A SEMILINEAR SYSTEM OF SCHRÖDINGER-MAXWELL EQUATIONS

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Abstract. In this paper we are going to prove existence and regularity results for positive solutions of the following elliptic system:

$$\begin{cases} -\operatorname{div}(M(x)\nabla u) + r\,\varphi\,u^{r-1} = f + \varphi^r\,, \\ -\operatorname{div}(M(x)\nabla \varphi) + r\,u\,\varphi^{r-1} = u^r\,. \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, M is a bounded, uniformly elliptic matrix, $r > 1$, and $f \geq 0$ belongs to some Lebesgue space $L^m(\Omega)$, with $m \geq 1$. We will also prove the relationships of the solutions of the system with saddle points of the integral functional

$$J(v,\psi) = \frac{1}{2}\int_\Omega M(x)\nabla v\cdot\nabla v - \frac{1}{2}\int_\Omega M(x)\nabla\psi\cdot\nabla\psi + \int_\Omega |v|^r\,\psi - \int_\Omega |\psi|^r\,v - \int_\Omega f\,v$$

1. Introduction

In this paper we study the existence and main properties of the weak (or distributional) solutions of the semilinear elliptic system

$$\begin{cases} u \in W_0^{1,2}(\Omega): \ -\operatorname{div}(M(x)\nabla u) + r\,\varphi\,u^{r-1} = f(x) + \varphi^r\,, \\ \varphi \in W_0^{1,2}(\Omega): \ -\operatorname{div}(M(x)\nabla \varphi) + r\,u\,\varphi^{r-1} = u^r\,, \end{cases} \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $N > 2$, $r > 1$, f is a positive function belonging to some Lebesgue space $L^m(\Omega)$, with $m \geq 1$, and $M(x)$ is a measurable matrix such that (for $0 < \alpha \leq \beta$)

$$M(x)\xi\cdot\xi \geq \alpha|\xi|^2\,, \qquad \text{a.e. in } \Omega, \text{ for every } \xi \text{ in } \mathbb{R}^N, \tag{1.2}$$

$$|M(x)| \leq \beta\,, \qquad \text{a.e. in } \Omega. \tag{1.3}$$

The motivations of the interest for the above system come from the paper [1], while existence and properties of solutions of systems of Schrödinger-Maxwell equations have also been studied in [4], [8], [11], where only the first equation of the system is semilinear.

A second motivation comes from a geometrical point of view: the solution (u, φ) of (1.1) are saddle points of the integral functional

$$J(v,\psi) = \frac{1}{2}\int_\Omega M(x)\nabla v\cdot\nabla v - \frac{1}{2}\int_\Omega M(x)\nabla\psi\cdot\nabla\psi + \int_\Omega |v|^r\,\psi - \int_\Omega |\psi|^r\,v - \int_\Omega f\,v\,,$$

even if, at first sight, this fact is not so evident since some terms may not be well defined; in this case, the regularizing effect on the solutions of the coupling of the equations in system (1.1) plays an important role.

Our main results on system (1.1) are proved in the following theorem.

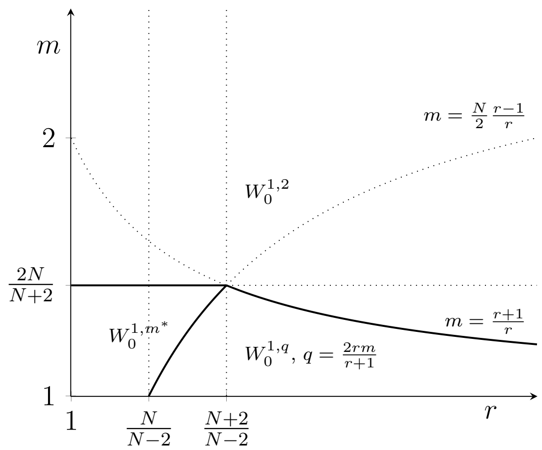
THEOREM 1.1. *Let $r > 1$ and let $f \geq 0$ be a function in $L^m(\Omega)$, with $m \geq 1$. Then there exist solutions u and φ of system (1.1), with $\varphi \leq u$, and the following Sobolev regularities depending on the values of r and m:*

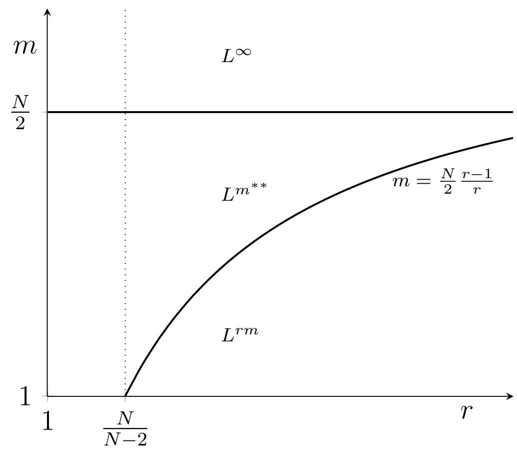
- *if $1 < r \leq \frac{N}{N-2}$:*
  - *if $m \geq \frac{2N}{N+2}$ then u and φ belong to $W_0^{1,2}(\Omega)$;*
  - *if $1 < m < \frac{2N}{N+2}$ then u and φ belong to $W_0^{1,m^*}(\Omega)$, where $m^* = \frac{Nm}{N-m}$;*
  - *if $m = 1$ then u and φ belong to $W_0^{1,q}(\Omega)$, for every $q < \frac{N}{N-1}$;*
- *if $\frac{N}{N-2} < r < \frac{N+2}{N-2}$:*
  - *if $m \geq \frac{2N}{N+2}$ then u and φ belong to $W_0^{1,2}(\Omega)$;*
  - *if $\frac{N}{2}\frac{r-1}{r} \leq m < \frac{2N}{N+2}$ then u and φ belong to $W_0^{1,m^*}(\Omega)$;*
  - *if $1 < m < \frac{N}{2}\frac{r-1}{r}$ then u and φ belong to $W_0^{1,q}(\Omega)$ with $q = \frac{2rm}{r+1}$;*
  - *if $m = 1$ then u and φ belong to $W_0^{1,q}(\Omega)$, for every $q < \frac{2r}{r+1}$;*
- *if $r \geq \frac{N+2}{N-2}$:*
  - *if $m \geq \frac{r+1}{r}$ then u and φ belong to $W_0^{1,2}(\Omega)$;*
  - *if $1 < m < \frac{r+1}{r}$ then u and φ belong to $W_0^{1,q}(\Omega)$ with $q = \frac{2rm}{r+1}$;*
  - *if $m = 1$ then u and φ belong to $W_0^{1,q}(\Omega)$, for every $q < \frac{2r}{r+1}$.*

Furthermore, we have the following results for Lebesgue summability:

- *if $1 < r < \frac{N}{N-2}$:*
  - *if $m > \frac{N}{2}$ then u and φ belong to $L^\infty(\Omega)$;*
  - *if $1 \leq m < \frac{N}{2}$ then u and φ belong to $L^{m^{**}}(\Omega)$, with $m^{**} = \frac{Nm}{N-2m}$;*
  - *if $m = 1$ then u and φ belong to $L^s(\Omega)$, for every $s < \frac{N}{N-2}$;*
- *if $r \geq \frac{N}{N-2}$:*
  - *if $m > \frac{N}{2}$ then u and φ belong to $L^\infty(\Omega)$;*
  - *if $\frac{N}{2}\frac{r-1}{r} < m < \frac{N}{2}$ then u and φ belong to $L^{m^{**}}(\Omega)$;*
  - *if $1 \leq m \leq \frac{N}{2}\frac{r-1}{r}$ then u and φ belong to $L^{rm}(\Omega)$.*

We point out that the results on u and φ in $L^{rm}(\Omega)$ are strongly related to some results of [9] (see also [7]).





Dependence on r and m of the Sobolev and Lebesgue regularities of u and φ

The plan of the paper is as follows: in the next section we will prove an existence result for bounded solutions u_n and φ_n of a "truncated" system which approximates (1.1), while in Section 3 we will prove a result which will be fundamental in the sequel:

namely, that $\varphi_n \le u_n$ for every n. Sections 4 and 5 will be devoted to the proof of Theorem 1.1 in the case $m > \frac{N}{2}$ and $1 < m < \frac{N}{2}$ respectively, while the proof of the case $m = 1$ of Theorem 1.1 is given in Section 6. Finally, in Section 7 we deal with the case of measure data, and in Section 8 we prove that if $m \ge \frac{2N}{N+2}$ the solution of system (1.1) given by Theorem 1.1 is a saddle point of the functional J defined above.

2. Existence of approximating solutions

We begin with an existence result for a problem which approximates (1.1). For n in $\mathbb{N}$, let us define the truncation at levels $\pm n$ as

$$T_n(s) = \max(-n, \min(s, n))\,.$$

Theorem 2.1. *Let $r > 1$, let $F \ge 0$ be in $L^m(\Omega)$, with $m > \frac{N}{2}$, and let n in $\mathbb{N}$. Then there exist solutions u_n and φ_n of the following system:*

$$\begin{cases} 0 \le u_n \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega): \ -\mathrm{div}(M(x)\nabla u_n) + r\,\varphi_n\, u_n^{r-1} = F + T_n(\varphi_n^r)\,, \\ 0 \le \varphi_n \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega): \ -\mathrm{div}(M(x)\nabla \varphi_n) + r\, u_n\, \varphi_n^{r-1} = T_n(u_n^r)\,. \end{cases} \tag{2.1}$$

Furthermore, u_n and φ_n belong to $L^\infty(\Omega)$.

Proof. Let n in $\mathbb{N}$, and let $v \ge 0$ be a function in $L^2(\Omega)$. Then there exists (see [2]) a unique weak solution ψ in $W_0^{1,2}(\Omega)$ of

$$-\mathrm{div}(M(x)\nabla\psi) + r\,v\,|\psi|^{r-2}\psi = T_n(v^r)\,,$$

that is

$$\int_\Omega M(x)\nabla\psi\cdot\nabla w + r\int_\Omega v\,|\psi|^{r-2}\psi\,w = \int_\Omega T_n(v^r)\,w\,, \qquad \forall w \in W_0^{1,2}(\Omega)\,.$$

It is easy to see, using the fact that $v \ge 0$, that $\psi \ge 0$, so that ψ is the solution of

$$-\mathrm{div}(M(x)\nabla\psi) + r\,v\,\psi^{r-1} = T_n(v^r)\,. \tag{2.2}$$

Furthermore, by the results of Stampacchia (see [12]), ψ belongs to $L^\infty(\Omega)$ and is such that

$$\|\psi\|_{W_0^{1,2}(\Omega)} \le C\,n\,, \tag{2.3}$$

for some positive constant C. Given ψ, let z be the unique weak solution in $W_0^{1,2}(\Omega)$ of

$$-\mathrm{div}(M(x)\nabla z) + r\,\psi\,|z|^{r-2}z = F + T_n(\psi^r)\,,$$

that is

$$\int_\Omega M(x)\nabla z\cdot\nabla w + r\int_\Omega \psi\,|z|^{r-2}z\,w = \int_\Omega F\,w + \int_\Omega T_n(\psi^r)\,w\,, \qquad \forall w \in W_0^{1,2}(\Omega)\,.$$

Using again the fact that both F and ψ are positive, z is positive as well, so that it solves

$$-\mathrm{div}(M(x)\nabla z) + r\,\psi\,z^{r-1} = F + T_n(\psi^r)\,. \tag{2.4}$$

Using once again the results by Stampacchia, one has that z belongs to $L^\infty(\Omega)$ and is such that

$$\|z\|_{W_0^{1,2}(\Omega)} \le C\left(\|F\|_{L^2(\Omega)} + n\right). \tag{2.5}$$

By Poincaré inequality, one thus has that

$$\|z\|_{L^2(\Omega)} \le C\left(\|F\|_{L^2(\Omega)} + n\right) = R\,,$$

which implies that the intersection of the positive cone of $L^2(\Omega)$ with the ball $B_R(0)$ of $L^2(\Omega)$ is invariant for the map $S : v \mapsto z$. We give now the (easy) proof that S is both continuous and compact, so that it will have a fixed point by Schauder theorem. As for compactness in $L^2(\Omega)$, it easily follows from (2.5) and from Rellich theorem, so that we only have to prove the continuity of S. To this aim, let v_k be a sequence of positive functions in $L^2(\Omega)$ strongly convergent to v in the same space. If ψ_k is the solution of

$$-\mathrm{div}(M(x)\nabla\psi_k) + r\, v_k\, \psi_k^{r-1} = T_n(v_k^r)\,,$$

then by (2.3) we have that ψ_k is bounded in $W_0^{1,2}(\Omega)$. Therefore, up to subsequences, still denoted by ψ_k, it converges weakly in $W_0^{1,2}(\Omega)$ and strongly in $L^2(\Omega)$ to some function ψ. Since the sequence $T_n(v_k^r)$ is strongly convergent to $T_n(v^r)$ in $L^p(\Omega)$ for every $1 \leq p < +\infty$ it is easy to see that ψ is the solution of

$$-\mathrm{div}(M(x)\nabla\psi) + r\, v\, \psi^{r-1} = T_n(v^r)\,.$$

Let now $z_k = S(v_k)$, which is the solution of

$$-\mathrm{div}(M(x)\nabla z_k) + r\, \psi_k\, z_k^{r-1} = F + T_n(\psi_k^r)\,.$$

Using again (2.5), we have that z_k is bounded in $W_0^{1,2}(\Omega)$, so that, up to subsequences still denoted by z_k, it converges weakly in $W_0^{1,2}(\Omega)$ and strongly in $L^2(\Omega)$ to some function z. Due to the strong convergence, in $L^m(\Omega)$, with $m > \frac{N}{2}$, of the right hand side $F + T_n(\psi_k^r)$, it is again easy to see that z is the solution of

$$-\mathrm{div}(M(x)\nabla z) + r\, \psi\, z^{r-1} = F + T_n(\psi^r)\,,$$

that is, $z = S(v)$. Since the limit is independent of all the extracted subsequences, the whole sequence $S(v_k)$ strongly converges in $L^2(\Omega)$ to $S(v)$, and so S is continuous, as desired.

Thus, by Schauder theorem, there exists a fixed point u_n of S. If we define φ_n as the positive weak solution of

$$\varphi_n \in W_0^{1,2}(\Omega) : \; -\mathrm{div}(M(x)\nabla\varphi_n) + r\, u_n\, \varphi_n^{r-1} = T_n(u_n^r)\,,$$

we than have that $u_n \geq 0$ solves

$$u_n \in W_0^{1,2}(\Omega) : \; -\mathrm{div}(M(x)\nabla u_n) + r\, \varphi_n\, u_n^{r-1} = F + T_n(u_n^r)\,,$$

as desired, and that both u_n and φ_n belong to $L^\infty(\Omega)$.

□

3. Comparison between u_n and φ_n

Aim of this sections is the proof of the following result, which will be fundamental in the sequel.

Theorem 3.1. *Let $r \geq 2$, and let u_n and φ_n be the weak solutions of system* (2.1) *given by Theorem 2.1. Then*

$$\varphi_n \leq u_n\,, \qquad \forall n \in \mathbb{N}. \tag{3.1}$$

Proof. Subtracting the two equations, we have

$$-\mathrm{div}(M(x)\nabla(\varphi_n - u_n)) + r\varphi_n u_n[\varphi_n^{r-2} - u_n^{r-2}] + T_n(\varphi_n^r) - T_n(u_n^r) = -F\,.$$

Taking $(\varphi_n - u_n)_+$ as test function in the weak formulation of this equation, we have

$$\int_\Omega M(x)\nabla(\varphi_n - u_n)\cdot\nabla(\varphi_n - u_n)_+ + r\int_\Omega \varphi_n u_n[\varphi_n^{r-2} - u_n^{r-2}](\varphi_n - u_n)_+ \\ + \int_\Omega [T_n(\varphi_n^r) - T_n(u_n^r)](\varphi_n - u_n)_+ = -\int_\Omega F(\varphi_n - u_n)_+\,.$$

We remark now that the right hand side is negative (since $F \geq 0$), while the second and third term of the left hand side are positive (since $u_n \geq 0$, $\varphi_n \geq 0$, and $r \geq 2$). Therefore, by (1.2) we have that

$$\alpha\int_\Omega |\nabla(\varphi_n - u_n)_+|^2 \leq \int_\Omega M(x)\nabla(\varphi_n - u_n)\cdot\nabla(\varphi_n - u_n)_+ \leq 0\,,$$

which implies that $(\varphi_n - u_n)_+ = 0$, and so $\varphi_n \leq u_n$, as desired. □

4. Proof of Theorem 1.1 if $m > \frac{N}{2}$

We begin this section proving that the sequences $\{u_n\}$ and $\{\varphi_n\}$ given by Theorem 2.1 are bounded in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.

THEOREM 4.1. *Let $r \geq 2$ and let $F \geq 0$ be a function in $L^m(\Omega)$, with $m > \frac{N}{2}$. Let u_n and φ_n be the solutions of system* (2.1) *given by Theorem 2.1. Then there exists a constant $C > 0$ such that*

$$\|u_n\|_{W_0^{1,2}(\Omega)} + \|\varphi_n\|_{W_0^{1,2}(\Omega)} + \|u_n\|_{L^\infty(\Omega)} + \|\varphi_n\|_{L^\infty(\Omega)} \leq C\,\|F\|_{L^m(\Omega)}\,. \tag{4.1}$$

Proof. We choose u_n as test function in the weak formulation of the first equation of (2.1), and φ_n as test function in the second. We have

$$\int_\Omega M(x)\nabla u_n\cdot\nabla u_n + r\int_\Omega \varphi_n u_n^r = \int_\Omega F u_n + \int_\Omega T_n(\varphi_n^r)u_n \leq \int_\Omega F u_n + \int_\Omega \varphi_n^r u_n\,,$$

and

$$\int_\Omega M(x)\nabla\varphi_n\cdot\nabla\varphi_n + r\int_\Omega u_n\varphi_n^r = \int_\Omega T_n(u_n^r)\varphi_n \leq \int_\Omega u_n^r\varphi_n\,.$$

Using (1.2), and summing the two inequalities, we obtain

$$\alpha\int_\Omega |\nabla u_n|^2 + \alpha\int_\Omega |\nabla\varphi_n|^2 + (r-1)\int_\Omega \varphi_n u_n^r + (r-1)\int_\Omega u_n\varphi_n^r \leq \int_\Omega F u_n\,, \tag{4.2}$$

which implies, dropping some positive terms, that

$$\alpha\int_\Omega |\nabla u_n|^2 \leq \int_\Omega F u_n\,.$$

From this inequality, and since $m > \frac{N}{2} > \frac{2N}{N+2}$, it is well known that one can prove there exists a constant $C > 0$ such that

$$\|u_n\|_{W_0^{1,2}(\Omega)} \leq C\,\|F\|_{L^m(\Omega)}\,, \tag{4.3}$$

so that (recalling (4.2)), one also has

$$\|\varphi_n\|_{W_0^{1,2}(\Omega)} \leq C\,\|F\|_{L^m(\Omega)}\,. \tag{4.4}$$

As for the estimate in $L^\infty(\Omega)$ of (4.1), we choose $G_k(u_n) = (u_n - k)_+$ as test function in the first equation to obtain, after using (1.2),

$$\begin{aligned}\alpha\int_\Omega |\nabla G_k(u_n)|^2 + r\int_\Omega \varphi_n u_n^{r-1} G_k(u_n) &= \int_\Omega F\,G_k(u_n) + \int_\Omega T_n(\varphi_n^r)\,G_k(u_n)\\ &\leq \int_\Omega F\,G_k(u_n) + \int_\Omega \varphi_n^r\,G_k(u_n)\\ &= \int_\Omega F\,G_k(u_n) + \int_\Omega \varphi_n\,\varphi_n^{r-1}\,G_k(u_n)\\ &\leq \int_\Omega F\,G_k(u_n) + \int_\Omega \varphi_n\,u_n^{r-1}\,G_k(u_n)\,,\end{aligned}$$

where in the last passage we have used that $\varphi_n \leq u_n$ thanks to Theorem 3.1. Thus, we have that

$$\alpha\int_\Omega |\nabla G_k(u_n)|^2 \leq \alpha\int_\Omega |\nabla G_k(u_n)|^2 + (r-1)\int_\Omega \varphi_n\,u_n^{r-1}\,G_k(u_n) \leq \int_\Omega F\,G_k(u_n)\,. \tag{4.5}$$

Using this inequality, Stampacchia proved in [12] that there exists a constant $C > 0$ such that

$$\|u_n\|_{L^\infty(\Omega)} \leq C\,\|F\|_{L^m(\Omega)}\,.$$

From this estimate, and since $\varphi_n \leq u_n$ by Theorem 3.1, we also have that

$$\|\varphi_n\|_{L^\infty(\Omega)} \leq C\,\|F\|_{L^m(\Omega)}\,,$$

as desired. □

As a consequence of this result, one can prove Theorem 1.1 in the case $r \geq 2$ and $m > \frac{N}{2}$.

Proof of Theorem 1.1, case $r \geq 2$ and $m > \frac{N}{2}$. Thanks to the estimate (4.1), if n is large enough one has that $T_n(u_n^r) = u_n^r$ and $T_n(\varphi_n^r) = \varphi_n^r$. Therefore, for such n, one has that $u \stackrel{\text{def}}{=} u_n$ and $\varphi \stackrel{\text{def}}{=} \varphi_n$ are solutions of the equations in (1.1), and both belong to $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. The fact that $\varphi \leq u$ follows from (3.1). □

Up to now, we have no results if $1 < r < 2$. The next proof fills the gap.

Proof of Theorem 1.1, case $1 < r < 2$ and $m > \frac{N}{2}$. We go back to the approximating solutions u_n and φ_n of (2.1):

$$\begin{cases} 0 \leq u_n \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega) : \ -\text{div}(M(x)\nabla u_n) + r\,\varphi_n\,u_n^{r-1} = F + T_n(\varphi_n^r)\,,\\ 0 \leq \varphi_n \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega) : \ -\text{div}(M(x)\nabla \varphi_n) + r\,u_n\,\varphi_n^{r-1} = T_n(u_n^r)\,. \end{cases} \tag{4.6}$$

Repeating the same steps of the proof of Theorem 4.1 we arrive (*without using that* $\varphi_n \leq u_n$) to (4.3) and (4.4): that is, both u_n and φ_n are bounded in $W_0^{1,2}(\Omega)$. Therefore, there exist u and φ, their weak limits in $W_0^{1,2}(\Omega)$ up to subsequences. Since $r < 2 < 2^*$, this means (by Rellich theorem) that, always up to subsequences,

$$u_n^r \text{ strongly converges to } u^r \text{ in } L^1(\Omega), \quad \varphi_n^r \text{ strongly converges to } \varphi^r \text{ in } L^1(\Omega),$$

and similar convergences hold for both $\varphi_n\,u_n^{r-1}$ and $u_n\,\varphi_n^{r-1}$. Thus, it is possible to pass to the limit in the weak formulations of (4.6) to find that u and φ belong to $W_0^{1,2}(\Omega)$, and are such that

$$\int_\Omega M(x)\nabla u\cdot\nabla w + r\int_\Omega \varphi\, u^{r-1}\, w = \int_\Omega F\, w + \int_\Omega \varphi^r\, w\,, \qquad \forall w\in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)\,,$$

and

$$\int_\Omega M(x)\nabla \varphi\cdot\nabla w + r\int_\Omega u\,\varphi^{r-1}\, w = \int_\Omega u^r\, w\,, \qquad \forall w\in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)\,.$$

Subtracting the second equation from the first, and choosing $w = T_1((\varphi-u)_+)$ as test function (which is an admissible choice since it is in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$), we obtain, after using (1.2),

$$\begin{aligned}\alpha\int_\Omega |\nabla T_1((\varphi-u)_+)|^2 &+ r\int_\Omega u\,\varphi^{r-1}\,T_1((\varphi-u)_+) - r\int_\Omega \varphi\, u^{r-1}\, T_1((\varphi-u)_+)\\ &\le \int_\Omega u^r\, T_1((\varphi-u)_+) - \int_\Omega \varphi^r\, T_1((\varphi-u)_+) - \int_\Omega F\, T_1((\varphi-u)_+)\,,\end{aligned}$$

which can be rewritten as

$$\begin{aligned}\alpha\int_\Omega |\nabla T_1((\varphi-u)_+)|^2 &+ \int_\Omega [\varphi^r + r\,u\,\varphi^{r-1} - r\,\varphi\,u^{r-1} - u^r]\,T_1((\varphi-u)_+)\\ &\le -\int_\Omega F\,T_1((\varphi-u)_+) \le 0\,,\end{aligned} \tag{4.7}$$

where the last inequality is due to the fact that both F and $T_1((\varphi-u)_+)$ are positive functions. Let us define the function

$$G(\varphi,u) = \varphi^r + r\,u\,\varphi^{r-1} - r\,\varphi\,u^{r-1} - u^r\,,$$

which we will only consider on the set where $\varphi\ge u$. Since $Q(t) = t^r$ is convex being $r>1$, we have, for every s and t in $\mathbb{R}$,

$$Q(t)\ge Q(s) + Q'(s)\,(t-s)\,,$$

so that

$$\varphi^r \ge u^r + r\,u^{r-1}\,(\varphi-u)\,.$$

Therefore,

$$\begin{aligned}G(\varphi,u) &= \varphi^r + r\,u\,\varphi^{r-1} - r\,\varphi\,u^{r-1} - u^r\\ &\ge u^r + r\,u^{r-1}\,(\varphi-u) + r\,u\,\varphi^{r-1} - r\,\varphi\,u^{r-1} - u^r\\ &= r\,u\,\varphi^{r-1} - r\,u^r = r\,u\,(\varphi^{r-1} - u^{r-1}) \ge 0\,,\end{aligned}$$

where $\varphi\ge u\ge 0$, since $t\mapsto t^{r-1}$ is increasing. Thus, from (4.7) we have that

$$\alpha\int_\Omega |\nabla T_1((\varphi-u)_+)|^2 \le \alpha\int_\Omega |\nabla T_1((\varphi-u)_+)|^2 + \int_\Omega G(\varphi,u)\,T_1((\varphi-u)_+) \le 0\,,$$

which implies that $\varphi\le u$.

Once we have that $\varphi\le u$, it follows that both u and φ belong to $L^\infty(\Omega)$. Indeed, if $w\ge 0$ is a function in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$, we have

$$\int_\Omega M(x)\nabla u\cdot\nabla w + r\int_\Omega \varphi\,u^{r-1}\,w = \int_\Omega F\,w + \int_\Omega \varphi^r\,w \le \int_\Omega F\,w + \int_\Omega \varphi\,u^{r-1}\,w\,,$$

so that

$$\int_\Omega M(x)\nabla u\cdot\nabla w + (r-1)\int_\Omega \varphi\, u^{r-1}\, w \le \int_\Omega F\, w\,, \quad \forall w\ge 0\,,\ w\in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)\,.$$

Since u, φ and w are positive, one thus has that

$$\int_\Omega M(x)\nabla u\cdot\nabla w \le \int_\Omega F\, w\,, \qquad \forall w\ge 0\,,\ w\in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)\,.$$

Recalling that u is in $W_0^{1,2}(\Omega)$, and F is in $L^m(\Omega)$, with $m > \frac{N}{2} > \frac{2N}{N+2}$, we have by density that

$$\int_\Omega M(x)\nabla u\cdot\nabla w \le \int_\Omega F\, w\,, \qquad \forall w\ge 0\,,\ w\in W_0^{1,2}(\Omega)\,.$$

Choosing $w = G_k(u) = (u-k)_+$ we can then proceed as in [12] to prove that u belongs to $L^\infty(\Omega)$, and then that φ belongs to $L^\infty(\Omega)$ as well since $\varphi\le u$. This concludes the proof. □

5. Proof of Theorem 1.1 if $1 < m < \frac{N}{2}$

In this section we are going to prove Theorem 1.1 if f belongs to $L^m(\Omega)$, with $1 < m < \frac{N}{2}$. Our first result yields *a priori* estimates on the solutions u and φ given by Theorem 1.1 for $L^\infty(\Omega)$ data, depending on the norm of f in $L^m(\Omega)$. To obtain such estimates, will use some techniques used in [11] in a similar framework.

Theorem 5.1. *Let $r > 1$, let $F\ge 0$ be a function in $L^\infty(\Omega)$, and let $m > 1$. Then, if u and φ are the solutions of* (1.1) *given by Theorem 1.1, the following holds:*

a) *if $1 < m < \frac{N}{2}$, then there exists a constant $C > 0$ such that*

$$\|u\|^r_{L^{r\,m}(\Omega)} + \|\varphi\|^r_{L^{r\,m}(\Omega)} \le C\,\|F\|_{L^m(\Omega)}\,. \tag{5.1}$$

b) *if $\frac{r+1}{r}\le m < \frac{N}{2}$, then there exists a constant $C > 0$ such that*

$$\|u\|^2_{W_0^{1,2}(\Omega)} + \|\varphi\|^2_{W_0^{1,2}(\Omega)} \le C\,\|F\|^{\frac{r+1}{r}}_{L^m(\Omega)}\,; \tag{5.2}$$

c) *if $1 < m < \frac{r+1}{r}$, then there exists a constant $C > 0$ such that*

$$\|u\|^q_{W_0^{1,q}(\Omega)} + \|\varphi\|^q_{W_0^{1,q}(\Omega)} \le C\,\|F\|^m_{L^m(\Omega)}\,, \qquad \text{with } q = \frac{2rm}{r+1}\,. \tag{5.3}$$

Proof. Let $\gamma > 0$, let $\varepsilon > 0$, and choose $v = [(u+\varepsilon)^\gamma - \varepsilon^\gamma]$ as test function in the first equation of (1.1). We obtain

$$\begin{aligned}\gamma\int_\Omega M(x)\nabla u\cdot\nabla u\,(u+\varepsilon)^{\gamma-1} &+ r\int_\Omega \varphi\, u^{r-1}[(u+\varepsilon)^\gamma - \varepsilon^\gamma] \\ &= \int_\Omega F\,[(u+\varepsilon)^\gamma - \varepsilon^\gamma] + \int_\Omega \varphi^r[(u+\varepsilon)^\gamma - \varepsilon^\gamma]\,.\end{aligned} \tag{5.4}$$

Recalling that $\varphi\le u$ and that $F\ge 0$, we have that

$$\int_\Omega F\,[(u+\varepsilon)^\gamma - \varepsilon^\gamma] + \int_\Omega \varphi^r[(u+\varepsilon)^\gamma - \varepsilon^\gamma] \le \int_\Omega F\,(u+\varepsilon)^\gamma + \int_\Omega \varphi\, u^{r-1}[(u+\varepsilon)^\gamma - \varepsilon^\gamma]\,,$$

so that (5.4) becomes, after using (1.2),

$$\begin{aligned}
&\alpha\,\gamma\int_\Omega|\nabla u|^2(u+\varepsilon)^{\gamma-1}+(r-1)\int_\Omega\varphi^r(u+\varepsilon)^\gamma\\
&\quad\le\alpha\,\gamma\int_\Omega|\nabla u|^2(u+\varepsilon)^{\gamma-1}+(r-1)\int_\Omega\varphi\,u^{r-1}(u+\varepsilon)^\gamma\\
&\quad\le\int_\Omega F\,(u+\varepsilon)^\gamma+(r-1)\varepsilon^\gamma\int_\Omega\varphi\,u^{r-1}\,.
\end{aligned}\tag{5.5}$$

Choosing $\varphi\,(u+\varepsilon)^{\gamma-1}$ as test function in the second equation yields (always using (1.2)) that

$$\alpha\int_\Omega|\nabla\varphi|^2(u+\varepsilon)^{\gamma-1}+r\int_\Omega u\varphi^r(u+\varepsilon)^{\gamma-1}\le\int_\Omega\varphi u^r(u+\varepsilon)^{\gamma-1}+C\int_\Omega|\nabla\varphi||\nabla u|(u+\varepsilon)^{\gamma-2}\varphi\,.$$

We now have

$$\int_\Omega\varphi u^r(u+\varepsilon)^{\gamma-1}=\int_\Omega\varphi\frac{u^r}{u+\varepsilon}(u+\varepsilon)^\gamma\le\int_\Omega\varphi\,u^{r-1}(u+\varepsilon)^\gamma\,,$$

and

$$\int_\Omega|\nabla\varphi||\nabla u|(u+\varepsilon)^{\gamma-2}\varphi\le\int_\Omega|\nabla\varphi||\nabla u|(u+\varepsilon)^{\gamma-1}\frac{\varphi}{u+\varepsilon}\le\int_\Omega|\nabla\varphi||\nabla u|(u+\varepsilon)^{\gamma-1}\,,$$

since $\varphi\le u\le u+\varepsilon$. Thus, by Young inequality, and dropping a positive term, we have

$$\begin{aligned}
\alpha\int_\Omega|\nabla\varphi|^2(u+\varepsilon)^{\gamma-1}&\le C\int_\Omega F\,(u+\varepsilon)^\gamma+C\,\varepsilon^\gamma\int_\Omega\varphi\,u^{r-1}\\
&\quad+\frac{\alpha}{2}\int_\Omega|\nabla\varphi|^2(u+\varepsilon)^{\gamma-1}+C\int_\Omega|\nabla u|^2(u+\varepsilon)^{\gamma-1}\\
&\le C\int_\Omega F\,(u+\varepsilon)^\gamma+C\,\varepsilon^\gamma\int_\Omega\varphi\,u^{r-1}+\frac{\alpha}{2}\int_\Omega|\nabla\varphi|^2(u+\varepsilon)^{\gamma-1}\,.
\end{aligned}$$

Therefore,

$$\frac{\alpha}{2}\int_\Omega|\nabla\varphi|^2(u+\varepsilon)^{\gamma-1}\le C\int_\Omega F\,(u+\varepsilon)^\gamma+C\,\varepsilon^\gamma\int_\Omega\varphi\,u^{r-1}\,.\tag{5.6}$$

We follow now the ideas of [11], and use $(u+\varepsilon)^\gamma-\varepsilon^\gamma$ as test function in the second equation of (1.1) to obtain, after reversing the identities

$$\int_\Omega u^r[(u+\varepsilon)^\gamma-\varepsilon^\gamma]=r\int_\Omega u\,\varphi^{r-1}[(u+\varepsilon)^\gamma-\varepsilon^\gamma]+\gamma\int_\Omega M(x)\nabla\varphi\cdot\nabla u\,(u+\varepsilon)^{\gamma-1}\,.$$

By Young inequality with exponents r and r' and by (5.5) we have

$$\begin{aligned}
r\int_\Omega u\,\varphi^{r-1}[(u+\varepsilon)^\gamma-\varepsilon^\gamma]&\le\frac{1}{2}\int_\Omega u^r[(u+\varepsilon)^\gamma-\varepsilon^\gamma]+C\int_\Omega\varphi^r[(u+\varepsilon)^\gamma-\varepsilon^\gamma]\,,\\
&\le\frac{1}{2}\int_\Omega u^r[(u+\varepsilon)^\gamma-\varepsilon^\gamma]+C\int_\Omega F\,(u+\varepsilon)^\gamma\,,
\end{aligned}$$

and, by (5.5) and (5.6),

$$\left|\gamma\int_\Omega M(x)\nabla\varphi\cdot\nabla u\,(u+\varepsilon)^{\gamma-1}\right|\le C\int_\Omega F\,(u+\varepsilon)^\gamma+C\,\varepsilon^\gamma\int_\Omega\varphi\,u^{r-1}\,.$$

Therefore, we have

$$\frac{1}{2}\int_\Omega u^r[(u+\varepsilon)^\gamma-\varepsilon^\gamma]\le C\int_\Omega F\,(u+\varepsilon)^\gamma+C\,\varepsilon^\gamma\int_\Omega\varphi\,u^{r-1}.$$

Letting ε tend to zero, and using both Fatou lemma and Lebesgue theorem (note that every function in the above inequality is positive and bounded), we arrive at

$$\int_\Omega u^{r+\gamma}\le C\int_\Omega F\,u^\gamma\le C\,\|F\|_{L^m(\Omega)}\Big(\int_\Omega u^{\gamma\,m'}\Big)^{\frac{1}{m'}}.$$

Choosing γ so that $r+\gamma=\gamma\,m'$ yields $\gamma=r(m-1)$, which is positive by the assumptions on r and m. Therefore, we have (using that $\varphi\le u$) that

$$\int_\Omega\varphi^{rm}\le\int_\Omega u^{rm}\le C\,\|F\|^m_{L^m(\Omega)},$$

which is (5.1). Letting ε tend to zero in (5.5) and (5.6), we obtain (dropping some positive terms), recalling the value of γ, and using (5.1)

$$\int_\Omega|\nabla u|^2\,u^{r(m-1)-1}+\int_\Omega|\nabla\varphi|^2\,u^{r(m-1)-1}\le C\int_\Omega F\,u^{r(m-1)}\le C\,\|F\|^m_{L^m(\Omega)}. \tag{5.7}$$

Suppose now that $r(m-1)<1$, which is true if and only if $1<m<\frac{r+1}{r}$, and let $q<2$. We then have, following [6],

$$\int_\Omega|\nabla u|^q=\int_\Omega\frac{|\nabla u|^q}{u^{\frac{q}{2}(1-r(m-1))}}\,u^{\frac{q}{2}(1-r(m-1))}\le\Big(\int_\Omega\frac{|\nabla u|^q}{u^{1-r(m-1)}}\Big)^{\frac{q}{2}}\Big(\int_\Omega u^{\frac{q}{2-q}(1-r(m-1))}\Big)^{\frac{2-q}{2}}.$$

Let q be such that

$$\frac{q}{2-q}(1-r(m-1))=r\,m\qquad\Longleftrightarrow\qquad q=\frac{2rm}{r+1},$$

and note that $q<2$ since $m<\frac{r+1}{r}$. Thus, by (5.1) and by (5.7),

$$\int_\Omega|\nabla u|^q\le C\,\|F\|^{\frac{m\,q}{2}}_{L^m(\Omega)}\Big(\int_\Omega u^{r\,m}\Big)^{\frac{2-q}{q}}\le C\,\|F\|^{\frac{m\,q}{2}}_{L^m(\Omega)}\,\|F\|^{\frac{m\,(2-q)}{2}}_{L^m(\Omega)}=C\,\|F\|^m_{L^m(\Omega)},$$

which is half of (5.3). The other half, the estimate on φ in $W^{1,q}_0(\Omega)$ for the same value of q, can be obtained in the same way starting from (5.7).

Suppose now that $r(m-1)=1$, that is $m=\frac{r+1}{r}$. In this case, (5.7) becomes

$$\int_\Omega|\nabla u|^2+\int_\Omega|\nabla\varphi|^2\le C\,\|F\|^{\frac{r+1}{r}}_{L^{\frac{r+1}{r}}(\Omega)}, \tag{5.8}$$

which is (5.2) in this case. If $r(m-1)>1$, that is if $m>\frac{r+1}{r}$, then by Hölder inequality one has

$$\|F\|_{L^{\frac{r+1}{r}}(\Omega)}\le C\,\|F\|_{L^m(\Omega)}.$$

Thus from (5.8), and the above inequality, it follows that

$$\int_\Omega|\nabla u|^2+\int_\Omega|\nabla\varphi|^2\le C\,\|F\|^{\frac{r+1}{r}}_{L^m(\Omega)},$$

which is (5.2) in this case. This concludes the proof. □

Once the *a priori* estimates have been obtained, we can prove the existence result of Theorem 1.1 in the case $1<m<\frac{N}{2}$, reasoning by approximation.

Proof of Theorem 1.1, case $1 < m < \frac{N}{2}$. Given f as in the statement, let $F_n = T_n(f)$. Then F_n belongs to $L^\infty(\Omega)$, and strongly converges to f in $L^m(\Omega)$. Let u_n and φ_n be the solutions of (1.1) with data F_n, whose existence is guaranteed by Theorem 1.1. Thanks to the results of Theorem 5.1, the sequences u_n and φ_n are bounded in $L^{r\,m}(\Omega)$, and in either $W_0^{1,2}(\Omega)$ (if $m \geq \frac{r+1}{r}$) or in $W_0^{1,q}(\Omega)$, with $q = \frac{2rm}{r+1}$ (if $1 < m < \frac{r+1}{r}$).

On the other hand, since both $\{u_n^r\}$ and $\{\varphi_n^r\}$ are bounded in $L^m(\Omega)$, as is $\{F_n\}$, classical elliptic regularity results (see for example [12], [5], [6]) yield that both $\{u_n\}$ and $\{\varphi_n\}$ are bounded in $W_0^{1,2}(\Omega)$ if $m \geq \frac{2N}{N+2}$, and in $W_0^{1,m^*}(\Omega)$ if $1 < m < \frac{2N}{N+2}$; as for Lebesgue summability, both the sequences $\{u_n\}$ and $\{\varphi_n\}$ are bounded in $L^{m^{**}}(\Omega)$.

Therefore, comparing the exponents, we have the following results depending on the values of r:

- if $1 < r \leq \frac{N}{N-2}$, then $\{u_n\}$ and $\{\varphi_n\}$ are bounded in $W_0^{1,2}(\Omega) \cap L^{m^{**}}(\Omega)$ if $m \geq \frac{2N}{N+2}$, and in $W_0^{1,m^*}(\Omega)$ if $1 < m < \frac{2N}{N+2}$;
- if $\frac{N}{N-2} < r < \frac{N+2}{N-2}$, then $\{u_n\}$ and $\{\varphi_n\}$ are bounded in $W_0^{1,2}(\Omega) \cap L^{m^{**}}(\Omega)$ if $m \geq \frac{2N}{N+2}$, in $W_0^{1,m^*}(\Omega)$ if $\frac{N}{2}\frac{r-1}{r} \leq m < \frac{2N}{N+2}$, and in $W_0^{1,q}(\Omega) \cap L^{rm}(\Omega)$, with $q = \frac{2rm}{r+1}$, if $1 < m < \frac{N}{2}\frac{r-1}{r}$;
- if $r \geq \frac{N+2}{N-2}$, then $\{u_n\}$ and $\{\varphi_n\}$ are bounded in $W_0^{1,2}(\Omega) \cap L^{rm}(\Omega)$ if $m \geq \frac{r+1}{r}$, and in $W_0^{1,q}(\Omega) \cap L^{rm}(\Omega)$, with $q = \frac{2rm}{r+1}$, if $1 < m < \frac{r+1}{r}$.

Thus, up to subsequences, they weakly converge respectively to u and φ in the various Sobolev and Lebesgue spaces where they are bounded, and almost everyhwere. Thanks to these convergences, and to the fact that $m > 1$, the sequence u_n^r is strongly convergent in $L^1(\Omega)$ to u^r; since it dominates the sequences φ_n^r, $u_n\,\varphi_n^{r-1}$ and $\varphi_n\,u_n^{r-1}$ (being $\varphi_n \leq u_n$), we have strong convergence in $L^1(\Omega)$ of φ_n^r, $u_n\,\varphi_n^{r-1}$ and $\varphi_n\,u_n^{r-1}$ to their respective almost everyhwere limits. All these results allow to pass to the limit in the weak formulation of the equations for u_n and φ_n, that is:

$$\int_\Omega M(x)\nabla u_n \cdot \nabla w + r\int_\Omega \varphi_n\,u_n^{r-1} w = \int_\Omega F_n\,w + \int_\Omega \varphi_n^r w\,, \qquad \forall w \in C_0^1(\Omega)\,,$$

and

$$\int_\Omega M(x)\nabla \varphi_n \cdot \nabla w + r\int_\Omega u_n\,\varphi_n^{r-1} w = \int_\Omega u_n^r w\,, \qquad \forall w \in C_0^1(\Omega)\,,$$

to have that

$$\int_\Omega M(x)\nabla u \cdot \nabla w + r\int_\Omega \varphi\,u^{r-1}\,w = \int_\Omega f\,w + \int_\Omega \varphi^r w\,, \qquad \forall w \in C_0^1(\Omega)\,,$$

and

$$\int_\Omega M(x)\nabla \varphi \cdot \nabla w + r\int_\Omega u\,\varphi^{r-1} w = \int_\Omega u^r w\,, \qquad \forall w \in C_0^1(\Omega)\,,$$

as desired. □

Remark 5.2. If f belongs to $L^m(\Omega)$, with m such that u and φ are in $W_0^{1,2}(\Omega)$, then we can choose test functions w in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ in the formulations for u and φ, which thus become weak solutions (instead of "only" distributional ones).

6. Proof of Theorem 1.1 if $m = 1$

In this section we prove Theorem 1.1 in the case $m = 1$. We begin, as usual, with some *a priori* estimates on the solutions u and φ given by Theorem 1.1 with $L^\infty(\Omega)$ data.

Theorem 6.1. *Let $r > 1$ and $F \geq 0$ be a function in $L^\infty(\Omega)$. Then, if u and φ are the solutions of* (1.1) *given by Theorem 1.1, the following holds:*

$$\|u\|_{W_0^{1,q}(\Omega)} + \|\varphi\|_{W_0^{1,q}(\Omega)} + \|u\|_{L^r(\Omega)} + \|\varphi\|_{L^r(\Omega)} \leq C\,\|F\|_{L^1(\Omega)}\,, \qquad \forall q < \frac{2r}{r+1}\,.$$

Proof. We choose $T_k(u)$ as test function in the first equation; after using (1.2), and that $\varphi \leq u$, we obtain

$$\alpha\int_\Omega |\nabla T_k(u)|^2 + (r-1)\int_\Omega \varphi^r T_k(u) \leq \alpha\int_\Omega |\nabla T_k(u)|^2 + (r-1)\int_\Omega \varphi u^{r-1} T_k(u) \leq \int_\Omega F\,T_k(u).$$

Dividing by k, we have that

$$\frac{1}{k}\int_\Omega |\nabla T_k(u)|^2 + \int_\Omega \varphi^r\,\frac{T_k(u)}{k} \leq \int_\Omega F\frac{T_k(u)}{k}\,. \tag{6.1}$$

Choosing $T_k(\varphi)$ as test function in the second equation yields, after dividing by k, using (1.2) and the fact that $\varphi \leq u$, that

$$\frac{\alpha}{k}\int_\Omega |\nabla T_k(\varphi)|^2 + r\int_\Omega u\,\varphi^{r-1}\frac{T_k(\varphi)}{k} \leq \int_\Omega u^r\frac{T_k(\varphi)}{k} \leq \int_\Omega u^r\frac{T_k(u)}{k}\,.$$

Therefore,

$$\frac{\alpha}{k}\int_\Omega |\nabla T_k(\varphi)|^2 \leq \int_\Omega u^r\frac{T_k(u)}{k}\,. \tag{6.2}$$

We now choose $T_k(u)$ as test function in the second equation of system (1.1); after reversing the identity, and dividing by k, we have

$$\int_\Omega u^r\frac{T_k(u)}{k} = r\int_\Omega u\,\varphi^{r-1}\frac{T_k(u)}{k} + \frac{1}{k}\int_\Omega M(x)\nabla\varphi\cdot\nabla T_k(u)\,.$$

We now remark that the last integral is on the set $\{0 \leq u \leq k\}$; since $\varphi \leq u$, on this set we have $\varphi \leq k$ too, so that $\nabla\varphi = \nabla T_k(\varphi)$. Therefore, using Young inequality twice, we have

$$\int_\Omega u^r\frac{T_k(u)}{k} \leq \frac{1}{4}\int_\Omega u^r\frac{T_k(u)}{k} + C\int_\Omega \varphi^r\frac{T_k(u)}{k} + \frac{\alpha}{4k}\int_\Omega |\nabla T_k(\varphi)|^2 + \frac{C}{k}\int_\Omega |\nabla T_k(u)|^2\,.$$

Using both (6.1) and (6.2) we thus have

$$\frac{1}{2}\int_\Omega u^r\frac{T_k(u)}{k} \leq C\int_\Omega F\frac{T_k(u)}{k}\,. \tag{6.3}$$

Letting k tend to zero, we obtain

$$\int_\Omega u^r \leq C\,\|F\|_{L^1(\Omega)}\,, \tag{6.4}$$

which then implies (since $\varphi \leq u$) that

$$\int_\Omega \varphi^r \leq C\,\|F\|_{L^1(\Omega)}\,. \tag{6.5}$$

Now we follow [3]: recalling (6.1) and dropping a positive term, we have that

$$\int_\Omega |\nabla T_k(u)|^2 \leq C\,k\,\|F\|_{L^1(\Omega)}\,. \tag{6.6}$$

Let $\lambda > 0$. Then

$$\{|\nabla u| \geq \lambda\} = \{|\nabla u| \geq \lambda\,,\ u \leq k\} \cup \{|\nabla u| \geq \lambda\,, u \geq k\}\,.$$

Therefore,

$$\begin{aligned} |\{|\nabla u| \geq \lambda\}| &\leq |\{|\nabla u| \geq \lambda\,,\ u \leq k\}| + |\{|\nabla u| \geq \lambda\,, u \geq k\}| \\ &\leq |\{|\nabla u| \geq \lambda\,,\ u \leq k\}| + |\{u \geq k\}|\,. \end{aligned}$$

Thus, from (6.4) and (6.6) it follows that

$$|\{|\nabla u| \geq \lambda\}| \leq C\,\|F\|_{L^1(\Omega)}\,\frac{k}{\lambda^2} + C\,\|F\|_{L^1(\Omega)}\,\frac{1}{k^r} \leq C\,\|F\|_{L^1(\Omega)}\left(\frac{k}{\lambda^2} + \frac{1}{k^r}\right).$$

Choosing $k^{r+1} = \lambda^2$, we obtain

$$|\{|\nabla u| \geq \lambda\}| \leq C\,\|F\|_{L^1(\Omega)}\,\frac{1}{\lambda^{\frac{2r}{r+1}}}\,,$$

which yields an estimate on ∇u in the Marcinkiewicz space $M^{\frac{2r}{r+1}}(\Omega)$; thanks to the well known inclusions between Marcinkiewicz and Lebesgue spaces, we thus have

$$\|u\|^q_{W^{1,q}_0(\Omega)} \leq C\,\|F\|_{L^1(\Omega)}\,, \qquad \forall q < \frac{2r}{r+1}\,,$$

as desired. The analogous estimate for φ is proved in the same way, using that $\varphi \leq u$. □

We can now conclude the proof of Theorem 1.1.

Proof of Theorem 1.1, case $m = 1$. Let $f \geq 0$ be as in the statement, and let $F_n = T_n(f)$. Then, if u_n and φ_n are the solutions of (1.1) with data F_n, from Theorem 6.1 it follows that

$$\|u_n\|^q_{W^{1,q}_0(\Omega)} + \|\varphi_n\|^q_{W^{1,q}_0(\Omega)} \leq C\,\|f\|_{L^1(\Omega)}\,, \quad \int_\Omega u_n^r + \int_\Omega \varphi_n^r \leq C\,\|f\|_{L^1(\Omega)}\,, \tag{6.7}$$

for every $q < \frac{2r}{r+1}$.

On the other hand, since both $\{u_n^r\}$ and $\{\varphi_n^r\}$ are bounded in $L^1(\Omega)$, as it is $\{F_n\}$, classical elliptic results (see for example [5], [6]) yield that both $\{u_n\}$ and $\{\varphi_n\}$ are bounded in $W^{1,q}_0(\Omega)$, for every $q < \frac{N}{N-1}$. Therefore, comparing the exponents, we have the following, depending on the values of r:

- if $1 < r < \frac{N}{N-2}$, then both $\{u_n\}$ and $\{\varphi_n\}$ are bounded in $W^{1,q}_0(\Omega) \cap L^s(\Omega)$, for every $q < \frac{N}{N-1}$, and every $s < \frac{N}{N-2}$;
- if $r \geq \frac{N}{N-2}$, then both $\{u_n\}$ and $\{\varphi_n\}$ are bounded in $W^{1,q}_0(\Omega) \cap L^r(\Omega)$, for every $q < \frac{2r}{r+1}$.

Thus, there exist u and φ, such that u_n and φ_n weakly converge, up to subsequences, to u and φ respectively, in the Sobolev and Lebesgue spaces as above, as well as almost everyhwere in Ω.

From the boundedness of u_n in $L^r(\Omega)$ it follows that

$$\lim_{h\to+\infty} |\{u_n \geq h\}| = 0\,, \qquad \text{uniformly in } n \text{ in } \mathbb{N},$$

which then, since f belongs to $L^1(\Omega)$, implies that

$$\lim_{h\to+\infty} \int_{\{u_n \geq h\}} f = 0\,, \qquad \text{uniformly in } n \text{ in } \mathbb{N}. \tag{6.8}$$

Going back to (6.3) written for u_n we have, since $F_n \leq f$, and since $T_k(u_n) = k$ where $u_n \geq k$,

$$\int_{\{u_n \geq k\}} u_n^r \leq \int_\Omega u_n^r \frac{T_k(u_n)}{k} \leq C \int_\Omega F_n \frac{T_k(u_n)}{k} \leq C \int_\Omega f \frac{T_k(u_n)}{k} \tag{6.9}$$

Let now $0 < \delta < 1$; then

$$\int_\Omega f \frac{T_k(u_n)}{k} = \int_{\{u_n \leq k\,\delta\}} f \frac{T_k(u_n)}{k} + \int_{\{u_n \geq k\,\delta\}} f \frac{T_k(u_n)}{k} \leq \delta \int_\Omega f + \int_{\{u_n \geq k\,\delta\}} f\,.$$

Using (6.8), we have that

$$\lim_{k \to +\infty} \int_{\{u_n \geq k\,\delta\}} f = 0\,, \qquad \text{uniformly in } n \text{ in } \mathbb{N},$$

so that

$$\limsup_{k \to +\infty} \int_\Omega f \frac{T_k(u_n)}{k} \leq \delta \int_\Omega f\,, \qquad \text{uniformly in } n \text{ in } \mathbb{N}.$$

Since δ is arbitrary,

$$\lim_{k \to +\infty} \int_\Omega f \frac{T_k(u_n)}{k} = 0\,, \qquad \text{uniformly in } n \text{ in } \mathbb{N}.$$

Using this fact, let $\varepsilon > 0$, and let k be such that

$$\int_\Omega f \frac{T_k(u_n)}{k} \leq \varepsilon\,, \qquad \forall n \in \mathbb{N}\,.$$

Therefore, thanks to (6.9), and if E is a measurable subset of Ω with $|E| < k^{-r}\,\varepsilon$, we have

$$\int_E u_n^r = \int_{E \cap \{u_n \leq k\}} u_n^r + \int_{E \cap \{u_n \geq k\}} u_n^r \leq k^r |E| + C \int_\Omega f \frac{T_k(u_n)}{k} \leq (C+1)\varepsilon\,, \quad \forall n \in \mathbb{N}\,.$$

From the previous inequality, it follows that the sequence $\{u_n^r\}$ is uniformly equi-integrable; since it is almost everywhere convergent to u^r, by Vitali theorem we have that

$$u_n^r \text{ strongly converges to } u^r \text{ in } L^1(\Omega).$$

Since $\varphi_n \leq u_n$, we have that u_n^r dominates the sequences φ_n^r, $u_n\,\varphi_n^{r-1}$ and $\varphi_n\,u_n^{r-1}$, which are thus strongly convergent in $L^1(\Omega)$ to their respective limits by the generalized Lebesgue theorem. These convergences, together with the weak convergence of u_n to u in $W_0^{1,q}(\Omega)$, for every $q < \frac{2r}{r+1}$, and of φ_n to φ in the same spaces, allow to pass to the limit in the distributional formulation of the two equations of (1.1):

$$\int_\Omega M(x)\nabla u_n \cdot \nabla w + r \int_\Omega \varphi_n\,u_n^{r-1} w = \int_\Omega F_n\,w + \int_\Omega \varphi_n^r w\,, \qquad \forall w \in C_0^1(\Omega)\,,$$

and

$$\int_\Omega M(x)\nabla \varphi_n \cdot \nabla w + r \int_\Omega u_n\,\varphi_n^{r-1} w = \int_\Omega u_n^r w\,, \qquad \forall w \in C_0^1(\Omega)\,,$$

to have that

$$\int_\Omega M(x)\nabla u \cdot \nabla w + r \int_\Omega \varphi\,u^{r-1}\,w = \int_\Omega f\,w + \int_\Omega \varphi^r\,w\,, \qquad \forall w \in C_0^1(\Omega)\,,$$

and

$$\int_\Omega M(x)\nabla \varphi \cdot \nabla w + r \int_\Omega u\,\varphi^{r-1} w = \int_\Omega u^r w\,, \qquad \forall w \in C_0^1(\Omega)\,,$$

as desired. □

7. Measure data

In this section we briefly deal with existence and nonexistence of solutions for system (1.1) if the datum f is a bounded positive Radon measure, instead of a function in $L^1(\Omega)$.

We begin with the existence result, which holds if r is "small".

THEOREM 7.1. *Let* $1 < r < \frac{N}{N-2}$, *and let* $\mu \geq 0$ *be a bounded Radon measure on* Ω. *Then there exist distributional solutions* u *and* φ *of* (1.1) *with datum* $f = \mu$. *Furthermore,* u *and* φ *belong to* $W_0^{1,q}(\Omega)$, *for every* $q < \frac{N}{N-1}$.

Proof. As for the case of $L^1(\Omega)$ data, we reason by approximation; let $\{f_n\}$ be a sequence of positive $L^\infty(\Omega)$ functions which converges to μ in the weak$*$ topology of measures, and let u_n and φ_n be the solutions of (1.1) with data f_n. Then, as in the proof of Theorem 1.1 in the case $m = 1$, we have that

$$\|u_n\|^q_{W_0^{1,q}(\Omega)} + \|\varphi_n\|^q_{W_0^{1,q}(\Omega)} \leq C\,\|f_n\|_{L^1(\Omega)} \leq C\,,$$

since the sequence $\{f_n\}$ is bounded in $L^1(\Omega)$. Therefore, up to subsequences, u_n and φ_n converge to some functions u and φ weakly in $W_0^{1,q}(\Omega)$, for every $q < \frac{N}{N-1}$, and, thanks to Rellich theorem, strongly in $L^s(\Omega)$ for every $s < \frac{N}{N-2}$. In particular, u_n and φ_n strongly converge to u and φ in $L^r(\Omega)$, so that u_n^r, φ_n^r, $\varphi_n\,u_n^{r-1}$ and $u_n\,\varphi_n^{r-1}$ strongly converge in $L^1(\Omega)$ to their respective limits.

These convergences are enough in order to pass to the limit in the distributional formulation of the two equations in (1.1) with data f_n, so that u and φ are distributional solutions of (1.1) with datum μ, as desired. □

We prove now that if r is "large", and the datum μ is the Dirac delta concentrated at a point x_0 in Ω, then existence of solutions for (1.1) fails.

THEOREM 7.2. *Let* $r > \frac{N}{N-2}$, *let* x_0 *in* Ω, *and let* $\{f_n\}$ *be a sequence of positive* $L^\infty(\Omega)$ *functions which converges to the Dirac delta concentrated at* x_0, *that is,*

$$\lim_{n\to+\infty} \int_\Omega f_n\,\psi = \psi(x_0)\,, \qquad \forall \psi \in C^0(\overline{\Omega})\,.$$

Let u_n *and* φ_n *be the solutions of* (1.1) *with data* f_n. *Then* u_n *and* φ_n *weakly converge to zero in* $W_0^{1,q}(\Omega)$, *for every* $q < \frac{2r}{r+1}$, *and there is no solution obtained by approximation of* (1.1) *if the datum is the Dirac delta concentrated at* x_0.

Proof. Recalling (6.7), we have that $\{u_n\}$ and $\{\varphi_n\}$ are bounded in $W_0^{1,q}(\Omega)$, for every $q < \frac{2r}{r+1}$, so that, up to subsequences, they weakly converge to u and φ in the same spaces. We recall that $\{x_0\}$ has zero $W_0^{1,p}$-capacity for every $1 < p < N$ (see e.g. [10]); since $\frac{2r}{r-1} < N$ by the assumptions on r, $\{x_0\}$ has zero s-capacity, for some $s > \frac{2r}{r-1}$. Therefore (see [10]), for every $\delta > 0$ there exists a function ψ_δ in $C_0^\infty(\Omega)$, such that

$$\int_\Omega |\nabla\psi_\delta|^s \leq \delta\,, \qquad 0 \leq \psi_\delta \leq 1\,, \qquad \psi_\delta(x_0) = 1\,. \tag{7.1}$$

We now choose $T_1(u_n)\,(1-\psi_\delta)$ as test function in the first equation of (1.1). We obtain

$$\begin{aligned}&\int_\Omega M(x)\nabla u_n\cdot\nabla T_1(u_n)\,(1-\psi_\delta)-\int_\Omega M(x)\nabla u_n\cdot\nabla\psi_\delta\,T_1(u_n)\\&\qquad+r\int_\Omega\varphi_n\,u_n^{r-1}\,T_1(u_n)\,(1-\psi_\delta)=\int_\Omega f_n\,T_1(u_n)\,(1-\psi_\delta)+\int_\Omega\varphi_n^r\,T_1(u_n)\,(1-\psi_\delta)\,.\end{aligned}$$

Recalling that $\varphi_n\le u_n$, we have that

$$\int_\Omega\varphi_n^r\,T_1(u_n)\,(1-\psi_\delta)\le\int_\Omega\varphi_n\,u_n^{r-1}\,T_1(u_n)\,(1-\psi_\delta)\,,$$

so that we have, also using (1.2) and that u_n and φ_n are positive,

$$\begin{aligned}&\alpha\int_\Omega|\nabla T_1(u_n)|^2(1-\psi_\delta)+(r-1)\int_\Omega\varphi_n\,u_n^{r-1}\,T_1(u_n)\,(1-\psi_\delta)\\&\qquad\le\int_\Omega f_n\,(1-\psi_\delta)+\int_\Omega M(x)\nabla u_n\cdot\nabla\psi_\delta\,T_1(u_n)\,.\end{aligned}\tag{7.2}$$

We now recall that $T_1(u_n)$ is bounded in $W_0^{1,2}(\Omega)$ (see (6.6)). Therefore, up to subsequences, it weakly converges in $W_0^{1,2}(\Omega)$ to $T_1(u)$. Passing to the limit in (7.2) as n tends to infinity, and dropping a positive term, we therefore obtain

$$\alpha\int_\Omega|\nabla T_1(u)|^2(1-\psi_\delta)\le(1-\psi_\delta(x_0))+\int_\Omega M(x)\nabla u\cdot\nabla\psi_\delta\,T_1(u)\,.$$

Recalling that $\psi_\delta(x_0)=1$, we thus have that

$$\alpha\int_\Omega|\nabla T_1(u)|^2(1-\psi_\delta)\le\int_\Omega M(x)\nabla u\cdot\nabla\psi_\delta\,T_1(u)\,.$$

Now we let δ tend to zero. Since $|\nabla u|$ belongs to $L^q(\Omega)$ for every $q<\frac{2r}{r+1}$, and $|\nabla\psi_\delta|$ tends to zero in $L^s(\Omega)$ for some $s>\frac{2r}{r-1}=\left(\frac{2r}{r+1}\right)'$, we have that

$$\lim_{\delta\to0^+}\int_\Omega M(x)\nabla u\cdot\nabla\psi_\delta\,T_1(u)=0\,,$$

which implies that

$$0\le\alpha\int_\Omega|\nabla T_1(u)|^2\le\lim_{\delta\to0^+}\alpha\int_\Omega|\nabla T_1(u)|^2(1-\psi_\delta)\le0\,.$$

Thus $T_1(u)=0$, which implies that $u=0$. We have therefore proved that u_n tends to zero. Since $0\le\varphi_n\le u_n$, we have that φ_n tends to zero too. Clearly, the functions $u=0$ and $\varphi=0$ solve problem (1.1) with datum $f=0$, and do not solve problem (1.1) with datum the Dirac delta concentrated at x_0: such a problem does not therefore have a solution obtained by approximation. □

Remark 7.3. As a consequence of the fact that u_n and φ_n converge to zero, one also has

$$\lim_{n\to+\infty}\int_\Omega[r\,\varphi_n\,u_n^{r-1}-\varphi_n^r]\,\psi=\psi(x_0)\,,\qquad\forall\psi\in C_0^\infty(\Omega)\,.$$

In other words, the Dirac delta disappears as n tends to infinity, since it is "hidden" in the lower order terms.

8. Saddle points of J

If f belongs to $L^m(\Omega)$, with $m \geq \frac{2N}{N+2}$, then there exist solutions u and φ of (1.1), with both u and φ in $W_0^{1,2}(\Omega) \cap L^{r\,m}(\Omega)$. We are going to show that, in this case, (u, φ) is a saddle point of the functional J defined in the introduction, in the sense that

$$J(u, \psi) \leq J(u, \varphi) \leq J(v, \varphi)\,,$$

for every v and ψ in $W_0^{1,2}(\Omega)$ such that

$$\int_\Omega \varphi\,|v|^r \text{ and } \int_\Omega u\,|\psi|^r \text{ are finite.} \tag{8.1}$$

Note that since u^r, φ^r, $\varphi\,u^{r-1}$ and $u\,\varphi^{r-1}$ belong to $L^m(\Omega)$, the assumption $m \geq \frac{2N}{N+2}$ implies that

$$\int_\Omega u^r\,\psi\,, \int_\Omega \varphi\,u^{r-1}\,v\,, \int_\Omega u\,\varphi^{r-1}\,\psi \text{ and } \int_\Omega \varphi^r\,v \text{ are finite for every } v,\ \psi \text{ in } W_0^{1,2}(\Omega),$$

so that both $J(u, \psi)$ and $J(v, \varphi)$ are well defined if (8.1) holds.

Recall that u is a solution of

$$-\text{div}(M(x)\nabla u) + r\,\varphi\,u^{r-1} = f + \varphi^r\,.$$

Choosing $u - v$ as test function, with v in $W_0^{1,2}(\Omega)$ such that (8.1) holds, we have that

$$\int_\Omega M(x)\nabla u \cdot \nabla(u - v) + r\int_\Omega \varphi\,u^{r-1}\,(u - v) = \int_\Omega f\,(u - v) + \int_\Omega \varphi^r\,(u - v)\,. \tag{8.2}$$

We observe now that we have, since M is uniformly elliptic,

$$\begin{aligned} \int_\Omega M(x)\nabla u \cdot \nabla(u - v) &= \frac{1}{2}\int_\Omega M(x)\nabla u \cdot \nabla u - \frac{1}{2}\int_\Omega M(x)\nabla v \cdot \nabla v \\ &\quad + \frac{1}{2}\int_\Omega M(x)\nabla(u - v) \cdot \nabla(u - v) \\ &\geq \frac{1}{2}\int_\Omega M(x)\nabla u \cdot \nabla u - \frac{1}{2}\int_\Omega M(x)\nabla v \cdot \nabla v\,. \end{aligned} \tag{8.3}$$

On the other hand, since $t \mapsto |t|^r$ is convex, and $u \geq 0$, we have

$$|v|^r \geq |u|^r + r\,|u|^{r-2}\,u\,(v - u) = u^r + r\,u^{r-1}\,(v - u)\,,$$

which is equivalent to

$$r\,u^{r-1}\,(u - v) \geq u^r - |v|^r\,.$$

Therefore, since φ is positive, we have

$$r\int_\Omega \varphi\,u^{r-1}\,(u - v) \geq \int_\Omega \varphi\,u^r - \int_\Omega \varphi\,|v|^r\,, \tag{8.4}$$

with the last integral being finite thanks to (8.1). Recalling (8.2), we thus have from (8.3) and (8.4) that

$$\begin{aligned} &\frac{1}{2}\int_\Omega M(x)\nabla u \cdot \nabla u + \int_\Omega \varphi\,u^r - \int_\Omega u\,\varphi^r - \int_\Omega f\,u \\ &\quad \leq \frac{1}{2}\int_\Omega M(x)\nabla v \cdot \nabla v + \int_\Omega \varphi\,|v|^r - \int_\Omega v\,\varphi^r - \int_\Omega f\,v\,, \end{aligned}$$

for every v in $W_0^{1,2}(\Omega)$ such that (8.1) holds. Thus, $J(u,\varphi) \leq J(v,\varphi)$ for every v in $W_0^{1,2}(\Omega)$ such that (8.1) holds. A similar technique yields that $J(u,\psi) \leq J(u,\varphi)$ for every ψ in $W_0^{1,2}(\Omega)$ such that (8.1) holds, as desired.

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