richter. Testing strictly concave rationality. *Journal of Economic Theory*, 53(2):287–303, 1991.

[70] E. C. Merkle and R. Hartman. Weighted Brier score decomposition for topically heterogenous forecasting tournaments. *Judgment and Decision Making*, 13(2):185–201, 2018.

[71] H. Nishimura, E. A. Ok, and J. K.-H. Quah. A comprehensive approach to revealed preference theory. *American Economic Review*, 107(4):1239–1263, 2017.

[72] R. J. Nunke and L. J. Savage. On the set of values of a nonatomic, finitely additive, finite measure. *Proceedings of the American Mathematical Society*, 3(2):217–218, 1952.

[73] G. Pasini. Principal component analysis for stock portfolio management. *International Journal of Pure and Applied Mathematics*, 115(1):153–167, 2017.

[74] J. Pfanzagl. Subjective probability derived from the Morgenstern-von Neumann utility theory. In M. Shubik, editor, *Essays in mathematical economics in honor of Oskar Morgenstern*, pages 237–251. Princeton University Press, Princeton, 1967.

[75] G. Pompilj. On intrinsic independence. *Bulletin of the International Statistical Institute*, 35(2):91–97, 1957.

[76] C. R. Rao. The use and interpretation of principal component analysis in applied research. *Sankhya: the Indian Journal of Statistics, Series A*, 26(4):329–358, 1964.

[77] E. Regazzini. Finitely additive conditional probabilities. *Rendiconti del Seminario Matematico e Fisico di Milano*, 55(1):69–89, 1985.

[78] R. T. Rockafellar, S. Uryasev, and M. Zabarankin. Generalized deviations in risk analysis. *Finance and Stochastics*, 10:51–74, 2006.

[79] R. T. Rockafellar, S. Uryasev, and M. Zabarankin. Optimality conditions in portfolio analysis with general deviation measures. *Mathematical Programming*, 108(2-3):515–540, 2006.

[80] P. A. Samuelson. Consumption theory in terms of revealed preference. *Economica*, 15(60):243–253, 1948.

[81] D. Schmeidler. Subjective probability and expected utility without additivity. *Econometrica*, 57(3):571–587, 1989.

[82] P. Slovic, B. Fischhoff, and S. Lichtenstein. Behavioral decision theory. *Annual Review of Psychology*, 28:1–39, 1977.

[83] W. D. Sudderth. Optimal markov strategies. *Decisions in Economics and Finance*, 43(1):43–54, 2020.

[84] H. R. Varian. The nonparametric approach to demand analysis. *Econometrica*, 50(4):945–973, 1982.

[85] H. R. Varian. Non-parametric tests of consumer behaviour. *The Review of Economic Studies*, 50(1):99–110, 1983.

[86] J. von Neumann. Examples of continuous geometries. *Proceedings of the National Academy of Sciences of the United States of America*, 22(2):101–108, 1936.

[87] D. von Rosen. Maximum likelihood estimators in multivariate linear normal models. *Journal of Multivariate Analysis*, 31(2):187–200, 1989.

[88] H. Wold, G. L. S. Shackle, and L. J. Savage. Ordinal preferences or cardinal utility? *Econometrica*, 20(4):661–664, 1952.

[89] M. E. Yaari. The dual theory of choice under risk. *Econometrica*, 55(1):95–115, 1987.